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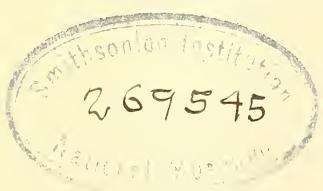
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VOL. XLII.

1921-1922.



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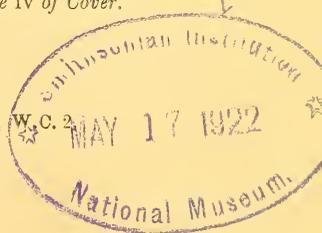
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PROCEEDINGS
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I.—On Tubes of Electromagnetic Force. By Professor
E. T. Whittaker, F.R.S.

(Read November 7, 1921. MS. received November 11, 1921.*)

§ 1. INTRODUCTION.

THE object of the present paper is to introduce certain surfaces, which will be shown to play the same part in the general electromagnetic field as Faraday's tubes of force do in electrostatic and magnetostatic fields.

The value of Faraday tubes in electrostatics and magnetism has led many investigators to seek for them a wider application, in connection with variable electromagnetic fields. In such investigations it has generally been assumed that the tubes are to be defined in much the same way as in static fields, but that instead of being at rest they are in motion. It cannot be said, however, that researches on these lines have led to completely satisfactory results: and the reason for this comparative failure is supplied by the Principle of Relativity. For, in the case of a purely electrostatic field, we can imagine an observer who is at rest relative to the electric charges: the electric force, as measured by this observer, depends only on the charges and their position, so that the Faraday tubes based on his measurements furnish a representation of the field which does not involve any arbitrary foreign element. If, however, we consider a field in which a number of charges and magnets are moving independently of each other, it is not possible for an observer to be at rest relative to all of them, and therefore all observers are on a footing

* The results of this paper were communicated to Section A of the British Association at its meeting in Edinburgh on September 8, 1921.



of equality, whatever be their velocities. But if we take two observers who are at any instant at the same point, but who are moving with different velocities, their measures of the electric and magnetic forces will not be concordant: and therefore it is not possible to construct a system of Faraday tubes which is purely objective, and independent of the arbitrary choice of an observer. A system of tubes of force which is to be really objective (*i.e.* depending only on the field and not at all on the peculiarities of the person observing it) can exist only in the four-dimensional world of space-time with which the Theory of Relativity has made us familiar. This is the standpoint of the present paper.

The surfaces here introduced, to which I give the name *calamoids*, may be regarded as a direct generalisation of Faraday's tubes, since they reduce to Faraday's electric tubes when the field is purely electrostatic, and they reduce to Faraday's magnetic tubes when the field is purely magnetostatic: and between these extreme types they provide a continuous transition. The way in which the generalisation is made may be illustrated by the following statement: With Faraday's electrostatic tubes, as we proceed along a tube, the magnitude of the electric force at any point is inversely proportional to the area of the cross-section of the tube (the cross-section being part of an equipotential surface), and the three components of the electric force are to each other in the same ratios as the areas of the projections of this cross-section on the three co-ordinate planes. With the calamoids introduced in the present paper, the quantity

$$\sqrt{\{(\text{Electric Force})^2 - (\text{Magnetic Force})^2\}}$$

(which, as is well known, is covariant with respect to all Lorentz transformations) is inversely proportional to the area of the cross-section of the calamoid (the cross-section being part of what is here called an electropotential surface), and the six components of the electric and magnetic forces are connected in a simple way with the areas of the six projections of this cross-section on the six co-ordinate planes of xy, yz, zx, xt, yt, zt . The calamoids are covariant with respect to the Lorentz transformations—which implies that they are the same, whatever be the observer whose measures of electric and magnetic force are used in constructing them.

The circumstance that the two kinds of Faraday tubes (electric and magnetic) are particular cases of calamoids seems to terminate the rivalry, so to speak, which has existed between them, and which has manifested itself in such questions as whether both of them should be regarded as physically existent, or only one kind, and if so, which kind.

The ordinary Faraday tubes not only furnish us with a graphical representation of the state of the field, but they also enable us, by comparing two regions at a distance from one another along the same tube, to establish direct connections between the fields in these distant regions: they enable us, in fact, to integrate the differential equations of the electrostatic field in an intuitive geometrical fashion. Similarly, the calomoids not only provide a graphical representation of the state and history of the whole field, but they also enable us to integrate the general differential equations of the field (the Maxwell-Lorentz equations) in an intuitive geometrical fashion.

