PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

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OF

EDINBURGH.

VOL. XL.

1919-1920.

EDINBURGH:
PRINTED BY NEILL AND COMPANY, LIMITED.
MDCCCCXXI.

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OF THE

ROYAL SOCIETY OF EDINBURGH.

SESSION 1919–20

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EDINBURGH:

PUBLISHED BY ROBERT GRANT & SON, 107 PRINCES STREET, AND
WILLIAMS & NORGATE, 14 HENRIETTA STREET, COVENT GARDEN, LONDON, W.C. 2.

Price Eleven Shillings.

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PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. XL. 1919–20.

I.—Notices of Fellows, Honorary and Ordinary, recently deceased. By The General Secretary.

Emil Fischer—the distinguished chemist—was born at Euskirchen on October 9, 1852. He was educated at the Bonn Gymnasium and afterwards at the Universities of Bonn and Strassburg. After occupying the Chairs of Chemistry at Munich, Erlangen, and Würzburg, he finally settled in Berlin where the greater part of his far-reaching work in chemistry was done. He received the Davy Medal in 1890 from the Royal Society of London, and was awarded the Nobel Prize in 1902.

He was elected an Honorary Fellow of the Royal Society of Edinburgh in 1908, and died on July 18, 1919.

Edward Charles Pickering was born in Boston on July 19, 1846. He was educated at the Boston Latin School and at Harvard University, where, after graduation, he was made an instructor in mathematics. Two years later he was appointed Thayer Professor of Physics at the Massachusetts Institute of Technology. He was the first to establish a physical laboratory in the United States, and rapidly gained a high reputation as a scientific worker. His career may be said to have begun in 1869 when he accompanied the Nautical Almanac Party to observe a total solar eclipse. In 1876 he was appointed the Director of the Harvard Observatory, and a year later moved with the observatory to Cambridge, Mass., and assumed the duties which were to be his for forty-two years. Much of the work he began here has become classical, especially photographic magnitudes, classification of spectra, and discussion of variable stars. More than eighty volumes of the *Annals of the Harvard Observatory* were prepared and distributed under his directorate.

He was elected an Honorary Fellow of the Royal Society of Edinburgh in 1916, and died on February 3, 1919.

Lord Rayleigh, John William Strutt, P.C., D.C.L., LL.D., F.R.S., was born on November 12, 1842, and graduated Senior Wrangler and Smith's Prizeman in 1865. In 1879 he succeeded Clerk Maxwell as Professor of Experimental Physics in Cambridge. He retired from this Chair in 1884. For nine years—1887–96—he acted as Secretary of the Royal Society of London, and subsequently became its President (1905-08). He was also Professor of Natural Philosophy at the Royal Institution from 1887–1905. Among his other honours are: Order of Merit, the Nobel Laureateship, and Officer Legion of Honour. His chief publication was the *Theory of Sound*, in two volumes, and his numerous scientific papers, elucidating many difficult points in physics and always advancing scientific knowledge, have been recently republished in five volumes.

He was elected an Honorary Fellow of our Society in 1886, and died on June 30, 1919.

M. Gustaf Retzius, anatomist and anthropologist, was born at Stockholm on October 17, 1842. He was educated at the Stockholm Gymnasium, the University of Upsala, and the Caroline Medico-Chirurgical Institute, Stockholm. His father—Anders Retzius—was also a famous anatomist and preceded his son in the Chair of Anatomy in the Caroline Medico-Chirurgical Institute, Stockholm. Gustaf Retzius worked out by approved methods lines of research commenced by his father, especially in regard to the description of heads and skulls. In 1902 he published, along with his colleague Professor Karl Fürst, an exhaustive work on the anthropology of Sweden.

He was elected an Honorary Fellow of the Royal Society of Edinburgh in 1908, and died on July 21, 1919.

John Mackay Bernard, of Dunsinnan, B.Sc., was born in 1857, and although marked out for a commercial career and large business responsibilities he studied chemistry and natural sciences in the University of Edinburgh, and graduated B.Sc. in 1887. He was very much interested in all applications of physical and mathematical science, and was especially prominent as a strong supporter of the Ben Nevis Observatory. During the last few years of the existence of the High-Level Observatory on

Ben Nevis Mr Mackay Bernard contributed no less than £2000 to its upkeep. He was an active member of the Meteorological Society, and filled the office of President from 1912–15.

He was elected a Fellow of the Royal Society of Edinburgh in 1887, and died on April 19, 1919.

Mr Thomas Fairley was born in Glasgow in 1843, and received the greater part of his education at Edinburgh. He displayed a remarkable aptitude for science, especially in relation to chemistry, which he had the good fortune to study under the late Lord Playfair, who was then Professor of Chemistry at Edinburgh University. After acting as assistant to Professor Playfair for some years, he became lecturer in chemistry at the Leeds School of Medicine, and science master at the Leeds Grammar School. He gradually devoted himself to the general practice of an analytical and consulting chemist. In 1873 he was appointed public analyst for Leeds and the North Riding, and chemist to the Yorkshire Agricultural Society. Many honorary yet important offices were filled by Mr Fairley, among them being that of the president of the Leeds Naturalists' Field Association. He served on different committees of the Leeds Institute of Science, Art, and Literature, and shortly after the transfer of the educational work of that institution to the Leeds Corporation he became chairman, the presidency being accepted by the late Lord Airedale. During his chairmanship of the Institute he was elected Chairman of the Council of the Association of Technical Institutions of Great Britain and Ireland for the year 1908.

One of the early members of the Institute of Chemistry of Great Britain and Ireland, he served at various times as a member of the Council and as one of the examiners of candidates for the Institute's Diploma of Associateship. Another scientific body in which he took special interest was the Society of Chemical Industry, of the Council of which he was a member, as well as chairman and secretary of the Yorkshire section of the society, resigning the secretaryship last year. He became a Fellow of the Royal Society of Edinburgh in 1875.

Mr Fairley's lectures on scientific subjects were always lucid and informing, while his contributions to scientific publications were valuable records. He wrote for the journal of the Chemical Society and the journal of the Society of Chemical Industry, and was the author of articles in Thorpe's new *Dictionary of Applied Chemistry*. His active interest in the British Association extended over a long period, and he frequently contributed papers or took part in the discussions, particularly in the physics or chemistry section.

Mr Fairley had been in failing health for some time before his death, which took place at his residence, Newton Grove, Chapeltown, Leeds, on February 21, 1919.

W. S. Greenfield, M.E., F.R.C.P.E., LL.D., was born at Salisbury in 1846, and graduated M.B. and B.S. of London University in 1872. In 1874 he became Demonstrator in Morbid Anatomy and Pathology at St Thomas's Hospital, and in 1878 succeeded Dr Burdon Sanderson in the Chair of Pathology at the Brown Institution, University of London. His investigations in human and comparative pathology had already established his reputation as an original investigator when he was appointed Professor of Pathology in the University of Edinburgh in 1881. Here he greatly raised the standard of pathological teaching, both in systematic lectures and in his clinical work.

He was elected a Fellow of the Royal Society of Edinburgh in 1886, and served on its Council from 1887–90. He retired from his active work in 1912, and after a well-earned rest at his house at Elie, died on August 12, 1919.

[*Contributed by* Mr D. B. Dott.]

After the usual course of education Mr William Lamond Howie served an apprenticeship in pharmacy, and subsequently acted as assistant to a medical practitioner in Glasgow. He studied chemistry under Dr Penny in the old Andersonian College, and passed the major examination of the Pharmaceutical Society in 1870. He contributed several papers on pharmaceutical subjects at the evening meetings of the Pharmaceutical Society in Edinburgh, and also at the annual meetings of the British Pharmaceutical Conference. Although originally devoted to laboratory work, his business ability was soon discovered, so that he became more and more absorbed in the commercial side of affairs, finding little time for experimental work. Yet, in the midst of business preoccupation, he found time to invent and patent a system of railway fencing which was the subject of a paper contributed to the Society of Arts. Mr Howie became an expert photographer. Many of his pictures have been reproduced in standard works on geology and geography, and he was a contributor in this department to the *Proceedings of the British Association*. In recreation his favourite pastime was mountaineering, so that it naturally evolved, by a combination of artistic photography and experience in wandering among beautiful scenery, that his lectures on "Our Scottish Alps" and "The Swiss Alps" were in great request. In 1899 Mr Howie removed permanently to London, where he was actively engaged down to the day of his death in the direction of the affairs of the extensive drug business with which he was latterly identified. Mr Howie was endowed with a charming personality, and was endeared to a large circle of friends. He was elected a Fellow of the Chemical Society in 1876, and of the Royal Society of Edinburgh in 1899. Mr Howie suddenly died of heart failure on the morning of December 17, 1918, in the seventy-third year of his age.

George William Jones, M.A., B.Sc., LL.B., was born in Dundee on November 17, 1878, and was educated in Morgan Academy. In 1897 he was appointed to a University bursary at Edinburgh, and in that year he began a most distinguished career as student of mathematics and natural philosophy. In due course he graduated with first-class honours in these subjects, and also as B.Sc. in mathematics, natural philosophy, and chemistry. He also gained the Baxter Scholarship in the University and the first MacLaren Scholarship in the Normal Training College. He organised, along with Mr Laing, the Scottish Tutorial and Educational Institute, with headquarters in Edinburgh and an important branch in Glasgow. In May 1917 he was appointed sub-lieutenant in the R.N.V.R., and in August 1918 entered the meteorological staff of the R.A.F. with the rank of captain. He was carrying on important investigations in this capacity when he succumbed to an attack of pneumonia.

He was elected a Fellow of the Royal Society of Edinburgh in 1905, and died on November 4, 1918.

Andrew King, M.A., F.I.C., who died on February 19, 1919, at the age of 56, was well known in scientific circles in Edinburgh as an able analytical chemist. He was on the staff of the Heriot-Watt College, and was for many years in charge of J. Y. Buchanan's private laboratory in Edinburgh. He was co-author of an important paper on "The Temperatures, Specific Gravities, and Salinities of the Weddell Sea and of the North and South Atlantic Ocean."

He was elected a Fellow of the Royal Society of Edinburgh in 1917.

The Right Hon. Sir John Hay Athole Macdonald, P.C., K.C., G.C.B., LL.D., was born in 1836, and received his education at the Edinburgh Academy and at the Universities of Edinburgh and Basle. In 1859 he was enrolled as an advocate, and during his early years of practice at the bar published a useful work on Criminal Law and Procedure. Always strongly Conservative in politics, his opportunity came with Disraeli's return to power in 1874, subsequently filling the important offices of

Solicitor-General and Lord-Advocate (1885). In 1887 he carried the Act for the Amendment of Criminal Procedure in Scotland, an important measure on which his fame as a Lord-Advocate chiefly rests. In 1888 he succeeded Lord Moncrieff as one of the Lords of Session, and his political career automatically came to an end. As Lord Justice-Clerk he assumed the title of Lord Kingsburgh, by which name he was known till 1915, when, on retirement from his judicial duties, he became familiarly known again as Sir John Macdonald.

Sir John was a man of many interests, two of which call for special notice. He was keenly interested in all that pertains to the art of war, being himself not merely an enthusiastic volunteer but one who by his suggestions and writings had an important influence on the methods of training in the regular Army. For many years he captained the Scottish Twenty Team in the National Challenge Trophy Competition, and was also Captain of the British Rifle Team which took part in an international match at the Philadelphia Centenary Exhibition. He filled the highest offices in the Volunteer Service, received the Volunteer Decoration in 1892; and on the occasion of Queen Victoria's Diamond Jubilee was awarded Commemoration Medals as Adjutant-General of the Royal Company of Archers and as Brigadier-General in command of the troops on duty at the Royal Procession. He was the author of numerous books on drill and infantry tactics.

Sir John was also greatly interested in applied science, especially in the development of motor traction, and in all things connected with the use of electrical power. As a Member of the Institution of Electrical Engineers he took an active part in the work of that Society, and was himself an inventor of several ingenious applications. He early divined the importance of motor traction, and was one of the first persons in this country to possess and to drive his own motor car—its number was S 1. He was President of the Scottish Automobile Club; and so convinced was he of the growing importance of the motor vehicle that he very early argued for the disuse of all tramway systems, whether cable or electrical, and for the substitution in their place of the motor bus.

As one of the Vice-Presidents of the Royal Society of Edinburgh, he was a regular attendant at the meetings, and his scientific and legal knowledge proved of great value in the deliberations and decisions of the Council.

Sir John Macdonald was a delightful companion, full of human interest, with a fine humour and large repertoire of racy stories. Many of these will be found in his *Life Jottings of an Old Edinburgh Citizen* (1915), a charming example of the best type of light literature. Under the pseudonym of "Jean Jambon" (John H.A.M.) he published in the late seventies *Our Trip to Blunderland*—a book written for the amusement of his own boys and frankly modelled on the classics of Lewis Carroll. The Scottish bench is rich in its memory of great lawyers who well knew the Horatian maxim, *Dulce est desipere in loco*. Sir John may not be ranked with the greatest of these; but few could show greater versatility in the multiplicity of interests and in the faculty for pure enjoyment of life.

Sir John Macdonald retained his faculties in full activity to the end of his long life, having visited London on Committee work only the week before his last brief illness. He died in Edinburgh on May 9, 1919, on the very day on which the freedom of his native city was to have been conferred upon him in recognition of his many services to his country and to his King.

R. C. Maclagan, M.D., F.R.C.P.E., was born in Edinburgh in 1839, and was the son of the late Sir Douglas Maclagan, Professor of Medical Jurisprudence in the Edinburgh University. He was educated at the High School, the University of Edinburgh, and at Berlin and Vienna. He practised as a physician in Edinburgh for a number of years, but finally became an active partner in the firm of A. B. Fleming & Co. of Edinburgh and London, and was chairman for over thirty years. He was a keen volunteer and gained two silver cups for shooting. He was the author of several books on folk-lore and anthropological subjects, and communicated to our Society a paper on "Two Historical Fallacies: Heather Beer and Uisge Beithe." He was elected a Fellow of the Royal Society of Edinburgh in 1869, and died on July 12, 1919.

R. Sydney Marsden, M.D., C.M., D.Sc., D.P.H., M.R.I.A., F.I.C., was the son of the late Robert Marsden, manufacturer, Sheffield. He was born at Sheffield in 1856, and was educated at Wesley College in that city, and also at the Universities of Edinburgh, Göttingen, Berlin, and Paris. In 1879 he was Lecturer on Chemistry at University College, Bristol. He was elected President of the Royal Medical Society in 1881, and went to Birkenhead in 1891, where he was thrice President of the Birkenhead Medical Society. Deeply interested in research work, he was the first person to produce the artificial diamond. In order to continue original research, he received grants from the Government Research Fund and the Royal and Medical Societies of London. He was called upon to give evidence before several Royal Commissions and Parliamentary Committees. An acknowledged authority on all matters dealing with public health, as Medical Officer of Birkenhead he was responsible for many improvements affecting the health of the borough. He revolutionised the system of sewer ventilation in the town, and was instrumental in securing the substitution of setts for the round boulders with which many years ago several thoroughfares were paved. He leaves a widow and daughter.

He was elected a Fellow of the Royal Society of Edinburgh in 1880, and died of pneumonia on March 8, 1919.

Sir Mitchell Mitchell-Thomson, Bart., was born in 1846 at Alloa, where he received his early education, proceeding afterwards to the Edinburgh Institution. After a business training of four years he established, along with his brothers, the firm of Messrs Mitchell-Thomson & Co., timber merchants, Granton, Edinburgh. In 1890 he entered the Town Council of Edinburgh, in the affairs of which he took a prominent part, devoting himself with keen energy and public spirit to all questions connected with the welfare of the city. He served as Lord Provost from 1897 to 1900, and was elected a Fellow of the Royal Society of Edinburgh in 1899. In 1900 he was created a Baronet, and was also a Knight of Grace of the Order of St John of Jerusalem. He died in Edinburgh on November 15, 1918.

Frederick Phillips, M.Sc., was born in 1879 at Woolwich, where he was educated, and in due course entered the Woolwich Arsenal in the Engineering Department. He determined, however, to follow the teaching profession, and after gaining a King's Scholarship at Peckham Pupil Teachers' School, he entered Boro Road Training College, Isleworth, and subsequently attended the Royal College of Science, South Kensington. He graduated M.Sc. in 1917, and after a year and a half spent as Chemistry Master in Ilford Secondary School he was appointed Lecturer in Pure and Applied Mathematics in University College, Southampton. He was a skilled mechanic and engineer, and was much interested in wireless telegraphy. During the war he carried out important investigations in the tracking of aeroplanes. He was elected a Fellow of the Royal Society of Edinburgh in 1917, and died at Southampton on October 14, 1918.

Sir Boverton Redwood, Bt., D.Sc. (Hon.), F.I C., F.C.S., A.Inst.C.E., was born in London, April 26, 1846, and was educated at the University College School. He was trained as a pharmaceutical chemist, but turned his attention early to the production and use of petroleum. From 1886, when he delivered the Cantor Lectures on Petroleum and its Products, he was prominent in all public questions regarding petroleum and connected subjects, was the author of many articles and books relating to the subject, and served on several important Commissions.

He was elected a Fellow of the Royal Society of Edinburgh in 1889, and was also a member of many other societies in pure and applied science. He was knighted in 1905, and died on June 4, 1919.

Edwin O. Sachs, Architect, A.Inst.M.E., F.R.G.S., was born in London on April 5, 1870. He was Chairman of the British Fire Prevention Committee, which he founded in 1897. The first fire-testing station established in Europe was also founded by him in 1899. He organised the International Fire Prevention Congress in London in 1903, as also the Special Fire Survey Force in connection with the war, acting himself as its commissioner. He was Vice-President of the International Fire Service Council and Hon. Member of the Imperial Russian Society of Architects. Many foreign countries bestowed their honours upon him, such as France, Italy, Belgium, and Russia, and France also made him an Officier d'Académie. He was the author of numerous articles on Fire and Fire Preventions, etc.

He was elected a Fellow of the Royal Society of Edinburgh in 1904, and died in London on September 9, 1919.

Sir James Sawyer, Kt., M.D., F.R.C.P., F.S.A., J.P., consulting physician to Queen's Hospital, Birmingham, was born at Carlisle on August 11, 1844. He was educated at Queen's College, Birmingham, and at London University. In 1875 he became Professor of Pathology in Birmingham ; in 1878 Professor of Materia Medica, and in 1885 Professor of Medicine. He was an authority on the diseases of the lungs, heart, and kidneys, and published many papers in medical periodicals. He was knighted in 1885 ; elected a Fellow of the Royal Society of Edinburgh in 1891, and died on January 27, 1919.

E. Wynston Waters was a student of medicine in Edinburgh and gained his diploma at the Royal College of Surgeons in the early eighties. He spent the greater part of his life in South Africa engaged in medical practice, and died in 1918 while employed in medical work in the army.

He was elected a Fellow of the Royal Society of Edinburgh in 1907.

(*Issued separately January* 31, 1920.)

II.—The Cooling of the Soil at Night, with Special Reference to Late Spring Frosts (II). By Captain T. Bedford Franklin, B.A. (Cantab.). *Communicated by* The General Secretary.

(MS. received September 26, 1919. Read November 3, 1919.)

SUMMARY.

I. Introduction.*

In my previous paper to the Society on the same subject I came to the conclusion that the temperature of the surface of open cultivated soil fell rapidly at the beginning of a calm clear night until it was such a number of degrees below the temperature at the 4-in. depth as to make the upward conduction from that depth to the surface balance the radiation. After this stage was reached the surface and 4-in. temperatures fell at the same rate.

If, therefore, the temperatures of the surface and 4-in. depth and the conductivity of the layer of soil between the 4-in. depth and the surface were known from readings of electrical resistance thermometers,

* The temperatures throughout this paper are in degrees centigrade.

and the rate of radiation was calculated from the value of the relative humidity, I suggested that it might be possible to forecast the minimum soil temperature for a calm clear night as early as the previous afternoon.

My observations have now extended over a sufficient period to enable me to give a formula for forecasting the minimum surface-soil temperature and to compare the minima, so forecasted, with the observed minima on a number of calm clear nights.

The observations given in my previous paper were made during the winter of 1918–19, and I assumed that the conductivity of the soil was uniformly of the value $K = \cdot 004$. This, of course, could only be justified if the soil was constantly wet after frequent rain—a state of affairs that actually did exist, as can be seen from the following extracts from the Monthly Weather Report of the Meteorological Office:—

November 1918 . . .	14 days of precipitation.
December 1918 . . .	21 " "
January 1919 . . .	24 " "
February 1919 . . .	16 " "
March 1919 . . .	22 " "

Thus during the whole period of observation the soil never had an opportunity of drying, and the value of the conductivity on any day was probably very close to the assigned average value.

The advent of fine weather in April and May 1919 made it impossible to rely any longer on an average value for the conductivity, as it was at once apparent, as soon as continuous records were kept of the readings of the electrical resistance thermometers, that the conductivity of the layer of soil between the 4-in. depth and the surface varied from day to day with the degree of wetness of the soil. As soon as a mulch of dry soil had formed on the surface, the conductivity diminished rapidly, and it appeared that the depth of this mulch determined the conductivity of the layer of soil down to the 4-in. depth.

I was thus dealing with a layer of soil which was not uniformly wet from top to bottom, and the values of the conductivity obtained were more apparent than real; to avoid this ambiguity, I preferred to employ the ratio of the ranges of temperature at the 4-in. depth and at the surface $\left(\frac{R_4}{R_0}\right)$ as my standard rather than the conductivity—the more so as this ratio is easily observed with considerable accuracy, and is an essential part of my forecast equation.

The quantities required for the forecast equation are:—

(1) The value of $\frac{R_4}{R_0}$ from minimum to maximum; this can be obtained by about 5 to 6 p.m. (see Tables I, II, III).

(2) The lag on the day in question; this may be observed by about 5 to 6 p.m., or may be found from the value of $\frac{R_4}{R_0}$ (see Table IV).

(3) The estimated relative humidity of the coming night; this requires a certain amount of practice and knowledge of local conditions, but fortunately a considerable error does not involve a correspondingly large error in the forecasted minimum.

(4) The number of degrees (θ) which the surface can fall below the 4-in. temperature before the upward conduction balances the radiation; this depends on $\frac{R_4}{R_0}$ and the relative humidity (see Table V).

(5) The probable difference between the air minimum over open-soil and the surface-soil minimum (see Table VI).

Before discussing the forecast equation, I propose to deal with these in turn. The values given in Sections II, III, IV, V are all from my own observations during the last five months—March–July 1919,—and of course are probably only strictly accurate for the particular location and soil in which the observations were made.

Though considerable alterations might be found in these values on moving from loam to clay, chalk, or sandy soils, I do not believe more than slight differences would be found to occur from place to place in the same type of soil. This is a question that requires further investigation, and I propose to make a series of observations in different types of soil this autumn; from a few observations made in September 1918 and August 1919, it appears that while the values of $\frac{R_4}{R_0}$ for loam may vary between ·44 and ·28, the values for sand may lie between ·60 and ·25, and for clay between ·41 and ·35 under similar conditions of weather.

II. The Values of $\frac{R_4}{R_0}$ in Soil of Varying Degrees of Wetness.

The value of $\frac{R_4}{R_0}$ is so essential a part of the forecast equation that it is important to know it exactly. This entails reading the minimum as well as the maximum at both the surface and 4-in. depth, a matter of no difficulty as far as the maximum is concerned, since it occurs in the afternoon, but a more serious item as regards the minimum readings, which must be made in the early morning.

With a view to obtaining an approximate value for $\frac{R_4}{R_0}$ without the strain of continuously getting up early enough to read the minima, I have tabulated a number of observed values of $\frac{R_4}{R_0}$ under varying conditions of wetness of the soil. These values, given in Tables I, II, III, should serve as a guide to obtain an estimate of $\frac{R_4}{R_0}$ with a fair degree of accuracy from the condition of the surface layer of soil and the known number of previous fine days.

(a) *Soil Wet just after Rain.*

Table I gives the values of $\frac{R_2}{R_0}$ and $\frac{R_4}{R_0}$ immediately after rain—the amount of which in the previous 24 hours is given.

Table I.—Values of $\frac{R_2}{R_0}$ and $\frac{R_4}{R_0}$.

Date.	$\frac{R_2}{R_0}$.	$\frac{R_4}{R_0}$.	Amount of Rain in previous 24 hours. Millimetres.
1919.			
April 11, 12, 13, 14, 15	·66	·42	1 to 3 mm. daily
„ 16 . . .	·66	·44	20 mm
„ 20 . . .	·65	·40	2 „
„ 29 . . .	·66	·43	7 „
„ 30 . . .	·66	·43	1 „
May 3 . . .	·64	·40	1 „
„ 4 . . .	·66	·42	2 „
„ 6 . . .	·65	·41	1 „
„ 7 . . .	·67	·43	9 „
„ 18 . . .	·66	·43	5 „
June 4 . . .	...	·44	3 „
„ 5 . . .	...	·41	2 „
„ 8 . . .	...	·42	3 „
„ 10 . . .	...	·43	6 „
„ 11 . . .	...	·41	10 „
„ 12 . . .	...	·40	1 „
„ 13 . . .	...	·43	7 „
„ 21, 26, 30 . .	...	·40	1 to 2 mm. daily
Average value .	·66	·42	

(b) Soil drying after a Number of Fine Days.

Table II gives the value of $\frac{R_4}{R_0}$ after a number of days of fine weather; it will be seen that the values differ considerably in sunny and overcast weather.

Table II.—Values of $\frac{R_4}{R_0}$.

Date.	Weather.	Number of Previous Days without Rain.															
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1919.																	
April 17	Overcast, windy	·40															
„ 18	„		·37														
„ 19	„			·36													
May 8	Overcast	·41															
„ 9	„		·40														
„ 10	Sunny			·38													
„ 11	„				·37												
„ 12	„					·36											
„ 13	„						·34										
„ 14	„							·32									
„ 15	Foggy, then sunny								·32								
„ 16	„									·31							
„ 17	„			¾-in.	dry	mul	ch				·30						
„ 19	Sunny	·40															
„ 20	„		·37														
„ 21	Overcast			·35													
„ 22	Sunny				·35												
„ 23	„					·32											
„ 24	„						·30										
„ 25	„							·30									
„ 26	„								·30								
„ 27	„									·29							
„ 28	„										·29						
„ 29	Foggy and overcast											·29					
„ 30	Sunny												·29				
„ 31	„													·28			
June 1	„														·28		
„ 2	„															·28	
„ 3	„					1½-	in.	dry	mu	lch							·28
„ 14	„	·40															
„ 15	„		·36														
„ 16	„			·35													
„ 17	„				·34												
„ 18	„					·32											
„ 19	Overcast, windy	¾-	in. d	ry m	ulch		·31										
July 5	Overcast	·40															
„ 6	„		·38														
„ 7	Sunny			·36													
„ 8	„				·34												
„ 9	„					·32											
„ 10	„						·31										
„ 11	„	1-	in. d	ry m	ulch			·30									

(c) *Soil Dry.*

To measure the values of $\frac{R_4}{R_0}$ and the lag in completely dry soil, I prepared a bed from soil which had been kept under cover for six months and installed thermometers at the surface and at a depth of 4 in. in this bed, which had a total depth of dry soil of 8 in.

As capillary action is very slow in dry soil, I hoped to obtain good results in the first few days of the trial, so long as the weather remained fine and no rain fell. Every day but one of the trial was sunny and no rain fell; the results are given in Table III.

Table III.—Values of $\frac{R_4}{R_0}$ and Lag.

Day.	$\frac{R_4}{R_0}$.	Lag.	Remarks.
1	·19	$5\frac{1}{2}$ hrs.	Sunny
2	·20	$5\frac{1}{2}$ "	"
3	·20	$5\frac{1}{2}$ "	Overcast, foggy
4	·25	$5\frac{1}{4}$ "	Sunny } Moisture rising by capillary action
5	·28	5 "	" }
6	·28	5 "	" }

III. Values of the Lag corresponding to Various Values of $\frac{R_4}{R_0}$.

On all days when the values of $\frac{R_4}{R_0}$ were found, the approximate value of the lag of the maximum and minimum at the 4-in. depth was observed. When $\frac{R_4}{R_0}$ had large values it was possible to read the lag to the nearest 15 minutes, but on those days when $\frac{R_4}{R_0}$ was small the temperature at the 4-in. depth arrived at and departed from its maximum and minimum values so slowly that the error in reading the lag may amount to 30 or 45 minutes.

Table IV gives the average values of the lag corresponding to various values of $\frac{R_4}{R_0}$, and shows the number of observations on which these average values are based.

Table IV.—Value of Lag in Hours.

$\frac{R_4}{R_0}=$	·44	·42	·40	·38	·36	·34	·32	·30	·28	·26	·24	·22	·20
Average value of lag in hours	$3\frac{1}{4}$	$3\frac{1}{4}$	$3\frac{1}{2}$	$3\frac{3}{4}$	4	$4\frac{1}{4}$	$4\frac{1}{4}$	$4\frac{1}{2}$	$4\frac{3}{4}$ or 5	$5\frac{1}{4}$	$5\frac{1}{4}$	$5\frac{1}{2}$	$5\frac{1}{2}$
No. of days on which average values are based	2	6	5	9	5	1	4	7	5	1	1	...	3

IV. Connection between $\frac{R_4}{R_0}$, the Relative Humidity, and the Number of Degrees Centigrade (θ) which the Surface can fall below the Temperature of the 4-in. Depth before the Upward Conduction balances the Radiation.

Table V gives the value of θ on clear nights of which the relative humidity was known compared with the value of $\frac{R_4}{R_0}$. The starred figures are the average of the values actually obtained by observation, the other figures are interpolated values.

Table V.—Values of θ in °C.

Relative Humidity.	Values of $\frac{R_4}{R_0}$.							
	·30	·32	·34	·36	·38	·40	·42	·44
90 per cent. .	8·0	7·5	7·0	6·5	6·0 *	5·5 *	5·0 *	4·6
80 " .	8·7 *	8·1	7·6 *	7·0	6·5	6·0 *	5·5 *	5·0 *
70 " .	9·5 *	8·9	8·3 *	7·7 *	7·1	6·6 *	6·1 *	5·5
60 " .	10·2	9·6	9·0	8·4	7·8	7·2 *	6·6 *	6·1

V. Differences between the Air Minimum over Open-Soil and Surface-Soil Minimum on Clear Nights.

Table VI gives the average and greatest and least observed differences—

(a) On 10 nights when surface froze.

(b) On 28 nights when surface did not freeze.

Table VI.—Differences of Air and Surface Minimum ° C.

	(a) Surface frozen. (10 nights.)	(b) Surface not frozen. (28 nights.)
Average difference between air minimum and surface-soil minimum	−2·7° C.	+ ·4° C.
Greatest and least differences of above	−3·2° C. ; −·3° C.	+1·0° C. ; −·1·0° C.

If the surface does not freeze it would appear that the air minimum over open soil follows the surface-soil minimum very closely.

VI. The Forecast Equation.

Let us suppose we wish to forecast the minimum surface temperature to-night, the conditions being favourable for a calm clear night. By about 6 p.m. we should know—

(1) The maximum surface temperature $= M_0$.

(2) The maximum 4-in. depth temperature $= M_4$.

(3) The ratio of ranges of surface and 4-in. temperature from minimum to maximum $= \alpha$.

(4) The probable average relative humidity of the coming night.

(5) The number of degrees which the surface temperature can fall below the 4-in. depth temperature read off Table V $= \theta$.

If R_0 and R_4 are the ranges from maximum to minimum on the coming night, we may assume that $R_4 = \alpha R_0$. When the surface has reached its minimum the 4-in. depth is still considerably above its minimum, the amount depending on the value of the lag.

Thus the surface temperature *at time of minimum* $= M_0 - R_0$

The 4-in. depth „ „ „ $= M_4 - \beta R_4$

$= M_4 - \alpha\beta R_0$

where β is a fraction depending on the value of the lag.

Now the surface temperature cannot fall more than $\theta°$ below the 4-in. temperature,

$$\therefore \quad M_0 - R_0 = M_4 - \alpha\beta R_0 - \theta,$$

$$\therefore \qquad R_0 = \frac{M_0 - M_4 + \theta}{1 - \alpha\beta} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If we assume that for an average spring day the maximum at the surface occurs about 2.30 p.m. and the minimum at about 4.30 a.m.—

a total period of 14 hours,—then the value of β corresponding to any value of $\frac{R_4}{R_0}$ may be found as follows:—

$$\alpha = \frac{R_4}{R_0} = \cdot 36 \text{ (say).}$$

The lag is 4 hours for this value of $\frac{R_4}{R_0}$ from Table IV. The total period is 14 hours.

$$\therefore \quad \beta = \frac{14-4}{14} = \frac{5}{7}.$$

$$\therefore \quad \alpha\beta = \frac{5}{7} \text{ of } \cdot 36$$

$$= \cdot 26.$$

$$\therefore \quad 1 - \alpha\beta = \cdot 74$$

and so on for other values of the lag and $\frac{R_4}{R_0}$.

Table VII gives the average values of $1 - \alpha\beta$ for various values of $\frac{R_4}{R_0}$.

TABLE VII.—VALUES OF $1 - \alpha\beta$.

	Values of $\frac{R_4}{R_0}$.										
	·44	·42	·40	·38	·36	·34	·32	·30	·28	·26	·24
Average value of $1 - \alpha\beta$.	·66	·68	·70	·72	·74	·76	·78	·80	·82	·84	·85

We now have everything necessary to compute the value of R_0 from the forecast equation.

For example, if

$$M_0 = 18° \text{ C.}$$
$$M_4 = 12° \text{ C.}$$
$$\frac{R_4}{R_0} = \alpha = \cdot 40,$$

relative humidity = 80 per cent.

Then, from Table V,

$$\theta = 6 \cdot 0°,$$

and from Table VII,

$$1 - \alpha\beta = \cdot 70.$$

$$\therefore \quad R_0 = \frac{M_0 - M_4 + \theta}{1 - \alpha\beta}$$

$$= \frac{18 - 12 + 6}{\cdot 70}$$

$$= \frac{12}{\cdot 7}$$

$$= 17 \cdot 1° \text{ C.}$$

Therefore, the temperature of the surface can only fall 17·1° from a maximum of 18°; in other words, the minimum surface temperature will be ·9° C. and there will be no frost.

The temperature so obtained by the formula is, of course, the minimum temperature of the surface under ideal conditions of perfectly clear sky, no wind, and dry air. A slight breeze, a temporary clouding of the sky, or the formation of a heavy dew will prevent the surface falling to as low a temperature as will be expected from the forecast equation. This can be clearly seen in Table VIII.

VII. A Comparison of the Forecasted and Observed Minimum Temperatures of Open Soil on Clear Nights.

When the surface temperature of a soil of high-water content has fallen to 0° C., latent heat is liberated in freezing the surface layer, and the surface-soil temperature does not fall below 0° C. until the surface is completely frozen—a process that takes some considerable time.

As the forecast equation does not take this liberation of latent heat into account, *it is only useful to forecast the occurrence, not the intensity*

Table VIII.—Forecasted and Observed Minima.

Date. (1)	M_0. (2)	M_4. (3)	$\frac{R_4}{R_0}=a$. (4)	Estimated Relative Humidity. (5)	6 p.m. Forecast of Surface Minimum. (6)	Observed Surface Minimum. (7)	Difference of Columns (6) and (7). (8)	Weather. (9)
1919.				per cent.				
April 9/10	14·0	8·0	·40	90	Frost	0·0	...	Clear, calm, high barometer
„ 12/13	16·6	9·6	·42	90	„	+ ·4°	...	„
„ 19/20	15·0	11·2	·40	75	+ 0·6°	+ 0·6	0·0°	„
„ 20/21	18·0	11·8	·42	75	+ 0·4	+ 0·6	− ·2	„
May 18/19	23·8	16·5	·43	80	+ 5·1	+ 5·7	− ·6	„
„ 19/20	29·8	18·6	·40	75	+ 4·7	+ 4·4	+ ·3	„
„ 26/27	30·8	19·0	·30	75	+ 4·7	+ 4·3	+ ·4	„
June 2/3	28·2	18·0	·28	80	+ 4·2	+ 4·0	+ ·2	„
„ 7/8	32·7	22·2	·42	80	+ 9·1	+ 9·1	0·0	„
„ 16/17	21·7	16·9	·40	85	+ 6·6	+ 7·0	− ·4	„
„ 25/26	28·3	17·6	·42	70	+ 3·6	+ 3·8	− ·2	„
July 12/13	29·0	17·6	·40	70	+ 4·6	+ 5·0	− ·4	„
„ 13/14	28·3	17·5	·40	70	+ 3·6	+ 3·8	− ·2	„
Aug. 12/13	33·3	23·3	·31	85	+10·5	+11·1	− ·6	„
„ 21/22	29·0	18·0	·42	75	+ 4·3	+ 4·0	+ ·3	„
Sept. 27/28	10·2	8·2	·40	80	Frost	0·0	...	„
Oct. 12/13	10·2	7·8	·40	80	„	0·0	...	„
„ 13/14	7·1	5·6	·40	75	„	− ·3	...	„
„ 14/15	5·3	4·6	·40	75	„	− ·5	...	„

Average difference (13 occasions), ·3°.

Table VIII—*continued.*

Date. (1)	M_0. (2)	M_4. (3)	$\frac{R_4}{R_0}=\alpha$. (4)	Estimated Relative Humidity. (5)	6 p.m. Forecast of Surface Minimum. (6)	Observed Surface Minimum. (7)	Difference of Columns (6) and (7). (8)	Weather. (9)
1919.				per cent.				
April 8/9	18·0	10·8	·44	90	+ ·3°	+ 2·0°	−1·7°	Clear, windy
„ 12/13	16·6	9·6	·42	90	Frost	+ 1·0	...	„ „
June 8/9	26·7	19·2	·42	70	+ 6·7°	+ 7·8	−1·1	„ „
„ 9/10	31·1	20·0	·45	85	+ 7·0	+ 7·9	− ·9	„ „
„ 13/14	26·7	17·8	·42	80	+ 5·5	+ 6·0	− ·5	„ „
„ 19/20	23·3	17·5	·36	75	+ 5·6	+ 6·6	−1·0	„ „
„ 21/22	34·0	21·4	·40	75	+ 7·0	+ 8·6	−1·6	„ „
„ 28/29	30·6	20·1	·34	75	+ 6·3	+ 7·0	− ·7	„ „
May 1/2	12·8	9·3	·38	90	Frost	+ 0·4	...	Clear most of night, no wind
„ 2/3	14·4	9·4	·42	90	„	+ 0·5	...	„
June 6/7	26·6	19·6	·40	85	+ 8·5°	+ 9·5	−1·0	„
„ 30/1	13·3	12·4	·40	90	+ 4·2	+ 5·0	− ·8	„

Average difference (9 occasions), −1·0°.

of frost; a very low forecasted minimum would certainly imply a considerable intensity of frost, but it would not be safe to assume that the surface would fall below 0° C. if the forecasted minimum was only a few degrees below that temperature. Thus in Table VIII, where the forecasted and observed minima are contrasted, I have not given the actual figures of the forecasted minimum when it was below 0° C., but have simply stated that frost was expected. The comparison is for calm clear nights between April 9 and October 16, 1919; before and after these dates the 4-in. depth temperatures were so low as to assure the occurrence of frost every calm clear night.

VIII. Conclusions.

It is customary in most text-books and articles dealing with the cooling by radiation at night to ignore the temperature and upward conduction of the soil entirely, or at least to state that it is of little account.

This may have some justification when dealing with the cooling over a large area probably covered with vegetation, but it seems to have no justification when we are considering the minimum temperature at night on plots of cultivated land on which the vegetation covers only a small fraction of the whole.

It is interesting to note that Professor H. J. Cox, in his exhaustive study of temperature conditions in the cranberry marshes of Wisconsin, upholds the view that the temperature and conductivity of the soil play a most important part in determining the minimum temperature at night.*

Thus in various places in his paper he states:—

(1) In 1906 and 1907, on a newly sanded thinly vined section of the marsh, the 3-in. depth soil maximum averaged highest and the soil minimum averaged lowest, *but the surface-soil minimum and air minimum averaged highest.*

(2) There is, of course, a direct relation between the soil temperature and the ensuing air temperature. An increase of heat during the day serves to raise the temperature of the soil, and this, in turn, prevents low minimum temperatures at night. When the soil temperature is high, more heat may be lost by conduction and radiation from the soil before the critical air temperature at or immediately above the soil is reached, even if the loss by radiation be at the same rate at all locations.

(3) The relation between the temperature of the soil and the occurrence of frost is noticeable in that it is practically impossible for frost to occur in sanded sections of the bog on the first cold night after a warm spell; but it is likely, if conditions are favourable, on the second night when the soil has become cold.

(4) The average increase in minimum air temperature on clear nights during September 1906 at Berlin, Wisconsin, on a clean sanded location where earth was warm, over a clean peat location where earth was relatively cold, was:—

(*a*) At surface 6·7° F.,

(*b*) At 5 in. height 2·3° F.,

(*c*) At 36 in. height ·5° F.,

showing that the warming effect of the heat of the soil reached even to a height of 36 inches.

These statements are fully borne out by my own observations, the tendency for low minimum temperatures when the soil is cool at the 4-in. depth being at once apparent from Table VIII. But in addition we see that low minimum temperatures may be expected even with relatively high 4-in. depth temperatures if the ground is very dry, as the heat is then prevented from being readily conducted to the surface.

If now we examine the results given in Table VIII a little more critically it will be noticed that these results may be divided into two groups: (*a*) when conditions were ideal, the sky being clear all night, with no wind; (*b*) when the conditions were only semi-ideal, the sky

* "Frost and Temperature Conditions in the Cranberry Marshes of Wisconsin," by H. J. Cox, U.S. Department of Agriculture, Weather Bureau, Bulletin T, 1910.

being clear only the majority of the night, or, if totally clear, a wind blowing. The average error in the forecast under these conditions was (*a*) ·30° C., (*b*) 1·0° C.

We thus see that the forecast equation does give us the ensuing minimum temperature of the surface of the soil to a very close degree of approximation when ideal conditions exist; and, as we should expect, that the forecasted minimum is low compared to the observed minimum when conditions are not ideal.

From Table VI we see that the air temperature over open cultivated soil follows the surface-soil temperature very closely so long as the soil does not freeze; it appears, therefore, from the above results, *that we can forecast the minimum temperature over open soil on calm clear nights, so far as to say whether there will be a frost or not, with a remarkable degree of exactitude.*

(*Issued separately February* 3, 1920.)

III.—Note on the Determinant whose Matrix is the Sum of Two Circulant Matrices. By Sir Thomas Muir, LL.D.

(MS. received October 11, 1919. Read November 3, 1919.)

(1) Perhaps the most interesting theorem as yet known in regard to circulants concerns the determinant whose matrix is the sum of two circulant matrices, one of the latter being taken symmetric with respect to the primary diagonal and the other with respect to the secondary diagonal: for example, the determinant

$$\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_3+b_2 & a_1+b_3 & a_2+b_1 \\ a_2+b_3 & a_3+b_1 & a_1+b_2 \end{vmatrix}$$

whose matrix is the sum of the matrices

$$\begin{matrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{matrix} \qquad \text{and} \qquad \begin{matrix} b_1 & b_2 & b_3 \\ b_2 & b_3 & b_1 \\ b_3 & b_1 & b_2 , \end{matrix}$$

that is to say, the matrices of

$$C(a_1, a_2, a_3) \quad \text{and} \quad (-1)^{\frac{1}{2}(3-1)(3-2)} . C(b_1, b_2, b_3).$$

(2) The property in question is that *such a determinant of the* n*th* *order has the quadratic factor*

$$(a_1 + a_2\omega + a_3\omega^2 + \dots)(a_1 + a_2\omega^{-1} + a_3\omega^{-2} + \dots) - (b_1 + b_2\omega + \dots)(b_1 + b_2\omega^{-1} + \dots)$$

where ω *is a primitive* n*th* *root of* 1. The proof originally given was of that indirect character which consists in the finding of two different forms of the resultant of a set of linear equations and thereby arriving at an equality. This was first published thirty-eight years ago (*Messenger of Math.*, xi, pp. 105–108), and up to the present no direct proof has been arrived at,—no proof, that is to say, resting on mere transformation in accordance with the ordinary elementary properties of determinants.

(3) If we restrict ourselves, merely for ease in writing, to the case where n is 5, what we have thus got to show is that, ϵ being a primitive 5th root of 1,

$$(a_1 + a_2\epsilon + \dots + a_5\epsilon^4)(a_1 + a_2\epsilon^{-1} + \dots + a_5\epsilon^{-4}) - (b_1 + b_2\epsilon + \dots + b_5\epsilon^4)(b_1 + b_2\epsilon^{-1} + \dots + b_5\epsilon^{-4})$$

is a factor of

$$\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 & a_4+b_4 & a_5+b_5 \\ a_5+b_2 & a_1+b_3 & a_2+b_4 & a_3+b_5 & a_4+b_1 \\ a_4+b_3 & a_5+b_4 & a_1+b_5 & a_2+b_1 & a_3+b_2 \\ a_3+b_4 & a_4+b_5 & a_5+b_1 & a_1+b_2 & a_2+b_3 \\ a_2+b_5 & a_3+b_1 & a_4+b_2 & a_5+b_3 & a_1+b_4 \end{vmatrix},$$

or, say

$$F_a G_a - F_b G_b \quad \text{a factor of } \Delta_5.$$

Increasing the first column by ϵ times the second, ϵ^2 times the third, and so forth, we obtain for our new first column

$$\left.\begin{matrix} F_a + F_b \\ \epsilon F_a + \epsilon^4 F_b \\ \epsilon^2 F_a + \epsilon^3 F_b \\ \epsilon^3 F_a + \epsilon^2 F_b \\ \epsilon^4 F_a + \epsilon F_b \end{matrix}\right\}, \text{ so that there thus results}$$

$$\Delta_5 = F_a . \begin{vmatrix} 1 & a_2+b_2 & \cdots & a_5+b_5 \\ \epsilon & a_1+b_3 & \cdots & a_4+b_1 \\ \epsilon^2 & a_5+b_4 & \cdots & a_3+b_2 \\ \epsilon^3 & a_4+b_5 & \cdots & a_2+b_3 \\ \epsilon^4 & a_3+b_1 & \cdots & a_1+b_4 \end{vmatrix} + F_b . \begin{vmatrix} 1 & a_2+b_2 & \cdots & a_5+b_5 \\ \epsilon^4 & a_1+b_3 & \cdots & a_4+b_1 \\ \epsilon^3 & a_5+b_4 & \cdots & a_3+b_2 \\ \epsilon^2 & a_4+b_5 & \cdots & a_2+b_3 \\ \epsilon & a_3+b_1 & \cdots & a_1+b_4 \end{vmatrix}$$

or, say, for shortness' sake

$$\Delta_5 = F_a . U + F_b . V.$$

If we write ϵ^{-1} for ϵ in this, F_a, F_b become G_a, G_b respectively, U and V are interchanged, and of course Δ_5 is unaltered. We thus have the companion equality

$$\Delta_5 = G_b . U + G_a . V,$$

and from the pair derive

$$(F_a - G_b)\Delta_5 = (F_a G_a - F_b G_b)V,$$

which gives us what we desire. For, the linear expression $F_a - G_b$ not being an exact divisor of $F_a G_a - F_b G_b$, the latter must be an exact divisor of Δ_5.

(4) From the same final equality there clearly comes the further result that $F_a - G_b$ is *an exact divisor of the determinant* V. Indeed, the equality leads us to all the divisors of V as well as to those of Δ_5. Thus, the factors of Δ_5 are seen to be three in number, two quadratics and one linear, one of the quadratics being $F_a G_a - F_b G_b$ or Q_1 say, the other Q_2 got therefrom by changing ϵ into ϵ^2, and the linear being evidently $\Sigma a + \Sigma b$. Substituting therefore

$$\left(\sum a + \sum b\right) Q_1 Q_2 \quad \text{for} \quad \Delta_5$$

in the said equality and removing Q_1 from both sides we obtain

$$V = (F_a - G_b)\left(\sum a + \sum b\right)Q_2;$$

in other words, the factors of V are the same as those of Δ_5, save that in V the linear $F_a - G_b$ takes the place of the quadratic Q_1.

Similarly, or by mere change of ϵ into ϵ^{-1}, we have

$$U = (G_a - F_b)\left(\sum a + \sum b\right)Q_2.$$

(5) By taking the sum of all the rows in any one of the determinants Δ_5, U, V we not only see that $\Sigma a + \Sigma b$ is a factor, but we can *remove* the factor and have the co-factor left expressed as a determinant. There is thus suggested for the future the problem of removing in some analogous fashion the linear factors $F_a - G_b$, $G_a - F_b$ from V and U respectively; or, better, from $V/(\Sigma a + \Sigma b)$ and $U/(\Sigma a + \Sigma b)$ respectively, and so obtain a determinant form for Q_2, and therefore for the other quadratic factor also.

In this connection we may note that by merely performing the multiplications and condensing we obtain

$$Q_1 = \left(\sum a_1^2 - \sum b_1^2\right) + \left(\mathring{\sum} a_1a_2 - \mathring{\sum} b_1b_2\right)(\epsilon + \epsilon^4) + \left(\mathring{\sum} a_1a_3 - \mathring{\sum} b_1b_3\right)(\epsilon^2 + \epsilon^3);$$

that therefore

$$Q_2 = \left(\sum a_1^2 - \sum b_1^2\right) + \left(\mathring{\sum} a_1a_2 - \mathring{\sum} b_1b_2\right)(\epsilon^2 + \epsilon^3) + \left(\mathring{\sum} a_1a_3 - \mathring{\sum} b_1b_3\right)(\epsilon + \epsilon^4),$$

and that, again by mere multiplication and condensation, there results

$$Q_1Q_2, \quad i.e. \quad \Delta_5\Big/\left(\sum a + \sum b\right) = A^2 - B^2 - C^2 - AB - AC + 3BC,$$

where

$$\begin{array}{lll} A & \text{stands for} & \sum a_1^2 - \sum b_1^2, \\ B & \quad ,, & \mathring{\sum} a_1a_2 - \mathring{\sum} b_1b_2, \\ C & \quad ,, & \mathring{\sum} a_1a_3 - \mathring{\sum} b_1b_3. \end{array}$$

(6) This association of two circulants that are symmetric with respect to different diagonals provides the solution, when n is prime, of a very interesting problem of arrangement; namely, the problem of *placing the* n^2 *elements of* $|\, a_1b_2c_3 \ldots$ *in an* n*-line square array so as to have each row containing every letter and each column every suffix.*

The problem was first proposed twenty-five years ago by J. D. Loriga in the *Intermédiaire des Math.*, i, pp. 146–147, and no solution has as yet appeared.

(7) At the outset it is clear that in the desired square array—or "solution" as we may call it—each row and each column must contain

the elements of a term of the given determinant. Further, it is seen that the $2n$ terms thus connected with a solution cannot be altered by transposition of rows or by transposition of columns; and that therefore two solutions that do not involve in their construction the same set of $2n$ terms are essentially different solutions.

(8) When n is prime a solution is always got by taking the array of elements whose array of letters is identical with $C(a, b, c, \ldots)_n$ and whose array of suffixes becomes identical with $C(1, 2, 3, \ldots, n)$ after the last $n-1$ rows of the latter have been reversed in order. For example, when n is 5, a solution is

$$\begin{array}{ccccc} a_1 & b_2 & c_3 & d_4 & e_5 \\ e_2 & a_3 & b_4 & c_5 & d_1 \\ d_3 & e_4 & a_5 & b_1 & c_2 \\ c_4 & d_5 & e_1 & a_2 & b_3 \\ b_5 & c_1 & d_2 & e_3 & a_4 . \end{array}$$

In other words, we take for our first row the diagonal of the given determinant, and thence get the other rows by cyclical forward movement of the letters and cyclical backward movement of the suffixes.

(9) Other solutions are got by starting with a first row whose elements belong to an as yet unused term of the given determinant and permuting the letters and the suffixes separately and in different ways as before. In the case where n is 5 there are twelve solutions, which, by reason of the nature of the law of formation, we can specify sufficiently by giving merely the first row of suffixes, namely,

$$\begin{array}{ccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & \quad & 1 & 3 & 2 & 4 & 5 & \quad & 1 & 4 & 2 & 3 & 5 \\ 1 & 2 & 3 & 5 & 4 & & 1 & 3 & 2 & 5 & 4 & & 1 & 4 & 3 & 2 & 5, \\ 1 & 2 & 4 & 3 & 5 & & 1 & 3 & 4 & 2 & 5 & & & & & & \\ 1 & 2 & 4 & 5 & 3 & & 1 & 3 & 4 & 5 & 2 & & & & & & \\ 1 & 2 & 5 & 3 & 4 & & & & & & & & & & & & \\ 1 & 2 & 5 & 4 & 3 & & & & & & & & & & & & \end{array}$$

it being understood that the first row of letters is always

$$a \quad b \quad c \quad d \quad e .$$

That these solutions are essentially different from one another is evident from the fact that in their construction not a single term of the given determinant appears twice, every additional solution using up, as it were, ten new terms, and the number of solutions thus being

$$\frac{n!}{2n}, \quad i.e. \quad \frac{(n-1)!}{2} .$$

(10) The law of formation from the first row shows that the signs of the terms used in connection with any solution must be all alike, so that we have six solutions involving positive terms and six involving negative terms.

(11) When n is composite no solution is obtained in this way, the fault being that it leads to half of the elements being used more than once and the others not at all. For example, when n is 4 it gives

$$\begin{matrix} a_1 & b_2 & c_3 & d_4 \\ d_2 & a_3 & b_4 & c_1 \\ c_3 & d_4 & a_1 & b_2 \\ b_4 & c_1 & d_2 & a_3 , \end{matrix}$$

in which $a_2, a_4, \ldots$ do not appear.

In this case, however, there actually can be found a proportionately greater number of solutions than in the case where n is 5. In the first place, the array of letters is in every solution

$$\begin{matrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a ; \end{matrix}$$

that is to say, the circulant array of a pair of two-line circulant arrays. In the second place, the suffixes of the first row may be any one of the six sets

$$\begin{matrix} 1 & 2 & 3 & 4 & \quad & 1 & 2 & 4 & 3 \\ 1 & 3 & 4 & 2 & \quad & 1 & 4 & 3 & 2 \\ 1 & 4 & 2 & 3, & \quad & 1 & 3 & 2 & 4 ; \end{matrix}$$

and if the set taken be

$$\mu \quad \nu \quad \rho \quad \sigma$$

then the set for the third row is

$$\nu \quad \mu \quad \sigma \quad \rho ,$$

and the sets for the second and fourth rows the reverses of these. For example,

$$\begin{matrix} a_1 & b_3 & c_4 & d_2 \\ b_2 & a_4 & d_3 & c_1 \\ c_3 & d_1 & a_2 & b_4 \\ d_4 & c_2 & b_1 & a_3 . \end{matrix}$$

In the case of each solution belonging to the first triad the involved terms of the given determinant are all positive, each pair of solutions having four terms in common: in the case of the other triad the terms are all negative. In the full set of solutions each term is thus involved twice, and consequently the number of solutions is relatively double the number found for the case where n is 5.

(12) The peculiar determinant form referred to in the preceding paragraph, namely, the circulant of the two circulant arrays,

$$\begin{matrix} a & b & & c & d \\ b & a & & d & c \end{matrix},$$

is the simplest possible case of a *block* circulant. The cyclically permuted arrays, however, need not be themselves circulant in form, but may be the arrays of general determinants so long as they are all of the same order, q say; and the number of them may also be unrestricted, p say; the resulting determinant being thus of the $(pq)^{\text{th}}$ order.

(13) One of the most important properties of such determinants is that they are homogenetic; in other words, that *the product of two block circulants, each of* p q*-by-*q *arrays, is itself a block circulant of the same type and dimensions.*

As a consequence of the definition the $(i, j)^{\text{th}}$ element in the first of the p arrays—and any one of the p arrays may be made the first without departing from the circulant form—occurs again in the places

$$(q+i, q+j), \quad (2q+i, 2q+j), \quad . \; . \; .$$

and what we have therefore got to prove is that in the product-determinant the elements in the places

$$(i, j), \quad (q+i, q+j), \quad (2q+i, 2q+j), \quad . \; . \; .$$

are identical. Further, if we denote the rows of the multiplicand by r_1, r_2, . . . , and those of the multiplier by ρ_1, ρ_2, . . . , this is the same as to say that we have got to show that

$$r_i \cdot \rho_j = r_{q+i} \cdot \rho_{q+j} = r_{2q+i} \cdot \rho_{2q+j} = \; . \; . \; . \; .$$

The lines on which such a proof may be effected are readily grasped by taking for the multiplicand the 9-line block circulant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 \\ a_7 & a_8 & a_9 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_7 & b_8 & b_9 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_7 & c_8 & c_9 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_1 & a_2 & a_3 \\ b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_1 & b_2 & b_3 \\ c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_1 & c_2 & c_3 \end{vmatrix}$$

whose permuted arrays are those of

$$|\,a_1 b_2 c_3\,|, \quad |\,a_4 b_5 c_6\,|, \quad |\,a_7 b_8 c_9\,|;$$

and for the multiplier the determinant similarly formed from

$$|\,\alpha_1 \beta_2 \gamma_3\,|, \quad |\,\alpha_4 \beta_5 \gamma_6\,|, \quad |\,\alpha_7 \beta_8 \gamma_9\,|;$$

and comparing the original forms of three of the elements of the product-determinant which, in accordance with our assertion, are expected to be identical, say the $(2, 3)^{\text{th}}$, the $(5, 6)^{\text{th}}$, and the $(8, 9)^{\text{th}}$. The $(2, 3)^{\text{th}}$ being the product of the second row of the multiplicand by the third row of the multiplier, *i.e.* $r_2 . \rho_3$, we may conveniently write in a form showing the rows in question, namely,

$$\frac{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9}{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9},$$

this being therefore used temporarily to stand for

$$b_1\gamma_1 + b_2\gamma_2 + \ . \ . \ . \ + b_9\gamma_9 .$$

Now the cyclical movements which change r_2 into r_5 are exactly those which change ρ_3 into ρ_6, so that in the $(5, 6)^{\text{th}}$ element of the product each γ must find itself still directly under the corresponding b, the element in fact being

$$\frac{b_7, b_8, b_9, b_1, b_2, b_3, b_4, b_5, b_6}{\gamma_7, \gamma_8, \gamma_9, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6}.$$

Similarly we may reason regarding r_8 and ρ_9, the $(8, 9)^{\text{th}}$ element being

$$\frac{b_4, b_5, b_6, b_7, b_8, b_9, b_1, b_2, b_3}{\gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_1, \gamma_2, \gamma_3}.$$

(14) Another theorem of like character is that *the adjugate of a block circulant is itself a block circulant of the same type and dimensions.* A proof of this may be worked up from considerations similar to those in the preceding paragraph. What we have got to show is that any element of the particular block circulant written at full length above, say the element b_6, has the same complementary minor in each of its three positions, namely, the $(2, 6)^{\text{th}}$ place, the $(5, 9)^{\text{th}}$, and the $(8, 3)^{\text{th}}$. Now the facts are that the complementary of the $(2, 6)^{\text{th}}$ element being written, we obtain from it the $(5, 9)^{\text{th}}$ by merely advancing its last set of three rows over all the other rows, and then doing the like with the last three columns; and that the $(8, 3)^{\text{th}}$ is similarly obtained by advancing its last set of six rows over the others, and then the last set of six columns.

(15) The opportunity may now be taken to add a few fresh facts on the *factorisation* of ordinary circulants.

In the first place, it deserves to be noted that *the* n*-line circulant has* *factors corresponding not only to the prime factors of* $x^n - 1$ *but to all the composite factors as well.* For example, when n is 6, not only have we

$$a+b+c+d+e+f, \qquad a-b+c-d+e-f,$$

$$\begin{vmatrix} a+d-b-e & b+e-c-f \\ b+e-c-f & c+f-d-a \end{vmatrix}, \qquad \begin{vmatrix} a-d+b-e & b-e+c-f \\ b-e+c-f & c-f+d-a \end{vmatrix},$$

corresponding to the prime factors

$$x-1, \quad x+1, \quad x^2+x+1, \quad x^2-x+1;$$

but we have

$$\begin{vmatrix} a+c+e & b+d+f \\ b+d+f & c+e+a \end{vmatrix}, \qquad \begin{vmatrix} a+d & b+e & c+f \\ b+e & c+f & d+a \\ c+f & d+a & e+b \end{vmatrix},$$

$$\begin{vmatrix} a+2b+2c+d & b+2c+2d+e & c+2d+2e+f \\ b+2c+2d+e & c+2d+2e+f & d+2e+2f+a \\ c+2d+2e+f & d+2e+2f+a & e+2f+2a+b \end{vmatrix}, \qquad \begin{vmatrix} a-2b+2c-d & \cdots \\ b-2c+2d-e & \cdots \\ c-2d+2e-f & \cdots \end{vmatrix},$$

$$\begin{vmatrix} a-d & b-e & c-f \\ b-e & c-f & d-a \\ c-f & d-a & e-b \end{vmatrix}, \qquad \begin{vmatrix} a-c & b-d & c-e & d-f \\ b-d & c-e & d-f & e-a \\ c-e & d-f & e-a & f-b \\ d-f & e-a & f-b & a-c \end{vmatrix},$$

corresponding to the bifid composites

$$x^2-1, \quad x^3-1,$$
$$x^3-2x^2+2x-1, \quad x^3+2x^2+2x+1,$$
$$x^3+1, \quad x^4+x^2+1;$$

and

$$\begin{vmatrix} a+b+c & b+c+d & c+d+e & d+e+f \\ b+c+d & c+d+e & d+e+f & e+f+a \\ c+d+e & d+e+f & e+f+a & f+a+b \\ d+e+f & e+f+a & f+a+b & a+b+c \end{vmatrix}, \qquad \begin{vmatrix} a-b+c & \cdots & d-e+f \\ b-c+d & \cdots & e-f+a \\ c-d+e & \cdots & f-a+b \\ d-e+f & \cdots & a-b+c \end{vmatrix},$$

$$\begin{vmatrix} a+b & b+c & c+d & d+e & e+f \\ b+c & c+d & d+e & e+f & f+a \\ c+d & d+e & e+f & f+a & a+b \\ d+e & e+f & f+a & a+b & b+c \\ e+f & f+a & a+b & b+c & c+d \end{vmatrix}, \qquad \begin{vmatrix} a-b & \cdots & e-f \\ b-c & \cdots & f-a \\ c-d & \cdots & a-b \\ d-e & \cdots & b-c \\ e-f & \cdots & c-d \end{vmatrix},$$

corresponding to the trifid composites

$$x^4-x^3+x-1, \quad x^4+x^3-x-1,$$
$$x^5-x^4+x^3-x^2+x-1, \quad x^5+x^4+x^3+x^2+x+1.$$

(16) The noteworthy point in regard to the determinant factors here is that they are without exception *persymmetric*, and that the existence of the persymmetry is not readily recognisable from an examination of

the forms in which they are first obtained. For example, if the factor corresponding to $x^4 - x^3 + x - 1$ be wanted, we take the equations

$$\left.\begin{aligned} ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \\ x^4 - x^3 + x - 1 = 0 \end{aligned}\right\}$$

and seek their eliminant. Using the second equation to free the first from the terms in x^5 and x^4, we obtain

$$(c + b + a)x^3 + (d - a)x^2 + (e - b)x + (f + b + a) = 0,$$

whence three other equations are got by cyclical substitution, and then from the set of four the result

$$\begin{vmatrix} c+b+a & d-a & e-b & f+b+a \\ d+c+b & e-b & f-c & a+c+b \\ e+d+c & f-c & a-d & b+d+c \\ f+e+d & a-d & b-e & c+e+d \end{vmatrix}.$$

The persymmetry latent in this has to be brought into evidence by performing the operations

$$\begin{gathered} \mathrm{col}_4 + \mathrm{col}_3 + \mathrm{col}_2 \\ \mathrm{col}_3 + \mathrm{col}_2 + \mathrm{col}_1 \\ \mathrm{col}_2 + \mathrm{col}_1 . \end{gathered}$$

(17) When, however, the persymmetric form is known, we can readily utilise our knowlege to evolve it directly from the circulant itself, and thereby simultaneously obtain the cofactor. Thus, performing on

$$\begin{vmatrix} a & b & c & d & e & f \\ b & c & d & e & f & a \\ c & d & e & f & a & b \\ d & e & f & a & b & c \\ e & f & a & b & c & d \\ f & a & b & c & d & e \end{vmatrix}$$

the operations

$$\mathrm{col}_1 + \mathrm{col}_2 + \mathrm{col}_3, \quad \ldots, \quad \mathrm{col}_4 + \mathrm{col}_5 + \mathrm{col}_6$$

we obtain

$$\begin{vmatrix} a+b+c & b+c+d & c+d+e & d+e+f & e & f \\ b+c+d & c+d+e & d+e+f & e+f+a & f & a \\ c+d+e & d+e+f & e+f+a & f+a+b & a & b \\ d+e+f & e+f+a & f+a+b & a+b+c & b & c \\ e+f+a & f+a+b & a+b+c & b+c+d & c & d \\ f+a+b & a+b+c & b+c+d & c+d+e & d & e \end{vmatrix},$$

the last two rows of which, by the performance of the operations

$$\begin{gathered} \mathrm{row}_6 - \mathrm{row}_5 + \mathrm{row}_3 - \mathrm{row}_2 \\ \mathrm{row}_5 - \mathrm{row}_4 + \mathrm{row}_2 - \mathrm{row}_1 \end{gathered}$$

become

$$\begin{matrix} 0 & 0 & 0 & 0 & c-b+f-e & d-c+a-f \\ 0 & 0 & 0 & 0 & d-c+a-f & e-d+b-a \end{matrix}$$

and so by a final step we reach the full resolution corresponding to the resolution

$$x^6-1=(x^4-x^3+x-1)(x^2+x+1),$$

and with both determinant factors in persymmetric form.

(18) A little additional interest attaches to this mode of resolution when the two factors are of the same degree; for example, the resolution

$$P(a+2b+2c+d, \ \ldots\ ) \ . \ P(a-2b+2c-d, \ \ldots\ )$$

corresponding to

$$x^6-1=(x^3-2x^2+2x-1)(x^3+2x^2+2x+1).$$

In such a case any element of the one persymmetric determinant and the corresponding element of the other are such that it is possible to view them as the sum and difference respectively of one and the same pair of quantities; for example, the elements in the $(11)^{\text{th}}$ places are

$$(a+2c)+(b+2d), \quad (a+2c)-(b+2d).$$

This peculiarity at once recalls Zehfuss' theorem regarding centro-symmetric determinants, from which we at once obtain

$$\begin{vmatrix} a+2c & b+2d & c+2e & f+2d & e+2c & d+2b \\ b+2d & c+2e & d+2f & a+2e & f+2d & e+2c \\ c+2e & d+2f & e+2a & b+2f & a+2e & f+2d \\ f+2d & a+2e & b+2f & e+2a & d+2f & c+2e \\ e+2c & f+2d & a+2e & d+2f & c+2e & b+2d \\ d+2b & e+2c & f+2d & c+2e & b+2d & a+2c \end{vmatrix}$$

as an equivalent of the product of the two P's, and therefore an equivalent of

$$C(a, b, c, d, e, f).$$

(19) Lastly, it has to be noted that each one of the persymmetric determinant factors has alternative forms which are obtainable by cyclical substitution; for example, corresponding to x^3-1 we have

$$\begin{vmatrix} a-d & b-e & c-f \\ b-e & c-f & d-a \\ c-f & d-a & e-b \end{vmatrix} = - \begin{vmatrix} b-e & c-f & d-a \\ c-f & d-a & e-b \\ d-a & e-b & f-c \end{vmatrix} = \ \ldots\ .$$

In the case of an n-line circulant with an m-line persymmetric factor the number of such alternatives is n if $2m-1$ is prime to n; otherwise it is less.

Rondebosch, S.A.,
11th September 1919.

(Issued separately February 3, 1920.)

IV.—Note on Mr Quilter's Photographs of Mirage. By The General Secretary.

In a paper published in the *Proceedings* of the Royal Society of Edinburgh (vol. xxxviii, pp. 166–168), Mr Alexander G. Ramage described cases of mirage as seen on the Queensferry Road, near Edinburgh. Similar appearances have been since then observed and reported from different parts of the country, and the photographs of these by Mr G. F. Quilter, Ingatestone, are of some interest. Prints of five photographs taken by Mr Quilter were shown at the meeting of the Society held on 3rd November 1919; four were taken at one place about 100 yards' distance from where the dry road appeared to be a pool. The camera looked along the road in a direction E.N.E., the road rising slightly towards the position where the mirage was seen. The best photographs, which were taken on 15th June 1919, at 1.30 p.m., showed certain posts on the north side of the road appearing distinctly reflected in the mirage pool. In the fifth photograph, taken at a different locality in August 1919, the shaded side of a telegraph post with neighbouring trees and a white-walled house beyond were distinctly seen as if reflected from a pool in the middle of the road. The main interest lies in the fact that the photographs bring out the phenomenon quite clearly.

V.—The Absorption of X-Rays. By Dr R. A. Houstoun, Lecturer on Physical Optics in the University of Glasgow.

(MS. received April 19, 1919. Read June 2, 1919. Revised MS. received October 14, 1919.)

§ 1. Let us suppose that a light wave is being propagated and absorbed in a homogeneous medium. Take OY as the direction of propagation, and consider a slice of the medium bounded by two planes at right angles to the direction of propagation and distant dy apart; dy is small in comparison with the wave-length. Let $x=f(t)$ denote the mean displacement of the electrons, and let $X=A\cos gt$ denote the electric intensity of the light wave in the slice, measured in electrostatic units. Then the average rate at which work is being done on an electron is

$$-Xe\frac{dx}{dt}=-Aef'(t)\cos gt.$$

Let there be N electrons per unit volume. Then if a portion of the slice of unit cross-sectional area is considered, the average rate at which work is done in it in time τ is

$$-\frac{NAedy}{\tau}\int^{\tau}f'(t)\cos gt\,dt.$$

The average rate at which energy flows through unit area is by Poynting's theorem $c\nu A^2/8\pi$, where c is the velocity of light *in vacuo* and ν the index of refraction, provided that the absorption is small. Hence if $E=E_0e^{-4\pi\kappa y/\lambda}$ represents the rate of diminution of energy in the wave where λ is the wave-length *in vacuo*, the average value of κ during τ is given by

$$\frac{4\pi\kappa}{\lambda}=-\frac{8\pi Ne}{\tau c\nu A}\int_0^{\tau}f'(t)\cos gt\,dt \quad . \quad . \quad . \quad . \quad (1)$$

Let us suppose that the electrons are subject to an equation of the type

$$m\frac{d^2x}{dt^2}+h\frac{dx}{dt}+fx=-Ae\cos gt,$$

and that the free vibrations have died down. Then

$$x=\frac{-Ae\cos(gt-\tan^{-1}hg/(f-mg^2))}{((f-mg^2)^2+h^2g^2)^{\frac{1}{2}}} \quad . \quad . \quad . \quad (2)$$

Differentiate x with respect to t, substitute the result for $f'(t)$ in (1), and we obtain

$$\left.\begin{aligned} \frac{4\pi\kappa}{\lambda} &= -\frac{8\pi \mathrm{N} e^2 g}{\tau c \nu}\int_0^\tau \cos gt \frac{\sin\left(gt - \tan^{-1} hg/(f-mg^2)\right)}{((f-mg^2)^2+h^2g^2)^{\frac{1}{2}}}dt \\ 2\nu\kappa &= \frac{8\pi \mathrm{N} e^2}{\tau}\int_0^\tau \cos^2 gt \frac{hg}{(f-mg^2)^2+h^2g^2}dt \\ &= \frac{4\pi \mathrm{N} e^2 hg}{(f-mg^2)^2+h^2g^2} \end{aligned}\right\} \quad . \quad . \quad (3)$$

if τ is very large compared with the period. (3) is the ordinary expression used to represent the variation of the coefficient of absorption through an absorption band. The advantage of the above method of deriving it over the conventional method, which is given in the same notation in chapter xxiv of my book on Light, is that the conventional proof requires that there should be many electrons to the wave-length, and consequently is not suitable for wave-lengths as short as those of X-rays. The above proof is free from this restriction.

Let us assume now that ν differs very slightly from unity, and that κ is plotted as a function of g; g is, of course, $2\pi c/\lambda$. It will be found that the second term in the denominator affects the shape of the curve very slightly; we may consequently write instead of (3)

$$\kappa = \frac{2\pi \mathrm{N} e^2 hg}{(f-mg^2)^2+h^2{g_m}^2} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where g_m is the value of g corresponding to the maximum value of κ. g_m is given by $f - m{g_m}^2 = 0$. The curve falls away to zero on both sides of this maximum. The area of the curve is given by

$$\int_0^\infty \frac{2\pi \mathrm{N} e^2 hg\,dg}{(f-mg^2)^2+h^2{g_m}^2} = \frac{\pi \mathrm{N} e^2}{mg_m}\left(\frac{\pi}{2} + \tan^{-1}\frac{f}{hg_m}\right) = \frac{\pi^2 \mathrm{N} e^2}{mg_m},$$

since f is large compared with hg_m. Now e and m are constants of nature, and g_m can be determined from the curve; hence if an absorption band has the shape required by theory, and we graph κ as a function of g, the number of electrons per unit volume concerned in its production is $mg_m/(\pi^2 e^2)$ times the area of the curve. But, what is more, any irregularly shaped band can be regarded as due to the superposition of a number of component bands, each of which has the theoretical shape, and for which the value of g_m does not vary much. The number of electrons concerned in the production of the resultant band is equal to the sum of the numbers of electrons concerned in the production of each of the component bands. *Thus the above rule holds for an absorption band of any shape whatever.*

§ 2. Professor Barkla has discovered absorption bands or "fluorescent absorption" in the X-ray region. These possess certain similarities to absorption bands in the visible spectrum, such as those of the aniline colouring matters. They also possess certain well-marked differences. So doubt might be expressed as to whether the application of the above rule to them is justifiable. However, I have applied it to them with striking numerical results.

The absorption bands in the X-ray region are divided into four classes, J, K, L, and M bands, in the order of increasing wave-length. They shift progressively along the spectrum in the direction of decreasing wave-lengths, as the atomic number of the element increases. The data on K and L bands are reproduced in Kaye's *X-Rays*; the data on J bands are given in a paper by Barkla and White.* The tops of the M bands have not been reached yet; we have only measurements of the slope on the side of decreasing wave-length. X-ray absorption spectra are characteristic of the atom, and strictly additive; there is no disturbing "constitutive influence."

The absorption in the X-ray region is measured by the mass absorption coefficient μ/ρ, where ρ is the density of the substance. μ is connected with κ, the coefficient of absorption in optics, by the equation

$$\mu = \frac{4\pi\kappa}{\lambda}.$$

If we take our rule and translate it into terms of μ and λ, it runs: if $\mu\lambda$ be graphed as a function of $1/\lambda$, and the area of the absorption band measured, then the number of electrons per unit volume concerned in its production is $(mc^2)/(\lambda_m \pi e^2)$ times the area of the band, where λ_m is the wave-length of the maximum of the band. The number of atoms per unit volume is $\rho/(wm_{\text{H}})$, where w is the atomic weight and m_{H} is the mass of the hydrogen atom. Consequently the number of electrons per atom is

$$\frac{mc^2}{\lambda_m \pi e^2} \frac{wm_{\text{H}}}{\rho} \times \text{area} \quad \text{or} \quad \left(\frac{m}{e}\right)\left(\frac{m_{\text{H}}}{e}\right)\frac{w}{\rho\lambda_m\pi} \times \text{area} \quad . \quad . \quad . \quad (5)$$

if e be changed to electromagnetic units. On substituting the numerical values (5) becomes

$$\frac{1}{5{\cdot}56 \times 10^{11}} \frac{w}{\rho\lambda_m} \times \text{area} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If we are dealing with a single absorption band, another formula may be used instead of the above. For, starting from (4), we find that the maximum value of κ is given by

$$\kappa_m = \frac{2\ Ne^2}{hg_m},$$

* *Phil. Mag.*, xxxiv, p. 270, 1917.

and that g_1, the value of g for which κ has half its maximum value, is given by

$$f - mg_1^2 = \pm hg_m \quad \text{or} \quad m(g_m^2 - g_1^2) = \pm hg_m.$$

Hence

$$m(g_m^2 - g_1^2) = \pm \frac{2\pi Ne^2}{\kappa_m}.$$

On substituting for g this equation gives

$$N = \pm \frac{2\pi mc^2\kappa_m}{e^2}\left(\frac{1}{\lambda_m^2} - \frac{1}{\lambda_1^2}\right),$$

which becomes

$$N = \pm \frac{m\pi}{e^2}\,\frac{4\kappa_m(\lambda_1 - \lambda_m)}{\lambda_m^3} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

on changing to electromagnetic units and remembering that λ_1 does not differ much from λ_m. On substituting

$$N = \frac{\rho}{wm_H} \quad \text{and} \quad \mu = \frac{4\pi\kappa}{\lambda}$$

in (7), we obtain for the number of electrons per atom

$$\frac{1}{1{\cdot}772 \times 10^{11}}\left(\frac{\mu_m}{\rho}\right)\frac{(\lambda_1 - \lambda_m)w}{\lambda_m^2} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In this formula μ_m is the maximum value of the linear absorption coefficient, λ_m the position of the maximum, and λ_1 a wave-length at which μ has half its maximum value.

On applying (6) and (8) to the experimental data we obtain the results given in the following table. The wave-lengths in the case of the K and L bands were obtained from Moseley's paper,* using the values for the Kα and Lα radiations, and interpolating and extrapolating when the radiations in question were not given in the paper.

Substance.	w.	$\mu_m\ \rho$.	λ_m.	$\lambda_1 - \lambda_m$.	No. of electrons per atom	
					by (8).	by (6).
Fe . . .	56	314	1·66	·24	·798	1·01
Ni . . .	59	265	1·44	·22	·935	·84
CuK . . .	64	176	1·20	·24	1·06	·66
CuL . . .	64	177	1·316	·25	·923	·70
Zn . . .	65	203	1·20	·26	1·34	2·01
Ag . . .	108	56·1	·462	·06	·959	1·08
C_2H_5Br . .	109	66·3	·88	·15	·790	·784
Pt . . .	195	180	·88	·18	4·60	1·94
Au . . .	197	160	·88	·22	4·01	1·76
Al . . .	27	·853	·387	·024	·00208	
H_2O . . .	18	·335	·376	·037	·00089	

* *Phil. Mag.*, xxvii, 1914, p. 703.

In the case of C_2H_5Br and H_2O it is the molecular weight that is given by w. The two entries marked CuK and CuL are independent determinations for the same band—in the one case based on observations made with the K radiations, and in the other case on observations made with the L radiations. The second last column gives the number of electrons per atom, or, in the case of a compound, per molecule, calculated by formula (8), and the last column the same quantity calculated by formula (6). The third, fourth, and fifth columns give the data used for the calculations by formula (8). In using (8), λ_1 was determined on both sides of the band, and half the difference taken as $\lambda_1 - \lambda_m$; λ_1 and λ_m are, of course, measured in 10^{-8} cms. Strictly speaking, formula (8) is not applicable, as the bands have not the simple form required by theory; but in practice it appears to give better results than the other formula. The difference between the last two columns is due to there not being sufficient points on the curves to determine their shapes with accuracy; it was necessary to extrapolate, and I apparently did this in a different manner each time.

Almost ten years ago * I applied a formula similar to (8) to the absorption bands of aniline colouring matters and inorganic salts. I found then that in the aniline colouring matters there was one electron per atom concerned in the production of each absorption band, and the values obtained for inorganic solutions suggested ions. But in many cases the results suggested neither atoms nor ions; the absorption band corresponded apparently not to the vibrations of a single electron in the atom, but to a degree of freedom of a group of electrons or ions.

The first seven results, which are all for K bands, do not differ much from I. Their mean value according to (8) is ·972, and according to (6) is 1·01. Hence in this case the absorption band is due to the vibration of one electron per atom or per molecule. The results in the case of Pt and Au are for L bands. There are not sufficient observations to speak with certainty, but these bands appear to be double. The estimate by (8) was made for one component on this assumption; the estimate by (6) was made for the whole band. Hence the discrepancy in this case is greater than it looks; there may be, however, one electron to each component. The results in the case of Al and H_2O are for J bands, and suggest on first sight that there is only one active electron in each 481 atoms and each 1124 molecules respectively. There is, however, a much more plausible interpretation than this. If we assume that the band is due to a charged hydrogen atom or an α particle, *i.e.* a helium atom with two

* *Proc. Roy. Soc.*, A, 82, 1909, p. 606.

charges, then the value in the second last column should be ·000546 or ·000273 respectively. The J bands are so faint and difficult to measure that the experimental values can be said to agree with the larger of the above figures within the error of observation.

It is straining the laws of probability altogether too far to assume that the results in the foregoing table are chance coincidences. Even if the mean values ·972 and 1·01 were accidental, we should still have to explain why the results for the J bands come so close to the value for α particles. Also, if the agreement for the K bands was accidental, we would expect a progessive falling away from unity as the atomic weight increases; but the result for silver, the element on the list with the highest atomic weight, agrees well with the mean, the large variations in λ_m, $\lambda_1 - \lambda_m$, and μ_m all compensating one another. It would be interesting to get measurements on the K bands of elements heavier than silver and lighter than iron; we could then test (8) over a wider range.

The agreement, such as it is, of the first seven results with unity verifies the wave-lengths of X-rays, which have hitherto been determined from (i) the theory of the crystal lattice, and (ii) the quantum theory, in a new and independent manner. The agreement also tells against the Bohr atom model, for there is no place in that model for electrons and atoms vibrating in the manner contemplated. But it suggests a connection with Soddy's theory of the relation between atomic disintegration and the periodic classification of the elements; if electrons and alpha particles are ejected from the atom, why should we not detect them vibrating in the atom?

§ 3. The difficulty in accepting the foregoing results seems to be that the foregoing theory apparently makes all the atoms absorb the X-rays, whereas in reality it is only a few atoms that do the absorbing. This difficulty can be diminished, but not removed, by two different arguments.

In the first place, it is nowhere assumed that all the atoms do the absorbing. N is not the number of electrons actually present in unit volume, but the limiting value of the ratio of number of electrons to element of volume, as the latter is made indefinitely small, at the place where the absorption is taking place. It is not necessary as regards the absorption (or the scattering) that the X radiation should go through the substance as a uniform wave-front; it may go through in streaks separated by dark spaces.

Again, although in light as a matter of convention we deal with uniform waves, they constitute an ideal case. The uniform spherical wave spreading out from a point source is a mathematical fiction. What we really have is a very great number of spherical wavelets, each diverging from a different electron, criss-crossing in various directions, and consequently interfering with one another. For example, suppose there are n electrons in the source, all close together, and that the intensity of radiation is required at a point P at a distance r great in comparison with the linear dimensions of the source, and so sensibly the same for all the electrons. Let the intensity at P due to a single electron be I/r^2. Then the resultant intensity may be anything from 0 to $n^2\mathrm{I}/r^2$, according to the number of wavelets coincident in phase at P, the lower values predominating. If the phases of all the different waves are absolutely at random, the problem reduces to a celebrated one solved by Lord Rayleigh, and the chance of a particular intensity J is

$$\frac{r^2}{\mathrm{I}n}e^{-\mathrm{J}r^2/\mathrm{I}n}\,d\mathrm{J}.$$

If there is any regularity of structure in the source, Lord Rayleigh's expression may not do justice to the higher intensities.

Thus even if all the atoms absorb, they do not absorb to the same extent; some absorb much more than others. This follows simply from the laws of chance.

§ 4. Objection may be taken to the friction term in the equation of the electron. We cannot speak of frictional forces in connection with electrons; the term represents loss of energy into the free vibrations of the electrons in a somewhat unsatisfactory manner.