§ 2. THE ELECTROPOTENTIAL SURFACES.

We shall, as the fundamental case, study a field free from ordinary ponderable matter, so that we have to consider only a region of free space with solitary electrons dispersed in it. (The formulæ for the more general case in which ponderable matter is present may be derived from the formulæ for this fundamental case, by supposing that each molecule of a ponderable dielectric contains an electric doublet, etc.) Since the electrons are singularities of the tubes of force, we shall for the present not consider the immediate neighbourhood of electrons, but shall investigate the tubes of force in free space, where their behaviour is regular. Moreover, we shall suppose the units so chosen that the velocity of light is unity. If, then, the three components of the electric vector are denoted by (d_x, d_y, d_z) , and the three components of the magnetic vector by (h_x, h_y, h_z) , we have the usual Maxwellian equations of the electromagnetic field

$$\left. \begin{aligned} \frac{\partial h_z}{\partial y} - \frac{\partial h_y}{\partial z} &= \frac{\partial d_x}{\partial t} \\ \frac{\partial h_x}{\partial z} - \frac{\partial h_z}{\partial x} &= \frac{\partial d_y}{\partial t} \\ \frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y} &= \frac{\partial d_z}{\partial t} \\ \frac{\partial d_x}{\partial x} + \frac{\partial d_y}{\partial y} + \frac{\partial d_z}{\partial z} &= 0 \end{aligned} \right\} \dots \dots \dots (1)$$

$$\left. \begin{aligned} \frac{\partial d_z}{\partial y} - \frac{\partial d_y}{\partial z} &= -\frac{\partial h_x}{\partial t} \\ \frac{\partial d_x}{\partial z} - \frac{\partial d_z}{\partial x} &= -\frac{\partial h_y}{\partial t} \\ \frac{\partial d_y}{\partial x} - \frac{\partial d_x}{\partial y} &= -\frac{\partial h_z}{\partial t} \\ \frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z} &= 0 \end{aligned} \right\} \dots \dots \dots (2)$$

It will appear that the analytical expressions relating to tubes of force are considerably simplified when (as is often the case in radiation fields) the electric and magnetic vectors are everywhere at right angles to each other: we shall take this case first, and so shall for the present assume the relation

$$d_x h_x + d_y h_y + d_z h_z = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This relation is, of course, satisfied not only in the radiation fields already referred to, but also in electrostatic and magnetostatic fields.

Now consider the set of total differential equations

$$\left. \begin{aligned} h_z dy - h_y dz + d_x dt &= 0 \\ -h_z dx + h_x dz + d_y dt &= 0 \\ h_y dx - h_x dy + d_z dt &= 0 \\ -d_x dx - d_y dy - d_z dz &= 0 \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is a covariant set of differential equations: for if we denote the left-hand members of the four equations by A_x, A_y, A_z, A_t respectively, then in Einstein's terminology (A_x, A_y, A_z, A_t) is a covariant four-vector.

This implies that when we perform the Lorentz transformation

$$\left\{ \begin{aligned} x &= x' \cosh \alpha + t' \sinh \alpha \\ y &= y' \\ z &= z' \\ t &= x' \sinh \alpha + t' \cosh \alpha, \end{aligned} \right.$$

then the four-vector (A_x, A_y, A_z, A_t) is transformed according to the formulæ

$$\left\{ \begin{aligned} A_x &= A_x' \cosh \alpha - A_t' \sinh \alpha \\ A_y &= A_y' \\ A_z &= A_z' \\ A_t &= A_t' \cosh \alpha - A_x' \sinh \alpha. \end{aligned} \right.$$

Let us first find how many of the four equations (4) are independent. If we eliminate dy between the first and fourth of them (making use of (3)), we obtain the second; and if we eliminate dz between the first and fourth, we obtain the third. Thus only two of the equations (4) are independent.

Now it is well known that if two total equations in four variables are given, say

$$\left. \begin{aligned} dz &= U_{11} dx + U_{12} dy \\ dt &= U_{21} dx + U_{22} dy \end{aligned} \right\},$$

then this system of total equations is unconditionally integrable (*i.e.* yields a pair of integral equations $\phi(x, y, z, t) = a, \psi(x, y, z, t) = b$) when, and only when, the following two conditions are satisfied:

$$\left. \begin{aligned} \left(\frac{\partial}{\partial x} + U_{11} \frac{\partial}{\partial z} + U_{21} \frac{\partial}{\partial t} \right) U_{12} - \left(\frac{\partial}{\partial y} + U_{12} \frac{\partial}{\partial z} + U_{22} \frac{\partial}{\partial t} \right) U_{11} &= 0 \\ \left(\frac{\partial}{\partial x} + U_{11} \frac{\partial}{\partial z} + U_{21} \frac{\partial}{\partial t} \right) U_{22} - \left(\frac{\partial}{\partial y} + U_{12} \frac{\partial}{\partial z} + U_{22} \frac{\partial}{\partial t} \right) U_{21} &= 0 \end{aligned} \right\}.$$

On applying this criterion to any two of the equations (4), we find that these conditions of integrability are satisfied, by virtue of the equations (1), (2), (3): and therefore the solution of the total differential equations (4) is represented by a pair of integral equations

$$\phi(x, y, z, t) = a, \quad \psi(x, y, z, t) = b \quad (5)$$

where a and b denote constants of integration.

The functions ϕ and ψ are solutions of the set of partial differential equations "adjoint" to the set of total differential equations: these adjoint equations in our case are any two of the equations

$$\left. \begin{aligned} d_z \frac{\partial f}{\partial y} - d_y \frac{\partial f}{\partial z} + h_z \frac{\partial f}{\partial t} &= 0 \\ -d_z \frac{\partial f}{\partial x} + d_x \frac{\partial f}{\partial z} + h_y \frac{\partial f}{\partial t} &= 0 \\ d_y \frac{\partial f}{\partial x} - d_x \frac{\partial f}{\partial y} + h_z \frac{\partial f}{\partial t} &= 0 \\ h_x \frac{\partial f}{\partial x} + h_y \frac{\partial f}{\partial y} + h_z \frac{\partial f}{\partial z} &= 0 \end{aligned} \right\}$$

Let us now express our result in the language of geometry. Let us use the word *surface* to mean a two-dimensional continuum in the four-dimensional hyperspace in which x, y, z, t are co-ordinates, so that a surface is defined by two equations between x, y, z, t . Then evidently we have proved that *the solution of the set of total differential equations (4) represents a family of ∞^2 surfaces* in the four-dimensional hyperspace in which (x, y, z, t) are the co-ordinates. These will be called the *electro-potential surfaces* of the electromagnetic field.

The statement that there are ∞^2 surfaces means that there are two arbitrary parameters in the equation of one of these surfaces, namely, the a and b of equations (5).

In order to understand the nature of these electropotential surfaces, let us consider for a moment the particular case of a purely electrostatic field, for which the magnetic vector everywhere vanishes and the components of the electric vector are the derivatives of an electrostatic potential $V(x, y, z)$: thus,

$$h_x = 0, \quad h_y = 0, \quad h_z = 0, \quad d_x = -\frac{\partial V}{\partial x}, \quad d_y = -\frac{\partial V}{\partial y}, \quad d_z = -\frac{\partial V}{\partial z}.$$

The equations (4) now reduce to the following:

$$\begin{cases} dt = 0 \\ \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = 0, \end{cases}$$

of which the integrals are

$$\begin{cases} t = \text{Constant} \\ V(x, y, z) = \text{Constant.} \end{cases}$$

But the latter equations define the ordinary equipotential surfaces of the electrostatic field: and thus we see that *the electropotential surfaces are a covariant family of ∞^2 surfaces, which exist in electromagnetic fields and which become the ordinary equipotential surfaces when the field is purely electrostatic.*

§ 3. THE ABSOLUTE.

In the course of this paper there will often be occasion to use terms belonging to metrical geometry, such as *distance, parallel, perpendicular.* It may be well at this stage to explain precisely what is meant by them.

The metrical properties of any kind of space are determined by what is called the *absolute* of that space. In the ordinary Euclidean geometry of the plane the absolute consists of a pair of imaginary points, namely, the "circular points at infinity"; all metrical properties can be defined in terms of these points, *e.g.* if OI, OJ are the lines drawn from a point O to the circular points at infinity, then two lines OA, OB through O are *perpendicular* when the lines OA, OB, OI, OJ form a harmonic pencil.