Let us try to construct an expression for the absorption band without using the conventional friction term at all. Suppose that when $t=0$ both x and dx/dt are equal to zero, a reasonable enough supposition, since, as the motion is then random, the mean values must be zero, and let

$$\frac{d^2x}{dt^2}+p^2x=-\frac{\mathrm{A}e}{m}\cos gt \qquad (9)$$

represent the motion of the electrons. Then, considering both free and forced vibrations and determining the constants from the initial conditions,

$$f(t)=x=\frac{-\mathrm{A}e}{m(p^2-g^2)}(\cos gt-\cos pt).$$

Thus

$$\int_0^\tau f'(t)\cos gt\,dt=\frac{-\mathrm{A}e}{m(p^2-g^2)}\int_0^\tau(p\sin pt-g\sin gt)\cos gt\,dt$$

$$=\frac{-\mathrm{A}e}{2m(p^2-g^2)}\left(\tfrac{1}{2}\cos 2gt-\frac{p}{p+g}\cos(p+g)t-\frac{p}{p-g}\cos(p-g)t\right)_0^\tau \qquad (10)$$

Since in an absorption band g is nearly equal to p, the third term in the bracket is the largest, and we can neglect the others. The expression for κ thus becomes

$$\kappa = \frac{\lambda N e^2 p}{\tau c v m(p^2 - g^2)(p-g)}(1 - \cos(p-g)t) \quad . \quad . \quad . \quad (11)$$

As τ increases the bracket oscillates between 0 and 2, but the denominator steadily increases. Thus for a large value of τ we have $\kappa = 0$. Half the time the light wave does work on the electron, and half the time the electron does work on the wave. Consequently with an undisturbed motion we get no absorption at all.

To account for absorption we must assume impacts, stoppages, and interruptions of the motion. If these disturbances were absolutely at random, they would give as much energy to the electron as they would take from it; hence they must vary according to some definite law. The energy lost by the electron goes presumbly into fluorescent and corpuscular radiation and the heat motion of the atom, but how it gets there we do not know. The law according to which the impacts and stoppages take place is also unknown to us, but the simplest assumption to make is that the electron stops dead at fixed intervals τ, no matter what the phase of the impressed force.

The above discussion assumes that the electric intensity is a maximum when the electron starts to vibrate. If it vibrates for intervals of τ seconds duration, the phase of the electric intensity must have all possible values when the vibration starts. Consequently the above discussion requires amplification.

Retain (9), but assume now that x and dx/dt are equal to zero when $t = t_0$. Then the solution becomes

$$f(t) = x = \frac{-Ae}{m(p^2 - g^2)}\left(\cos gt - \cos gt_0 \cos p(t - t_0) + \frac{g}{p}\sin gt_0 \sin p(t - t_0)\right)$$

$$= \frac{-Ae}{m(p^2 - g^2)}(\cos gt - R\cos\{p(t - t_0) + \phi\}),$$

where $R^2 = \cos^2 gt_0 + (g/p)^2 \sin^2 gt_0$ and $\tan\phi = (g/p)\tan gt_0$. Differentiate this, multiply by $\cos gt$, and integrate, retaining only the term multiplied by $1/(p-g)$. Then we obtain instead of (10)

$$\frac{AeRp}{m(p^2 - g^2)(p-g)}\sin\left(\tfrac{1}{2}(p-g)\tau + \phi - gt_0\right)\sin\tfrac{1}{2}(p-g)\tau.$$

Since g is nearly equal to p we can write $\phi = gt_0$ and $R = 1$. Hence we obtain

$$\kappa = \frac{2\lambda N e^2 p}{\tau c v m(p^2 - g^2)(p-g)}\sin^2\tfrac{1}{2}(p-g)\tau \quad . \quad . \quad . \quad . \quad (12)$$

the same result as (11). Neglecting the factors that vary slowly and putting $\nu=1$, this becomes

$$\kappa=\frac{\tau\lambda Ne^2}{4cm}\left(\frac{\sin\frac{1}{2}(p-g)\tau}{\frac{1}{2}(p-g)\tau}\right)^2 . \qquad (13)$$

The expression in the bracket is of the same form as the solution of a well-known diffraction problem. Its maximum value is 1. Hence

$$\kappa_m=\frac{\tau\lambda_m Ne^2}{4cm} \qquad (14)$$

κ has half its maximum value for the values of g given by

$$(p-g)\tau=2\sqrt{2}\sin\tfrac{1}{2}(p-g)\tau.$$

The value of $(p-g)\tau$ satisfying this equation is found to be 2·232. Consequently

$$\frac{\kappa_m(\lambda_1-\lambda_m)}{\lambda_m^{\ 3}}=\frac{\tau\lambda Ne^2}{4cm}\,\frac{2\cdot 232}{2\pi c\tau\lambda}=\frac{\cdot 28 Ne^2}{\pi m}$$

approximately, if e be changed to electromagnetic units. But according to equation (7)

$$\frac{\kappa_m(\lambda_1-\lambda_m)}{\lambda_m^{\ 3}}=\frac{Ne^2}{4\pi m},$$

i.e. we have now ·28 instead of ·25. So that the new treatment leads to substantially the same result for the number of electrons per atom as the former treatment, which was based on the ordinary friction term.

It is obvious from (13) that the greater τ is, the narrower is the band. Also, if (13) gave κ accurately, there should be small maxima on each side of the principal one. These are, however, not to be expected on account of the neglected terms in (10), and also on account of variations in the value of τ.

It is obviously not necessary in the above discussion that the intervals τ should follow consecutively. There may be — indeed the numerical values of the quantities involved demand that there should be—intervals of irregular motion, *i.e.* of no absorption, in between the intervals τ. For example, in the case of the FeK band τ is about 2·5 times the free period of the κ electron, but the intervals τ constitute only about four-fifths of the total time. During the rest of the time the electron is giving back as much energy to the wave as it is taking from it. And the value of τ above is an average one; an exceptionally large value of τ would permit of a correspondingly longer period of no absorption. This consideration may help to remove the difficulty alluded to in § 3.

(*Issued separately February* 23, 1920.)

VI.—Note on the Scattering of X-Rays. By Dr R. A. Houstoun, Lecturer on Physical Optics in the University of Glasgow.

(MS. received April 19, 1919. Read June 2, 1919. Revised October 14, 1919.)

§ 1. The scattering of X-rays is one of the most promising methods of approaching the constitution of the atom. Sir J. J. Thomson's theory undoubtedly gives the main features of the phenomenon. But there are certain facts, namely:

(i) the increase of scattering with heavy elements for soft rays,
(ii) the extra-radiation, and
(iii) the low value of the scattering coefficient for γ-rays,

for which it affords no explanation. It seems worth while trying to extend the theory so as to cover these facts also.

Professor Barkla has suggested that (i) is to be explained by interference between the different electrons of the atom, by the atom scattering as a whole in the case of the longer wave-lengths. We shall formulate this idea mathematically.

Let us suppose there are n electrons in the atom, and let P denote

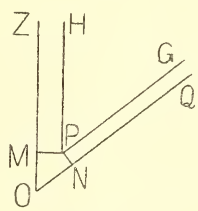


Fig. 1.

the position of one of them. Take an origin O in the atom, say at its centroid, and let a homogeneous radiation be incident in direction ZO. The intensity of the radiation scattered by the n electrons is required in the direction OQ. Let $\angle ZOQ = \theta$. P is generally not in the plane ZOQ. The intensity of the radiation scattered by a single electron in the direction OQ is, in the usual notation, $I_{\pi/2}(1+\cos^2\theta)$.

The length of the actual path taken by the ray HPG varies with the position of P. We shall specify it by its difference from the path ZOQ. Draw PM perpendicular to ZO and PN perpendicular to OQ. The actual path is shorter than the reference path by OM + ON. Hence the amplitude of the ray traversing it is proportional to

$$\cos\{gt + k(\mathrm{OM} + \mathrm{ON})\},$$

where $k = 2\pi/\lambda$ and $g = 2\pi/\tau$, λ and τ denoting wave-length and period. Form similar terms for the other electrons, take the sum, and we obtain

$$\cos gt \ \sum \cos k(\mathrm{OM} + \mathrm{ON}) - \sin gt \ \sum \sin k(\mathrm{OM} + \mathrm{ON}).$$

The intensity of this wave varies as

$$\left.\begin{array}{l} [\sum \cos k(\mathrm{OM} + \mathrm{ON})]^2 + [\sum \sin k(\mathrm{OM} + \mathrm{ON})]^2 \\ = n + \sum 2[\cos k(\mathrm{OM}_a + \mathrm{ON}_a) \cos k(\mathrm{OM}_b + \mathrm{ON}_b) + \sin k(\mathrm{OM}_a + \mathrm{ON}_a) \sin k(\mathrm{OM}_b + \mathrm{ON}_b)] \\ = n + \sum 2 \cos k(\mathrm{OM}_a - \mathrm{OM}_b + \mathrm{ON}_a - \mathrm{ON}_b) \end{array}\right\} \quad (1)$$

where the suffixes a and b refer to the a^{th} and b^{th} electrons in the atom, and the summation is taken over all the $\frac{1}{2}n(n-1)$ pairs of electrons in the atom.

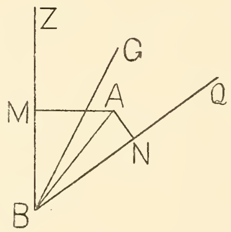


Fig. 2.

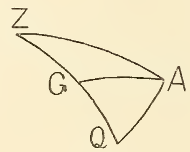


Fig. 3.

Let us consider the pair formed by the a^{th} and b^{th} electrons a little more closely. Let A and B denote their positions in space, and let L be the distance between them. Let ZB and BQ denote the directions of the incident and scattered light. Let BG bisect $\angle$ ZBQ. Draw AM perpendicular to BZ and AN perpendicular to BQ. Then the typical term under the summation sign becomes

$$\cos k(\mathrm{BM} + \mathrm{BN}) = \cos k\mathrm{L}\,(\cos \mathrm{ZBA} + \cos \mathrm{ABQ}).$$

In fig. 3, Z, G, A, and Q represent the intersections of the lines BZ, BG, BA, and BQ with a sphere of unit radius with its centre at B. Then by a formula in spherical trigonometry

$$\cos \mathrm{ZBA} = \cos \mathrm{ZG} \cos \mathrm{GA} + \sin \mathrm{ZG} \sin \mathrm{GA} \cos \mathrm{ZGA}$$

and

$$\cos \mathrm{ABQ} = \cos \mathrm{GQ} \cos \mathrm{GA} + \sin \mathrm{GQ} \sin \mathrm{GA} \cos \mathrm{AGQ}.$$

But $\mathrm{ZG} = \mathrm{GQ} = \frac{1}{2}\theta$. Hence, by adding

$$\cos \mathrm{ZBA} + \cos \mathrm{ABQ} = 2 \cos \tfrac{1}{2}\theta \cos \mathrm{GA} = 2 \cos \tfrac{1}{2}\theta \cos \phi,$$

if $\angle$ GBA be denoted by ϕ. Equation (1) thus takes the form

$$n+\sum 2\cos(2k\mathrm{L}\cos\tfrac{1}{2}\theta\cos\phi) \quad . \quad . \quad . \quad . \quad (2)$$

Now BZ, BG, and BQ are fixed directions, but the atom may be orientated in any direction whatever with reference to the incident light, *i.e.* all directions of BA are equally possible. We consequently require the average of the term under the summation sign in (2) for all the different values of ϕ. The number of times ϕ has one particular value is proportional to the area of the zone comprised between ϕ and $\phi+d\phi$ on the sphere of unit radius, *i.e.* to $2\pi\sin\phi\, d\phi$. The average is therefore

$$\begin{aligned}&\frac{1}{4\pi}\int_0^\pi \cos(2k\mathrm{L}\cos\tfrac{1}{2}\theta\cos\phi)2\pi\sin\phi\, d\phi\\ &=\tfrac{1}{2}\int_{-1}^{+1}\cos(2k\mathrm{L}x\cos\tfrac{1}{2}\theta)dx\\ &=\frac{\sin(2k\mathrm{L}\cos\frac{1}{2}\theta)}{2k\mathrm{L}\cos\frac{1}{2}\theta}.\end{aligned}$$

Substituting this value in (2), and at the same time allowing for the variation in the scattering from a single electron, we obtain finally

$$\mathrm{I}=\mathrm{I}'_{\pi/2}(1+\cos^2\theta)\left(n+\sum\frac{\sin(2k\mathrm{L}\cos\frac{1}{2}\theta)}{k\mathrm{L}\cos\frac{1}{2}\theta}\right) \quad . \quad . \quad . \quad (3)$$

for the energy scattered in direction θ. $\mathrm{I}'_{\pi/2}$ has been distinguished with a dash, because it no longer represents the energy radiated in direction $\frac{1}{2}\pi$. The latter is now given by

$$\mathrm{I}_{\pi/2}=\mathrm{I}'_{\pi/2}\left(n+\sum\frac{\sin(2k\mathrm{L}\cos\frac{1}{4}\pi)}{k\mathrm{L}\cos\frac{1}{4}\pi}\right) \quad . \quad . \quad . \quad . \quad (4)$$

There are $\frac{1}{2}n(n-1)$ possible pairs of electrons, *i.e.* $\frac{1}{2}n(n-1)$ terms under the summation sign or in the case of lead $\frac{1}{2}82\times 81=3321$ terms, and before the summation can be made we must know the value of L for each of these pairs. If the wave-length is very long, $k=0$, and

$$\left.\begin{aligned}\mathrm{I}&=\mathrm{I}'_{\pi/2}(1+\cos^2\theta)\left(n+\sum 2\frac{\sin 0}{0}\right)=\mathrm{I}'_{\pi/2}(1+\cos^2\theta)(n+n(n-1))\\ &=\mathrm{I}'_{\pi/2}n^2(1+\cos^2\theta)\end{aligned}\right\} \quad . \quad (5)$$

i.e. the energy radiated is n^2 times greater than the original formula would lead us to expect from a single electron, and n times greater than it would lead us to expect from the atom. This result has already been anticipated.*

§ 2. In order to study the possibilities of formula (3) it is advantageous to take the case of $n=4$ and L the same for all six possible pairs of electrons.

* Barkla and Dunlop, *Phil. Mag.*, xxxi, 1916, p. 231.

This would correspond to the case in which the electrons were arranged at the corners of a tetrahedron of length of edge equal to L. Then (3) becomes

$$I = I'_{\pi/2}(1+\cos^2\theta)\left(4 + 12\frac{\sin(2k\mathrm{L}\cos\frac{1}{2}\theta)}{2k\mathrm{L}\cos\frac{1}{2}\theta}\right) \quad . \quad . \quad . \quad (6)$$

In fig. 4 this expression has been graphed for different values of θ. The curve A gives the first factor alone, the distribution of the scattered

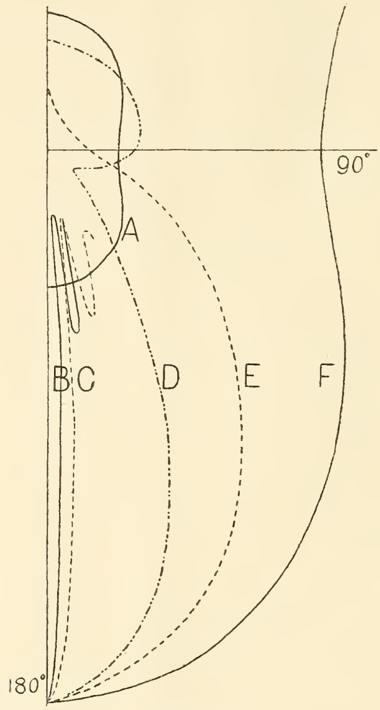


Fig. 4.

energy on the assumption that the four electrons scatter independently. The curves B, C, D, E, and F give the expression for $2k\mathrm{L} = 100$, 50, 10, 5, and $\frac{1}{5}$ respectively. They explain the extra-radiation quite satisfactorily; when the wave-length is very short there is an extra-radiation confined to a small region round about $\theta = 180°$; as the wave-length increases this extra-radiation widens out, until in the case of F it spreads over the whole 360° and we pass over to the case represented by formula (5). The radii of F are four times the radii of A within the limits of accuracy of the diagram. No matter what the value of $2k\mathrm{L}$, the radius has always the same value for $\theta = 180°$.

In the ideal case represented in the diagram L has the same value for each pair of electrons. In the real case L varies from pair to pair. It is easily seen that the effect of this variation will be obtained by adding the radii of different curves of the type shown in the diagram. All such curves coincide at $\theta = 180°$, so we shall always have a well-marked extra-radiation there. But the oscillations shown by B and C will occur at different values of θ in the different cases. Hence these oscillations will annul one another, and, if $2kL$ is large, we shall simply have the extra-radiation passing smoothly into the simple curve A. And, of course, the radii of the curve F are always n times the radii of the curve A.

To show that the curves in fig. 4 are in accordance with experiment, we may quote from a paper by E. A. Owen:* "The distribution on the incident side of the radiator in each case agrees closely with the theoretical distribution given by $I_\theta = I_{\pi/2}(1 + \cos^2\theta)$. In the case of the hardest rays the same theoretical distribution is also obtained on the emergent side for the positions examined (*i.e.* to 150°). A dissymmetry, however, appears when the rays become softer, and this increases with the softness of the rays." And, to quote from another paper by the same author, p. 530 in the same volume: "The total excess radiation round the radiator decreases as the primary beam becomes harder and increases with the atomic weight of the radiator."

It is not possible to derive a value, or even an average value, of L from the observations already made on the extra-radiation. They do not come close enough to 180°.

Let us now consider the numerical value of $I_{\pi/2}$ for the same ideal case of the tetrahedral atom. We have

$$I_{\pi/2} = I'_{\pi/2}\left(4 + 12\frac{\sin(2kL\cos\frac{1}{4}\pi)}{2kL\cos\frac{1}{4}\pi}\right).$$

This is represented as a function of kL in fig. 5, the horizontal line giving the value for no interference. The curve starts at an ordinate four times as high as the straight line, decreases towards the latter, and approaches it in a series of diminishing oscillations. For an actual atom owing to L having different values for the different pairs, the oscillations take place at different points when plotted as a function of k, and hence annul one another. In the case of lead the value of the ordinate at the origin should be eighty-two times the end value.

§ 3. We have now to consider the low value of the scattering coefficient for γ-rays. This is a great difficulty. According to the formula given above, the scattering should diminish until the molecule is large compared

* *Proc. Camb. Phil. Soc.*, 16, 1910–12, p. 161.

with the wave-length, until kL equals perhaps 6π, or $\lambda = \frac{1}{3}L$, but to all further diminutions of λ it should remain constant. J. J. Thomson's formula offers an irreducible minimum. There appears to be no way of getting below it except by reducing the number of scattering electrons, *i.e.* by making this less than the atomic number. So the question must be left open at present. Apart from this difficulty, formula (3) seems to be useful in explaining the scattering of γ-rays. The curves on p. 281 of Rutherford's *Radioactive Substances and their Radiations* are not unlike those in fig. 4.

If the atoms in aluminium were arranged cubically, they would be $2{\cdot}56 \times 10^{-8}$ cm. apart. J. J. Thomson's formula gives about ·19 for the mass-scattering coefficient of aluminium. This is the experimental value

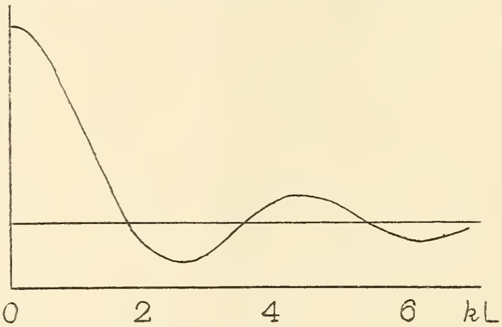


Fig. 5.

for $\lambda = {\cdot}166 \times 10^{-8}$ cm. Consequently the average value of L should be about $3 \times {\cdot}166 \times 10^{-8} = {\cdot}5 \times 10^{-8}$ cm. Thus if there are thirteen electrons in the aluminium atom, there is quite sufficient space to get them all in.

J. J. Thomson's theory of scattering is inconsistent with the Bohr atom model. For, according to the latter, the electrons revolve in rings, yet in defiance of all the laws of electromagnetism the acceleration necessary to keep them in their orbits is unaccompanied by radiation. But according to the theory of scattering the acceleration produced by the light wave is accompanied by radiation. We cannot, without further explanation, maintain at the same time that acceleration is and is not accompanied by radiation.

§ 4. J. J. Thomson's theory of scattering was written before the nature of X-rays was ascertained, and he assumes that they consist of short, sharp pulses.* We know now that they consist of harmonic trains distinguished only from ordinary light by their extremely short wave-length. It is consequently necessary to examine the derivation of the formula afresh.

* *Conduction of Electricity through Gases*, p. 269.

Let

$$m\frac{d^2x}{dt^2}+fx=-eA\cos gt \quad . \quad . \quad . \quad . \quad . \quad (7)$$

represent the equation of motion of the electron under the influence of the impressed wave. Assume that the vibration is forced. Then

$$x=-\frac{eA}{f-mg^2}\cos gt.$$

If we apply the usual formula for a Hertzian doublet, the rate of radiation from this electron is found to be

$$\frac{g^4c}{3}\frac{e^4A^2}{(f-mg^2)^2}.$$

By means of Poynting's theorem the amount of energy flowing through unit area in unit time is $cA^2/8\pi$, where A is measured electrostatically. Hence the fraction radiated is

$$\frac{8\pi}{3}\frac{e^4g^4}{(f-mg^2)^2}.$$

If there are N electrons per unit volume, the fraction radiated becomes N times as great. Hence, if $E=E_0e^{-hy}$ represents the diminution of energy in the wave,

$$\left.\begin{aligned} h&=\frac{8\pi}{3}\frac{Ne^4g^4}{(f-mg^2)^2} \\ &=\frac{8\pi}{3}\frac{Ne^4}{m^2\left(1-\frac{\lambda^2}{\lambda_0^2}\right)^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where λ_0 is the wave-length of the free vibration of the electron. This expression reduces to the usual formula when λ^2/λ_0^2 is small in comparison with I. This assumption is justified for the regions of the spectrum in which the scattering has been investigated.

It has been assumed above for simplicity that all the electrons in the atom have the same free period. If each electron has its own free period, a Σ must be prefixed to the expression in (8) and the summation taken over all the electrons in the atom; N is then the number of atoms per unit volume. According to my paper on the absorption of X-rays, there is one K electron in each atom which vibrates about a position of equilibrium in the atom, its period coinciding with the period of the K absorption band. As we approach the K absorption band from the side of shorter wave-lengths, the λ^2 in the denominator of (8) increases and the denominator decreases in value. Consequently the scattering due to this electron should increase. As a matter of fact, the scattering increases as we approach the K bands for the metals actually investigated—copper, silver, tin, and lead. But a numerical investigation shows that the proportion of this increase due to the free period of the K electron must be a small one; the increase is nearly all due to interference between the different electrons of the atom according to formula (3).

Equation (7) is not quite accurate, because there is a resistance to the motion due to the reaction of the radiation on the electron. This resistance introduces an additional term

$$-\frac{2}{3}\frac{e^2}{c^3}\frac{d^3x}{dt^3}$$

on the left side of the equation (e in electrostatic units). This additional term diminishes the scattering; owing to the reaction of the radiation the electron neither moves so far nor so fast. But the additional term is a small one, and does not become appreciable until the wave-length diminishes to 10^{-11} cm.

I am greatly indebted to Professor Barkla for pointing out an important error in my first calculations on this subject.

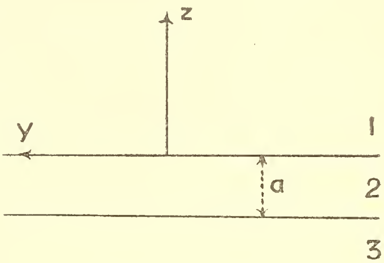
(*Issued separately March* 25, 1920.)

VII.—An Unnoticed Point in the Theory of Newton's Rings. By J. Marshall, M.A., B.Sc., University College, Nottingham. *Communicated by* Professor W. Peddie, D.Sc.

(MS. received November 28, 1919. Read January 12, 1920.)

The consideration of the passage of light waves through three media elucidates an interesting point in the Theory of Newton's Rings.

Let the three media be denoted by the numerals 1, 2, 3, and let $z=0$ be the plane of separation between media 1 and 2 and $z=-a$ the plane of separation between media 2 and 3. Also let the axis of x in the plane $z=0$ be parallel to the wave front of the incident plane waves.



If $f(my+nz+ct)$ be taken to represent the wave form, the electromagnetic equations

$$\text{curl E} = -\mu\dot{\text{H}}$$

and

$$\text{curl H} = \frac{\text{K}}{c^2}\dot{\text{X}}$$

become

1 . . . . . . $\begin{cases} m\text{Z} - n\text{Y} = -\mu c\alpha \\ n\text{X} = -\mu c\beta \\ -m\text{X} = -\mu c\gamma \end{cases}$

and

2 . . . . $\begin{cases} m\gamma - n\beta = \dfrac{\text{K}}{c}\text{X} \\ n\alpha = \dfrac{\text{K}}{c}\text{Y} \\ -m\alpha = \dfrac{\text{K}}{c}\text{Z}. \end{cases}$

Equations 1 and 2 can be replaced by

3 . . . . $\begin{cases} \beta = -\dfrac{n}{\mu c}\text{X} \\ \gamma = +\dfrac{m}{\mu c}\text{X} \end{cases}$

4 . . . . $\begin{cases} \text{Y} = \dfrac{cn}{\text{K}}\alpha \\ \text{Z} = -\dfrac{cm}{\text{K}}\alpha \end{cases}$

and

5 . . . . $m^2 + n^2 = \mu\text{K}.$

If ϕ be the angle of incidence we have

6 $$\frac{m}{\sin\phi} = \frac{n}{\cos\phi} = \sqrt{\mu \mathrm{K}}.$$

It is evident from the form of equations 3, 4, and 5 that if X and α are given for any medium the electric and magnetic vectors in that medium are completely specified. In the three media therefore we have respectively

7 $$\begin{cases} \mathrm{X}_1 = \mathrm{A}e^{i\kappa(my+n_1z+ct)} + \mathrm{A}'e^{i\kappa(my-n_1z+ct)} \\ \mathrm{X}_2 = \mathrm{A}_1e^{i\kappa(my+n_2z+ct)} + \mathrm{A}_2e^{i\kappa(my-n_2z+ct)} \\ \mathrm{X}_3 = \mathrm{A}_3e^{i\kappa(my+n_3\overline{z+a}+ct)} \end{cases}$$

8 $$\alpha_1 = \mathrm{B}e^{i\kappa(my+n_1z+ct)} + \mathrm{B}'e^{i\kappa(my-n_1z+ct)},$$

with similar forms for α_2 and α_3.

Using equation 3, we have the following form for β_1 and similar forms for β_2 and β_3:

9 $$\beta_1 = -\frac{n_1}{\mu_1 c}\mathrm{A}e^{i\kappa(my+n_1z+ct)} + \frac{n_1}{\mu_1 c}\mathrm{A}'e^{i\kappa(my-n_1z+ct)}.$$

The boundary conditions are

10 $$\mathrm{X}_1 = \mathrm{X}_2, \quad \mathrm{Y}_1 = \mathrm{Y}_2, \quad \alpha_1 = \alpha_2, \quad \beta_1 = \beta_2, \quad \text{at } z = 0.$$

11 $$\mathrm{X}_2 = \mathrm{X}_3, \quad \mathrm{Y}_2 = \mathrm{Y}_2, \quad \alpha_2 = \alpha_3, \quad \beta_2 = \beta_3, \quad \text{at } z = -a.$$

From these equations we obtain

12 $$\mathrm{A} + \mathrm{A}' = \mathrm{A}_1 + \mathrm{A}_2\,; \quad p_1(\mathrm{A} - \mathrm{A}') = p_2(\mathrm{A}_1 + \mathrm{A}_2)$$

and

13 $$\mathrm{A}_1e^{-i\theta} + \mathrm{A}_2e^{i\theta} = \mathrm{A}_3\,; \quad p_2(\mathrm{A}_1e^{-i\theta} - \mathrm{A}_2e^{i\theta}) = p_3\mathrm{A}_3$$

where

$$p_1 = \frac{n_1}{\mu_1 c}, \text{ etc.}$$

and

$$\theta = \kappa a n_2 = \frac{2\pi}{\lambda} \cdot a n_2.$$

Similar equations hold for the "B" coefficients if p_1, p_2, p_3 are replaced by q_1, q_2, q_3 where

$$q_1 = \frac{cn_1}{\mathrm{K}_1}, \text{ etc.}$$

Solving equations 12 and 13, we obtain

14 $$\frac{p_1\mathrm{A}}{p_2(p_1+p_3)\cos\theta + i(p_1p_3+p_2^2)\sin\theta} = \frac{p_1\mathrm{A}'}{p_2(p_1-p_3)\cos\theta + i(p_1p_3-p_2^2)\sin\theta}$$
$$= \frac{\mathrm{A}_1}{(p_2+p_3)e^{i\theta}} = \frac{\mathrm{A}_2}{(p_2-p_3)e^{-i\theta}} = \frac{\mathrm{A}_3}{2p_2}$$

We require to examine the reflected light whose amplitude depends on $|A'|$ and $|B'|$. From equation 14 we have for normal incidence

$$15 \quad . \quad . \quad \frac{A'}{A} = \frac{P + iQ}{L + iM} = -\frac{B'}{B} \quad \text{and} \quad \left|\frac{A'}{A}\right|^2 = \left|\frac{B'}{B}\right|^2 = \frac{P^2 + Q^2}{L^2 + M^2}$$

where

$$16 \quad . \quad . \quad . \quad . \quad \begin{cases} P = r_2(r_1 - r_3)\cos\theta \\ Q = (r_1 r_3 - r_2^2)\sin\theta \\ L = r_2(r_1 + r_3)\cos\theta \\ M = (r_1 r_3 + r_2^2)\sin\theta \end{cases}$$

since

$$p_1 = \frac{n_1}{\mu_1 c} = \frac{\sqrt{\mu_1 K_1}}{\mu_1 c} = \frac{r_1}{c} \quad \text{where } r_1 = \text{refractive-index of medium 1}$$

and

$$q_1 = \frac{cn_1}{K_1} = \frac{c\sqrt{\mu_1 K_1}}{K_1} = \frac{c}{r_1}.$$

In each case we have taken unity as the measure of permeability.

The special case of Newton's Rings follows at once from this analysis. When the media are glass, air, glass respectively, $r_1 = r_3$, and P is therefore zero. Hence $|A'|$ and $|B'|$ will both vanish when

$$\sin\theta = 0,$$

i.e. when

$$\theta = k\pi, \quad \text{where } k = 0, 1, 2, 3 \ldots$$

i.e. when

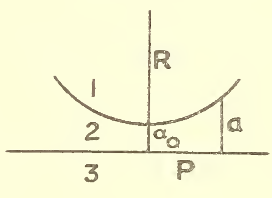
$$\frac{2\pi}{\lambda} \cdot an_2 = \frac{2\pi}{\lambda} \cdot ar_2 = k\pi,$$

i.e.

$$a = \frac{k}{2}\lambda, \quad \text{since } r_2 = 1 \text{ for air.}$$

If the radius of curvature of the surface of medium 1 be R, and the radius of a dark ring be ρ, we have

$$a_0 + \frac{\rho^2}{2R} = k\frac{\lambda}{2}$$



where a_0 = shortest distance between media 1 and 3.

When media 1, 2, 3 are different, $r_1 \neq r_2 \neq r_3$.

(*a*) Since $r_1 \neq r_3$, $P = 0$ when $\cos\theta = 0$.

Also, if $r_1 r_3 \neq r_2^2$, $Q = 0$ only when $\sin\theta = 0$.

Hence P and Q can *never* be zero together.

Similarly, L and M can *never* be zero together.

$\therefore$ $|A'|$ and $|B'|$ will *never* be zero when $r_1 \neq r_3$ and $r_1 r_3 \neq r_2^2$.

In this case, therefore, there will be no ring formation due to interference, although there may be rings due to straining of the media at the surfaces of contact (vide Stokes' *Papers*, vol. ii, p. 86).

(*b*) There is one case in which $|A'|$ and $|B'|$ will vanish, however, viz. when $P=Q=0$ simultaneously and either $R\neq 0$ or $S\neq 0$.

This case will exist when

$$\cos\theta=0, \qquad \therefore \quad P=0;$$

and when

$$r_1 r_3 = r_2^2, \qquad \therefore \quad Q=0;$$

i.e. when

$$\kappa a n_2 = (2k+1)\frac{\pi}{2}, \text{ where } k=0, 1, 2, 3 \ldots$$

i.e. when

$$a_0 + \frac{\rho^2}{2R} = (2k+1)\frac{\lambda}{4r_2}.$$

(i) When $\theta=0$, *i.e.* for $a_0=0$,

$$|A'|\neq 0 \qquad |B'|\neq 0.$$

$\therefore$ In this case we shall have a bright spot at the centre and interference bands given in radii by the equation

$$2r_2\rho^2 = R(2k+1)\lambda.$$

(ii) If, however, $a_0\neq 0$, $|A'|$ and $|B'|$ are each zero when $4r_2a_0=\lambda$, and in this case we shall have a *dark* spot at the centre and interference rings given by the equation

$$r_2\rho^2 = Rk\lambda.$$

(iii) When $a_0\neq 0$ and $4r_2a_0\neq\lambda$, $|A'|\neq 0$ and $|B'|\neq 0$ at the centre, and we have a bright spot at the centre and interference rings given by the equation

$$4r_2\left(a_0+\frac{\rho^2}{2R}\right) = (2k+1)\lambda.$$

It is stated usually that Brewster verified the existence of a bright spot at the centre of Newton's Rings when the refractive index of medium 2 is intermediate between that of medium 1 and medium 3. This relation is not incompatible with the condition $r_1r_3\neq r_2^2$ or with the condition $r_1r_3=r_2^2$.

When $r_1r_3\neq r_2^2$ we have shown in section (*a*) that no interference rings are obtainable.

When, however, $r_1r_3=r_2^2$ and $4a_0r_2\neq\lambda$, we shall have a system of interference rings with a bright spot at the centre, but in the case when $r_1r_3=r_2^2$ and $4a_0r_2=\lambda$ we shall have a system of interference rings with a dark spot at the centre.

Brewster took air, soap solution, and diamond for his three media, giving respectively

$r_1 = 1$ for air.
$r_2 = 1 \cdot 475$ for soap solution.
$r_3 = 2 \cdot 44$ for diamond.

In this case we have

$$r_1 < r_2 < r_3.$$

Also

$$r_1 r_3 = 2 \cdot 44$$

and

$$r_2^2 = 2 \cdot 176.$$

Hence

$$r_1 r_3 - r_2^2 = \cdot 164.$$

This condition approximates closely to the case in which $r_1 r_3 = r_2^2$, and explains why Brewster obtained a bright spot at the centre and variations in intensity approximating to darkness when $\cos\theta = 0$, since case (a) is avoided and case (b) (ii) is highly exceptional.

(*Issued separately March* 25, 1920.)

VIII.—The Effect of Weather Changes on Soil Temperatures. By Captain T. Bedford Franklin, B.A. (Cantab.).

(MS. received January 27, 1920. Read March 15, 1920.)

SUMMARY.

Section I.—Introduction.

The temperature of the air depends on so many varying factors that its prediction is a matter of considerable difficulty, and can only be made with any degree of certainty when the minimum number of these factors is at work. Underground temperatures are dependent not only on the same factors as affect the air temperature, but, in addition, are much more affected by conductivity, rainfall, evaporation, latent heat, etc., and thus the changes in temperature beneath the surface of the soil constitute a much more complex problem.

In my previous papers I have discussed the question of underground temperature under very special conditions only—namely, during calm, clear nights; but as it is a sound principle to assume that the points of even the most simple aspect of a problem cannot be appreciated fully until the whole complex problem has been investigated, I now propose to attempt to give a review of all the changes in underground temperatures brought about by weather changes at the surface.

In comparison with the variation of surface temperature, the regular pulsations of temperature underground follow well-known laws for amplitude and retardation according to depth; but in these regular pulsations there are minor fluctuations which occur either day by day, or at irregular intervals, according to the weather and the state of the surface soil.

Of these minor fluctuations the most important are due to the percolation of rain, the movements of soil air and moisture, the presence of a dry surface mulch, the prevalence of strong winds of low relative humidity, or the occurrence of frost and snow. We have thus several agencies at work tending to disturb the regular changes of temperature underground, and I propose to deal with each of these in turn in the light of my observations with electrical resistance thermometers during the past year. These observations have been kept in graphical form, and it is thus a simple matter to note the combined effect of the various agencies at work, and it is possible in favourable circumstances to assign to each agency its own share in the fluctuations.

For a discussion such as this, mean daily values of underground temperature would be useless, as frequently the agency at work causing the fluctuations from the normal is of a temporary nature, and its effect would only come to view in hourly values during the period of its activity; the whole of this paper is therefore based on the records of hourly values taken during the day and on selected nights during 1919.

Throughout this paper I have used the ratio of the ranges of temperature at the 4-inch depth and at the surface $\left(\frac{R_4}{R_0}\right)$ as my standard; this is a purely arbitrary choice, as the ratio of the ranges at the 8-inch depth and at the surface $\left(\frac{R_8}{R_0}\right)$ would have served the same purpose, as similar fluctuations—though in a less degree—are shown by the values of $\frac{R_8}{R_0}$. Below the 8-inch depth comparison is not possible, as the soil at that depth in the garden where the observations were made is a stony subsoil of quite a different nature to the surface soil.

Section II.—Sunshine and Shade.

(a) *Movements of Soil Air and Water.*

It is an undoubted fact that changes in temperature of the surface soil bring about considerable movements of the soil air; these air movements have been discussed by Professor King in his book *The Soil*, where he gives particulars of their influence on the height of water in wells and on the flow of springs.* If the downward expansion of air—heated by the action of the sun on the surface—is sufficient to bring about these movements of soil water, it is only to be expected that it should also affect the underground temperatures.

When we examine closely the temperature curve for the 4-inch depth on sunny and overcast days, two points of difference are at once apparent:—

(1) On sunny days, as soon as the surface begins to rise rapidly in temperature a slight convexity appears in the 4-inch depth curve which is not noticeable on an overcast day. This convexity cannot be due to conduction, as it appears very soon after the first rapid rise of the surface, whilst any change due to conduction could only show after a number of hours equal to the lag on the day in question. Moreover, the surface temperature curve is almost invariably concave, and so could not give rise to any convexity in the 4-inch depth curve. A corresponding but slighter concavity appears early in the afternoon, when the surface falls rapidly in temperature.

It appears that these changes in the shape of the 4-inch depth temperature curve can be explained in part by the fact that as soon as the surface becomes hot the soil air expands and pushes downwards, displacing the air round the 4-inch depth thermometer and causing a small rise of temperature there; whilst in the early afternoon, when the surface cools again and the soil air contracts, this displaced air rises and regains its former position. This would only be noticeable on sunny days, when the surface changes of temperature are large and rapid.

Recent observations seem to point to the possibility of these variations being brought about in part by renewed percolation of water due to the reduced viscosity and surface tension of the soil water consequent on the rapid rise of temperature of the soil.

Under strong sun a temperature of 30° C. at the surface, and from 15°–20° C. at the 4-inch depth, may often be attained, and the very considerable reduction in viscosity entailed must result in renewed percolation of soil water, with a corresponding rise in the 4-inch depth temperature. This will be further discussed in Section V.

(2) On overcast days or in shady places, owing to the absence of the above-mentioned convexity, the range of temperature at the 4-inch depth is diminished, and the value of $\frac{R_4}{R_0}$—the ratio of the ranges at the 4-inch

* *The Soil*, by F. H. King. The Rural Science Series. Macmillan, 1917.

depth and at the surface—is below the normal from minimum to maximum, and lies between the two values from maximum to minimum on either side of it; whilst on sunny days the corresponding value of $\frac{R_4}{R_0}$ is normal and follows closely the two values from maximum to minimum on either side of it, only showing a very slight downward tendency owing to reduced conductivity from the drying of the soil.

This is clearly shown in Table I.

Table I.—Relation of $\frac{R_4}{R_0}$ to the Normal—Sunny and Dull Days.

Date.	Sunny Days.			Dull Days.		
	Previous $\frac{R_4}{R_0}$.	$\frac{R_4}{R_0}$ Minimum to Maximum.	Subsequent $\frac{R_4}{R_0}$.	Previous $\frac{R_4}{R_0}$.	$\frac{R_4}{R_0}$ Minimum to Maximum.	Subsequent $\frac{R_4}{R_0}$.
1919.						
April 19	...	...	...	·42	·40	·44
,, 20	...	...	...	·44	·40	·42
,, 22	...	...	...	·38	·35	·38
May 2	...	...	...	·41	·36	·39
,, 3	·39	·38	·39	...	...	...
,, 16	·31	·31	·31	...	...	...
,, 19	·40	·39	·38	...	...	...
,, 21	...	...	...	·35	·33	·35
,, 24	·30	·30	·30	...	...	...
,, 26	·30	·31	·30	...	...	...
,, 27	·30	·30	·30	...	...	...
,, 30	...	...	...	·32	·29	·30
June 1	·28	·28	·28	...	...	...
,, 8	...	...	...	·42	·39	·42
,, 16	...	...	...	·35	·35	·39
,, 18	·32	·32	·32	...	...	...
,, 27	·36	·37	·35	...	...	...
Oct. 9	...	...	...	·34	·29	·34
,, 15	...	...	...	·37	·30	·36
Averages	·33	·33	·325	·38	·345	·38

(*b*) *Diffusion of Water Vapour.*

Dr John Aitken has clearly shown that diffusion of water vapour takes place in the soil, downwards by day when the surface is hot, and upwards at night when it is cold compared to the underground layers. This upward diffusion at night he shows to be one of the prime causes of the formation of dew.*

* "On Dew," by John Aitken, LL.D., F.R.S., *Trans. Roy. Soc. Edin.*, vol. xxxiii.

On clear nights, when the surface of the soil is of much the same temperature as the air above it, some of this rising vapour must be trapped in the rough surface layer and condensed there: one has only to examine the surface on a frosty morning, and notice the hoar frost on the under side of every little earth clod, to see that this is so.

Now this condensation of vapour in the surface must give rise to the liberation of a considerable amount of latent heat which will go towards balancing the outgoing radiation, and will prevent the surface falling to as low a temperature as it otherwise would have done; we should thus expect that, since R_0 was diminished, the value of $\frac{R_4}{R_0}$ on such nights would be above the normal, and would be greater than the values of $\frac{R_4}{R_0}$ from minimum to maximum on either side of it.

This is shown in Table II: the variation will, of course, be noticed

Table II.—Relation of $\frac{R_4}{R_0}$ to Normal on Clear Nights when Risen Vapour condenses in Surface.

Date.	Previous $\frac{R_4}{R_0}$.	$\frac{R_4}{R_0}$ Maximum to Minimum on Clear Nights.	Subsequent $\frac{R_4}{R_0}$.	Fall of Surface Temperature below 4-inch Depth Temperature.
1919.				
Apr. 12/13	·22	·28	·23	4·9° C.
„ 19/20	·40	44	·40	7·0° C.
„ 20/21	·40	·42	·40	6·0° C.
May 1/2	·38	·41	·36	6·3° C.
„ 2/3	·36	·39	·38	5·5° C.
Oct. 8/9	·26	·34	·29	5·5° C.
„ 12/13	·30	·37	·30	5·3° C.
„ 15/16	·30	·36	33	4·0° C.
Averages	·33	·375	·34	5·6° C.

only on nights when the surface temperature falls so much lower than the 4-inch depth temperature as to be below the dew point of the rising vapour.

(c) *A Dry Surface Mulch.*

Dry soil contains a large percentage of air, and as air is an even worse conductor than water, the conductivity of soil is increased by wetting, and is at its maximum just after rain. As soon as a period of settled dry weather sets in, the value of $\frac{R_4}{R_0}$ falls from the normal value of ·42 in wet soil to somewhere about ·28, the latter value depending on the length of the dry period and the depth of the mulch formed on the surface.*

That this does not entail a decided check in the gradual warming up of the underground layers is due to the fact that a dry soil rises to a much higher maximum temperature than a wet one, for in the wet surface a considerable amount of the radiant energy of the sun is used up in evaporation rather than in raising the temperature of the surface. Whilst a surface range of 20° C. in wet soil when $\frac{R_4}{R_0} = \cdot 40$ produces a 4-inch depth range of 8° C., a surface range of 27° C. in a dry soil when $\frac{R_4}{R_0} = \cdot 30$ will produce a 4-inch depth range of 8·1° C., *i.e.* the temperature ranges at the 4-inch depth are practically the same.

This automatic adjustment of the underground range is well seen in observations made during dry spells of 8 days in May 1919 and 16 days in May and June 1919, shown in Table III.

Table III.—Ranges at Surface and 4-inch Depth during Dry Periods.

Date.	Number of Previous Dry Days.	$\frac{R_4}{R_0}$.	Surface Range.	4-inch Range.	Hours of Sunshine.	Remarks.
1919.						
May 10	0	·40	13·0° C.	5·2° C.	8·2 hrs.	After 5 millimetres of rain on May 18, $\frac{R_4}{R_0}$ rose to ·43, and the surface range fell to 18° C.
„ 12	2	·36	14·2° C.	5·1° C.	5·0 „	
„ 15	5	·32	19·0° C.	6·0° C.	12·8 „	
„ 17	7	·30	20·0° C.	6·0° C.	7·0 „	
May 18	0	·43	18·0° C.	7·8° C.	7·0 hrs.	On June 3, after the 16 dry days a $1\frac{1}{2}''$ dry mulch had formed, but when 4 millimetres of rain fell, $\frac{R_4}{R_0}$ rose to ·40, and the surface range fell to 18·5° C.
„ 22	4	·35	24·0° C.	8·4° C.	11·8 „	
„ 26	8	·30	27·0° C.	8·1° C.	11·4 „	
„ 28	10	·29	28·0° C.	8·1° C.	13·2 „	
June 2	15	·28	25·4° C.	7·1° C.	9·7 „	

It will thus be seen that, while a dry surface mulch has a decided effect on the values of $\frac{R_4}{R_0}$ and the conductivity of the soil, owing to the increased range of the surface when the soil is dry, there is no great corresponding effect on the 4-inch depth temperature under a mulch of $1\frac{1}{2}$ inches. The underground layers thus tend to rise fairly steadily in the spring and early summer, under normal weather conditions, in which deep mulches are not likely to form, so long as sunshine is not deficient.

* "The Cooling of the Soil at Night, with special reference to Spring Frosts," by the Author, *Proc. Roy. Soc. Edin.*, vol. xl, 1919–20, part i, No. 2.

The very gradual reduction day by day in the value of $\frac{R_4}{R_0}$ when the surface mulch is over 1 inch deep shows how efficient even so slight a depth of dry soil is in reducing loss of water by evaporation, and it is doubtful if much advantage is obtained, as far as checking evaporation is concerned, by further depths of dry soil. Yet King has proved the value of a 4-inch mulch in conserving water in the soil,* and possibly it is the fact that such a depth of dry soil, acting as a poor conductor, prevents any great rise in temperature of the soil under the mulch, and so reduces the possibility of renewed percolation of water, due to the reduced viscosity and surface tension of the soil water consequent on high underground temperatures.

Section III.—Rain.

(a) *Percolation of Water.*

The normal value of $\frac{R_4}{R_0}$ in wet soil a few hours after rain has ceased is about ·42, with a corresponding lag of about $3\frac{1}{4}$ hours; but these values are subject to great fluctuations during the actual period of rainfall, as may be seen from Table IV.

Although such high values of $\frac{R_4}{R_0}$ prevail during actual rainfall, the

Table IV.—Values of $\frac{R_4}{R_0}$ and Lag during Rain.

Date.	Value of $\frac{R_4}{R_0}$.	Value of Lag.	Temperature Conditions of Surface and 4-inch Depth.	Time of Day Rain fell.
1919.				
Apr. 10	·55	$2\frac{1}{2}$ hrs.	Surface warmer than 4-inch depth	11 a.m.
May 4	·50	3 ,,	,, ,, ,,	11 a.m. and 2–3 p.m.
June 4	·60	2 ,,	,, ,, ,,	11 a.m.–3 p.m.
,, 13	·85	50 mins.	,, ,, ,,	8 a.m.–12 noon.
Oct. 29	·50	$2\frac{1}{2}$ hrs.	,, ,, ,,	8 a.m.–12 noon.
Nov. 17	...	50 mins.	,, ,, ,,	8 a.m.–3 p.m., after snow lying.
,, 18	·55	$2\frac{1}{4}$ hrs.	,, ,, ,,	2 p.m.–8 p.m.
Apr. 13	·70	1 hr.	Surface colder than 4-inch depth	Noon.
May 2	·75	$1\frac{1}{2}$ hrs.	,, ,, ,,	Night.
,, 17	·50	$2\frac{1}{2}$,,	,, ,, ,,	5 p.m. onwards.
June 7	·75	1 hr.	,, ,, ,,	5 p.m.–6.30 p.m.
,, 12	·70	...	,, ,, ,,	6 p.m.–7 p.m.
,, 25	·60	$1\frac{1}{2}$ hrs.	,, ,, ,,	3.30 p.m. 5.30 p.m.
Oct. 23	·71	1 hr.	,, ,, ,,	2 p.m.–10 p.m.
Dec. 21	·68	$1\frac{3}{4}$ hrs.	,, ,, ,,	12 noon onwards
,, 24	·60	$1\frac{3}{4}$,,	,, ,, ,,	2 p.m.–5 p.m.

* F. H. King, *Physics of Agriculture*, p. 186.

values tend to return to normal so soon after rain has ceased that it seems reasonable to suppose that the percolation of water rather than a change in the conductivity of the soil is the cause of these fluctuations.

Rain is thus a great equaliser of temperature between the surface and underground—in fact, it is probable that the first rise of temperature underground from the low level of winter is due to warm spring rains. It is interesting to note in this connection that the time of day when most rain falls at Kew Observatory is as follows:—*

April–August: Noon–6 p.m., when surface is warm.

September–March: 6 p.m.–6 a.m., when surface is cold.

(b) *Rapid Changes in Underground Temperature after Heavy Rain.*

Given heavy rainfall, and a rapid subsequent change from cloudy to clear sky, we should expect these exceptional values of $\frac{R_4}{R_0}$ and the lag to cause very marked rapid changes in underground temperature. For if the rain has been heavy enough to saturate the soil and form in pools on the surface, the surface water must be considerably warmed in the sun or cooled under the clear night sky, before it succeeds in draining away, and this heating or cooling will in turn be communicated to the underground layers through which it percolates.

Moreover, the soil air which has been rapidly driven downwards before the percolating water must be renewed from the surface as soon as the rain ceases, and it too will be warmer or colder than the air whose place it takes, according to the time of day at which the sudden change from rain to clear sky takes place.

Four instances of this exceptional result are given in Table V; the

Table V.—Rapid Changes of Underground Temperature after Heavy Rain.

Date.	Duration of Rain.	Subsequent Change in Underground Temperature.
1918.		
Dec. 5	10 a.m.–12.30 p.m.	6-inch depth temperature rose 2·5° C. in 2 hours.
„ 10	2–7 p.m.	„ „ „ fell 2·2° C. „
1919.		
June 7	5–6.30 p.m., very heavy.	4-inch depth temperature fell 2·5° C. in 1 hour.
„ 13	8 a.m.–12 noon.	„ „ „ rose 3·0° C. „

temperature curves of the surface and 4-inch depth temperatures on June 13, 1919 show the following remarkable features:—

* *Meteorological Glossary*, 1918. M.O. 225, ii, p. 230.

(1) The surface and 4-inch depth curves were both at about 14° C. for some hours during the rainfall.

(2) The 4-inch depth temperature rose 3° C. between 1 and 2 p.m., the rise beginning only 50 minutes after the rise at the surface.

Section IV.—The Relation of the Above.

(a) *Various Soils—Sand, Loam, and Clay.*

If we compare sand, loam, and clay in a similar wet condition after rain, we find considerable difference in the values of $\frac{R_4}{R_0}$ from one to another. Thus in sand $\frac{R_4}{R_0}$ seems to have a value of about ·60, in loam of ·42, and in clay of ·40.

As the soil dries these values decrease—very rapidly in sand to ·23, fairly rapidly in loam to ·28, and slowly in clay to ·37 after a considerable period of dry, sunny weather.

At the beginning of the period the surface ranges should have been the greatest in sand and the least in clay, with an intermediate value for loam—due to the specific heats of equal volumes of wet sand, loam, and clay being ·34, ·53, and ·61 respectively. But when a dry mulch had formed the ranges should be in the following order—sand, clay, loam, since the specific heats dry are ·12 for sand, ·15 for clay, and ·18 for loam. This ideal state of things is upset by two causes—the disinclination of clay to form a dry surface mulch, and the fact that both sand and clay, owing to their light colour, reflect back some of the radiant energy of the sun which is absorbed by the darker loam.

In effect we find that the ranges at the surface are greatest in sand, though not so much greater as we might expect from its low specific heat, intermediate in loam, and least in clay, under practically all field conditions.

In Table VI the surface and 4-inch depth ranges for these three soils are compared during a drought of 16 days in May and June 1919, and it will be seen that clay suffers in comparison to sand and loam by reason of its low value of $\frac{R_4}{R_0}$ when thoroughly wet—possibly due to the restricted movements of soil air and water—and later on, when its value of $\frac{R_4}{R_0}$ is greater than for either sand or loam, by its disinclination to form a dry surface and so attain a high maximum temperature in the surface. It does not compare equally with the other soils until after a period of about ten days' dry weather—a state of affairs that is unlikely to occur frequently in spring; moreover, while warm spring rains percolate very rapidly into sand or loam and so carry their warmth with them, clay absorbs water so slowly that a large proportion of the water runs off down furrows, etc., and the residue takes so long to percolate that its heating effect is lost.

Table VI.—Rise of Underground Temperature in Sand, Loam, and Clay.

Date.	Number of Previous Dry Days.	Sand. $\frac{R}{R_0}$.	Sand. Surface Range.	Sand. 4-inch Range.	Loam. $\frac{R_4}{R_0}$.	Loam. Surface Range.	Loam. 4-inch Range.	Clay. $\frac{R_4}{R_0}$.	Clay. Surface Range.	Clay. 4-inch Range.
1919.										
May 18	0	·60	19·0° C.	11·4° C.	·43	18·0° C.	7·8° C.	·40	16·0° C.	6·4° C.
„ 19	1	·52	26·0° C.	13·5° C.	·40	24·0° C.	9·6° C.	·40	17·0° C.	6·8° C.
„ 22	4	·40	25·0° C.	10·0° C.	·35	24·0° C.	8·4° C.	·39	17·0° C.	6·6° C.
„ 26	8	·30	28·0° C.	8·4° C.	·30	27·0° C.	8·1° C.	·38	19·0° C.	7·2° C.
„ 28	10	·27	29·0° C.	7·8° C.	·29	28·0° C.	8·1° C.	·37	20·0° C.	7·4° C.
June 2	15	·23	26·0° C.	6·0° C.	·28	25·4° C.	7·1° C.	·37	18·0° C.	6·7° C.

May 18–28. Rise of 4-inch depth temperature :—
Sand +8·2° C. Loam +6·6° C. Clay +4·2° C.

Clay soil is thus cold and late in spring compared to the others, and, conversely, warm in autumn after the other two have cooled down by radiation and percolation of cold rains.

The following figures for the difference in soil temperature at a depth of 6 inches in a sandy loam and clay soil, only about 200 yards distant from each other, are given in *Science and Fruit Farming*, by the Duke of Bedford and Spencer Pickering:—*

	Sandy Loam.	Clay.	
January	...	+0·9° F.	Mean of 20 years 1895–1915.
February	...	+0·8° F.	
March	...	+0·7° F.	
April	...	+0·2° F.	
May	+0·8° F.	...	
June	+1·2° F.	...	
July	+1·2° F.	...	
August	+0·3° F.	...	
September	+0·0° F.	+0·0° F.	
October	...	+0·7° F.	
November	...	+1·0° F.	
December	...	+1·2° F.	

* *Science and Fruit Farming*, by the Duke of Bedford and Spencer Pickering. Macmillan.

(b) The Seasonal Changes of Conductivity.

If we combine the various effects of the agencies already considered in a general review of the year, we should expect to find that $\frac{R_4}{R_0}$ had the following values during the seasons:—

(1) A high value in winter, due to percolation of water—except during periods of frost and lying snow, which will reduce the value in the late winter.

(2) A rise in early spring after frost is over and when the longer sunny days with frequent showers have become a feature of the weather.

(3) A fall in summer due to the effect of a dry surface mulch.

(4) A second rise in autumn when damper conditions have set in and the sun is still strong.

(5) A return to the normal winter value in November.

In Table VII are given the mean values of $\frac{R_4}{R_0}$ for each month compared with the monthly rainfall; there appears to be a very close connection between the value of $\frac{R_4}{R_0}$ and the rainfall, and perhaps an even closer connection between it and the days of precipitation; this is particularly noticeable in January and October 1919, when 50 millimetres fell in both months, but on 24 days in the former and 10 days in the latter month—the corresponding values of $\frac{R_4}{R_0}$ being ·40 and ·34 respectively.

Table VII.—Average Monthly Values of $\frac{R_4}{R_0}$ and the Rainfall.

1919. Month.	$\frac{R_4}{R_0}$.	Rainfall in Millimetres.	No. of Days of Precipitation.	Remarks.
January	·40	50	24	
February . . .	·36	50	16	Hard frost.
March	·30	32	22	" "
April	·35	70	16	
May	·34	17	10	Sunny and warm.
June	·35	40	17	
July	·32	17	8	Sunny and dry.
August	·33	40	13	Sunny.
September . . .	·35	52	13	
October	·34	52	10	
November . . .	·42	115	24	Very wet.
December . . .	·40	95	23	

Section V.—Heat Transfer in the Soil.

With the evidence of the preceding sections before us, there seems no doubt that percolation plays a very large part in the transfer of heat in the soil, not only during or soon after actual rainfall, when water is percolating freely through the soil, but at other times when for any reason percolation starts afresh in soil that apparently had given up all its gravitational water.*

If a sudden increase of temperature in the soil does bring about renewed percolation, it seemed reasonable to suppose that the effect of this percolation would be felt more fully and rapidly vertically underneath the source of the heat than at an equal lateral distance from it. To test this, I buried a hollow metal sphere in the soil and arranged thermometers 3 inches vertically above and below it, and 3 inches laterally from it; the sphere was then filled with boiling water and the behaviour of the thermometers noted. The thermometers below and to the side of the sphere were put in position without disturbing the soil; the thermometer above the sphere was put in the replaced surface soil which had been carefully removed before digging the sphere in. The whole area was protected from atmospheric influence by a layer of sacks and earth; the rise in temperature and lags of the various thermometers are given below:—

Experiment I.

3 inches below sphere . .	Rise = 5·6° C.	Lag = 2·8 hrs.
„ above sphere . .	„ = 4·0° C.	„ = 3·5 „
„ to side of sphere .	„ = 4·3° C.	„ = 3·2 „

Experiment II.

3 inches below sphere . .	Rise = 4·6° C.	Lag = 2·5 hrs.
„ above sphere . .	„ = 3·2° C.	„ = 3·2 „
„ to side of sphere .	„ = 3·3° C.	„ = 3·1 „

Both experiments were made when the soil was in a drying condition after several dry days; I thus hoped to catch the moment when there was a certain amount of "transfer resistance" to heat due to discontinuities in the water films round the soil particles, though the soil was not yet too dry to prevent fresh percolation when these discontinuities disappeared in virtue of the decreased viscosity and surface tension of the soil water due to the applied heat.

Whether I actually did catch the auspicious moment is difficult to tell, but the results show that in both experiments the heat from the sphere travelled more easily and rapidly vertically below the heat source than in any other direction, and I venture to think that this result must have been due to renewed percolation.

* F. H. King, *The Soil*; A. D. Hall, *The Soil.*

The conditions under which heat transfer takes place, and the rapidity of this transfer, in soil of varying degrees of wetness would seem to be as follows:—

TABLE VIII.—TRANSFER OF HEAT IN LIGHT LOAM SOIL.

Condition of Soil.	Observed Values of $\frac{R_4}{R_0}$.	Mode of Heat Transfer.	Remarks.
I. Very dry . . .	·19	By conduction from particle to particle, with considerable discontinuity owing to small number of points of contact and many air gaps.	See Section II (*c*).
II. Increasing wetness up to point when free percolation begins.	·19 to ·42	By conduction from particle to particle, with gradually lessening discontinuity owing to growing water films. During this stage, either before free percolation begins or after it has ceased, fresh percolation may be induced by a rise in temperature of the soil.	See Sections II (*a*) and III (*a*).
III. Further addition of water, so that—			
(*a*) There is free percolation; or	·42 to ·85	(*a*) By conduction and percolation.	See Sections III(*a*), III(*b*), and IV (*b*).
(*b*) There is water-logging.	·42 to ·30	(*b*) By conduction only, gradually diminishing in value to conduction of water.	

SECTION VI.—WIND AND RELATIVE HUMIDITY.

The surface of the soil is almost always damp, and on overcast days or in the shade the surface temperature follows the *wet*-bulb temperature very closely, and during the prevalence of strong winds of low relative humidity is often considerably below the air temperature as given by the dry bulb.

In spring, particularly when the mean relative humidity is low and strong winds are frequent, the surface and underground temperatures may be reduced by excessive evaporation below the normal even though the air temperature as shown by the dry bulb is equal to or above the normal.

Thus, under exceptional circumstances, the rule that the relation of the mean air temperature to the normal is a good index of the relation of underground temperature to its normal breaks down; this happened at Hodsock Priory in April 1915, when an unusually low relative humidity with an excess of N.E. winds of more than average velocity was recorded, and underground temperatures were low in spite of the air temperature being above the normal. A similar result, though in a less degree, was noticeable in April 1911 and 1916.

Some observations made in the spring and autumn of 1919 on the difference of temperature between the air and the surface under varying conditions of wind and humidity are given in Table IX, and it will be seen

TABLE IX.—DIFFERENCES BETWEEN AIR AND SURFACE TEMPERATURE IN WINDS OF VARYING FORCE AND HUMIDITY.

	7 a.m.			1 p.m.			6 p.m.		
Date.	Wind.	Humidity.	Fall of Surface below Air Temperature.	Wind.	Humidity.	Fall of Surface below Air Temperature.	Wind.	Humidity.	Fall of Surface below Air Temperature.
1919.									
Apr. 2	...	...	...	W. 5	65	3·6° C.	W. 4	85	1·3° C.
„ 3	0	92	1·0° C.	E.N.E. 2	92	2·0° C.	E. 2	92	·4° C.
„ 4	0	98	·2° C.	W.S.W. 6	65	3·4° C.	W.S.W. 6	87	·7° C.
„ 5	W.S.W. 6	87	·8° C.	W.S.W. 6	75	2·0° C.	W.S.W. 6	75	1·6° C.
„ 6	W.S.W. 4	87	·5° C.	W.S.W. 5	65	2·4° C.	S.W. 4	80	·8° C.
„ 7	S.W. 4	75	1·0° C.	S.W. 4	65	1·6° C.	S.W. 3	65	1·3° C.
Oct. 25	N.E. 4	75	1·0° C.	N.N.E. 1	75	1·2° C.	W.S.W. 1	75	·9° C.
„ 26	N.W. 4	75	1·0° C.	N.W. 6	65	3·2° C.	N. 5	75	2·0° C.
„ 28	N. 4	82	·4° C.	N.N.E. 4	75	2·3° C.	N.E. 2	92	0·0° C.
Nov. 18	W.N.W. 4	82	1·4° C.	W.S.W. 6	75	2·5° C.	S.W. 7	82	1·2° C.
„ 21	W.N.W. 4	75	1·7° C.	N.W. 3	65	2·5° C.	N. 3	75	1·4° C.
„ 22	W.S.W. 4	92	1·0° C.	W. 8	75	2·2° C.	W. 4	82	1·5° C.
Dec. 19	N.W. 5	55	2·6° C.	W.N.W. 5	60	2·5° C.	W. 3	90	1·3° C.
1920.									
Jan. 7	S.W. 3	90	·8° C.	S.W. 6	85	1·8° C.	S.W. 8	80	1·5° C.
„ 27	S.W. 6	75	2·1° C.	S. 8	72	3·6° C.	N.W. 6	65	2·2° C.

that the differences vary between 0° C. and 3·6° C. under the extreme conditions of no wind and practically saturated air, and strong wind and dry air, respectively. A difference of 3·6° C. at the surface corresponds to a difference of about 1·5° C. at the 4-inch depth, showing how potent an influence wind and humidity are on underground temperatures.

SECTION VII.—FROST.

(a) *The Effect of Frost.*

At the beginning of a spell of frost, and before the surface actually becomes frozen, the fall in the underground temperature is of course fairly rapid, as it follows with normal amplitude and retardation the fall of the surface temperature. As soon as the surface freezes, however, this rapid fall is replaced by a slow, steady decrease in temperature, which is strongly in contrast to its former rapid fall.

The small changes in underground temperature during a period of frost are due to the fact that when once the soil is frozen a layer of invariable temperature—the bottom of the frozen soil, where the temperature is always 0° C.—lies between the surface and the lower depths. During a spell of frost the surface may fall to several degrees below 0° C., but, owing to the latent heat liberated at the bottom of the frozen layer, the temperature gradient in the frozen soil is much greater than in the unfrozen soil beneath, and the temperature variations in the surface are not conducted uniformly downwards, the only result of low surface temperatures being to slowly increase the depth of the frozen soil.

Thus, during the process of freezing and thawing until the whole frozen depth is thawed and the temperature gradient again becomes uniform, surface variations of temperature hardly affect the 4-inch thermometer at all.

These contrasted fluctuations may be well seen in Table X, where the movements of underground temperature at the 4-inch depth before and during spells of frost are given. No snow was on the ground on the dates considered; the effect of a covering of snow will be dealt with later.

Table X.—Fluctuations in Temperature at 4-inch Depth just before and after Surface is Frozen.

Date.	Air over Soil. Minima.	Surface. Minima.	4-inch Depth. Before Surface froze. Fall and Rate.	4-inch Depth. After Surface froze. Fall and Rate.
1919. Mar. 20–26	−2·0° C. −3·0° C. −5·5° C. −3·5° C. −5·0° C. −5·0° C.	0·0° C. 0·0° C. −2·8° C. −1·5° C. −1·6° C. −2·2° C.	1·6° C. in 12 hours.	1·0° C. in 120 hours.
Apr. 19–20	1·2° C.	0·0° C.	3·9° C. in 12 hours.	...
Oct. 12–13	−0·4° C.	0·0° C.	3·6° C. in 18 hours.	...
Nov. 7–12	−1·4° C. 0·0° C. −0·8° C. −2·4° C.	−1·0° C. −1·2° C. 0·0° C. −0·8° C.	2·2° C. in 18 hours.	1·1° C. in 84 hours.
Dec. 6–9	−0·2° C. −1·1° C. −3·0° C. −4·0° C.	−0·4° C. −1·0° C. −1·2° C. −1·6° C.	1·1° C. in 12 hours.	0·7° C. in 36 hours.
„ 24–26	−1·0° C. −5·3° C.	0·0° C. −2·4° C.	1·4° C. in 24 hours.	0·5° C. in 24 hours.
„ 31–Jan. 4, 1920	−4·1° C. −2·4° C. −1·0° C. −2·0° C.	−2·1° C. −2·6° C. −1·8° C. −0·5° C.	1·7° C. in 24 hours.	0·4° C. in 72 hours.

(b) The Conductivity of Frozen Soil, and the Depth frozen for Various Surface Temperatures.

As soon as we begin to deal with frozen soil the question of latent heat becomes all-important. The soil in which my observations have been made was found, after many experiments in November 1918, and again in November 1919, to contain about 45 per cent. by volume of water in its normal winter condition. Thus one cubic centimetre of soil contains ·45 c.c. of water, and 36 calories are liberated in freezing it.

If K' = the conductivity of frozen soil,
x = total depth frozen,
h = hours of frost,
θ_0 = average temperature of surface during h,

then, when once the surface is frozen, the heat escaping from the bottom of the frozen layer to the surface must equal the sum of the latent heat liberated and the heat conducted to the bottom of the frozen layer from the 4-inch depth. From this relation the following values were obtained for K':—

1919.		
March 22/23	. . . .	$K' = \cdot 0046$
„ 23/24	. . . .	$K' = \cdot 0040$
„ 24/25	. . . .	$K' = \cdot 0050$
„ 25/26	. . . .	$K' = \cdot 0048$
„ 28/29	. . . .	$K' = \cdot 0040$
Nov. 13/14	. . . .	$K' = \cdot 0044$
„ 14/15	. . . .	$K' = \cdot 0043$

giving an average value of $K = \cdot 0045$.

If now we neglect the upward conduction from the 4-inch depth, which must in any case be small, then the latent heat liberated in freezing a small depth dx must be conducted upwards from the frozen layer to the surface in time dt.

$$\therefore \quad \frac{K'\theta_0}{x}\,dt = 36dx$$

$$\therefore \quad 36x\,dx = K'\theta_0 dt$$

$$\text{Integrating,} \quad x^2 = \frac{K'\theta_0}{18}t = \cdot 9\theta_0 h,$$

giving us the depth frozen in terms of the mean surface temperature below 0° C. and the duration of the frost.

In Table XI the calculated and observed values of the depth frozen on several dates are compared; as we have neglected the upward conduction from the 4-inch depth, we should expect the calculated values to be slightly too large, and this is found to be the case.

Table XI.—Calculated and Observed Values of Depth Frozen.

Date. (1)	h. (2)	θ_0. (3)	$x^2 = \cdot 9\theta_0 h$. (4)	Calculated Value of Depth frozen in cms. (5)	Observed Value of Depth frozen in cms. (6)	Difference of Columns (5) and (6) in cms. (7)
1919.						
Mar. 22/23	10	1·15	10·35	3·2	3·0	+·2
„ 23/24	10	·53	4·7	2·15	1·8	+·35
„ 24/25	11	·75	7·4	2·7	2·6	+·1
„ 25/26	11	·95	9·4	3·1	3·0	+·1
„ 28/29	10	1·23	11·0	3·3	3·0	+·3
Apr. 2/3	10	·20	1·8	1·35	1·0	+·35
„ 3/4	12	·75	8·1	2·8	2·5	+·3
„ 4/5	10	·80	7·2	2·7	2·3	+·4
Nov. 13/14	10	5·1	45·9	6·8	6·4	+·4
„ 14/15	12	2·5	27·0	5·2	5·0	+·2
„ 29/30	12	·8	8·6	2·9	2·5	+·4
Dec. 25/26	19	1·2	20·5	4·5	4·2	+·3
Dec. 31/Jan. 1	11	1·0	9·9	3·2	3·0	+·2
1920.						
Jan. 1/2	34	1·7	52·0	7·2	6·6	+·6

From the formula a good idea may be formed of the risk crops—such as potatoes—run from various degrees of frost at the surface. For frost to reach a depth of 4 inches in one night of 12 hours, an average surface temperature of $-9{\cdot}2°$ C. is needed; but in a continuous frost lasting over four days an average surface temperature of only $-1{\cdot}1°$ C. is required to reach the same depth.

During the continuous frost from December 31, 1919, to January 5, 1920, the frost reached the 4-inch depth after 100 hours, during which the average surface temperature was $-1{\cdot}24°$ C.

Section VIII.—Snow.

The effect of a covering of snow is to reduce the changes in underground temperature even more than frost does. When snow falls on unfrozen soil the surface now becomes the layer of invariable temperature; and even when the snow falls on ground already frozen, the protecting influence of the snow causes the ground to thaw gradually, so that in either case the layer of invariable temperature tends to rise rather than to fall as in the case of frost.

Snow is an extremely bad conductor, and as little as 4 inches of snow provides a complete protection for the surface from very large variation of temperature above the snow surface. This is well shown in Table XII, which gives the temperature of the air above the snow, the soil surface, and the 4-inch and 8-inch depths during the severe snow and subsequent frost of November 12–16, 1919. During the whole period the surface of the soil remained unfrozen, though the air temperature over the snow fell to −15° C. on one night, and averaged −8° C. over the whole period; the 4-inch temperature fell 0·3° C., and the 8-inch temperature 0·5° C., while the snow was on the ground and before the rain came on November 17.

Table XII.—Fluctuations of Underground Temperature during Lying Snow.

Date and Hour.		Air over Snow.	Soil Surface under Snow.	4-inch Depth.	8-inch Depth.	Depth of Snow lying.
1919. Nov. 12	6 a.m.	− 0·4° C.	0·0° C.	1·1° C.	2·1° C.	Snow ceased 8 p.m., lying 6 inches deep.
	noon.	− 0·4° C.	0·0° C.	1·0° C.	2·1° C.	
	6 p.m.	− 0·4° C.	0·0° C.	1·0° C.	2·1° C.	
	midnight.	− 0·6° C.	0·0° C.	1·0° C.	2·1° C.	
Nov. 13	6 a.m.	− 8·0° C.	0·0° C.	1·0° C.	2·1° C.	4 inches deep.
	noon.	− 2·8° C.	0·0° C.	1.0° C.	2·1° C.	
	6 p.m.	−10·6° C.	0·0° C.	1·0° C.	2.1° C.	
	midnight.	−13·6° C.	0·0° C.	1·0° C.	2·1° C.	
Nov. 14	6 a.m.	−15·0° C.	0·0° C.	1·0° C.	2·1° C.	3 inches deep.
	noon.	− 8·6° C.	0·0° C.	1·0° C.	2·1° C.	
	6 p.m.	− 9·6° C.	0·0° C.	1·0° C.	2·1° C.	
	midnight.	− 7·7° C.	0·0° C.	1·0° C.	2·1° C.	
Nov. 15	6 a.m.	−13·7° C.	0·0° C.	0·9° C.	2·0° C.	$2\frac{3}{4}$ inches deep.
	noon.	− 4·6° C.	0·0° C.	0·9° C.	1·8° C.	
	6 p.m.	− 6·5° C.	0·0° C.	0·9° C.	1·8° C.	
	midnight.	− 9·5° C.	0·0° C.	0·9° C.	1·8° C.	
Nov. 16	6 a.m.	− 7·0° C.	0·0° C.	0·8° C.	1·7° C.	$2\frac{1}{4}$ inches deep.
	noon.	− 1·6° C.	0·0° C.	0·8° C.	1·7° C.	
	6 p.m.	0·0° C.	0·0° C.	0·8° C.	1·7° C.	
	midnight.	0·0° C.	0·0° C.	0·7° C.	1·6° C.	
Nov. 17	noon.	7·0° C.	0·4° C.	0·5° C.	1·4° C.	All melted.

At 7 a.m. on November 14, 1919, when the air temperature was −15° C. and the 4-inch depth temperature was 1° C., the soil surface was exposed and found to be unfrozen. Thus the heat conducted from the surface of the soil through the snow must have been balanced by the heat conducted to the surface from the 4-inch depth.

If K and K_s are the conductivities of unfrozen soil and snow respectively,

$$\text{Then } \frac{K \times 1}{10} = \frac{K_s \times 15}{10}$$

$$\therefore\ K_s = \frac{1}{15} K$$

$$= \frac{1}{15} \times \cdot 004$$

$$= \cdot 00027.$$

It is obvious that so long as the air temperature was above -15° C. thawing must have been going on constantly at the bottom of the snow layer, this thawing being the more rapid the more the air temperature rose. From Table XII we see that between 6 p.m. November 12 and noon November 13, with an average air temperature of -3° C., the snow diminished 2 inches in depth; but between 6 p.m. November 13 and noon November 14, with an average air temperature of $-12{\cdot}0^\circ$ C., only 1 inch of snow was melted.

Section IX.—The Coming of Spring.

Spring, considered as the beginning of the period of active growth of crops, depends in the main on the soil temperature at the depth at which these crops grow. At first sight it would appear a simple matter—with the meteorological data for any year before us—to say whether that year had enjoyed an early or late spring; but when we study the phenological returns for the last thirty years the question assumes a more complex appearance.

If we plot the deviations from the normal of the date of first flowering of any plant against the corresponding deviations from normal for some chosen preceding period of the values of the mean maximum, minimum, or soil temperatures, we do not find the closely corresponding results that we may have expected. As, however, mean values cover a multitude of variations, this is not to be wondered at, and it is doubtful if any good result can be obtained from monthly mean values at all. For let us consider two extreme months—

(1) With a very small range of temperature and a mean *equal to the average.*

(2) With a spell of frost for, say, 15 days and very warm weather for 15 days, and a mean *above the average.*

Now, if the growing temperature for the plant considered was below the mean for the month, No. 1 would give nearly 30 growing days, and the plant would appear in flower earlier than in No. 2, which gives only

15 growing days, *although the mean temperature of No. 2 is higher than of No. 1.*

Thus not only the growing temperature of the plant, but the daily values that make up the mean, have to be known before we can make any rule for the behaviour of the plant under the given conditions, and the date when the minimum soil temperature at from 4 to 6 inches depth passes the growing temperature of the proposed crop is the real question that concerns the cultivator.

If we take 5·5° C. as the average growing temperature of most crops, it is of interest to note how weather changes help forward or retard the arrival of this temperature in the soil at 4 inches depth.

In March and April 1919 there was a succession of spells of very different weather, which showed by their influence on the 4-inch depth temperature that it is the overcast weather, with bright intervals and overcast nights, that most rapidly increases the temperature underground, and not, as might be expected, the bright, sunny weather with clear sky and low relative humidity.

This is shown in Table XIII.

Table XIII.—Types of Weather and the Arrival of Spring Underground.

Date.	Mean 4-inch Depth Temperature.	No. of Days when Minimum 4-inch Depth Temperature was above 5·5° C.	Type of Weather.
1919.			
Mar. 21–26	Fell from 1·9° C. to 0·6° C.	0	Snow showers, cold E. wind, clear nights with hard frost.
Apr. 2–8	Rose from 1·7° C. to 7·8° C.	5	Overcast, fair intervals, W. or S.W. wind, overcast nights.
„ 9–15	Fell from 7·8° C. to 4·5° C.	2	Clear sky, bright sunshine, clear nights, N.W. wind.
„ 16–19	Rose from 4·5° C. to 10·4° C.	4	Overcast, fair intervals, overcast nights, W. or S.W. wind.
„ 20–23	Fell from 10·4° C. to 8·4° C.	1	Clear sky, bright sunshine, clear nights.

Thus, in 9 out of 11 overcast days, but in only 3 out of 11 clear days, the minimum temperature at the 4-inch depth was above 5·5° C.

These figures emphasise the importance of the effect of frost and clear nights on underground temperatures; in fact, we may say that the soil temperature for any period in winter or spring is mainly dependent on the number of frosts which occur while the ground is open without deep-lying snow during that period. Frosts that occur when several inches of

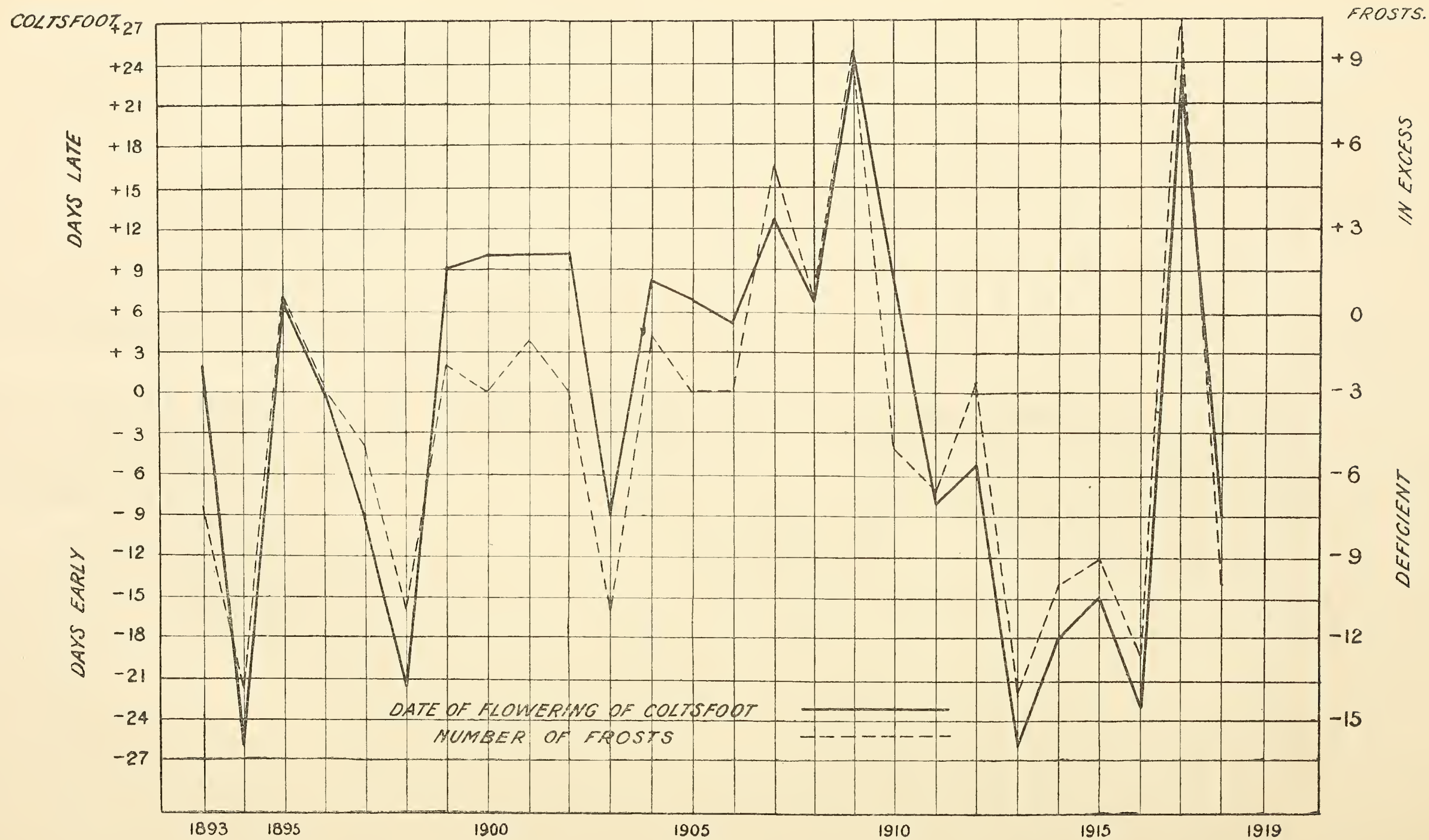

COLTSFOOT
FROSTS.
+27
+24
+21
+18
+15
+12
+ 9
+ 6
+ 3
0
− 3
− 6
− 9
−12
−15
−18
−21
−24
−27
DAYS LATE
DAYS EARLY
+ 9
+ 6
+ 3
0
− 3
− 6
− 9
−12
−15
IN EXCESS
DEFICIENT
DATE OF FLOWERING OF COLTSFOOT
NUMBER OF FROSTS
1893
1895
1900
1905
1910
1915
1919

Fig. 1.

snow is lying have, as we have already seen, little effect on soil temperature, owing to the great protection given by the snow.

In fig. 1 I have plotted the deviation from the mean date of flowering of Coltsfoot, at Hodsock Priory, against the deviation from the mean of the number of frosts when deep snow was not lying during the two months previous to the plant flowering. It will be seen that there is a remarkable agreement between the two curves, and in fact the average error over twenty-six years in predicting the date of first flowering of Coltsfoot *from the frost curve* is only four days. Coltsfoot was chosen for the purpose as it is a plant which flowers before its leaves appear, and its date of flowering would appear to be entirely dependent on the underground temperature down to the depth of its roots.

The zeros of the two curves do not coincide. This was to be expected, as the average number of days' snow in January and February is 10, and on some of these frost was bound to occur when deep snow was lying. The curves are plotted from the observations of Mr Henry Mellish, Hodsock Priory, who kindly provided me with the requisite data.*

Section X.—Weather Changes of Long Period.

Daily temperature changes are in the main periodic, the early morning minimum and afternoon maximum occurring with great regularity in spite of weather changes. But there have been several occasions during 1919—associated with the passage north of these islands of deep cyclonic depressions—when a strong warm west wind has blown for some days and the intervals between the minimum and maximum temperatures have been as much as 54 hours. From the mathematical theory of conduction we should expect that such a state of affairs would produce large values of $\frac{R_4}{R_0}$, with a consequent considerable rise in underground temperature; of course the passage of such depressions is often accompanied by rain, which, we have already seen, causes large values for $\frac{R_4}{R_0}$, but the values have been large even when no rain has fallen, as will be seen from Table XIV.

On November 22, 1919, the surface minimum was 0·0° C., and the 4-inch depth minimum 0·4° C.; 34 hours later the surface temperature had risen to a maximum of 9·8° C., and the 4-inch depth temperature stood at 7·1° C.

Such weather in spring is obviously of the greatest possible assistance

* *The Weather at Hodsock Priory*, Henry Mellish.

in warming up the underground layers in a very short time; on April 2, 1919, the 4-inch depth temperature, which for some weeks had been near freezing-point, rose in the period of 26 hours to 5·0° C., and, thanks to a succession of overcast nights, mounted steadily to 9·4° C. on April 8,

Table XIV.—Values of $\frac{R_4}{R_0}$ during Long-Period Temperature Changes.

Date.	Period Minimum to Maximum.	Value of $\frac{R_4}{R_0}$.	Position of Cyclonic Depression.	Weather.	Rain during Period.
1919.					
April 2	26 hours	·55	Scandinavia . . .	Overcast, S.W. wind.	nil.
" 17	33 "	·42	Iceland	"	1 mm.
Sept. 29	54 "	·57	N. Scotland . . .	"	2 mm.
Oct. 16	31 "	·50	Scandinavia and Iceland .	"	nil.
Nov. 22	34 "	·68	Iceland	"	"
Dec. 19	26 "	·52	Scandinavia and Iceland .	"	"
1920.					
Jan. 15	28 "	·58	Iceland	"	"

thus providing a much-needed growing period during that time. Two clear nights on April 8 and 9 reduced the 4-inch depth temperature to 4·2° C., but a second long-period spell of cyclonic weather on April 17 brought it up to 9·6° C. again, and so prevented any decided check to the growth of vegetation.

Section XI.—Conclusions.

1. The values of $\frac{R_4}{R_0}$ have a wide range of variation, from ·19 in very dry soil to ·85 during heavy rain; the most common value is about ·40. The monthly mean values showed a decided connection between $\frac{R_4}{R_0}$ and the frequency of rainfall; in fact, percolation of rain seems to be the dominating factor in deciding the value of $\frac{R_4}{R_0}$.

This is also borne out by the different values of $\frac{R_4}{R_0}$ in various soils according to their behaviour with regard to water; in sand the values change with mercurial rapidity, due to the easy percolation of rain and subsequent rapid drying, while in clay they change but sluggishly, since clay takes up and parts with water with difficulty.

2. In view of the fact that the values of $\frac{R_4}{R_0}$, and therefore the values

of the diffusivity of the soil, are so dependent on the percolation of rain, it is possible that the values commonly given for the diffusivity of the surface layers of the earth need revision.

3. Underground temperatures are also considerably affected by—

(*a*) Strong winds of low relative humidity.

(*b*) The frequency and intensity of frost when the soil has no snow covering.

(*c*) The depth of snow.

(*d*) Weather changes of long period.

4. The date of flowering of Coltsfoot appears to bear little relation to the monthly mean values of temperature, but is closely related to the number of frosts on open soil not covered with deep snow. It is possible that good results would be obtained by comparing the phenological returns of the last thirty years with the accumulated temperature *underground* above the growing temperature for each plant considered.

(*Issued separately March* 25, 1920.)

IX.—On the Quaternionic System as the Algebra of the Relations of Physics and Relativity. By Professor W. Peddie.

(MS. received February 2, 1920. Read February 2, 1920.)

In a recently published paper (*Proc. R.S.*, Dec. 1919) Mr W. J. Johnston makes a brilliant use of Clifford's extension of Hamilton's quaternions to the case of four (or more) variables. He proceeds, by adding a fourth independent and real unit vector and making the corresponding tensor imaginary, to modify the quaternionic ∇ so as to make it applicable to physical problems on the basis of relativity. In this respect his vectors are simply the direct extension of Hamilton's bivectors.

In an intensely illuminative paper on Generalised Relativity, immediately following the one referred to, Sir Joseph Larmor simplifies the expressions by making the added unit vector itself imaginary, so that its square becomes positive unity.

Hamilton did not himself generalise his algebra, though no one could have done it more ably than he could have done. The latter part of his life was devoted to the development of quaternions in their relation to ordinary geometry of three dimensions. In these dimensions the vector product ijk has no meaning. The vector product ij can only indicate the plane ij, which is specifiable by k. Nothing is therefore gained by not identifying them and so putting $ijk=-1$. If we put $ijk=-\Omega$, we find $\Omega^2=+1$, $i\Omega=\Omega i$, so that Ω has no properties different from those of positive unity.

If we deal with a system of four unit vectors (i, j, k, l) and put $ijkl=-\Omega$, we find that while, as in the case of three, $\Omega^2=+1$, Ω is itself non-commutative with the other units. In fact $\Omega i=-i\Omega$. Thus we cannot identify Ω with positive unity. It possesses half of the properties of a unit vector, and can take the place of l in the vector $\rho=xi+yj+zk+ul$. Thus $ijkl=-\Omega$, $ijk=\Omega l$, $l=\Omega ijk=-ijk\Omega$, and ρ can be expressed as $xi+yj+zk+u\Omega ijk$. It is true that relations such as $ij=\Omega lk$ subsist, so that the 15 possible vector combinations are not all independent. Eight only remain. But these are sufficient for the statement of the result of any fourth dimensional effect upon a tridimensional system. The Hamilton-Clifford symbol Ω, added to the quaternionically conditioned system i, j, k, is sufficient.

As an example, we may consider Johnston's problem. The four-dimensional operator

$$\nabla_\Omega = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} + l\frac{\partial}{\partial u}\right)$$

has to be modified to suit a tridimensional intelligence. As above, the substitution for l is Ωijk. The quantity u is a four-dimensional scalar whose numerical magnitude is ct. If u be an imaginary scalar, the operator remains homogeneous if l be made imaginary. It may then be replaced by Ω. Thus

$$\nabla_\Omega = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} + c^{-1}\Omega\frac{\partial}{\partial t}.$$

Operating by this upon Johnston's vector U written in the form

$$c(i\mathrm{F} + j\mathrm{G} + k\mathrm{H}) - \Omega\phi,$$

where the vector in the bracket is the vector potential and ϕ is the scalar potential, we find

$$\mathrm{S}\nabla_\Omega \mathrm{U} = -c\left(\frac{\partial \mathrm{F}}{\partial x} + \frac{\partial \mathrm{G}}{\partial y} + \frac{\partial \mathrm{H}}{\partial z}\right) - c^{-1}\frac{\partial\phi}{\partial t}$$

$$\mathrm{V}\,.\,\nabla_\Omega \mathrm{U} = jk\,.\,\alpha + ki\,.\,\beta + ij\,.\,\gamma + (i\Omega\mathrm{X} + j\Omega\mathrm{Y} + k\Omega\mathrm{Z})$$

(α, β, γ) and (X, Y, Z) being the magnetic and electric force components respectively, each measured in their own distinctive units.

The vanishing of $\mathrm{S}\,.\,\nabla_\Omega \mathrm{U}$ gives the electrostatic condition for the distribution of electricity and electric force. The vector part is a Minkowskian "six-vector." Operating upon it by ∇_Ω we get

$$\begin{aligned}\nabla_\Omega \mathrm{V}\nabla_\Omega \mathrm{U} = ijk\left(\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial y} + \frac{\partial\gamma}{\partial z}\right) - \Omega\left(\frac{\partial \mathrm{X}}{\partial x} + \frac{\partial \mathrm{Y}}{\partial y} + \frac{\partial \mathrm{Z}}{\partial z}\right)\\ + i\left(\frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z} - c^{-1}\dot{\mathrm{X}}\right) + \ \ldots\\ + \Omega ij\left(\frac{\partial \mathrm{Y}}{\partial x} - \frac{\partial \mathrm{X}}{\partial y} + c^{-1}\dot{\gamma}\right) + \ \ldots\end{aligned}$$

the vanishing of which gives the conditions obtaining in the electromagnetic field in free space, and exhibits them as a consequence of the condition of relativity, as Johnston has shown.

In analogy with the previous procedure, if we had to deal in three-dimensional space with some effect in five-dimensional space in which the more limited space existed, we should have to introduce Ω_5, where

$$ijklm = -\Omega_5.$$

We find that it obeys the commutative law and that its square is negative unity.

So $$ijkl = \Omega_5 m,$$

$$-\Omega_5 ijkl = m.$$

Hence, substituting for l, from the previous condition, we get

$$\Omega_5\Omega_4 = m,$$

and readily prove

$$\Omega_5\Omega_4 = -\Omega_4\Omega_5.$$

The general conditions are that Ω_n is commutative with linear vectors where n is odd, and non-commutative with them where n is even; while its square is negative unity when n has the form $(4\mu+1)$ or $(4\mu+2)$, and is positive unity when n has the form $(4\mu+3)$ or $(4\mu+4)$, μ being an integer, zero included.

In all cases in which our observations are upon directed phenomena occurring in tridimensional space, but which are actually, or merely descriptively, to be regarded as influenced by the existence of that space in space of a higher order, the appropriate algebra to be used in their investigation is that of quaternions with the addition of the symbol Ω of the higher space involved.

(*Added February* 25, 1920.)

It is not necessary to use as above the Hamilton-Clifford extension of quaternions in the treatment of relativity problems. Hamilton's biquaternions, so exquisitely developed in his tremendous Lecture VII, are directly applicable (see Silberstein's *Theory of Relativity*).

(*Issued separately May* 29, 1920.)

X.—Note on Pfaffians with Polynomial Elements.

By Sir Thomas Muir, F.R.S.

(MS. received January 28, 1920. Read March 1, 1920.)

(1) So far back as March 1855 Brioschi in effect formulated the theorem that *any even-ordered determinant is expressible as a Pfaffian*; for example, in the case of the fourth order he gave the equality

$$|a_1b_2c_3d_4| = \begin{vmatrix} (1,2) & (1,3) & (1,4) \\ & (2,3) & (2,4) \\ & & (3,4) \end{vmatrix}$$

where (h, k) stands for

$$a_hb_k - b_ha_k + c_hd_k - d_hc_k,$$

this expression being obtained by multiplying the h^{th} column

$$a_h, \quad b_h, \quad c_h, \quad d_h$$

of the given determinant by the k^{th} column

$$b_k, -a_k, \quad d_k, -c_k$$

of an equivalent determinant.

(2) It is manifest, however, that we may think of (h, k) as arising in a totally different manner, namely, through the addition of two determinants of the second order

$$|a_hb_k|, \quad |c_hd_k|;$$

and it is thus suggested to ascertain the consequences of such a change in the point of view.

(3) *The Pfaffian whose every element is the sum of the corresponding elements of the identically vanishing Pfaffians*

$$\begin{vmatrix} |a_1b_2| & |a_1b_3| & |a_1b_4| \\ & |a_2b_3| & |a_2b_4| \\ & & |a_3b_4| \end{vmatrix}, \quad \begin{vmatrix} |c_1d_2| & |c_1d_3| & |c_1d_4| \\ & |c_2d_3| & |c_2d_4| \\ & & |c_3d_4| \end{vmatrix}, \quad \begin{vmatrix} |e_1f_2| & |e_1f_3| & |e_1f_4| \\ & |e_2f_3| & |e_2f_4| \\ & & |e_3f_4| \end{vmatrix}, \ldots$$

is expressible as a sum of 4-line determinants, namely, the sum

$$|a_1b_2c_3d_4| + |a_1b_2e_3f_4| + |c_1d_2e_3f_4| + \ldots,$$

where, if the number of vanishing Pfaffians be m, *the number of determinants is* $\frac{1}{2}\mathrm{m}(\mathrm{m}-1)$.

The reason for this is that the Pfaffian

$$\begin{vmatrix} |a_1b_2|+|c_1d_2|+\ldots & |a_1b_3|+|c_1d_3|+\ldots & |a_1b_4|+|c_1d_4|+\ldots \\ & |a_2b_3|+|c_2d_3|+\ldots & |a_2b_4|+|c_2d_4|+\ldots \\ & & |a_3b_4|+|c_3d_4|+\ldots \end{vmatrix}$$

is expressible as a sum of m^2 Pfaffians, of which m are the given vanishing Pfaffians, and the remaining m^2-m combine in pairs to form the equivalents of $\frac{1}{2}(m^2-m)$ determinants, thus

$$\left|\begin{array}{ccc} |a_1b_2| & |a_1b_3| & |a_1b_4| \\ & |c_2d_3| & |c_2d_4| \\ & & |c_3d_4| \end{array}\right| + \left|\begin{array}{ccc} |c_1d_2| & |c_1d_3| & |c_1d_4| \\ & |a_2b_3| & |a_2b_4| \\ & & |a_3b_4| \end{array}\right| = |a_1b_2c_3d_4|\,.$$

(4) The case of this theorem where m is 2 is essentially identical with Brioschi's theorem. The vanishing Pfaffians which are taken as the foundation of it are those whose elements are the determinants of the arrays

$$\left\|\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{array}\right\|,\quad \left\|\begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{array}\right\|,\quad \ldots;$$

so that, for shortness' sake, we may appropriately speak of

$$\left|\begin{array}{ccc} |a_1b_2| & |a_1b_3| & |a_1b_4| \\ & |a_2b_3| & |a_2b_4| \\ & & |a_3b_4| \end{array}\right| \quad \text{as the Pfaffian of} \quad \left\|\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{array}\right\|.$$

(5) *The Pfaffian whose every element is the sum of the corresponding elements of the Pfaffians of the arrays*

$$\left\|\begin{array}{cccc} a_1 & a_2 & \ldots & a_6 \\ b_1 & b_2 & \ldots & b_6 \end{array}\right\|,\quad \left\|\begin{array}{cccc} c_1 & c_2 & \ldots & c_6 \\ d_1 & d_2 & \ldots & d_6 \end{array}\right\|,\quad \left\|\begin{array}{cccc} e_1 & e_2 & \ldots & e_6 \\ f_1 & f_2 & \ldots & f_6 \end{array}\right\|,\quad \left\|\begin{array}{cccc} g_1 & g_2 & \ldots & g_6 \\ h_1 & h_2 & \ldots & h_6 \end{array}\right\|,\ \ldots$$

is expressible as a sum of 6-*line determinants, namely, the sum*

$$|a_1b_2c_3d_4e_5f_6| + |a_1b_2c_3d_4g_5h_6| + |a_1b_2e_3f_4g_5h_6| + |c_1d_2e_3f_4g_5h_6| + \ldots$$

where, if the number of vanishing Pfaffians be m, *the number of determinants is* $\frac{1}{6}\mathrm{m}(\mathrm{m}-1)(\mathrm{m}-2)$.

Taking the case where m is 3, and using repeatedly the corresponding case of the preceding theorem, we obtain in the first place the given Pfaffian

$$\left|\begin{array}{ccccc} |a_1b_2|+|c_1d_2|+|e_1f_2| & \Sigma|a_1b_3| & \Sigma|a_1b_4| & \Sigma|a_1b_5| & \Sigma|a_1b_6| \\ & \Sigma|a_2b_3| & \Sigma|a_2b_4| & \Sigma|a_2b_5| & \Sigma|a_2b_6| \\ & & \Sigma|a_3b_4| & \Sigma|a_3b_5| & \Sigma|a_3b_6| \\ & & & \Sigma|a_4b_5| & \Sigma|a_4b_6| \\ & & & & \Sigma|a_5b_6| \end{array}\right|$$

$$\begin{aligned} &= \{|a_1b_2|+|c_1d_2|+|e_1f_2|\}\,.\,\{|a_3b_4c_5d_6|+|a_3b_4e_5f_6|+|c_3d_4e_5f_6|\} \\ &- \{|a_1b_3|+|c_1d_3|+|e_1f_3|\}\,.\,\{|a_2b_4c_5d_6|+|a_2b_4e_5f_6|+|c_2d_4e_5f_6|\} \\ &+ \{|a_1b_4|+|c_1d_4|+|e_1f_4|\}\,.\,\{|a_2b_3c_5d_6|+|a_2b_3e_5f_6|+|c_2d_3e_5f_6|\} \\ &- \{|a_1b_5|+|c_1d_5|+|e_1f_5|\}\,.\,\{|a_2b_3c_4d_6|+|a_2b_3e_4f_6|+|c_2d_3e_4f_6|\} \\ &+ \{|a_1b_6|+|c_1d_6|+|e_1f_6|\}\,.\,\{|a_2b_3c_4d_5|+|a_2b_3e_4f_5|+|c_2d_3e_4f_5|\}\,. \end{aligned}$$

Next, if the five multiplications here indicated be performed, there result

3.3.5 products which, preparatory to the use of Laplace's theorem, can be recombined in 9 sets of 5, the first and the last set being equal to

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ \cdot & a_2 & a_3 & a_4 & a_5 & a_6 \\ \cdot & b_2 & b_3 & b_4 & b_5 & b_6 \\ \cdot & c_2 & c_3 & c_4 & c_5 & c_6 \\ \cdot & d_2 & d_3 & d_4 & d_5 & d_6 \end{vmatrix}, \quad \begin{vmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_5 \\ \cdot & c_2 & c_3 & c_4 & c_5 & c_6 \\ \cdot & d_2 & d_3 & d_4 & d_5 & d_6 \\ \cdot & e_2 & e_3 & e_4 & e_5 & e_6 \\ \cdot & f_2 & f_3 & f_4 & f_5 & f_6 \end{vmatrix}$$

respectively. Lastly, as these two and four others like them are seen to vanish, our expression is reduced to

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ \cdot & c_2 & c_3 & c_4 & c_5 & c_6 \\ \cdot & d_2 & d_3 & d_4 & d_5 & d_6 \\ \cdot & e_2 & e_3 & e_4 & e_5 & e_6 \\ \cdot & f_2 & f_3 & f_4 & f_5 & f_6 \end{vmatrix} + \begin{vmatrix} \cdot & a_2 & a_3 & a_4 & a_5 & a_6 \\ \cdot & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ \cdot & e_2 & e_3 & e_4 & e_5 & e_6 \\ \cdot & f_2 & f_3 & f_4 & f_5 & f_6 \end{vmatrix} + \begin{vmatrix} \cdot & a_2 & a_3 & a_4 & a_5 & a_6 \\ \cdot & b_2 & b_3 & b_4 & b_5 & b_6 \\ \cdot & c_2 & c_3 & c_4 & c_5 & c_6 \\ \cdot & d_2 & d_3 & d_4 & d_5 & d_6 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \end{vmatrix}$$

which manifestly equals

$$| a_1 b_2 c_3 d_4 e_5 f_6 | .$$

When m is greater than 3 the same procedure suffices for proof.

(6) Just as we had for our groundwork in § 3 Pfaffians of 2-by-4 arrays, and in § 5 Pfaffians of 2-by-6 arrays, so we might now proceed with like results to use Pfaffians of 2-by-8 arrays, and so on. When the number of 2-by-$2m$ arrays is less than m the result is 0: when the number is m the result is the $2m$-line determinant formed by combining the arrays into one: and when the number is greater than m, say z, the result is the sum of $C_{z,m}$ determinants of the $(2m)^{\text{th}}$ order.

(7) It is important to note that both members of the equality may be transformed into an aggregate of terms of the form

$$| a_m b_n | . | c_p d_q | . | e_r f_s | \ . \ . \ .$$

and that a comparison of the two aggregates is instructive. Taking the case of § 5, we observe first that the Pfaffian under consideration being of six lines the number of terms in its ordinary expansion is 1.3.5, and in the second place that these being of the type

$$\{ | a_1 b_2 | + | c_1 d_2 | + | e_1 f_2 | \} . \{ | a_3 b_4 | + | c_3 d_4 | + | e_3 f_4 | \} . \{ | a_5 b_6 | + | c_5 d_6 | + | e_5 f_6 | \},$$

the total number of terms which are products of two-line determinants is $1.3.5.3^3$, *i.e.* 405. On the other hand, the member on the right being a six-line determinant, Laplace's expansion of it as an aggregate of such products consists of $C_{6,2} . C_{4,2}$, *i.e.* 90 terms. The number of terms which thus must disappear is 315; and, this not being a multiple of 90, it cannot be that their aggregate is representable as a sum of vanishing determinants —at any rate, of determinants that vanish because of having two rows alike.*

(8) The occurrence of Pfaffians with polynomial elements of *special* form naturally turns the mind to polynomial elements that are unspecialised, and to the question of the existence of an analogue to Albeggiani's or other like theorem in determinants. Such a theorem, when freed of details, is the following:—*Any* 2n-*line Pfaffian with* k-*termed elements is equal to the sum of* k^n *Pfaffians with monomial elements.* The simplest case is in full

$$\begin{vmatrix} a_1+\alpha_1 & a_2+\alpha_2 & a_3+\alpha_3 \\ & b_2+\beta_2 & b_3+\beta_3 \\ & & c_3+\gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ & b_2 & b_3 \\ & & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ & \beta_2 & \beta_3 \\ & & \gamma_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ & b_2 & b_3 \\ & & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ & \beta_2 & \beta_3 \\ & & \gamma_3 \end{vmatrix}.$$

If we view the triangular array of such Pfaffians as composed of two parts, and use i to stand for "Italic" and g for "Greek," this expansion may be greatly abbreviated,—thus

$$ii + ig + gi + gg\,.$$

Taking a further step towards ultra-symbolism, we may use for this

$$(i+g)(i+g)\,,$$

or even, with a proviso, $(i+g)^2$. When the elements are trinomial the result is

$$\begin{vmatrix} a_1+\alpha_1+A_1 & a_2+\alpha_2+A_2 & a_3+\alpha_3+A_3 \\ & b_2+\beta_2+B_2 & b_3+\beta_3+B_3 \\ & & c_3+\gamma_3+C_3 \end{vmatrix}$$

$$= ii+ig+ic+gi+gg+gc+ci+cg+cc$$

$$= (i+g+c)(i+g+c) = (i+g+c)^2,$$

where, in similar fashion, c is used for "capital."

* This is at variance with a statement of Pascal's when dealing with Brioschi's theorem above referred to. In his paper on "Un teorema sui determinanti di ordine pari," he says, "Gli altri termini dello sviluppo del prodotto (7) sono similmente termini dello sviluppo di determinanti come D dove però alcune coppie di linee sono ripetute e che quindi sono zero" (see *Rendic.... Accad.... Napoli*, xxv, 1919). The case where m is 2 in § 3 provides the simplest test: for then the number of terms on the left is twelve, and the number on the right is 6, and the 6 on the left whose aggregate must vanish are

$$|a_1b_2|\cdot|a_3b_4|,\quad -|a_1b_3|\cdot|a_2b_4|,\quad |a_1b_4|\cdot|a_2b_3|,$$
$$|c_1d_2|\cdot|c_3d_4|,\quad -|c_1d_3|\cdot|c_2d_4|,\quad |c_1d_4|\cdot|c_2d_3|,$$

which, instead of forming a vanishing determinant, form two vanishing Pfaffians.

For the Pfaffian of the next higher order we have

$$\left|\begin{array}{ccccc} a_1+\alpha_1 & a_2+\alpha_2 & a_3+\alpha_3 & a_4+\alpha_4 & a_5+\alpha_5 \\ & b_2+\beta_2 & b_3+\beta_3 & b_4+\beta_4 & b_5+\beta_5 \\ & & c_3+\gamma_3 & c_4+\gamma_4 & c_5+\gamma_5 \\ & & & d_4+\delta_4 & d_5+\delta_5 \\ & & & & e_5+\epsilon_5 \end{array}\right.$$

$$\begin{aligned} &= iiii + iigg + igig + iggi + gggg + ggii + gigi + giig \\ &= (ii+gg)^2 + (ig+gi)^2, \\ &= \begin{array}{cc|cc} i & g & i & g \\ \hline i & g & i & g \\ g & i & g & i \end{array}; \end{aligned}$$

and when the elements are trinomial

$$(ii+gg+cc)^2 + (ig+gc+ci)(ic+gi+cg) + (ic+gi+cg)(ig+gc+ci)$$

$$\text{i.e.}\quad \begin{array}{ccc|ccc} i & g & c & i & g & c \\ \hline i & g & c & i & g & c \\ g & c & i & c & i & g \\ c & i & g & g & c & i \end{array}.$$

(9) It has to be observed that in the foregoing theorem the degeneration from elements of k terms to elements of $k-1$ terms does not take place with the readiness found in the case of the analogous theorem in determinants. Thus the twenty-seven Pfaffians in the example just given do not, on substituting zeros for the capital letters, at once reduce to the eight of the preceding example—a fact which is manifest on noting that two of the eight, $igig$ and $gigi$, do not appear among the twenty-seven at all. What actually takes place is the immediate vanishing of fifteen of the twenty-seven, and the partial vanishing of six, the latter group being

$$igi. + ig.g + i.ig + gig. + gi.i + g.gi,$$

which condenses * into

$$igig + gigi$$

as it ought.

(10) The like is true in regard to degeneracy from Pfaffians of one order to those of the next lower order. For example, if in the last equality of § 8 we equate cofactors of e_5 in the hope of obtaining the second equality of the same paragraph, we are disappointed; an additional transformation dependent on the manifest equality

$$\left|\begin{array}{ccc} A_1 & A_2 & A_3 \\ & B_2 & B_3 \\ & & C_3 \end{array}\right| + \left|\begin{array}{ccc} A_1 & A_2 & A_3 \\ & b_2 & b_3 \\ & & c_3 \end{array}\right| = \left|\begin{array}{ccc} A_1 & A_2 & A_3 \\ & B_2 & B_3 \\ & & c_3 \end{array}\right| + \left|\begin{array}{ccc} A_1 & A_2 & A_3 \\ & b_2 & b_3 \\ & & C_3 \end{array}\right|$$

is necessary.

* *Trans. S. African Phil. Soc.*, xv, pp. 35–41, §§ 6, 7.

(11) It only remains now to examine whether anything is to be learned by trying to connect the subject of §§ 1–7 with that of §§ 8–10. The latter being the expression of a Pfaffian with *general* polynomial elements as a sum of *Pfaffians,* and the former the expression of a Pfaffian with *special* polynomial elements as a sum of *determinants,** the possibility is suggested that the substitution of the particular elements

$$\begin{array}{ccc} |a_1b_2|\,, & |a_1b_3|\,, & \cdots \\ & |a_2b_3|\,, & \cdots \\ & & \cdots \end{array}$$

for the general elements

$$\begin{array}{ccc} a_1\,, & a_2\,, & \cdots \\ & b_2\,, & \cdots \\ & & \cdots \end{array}$$

might lead to a second mode of establishing the less general result.

The details of the examination need not be entered on. Suffice it to say that they seem to shut out all hope of any mode of proof more attractive than that originally given.

* In view of the footnote above (§ 7), it is fitting that I should here point out an incorrect footnote of my own bearing on a determinant and a sum of Pfaffians. See *Trans. Roy. Soc. Edin.*, xlv, 1906-7, p. 313.

RONDEBOSCH, S.A.,
5th January 1920.

(*Issued separately May* 29, 1920.)

XI.—The Daily Temperature Curve. By L. Becker, M.A., Ph.D., Professor of Astronomy in the University of Glasgow.

(MS. received February 17, 1920. Read March 15, 1920.)

(1) *Outline of Investigation.*—During the last half-century the temperature of the air has been recorded by thermographs at a number of places, and the registered graphs have been measured and mean daily temperature curves have been calculated from them for each month of the year. The amount of labour entailed in these operations is enormous, and it may well be asked whether the figures cannot be put to some further use. It appears to me that a comparison of the daily curves obtained at various places would furnish important information in regard to climate. To effect this the curves may be represented by Fourier's Series, and the values of the constants at the various places may be compared. Another method of comparison, however, seems to me more promising. It has been suggested that the daily temperature curve might be represented by the differential equation

$$\text{(I)} \quad . \quad . \quad . \quad . \quad . \quad \ddot{\tau} + a\dot{\tau} + b(\tau - c_0) = \mathrm{F}(z),$$

which is based on physical considerations. (Its derivation is given in section 2.) The constants, a, b, and c_0, the function, $\mathrm{F}(z)$, of the zenith distance, z, of the sun, and the two constants of integration are the quantities whose values at different places may be compared. The object of this paper is to show how the numerical values of these quantities may be calculated from the registered figures, and to prove that the observations are sufficiently well represented by the integral of equation (I).

I have used the twelve curves, one for each month, which were calculated from the temperatures recorded at Glasgow Observatory in the forty-five years 1868 to 1912, on behalf of the Meteorological Council, London.

In my method I determine the constants a and b and the function F(z) from the numerical values of τ, $\dot{\tau}$, $\ddot{\tau}$ *as obtained from the curves.* The constant c_0 and the constants of integration, c_1 and c_2, are calculated from the integral, (II), of equation (I). The integrals T_1 and T_2 extend from the time of rising of the sun, t_r, and are calculated by mechanical quadrature.

$$\text{(II)} \quad . \quad . \quad . \quad . \quad . \quad \tau = \mathrm{C}_t + \mathrm{T}_1 - \mathrm{T}_2,$$

where

$$\alpha^2 = \tfrac{1}{4}a^2 - b\,, \quad \beta = \tfrac{1}{2}a - \alpha\,, \quad \gamma = \tfrac{1}{2}a + \alpha$$
$$C_t = c_0 + c_1 e^{-\beta t} + c_2 e^{-\gamma t}$$

$$X_1 = e^{\beta x}(2\alpha)^{-1}F(z) \quad \text{and} \quad X_2 = e^{\gamma x}(2\alpha)^{-1}F(z)$$
$$T_1 = e^{-\beta t}\int_{tr}^{t} X_1 dx \quad \text{and} \quad T_2 = e^{-\gamma t}\int_{tr}^{t} X_2 dx.$$

(2) *Derivation of the Differential Equation* (*I*).—I have adopted for the night-temperature curves the differential equations proposed by Weilenmann. These equations are reproduced in Dr Halm's memoir on the daily temperature curve.* For the temperatures at day time Dr Halm has added to these equations a term which represents solar radiation. I also have introduced solar terms, but I have chosen another expression for them and have treated the problem in a different way.

We have to represent the average temperature curve, and may suppose that it belongs to an average state of atmosphere characterised by a certain degree of transparency. In such an average state the transparency is less than that of a clear sky, and the radiation of the soil and lower strata will be less than in that ideal case. I assume with Weilenmann as a working hypothesis that radiation takes place towards and from a layer in the upper strata of the atmosphere where the temperature, τ_m, remains constant during the day, though it may change in the course of the year. Let τ' designate the temperature of the soil in the neighbourhood of the thermometer. The amount of heat gained on unit area of soil by radiation and also the corresponding rise in temperature of the soil are nearly proportional to $(\tau_m - \tau')$. Heat is also transferred between these two places by convection, and the corresponding increase is given by a similar term. The corresponding gain in the temperature of the soil thus becomes $\Sigma a_m(\tau_m - \tau')$. The soil also gains heat by convection and radiation from the air at the place where the thermometer is placed; the rise in temperature is proportional to $\tau - \tau'$, τ designating the temperature at the thermometer. The rate, $\dot{\tau}'$, at which the temperature of the soil rises becomes $(-p'\tau' + q'\tau - m')$, apart from solar radiation.

In like manner the rate, $\dot{\tau}$, at which the air temperature changes is $(-p\tau + q\tau' - m)$. p, q, m, p', q', m' are constant quantities.

These expressions also account for any change in temperature arising from radiation or convection from or to a region where the temperature is proportional to either τ or τ'.

The two values of $\dot{\tau}'$ and $\dot{\tau}$ furnish the two equations proposed by

* *Nova Acta der Ksl. Leop. Carol. Deutschen Akademie der Naturforscher*, Bd. lxvii, No. 2, p. 10.

Weilenmann, with this difference, that he assumes equal values for the constants, ($p=q=q'=h$, $p'=2h$).

As to solar radiation, let J be proportional to the quantity of heat radiated per unit time from the sun on unit area placed at right angles to the rays' path. The heat received by direct solar radiation on unit area of soil is proportional to (J cos z), where z designates the zenith distance of the sun. z at apparent solar time, t^{H}, depends on the geographical latitude, ϕ, the declination, δ, and the hour-angle, t^{h}, of the sun, and is calculated from

$$(1) \quad \cos z = \sin\phi\sin\delta + \cos\phi\cos\delta\cos t.$$

Again, solar heat is reflected to the soil by clouds, the amount of heat being proportional to J. The corresponding change in temperature per unit time is given in the right-hand side of equation (2).

Unit volume of air receives solar heat both directly from the sun and also through reflection by soil and clouds. Both amounts are proportional to J. The corresponding change in temperature of the air per unit time is again proportional to J, and its amount is entered in the right-hand side of equation (3).

$$(2) \quad \dot{\tau}' + p'\tau' - q'\tau - m' = k'\cos z\,\mathrm{J} + l'\mathrm{J}.$$
$$(3) \quad \dot{\tau} + p\tau - q\tau' - m = \qquad + l\mathrm{J}.$$

Eliminate τ' from these equations and find the formula (I) in section 1, where

$$a = p + p',\ b = pp' - qq',\ bc_0 = qm' + p'm,$$

and

$$(4) \quad \mathrm{F}(z) = [qk'\cos z + (ql' + p'l)]\mathrm{J} + l\frac{d\mathrm{J}}{dt}.$$

(3) *The Values of J and F(z).*—Consider, in the atmosphere, a tube of unit section curved along the path of the sun's rays. Let s designate the total mass of air in the tube. The intensity, J, increases at each step, ds, by d$\mathrm{J} = -n\mathrm{J}\,ds$, where n expresses the fraction absorbed by unit mass. Hence J is proportional to 10^{-ns}, and $(-n)$ may thus also be defined as the logarithm of the coefficient of transmission (*i.e.* ratio of intensity after passing through unit mass and original intensity). Laplace's well-known formula gives the following value of s which is expressed in units of the mass of air in a tube directed towards the zenith :—

$$(5) \quad s = \frac{1}{\alpha_0}\frac{\mathrm{R}}{\sin z} = \frac{\alpha_z}{\alpha_0}\sec z,$$

where $\mathrm{R}(=\alpha_z\tan z)$ is the value of the atmospheric refraction at zenith distance, z, and a_0 is the constant coefficient of tan z in the power series in (tan z) which expresses the refraction.* Hence by differentiation

$$(6) \quad . \quad . \quad \begin{cases} -\dfrac{ds}{d(\cos z)} = \dfrac{a_z}{a_0}\sec^2 z\left[1 + \dfrac{\cot z}{\text{Mod. sin } 1'}\cdot\dfrac{d\log a_z}{dz}\right] & \text{for } z < 85° \\ -\dfrac{ds}{d(\cos z)} = \text{cosec}^2 z(1{\cdot}7747)\dfrac{dR}{dz} - \ldots & \text{for } z > 85° \end{cases}$$

The differential quotients on the right-hand sides are supposed to be expressed in seconds of arc per 1′ of zenith distance. They are tabulated in refraction tables.†

Substitute the above values in (4) and introduce the notation

$$(7) \quad . \quad . \quad F = 10^{-ns}, \quad E = F\cos z, \quad G = F\left(\frac{-ds}{d(\cos z)}\right).$$

According to section 5 the numerical values of the coefficients of E, F, and (cos δ sin t)G differ for the twelve curves (one for each month). I therefore introduce instead of each coefficient two factors, the one (e, f, g) being constant, the other (μ) being variable, viz. $\mu_2^{-1}e$, $\mu_2^{-1}f$, $\mu_1^{-1}g$. Formula (4) then becomes

$$(\text{I}^*) \quad . \quad . \quad . \quad F(z) = \mu_2^{-1}(eE + fF) - \mu_1^{-1}g(\cos\delta\sin t)G.$$

(4) *Preliminary Calculations.*—In the forty-five years 1868·0 to 1913·0 the temperature was registered at Glasgow Observatory by a photo-thermograph, and the curves were measured at points belonging to the thermograph hours, i, which are two minutes earlier than the respective hours of Greenwich mean time. These measurements were then reduced to the readings of the standard thermometer. Means were taken of all the figures obtained in the forty-five years at the same hour and in the same month, all incomplete days, 2·7 per cent. of the total number, being discarded. In this way the twelve mean daily temperature curves published in Table I were obtained.

I smooth these curves so as to be better able to interpolate. But the corrections do not exceed 0°·05 F., and generally amount to 0°·02 or less.

Let at t^{H} apparent solar time, E denote the equation of time, λ the longitude (West), and let both together with the hour-angle be expressed in "hours" (h) of angle. Hence $t^{\text{h}} = i^{\text{h}} - j^{\text{h}}$, and $j^{\text{h}} = 0^{\text{h}}{\cdot}03 + (E + \lambda)^{\text{h}}$. I interpolate τ for $i = t + j$, where t has the values 0^{h}, 1^{h}, 2^{h}, . . . and thus get the temperature curves at full hours of t.

* Chauvenet, *Spherical Astronomy.*
† *Ibid.*

I write out the successive first to fifth differences of $\tau, f, \ldots f^{\mathrm{V}}$, and calculate $\dot{\tau}$ and $\ddot{\tau}$ at each full hour of t from the formulæ *

$$\dot{\tau} = f^{\mathrm{I}} - \frac{1}{6}f^{\mathrm{III}} + \frac{1}{30}f^{\mathrm{V}}$$

$$\ddot{\tau} = f^{\mathrm{II}} - \frac{1}{12}f^{\mathrm{IV}}.$$

The zenith distance, z, of the sun is computed from the formulæ (I) with $t = 0^{\mathrm{h}}, 1^{\mathrm{h}}, \ldots 23^{\mathrm{h}}$, and the declination of the sun at the middle of the month. The values t_r of t at sunrise and sunset are also calculated.

(5) *The Cloud-Factors, μ_1 and μ_2, for each month.*—Let D stand for half the difference and S for half the sum of the observed temperatures or their differential quotients at corresponding hours in the afternoon and forenoon when the sun has the same zenith distance. Formula (I) then becomes with regard to (I*)—

(III) . . . $\mu_1 D_1(\ddot{\tau}) + \mu_1 a D_1(\dot{\tau}) + \mu_1 b D_1(\tau) = -gG$

where

$$D_1 = D/(\cos\delta \ \sin t).$$

(IV) . . $\mu_2 S(\ddot{\tau}) + \mu_2 a S(\dot{\tau}) + \mu_2 b S(\tau) - \mu_2 b c_0 = eE + fF.$

D_1 and S are calculated from each two corresponding points of each curve.

In accordance with these two formulæ, which contain only functions of z on the right-hand side (see (7)), $\mu_1 D_1(\tau)$ must have the same values at the same zenith distance in all the twelve months, and the same holds good for S and the arguments $\dot{\tau}$ and $\ddot{\tau}$. Therefore I may interpolate $D_1(\tau)$ for the zenith distances of 90°, 85°, 80° . . . With this object in view I draw for January a curve whose ordinates are $D_1(\tau)$ and whose abscissæ are their corresponding zenith distances, and obtain from this graph $D_1(\tau)$ for the zenith distances 90°, 85°, 80° . . . Similarly the other quantities are found for every fifth degree of zenith distance. This graphical interpolation is repeated for the other months.

I arbitrarily assume that the average value of μ_1 for the period April to September equals unity.

To determine μ_1 in this unit, I compile the twelve monthly values of $D_1(\tau)$ belonging to the zenith distance of 90°, take the mean of the six values for April to September, and compute the ratios (μ_1) of this mean and all the individual values. Other determinations of μ_1 are carried out for the other zenith distances, 85°, 80°, . . . 55°. For smaller zenith distances there are obviously not six values available. Thus there are 3 determinations of μ_1 for December and January, 4 for November, 5 for February, 6 for October, 7 for March, and 8 for each of the other six months.

* Chauvenet, *Spherical Astronomy.*

To determine μ_2, the term $\mu_2 bc_0$, which has the same value for points of the same curve, must be eliminated from (IV). Therefore deduct from each equation (IV) belonging to a month the mean of its three equations appertaining to the zenith distances of 90°, 85°, 80°, and then proceed as for D_1.

The values of μ_1 and μ_2 are compiled below. The agreement between the two columns is so good that I assume their mean values, μ, for both.

	μ_1	μ_2	μ
January . . .	2·30	1·91	2·10
February. . .	1·67	1·58	1·63
March . . .	1·31	1·34	1·33
April . . .	1·00	1·09	1·04
May . . .	0·89	0·91	0·90
June . . .	0·88	0·87	0·88
July . . .	1·01	0·96	0·98
August . . .	1·11	1·03	1·07
September . .	1·11	1·14	1·12
October . . .	1·27	1·28	1·28
November . .	1·63	1·63	1·63
December . .	2·07	1·94	2·00

The factors may be determined in the same way from the corresponding quantities for $\dot{\tau}$ or $\ddot{\tau}$. I find that in every case they differ less than five per cent. from the values in the table.

I now calculate $\mu D_1(\tau)$ . . . for each month. The means of these monthly values are printed in section 7 in the first three columns. In formula (I*) the factor μ may be combined with the quantities E, F, G, which depend on the solar intensity. It thus appears that only part of the solar radiation, and not the same parts in the several months, has been effective. This may be explained by the variability, from month to month, of the average cloudiness in that region of the sky where the sun stands. Let c be the fraction of the sky that is clouded over, and f a constant during the year, then $c = 1 - f/\mu$. In Glasgow direct observation gives the value 0·7 to the average cloudiness, which nearly belongs to unit value of μ. Hence $f = 0·3$, to which belongs an average cloudiness of 0·86 in January, and of 0·65 in June, according to the values of μ in the table above.

(6) *n and the Coefficient of Transmission in the Atmosphere.*—I employ formula (III) for the determination of n, and obtain the value by trial and error. I choose successively the values $n = 0·10$, 0·15, 0·16, 0·17, 0·20, 0·25, and calculate the numerical values of G from formulæ (5), (6), (7). Two of the unknowns, a and g, are determined in each case

as functions of b (*cf.* next section), and the residuals are then calculated also in terms of b. These residuals turn out smallest for $n=0{\cdot}16$ and values of b not exceeding 0·10. This value, $n=0{\cdot}16$, is adopted in the final calculations. Hence the coefficient of transmission (§ 3) has the value 0·69 at Glasgow (log 0·69 = −0·16).

(7) *The Values of the Coefficients.*—The right-hand sides of the equations (III) and (IV*) can now be computed from (5), (6), (7). The means of their monthly values are contained in the last two columns of the equations below.

z	Equations (III).	Residuals.
90°	$-0{\cdot}15-0{\cdot}75\,a+2{\cdot}64\,b=-0{\cdot}00\,g$	$-0{\cdot}39-0{\cdot}68\,b$
85	$-0{\cdot}43-1{\cdot}03\,a+3{\cdot}08\,b=-2{\cdot}27\,g$	$-0{\cdot}06+0{\cdot}36\,b$
80	$-0{\cdot}65-1{\cdot}28\,a+3{\cdot}47\,b=-3{\cdot}87\,g$	$+0{\cdot}13+0{\cdot}93\,b$
75	$-0{\cdot}62-1{\cdot}41\,a+3{\cdot}66\,b=-3{\cdot}50\,g$	$0{\cdot}00+0{\cdot}24\,b$
70	$-0{\cdot}49-1{\cdot}44\,a+3{\cdot}68\,b=-2{\cdot}85\,g$	$-0{\cdot}08-0{\cdot}39\,b$
65	$-0{\cdot}30-1{\cdot}37\,a+3{\cdot}59\,b=-2{\cdot}31\,g$	$-0{\cdot}03-0{\cdot}61\,b$
60	$-0{\cdot}17-1{\cdot}25\,a+3{\cdot}46\,b=-1{\cdot}90\,g$	$+0{\cdot}01-0{\cdot}54\,b$
55	$-0{\cdot}11-1{\cdot}12\,a+3{\cdot}33\,b=-1{\cdot}59\,g$	$+0{\cdot}02-0{\cdot}34\,b$
50	$-0{\cdot}08-1{\cdot}02\,a+3{\cdot}20\,b=-1{\cdot}36\,g$	$+0{\cdot}02-0{\cdot}22\,b$
45	$-0{\cdot}08-0{\cdot}93\,a+3{\cdot}11\,b=-1{\cdot}20\,g$	$0{\cdot}00-0{\cdot}04\,b$
40	$-0{\cdot}06-0{\cdot}82\,a+3{\cdot}01\,b=-1{\cdot}05\,g$	$0{\cdot}00+0{\cdot}23\,b$
35	$-0{\cdot}08-0{\cdot}77\,a+2{\cdot}99\,b=-0{\cdot}95\,g$	$-0{\cdot}03+0{\cdot}35\,b$

z	Equations (IV*).	Residuals.
90°	$+0{\cdot}08-0{\cdot}23\,a-0{\cdot}60\,b=-0{\cdot}008\,e-0{\cdot}052\,f$	$-0{\cdot}03-0{\cdot}51\,b$
85	$+0{\cdot}03-0{\cdot}04\,a-0{\cdot}04\,b=-0{\cdot}006\,e-0{\cdot}028\,f$	$0{\cdot}00+0{\cdot}37\,b$
80	$-0{\cdot}11+0{\cdot}28\,a+0{\cdot}65\,b=+0{\cdot}015\,e+0{\cdot}079\,f$	$+0{\cdot}04+0{\cdot}24\,b$
75	$-0{\cdot}31+0{\cdot}63\,a+1{\cdot}45\,b=+0{\cdot}056\,e+0{\cdot}197\,f$	$+0{\cdot}02+0{\cdot}11\,b$
70	$-0{\cdot}44+0{\cdot}85\,a+2{\cdot}22\,b=+0{\cdot}110\,e+0{\cdot}293\,f$	$0{\cdot}00-0{\cdot}07\,b$
65	$-0{\cdot}57+1{\cdot}09\,a+3{\cdot}02\,b=+0{\cdot}170\,e+0{\cdot}370\,f$	$-0{\cdot}04+0{\cdot}23\,b$
60	$-0{\cdot}58+1{\cdot}16\,a+3{\cdot}61\,b=+0{\cdot}232\,e+0{\cdot}429\,f$	$-0{\cdot}02-0{\cdot}02\,b$
55	$-0{\cdot}61+1{\cdot}23\,a+4{\cdot}22\,b=+0{\cdot}294\,e+0{\cdot}475\,f$	$-0{\cdot}03+0{\cdot}05\,b$
50	$-0{\cdot}58+1{\cdot}28\,a+4{\cdot}68\,b=+0{\cdot}354\,e+0{\cdot}512\,f$	$-0{\cdot}01+0{\cdot}03\,b$
45	$-0{\cdot}57+1{\cdot}31\,a+5{\cdot}08\,b=+0{\cdot}411\,e+0{\cdot}542\,f$	$-0{\cdot}01+0{\cdot}01\,b$
40	$-0{\cdot}54+1{\cdot}33\,a+5{\cdot}43\,b=+0{\cdot}463\,e+0{\cdot}565\,f$	$+0{\cdot}01+0{\cdot}05\,b$
35	$-0{\cdot}51+1{\cdot}36\,a+5{\cdot}61\,b=+0{\cdot}512\,e+0{\cdot}585\,f$	$+0{\cdot}03+0{\cdot}03\,b$
30	$-0{\cdot}48+1{\cdot}34\,a+5{\cdot}84\,b=+0{\cdot}557\,e+0{\cdot}601\,f$	$+0{\cdot}02-0{\cdot}07\,b$

I take the sums of the equations for zenith distances 85° to 60° and the sums of the equations foı smaller zenith distances. The two equations resulting from (III) give $a=+0{\cdot}317+4{\cdot}43\,b$; $g=+0{\cdot}306+0{\cdot}809\,b$. The two equations derived from (IV*) give, with the above value of a, $e=+0{\cdot}803-1{\cdot}94\,b$; $f=-0{\cdot}880+21{\cdot}52\,b$.

The value of b is not well found from the equations. Indeed, any value of b between zero and 0·10 might be employed. Though zero value of b gives the smallest residuals, I arbitrarily adopt 0·05, which is a better value for the night curves than zero, and will be used for them (§ 9). The residuals for the value of 0·05 are only slightly larger than for zero value.

Therefore the values of the constants become

$$a = 0{\cdot}538,\ b = 0{\cdot}050,\ e = 0{\cdot}706,\ f = 0{\cdot}196,\ g = 0{\cdot}346,$$

and herewith

$$\alpha = 0{\cdot}1495,\ \log e^{-\beta t} = -\beta \text{ Mod. } t = -0{\cdot}0519\,t$$
$$\log e^{-\gamma t} = -\gamma \text{ Mod. } t = -0{\cdot}1818\,t.$$

(8) *The Values of the Constant c_0 and of the Constants of Integration.*—I calculate the values of s from (5) for each full hour of t at which the sun stands above the horizon and for each month. I take the average temperature and pressure of the air into account (by means of factor Δ in Table II, which gives the principal results for June), though this is perhaps an unnecessary complication. With these values of s I calculate further E, F, G from (6) and (7), which I substitute in (I*), and find $(2\alpha)^{-1}F(z)$ from $(2\alpha)^{-1}F(z) = (0{\cdot}3731)\mu^{-1}E + (9{\cdot}8166 - 10)\mu^{-1}F - (0{\cdot}0634)\mu^{-1}\cos\delta \sin t\ G$, the figures enclosed in brackets being logarithms.

The functions X_1 and X_2 are then found, according to formulæ (II), by adding $0{\cdot}0519\,t$ and $0{\cdot}1818\,t$. The integration is effected by mechanical quadrature, the lower limit t_r being the full hour nearest sunrise. The numerical values of T being thus known (*cf.* Tables II and III), the equations (II) furnish for each month from seven (for December) to nineteen (for June) linear equations for the determination of the unknowns c_0, c_1, c_2. The solution is best made by trial and error, c_0 being suitably chosen and the other two unknowns being calculated.

(9) *The Night Curves.*—The differential equation is again given by (I) if $F(z)$ be replaced by zero. Its integral is formula (II), where $T = 0$.

Also in this case the constants a and b are determined from the linear equation (I), in which τ, $\dot{\tau}$, $\ddot{\tau}$ are numerically known. The calculation is simple. c_0 is eliminated from the equations of the same month by deducting from each equation the mean of all the equations. Then take the sum of all the equations belonging to the same month. Combine these in two groups, which are:

April to September .	$a = 0{\cdot}47 + 6{\cdot}7\,b$
October to March . .	$a = 0{\cdot}19 + 7{\cdot}3\,b$
I adopt	$a = 0{\cdot}32 + 7{\cdot}0\,b$
as compared with . .	$a = 0{\cdot}32 + 4{\cdot}4.\,b$ for the day curves (§ 6).

The value of b is again not accurately determinable. If a be substituted in

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OF THE

ROYAL SOCIETY OF EDINBURGH.

SESSION 1919–20

Part II] VOL. XL. [Pp. 97–263

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the twelve equations and the sum be taken, the result is $b=0{\cdot}08$. I adopt 0·05 (see § 7). Hence $a=0{\cdot}680$, $b=0{\cdot}050$, $\alpha=0{\cdot}2561$, $\log e^{-\beta t}=-0{\cdot}0364\,t$, $\log e^{-\gamma t}=-0{\cdot}2589\,t$. It appears from the last paragraph in section 2 that the values of a and b at night and day need not be the same. The day curves demand a small positive value of b, since otherwise f (*cf.* § 7), which is proportional to $(ql'+p'l)$, would be negative.

c_0 and the constants of integration are determined in the manner set forth in section 8. In deriving their values I have used, in formula (II), $t-t_s$ instead of t, thus counting the hour angles from sunset.

(10) *The Tables.*—Table I contains the observed average hourly temperatures as already stated in section 4.

Tables II and III give the results of the various calculations for the month of June, as an illustration of the method.

Table IV contains the values of the constants c_0 and of the constants of integration, c_1 and c_2. These constants, together with the cloud-factor (§ 5), vary during the year, while the other constants (§ 7) maintain their values throughout the year.

c_0 of the day curves exceeds the minimum temperature, while c_0 of the night curves is smaller than the minimum temperature. This discrepancy suggests that c_0 is actually not a constant, but varies with a daily period. To take this into account a third differential equation has to be added to formulæ (3) and (4), which would, however, be a serious complication of the problem. The representation of the observations, as it is, is quite satisfactory.

The constants of integration also differ for day and night curves even though the same values of the constants a and b be used and the hour angles be counted from the same zero. It appears to me that the cause must be sought in the differential equation, which does not take all the agencies into account.

Table V gives a comparison of the observed and calculated temperatures. In the bottom line the average departures are given, which range between 0°·18 in May and 0°·04 in November for the day curves, and are less than 0°·05 for all the night curves. The departures of the calculated temperatures are systematic, and largest near sunrise and sunset. As for the large departures of the night curves in December and January, it will be seen by reference to Table I that the temperatures begin to rise several hours before sunrise. This again proves that the differential equation (I) does not perfectly represent the case. At the same time the agreement between calculated and observed figures is quite satisfactory for the purpose for which the work has been undertaken.

TABLE I.—AVERAGE TEMPERATURES, 1868 to 1913, AT GLASGOW OBSERVATORY AT THERMOGRAPH-HOURS *i*.

i.	Jan.	Feb.	Mar.	Apr.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
h	°	°	°	°	°	°	°	°	°	°	°	°
1	38·15	38·08	38·49	41·58	45·58	50·77	53·80	53·53	50·68	45·70	41·39	38·73
2	38·09	37·92	38·23	41·17	45·10	50·32	53·37	53·21	50·37	45·44	41·22	38·58
3	38·02	37·75	37·99	40·81	44·70	49·90	53·04	52·87	50·08	45·24	41·10	38·52
4	37·93	37·65	37·78	40·51	44·34	49·64	52·77	52·59	49·80	45·06	40·96	38·42
5	37·86	37·55	37·65	40·30	44·34	49·91	52·84	52·39	49·58	44·97	40·88	38·38
6	37·83	37·43	37·51	40·34	45·02	50·78	53·51	52·62	49·44	44·83	40·83	38·31
7	37·86	37·49	37·63	41·29	46·59	52·22	54·85	53·73	49·92	44·89	40·83	38·39
8	37·94	37·57	38·27	42·73	48·17	53·81	56·33	55·18	51·07	45·33	40·89	38·45
9	38·02	37·98	39·44	44·37	49·71	55·45	57·85	56·79	52·69	46·40	41·23	38·54
10	38·35	38·73	40·64	45·82	51·05	56·80	59·02	58·11	54·12	47·63	41·87	38·86
11	38·95	39·71	41·92	47·09	52·31	57·95	60·09	59·32	55·41	48·90	42·77	39·45
Noon	39·55	40·56	42·87	48·08	53·31	58·88	60·93	60·20	56·37	49·84	43·49	39·96
1	40·02	41·22	43·51	48·88	54·12	59·68	61·73	60·87	57·08	50·47	44·01	40·42
2	40·21	41·49	43·92	49·35	54·61	60·22	62·18	61·30	57·48	50·70	44·16	40·57
3	40·12	41·51	43·99	49·49	54·99	60·50	62·47	61·40	57·48	50·56	43·98	40·41
4	39·76	41·15	43·72	49·38	54·95	60·53	62·47	61·23	57·18	50·00	43·49	40·10
5	39·38	40·56	43·14	48·89	54·52	60·05	62·04	60·65	56·44	49·08	42·97	39·83
6	39·12	39·95	42·17	47·79	53·52	59·16	61·38	59·71	55·27	48·18	42·62	39·62
7	38·86	39·50	41·17	46·23	51·86	57·66	59·71	58·17	53·92	47·52	42·33	39·44
8	38·69	39·13	40·46	44·92	50·08	55·84	58·10	56·63	53·02	47·02	42·06	39·28
9	38·54	38·87	39·95	43·95	48·62	54·17	56·55	55·57	52·30	46·66	41·84	39·17
10	38·43	38·63	39·54	43·24	47·59	52·97	55·57	54·80	51·75	46·38	41·64	39·04
11	38·31	38·40	39·14	42·64	46·87	52·05	54·84	54·28	51·33	46·06	41·47	38·95
12	38·23	38·18	38·82	42·15	46·24	51·45	54·25	53·81	50·92	45·76	41·27	38·81

TABLE II.

June.

$\delta = +23^\circ{\cdot}1$, $E + \lambda = +0^h{\cdot}29$, $t_s = 8^h{\cdot}70$ at sunset.

t.	Observations.			z.	Logarithms.				
	τ.	$\dot{\tau}$.	$\ddot{\tau}$.		s.	Δ.	$F = 10^{-0{\cdot}16s}$.	$\dfrac{-ds}{d(\cos z)}$.	$\cos\delta \sin t$.
h	°	°	°						
−11	50·65	−0·52	+0·05						
−10	50·16	0·45	0·10	°					
− 9	49·78	−0·28	0·27	91·2	1·75	−20	1·1	3·0	9·8132$_n$
− 8	49·68	+0·13	0·55	85·9	1·076	−15	8·102	2·086	9·9012$_n$
− 7	50·12	0·77	0·67	78·8	0·6993	−10	9·2013	1·3851	9·9486$_n$
− 6	51·20	1·34	0·44	71·0	0·4831	+ 1	9·5133	0·9610	9·9637$_n$
− 5	52·71	1·63	+0·14	62·7	0·3365	15	9·6516	0·6717	9·9486$_n$
− 4	54·36	1·63	−0·12	54·3	0·2329	30	9·7246	0·4669	9·9012$_n$
− 3	55·90	1·42	0·26	46·3	0·1600	44	9·7664	0·3229	9·8132$_n$
− 2	57·19	1·17	0·21	39·5	0·1122	54	9·7902	0·2284	9·6627$_n$
− 1	58·27	0·99	0·18	34·5	0·0837	64	9·8032	0·1725	9·3767$_n$
Noon	59·17	0·80	0·20	32·8	0·0752	71	9·8066	0·1562	...
1	59·87	0·60	0·22	34·5	0·0837	77	9·8025	0·1738	9·3767
2	60·35	0·35	0·28	39·5	0·1122	81	9·7890	0·2311	9·6627
3	60·55	+0·04	0·34	46·3	0·1600	84	9·7642	0·3269	9·8132
4	60·41	−0·34	0·44	54·3	0·2329	81	9·7214	0·4720	9·9012
5	59·83	0·83	0·54	62·7	0·3365	75	9·6468	0·6777	9·9486
6	58·72	1·39	0·53	71·0	0·4831	66	9·5059	0·9675	9·9637
7	57·10	1·80	−0·23	78·8	0·6993	51	9·1900	1·3912	9·9486
8	55·27	1·76	+0·31	85·9	1·076	+27	8·083	2·086	9·9012
9	53·72	1·32	0·49	91·2	1·75	−19	1·1	3·0	9·8132
10	52·63	0·91	0·30	...	...	unit	deduct		deduct
11	51·84	0·70	0·14	...	...	4th	10		10
12	51·20	−0·59	+0·09	...	...	decim.			

TABLE III.

June.

t.	$(2a)^{-1}F(z)$.				X_1.	X_2.	T_1.	T_2.	T_1-T_2.	C_t.	τ_c.	O – C.
	e-term.	f-term.	g-term.	Sum.								
−11	...	...	...	...	...	...	...	...	0	50°·64	50°·64	+0°·01
−10	...	...	...	...	...	...	...	...	0	50·16	50·16	0·00
− 9	0·000	0·000	0·000	0·000	0·000	0·000	0·00	0·00	0·00	50·08	50·08	−0·30
− 8	+0·009	+0·002	+1·615	+1·626	+0·625	+ 0·057	+ 0·57	+0·48	+0·09	49·56	49·65	+0·03
− 7	0·118	0·083	4·507	4·708	2·039	0·252	3·67	3·20	0·47	49·33	49·80	+0·32
− 6	0·243	0·285	3·605	4·133	2·018	0·336	7·61	5·83	1·78	49·28	51·06	+0·14
− 5	0·334	0·552	2·459	3·345	1·840	0·413	10·23	6·85	3·38	49·34	52·72	−0·01
− 4	0·395	0·830	1·628	2·853	1·769	0·535	11·97	7·00	4·97	49·45	54·42	−0·06
− 3	0·435	1·082	1·051	2·568	1·794	0·732	13·16	6·81	6·36	49·61	55·97	−0·07
− 2	0·460	1·277	0·631	2·368	1·865	1·025	14·00	6·49	7·51	49·77	57·28	−0·09
− 1	0·474	1·405	+0·296	2·175	1·930	1·432	14·57	6·12	8·44	49·93	58·37	−0·10
Noon	0·477	1·445	0·000	1·922	1·922	1·922	14·86	5·70	9·16	50·09	59·25	−0·08
1	0·473	1·403	−0·296	1·580	1·781	2·402	14·84	5·18	9·66	50·24	59·90	−0·03
2	0·458	1·274	0·633	1·099	1·396	2·538	14·44	4·50	9·94	50·38	60·32	+0·03
3	0·433	1·077	1·055	+0·455	+0·651	+ 1·597	13·56	3·59	9·96	50·50	60·46	+0·09
4	0·392	0·824	1·635	−0·419	−0·676	− 2·235	12·06	2·37	9 69	50·61	60·30	+0·11
5	0·330	0·546	2·466	1·590	2·890	12·89	9·77	+0·72	9·05	50·72	59·77	+0·06
6	0·239	0·280	3·597	3·078	6·305	37·9	6·51	−1·39	7·90	50·81	58·71	+0·01
7	0·115	0·081	4·453	4·257	9·83	79·7	+ 2·06	4·22	6·28	50·90	57·18	−0·08
8	+0·009	+0·002	−1·546	−1·535	−3·99	−43·7	− 1·00	5·13	4·13	50·96	55·09	+0·18
9	0·000	0·000	0·000	0·000	0·000	0·0	− 1·42	−3·81	+2·39	51·03	53·42	+0·30
10	...	...	...	...	...	...	...	...	0	52·62	52·62	+0·01
11	...	...	...	...	...	...	...	...	0	51·84	51·84	0·00
12	...	...	...	...	...	...	...	...	0	51·20	51·20	0·00

Table IV.—c_0 and the Constants of Integration.

	Day Curves.			Night Curves.		
	c_0.	c_1.	c_2.	c_0.	c_1.	c_2.
January	40·15	−1·35	0·00	37·28	+1·81	+2·87
February	40·26	1·60	0·00	36·34	3·26	3·92
March	40·88	2·00	+0·05	35·38	5·44	5·50
April	44·06	2·55	0·14	36·57	8·05	7·49
May	46·94	2·00	0·11	39·68	8·67	7·26
June	51·55	1·53	0·07	45·02	7·94	6·53
July	54·73	1·40	+0·07	49·17	6·43	5·56
August	55·00	1·10	0·00	49·83	5·52	5·17
September	53·21	1·75	0·00	47·08	6·03	6·13
October	48·65	2·05	0·00	43·31	4·38	5·34
November	44·06	1·95	0·00	40·08	2·48	3·98
December	41·49	−2·00	0·00	37·65	+2·30	+3·84

Table V.—Observed Temperatures minus Calculated Temperatures.
Unit=0°·01 F.

t.	Jan.	Feb.	Mar.	Apr.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
−11	0	0	− 2	− 2	− 1	+ 1	0	+ 5	+ 5	+ 2	+ 3	0
10	+ 1	0	− 2	− 3	− 1	0	0	+ 7	+ 6	+ 1	+ 5	− 3
9	− 1	− 1	− 2	0	+ 2	−30	0	− 2	+ 2	0	0	− 4
8	− 4	− 3	− 1	+ 3	−50	+ 3	−34	−10	− 3	− 1	− 4	− 6
7	− 4	− 3	+ 2	−49	+ 6	+32	+12	− 1	− 5	− 2	− 5	− 4
6	+ 2	+ 3	+15	+ 2	+14	+14	+ 9	+ 7	− 5	+ 2	− 2	+ 2
5	+ 9	+15	+19	+ 1	+12	− 1	+ 7	− 9	+ 3	0	+ 3	+11
4	− 2	+ 5	+ 4	+ 3	− 2	− 6	+ 5	− 8	− 3	+ 8	0	+22
3	− 8	− 4	− 6	+ 2	−18	− 7	0	− 2	0	− 5	− 3	− 1
2	− 6	− 8	− 5	− 1	−23	− 9	− 6	+ 2	+ 1	− 4	− 4	0
− 1	+ 3	− 3	− 1	− 3	−19	−10	−12	+ 2	+ 1	+ 3	0	+ 3
Noon	+ 9	+ 3	+ 1	− 3	−16	− 8	− 9	+ 3	− 1	+ 6	+ 5	+ 6
1	+12	+ 4	− 1	− 4	−12	− 3	− 5	+ 6	+ 2	+ 5	+ 6	+ 7
2	+ 6	− 1	− 4	− 3	− 3	+ 3	0	+ 5	− 1	− 2	+ 4	+ 2
3	− 2	− 8	− 5	− 2	+ 7	+ 9	+ 4	+ 1	− 5	− 3	− 5	−16
4	−10	− 7	− 2	+ 1	+15	+11	+ 8	− 4	0	−11	− 7	+ 3
5	− 1	+ 3	+ 2	+ 8	+19	+ 6	+ 5	− 2	− 2	+ 4	−12	+ 2
6	+ 1	− 3	+28	+ 1	+20	+ 1	− 4	−13	+27	0	+ 1	0
7	0	0	+ 1	+33	+14	− 8	−20	+ 3	+ 3	− 1	+ 6	+ 1
8	0	+ 3	− 1	− 1	+52	+18	+10	+ 8	− 2	0	+ 6	+ 3
9	+ 1	+ 4	+ 2	− 2	0	+30	+ 1	− 2	− 3	+ 1	+ 3	+ 2
10	0	+ 2	+ 2	0	− 1	+ 1	− 2	− 4	− 4	+ 1	− 1	+ 4
11	− 2	0	+ 1	+ 1	+ 2	0	− 1	− 2	− 4	+ 1	− 4	+ 3
12	− 2	− 1	− 1	+ 2	+ 1	0	− 1	+ 1	0	+ 1	− 1	+ 2
Errors												
Day	± 6	± 6	± 7	± 8	±18	±11	± 9	± 5	± 4	± 5	± 4	± 5
Night	± 2	± 2	± 2	± 2	± 1	0	± 1	± 4	± 3	± 1	± 4	± 4

See Tables II and III for details of the calculation of the figures for June.

(Issued separately August 4, 1920.)

XII.—Molecular Energy in Gases. By Sir J. A. Ewing, K.C.B., F.R.S.

(Abstract of an Address delivered to the Royal Society of Edinburgh by request of the Council, May 3, 1920.)

(MS. received May 3, 1920. Read May 3, 1920.)

Experiments on the specific heats of gases and on their infra-red emission and absorption have gone a good way towards showing how the molecules take up energy when a gas is heated, and have raised questions as to how far the observed facts may be explained on the basis of "Newtonian dynamics," and what particulars suggest or necessitate a resort to the Quantum Theory. It may be useful to state some of these questions without attempting categorical answers, which indeed cannot be offered until the physical concepts underlying the Quantum Theory have become more definite.

It is now established that when diatomic and polyatomic gases are strongly heated the specific heats become notably greater, and that the increase of specific heat is associated with infra-red radiation. Both of these effects are conspicuous when an explosive mixture is fired in a closed combustion vessel or in the cylinder of a gas-engine.

In the explosion, for instance, of a rich mixture of coal-gas and air, the highest temperature reached is only about 2000° C., although the heat developed by the explosion would be sufficient to raise the products of combustion to a very much higher temperature if the specific heat of the products remained the same as when they are cold. The experiments of B. Hopkinson and others show that nearly all the heat has been generated by the time the maximum temperature is reached. It is recognised that no substantial part of the effect can be ascribed to "afterburning" or delayed combustion. The observed defect of temperature is sufficiently explained by increase of specific heat as the gas becomes hot. The internal energy E, instead of being proportional to the temperature, increases at an augmenting rate. This is well illustrated by data contained in the First Report of the British Association Committee on Gaseous Explosions (1908). A curve is given there for the internal energy of a gaseous mixture containing 5 per cent. by volume of carbon dioxide and 12 per cent. of water-vapour, the remainder being nearly all nitrogen. The curve is based on experimental results obtained by several independent observers using different methods.

Examination of the curve shows that it is well represented by the empirical equation

$$E = 5{\cdot}2t + 0{\cdot}00043t^2 + 0{\cdot}0000002t^3,$$

where the energy E is reckoned per gramme-molecule from 0° C., and t is the Centigrade scale temperature, or $T - 273{\cdot}1°$.

This gives, for the specific heat at constant volume per gramme-molecule,

$$C_v = 5{\cdot}2 + 0{\cdot}00086t + 0{\cdot}0000006t^2.$$

At 2000° C., therefore, the specific heat of this mixture is 9·3 as against 5·2 at 0° C., and the energy is about one-third more than it would have been, at the same temperature, if the specific heat had not increased. A great part, but by no means all, of this increase is due to the triatomic constituents of the mixture. In them the specific heat is much more affected by temperature than it is in oxygen or nitrogen. But the researches of Nernst and others have made it clear that the specific heat of diatomic gases also is increased by heating, though to a less degree. On the other hand, in monatomic gases (helium, argon, mercury-vapour) no increase whatever in the specific heat has been observed, even at temperatures exceeding 2000° C.

It is also well known that much energy is radiated from a non-luminous flame or from an exploded gas mixture, in the form of infra-red bands, whose wave-lengths correspond to those of the absorption bands of the products of combustion. Thus a carbonic oxide flame is observed to radiate energy in bands whose wave-lengths have their maxima about $2{\cdot}7\mu$, $4{\cdot}3\mu$, and $14{\cdot}7\mu$, corresponding to the three conspicuous absorption bands of carbon dioxide.

In any gas, at temperatures such as 2000° C., the short-period vibrations which produce the lines of the visible spectrum make no more than a negligible contribution to the energy. This is illustrated by the absence of any measurable increase in the specific heat of monatomic gases at high temperatures.

These facts are obviously consistent with what is, I think, the generally accepted view that in any gas the lines of the visible spectrum are due to vibrations within the atom; and that the longer-period vibrations which produce the infra-red bands, and do make a substantial addition to the energy of a hot diatomic or polyatomic gas, consist of to-and-fro movements of the atoms that compose the molecule.

It is well known that the measured specific heat of a monatomic gas is fully accounted for, on the kinetic theory, by the energy of translation of its molecules, C_v being practically equal to $\frac{3}{2}R$, C_p to $\frac{5}{2}R$, and γ to $1\frac{2}{3}$, when the condition as to density is such that the postulates of the kinetic theory approximately apply. In any gas for which these postulates hold, we may write

$$E = E' + E'' + E''',$$

where E is the whole communicable energy, E′ is the part of it which is due to translation of the molecules, E″ the part due to rotation of the molecules, and E‴ the part due to vibration within the molecules. C_v is of course dE/dT, and E′ is $\frac{3}{2}RT$.

The fact that in all monatomic gases γ is found to be exactly or very nearly equal to $1\frac{2}{3}$, and to retain that value even when the gas is strongly heated, shows that in such gases E″ as well as E‴ is negligible. In the course of its encounters the molecule acquires no rotation, or at least none that contributes appreciably to the energy. Although, according to prevailing modern conceptions of atomic structure, the moment of inertia of a single-atom molecule about any axis through its centre is excessively small, owing to concentration of the mass in the positive nucleus, it must be regarded as finite, and the principle of equipartition might seem to demand that each of the three degrees of freedom of rotation should take up an amount of energy equal to $\frac{1}{2}RT$. The aid of the Quantum Theory has been invoked to explain why this does not occur.* According to that explanation the least amount of energy of rotation which a molecule can acquire during an encounter is $h\nu$, where ν is the number of turns per second and h is Planck's constant; hence the least possible speed of rotation ν_0 is given by the equation

$$h\nu_0 = 2\pi^2\nu_0{}^2 I,$$

from which

$$\nu_0 = \frac{h}{2\pi^2 I},$$

and the quantum of energy of rotation is

$$\frac{h^2}{2\pi^2 I}.$$

In a monatomic molecule the moment of inertia I is excessively small and the quantum is consequently so large as greatly to exceed the whole energy of the striking molecule, with the result that no energy of rotation passes during the encounter. This extension of the theory of quanta to the rotational kinetic energy of a body that is free to revolve in space without constraints seems to me unnecessary. For the forces in and about an atom

* *E.g.* in Perrin's *Les Atomes*, art. 94.

may be presumed to be such that the atom is dynamically equivalent to a concentrated massive particle surrounded by a massless quasi-elastic envelope or fender. In an encounter the fenders take the blow and prevent the massive nuclei from close approach.* An encounter between two monatomic molecules whose masses are thus protected cannot be expected to communicate any rotational energy, if we make the very natural assumption that the fender does not transmit shearing stress. In that case the stress due to a blow is radial, and energy of translation only is taken up.

Consider next a diatomic gas. The mass of the molecule is made up, we may suppose, of two concentrated particles held some way apart, and the "fender" of the molecule as a whole will have a surface of revolution about the line joining the two particles. Here again, if we assume the fender to be incapable of transmitting shearing stress, no energy of rotation about that line can be communicated by an encounter though the molecule is free to take up, and will take up, energy of rotation about any transverse axis. Thus one of its three degrees of rotational freedom is ineffective, the other two are effective, and the principle of equipartition gives the familiar result that, considered as a rigid body, its whole average energy, $E'+E''$, is $\frac{5}{3}RT$, making $C_v=\frac{5}{3}R$, $C_p=\frac{7}{3}R$, and $\gamma=1{\cdot}4$. This is in accordance with the experimental fact that in general (subject to exceptions which will be referred to presently) a diatomic gas does exhibit very nearly these values of C_v, C_p, and γ. The slightly greater value of γ which is observed in normal air, etc. (1·402 or so), is readily accounted for by departure from the conditions postulated in the kinetic theory of an ideal gas, which make $PV=RT$.

In this case also the Quantum Theory has been called in to explain a result which seems intelligible enough without it. The moment of inertia of a diatomic molecule about the line joining the two centres is of a much smaller order of magnitude than the moment about a transverse axis: it is of an order so small that the quantum of energy $h^2/2\pi^2I$ with respect to rotation about that line is very large in comparison with the average energy of the blows which other molecules deliver, even when the gas is strongly heated. According to the Quantum Theory, therefore, energy of rotation about that line would not become established by encounters. About either transverse axis, however, the quantum of energy $h^2/2\pi^2I$ is far smaller, and

* *Cf.* Perrin (*loc. cit.*): "Quant au rayon de protection, distance des centres au moment du choc, il définit une distance pour laquelle la substance de l'atome exerce une force répulsive énorme sur la substance d'un autre atome. . . . En d'autres termes, chaque atome est condensé au centre d'un mince *armure* sphérique, relativement très vaste, qui le protège contre l'approche des autres atomes."

this, or a multiple of it, is readily supplied by the blows of other molecules. Hence the sum of E′ and E″ should remain equal to $\frac{5}{2}$RT, at least up to temperatures as high as are reached in experiments on the specific heats of gases. But the same result, as we have seen, follows from more ordinary dynamics.

An apparently stronger case for the application of the Quantum Theory to the rotation of molecules in a gas is presented in the remarkable fact, discovered by Eucken,* that when hydrogen is exposed to extreme cold (but is still a gas) its specific heat falls to a value which indicates that its molecules have ceased to take up energy of rotation. Eucken found that C_v in hydrogen fell from 4·94 per gramme-molecule at 0° C. (approximately $\frac{5}{2}$R) to about 3·1 when the temperature was reduced below −200° C. This has been confirmed by independent measurements of C_p † and of γ ‡. As the temperature falls, C_v approaches a limit of $\frac{3}{2}$R, and γ approaches a limit of $1\frac{2}{3}$, showing that at a temperature of 30° or 40° absolute the molecule of hydrogen takes up energy of translation only, behaving as if it were a single-atom molecule of double mass. On the Quantum Theory this is accounted for by saying that though the molecule retains its "dumb-bell" form its energy-quantum with respect to rotation about a transverse axis is sufficiently large to exceed the energy of the blows which it receives at very low temperatures. Thus if we conceive the hydrogen molecule to be made up of a pair of masses, each equal to $1{\cdot}62 \times 10^{-24}$ gr., at points 10^{-8} cm. apart, the moment of inertia I about a transverse axis is $8{\cdot}1 \times 10^{-41}$, and the energy-quantum $h^2/2\pi^2 I$ is $2{\cdot}6 \times 10^{-14}$ ergs, when h is taken as $6{\cdot}5 \times 10^{-27}$ erg-seconds. To compare this with the energy that may be delivered by a blow, we have R for any gas equal to $1{\cdot}35 \times 10^{-16}$ per molecule, the gas-constant being 1·985 in thermal units per gramme-molecule, and Avogadro's constant being $6{\cdot}16 \times 10^{-23}$. Hence the average energy of translation per molecule, $\frac{3}{2}$RT, is $2{\cdot}02 \times 10^{-16}$T, which becomes $5{\cdot}5 \times 10^{-14}$ ergs at 0° C. and $0{\cdot}6 \times 10^{-14}$ ergs at 30° absolute. The latter figure is well below the estimated value of the energy-quantum. So far as it goes, therefore, the general result of this calculation is in accord with the Quantum Theory.

It is extremely difficult, however, to believe that a body of the assumed form, free to move and turn in space without constraint, and exposed to entirely casual blows from other like bodies—which may hit it anywhere and at any angle—can refuse to accept angular momentum from such blows

* A. Eucken, *Sitzungsb. d. k. preuss. Akad.*, Feb. 1912.

† Scheel and Heuse, *Sitzungsb. d. k. preuss. Akad.*, Jan. 1915; *Ann. d. Physik*, 1913, vol. xl, p. 473.

‡ M. C. Shields, *Phys. Rev.*, Nov. 1917.

except in finite quantities equal to h/π or an integral multiple of h/π.* A solution may conceivably be found in the idea that the molecule is not really free to turn—that there is a constraint to be overcome, an action say, between it and the ether due to forces on the electrons. It may be admitted, from the known facts about the absorption and emission of radiant energy, that the ether must be regarded as forming part of the dynamical system. But one naturally seeks an escape from the application of the Quantum Theory to the rotation of the molecules, and the following suggestion is tentatively offered as an alternative. Eucken's result would be explicable if we might suppose that under extreme cold the hydrogen molecule tends to assume a different form, by a coming together of the two atoms which are held apart at ordinary temperatures. Imagine the pair of massive particles to come together, without coalescing, but so close as to constitute what is virtually a single-atom molecule of double mass: the properties of a monatomic gas would then be simulated, and there would be no communicable rotation. The specific heats would then depend only on translation, and their theoretical values would agree with the observed values. If we accept this view it may be conjectured that the paired atoms come together when the rotational speed of the diatomic molecule falls below a certain limit, perhaps like the balls of an unstable centrifugal governor, and remain together until they are forced apart by a violent encounter. There are obvious difficulties in such a hypothesis, both as to the coming together of the atoms and their separation to restore the usual diatomic form to the molecule when the gas is reheated; but they are perhaps less formidable than those that are involved in extending the idea of quanta to the angular momentum of a free body.

A curious point may be noted in passing. If the hypothesis here suggested be valid, it appears likely that there may be hysteresis in the relation of the specific heat to the temperature when hydrogen is cooled to and warmed from a very low temperature. The dynamical conditions that would bring the pair of atoms in a molecule close together are different from those that would set them apart again; the parting would probably require a higher temperature. One might expect, therefore, to find the change of structure exhibit hysteresis, resembling that which occurs in the passage of iron from one to another of two allotropic states.

* Taking $h\nu$ as the quantum of kinetic energy of rotation, the corresponding quantum of angular momentum is h/π. The theory requires that no blow should communicate angular momentum at all unless it communicates as much as this, whether we regard the indivisible quantum concerned in the operation as so much energy $h\nu$ or as so much "action" h, the dimensions of which are those of angular momentum.

Returning now to the normal action of a diatomic gas such as N_2, O_2, CO, or NO, in which the observed specific heat at ordinary temperatures is completely accounted for by molecular movements of translation and rotation, it is clear that any increase of specific heat when the temperature is raised implies that additional energy is then taken up in establishing vibration, and that the principle of equipartition thereby fails. At ordinary temperatures the terms E' and E'' make up substantially all the energy; the vibration term E''', representing to-and-fro movements of the pair of atoms, is generally insignificant. At high temperatures E''' becomes relatively larger, with the effect that the specific heat C_v increases above $\frac{5}{2}R$, and γ is reduced below 1·4. In the molecules of these gases, energy of vibration, in any amount that is sufficient to affect measurably the specific heat, is produced only by comparatively violent encounters. But in some diatomic gases, notably the vapours of the halogen elements, it is observed that at ordinary or moderate temperatures C_v exceeds $\frac{5}{2}R$ and γ is less than 1·4. There the atoms are heavy and presumably the bonds between the pair are relatively weak, so that vibrations which contribute a substantial part of the whole energy are set up by low-temperature encounters, vibrations of relatively long period being excited at relative low temperature.

In a triatomic gas the atoms presumably group themselves with the massive particles at the corners of a triangle, and in a polyatomic gas at the corners of a polyhedron. In either case there are clearly three effective freedoms of rotation, and the principle of equipartition applied to them makes E'' equal to $\frac{3}{2}RT$. Hence, in such gases, so long as the vibratory energy is negligible we should have $C_v = 3R$ and $\gamma = 1\frac{1}{3}$. The fact that, when the specific heat of such gases is measured in the normal state, C_v is found to be greater than 3R, and γ always less than $1\frac{1}{3}$, means that there is an appreciable amount of energy of vibration, even at ordinary temperatures. This is especially noticeable in polyatomic gases, where the possible modes of to-and-fro vibration of atoms are more numerous. In CO_2 there are, as we saw, three characteristic modes;* one at least of these modes is excited by low-temperature encounters sufficiently to contribute an amount of energy which reduces γ from 1·333 to about 1·30. In polyatomic gases values of γ as low as 1·13 or lower are found at ordinary temperatures.† The increase in specific heat which is produced by heating shows that the

* An interesting paper by Bjerrum (*Vorhandl. d. deutsch. Phys. Gesellschaft*, 1914, p. 737) suggests various configurations of a CO_2 molecule to give the three observed periods of vibration.

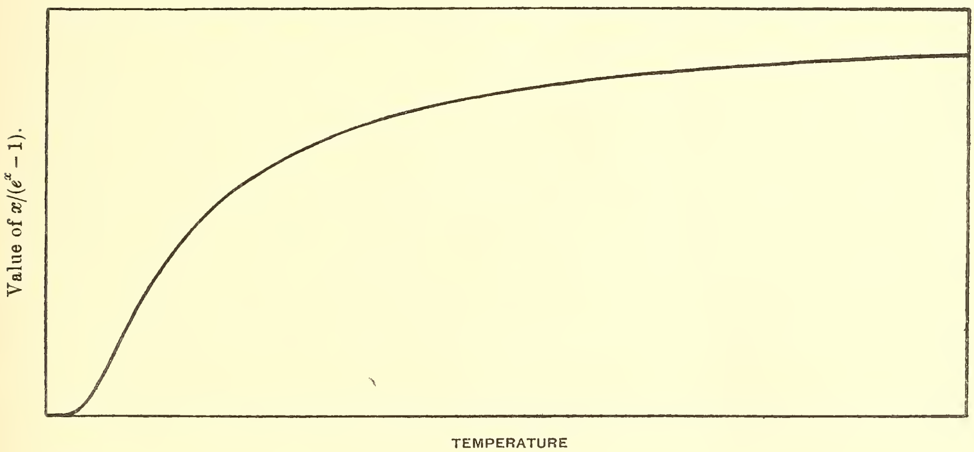
† Capstick, *Phil. Trans.*, A, 1895, vol. 186, p. 588.

various modes of vibration take up a more and more important part of the total energy as the temperature rises. And measurements of the radiation show that in this progressive development of E''', vibrations of long period begin to contribute substantially while the gas is still comparatively cold, whereas those of shorter period begin to contribute only when the gas is more strongly heated.

How does this progression occur? If we accept the Quantum Theory, the answer is at once given by Planck's formula

$$E_\nu''' = \frac{x}{e^x - 1} RT,$$

where E_ν''' is that part of the average energy of vibration, per molecule, which is contributed by a mode of vibration whose frequency is ν, and x is



$h\nu/RT$. In a gas whose molecules are capable of more than one mode of vibration the whole vibrational energy E''' would be the sum of as many terms, in the above form, as there are modes. For any given frequency the factor $x/(e^x - 1)$ tends towards the value 1 as T is indefinitely increased, but is insignificantly small when T is low. Plotted in relation to T it gives a curve, shown in the figure, which closely resembles the typical curve of magnetisation of a ferromagnetic substance under a steadily increasing magnetic force, with its very gradual beginning, its subsequent rapid rise, and its final asymptotic approach to a limiting value.

So little is known as to the mechanism of vibration in the molecules of a gas that one may be less disinclined to apply the Quantum Theory to the vibrations of the molecules than to their rotations, especially because in dealing with vibrations the energy-quantum involves a frequency which is determined by the nature of the vibrating body.