If (x, y) are ordinary rectangular co-ordinates in the plane, and if we write

$$x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3},$$

so that (x_1, x_2, x_3) are the homogeneous co-ordinates of a point, then the two circular points at infinity are represented by the equations

$$x_1^2 + x_2^2 = 0, \quad x_3 = 0.$$

Similarly, in Euclidean geometry of three dimensions the absolute on which all metrical properties depend is an imaginary circle at infinity, represented by the equations (in homogeneous co-ordinates)

$$x_1^2 + x_2^2 + x_3^2 = 0, \quad x_4 = 0,$$

and in the Euclidean geometry of four dimensions the absolute is an imaginary sphere at infinity represented by the equations

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, \quad x_5 = 0.$$

The four-dimensional hyperspace with which we are concerned in the present paper is formed of the aggregate of all the three-dimensional "instantaneous spaces" perceived by an observer at successive instants. This aggregate is the same for all observers, although from the Theory of Relativity we know that its dissection into instantaneous spaces is different for different observers. This hyperspace is not Euclidean, because the time t is different in character from the three space co-ordinates x, y, z . Writing

$$x = \frac{x_1}{x_5}, \quad y = \frac{x_2}{x_5}, \quad z = \frac{x_3}{x_5}, \quad t = \frac{x_4}{x_5},$$

so that $(x_1, x_2, x_3, x_4, x_5)$ are homogeneous co-ordinates, we can express the difference in character between the time co-ordinate and the space co-ordinate by taking the absolute to be

$$x_1^2 + x_2^2 + x_3^2 - c^2 x_4^2 = 0, \quad x_5 = 0,$$

where c denotes the velocity of light.* For simplicity, we choose our units so that $c=1$, and the absolute will therefore be taken to be

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0, \quad x_5 = 0.$$

The negative sign which affects x_4^2 is the analytical way of expressing the difference between time and space.

The transformations, which are called "Lorentz transformations" in electromagnetic theory, transform this absolute into itself: and the "restricted theory of relativity" is nothing but the invariant theory of the four-dimensional world of space and time with respect to the transformations of this group.

With this absolute, geometry is non-Euclidean and of the "hyperbolic" type: the *distance* between two points whose co-ordinates are (x, y, z, t) and (x', y', z', t') is

$$\sqrt{\{-(x-x')^2 - (y-y')^2 - (z-z')^2 + (t-t')^2\}}.$$

The *area* of a region on a surface which is defined by the equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad t = t(u, v)$$

is

$$\iint \left[\left\{ \frac{\partial(x, y)}{\partial(u, v)} \right\}^2 + \left\{ \frac{\partial(x, z)}{\partial(u, v)} \right\}^2 + \left\{ \frac{\partial(y, z)}{\partial(u, v)} \right\}^2 - \left\{ \frac{\partial(x, t)}{\partial(u, v)} \right\}^2 - \left\{ \frac{\partial(y, t)}{\partial(u, v)} \right\}^2 - \left\{ \frac{\partial(z, t)}{\partial(u, v)} \right\}^2 \right]^{\frac{1}{2}} du dv.$$

Let us now introduce the notion of parallelism. Consider two planes. Each plane meets the hyperplane † at infinity, $x_5=0$, in a line. If these lines do not intersect, the planes are said to be *not parallel*. If the lines intersect in a point, the planes are said to be *half-parallel*. If the lines are coincident, the planes are said to be *absolutely parallel*. Thus in four-dimensional hyperspace there are two kinds of parallelism.

Next, consider perpendicularity. Two lines are said to be *orthogonal* or *perpendicular* when the points in which they meet the hyperplane at infinity are conjugate with respect to the absolute. With regard to planes, there are two possible degrees of orthogonality, just as there are two degrees of parallelism; two planes may be either—

* So far as I know, this remark was first made by Klein.

† The locus of points whose co-ordinates satisfy a linear equation

$$ax + by + cz + dt + e = 0$$

is called a *hyperplane*. The intersection of two hyperplanes is in general a plane, and the intersection of three hyperplanes is in general a straight line.

(1) *absolutely orthogonal*, namely, when every line which lies in one plane is orthogonal to every line which lies in the other plane;

or

(2) *half-orthogonal*, namely, when one of the planes contains a line which is orthogonal to all the lines in the other plane;

or

(3) *not orthogonal at all*.

§ 4. PROPERTIES OF THE ELECTROPOTENTIAL SURFACES.

(i) In electrostatics the equipotential surfaces have the property that the projections of any surface-element on the three co-ordinate planes are proportional to the three components of the electric vector. We shall now show that in the general electromagnetic field the electropotential surfaces have the property that the projections of any surface-element on the six co-ordinate planes are proportional to the six components of the electric and magnetic vectors.