The Planck formula is undoubtedly in good general agreement with what is known about the specific heat of gases, both as to the effect of vibrations of the infra-red type in contributing to the total energy, and as to the practically negligible effect of those of high frequency. The measured values of the specific heat in gases are scarcely definite enough to supply material for a quantitative comparison between them and the values that are deduced from the formula. But it may be questioned whether even a quantitative agreement would be decisive, for the general type of curve that is given by Planck's formula is found in other physical phenomena—as for instance in the phenomena of magnetisation, where the constraint of the molecules resulting from their mutual forces accounts for similar characteristics in the curve. It would be easy, therefore, to base too much on the fact that the observed changes in molecular energy are adequately represented by means of such a curve. On the other hand, it may be pointed out that such a curve is found in the process of magnetisation only because that process results from the action of indivisible units or quanta, namely, the molecular magnets. This consideration suggests that the apparent antagonism between the older dynamics and the Quantum Theory may not impossibly disappear when the real nature of the electrical or magnetic atomicity which determines the quantum is understood.

Summary.

A review of known facts about the specific heat of gases, at temperatures ranging from low values up to 2000° C., points to the following conclusions:—

1. The increase in specific heat which is observed to occur in most gases when they are heated is due to the setting up of to-and-fro vibrations of the atoms composing the molecules. The principle of equipartition does not apply to these vibrations.

2. In monatomic gases substantially all the energy, so far as that is communicable, consists of energy of translation. Accordingly the specific heat is sensibly constant, C_v having the value $\frac{3}{2}R$ and γ the value $1\frac{2}{3}$.

3. In diatomic gases, under normal conditions, the energy consists mainly of energy of translation and energy of rotation about axes transverse to the line joining the two atoms. The specific heat C_v is approximately $\frac{5}{2}R$, and two-fifths of this quantity is accounted for by there being two (and only two) effective degrees of freedom of rotation. The normal

value of γ is accordingly $1\frac{2}{5}$. When the gas is cold the energy of vibration is nearly negligible, except in certain gases such as the vapours of the halogen elements, where it forms an appreciable part of the whole energy even at ordinary temperatures. When a diatomic gas is heated the to-and-fro vibration of the atoms comes increasingly into play and contributes a substantial addition to the energy, with the result that the specific heat rises and γ falls below $1\frac{2}{5}$.

4. In all gases at all temperatures there is equipartition of energy between each degree of freedom of translation and each effective degree of freedom of rotation.

5. The abnormal behaviour of hydrogen at very low temperatures, discovered by Eucken, may conjecturally be accounted for by supposing a change of molecular structure to occur which deprives the hydrogen molecule of its two normal degrees of freedom of rotation. If such a change of structure occurs it may be expected to exhibit hysteresis in relation to the temperature.

6. In triatomic and polyatomic gases there are three effective degrees of freedom of rotation which, along with the three degrees of freedom of translation, would make C_v equal to 3R and γ equal to $1\frac{1}{3}$ if there were no energy of vibration. But in addition there is in general a considerable amount of energy of vibration, resulting from to-and-fro movements of the atoms within the molecule, to which the principle of equipartition does not apply. Vibrations of relatively long period become important at relatively low temperatures. This makes the specific heat actually greater than 3R and γ less than $1\frac{1}{3}$, especially at high temperatures, when the energy of vibration becomes a large part of the whole energy.

7. It does not appear to be necessary to have recourse to the Quantum Theory in dealing with molecular rotations in gases: at the same time, the observed facts do not conflict with the theory.

8. The general effect of to-and-fro vibrations of atoms within the molecules of a gas is satisfactorily expressed in terms of the Quantum Theory. The resemblance which exists between the type of curve given by Planck's theory and the curve of magnetisation of a ferromagnetic substance suggests that if the nature of the atoms and their constraints were better understood the results might admit of interpretation in terms which would not be inconsistent with the older dynamics.

(*Issued separately August* 4, 1920.)

XIII.—On a Class of Graduation Formulæ.

By Catherine W. M. Sherriff, M.A., B.Sc.
Communicated by Professor E. T. Whittaker, F.R.S

(MS. received March 15, 1920. Read March 15, 1920.
Received in revised form June 17, 1920.)

§ 1. Introduction.

The problem of *Graduation* or *Adjustment*, with which the present paper is concerned, may be defined as follows. Let a number u be a function of a number x: and suppose that, corresponding to the values . . . $-3, -2, -1, 0, 1, 2, 3$ . . . of x, we have obtained, as a result of observation, values . . . $u_{-3}, u_{-2}, u_{-1}, u_0, u_1, u_2, u_3$ . . . for u. Owing to errors of observation, these observed values when plotted against the corresponding values of x do not lie on a smooth curve, although for theoretical reasons we believe that they would do so if freed from errors. The problem is to determine the most probable set of "graduated" or "adjusted" values

$$\ldots\ u_{-3}',\, u_{-2}',\, u_{-1}',\, u_0',\, u_1',\, u_2',\, u_3'\ \ldots$$

which differ only slightly from the above observed values, and which lie on a smooth curve.

In many cases the lack of regularity in the u's arises not exactly from errors of observation, but from the circumstance that u is a statistical quantity which has been derived from enumerations taken over a comparatively small field. Thus if u_x represents the number of persons dying between ages x and $x+1$ per 10,000 population, the sequence of u's will show irregularities which will be more marked the smaller the population from which the statistics are derived.

§ 2. Summation Formulæ of Graduation.

The problem of "graduating" scientific observations is frequently performed by plotting u_x against x and drawing a smooth freehand curve to pass as nearly as possible through the plotted points.* In the construction of mortality tables, however, actuaries have long been accustomed, in cases where direct curve fitting is impracticable, to effect the necessary graduations by formulæ. The formulæ used by actuaries belong to what

* *Journ. Inst. Act.*, 22, p. 320, and 30, p. 162.

may be called the "summation" type, which is based on the following principle. Let Δ denote the operation of differencing, so that $\Delta u_n = u_{n+1} - u_n$, and let $[2m+1]\,u_n$ denote the sum of $(2m+1)$ u's of which u_n is the middle one, so that

$$[2m+1]u_0 = u_{-m} + u_{-m+1} + \ .\ .\ .\ + u_{-1} + u_0 + u_1 + \ .\ .\ .\ + u_{m-1} + u_m.$$

Then it is possible to find combinations of these operations Δ and [ ], which, when differences above a certain order are neglected, merely reproduce the functions operated on; so that we have (say)

$$f\{\Delta, [\quad]\}\,u_x = u_x + \text{high differences.}$$

We now take $f\{\Delta, [\quad]\}u_x$ to be the graduated value of u_x; that is,

$$u_x' = f\{\Delta, [\quad]\}u_x:$$

the merit of this u_x' depending on the circumstance, that $f\{\Delta, [\quad]\}\,u_x$ involves a large number of the observed u's, whose errors to a considerable extent neutralise each other, and so produce a smoothed value u_x' in place of u_x.

Perhaps the best of the summation formulæ of graduation correct to third differences is that of J. Spencer,* namely,

$$u_0' = \frac{[5][5][7]}{5\,.\,5\,.\,7}(1 - 4\partial^2 - 3\partial^4 - \tfrac{1}{2}\partial^6)u_0,$$

where $\partial^2 u_0$ stands for $u_{-1} - 2u_0 + u_1$. Written in full, Spencer's formula is

$$u_0' = \tfrac{1}{350}\{60u_0 + 57(u_1 + u_{-1}) + 47(u_2 + u_{-2}) + 33(u_3 + u_{-3}) + 18(u_4 + u_{-4}) + 6(u_5 + u_{-5}) \\ - 2(u_6 + u_{-6}) - 5(u_7 + u_{-7}) - 5(u_8 + u_{-8}) - 3(u_9 + u_{-9}) - (u_{10} + u_{-10})\}.$$

It is a case of a more general formula, which may be written

$$u_0' = \frac{[p][q][r]\ .\ .\ .}{p.\,q.\,r\ .\ .\ .}\left\{1 - \frac{\Sigma(p^2-1)}{24}\partial^2 + \text{any}\dagger\text{ multiples of higher even central differences}\right\}u_0,$$

where p, q, r are any whole numbers, of which an even number (or zero) must be even, and any number may be odd.

§ 3. Formulæ Based on the Method of Least-Squares.

The formulæ of graduation which are studied in the present paper are obtained by a wholly different method from the summation formulæ which have just been referred to. Supposing the ungraduated values u_x to be plotted as points against the corresponding values of x, we shall fit a parabolic curve of some assigned degree j to successive sets of $(2m+1)$ points $(u_{-m}, u_{-m+1}\ .\ .\ .\ u_0\ .\ .\ .\ u_m)$, determining the constants of the curve

* *Journ. Inst. Act.*, 38, p. 334. † Usually small multiples.

by the method of least-squares, and we shall then take the successive central ordinates of these curves as the graduated value of u_0. This method has not hitherto been adopted in actuarial practice, and indeed the detailed graduation formulæ based on it do not appear to have been given previously *in extenso*, except for small values of j, although the theory of the method has been discussed, examples given, and many interesting properties established in some valuable papers by W. F. Sheppard.*

Among the questions which present themselves naturally, but have not as yet been solved, may be mentioned the following. What is the connection between summation formulæ and least-square formulæ? and, in particular, is it possible for a formula to belong to both types—or, in other words, are there any formulæ which can be justified by the least-square principle, and can moreover be represented by the summatory notation? Such formulæ would, if they exist, be of particular interest, since the process of summation is so easily performed by the computer that summatory formulæ are the simplest to use, whereas the least-square formulæ appear to have the sounder theoretical basis. Again, is it a fact that least-square formulæ give better graduation than summation formulæ? Or are the least-square formulæ better adapted for dealing with certain classes of problems and the summation formulæ better adapted for dealing with other classes? These and similar questions provided the motive for the present investigation.

Let u_x be the ungraduated value of u corresponding to the value x, and let $u'(x)$ be the ordinate of the parabolic curve which is fitted to the data, so that the problem is to find

$$u'(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ . \ . \ . \ + c_j x^j,$$

for which $\sum_{p=-m}^{m} \{u_p - u'(p)\}^2$ is a minimum.

The "equations of condition" are

$$\begin{array}{lll}
c_0 + c_1 m + c_2 m^2 + c_3 m^3 + \ . \ . \ . & + c_j m^j & = u_m \\
c_0 + c_1(m-1) + c_2(m-1)^2 + c_3(m-1)^3 + \ . \ . \ . & + c_j(m-1)^j & = u_{m-1} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \\
c_0 + c_1 . 1 + c_2 . 1^2 + c_3 . 1^3 & + c_j . 1^j & = u_1 \\
c_0 & & = u_0 \\
c_0 - c_1 . 1 + c_2 . 1^2 - c_3 . 1^3 & + (-1)^j c_j 1^j & = u_{-1} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \\
c_0 - c_1(m-1) + c_2(m-1)^2 - c_3(m-1)^3 + \ . \ . \ . & + (-1)^j c_j(m-1)^j & = u_{-(m-1)} \\
c_0 - c_1 m + c_2 m^2 - c_3 m^3 + \ . \ . \ . & + (-1)^j c_j m^j & = u_{-m}.
\end{array}$$

* *Proc. of the Vth Internat. Congress of Mathematicians, Cambridge*, 1912, ii, p. 348; *Proc. of the London Mathematical Society*, ser. 2, vol. xiii, part ii; *Journ. Inst. Act.*, **48**, p. 171; **48**, p. 390; **49**, p. 148.

Since the sums of the odd powers over all the values from $-m$ to $+m$ vanish, it is obvious that only half of the "normal equations" will contain c_0. Denoting $(-m)^r+(-m+1)^r+\ldots+(-1)^r+(1)^r+\ldots+(m-1)^r+m^r$ by Σ_r for all values of r except zero, the normal equations containing c_0 are

$$\begin{array}{l}
\Sigma_0 c_0 + \Sigma_2 c_2 + \Sigma_4 c_4 + \ldots + \Sigma_{2k} c_{2k} = \Sigma u_s \\
\Sigma_2 c_0 + \Sigma_4 c_2 + \Sigma_6 c_4 + \ldots + \Sigma_{2k+2} c_{2k} = \Sigma s^2 u_s \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\Sigma_{2k} c_0 + \Sigma_{2k+2} c_2 + \Sigma_{2k+4} c_4 + \ldots + \Sigma_{4k} c_{2k} = \Sigma s^{2k} u_s, \quad \text{where } j = 2k \text{ or } 2k+1,
\end{array}$$

$\Sigma_0 = 2m+1$, and $\Sigma s^p u_s = (-m)^p u_{-m} + (-m+1)^p u_{-m+1} + \ldots + (m-1)^p u_{m-1} + m^p u_m$.

Solving these equations determinantally, we have as the value of c_0, which is equal to $u'(0)$, the graduated value of u_0:

$$\frac{\begin{vmatrix} \Sigma u_s & \Sigma s^2 u_s & \Sigma s^4 u_s & \ldots & \Sigma s^{2k} u_s \\ \Sigma_2 & \Sigma_4 & \Sigma_6 & \ldots & \Sigma_{2k+2} \\ \Sigma_4 & \Sigma_6 & \Sigma_8 & \ldots & \Sigma_{2k+4} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \Sigma_{2k} & \Sigma_{2k+2} & \Sigma_{2k+4} & \ldots & \Sigma_{4k} \end{vmatrix}}{\begin{vmatrix} \Sigma_0 & \Sigma_2 & \Sigma_4 & \ldots & \Sigma_{2k} \\ \Sigma_2 & \Sigma_4 & \Sigma_6 & \ldots & \Sigma_{2k+2} \\ \Sigma_4 & \Sigma_6 & \Sigma_8 & \ldots & \Sigma_{2k+4} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \Sigma_{2k} & \Sigma_{2k+2} & \Sigma_{2k+4} & \ldots & \Sigma_{4k} \end{vmatrix}}, \quad . \quad . \quad . \quad (A),$$

a linear combination of the quantities $\Sigma u_s, \Sigma s^2 u_s \ldots \Sigma s^{2k} u_s$.

If it be supposed that each of the given data is weighted, we may take $w_{-m}, w_{-m+1}, w_{-m+2} \ldots w_{-1}, w_0, w_1 \ldots w_{m-1}, w_m$ to be the weights, respectively, of $u_{-m}, u_{-m+1}, u_{-m+2} \ldots u_{-1}, u_0, u_1 \ldots u_{m-1}, u_m$; the normal equation in c_0 will be

$$\begin{aligned} c_0\{w_{-m}+w_{-m+1}+\ldots+w_m\} + c_1\{-mw_{-m}-(m-1)w_{-m+1}-\ldots+(m-1)w_{m-1}+mw_m\} \\ + c_2\{m^2 w_{-m}+(m-1)^2 w_{-m+1}+\ldots+m^2 w_m\}+\ldots \\ = w_{-m}u_{-m}+w_{-m+1}u_{-m+1}+\ldots+w_m u_m. \end{aligned}$$

It follows that when $w_m = -w_m$, etc., the quotient of the two determinants (A) will still represent the graduated value of u_0, provided the notation in the case of weighted data is understood to be

$$\begin{aligned}
\Sigma_0 &= w_{-m}+w_{-m+1}+\ldots+w_0+\ldots+w_{m-1}+w_m. \\
\Sigma_r &= (-m)^r w_{-m}+(-m+1)^r w_{-m+1}+\ldots+(m-1)^r w_{m-1}+m^r w. \\
\Sigma u_s &= w_{-m}u_{-m}+w_{-m+1}u_{-m+1}+\ldots+w_0 u_0+\ldots+w_{m-1}u_{m-1}+w_m u_m. \\
\Sigma s^p u_s &= (-m)^p w_{-m}u_{-m}+(-m+1)^p w_{-m+1}u_{-m+1}+\ldots+(m-1)^p u_{m-1}w_{m-1}+m^p u_m w_m.
\end{aligned}$$

The quotient of the two determinants (A), when expanded by Schwein's theorem, gives

$$\frac{\Sigma u_s}{\Sigma_0}+\frac{\Sigma_2\begin{vmatrix}\Sigma u_s & \Sigma s^2u_s\\ \Sigma_0 & \Sigma_2\end{vmatrix}}{\Sigma_0\begin{vmatrix}\Sigma_0 & \Sigma_2\\ \Sigma_2 & \Sigma_4\end{vmatrix}}+\frac{\begin{vmatrix}\Sigma_2 & \Sigma_4\\ \Sigma_4 & \Sigma_6\end{vmatrix}\begin{vmatrix}\Sigma u_s & \Sigma s^2u_s & \Sigma s^4u_s\\ \Sigma_0 & \Sigma_2 & \Sigma_4\\ \Sigma_2 & \Sigma_4 & \Sigma_6\end{vmatrix}}{\begin{vmatrix}\Sigma_0 & \Sigma_2\\ \Sigma_2 & \Sigma_4\end{vmatrix}\begin{vmatrix}\Sigma_0 & \Sigma_2 & \Sigma_4\\ \Sigma_2 & \Sigma_4 & \Sigma_6\\ \Sigma_4 & \Sigma_6 & \Sigma_8\end{vmatrix}}+\frac{\begin{vmatrix}\Sigma_2 & \Sigma_4 & \Sigma_6\\ \Sigma_4 & \Sigma_6 & \Sigma_8\\ \Sigma_6 & \Sigma_8 & \Sigma_{10}\end{vmatrix}\begin{vmatrix}\Sigma u_s & \Sigma s^2u_s & \Sigma s^4u_s & \Sigma s^6u_s\\ \Sigma_0 & \Sigma_2 & \Sigma_4 & \Sigma_6\\ \Sigma_2 & \Sigma_4 & \Sigma_6 & \Sigma_8\\ \Sigma_4 & \Sigma_6 & \Sigma_8 & \Sigma_{10}\end{vmatrix}}{\begin{vmatrix}\Sigma_0 & \Sigma_2 & \Sigma_4\\ \Sigma_2 & \Sigma_4 & \Sigma_6\\ \Sigma_4 & \Sigma_6 & \Sigma_8\end{vmatrix}\begin{vmatrix}\Sigma_0 & \Sigma_2 & \Sigma_4 & \Sigma_6\\ \Sigma_2 & \Sigma_4 & \Sigma_6 & \Sigma_8\\ \Sigma_4 & \Sigma_6 & \Sigma_8 & \Sigma_{10}\\ \Sigma_6 & \Sigma_8 & \Sigma_{10} & \Sigma_{12}\end{vmatrix}}+\ldots,$$

a form more suitable for computation.

If we take the first term only we obtain the graduation formula which would be obtained by fitting a straight line to each set of $2m+1$ observations; the first two terms give graduation formulæ obtained by fitting a parabola of the second or third order; and, in general, from r terms we derive the graduation formulæ obtained by fitting a parabola of the $(2r-1)^{\text{th}}$ or $(2r-2)^{\text{th}}$ order to each successive set of $2m+1$ observations.

The results of computations based on the above expansion are collected in the following table (Table I), which gives the coefficients $\mathrm{M}, p_0, p_1, p_2 \ldots$ in the least-square graduation formulæ

$$\mathrm{M}u_0'=p_0u_0+p_1(u_1+u_{-1})+p_2(u_2+u_{-2})+\ \ldots\ +p_m(u_m+u_{-m}),$$

formed on the assumptions $k=0, 1, 2, 3, 4$ and for a considerable range of m.* The table, in fact, gives explicitly all the graduation-formulæ of the least-square type which are ever likely to be used in practice.

* I must express my obligations to Mr John Maclean, M.A., B.Sc., Professor of Mathematics in Wilson College, Bombay, and to Mr Jason J. Nassau, C.E., M.Sc., Instructor in Mathematics in Syracuse University, N.Y., U.S.A., who took a great share in performing the computations, which were carried out in the Mathematical Institute of Edinburgh University. It should also be mentioned that Mr W. F. Sheppard (*loc. cit.*) had previously discussed the cases $k=0, 1, 2$; but his formulæ, being expressed in terms of c_0 (Sheppard's b_0), ∂^2c_0, $\partial^4c_0 \ldots$ in one paper and in terms of central sums in another paper, are not the same as the formulæ found above under the headings $k=0, 1, 2$, although, of course, they may be shown to be equivalent to them.

TABLE I.

CASE I.—$k=0$, i.e. $j=1$.

$$u_0' = \frac{1}{2m+1}(u_0 + \overline{u_1 + u_{-1}} + \overline{u_2 + u_{-2}} + \ . \ . \ . \ + \overline{u_m + u_{-m}}).$$

CASE II.—$k=1$, i.e. $j=2$ or 3.

No. of Terms in Formula $=2m+1$.	M $=\Sigma p$.	p_0.	p_1.	p_2.	p_3.	p_4.	p_5.	p_6.	p_7.	p_8.	p_9.	p_{10}.	p_{11}.	p_{12}.	p_{13}.	p_{14}.	p_{15}.
3	1	1															
5	35	17	12	−3													
7	21	7	6	3	−2												
9	231	59	54	39	14	−21											
11	429	89	84	69	44	9	−36										
13	143	25	24	21	16	9	0	−11									
15	1105	167	162	147	122	87	42	−13	−78								
17	323	43	42	39	34	27	18	7	−6	−21							
19	2261	269	264	249	224	189	144	89	24	−51	−136						
21	3059	329	324	309	284	249	204	149	84	9	−76	−171					
23	805	79	78	75	70	63	54	43	30	15	−2	−21	−42				
25	5175	467	462	447	422	387	342	287	222	147	62	−33	−138	−253			
27	1305	109	108	105	100	93	84	73	60	45	28	9	−12	−35	−60		
29	8091	629	624	609	584	549	504	449	384	309	224	129	24	−91	−216	−351	
31	9889	719	714	699	674	639	594	539	474	399	314	219	114	−1	−126	−261	−406

CASE III.—$k=2$, i.e. $j=4$ or 5.

No. of Terms in Formula.	M.	p_0.	p_1.	p_2.	p_3.	p_4.	p_5.	p_6.	p_7.	p_8.	p_9.	p_{10}.	p_{11}.	p_{12}.	p_{13}.	p_{14}.	p_{15}.
5	1	1															
7	231	131	75	−30	5												
9	429	179	135	30	−55	15											
11	429	143	120	60	−10	−45	18										
13	2431	677	600	390	110	−135	−198	110									
15	46189	11063	10125	7500	3755	−165	−2937	−2860	2145								
17	4199	883	825	660	415	135	−117	−260	−195	195							
19	7429	1393	1320	1110	790	405	18	−290	−420	−255	340						
21	260015	44003	42120	36660	28190	17655	6378	−3940	−11220	−13005	−6460	11628					
23	2185	337	325	290	235	165	87	10	−55	−95	−95	−38	95				
25	30015	4253	4125	3750	3155	2385	1503	590	−255	−915	−1255	−1122	−345	1265			
27	930465	121943	118800	109560	94790	75435	52818	28640	4980	−15705	−30580	−36432	−29670	−6325	37950		
29	445005	54251	53040	49470	43730	36135	27126	17270	7260	−2085	−9820	−14874	−16050	−12025	−1350	17550	
31	29667	3381	3315	3120	2805	2385	1881	1320	735	165	−345	−744	−975	−975	−675	0	1131

CASE IV.—$k=3$, i.e. $j=6$ or 7.

No. of Terms in Formula.	M.	p_0.	p_1.	p_2.	p_3.	p_4.	p_5.	p_6.	p_7.	p_8.	p_9.	p_{10}.
7	1	1										
9	1287	797	392	−196	56	−7						
11	2431	1157	784	28	−308	161	−28					
13	46189	18063	14000	4550	−3500	−3605	3388	−770				
15	138567	46189	38500	19250	−1540	−11935	−5236	10010	−2860			
17	96577	28109	24500	15050	3500	−5215	−6916	−910	6500	−2275		
19	37145	9605	8624	5978	2492	−679	−2380	−1918	344	2261	−952	
21	334305	77821	71344	53508	28812	3801	−14364	−19908	−11016	7021	18088	−9044

CASE V.—$k=4$, i.e. $j=8$ or 9.

No. of Terms in Formula.	M.	p_0.	p_1.	p_2.	p_3.	p_4.	p_5.	p_6.	p_7.	p_8.	p_9.	p_{10}.
9	1	1										
11	46189	30313	13230	−7560	2835	−630	63					
13	4199	2183	1350	−135	−465	360	−117	15				
15	96577	42109	30375	5625	−9885	−3870	8109	−3705	585			
17	7429	2803	2205	819	−399	−630	63	525	−333	63		
19	22287	7429	6174	3087	−147	−1764	−1071	777	1206	−1071	238	
21	3231615	966073	833490	493920	97755	−178290	−219177	−48930	148410	123165	−158270	40698

§ 4. The Curves of Coefficients.

Let

$$u_0' = p_0 u_0 + p_1(u_1 + u_{-1}) + p_2(u_2 + u_{-2}) + \ . \ . \ .$$

be a graduation formula; by plotting the coefficients p_s as ordinates against their suffixes s as abscissæ, we obtain a set of points which, when joined so as to form a smooth curve, may be called the *curve of coefficients.* It has been remarked by G. J. Lidstone * that for the best graduation formulæ of the summation type the curve of coefficients resembles the curve of coefficients for Spencer's formula, which is the fifth curve in the annexed diagram (fig. 1). Now, in the case of the least-square graduation formulæ, we readily see that coefficients of the $k=1$, 2, 3, and 4 formulæ have respectively their second, fourth, sixth, and eighth differences constant: which shows us that the curves of coefficients in the case of $k=1, 2, 3$, and 4 are parabolæ of the second, fourth, sixth, and eighth order respectively. The equations to these curves for $m=10$ are, in fact,

$$3059y = 329 - 5x^2 \text{ when } k=1,$$

$$260015y = 44003 - \frac{7595}{4}x^2 + \frac{63}{4}x^4 \text{ when } k=2,$$

$$334305y = 77821 - \frac{1190357}{180}x^2 + \frac{24640}{180}x^4 - \frac{143}{180}x^6 \text{ when } k=3,$$

$$3231615y = 966073 - \frac{79321660}{576}x^2 + \frac{2994915}{576}x^4 - \frac{41250}{576}x^6 + \frac{187}{576}x^8 \text{ when } k=4,$$

where x has values ranging from -10 to $+10$.

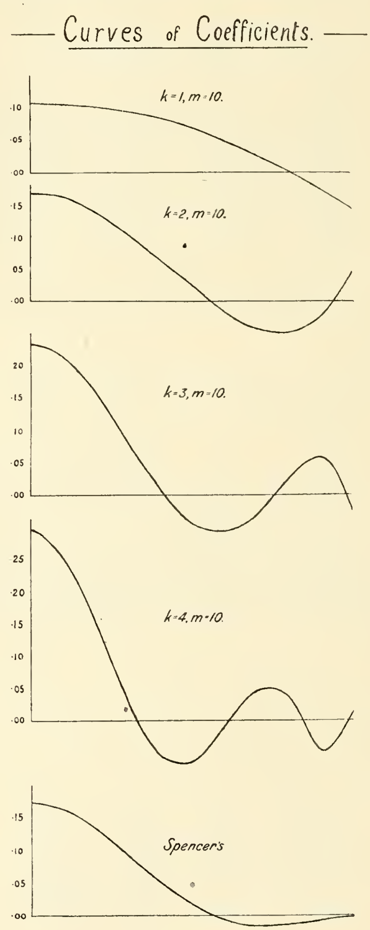


Fig. 1.

These curves are also represented in the diagram (fig. 1). It is evident that *the curves of coefficients for least-square formulæ do not resemble the curves of coefficients for the summation formulæ, the differences between the curves being so fundamental as to indicate an essential difference between the two classes of formulæ.*

* *Journ. Inst. Act.*, **41**, p. 348. See also Karup (J.), *Trans. of 2nd International Actuarial Congress*, p. 47, and translation, p. 93.

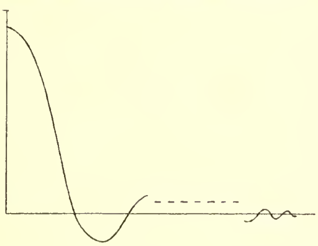
Considering the graduated value of u_0 as the quotient of the two determinants (A), it is obvious * that in the general case the curve of coefficients will have as its equation

$$\begin{vmatrix} \Sigma_0 & \Sigma_2 & \Sigma_4 & \cdots & \Sigma_{2k} \\ \Sigma_2 & \Sigma_4 & \Sigma_6 & \cdots & \Sigma_{2k+2} \\ \vdots & \vdots & \vdots & & \vdots \\ \Sigma_{2k} & \Sigma_{2k+2} & \Sigma_{2k+4} & \cdots & \Sigma_{4k} \end{vmatrix} y = \begin{vmatrix} \Sigma_4 & \Sigma_6 & \cdots & \Sigma_{2k+2} \\ \Sigma_6 & \Sigma_8 & \cdots & \Sigma_{2k+4} \\ \vdots & \vdots & & \vdots \\ \Sigma_{2k+2} & \Sigma_{2k+4} & \cdots & \Sigma_{4k} \end{vmatrix} - \begin{vmatrix} \Sigma_2 & \Sigma_6 & \Sigma_8 & \cdots & \Sigma_{2k+2} \\ \Sigma_4 & \Sigma_8 & \Sigma_{10} & \cdots & \Sigma_{2k+4} \\ \vdots & \vdots & \vdots & & \vdots \\ \Sigma_{2k} & \Sigma_{2k+4} & \Sigma_{2k+6} & \cdots & \Sigma_{4k} \end{vmatrix} x^2$$

$$+ \begin{vmatrix} \Sigma_2 & \Sigma_4 & \Sigma_8 & \Sigma_{10} & \cdots & \Sigma_{2k+2} \\ \Sigma_4 & \Sigma_6 & \Sigma_{10} & \Sigma_{12} & \cdots & \Sigma_{2k+4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \Sigma_{2k} & \Sigma_{2k+2} & \Sigma_{2k+6} & \Sigma_{2k+8} & \cdots & \Sigma_{4k} \end{vmatrix} x^4 - \dots$$

$$+ (-1)^k \begin{vmatrix} \Sigma_2 & \Sigma_4 & \cdots & \Sigma_{2k} \\ \Sigma_4 & \Sigma_6 & \cdots & \Sigma_{2k+2} \\ \vdots & \vdots & & \vdots \\ \Sigma_{2k} & \Sigma_{2k+2} & \cdots & \Sigma_{4k-2} \end{vmatrix} x^{2k} \quad . \quad . \quad . \quad . \quad . \quad \text{(B)},$$

and will be of this form:—



Substituting the values 1 and 2 for k in equation (B), we have

(α) when $k=1$, $\begin{vmatrix} \Sigma_0 & \Sigma_2 \\ \Sigma_2 & \Sigma_4 \end{vmatrix} y = \Sigma_4 - \Sigma_2 x^2$,

or

$$(2m-1)(2m+1)(2m+3)y = 3(3m^2+3m-1) - 15x^2;$$

(β) when $k=2$, $\begin{vmatrix} \Sigma_0 & \Sigma_2 & \Sigma_4 \\ \Sigma_2 & \Sigma_4 & \Sigma_6 \\ \Sigma_4 & \Sigma_6 & \Sigma_8 \end{vmatrix} y = \begin{vmatrix} \Sigma_4 & \Sigma_6 \\ \Sigma_6 & \Sigma_8 \end{vmatrix} - \begin{vmatrix} \Sigma_2 & \Sigma_6 \\ \Sigma_4 & \Sigma_8 \end{vmatrix} x^2 + \begin{vmatrix} \Sigma_2 & \Sigma_4 \\ \Sigma_4 & \Sigma_6 \end{vmatrix} x^4$,

or

$$4(2m-3)(2m-1)(2m+1)(2m+3)(2m+5)y = 15(15m^4+30m^3-35m^2-50m+12) - 525(2m^2+2m-3)x^2 + 945x^4,$$

which give the following equations to the parabolæ for different values of m.

(α) $k=1$.

$m=11$, $805y = 79 - x^2$, where x has values ranging from -11 to $+11$
$m=12$, $5175y = 467 - 5x^2$, ,, ,, ,, -12 ,, $+12$
$m=13$, $1305y = 109 - x^2$, ,, ,, ,, -13 ,, $+13$
$m=14$, $8091y = 629 - 5x^2$, ,, ,, ,, -14 ,, $+14$
$m=15$, $9889y = 719 - 5x^2$, ,, ,, ,, -15 ,, $+15$.

* As is easily seen by expanding the elements in the first row of the numerator determinant in equation (A).

(β) $k=2$.

$m=11$,	$2185y=337-\frac{145}{12}x^2+\frac{x^4}{12}$,	where x has values ranging from	-11 to	$+11$
$m=12$,	$30015y=4253-\frac{515}{4}x^2+\frac{3}{4}x^4$,	,, ,, ,,	-12 ,,	$+12$
$m=13$,	$930465y=121943-\frac{12635}{4}x^2+\frac{63}{4}x^4$,	,, ,, ,,	-13 ,,	$+13$
$m=14$,	$445005y=54251-\frac{4865}{4}x^2+\frac{21}{4}x^4$,	,, ,, ,,	-14 ,,	$+14$
$m=15$,	$29667y=3381-\frac{265}{4}x^2+\frac{x^4}{4}$,	,, ,, ,,	-15 ,,	$+15$

From these equations the formulæ for $k=1$ and 2, $m=11$ . . . 15, given in Table I, were derived.

§ 5. Incompatibility of Least-Square and Summation Formulæ.

Although the considerations adduced in § 4 made it tolerably certain that the least-square formulæ (at any rate when the weights of the data are supposed equal) cannot be derived by summation processes, it seemed desirable to establish this fact directly, which may be done in the following way. Let E be, as usual, the symbolic operator defined by the equation $\mathrm{E}u_x=u_{x+1}$, and write $\mathrm{E}=e^{2i\phi}$ so

$$\begin{aligned}
&u_{-m}+u_{-m+1}+\ .\ .\ .\ +u_{-1}+u_0+u_1+\ .\ .\ .\ +u_m\\
&=\left\{e^{-2mi\phi}+e^{-(2m-1)i\phi}+\ .\ .\ .\ +e^{-2i\phi}+1+e^{2i\phi}+\ .\ .\ .\ +e^{2mi\phi}\right\}u_0\\
&=(1+2\cos 2\phi+2\cos 4\phi+\ .\ .\ .\ +2\cos 2m\phi)u_0\\
&=\frac{\sin(2m+1)\phi}{\sin\phi}u_0.
\end{aligned}$$

Hence if a graduation formula be written

$$u_0'=p_0u_0+p_1(u_1+u_{-1})+p_2(u_2+u_{-2})+\ .\ .\ .,$$

and if the right-hand side of this equation is capable of being expressed as the result of performing the operation $[n]$ on a simpler expression, then when in this right-hand side we replace u_n by $e^{2ni\phi}u_0$ we must obtain a function of ϕ, which vanishes when $\frac{\sin n\phi}{\sin\phi}$ vanishes, *i.e.* when

$$\phi=\frac{\pi}{n},\ \frac{2\pi}{n},\ \frac{3\pi}{n}\ .\ .\ .\ :$$

that is

$$p_0+2p_1\cos 2\phi+2p_2\cos 4\phi+\ .\ .\ .\ \text{must vanish, for } \phi=\frac{\pi}{n},\ \frac{2\pi}{n}\ .\ .\ .$$

By the application of this test to each of the least-square formulæ it was found that *in no case can any of the least-square formulæ be derived* *by the summatory process.* The difference between the two classes of formulæ is therefore fundamental.

In applying the ϕ method of testing whether a least-square formula of graduation can be represented by a summation formula, we may note that the terms of the least-square formula may readily be summed with respect to ϕ and thus presented in a compact form which is readily tested: for example, the least-square formula of graduation corresponding to $k=1$, $m=10$ may be written

$$u_0'=\frac{1}{3059}\left(-\frac{221\sin 21\phi}{\sin\phi}-\frac{105}{2}\quad\frac{\cos 21\phi\cos\phi}{\sin^2\phi}+\frac{5}{2}\quad\frac{\sin 21\phi\cos^2\phi}{\sin^3\phi}\right)u_0$$

where $e^{2ni\phi}u_0$ represents u_n. It is only the work of a moment to show that this does not vanish when $\phi=\frac{\pi}{n}$, where n is a given whole number. On the other hand, in considering Spencer's formula and other summation formulæ, it was found that the series

$$p_0+2p_1\cos\frac{2\pi}{n}+2p_2\cos\frac{4\pi}{n}+2p_3\frac{\cos 6\pi}{n}+\ .\ .\ .$$

took the forms

$$k_1\left[1+2\left(\cos\frac{2\pi}{3}\right)\right];\quad k_2\left[1+2\left(\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}\right)\right];$$

$$k_3\left[1+2\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\right)\right]\ .\ .\ .\ k_n\left[1+2\left(\cos\frac{2\pi}{2n+1}+\cos\frac{4\pi}{2n+1}+\ .\ .\ .\ +\cos\frac{2n\pi}{2n+1}\right)\right]$$

where the k's are numerical quantities, when it had been obtained by the operations [3], [5], [7], . . . $[2n+1]$ respectively on simpler expressions. Each of these expressions vanish as the part in the round brackets in every case equals $-\frac{1}{2}$.

A more straightforward test* is one obtained from the application of the remainder theorem; *e.g.* Spencer's formula would be looked upon as corresponding to the algebraic expression

$$-1-3x-5x^2-5x^3-2x^4+6x^5+18x^6+33x^7+47x^8+57x^9+60x^{10}+57x^{11}+47x^{12}+33x^{13}$$
$$+18x^{14}+6x^{15}-2x^{16}-5x^{17}-5x^{18}-3x^{19}-x^{20}.$$

If Spencer's formula can be derived by performing the operation [5] on a simpler expression, then this algebraic analogue of Spencer's formula must be divisible by $1+x+x^2+x^3+x^4$, *i.e.* divisible by $\frac{1-x^5}{1-x}$ For the test substitute x^5 for 1, arrange the expression so obtained in columns of x^0, x, x^2, x^3, and x^4, and add the members in each column, *e.g.*:—

x^0.	x.	x^2.	x^3.	x^4.
−1 +6 +60 +6 −1	−3 +18 +57 −2	−5 +33 +47 −5	−5 +47 +33 −5	−2 +57 +18 −3
+70	+70	+70	+70	+70

* See also *Trans. Act. Soc. of America*, vol. xix, pp. 302–3.

For exact divisibility the coefficients of the various powers of x in the expression so obtained must be equal.

The test, then, for finding whether an expression can be obtained by performing the operation $[n]$ on a simpler expression, is to arrange the coefficients of the various terms of the given expression in order in n columns; the sum of the members in each column is taken, and if these sums are equal, the expression can be obtained by summation.

It may be pointed out that these sums give a clue to other summatory factors which the expression may have, *e.g.* in the case worked out we observe that 70 has as its factors 2, 5, 7. Hence it is probable that the expression may have been obtained by performing the operation [2], [7], and a repeated [5] on a simpler expression. Before testing for the second [5] it is necessary to eliminate the first by a process of division. It will be found that Spencer's formula is not obtained by an operation [2], but is obtained by an operation [7] on a simpler expression. The sums of the members in the various columns contain all remaining summatory factors, but they may have factors which do not belong to the summatory process.

§ 6. Tests Performed on Actual Data.

We shall now consider the relative merits of summation formulæ and least-square formulæ as tested by their performance when applied to definite numerical data.

We have first to decide what is to be accepted as the measure of good performance in a graduation. Some previous writers have taken the criterion of goodness to be the smallness of the third differences of the graduated values. It is evident however that this, taken by itself, is worthless as a criterion, since it takes no account of the magnitude of the differences between the graduated and ungraduated values—in other words, it takes no account of the sacrifices that have had to be made in departing from the original data. We can, in fact, quite easily devise a graduation formula which will make the third differences of the graduated values as small as we please, or which will even make all of them absolutely zero, provided we are willing to sanction sufficiently large differences between the graduated and ungraduated values. Such a formula would come out first in order of merit when judged by the criterion of small third differences, although its absurdity would be obvious. If we are to have a criterion in any way resembling this, it would seem best to adopt that which has been proposed by Professor Whittaker, and which may be stated thus: Let u_s' be the graduated value corresponding to the ungraduated value u_s: then the merit of a graduation is to be estimated by the smallness of $\Sigma(u_s'-u_s)^2+\lambda\Sigma(\Delta^3 u_s')^2$, where λ is a constant whose magnitude depends on the weight of the observations, *i.e.* λ measures the extent to which we are willing to modify the original data for the sake of obtaining smoothness.

In the present paper, however, I have used a test of a somewhat different type, which so far as I know has not been applied before, but which seems to have some advantages. It may be described thus:

Consider some known analytic function of x, such as $\log x$, of which tables accurate to (say) 6 places are available. A 4-place table of this function may be prepared by omitting the last two digits (which will be called the *tail*) and "*forcing*," *i.e.* increasing the last retained digit by unity when the omitted tail begins with one of the digits 5, 6, 7, 8, or 9. We can regard the values of $\log x$ given by the 4-place table as affected with "errors," namely, the errors which have been produced by omitting the tails. Let us now take a sequence of these 4-place values, and graduate them by the graduation formula which is to be tested; the effect of the graduation should be to smooth out the "errors" and restore, to some extent at least, the more accurate values of the 6-place table. The success with which this is performed may be taken as a measure of the merit of the graduation formula: for it must be remembered that the true purpose of a graduation formula is precisely to reduce the magnitude of accidental errors. The advantage of using a known function, such as $\log x$, for the test is that we can be certain that the errors (viz. the tails) are accidental, *i.e.* non-systematic. It may, I think, be objected to the work of some previous writers on the subject of graduation that they have not attended sufficiently to the all-important distinction between accidental and systematic errors, and have not adequately guarded themselves against achieving smoothness by what amounts to the introduction of new systematic errors into the graduated values.

α. In Table II this method of testing is applied to the function $u(x) = 0{\cdot}01x^3 + 0{\cdot}13$; the last two digits are omitted, and the forced numbers so obtained are taken as the ungraduated data. Spencer's formula and the least-square formula $k=1$, $m=10$ are applied, and the graduated values are calculated to the same number of digits as the true values. The merits of the graduated values are easily obtained by comparing columns 9, 10, and 11; the result is that the sum of the squares of the residual errors is 1082 when the least-square formula is used, and 1712 when Spencer's formula is used.

β. In Table III the method of testing is applied to the function $\frac{10^7}{x} - 39{,}999{\cdot}95$. Spencer's formula and the least-square formula $k=1$, $m=10$ are used. The merits of the graduated values are obtained by comparing columns 9, 10, and 11: the result is that the sum of the squares of the residual errors is 873 when the least-square formula is used, and 1327 when Spencer's formula is used.

γ. In Table IV twice the fifth powers of certain numbers are forced, and these are the ungraduated values. In this case formulæ correct to fifth differences must be used. As there is no existing summation formula correct to fifth differences which has the same outstanding merit as Spencer's formula has among formulæ correct to third differences, two fifth-difference formulæ, due to G. King (*J.I.A.*, 41, pp. 544 and 553), have been used along with the least-square formula $k=2$, $m=8$. These formulæ, here denoted by (a) and (b) respectively, are as follows:—

$$(a)\quad 2187u_0' = [3]^5\{81[1]+3[3]-21[3]^2+4[3]^3\}u_0$$
$$= 729u_0+590u_{\pm1}+250u_{\pm2}-85u_{\pm4}-41u_{\pm5}+11u_{\pm7}+4u_{\pm8}.$$

$$(b)\quad 2187u_0' = [3]^6\{84[1]-48[3]+7[3]^2\}u_0$$
$$= 729u_0+560u_{\pm1}+280u_{\pm2}-70u_{\pm4}-56u_{\pm5}+8u_{\pm7}+7u_{\pm8}.$$

The merits of the graduated values are obtained by comparing columns 11, 12, 13, and 14: the result is that the sum of the squares of the residual errors is 1669 when the least-square formula is used, 2327 when the summation formula (a) is used, and 2162 when the summation formula (b) is used.

The general conclusion we have reached may be expressed thus:—*the least-square formulæ of graduation remove accidental errors more successfully than the summation formulæ of graduation; but in many classes of investigations, the superiority of the least-square formulæ is perhaps not so marked as to compensate for the somewhat greater amount of arithmetical calculation involved*, especially when (as in actuarial applications) the graduation extends to a series of nearly 100 values.

§ 7. Sums of Powers.

Incidentally, in the course of the calculations it was necessary to calculate values of the sums of the powers of the natural numbers. As the results may be of use to other workers, they are given in Table V below.

In conclusion, I have to thank Professor Whittaker, at whose suggestion this investigation was begun, and Mr G. J. Lidstone, F.I.A., for valuable help during its progress.

Table II.—Graduation of the Cubes of Numbers by Spencer's Formula and the Least-Square Formula $k=1$, $m=10$.

No. x.	$0·01x^3+0·13=U$.	Forced Value of $U=u$.	Graduated Value of $u=u'$.		Difference between Ungraduated and Graduated Values, and True Values $\times 10^2$.			$(U-u)^2$. $(\text{Col. } 6)^2$.	$(U-u')^2$.	
			By Least-Square Formula $k=1$, $m=10$.	By Spencer's Formula.	Col. 2–Col. 3.	Col. 2–Col. 4.	Col. 2–Col. 5.		$(\text{Col. } 7)^2$.	$(\text{Col. } 8)^2$.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
60	2160·13	2160	...	...	...	...	...	...	...	...
61	2269·94	2270	...	...	...	...	...	...	...	...
62	2383·41	2383	...	...	...	...	...	...	...	...
63	2500·60	2501	...	...	...	...	...	...	...	...
64	2621·57	2622	...	...	...	...	...	...	...	...
65	2746·38	2746	...	...	...	...	...	...	...	...
66	2875·09	2875	...	...	...	...	...	...	...	...
67	3007·76	3008	...	...	...	...	...	...	...	...
68	3144·45	3144	...	...	...	...	...	...	...	...
69	3285·22	3285	...	...	...	...	...	...	...	...
70	3430·13	3430	3430·03	3429·99	13	10	14	169	100	196
71	3579·24	3579	3579·11	3579·10	24	13	14	576	169	196
72	3732·61	3733	3732·50	3732·48	−39	11	13	1521	121	169
73	3890·30	3890	3890·19	3890·19	30	11	11	900	121	121
74	4052·37	4052	4052·32	4052·29	37	5	8	1369	25	64
75	4218·88	4219	4218·91	4218·84	−12	−3	4	144	9	16
76	4389·89	4390	4389·90	4389·90	−11	−1	−1	121	1	1
77	4565·46	4565	4565·50	4565·52	46	−4	−6	2116	16	36
78	4745·65	4746	4745·73	4745·76	−35	−8	−11	1225	64	121
79	4930·52	4931	4930·58	4930·66	−48	−6	−14	2304	36	196
80	5120·13	5120	5120·21	5120·28	13	−8	−15	169	64	225
81	5314·54	5315	5314·61	5314·68	−36	−7	−14	1296	49	196
82	5513·81	5514	5513·87	5513·91	−19	−6	−10	361	36	100
83	5718·00	5718	5718·08	5718·06	0	−8	−6	0	64	36
84	5927·17	5927	5927·25	5927·19	17	−8	−2	289	64	4
85	6141·38	6141	6141·43	6141·37	38	−5	1	1444	25	1
86	6360·69	6361	6360·77	6360·67	−31	−8	2	961	64	4
87	6585·16	6585	6585·23	6585·15	16	−7	1	256	49	1
88	6814·85	6815	6814·86	6814·87	−15	−1	−2	225	1	4
89	7049·82	7050	7049·84	7049·87	−18	−2	−5	324	4	25
90	7290·13	7290	...	...	...	...	...	...	...	...
91	7535·84	7536	...	...	...	...	...	...	...	...
92	7787·01	7787	...	...	...	...	...	...	...	...
93	8043·70	8044	...	...	...	...	...	...	...	...
94	8305·97	8306	...	...	...	...	...	...	...	...
95	8573·88	8574	...	...	...	...	...	...	...	...
96	8847·49	8847	...	...	...	...	...	...	...	...
97	9126·86	9127	...	...	...	...	...	...	...	...
98	9412·05	9412	...	...	...	...	...	...	...	...
99	9703·12	9703	...	...	...	...	...	...	...	...
Total for columns 9, 10, and 11 . . .								15770	1082	1712

TABLE III.—GRADUATION OF THE RECIPROCALS OF NUMBERS BY SPENCER'S FORMULA AND THE LEAST-SQUARE FORMULA $k=1$, $m=10$.

No. x.	$10^7/x - 39{,}999{\cdot}95 = \mathrm{U}$.	Forced Value of $\mathrm{U} = u$.	Graduated Value of $u=u'$. By Least-Square Formula $k=1$, $m=10$.	Graduated Value of $u=u'$. By Spencer's Formula.	Difference between Ungraduated and Graduated and True Values $\times 10^2$. Col. 2–Col. 3.	Col. 2–Col. 4.	Col. 2–Col. 5.	$(\mathrm{U}-u)^2$. $(\text{Col. }6)^2$.	$(\mathrm{U}-u')^2$. $(\text{Col. }7)^2$.	$(\text{Col. }8)^2$.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
201	9751·29	9751	...	...	...	...	...	...	...	...
202	9505·00	9505	...	...	...	...	...	...	...	...
203	9261·13	9261	...	...	...	...	...	...	...	...
204	9019·66	9020	...	...	...	...	...	...	...	...
205	8780·54	8781	...	...	...	...	...	...	...	...
206	8543·74	8544	...	...	...	...	...	...	...	...
207	8309·23	8309	...	...	...	...	...	...	...	...
208	8076·97	8077	...	...	...	...	...	...	...	...
209	7846·94	7847	...	...	...	...	...	...	...	...
210	7619·10	7619	...	...	...	...	...	...	...	...
211	7393·41	7393	7393·36	7393·29	41	5	12	1681	25	144
212	7169·86	7170	7169·80	7169·73	−14	6	13	196	36	169
213	6948·41	6948	6948·34	6948·26	41	7	15	1681	49	225
214	6729·02	6729	6728·93	6728·94	2	9	8	4	81	64
215	6511·68	6512	6511·63	6511·64	−32	5	4	1024	25	16
216	6296·35	6296	6296·31	6296·36	35	4	−1	1225	16	1
217	6083·00	6083	6082·99	6083·05	0	1	−5	0	1	25
218	5871·61	5872	5871·58	5871·69	−39	3	−8	1521	9	64
219	5662·15	5662	5662·15	5662·23	15	0	−8	225	0	64
220	5454·60	5455	5454·64	5454·67	−40	−4	−7	1600	16	49
221	5248·92	5249	5248·97	5248·97	−8	−5	−5	64	25	25
222	5045·10	5045	5045·14	5245·11	10	−4	−1	100	16	1
223	4843·10	4843	4843·10	4843·09	10	0	1	100	0	1
224	4642·91	4643	4642·88	4642·88	−9	3	3	81	9	9
225	4444·49	4444	4444·46	4444·46	49	3	3	2401	9	9
226	4247·84	4248	4247·80	4247·79	−16	4	5	256	16	25
227	4052·91	4053	4052·84	4052·86	−9	7	5	81	49	25
228	3859·70	3860	3859·59	3859·64	−30	11	6	900	121	36
229	3668·17	3668	3668·08	3668·10	17	9	7	289	81	49
230	3478·31	3478	3478·19	3478·23	31	12	8	961	144	64
231	3290·09	3290	3289·99	3290·00	9	10	9	81	100	81
232	3103·50	3103	3103·44	3103·40	50	6	10	2500	36	100
233	2918·50	2919	2918·47	2918·41	−50	3	9	2500	9	81
234	2735·09	2735	...	...	...	...	...	...	...	...
235	2553·24	2553	...	...	...	...	...	...	...	...
236	2372·93	2373	...	...	...	...	...	...	...	...
237	2194·14	2194	...	...	...	...	...	...	...	...
238	2016·86	2017	...	...	...	...	...	...	...	...
239	1841·05	1841	...	...	...	...	...	...	...	...
240	1666·72	1667	...	...	...	...	...	...	...	...
241	1493·83	1494	...	...	...	...	...	...	...	...
242	1322·36	1322	...	...	...	...	...	...	...	...
243	1152·31	1152	...	...	...	...	...	...	...	...
Total for columns 9, 10, and 11 . . .								19471	873	1327

Table IV.—Graduation of the Fifth Powers of Numbers by Summation Formulæ (a) and (b) and the Least-Square Formula $k=2$, $m=8$.

No. $10^2\times x$.	$10^8\times 2x^5=$ U.	Forced Value of U $=u$.	Graduated Value of $u=u'$. By Least-Square Formula $k=2$, $m=8$.	By Summation Formula (a).	By Summation Formula (b).	Difference between Ungraduated, Graduated, and True Values $\times 10^2$. Col. 2–Col. 3.	Col. 2–Col. 4.	Col. 2–Col. 5.	Col. 2–Col. 6.	$(\mathrm{U}-u)^2 =$ (Col. 7)^2.	$(\mathrm{U}-u')^2$. (Col. 8)^2.	(Col. 9)^2.	(Col. 10)^2.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
51	6900505·02	6900505	...	...	...	...	...	...	...	...	...	...	...
52	7604080·64	7604081	...	...	...	...	...	...	...	...	...	...	...
53	8363909·86	8363910	...	...	...	...	...	...	...	...	...	...	...
54	9183300·48	9183300	...	...	...	...	...	...	...	...	...	...	...
55	10065687·50	10065688	...	...	...	...	...	...	...	...	...	...	...
56	11014635·52	11014636	...	...	...	...	...	...	...	...	...	...	...
57	12033841·14	12033841	...	...	...	...	...	...	...	...	...	...	...
58	13127135·36	13127135	...	...	...	...	...	...	...	...	...	...	...
59	14298485·98	14298486	14298485·93	14298485·87	14298485·88	−2	5	11	10	4	25	121	100
60	15552000·00	15552000	15551999·97	15551999·98	15551999·97	0	3	2	3	0	9	4	9
61	16891926·02	16891926	16891926·04	16891926·09	16891926·09	2	−2	−7	−7	4	4	49	49
62	18322656·64	18322657	18322656·67	18322656·74	18322656·73	−36	−3	−10	−9	1296	9	100	81
63	19848730·86	19848731	19848731·01	19848730·95	19848730·96	−14	−15	−9	−10	196	225	81	100
64	21474836·48	21474836	21474836·68	21474836·60	21474836·60	48	−20	−12	−12	2304	400	144	144
65	23205812·50	23205813	23205812·61	23205812·67	23205812·67	−50	−11	−17	−17	2500	121	289	289
66	25046651·52	25046652	25046651·52	25046651·67	25046651·65	−48	0	−15	−13	2304	0	225	169
67	27002502·14	27002502	27002502·19	27002502·17	27002502·17	14	−5	−3	−3	196	25	9	9
68	29078671·36	29078671	29078671·38	29078671·27	29078671·27	36	−2	9	9	1296	4	81	81
69	31280626·98	31280627	31280626·93	31280626·87	31280626·88	−2	5	11	10	4	25	121	100
70	33614000·00	33614000	33613999·97	33613999·98	33613999·97	0	3	2	3	0	9	4	9
71	36084587·02	36084587	36084587·04	36084587·09	36084587·09	2	−2	−7	−7	4	4	49	49
72	38698352·64	38698353	38698352·67	38698352·74	38698352·73	−36	−3	−10	−9	1296	9	100	81
73	41461431·86	41461432	41461432·01	41461431·95	41461431·96	−14	−15	−9	−10	196	225	81	100
74	44380132·48	44380132	44380132·68	44380132·60	44380132·60	48	−20	−12	−12	2304	400	144	144
75	47460937·50	47460938	47460937·61	47460937·67	47460937·67	−50	−11	−17	−17	2500	121	289	289
76	50710507·52	50710508	50710507·52	50710507·67	50710507·65	−48	0	−15	−13	2304	0	225	169
77	54135683·14	54135683	54135683·19	54135683·17	54135683·17	14	−5	−3	−3	196	25	9	9
78	57743487·36	57743487	57743487·38	57743487·27	57743487·27	36	−2	9	9	1296	4	81	81
79	61541127·98	61541128	61541127·93	61541127·87	61541127·88	−2	5	11	10	4	25	121	100
80	65536000·00	65536000	...	...	...	...	...	...	...	...	...	...	...
81	69735688·02	69735688	...	...	...	...	...	...	...	...	...	...	...
82	74147968·64	74147969	...	...	...	...	...	...	...	...	...	...	...
83	78780812·86	78780813	...	...	...	...	...	...	...	...	...	...	...
84	83642388·48	83642388	...	...	...	...	...	...	...	...	...	...	...
85	88741062·50	88741063	...	...	...	...	...	...	...	...	...	...	...
86	94085403·52	94085404	...	...	...	...	...	...	...	...	...	...	...
87	99684184·14	99684184	...	...	...	...	...	...	...	...	...	...	...
					Total for columns 11, 12, 13 and 14 . .					20204	1669	2327	2162

Table V.—Sums of Powers of the Natural Numbers.

$\Sigma_r = (-m)^r + (-m+1)^r + \ldots + (m-1)^r + m^r$.

m.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	m.
Σ_2	2	10	28	60	110	182	280	408	570	770	Σ_2
Σ_4	2	34	196	708	1958	4550	9352	17544	30666	50666	Σ_4
Σ_6	2	130	1588	9780	41030	134342	369640	893928	1956810	3956810	Σ_6
Σ_8	2	514	13636	144708	925958	4285190	15814792	49369224	135462666	335462666	Σ_8
Σ_{10}	2	2050	120148	2217300	21748550	142680902	707631400	2855115048	9828683850	29828683850	Σ_{10}
Σ_{12}	2	8194	1071076	34625508	522906758	4876471430	32559045832	169997999304	734857072266	2734857072266	Σ_{12}
Σ_{14}	2	32770	9598708	546469620	12753500870	169481829062	1525927974760	10322020996968	56075605906890	256075605906890	Σ_{14}
Σ_{16}	2	131074	86224516	8676159108	313851940358	5956071755270	72421932894472	635371886315784	4341412264019466	24341412264019466	Σ_{16}

(*Issued separately August* 4, 1920.)

XIV.—An Identical Relation connecting Seven Vectors.

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(MS. received April 23, 1920. Read June 7, 1920.)

1. Utility of Vector Identities.

Among the most useful results of quaternion or vectorial algebra we may count the identities and transformations worked out by Hamilton and Tait and their all too few followers. It is not merely that a vector identity is equivalent to three scalar identities—a fact which aids us greatly to condense our calculations in respect to bulk. Yet more important is the greater fruitfulness of a vectorial relation in giving rise to derived relations which we would be less likely to perceive from a purely scalar analysis.

In this paper I propose, first, to develop an identical equation satisfied by any seven vectors and also satisfied by an arbitrary quadratic function; second, to exemplify the methods by which new identities can be derived from this (or from any) vector identity.

2. Fundamental Identities.

Hamilton and Tait make constant use of certain fundamental relations connecting three and four vectors. There are also, among the examples in Tait's *Quaternions* and in Kelland and Tait's *Introduction to Quaternions,* a considerable number of identities connecting five or more vectors. Most frequently used are the two following:

$$\rho S\alpha\beta\gamma = \alpha S\beta\gamma\rho + \beta S\gamma\alpha\rho + \gamma S\alpha\beta\rho \quad . \quad . \quad . \quad . \quad (1)$$

which expresses any fourth vector ρ in terms of its components along three other vectors α, β, γ; and

$$V\alpha V\beta\gamma = \gamma S\alpha\beta - \beta S\alpha\gamma \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is of constant utility.

3. The Quadric Cone through Five Vectors.

The present paper deals with certain expressions which are quadratic forms in the sense that they are homogeneous of degree two in various sets of quantities. We begin with Hamilton's expression

$$S\,.\,V(V\alpha\beta V\delta\epsilon)V(V\beta\gamma V\epsilon\rho)V(V\gamma\delta V\rho\alpha) \quad . \quad . \quad . \quad (3)$$

which vanishes when the six vectors lie on a quadric cone (see Tait, *Quaternions*, Art. 261 of the 3rd edition). In other words, if ρ be regarded as variable, the vanishing of (3) defines a quadric cone through the other five vectors. The order of vectors in (3) may be easily remembered if we first establish the cyclic order

$$\alpha\beta\gamma\delta\epsilon\rho\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and then note that, in (3), each of the three expressions in parentheses is formed as if the second V stood for an omitted vector: we do not have $V\alpha\beta\gamma\delta$ but $V\alpha\beta V\delta\epsilon$, etc.

We note also that the three groups of four vectors in parentheses begin respectively with α, β, γ, the first three vectors as laid down in (4). If we had begun the groups with β, γ, and δ, leaving the cyclic order unchanged, thus:

$$S.V(V\beta\gamma V\epsilon\rho)V(V\gamma\delta V\rho\alpha)V(V\delta\epsilon V\alpha\beta) \quad . \quad . \quad . \quad (5)$$

the sign of the result would have been changed, for

$$V(V\delta\epsilon V\alpha\beta) = -V(V\alpha\beta V\delta\epsilon) \quad . \quad . \quad . \quad . \quad (6)$$

but the expression is otherwise unaltered.

To get further light on the changes of sign produced by change in the order of vectors, we may expand (3) by repeated use of (2) and obtain easily

$$S\gamma\alpha\beta S\delta\epsilon\beta S\gamma\delta\rho S\alpha\epsilon\rho - S\gamma\delta\beta S\alpha\epsilon\beta S\gamma\alpha\rho S\delta\epsilon\rho \quad . \quad . \quad . \quad (7)$$

identically equal to (3). Now by inspection (7) changes sign when γ and ϵ are interchanged, or if β and ρ are interchanged, or if α and δ are interchanged. Similarly, we may show that (3) changes sign when any two vectors are interchanged.

It follows that all expressions similar to (3) which can be made from the six vectors are equal except in sign. To determine in any given case whether the sign is the same as that of (3), we have first to count the number of interchanges needed to bring the cyclic order into agreement with (4), and second to note whether, after these interchanges have been made, the first vector in the first group is of odd or of even number, when α is counted as number one: if even, an additional change of sign must be counted.

4. Expressions formed by selecting Six out of Seven Vectors.

Let us now adjoin a seventh vector σ and lay down the cyclic order

$$\alpha\beta\gamma\delta\epsilon\rho\sigma\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

It is evident that by dropping any one of the seven vectors we may form expressions like (3) out of the remaining six. If it be agreed that the manner of writing shall be as in (3), it will be sufficient to specify the omitted vector. Thus (3) may be abbreviated as $C(\sigma)$ or simply as $C\sigma$. Similarly, if δ be omitted, we may write

$$C\delta = S \,.\, V(V\epsilon\rho V\alpha\beta)V(V\rho\sigma V\beta\gamma)V(V\sigma\alpha V\gamma\epsilon) \quad . \quad . \quad . \quad (9)$$

with a similar convention for the other expressions of like form. When necessary to avoid ambiguity, the vectors which occur may be designated in their order, thus:

$$C\delta = C(\epsilon\rho\sigma\alpha\beta\gamma) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

or as a further illustration:

$$C(\beta\rho\lambda\phi\tau\mu) = S \,.\, V(V\beta\rho V\phi\tau)V(V\rho\lambda V\tau\mu)V(V\lambda\phi V\mu\beta) \quad . \quad . \quad (11)$$

where β, ρ, λ, ϕ, τ, and μ are any six vectors whatever. This may serve as a definition of C.

5. Identical Relation connecting Seven Vectors.

Let now $F(\alpha)$ denote any function homogeneous of the second degree in α in the sense that the scalar components of $F(\alpha)$ are quadratic polynomials in the components of α. $F(\alpha)$ may be a vector or a quaternion, or, of course, a scalar. Let $C\alpha$, $C\beta$, $C\gamma$, etc., be the seven expressions formed as in (3) or in (9) by omitting the designated vector from the cycle (8). The following equation must hold for all values of the seven vectors and for any quadratic function F:

$$F(\alpha)C\alpha + F(\beta)C\beta + \ . \ . \ . \ + F(\sigma)C\sigma = 0 \quad . \quad . \quad . \quad (12)$$

The proof of this relation depends on the following:

Lemma.—If two of the seven vectors coincide, the corresponding C's become equal in absolute value but opposite in sign.

For example, if β be allowed to coincide with a given α, the expression $C\beta$, which does not contain β, is unaltered, but $C\alpha$, which was originally

$$S \,.\, V(V\beta\gamma V\epsilon\rho)V(V\gamma\delta V\rho\sigma)V(V\delta\epsilon V\sigma\beta),$$

now becomes

$$S \,.\, V(V\alpha\gamma V\epsilon\rho)V(V\gamma\delta V\rho\sigma)V(V\delta\epsilon V\sigma\alpha).$$

This is the negative of $C\beta$, for the first vector of the first group is α, while in forming $C\beta$ the first vector would be γ, which, counted as number one, makes α the sixth in order. Similarly, we may prove the lemma for any pair of C's.

It follows at once that (12) holds true when β is made to coincide with α. For $C\gamma$, $C\delta$, $C\epsilon$, $C\rho$, and $C\sigma$ all vanish, and the first two terms become equal in absolute value and opposite in sign.

Similarly, (12) holds true when γ, δ, ϵ, ρ, or σ coincides with α. But

(12) is a quadratic expression in a. It cannot vanish for more than five arbitrary values of a unless it vanishes identically. For any scalar component of (12) may be regarded as defining by its vanishing a quadric cone through six arbitrary vectors β, γ, δ, ϵ, ρ, σ, with variable a. But this is impossible, that is, each component of (12) vanishes identically.

6. Methods for deriving New Identities.

An almost unlimited number of new identities may be obtained from (12). Among the more obvious methods we may consider the following three:

1. Giving particular values to the seven vectors, and special forms to the function F.
2. Acting on (12) by various differential operators.
3. Replacing one or more vectors by the operator ∇.

These methods I shall now briefly exemplify.

7. Identities obtained from (12) without Differentiation.

As a first example let $F(a) = a^2$. Then

$$a^2 Ca + \beta^2 C\beta + \gamma^2 C\gamma + \ldots \sigma^2 C\sigma = 0 \qquad (13)$$

or briefly $\Sigma a^2 Ca = 0$. This is, of course, a scalar identity.

Again, let $F(a) = aSa\tau$, where τ is any vector whatever. This gives the vector identity, linear in τ, but quadratic in the other seven vectors,

$$aSa\tau Ca + \beta S\beta\tau C\beta + \ldots + \sigma S\sigma\tau C\sigma = 0 \qquad (14)$$

or $\Sigma aSa\tau Ca = 0$ where the summation is with respect to the seven vectors of (8).

Again, the symbol S in (14) may be replaced by V, giving

$$CaVaVa\tau + C\beta V\beta V\beta\tau + \ldots + C\sigma V\sigma V\sigma\tau = 0 \qquad (15)$$

To the vector τ occurring in (14) and (15) special values may be given. Thus if $\tau = V\rho\sigma$ we have from (14)

$$aSa\rho\sigma Ca + \beta S\beta\rho\sigma C\beta + \gamma S\gamma\rho\sigma C\gamma + \delta S\delta\rho\sigma C\delta + \epsilon S\epsilon\rho\sigma C\epsilon = 0 \qquad (16)$$

for the last two terms of (12) now vanish.

Multiplying (16) into $\delta\epsilon$ and taking scalars gives

$$Sa\delta\epsilon Sa\rho\sigma Ca + S\beta\delta\epsilon S\beta\rho\sigma C\beta + S\gamma\delta\epsilon S\gamma\rho\sigma C\gamma = 0 \qquad (17)$$

a scalar relation connecting the three expressions Ca, $C\beta$, and $C\gamma$. In a similar manner we may obtain these two other relations which involve the same C's:

$$Sa\delta\rho Sa\epsilon\sigma Ca + S\beta\delta\rho S\beta\epsilon\sigma C\beta + S\gamma\delta\rho S\gamma\epsilon\sigma C\gamma = 0 \qquad (18)$$

$$Sa\delta\sigma Sa\epsilon\rho Ca + S\beta\delta\sigma S\beta\epsilon\rho C\beta + S\gamma\delta\sigma S\gamma\epsilon\rho C\gamma = 0 \qquad (19)$$

The last three results, of course, involve the identical vanishing of the determinant

$$\begin{vmatrix} S\alpha\delta\epsilon S\alpha\rho\sigma, & S\beta\delta\epsilon S\beta\rho\sigma, & S\gamma\delta\epsilon S\gamma\rho\sigma \\ S\alpha\delta\rho S\alpha\epsilon\sigma, & S\beta\delta\rho S\beta\epsilon\sigma, & S\gamma\delta\rho S\gamma\epsilon\sigma \\ S\alpha\delta\sigma S\alpha\epsilon\rho, & S\beta\delta\sigma S\beta\epsilon\rho, & S\gamma\delta\sigma S\gamma\epsilon\rho \end{vmatrix} \quad . \quad . \quad . \quad (20)$$

which may be directly verified by noting that the determinant is quadratic in α and vanishes when α coincides with any one of the other six vectors. It is easy to show by expansions like (7) that any of the two-row minors from the elements of the second and third columns is equal in absolute value to $C\alpha$, similarly for $C\beta$ and $C\gamma$; in other words, the identities (17), (18), (19) are equivalent to expansion of the determinant by the elements of the respective rows; this is also evident from equations (17)–(19).

Again, in (14) let $\tau = \sigma$, and we have

$$\alpha S\alpha\sigma C\alpha + \beta S\beta\sigma C\beta + \gamma S\gamma\sigma C\gamma + \delta S\delta\sigma C\delta + \epsilon S\epsilon\sigma C\epsilon + \rho S\rho\sigma C\rho = -\sigma^3 C\sigma \quad . \quad (21)$$

It is evident that by assigning various forms to the function F we may obtain such relations in large number.

8. Application to the General Theory of Quadratic Vectors.

Suppose that, in (12), F stands for any quadratic vector function, and let the seven vectors $\alpha, \beta, \ldots \sigma$ be the seven axes of the function, so that $F(\alpha) = a_1\alpha$, $F(\beta) = a_2\beta$, etc.; we have

$$a_1\alpha C\alpha + a_2\beta C\beta + \ldots + a_7\sigma C\sigma = 0 \quad . \quad . \quad . \quad . \quad (22)$$

which is an illustration of the fact that the scalars a_1, a_2, etc., cannot be assigned arbitrarily when the axes have been chosen—a noteworthy difference between quadratic and linear vector functions.

Again, let σ be the variable point vector, and the other six vectors be any assigned constant vectors which do not lie on a quadric cone, so that $C\sigma$ is not zero. (12) then shows that any quadratic vector $F(\sigma)$ is fully determined by its effect on any six vectors not on a quadric cone, a fact that might perhaps be anticipated, since the quadratic vector depends on eighteen scalar constants. (I have elsewhere shown that the quadratic vector is determined by the directions of its seven axes, aside from a term of the form $\rho S\eta\rho$ and a multiplicative scalar constant. See *Proc. Amer. Acad. Arts and Sci.*, vol. lii, No. 7 (Jan. 1917), p. 377.)

As another illustration, take $\beta = F(\alpha)$ and $\gamma = F[F(\alpha)]$, which we may abbreviate as $\gamma = F^2(\alpha)$. Let $\delta = F^3(\alpha)$, etc., up to $\sigma = F^6(\alpha)$. We then have by (12) an identical relation which must be satisfied by the seven vectors $F(\alpha), F^2(\alpha), \ldots F^7(\alpha)$. We have evidently an analogy here with Hamilton's well-known cubic satisfied by a linear vector function. Of course, to make the analogy complete it must be shown that the coefficients $C\alpha \ldots C\sigma$ become invariants (or covariants) when the vectors $\beta, \gamma, \ldots \sigma$ are chosen as above. The investigation of this interesting question must be reserved for a separate paper.

9. Identities obtained from (12) by Differential Operators.

An almost unlimited number of identities may be obtained from (12) by first multiplying through by any function of ∇ and then allowing ∇ to act on one of the vectors, as σ, or on the constituents of F.

For example, multiply into ∇, take scalars, and allow ∇ to act on σ. We have, if F does not contain σ implicitly,

$$SF(\alpha)\nabla . C\alpha + SF(\beta)\nabla . C\beta + \ldots + SF(\rho)\nabla . C\rho + C\sigma S\nabla F(\sigma) = 0 \quad . \quad (23)$$

As a special case, let $F(\alpha) = \alpha S\alpha\tau$ as in (14), when (23) gives

$$S\alpha\tau S\alpha\nabla . C\alpha + S\beta\tau S\beta\nabla . C\beta + \ldots + S\rho\tau S\rho\nabla . C\rho - 4C\sigma S\sigma\tau = 0 \quad . \quad (24)$$

where ∇ is understood to act on σ alone.

The expressions $S\alpha\nabla . C\alpha$, $S\beta\nabla . C\beta$, etc., which occur in this result deserve a moment's special consideration. Since $C\alpha$ is a homogeneous quadric function of σ whose vanishing, when σ is the variable point vector, defines a quadric cone, it is clear that $S\alpha\nabla . C\alpha$ defines, by its vanishing, the polar plane of α with respect to this cone. The importance of such polar relations is too well known to require emphasis here. It is evident that, by acting on the various C's with operators of the form $S\lambda\nabla$ and letting ∇ act on any one of the vectors present, we may obtain a very great number of such polar planes. A systematic notation is desirable. It is apparently sufficient to designate, in order, the omitted vector, the vector whose polar is being taken, and the operand. Thus we may write $S\alpha\nabla . C\alpha = -L[\alpha, \alpha\sigma]$ when ∇ acts on σ. As a further illustration, if we start with the expression (3), act on it with $S\lambda\nabla$, and regard ρ as the variable point vector, we may write

$$S\lambda\nabla . C\sigma = -L[\sigma, \lambda\rho] \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where we understand:

1. The first vector in brackets denotes the omitted vector.

2. The second vector occurs in the operator, *i.e.* we are finding the polar of this vector.

3. The last vector in brackets is the operand or variable. The letter L may remind us the expression is linear in the vectors λ and ρ. It is symmetrical in these vectors.

This notation is doubtless far from ideal, but is compact, and sufficiently specific for our present purpose. When necessary to prevent ambiguity, we may designate the vectors which define the quadric cone, in their cyclic order, instead of designating the omitted vector. Thus we might have written

$$S\lambda\nabla' . C(\alpha\beta\gamma\delta\epsilon\rho') = -L(\alpha\beta\gamma\delta\epsilon[\lambda\rho]) \quad . \quad . \quad . \quad . \quad (25)$$

where the vectors λ and ρ may be interchanged without altering the value of either side.

If we wish to expand the expression in explicit form we may return to (7) and obtain by actual differentiation

$$\begin{aligned} L[\sigma, \lambda\rho] = L(\alpha\beta\gamma\delta\epsilon[\lambda\rho]) &= S\gamma\alpha\beta S\delta\epsilon\beta(S\gamma\delta\rho S\alpha\epsilon\lambda + S\gamma\delta\lambda S\alpha\epsilon\rho) \\ &\quad - S\gamma\delta\beta S\alpha\epsilon\beta(S\gamma\alpha\rho S\delta\epsilon\lambda + S\gamma\alpha\lambda S\delta\epsilon\rho) \quad . \quad . \quad (26) \end{aligned}$$

which may serve as definition of the function L of seven vectors.

With these conventions (24) would be written

$$S\alpha\tau L[\alpha, \alpha\sigma] + S\beta\tau L[\beta, \beta\sigma] + \ . \ . \ . + S\rho\tau L[\rho, \rho\sigma] + 4C\sigma S\sigma\tau = 0 \quad . \quad (27)$$

We may now obtain identities connecting the original six vectors of (4) by allowing σ to coincide (after the differentiation) with one of these, and also giving special values to τ. If, for example, we let σ coincide with α we shall have $C\sigma = -C\alpha$ by the former lemma, while $L[\alpha, \alpha\sigma] = 2C\alpha = -2C\sigma$. Furthermore, $L[\beta, \beta\sigma]$ becomes $-L[\sigma, \alpha\beta]$, and similarly for the other L's. Hence (27), when σ coincides with α, gives

$$2C\sigma S\alpha\tau = S\beta\tau L[\sigma, \alpha\beta] + S\gamma\tau L[\sigma, \alpha\gamma] + \ . \ . \ . + S\rho\tau L[\sigma, \alpha\rho] \quad . \quad . \quad (28)$$

from which σ is absent. By giving to τ the value $V\epsilon\rho$ this gives

$$2C\sigma S\alpha\epsilon\rho = S\beta\epsilon\rho L[\sigma, \alpha\beta] + S\gamma\epsilon\rho L[\sigma, \alpha\gamma] + S\delta\epsilon\rho L[\sigma, \alpha\delta] \quad . \quad . \quad (29)$$

a useful identity connecting three L's with one another. We note that these three L's involve only the six vectors of the cycle (4). By letting τ have the value $V\rho\alpha$ we should have a relation between four of the L's not involving the quantity $C\sigma$.

These last results were obtained by acting on (12) with $S\nabla$. Slightly more general results are obtained by acting with $S\lambda\nabla$, where λ is any vector whatever. After the differentiation special values may, of course, be given to λ, to the function F, and to the vector acted on by ∇. The number of identities thus obtainable is practically unlimited.

A result which I have found of frequent utility in the general theory of quadratic vectors is obtained as follows:—Multiply (14) by $\alpha\nabla$ and take scalars, giving

$$S\beta\tau S\beta\alpha\nabla . C\beta + S\gamma\tau S\gamma\alpha\nabla . C\gamma + \ . \ . \ . + S\sigma\tau S\sigma\alpha\nabla . C\sigma = 0 \quad . \quad (30)$$

Now allow ∇ to operate on σ, and we have

$$S\beta\tau L[\beta, V\beta\alpha, \sigma] + S\gamma\tau L[\gamma, V\gamma\alpha, \sigma] + \ . \ . \ . + S\rho\tau L[\rho, V\rho\alpha, \sigma] + C\sigma S\sigma\alpha\tau = 0 \quad (31)$$

Next let σ coincide with a, giving

$$S\beta\tau L(\gamma\delta\epsilon\rho[V\beta a, a]a) + S\gamma\tau L(\delta\epsilon\rho[V\gamma a, a]a\beta) + \ldots + S\rho\tau L([V\rho a, a]a\beta\gamma\delta\epsilon) = 0 \quad (32)$$

for the last term of (31) vanishes when σ coincides with a. Finally, let τ have the value $V\epsilon\rho$, and we have the following relation which connects three L's:

$$S\beta\epsilon\rho L(\gamma\delta\epsilon\rho[V\beta a, a]a) + S\gamma\epsilon\rho L(\delta\epsilon\rho[V\gamma a, a]a\beta) + S\delta\epsilon\rho L(\epsilon\rho[V\delta a, a]a\beta\gamma) = 0 \quad (33)$$

To interpret the meaning of these three L's, we note that they are derivable from the expression $C\sigma$ by regarding β, γ, and δ, respectively, as variable. Thus, for example,

$$\begin{aligned} L(\gamma\delta\epsilon\rho[V\beta a, a]a) &= -L(a[V\beta a, a]\gamma\delta\epsilon\rho) \\ &= -S\beta a\nabla \, . \, Sa\nabla \, . \, C\sigma \quad . \quad . \quad . \quad (34) \end{aligned}$$

where both ∇'s act on β only. The vanishing of this expression would imply that the polar plane of the vector $V\beta a$ with respect to the cone $C\sigma = 0$ (where β is the variable point vector) passes through the vector a.

It is probable, however, that the most interesting results come from the use of the operator ∇^2. The expressions obtained by acting with ∇^2 on one of the C's have remarkable properties. The easiest way to operate with ∇^2 is, usually, to act first with $S\lambda\nabla$, then write $\lambda = \nabla$ and operate again. Thus, if ∇ acts on ρ we have

$$S\lambda\nabla(S\mu\rho S\nu\rho) = -S\mu\lambda S\nu\rho - S\mu\rho S\nu\lambda \quad . \quad . \quad . \quad . \quad (35)$$

whence writing ∇ for λ we have

$$\nabla^2(S\mu\rho S\nu\rho) = 2S\mu\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

With this result in mind, we see by inspection of (7) that

$$\nabla^2 C\sigma = 2S\gamma a\beta S\delta\epsilon\beta SV\gamma\delta Va\epsilon - 2S\gamma\delta\beta Sa\epsilon\beta SV\gamma aV\delta\epsilon \quad . \quad . \quad (37)$$

Since the original expression $C\sigma$ was proved to be unaltered in absolute value by interchanging any pair of vectors, the same must hold for the right of (37), and the rules for changes of sign must also hold. It follows that the expression can be expanded in a great variety of ways, according to the original order of writing in expressions like (3). The reader may be interested to examine what relation, geometrically, holds among the five vectors which are present, when the right of (37) vanishes.

If we operate again with ∇^2, letting ∇ act this time on β, the result is zero; that is, the right of (37) is a harmonic function of β. Whence it is also a harmonic function of a, γ, δ, and ϵ. This suggests the notation (when ∇ acts on ρ)

$$\nabla^2 C\sigma = H[\sigma\rho] \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

where the first vector in brackets denotes the omitted vector, and the second denotes the one which was acted on by ∇. Similarly, if ∇ acts on σ,

$$\nabla^2 C\rho = H[\rho\sigma] \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and it is evident in virtue of the lemma above that

$$H[\sigma\rho] = -H[\rho\sigma] \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Where needed to avoid ambiguity, we may designate the vectors which occur in the result, thus:

$$H(\alpha\beta\gamma\delta\epsilon) = 2S\alpha\beta\epsilon S\gamma\delta\epsilon S \,.\, V\alpha\gamma V\beta\delta - 2S\alpha\gamma\epsilon S\beta\delta\epsilon S \,.\, V\alpha\beta V\gamma\delta \quad . \quad . \quad (41)$$

which may serve as definition of the function H of five vectors. The factor 2 is retained for convenience.

Returning now to our general identity (12) and operating with ∇^2, σ being the operand, we obtain

$$F(\alpha)H[\alpha\sigma] + F(\beta)H[\beta\sigma] + \ . \ . \ . \ + F(\rho)H[\rho\sigma] + C\sigma\nabla^2 F(\sigma) = 0 \quad . \quad (42)$$

which expresses $\nabla^2 F(\sigma)$ in terms of $F(\alpha)$, $F(\beta)$, etc. It is clear that this is an identity of the same sort as the familiar

$$\nabla\phi\rho = \lambda\mu\phi\nu + \mu\nu\phi\lambda + \nu\lambda\phi\mu,$$

where ϕ is a linear vector function and ∇ acts on ρ: the effect of ∇ on the function is expressed in terms of functional operations.

By giving to F various values we obtain numerous relations connecting the H's. Thus, if $F(\alpha) = \alpha^2$ we have

$$\alpha^2 H[\alpha\sigma] + \beta^2 H[\beta\sigma] + \ . \ . \ . \ + \rho^2 H[\rho\sigma] + 6C\omega = 0 \quad . \quad . \quad (43)$$

which expresses one of the C's in terms of six H's. We note that this identity involves only six vectors.

10. Identities obtained by replacing Vectors by ∇.

We come now to the third of the three methods mentioned in Art. 6, namely, replacing one or more vectors by the operator ∇. Since ∇ is a vector, we may, in any of the foregoing identities, let any one of the vectors present become ∇ and we shall have an *operator* which vanishes identically, and which, when applied to any expression whatever, yields an identically vanishing result.

To illustrate, let $F(\alpha) = \alpha S\alpha\tau$ and use (42). We have first

$$\alpha S\alpha\tau H[\alpha\sigma] + \beta S\beta\tau H[\beta\sigma] + \ . \ . \ . \ + \rho S\rho\tau H[\rho\sigma] + C\sigma\nabla^2(\sigma S\sigma\tau) = 0 \quad . \quad (44)$$

where ∇ in the last term acts on σ. Now

$$S\lambda\nabla(\sigma S\sigma\tau) = -\lambda S\sigma\tau - \sigma S\lambda\tau \quad . \quad . \quad . \quad . \quad (45)$$

so that writing ∇ for λ we have

$$\begin{aligned}\nabla^2(\sigma S\sigma\tau) &= -\nabla S\sigma\tau - S\tau\nabla\,.\,\sigma\\ &= 2\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (46)\end{aligned}$$

whence (44) becomes

$$\alpha S\alpha\tau H[\alpha\sigma] + \beta S\beta\tau H[\beta\sigma] + \ .\ .\ .\ + \rho S\rho\tau H[\rho\sigma] + 2\tau C\sigma = 0 \qquad . \quad (47)$$

which so far is merely an identity connecting the six vectors of the cycle (4) with the vector τ which occurs in a linear manner. In other words, this identity is of the same sort as (14), but simpler in that σ no longer enters, and the H's involve only five vectors. If we now write ∇ for τ we have a relation expressing ∇ in terms of the six operations $S\alpha\nabla$, $S\beta\nabla$, . . . $S\rho\nabla$, where the six vectors α, β, . . . ρ are any vectors whatever, namely,

$$H[\alpha\sigma]\alpha S\alpha\nabla\,.\, + H[\beta\sigma]\beta S\beta\nabla\,.\, + \ .\ .\ .\ + H[\rho\sigma]\rho S\rho\nabla\,.\, = -2C\sigma\,.\,\nabla \quad . \quad (48)$$

If we multiply into ∇ we have ∇^2 expressed in terms of the squares of the same operators, thus:

$$H[\alpha\sigma]S^2\alpha\nabla\,.\, + H[\beta\sigma]S^2\beta\nabla\,.\, + \ .\ .\ .\ + H[\rho\sigma]S^2\rho\nabla\,.\, = -2C\sigma\,.\,\nabla^2 \quad . \quad (49)$$

As a final illustration I shall derive an identity which I have found useful in working with quadratic vectors. Beginning with (47), multiply through with $\rho\tau$ and take scalars. The last two terms disappear, giving the following five-term identity:

$$S\alpha\rho\tau S\alpha\tau H[\alpha\sigma] + S\beta\rho\tau S\beta\tau H[\beta\sigma] + \ .\ .\ .\ + S\epsilon\rho\tau S\epsilon\tau H[\epsilon\sigma] = 0 \qquad . \quad (50)$$

Next write ∇ for τ, and we have the identically vanishing operator

$$S\alpha\rho\nabla\,.\,S\alpha\nabla\,.\,H[\alpha\sigma] + S\beta\rho\nabla\,.\,S\beta\nabla\,.\,H[\beta\sigma] + \ .\ .\ .\ + S\epsilon\rho\nabla\,.\,S\epsilon\nabla\,.\,H[\epsilon\sigma] = 0 \quad (51)$$

which involves only the six vectors of the cycle (4). Let the left side of (51) operate on our original expression (3), ρ being taken as the variable, that is, ρ is acted on by ∇ wherever it occurs in (3) but not where it occurs in the operator itself. Note that by our definition of the function L we have

$$S\alpha\rho\nabla\,.\,S\alpha\nabla\,.\,C\sigma = L(\alpha\beta\gamma\delta\epsilon[V\alpha\rho\,,\,\alpha]) \quad . \quad . \quad . \quad . \quad (52)$$

with similar expressions for the other terms of the result. We then have when (51) is multiplied into (3)

$$\begin{aligned}H[\alpha\sigma]\,.\,L(\alpha\beta\gamma\delta\epsilon[V\alpha\rho,\,\alpha]) + H[\beta\sigma]\,.\,L(\alpha\beta\gamma\delta\epsilon[V\beta\rho,\,\beta]) + \ .\ .\ .\\ + [H\epsilon\sigma]\,.\,L(\alpha\beta\gamma\delta\epsilon[V\epsilon\rho,\,\epsilon]) = 0\,. \quad . \quad . \quad (53)\end{aligned}$$

This is a scalar identity involving the six vectors of (4), for the H's are specified, for brevity, by writing the vectors which do not occur. (We obtained $H[\alpha\sigma]$ from $C\alpha$ by letting ∇ act twice on σ.) The five L's which occur in this identity are of the same character as those in (33), but differ in that they are all obtained from $C\sigma$ by using ρ as operand. They are all linear in ρ, while each is of the fourth degree in one vector and quadratic in the rest. The identity as a whole is thus of the fourth degree in α, β, γ, δ, ϵ, and of the third degree in ρ. Expressions like the L's of (53) present themselves at once when we attempt to solve the equation $V\nabla F\rho = 0$, where $F\rho$ is a quadratic vector function of ρ to be specified by its axes. The details of this question must be reserved for a future work.

11. Extension of the Method.

In the foregoing paper our results have been obtained from the identity (12) connecting an arbitrary quadratic function, of seven different vectors successively, with scalars which vanish when six of these vectors lie on a quadric cone.

If we can form scalar functions of ten vectors which vanish when these lie on a cubic cone, we may evidently extend our results to an arbitrary function of the third degree, and so on.

In particular, if for the vectors in question we choose $F(\alpha)$, $F^2(\alpha)$, $F^3(\alpha)$, etc., we shall have an identity satisfied by the functional operation of whatever degree. How far the coefficients are invariant under varying α appears a question of interest. The form of the coefficients may be determined by the methods of Art. 7. Taking the cubic case as an illustration, assume that any function F homogeneous of degree three in one vector satisfies an identity of the form

$$F(\alpha_1)C_1 + F(\alpha_2)C_2 + \ldots + F(\alpha_{11})C_{11} = 0 \quad . \quad . \quad . \quad (54)$$

where the α's are any eleven vectors whatever and the C's are to be determined. Let $F(\alpha_1) = \alpha_1 S\alpha_1\tau S\alpha_1\pi$, where τ and π are any two vectors. The identity becomes

$$\alpha_1 S\alpha_1\tau S\alpha_1\pi C_1 + \alpha_2 S\alpha_2\tau S\alpha_2\pi C_2 + \ldots + \alpha_{11} S\alpha_{11}\tau S\alpha_{11}\pi = 0 \quad . \quad . \quad (55)$$

Let $\pi = V\alpha_{10}\alpha_{11}$ and $\tau = V\alpha_8\alpha_9$, also multiply into $V\alpha_6\alpha_7$ and take scalars, giving the five-term identity

$$S\alpha_1\alpha_6\alpha_7 S\alpha_1\alpha_8\alpha_9 S\alpha_1\alpha_{10}\alpha_{11} C_1 + \ldots + S\alpha_5\alpha_6\alpha_7 S\alpha_5\alpha_8\alpha_9 S\alpha_5\alpha_{10}\alpha_{11} C_5 = 0 \quad . \quad (56)$$

We may at once write down as many equations of this form as there are ways of pairing off the six vectors $\alpha_6, \alpha_7, \ldots \alpha_{11}$. Thus the first five C's can be found, and similarly the others. To prove that the C's are uniquely determined in this manner would too far lengthen the discussion. It is probable that the C's in the third and higher degrees may be much more neatly expressed by products analogous with Hamilton's expression (3).

(*Issued separately October* 13, 1920.)

XV.—The Harmala Alkaloids in Malaria.

By **J. A. Gunn**, M.D., D.Sc., Professor of Pharmacology in the University of Oxford; and **Lt.-Col. D. G. Marshall**, M.B., Lecturer on Tropical Medicine in the University of Edinburgh.

(MS. received June 21, 1920. Read June 21, 1920.)

A.—HARMALINE IN TRYPANOSOMIASIS AND MALARIA.
(J. A. Gunn.)

1. Introductory.

In two former contributions to this Society I described the pharmacological actions of harmaline * and harmine,† the two alkaloids found in the seeds of *Peganum harmala*. In regard to harmaline I came to the following conclusions:—"It differs from most alkaloids in that it does not exert, to the same extent as they do, a selective action on one kind of tissue. It attacks not only highly specialised tissues such as voluntary muscle, muscle of the heart, blood-vessels and uterus, and cells of the central nervous system, but also less highly differentiated cells, such as pigment cells, protozoa (*Raab*), and ciliated epithelium (*Jacobson*).

"In this account of its pharmacology the actions of harmaline have been shown to resemble very closely those of another alkaloid, of which the above type of non-selective action is also true, namely, quinine. As a pharmacological agent, harmaline ought to be grouped with quinine, and therefore with those substances which are conveniently, if somewhat indefinitely, termed protoplasmic poisons.

"Considering the close resemblance in the pharmacological actions of harmaline and quinine, one is led to anticipate some corresponding similarity in their therapeutic effects. With this subject I hope to deal on a future occasion."

In regard to harmine, it was found that "the pharmacological actions of harmine resemble very closely those of harmaline, in so far as the symptoms produced in the intact animal and the effects produced on the various systems and tissues are qualitatively the same in the case of both alkaloids. Harmine is, however, only about half as toxic as harmaline."

Upon this pharmacological basis I have since then endeavoured to obtain evidence as to the value of these alkaloids in protozoal diseases,

* Gunn, *Trans. Roy. Soc. Edin.*, xlvii, 1909, pp. 245–272.

† Gunn, *ibid.*, xlviii, 1911, pp. 83–96.

especially in malaria. It is important that evidence of this kind should be put on record, not only with a view to the possible improvement of the treatment of known protozoal diseases, but also with the set intention of keeping such pharmacological agents in mind for trial in such new diseases due to protozoa as are, no doubt, still to be discovered.

The term "general protoplasmic poison" is apt to be misleading, for it tends to obscure the fact that the substances included under this designation exhibit selective action of a high order. The comparative insusceptibility of the salt-water as compared with the fresh-water amœba is only one well-known instance of this. This fact is mentioned merely because it explains why the employment of such substances in protozoal diseases, based only upon general pharmacological evidence, is within limits empirical. In the present stage of knowledge, one cannot, from the fact that a substance is highly toxic to one, or even many, species of protozoa, predict that it will be comparably toxic to the pathogenic protozoon of a particular disease. Generally speaking, different protoplasmic poisons have been found necessary for the treatment of different protozoal diseases (malaria, amœbic dysentery, syphilis, etc.), although a remedy which is specific for one disease may have a slighter effect on another, especially if caused by a closely allied type of protozoon.

The minimum lethal dose of harmaline by subcutaneous injection was found to be 0·1 gramme per kilogramme for the rabbit, guinea-pig, rat, and cat. If the whole dose were in the blood at one time, this would give a concentration of about 1 in 500 to 800, allowing the blood-volume to be somewhere between one-twelfth and one-twentieth of the body-weight. Raab showed that a solution of harmaline 1 in 20,000 kills paramœcia in 8 to 20 minutes. If, therefore, one gave one-fifth of the M.L.D. by subcutaneous injection in a mammal, a concentration sufficient to kill paramœcia in 8 to 20 minutes (1 in 20,000) would be found in the blood if only one-fifth to one-eighth of this dose were in the blood at one time.

The use of harmaline in a protozoal disease was therefore well within the range of therapeutic possibility, although, as has been explained, such experiments can give nothing more than an indication of a possible therapeutic value in any particular disease.

2. Experiments on Trypanosomiasis.

Professor James Ritchie kindly gave me a strain of "surra" for investigating the action of harmaline on trypanosomes. It was found in twelve control experiments that this strain killed rats by subcutaneous injection in five to seven days. Three types of experiment were performed.

(*a*) *Experiments* in vitro.

The following experiment will serve for illustration:—

3rd March 1911. A rat which had received an injection of surra blood four days previously was killed, and 0·4 c.c. of blood drawn off from the ventricle. 0·2 c.c. of this blood was added straightway to 0·2 c.c. of citrated saline solution, and the other 0·2 c.c. of blood to 0·2 c.c. of citrated saline solution containing 1 in 2000 of harmaline hydrochloride. Samples from the two tubes were examined microscopically at intervals to determine the condition of the trypanosomes. The observations made are summarised in the adjoining table.

Time.	Tube A.	Tube B.
2.30	0·2 c.c. surra blood + 0·2 c.c. citrated saline solution.	0·2 c.c. surra blood + 0·2 c.c. citrated saline solution containing 1 in 2000 harmaline.
2.45	All trypanosomes active.	All trypanosomes active.
3.0	...	Some active, some motionless.
4.0	...	Very few active, majority sluggish, and some motionless.
6.0	...	No change.
8.0	All moving, most of them actively.	Most quite motionless, but one or two normally active.

The concentration of harmaline in tube B, after mixing the blood and saline, was 1 in 4000. This failed to kill all the trypanosones in five and a half hours, though it killed, or at least rendered motionless, the great majority of them.

In a similar experiment a concentration of harmaline 1 in 1000 in one hour stopped the movement of all but a few of the trypanosomes, and they became adherent to one another in groups. A few of them still continued to execute undulatory movements, but these movements did not lead to any forward progression of the trypanosome.

It is clear from these experiments that this type of trypanosome is far less susceptible to the toxic action of harmaline than is, for example, paramœcium. One point which was consistently observed in these experiments is that a concentration of harmaline which would render the great majority of the trypanosomes completely inactive and apparently dead would leave a few of them apparently unscathed, and the latter would move rapidly across the field of the microscope when their companions had ceased to execute even undulatory movements. This effect could not have been due to incomplete mixing of the harmaline solution with the blood. In the experiment quoted some of the trypanosomes had ceased to move in half an hour, whereas a few of them were actively, and to all appearance normally, motile five hours later; the organisms in the control tube being meanwhile uniformly active. It is difficult to avoid the conclusion that some of the trypanosomes are markedly less susceptible to harmaline than others. The phenomenon may be similar to that observed in partial hæmolysis, where blood corpuscles, exposed to the same concentration of a hæmolytic agent, may be hæmolysed to the extent, for example, of ninety per cent., the remaining corpuscles not being hæmolysed. This type of effect cannot well be due, in the experiment quoted, to some of the organisms taking up the poison before the others and leaving too little in the solution to kill their fellows, because in that case the effect would not, as it does, happen with much stronger solutions containing proportionately four times the amount of harmaline to the same number of trypanosomes.

Possibly these facts may have some bearing upon the problem of the treatment of protozoal diseases. For, if this difference in susceptibility between the fully developed organisms which is shown to occur *in vitro* occurs also *in vivo*—as there is every reason to believe it must,—then it is clearly possible for a dose of a therapeutic agent to kill off the great majority of those organisms and yet leave a few unharmed; and this survival of a few may occur after a much larger dose of the therapeutic agent than is necessary to kill the majority of the organisms. If this be true, relapses of malaria or trypanosomiasis occurring after treatment, especially relapses after a short interval, may be due merely to the survival of a few more resistant adult organisms, and not necessarily in every case to the persistence of spore forms.

Experiments in vivo.

Two types of experiment were employed. In the first, the surra blood was exposed *in vitro* to a solution of harmaline before injection into the animal; in the second, the surra blood and the harmaline were both injected subcutaneously into opposite flanks of the animal. Both sets of experimenis were suitably controlled.

(*b*) 27th February 1911. A rat that had been injected with surra four days previously was killed, and its blood made up to 8 c.c. with citrated saline. Into a small tube A was put 1 c.c. of this blood mixture + 0·5 c.c. citrated saline; into a tube B, 1 c.c. of the same blood mixture + 0·5 c.c. of a solution of the harmaline 1 in 1000 dissolved in the citrated saline solution.

The tubes were incubated for an hour at 37° C. The contents of the tubes were therefore in other respects the same, except that one contained harmaline and the other not. The object was to determine whether this exposure to harmaline so diminished the vitality of the organisms as to render them less virulent.

Two rats belonging to the same litter and each weighing 29 grammes were selected; one received by subcutaneous injection the contents of tube A, and the other the contents of tube B. Both rats died on the fifth day after injection, the blood in each being found to swarm with trypanosomes. There was therefore no appreciable diminution in virulence of the trypanosomes produced by exposure to harmaline 1 in 3000 for one hour at blood temperature.

(*c*) In a second type of experiment two rats of the same weight were injected each with 1 c.c. of the same mixture of surra blood in citrated saline, the injections being given subcutaneously into the right flank. One of them received at the same time a dose of harmaline hydrochloride 0·025 gramme per kilogramme, or one-quarter of the minimum lethal dose, the injection being given into the left flank. Both rats died on the sixth day after injection, with their blood thronged with trypanosomes. This dose of harmaline, proportionately as large as one would probably care to employ therapeutically, had no effect in preventing death from this infection with trypanosomes.

It is clear from these experiments that, though harmaline does have a toxic action on *Trypanosoma Evansi*, this action is not sufficiently powerful to suggest that it would be of practical use in the treatment of diseases due to this type of organism. Possibly better results might have been obtained if the infected rat in the third experiment had received daily-repeated injections of harmaline rather than one single dose on the first day; because in the last three days there would be little harmaline left in the blood. But at the time it seemed hardly worth while to prosecute the experiments further, especially as alkaloids of the quinine type have not so far proved of much value in the treatment of diseases due to trypanosomes.*

* Since the above was written, my friend Mr Hilton-Simpson has kindly shown me the unpublished notes recently collected by him on the native medicines used by the inhabitants of Shawia, in Algeria. Of special interest in connection with this communication are the observations he has made on the uses of *Peganum harmala*, of which observations the following is an extract:—

"Seeds of *Peganum harmala* are collected in summer and can be kept for four years, but not longer, because after that they deteriorate. These serve for a variety of treatments and are the best remedy for syphilis. The seeds are ground to a very fine powder and made into pills with honey, and one pill is taken night and morning for fifteen days, when a cure is affected: a purge, preferably castor oil, is taken before the treatment begins. This 'harmel

3. Harmaline in Malaria.

In the autumn of 1911, by the kindness of Lt.-Col. Marshall, I got in touch with a case of malaria, which I treated successfully with harmaline. This patient was a discharged soldier suffering from relapsing malaria acquired in India the previous year. He himself was convinced that harmaline acted much more efficiently than did quinine, with which he had been treated in India. Unfortunately, the conditions of the trial were unavoidably too unfavourable to allow the experiment to be regarded as evidence. It was, however, sufficient to encourage a serious trial of harmaline under more favourable experimental conditions

By arrangements that were made for me by Col. (now Sir R. H.) Firth, harmaline was tried on eleven cases of malaria in India by an officer of the R.A.M.C., and the following is an extract from his report:—

During the months of July and August eleven cases of malaria were treated with harmaline in the hospital at Peshawar. Ten were cases of benign tertian, and one of malignant tertian.

In five cases harmaline brought down the temperature and caused parasites to disappear from the blood in from three to six days. In one case harmaline produced an apparent cure, but a relapse with rigors occurred which did not yield to harmaline but yielded readily to quinine. In five cases harmaline failed to check the disease, but quinine subsequently caused an immediate fall of temperature. In six cases harmaline caused symptoms of nausea and giddiness. In all the cases the dose to commence with was 2 grains once a day, which, if well tolerated, was increased after two days to 2 grains twice daily.

Fig. 1 is a temperature chart of one of the cases successfully treated with harmaline. The patient was admitted with acute malaria and a temperature of 105·5°. Harmaline hydrochloride, 2 grains a day, was

powder intoxicates like alcohol, and is so strong that the seeds have to be soaked in water to weaken them for women's use, and they are then given unpowdered. In mild cases a little of the harmel powder is placed under the tongue once daily for forty days.

"The whole plant is sometimes used. It is washed in cold water, cut up, and boiled in water for two to three hours, and the liquid bottled. This is used as a sole beverage for syphilis.

"Harmel is also used for 'fever'; usually the patient fumigates himself with the smoke of the burning seeds. It is also used as an ointment for septic wounds."

It is interesting to find that the plant from which the alkaloids harmaline and harmine are obtained is held in such high repute as a curative agent. It is possible that, especially if given over a long period and in large doses, it may have some value in the treatment of syphilis. It is also possible that in large doses it will produce a kind of intoxication in man, seeing that it has been found to produce a pronounced stimulation of the cerebrum in the lower animals. That it will have an antiseptic action is almost certain.

administered from the following morning onwards. On the third day his temperature rose only to 100°, and he had no further fever.

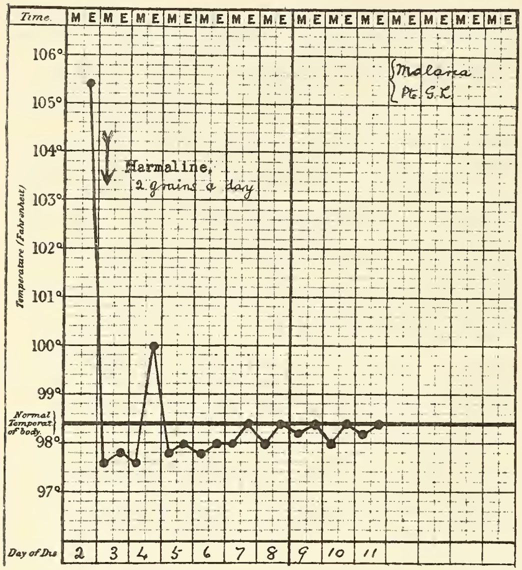

FIG. 1.—Temperature chart of a case of Acute Malaria successfully treated by Harmaline.

Fig. 2 shows the temperature chart of Pte. E. S., and illustrates a case unsuccessfully treated by harmaline. Harmaline may in this case have somewhat allayed the fever, but it did not do so completely, whereas the subsequent administration of quinine quickly brought down the temperature.

These experiments show that harmaline has a definite curative effect in malaria, but that in the doses given it is not so effective as quinine. On the other hand, the results obtained with harmaline were remarkably good for a preliminary series, if account is taken of the fact that one had little previous experience of its administration or dosage. The results were probably better than could have been obtained with anything except one of the cinchona alkaloids. As a comparison between harmaline and quinine the experiments were favourable to quinine for two reasons: in the first place, quinine was tried after the patients had had harmaline, so that any beneficial effects previously produced by harmaline told in favour of quinine; secondly, the dose of quinine was relatively higher,

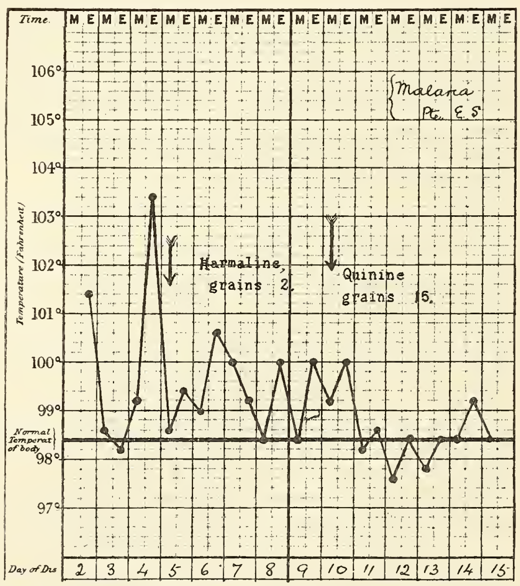

FIG. 2.—Temperature chart of a case of Acute Malaria unsuccessfully treated by Harmaline.

for, though harmaline is about four and a half times as toxic as quinine (per kilogramme for laboratory animals), the dosage used of quinine was seven and a half times that of harmaline.

One conclusion can be fairly drawn, and that is that harmaline is a remedy of some value for malaria; and this conclusion is at least of academic interest as substantiating a hypothesis in regard to therapeutic use based entirely on pharmacological evidence. The use of quinine has centuries of clinical experience behind it, and even yet there is disagreement as to how best to obtain the maximum curative effects from it; so that a new remedy requires an extended trial before it is possible to compare its therapeutic value with that of quinine. Three points deserve to be further investigated:—Is harmaline of value as an adjuvant to quinine? Is harmaline of value in types of malaria which quinine fails to cure? What effect has harmine on malaria?

Harmaline has the formula $C_{13}H_{14}N_2O$, and harmine $C_{13}H_{12}N_2O$. They are closely related chemically and pharmacologically. One would expect them to have similar effects in malaria, though either might easily be considerably more effective than the other. Harmine is only about half as toxic as harmaline, so that a larger dose of it could be given. During the war Lt.-Col. Marshall had charge of a large number of malarial cases, and he offered to try harmine in some of them. The supply of the alkaloid was limited. His report on the cases follows.

B.—HARMINE IN MALARIA. (D. G. Marshall.)

"In November 1918 Professor Gunn kindly sent me some harmine for experimental work in cases of malaria. Unfortunately, the supply of the drug was small, and consequently the observations were somewhat restricted. It was decided to employ harmine to determine its efficacy, as compared with quinine, on two lines: (*a*) its power to cut short acute attacks, and (*b*) its power to prevent relapses.

"(*a*) In all acute cases where it was employed it proved ineffectual, and quinine had to be substituted.

"(*b*) The results here were more satisfactory. Three cases were selected for trial. In all of them quinine with arsenic had been administered in full doses but had failed to prevent relapses. One was benign tertian, one mixed infection (benign tertian and malignant), the third malignant.

"1st case. T. H., benign tertian, had suffered five severe relapses between February and June (126 days). The administration of harmine was commenced on the 7th July, and he was kept under observation for 84 days.

No relapse occurred in this time, and he was discharged apparently cured on the 30th September. See fig. 3.

"2nd case. H. S., mixed infection, had suffered eight relapses in 148 days, from 6th April to September, though under regular quinine treatment, with iron and arsenic. Administration of harmine was commenced towards the end of September. The case was kept under observation for 119 days, to

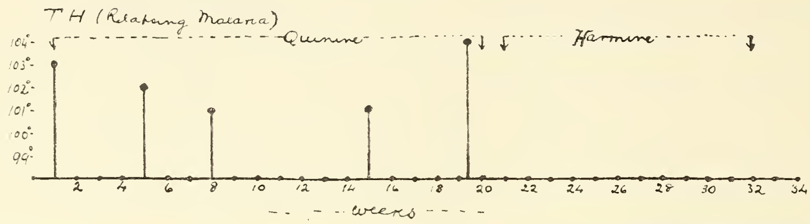

Fig. 3.—Diagrammatic representation of a case of Relapsing Malaria, treated first by Quinine and then by Harmine.

the 25th January, when he was discharged apparently cured, no relapse having occurred during this period. See fig. 4.

"In this case it is interesting to note that, though harmine was given from the 21st of September, the patient suffered a relapse on the 24th. Apparently the amount of harmine present in the circulation was insufficient to prevent the development of the parasites, though later, after regular continuous dosage, no relapse occurred.

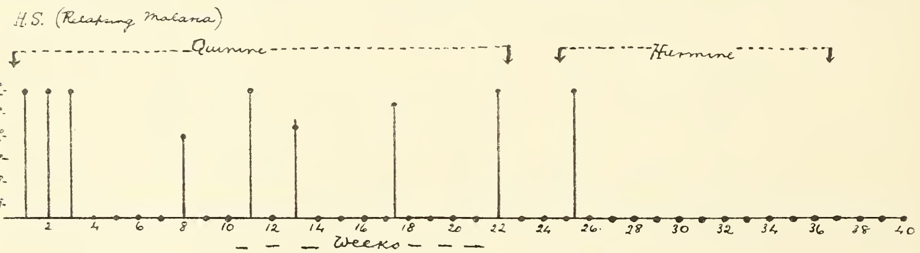

Fig. 4.—Diagrammatic representation of a case of Relapsing Malaria, treated first by Quinine and then by Harmine.

"The third case was one of severe malignant infection with frequent relapses. Unfortunately the chart is not available at present, for it was sent to the Research Committee and has not yet been returned. The usual treatment by quinine and arsenic having failed to prevent the relapses, harmine was substituted for it at the end of a relapse. The patient rapidly improved, and no relapse occurred for about six weeks. I found him one morning suffering from a relapse, and was informed that the supply of harmine had become exhausted and that he had had no harmine for about a week."

Harmine is about twice as toxic as quinine for laboratory mammals, and therefore could probably be given to man in about half the dosage of quinine. The dose employed of harmine, 3 grains per day, would, unless it was proportionately much more toxic than quinine to the malarial parasite, almost certainly be insufficient to cut short acute attacks. It would be interesting to know whether larger doses of harmine would be more successful for this purpose.

In the three cases of relapsing malaria it seemed to act much better than quinine. In these cases quinine had been given over long periods and had failed to prevent relapses. It would be remarkable if the relapses ceased in each of the only three cases tried precisely upon the administration of harmine, without the result being in part at least due to the action of this alkaloid. The results so far obtained are, however, sufficiently encouraging to warrant further trial of harmaline and harmine in malaria and other protozoal diseases. The plant from which they are obtained is very common and widely distributed, and if it proved of therapeutic value the alkaloids could be obtained from it cheaply and in quantity. While many more experiments would be required to delimit the value of these harmala alkaloids in malaria, perhaps the results so far obtained and here recorded will induce others, who may have the opportunity, to endeavour to obtain more decisive results based upon wider experience.

Summary of Results.

Harmaline has been tried in experimental trypanosomiasis in rats. While it has a toxic action on the trypanosomes *in vitro*, it was not found to have any curative effect on the disease in the doses given.

Harmaline has been tried in acute malaria. In half the cases it caused disappearance of the parasites from the peripheral blood, fall of temperature, and relief of symptoms. In the other half of the cases it failed to check the disease, though quinine subsequently did. In the doses given it is inferior to quinine for acute malaria.

Harmine has been tried both in acute and in relapsing malaria. It failed to cut short acute attacks. In three cases of relapsing malaria it proved remarkably successful, preventing the recurrence of attacks when long-continued quinine administration had failed to do so.

Both alkaloids could with safety be given in much larger doses, and probably better results could be obtained in this way, both in acute and in relapsing malaria.

(*Issued separately October* 13, 1920.)

XVI.—A Law of Force giving Stability to the Rutherford Atom.

By J. Marshall, M.A., B.Sc., University College, Nottingham.

(Read June 21, 1920. Revised MS. received July 13, 1920.)

Introduction.

There is a large accumulation of evidence in support of Sir Ernest Rutherford's conception of atomic structure. According to this conception an uncharged atom consists of certain groupings of negative electrons rotating about a positively charged nucleus. Dr Bohr's theory of line spectra, which is based on this structure, frankly discards dynamical conditions of stability. There is no doubt, however, that this theory has been very successful in accounting for the numerical relationships between the frequencies of spectral lines.

In this paper it is proposed to obtain a law of force which will preserve the stability of a group of electrons rotating in the same plane about a positive nucleus in the simple cases in which the Bohr theory is applicable. In order to account for atomic behaviour, we shall assume a law of force which for distances comparable with the radius of the atom is at variance with the inverse square law, but which for distances great in comparison with atomic dimensions differs by a negligible quantity from this law.

Analysis.

Let the positive charge on the nucleus be se and the number of electrons in the ring rotating round the nucleus as centre be p.

In equilibrium the cylindrical co-ordinates of the pth electron will be $(r, \theta_p, 0)$ referred to the positive nucleus as centre. In the disturbed state the co-ordinates of the pth electron will be $(r+r_p, \theta_p+\phi_p, x_p)$.

Let the force on the pth electron due to the positive nucleus be $\sum' \frac{se^2k_n}{r^n}$, where Σ' is a summation with respect to n, where n can have any value $\not<2$. The forces acting on the pth electron can therefore be derived from the potential function

$$V = \sum_{t=1}^{p-1} \frac{e^2}{\{(r+r_t)^2+(r+r_p)^2+(x_p-x_t)^2-2(r+r_t)(r+r_p)\cos(\theta_p+\phi_p-\theta_t-\phi_t)\}^{\frac{1}{2}}}$$
$$-\sum{}' \frac{se^2k_n}{(n-1)\{(r+r_p)^2+x_p{}^2\}^{(n-1)/2}}.$$

Hence for displacements perpendicular to the plane of the orbit we have the equation of motion

1. $$m\ddot{x}_p = -\frac{\partial V}{\partial x_p}.$$

We shall put

$$\theta_p - \theta_t = 2\pi - 2ta, \quad \text{where } a = \pi/p.$$

Neglecting quantities of the second order, we have on expansion of the right side of equation 1:

1·1. $$m\ddot{x}_p = \sum \frac{e^2}{8r^3 \sin^3 ta}(x_p - x_t) - \sum{}' \frac{se^2 k_n}{r^{n+1}} x_p$$

$$= x_p\left\{\sum \frac{e^2}{8r^3}\operatorname{cosec}^3 ta - \sum{}' \frac{se^2 k_n}{r^{n+1}}\right\} - \sum \frac{e^2}{8r^3}\operatorname{cosec}^3 ta x_t.$$

If we put $x_2 = \beta x_1$, $x_3 = \beta x_2$, $x_4 = \beta x_3$, . . . $x_p = \beta x_{p-1}$, and $x = \beta x_p$, we have

$$\beta^p = 1 = \cos 2k\pi + i \sin 2k\pi$$
$$\beta = \cos 2ka + i \sin 2ka, \quad \text{where } k = 0,\ 1,\ 2,\ .\ .\ .\ p-1.$$

For a vibration which involves e^{iqt} as a time factor, we have

1·2. $$q^2 = \sum{}' \frac{se^2 k_n}{mr^{n+1}} - \sum \frac{e^2}{8mr^3}\operatorname{cosec}^3 ta + \sum \frac{e^2}{8mr^3}\operatorname{cosec}^3 ta \cos 2kta,$$

since $x_t = \beta^t x_p$ and

$$\sum_{t=1}^{p-1} \sin 2kta \operatorname{cosec}^3 ta = 0.$$

Displacements perpendicular to the plane of the orbit will be stable if

1·3. $$\sum{}' \frac{se^2 k_n}{mr^{n+1}} > \sum \frac{e^2}{8mr^3}\operatorname{cosec}^3 ta(1 - \cos 2kta).$$

For radial displacements we have the equation

2. $$m\ddot{r}_p - mr\omega^2 - 2mr\omega\dot{\phi}_p - mr_p\omega^2 = -\frac{\partial V}{\partial r_p}.$$

On expanding the right side of equation 2 and neglecting terms of the second order, we have

2·1. $$m\ddot{r}_p - mr\omega^2 - 2mr\omega\dot{\phi}_p - mr_p\omega^2$$
$$= \sum \frac{e^2}{4r^2}\operatorname{cosec} ta - \sum{}' \frac{se^2 k_n}{r^n}$$
$$+ \sum \frac{e^2}{8r^2}\cos ta \operatorname{cosec}^2 ta(\phi_p - \phi_t)$$
$$- \sum \frac{e^2}{8r^3}(3 \operatorname{cosec} ta - \operatorname{cosec}^3 ta) r_p$$
$$- \sum \frac{e^2}{8r^3}(\operatorname{cosec} ta + \operatorname{cosec}^3 ta) r_t$$
$$+ \sum{}' \frac{se^2 k_n n r_p}{r^{n+1}}.$$

For steady motion we have

2·2. $$mr\omega^2 = \sum{}' \frac{se^2 k_n}{r^n} - \sum \frac{e^2}{4r^2}\operatorname{cosec} ta.$$

Using this condition which determines ω, we obtain

2·3.
$$m\ddot{r}_p - 2mr\omega\dot{\phi}_p$$
$$= -\sum\frac{e^2}{8r^2}\cos ta \operatorname{cosec}^2 ta\phi_t - r_p\left\{\sum\frac{e^2}{8r^3}(5\operatorname{cosec} ta - \operatorname{cosec}^3 ta)\right.$$
$$\left.-\sum{}'\frac{(n+1)se^2k_n}{r^{n+1}}\right\} - \sum\frac{e^2}{8r^3}(\operatorname{cosec} ta + \operatorname{cosec}^3 ta)r_t$$

since $\sum_{t=1}^{p-1}\cos ta \operatorname{cosec}^2 ta = 0$.

For tangential displacements we have

3.
$$mr\ddot{\phi}_p + 2m\dot{r}_p\omega = -\frac{\partial V}{(r+r_p)\partial\theta_p}.$$

On expanding the right-hand side of equation 3, we have

3·1.
$$mr\ddot{\phi}_p + 2m\omega\dot{r}_p$$
$$= r_p\sum\frac{3e^2}{8r^3}\frac{\cos ta}{\sin^2 ta} + \sum\frac{e^2}{8r^3}\frac{\cos ta}{\sin^2 ta}r_t$$
$$-\phi_p\sum\frac{e^2}{4r^2}\frac{\cos ta}{\sin^2 ta}(\cot ta + \tfrac{1}{2}\tan ta)$$
$$+\sum\frac{e^2}{4r^2}\frac{\cos ta}{\sin^2 ta}(\cot ta + \tfrac{1}{2}\tan ta)\phi_t - \sum\frac{e^2}{4r^2}\frac{\cos ta}{\sin^2 ta}.$$

Since $\sum_{t=1}^{p-1}\frac{\cos ta}{\sin^2 ta} = 0$, equation 3·1 becomes

3·2.
$$mr\ddot{\phi}_p + 2m\omega\dot{r}_p$$
$$= -\phi_p\sum\frac{e^2}{4r^2}\frac{\cos ta}{\sin^2 ta}(\cot ta + \tfrac{1}{2}\tan ta)$$
$$+\sum\frac{e^2}{4r^2}\frac{\cos ta}{\sin^2 ta}(\cot ta + \tfrac{1}{2}\tan ta)\phi_t$$
$$+\sum\frac{e^2}{8r^3}\frac{\cos ta}{\sin^2 ta}r_t.$$

The equations 2·3 and 3·2, which determine the stability of the configuration in the plane of the orbit, are of the form

4·1. $\ddot{r}_p - 2r\omega\dot{\phi}_p = -\Sigma rA_t\phi_t - Br_p - \Sigma C_t r_t \qquad p = 1, 2, 3 \ldots p.$

4·2. $r\ddot{\phi}_p + 2\omega\dot{r}_p = -Dr\phi_p + \Sigma D_t r\phi_t + \Sigma A_t r_t \qquad p = 1, 2, 3 \ldots p,$

where

4·3.
$$\begin{cases} A_t = \dfrac{e^2}{8mr^3}\cos ta \operatorname{cosec}^2 ta \\ B = \displaystyle\sum_{t=1}^{p-1}\frac{e^2}{8mr^3}(5\operatorname{cosec} ta - \operatorname{cosec}^3 ta) - \sum{}'\frac{(n+1)se^2k_n}{mr^{n+1}} \\ C_t = \dfrac{e^2}{8mr^3}(\operatorname{cosec} ta + \operatorname{cosec}^3 ta) \\ D_t = \dfrac{e^2}{4mr^3}\dfrac{\cos ta}{\sin^2 ta}(\cot ta + \tfrac{1}{2}\tan ta) \\ D = \Sigma D_t. \end{cases}$$

In order to solve the equations 4·1 and 4·2, we use the artifice

$$r_p = \beta r_{p-1}\,;\; r_{p-1} = \beta r_{p-2}\,,\; . \; . \; . \qquad r_1 = \beta r_p.$$

$$\therefore \quad \beta^p = 1 = \cos 2k\pi + i \sin 2k\pi,$$

$$i.e. \quad \beta = \cos 2ka + i \sin 2ka. \qquad k = 0,\ 1,\ 2,\ .\ .\ .\ p-1.$$

A similar set of equations is used for the ϕ's, viz.:

$$\phi_p = \beta\phi_{p-1}\,;\; \phi_{p-1} = \beta\phi_{p-2}\,;\; . \; . \; . \qquad \phi_1 = \beta\phi_p.$$

If the time factor enters in the form e^{iqt} in each variable, we have

4·4. $$(\mathrm{B} - q^2 + \Sigma \mathrm{C}_t\beta^t)r_p = (2\omega iq - \Sigma \mathrm{A}_t\beta^t)r\phi_p.$$

4·5. $$(\mathrm{D} - q^2 - \Sigma \mathrm{D}_t\beta^t)r\phi_p = (-2\omega iq + \Sigma \mathrm{A}_t\beta^t)r_p.$$

From equations 4·4 and 4·5 we obtain an equation determining q.

4·6. $$(\mathrm{B} - q^2 + \Sigma \mathrm{C}_t\beta^t)(\mathrm{D} - q^2 - \Sigma \mathrm{D}_t\beta^t) = -(2\omega iq - \Sigma \mathrm{A}_t\beta^t)^2.$$

This equation involves imaginary coefficients of q. But

$$\Sigma \mathrm{A}_t\beta^t = \Sigma\{\mathrm{A}_t \cos 2kta + i\mathrm{A}_t \sin 2kta\}.$$

But

$$\Sigma \frac{\cos ta}{\sin^2 ta} \cos 2kta = 0$$

And it can be shown that

$$\Sigma \mathrm{A}_t\beta^t = i\Sigma \mathrm{A}_t \sin 2kta,$$
$$\Sigma \mathrm{C}_t\beta^t = \Sigma \mathrm{C}_t \cos 2kta,$$
$$\Sigma \mathrm{D}_t\beta^t = \Sigma \mathrm{D}_t \cos 2kta.$$

The equation for the frequency q becomes

4·7. $$(\mathrm{B} + \Sigma \mathrm{C}_t \cos 2kta - q^2)(\mathrm{D} - \Sigma \mathrm{D}_t \cos 2kta - q^2) - (2\omega q - \Sigma \mathrm{A}_t \sin 2kta)^2 = 0.$$

This equation is a quartic, and will in general give 4 values of q.

k has the p values $0, 1, 2, 3, \; . \; . \; . \; p-1$.

If $p-k$ is written for k in 4·7, the values of q obtained merely differ in sign.

If p is odd we obtain $\frac{p-1}{2}$ equations of the form 4·7. Also for the value $k=0$, $\Sigma \mathrm{A}_t \sin 2kta = 0$, so that we have a quadratic in q^2.

There are therefore $4 \cdot \frac{p-1}{2} + 2 = 2p$ values of the frequency corresponding to $2p$ degrees of freedom. When p is even we obtain $\frac{p}{2} - 1$ equations of the form 4·7 giving $2p-4$ frequencies.

When $k=0$, as before we get another two values for q; and when $k = \frac{p}{2}$, $\Sigma \mathrm{A}_t \sin 2kta = \Sigma \mathrm{A}_t \sin pta = \Sigma \mathrm{A}_t \sin t\pi = 0$, so that we obtain another two values for q. The total number of values for q is $2p - 4 + 2 + 2 = 2p$, corresponding to the $2p$ degrees of freedom of the rotating group in the plane.

Let

$$
4\cdot 8. \quad \left\{
\begin{aligned}
F &= \frac{mr^3}{e^2}(B + \Sigma C_t \cos 2kta) \\
&= \Sigma \tfrac{1}{8}[5 \operatorname{cosec} ta - \operatorname{cosec}^3 ta + \cos 2kta\,(\operatorname{cosec} ta + \operatorname{cosec}^3 ta)] \\
&\qquad\qquad - s[3 - \Sigma'(n+1)K_{n-2}] \\
&\qquad \text{where } n > 2, \text{ and } 3 - \Sigma'(n+1)K_{n-2} = \Sigma' \frac{(n+1)k_n}{r^{n-2}} \\
G &= \frac{mr^3}{e^2}(D - \Sigma D_t \cos 2kta) \\
&\doteq \Sigma \tfrac{1}{2} \frac{\sin^2 kta}{\sin^2 ta}(\cot ta + \tfrac{1}{2} \tan ta) \cos ta \\
H &= \frac{mr^3}{e^2} \Sigma A_t \sin 2kta \\
&= \Sigma \tfrac{1}{8} \cos ta \operatorname{cosec}^2 ta \sin 2kta \\
J &= \sqrt{\frac{mr^3}{e^2}}\, 2\omega \\
q' &= q\sqrt{\frac{mr^3}{e^2}}\,.
\end{aligned}
\right.
$$

The equation 4·7 now takes the form

$$
4\cdot 9. \qquad (F - q'^2)(G - q'^2) - (H - Jq')^2 = 0.
$$

By consideration of signs of the left side of this equation it is easy to show that q' will have four real values if F and G are positive and each is greater than $(H/J)^2$.

For stability we must also satisfy the inequalities obtained in equation 1·3 and 2·2, viz.:

$$
5. \qquad \left\{
\begin{aligned}
4s(1 - \Sigma' K_{n-2}) &> \Sigma \operatorname{cosec}^3 ta \sin^2 kta. \\
4s(1 - \Sigma' K_{n-2}) &> \Sigma \operatorname{cosec} ta.
\end{aligned}
\right.
$$

This latter condition is necessary in order that ω^2 given by the equation

$$
\frac{4mr^3}{e^2}\omega^2 = 4s(1 - \Sigma' K_{n-2}) - \Sigma \operatorname{cosec} ta
$$

may be real.

$$
\therefore \qquad J^2 = 4s(1 - \Sigma' K_{n-2}) - \Sigma \operatorname{cosec} ta.
$$

Particular Cases.

In the previous work we have developed the conditions of stability on the assumption that the force between the positive nucleus and the negative electron is represented by a function of the form $\frac{1}{r^2} - \sum' \frac{k_n}{r^n}$, where $n > 2$.

We shall now apply our results to particular cases. If we simplify the law of force to the form $\frac{1}{r^2}-\frac{b^{n-2}}{r^n}$, it is evident that the constant b will give a measure of the closest approach of the electron to the positive nucleus, when there are no other forces acting on the electron.

Let

$$K=\frac{b^{n-2}}{r^{n-2}}.$$

One necessary condition that this law must satisfy is, that for distances large relatively to the radius of an atom the law of the inverse square of the distance must hold. This condition is evidently satisfied, since b is a small quantity and n is $\nless 3$.

Special Cases.

(1) $p=s=1,\ t=0.$

The condition for stability is

$$1>K>\frac{3}{n+1}$$

and is satisfied $n\nless 3$.

(2) $p=s=2,\ t=1,\ a=\frac{\pi}{2},\ k=0$ and 1.

$H=0$ in each case, and G is positive. Stability is obtainable if

$$\cdot 87>K>\frac{2\cdot 87}{n+1}.$$

If $n\nless 3$, F is positive, and F and G are each greater than $(H/J)^2$.

(3) $p=s=3,\ t=2,\ a=\frac{\pi}{3},\ k=0,\ 1,$ and 2.

In each case F and G are positive and greater than $(H/J)^2$ if

$$\cdot 8>K>\frac{2\cdot 76}{n+1},$$

i.e. if $n\nless 3$.

(4) $p=s=4,\ t=3,\ a=\frac{\pi}{4},\ k=0,\ 1,\ 2,$ and 3.

In each case F and G are positive and greater than $(H/J)^2$ if

$$\cdot 65>K>\frac{2\cdot 81}{n+1},$$

i.e. if $n\nless 4$.

(5) $p=s=5,\ t=4,\ a=\frac{\pi}{5},\ k=0,\ 1,\ 2,\ 3,$ and 4.

In each case F and G are positive and greater than $(H/J)^2$ if

$$\cdot 51 > K > \frac{2\cdot 86}{n+1},$$

i.e. if $n \nless 5$.

(6) $p=s=6$, $t=5$, $a=\frac{\pi}{6}$, $k=0$, 1, 2, 3, 4, and 5.

In each case F and G are positive and greater than $(H/J)^2$ if

$$\cdot 29 > K > \frac{3\cdot 01}{n+1},$$

i.e. if $n \nless 10$.

(7) $p=s=7$, $t=6$, $a=\frac{\pi}{7}$, $k=0$, 1, 2, 3, 4, 5, and 6.

In each case F and G are positive and greater than $(H/J)^2$ if

$$\cdot 10 > K > \frac{3\cdot 13}{n+1},$$

i.e. if $n \nless 32$.

(8) $p=s=8$, $t=7$, $a=\frac{\pi}{8}$, $k=0$, 1, 2, 3, 4, 5, 6, and 7.

It is not necessary to discuss the conditions of stability in full, as the condition for displacements *perpendicular to the plane of the orbit* is not satisfied, since

$$4s(1-K) > \Sigma \operatorname{cosec}^3 ta \sin^2 kta$$

gives for $k=3$ or 5,

$$1-K > 1\cdot 08.$$

Since G, H, and J are independent of n for a fixed value of K, and F increases as n increases, it follows that if stability is obtainable for *any* value of n it is obtainable for *all larger* values of n.

Conclusions.

The preceding analysis shows that if a law of force between a positive nucleus and a negative electron be of the form $\frac{1}{r^2}\left(1-\frac{b^{n-2}}{r^{n-2}}\right)$, an n can be found which will preserve the stability of a group of electrons, *not exceeding seven in number*, rotating in a circular orbit round a positive nucleus. Since b is smaller than the radius of an atom, for distances large in comparison with the radius of an atom this law of force will differ from the inverse square law by a negligible quantity. Stability for the simple cases of the Rutherford atom will therefore be established.

The question of the stability of a series of rings of electrons rotating about a positive nucleus introduces considerable complexity into the rigorous analysis required. If we make the tentative assumption that, so far as the effect of an inner set of rings on the outer ring is concerned, we can replace the inner set by an equivalent charge at the centre of the atom, the conditions for stability of an outer ring of p electrons rotating in a circular orbit will be determined by the equations

$$4p\left(1-\frac{s\mathrm{K}}{p}\right)>\sum_{t=1}^{p-1}\operatorname{cosec}^3 ta\sin^2 kta,$$

and the reality of the roots of the equation

$$(\mathrm{F}-q'^2)(\mathrm{G}-q'^2)=(\mathrm{H}-\mathrm{J}q')^2,$$

where

$$\mathrm{F}=\sum_{t=1}^{p-1}\tfrac{1}{8}[5\operatorname{cosec} ta-\operatorname{cosec}^3 ta+\cos tka(\operatorname{cosec} ta+\operatorname{cosec}^3 ta)]-p\left[3-(n+1)\frac{s\mathrm{K}}{p}\right],$$

and G and H have the same values as before,

$$\mathrm{J}^2=4p\left(1-\frac{s\mathrm{K}}{p}\right)-\Sigma\operatorname{cosec} ta.$$

In these conditions $\frac{s\mathrm{K}}{p}$ replaces K; and since $s>p$, the new value of K will be less than the previous value of K. This can be obtained since r is greater for the outer set than for any one of the values of r for the inner set, the value of n being kept the same; or we may increase n as well as r, and the increased value of n will still maintain the stability of the inner set.

The displacements of the electrons in the outer ring, perpendicular to the plane of the orbit, will be unstable, however, when p exceeds seven.

This would seem to indicate that the atom could be built up of a series of rings of *seven* electrons, and we should expect a periodicity in the chemical properties of the atoms corresponding to Mendeléeff's classification, which was stated by Newlands in 1864 in the form, "the eighth element starting from a given element is a kind of repetition of the first."

(*Issued separately October* 12, 1920.)

XVII.—The Explanation of an apparent Anomaly outstanding in the Results of Measurement of Dissociation Pressures. By Professor Alan W. C. Menzies.

(MS. received May 4, 1920. Read June 21, 1920.
Revised MS. received September 6, 1920.)

(Abstract.)

In 1888 Tammann,* applying a form of the gas current saturation method to the measurement of the dissociation pressures of salt hydrates, obtained results which were uniformly higher by from 2 to 5 per cent. than the results obtained by Frowein † with the tensimeter. This anomalous behaviour was confirmed, in Nernst's laboratory, by Schottky,‡ who reported that the initial dissociation pressures developed in tensimetric measurements were, in cited cases, higher than the equilibrium values. In 1911 Partington,§ re-studying certain of the facts, added further confirmation by the use of the gas current saturation method in a somewhat altered form.

Thoughtful elucidations and critical discussions of this apparent anomaly have been offered by Tammann, Nernst,‖ Partington, Brereton Baker,¶ and Campbell,** those of Nernst and of Campbell being especially instructive.

In view of notorious precedent in the case of Charles II and the Royal Society of London, it seemed worth while, despite the weight of triple authority cited above, to seek a still simpler explanation of the anomaly than those hitherto suggested by once again examining the experimental facts. The dissociation pressure of the same sample of cupric sulphate pentahydrate in equilibrium with trihydrate was accordingly measured, near 25°, by both the gas current saturation method and by the tensimetric method. Certain precautions were adopted so as to avoid errors that were suspected to have been incident to the work of the earlier experimenters.

The only two non-preliminary measurements by the gas current saturation method gave 7·83 and 7·78 mm. mercury at 0° as the dissociation pressure at 25·00° of the system $CuSO_4(5–3)H_2O$ and vapour: average,

* *Ann. Physik*, xxxiii, p. 322 (1888).
† *Z. physik. Chem.*, i, p. 5 (1887).
‡ *Z. physik. Chem.*, lxiv, p. 415 (1908).
§ *J. Chem. Soc.*, xcix, p. 466 (1911).
‖ *Z. physik. Chem.*, lxiv, p. 415 (1908).
¶ *Ann. Rep. Progress Chem.*, viii, p. 34 (1912).
** *Trans. Faraday Soc.*, x, p. 195 (1914).

7·80 mm. The only two non-preliminary measurements by the tensimetric method gave 7·72 and 7·76 mm.: average, 7·74 mm. In spite of precautions to avoid condensation * of saturated water vapour before reaching the absorption tubes, the gas current saturation results would be expected to turn out high by a fraction of 1 per cent., in conformity with the consistent results of the extremely careful work of Berkeley and Hartley.† As the results stand, therefore, it is obvious that the discrepancy is well within the experimental error of such measurements. And thus the anomaly disappears.

In the interest of progress, it is perhaps allowable briefly to indicate one or two points at which error may have entered into the experimental work of previous investigators in this field. Too much confidence has perhaps been placed upon the tensimetric results of Frowein, which fall over 5 per cent. below those given above. A comparison of such other tensimetric measurements as have been published since Frowein's time points in the same direction, thus indicating, in the writer's opinion, that Frowein too lightly assumed the absence of permanent gas from his tensimeters. The ingenious suggestion of Campbell, based on the fact that the observed vapour pressure of water in air is less than in a vacuum, thus overshoots the mark in harmonising the results of Tammann with those of Frowein, for Frowein's figures are unquestionably low. Schottky seems to have understood the danger of the presence of permanent gas, for he allowed his tensimeters to lie in the horizontal position, with the opposite sides in free communication, for twelve hours prior to erection for observation. When, after erection, one bulb of the tensimeter is raised to the higher temperature, as was done in his work, not only will the dissociation pressure on that side increase, but so also will the air pressure. But air is soluble in paraffin oil, which Schottky used, as well as in all common manometric liquids except mercury. One would therefore predict, in accordance with Henry's law, that Schottky should find, exactly what he did find, an initial excess of pressure that slowly fell off to the true equilibrium value as the air pressures on opposite sides slowly became equalised by the mechanism of solution and diffusion of air in the oil. Others also, up to the present time, continue to fail to recognise this consequence of Henry's law.‡ With regard to Partington's work, it is suggested that glass-wool plugs § should not be used to filter rather small volumes of air of 100 per cent. humidity, if the humidity is later to be

* *Cf.* Menzies, *J. Amer. Chem. Soc.*, xlii, p. 978 (1920).

† *Trans. Roy. Soc. London*, A, ccix, p. 177 (1909).

‡ The matter will be referred to elsewhere. § *Cf.* Menzies, *loc. cit.*

assumed as 100 per cent. Partington does not state whether he corrected for the error that so long escaped the vigilance of Berkeley,* who pointed it out some years after Partington's work; nor even whether the more obvious correction for change of air pressure due to head of water in his saturator was duly applied.

Princeton University,
August 1920.

* *Nature*, xcv, p. 34 (1915).

(*Issued separately October* 12, 1920.)

XVIII.—The Musical Scale. By Joseph Goold.

Communicated by C. R. Gibson.

(MS. received January 1, 1920. Read June 21, 1920.)

(Abstract.)

I. The Tonal Order of the Chromatic Scale.

The exact ratios of the chromatic scale were first made known by Claudius Ptolemy of Alexandria about A.D. 150. We give them here as he found them, unaltered and unalterable.

Diagram I.

Ptolemy's Scale-Ratios.

Scale-names	C	D♭	D	E♭	E	F	F♯	G	A♭	A	B♭	B	C
Scale-ratios	$\frac{1}{1}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{15}{8}$	$\frac{2}{1}$

It is a most remarkable fact that while the ratios of the chromatic scale have thus been known for nearly two thousand years, its mathematical symmetry and origin have remained unrecognised down to the present day.

For the concrete musical faculty is compounded of two distinct elements, physical and mathematical.

In its physical aspect (with which we are familiar), the scale is a series of notes arranged in order of pitch upwards from the tonic (or keynote).

In its mathematical aspect, the scale is a system of intervals measured symmetrically upwards and downwards from tonic and dominant as central interval (or tonal centre).

The physical order of the scale is that which we most easily appreciate, but the mathematical order is that in which it originates. The chromatic scale therefore is, *primarily*, not a series of notes but a system of intervals (or tonal system).

Diagram I shows the scale in its familiar order of pitch, and it is remarkable how effectually this physical order conceals the mathematical from view.

The mathematical order of the scale may be demonstrated in several different ways, but in each and all alike we shall find the principles of *symmetry and simplicity* dominant throughout.

An octave is the simplest of all possible intervals, and therefore it is universally recognised as the fundamental musical unit. The chromatic scale (or tonal system) is the simplest possible division of the octave (or of multiple octaves).

We have already said that the scale is *primarily* a system of intervals rather than a series of notes; for though its divisions are *marked* by notes (just as the divisions of a ruler are marked by lines), the divisions themselves are not notes but intervals. These intervals, moreover, are perfect fifths and major thirds, and this is so because they are the simplest natural intervals by which the octave can be divided.

It might appear to some that the simplest procedure would be to divide the octave by the interval of a semitone; this, of course, is the principle of equal temperament, adopted in our pianos and organs, which, though of great practical value, is entirely inadmissible for scientific purposes, being both artificial and inaccurate.

Again, it might be supposed that the perfect fifth alone would suffice as a divisor for the octave, and indeed such a suggestion was made by Pythagoras, but the result was unsatisfactory: the fifth with the octave alone will not yield the notes of the perfect scale, whereas the natural scale may be obtained symmetrically by fifths and thirds from a central interval.

The Tonal System.

Let us now proceed to build up the chromatic scale on these lines.

Take any convenient musical sound to begin with. We will call it C.

Before the unprepared ear can recognise it as C (or as any note whatever), some other musical sound must be associated with it.

Our scale, however, is bound to start with an interval, and the simplest interval (next to the octave) is the perfect fifth, C G, and this we take as the central interval (or tonal centre).

If we now add to this central interval the two simplest and most symmetrical measurements possible (fifths above and below), we get the series of perfect fifths:—

F, C, G, D.

These are the four fundamental notes of the scale, belonging equally to the major and minor systems.

Take now the interval of the major third above and below each of these four notes, that is, the ratios 5/4, 4/5, and we get the complete chromatic scale (or tonal system), as follows:—

Diagram II.

Chromatic Scale of C.

A	E	B	F♯
F	**C**	**G**	D
D♭	A♭	E♭	B♭

Tonal system measured symmetrically by fifths horizontally and thirds vertically from central interval.

Diagram II is of great importance, and will repay careful study, for it represents in simple and graphic form all the principal facts of scale-structure associated with the tonic C and its dominant G.

It shows the symmetrical derivation both of the chromatic and diatonic scales; it discloses the true harmonic order of the chromatic scale; it is an index of tonality, and (in its extended form, given below) it explains the phenomena of the relative major and minor keys.

For the present we will confine our attention to the diatonic major and minor scales.

Let it be observed that the major scale consists of the four fundamentals or their octaves (F, C, G, D), together with the three upward measured thirds or their octaves (A, E, B); whilst the minor scale consists of the same fundamentals in association with the three downward measured thirds (A♭, E♭, B♭).

Now observe the position of these scale-notes in relation to the tonal centre (C G), according to the measurements shown in Diagram II, and it will be seen that the two scales are perfectly symmetrical, and that the minor scale is therefore the exact mathematical inversion of the major, its natural form being downward from the dominant, just as that of the major scale is upward from the tonic.

The Mathematical System.

The scale-notes being derived by measurements of fifths and thirds from the tonal centre, their position on the diagram (No. II) enables us to calculate the ratio of each pair.

For example, taking C as the base of measurement, the note B is one-fifth and one-third above; its ratio therefore is

$$\frac{3}{2} \times \frac{5}{4} = \frac{15}{8}$$

In this way we can prove all our scale-ratios to be identical with those of the Ptolemaic series (Diagram I).

In doing this we must realise that the primary measurements of these intervals actually extend over a compass of more than nine octaves; as, however, the character of an interval remains the same regardless of octave-measurement, we reduce the primary ratios to scale-limit, multiplying or dividing them by powers of 2, according to the number of octaves we require to raise or lower them.

This arithmetical process applied to Diagram II gives Diagram III.

Diagram III.

$\frac{5}{3}$	$\frac{5}{4}$	$\frac{15}{8}$	$\frac{45}{32}$
$\frac{4}{3}$	1	$\frac{3}{2}$	$\frac{9}{8}$
$\frac{16}{15}$	$\frac{8}{5}$	$\frac{6}{5}$	$\frac{9}{5}$

Scale-ratios of the tonal system identical with those of the Ptolemaic series.

II. The Harmonic Order of the Scale.

We have said that the mathematical order of the scale may be demonstrated in several different ways, and we have already viewed it as a symmetrical system based upon a central interval; that is, its tonal order: we are now to see that it has another symmetry, equally perfect, but based upon its harmonic order, and measured from its harmonic extremes.

It is a well-known fact that the natural harmonic series is included in a series of whole multiples of 1, *only a definite selection of these multiples having any place in the scale.* We shall see, for example, that in the chromatic scale we have only 12 out of 675 possible harmonics; and although in the extended scale there is an indefinite number, *they are all of them multiple powers of the two prime numbers* 3 *and* 5. In fact, all ratios of notes and intervals that are used in music are products of the powers of the three simplest prime numbers 2, 3, and 5.

We will demonstrate this harmonic scale-order mathematically.

Let the indices of the simplest positive powers of 3 and 5 be arranged in a series of twelve pairs, in order of magnitude:—

Powers of 5 . .	0	0	1	0	1	2	0	1	2	1	2	2
,, 3 . .	0	1	0	2	1	0	3	2	1	3	2	3

Under these pairs of indices we now write the corresponding product in whole numbers:—

Harmonics .	1	3	5	9	15	25	27	45	75	135	225	675

Here we have the select series of harmonics constituting the chromatic scale, viz. the twelve simplest available powers of the two simplest prime numbers which can divide the octave. *These are the root-ratios of the* C *scale, expressed as harmonics of* D♭.

Ratios . .	$\frac{1}{1}$	$\frac{3}{1}$	$\frac{5}{1}$	$\frac{9}{1}$	$\frac{15}{1}$	$\frac{25}{1}$	$\frac{27}{1}$	$\frac{45}{1}$	$\frac{75}{1}$	$\frac{135}{1}$	$\frac{225}{1}$	$\frac{675}{1}$
Scale-names .	D♭	A♭	F	E♭	C	A	B♭	G	E	D	B	F♯

If, now, we divide this series by 15 (to make C the unit of measurement), we shall at once recognise the primary ratios of the C scale, though still in their harmonic order:—

$\frac{1}{15}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{3}{5}$	$\frac{1}{1}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{3}{1}$	$\frac{5}{1}$	$\frac{9}{1}$	$\frac{15}{1}$	$\frac{45}{1}$
D♭	A♭	F	E♭	C	A	B♭	G	E	D	B	F♯

In this form these ratios occupy a compass of nine octaves and one augmented third: reduced to scale-limit and arranged in scale-order, we recognise them as Ptolemy's ratios!

1	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{15}{8}$	2
C	D♭	D	E♭	E	F	F♯	G	A♭	A	B♭	B	C

The fact of outstanding significance is the mathematical identity which is thus revealed between the harmonic and the tonal order of the chromatic scale.

A diagram will illustrate this.

Diagram IV.

Chromatic Scale of C.

25	75	225	675
5	15	45	135
1	3	9	27

Harmonic order measured upwards from D♭. Identical with tonal order, measured upwards and downwards from central interval.

It should be observed that these harmonic measurements bear exactly the same relation to one another, upwards or downwards from either extreme.

The Symmetric Scale.

It is therefore evident, both from the tonal and the harmonic order of the scale, that its twelve notes occupy positions mathematically opposed to one another, the whole series comprising six symmetric intervals.

Diagram V.

Symmetric Intervals in Harmonic Order.

1 D♭ — F♯ 675
3 A♭ — B 225
5 F — D 135
9 E♭ — E 75
15 C — G 45
25 A — B♭ 27

Diagram V shows the mathematical symmetry of the scale-notes very clearly, and its importance will soon appear in regard to their functional characters, and later in regard to their tonality.

If, now, we rearrange these symmetric intervals in pitch order, we shall have the *symmetrical chromatic scale*:—

SCALE-NAMES.

B♭	B	C	D♭	D	E♭	E	F	F♯	G	A♭	A

RATIOS.

$\frac{9}{5}$	$\frac{15}{8}$	1 or $\frac{}{2}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{45}{32}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$

SEMITONE-RATIOS.

$\frac{25}{24}$	$\frac{16}{15}$	$\frac{16}{15}$	$\frac{135}{128}$	$\frac{16}{15}$	$\frac{25}{24}$	$\frac{16}{15}$	$\frac{135}{128}$	$\frac{16}{15}$	$\frac{16}{15}$	$\frac{25}{24}$

It will be seen that the semitone-ratios have the same order when read from either end, which proves that the scale-ratios themselves are symmetrical. The symmetry which thus appears only in our third line is, of course, involved equally in both the others, but is effectually obscured by making C the centre of measurement, whereas the true centre is the interval C G.

It now remains for us to examine the functional character of the scale-notes; to show the order of the extended scale, and the twelve-note limit of the tonal system; to consider the subjects of tonality and key-relationship (with special regard to the relative minor key), and to mention very briefly the subject of comparative magnitudes.

III. Functional Characters of the Scale-notes.

As we saw in its symmetric order, the chromatic scale is divisible into two equal and similar parts, in each of which (so far as the mathematical element is concerned) every note has a function corresponding to that of its fellow (Diagram V). The twelve notes, therefore, comprise six functional pairs. It is not an easy matter to find simple and adequate names for these co-functionaries, but their order and character may be conveniently intimated as follows:—

Diagram VI.

Mathematical Co-functionaries of the Scale.

Scale-names.	Functional Characters.
C G	Dominators.
E E♭	Mediants.
B A♭	Leaders.
F D	Ultra-dominators.
A B♭	Ultra-mediants.
F♯ D♭	Ultra-leaders.

The character of the first three pairs is more definite than that of the last three, and their designation is sufficiently clear. For the sake of simplicity we use the same terms to describe the last three pairs, only adopting the prefix "ultra." It will be observed that these latter co-functionaries are, mathematically, the exterior notes of the scale, removed one-fifth upward or downward from the other six. (See Diagram II.)

Let us now consider these co-functionaries a little more closely.

Dominators C G.—These two notes, forming the central interval, are the basis of measurement for all others in the scale, C dominating the major, and G dominating the minor, system. This would seem to explain the peculiar pre-eminence of the dominant in the minor key.

Mediants E E♭.—These notes, so familiar as major and minor thirds of the common chord, are essentially major thirds (root measures) upward and downward from their respective bases, C and G. The two mediants are consonant with both dominators. They are distinctive also as relatives, E being dominant of the relative minor, and E♭ tonic of the relative major, key.

Leaders B A♭.—Here again we are familiar enough with the major functionary (B), and need only to observe how the same character of leader is natural to A♭ in the minor key; just as the upward measured seventh leads to its dominator, C, so the downward measured seventh leads to its dominator, G.

Relatives of the key: A being tonic of the relative minor, and B♭ *dominant of the relative major.*—These ultra-mediants A and B♭ are remarkable as creating a unique interval in the chromatic scale; an interval neither diatonic nor chromatic, but essentially harmonic in character and in fact. This interval is not diatonic, because the two notes A and B♭ do not belong to any single diatonic scale;* it is not chromatic, because they constitute different degrees in the chromatic scale and have not the same letter name: it is harmonic, because it constitutes the centre of the harmonic scale of C, based on D♭ (Diagrams IV and V). This unique semitone between A and B♭ has ratio 27 : 25, and is the largest in the chromatic scale.

Ultra-leaders F♯ and D♭.—These are the chromatic extremes (the flattest and sharpest notes in the key); their character as ultra-leaders is very distinctive, F♯ leading up to the tonic of the new sharp key, and D♭ leading

* In the scales of F and B♭ major, D and G minor, *either* the A *or* the B♭ is a *different note* from the A and B♭ in scale of C. (See below, Diagram VII.)

down to the dominant of the new flat key. Though not included in the major and minor systems (as familiarly used), they are capable of diatonic employment; indeed, this is implied by their appearance in "chromatic" chords.

Before leaving this subject, we must mention a fact of special interest involved in the mathematical symmetry of the scale, viz. that for every possible melody or melodic phrase the scale affords a corresponding antithetic melody or phrase which is a mathematical inversion of the first.

We must remember, however, both in regard to this and all we have been saying of functional characters, that in practice the physical element asserts itself and modifies the effect of these mathematical relationships, making them difficult to appreciate or even obscuring them altogether.

IV. The Unlimited (or Extended) Scale.*

Hitherto we have found it convenient to discuss the principles of scale-order only in connection with the key of C. These principles, of course, apply equally to all keys.

For example, any note † in the chromatic scale of C may become the keynote of another scale, identical in order, symmetry, and ratio. And this process might be repeated to any extent, for the musical scale is quite unlimited, extending in fifths and thirds indefinitely, its universal order and symmetry transcending all tonal systems.

The *nucleus* of this unlimited scale is the twelve-note chromatic scale (or tonal system), the boundary of which (as we shall see) is very strictly determined. Its measurements of fifths and thirds, indefinitely extended, create innumerable tonal systems which interpenetrate one another, and therefore every tonal system is surrounded by a field of notes, extraneous to itself, but having a definite relationship and affording unlimited scope for modulation.

* *I.e.* "unlimited" mathematically, though in practice merely "extended."

† Excepting, of course, the keynote C.

[Diagram

Diagram VII.

The Extended Scale.

The numbers give the pitch of each note in 600ths of an octave, as from centre (C).

G̀♯ 375	D̀♯ 126	À♯ 477	E♯ 228	B♯ 579	F× 330	C× 681	Ǵ× 432	D́× 183	Á× 534
È 182	B̀ 533	F̀♯ 284	C♯ 35	G♯ 386	D♯ 137	A♯ 488	É♯ 239	B́♯ 590	F́× 341
C̀ 589	G̀ 340	D̀ 91	A 442	E 193	B 544	F♯ 295	Ć♯ 46	Ǵ♯ 397	D́♯ 148
À♭ 396	È♭ 147	B̀♭ 498	F 249	C 0	G 351	D 102	Á 453	É 204	B́ 555
F̀♭ 203	C̀♭ 554	G̀♭ 305	D♭ 56	A♭ 407	E♭ 158	B♭ 509	F́ 260	Ć 11	Ǵ 362
D̀♭♭ 10	À♭♭ 361	È♭♭ 112	B♭♭ 463	F♭ 214	C♭ 565	G♭ 316	D́♭ 67	Á♭ 418	É♭ 169
B̀♭♭♭ 417	F̀♭♭ 168	C̀♭♭ 519	G♭♭ 270	D♭♭ 121	A♭♭ 372	E♭♭ 123	B́♭♭ 474	F́♭ 225	Ć♭ 576

Showing part of the unlimited musical scale, extending in fifths and thirds from the central nucleus, and comprising thirty-five tonal systems.

Every note represented is different in pitch, and all lie within the compass of one octave. Those marked (\`) are flatter and (´) sharper by one "comma" (one-fifth of a semitone) than the notes of corresponding name in the centre. (See note above diagram.)

The diagram indicates the relationship of all keys in general use, and shows at a glance the true notes of each chromatic scale.

Tonality.

We have said that the scale-diagram is an index of tonality, and we must now explain this, but we shall not discuss the subject any further than that purpose requires.

Tonality may be defined as that predominating relationship (of notes, chords, and keys) to the tonal centre which gives coherence and stability to a musical composition. The essence of tonality, therefore, is simplicity of tonal measurement.

Let us examine the tonality of the scale-notes on this basis.

The notes of strongest tonality will certainly be those of the central interval (C G, Diagram II), and of these two the keynote (C) is the stronger; we therefore mark them 1, 2, as in the centre of Diagram VIII.

Next will come the fundamentals F and D; we take F first, as measured immediately from the tonic, and D second, as measured from the dominant; 3, 1, 2, 4, as shown in the middle row.

Now come the mediants, E and E♭, or 5, 6, first the upward-measured third and then its co-functionary.

Then the leaders A♭ and B, or 7, 8.

There now remain the extremes, and we deal with them in the same way, taking first the more upward measurement from the tonic (A), and then its co-functionary (B♭), 9, 10 in the diagram.

And lastly, the chromatic extremes, D♭ and F♯, measuring first from tonic and then from dominant, 11, 12 as in the diagram.

Diagram VIII.

Tonality of Scale-notes.

1. Tonic . . C
2. Dominant . G
3. Subdominant . F
4. Supertonic . D
5. Major mediant E
6. Minor mediant E♭

9	5	8	12
3	1	2	4
11	7	6	10

7. Minor leader . A♭
8. Major leader . B
9. Submediant . A
10. Minor seventh B♭
11. Flattest note D♭
12. Sharpest note F♯

The whole subject of related keys, major and minor, may be studied with great advantage along the lines of Diagram VII.

The fact of supreme significance, which we have tried to expound in this short treatise, is the tonal order of the chromatic scale, and its identity, on one hand, with the Ptolemaic ratios, and on the other, with the select harmonic series.

We have indicated only a small part of all that this involves, but enough, we trust, to invite a wider study of the musical scale, and a closer inspection of "the wonders of its hidden order."

(Issued separately January 22, 1921.)

XIX.—**Thermometer Screens.** By the late **Dr John Aitken**, F.R.S. (The paper was finished shortly before his death.) *Communicated by* Dr C. G. Knott, F.R.S.

(MS. received September 10, 1920. Read January 10, 1921.)

It is now thirty-two years since I last brought this subject before the Royal Society of Edinburgh. Since then nothing has been done to remedy the imperfections which were then shown to characterise the thermometer screens used throughout the country. In some respects these imperfections may seem to be of small importance. Yet in these days when accuracy of measurement is being more and more insisted on, and particularly now when reconstruction is everywhere in vogue, it does seem desirable that apparatus largely used by thousands of observers should be as perfect as it is possible to make it.

In my previous papers * it was shown that in sunshine the Stevenson screen always gave too high readings. The trouble taken to get Kew corrections for thermometers whose errors in that respect rarely exceed 0°·2 or 0°·3, and the application of the corrections to readings obtained under conditions in which they may be easily 2° or 3° in error, do not commend themselves as worthy examples of scientific method—it is very like a case of straining out a gnat and swallowing a camel.

Having changed to other surroundings since the former series of observations were carried out, I determined to repeat the work in the altered conditions; but it was not till after the middle of August of this year (1919) that I was able to instal all the necessary instruments and make comparisons. Soon after I came to Ardenlea, the Stevenson screen had been fitted up on a lawn-tennis green more or less surrounded by trees, but subject nevertheless to a fairly free circulation of wind. On this green the new tests and comparisons were made.

Taking the temperature of the air seems to be, at first sight, a very simple matter. It is, however, very far from being simple; and, as a matter of fact, no one has ever attempted to define what we are to measure.

The fine-bulb † thermometer used in previous tests showed that, when

* *Proc. Roy. Soc. Edin.*, vol. xii, 1882–84 ; vol. xiii, 1884–86 ; vol. xiv, 1886–87.

† The bulb is cylindrical, with a length of 25 mm. and a diameter of a little over 1 mm. It is provided with a sheath of pure silver which fits it closely. The readings of this instrument when simply placed under a sunshade were found to agree very closely with those given by shaded thermometers in a steady draught of air. It may be regarded as giving very nearly the true temperature of the air. See *Proc. R.S.E.*, vol. xii, p. 688, 1882–84.

there is any radiation, the air passing the thermometer bulb is very variable as regards temperature, rising and falling quickly, changes of 1° F. being frequently observed to occur in the thermometer during five seconds. A thermometer with an ordinary-sized bulb similarly placed responds very slowly to these fluctuations of temperature in the air. Even the fine-bulb thermometer does not, in fact, show the extreme variations of temperature in the air. Sensitive though it is, it has what might be called a "thermal inertia" * of appreciable amount. If the thermometer were smaller and of less thermal inertia, its up-and-down movements would be greater. Probably an electric resistance thermometer with a very fine short wire would respond more quickly to the variations of air temperature. Every thermometer will have its own thermal inertia determining how quickly it will respond to, and how far it will follow after, a temperature change in the air. With solar radiation heating all kinds of matter at the earth's surface, the air is full of heated currents rising and spreading more or less irregularly. What are we then to understand by the temperature of the air at a given time and place, and how are we to measure it? It is obviously impossible to measure the temperature of the hottest whiff of air; for a registering thermometer of no thermal inertia is unthinkable. Every thermometer must give its own averaging of the varying temperatures to which it responds more or less sluggishly according to its own thermal inertia.

For example, let the fine-bulb thermometer and an ordinary thermometer with a bulb of 5 or 6 mm. diameter be exposed under a sunshade and let their indications be observed for a few minutes. While the fine-bulb readings are subject to continual ups and downs, the other responds much more slowly, neither going so high nor falling so low. If the temperature is rising on the whole through a series of variations, the average of the fine-bulb readings will be above the average of the other; and if the temperature is falling as a whole, the fine-bulb thermometer will give the lower average. (See paper on "Thermometer Screens," part iv, *Proc. Roy. Soc. Edin.*, vol. xiv, 1886–87, for detailed experiments.)

Thus there is nothing really determinate until we fix on a definite size of bulb for the thermometer, and make the thermal inertia of the screen as small as possible. If the thermal inertia of thermometer and screen is very small, the thermometer will give nearly the average temperature of a small

* [Dr Aitken introduced the word "inertia" to mean the slowness of response of any body to a heating or cooling process. It depends on many factors, such as thermal capacity, absorption, radiation, conductivity, etc. I have taken the liberty of prefixing the word "thermal" in all cases, so as to prevent confusion.—C. G. K.]

amount of air. If the thermal inertia is very large, the thermometer will give roughly the average temperature of a greater amount of air than in the former case. Large bulbs and certain forms of screen occasion a further source of error on account of radiation effects.

With these considerations as a guide, two exactly similar Fahrenheit thermometers with bulbs 10 mm. in diameter were used. The scales were wide and easily read, and the Kew corrections were very small, never more than 0°·1 at any part of the scale. They were placed under two sunshades of my own design, of which one variety is shown in the accompanying figure (p. 176).

This screen consists essentially of two horizontal boards one above the other, with an air space between, as indicated in the sketch. The thermometer bulb lies a short distance below the lower board, and the stem passes upwards through holes pierced in the boards. This form of screen is more efficient than the Stevenson arrangement. A thermometer protected by it always reads lower than the thermometer in the Stevenson screen when there is any radiation, although the latter may read as low as the former at the beginning of the rise in the presence of sunshine. This is owing to the great thermal inertia of the Stevenson screen. Further, the errors in the readings of the Stevenson screen thermometer go on increasing with the duration of the radiation. There is no such effect in the case of the other form of screen, and all maximum readings taken under it are lower than those taken in the Stevenson screen.

There are several reasons for the temperatures within the Stevenson screen being recorded too high. First, the air comes into the screened region over louvres heated by radiation. Second, the open bottom of the screen allows radiant heat from the soil to enter. Third, the thermometer readings are also raised by the heating of the inside of the roof during continuous radiation. This may be easily remedied by fitting to the screen another cover, with an air space between.

We return to the experiments with the two thermometers. These were placed with their bulbs about 4 cms. below the underside of the screen. The readings were always the same even under the condition of strong sunshine.

The bulb of one of the thermometers was now coated with aluminium paint, which is known to have small absorption powers for heat, and placed under the sunshade alongside the other thermometer. The readings of the two remained practically the same. Gold paint was then tried, with the result that the reading of the thermometer with the gilded bulb fell to nearly one degree below the other, showing therefore considerably less absorption of radiant heat. Compared with the silver-sheathed fine-bulb thermometer, the gilded-bulb instrument read about 0°·5 higher. The next protection tried was gold leaf. The bulb and part of the stem were dipped into a thin solution of spirit varnish, taken out, shaken to get rid of the last drop, and then turned bulb up so as to have as thin a layer of varnish as possible. A piece of gold leaf previously prepared was placed on the bulb, and was at once drawn round it by the capillarity of the varnish. A blast of air immediately applied caused the creases to fold closer to the glass. After drying for two or three days, the gilded bulb was rubbed with a fine brush and the superfluous leaf removed. This gold-gilt thermometer when tested against the silver-sheathed fine-bulb instrument gave the same reading when the air temperature was fairly steady. It was not, of course, so sensitive to rapid changes.

In the following tests the gold-gilt thermometer was used as a standard, since its thermal inertia was about the same as that of the other thermometers in use. The silver-sheathed thermometer was, however, always placed alongside as a check on the gold-gilt one.

The thermal inertia of the Stevenson screen is very marked. It lags behind both in a rising and in a falling temperature, the enclosed thermometer reading lower in the former case and higher in the latter than a freely exposed shaded thermometer.

There is also increased thermal inertia in the maximum and minimum thermometers, which are usually constructed with heavy metal frames. These are nearly in contact with the bulbs, thereby checking the free circulation of the air. This adds needlessly to the thermal inertia, as direct testing clearly proves. Maximum and minimum thermometers should have the frames cut away all round the bulb to give free access of air.

All screens must have more or less thermal inertia, but we should endeavour to keep this factor as low as possible and effect a reasonable compromise. The thing to be avoided is too great thermal inertia, especially when combined, as in the case of the Stevenson screen, with the "bottling up" of heat consequent on the long-continued action of radiation.

As clearly established by prolonged observations, the simple double-board sunshade with the thermometer bulb freely exposed under it, as shown in fig. 1, is more efficient than the Stevenson screen, besides being very much simpler in construction. It may, however, be greatly improved. Thus radiation from below can be intercepted by a small horizontal screen *a* placed under the bulb, as shown in fig. 2; and secondary radiation from the lower surface of the shade proper can be intercepted by a small screen, similar to *a*, and placed above the bulb. Finally, the radiation from more distant objects or from the sky on the sides of the apparatus can be intercepted by means of a small annular piece with wide central hole surrounding the bulb, as shown in fig. 3. It should be grooved on the outside vertical rim so as to prevent the air heated on it from entering the bulb enclosure. Another obvious improvement would be to make each of the small disks *a* of two thin parallel disks with air space between, in place of one thickish disk of wood.

With all these precautions there still remains some secondary radiation which, falling on the bulb, causes the thermometer to read too high by an amount depending on the size of the bulb and the nature and thickness of the glass. No amount of air circulation can check this heating effect. For example, if we take two similar thermometers and protect one with a silver sheath, and then sling them in the usual manner so as to cause them to move rapidly through the air, we find that the unprotected thermometer may read as much as a degree higher than the other. This is due to radiation entering the unprotected bulb. The best protection is, of course, to gild the bulb. This plan, though satisfactory for experimental purposes, must, I fear, be dismissed as unpractical under ordinary working conditions, since it would be impossible to keep all the bulbs constantly bright.

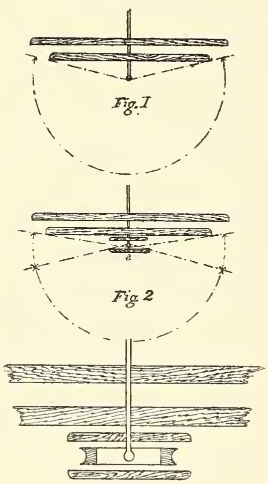

Fig. 3.

In addition to the ordinary measurements of temperature by means of shaded or protected thermometers, it is important to record measurement of radiation of direct sunshine. This is usually done by means of a black-bulb thermometer; but there are advantages in the use of an instrument first described in my early papers on Thermometer Screens already referred to, and used in connection with my researches on Dew. (See "Thermometer Screens," *Proc. R.S.E.*, vol. xii, 1883–84; also paper "On Dew," *Trans. R.S.E.*, vol. xxxiii, 1885–86.) This instrument is called the radiation box, R.B., and its indications are compared with the readings of the black-bulb thermometer, B.B.

One of the radiation boxes with its enclosed maximum and minimum thermometers is placed horizontally on a light wood frame with its surface at the same height as the thermometers in the screen. During the day the maximum radiation temperature is recorded, and the minimum during the night. During summer its average maximum is higher than the black bulb *in vacuo*; but in winter it is lower owing to the low sun, so that the rays fall very obliquely on the surface of the radiation box. For no particular height of the sun, however, do these two types of instrument agree in their maximum readings, since the radiation box is greatly affected by the wind. The temperature of the radiation box is also raised by long-wave radiations from sky, cloud, and surrounding objects. As already pointed out in previous communications, the indications of the radiation box are more in accordance with climate than are the readings of the black bulb.

The accompanying table (p. 180) contains observations from August 17, 1919, to October 31 of the same year.* A brief discussion of these will illustrate the points now being emphasised.

To all temperature readings the Kew corrections of the thermometers have been applied. The observations here recorded were all made at 9.30 a.m. The first item entered for each day is the direction and strength of wind. Then follow the temperature readings of the thermometers in (1) the Stevenson screen; (2) the screen C as shown in fig. 2, without the annular piece x; (3) the difference between these; (4) the temperature registered by the thermometer in screen D, which was provided with a vertical draft tube. Then follow the readings of the radiation box, R.B., and the black bulb *in vacuo*, B.B.; and finally a general description of the weather for the day.

An inspection of the table will establish clearly the following points:—

1. The thermometer in the Stevenson screen reads always higher than the others.
2. The error of the Stevenson screen thermometer is greatest when the R.B. and B.B. readings are high and there is an absence of wind.
3. The error is small when the weather is dull and the R.B. and B.B. readings are slightly above the temperature of the air.

These features are brought out very strikingly in the week beginning September 20. The fine, sunny, comparatively calm weather of the 21st and 23rd days are associated with large errors in the Stevenson screen thermometer; while the errors are very small on the contiguous days of cloudy and stormy weather. The long stretch of fine weather beginning October 6 and ending October 16 is characterised by large errors of the Stevenson screen thermometer and high readings of the radiation box.

[The average results may be brought out very simply by forming the data into three groups: (1) those in which the error of the Stevenson screen thermometer is 1·5 and upwards; (2) those in which the error lies

[* In the original notebook from which these observations are taken, the daily observations continue till November 8, six days before Dr Aitken's death.]

between 1·0 and 1·5; (3) those in which the error is less than 1. Tabulate alongside each of these numbers the corresponding excess of the R.B. reading over the air temperature. Adding and taking the average values for each group, we find as follows:—

	Average Error.	Average Excess.	No. of Cases.
Group (1) .	2·8	47	33
Group (2) .	1·1	36	7
Group (3) .	0·4	22	30

Comparing groups (1) and (3), we see that with an average increase of 25° in the excess of the R.B. temperature above the air temperature, the average error of the Stevenson screen thermometer is increased sevenfold. From the middle of August to the end of October nearly 50 per cent. of the morning readings of the thermometer in the Stevenson screen were in error by more than 1°·5, the average being 2°·8.—C. G. K.]

In addition to the regular morning and evening readings, many others were taken at different hours during the rise of temperature, and these all tell the same tale as to the lag in the records of the Stevenson screen thermometer.

A comparison of the figures given here with the results given in previous papers shows that the conditions at Ardenlea were much more trying for the Stevenson screen than those at Darroch, where the early observations were made. The situation at Darroch had a much freer exposure; but there are sure to be some situations with a worse exposure than the Ardenlea tennis lawn. It is only on the top of a hill or on a wide perfectly open space that the temperature within the Stevenson screen can be kept down to near the true air temperature. Even at the Ben Nevis Observatory the tests showed that the Stevenson screen was influenced by radiation.

There are still two points to be settled with regard to the C screen, namely, the size of the upper screen and the size of the bulb of the thermometer. Both must be standardised if different screens are to give comparable and concordant readings. The screen used in the present investigation was small, the upper protecting screen being a circular disk 19 inches (48 cm.) in diameter, and the lower 14 inches (36 cm.). The two under screens *a* and *b* were 4·6 inches (11·6 cm.) in diameter. With so small a screen the sun's rays at early morning and late evening will fall on the bulbs and tend to heat them. This does not, however, happen near the hour of maximum temperature for the day; and up to November it had no influence on the maximum recorded during the heat of the day. A low midwinter sun might possibly have some effect.

To test the point here raised a larger rectangular screen was set up. The top part measured 30 by 21 inches (76 by 53 cm.) and the under part 24 by 15 inches (61 by 38 cm.). The screen was placed with its length E. and W. to shut out the morning and evening rays. The small under screens were the same as in the other case. So far very few observations have been made; but there is no evidence of any difference in the behaviour of the large and small screens.*

Before concluding I would like to refer again to night temperatures taken on grass. The present haphazard method is of little value if the measurements made at different stations are to be taken as true indications of the intensity of the night radiation at these stations. The variables are far too numerous. Thus a thermometer placed on a small grass surface surrounded by bare ground reads higher than one placed on a large grass surface, owing to the drifting in of the warmer air over the bare ground. A thermometer placed on thin turf reads higher than one placed on thick, mossy turf, because it receives more heat from underneath. When placed on turf protected from wind, the thermometer reads lower than one more freely exposed to air currents, which mix the higher warmer air with the colder air at the grass level. Since it is impossible to have the same conditions at all stations, some less objectionable method is desirable.

I have previously advocated the use of the radiation box placed at the height of the screen thermometers. It is not affected by variations in the amount of heat received from beneath the grass thermometers. It minimises the effect of the variations due to the extent of surface under grass and the variations due to wind. At night the R.B. always reads much lower than the grass thermometer. It must however be standardised. All radiation boxes must be of equal size. The ones I use are 36 cm. to the side, for the simple reason that tin plates can be easily got of that size. Any other convenient size might, however, be adopted. Again, the weight and kind of metal used for the top surface and for the tubes holding the thermometers must also be standardised, so that the thermal inertia may be a definite quantity; and similar precautions must be taken as regards the depth of the box, the amount and nature of the non-conducting packing, the size and shape of the thermometers, and the like. These precautions were taken in the construction of a number of boxes used in this district, and all gave the same radiation temperatures.

[* Further work as planned was interrupted by Dr Aitken's death.]

Table comparing the Action of the Stevenson Screen with that of Aitken's Screen C, etc.

Date.	Wind.	Steven-son.	C.	Differ-ence.	R.B.	B.B.	Weather.
Aug. 17	SW ·5	73·7	72·0	1·7	130·1	123·5	Fine, sunny, strong SW
„ 18	SSW 1·	66·8	65·4	1·4	117·0	119·0	Cloud and sun, strong SW
„ 19	SW 1·	67·7	66·0	1·7	130·6	124·8	Cloudy to fine, SW 2
„ 20	Calm	65·8	64·0	1·8	120·5	117·0	Variable, showery, WSW 1·
„ 21	SW ·1	61·9	59·6	2·3	121·5	118·0	Cloudy to fine, SW 2·
„ 22	SW ·5	65·8	63·9	1·9	127·2	117·6	„ „ SW 3·
„ 23	SW ·2	64·8	63·0	1·8	124·0	117·4	„ „ WSW ·5
„ 24	Calm	66·7	62·4	4·3	138·7	121·0	„ „ calm
„ 25	E ·1	54·3	54·0	0·3	67·0	69·0	Light drizzle all day, ENE ·2
„ 26	Calm	60·0	59·4	0·6	84·5	86·8	Variable, SW ·2
„ 27	NNW ·5	59·6	58·4	1·2	103·0	110·0	„ , little sun, WSW 1·
„ 28	Calm	54·7	53·3	1·4	89·2	80·0	Cloudy, calm
„ 29	WSW ·5	62·9	60·5	1·4	123·0	117·0	Cloudy to fine, WNW 1·
„ 30	SW 1·	60·2	58·3	1·9	113·0	113·0	„ „ W ·5
„ 31	N ·5	62·9	60·1	2·8	131·8	122·0	„ „ W 1·
Sept. 5	S ·5	72·0	70·0	2·0	119·5	118·2	Fine but stormy, S
„ 6	SW ·5	64·8	63·3	1·5	102·0	114·0	Cloudy, SW 1·
„ 7	WSW 1·	66·7	65·0	1·7	121·2	120·0	Fine, SW 1·
„ 8	SW ·5	60·7	...	...	83·2	81 6	Dull all day, S ·5
„ 9	SW ·5	63·9	63·3	0·6	82·4	78·2	Dull, SW ·5
„ 10	SSW 1·	69·8	69·1	0·7	111·2	117·4	Fine, SW 1·
„ 11	Calm	65·9	65·1	0·8	90·0	85·3	Dull
„ 12	E ·2	50·6	50·9	−0·3	57·0	59·5	Very dull, drizzling. Dull till eve, E ·2
„ 13	Calm	56·4	56·4	0·0	95·2	103·0	Changeable
„ 14	„	64·8	61·1	3·7	115·2	102·3	Fine, calm, cloudless
„ 15	„	67·8	64·5	3·3	117·0	106·2	Cloudless, SW ·2
„ 16	SSW ·2	64·9	63·6	1·3	106·4	102·5	Fine but cloudy
„ 17	SW 1·	63·1	62·8	0·3	95·0	92·0	Dull, SW 1·
„ 18	SSW 2·	62·6	61·2	1·4	102·6	105·0	Showers and stormy
„ 19	N 1·	52·9	...	...	104·0	104·0	Cloudy to fine, SW
„ 20	WSW ·5	55·7	55·0	0·7	109·1	105·5	Cloudy to fine
„ 21	W ·2	59·9	56·3	3 6	111·3	107·2	Fine, sunny, W ·2
„ 22	S 1·	55·6	55·4	0·2	92·2	60·0	Raining, stormy
„ 23	WSW ·2	58·6	55·6	3·0	113·6	106 3	Fine, sunny, W ·2
„ 24	SW 2·	58·3	58·0	0·3	78·9	86·2	Showery, stormy, SW
„ 25	WSW 3·	59·6	59·2	0·4	68·2	79·2	Drizzling, dull, SSW ·4
„ 26	SW 1·	54·5	54·7	−0·2	94·2	101·7	Cloudy, stormy
„ 27	W ·2	58·5	56·0	2·5	103 5	105·0	Fine, W ·2
„ 28	Calm	55·5	54·0	1·5	105·7	101·0	Fine a.m., dull p.m.
„ 29	SW 1·	52·6	52·7	−0·1	69·5	63·0	Dull, drizzling all day
„ 30	SSW 1·	58·9	58·2	0·7	86·6	87·6	Dull day
Oct. 1	Calm	53·5	53·0	0·5	74·5	71·0	Dull, drizzling, SW ·2
„ 2	SW ·5	60·5	58·7	1·8	108·4	103·0	Fine, SW 1·
„ 3	SW ·5	58·2	58·0	0·2	67·0	75·0	Dull all day, SW ·2
„ 4	SSW 1·	64·8	64·0	0·8	99·5	104·0	Hazed sky; calm

Date.	Wind.	Stevenson.	C.	Difference.	R.B.	B.B.	Weather.
Oct. 5	SW ·2	61·7	60·7	1·0	76·2	74·4	Dull day
„ 6	SW 1·	68·1	66·2	1·9	102·0	104·0	Fine, E ·5
„ 7	Calm	60·0	56·5	3·5	83·0	85·5	Fine, calm
„ 8	SW ·5	66·4	64·4	2·0	99·0	106·0	Cloudy to fine, W ·5
„ 9	W ·2	57·6	53·0	4·6	100·0	96·0	Fine, calm
„ 10	SSW ·5	56·1	54·0	2·1	85·0	80·0	Fine, E ·2
„ 11	W ·2	54·9	51·2	3·7	102·1	95·2	Fine, calm
„ 12	W ·2	50·6	50·8	−0·2	72·7	89·4	Fine a.m., dull p.m., NW ·5
„ 13	N ·2	55·6	54·2	1·4	94·4	92·5	Most dull and calm
„ 14	W ·2	52·6	49·3	3·3	87·0	86·4	Fine a.m., cloudy p.m., N ·2
„ 15	W ·2	52·1	49·0	3·1	80·0	90·0	Fine, NW ·1
„ 16	W ·2	52·6	50·2	2·4	91·0	89·4	Fine a.m., Dull p.m., WNW ·5
„ 17	SW 1·	58·6	57·8	0·8	83·0	86·5	Fine to cloudy, W 1·
„ 18	SW ·5	53·8	53·6	0·2	72·0	73·0	Dullish all day, SW ·5
„ 19	Calm	58·6	57·6	1·0	74·4	82·0	Dullish, SSW ·5
„ 20	SSW ·5	55·7	55·6	0·1	63·2	66·0	Cloudy a.m., rain p.m., SSW ·5
„ 21	SSW ·2	59·3	58·5	0·8	70·3	70·3	Dull, calm, foggy
„ 22	SSW ·2	58·8	58·1	0·7	71·2	73·1	Dullish
„ 23	Calm	55·6	55·5	0·1	59·6	64·0	Very dull, dark day
„ 24	NE ·5	46·8	46·7	0·1	56·7	55·0	Dull day, NNE ·5
„ 25	Calm	52·7	48·2	4·5	76·2	86·4	Fine, calm, cloudless
„ 26	W ·5	52·4	50·0	2·4	75·6	89·3	Fine, NW ·5
„ 27	N 1·	45·9	45·1	0·8	54·2	64·0	Cloudy
„ 28	NW ·5	49·7	47·1	2·6	76·0	77·2	Fine, NW ·5
„ 29	NE ·5	50·4	47·9	2·5	62·3	73·0	Sunny to cloudy, N 1·
„ 30	N ·5	46·5	46·3	0·2	51·5	52·2	Dull day, ENE, 1·
„ 31	Calm	49·9	49·1	0·8	77·3	81·6	Mostly cloudy, NE ·2

(*Issued separately January* 24, 1921.)

OBITUARY NOTICES.

Sir John Jackson, Kt., LL.D. By David Alan Stevenson, B.Sc., M.Inst.C.E.

(MS. received and read June 21, 1920.)

Sir John Jackson, the head of the well-known firm of "Sir John Jackson, Limited," was born at York in 1851. Immediately on leaving school he entered workshops in Newcastle to learn the mechanical side of the profession that he was ultimately to follow, and from there he proceeded to Edinburgh University to study science and engineering generally. At the university he came under the influence of the late Professor Tait, and so impressed was he with the value of the training he received in Tait's laboratory that, when Tait died, Sir John founded the "Tait Memorial Fund" to encourage physical research.

At the early age of twenty-five he obtained his first contract, and soon afterwards he was entrusted with the contract for the completion of Stobcross Docks, Glasgow, a difficult and important work involving the sinking of cylinders in soft ground, at that time a novel form of constructing foundations. This was rapidly followed by such important undertakings as Middlesbrough, Hartlepool, and North Sunderland Docks, the extension of the Admiralty Pier at Dover, the completion of the last eight miles of the Manchester Ship Canal, in connection with which he was knighted, the Dover Commercial Harbour, the foundations of the Tower Bridge, docks at Swansea, Methil, and Burntisland, and a deep lock at Barry. The greatest undertaking in this country he was responsible for was undoubtedly the extension of the Admiralty Works at Keyham, a work of peculiar engineering difficulty on account of the great depth to which the foundations had to be carried, which took ten years to complete and cost nearly £4,000,000.

Abroad, no less than at home, he was entrusted with the execution of important undertakings. He constructed the Naval Harbour and Graving Dock at Simon's Town in Cape Colony, important harbour works at Singapore, and a breakwater at Victoria in British Columbia. He advised the Austro-Hungarian Government on the extension of the Arsenal Works at

Pola, and made for the Russian Government surveys and proposals for a second Trans-Siberian Railway, estimated to cost £20,000,000, which was not proceeded with on account of the outbreak of war. For the Spanish Government he constructed the arsenal at Ferrol, for the Chilian Government a metre-gauge railway across the Andes, joining Arica and La Paz in Bolivia, through difficult country with a summit level of 14,000 ft. For the Turkish Government he built the great Hindiad Barrage across the Euphrates, a masonry dam with sluice gates for irrigation, and a navigation lock, which were intended to be the first instalments of a great Mesopotamian scheme, estimated to cost £18,000,000, but which was brought to a stop by the War.

When the question of constructing a bridge over the Channel between Dover and Calais was under discussion, the French engineers who were entertaining the project invited Sir John to confer with them, and he was also consulted in connection with the Nicaraguan Canal scheme. When Messrs D. & C. Stevenson, M.Inst.C.E., brought their scheme for a Forth and Clyde Ship Canal by the Loch Lomond route before him, he entered into it with characteristic energy, and to the end of his life was a strong supporter and able advocate of that scheme. He served on the Royal Commission that was appointed to inquire into the military preparation and other matters connected with the War in South Africa.

When the Great War broke out, he offered his own and his staff's services to the War Office, and his firm were appointed Superintending Engineers to the War Department. Lord Kitchener gave Sir John a practically free hand, with the result that, in an extraordinary short time, quarters for enormous bodies of troops and cavalry were established at such places as Salisbury Plain, Grantham, Purfleet, and Ormskirk, involving the provision of water supplies, electric light installations, roads, railways, sidings, and bridges.

In addition to the knighthood conferred on him in connection with the completion of the Manchester Ship Canal, the University of Edinburgh, his Alma Mater, conferred on him the degree of LL.D. The King of Spain presented him with the Grand Cross of Naval Merit, the Chilian Government with the Order of Merit First Class, and the Duke of Connaught conferred on him the Victorian Order.

He represented Devonport in Parliament as a Unionist from 1910 to 1918, when, Devonport having become a single-member constituency, he resigned his seat.

With such a record of work performed, it is hardly necessary to say that Sir John was a man of great energy and an untiring worker. He was gifted with strong common sense, which made him an efficient organiser and a sound man of business. He had an extraordinary aptitude for getting down to the bottom of any question that came before him, and for facing and overcoming any difficulty, whether engineering or administrative, that he encountered in the course of his work. He had the faculty of drawing round him men of great ability to assist him in his work, and of getting from them ready and loyal assistance. He was deeply interested in the welfare of his workmen, of whom he often had thousands in his employment at one time.

In a word, he was a great and successful Contractor, and the works he executed will form a worthy and lasting memorial of his useful and busy life.

In his private life his friends found him broad-minded, kind-hearted, and generous. He had a large circle of friends to whom he was fond of extending hospitality when occasion offered, and by whom he will be greatly missed. Rowing, yachting, and small-boat sailing were his favourite pastimes, and he was a member of the Royal Yacht Squadron and other yacht clubs.

After the Armistice, Sir John had begun to gather up again the threads of business and prepare for undertaking further large schemes of work, when he died somewhat suddenly on December 14, 1919. He was elected a Fellow of the Society in 1894.

David Berry Hart, M.D., F.R.C.P.E., Librarian, Royal College of Physicians, Edinburgh. By J. W. Ballantyne, M.D., F.R.C.P.E.

(MS. received July 13, 1920.)

It was the scientific aspects of medicine which appealed most strongly to David Berry Hart, and his interest was not limited to the subjects which are usually classified as purely medical; he was therefore in a real sense at home in the rooms of the Royal Society of Edinburgh, and he enjoyed to the full the intercourse with scientists of all departments which was to be had there. From the very first true science dominated all Dr Hart's contributions to medicine, and he was quick to apply every advance in cognate subjects to the elucidation of the problems of obstetrics and gynecology.

Dr Berry Hart was born in Edinburgh in October 1851, both his father's and his mother's families being well known in the city; and he died there on June 10, 1920, after an illness which had only been recognised as serious a few days previously. He was educated in his native town, and he passed through the medical curriculum in the university with distinction, graduating as M.B. and C.M. in 1877. After a short visit to Vienna he began his professional career as Assistant in the Midwifery Department of the University under Professor Alexander Russell Simpson, and at the same time he assisted him in his private practice. Shortly afterwards the association thus set up between Sir Alexander and Dr Hart was firmly established by the publication of a joint work upon an aspect of pelvic anatomy and physiology which was of great importance. From 1877 onwards there flowed from Dr Hart's pen a long series of articles on obstetrics and gynecology, scarcely one of which could be regarded as mediocre and many of which would have sufficed alone to have established his reputation as a scientific worker and deep thinker of a high order. They numbered by the time of his death over seventy, and whilst, as has been said, all were worthy of study, those upon the anatomy and physiology of the female pelvic floor, upon the physiology of the third stage of labour, and upon ectopic pregnancy were outstanding contributions which decided the trend of medical opinion all over the world. In his later years Dr Hart dealt with the fascinating problems of heredity as they appeared in connection with Mendelism, hermaphroditism, and teratology; here he was treating abstruse, difficult, and more purely theoretical questions, but what he had to say was always well worth attending to.

Meanwhile Berry Hart was receiving distinctions of various kinds. At his graduation as M.D. in 1880 his Alma Mater awarded him a gold medal and the Syme Surgical Fellowship, both of them highly coveted honours. He began to lecture outside the university at Surgeons' Hall in 1883, and he continued to do so there till within a few days of his death. He always had a good class, and the thinkers among the undergraduates found in him a most inspiring teacher who spared no pains to make clear the fundamental principles of obstetrics and gynecology. In 1884, when he was appointed Assistant Physician to the Edinburgh Royal Maternity Hospital, and in 1886, when he became Assistant Gynecologist to the Royal Infirmary, the sphere of his activities was immensely enlarged and the clinical aspect of his speciality engaged his attention, with immediate results in his literary output, his contributions to the Obstetrical Society (of which he was President in 1890) coming to be of a more directly practical kind. He had been elected a Fellow of the Royal College of Physicians in 1880, and one of the posts which he held in his later years with great acceptance was that of the Librarianship of the College. He wrote two text-books which had each a large circulation and brought fame to the Edinburgh Medical School. One was his *Guide to Midwifery*, noteworthy for the novelty of its arrangement and views; the other, in the writing of which he enjoyed the collaboration of his friend Dr Freeland Barbour, was the famous *Manual of Gynecology*, which did so much to establish Gynecology as a scientific speciality, and which was translated into three foreign languages.

In addition to the honours which Dr Berry Hart received at home, there were the distinctions which were conferred upon him abroad: America made him an Honorary Fellow of her Gynecological Society, and Germany gave him similar distinctions. He was several times asked to open discussions at annual meetings of the British Medical Association, and he was widely recognised as one of Edinburgh's most learned obstetricians. In addition to these many sides of his life there were others, such as his private practice in his speciality, his examining work both in Edinburgh and at other universities, and his post-graduate teaching; but, above all, he impressed his fellows with the extraordinarily wide outlook of his mind upon all the problems of life, and especially upon those concerned with embryology and generation. He was an attractive speaker upon these subjects; his interest in them was intense; and he brought to their elucidation a contagious enthusiasm and an immense knowledge of recondite matters, the product of a lifetime of close reading of stimulating books in every department not only of medicine but of all the sciences. This personal equipment was easily recognised in the contribution which he made to the *Proceedings* of this Society on the subject of the "free-martin."

Surgeon-General W. C. Gorgas. By Major-General W. B. Bannerman, M.D., D.Sc.

(MS. received November 11, 1920. Read November 22, 1920.)

William Crawford Gorgas was born at Mobile, Alabama, on 3rd October 1854. He was son of Joshua Gorgas, a General of Ordnance in the Confederate States Army, and his mother was also of Southern stock. He was educated at the Southern University, Tennessee, where he graduated A.B. 1875. His medical education was taken at Bellevue Hospital Medical College, New York, where he graduated M.D. in 1879. After a residentship at this college he passed into the U.S. Army in 1880. His first term of service was in Western Texas, where he early came in contact with yellow fever, and himself suffered from an attack of this disease. He became Captain Assistant Surgeon in 1885, Surgeon-Major and Chief Sanitary Officer, Havana, in 1898. For his great work in clearing Havana of yellow fever he was, by special act of Congress, promoted Colonel and Assistant Surgeon-General U.S. Army, and in 1914 he was promoted to be Surgeon-General, and was given the rank of Major-General in 1915. He conducted the administration of the Surgeon-General's office in Washington during the war, and visited France along with the Secretary of War before his retirement under the age rule in 1918.

He was on his way to the west coast of Africa to study yellow fever there when he was stricken down by cerebral hæmorrhage in London in May last. He was treated in the officers' wards of the Queen Alexandra Military Hospital, Millbank, where he was visited by King George, who decorated him with the insignia of K.C.M.G. for his great work in rendering the tropics fit for white men to live in, and there he died in the early morning of 4th July 1920. He was accorded a public military funeral at St Paul's on 9th July, which was attended by representatives of the King, the Army Medical Department, the Medical Department of the Navy, the Presidents of the Royal Colleges of Physicians and Surgeons, the Presidents of the Royal Society of Medicine and the Royal Society of Tropical Medicine and Hygiene, and representatives of many other learned societies. This is probably a unique tribute to one who was not a British subject, but who had made himself worthy of it by conquering yellow fever and malaria, and showing by practical illustration how the tropics may be made fit for a white population to live and work in.

Gorgas came first into public notice on account of his work in clearing Havana of yellow fever after the Spanish-American War. The large influx of non-immune strangers during the war resulted in an outbreak of yellow fever. In 1900 an American Army Commission was sent out to study the disease, of which Dr Walter Reed was the best-known member. Taking their cue from Ross, who four years before had proved that malaria was transmitted from man to man by the bites of Anopheles mosquitoes, themselves infected by sucking the blood of malaria-stricken men, Dr Reed and his assistants turned their attention to the local mosquitoes, which had already, in 1881, been suspected by Dr Charles Finlay of being the cause of spread of yellow fever in Havana. By experiments on themselves they proved that yellow fever was due to the bites of a species of mosquito, *Stegomyia fasciata,* and not to contact with the patient, his clothes, or fomites. They found that the Stegomyia lives for three months. It becomes dangerous only by imbibing the blood of man during the first three days of an attack of yellow fever, and even then twelve days elapse before its bite is infectious; six days after a person has been bitten by an infectious Stegomyia he develops yellow fever, and for the next three days his blood is capable of infecting a Stegomyia. In no other way can yellow fever be spread. The Stegomyia breeds chiefly in or near towns, being a domestic mosquito, in contradistinction to the malaria-carrying Anopheles, which loves clean water and jungle streams. The Stegomyia breeds on the surface of cisterns, stagnant pools, or in old pots in neglected backyards.

This discovery was at once seized on by Gorgas, who was Chief Sanitary Officer of Havana. Fortunately the Governor was General Wood, who had studied medicine for several years, and he gave Gorgas a free hand, and he at once turned the whole energies of the organised sanitary department to rid Havana of the infected Stegomyia. The patients were screened in hospitals, so that mosquitoes could not get at them. All breeding-places, such as cisterns, were screened, tubs and tanks, old pots, etc., were emptied, and severe fines were imposed on persons with open water on their premises. The methods were described by Gorgas' opponents as ruthless, but in a few months Havana was free of yellow fever. It remained free for three years. "The government was then turned over to the Cubans," and yellow fever reappeared. In 1906 the United States again assumed control, the same sanitary methods were reapplied, and yellow fever again disappeared.

For this triumph of applied science Gorgas was promoted to be Colonel and Assistant Surgeon-General by a special act of Congress.

When the United States took possession of the Panama Canal Zone in 1904 the great problem was, How can the health of the thousands of workmen be preserved? As Gorgas said, the experience of the French and other predecessors proved that "unless we could protect our force against yellow fever and malaria, we would be unable to accomplish the work" of constructing the canal.

The problem of the Panama Zone differed from that of Havana, in that the canal extended for some fifty miles from north to south, through country covered with swamps and tropical vegetation, and with two good-sized towns, Colon and Panama, at either end. Yellow fever was to be feared in the towns where the Stegomyia breeds, and malaria in the country districts where the Anopheles abounds. It took sixteen months to rid the Zone of yellow fever as against seven in Havana, and Gorgas points out that this was due to the strictness of military rule in Havana, and the absence of this at first in Panama. It was not until President Roosevelt peremptorily interfered, and sent Judge Magoon with arbitrary powers to help Gorgas, that real progress was made, and this was only done under pressure when yellow fever broke out in the administrative building at Panama, and the Americans in panic were leaving in large numbers. When Gorgas was given a free hand things began to move, and at one time an army of 4100 men was employed in the sanitary department. Four hundred men were employed in Panama alone in fumigating the houses, which were done three times, 200,000 pounds of pyrethrum powder and 400,000 pounds of sulphur being used. The entire surface of the ground was paved or cemented, the houses were screened and properly drained, a constant water-supply was instituted, and no cisterns were allowed. So strict were the rules that even water-jugs were prohibited, all water being drawn direct from the tap. In a few months yellow fever disappeared, and has not since made its appearance in the Zone.

In the same energetic way the malaria problem of the intervening country was tackled. Swamps were drained, ditches lined with cement, underground tile drains laid, water that could not be removed was oiled, houses were screened, and the population treated with quinine. In this way Gorgas converted the "white man's grave," as Panama has been called, into one of the healthiest parts of the world, where the death-rate from all causes was about 10 per 1000 in 1908.

As Gorgas himself says, "I think the sanitarian can now show that any population coming into the tropics can protect itself against those two diseases [malaria and yellow fever] by measures that are both simple and inexpensive; that with these two diseases eliminated, life in the tropics for the Anglo-Saxon will be more healthful than in the temperate zones; that gradually within the next two or three centuries tropical countries, which offer a much greater return for man's labour than do the temperate zones, will be settled up by the white races; and that again the centres of wealth, civilisation, and population will be in the tropics, as they were in the dawn of man's history, rather than in the temperate zone as at present."

Whether this theory will prove correct time alone can show, but that it has been put forward shows what the labour of organised sanitation can do when directed by such a man as Gorgas.

Gorgas was a member of many scientific societies, and received many honours during his life. He was awarded the Kingsley Gold Medal by the Liverpool School of Tropical Medicine in 1907; he was made Honorary D.Sc. of Oxford in 1914; Cambridge made him an LL.D. this year; and a few weeks before his death the King knighted him.

He was made Hon. Fellow of our Society in 1916.

He was not a scientist in the strict sense of the word, perhaps, and he calls himself "a working doctor," yet he accomplished more than many an original investigator. He drove yellow fever out of Havana; he made Panama, "the white man's grave," one of the most healthful places in the world; and he was responsible for the medical administration of the American Army, with one of the lowest mortality records heretofore achieved. Yet with all this driving force and fighting ability "he was a man of soft word and kind thought, seeking counsel and gathering around himself associates whom he trusted and who put their trust in him. He made many friends through his ever-gracious manner."

Obituary Notices of Fellows, Honorary and Ordinary.
By The Assistant Secretary.

Erskine Beveridge, LL.D. (St Andrews), was educated at Edinburgh University, and succeeded his late father as principal of the linen manufacturing firm of Erskine, Beveridge & Co., Ltd., Dunfermline. Following upon extensive research in the Western Islands of Scotland, one of which —Vallay, in the Outer Hebrides—he subsequently purchased, he published in 1903 *Coll and Tiree,* in which he dealt in exhaustive and interesting fashion with the prehistoric forts and ecclesiastical antiquities of these places, and gave particulars with regard to ancient remains in the Threshnish Isles. He also published in 1893 *The Churchyard Memorials of Crail,* and was the author of *A Bibliography of Dunfermline and West Fife,* and *Burgh Records of Dunfermline from 1488 to 1584.*

He was elected a Fellow of the Royal Society of Edinburgh in 1904, and died on August 10, 1920.

J. Hamilton Fullarton, D.Sc., was born in Arran in 1856. Deceased was one of Glasgow University's most distinguished students, and later a Lecturer in Biology in the University. He was a recognised expert in all fishery matters, his writings on the subject being translated into almost every European language. For some time he acted as Superintendent of the Scottish Fishery Board Laboratory. Dr Fullarton was widely travelled, but in later years had settled in London, where he built up a large consulting practice.

Dr Fullarton was a man of the highest character, proud of his Highland birth, and a great fighter for the cause of the crofter. He was elected a Fellow of the Royal Society of Edinburgh in 1891, and died in London on May 6, 1920.

Richard D. Graham was elected a Fellow of the Royal Society of Edinburgh in 1891, and died at 12 Strathearn Road, Edinburgh, in his eighty-fifth year, on September 12, 1920.

Charles Edward Green, head of the well-known publishing firm of Messrs William Green & Son, was born in Edinburgh fifty-four years ago. He began his career as a medical student at Edinburgh University, but the death of his father made it necessary for him to abandon the medical course and devote his energies to business. The small concern of which he assumed the management was, by his ability and enterprise, steadily extended and developed until it became a great law publishing house. His encyclopædias of law and medicine are familiar to all practising lawyers and physicians. His valuable collection of law reports has a place in every legal library, and many standard works written in recent years on law, medicine, accounting, and agriculture were published by him. He was the founder and proprietor of the *Scots Law Times*, the *Juridical Review* (which journal he also edited), and the *Veterinary Review*.

A survival of his early training was shown in his researches into the origin of cancer. He devoted many years to this inquiry, the results of which are embodied in a monograph, *The Cancer Problem*, which aroused great interest among the Faculty and passed through several editions. His contributions to lighter literature are entitled: *Lives in a Lowland Parish* and a *County History of East Lothian*.

He was elected a Fellow of the Royal Society of Edinburgh in 1910, and died at Liberton on 6th January 1920.

[*Contributed by his Widow.*]

Allan M'Lane Hamilton, M.D., was born in Brooklyn, New York, October 6, 1848, the son of Philip Hamilton, the youngest son of Alexander Hamilton the great Federalist, and Rebecca, eldest daughter of Louis M'Lane of Delaware, who was Secretary of the Treasury and of State, and Minister to England, in the cabinets of Andrew Jackson and Martin van Buren.

Dr Hamilton's choice of medicine as a profession dated from a trip to South America made in company with Louis Agassiz in 1865. In 1870 he graduated from the College of Physicians and Surgeons in New York, taking the First Faculty and Harsen prizes. His master was the late Henry B. Sands. From the first Dr Hamilton was interested in the study of nervous diseases and insanity. He was made Physician-in-Charge of the Hospital for Nervous Diseases, and later occupied such posts as Attending Physician to the Hospital for Paralysed and Epileptics on Blackwell's Island, New York, his colleagues being the late Dr E. C. Seguin and E. G. Janeway. For the first twelve years of practice he was on the Board of Health, and gained a large and varied experience in smallpox and cholera epidemics. In 1899 he took the first prize given by the American Medical Association for an essay upon "Diseases of the Lateral Column of the Spinal Cord."

He founded the New York Psychiatrical Society a few years ago;

and was made a Fellow of the Royal Society of Edinburgh and a Corresponding Fellow of the Medical Society of London.

For several years he was Professor of Psychiatry at Cornell Medical College in New York, and was Honorary Member of the American Neurological Association and the New York Psychiatrical Society at the time of his death. A few years ago Hamilton College bestowed a degree upon him.

For many years Dr Hamilton's name was connected with most of the great criminal cases and prominent trials of will cases, and he became an authority on medical jurisprudence. He was also engaged as expert in many railroad litigations due to accidents, and wrote a book on *Railway Accidents and Other Injuries.* He was the author of *A System of Legal Medicine,* widely used both in the States and abroad, and was a frequent contributor to the *North American Review,* the *Century,* and *Scribner's Magazine.* He used every effort to reform expert testimony, and to the day of his death maintained that the famous Thaw trial had killed all expert testimony and ruined the career of alienists. He believed that all experts should be selected by Government as purely disinterested witnesses.

Dr Hamilton's first great case as an alienist was that of Guiteau, murderer of President Garfield. Hamilton was called to testify for the Government, and Guiteau was adjudged sane. He was also a prominent figure in the celebrated Maria Barbieri case, the Patrick will case, the Molineux poison case, and that of Csolgosz, the assassin of President M'Kinley.

In the summer of 1907 Dr Hamilton was sent for by Mrs Mary Baker Eddy, founder of Christian Science, to examine her mental condition with a view to testifying as to her capacity to manage her financial and other affairs; and the account of the interviews he had with her is told in his last book, *Recollections of an Alienist.*

For fifty years Dr Hamilton worked in New York, and he had a large private practice for many years until he became absorbed in the larger interests in the law courts.

In 1910 he published the *Intimate Life of Alexander Hamilton.*

At the time of his death last November, he had in mind further "Recollections," and amongst his papers are jottings and notes for an essay on "Four Great Wits."

Dr Hamilton wrote and spoke most strongly against abuses in insane asylums, and after seeing three executions in Sing Sing Prison was an ardent advocate of the abolition of the electric chair. In 1902 he suffered from a severe illness, which led, after years of pain and discomfort, to a major operation in 1913, these being almost the busiest years of his whole career. From 1913 on he never knew what freedom from suffering was, and gradually failed until his release came, November 23, 1919.

As host, friend, companion, and physician, few men could compare with him, and his loss has been deeply felt by those who knew him in Great Britain and America. For thirty years Dr Hamilton had been a member of the Beefsteak Club in London, and one of his greatest griefs in the last year was that he might never again see his old friends there.

Alexander Macalister, M.D., LL.D., F.R.S., was born in Dublin in 1844. He became a Demonstrator in Anatomy at the Royal College of Surgeons in Dublin before he was seventeen years of age. At the age of twenty-five he became Professor of Zoology at Trinity College, Dublin, and eight years later succeeded to the Chair of Anatomy and Chirurgery. In 1883 he succeeded Sir George Humphrey as Professor of Anatomy at the University of Cambridge; the great anthropological collection which he made there will always remain as a memorial of his zeal and energy. Professor Macalister spent his life in amassing facts, and avoided generalisations and the formulation of explanations or theories. His publications are as follows:—*Introduction to Animal Morphology*, 1876; *Morphology of Vertebrate Animals*, 1878; *Text-Book of Human Anatomy*, 1889; *Evolution in Church History*, 1879; *Memoir of James Macartney*, 1900; besides numerous papers and smaller text-books for students in Zoology and Physiology.

He was elected an Honorary Fellow of the Royal Society of Edinburgh in 1916, and died in September 1919.

Edward Pierson Ramsay, LL.D., was born in Australia on December 3, 1842. He was educated in Australia, and was appointed Curator of the Australian Museum, Sydney, in 1874, which post he held with distinction until 1894, when he retired owing to ill health. Dr Ramsay was the author of over one hundred and twenty papers in periodicals, including the *Journal of the Linnean Society of N.S.W.*, *Proceedings of the Zoological Society of London*, and the *Records of the Australian Museum, Sydney.*

He was elected a Fellow of the Royal Society of Edinburgh in 1884, and died in Australia in 1917.

Reginald L. A. E. Westergaard, Ph.D., was a native of Denmark, and a student of the University of Copenhagen. He came to Scotland as scientific expert to the firm of Messrs M'Ewan & Co., Ltd., brewers, and in 1905 was appointed Lecturer in, and later Professor of, Technical Mycology in the Heriot Watt College, Edinburgh. He held this appointment until 1918, when he retired with the intention of again taking up the commercial side of his subject.

Dr Westergaard was elected a Fellow of the Royal Society of Edinburgh in 1909, and published a short paper in the *Proceedings*, vol. xxix, p. 748, "On the Development of Mixed Cultures of Bacteria and Lower Fungi in Liquid and Solid Media." He died at Lundin Links, Fife, on June 19, 1920.

John Hardie Wilson, D.Sc., Lecturer in Agriculture and Rural Economy, St Andrews University, was a native of St Andrews, and from boyhood was an enthusiastic botanist. Several years ago he published a volume entitled *Rambles round St Andrews*, which is a valuable guide to the flora of the district. He laid out the first botanic garden at St Andrews, and was the first Lecturer in Botany at the University. After an interval spent as Botanic Lecturer in Leeds, he returned to his Alma Mater. For the past six years he had been engaged in research work for the Board of Agriculture in Scotland for the improvement of potatoes, turnips, and grain. He was a member of a Scottish Commission which investigated the conditions of agriculture in Australia a number of years ago.

He was elected a Fellow of the Royal Society of Edinburgh in 1891, and died at St Andrews on January 13, 1920.

APPENDIX.

CONTENTS.

PROCEEDINGS OF THE STATUTORY GENERAL MEETING

Beginning the 137th Session, 1919–1920.

At the Statutory Meeting of the Royal Society of Edinburgh, held in the Society's Lecture Room, 24 George Street, on Monday, October 27, 1919, at 4.30 p.m.,

Dr John Horne, F.R.S., F.G.S., President, in the Chair,

the Minutes of the last Statutory Meeting of October 28, 1918, were read, approved, and signed.

The Chairman nominated as Scrutineers, the Rev. R. S. Calderwood and Dr A. Morgan.

The Ballot for the Election of Office-Bearers and Members of Council was then taken.

The Secretary submitted the following Report:—

Since the last Report was submitted the war, which influenced in many ways the work of the Society, has come to an end; one important consequence of this is the return of Mr George Stewart, Librarian and Assistant Secretary, to his work in the Society. With the rearrangement of duties Miss Le Harivel, who has acted during the war as temporary Librarian and Assistant Secretary, has been officially appointed Assistant Librarian. Most of our activities are proceeding very much as during the war. The number of papers read at our meetings during Session 1918–1919 was 23, as compared with 25 the preceding year. Of these 13 have been, or are being, published in the *Proceedings* and 6 in the *Transactions*. Of the papers read 4 were in mathematics, 5 in physics, 2 in meteorology, 5 in chemistry, 3 in zoology, and 4 in botany. An address on Colour Blindness, with Lantern Demonstration, was given by Mr C. R. Gibson of Glasgow.

Last year the Society elected 11 new Fellows, and we lost by death 15 Ordinary Fellows and 4 Honorary Fellows.

The Makdougall-Brisbane Prize was awarded to Professor A. A. Lawson.

With the great increase in the cost of publication, and the loss the Society sustained some years ago by fire, it was evident to the Council that, with the ordinary output of papers and publications, there would be a serious deficit during the Session. At the beginning of the Session there was already £200 of debt to clear off incurred by the fire, and a careful estimate showed that the Council could only use £600 for publication purposes, and would be compelled to postpone the binding of serials and journals. These considerations induced the Council to approach the Chancellor of the Exchequer and ask for an increased annual grant. Before presenting their Memorandum the Council had a meeting with the Secretary for Scotland, and encouraged by the reception given by him, the Council prepared the following Memorandum:—

ROYAL SOCIETY OF EDINBURGH.

Application to the Treasury for an Increased Government Grant.

Memorandum.

The Royal Society of Edinburgh was founded by Royal Charter in 1783, in the reign of George III; *ad Statum illius partis Imperii nostri quæ Scotia vocatur accommodata*. According to the Charter the work of the Society was to include Mathematics, Physics, Chemistry, Natural History, Archæology, Philology, and Literature; but, in comparison with the scientific subjects specified, Literature has long occupied a subordinate position. The Society takes rank as the National Scientific Academy in Scotland, in the same manner as the French Academy of Sciences, the Royal Society of London, the Royal Irish Academy, the National Academy of Sciences of Washington, and other like Institutions in their respective countries. It has been, since its foundation, the centre of scientific activity for the whole of Scotland, and has included among its Presidents such distinguished men as Sir James Hall, Sir Walter Scott, the eighth Duke of Argyll, Sir David Brewster, Lord Kelvin, Sir William Turner, and Professor James Geikie.

The chief aim of the Society is the development of original research in Scotland. Its success in this respect has been remarkable. The *Transactions* and *Proceedings* of the Society have always contained papers of high scientific value, some of which have laid the foundations of new branches of science. In recent years there has been a large increase in its publications, especially in those relating to Natural Science.

From 1826 to 1909 the Society occupied rooms in the Royal Institution. These rooms being required for an extension of the Royal Scottish Academy, the Government purchased and equipped the building in George Street at present occupied by the Society, and agreed to allot an annual grant of £600 to assist its scientific work.

The other main source of revenue of the Society is derived from the contributions of its Fellows. These amount to about £890 per annum. The total income is about £2100. The expenditure may be arranged under three heads:—

(1) Publication of scientific researches.

(2) Upkeep of Library, which, with the exception of that belonging to the Royal Society of London, is the most complete library of scientific reference in the United Kingdom.

(3) Salaries and current expenses.

During the twelve years 1902–1914 the annual sum spent on the *Transactions* and *Proceedings* averaged £1050. Before 1906 there was a constant excess of payments over receipts; the balance being met by gifts and bequests from friends of the Society. From 1906 until the beginning of the war the Society was able, with the aid of its grant from the Government, to cope with the papers presented to it for publication; but since 1914 printing charges have increased 125 per cent., and are not likely to diminish, so that the Society is no longer able to carry on this work efficiently. During the session ending September 30, 1918, the Council has been compelled to decline papers offered to the Society owing to lack of funds available for their publication. This is the more to be regretted since one result of the war has been to show the necessity of encouraging original scientific research in every possible way.

The Council, recognising this necessity, now beg to appeal for an increased grant from the Treasury to enable the Society to meet the enhanced demands made upon its resources. They calculate that an additional sum of £1000 a year will be required in the immediate future, and they therefore ask that the Treasury Grant be increased from £600 to £1600. In this connection it may be pointed out that the Royal Irish Academy, which occupies the same position in Ireland that the Royal Society of Edinburgh occupies in Scotland, receives an annual Government grant of the above amount (£1600). Including this grant, it has a total revenue of nearly £2500, of which about £470 is spent on the publication of scientific papers, and about £400 on literary researches and publications. With a smaller total revenue and a far smaller Government grant, the Royal Society of Edinburgh has expended a much larger amount on the publication of scientific researches. In order to maintain, and still more to improve, its position as a Society for publishing original scientific research in Scotland, an increased grant from the Treasury is essential.

Royal Society of Edinburgh,
22 George Street,
December 30, 1918.

This Memorandum along with a covering letter was sent to all our Ordinary and Honorary Fellows resident in London, also to all the Scottish Members of Parliament. Many cordial replies were received, and generous offers made to help the Society in presenting its case to the Chancellor of the Exchequer. In due course the Council received the following reply from the Chancellor of the Exchequer:—

COPY.

Treasury Chambers,
Whitehall, S.W., 1,
22*nd March*, 1919.

Dear Sir,—The Chancellor of the Exchequer desires me to express his regret that he has not been in a position to reply at an earlier date to your letter of the 21st January last, asking him to receive a deputation from the Royal Society of Edinburgh to urge the increase of the Government grant made to them from £600 to £1600 per annum.

Mr Chamberlain notes that this request is made on account of the increased cost of printing. Increased grants have not, however, been given on this ground in other similar cases.

As regards the comparison with the Royal Irish Academy, he must point out that the grant made to them is aid, not merely of their publication of scientific researches, but also of their literary activities, such as the publication of Irish manuscripts. It should further be borne in mind that your Society's existing grant is more than proportionate to that assigned to the English Royal Society for similar purposes.

Mr Chamberlain finds that whereas until 1907 the Government grant to the Royal Society of Edinburgh was only £300, and was given to cover the rent paid by them, it has since been increased to £600, although the Society has been housed free at a cost of £20,000 from funds provided by the late Board of Manufactures.

In these circumstances, and having regard to the present condition of national finance, Mr Chamberlain regrets that he cannot propose to Parliament an increase in the grant to the Society at the present time, though he would be ready to reconsider the question along with other similar claims when the financial situation is more favourable; and he fears that no useful purpose would be served by his consenting to receive a deputation on this subject.—Yours faithfully,

R. P. M. Gower.

The General Secretary,
Royal Society of Edinburgh.

The Council replied to this communication in the following terms:—

ROYAL SOCIETY OF EDINBURGH.

COPY.

22 George Street,
April 8, 1919.

Sir,—I am instructed by the Council of the Royal Society of Edinburgh to acknowledge the receipt of your letter dated March 22.

The Council regret that you are unable at the present time to propose an increase of the Government Grant made to the Society, and especially that you decline to receive a deputation on the subject.

I am directed by the Council to point out, in reply to the remarks in your letter about the Royal Irish Academy, that the Council fail to see any difference in principle between the publication of scientific and literary research.

With regard to the other point mentioned, the amount assigned to the Royal Society of London is £1000 for publication and £4000 for direct aid in scientific research. The Royal Society of London have, moreover, much larger invested funds to help in their publication of scientific work. The Council is, therefore, unable to understand how the existing amount granted to the Royal Society of Edinburgh is more than proportionate to that assigned to the English Royal Society for similar purposes. I am further instructed to emphasise the fact that the Society spends all its grant on research—publication being a necessary part of research—and scientific research is acknowledged to be one of the crying needs of the nation. In the meantime all the available funds (for this purpose) are exhausted, and valuable work cannot be published for lack of means.

It should be noted that previously to 1907 the Royal Society of Edinburgh was housed, rent free, in one of the finest buildings of the city, on a commanding site in Princes Street, and was provided with the present house in place of that of which they were then dispossessed.

I have the honour to be, sir, your obedient servant,

C. G. Knott,
General Secretary, R.S.E.

The Rt. Hon. Austen Chamberlain, M.P.,
Treasury Chambers, Whitehall, S.W.

Since it was evident that nothing could in the meantime be effected in the way proposed, one of our Members, resident in London, suggested that the Society should make an appeal to its Fellows to subscribe to a special fund to help the Society over its present difficulties, and at the same time sent £100 as a first contribution. Acting on this suggestion, the Council sent an appeal to all the Fellows of the Society in the following terms:—

ROYAL SOCIETY OF EDINBURGH.

22 George Street,
May 1919.

Urgent.

Dear Sir,—The Council of the Royal Society of Edinburgh made an Appeal last January to the Chancellor of the Exchequer for an increase in the annual grant, in order to enable the Society to publish scientific papers committed to its care. In his reply to our memorandum the Chancellor of the Exchequer referred to several of the important points, and concluded in these words:—"In these circumstances, and having regard to the present condition of national finance, Mr Chamberlain regrets that he cannot propose to Parliament an increase in the grant to the Society at the present time, though he would be ready to reconsider the question along with other similar claims when the financial situation is more favourable."

The great increase in the cost of publication is interfering seriously with the normal work of the Society, and the Council have been considering anxiously how best to meet the situation.

One of our Fellows, resident in London, recently gave a generous donation of £100 so as to help to relieve the financial difficulties in which we find ourselves, and suggested that other Fellows might be in a position to follow his example. The Council have resolved to approach the Fellows in regard to this matter, and have decided to ask for voluntary contributions towards a Special Subscription Fund. They trust that the response to this appeal will be such as to ensure the continued publication of scientific papers during the present session.—Yours very truly,

C. G. Knott,
General Secretary, R.S.E.

Appended is a statement of the amount of money immediately required to place the Society on a satisfactory basis:—

1. Printing and distributing papers still to be published this session	£180	0	0
2. Neill & Co.'s account, 1917—remaining portion of debt	200	0	0
3. Completing the serials got from enemy countries	450	0	0
4. Despatch of *Transactions* and *Proceedings* held up during war	60	0	0
	£890	0	0

It should be further noted that the Binding of Serials is very much in arrears, to meet which a large additional sum will be required.

By September 1919 the whole amount subscribed to the "Special Subscription Fund" was £775, 16s. 6d., and after a second appeal this was increased to £1072, 17s. 6d.

Owing to the generosity of many of its Fellows, the Society has now been able to clear off the debts of former years, and to meet our present expenses for publication. Of the whole sum subscribed to the Special Subscription Fund there remains in hand on October 27, 1919, the sum of £771, which will be carried over to ease the financial stress in meeting the expenses specially referred to in the Second Appeal, viz.:—Completing the Serials from Foreign Countries; the despatch of *Transactions* and *Proceedings* held up during the war, and the printing and distribution of the remaining papers which belong to the last session. It should be noted, however, that the general situation remains as before, viz. that in view of the continuing high prices the present income of the Society is not sufficient for it to continue to publish to nearly the same extent as heretofore the results of scientific research.

During the year the Society appointed two delegates, Sir E. Sharpey Schafer and Dr C. G. Knott, to the Conference of the International Association. The first meeting was held in Paris towards the end of November 1918, and the arrangements made at that time have resulted in the formation of two important International Unions, viz. that of Astronomy and Geophysics. Professor George Forbes and Mr M'Ewan were appointed delegates to the Union of Astronomy, and Dr Knott and Dr Crichton Mitchell to the Union of Geophysics.

Professor Lapworth and Professor Hudson Beare were chosen as representatives of the Society to the Watt Centenary Celebration held in Birmingham in September last.

The Treasurer in submitting his Report for the year compared the Income and Expenditure with those of the previous year, and called attention to the fact that the deficit of £301, 16s. 11d. on the year's working had been met by transferring that sum from the "Special Subscription Fund" to the "General Fund."

Dr E. M. Wedderburn moved the adoption of the Reports, and the reappointment of Messrs Lindsay, Jamieson & Haldane, C.A., as auditors of the accounts for the ensuing Session.

This was unanimously agreed to.

The Scrutineers reported that the Ballot Papers were in order, and that the following had been elected as Office Bearers and Members of Council:—

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President.

Professor G. A. Gibson, M.A., LL.D.,
Robert Kidston, LL.D., F.R.S., F.G.S.,
Professor D. Noël Paton, M.D., B.Sc., LL.D., F.R.C.P.E., F.R.S.,
Professor Arthur Robinson, M.D., M.R.C.S.,
Sir George A. Berry, M.B., C.M., LL.D., F.R.C.S.E.,
Professor William Peddie, D.Sc.,
} Vice-Presidents.

Cargill G. Knott, D.Sc., LL.D., General Secretary.

Professor E. T. Whittaker, Sc.D., F.R.S.,
J. H. Ashworth, D.Sc., F.R.S.,
} Secretaries to Ordinary Meetings.

James Currie, M.A., LL.D., Treasurer.

A. Crichton Mitchell, D.Sc., Hon. D.Sc. (Geneva), Curator of Library and Museum.

ORDINARY MEMBERS OF COUNCIL.

Professor P. T. Herring, M.D., F.R.C.P.E.
Professor T. J. Jehu, M.A., M.D., F.G.S.
Alexander Lauder, D.Sc., F.I.C.
The Hon. Lord Guthrie, LL.D.
Professor R. A. Sampson, M.A., D.Sc., F.R.S.
Professor J. Lorrain Smith, M.A., M.D., F.R.S.
W. A. Tait, D.Sc., M.Inst.C.E.
Maj.-General W. B. Bannerman, C.S.I., I.M.S., M.D., D.Sc.
Henry Moubray Cadell, of Grange, B.Sc.
Professor Arthur Robertson Cushny, M.A., M.D., LL.D., F.R.S.
Principal Sir James Alfred Ewing, K.C.B., M.A., B.Sc., LL.D., M.Inst.C.E., F.R.S.
George James Lidstone, F.F.A., F.I.A.

SOCIETY'S REPRESENTATIVE ON GEORGE HERIOT'S TRUST.

W. A. Tait, D.Sc., M.Inst.C.E.

The Chairman, in the name of the Society, thanked the Scrutineers for their services.

PROCEEDINGS OF THE ORDINARY MEETINGS,
Session 1919–1920.

FIRST ORDINARY MEETING.

Monday, November 3, 1919.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

The President delivered a short Address.

The following Communications were submitted:—

1. The Cooling of the Soil at Night, with Special Reference to Late Spring Frosts. By Capt. T. BEDFORD FRANKLIN. Communicated by THE GENERAL SECRETARY. *Proc.*, vol. xl, pp. 10–22.
2. Note on the Determinant whose Matrix is the Sum of Two Circulant Matrices. By Sir THOMAS MUIR, LL.D., F.R.S. *Proc.*, vol. xl, pp. 23–32.
3. Note and Exhibition of Photographs of Mirage at Ingatestone. By G. F. QUILTER, Esq. Communicated by THE GENERAL SECRETARY. *Proc.*, vol. xl, p. 33.

Mr B. D. PORRITT signed the Roll and was duly admitted a Fellow of the Society.

SECOND ORDINARY MEETING.

Monday, December 1, 1919.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

The following Communications were submitted:—

On Old Red Sandstone Plants showing Structure from the Rhynie Chert Bed, Aberdeenshire. Part III.: *Asteroxylon Mackiei*, Kidston and Lang. By Dr R. KIDSTON, F.R.S., and Professor W. H. LANG, D.Sc., F.R.S. *Trans.*, vol. lii, pp. 643–680.

Two Special Meetings of the Royal Society of Edinburgh were held in the Society's House, 24 George Street, on Monday, December 8, 1919, and on Tuesday, December 9, 1919, at 5 p.m.

Professor William Peddie, D.Sc., Vice-President, in the Chair.

By request of the Council, Professor R. A. SAMPSON, M.A., D.Sc., F.R.S., gave addresses on "The Theory of Gravitation."

8th December, at 5 *p.m.*

1. Newton's Views and their Subsequent History.

9th December, at 5 *p.m.*

2. The Theory of Einstein, and its Observational Tests.

THIRD ORDINARY MEETING.

Monday, January 12, 1920.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

The following Communications were submitted:—

1. Obituary Notices of Fellows deceased during the Session 1918–1919. By THE GENERAL SECRETARY.
2. The Atomic Space Lattice in Magnetite. By Professor W. PEDDIE, D.Sc.
3. An Unnoticed Point in the Theory of Newton's Rings. By JOHN MARSHALL, M.A., B.Sc. Communicated by Professor W. PEDDIE, D.Sc. *Proc.*, vol. xl, pp. 51–55.
4. Theoretical Determination of the Longitudinal Seiches of Lake Geneva. By Messrs A. T. DOODSON, R. M. CAREY, and R. BALDWIN. Communicated by Dr E. M. WEDDERBURN, M.A., LL.B. *Trans.*, vol. lii, pp. 629–642.

FOURTH ORDINARY MEETING.

Monday, February 2, 1920.

Sir George A. Berry, M.B., C.M., LL.D., F.R.C.S.E., Vice-President, in the Chair.

The following Communications were submitted:—

1. Magnetic Strains in Nickel Steel Tubes. By Dr C. G. Knott and Miss Dallas.
2. The Adequacy of the Young-Helmholtz Theory of Colour-Vision and Colour-Blindness. By Professor W. Peddie, D.Sc.
3. On the Quaternionic System as the Algebra of the Relations of Physics and Relativity. By Professor W. Peddie, D.Sc. *Proc.*, vol. xl, pp. 80–82.

FIFTH ORDINARY MEETING.

Monday, March 1, 1920.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

The Annual Election of Fellows took place. The following were elected:—Herbert Stanley Allen, Ernest Masson Anderson, Richard Siddoway Bagnall, Edward Battersby Bailey, R. G. Carruthers, William Grant Craib, Thomas Bedford Franklin, Sudhamoy Ghosh, William Gordon, Alexander Robert Horne, James Gall Inglis, John Charles Lamont, Thomas A. Lindsay, Stuart M'Donald, John Marshall, George Leslie Purser, William Robert Smellie, John Barber Todd, Thomas Walmsley, Malcolm Wilson.

The following Communications were submitted:—

1. New Stelar Facts, and their Bearing on Stelar Theories for the Ferns. By Dr J. M'Lean Thompson. *Trans.*, vol. lii, pp. 715–735.
2. On the Nestling Feathers of Birds. By Professor J. C. Ewart, F.R.S.
3. Note on Pfaffians with Polynomial Elements. By Sir Thomas Muir, C.M.G., F.R.S. *Proc.*, vol. xl, pp. 83–88.

SIXTH ORDINARY MEETING.

Monday, March 15, 1920.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

The following Communications were submitted:—

1. The Effect of Weather Changes on Soil Temperatures. By Captain T. Bedford Franklin. *Proc.*, vol. xl, pp. 56–79.

2. (*a*) Geological Observations in the South Shetlands, the Palmer Archipelago, and the Danco Coast, Grahamland. By D. Ferguson, M.I.Min. Eng. Communicated by G. W. Tyrrell, A.R.C.Sc., F.G.S. *Trans.*

(*b*) A Contribution to the Petrography of the South Shetlands, the Palmer Archipelago, and the Danco Coast, Grahamland. By G. W. Tyrrell. A.R.C.Sc., F.G.S. *Trans.*

(*c*) Petrographical Notes on Rocks from Deception Island and Roberts Island (South Shetlands), the Danco Coast and Adjacent Islands, Grahamland. By H. H. Thomas, D.Sc., F.G.S. Communicated by G. W. Tyrrell, A.R.C.Sc., F.G.S. *Trans.*

3. On a Class of Graduation Formulæ. By Miss C. W. M. Sherriff. Communicated by Professor E. T. Whittaker, F.R.S. *Proc.*, vol. xl, pp. 112–128.

4. The Daily Temperature Curve. By Professor L. Becker, Ph.D. *Proc.*, vol. xl, pp. 83–101.

The following recently elected Fellows signed the Roll and were duly admitted to the Society:—E. M. Anderson, E. B. Bailey, W. G. Craib, T. B. Franklin, W. Gordon, J. G. Inglis, G. L. Purser, J. B. Todd, and M. Wilson.

The Council gave notice that the following changes of rule will be submitted for discussion by the Society at an early date:—

Law VIII.

Honorary Fellows shall be persons eminently distinguished in Science or Literature. They shall not be liable to contribute to the Society's Funds. Personages of the Blood Royal may be elected Honorary Fellows at any time on the nomination of the Council, and without regard to the limitation of numbers specified in Law I.

Law IX.

Honorary Fellows shall be proposed by the Council. The nominations shall be announced from the Chair at the first Ordinary Meeting after their selection. The names shall be printed in the circular for the last Ordinary Meeting of the Session, when the election shall be by Ballot, after the manner prescribed in Laws III and IV for the Election of Fellows.

Law XIII.

Meetings for reading and discussing communications and for general business, herein termed Ordinary Meetings, shall be held, when convenient, on the first and third Mondays of each month from November to July inclusive, with the exception that in January the meetings shall be held on the second and fourth Mondays.

SEVENTH ORDINARY MEETING.

Monday, May 3, 1920.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

The amendments to Laws VIII, IX, and XIII as stated in the Minutes of the Meeting of March 15, 1920, were proposed from the Chair and carried.

By request of the Council, Principal Sir James Alfred Ewing, K.C.B., LL.D., F.R.S., delivered an address on "The Molecular Energy of Gases." *Proc.*, vol. xl, pp. 102–111.

A vote of thanks to Sir Alfred Ewing was proposed by Professor E. T. Whittaker, seconded by Dr H. S. Allen, and carried unanimously.

The following nominations to the Honorary Fellowship of the Society were announced from the Chair:—

William Wallace Campbell, Charles Emile Picard, Hendrik Anton Lorentz, Charles Richet, Yves Delage, Alfred Gabriel Nathorst, and Georg Ossian Sars.

Dr H. S. Allen signed the Roll and was duly admitted a Fellow of the Society.

EIGHTH ORDINARY MEETING.

Monday, June 7, 1920.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

H.R.H. The Prince of Wales was elected an Honorary Fellow of the Society.

The Keith Prize for the Biennial Period 1917–1919 was awarded to Professor John Stephenson, Lieut.-Colonel Indian Medical Service. He joined the service in 1895, and was for eleven years employed on military duty, on plague service, and as civil surgeon. In 1906 he was appointed Professor of Biology in the Government College, Lahore, and for some years he has been Principal of the College. Soon after his appointment as Professor he began to publish the results of his studies on Indian Oligochæta, and the twenty-six papers published during the period 1907–1919 bear witness to his devotion to research and his untiring industry.

The majority of these papers are of a systematic character, and by his detailed study and careful description of mature examples Professor Stephenson has added much to our knowledge of the structure and comparative anatomy of Oligochæta, both fresh-water and terrestrial. But besides these studies of anatomical and systematic features, the author has kept in view the broader issues of his work and has drawn attention to many matters of general interest connected with the zoogeography and the phylogeny of Oriental Oligochæta. Special mention should be made of Professor Stephenson's memoirs on gill-bearing aquatic Oligochæta.

Finally, reference should be made to his observations on the phenomena of anti-peristalsis in Oligochæta, first presented in his thesis for the degree of Doctor of Science in 1909 and afterwards extended into a memoir published in the *Transactions* of this Society in 1913. This memoir makes a substantial contribution to our knowledge of anti-peristalsis and intestinal respiration in Oligochæta and other Annelids.

The Neill Prize for the Biennial Period 1917–1919 was awarded to Professor JOHN TAIT (and received on his behalf by Dr Bowie) for his work on Crustacea. His first observations, twelve years ago, were upon the coagulation of the blood of Lygia and Gammarus, but from these he was led to study other aspects of the physiology and also the morphology of certain Crustacea. Five papers of this nature were published in the *Proceedings* of the Society in 1917. Of these the most noteworthy is an account of the structural features of Glyptonotus, based on the rich material collected by the Scottish National Antarctic Expedition, now in the Scottish Oceanographical Laboratory. Special mention should be made of two papers published by Professor Tait in the *Quarterly Journal of Experimental Physiology* in 1918. One of these, written in collaboration with Dr J. A. Gunn, is devoted to a study of the blood of the Crayfish, and contains a detailed account of the structure and physiology of the blood-corpuscles—three types of which are carefully described. Dr W. B. Hardy showed in 1892 that coagulation of the blood of the Crayfish follows cytolysis of certain corpuscles which he termed "explosive corpuscles." Drs Tait and Gunn found that another type of blood-corpuscle—which they termed a thigmocyte—also undergoes cytolysis, and thus an explanation was afforded of the fact already known that there are two successive coagulations separated by an interval of time. The other paper by Professor Tait is devoted to the consideration of phagocytosis, amœboid movement, and a number of related problems of great general interest, and this paper also is marked by the same combination of careful observation and suggestive interpretation.

The following communications were submitted:—

1. Petrography of the Intrusive Igneous Rocks in the neighbourhood of Dundee. By DAVID BALSILLIE, F.G.S. *Trans.*

2. An Identical Relation connecting Seven Vectors. By Mr F. L. HITCHCOCK. Communicated by THE GENERAL SECRETARY. *Proc.*, vol. xl, pp. 129–139.

Dr J. C. LAMONT and Professor STEPHENSON (elected 1912) signed the Roll and were duly admitted as Fellows.

NINTH ORDINARY MEETING.

Monday, June 21, 1920.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

The Ballot for the election of Honorary Fellows was taken, Mr W. A. TAIT and Dr A. CRICHTON MITCHELL being nominated as Scrutineers of the Ballot. After announcement by the Scrutineers the Chairman declared that the following had been elected Foreign Honorary Fellows:—

WILLIAM WALLACE CAMPBELL, Director of the Lick Observatory.
YVES DELAGE, Professor of Zoology, Faculty of Sciences, Paris.
HENDRIK ANTON LORENTZ, Professor of Physics, Leiden University.
ALFRED GABRIEL NATHORST, Stockholm.
CH. EMILE PICARD, Perpetual Secretary, Academy of Sciences, Paris.
CHARLES RICHET, Professor of Physiology, Faculty of Medicine, Paris.
GEORG OSSIAN SARS, formerly Professor of Zoology, Christiania, and Director of Norwegian Fisheries.

The following Communications were submitted:—

1. Obituary Notice of Sir John Jackson, Kt., LL.D. By D. A. STEVENSON, B.Sc., M.Inst.C.E. *Proc.*, vol. xl, pp. 182–184.

2. The Harmala Alkaloids in Malaria. By Professor J. A. GUNN, D.Sc., and Lieut.-Col. D. G. MARSHALL, M.B. *Proc.*, vol. xl, pp. 140–149.

3. The Musical Scale. By Mr JOSEPH GOOLD, Nottingham. Communicated by Mr C. R. GIBSON. *Proc.*, vol. xl, pp. 161–171.

4. The Explanation of an Apparent Anomaly in the Results of Measurement of Dissociation Pressures. By Professor A. W. C. MENZIES, M.A., B.Sc. *Proc.*, vol. xl, pp. 158–160.

5. A Law of Force giving Stability to the Rutherford Atom. By JOHN MARSHALL, M.A., B.Sc. *Proc.*, vol. xl, pp. 150–157.

PROCEEDINGS OF THE STATUTORY GENERAL MEETING
Ending the 137th Session, 1919–1920.

At the Statutory Meeting of the Royal Society of Edinburgh, held in the Society's Lecture Room, 24 George Street, on Monday, October 25, 1920, at 4.30 p.m.,

Professor Frederick O. Bower, President, in the Chair,

the Minutes of Meeting of October 27, 1919, were read, approved, and signed.

The Chairman nominated as Scrutineers, Dr Inglis Clark and Mr Gall Inglis.

The ballot for the election of Office-Bearers and Members of Council was then taken.

The Secretary submitted the following Report:—

The number of papers read at our Meetings during the last Session was 27, as compared with 23 the previous Session. Of these, three were Addresses, of which two were delivered by Professor R. A. Sampson, "On Gravitation and Relativity," and one by Principal Sir James Alfred Ewing, "On the Molecular Energy of Gases." Of the others, 4 were in Mathematics, 7 in Physics, 4 in Meteorology, 1 in Chemistry, 1 in Zoology, 1 in Botany, 5 in Geology, and 1 in Medicine. 23 have been, or are being, published in the *Proceedings* or *Transactions.*

Last Session the Society elected 20 new Fellows. Two Fellows have resigned, and we have lost by death 3 Honorary Fellows and 16 Ordinary Fellows.

The Keith Prize was awarded to Professor John Stephenson, Lt. Col. I.M.S., and the Neill Prize to Professor John Tait.

Owing to the continued increase in the cost of publication the Council has to exercise great care in order to meet the necessary expenses. The income of the Society remains practically the same; but the Special Fund to which many Fellows contributed has enabled us to meet the comparatively small deficit this year. An Appeal by the Council to the Carnegie United Kingdom Trust for aid in binding the loose journals in the Library was very generously met by the Trust, and already £1500 (half the promised grant) has been received from the Trust and paid over to the binders. The Society takes this opportunity of expressing its thanks to the Trust for their generous help.

An interesting event of the year was the election of the Prince of Wales as an Honorary Fellow Also the names of seven distinguished scientific men were added to the roll of Foreign Honorary Fellows.

All who have had occasion to use the Library will hear with regret that Miss Le Harivel has been obliged, for reasons of health, to resign her position as Assistant Librarian. Steps are being taken to fill up the vacancy thus occasioned.

The Society is being called upon to take an increasing share in national developments. Thus Dr E. M. Wedderburn has been appointed the Society's representative on the newly constituted Meteorological Committee under the Air Ministry. In connection with the International Association of Scientific Bodies, Professor Whittaker and Dr Knott have been appointed representatives on the National Union of Mathematics, Dr H. R. Mill on the National Union of Geography, and Dr Crichton Mitchell and Dr Erskine Murray on the National Union of Radio-Telegraphy.

It will interest the Fellows to know that the Society is associated with the Municipality and the University in preparing for the Meeting of the British Association to be held in Edinburgh in September 1921.

In order to give a distinct scientific flavour to the Statutory Meeting, the Council has adopted a suggestion made last year by the previous President, Dr Horne, and asked the President to deliver his Address at this Meeting instead of a week later at the first Ordinary Meeting of the Session.

The Treasurer, in submitting his Report for the year, compared the income and expenditure with those of the previous year, and called attention to the fact that the deficit of £34, 9s. 5d. on the year's working had been met from the Special Subscription Fund, which now stands at £792, 11s. 5d. The full list of subscribers up to the present is given on pp. 208–209.

Dr John Horne moved the adoption of the Reports, and the reappointment of Messrs Lindsay, Jamieson & Haldane, C.A., as auditors of the accounts for the ensuing Session.

This was unanimously agreed to.

The Scrutineers reported that the Ballot Papers were in order, and that the following Office-bearers and Members of Council had been elected :—

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President.

Vice-Presidents:
Professor D. Noël Paton, M.D., B.Sc., LL.D., F.R.C.P.E., F.R.S.,
Professor Arthur Robinson, M.D., M.R.C.S.,
Sir George A. Berry, M.B., C.M., LL.D., F.R.C.S.E.,
Professor William Peddie, D.Sc.,
Principal Sir James Alfred Ewing, K.C.B., M.A., B.Sc., LL.D., M.Inst.C.E., F.R.S.,
Professor John Walter Gregory, D.Sc., F.R.S.,

Cargill G. Knott, D.Sc., LL.D., F.R.S., General Secretary.

Secretaries to Ordinary Meetings:
Professor E. T. Whittaker, Sc.D., F.R.S.,
Professor J. H. Ashworth, D.Sc., F.R.S.,

James Currie, M.A., LL.D., Treasurer.
A. Crichton Mitchell, D.Sc., Hon. D.Sc. (Geneva), Curator of Library and Museum.

ORDINARY MEMBERS OF COUNCIL.

Professor R. A. Sampson, M.A., D.Sc., F.R.S.
Professor J. Lorrain Smith, M.A., M.D., F.R.S.
W. A. Tait, D.Sc., M.Inst.C.E.
Maj.-General W. B. Bannerman, C.S.I., I.M.S., M.D., D.Sc.
Henry Moubray Cadell, of Grange, B.Sc.
Professor Arthur Robertson Cushny, M.A., M.D., LL.D., F.R.S.
Professor Francis Gibson Baily, M.A., M.Inst.E.E.
George James Lidstone, F.F.A., F.I.A.
Robert Campbell, D.Sc.
Professor James Colquhoun Irvine, C.B.E. Ph.D., D.Sc., F.R.S.
The Hon. Lord Salvesen.
Professor J. Arthur Thomson, M.A., LL.D.

SOCIETY'S REPRESENTATIVE ON GEORGE HERIOT'S TRUST.

W. A. Tait, D.Sc., M.Inst.C.E.

The President, in the name of the Society, thanked the Scrutineers for their services.

LIST OF CONTRIBUTIONS TO SPECIAL SUBSCRIPTION FUND.

Ordinary Fellows.

Rt. Hon. Lord Abercromby, LL.D.	£21	0	0
Sir J. O. Affleck, M.D.	5	5	0
(Late) Dr John Aitken, F.R.S.	52	10	0
R. G. Alford, Esq., M.Inst.C.E.	2	2	0
Dr Daniel E. Anderson	0	10	0
Sir R. Rowand Anderson	2	2	0
James Archibald, Esq., M.A.	1	1	0
Prof. J. H. Ashworth, F.R.S.	3	3	0
Maj.-Gen. W. B. Bannerman, M.D.	2	2	0
Dr A. H. F. Barbour	10	10	0
Dr H. Barnes, O.B.E.	1	1	0
Sir James Barr, M.D.	5	5	0
(Late) Dr J. G. Bartholomew	10	10	0
Prof. Hudson Beare, M.Inst.C.E.	2	2	0
F. Faithfull Begg, Esq.	1	1	0
J. P. F. Bell, Esq., F.Z.S.	1	1	0
Sir Geo. A. Berry	5	5	0
Prof. A. A. Boon, D.Sc.	1	1	0
R. Le Boutillier (enclosed by Prof. Macfarlane)	3	3	0
Prof. F. O. Bower, F.R.S.	1	1	0
Principal O. C. Bradley, M.D.	1	1	0
Prof. Henry Briggs, D.Sc.	1	1	0
Carry forward	£133	17	0

Ordinary Fellows.

Brought forward	£133	17	0
Sir Charles Bright, M.Inst.C.E.	25	0	0
Em.-Prof. Crum Brown, F.R.S.	2	2	0
David Brown, Esq., F.C.S.	2	0	0
Dr Wm. Spiers Bruce	2	2	0
Prof. T. H. Bryce, M.D.	5	5	0
Dr Thos. Lowe Bunting	1	1	0
H. M. Cadell, Esq., of Grange, B.Sc.	100	0	0
Col. David Carnegie, M.Inst.C.E.	10	10	0
Wm. Allan Carter, Esq., O.B.E.	2	2	0
C. Carus-Wilson, Esq., F.G.S.	5	5	0
Em.-Prof. J. Chiene, C.B.	2	2	0
Dr Wm. Inglis Clark	5	5	0
Prof. J. Norman Collie, F.R.S.	3	3	0
Dr Walter Colquhoun	2	2	0
Alexander Cowan, Esq.	5	0	0
Prof. L. Crawford, D.Sc.	1	1	0
Dr James Currie, M.A.	20	0	0
Sir Francis Darwin, Kt.	5	5	0
T. Cuthbert Day, Esq.	5	5	0
Sir Archibald Denny, LL.D.	3	3	0
Sir James Dewar, Kt.	10	10	0
Dr T. W. Dewar	2	2	0
Carry forward	£354	2	0

Ordinary Fellows.

	£	s.	d.
Brought forward	£354	2	0
Sir J. J. Dobbie, Kt.	2	2	0
W. J. Dundas, Esq., W.S., LL.D.	21	0	0
Sir F. W. Dyson, Kt.	2	0	0
Dr. J. Edwards	3	3	0
Dr J. W. H. Eyre	3	3	0
Charles A. Fawsett, Esq.	5	5	0
Dr J. Haig Ferguson	2	2	0
Sir John R. Findlay, K.B.E.	10	10	0
John S. Ford, Esq., F.C.S.	2	2	0
Alex. Fraser, Esq., F.F.A.	2	2	0
William Fraser, Esq.	21	0	0
T. Lindsay Galloway, Esq., M.A.	3	3	0
Rt. Hon. Sir Auckland C. Geddes, K.C.B.	10	10	0
Dr J. E. Gemmell	2	2	0
Brig.-Gen. Sir Alex. Gibb, K.B.E.	52	10	0
C. R. Gibson, Esq.	3	3	0
Prof. Geo. A. Gibson, LL.D.	5	0	0
(Late) Richard D. Graham, Esq.	2	2	0
Prof. J. G. Gray	2	2	0
Prof. J. W. Gregory, F.R.S.	2	2	0
(Late) The Hon. Lord Guthrie, LL.D.	10	10	0
John Harrison, Esq., O.B.E., LL.D.	2	0	0
Thos. Heath, Esq., B.A.	1	1	0
Dr J. Fletcher Horne	1	1	0
Dr John Horne, F.R.S.	5	5	0
Dr R. A. Houstoun	3	3	0
Dr Wm. Evans Hoyle	2	2	0
Sir John Jackson, LL.D.	10	10	0
Prof. T. J. Jehu, M.D.	2	2	0
Col. H. H. Johnston, C.B.	1	1	0
*Sir J. H. R. Kemnal	100	0	0
Dr Foster Kennedy	3	0	0
A. W. Kerr, Esq.	1	1	0
Dr Robert Kidston, F.R.S.	5	5	0
Dr C. G. Knott, F.R.S.	3	3	0
Prof. P. R. Scott Lang, M.A.	2	2	0
Dr Alex. Lauder	2	2	0
Dr David Lawson	5	5	0
Forrest H. Lightbody, Esq.	1	1	0
Rev. Dr James Lindsay	1	1	0
The Most Hon. the Marquis of Linlithgow	5	0	0
Geo. M. Low, Esq.	10	10	0
Dr D. F. Lowe	2	2	0
Dr Graham Lusk	5	0	0
Dr W. J. Macdonald	1	1	0
Prof. R. Stewart MacDougall, D.Sc.	2	2	0
Prof. J. M. Macfarlane, D.Sc.	3	4	0
Dr George M'Gowan	2	2	0
J. W. M'Intosh, Esq., A.R.C.V.S.	1	1	0
Dr John Macintyre	2	2	0
Em.-Prof. John G. M'Kendrick, F.R.S.	2	2	0
Dr Alister Mackenzie	2	2	0
Sir Wm. Leslie Mackenzie, LL.D.	1	1	0
The late Dr Maclagan	1	1	0
Prof. C. R. Marshall, M.D.	2	2	0
Prof. D. H. Marshall	1	0	0
(Late) Sir T. Carlaw Martin	2	2	0
Dr Hugh Robert Mill, LL.D.	2	2	0
William Milne, M.A.	1	1	3
Carry forward	£722	17	3

Ordinary Fellows.

	£	s.	d.
Brought forward	£722	17	3
Dr A. Crichton Mitchell	5	0	0
George A. Mitchell, Esq., M.A.	5	5	0
Rev. A. Moffat, M.A.	1	1	0
R. L. Mond, Esq., M.A.	105	0	0
Dr R. Owen Morris	1	1	0
R. C. Mossman, Esq.	2	2	0
Prof. R. Muir, F.R.S.	2	2	0
Sir Thomas Muir, LL.D.	5	5	0
(Late) Dr Robert Munro	5	5	0
Dr J. Erskine Murray	1	1	0
Sir F. Grant Ogilvie, C.B.	1	1	0
Dr John Wm. Pare	3	3	0
Prof. D. Noël Paton, F.R.S.	5	0	0
Dr T. S. Patterson	2	2	0
Sir David Paulin	2	2	0
John T. Pearce, Esq., B.A., B.Sc.	1	1	0
Prof. Wm. Peddie, D.Sc.	3	3	6
A. G. Perkin, Esq., F.R.S.	2	2	0
Sir Robert Philip, M.D.	1	1	0
Major C. E. S. Phillips, O.B.E.	5	5	0
Basil A. Pilkington	2	2	0
Lt.-Col. Sir David Prain, F.R.S.	2	2	0
Dr C. S. du Riche Preller	1	1	0
Laurence Pullar, Esq.	50	0	0
Dr Harry Rainy	1	1	0
Alex. G. Ramage, Esq.	2	2	0
Dr J. R. Ratcliffe	1	1	0
Ralph Richardson, Esq., W.S.	5	5	0
Alfred H. Roberts, Esq., O.B.E.	1	1	0
James Russell, Esq.	2	2	0
Hon. Lord Salvesen	5	5	0
T. E. Salvesen, Esq.	10	10	0
Prof. R. A. Sampson, F.R.S.	5	5	0
Prof. Sutherland Simpson, M.D.	1	1	0
R. T. Skinner, Esq., M.A.	1	1	0
C. Michie Smith, C.I.E., B.Sc.	10	0	0
Prof. Wm. Somerville, D.Sc.	2	2	0
James Sorley, Esq.	10	10	0
Frank Spence, Esq., M.A.	1	1	0
(Late) Dr T. B. Sprague	10	10	0
Prof. John Stephenson, M.B., D.Sc.	5	5	0
D. R. Steuart, Esq., F.I.C.	3	3	0
John W. Tait, Esq., B.Sc.	1	1	0
W. A. Tait, Esq., D.Sc., M.Inst.C.E.	5	5	0
W. Thorneycroft, Esq., J.P.	5	5	0
Dr A. F. Tredgold	1	1	0
Sir James Walker, F.R.S.	10	10	0
(Late) Dr Robert Walker	1	1	0
Prof. R. Wallace, F.L.S.	1	1	0
Andrew Watt, Esq., M.A.	1	1	0
Rev. Dr Lauchlan Maclean Watt	1	1	0
Dr J. C. Webster	3	3	0
Dr E. M. Wedderburn, M.A., LL.B.	10	10	0
Wm. Williamson, Esq., F.L.S.	5	5	0
J. C. Wright, Esq.	1	1	0
Anonymous	52	10	0

Honorary Fellows.

	£	s.	d.
Prof. J. N. Langley, F.R.S.	3	3	0
Sir Joseph Larmor, F.R.S.	5	5	0
Em. Prof. A. Liversidge, F.R.S.	5	5	0
Total	£1128	17	9

* It was this generous subscription, and the accompanying suggestion from the donor, which prompted the Council to initiate the "Special Subscription Fund."

THE KEITH, MAKDOUGALL-BRISBANE, NEILL, GUNNING VICTORIA JUBILEE, AND JAMES SCOTT PRIZES.

The above Prizes will be awarded by the Council in the following manner:—

I. KEITH PRIZE.

The Keith Prize, consisting of a Gold Medal and from £40 to £50 in Money, will be awarded in the Session 1921–1922 for the "best communication on a scientific subject, communicated,* in the first instance, to the Royal Society of Edinburgh during the Sessions 1919–1920 and 1920–1921." Preference will be given to a paper containing a discovery.

II. MAKDOUGALL-BRISBANE PRIZE.

This Prize is to be awarded biennially by the Council of the Royal Society of Edinburgh to such person, for such purposes, for such objects, and in such manner as shall appear to them the most conducive to the promotion of the interests of science; with the *proviso* that the Council shall not be compelled to award the Prize unless there shall be some individual engaged in scientific pursuit, or some paper written on a scientific subject, or some discovery in science made during the biennial period, of sufficient merit or importance in the opinion of the Council to be entitled to the Prize.

1. The Prize, consisting of a Gold Medal and a sum of Money, will be awarded before the close of the Session 1922–1923, for an Essay or Paper having reference to any branch of scientific inquiry, whether Material or Mental.

2. Competing Essays to be addressed to the Secretary of the Society, and transmitted not later than 8th July 1922.

3. The Competition is open to all men of science.

4. The Essays may be either anonymous or otherwise. In the former case, they must be distinguished by mottoes, with corresponding sealed billets, superscribed with the same motto, and containing the name of the Author.

5. The Council impose no restriction as to the length of the Essays, which may be, at the discretion of the Council, read at the Ordinary Meetings of the Society. They wish also to leave the property and free disposal of the manuscripts to the Authors; a copy, however, being deposited in the Archives of the Society, unless the paper shall be published in the Transactions.

* For the purposes of this award the word "communicated" shall be understood to mean the date on which the manuscript of a paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

6. In awarding the Prize, the Council will also take into consideration any scientific papers presented * to the Society during the Sessions 1918–19, 1919–20, whether they may have been given in with a view to the prize or not.

III. NEILL PRIZE.

The Council of the Royal Society of Edinburgh having received the bequest of the late Dr Patrick Neill of the sum of £500, for the purpose of "the interest thereof being applied in furnishing a Medal or other reward every second or third year to any distinguished Scottish Naturalist, according as such Medal or reward shall be voted by the Council of the said Society," hereby intimate:

1. The Neill Prize, consisting of a Gold Medal and a sum of Money, will be awarded during the Session 1921–1922.

2. The Prize will be given for a Paper of distinguished merit, on a subject of Natural History, by a Scottish Naturalist, which shall have been presented * to the Society during the two years preceding the fourth Monday in October 1921,—or failing presentation of a paper sufficiently meritorious, it will be awarded for a work or publication by some distinguished Scottish Naturalist, on some branch of Natural History, bearing date within five years of the time of award.

IV. GUNNING VICTORIA JUBILEE PRIZE.

This Prize, founded in the year 1887 by Dr R. H. Gunning, is to be awarded quadrennially by the Council of the Royal Society of Edinburgh, in recognition of original work in Physics, Chemistry, or Pure or Applied Mathematics.

Evidence of such work may be afforded either by a Paper presented to the Society, or by a Paper on one of the above subjects, or some discovery in them elsewhere communicated or made, which the Council may consider to be deserving of the Prize.

The Prize consists of a sum of money, and is open to men of science resident in or connected with Scotland. The first award was made in the year 1887. The next award will be made in 1924–1925.

In accordance with the wish of the Donor, the Council of the Society may on fit occasions award the Prize for work of a definite kind to be undertaken during the three succeeding years by a scientific man of recognised ability.

V. JAMES SCOTT PRIZE.

This Prize, founded in the year 1918 by the Trustees of the James Scott Bequest, is to be awarded triennially, or at such intervals as the Council of the Royal Society of Edinburgh may decide, "for a lecture or essay on the fundamental concepts of Natural Philosophy."

The first award will be in the year 1922.

* For the purposes of this award the word "presented" shall be understood to mean the date on which the manuscript of a paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

RESOLUTIONS OF COUNCIL IN REGARD TO THE MODE OF AWARDING PRIZES.

(*See Minutes of Meeting of January* 18, 1915.)

I. With regard to the Keith and Makdougall-Brisbane Prizes, which are open to all Sciences, the mode of award will be as follows :—

1. Papers or essays to be considered shall be arranged in two groups, A and B, —Group A to include Astronomy, Chemistry, Mathematics, Metallurgy, Meteorology and Physics; Group B to include Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, and Zoology.
2. These two Prizes shall be awarded to each group in alternate biennial periods, provided papers worthy of recommendation have been communicated to the Society.
3. Prior to the adjudication the Council shall appoint, in the first instance, a Committee composed of representatives of the group of Sciences which did not receive the award in the immediately preceding period. The Committee shall consider the Papers which come within their group of Sciences, and report in due course to the Council.
4. In the event of the aforesaid Committee reporting that within their group of subjects there is, in their opinion, no paper worthy of being recommended for the award, the Council, on accepting this report, shall appoint a Committee representative of the alternate group to consider papers coming within their group and to report accordingly.
5. Papers to be considered by the Committees shall fall within the period dating from the last award in groups A and B respectively.

II. With regard to the Neill Prize, the term "Naturalist" shall be understood to include any student in the Sciences composing group B, namely, Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, Zoology.

AWARDS OF THE KEITH, MAKDOUGALL-BRISBANE, NEILL, AND GUNNING PRIZES.

I. KEITH PRIZE.

1st Biennial Period, 1827-29.—Dr Brewster, for his papers "on his Discovery of Two New Immiscible Fluids in the Cavities of certain Minerals," published in the Transactions of the Society.

2nd Biennial Period, 1829-31.—Dr Brewster, for his paper "on a New Analysis of Solar Light," published in the Transactions of the Society.

3rd Biennial Period, 1831-33.—Thomas Graham, Esq., for his paper "on the Law of the Diffusion of Gases," published in the Transactions of the Society.

4th Biennial Period, 1833-35.—Professor J. D. Forbes, for his paper "on the Refraction and Polarization of Heat," published in the Transactions of the Society.

5th Biennial Period, 1835-37.—John Scott Russell, Esq., for his researches "on Hydrodynamics," published in the Transactions of the Society.

6th Biennial Period, 1837-39.—Mr John Shaw, for his experiments "on the Development and Growth of the Salmon," published in the Transactions of the Society.

7th Biennial Period, 1839-41.—Not awarded.

8th Biennial Period, 1841-43.—Professor James David Forbes, for his papers "on Glaciers," published in the Proceedings of the Society.

9th Biennial Period, 1843-45.—Not awarded.

10th Biennial Period, 1845-47.—General Sir Thomas Brisbane, Bart., for the Makerstoun Observations on Magnetic Phenomena, made at his expense, and published in the Transactions of the Society.

11th Biennial Period, 1847-49.—Not awarded.

12th Biennial Period, 1849-51.—Professor Kelland, for his papers "on General Differentiation, including his more recent Communication on a process of the Differential Calculus, and its application to the solution of certain Differential Equations," published in the Transactions of the Society.

13th Biennial Period, 1851-53.—W. J. Macquorn Rankine, Esq., for his series of papers "on the Mechanical Action of Heat," published in the Transactions of the Society.

14th Biennial Period, 1853-55.—Dr Thomas Anderson, for his papers "on the Crystalline Constituents of Opium, and on the Products of the Destructive Distillation of Animal Substances," published in the Transactions of the Society.

15th Biennial Period, 1855-57.—Professor Boole, for his Memoir "on the Application of the Theory of Probabilities to Questions of the Combination of Testimonies and Judgments," published in the Transactions of the Society.

16th Biennial Period, 1857-59.—Not awarded.

17th Biennial Period, 1859-61.—John Allan Broun, Esq., F.R.S., Director of the Trevandrum Observatory, for his papers "on the Horizontal Force of the Earth's Magnetism, on the Correction of the Bifilar Magnetometer, and on Terrestrial Magnetism generally," published in the Transactions of the Society.

18th Biennial Period, 1861-63.—Professor William Thomson, of the University of Glasgow, for his Communication "on some Kinematical and Dynamical Theorems."

19th Biennial Period, 1863-65.—Principal Forbes, St Andrews, for his "Experimental Inquiry into the Laws of Conduction of Heat in Iron Bars," published in the Transactions of the Society.

20th Biennial Period, 1865-67.—Professor C. Piazzi Smyth, for his paper "on Recent Measures at the Great Pyramid," published in the Transactions of the Society.

21st Biennial Period, 1867-69.—Professor P. G. Tait, for his paper "on the Rotation of a Rigid Body about a Fixed Point," published in the Transactions of the Society.

22nd Biennial Period, 1869-71.—Professor Clerk Maxwell, for his paper "on Figures, Frames, and Diagrams of Forces," published in the Transactions of the Society.

23RD BIENNIAL PERIOD, 1871–73.—Professor P. G. TAIT, for his paper entitled "First Approximation to a Thermo-electric Diagram," published in the Transactions of the Society.

24TH BIENNIAL PERIOD, 1873–1875.—Professor CRUM BROWN, for his Researches "on the Sense of Rotation, and on the Anatomical Relations of the Semicircular Canals of the Internal Ear."

25TH BIENNIAL PERIOD, 1875–77.—Professor M. FORSTER HEDDLE, for his papers "on the Rhombohedral Carbonates," and "on the Felspars of Scotland," published in the Transactions of the Society.

26TH BIENNIAL PERIOD, 1877–79.—Professor H. C. FLEEMING JENKIN, for his paper "on the Application of Graphic Methods to the Determination of the Efficiency of Machinery," published in the Transactions of the Society; Part II having appeared in the volume for 1877–78.

27TH BIENNIAL PERIOD, 1879–81.—Professor GEORGE CHRYSTAL, for his paper "on the Differential Telephone," published in the Transactions of the Society.

28TH BIENNIAL PERIOD, 1881–83.—THOMAS MUIR, Esq., LL.D., for his "Researches into the Theory of Determinants and Continued Fractions," published in the Proceedings of the Society.

29TH BIENNIAL PERIOD, 1883–85.—JOHN AITKEN, Esq., for his paper "on the Formation of Small Clear Spaces in Dusty Air," and for previous papers on Atmospheric Phenomena, published in the Transactions of the Society.

30TH BIENNIAL PERIOD, 1885–87.—JOHN YOUNG BUCHANAN, Esq., for a series of communications, extending over several years, on subjects connected with Ocean Circulation, Compressibility of Glass, etc.; two of which, viz., "On Ice and Brines," and "On the Distribution of Temperature in the Antarctic Ocean," have been published in the Proceedings of the Society.

31ST BIENNIAL PERIOD, 1887–89.—Professor E. A. LETTS, for his papers on the Organic Compounds of Phosphorus, published in the Transactions of the Society.

32ND BIENNIAL PERIOD, 1889–91.—R. T. OMOND, Esq., for his contributions to Meteorological Science, many of which are contained in vol. xxxiv of the Society's Transactions.

33RD BIENNIAL PERIOD, 1891–93.—Professor THOMAS R. FRASER, F.R.S., for his papers on *Strophanthus hispidus*, Strophanthin, and Strophanthidin, read to the Society in February and June 1889 and in December 1891, and printed in vols. xxxv, xxxvi, and xxxvii of the Society's Transactions.

34TH BIENNIAL PERIOD, 1893–95.—Dr CARGILL G. KNOTT, for his papers on the Strains produced by Magnetism in Iron and in Nickel, which have appeared in the Transactions and Proceedings of the Society.

35TH BIENNIAL PERIOD, 1895–97.—Dr THOMAS MUIR, for his continued communications on Determinants and Allied Questions.

36TH BIENNIAL PERIOD, 1897–99.—Dr JAMES BURGESS, for his paper "on the Definite Integral $\frac{2}{\sqrt{\pi}}\int_0^t \epsilon^{-t^2}dt$, with extended Tables of Values," printed in vol. xxxix of the Transactions of the Society.

37TH BIENNIAL PERIOD, 1899–1901.—Dr HUGH MARSHALL, for his discovery of the Persulphates, and for his Communications on the Properties and Reactions of these Salts, published in the Proceedings of the Society.

38TH BIENNIAL PERIOD, 1901–03.—Sir WILLIAM TURNER, K.C.B., LL.D., F.R.S., etc., for his memoirs entitled "A Contribution to the Craniology of the People of Scotland," published in the Transactions of the Society, and for his "Contributions to the Craniology of the People of the Empire of India," Parts I, II, likewise published in the Transactions of the Society.

39TH BIENNIAL PERIOD, 1903–05.—THOMAS H. BRYCE, M.A., M.D., for his two papers on "The Histology of the Blood of the Larva of *Lepidosiren paradoxa*," published in the Transactions of the Society within the period.

40TH BIENNIAL PERIOD, 1905–07.—ALEXANDER BRUCE, M.A., M.D., F.R.C.P.E., for his paper entitled "Distribution of the Cells in the Intermedio-Lateral Tract of the Spinal Cord," published in the Transactions of the Society within the period.

41ST BIENNIAL PERIOD, 1907–09.—WHEELTON HIND, M.D., B.S., F.R.C.S., F.G.S., for a paper published in the Transactions of the Society, "On the Lamellibranch and Gasteropod Fauna found in the Millstone Grit of Scotland."

42ND BIENNIAL PERIOD, 1909–11.—Professor ALEXANDER SMITH, B.Sc., Ph.D., of New York, for his researches upon "Sulphur" and upon "Vapour Pressure," appearing in the Proceedings of the Society.

43RD BIENNIAL PERIOD, 1911–1913.—JAMES RUSSELL, Esq., for his series of investigations relating to magnetic phenomena in metals and the molecular theory of magnetism, the results of which have been published in the Proceedings and Transactions of the Society, the last paper having been issued within the period.

44TH BIENNIAL PERIOD, 1913–15.—JAMES HARTLEY ASHWORTH, D.Sc., for his papers on "Larvæ of Lingula and Pelagodiscus," and on "Sclerocheilus," published in the Transactions of the Society, and for other papers on the Morphology and Histology of Polychæta.

45TH BIENNIAL PERIOD, 1915–17.—ROBERT C. MOSSMAN, for his work on the Meteorology of the Antarctic Regions, which originated with the important series of observations made by him during the voyage of the "Scotia" (1902–1904), and includes his paper "On a Sea-Saw of Barometric Pressure, Temperature, and Wind Velocity between the Weddell Sea and the Ross Sea," published in the Proceedings of the Society.

46TH BIENNIAL PERIOD, 1917–19.—JOHN STEPHENSON, Lt.-Col. I.M.S., for his series of papers on the Oligochæta and other Annelida, several of which have been published in the Transactions of the Society.

II. MAKDOUGALL-BRISBANE PRIZE.

1ST BIENNIAL PERIOD, 1859.—Sir RODERICK IMPEY MURCHISON, on account of his Contributions to the Geology of Scotland.

2ND BIENNIAL PERIOD, 1860–62.—WILLIAM SELLER, M.D., F.R.C.P.E., for his "Memoir of the Life and Writings of Dr Robert Whytt," published in the Transactions of the Society.

3RD BIENNIAL PERIOD, 1862–64.—JOHN DENIS MACDONALD, Esq., R.N., F.R.S., Surgeon of H.M.S. "Icarus," for his paper "on the Representative Relationships of the Fixed and Free Tunicata, regarded as Two Sub-classes of equivalent value; with some General Remarks on their Morphology," published in the Transactions of the Society.

4TH BIENNIAL PERIOD, 1864–66.—Not awarded.

5TH BIENNIAL PERIOD, 1866–68.—Dr ALEXANDER CRUM BROWN and Dr THOMAS RICHARD FRASER, for their conjoint paper "on the Connection between Chemical Constitution and Physiological Action," published in the Transactions of the Society.

6TH BIENNIAL PERIOD, 1868–70.—Not awarded.

7TH BIENNIAL PERIOD, 1870–72.—GEORGE JAMES ALLMAN, M.D., F.R.S., Emeritus Professor of Natural History, for his paper "on the Homological Relations of the Cœlenterata," published in the Transactions, which forms a leading chapter of his Monograph of Gymnoblastic or Tubularian Hydroids—since published.

8TH BIENNIAL PERIOD, 1872–74.—Professor LISTER, for his paper "on the Germ Theory of Putrefaction and the Fermentive Changes," communicated to the Society, 7th April 1873.

9TH BIENNIAL PERIOD, 1874–76.—ALEXANDER BUCHAN, A.M., for his paper "on the Diurnal Oscillation of the Barometer," published in the Transactions of the Society.

10TH BIENNIAL PERIOD, 1876–78.—Professor ARCHIBALD GEIKIE, for his paper "on the Old Red Sandstone of Western Europe," published in the Transactions of the Society.

11TH BIENNIAL PERIOD, 1878–80.—Professor PIAZZI SMYTH, Astronomer-Royal for Scotland, for his paper "on the Solar Spectrum in 1877–78, with some Practical Idea of its probable Temperature of Origination," published in the Transactions of the Society.

12TH BIENNIAL PERIOD, 1880–82.—Professor JAMES GEIKIE, for his "Contributions to the Geology of the North-West of Europe," including his paper "on the Geology of the Faroes," published in the Transactions of the Society.

13TH BIENNIAL PERIOD, 1882–84.—EDWARD SANG, Esq., LL.D., for his paper "on the Need of Decimal Subdivisions in Astronomy and Navigation, and on Tables requisite therefor," and generally for his Recalculations of Logarithms both of Numbers and Trigonometrical Ratios, —the former communication being published in the Proceedings of the Society.

14TH BIENNIAL PERIOD, 1884–86.—JOHN MURRAY, Esq., LL.D., for his papers "On the Drainage Areas of Continents, and Ocean Deposits," "The Rainfall of the Globe, and Discharge of Rivers," "The Height of the Land and Depth of the Ocean," and "The Distribution of Temperature in the Scottish Lochs as affected by the Wind."

15TH BIENNIAL PERIOD, 1886–88.—ARCHIBALD GEIKIE, Esq., LL.D., for numerous Communications, especially that entitled "History of Volcanic Action during the Tertiary Period in the British Isles," published in the Transactions of the Society.

16TH BIENNIAL PERIOD, 1889–90.—Dr LUDWIG BECKER, for his paper on "The Solar Spectrum at Medium and Low Altitudes," printed in vol. xxxvi, Part I, of the Society's Transactions.

17th Biennial Period, 1890–92.—Hugh Robert Mill, Esq., D.Sc., for his papers on "The Physical Conditions of the Clyde Sea Area," Part I being already published in vol. xxxvi of the Society's Transactions.

18th Biennial Period, 1892–94.—Professor James Walker, D.Sc., Ph.D., for his work on Physical Chemistry, part of which has been published in the Proceedings of the Society, vol. xx, pp. 255–263. In making this award, the Council took into consideration the work done by Professor Walker along with Professor Crum Brown on the Electrolytic Synthesis of Dibasic Acids, published in the Transactions of the Society.

19th Biennial Period, 1894–96.—Professor John G. M'Kendrick, for numerous Physiological papers, especially in connection with Sound, many of which have appeared in the Society's publications.

20th Biennial Period, 1896–98.—Dr William Peddie, for his papers on the Torsional Rigidity of Wires.

21st Biennial Period, 1898–1900.—Dr Ramsay H. Traquair, for his paper entitled "Report on Fossil Fishes collected by the Geological Survey in the Upper Silurian Rocks of Scotland," printed in vol. xxxix of the Transactions of the Society.

22nd Biennial Period, 1900–02.—Dr Arthur T. Masterman, for his paper entitled "The Early Development of *Cribrella oculata* (Forbes), with remarks on Echinoderm Development," printed in vol. xl of the Transactions of the Society.

23rd Biennial Period, 1902–04.—Mr John Dougall, M.A., for his paper on "An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate," published in vol. xli of the Transactions of the Society.

24th Biennial Period, 1904–06.—Jacob E. Halm, Ph.D., for his two papers entitled "Spectroscopic Observations of the Rotation of the Sun," and "Some Further Results obtained with the Spectroheliometer," and for other astronomical and mathematical papers published in the Transactions and Proceedings of the Society within the period.

25th Biennial Period, 1906–08.—D. T. Gwynne-Vaughan, M.A., F.L.S., for his papers, 1st, "On the Fossil Osmundaceæ," and 2nd, "On the Origin of the Adaxially-curved Leaf-trace in the Filicales," communicated by him conjointly with Dr R. Kidston.

26th Biennial Period, 1908–10.—Ernest MacLagan Wedderburn, M.A., LL.B., for his series of papers bearing upon "The Temperature Distribution in Fresh-water Lochs," and especially upon "The Temperature Seiche."

27th Biennial Period, 1910–12.—John Brownlee, M.A., M.D., D.Sc., for his contributions to the Theory of Mendelian Distributions and cognate subjects, published in the Proceedings of the Society within and prior to the prescribed period.

28th Biennial Period, 1912–14.—Professor C. R. Marshall, M.D., M.A., for his studies "On the Pharmacological Action of Tetra-alkyl-ammonium Compounds."

29th Biennial Period, 1914–16.—Robert Alexander Houstoun, Ph.D., D.Sc., for his series of papers on "The Absorption of Light by Inorganic Salts," published in the Proceedings of the Society.

30th Biennial Period, 1916–18.—Professor A. Anstruther Lawson for his Memoirs on "The Prothalli of *Tmesipteris Tannensis* and of *Psilotum*," published in the Transactions of the Society, together with previous papers on Cytology and on The Gametophytes of various Gymnosperms.

III. THE NEILL PRIZE.

1st Triennial Period, 1856–59.—Dr W. Lauder Lindsay, for his paper "on the Spermogones and Pycnides of Filamentous, Fruticulose, and Foliaceous Lichens," published in the Transactions of the Society.

2nd Triennial Period, 1859–61.—Robert Kaye Greville, LL.D., for his contributions to Scottish Natural History, more especially in the department of Cryptogamic Botany, including his recent papers on Diatomaceæ.

3rd Triennial Period, 1862–65.—Andrew Crombie Ramsay, F.R.S., Professor of Geology in the Government School of Mines, and Local Director of the Geological Survey of Great Britain, for his various works and memoirs published during the last five years, in which he has applied the large experience acquired by him in the Direction of the arduous work of the Geological Survey of Great Britain to the elucidation of important questions bearing on Geological Science.

4th Triennial Period, 1865–68.—Dr William Carmichael M'Intosh, for his paper "on the Structure of the British Nemerteans, and on some New British Annelids," published in the Transactions of the Society.

5TH TRIENNIAL PERIOD, 1868–71.—Professor WILLIAM TURNER, for his papers "on the Great Finner Whale; and on the Gravid Uterus, and the Arrangement of the Fœtal Membranes in the Cetacea," published in the Transactions of the Society.

6TH TRIENNIAL PERIOD, 1871–74.—CHARLES WILLIAM PEACH, Esq., for his Contributions to Scottish Zoology and Geology, and for his recent contributions to Fossil Botany.

7TH TRIENNIAL PERIOD, 1874–77.—Dr RAMSAY H. TRAQUAIR, for his paper "on the Structure and Affinities of *Tristichopterus alatus* (Egerton)," published in the Transactions of the Society, and also for his contributions to the Knowledge of the Structure of Recent and Fossil Fishes.

8TH TRIENNIAL PERIOD, 1877–80.—JOHN MURRAY, Esq., for his paper "on the Structure and Origin of Coral Reefs and Islands," published (in abstract) in the Proceedings of the Society.

9TH TRIENNIAL PERIOD, 1880–83.—Professor HERDMAN, for his papers "on the Tunicata," published in the Proceedings and Transactions of the Society.

10TH TRIENNIAL PERIOD, 1883–86.—B. N. PEACH, Esq., for his Contributions to the Geology and Palæontology of Scotland, published in the Transactions of the Society.

11TH TRIENNIAL PERIOD, 1886–89.—ROBERT KIDSTON, Esq., for his Researches in Fossil Botany, published in the Transactions of the Society.

12TH TRIENNIAL PERIOD, 1889–92.—JOHN HORNE, Esq., F.G.S., for his Investigations into the Geological Structure and Petrology of the North-West Highlands.

13TH TRIENNIAL PERIOD, 1892–95.—ROBERT IRVINE, Esq., for his papers on the Action of Organisms in the Secretion of Carbonate of Lime and Silica, and on the solution of these substances in Organic Juices. These are printed in the Society's Transactions and Proceedings.

14TH TRIENNIAL PERIOD, 1895–98.—Professor COSSAR EWART, for his recent Investigations connected with Telegony.

15TH TRIENNIAL PERIOD, 1898–1901.—Dr JOHN S. FLETT, for his papers entitled "The Old Red Sandstone of the Orkneys" and "The Trap Dykes of the Orkneys," printed in vol. xxxix of the Transactions of the Society.

16TH TRIENNIAL PERIOD, 1901–04.—Professor J. GRAHAM KERR, M.A., for his Researches on *Lepidosiren paradoxa*, published in the Philosophical Transactions of the Royal Society, London.

17TH TRIENNIAL PERIOD, 1904–07.—FRANK J. COLE, B.Sc., for his paper entitled "A Monograph on the General Morphology of the Myxinoid Fishes, based on a Study of Myxine," published in the Transactions of the Society, regard being also paid to Mr Cole's other valuable contributions to the Anatomy and Morphology of Fishes.

1ST BIENNIAL PERIOD, 1907–09.—FRANCIS J. LEWIS, M.Sc., F.L.S., for his papers in the Society's Transactions "On the Plant Remains of the Scottish Peat Mosses."

2ND BIENNIAL PERIOD, 1909–11.—JAMES MURRAY, Esq., for his paper on "Scottish Rotifers collected by the Lake Survey (Supplement)," and other papers on the "Rotifera" and "Tardigrada," which appeared in the Transactions of the Society—(this Prize was awarded after consideration of the papers received within the five years prior to the time of award: see Neill Prize Regulations).

3RD BIENNIAL PERIOD, 1911–13.—Dr W. S. BRUCE, in recognition of the scientific results of his Arctic and Antarctic explorations.

4TH BIENNIAL PERIOD, 1913–15.—ROBERT CAMPBELL, D.Sc., for his paper on "The Upper Cambrian Rocks at Craigeven Bay, Stonehaven," and "Downtonian and Old Red Sandstone Rocks of Kincardineshire," published in the Transactions of the Society.

5TH BIENNIAL PERIOD, 1915–17.—W. H. LANG, F.R.S., M.B., D.Sc., for his paper in conjunction with Dr R. KIDSTON, F.R.S., on *Rhynia Gwynne-Vaughani*, Kidston and Lang, published in the Transactions of the Society, and for his previous investigations on Pteridophytes and Cycads.

6TH BIENNIAL PERIOD, 1917–19.—JOHN TAIT, D.Sc., M.D., for his work on Crustacea, published in the Proceedings of the Society, and for his papers on the blood of Crustacea.

IV. GUNNING VICTORIA JUBILEE PRIZE.

1ST TRIENNIAL PERIOD, 1884–87.—Sir WILLIAM THOMSON, Pres. R.S.E., F.R.S., for a remarkable series of papers "on Hydrokinetics," especially on Waves and Vortices, which have been communicated to the Society.

2nd Triennial Period, 1887–90.—Professor P. G. Tait, Sec. R.S.E., for his work in connection with the "Challenger" Expedition, and his other Researches in Physical Science.

3rd Triennial Period, 1890-93.—Alexander Buchan, Esq., LL.D., for his varied, extensive, and extremely important Contributions to Meteorology, many of which have appeared in the Society's publications.

4th Triennial Period, 1893–96.—John Aitken, Esq., for his brilliant Investigations in Physics, especially in connection with the Formation and Condensation of Aqueous Vapour.

1st Quadrennial Period, 1896–1900.—Dr T. D. Anderson, for his discoveries of New and Variable Stars.

2nd Quadrennial Period, 1900-04.—Sir James Dewar, LL.D., D.C.L., F.R.S., etc., for his researches on the Liquefaction of Gases, extending over the last quarter of a century, and on the Chemical and Physical Properties of Substances at Low Temperatures: his earliest papers being published in the Transactions and Proceedings of the Society.

3rd Quadrennial Period, 1904–08.—Professor George Chrystal, M.A., LL.D., for a series of papers on "Seiches," including "The Hydrodynamical Theory and Experimental Investigations of the Seiche Phenomena of Certain Scottish Lakes."

4th Quadrennial Period, 1908–12.—Professor J. Norman Collie, Ph.D., F.R.S., for his distinguished contributions to Chemistry, Organic and Inorganic, during twenty-seven years, including his work upon Neon and other rare gases. Professor Collie's early papers were contributed to the Transactions of the Society.

5th Quadrennial Period, 1912–16.—Sir Thos. Muir, C.M.G., LL.D., F.R.S., for his series of Memoirs upon "The Theory and History of Determinants and Allied Forms," published in the Transactions and Proceedings of the Society between the years 1872 and 1915.

ABSTRACT

OF

THE ACCOUNTS OF JAMES CURRIE, ESQ., LL.D.

As Treasurer of the Royal Society of Edinburgh.

SESSION 1919–1920.

I. GENERAL FUND.

CHARGE.

1. Arrears of Contributions at 30th September 1919			£91 7 0
2. Contributions for present Session :—			
1. 218 Fellows at £2, 2s. each		£457 16 0	
66 Fellows at £3, 3s. each		207 18 0	
		£665 14 0	
Deduct—			
Commutation of Contributions of two Fellows—proportion thereof included in above	£4 4 0		
Contribution for present Session included in 1918–1919 Accounts	2 2 0		
		6 6 0	
		£659 8 0	
2. Fees of Admission and Contributions of nineteen new Fellows at £4, 4s. each		79 16 0	
3. Commutation Fees in lieu of future Contributions of three Fellows		60 18 0	
			800 2 0
3. Interest received—			
Interest on £7830 five per cent. War Loan, 1929–47, Untaxed		£391 10 0	
Annuity from Edinburgh and District Water Trust, less Tax, £15, 15s.		36 15 0	
Interest on Deposit Receipts		47 5 10	
			475 10 10
4. Transactions and Proceedings			55 19 10
5. Annual Grant from Government			600 0 0
6. Income Tax repaid for year to 5th April 1920			15 15 0
7. Receipts from Sale of Napier Tercentenary Memorial Volume			15 0 11
Amount of the Charge			**£2053 15 7**

DISCHARGE.

1. TAXES, INSURANCE, COAL AND LIGHTING :—		
Inhabited House Duty	£0 6 3	
Insurance	24 7 1	
Coal, etc., to 28th July 1920	46 16 3	
Gas to 11th May 1920	4 1 0	
Electric Light to 30th April 1920	6 4 1	
Water, 1919–20	4 4 0	
		£85 18 8
Carry forward		£85 18 8

Brought forward . .			£85 18 8
2. SALARIES :—			
General Secretary, 1919–20		£100 0 0	
Librarian and Assistant Secretary		180 0 0	
Assistant Librarian		95 6 8	
Office Keeper		96 6 8	
Treasurer's Clerk		35 0 0	
			506 13 4
3. EXPENSES OF TRANSACTIONS :—			
Neill & Co., Ltd., Printers		£214 19 9	
Hislop & Day, Ltd., Engravers		6 4 2	
A. Ritchie & Son		3 10 0	
Werner & Winter		17 18 6	
The Zinco-Collotype Co.		201 0 0	
Less Receipts—		£443 12 5	
Grants by Carnegie Trustees towards Messrs Thompson's, Shann's, and Kidston's Papers	£107 6 6		
Professor Ashworth	0 13 0		
Dr Wedderburn	15 14 4		
		123 13 10	
			319 18 7
4. EXPENSES OF PROCEEDINGS :—			
Neill & Co., Ltd., Printers		£435 10 9	
Hislop & Day, Ltd., Engravers		9 4 3	
Alex. Dowell		0 13 0	
Less—		£445 8 0	
Lieut. Logie—Repaid by him	£34 5 3		
Royal Society, London—Grant towards Miss Haviland's Paper	25 0 0		
Carnegie Trustees—Grant towards Dr Knott's Paper	3 0 0		
		62 5 3	
			383 2 9
5. BOOKS, PERIODICALS, NEWSPAPERS, ETC. :—			
James Thin, Bookseller		£190 17 9	
R. Grant & Son, Booksellers		6 3 0	
W. Green & Son, Ltd., Booksellers		1 13 2	
Robertson & Scott, News Agents		7 18 6	
Ray Society, Subscription		1 1 0	
Palæontographical Society, Do.		1 1 0	
Board of Scientific Societies, London, Donation		5 0 0	
Williams & Norgate		1 18 0	
Harrison & Sons		0 4 8	
			215 17 1
6. ORROCK & SON, BOOKBINDERS		£500 0 0	
Less—Received from Carnegie Trust		500 0 0	
7. OTHER PAYMENTS :—			
Neill & Co., Ltd., Printers		£126 16 1	
E. Sawers, Purveyor		39 1 0	
S. Duncan, Tailor (uniforms)		10 13 0	
S. Heddle—Bonus		25 0 0	
Andrew H. Baird		7 10 0	
Lindsay, Jamieson & Haldane, C.A., Auditors		10 10 0	
Post Office Telephone Rent		12 0 0	
A. Cowan & Sons, Ltd.		18 6 0	
Orrock & Son, Bookbinders		40 18 9	
Gillies & Wright, Joiners		5 19 0	
R. Graham, Slater		17 10 6	
Burn Brothers, Plumbers		5 14 1	
A. Black & Co., Brushmakers		4 14 3	
Macmillan & Co., Ltd.,		4 12 0	
G. Waterston & Sons, Ltd.		2 12 9	
Hislop & Day, Ltd.		0 16 6	
Travelling Expenses of Delegates to London		41 16 0	
Petty Expenses, Postages, Carriage, etc.		94 1 8	
			468 11 7
Carry forward . .			£1980 2 0

Brought forward		£1980 2 0
8. ARREARS of CONTRIBUTIONS written off		15 15 0
9. ARREARS of CONTRIBUTIONS outstanding at 30th September 1920 :—		
Present Session	£60 18 0	
Previous Sessions	31 10 0	
		92 8 0
Amount of the Discharge . .		**£2088 5 0**

Amount of the Charge	**£2053 15 7**
Amount of the Discharge	**2088 5 0**
Excess of Payments over Receipts for 1919-1920 . .	**£34 9 5**
Less—Transferred from Special Subscription Fund	34 9 5

SPECIAL SUBSCRIPTION FUND.

To 30*th September* 1920.

CHARGE.

1. Balance at 30th September 1919 :—		
Due by Union Bank of Scotland, Ltd., on Account Current		£731 10 0
Due by Treasurer		13 8 7
		£744 18 7
2. Subscriptions received		82 2 3
		£827 0 10

DISCHARGE.

1. Amount transferred to General Fund to meet Deficiency on Accounts to 30th September 1920		£34 9 5
2. Balance at 30th September 1920 :—		
Due by Union Bank of Scotland, Ltd., on Deposit Receipt .	£700 0 0	
Due by Union Bank of Scotland, Ltd., on Account Current .	88 11 11	
Due by Treasurer	3 19 6	
		792 11 5
		£827 0 10

II. KEITH FUND

To 30*th September* 1920.

CHARGE.

1. BALANCE due by Union Bank of Scotland, Ltd., at 30th September 1919 :—		
On Deposit Receipt		£59 9 1
On Account Current		14 4 10
		£73 13 11
2. INTEREST RECEIVED :—		
On £650 five per cent. War Loan, 1929-47, Untaxed . .	£32 10 0	
On Deposit Receipt, Union Bank of Scotland, Ltd. . .	1 12 2	
		34 2 2
		£107 16 1

DISCHARGE.

1. Prof. John Stephenson—Money Portion of Prize, 1917-19	£44 1 3
2. Alex. Kirkwood & Son, Engravers, Gold Medal	24 0 0
3. BALANCE due by Union Bank of Scotland, Ltd., on Account Current at 30th September 1920	39 14 10
	£107 16 1

III. NEILL FUND

To 30th September 1920.

CHARGE.

1. Balance due by Union Bank of Scotland, Ltd., at 30th September 1919 :—		
On Deposit Receipt		£30 2 10
On Account Current		1 5 6
		£31 8 4
2. Interest Received :—		
On £300 five per cent. War Loan, 1929–47, Untaxed	£15 0 0	
On Deposit Receipt, Union Bank of Scotland, Ltd.	0 13 5	
		15 13 5
		£47 1 9

DISCHARGE.

1. Prof. John Tait—Money Portion of Prize, 1917–19	£7 11 3
2. Alex. Kirkwood & Son, Engravers, Gold Medal	24 0 0
3. Balance due by Union Bank of Scotland, Ltd., on Account Current at 30th September 1920	15 10 6
	£47 1 9

IV. MAKDOUGALL-BRISBANE FUND

To 30th September 1920.

CHARGE.

1. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1919	£35 15 6
2. Interest Received :—	
On £400 five per cent. War Loan, 1929–47, Untaxed	20 0 0
	£55 15 6

DISCHARGE.

1. Balance due by Union Bank of Scotland, Ltd., at 30th September 1920 :—	
On Deposit Receipt	£35 15 6
On Account Current	20 0 0
	£55 15 6

V. MAKERSTOUN MAGNETIC METEOROLOGICAL OBSERVATION FUND

To 30th September 1920.

CHARGE.

1. Balance due by Union Bank of Scotland, Ltd., on Account Current at 30th September 1919	£23 12 11
2. Interest Received :—	
On £250 five per cent. War Loan, 1929–47, Untaxed	12 10 0
	£36 2 11

DISCHARGE.

1. Balance due by Union Bank of Scotland, Ltd., on Account Current at 30th September 1920	£36 2 11

VI. GUNNING VICTORIA JUBILEE PRIZE FUND

To 30th September 1920.

(Instituted by Dr R. H. Gunning of Edinburgh and Rio de Janeiro.)

CHARGE.

1. Balance due by Union Bank of Scotland, Ltd., at 30th September 1919 :—		
On Deposit Receipt		£57 14 2
On Account Current		42 15 0
		£100 9 2
2. Interest Received :—		
On £570 five per cent. War Loan, 1929–47, Untaxed		28 10 0
		£128 19 2

DISCHARGE.

1. Balance due by Union Bank of Scotland, Ltd., at 30th September 1920 :—		
On Deposit Receipt		£57 14 2
On Account Current		71 5 0
		£128 19 2

VII. JAMES SCOTT PRIZE FUND

To 30th September 1920.

CHARGE.

1. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1919		£264 4 0

DISCHARGE.

1. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1920		£264 4 0

VIII. DR JOHN AITKEN FUND

(For Publication of his Scientific Work.)

To 30th September 1920.

CHARGE.

1. Legacy by the late Dr John Aitken, F.R.S., Ardenlea, Falkirk		£1000 0 0
2. Interest Received :—		
On Legacy to 2nd April 1920	£19 6 3	
On Deposit Receipts, Union Bank of Scotland, Ltd.	21 5 8	
		40 11 11
		£1040 11 11

DISCHARGE.

1. Bank Charge on Cheque for £1019, 6s. 3d.		£0 10 0
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1920		1040 1 11
		£1040 11 11

STATE OF THE FUNDS BELONGING TO THE ROYAL SOCIETY OF EDINBURGH

As at 30th *September* 1920.

1. GENERAL FUND—

1. £7830 five per cent. War Loan, 1929–47, at $84\frac{3}{4}$ per cent.		£6635 18 6
2. £52, 10s. Annuity of the Edinburgh and District Water Trust, equivalent to £875 at 98 per cent.		857 10 0
3. Deposit Receipt Union Bank of Scotland, Ltd., being balance of Legacy received during 1917–18, from the Trustees of the late Mr Robert Mackay Smith, £500 *less* legacy duty £50		450 0 0
4. Arrears of Contributions, as per preceding Abstract of Accounts		92 8 0
5. Balance of Special Subscription Fund		792 11 5
AMOUNT		£8828 7 11

Exclusive of Library, Museum, Pictures, etc., and Furniture in the Society's Rooms at George Street, Edinburgh.

2. KEITH FUND—

1. £650 five per cent. War Loan, 1929–47, at $84\frac{3}{4}$ per cent.		£550 17 6
2. Balance due by Union Bank of Scotland, Ltd., on Account Current		39 14 10
AMOUNT		£590 12 4

3. NEILL FUND—

1. £300 five per cent. War Loan, 1929–47, at $84\frac{3}{4}$ per cent.		£254 5 0
2. Balance due by Union Bank of Scotland, Ltd. on Account Current		15 10 6
AMOUNT		£269 15 6

4. MAKDOUGALL-BRISBANE FUND—

1. £400 five per cent. War Loan, 1929–47, at $84\frac{3}{4}$ per cent.		£339 0 0
2. Balance due by Union Bank of Scotland, Ltd. :—		
On Deposit Receipt	£35 15 6	
On Account Current	20 0 0	
		55 15 6
AMOUNT		£394 15 6

5. MAKERSTOUN MAGNETIC METEOROLOGICAL OBSERVATION FUND—

1. £250 five per cent. War Loan, 1929–47, at $84\frac{3}{4}$ per cent.		£211 17 6
2. Balance due by Union Bank of Scotland, Ltd., on Account Current		36 2 11
AMOUNT		£248 0 5

6. GUNNING VICTORIA JUBILEE PRIZE FUND—Instituted by Dr Gunning of Edinburgh and Rio de Janeiro—

1. £570 five per cent. War Loan, 1929–47, at $84\frac{3}{4}$ per cent.		£483 1 6
2. Balance due by Union Bank of Scotland, Ltd. :—		
On Deposit Receipt	£57 14 2	
On Account Current	71 5 0	
		128 19 2
AMOUNT		£612 0 8

7. JAMES SCOTT PRIZE FUND—

Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	£264 4 0

8. TAIT MEMORIAL FUND—

This Fund consists mainly of War Loan, and is to mature for a period of about ten years from 1918, when it is expected to yield about £75 per annum.

9. DR JOHN AITKEN FUND—

Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt . . . £1040 1 11

EDINBURGH, *15th October* 1920.—We have examined the preceding Accounts of the Treasurer of the Royal Society of Edinburgh for the Session 1919–1920, and have found them to be correct. The securities of the various Investments at 30th September 1920, as noted in the foregoing Statement of Funds (with the exception of No. 8), have been exhibited to us.

LINDSAY, JAMIESON & HALDANE, C.A.,
Auditors.

THE COUNCIL OF THE SOCIETY.

October 1920.

President.

Professor FREDERICK O. BOWER, M.A., D.Sc., LL.D., F.R.S., F.L.S.

Vice-Presidents.

Professor D. NOËL PATON, M.D., B.Sc., LL.D., F.R.C.P.E., F.R.S., Professor of Physiology, University, Glasgow.

Professor A. ROBINSON, M.D., M.R.C.S., Professor of Anatomy, University, Edinburgh.

Sir GEORGE A. BERRY, M.B., C.M., LL.D., F.R.C.S.E.

Professor WILLIAM PEDDIE, D.Sc., Professor of Natural Philosophy in University College, Dundee.

Principal Sir JAMES ALFRED EWING, K.C.B., M.A., B.Sc., LL.D., M.Inst.C.E., F.R.S., Principal, University of Edinburgh.

Professor JOHN WALTER GREGORY, D.Sc., F.R.S., Professor of Geology, University of Glasgow.

General Secretary.

CARGILL G. KNOTT, D.Sc., LL.D., F.R.S.

Secretaries to Ordinary Meetings.

Professor E. T. WHITTAKER, Sc.D., F.R.S., Professor of Mathematics, University, Edinburgh.

Professor J. H. ASHWORTH, D.Sc., F.R.S., Professor of Zoology, University, Edinburgh.

Treasurer.

JAMES CURRIE, M.A., LL.D.

Curator of Library and Museum.

A. CRICHTON MITCHELL, D.Sc., Hon. D.Sc. (Geneva).

Councillors.

Professor R. A. SAMPSON, M.A., D.Sc., F.R.S.

Professor J. LORRAIN SMITH, M.A., M.D., F.R.S.

W. A. TAIT, D.Sc., M.Inst.C.E.

Maj.-General W. B. BANNERMAN, C.S.I., I.M.S., M.D., D.Sc.

HENRY MOUBRAY CADELL, of Grange, B.Sc.

Professor ARTHUR ROBERTSON CUSHNY, M.A., M.D., LL.D., F.R.S.

Professor FRANCIS GIBSON BAILY, M.A., M.Inst.E.E.

GEORGE JAMES LIDSTONE, F.F.A., F.I.A.

ROBERT CAMPBELL, M.A., D.Sc., F.G.S.

Professor JAMES COLQUHOUN IRVINE, C.B.E., Ph.D., D.Sc., F.R.S.

The Hon. LORD SALVESEN.

Professor J. ARTHUR THOMSON, M.A., LL.D.

ALPHABETICAL LIST OF THE ORDINARY FELLOWS OF THE SOCIETY,

Corrected to January 31, 1921.

N.B.—*Those marked* are Annual Contributors.*

B. prefixed to a name indicates that the Fellow has received a Makdougall-Brisbane Medal.			
K.	,,	,,	,, Keith Medal.
N.	,,	,,	,, Neill Medal.
V. J.	,,	,,	,, the Gunning Victoria Jubilee Prize.
C.	,,	,,	,, contributed one or more Communications to the Society's TRANSACTIONS or PROCEEDINGS.

Date of Election.			Service on Council, etc.
1898	C.	* Abercromby, the Rt. Hon. Lord, LL.D., 62 Palmerston Place, Edinburgh	
1898		Adami, J. G., C.B.E., M.A., M.D. (Camb., M'Gill, and Belfast), LL.D., F.R.S., Vice-Chancellor of the University of Liverpool.	
1896		Affleck, Sir Jas. Ormiston, M.D., LL.D., F.R.C.P.E., 38 Heriot Row, Edinburgh	
1895		Alford, Robert Gervase, M.Inst.C.E., Three Gables, Woodburn Park Road, Tunbridge Wells, Kent	
1889		Alison, John, M.A., Head Master, George Watson's College, Edinburgh 5	
1894		Allan, Francis John, M.D., C.M. Edin., M.O.H. City of Westminster, Westminster City Hall, Charing Cross Road, London	
1888	C.	Allardice, R. E., M.A., Professor of Mathematics in Stanford University, Palo Alto, Santa Clara Co., California	
1920	C.	* Allen, Herbert Stanley, M.A. (Cambridge), D.Sc. (London), Lecturer in Natural Philosophy in the University of Edinburgh	
1906		Anderson, Daniel E., M.D., B.A., B.Sc., Green Bank, Merton Lane, Highgate, London, N.	
1920	C.	* Anderson, Ernest Masson, M.A., B.Sc., F.G.S., Geologist, H.M. Geological Survey of Scotland, 5 Riselaw Road, Edinburgh 10	
1883		Anderson, Sir Robert Rowand, LL.D., 16 Rutland Square, Edinburgh	
1905		* Anderson, William, M.A., Head Science Master, George Watson's College, Edinburgh, 6 Lockharton Crescent, Edinburgh	
1903		Anderson-Berry, David, M.D., LL.D., F.R.S.L., M.R.A.S., F.S.A. (Scot.), Versailles, Highgate, London, N.	
1905		* Andrew, George, M.A., B.A., H.M.I.S., Balgillo Cottage, Seafield Road, Broughty Ferry	
1881	C.	Anglin, A. H., M.A., LL.D., M.R.I.A., Professor of Mathematics, Queen's College, Cork 15	
1915		Anthony, Charles, M.Inst.C.E., M. Am. Soc. C.E., F.R.San.I., F.R.A.S., F.R.Met.S., F.R.M.S., F.C.S., General Manager, Water Works Company, Vieytes Esq. Gorriti, Bahia Blanca, Argentina	
1906		Appleton, Colonel Arthur Frederick, F.R.C.V.S., Nylstroom, Smoke Lane, Reigate	
1899		Appleyard, James R., Royal Technical Institute, Salford, Manchester	
1910	C.	Archibald, E. H., B.Sc., Professor of Chemistry, University of British Columbia, Vancouver, Canada	
1907		* Archibald, James, M.A., 31 Leamington Terrace, Edinburgh 20	
1911	C. K.	* Ashworth, James Hartley, D.Sc., F.R.S., Professor of Zoology, University of Edinburgh (SECRETARY), 69 Braid Avenue, Edinburgh	1912–14, 1915–18. Sec. 1918–

Date of Election			Service on Council, etc.
1907		* Badre, Muhammad, Ph.D., Almuneerah, Cairo, Egypt	
1920		* Bagnall, Richard Siddoway, Forgemaster, Director of Engineering Works, Rydal Mount, Blaydon-on-Tyne	
1920	C.	* Bailey, Edward Battersby, M.C., B.A., F.G.S., District Geologist, H.M. Geological Survey of Scotland, 23 Pentland Terrace, Edinburgh	
1896	C.	Baily, Francis Gibson, M.A., M.Inst.E.E., Professor of Electrical Engineering, Heriot-Watt College, Edinburgh, Newbury, Colinton, Midlothian 25	1909–12. 1920–
1877	C.	Balfour, Sir I. Bayley, K.B.E., M.A., Sc.D., M.D., LL.D., F.R.S., F.L.S., King's Botanist in Scotland, Professor of Botany in the University of Edinburgh and Keeper of the Royal Botanic Garden, Inverleith House, Edinburgh	1888–91.
1905	C.	Balfour-Browne, William Alexander Francis, M.A., Barrister-at-Law, Oaklands, Fenstanton, near St Ives, Hunts	
1892	C.	Ballantyne, J. W., M.D., F.R.C.P.E., 19 Rothesay Terrace, Edinburgh	
1918		* Balsillie, David, B.Sc., F.G.S., Lecturer in Chemistry in the University of Edinburgh, 14 Greyfriars Garden, St Andrews	
1902	C.	Bannerman, W. B., C.S.I., I.M.S., M.D., D.Sc., Maj.-General, Indian Medical Service, 11 Strathearn Place, Edinburgh 30	1919–
1889		Barbour, A. H. F., M.A., M.D., LL.D., F.R.C.P.E., 4 Charlotte Square, Edinburgh	
1886		Barclay, A. J. Gunion, M.A., 3 Chandos Avenue, Oakleigh Park, London, N.	
1883	C.	Barclay, G. W. W., M.A., Raeden House, Aberdeen	
1903		Bardswell, Noël Dean, M.D., M.R.C.P. Ed. and Lond., King Edward VII Sanatorium, Midhurst	
1914	C.	* Barkla, Charles Glover, M.A., D.Sc., F.R.S., Professor of Natural Philosophy in the University of Edinburgh, 20 Hermitage Drive, Edinburgh 35	1915–18.
1882	C.	Barnes, Henry, O.B.E., M.D., LL.D., 6 Portland Square, Carlisle	
1904		Barr, Sir James, C.B.E., M.D., LL.D., F.R.C.P. Lond., 72 Rodney Street, Liverpool	
1874		Barrett, Sir William F., F.R.S., M.R.I.A., formerly Professor of Physics, Royal College of Science, Dublin, 31 Devonshire Place, London, W. 1	
1895	C.	Barton, Edwin H., D.Sc., F.R.S., F.P.S.L., F.Inst.P., A.M.Inst.E.E., Professor of Experimental Physics, University College, Nottingham	
1904		* Baxter, William Muirhead, Glenalmond, Sciennes Gardens, Edinburgh 40	
1913		Beard, Joseph, F.R.C.S. (Edin.), M.R.C.S. (Eng.), L.R.C.P. (Lond.), D.P.H. (Camb.), Medical Officer of Health and School Medical Officer, City of Carlisle, 8 Carlton Gardens, Carlisle	
1888		Beare, Thomas Hudson, B.A., B.Sc., M.Inst.C.E., J.P., D.L., Professor of Engineering in the University of Edinburgh	1907–09. V-P 1909–15.
1897	C.	* Beattie, Sir John Carruthers, K.B., D.Sc., Vice-Chancellor and Principal, The University, Cape Town	
1892		Beck, Sir J. H. Meiring, Kt., M.D., M.R.C.P.E., Drostdy, Tulbagh, Cape Province, South Africa	
1893	C. B.	Becker, Ludwig, Ph.D., Regius Professor of Astronomy in the University of Glasgow, The Observatory, Glasgow 45	
1882	C.	Beddard, Frank E., M.A. Oxon., F.R.S., Prosector to the Zoological Society of London, Zoological Society's Gardens, Regent's Park, London	
1887		Begg, Ferdinand Faithfull, 46 Saint Aubyns, Hove, Sussex	
1906		Bell, John Patrick Fair, F.Z.S., Springbank, Ayton, Berwickshire	
1916		* Bell, Robert John Tainsh, M.A., D.Sc., Professor of Mathematics in the University of Otago, New Zealand	
1915		Bell, Walter Leonard, M.D.Edin., F.S.A.Scot., 123 London Road, North Lowestoft, Suffolk 50	
1893	C.	Berry, Sir George A., M.B., C.M., LL.D., F.R.C.S.E. (Vice-President), 31 Drumsheugh Gardens, Edinburgh	1916–19. V-P 1919–
1897	C.	Berry, Richard J. A., M.D., F.R.C.S.E., Professor of Anatomy in the University of Melbourne, Victoria, Australia	
1880	C.	Birch, De Burgh, C.B., M.D., Emeritus Professor of Physiology in the University of Leeds	
1907		* Black, Frederick Alexander, Solicitor, 59 Academy Street, Inverness	
1884	C.	Black, John S., M.A., LL.D., 125 St James' Court, London, S.W. 1 55	1891–94, 1916–18. Cur. 1906–16.
1897	C.	* Blaikie, Walter Biggar, LL.D., The Loan, Colinton	1914–17.
1904	C.	* Bles, Edward J., M.A., D.Sc., Elterholm, Cambridge	

Date of Election.			Service on Council, etc.
1918		* Blight, Francis James, Chairman and Managing Director of Charles Griffin & Co., Ltd., Publishers, Tregenna, Wembley, Middlesex	
1894		Bolton, Herbert, M.Sc., F.G.S., F.Z.S., Director of the Bristol Museum and Art Gallery, Bristol, 58 Coldharbour Road, Redland, Bristol	
1915		* Boon, Alfred Archibald, D.Sc., F.I.C., B.A., Professor of Chemistry, Heriot-Watt College, Edinburgh 60	
1872	C.	Bottomley, J. Thomson, M.A., D.Sc., LL.D., F.R.S., F.C.S., 13 University Gardens, Glasgow	
1886	C.	Bower, Frederick O., M.A., D.Sc., LL.D., F.R.S., F.L.S. (President), Regius Professor of Botany in the University of Glasgow, 1 St John's Terrace, Hillhead, Glasgow	1887–90, 1893–96, 1907–09, 1917–19 V-P 1910–16. P 1919–
1884	C.	Bowman, Frederick Hungerford, D.Sc., F.C.S. (Lond. and Berl.), F.I.C., A.Inst.C.E., A.Inst.M.E., M.Inst.E.E., etc., 76 Acomb Street, Whitworth Park, Manchester	
1901		Bradbury, J. B., M.D., Downing Professor of Medicine, University of Cambridge	
1916		Bradley, Francis Ernest, M.A., M.Com., LL.D., Barrister-at-Law, Examiner to the Council of Legal Education, Bank of England Chambers, Tib Lane, Manchester 65	
1903	C.	* Bradley, O. Charnock, M.D., D.Sc., Principal, Royal Dick Veterinary College, Edinburgh, President of the Royal College of Veterinary Surgeons, London	1907–10, 1915–17.
1886		Bramwell, Byrom, M.D., F.R.C.P.E., LL.D., 23 Drumsheugh Gardens, Edinburgh	1890–93.
1907		* Bramwell, Edwin, M.D., F.R.C.P.E., F.R.C.P. Lond., 23 Drumsheugh Gardens, Edinburgh	
1918		* Bremner, Alexander, M.A., D.Sc., Headmaster, Demonstration School, Training Centre, Aberdeen, 13 Belgrave Terrace, Aberdeen	
1912		Bridger, Adolphus Edward, M.D. (Edin.), F.R.C.P. (Edin.), B.Sc. (Paris), B.L. (Paris), Foley Lodge, Langham Street, London, W. 70	
1916	C.	* Briggs, Henry, D.Sc., A.R.S.M., Professor of Mining, Heriot-Watt College, Allermuir, Liberton, Midlothian	
1895		Bright, Sir Charles, M.Inst.C.E., M.Inst.E.E., F.R.Aë.S., F.Inst.Radio.E., F.R.A.S., F.R.G.S., Leigh Grange, Kent, and Athenæum Club, Pall Mall, London, S.W.	
1893		Brock, G. Sandison, M.D., 6 Corso d'Italia, Rome, Italy	
1901	C.	* Brodie, W. Brodie, M.B., Camden House, Bletchingley, Surrey	
1907		Brown, Alexander, M.A., B.Sc., Professor of Applied Mathematics, The University, Cape Town 75	
1864	C. K. B.	Brown, Alex. Crum, M.A., M.D., D.Sc., F.R.C.P.E., LL.D., F.R.S., Emeritus Professor of Chemistry in the University of Edinburgh, 8 Belgrave Crescent, Edinburgh	1865–68, 1869–72, 1873–75, 1876–78, 1911–13. Sec. 1879–1905. V-P 1905–11.
1898		* Brown, David, F.C.S., F.I.C., J.P., Willowbrae House, Willowbrae Road, Edinburgh	
1883	C.	Brown, J. J. Graham, M.D., F.R.C.P.E., 3 Chester Street, Edinburgh	
1885	C.	Brown, J. Macdonald, M.D., F.R.C.S., 64 Upper Berkeley Street, Portman Square, London, W.	
1909	B. C.	* Brownlee, John, M.A., M.D., D.Sc., the National Institute for Medical Research, Mount Vernon, Hampstead, N.W. 3 80	
1912		* Bruce, Alexander Ninian, D.Sc., M.D., 8 Ainslie Place, Edinburgh	
1906	C. N.	Bruce, William Speirs, LL.D., formerly Director of the Scottish Oceanographical Laboratory, Edinburgh, Antarctica, Joppa, Edinburgh	1909–12.
1898	C. K.	* Bryce, T. H., M.A., M.D. (Edin.), Professor of Anatomy in the University of Glasgow, 2 The University, Glasgow	1911–14.
1870	C. K.	Buchanan, John Young, M.A., F.R.S., Athenæum Club, Pall Mall, London, S.W.	1878–81, 1884–86.

Date of Election.			Service on Council, etc.
1905		Bunting, Thomas Lowe, M.D., 27 Denton Road, Scotswood, Newcastle-on-Tyne 85	
1902		* Burgess, A. G., M.A., Rector of The Academy, Rothesay, Blythswood, Rothesay	
1887		Burnet, Sir John James, LL.D., R.S.A., Architect, 239 St Vincent Street, Glasgow	
1888		Burns, Rev. T., D.D., F.S.A. Scot., Minister of Lady Glenorchy's Parish Church, Croston Lodge, Chalmers Crescent, Edinburgh	
1917		* Burnside, George Barnhill, Admiralty Experimental Station, Shandon, Dumbartonshire	
1915		* Butchart, Raymond Keiler, B.Sc., University College, Dundee, 5 Briarwood Terrace, West Park Road, Dundee 90	
1896		Butters, J. W., M.A., B.Sc., Rector of Ardrossan Academy	
1887	C.	Cadell, Henry Moubray, of Grange, B.Sc., Linlithgow	1919–
1910		* Calderwood, Rev. Robert Sibbald, Minister of Cambuslang, The Manse, Cambuslang, Lanarkshire	
1893	C.	Calderwood, W. L., Inspector of Salmon Fisheries of Scotland, South Bank, Canaan Lane, Edinburgh	
1894		Cameron, James Angus, M.D., Medical Officer of Health, Firhall, Nairn 95	
1905	C.	Cameron, John, M.D., D.Sc., M.R.C.S. Eng., Dalhousie University, Halifax, Nova Scotia	
1904		* Campbell, Charles Duff, Scottish Liberal Club, Princes Street, Edinburgh	
1918		* Campbell, John Menzies, L.D.S. (Glas.), D.D.S. (Toronto), L.D.S. (Ontario), 14 Buckingham Terrace, Glasgow, W.	
1915	C. N.	* Campbell, Robert, M.A., D.Sc., F.G.S., Lecturer in Petrology, University of Edinburgh, 2 Woodhall Road, Colinton	1920–
1899	C.	* Carlier, Edmund W. W., M.D., M.Sc., F.E.S., Professor of Physiology, University, Birmingham 100	
1910		Carnegie, Col. David, C.B.E., M.Inst.C.E., M.Inst.Mech.E., M.I.S.Inst., "Woodlands," Beckenham Hill, Kent	
1920	C.	* Carruthers, R. G., F.G.S., District Geologist, H.M. Geological Survey, 28 Jermyn Street, London	
1905	C.	* Carse, George Alexander, M.A., D.Sc., Lecturer on Natural Philosophy, University of Edinburgh, 3 Middleby Street, Edinburgh	
1901		Carslaw, H. S., M.A., D.Sc., Professor of Mathematics in the University of Sydney, New South Wales	
1905		Carter, Joseph Henry, F.R.C.V.S., Avalon, Western Road, Henley-on-Thames 105	
1898		* Carter, Wm. Allan, O.B.E., M.Inst. C.E., Stamford Hall, Gullane	1911–14.
1898		Carus-Wilson, Cecil, F.R.G.S., F.G.S., Waldegrave Park, Strawberry Hill, Middlesex, and Sandacres Lodge, Parkstone-on-Sea, Dorset	
1908		Cavanagh, Thomas Francis, M.D., The Hospital, Bella Coola, B.C., Canada	
1882		Cay, W. Dyce, M.Inst.C.E., Junior Carlton Club, Pall Mall, London, S.W. 1	
1899		Chatham, James, Actuary, c/o Robert Murrie, Esq., 28 St Andrew Square, Edinburgh 110	
1912		Chaudhuri, Banawari Lal, B.A. (Cal.), B.Sc. (Edin.), Assistant Superintendent, Natural History Section, Indian Museum, 120 Lower Circular Road, Calcutta, India	
1874		Chiene, John, C.B., M.D., LL.D., F.R.C.S.E., Emeritus Professor of Surgery in the University of Edinburgh, Barnton Avenue, Davidson's Mains	1884–86, 1904–06.
1891		Clark, John B., M.A., Head Master of Heriot's Hospital School, Lauriston, Garleffin, 146 Craiglea Drive, Edinburgh	
1911		* Clark, William Inglis, D.Sc., 22 Buckingham Terrace, Edinburgh	
1903		* Clarke, William Eagle, LL.D., F.L.S., Keeper of the Natural History Collections in the Royal Scottish Museum, Edinburgh, 35 Braid Road, Edinburgh 115	
1909		Clayton, Thomas Morrison, M.D., D.Hy., B.Sc., D.P.H., Medical Officer of Health, Gateshead, 13 The Crescent, Gateshead-on-Tyne	
1913		* Cleghorn, Alexander, M.Inst.C.E., Marine Engineer, 14 Hatfield Drive, Kelvinside, Glasgow	
1904	C.	Coker, Ernest George, M.A., D.Sc., Hon. D.Sc. (Sydney), F.R.S., M.Inst.C.E., M.Inst.E.E., Professor of Civil and Mechanical Engineering, University of London, University College, Gower Street, London, W.C.	
1904		Coles, Alfred Charles, M.D., D.Sc., York House, Poole Road, Bournemouth, W.	
1888	V. J. C.	Collie, John Norman, Ph.D., D.Sc., LL.D., F.R.S., F.C.S., F.I.C., F.R.G.S., Professor of Organic Chemistry in the University College, Gower Street, London 120	

Date of Election.			Service on Council, etc.
1904		* Colquhoun, Walter, M.A., M.B., 18 Walmer Crescent, Ibrox, Glasgow	
1909	C.	* Comrie, Peter, M.A., B.Sc., Head Mathematical Master, Boroughmuir Junior Student Centre, 19 Craighouse Terrace, Edinburgh	
1886		Connan, Daniel M., M.A.	
1905		* Corrie, David, F.C.S., 159 Lauderdale Mansions, Maida Vale, London, W. 9.	
1914		* Coutts, William Barron, M.A., B.Sc., 33 Dalhousie Terrace, Edinburgh, Royal Garrison Artillery, Ordnance College, Woolwich, London, S.E. 125	
1911		* Cowan, Alexander, Papermaker, Valleyfield, Penicuik, Midlothian	
1920		Craib, William Grant, M.A. (Aberdeen), Regius Professor of Botany in the University of Aberdeen	
1916	C.	Craig, E. H. Cunningham, B.A. (Cambridge), Geologist and Mining Engineer, The Dutch House, Beaconsfield	
1908		Craig, James Ireland, M.A., B.A., Woolwich House, The Drive, Sydenham, London, S.E. 26	
1875		Craig, William, M.D., F.R.C.S.E., Lecturer on Materia Medica to the College of Surgeons, 71 Bruntsfield Place, Edinburgh 130	
1903		Crawford, Lawrence, M.A., D.Sc., Professor of Pure Mathematics, The University, Cape Town	
1870		Crichton-Browne, Sir Jas., Kt., M.D., LL.D., D.Sc., F.R.S., Lord Chancellor's Visitor and Vice-President and Treasurer of the Royal Institution of Great Britain, 45 Hans Place, S.W., and Royal Courts of Justice, Strand, London	
1916		* Crombie, James Edward, M.A., LL.D., Millowner, Parkhill House, Dyce, Aberdeenshire	
1886		Croom, Sir John Halliday, Kt., M.D., F.R.C.P.E., Professor of Midwifery in the University of Edinburgh, late President, Royal College of Surgeons, Edinburgh, 25 Charlotte Square, Edinburgh	
1914		* Cumming, Alexander Charles, D.Sc., O.B.E., Lecturer in Chemistry, University, Edinburgh, 2 Relugas Road, Edinburgh 135	
1917		* Cunningham, Brysson, D.Sc., B.E., M.Inst.C.E., Civil Engineer, 16 Beechwood Road, Sanderstead, Surrey	
1898		* Currie, James, M.A. Cantab., LL.D. (TREASURER), Larkfield, Goldenacre, Edinburgh	Treas. 1906–
1919		* Cushny, Arthur Robertson, M.A., M.D., LL.D., F.R.S., Professor of Materia Medica and Pharmacology, University, Edinburgh	1919–
1904		* Cuthbertson, John, Secretary, West of Scotland Agricultural College, 6 Charles Street, Kilmarnock	
1885		Daniell, Alfred, M.A., LL.B., D.Sc., Advocate, The Athenæum Club, Pall Mall, London 140	
1884		Davy, R., F.R.C.S. Eng., Consulting Surgeon to Westminster Hospital, Burstone Manor, Bow, North Devon	
1917		* Day, T. Cuthbert, Partner of the firm of Hislop & Day, 36 Hillside Crescent, Edinburgh	
1894		Denny, Sir Archibald, Bart., LL.D., Cardross Park, Cardross, Dumbartonshire	
1869	C. V. J.	Dewar, Sir James, Kt., M.A., LL.D., D.C.L., D.Sc., F.R.S., V.P.C.S., Jacksonian Professor of Natural and Experimental Philosophy in the University of Cambridge, and Fullerian Professor of Chemistry at the Royal Institution of Great Britain, London	1872–74.
1905		* Dewar, James Campbell, C.A., 27 Douglas Crescent, Edinburgh 145	
1906		* Dewar, Thomas William, M.D., F.R.C.P., Kincairn, Dunblane	
1884		Dickson, the Right Hon. Charles Scott, Lord Justice-Clerk, K.C., LL.D., 22 Moray Place, Edinburgh	
1888	C.	Dickson, Henry Newton, C.B.E., M.A., D.Sc., Professor of Geography at University College, Reading, 160 Castle Hill, Reading.	
1876	C.	Dickson, J. D. Hamilton, M.A., Senior Fellow and formerly Tutor, St Peter's College, Cambridge	
1885	C.	Dixon, James Main, M.A., Litt. Hum. Doctor, Professor of English, University of Southern California, University Avenue, Los Angeles, California, U.S.A. 150	
1897		* Dobbie, James Bell, F.Z.S., 12 South Inverleith Avenue, Edinburgh	
1904		* Dobbie, Sir James Johnston, Kt., M.A., D.Sc., LL.D., F.R.S., formerly Principal of the Government Laboratories, London, 4 Vicarage Gate, Kensington, London, W.	1905–08.
1881	C.	Dobbin, Leonard, Ph.D., Lecturer in Chemistry in the University of Edinburgh, 6 Wilton Road, Edinburgh	1904–07, 1913–16.
1918		* Dodd, Alexander Scott, B.Sc., F.I.C., F.C.S., City Analyst for Edinburgh, 20 Stafford Street, Edinburgh	

Date of Election.			Service on Council, etc.
1905		* Donaldson, Rev. Wm. Galloway, J.P., F.R.G.S., F.E.I.S., The Manse, Forfar 155	
1882	C.	Dott, David B., F.I.C., Memb. Pharm. Soc., Ravenslea, Musselburgh	
1918		* Douglas, Carstairs Cumming, M.D., D.Sc., Professor of Medical Jurisprudence and Hygiene, Anderson's College, Glasgow, 2 Royal Crescent, Glasgow	
1910		* Douglas, Loudon MacQueen, Author and Lecturer, 29 W. Saville Terrace, Newington, Edinburgh	
1908	C.	Drinkwater, Harry, M.D., M.R.C.S. (Eng.), F.L.S., Lister House, Wrexham, North Wales	
1901		* Drinkwater, Thomas W., L.R.C.P.E., L.R.C.S.E., Chemical Laboratory, Surgeons' Hall, Edinburgh 160	
1917		* Dron, Robert W., A.M.Inst.C.E., 65 Renfield Street, Glasgow	
1919		* Dundas, William John, W.S., LL.D., Crown Agent for Scotland, 11 Drumsheugh Gardens, Edinburgh	
1904		* Dunlop, William Brown, M.A., 4a St Andrew Square, Edinburgh	
1903		Dunstan, John, M.R.C.V.S., Inversnaid, Liskeard, Cornwall	
1892	C.	Dunstan, M. J. R., M.A., F.I.C., F.C.S., Principal, South-Eastern Agricultural College, Wye, Kent 165	
1906	C.	Dyson, Sir Frank Watson, Kt., M.A., LL.D., F.R.S., Astronomer Royal, Royal Observatory, Greenwich	1907–10.
1893		Edington, Alexander, M.D., Howick, Natal	
1904		* Edwards, John, LL.D., 4 Great Western Terrace, Kelvinside, Glasgow	
1904		* Elder, William, M.D., F.R.C.P.E., 4 John's Place, Leith	
1875		Elliot, Daniel G., American Museum of Natural History, Central Park West, New York, N.Y., U.S.A. 170	
1921		* Elliot, George Francis Scott, M.A. (Cantab.), B.Sc., F.R.G.S., F.L.S., Drumwhill, Mossdale	
1906	C.	* Ellis, David, D.Sc., Ph.D., Lecturer in Botany and Bacteriology, Glasgow and West of Scotland Technical College, Glasgow	
1897	C.	* Erskine-Murray, James Robert, D.Sc., 16 Elmfield Road, Bromley, Kent	
1884		Evans, William, F.F.A., 38 Morningside Park, Edinburgh	
1879	C. N.	Ewart, James Cossar, M.D., F.R.C.S.E., F.R.S., F.Z.S., Regius Professor of Natural History, University of Edinburgh, Craigybield, Penicuik, Midlothian 175	1882–85, 1904–07. V-P 1907–12.
1902		* Ewen, John Taylor, B.Sc., M.I.Mech.E., H.M. Inspector of Schools, Clairmont, 54 Albert Drive, Pollokshields, Glasgow	
1878	C.	Ewing, Sir James Alfred, K.C.B., M.A., B.Sc., LL.D., M.Inst.C.E., F.R.S., J.P. (Vice-President), Principal of the University of Edinburgh, formerly Director of Naval Education, Admiralty, 16 Moray Place, Edinburgh	1888–91, 1919–20. V-P 1920–
1900	C.	Eyre, John W. H., M.D., M.S. (Dunelm), D.P.H. (Camb.), Professor of Bacteriology, Guy's Hospital, London	
1910		* Fairgrieve, Mungo M'Callum, M.A. (Glasg.), M.A. (Cambridge), Master at the Edinburgh Academy, 37 Queen's Crescent, Edinburgh	
1907	C.	Falconer, John Downie, M.A., D.Sc., F.G.S., Lecturer on Geography, The University, Glasgow 180	
1888	C.	Fawsitt, Charles A., Coney Park, Bridge of Allan	
1883	C.	Felkin, Robert W., M.D., F.R.G.S., Whare Ra, Havelock North, Hawk's Bay, New Zealand	
1899		* Fergus, Andrew Freeland, M.D., 22 Blythswood Square, Glasgow	
1907		* Fergus, Edward Oswald, c/o 22 Blythswood Square, Glasgow	
1904		* Ferguson, James Haig, M.D., F.R.C.P.E., F.R.C.S.E., 7 Coates Crescent, Edinburgh 185	
1898		* Findlay, Sir John R., K.B.E., M.A. Oxon., 3 Rothesay Terrace, Edinburgh	
1899		* Finlay, David W., B.A., M.D., LL.D., F.R.C.P., D.P.H., Emeritus Professor of Medicine in the University of Aberdeen, Honorary Physician to His Majesty in Scotland, Balgownie, Helensburgh	
1911		Fleming, John Arnold, F.C.S., etc., Pottery Manufacturer, 136 Glebe Street, St Rollox, Glasgow, Locksley, Helensburgh	
1906		* Fleming, Robert Alexander, M.A., M.D., F.R.C.P.E., Assistant Physician, Royal Infirmary, 10 Chester Street, Edinburgh	
1900	C. N.	* Flett, John S., M.A., D.Sc., LL.D., F.R.S., O.B.E., Director of the Geological Survey of Great Britain and of the Museum of Practical Geology, London, 28 Jermyn Street, S.W. 1 190	1916–19.
1872		Forbes, Professor George, M.A., M.Inst.C.E., M.Inst.E.E., F.R.S., F.R.A.S., 11 Little College Street, Westminster, S.W.	
1892		Ford, John Simpson, F.C.S., 7 Corrennie Drive, Edinburgh	

Date of Election.			Service on Council, etc.
1920	C.	* Franklin, Thomas Bedford, B.A. (Hons. Mathematics), Cambridge, Assistant Master, Fettes College, Edinburgh, 1 Moredun Crescent, Edinburgh	
1910		* Fraser, Alexander, Actuary, 17 Eildon Street, Edinburgh	
1896		Fraser, John, M.B., F.R.C.P.E., formerly one of H.M. Commissioners in Lunacy for Scotland, 54 Great King Street, Edinburgh 195	
1915		* Fraser, Rev. Joseph Robert, U.F. Manse, Kinneff, Bervie	
1914		* Fraser, William, Managing Director, Neill & Co., Ltd., Printers, 212 Causewayside, Edinburgh	
1891		Fulton, T. Wemyss, M.D., Scientific Superintendent Scottish Fishery Board, 41 Queen's Road, Aberdeen	
1907		* Galbraith, Alexander, "Ravenswood," Dalmuir, Dumbartonshire	
1918		* Galloway, T. Lindsay, M.A., F.G.S., Assoc. M.Inst.C.E., M.Inst.M.E., Coalmaster, Kilchrist, Campbeltown, Argyllshire 200	
1888	C.	Galt, Alexander, D.Sc., St Margaret's, Craiglockhart, Edinburgh	
1901		Ganguli, Sanjiban, M.A., Principal, Maharaja's College, and Director of Public Instruction, Jaipur State, Jaipur, India	
1899		Gatehouse, T. E., A.M.Inst.C.E., M.Inst.M.E., M.Inst.E.E., Fairfield, 128 Tulse Hill, London, S.W.	
1909	C.	* Geddes, Rt. Hon. Sir Auckland C., K.C.B., M.D., D.C.L., British Ambassador to the U.S.A., The British Embassy, Washington	
1880	C.	Geddes, Patrick, Professor of Botany in University College, Dundee, and Lecturer on Zoology, Ramsay Garden, University Hall, Edinburgh 205	
1861	C. B.	Geikie, Sir Archibald, O.M., K.C.B., D.C.L. Oxf., D.Sc., LL.D., Ph.D., Late Pres. R.S., Foreign Member of the Reale Accad. Lincei, Rome, of the National Acad. of the United States, of the Academies of Stockholm, Christiania, Göttingen, Corresponding Member of the Institute of France and of the Academies of Berlin, Vienna, Munich, Turin, Belgium, Philadelphia, New York, etc., Shepherd's Down, Haslemere, Surrey	1869–72, 1874–76, 1879–82.
1914		Gemmell, John Edward, M.B., C.M., Hon. Surgeon Hospital for Women and Maternity Hospital; Hon. Gynæcologist, Victoria Central Hospital, Liscard, 28 Rodney Street, Liverpool	
1909		* Gentle, William, B.Sc., 12 Mayfield Road, Edinburgh	
1920	C.	* Ghosh, Sudhamoy, M.Sc. (Cal.), D.Sc. (Edin.), F.C.S., Government Research Chemist, Medical College, Calcutta, 9/1 Rammohan Dutt Road, Bhowanipur, Calcutta, India	
1914		* Gibb, Sir Alexander, G.B.E., C.B., Director-General of Civil Engineering, Ministry of Transport, 6 Whitehall Gardens, London, S.W. 1 210	
1916		* Gibb, A. W., D.Sc., Lecturer in Geology, The University, Aberdeen, 1 Belvidere Street, Aberdeen	
1910	C.	* Gibb, David, M.A., B.Sc., Lecturer in Mathematics, Edinburgh University, 15 South Lauder Road, Edinburgh	
1917	C.	* Gibson, Alexander, M.B., Ch.B., F.R.C.S. (Eng.), Professor of Anatomy in the Medical College, Winnipeg, Canada	
1910		* Gibson, Charles Robert, Lynton, Mansewood, by Pollokshaws	
1890		Gibson, George A., M.A., LL.D. Professor of Mathematics in the University of Glasgow, 10 The University, Glasgow 215	1905–08, 1912–13. V-P 1917–20.
1911		Gidney, Henry A. J., L.M. and S. Socts. Ap. (Lond.), F.R.C.S. (Edin.), D.P.H. (Camb.), D.O. (Oxford), Army Specialist Public Health, c/o Thomas Cook & Sons, Ludgate Circus, London	
1900		Gilchrist, Douglas A., B.Sc., Professor of Agriculture and Rural Economy, Armstrong College, Newcastle-upon-Tyne	
1880		Gilruth, George Ritchie, Surgeon, Allanton House, Bridge of Allan	
1907		Gilruth, John Anderson, M.R.C.V.S., D.V.Sc. (Melb.), Administrator, Government House, Darwin Northern Territory, Australia	
1909		* Gladstone, Hugh Steuart, M.A., M.B.O.U., F.Z.S., Capenoch, Thornhill, Dumfriesshire 220	
1911		Gladstone, Reginald John, M.D., F.R.C.S. (Eng.), Lecturer and Senior Demonstrator of Anatomy, King's College, University of London, 22 Regent's Park Terrace, London, N.W.	
1898		* Glaister, John, M.D., F.R.F.P.S. Glasgow, D.P.H. Camb., Professor of Forensic Medicine in the University of Glasgow, 3 Newton Place, Glasgow	
1910		Goodall, Joseph Strickland, M.B. (Lond.), M.S.A. (Eng.), Lecturer on Physiology, Middlesex Hospital, London, Annandale Lodge, Vanbrugh Park, Blackheath, London, S.E.	
1901		Goodwillie, James, M.A., B.Sc., Liberton, Edinburgh	

Date of Election.			Service on Council, etc.
1920	C.	* Gordon, William, B.Sc., A.M.I.Mech.E., Lecturer in Engineering in the University of Edinburgh, 3 Wellington Street, Edinburgh 225	
1913	C.	* Gordon, William Thomas, M.A., D.Sc. (Edin.), B.A. (Cantab.), Lecturer in Geology, University of London, King's College, Strand, W.C.	
1897		Gordon-Munn, John Gordon, M.D., Heigham Hall, Norwich	
1898	C.	* Gray, Albert, A., M.D., 4 Clairmont Gardens, Glasgow	
1883	C.	Gray, Andrew, M.A., LL.D., F.R.S., Professor of Natural Philosophy in the University of Glasgow	1903–06. V-P 1906–09.
1910		Gray, Bruce M'Gregor, C.E., A.M.Inst.C.E., Westbourne Grove, Selby, Yorkshire 230	
1909	C.	* Gray, James Gordon, D.Sc., Professor of Applied Physics in the University of Glasgow, 11 The University, Glasgow	1913–15.
1918		* Gray, Wm. Forbes, F.S.A. (Scot.), Editor and Author, 8 Mansionhouse Road, Edinburgh	
1897		Greenlees, Thomas Duncan, M.D. Edin., Yèresco, Fordingbridge, Hants	
1905	C.	* Gregory, John Walter, D.Sc., F.R.S. (Vice-President), Professor of Geology in the University of Glasgow, 4 Park Quadrant, Glasgow	1908–11. V-P 1920–
1906		Greig, Edward David Wilson, C.I.E., M.D., D.Sc., Lt.-Col., H.M. Indian Medical Service, Pasteur Institute, Kasauli, India 235	
1905		Greig, Sir Robert Blyth, LL.D., F.Z.S., Board of Agriculture for Scotland, 29 St Andrew Square, Edinburgh	
1910		* Grimshaw, Percy Hall, Assistant Keeper, Natural History Department, The Royal Scottish Museum, 49 Comiston Drive, Edinburgh	
1899		* Guest, Edward Graham, M.A., B.Sc., 5 Newbattle Terrace, Edinburgh	
1907	C.	* Gulliver, Gilbert Henry, D.Sc., A.M.I.Mech.E., 99 Southwark Street, London, S.E.	
1911	C.	* Gunn, James Andrew, M.A., M.D., D.Sc., Professor of Pharmacology in the University of Oxford 240	
1888	C.	Guppy, Henry Brougham, M.B., Rosario, Salcombe, Devon	
1911		* Guy, William, F.R.C.S., L.R.C.P., L.D.S.Ed., Consulting Dental Surgeon, Edinburgh Royal Infirmary; Dean, Edinburgh Dental Hospital and School; Lecturer on Human and Comparative Dental Anatomy and Physiology, 11 Wemyss Place, Edinburgh	
1911		Hall-Edwards, John Francis, L.R.C.P. (Edin.), Hon. F.R.P.S., Senior Medical Officer in charge of X-ray Department, General Hospital, Birmingham, 141A and 141B Great Charles Street (Newhall Street), Birmingham	
1918		* Hardie, P. S., M.A., B.Sc., Lecturer in Physics, Sultania Training College, Cairo, Egypt	
1896	C.	Harris, David Fraser, B.Sc. (Lond.), D.Sc. (Birm.), M.D., F.S.A. Scot., Professor of Physiology in the Dalhousie University, Halifax, Nova Scotia 245	
1914		Harrison, Edward Philip, Ph.D., Professor of Physics, Presidency College, University of Calcutta, The Observatory, Alipore, Calcutta	
1917		* Harrison, John, C.B.E., J.P., LL.D., Chairman of the Edinburgh Public Library, Rockville, Napier Road, Edinburgh	
1914	C.	Harvey-Gibson, Robert John, C.B.E., M.A., D.L. and J.P. for the County Palatine of Lancaster, Mem. Roy. Dub. Soc., Professor of Botany, University of Liverpool, 18 Gambier Terrace, Liverpool	
1880	C.	Haycraft, J. Berry, M.D., D.Sc., Professor of Physiology in the University College of South Wales and Monmouthshire, Cardiff	
1892	C.	Heath, Thomas, B.A., formerly Assistant Astronomer, Royal Observatory, Edinburgh, 11 Cluny Drive, Edinburgh 250	
1893		Hehir, Sir Patrick, K.C.I.E., C.B., C.M.G., M.D., F.R.C.S.E., M.R.C.S., L.R.C.P.E., Retired Maj.-General I.M.S., 3 Nelson Terrace, Westward Ho! N. Devon	
1890	C.	Helme, T. Arthur, M.D., M.R.C.P., M.R.C.S., Tan y vron, Rhosneigr, Ty Croes, R.S.O., Anglesey	
1900		Henderson, John, D.Sc., A.Inst.E.E., Kinnoul, Warwick's Bench Road, Guildford, Surrey	
1908		* Henderson, William Dawson, M.A., B.Sc., Ph.D., Lecturer, Zoological Laboratories, University, Bristol	
1890	C.	Hepburn, David, C.M.G., M.D., Professor of Anatomy in the University College of South Wales and Monmouthshire, Cardiff 255	
1881	C. N.	Herdman, W. A., D.Sc., LL.D., F.R.S., Past Pres. L.S., Professor of Natural History in the University of Liverpool, Croxteth Lodge, Ullet Road, Liverpool	

Date of Election.			Service on Council, etc.
1916		* Herring, Percy Theodore, M.D., F.R.C.P.Ed., Professor of Physiology, University of St Andrews, Hepburn Gardens, St Andrews	1917–20.
1894		Hill, Alfred, M.D., M.R.C.S., F.I.C., Valentine Mount, Freshwater Bay, Isle of Wight	
1902		Hinxman, Lionel W., B.A., formerly of the Geological Survey of Scotland, Kensington House, Ringwood, Hants	
1904		Hobday, Major Frederick T. G., C.M.G., F.R.C.V.S., Officier du Merite Agricole, Cavaliere dei S.S. Maurizio e Lazaro, Hon. Veterinary Surgeon to H.M. the King, Editor of the Veterinary Journal, 165 Church Street, Kensington, London, W. 260	
1885		Hodgkinson, W. R., C.B.E., M.A., Ph.D., F.I.C., F.C.S., Professor of Chemistry and Physics at the Ordnance College, Woolwich, 89 Shooter's Hill Road, Blackheath, Kent	
1911		Holland, William Jacob, LL.D. St Andrews, etc., Director Carnegie Institute, Pittsburg, Pa., 5545 Forbes Street, Pittsburg, Pa., U.S.A.	
1920	C.	* Horne, Alexander Robert, O.B.E., B.Sc., M.I.Mech.E., A.M.I.C.E., Professor of Engineering, Robert Gordon's Technical College, Aberdeen, 374 Great Western Road, Aberdeen	
1881	C. N.	Horne, John, LL.D., F.R.S., F.G.S., formerly Director of the Geological Survey of Scotland, 20 Merchiston Gardens, Edinburgh	1902–05, 1906–07, 1914–15. V-P 1907–1913. P 1915–19.
1896		Horne, J. Fletcher, M.D., F.R.C.S.E., The Poplars, Barnsley 265	
1904	C.	* Horsburgh, Ellice Martin, M.A., D.Sc., Reader in Technical Mathematics, University of Edinburgh, 11 Granville Terrace, Edinburgh	
1897		Houston, Sir Alex. Cruikshanks, K.B.E., C.V.O., M.B., C.M., D.Sc., 19 Fairhazel Gardens, South Hampstead, London, N.W.	
1912	C. B.	* Houstoun, Robert Alexander, M.A., Ph.D., D.Sc., Lecturer in Physical Optics, University, Glasgow, 45 Kirklee Road, Glasgow	
1893		Howden, Robert, M.A., M.B., C.M., D.Sc., Professor of Anatomy in the University of Durham, 14 Burdon Terrace, Newcastle-upon-Tyne	
1883	C.	Hoyle, William Evans, M.A., D.Sc., M.R.C.S., Director of the Welsh National Museum : Crowland, Llandaff, Wales 270	
1910		Hume, William Fraser, D.Sc. (Lond.), Director, Geological Survey of Egypt, Helwân, Egypt	
1916		* Hunter, Charles Stewart, L.R.C.P.E., L.R.C.S.E., D.P.H., Walden, Anerley Road, London, S.E. 20	
1911		Hunter, Gilbert Macintyre, M.Inst.C.E , M.Inst.E.S., M.Inst.M.E., Resident Engineer Nitrate Railways, Iquique, Chile, and Maybole, Ayrshire	
1887	C.	Hunter, James, F.R.C.S.E., F.R.A.S., Rosetta, Liberton, Midlothian	
1887	C.	Hunter, William, M.D., M.R.C.P.L. and E., M.R.C.S., 103 Harley Street, London 275	
1908		Hyslop, Theophilus Bulkeley, M.D., M.R.C.P.E., 5 Portland Place, London, W.	
1920		* Inglis, James Gall, Publisher and Editor of Educational Works, Edinburgh, 36 Blacket Place, Edinburgh	
1912		* Inglis, Robert John Mathieson, A.M.Inst.C.E., 31 Buckingham Terrace, Glasgow, W. ; Tantah, Peebles	
1904	C.	Innes, R. T. A., Director, Government Observatory, Johannesburg, Transvaal	
1917		* Irvine, James Colquhoun, C.B.E., Ph.D., D.Sc., F.R.S., Principal of the University of St Andrews 280	1920–
1914		Jack, John Noble	
1875		Jack, William, M.A., LL.D., Emeritus Professor of Mathematics in the University of Glasgow	1888–91.
1889		James, Alexander, M.D., F.R.C.P.E., 9 Randolph Crescent, Edinburgh	
1901		* Jardine, Robert, M.D., M.R.C.S., F.R.F.P.S. Glas., 20 Royal Crescent, Glasgow	
1912	C.	* Jeffrey, George Rutherford, M.D. (Glasg.), F.R.C.P. (Edin.), etc., Bootham Park Private Mental Hospital, York 285	
1906	C.	* Jehu, Thomas John, M.A., M.D., F.G.S., Professor of Geology in the University of Edinburgh : 23 Great King Street, Edinburgh	1917–20.
1900		* Jerdan, David Smiles, M.A., D.Sc., Ph.D., Temora, Colinton, Midlothian	
1916		* Johnston, Col. Sir Duncan A., K.C.M.G., C.B.E., C.B., Colonel Royal Engineers, 8 Lansdowne Crescent, Edinburgh	

Date of Election.			Service on Council, etc.
1895		Johnston, Col. Henry Halcro, C.B., D.Sc., M.D., F.L.S., late A.M.S., Orphir House, Kirkwall, Orkney	
1903	C.	* Johnston, Thomas Nicol, M.B., C.M., Pogbie, Humbie, East Lothian 290	
1874		Jones, Francis, M.Sc., Lecturer in Chemistry, 17 Whalley Road, Whalley Range, Manchester	
1888		Jones, John Alfred, M.Inst.C.E., Fellow of the University of Madras, Sanitary Engineer to the Government of Madras, c/o Messrs Parry & Co., 70 Gracechurch Street, London	
1915		Kemnal, Sir James Hermann Rosenthal, Managing Director and Engineer-in-Chief of Babcock & Wilcox, Ltd., Kemnal Manor, Chislehurst, Kent	
1912		Kennedy, Robert Foster, M.D. (Queen's Univ., Belfast), M.B., B.Ch. (R.U.I.), Assistant Professor of Neurology, Cornell University, New York, 20 West 50th Street, New York, U.S.A.	
1909		Kenwood, Henry Richard, M.B., Chadwick Professor of Hygiene in the University of London, 126 Queen's Road, Finsbury Park, London, N. 295	
1908		* Kerr, Andrew William, F.S.A. Scot., 81 Great King Street, Edinburgh	
1891		Kerr, Joshua Law, M.D., Worthen, Shropshire	
1913		* Kerr, Walter Hume, M.A., B.Sc., Lecturer on Engineering Drawing and Structural Design in the University of Edinburgh	
1908		Kidd, Walter Aubrey, M.D., 2 Suffolk Square, Cheltenham	
1886	C. N.	Kidston, Robert, LL.D., F.R.S., F.G.S., 12 Clarendon Place, Stirling 300	1891–94, 1903–06. Sec. 1909–16. V-P 1917–20.
1907		* King, Archibald, M.A., B.Sc., formerly Rector of the Academy, Castle Douglas; Junior Inspector of Schools, La Maisonnette, Clarkston, Glasgow	
1880		King, W. F., Lonend, Russell Place, Trinity, Leith	
1918		* Kingon, Rev. John Robert Lewis, M.A. (Edin. and Cape of Good Hope), F.L.S., Missionary of the U.F. Church of Scotland, St Andrew's Manse, Port Elizabeth, C.P., South Africa	
1878		Kintore, The Right Hon. the Earl of, P.C., G.C.M.G., M.A. Cantab., LL.D. Cambridge, Aberdeen, and Adelaide, Keith Hall, Inverurie, Aberdeenshire	
1901		* Knight, Rev. G. A. Frank, M.A., 5 Granby Terrace, Hillhead, Glasgow 305	
1907		* Knight, James, M.A., D.Sc., F.C.S., F.G.S., Head Master, Queen's Park High School, The Shieling, Uddingston, by Glasgow	
1880	C. K.	Knott, C. G., D.Sc., LL.D., F.R.S., Reader in Applied Mathematics in the University of Edinburgh, formerly Professor of Physics, Imperial University, Japan (Gen. Secretary), 42 Upper Gray Street, Edinburgh	1894–97, 1898–1901, 1902–05. Sec. 1905–12. Gen. Sec. 1912–
1920		* Lamont, John Charles, Lieut.-Col., I.M.S. (retired), C.I.E., M.B., C.M. (Edin.), M.R.C.S. (Eng.), 7 Merchiston Park, Edinburgh	
1878	C.	Lang, Sir P. R. Scott, Kt., M.A., B.Sc., Professor of Mathematics, University of St Andrews	
1910	C.	* Lauder, Alexander, D.Sc., Lecturer in Agricultural Chemistry, Edinburgh and East of Scotland College of Agriculture, 13 George Square, Edinburgh 310	1917–20.
1885	C.	Laurie, A. P., M.A., D.Sc., J.P., Principal of the Heriot-Watt College, Edinburgh	1908–11, 1913–16.
1894	C.	Laurie, Malcolm, B.A., D.Sc., F.L.S., 4 Wordsworth Road, Harpenden, Herts	
1910	C. B.	* Lawson, A. Anstruther, B.Sc., Ph.D., D.Sc., F.L.S., Professor of Botany, University of Sydney, New South Wales, Australia	
1905		* Lawson, David, M.A., M.D., L.R.C.P. and S.E., Druimdarroch, Banchory, Kincardineshire	
1910	C.	* Lee, Gabriel W., D.Sc., Palæontologist, Geological Survey of Scotland, 33 George Square, Edinburgh 315	
1903		* Leighton, Gerald Rowley, O.B.E., M.D., Medical Officer, Scottish Board of Health, 125 George Street, Edinburgh	
1910		Levie, Alexander, F.R.C.V.S., D.V.S.M., The Nook, Belper Road, Derby	
1916	C.	* Levy, Hyman, M.A., B.Sc., Assistant Professor of Mathematics, Imperial College of Science and Technology, London, S.W. 7, "Eskbank," 105 Cambridge Road, Teddington, Middlesex	

Date of Election.			Service on Council, etc.
1914	C. N.	Lewis, Francis John, D.Sc., F.L.S., Professor of Biology, University of Alberta, Edmonton South, Alberta, Canada	
1918		* Lidstone, George James, F.F.A., F.I.A., Manager and Actuary of the Scottish Widows' Fund Life Assurance Society, 8 Eglinton Crescent, Edinburgh 320	1919–
1905		* Lightbody, Forrest Hay, 53 Queen Street, Edinburgh	
1889		Lindsay, Rev. James, M.A., D.D., B.Sc., F.R.S.L., F.G.S., M.R.A.S., Corresponding Member of the Royal Academy of Sciences, Letters and Arts, of Padua, Associate of the Philosophical Society of Louvain, Annick Lodge, Irvine	
1912		* Lindsay, John George, M.A., B.Sc. (Edin.), Rector of Dunfermline High School	
1920	C.	* Lindsay, Thomas A., M.A. (Hons.), B.Sc., Head Science Master, Madras College, St Andrews, and Lecturer in the Provincial College	
1912		* Linlithgow, The Most Honourable the Marquis of, Hopetoun House, South Queensferry 325	
1903		Liston, William Glen, M.D., Lt.-Col. Indian Medical Service, Director Bombay Bacteriological Laboratory, India, 33 Comely Bank, Edinburgh	
1903		* Littlejohn, Henry Harvey, M.A., M.B., B.Sc., F.R.C.S.E., Professor of Forensic Medicine, Dean of the Faculty of Medicine in the University of Edinburgh, 11 Rutland Street, Edinburgh	
1898		* Lothian, Alexander Veitch, M.A., B.Sc., Training College, Cowcaddens, Glasgow	
1884		Low, George M., Actuary, 11 Moray Place, Edinburgh	
1888		Lowe, D. F., M.A., LL.D., formerly Headmaster of Heriot's Hospital School, Lauriston, 19 George Square, Edinburgh 330	
1900		Lusk, Graham, Ph.D., M.A., Professor of Physiology, Cornell University Medical College, New York, N.Y., U.S.A.	
1894		Mabbott, Walter John, M.A., Rector of County High School, Duns, Berwickshire	
1887		M'Aldowie, Alexander M., M.D., 8 Holland Road, Cheltenham	
1917		* Macalister, Sir Donald, K.C.B., Principal of the University of Glasgow, The University, Glasgow	
1907		MacAlister, Donald Alexander, A.R.S.M., F.G.S., 10 St Alban's Road, Kensington, London, W. 8 335	
1883		M'Bride, P., M.D., F.R.C.P.E., 10 Park Avenue, Harrogate, and Hill House, Withypool, Dunster, Somerset	
1903		* M'Cormick, Sir W.S., M.A., LL.D., Chairman of the Advisory Council, Department of Scientific and Industrial Research, 16–18 Old Queen Street, Westminster, S.W. 1	1910–13.
1918		* M'Culloch, Rev. James David, D.D., 43 Brougham Street, Greenock	
1905		* Macdonald, Hector Munro, M.A., F.R.S., Professor of Mathematics, University of Aberdeen, 52 College Bounds, Aberdeen	1908–11.
1897	C.	* Macdonald, James A., M.A., B.Sc., H.M. Inspector of Schools, Stewarton, Kilmacolm 340	
1904		* Macdonald, John A., M.A., B.Sc., King Edward VII School, Johannesburg, Transvaal	
1920		* M'Donald, Stuart, M.D., F.R.C.P.E., M.A., Professor of Pathology, School of Medicine, Newcastle-on-Tyne	
1904		Macdonald, William, M.S.Agr., Sc.D., Ph.D., D.Sc., Editor, *Agricultural Journal* of South Africa, Rand Club, Johannesburg, Transvaal	
1886		Macdonald, William J., M.A., LL.D., 15 Comiston Drive, Edinburgh	
1901	C.	* MacDougall, R. Stewart, M.A., D.Sc., Professor of Biology, Royal Veterinary College, Edinburgh, 9 Dryden Place, Edinburgh 345	1914–17.
1910		Macewen, Hugh Allen, M.B., Ch.B., D.P.H. (Lond. and Camb.), Local Government Board, Whitehall, London, S.W.	
1888	C.	McFadyean, Sir John, Kt., M.B., B.Sc., LL.D., Principal, and Professor of Comparative Pathology in the Royal Veterinary College, Camden Town, London	
1885	C.	Macfarlane, J. M., D.Sc., Professor of Botany and Director of the Botanic Garden, University of Pennsylvania, Philadelphia, Pennsylvania, U.S.A.	
1897		* MacGillivray, Angus, C.M., M.D., D.Sc., F.S.A. (Scot.), 23 South Tay Street, Dundee	
1878		M'Gowan, George, F.I.C., Ph.D., 21 Montpelier Road, Ealing, London, W. 5 350	
1903		* M'Intosh, Donald C., M.A., D.Sc., Education Offices, Elgin	
1911		M'Intosh, John William, A.R.C.V.S., Dollis Hill Farm, Cricklewood, London, N.W. 2	
1869	C. N.	M'Intosh, William Carmichael, M.D., LL.D., F.R.S., F.L.S., Emeritus Professor of Natural History in the University of St Andrews, Pres. Ray Society, Nevay Park, Meigle	1885–88.
1895	C.	Macintyre, John, M.D., LL.D., 179 Bath Street, Glasgow	

Date of Election.			Service on Council, etc.
1914		* M'Kendrick, Archibald, F.R.C.S.E., D.P.H., L.D.S., 11 Rothesay Place, Edinburgh 355	
1873	C. B.	M'Kendrick, John G., M.D., F.R.C.P.E., LL.D., F.R.S., Emeritus Professor of Physiology in the University of Glasgow, Maxieburn, Stonehaven	1875–78, 1885–88, 1893–94, 1900–02. V-P 1894–1900.
1912	C.	M'Kendrick, Anderson Gray, M.B., Major, Indian Medical Service, Superintendent, Research Laboratory, Royal College of Physicians, Edinburgh	
1900	C.	* M'Kendrick, John Souttar, M.D., F.R.F.P.S.G., 2 Buckingham Terrace, Hillhead, Glasgow	
1910	C.	* Mackenzie, Alister, M.A., M.D., D.P.H., Principal, College of Hygiene and Physical Training, Dunfermline	
1916	C.	* Mackenzie, John E., D.Sc., Lecturer in Chemistry, University of Edinburgh, Major-Adjutant, O.T.C., 2A Ramsay Garden, Edinburgh 360	
1894		Mackenzie, Robert, M.D., Napier, Nairn	
1904	C.	* Mackenzie, Sir W. Leslie, M.A., M.D., D.P.H., LL.D., Medical Member of the Scottish Board of Health, 4 Clarendon Crescent, Edinburgh	
1918		* Mackie, Wm., M.A., M.D., D.P.H., 13 North Street, Elgin	
1910		* MacKinnon, James, M.A., Ph.D., Professor of Ecclesiastical History, Edinburgh University, 12 Lygon Road, Edinburgh	
1904		* Mackintosh, Donald James, C.B., M.V.O., M.B., C.M., LL.D., Supt. Western Infirmary, Glasgow 365	
1899		Maclean, Ewan John, M.D., M.R.C.P. Lond., 12 Park Place, Cardiff	
1888	C.	Maclean, Magnus, M.A., D.Sc., LL.D., M.Inst.C.E., M.I.E.E., Professor of Electrical Engineering in the Royal Technical College, 51 Kerrsland Terrace, Hillhead, Glasgow	1916–19.
1913		* M'Lellan, Dugald, M.Inst.C.E., District Engineer, Caledonian Railway, 20 Kingsburgh Road, Murrayfield, Edinburgh	
1916	C.	* M'Lintock, W. F. P., D.Sc. (Edin.), Royal Scottish Museum, Edinburgh	
1907	C.	* Macnair, Professor Peter, Curator of the Natural History Collections in the Glasgow Museums, Kelvingrove Museum, Glasgow 370	
1917		* Macpherson, Rev. Hector Copland, M.A., F.R.A.S., Minister of the U.F. Church of Scotland, Loudoun United Free Manse, Newmilns, Ayrshire	
1898	C.	Mahâlanobis, S. C., B.Sc., Professor of Physiology, Presidency College, Calcutta, India	
1913		Majumdar, Tarak Nath, D.P.H. (Cal.), L.M.S., F.C.S., Health Officer, Calcutta, IV, 37 Lower Chitpore Road, Calcutta, India	
1917		* Malcolm, Louis William Gunther, M.A. (Melbourne), Capt. R.G.A., Christ's College, Cambridge	
1908		Mallik, Devendranath, Sc.D., B.A., Professor of Mathematics, Astronomical Observatory, Presidential College, Calcutta, India 375	
1912		Maloney, William Joseph, M.D. (Edin.), Professor of Neurology at Fordham University, New York City, N.Y., U.S.A.	
1913		Marchant, Rev. James, C.B.E., LL.D., F.R.A.S., F.L.S., Director, National Council for Promotion of Race-Regeneration, 20 Bedford Square, London, W.C.	
1909	C. B.	* Marshall, C. R., M.D., M.A., Professor of Materia Medica, Marischal College, Aberdeen	1915–18.
1882	C.	Marshall, D. H., M.A., Em.-Professor of Physics, Queen's University, Elmtree House, Union Street, W., Kingston, Ontario, Canada	
1901	C.	Marshall, F. H. A., Sc.D., Lecturer on Agricultural Physiology in the University of Cambridge, Christ's College, Cambridge 380	
1920	C.	* Marshall, John, M.A., D.Sc. (St Andrews), B.A. (Cantab.), Senior Lecturer in Mathematics, University College, Swansea	
1913		Masson, George Henry, M.D., D.Sc., M.R.C.P.E., Port of Spain, Trinidad, British West Indies	
1885	C.	Masson, Orme, D.Sc., F.R.S., Professor of Chemistry in the University of Melbourne	
1898	C. B.	* Masterman, Arthur Thomas, M.A., D.Sc., F.R.S., Superintending Inspector, H.M. Board of Agriculture and Fisheries, 43 Parliament Street, London, S.W.	1902–04.
1911		Mathews, Gregory Macalister, F.L.S., F.Z.S., Foulis Court, Fair Oaks, Hants 385	
1906		* Mathieson, Robert, F.C.S., St Serf's, Innerleithen	
1902		Matthews, Ernest Romney, A.M.Inst.C.E., F.G.S., Chadwick Professor of Municipal Engineering in the University of London, University College, Gower Street, London, W.C.	
1917		* Maylard, A. Ernest, M.B., B.Sc. (Lond.), F.R.F.P.S. (Glasgow), 1 Windsor Terrace, W., Great Western Road, Glasgow	

Date of Election.			Service on Council, etc.
1901	C.	* Menzies, Alan W. C., M.A., B.Sc., Ph.D., F.C.S., Professor of Chemistry, Princeton University, Princeton, New Jersey, U.S.A.	
1917		* Merson, George Fowlie, Manufacturing Technical Chemist, 9 Hampton Terrace, Edinburgh 390	
1888		Methven, Cathcart W., M.Inst.C.E., F.R.I.B.A., Durham, Natal, S. Africa	
1902	C.	Metzler, William H., A.B., Ph.D., Corresponding Fellow of the Royal Society of Canada, Professor of Mathematics, Syracuse University, Syracuse, N.Y., U.S.A.	
1885	C. B.	Mill, Hugh Robert, D.Sc., LL.D., Hill Crest, Dorman's Park, E. Grinstead, London	
1908		* Miller, Alexander Cameron, M.D., F.S.A. Scot., Craig Linnhe, Fort-William, Inverness-shire	
1910		* Miller, John, M.A., D.Sc., Professor of Mathematics, Royal Technical College, 2 Northbank Terrace, North Kelvinside, Glasgow 395	
1909		Mills, Bernard Langley, M.D., F.R.C.S.E., M.R.C.S., D.P.H., Lt.-Col. R.A.M.C., formerly Army Specialist in Hygiene, c/o National Provincial Bank, Fargate, Sheffield	
1905		* Milne, Archibald, M.A., D.Sc., Lecturer on Mathematics and Science, Edinburgh Provincial Training College, 108 Comiston Drive, Edinburgh	
1905		* Milne, C. H., M.A., Head Master, Daniel Stewart's College, 19 Merchiston Gardens, Edinburgh	
1904	C.	* Milne, James Robert, D.Sc., Lecturer in Natural Philosophy, University of Edinburgh, 17 Manor Place, Edinburgh	
1886		Milne, William, M.A., B.Sc., 70 Beechgrove Terrace, Aberdeen 400	
1899		* Milroy, T. H., M.D., B.Sc., Professor of Physiology in Queen's College, Belfast	
1889	C.	Mitchell, A. Crichton, D.Sc., Hon. Doc. Sc. (Genève), formerly Director of Public Instruction in Travancore, India (CURATOR OF LIBRARY AND MUSEUM), The Observatory, Eskdalemuir, Langholm, Dumfriesshire	1915–16. Cur. 1916–
1897		Mitchell, George Arthur, M.A., 9 Lowther Terrace, Kelvinside, Glasgow	
1900		* Mitchell, James, M.A., B.Sc., Monydrain, Lochgilphead	
1911		Modi, Edalji Manekji, D.Sc., LL.D., Litt.D., F.C.S., etc., Proprietor and Director of Arthur Road Chemical Works, Meher Buildings, Tardeo, Bombay, India 405	
1906	C.	Moffat, Rev. Alexander, M.A., B.Sc., Professor of Physical Science, Christian College, Madras, India	
1890	C.	Mond, R. L., M.A. Cantab., F.C.S., Combe Bank, near Sevenoaks, Kent	
1887	C.	Moos, N. A. F., D.Sc., L.C.E., J.P., Director of Bombay and Alibag Observatories (retired), Gowalia Tank Road, Bombay, India	
1896		Morgan, Alexander, M.A., D.Sc., Principal, Edinburgh Provincial Training College, 1 Midmar Gardens, Edinburgh	
1919		* Morris, Robert Owen, M.A., M.D., C.M. (Edin.), D.P.H. (Liverpool), Tuberculosis Institute, Newtown, N. Wales 410	
1892	C.	Morrison, J. T., M.A., B.Sc., Professor of Physics and Chemistry, Victoria College, Stellenbosch, Cape Colony	
1914		Mort, Spencer, M.D., Ch.B., F.R.C.S.E., Lieut.-Col. R.A.M.C., Medical Officer in Charge, Edmonton Military Hospital, Silver Street, Upper Edmonton, London, N.	
1901		Moses, O. St John, I.M.S., M.D., D.Sc., F.R.C.S., Captain, Professor of Medical Jurisprudence, c/o Messrs King, Hamilton & Co., 4 and 5 Koila Ghat Street, Calcutta, India	
1892	C. K.	Mossman, R. C., Fernbank, South Morton Street, Joppa, Edinburgh	
1916		* Muir, Robert, M.A., M.D., Sc.D., F.R.S., Professor of Pathology, University of Glasgow, 16 Victoria Crescent, Dowanhill, Glasgow 415	
1874	C. K. V. J.	Muir, Sir Thomas, C.M.G., M.A., LL.D., F.R.S., Superintendent-General of Education for Cape Colony, Education Office, Cape Town, and Elmcote, Sandown Road, Rondebosch, South Africa	1885–88. V-P 1888–91.
1888	C.	Muirhead, George, Commissioner to His Grace the Duke of Richmond and Gordon, K.G., Speybank, Fochabers	
1907		Muirhead, James M. P., J.P., F.R.S.L., F.S.S., c/o Royal Societies Club, St James's Street, London, S.W.	
1887		Mukhopâdhyay, Asûtosh, M.A., LL.D., F.R.A.S., M.R.I.A., Professor of Mathematics at the Indian Association for the Cultivation of Science, 77 Russa Road North, Bhowanipore, Calcutta, India	
1896		Murray, Alfred A., M.A., LL.B., 20 Warriston Crescent, Edinburgh 420	
1907		Musgrove, James, M.D., F.R.C.S. Edin. and Eng., LL.D., Emeritus-Professor of Anatomy, University of St Andrews, The Swallowgate, St Andrews	
1888		Napier, A. D. Leith, M.D., C.M., M.R.C.P., 46 Austral Terrace, Malvern, S. Australia	

Date of Election.			Service on Council, etc.
1897		Nash, Hon. Alfred George, M.L.C., B.Sc., F.R.G.S., C.E., Belretiro, Mandeville, Jamaica, W.I.	
1898		Newman, Sir George, K.C.B., M.D., D.C.L., F.R.C.P., Chief Medical Officer of the Ministry of Health and the Board of Education, Whitehall, S.W. 1	
1884		Nicholson, J. Shield, M.A., D.Sc., Professor of Political Economy in the University of Edinburgh, 3 Belford Park, Edinburgh 425	1885–87, 1892–95, 1897–1900.
1880	C.	Nicol, W. W. J., M.A., D.Sc., 15 Blacket Place, Edinburgh	
1878		Norris, Richard, M.D., M.R.C.S. Eng., 3 Walsall Road, Birchfield, Birmingham	
1888		Ogilvie, Sir F. Grant, Kt., C.B., M.A., B.Sc., LL.D., Principal Assistant Secretary, Department of Scientific and Industrial Research, 15 Evelyn Gardens, London, S.W.	1901–03.
1888		Oliphant, James, M.A., 11 Heathfield Park, Willesden Green, London	
1886		Oliver, James, M.D., F.L.S., Physician to the London Hospital for Women, 123 Harley Street, London, W. 430	
1895	C.	Oliver, Sir Thomas, Kt., M.D., LL.D., F.R.C.P., Professor of Physiology in the University of Durham, 7 Ellison Place, Newcastle-upon-Tyne	
1915		* Orr, Lewis P., F.F.A., Manager of Scottish Life Assurance Co., 19 St Andrew Square, Edinburgh	
1914		* Oswald, Alfred, Lecturer in German, Glasgow Provincial Training College, 11 Nelson Terrace, Hillhead, Glasgow	
1908		Page, William Davidge, F.C.S., F.G.S., M.Inst.M.E., 10 Clifton Dale, York	
1905		Pallin, William Alfred, F.R.C.V.S., Veterinary-Major, Royal Horse Guards, London 435	
1914		Pare, John William, M.D., C.M., L.D.S., Lecturer in Dental Anatomy, National Dental Hospital, 9A Cavendish Square, London, W.	
1901		* Paterson, David, F.C.S., Lea Bank, Rosslyn, Midlothian	
1918		* Paterson, Rev. William Paterson, D.D., LL.D., Professor of Divinity, University, Edinburgh, 3 Royal Terrace, Edinburgh	
1886	C.	Paton, D. Noël, M.D., B.Sc., LL.D., F.R.C.P.E., F.R.S. (Vice-President), Professor of Physiology in the University of Glasgow, University, Glasgow	1894–97, 1904–06, 1909–12. V-P 1918–
1919	C.	* Patterson, Thomas Stewart, D.Sc. (London and Glasgow), Ph.D. (Heidelberg), Professor of Organic Chemistry in the University of Glasgow, 10 Oakfield Terrace, Hillhead, Glasgow 440	
1892		Paulin, Sir David, Actuary, 6 Forres Street, Edinburgh	
1881	C. N.	Peach, Benjamin N., LL.D., F.R.S., F.G.S., formerly District Superintendent and Acting Palæontologist of the Geological Survey of Scotland, 72 Grange Loan, Edinburgh	1905–08, 1911–12. V-P 1912–17.
1907		* Pearce, John Thomson, B.A., B.Sc., School House, Tranent	
1914		Pearson, Joseph, D.Sc., F.L.S., Director of the Colombo Museum, and Marine Biologist to the Ceylon Government, Colombo Museum, Ceylon	
1904		* Peck, James Wallace, C.B., M.A., Senior Assistant-Secretary, Ministry of Food, London, Caledonian Club, St. James's Sq., London, S.W. 1 445	
1889		Peck, Sir William, Kt., F.R.A.S., Town's Astronomer, City Observatory, Calton Hill, Edinburgh	
1887	C. B.	Peddie, Wm., D.Sc. (Vice-President), Professor of Natural Philosophy in University College, Dundee, The Weisha, Ninewells, Dundee	1904–07, 1908–11. V.P. 1919–
1893		Perkin, Arthur George, F.R.S., Grosvenor Lodge, Grosvenor Road, Leeds	
1913	C.	Philip, Alexander, M.A., LL.B., Writer, The Mary Acre, Brechin	
1889		Philip, Sir R. W., Kt., M.A., M.D., LL.D., F.R.C.P.E., Professor of Tuberculosis, University of Edinburgh, 45 Charlotte Square, Edinburgh 450	
1907	C.	Phillips, Major Charles E. S., O.B.E., 54 Bedford Gardens, London, W. 8.	
1914		* Pilkington, Basil Alexander, "Kambla," Davidson's Mains	
1905		* Pinkerton, Peter, M.A., D.Sc., Rector, High School, Glasgow, 7 Park Quadrant, Glasgow, W.	
1908	C.	* Pirie, James Hunter Harvey, B.Sc., M.D., F.R.C.P.E., Superintendent of the Routine Division of The South African Institute for Medical Research, P.O. Box 1038, Johannesburg, South Africa	
1911		* Pirie, James Simpson, M.Inst.C.E., 28 Scotland Street, Edinburgh 455	

Date of Election.			Service on Council, etc.
1906		Pitchford, Herbert Watkins, F.R.C.V.S.	
1919		* Porritt, B. D., M.Sc. (Lond.), F.I.C., Research Association of British Rubber and Tyre Manufacturers Chemistry Dept., University College, Gower Street, London, W.C. 1	
1888		Prain, Sir David, Lt.-Col., Indian Medical Service (retired), Kt., C.M.G., C.I.E., M.A., M.B., LL.D., F.L.S., F.R.S., For. Memb. K. Svensk. Vetensk. Akad. ; Hon. Memb. Soc. Lett. ed Arti d. Zelanti, Acireale ; Pharm. Soc. Gt. Britain ; Corr. Memb. Bayer Akad. Wiss., etc. ; Director, Royal Botanic Gardens, Kew, Surrey	
1902		* Preller, Charles du Riche, M.A., Ph.D., A.M.Inst.C.E., M.I.E.E., F.G.S., 61 Melville Street, Edinburgh	
1892	C.	Pressland, Arthur J., M.A. Camb., Edinburgh Academy 460	
1915		Price, Frederick William, M.D., M.R.C.P. Edin., Physician to the Great Northern Hospital, London, 133 Harley Street, London, W.	
1908		* Pringle, George Cossar, M.A., Organising Secretary of the Educational Institute of Scotland, 120 Polwarth Terrace, Edinburgh	
1903		* Pullar, Laurence, Dunbarney, Bridge of Earn, Perthshire	
1911		Purdy, John Smith, M.D., C.M. (Aberd.), D.P.H. (Camb.), F.R.G.S., Chief Health Officer for Tasmania, Islington, Hobart, Tasmania	
1920		* Purser, George Leslie, M.A. (Cantab.), Assistant in the Natural History Department of the University of Edinburgh, c/o Muir, 21 Buccleuch Place, Edinburgh 465	
1898		* Purves, John Archibald, D.Sc., 52 Queen Street, Exeter	
1897	C.	* Rainy, Harry, M.A., M.B., C.M., F.R.C.P.Ed., 16 Great Stuart Street, Edinburgh	
1899	C.	* Ramage, Alexander G., Marchfield, Davidson's Mains, Midlothian	
1914		* Ramsay, Peter, M.A., B.Sc., Head Mathematical Master, George Watson's College, 63 Comiston Drive, Edinburgh	
1911		* Rankin, Adam A., British Astronomical Association, West of Scotland Branch, 24 Woodend Drive, Jordanhill, Glasgow 470	
1891		Rankine, John, K.C., M.A., LL.D., Professor of the Law of Scotland in the University of Edinburgh, 23 Ainslie Place, Edinburgh	
1904		Ratcliffe, Joseph Riley, M.B., C.M., c/o The Librarian, The University, Birmingham	
1900		Raw, Nathan, C.M.G., M.D., M.P., 58 Harley Street, Cavendish Square, London, W. 1.	
1883	C.	Readman, J. B., D.Sc., F.C.S., Frankleigh House, Bradford-on-Avon, Wilts	
1902		Rees-Roberts, John Vernon, M.D., D.Sc., D.P.H., 11 Oak Hill Park, Hampstead, London, N.W. 3 475	
1902		Reid, George Archdall O'Brien, M.B., C.M., 9 Victoria Road South, Southsea, Hants	
1913		Reid, Harry Avery, F.R.C.V.S., D.V.H., Bacteriologist and Pathologist, Department of Agriculture, Wellington, New Zealand	
1908	C.	* Rennie, John, D.Sc., Lecturer on Parasitology and Experimental Zoology, University of Aberdeen, 60 Desswood Place, Aberdeen	
1914		Renshaw, Graham, M.D., M.R.C.S., L.R.C.P., L.S.A., Editor of the *Avicultural Magazine*, Sale Bridge House, Sale, Manchester	
1913		* Richardson, Harry, M.Inst.E.E., M.Inst.M.E., General Manager and Chief Engineer, Electricity Supply, Dundee and District, Dudhope Crescent Road, Dundee 480	
1908		Richardson, Linsdall, F.G.S., 10 Oxford Parade, Cheltenham, Glos.	
1875		Richardson, Ralph, W.S., 29 Eglinton Crescent, Edinburgh	
1916	C.	* Ritchie, James, M.A., D.Sc., Royal Scottish Museum, 20 Upper Gray Street, Edinburgh	
1914	C.	* Ritchie, James Bonnyman, D.Sc., Science Master, Kelvinside Academy, Glasgow	
1906	C.	* Ritchie, William Thomas, M.D., F.R.C.P.E., 14 Rothesay Place, Edinburgh 485	
1898	C.	Roberts, Alexander William, D.Sc., F.R.A.S., Lovedale, South Africa	
1919		* Roberts, Alfred Henry, O.B.E., M.Inst.C.E., Superintendent and Engineer, Leith Docks, 2 Cargil Terrace, Trinity, Edinburgh	
1880		Roberts, D. Lloyd, M.D., F.R.C.P.L., 23 St John's Street, Manchester	
1900		* Robertson, Joseph M'Gregor, M.B., C.M., 26 Buckingham Terrace, Glasgow	
1902	C.	* Robertson, Robert A., M.A., B.Sc., Lecturer on Botany in the University of St Andrews 490	
1919		* Robertson, William Alexander, F.F.A., Century Insurance Co., Ltd., 18 Charlotte Square ; 12 Lonsdale Terrace, Edinburgh	
1896	C.	Robertson, W. G. Aitchison, D.Sc., M.D., F.R.C.P.E., The Grange, Ashford, Middlesex	

Date of Election.			Service on Council, etc.
1910	C.	* Robinson, Arthur, M.D., M.R.C.S. (VICE-PRESIDENT), Professor of Anatomy, University of Edinburgh, 35 Coates Gardens, Edinburgh	1910–12. Sec. 1912–18. V-P 1918–
1916		* Ronald, David, Civil Engineer, Chief Engineer, Scottish Board of Health, 125 George Street, Edinburgh	
1881		Rosebery, The Right Hon. the Earl of, K.G., K.T., LL.D., D.C.L., F.R.S. Dalmeny Park, Edinburgh 495	
1909	C.	* Ross, Alex. David, M.A., D.Sc., F.R.A.S., Professor of Mathematics and Physics, University of Western Australia, Perth, Western Australia	
1906		* Russell, Alexander Durie, B.Sc., Mathematical Master, Falkirk High School, 14 Heugh Street, Falkirk	
1902	C. K.	* Russell, James, 22 Glenorchy Terrace, Edinburgh	
1906		Saleeby, Caleb William, M.D., 10 Campden Mansions, Kensington, London, W. 8	
1916		* Salvesen, The Hon. Lord, Judge of the Court of Session, Dean Park House, Edinburgh 500	1920–
1914		* Salvesen, Theodore Emile, 37 Inverleith Place, Edinburgh	
1912	C.	* Sampson, Ralph Allen, M.A., D.Sc., F.R.S., Astronomer Royal for Scotland, Professor of Astronomy, University, Edinburgh, Royal Observatory, Edinburgh	1912–15. 1919– V-P 1915–18.
1903		* Samuel, Sir John S., K.B.E., D.L., J.P., F.S.A. (Scot.), 13 Park Circus, Glasgow, W.	
1903		* Sarolea, Charles, Ph.D., D.Litt., Professor of French, University of Edinburgh, 21 Royal Terrace, Edinburgh	
1900	C.	* Schafer, Sir Edward Albert Sharpey, M.D., LL.D., D.Sc., F.R.S., Professor of Physiology in the University of Edinburgh 505	1900–03, 1906–09. 1918–19. V-P 1913–17.
1919		* Scott, Alexander, M.A., D.Sc., Carnegie Scholar, 1912–14; 1851 Exhibition Scholar, 1914–16; lectured (temp.) on Petrology, Oxford, 1914–15, and at Glasgow University, 1917–18; Physical Chemist in charge of Radiometric Laboratory, Glasgow University, 1916–18; Chief Assistant to Principal, Pottery Laboratory, Stoke-on-Trent	
1885	C.	Scott, Alexander, M.A., D.Sc., F.R.S., 34 Upper Hamilton Terrace, London, N.W.	
1919		* Scott, Alexander Ritchie, B.Sc. (Edin.), D.Sc. (Lond.), Director of Returns, Local Authorities Dept., Ministry of Food, Deputy Director of Statistics, Ministry of Food, 79 Fawnbrake Avenue, Herne Hill, London, S.E. 24	
1917		* Scott, Henry Harold, M.D., M.R.C.P., L.R.C.P. (London), M.R.C.S. (Eng.), D.P.H., Bacteriologist and Pathologist to the Government of Hong Kong	
1908		* Simpson, George Freeland Barbour, M.D., F.R.C.P.E., F.R.C.S.E., 43 Manor Place, Edinburgh 510	
1900	C.	* Simpson, James Young, M.A., D.Sc., Professor of Natural Science in the New College, Edinburgh, 25 Chester Street, Edinburgh	
1911	C.	Simpson, Sutherland, M.D., D.Sc. (Edin.), Professor of Physiology, Medical College, Cornell University, Ithaca, N.Y., U.S.A., 118 Eddy Street, Ithaca, N.Y., U.S.A.	
1900		Sinhjee, Sir Bhagvat, G.C.I.E., M.D., LL.D. Edin., H.H. the Thakur Sahib of Gondal, Gondal, Kathiawar, Bombay, India	
1903		* Skinner, Robert Taylor, M.A., J.P., Head Master, Donaldson's Hospital, Edinburgh	
1901		* Smart, Edward, B.A., B.Sc., Tillyloss, Tullylumb Terrace, Perth 515	
1920		* Smellie, William Robert, M.A., B.Sc., D.Sc., Geologist on the Staff of the Anglo-Persian Oil Company, Mayfield, Mossend, near Glasgow	
1891	C. K.	Smith, Alexander, B.Sc., Ph.D., LL.D., Department of Chemistry, Columbia University, New York, N.Y., U.S.A.	
1882	C.	Smith, C. Michie, C.I.E., B.Sc., F.R.A.S., formerly Director of the Kodaikânal and Madras Observatories, Winsford, Kodaikânal, South India, c/o Messrs H. S. King & Co., 9 Pall Mall, London, S.W. 1	
1917		* Smith, Sir George Adam, M.A., D.D., LL.D., Litt.D., Principal and Vice-Chancellor of Aberdeen University, Chanonry Lodge, Old Aberdeen	
1915		* Smith, James Lorrain, M.A., M.D., F.R.S., Professor of Pathology, University of Edinburgh, 9 Carlton Terrace, Edinburgh 520	1918–

Date of Election.			Service on Council, etc.
1911		* Smith, Stephen, B.Sc., Engineer, 31 Grange Loan, Edinburgh	
1907	C.	Smith, William Ramsay, D.Sc., M.D., C.M., Permanent Head of the Health Department, South Australia, Belair, South Australia	
1880		Smith, Sir William (Robert), M.D., D.Sc., LL.D., Principal of The Royal Institute of Public Health, Em.-Professor of Forensic Medicine and Toxicology in King's College, University of London, 36 Russell Square, London, W.C. 1	
1919		* Smith, William Wright, M.A. (Edin.), Assistant Keeper, Royal Botanic Garden, Edinburgh, 6 Lennox Row, Trinity, Edinburgh	
1899		Snell, Ernest Hugh, M.D., B.Sc., D.P.H. Camb., Medical Officer of Health, Coventry 525	
1880		Sollas, W. J., M.A., D.Sc., LL.D., F.R.S., Fellow of University College, Oxford, and Professor of Geology and Palæontology in the University of Oxford	
1910		* Somerville, Robert, B.Sc., Science Master, High School, Dunfermline, 31 Cameron Street, Dunfermline	
1889		Somerville, Wm., M.A., D.Sc., D.Oec., Sibthorpian Professor of Rural Economy and Fellow of St John's College in the University of Oxford, 121 Banbury Road, Oxford	
1911	C.	* Sommerville, Duncan M'Laren Young, M.A., D.Sc., Professor of Pure and Applied Mathematics, Victoria College, Wellington, New Zealand	
1882		Sorley, James, 73 Onslow Square, London, S.W. 7 530	
1896		Spence, Frank, M.A., B.Sc., 25 Craiglea Drive, Edinburgh	
1906		Squance, Major Thomas Coke, M.D., M.S., F.R.M.S., F.S.A.Scot., Physician and Pathologist in the Sunderland Infirmary, President Sunderland Antiquarian Society, Sunderland Naturalists' Association, The Cottage, Newbiggin, Aysgarth, S.O.	
1891		Stanfield, Richard, Professor of Mechanics and Engineering in the Heriot-Watt College, Edinburgh	
1915		* Steggall, John Edward Aloysius, M.A., Professor of Mathematics at University College, Dundee (St Andrews University), Woodend, Perth Road, Dundee	
1912	C. K.	Stephenson, John, M.B., D.Sc. (Lond.), Lt.-Col. I.M.S., Zoological Department, University, Edinburgh 535	
1910		* Stephenson, Thomas, F.C.S., Editor of the *Prescriber*, Examiner to the Pharmaceutical Society, 6 South Charlotte Street, Edinburgh	
1916		* Steuart, D. R., F.I.C., Chemist to the Broxburn Oil Company, 20 Hillview, Blackhall, Midlothian	
1886	C.	Stevenson, Charles A., B.Sc., M.Inst.C.E., 28 Douglas Crescent, Edinburgh	
1884		Stevenson, David Alan, B.Sc., M.Inst.C.E., 84 George Street, Edinburgh	
1919		* Stevenson, David Alan, B.Sc., A.M.Inst.C.E., Captain R.M.E., seconded to Admiralty, 28 Douglas Crescent, Edinburgh 540	
1888	C.	Stewart, Charles Hunter, D.Sc., M.B., C.M., Professor of Public Health in the University of Edinburgh, Usher Institute of Public Health, Warrender Park Road, Edinburgh	
1902		* Stockdale, Herbert Fitton, LL.D., Director of the Royal Technical College, Glasgow, Clairinch, Upper Helensburgh, Dumbartonshire	
1889	C.	Stockman, Ralph, M.D., F.R.C.P.E., Professor of Materia Medica and Therapeutics in the University of Glasgow	1903-05.
1906		Story, Fraser, Professor of Forestry, University College, Bangor, North Wales	
1907		* Strong, John, C.B.E., M.A., LL.D., Professor of Education in the University of Leeds 545	
1903		Sutherland, David W., M.D., M.R.C.P., C.I.E., Lt.-Col. I.M.S., Principal and Professor of Medicine, Medical College, Lahore, India	
1905		Swithinbank, Commander Harold William, Crag Head, Bournemouth, Hants	
1912		* Syme, William Smith, M.D. (Edin.), 11 Lynedoch Crescent, Glasgow	
1885	C.	Symington, Johnson, M.D., LL.D., F.R.C.S., F.R.S., formerly Professor of Anatomy in the Queen's University, 3 Chlorine Gardens, Belfast	1892-93.
1917	C. N	* Tait, John, D.Sc., M.D., Professor of Physiology, M'Gill University, Montreal, Canada 550	
1904		* Tait, John W., B.Sc., Rector of Leith Academy, 18 Netherby Road, Leith	
1898	C.	Tait, William Archer, D.Sc., M.Inst.C.E., 72A George Street, Edinburgh (Society's Representative on George Heriot's Trust)	1914-17, 1918-
1895		Talmage, James Edward, D.Sc., Ph.D., F.R.M.S., F.G.S., Professor of Geology, University of Utah, Salt Lake City, Utah, U.S.A.	
1890	C.	Tanakadate, Aikitu, Professor of Natural Philosophy in the Imperial University of Japan, Tokyo, Japan	

Date of Election.			Service on Council, etc.
1870		Tatlock, Robert R., F.C.S., City Analyst's Office, 156 Bath Street, Glasgow 555	
1899		* Taylor, James, M.A., Mathematical Master in the Edinburgh Academy	
1917	C.	* Taylor, William White, M.A., D.Sc., Lecturer on Chemical Physiology, University, Edinburgh, Park Villa, Liberton, Edinburgh	
1892		Thackwell, J. B., M.B., C.M., 423A Battersea Park Road, London, S.W.	
1885	C.	Thompson, D'Arcy W., C.B., D.Litt., F.R.S., Professor of Natural History, University, St Andrews, 44 South Street, St Andrews	1892–95, 1896–99, 1907–10, 1912–15. V-P 1916–19.
1917	C.	* Thompson, John M'Lean, M.A., D.Sc., F.L.S., Lecturer in Plant Morphology, Department of Botany, University, Glasgow, 2 Second Avenue, King's Park, Cathcart, Glasgow 560	
1905		* Thoms, Alexander, 7 Playfair Terrace, St Andrews	
1887		Thomson, Andrew, M.A., D.Sc., F.I.C., 145 Bruntsfield Place, Edinburgh	
1896		Thomson, George Ritchie, C.M.G., M.B., C.M., Professor of Surgery, University College, Johannesburg, Transvaal	
1903		Thomson, George S., F.C.S., Ferma Albion, Marculesci, Roumania	
1906		* Thomson, Gilbert, M.Inst.C.E., 164 Bath Street, Glasgow 565	
1887	C.	Thomson, J. Arthur, M.A., LL.D., Regius Professor of Natural History in the University of Aberdeen	1906–08, 1920–
1906	C.	Thomson, James Stuart, M.Sc., Ph.D., Zoological Department, University, Manchester	
1880		Thomson, John Millar, LL.D., F.R.S., Em.-Professor of Chemistry in King's College, London, 58 Bedford Gardens, Kensington, W.	
1899		* Thomson, R. Tatlock, F.C.S., 156 Bath Street, Glasgow	
1912	C.	Thomson, Robert Black, M.B. Edin., Professor of Anatomy, The University, Cape Town 570	
1870		Thomson, Spencer C., Actuary, 10 Eglinton Crescent, Edinburgh	
1882		Thomson, Wm., M.A., B.Sc., LL.D., Registrar, University of South Africa, Somerset House, Vermeulen Street, Pretoria	
1876	C.	Thomson, William, Royal Institution, Manchester	
1917		* Thorneycroft, Wallace, J.P., Coal and Iron Master, Plean House, Plean, Stirling	
1920		* Todd, John Barber, B.Sc., A.M.I.Mech.E., Lecturer in Engineering in the University of Edinburgh, 173 Bruntsfield Place, Edinburgh 575	
1917		* Tovey, Donald Francis, B.A., Professor of Music, University, Edinburgh, 2 St Margaret's Road, Edinburgh	
1914		Tredgold, Alfred Frank, M.D. (Durham), L.R.C.P., M.R.C.S., Hon. Consulting Physician to National Association for the Feeble-minded, 6 Dapdune Crescent, Guildford, Surrey	
1915		* Trotter, George Clark, M.D., Ch.B. Edin., D.P.H. (Aberdeen), Medical Officer of Health, Metropolitan Borough of Deptford, Public Health Department, Deptford Town Hall, New Cross, London, S.E. 14	
1905		* Turner, Arthur Logan, M.D., F.R.C.S.E., 27 Walker Street, Edinburgh	
1906	C.	* Turner, Dawson F. D., B.A., M.D., F.R.C.P.E., M.R.C.P., Lecturer on Medical Physics, Surgeons' Hall, Physician in charge of Radium Treatment, Royal Infirmary, Edinburgh, 37 George Square, Edinburgh 580	
1895		Turton, Albert H., M.I.M.M., 233 George Road, Erdington, Birmingham	
1898	C.	* Tweedie, Charles, M.A., B.Sc., formerly Lecturer on Mathematics in the University of Edinburgh, Duns, Berwickshire	
1918	C.	* Tyrrell, G. W., A.R.C.Sc., F.G.S., Chief Assistant and Lecturer in Petrology, Geological Department, University, Glasgow	
1910		Vincent, Swale, M.D. Lond., D.Sc. Edin., etc., Professor of Physiology in the University of London, Physiological Laboratory, Middlesex Hospital Medical School, Berners Street, London, W. 1	
1891	C. B.	Walker, Sir James, Kt., D.Sc., Ph.D., LL.D., F.R.S., Professor of Chemistry in the University of Edinburgh, 5 Wester Coates Road, Edinburgh 585	1903–05, 1910–13. V-P 1916–19.
1902		* Wallace, Alexander G., M.A., 56 Fonthill Road, Aberdeen	
1886	C.	Wallace, R., M.A., F.L.S., Professor of Agriculture and Rural Economy in the University of Edinburgh	
1898		Wallace, Wm., M.A., Belvedere, Alberta, Canada	
1891		Walmsley, R. Mullineux, D.Sc., Principal of the Northampton Institute, Clerkenwell, London	

Date of Election.			Service on Council, etc.
1920		* Walmsley, Thomas, M.D. (Glasgow), Professor of Anatomy in the University of Belfast, 59 South Street, Greenock 590	
1901	C.	* Waterston, David, M.A., M.D., F.R.C.S.E., Professor of Anatomy, University, St Andrews	1916–19.
1911		* Watson, James A. S., B.Sc., etc., Lecturer in Agriculture, University of Edinburgh, 30 Mayfield Terrace, Edinburgh	
1900		* Watson, Thomas P., M.A., B.Sc., Principal, Keighley Institute, Keighley	
1907		* Watt, Andrew, M.A., Meteorological Office, Air Ministry, 6 Woodburn Terrace, Edinburgh	1912–14.
1911		Watt, James, W.S., F.F.A., Craiglockhart House, Slateford, Edinburgh 595	
1911		* Watt, Rev. Lauchlan Maclean, D.D., Minister of St Stephen's Parish, 7 Royal Circus, Edinburgh	
1896		Webster, John Clarence, B.A., M.D., F.R.C.P.E., Professor of Obstetrics and Gynæcology, Rush Medical College, Shediac, N.B., Canada	
1907	B. C.	* Wedderburn, Ernest Maclagan, M.A., LL.B., W.S., D.Sc., 6 Succoth Gardens, Edinburgh	1913–16.
1903	C.	* Wedderburn, J. H. Maclagan, M.A., D.Sc., P.O. Box 53, Princeton, N.J., U.S.A.	
1904		Wedderspoon, William Gibson, M.A., LL.D., Indian Educational Service, Senior Inspector of Schools, Burma, The Education Office, Rangoon, Burma 600	
1896		Wenley, Robert Mark, M.A., D.Sc., D.Phil., Litt.D., LL.D., D.C.L., Professor of Philosophy in the University of Michigan, Ann Arbor, U.S.A.	
1916		* White, J. Martin, Esq., of Balruddery, Balruddery, near Dundee	
1896	C.	White, Philip J., M.B., Professor of Zoology in University College, Bangor, North Wales	
1911		* Whittaker, Charles Richard, F.R.C.S. (Edin.), F.S.A. (Scot.), Lynwood, Hatton Place, Edinburgh	
1912	C.	* Whittaker, Edmund Taylor, Sc.D., F.R.S., Professor of Mathematics in the University of Edinburgh (Secretary), 35 George Square, Edinburgh 605	1912–15. Sec. 1916–
1918		* Whyte, Rev. Charles, M.A., LL.D., F.R.A.S., U.F. Church Manse, Kingswells, Aberdeen	
1918		* Wight, John Thomas, General Manager, Hydraulic Gears, Ltd., Beavor Lane, Hammersmith, London, W. 6; Dartbeigh, Ascot, Berks.	
1879		Will, John Charles Ogilvie, of Newton of Pitfodels, M.D., 17 Bon-Accord Square, Aberdeen	
1908		* Williamson, Henry Charles, M.A., D.Sc., Naturalist to the Fishery Board for Scotland, Marine Laboratory, Aberdeen	
1910	C.	* Williamson, William, F.L.S., 79 Morningside Drive, Edinburgh 610	
1900		Wilson, Alfred C., F.C.S., Voewood Croft, Stockton-on-Tees	
1911		* Wilson, Andrew, M.Inst.C.E., 66 Netherby Road, Trinity, Edinburgh	
1902		* Wilson, Charles T. R., M.A., F.R.S., 14 Cranmer Road, Cambridge, Sidney Sussex College, Cambridge	
1895		Wilson-Barker, Sir David, Kt., R.N.R., F.R.G.S., late Captain-Superintendent Thames Nautical Training College, H.M.S. "Worcester," off Greenhithe, Kent, Flimwell Grange, near Hawkhurst, Kent	
1882		Wilson, George, M.A., M.B., LL.D. 615	
1920		* Wilson, Malcolm, D.Sc. (London), Lecturer in Mycology and Bacteriology in the University of Edinburgh, 31 Wardie Road, Trinity, Leith	
1908		* Wood, Thomas, M.D., Eastwood, 182 Ferry Road, Bonnington, Leith	
1886	C.	Woodhead, Sir German Sims, K.B.E., M.D., F.R.C.P.E., Professor of Pathology in the University of Cambridge	1887–90.
1884		Woods, G. A., M.R.C.S., 1 Hammelton Road, Bromley, Kent	
1911		* Wrigley, Ruric Whitehead, B.A. (Cantab.), Assistant Astronomer, Royal Observatory, Edinburgh 620	
1890		Wright, Johnstone Christie, Conservative Club, Edinburgh	
1896		Wright, Sir Robert Patrick, LL.D., Chairman of the Board of Agriculture for Scotland, Kingarth, Colinton, Midlothian	
1882		Young, Frank W., F.C.S., Principal Education Officer, Ministry of Labour, Appointments Department, Balmoral Hotel, Princes Street, Edinburgh	
1892		Young, George, Ph.D., "Bradda," Church Crescent, Church End, Finchley, London, N.	
1896	C.	Young, James Buchanan, M.B., D.Sc., Dalveen, Braeside, Liberton 625	
1904		Young, R. B., M.A., D.Sc., F.G.S., Professor of Geology and Mineralogy in the South African School of Mines and Technology, Johannesburg, Transvaal 626	

LIST OF HONORARY FELLOWS OF THE SOCIETY

At January 31, 1921.

HIS MOST GRACIOUS MAJESTY THE KING.

HIS ROYAL HIGHNESS THE PRINCE OF WALES.

FOREIGNERS (LIMITED TO THIRTY-SIX BY LAW I).

Elected

1916 Charles Barrois, Professor of Geology and Mineralogy, Université, Lille, France: 37 rue Pascal, Lille.

1905 Waldemar Christofer Brögger, Professor of Mineralogy and Geology, K. Frederiks Universitet, Christiania, Norway.

1916 Douglas Houghton Campbell, Professor of Botany, Leland Stanford Junior University, California, U.S.A.

1920 William Wallace Campbell, Director of the Lick Observatory, Mt. Hamilton, California, U.S.A.

1910 Hugo de Vries, Professor of Plant Anatomy and Physiology, Lunteren, Holland.

1916 Marcel Eugène Emile Gley, Professor of Physiology, Collège de France, Paris, Membre de l'Académie de Médecine, Paris: 14, rue Monsieur le Prince, Paris.

1910 Karl F. von Goebel, Professor of Botany, Universität, München, Germany.

1916 Camillo Golgi, Professor of Pathology, Università, Pavia, Italy.

1916 Gio. Battista Grassi, Professor of Comparative Anatomy, Regia Università, Roma, Italy: Via Agostino Depretis N. 91, Rome.

1905 Paul Heinrich von Groth, Professor of Mineralogy, Universität, München, Germany.

1913 George Ellery Hale, Director of Mount Wilson Solar Observatory (Carnegie Institution of Washington), Pasadena, California, U.S.A.

1883 Julius Hann, Emeritus Professor of Cosmical Physics, Universität, Wien, Austria.

1910 James Cornelius Kapteyn, Professor of Astronomy, Universiteit, Gröningen, Holland.

1897 Gabriel Lippmann, Professor of Physics, Université, Paris, France.

1920 Hendrik Antoon Lorentz, Professor of Physics, Leiden University.

1895 Carl Menger, Professor of Political Economy, Universität, Wien, Austria: Wien ix, Fuchstallerg 2, Austria.

1910 Albert Abraham Michelson, Professor of Physics, University, Chicago, U.S.A.

1897 Fridtjof Nansen, Professor of Oceanography, K. Frederiks Universitet, Christiania, Norway.

1920 Alfred Gabriel Nathorst, Em -Professor of Palæobotany at the Royal Swedish State Mus. of Nat. Hist. Kammakaregatan, 40, Stockholm, Sweden.

1908 Henry Fairfield Osborn, Professor of Zoology, Columbia University and American Museum of Natural History, New York, N.Y., U.S.A.

1908 Ivan Petrovitch Pawlov, Emeritus Professor of Physiology, Kais. Inst. Exper. Med., Petrograd: Wedenskaja Strasse 4, Petrograd, Russia.

1920 Ch. Emile Picard, Perpetual Secretary, Academy of Sciences, Paris.

1889 Georg Hermann Quincke, Emeritus Professor of Physics, Bergstrasse 41, Heidelberg, Germany.

1913 Santiago Ramón y Cajal, Professor of Histology and Pathological Anatomy, Universidad, Madrid. Spain.

1920 Charles Richet, Professor of Physiology, Faculty of Medicine, Paris.

1920 Georg Ossian Sars, formerly Professor of Zoology, Christiania, and Director of Norwegian Fisheries.

1913 Vito Volterra, Professor of Mathematical Physics, Regia Università, Rome, Italy.

1916 Eugenius Warming, Emeritus Professor of Botany at the University of Copenhagen and Director of the Botanical Garden: Bjerregaardsvej 5, Copenhagen, Valby.

Total, 28.

BRITISH SUBJECTS (LIMITED TO TWENTY BY LAW I).

1916 Sir Francis Darwin, Kt., D.Sc., M.B., F.R.S., Hon. Fellow; Christ's College, Cambridge, 10 Madingley Road, Cambridge.

1900 Sir David Ferrier, Kt., M.A., M.D., LL.D., F.R S., Emer.-Professor of Neuro-Pathology. King's College, London, 34 Cavendish Square, London, W.

1900 Andrew Russell Forsyth, M.A., Sc.D., LL.D., Math.D., F.R.S., Chief Professor of Mathematics in the Imperial College of Science and Technology, London, formerly Sadleirian Professor of Pure Mathematics in the University of Cambridge, Imperial College of Science and Technology, London, S.W.

Elected

1910 Sir James George Frazer, D.C.L., LL.D., Litt.D., F.B.A., Fellow of Trinity College, Cambridge, Professor of Social Anthropology in the University of Liverpool, Trinity College Cambridge.

1916 James Whitbread Lee Glaisher, M.A., Sc.D., F.R.S., Fellow of Trinity College, Cambridge.

1908 Sir Alexander B. W. Kennedy, Kt., LL.D., F.R.S., Past Pres. Inst. C.E., A7, Albany, Piccadilly, London, W.

1913 Horace Lamb, M.A., Sc.D., D.Sc., LL.D., F.R.S., Professor of Mathematics in the University of Manchester.

1916 John Newport Langley, Sc.D., LL.D., F.R.S., Fellow of Trinity College, Professor of Physiology in the University of Cambridge, Hedgerley Lodge, Madingley Road, Cambridge.

1908 Sir Edwin Ray Lankester, K.C.B., LL.D., F.R.S., 29 Thurloe Place, S. Kensington, London, S.W.

1910 Sir Joseph Larmor, Kt., M.A., D.Sc., LL.D., D.C.L., F.R.S., M.P. University of Cambridge since 1911, Lucasian Professor of Mathematics in the University of Cambridge, St John's College, Cambridge.

1900 Archibald Liversidge, M.A., LL.D., F.R.S., Emer.-Professor of Chemistry in the University of Sydney, Fieldhead, Coombe Warren, Kingston, Surrey.

1916 Sir Arthur Schuster, Ph.D., D.Sc., LL.D., D. ès Sc. Geneva, Vice-President of the Royal Society, Honorary Professor of Physics in the University of Manchester, Yeldall, Twyford, Berks.

1908 Charles Scott Sherrington, M.A., M.D., LL.D., P.R.S., Waynflete Professor of Physiology in the University of Oxford, Physiological Laboratory, Oxford.

1913 Sir William Turner Thiselton-Dyer, K.C.M.G., C.I.E., M.A., LL.D., F.R.S., formerly Director of the Royal Botanic Gardens, Kew: The Ferns, Witcombe, Gloucester.

1905 Sir Joseph John Thomson, Kt., O.M., D.Sc., LL.D., Past Pres.R.S., lately Cavendish Professor of Experimental Physics, University of Cambridge, Trinity College, Cambridge.

1900 Sir Thomas Edward Thorpe, Kt., C.B., D.Sc., LL.D., F.R.S., formerly Principal of the Government Laboratories, Imperial College of Science and Technology, South Kensington, London, S.W., Whinfield, Salcombe, South Devon.

Total, 16.

CHANGES IN FELLOWSHIP DURING SESSION 1919–1920.

ORDINARY FELLOWS OF THE SOCIETY ELECTED.

HERBERT STANLEY ALLEN.
ERNEST MASSON ANDERSON.
RICHARD SIDDOWAY BAGNALL.
EDWARD BATTERSBY BAILEY.
R. G. CARRUTHERS.
WILLIAM GRANT CRAIB.
THOMAS BEDFORD FRANKLIN.
SUDHAMOY GHOSH.
WILLIAM GORDON.
ALEXANDER ROBERT HORNE.
JAMES GALL INGLIS.
JOHN CHARLES LAMONT.
THOMAS A. LINDSAY.
STUART M'DONALD.
JOHN MARSHALL.
GEORGE LESLIE PURSER.
WM. ROBERT SMELLIE.
JOHN BARBER TODD.
THOMAS WALMSLEY.
MALCOLM WILSON.

HONORARY FELLOWS ELECTED.

7th June 1920.

H.R.H. THE PRINCE OF WALES.

21st June 1920.

WILLIAM WALLACE CAMPBELL.
YVES DELAGE.
HENDRIK ANTON LORENTZ.
ALFRED GABRIEL NATHORST.
CH. EMILE PICARD.
CHARLES RICHET.
GEORG OSSIAN SARS.

ORDINARY FELLOWS DECEASED.

JOHN AITKEN.
J. G. BARTHOLOMEW.
ERSKINE BEVERIDGE.
Sir THOMAS R. FRASER.
J. H. FULLARTON.
R. D. GRAHAM.
CHARLES E. GREEN.
The Hon. LORD GUTHRIE.
ALLAN M'LANE HAMILTON.
D. BERRY HART.
Sir JOHN JACKSON.
ROBERT MUNRO.
E. P. RAMSAY (died 1917).
R. L. A. E. WESTERGAARD.
JOHN H. WILSON.

HONORARY FELLOWS DECEASED.

YVES DELAGE (died 8th Oct. 1920).
WM. CRAWFORD GORGAS.
CHARLES LAPWORTH.
ALEXANDER MACALISTER (died 19th September 1919).
AUGUSTO RIGHI.

ORDINARY FELLOW RESIGNED.

ARNOLD HARTLEY GIBSON.

Additions to Library by Gift or Purchase.

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Astrographic Catalogue 1900·0. Hyderabad Section, Dec. $-16°$ to $-21°$. Vol. III. 4to. Edinburgh, 1920. (*Presented.*)

Autografos de Morelos. Los Publica el Archivo General de la Nacion como Homenaje al Heroe en el ciii Aniversario de su Muerte. Fol. Mexico, 1918. (*Presented.*)

Ballou, H. A. The Pink Boll Worm in Egypt in 1916–17 (*Gelechia gossypiella*, Saunders). (Ministry of Agriculture, Egypt.)

Biblioteca Interamericana:—

Vida Constitucional de los Estados Unidos. Por Benjamin Harrison. 8vo. New York, 1919.

Cuentos Clasicos del Norte. Primera Serie. Por Edgar Allan Poe. 8vo. New York, 1920.

——— Segunda Serie. Por Washington Irving, Nathaniel Hawthorne, and Edward Everett Hale. 8vo. New York, 1920.

(*All in Spanish.*) (*Presented.*)

Boletim Meteorologico, Observatorio Nacional do Rio de Janeiro, Anno de 1910. Fol. Brussels, Paris. n.d. (*Presented by R. C. Mossman.*)

Bright, Sir Charles. Inter-Imperial Communication through Cable, Wireless, and Air. (Paper read before Section F of the British Association, Sept. 12, 1919.) (*Presented by the Author.*)

Buchanan, J. Y. Accounts rendered of Work Done and Things Seen. 8vo. Cambridge, 1919. (*Presented by the Author.*)

Butler, Frank Hedges. Through Lapland with Skis and Reindeer. 8vo. London, 1919. (*Presented by the Author.*)

The Cambridge British Flora. By C. E. Moss. Vol. III: Portulacaceæ to Fumariaceæ. Plates and Text. Fol. Cambridge, 1920. (*Purchased.*)

Carnegie, David. Can Church and Industry Unite? 8vo. London, Edinburgh, New York, 1920.

Catalogue of the Printed Books in the Library of the University of Edinburgh. Vol. I (A–F). 4to. Edinburgh, 1918. (*Presented.*)

Catalogue of the Scientific Serial Publications in the Principal Libraries of Calcutta. Complied for the Asiatic Society of Bengal by Stanley Kemp. 8vo. Calcutta, 1918. (*Presented.*)

Census of the Commonwealth of Australia taken for the night between the 2nd and 3rd April 1911. Vol. I. Issued by G. H. Knibbs. Fol. Melbourne, 1917. (*Presented.*)

Clark, Alex. The Polarity of Matter. 8vo. London, Edinburgh, 1907. (*Presented by Mr. J. Gall Inglis.*)

De Forcrand, R. Cours de Chimie à l'usage des étudiants P.C.N. et S.P.C.N. 2e ed. Tom. i et ii. 8vo. Paris, 1918, 1919. (*Presented.*)

De Man, J. G. Die Frei in der Reinen Erde und im Süssen Wasser Lebenden Nematoden der Niederländischen Fauna. Text and Plates. 8vo and fol. Leiden, 1919. (*Presented by the Author.*)

Dehaut, E. G. Contributions à l'Etude de la Vie Vertébrée Insulaire dans la Région Méditerranéenne Occidentale et particulièrement en Sardaigne et en Corse. 8vo. Paris, 1920. (*Presented by the Author.*)

Diplomatic Correspondence between the United States and Belligerent Governments relating to Neutral Rights and Commerce. Special Supplements to the *American Journal of International Law.* Vols. IX–XI. La. 8vo. New York, 1915–1917. (*Presented.*)

Eclipses de Sol del 3 de Diciembre de 1918 y 29 de Mayo de 1919. (Instituto Meteorologico Nacional.) 4to. Montevideo, 1920. (*Presented by Mr R. C. Mossman.*)

Filow, Bogdan D. Early Bulgarian Art. 4to. Berne, 1919.

Gamble, J. S. Flora of the Presidency of Madras. Part III: Leguminosæ - Cæsalpinioideæ to Caprifoliaceæ. 8vo. London, 1919. (*Presented.*)

Ganguli, S. Lectures on the Theory of Plane Curves. Parts I and II. 8vo. Calcutta, 1919. (*Presented by the Author.*)

Gley, E. Quatre Leçons sur les Sécrétions Internes. 8vo. Paris, 1920. (*Presented by the Author.*)

Gunnerus, Johan Ernst. 1718—26 Februar—1918. Mindeblade utgit av Det Kongelige Norske Videnskabers Selskab, Trondhjem. 4to. Trondhjem, 1918. (*Presented.*)

Gutzmer, A. Zum Jubiläum der Logarithmen. 4to. Leipzig u. Berlin, 1914. (*Presented.*)

Harvey-Gibson, R. J. Outlines of the History of Botany. 8vo. London, 1919.

Herdenking van het 150-jarig Bestaan van het Bataafsch Genootschap der Proefondervindelijke Wijsbegeerte te Rotterdam. 1769—17 Augustus —1819. La. 8vo. Rotterdam, 1919. (*Presented.*)

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Indian Medicinal Plants. By Lt.-Col. K. R. Kirtikar and Major B. D. Basu. Text—parts i–ii. Plates—parts i–iv. 4to. Allahabad, 1918. (*Presented.*)

Iwakawa, T. Catalogue of Japanese Mollusca in the Natural History Department, Tokyo Imperial Museum. 8vo. Tokyo, 1919. (*Presented.*)

Jacobi Cuiacii Juris-Consulti Opera, Quae de Jure fecit, et edi volint. Hanoviae, 1602. (*Presented by Mr Orrock, Bookbinder.*)

Jennings, H. S. Life and Death. Heredity and Evolution in Unicellular Organisms. 8vo. Boston, 1920. (*Presented by the Wagner Free Inst. of Science.*)

Johnston, Sir Harry H. A Comparative Study of the Bantu and Semi-Bantu Languages. Vol. I. 4to. Oxford, 1919. (*Presented.*)

Kaye, G. R. A Guide to the Old Observatories at Delhi, Jaipur, Ujjain, and Benares. 8vo. Calcutta, 1920. (*Presented.*)

Kelvin, Lord. Notice Historique sur la vie et l'œuvre de. Par M. Emile Picard. (Institut de France, Académie des Sciences.) 4to. Paris, 1919. (*Presented.*)

Lavoisier, Antoine-Laurent. Mémoires sur la Respiration et la Transpiration des Animaux. 8vo. Paris, 1920. (*Presented.*)

Leenhardt, F. Etude Géologique de la Région du Mont Ventoux. 4to. Montpellier, Paris, 1883. (*Presented by Prof. J. Y. Simpson.*)

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Macpherson, Hector. Herschel. (Pioneers of Progress: Men of Science.) 8vo. London, 1919. (*Presented by the Author.*)

Makower, W., et H. Geiger. Mesures Pratiques en Radioactivité. 8vo. Paris, 1919. (*Presented.*)

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Negri, Galdino. Nueva Contribución a la Determinación Racional de Algunas Funciones Sísmicas. (*Anales de la Academia de Ciencias Exactas Físicas y Naturales.*) La. 8vo. La Plata, 1917.

Oersted, H. C. The Discovery of Electromagnetism. 8vo. Copenhagen, 1920. (*Presented by the Oersted Committee.*)

——— Scientific Papers. Collected Edition, with Two Essays on his Work by Kirstine Meyer. Vols. I, II, and III. 4to. Copenhagen, 1920. (*Ed. by Royal Danish Society of Sciences.*)

Przibram, Prof. Dr Ernst. Der Gegenwärtige Bestand der vorm. Královschen Sammlung von Mikroorganismen. 8vo. Wien, 1919. (*Presented.*)

Rangacharya, V. A Topographical List of the Inscriptions of the Madras Presidency. Vols. I–III. 4to. Madras, 1919. (*Presented.*)

Raspail, Xavier. Trente Ans de Critiques Médicales et Scientifiques, 1884–1914. 8vo. Paris, 1916. (*Presented by the Author.*)

Report (First) of the Committee of the Institution of Civil Engineers appointed to investigate the Deterioration of Structures of Timber, Metal, and Concrete exposed to the Action of Sea-Water. 8vo London, 1920. (*Presented.*)

Report of the Interdepartmental Committee on Research and Development in the Dependencies of the Falkland Islands, with Appendices, Maps, etc. April 1920. Fol. London, 1920. (*Presented by the Under-Secretary of State for the Colonies.*)

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Ritchie, James. The Influence of Man on Animal Life in Scotland. A Study in Faunal Evolution. 8vo. Cambridge, 1920.

Rolf, Bruno. Tables Psychrométriques Portatives, calculées selon la formule Svensson-Ekholm. (Station d'Explorations Scientifiques d'Abisko.) 8vo. Stockholm, 1919. (*Presented.*)

Scottish National Antarctic Expedition. Report of the Scientific Results of the Voyage of S.Y. "Scotia," 1902–04. Vol. VII: Zoology. 4to. Edinburgh, 1920. (*Presented.*)

Silberstein, L. Report on the Quantum Theory of Spectra. 8vo. London, 1920. (*Presented by Adam Hilger, Ltd.*)

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Stieltjes, Thos. Jan. Œuvres complètes de. Tom. II. (Publiées par les soins de la Société Mathématique d'Amsterdam.) 4to. Groningen. 1918. (*Presented by the above Society.*)

Tredgold, A. F. Mental Deficiency (Amentia). 3rd edition. 8vo. London, 1920. (*Presented by the Author.*)

Whitman, C. H. A Subject Index to the Poems of Edmund Spenser. (Published under the auspices of the Connecticut Academy of Arts and Sciences.) 8vo. New Haven, 1918. (*Presented.*)

LAWS OF THE SOCIETY.

Adopted July 3, 1916; amended December 18, 1916.

(Laws VIII, IX, and XIII amended May 3, 1920. Law VI amended February 7, 1921.)

I.

THE ROYAL SOCIETY OF EDINBURGH, which was instituted by Royal Charter in 1783 for the promotion of Science and Literature, shall consist of Ordinary Fellows (hereinafter to be termed Fellows) and Honorary Fellows. The number of Honorary Fellows shall not exceed fifty-six, of whom not more than twenty may be British subjects, and not more than thirty-six subjects of Foreign States.

Fellows only shall be eligible to hold office or to vote at any Meeting of the Society.

ELECTION OF FELLOWS.

II.

Each Candidate for admission as a Fellow shall be proposed by at least four Fellows, two of whom must certify from personal knowledge. The Official Certificate shall specify the name, rank, profession, place of residence, and the qualifications of the Candidate. The Certificate shall be delivered to the General Secretary before the 30th of November, and, subject to the approval of the Council, shall be exhibited in the Society's House during the month of January following. All Certificates so exhibited shall be considered by the Council at its first meeting in February, and a list of the Candidates approved by the Council for election shall be issued to the Fellows not later than the 21st of February.

III.

The election of Fellows shall be by Ballot, and shall take place at the first Ordinary Meeting in March. Only Candidates approved by the Council shall be eligible for election. A Candidate shall be held not elected, unless he is supported by a majority of two-thirds of the Fellows present and voting.

IV.

On the day of election of Fellows two scrutineers, nominated by the President, shall examine the votes and hand their report to the President, who shall declare the result.

V.

Each Fellow, after his election, is expected to attend an Ordinary Meeting, and sign the Roll of Fellows, he having first made the payments required by Law VI. He shall be introduced to the President, who shall address him in these words :

In the name and by the authority of THE ROYAL SOCIETY OF EDINBURGH, I admit you a Fellow thereof.

PAYMENTS BY FELLOWS.

VI.

Each Fellow shall, before he is admitted to the privileges of Fellowship, pay an admission fee of three guineas, and a subscription of three guineas for the year of election. He shall continue to pay a subscription of three guineas at the beginning of each session so long as he remains a Fellow.

Each Fellow who was elected subsequent to December 1916 and previous to December 1920 shall also pay a subscription of three guineas at the beginning of each session so long as he remains a Fellow.

Each Fellow who was elected previous to December 1916, and who has not completed his twenty-five annual payments,* shall, at the beginning of each session, pay three guineas or four guineas according as he has or has not made ten annual payments as reckoned from the year of election. Each Fellow who has completed or shall complete his payments shall be invited to contribute one guinea per annum or to pay a single sum of ten guineas.

A Fellow may compound for the annual subscriptions by a single payment of fifty guineas, or on such other terms as the Council may from time to time fix.

VII.

A Fellow who, after application made by the Treasurer, fails to pay any contribution due by him, shall be reported to the Council, and, if the Council see fit, shall be declared no longer a Fellow. Notwithstanding such declaration all arrears of contributions shall remain exigible.

* The following is an extract from the previous law affecting Annual Subscribers elected prior to December 1916 :—"Every Ordinary Fellow, within three months after his election, shall pay Two Guineas as the fee of admission, and Three Guineas as his contribution for the Session in which he has been elected ; and annually at the commencement of every Session, Three Guineas into the hands of the Treasurer. This annual contribution shall continue for ten years after his admission, and it shall be limited to Two Guineas for fifteen years thereafter."

ELECTION OF HONORARY FELLOWS.

VIII.

Honorary Fellows shall be persons eminently distinguished in Science or Literature. They shall not be liable to contribute to the Society's Funds. Personages of the Blood Royal may be elected Honorary Fellows at any time on the nomination of the Council, and without regard to the limitation of numbers specified in Law I.

IX.

Honorary Fellows shall be proposed by the Council. The nominations shall be announced from the Chair at the First Ordinary Meeting after their selection. The names shall be printed in the circular for the last Ordinary Meeting of the Session, when the election shall be by Ballot, after the manner prescribed in Laws III and IV for the Election of Fellows.

EXPULSION OF FELLOWS.

X.

If, in the opinion of the Council, the conduct of any Fellow is injurious to the character or interests of the Society, the Council may, by registered letter, request him to resign. If he fail to do so within one month of such request, the Council shall call a Special Meeting of the Society to consider the matter. If a majority consisting of not less than two-thirds of the Fellows present and voting decide for expulsion, he shall be expelled by declaration from the Chair, his name shall be erased from the Roll, and he shall forfeit all right or claim in or to the property of the Society.

XI.

It shall be competent for the Council to remove any person from the Roll of Honorary Fellows if, in their opinion, his remaining on the Roll would be injurious to the character or interests of the Society. Reasonable notice of such proposal shall be given to each member of the Council, and, if possible, to the Honorary Fellow himself. Thereafter the decision on the question shall not be taken until the matter has been discussed at two Meetings of Council, separated by an interval of not less than fourteen days. A majority of two-thirds of the members present and voting shall be required for such removal.

MEETINGS OF THE SOCIETY.

XII.

A Statutory Meeting for the election of Council and Office-Bearers, for the presentation of the Annual Reports, and for such other business as may be arranged by the Council, shall be held on the fourth Monday of October. Each Session of the Society shall begin at the date of the Statutory Meeting.

XIII.

Meetings for reading and discussing communications and for general business, herein termed Ordinary Meetings, shall be held, when convenient, on the first and third Mondays of each month from November to July inclusive, with the exception that in January the meetings shall be held on the second and fourth Mondays.

XIV.

A Special Meeting of the Society may be called at any time by direction of the Council, or on a requisition to the Council signed by not fewer than six Fellows. The date and hour of such Meeting shall be determined by the Council, who shall give not less than seven days' notice of such Meeting. The notice shall state the purpose for which the Special Meeting is summoned; no other business shall be transacted.

PUBLICATION OF PAPERS.

XV.

The Society shall publish Transactions and Proceedings. The consideration of the acceptance, reading, and publication of papers is vested in the Council, whose decision shall be final. Acceptance for reading shall not necessarily imply acceptance for publication.

DISTRIBUTION OF PUBLICATIONS.

XVI.

Fellows who are not in arrear with their Annual Subscriptions and all Honorary Fellows shall be entitled gratis to copies of the Parts of the Transactions and the Proceedings published subsequently to their admission.

Copies of the Parts of the Proceedings shall be distributed by post or otherwise, as soon as may be convenient after publication; copies of the Transactions or Parts thereof shall be obtainable upon application, either personally or by an authorised agent, to the Librarian, provided the application is made within five years after the date of publication.

CONSTITUTION OF COUNCIL.

XVII.

The Council shall consist of a President, six Vice-Presidents, a Treasurer, a General Secretary, two Secretaries to the Ordinary Meetings (the one representing the Biological group and the other the Physical group of Sciences),* a Curator of the Library and Museum, and twelve ordinary members of Council.

ELECTION OF COUNCIL.

XVIII.

The election of the Council and Office-Bearers for the ensuing Session shall be held at the Statutory Meeting on the fourth Monday of October. The list of the names recommended by the Council shall be issued to the Fellows not less than one week before the Meeting. The election shall be by Ballot, and shall be determined by a majority of the Fellows present and voting. Scrutineers shall be nominated as in Law IV.

XIX.

The President may hold office for a period not exceeding five consecutive years; the Vice-Presidents, not exceeding three; the Secretaries to the Ordinary Meetings, not exceeding five; the General Secretary, the Treasurer, and the Curator of the Library and Museum, not exceeding ten; and ordinary members of Council, not exceeding three consecutive years.

XX.

In the event of a vacancy arising in the Council or in any of the offices enumerated in Law XVII, the Council shall proceed, as soon as convenient, to elect a Fellow to fill such vacancy for the period up to the next Statutory Meeting.

* The Biological group includes Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, Zoology; the Physical group includes Astronomy, Chemistry, Mathematics, Metallurgy, Meteorology, Physics.

POWERS OF THE COUNCIL.

XXI.

The Council shall have the following powers:—(1) To manage all business concerning the affairs of the Society. (2) To decide what papers shall be accepted for communication to the Society, and what papers shall be printed in whole or in part in the Transactions and Proceedings. (3) To appoint Committees. (4) To appoint employees and determine their remuneration. (5) To award the various prizes vested in the Society, in accordance with the terms of the respective deeds of gift, provided that no member of the existing Council shall be eligible for any such award. (6) To make from time to time Standing Orders for the regulation of the affairs of the Society. (7) To control the investment or expenditure of the Funds of the Society.

At Meetings of the Council the President or Chairman shall have a casting as well as a deliberative vote.

DUTIES OF PRESIDENT AND VICE-PRESIDENTS.

XXII.

The President shall take the Chair at Meetings of Council and of the Fellows. It shall be his duty to see that the business is conducted in accordance with the Charter and Laws of the Society. When unable to be present at any Meetings or attend to current business, he shall give notice to the General Secretary, in order that his place may be supplied. In the absence of the President his duties shall be discharged by one of the Vice-Presidents.

DUTIES OF THE TREASURER.

XXIII.

The Treasurer shall receive the monies due to the Society and shall make payments authorised by the Council. He shall lay before the Council a list of arrears in accordance with Rule VII. He shall keep accounts of all receipts and payments, and at the Statutory Meeting shall present the accounts for the preceding Session, balanced to the 30th of September, and audited by a professional accountant appointed annually by the Society.

DUTIES OF THE GENERAL SECRETARY.

XXIV.

The General Secretary shall be responsible to the Council for the conduct of the Society's correspondence, publications, and all other business except that which relates to finance. He shall keep Minutes of the Statutory and Special

Meetings of the Society and Minutes of the Meetings of Council. He shall superintend, with the aid of the Assistant Secretary, the publication of the Transactions and Proceedings. He shall supervise the employees in the discharge of their duties.

DUTIES OF SECRETARIES TO ORDINARY MEETINGS.

XXV.

The Secretaries to Ordinary Meetings shall keep Minutes of the Ordinary Meetings. They shall assist the General Secretary, when necessary, in superintending the publication of the Transactions and Proceedings. In his absence, one of them shall perform his duties.

DUTIES OF CURATOR OF LIBRARY AND MUSEUM.

XXVI.

The Curator of the Library and Museum shall have charge of the Books, Manuscripts, Maps, and other articles belonging to the Society. He shall keep the Card Catalogue up to date. He shall purchase Books sanctioned by the Council.

ASSISTANT-SECRETARY AND LIBRARIAN.

XXVII.

The Council shall appoint an Assistant-Secretary and Librarian, who shall hold office during the pleasure of the Council. He shall give all his time, during prescribed hours, to the work of the Society, and shall be paid according to the determination of the Council. When necessary he shall act under the Treasurer in receiving subscriptions, giving out receipts, and paying employees.

ALTERATION OF LAWS.

XXVIII.

Any proposed alteration in the Laws shall be considered by the Council, due notice having been given to each member of Council. Such alteration, if approved by the Council, shall be proposed from the Chair at the next Ordinary Meeting of the Society, and, in accordance with the Charter, shall be considered and voted upon at a Meeting held at least one month after that at which the motion for alteration shall have been proposed.

INDEX.

Index of Papers published in the "Transactions" during Session 1919–20.

(Arranged under the Authors' Names.)

PRINTED IN GREAT BRITAIN BY NEILL AND CO., LTD., EDINBURGH.

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MODEL INDEX.

Schäfer, E. A.—On the Existence within the Liver Cells of Channels which can be directly injected from the Blood-vessels. Proc. Roy. Soc. Edin., vol. 1902, pp.

Cells, Liver,—Intra-cellular Canaliculi in.
E. A. Schäfer. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

Liver,—Injection within Cells of.
E. A. Schäfer. Proc. Roy. Soc. Edin., vol. , 1902, pp. .

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