To prove this, we remark that if a surface is defined by the equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad t = t(u, v),$$

then (the absolute being that described in § 3) the projections of a surface-element on the planes of

$$xy, \quad xz, \quad xt, \quad yz, \quad yt, \quad zt \quad \text{respectively}$$

are

$$\frac{\partial(x, y)}{\partial(u, v)} du dv, \quad \frac{\partial(x, z)}{\partial(u, v)} du dv, \quad i \frac{\partial(x, t)}{\partial(u, v)} du dv, \quad \frac{\partial(y, z)}{\partial(u, v)} du dv, \quad i \frac{\partial(y, t)}{\partial(u, v)} du dv, \quad i \frac{\partial(z, t)}{\partial(u, v)} du dv \quad (6)$$

Now for an equipotential surface we have from the first of equations (4)

$$h_z \frac{\partial y}{\partial u} - h_y \frac{\partial z}{\partial u} + d_x \frac{\partial t}{\partial u} = 0$$

$$h_z \frac{\partial y}{\partial v} - h_y \frac{\partial z}{\partial v} + d_x \frac{\partial t}{\partial v} = 0,$$

whence we have at once

$$\frac{h_z}{\frac{\partial(z, t)}{\partial(u, v)}} = \frac{h_y}{\frac{\partial(y, t)}{\partial(u, v)}} = \frac{d_x}{\frac{\partial(x, t)}{\partial(u, v)}}$$

and by use of the other equations (4) we extend this so as to obtain

$$\frac{d_z}{\frac{\partial(x, y)}{\partial(u, v)}} = \frac{-d_y}{\frac{\partial(x, z)}{\partial(u, v)}} = \frac{h_x}{\frac{\partial(x, t)}{\partial(u, v)}} = \frac{d_x}{\frac{\partial(y, z)}{\partial(u, v)}} = \frac{h_y}{\frac{\partial(y, t)}{\partial(u, v)}} = \frac{h_z}{\frac{\partial(z, t)}{\partial(u, v)}} \quad (7)$$

Comparing (6) and (7), we see that *the projections of any surface-element of an electropotential surface, on the six co-ordinate planes*

$$xy, \quad xz, \quad xt, \quad yz, \quad yt, \quad zt,$$

are respectively proportional to

$$d_z, \quad -d_y, \quad ih_x, \quad d_x, \quad ih_y, \quad ih_z.$$

Thus *the inclinations of the electropotential surface at any point to the co-ordinate planes specifies the nature of the electromagnetic field at that point.*

(ii) Consider the intersection of an electropotential surface with the three-dimensional "space" observed by an observer at any instant t_0 . This instantaneous space will be the hyperplane $t=t_0$. The intersection of the hyperplane $t=t_0$ with the electropotential surface will be a curve, and on putting $dt=0$ in the equations (4) which define the electropotential surface we see that for this curve we have

$$\frac{dx}{h_x} = \frac{dy}{h_y} = \frac{dz}{h_z}.$$

That is to say, *the intersections of the electropotential surfaces with the instantaneous space of an observer are the lines of magnetic force (in Faraday's sense) of that observer at that instant. In fact, each electropotential surface may be regarded as a single moving Faraday line of magnetic force.*

This applies to *any* observer, since the electropotential surfaces are covariant: and therefore we see that *the electropotential surfaces may be regarded as built up of the Faraday lines of magnetic force of the field, as perceived by different observers moving in all possible directions with all possible velocities.*

§ 5. THE MAGNETOPOTENTIAL SURFACES.

We shall next introduce another family of ∞^2 surfaces which exist in the electromagnetic field.

Consider the set of total differential equations

$$\left. \begin{aligned} d_z dy - d_y dz - h_x dt &= 0 \\ -d_z dx + d_x dz - h_y dt &= 0 \\ d_y dx - d_x dy - h_z dt &= 0 \\ h_x dx + h_y dy + h_z dz &= 0 \end{aligned} \right\} \dots \dots \dots (8)$$

By reasoning similar to that in § 2, we can prove that *the solution of this set of total differential equations represents a family of ∞^2 surfaces in the four-dimensional hyperspace. These will be called the magnetopotential surfaces of the electromagnetic field. As in §§ 2, 4,*

we can show that the magnetopotential surfaces are a covariant family of ∞^2 surfaces, which exist in electromagnetic fields, and which reduce to the ordinary equipotential surfaces when the field is purely magneto-static. In the general case, the intersections of the magnetopotential surfaces with the instantaneous space of an observer are the lines of electric force (in Faraday's sense) of that observer at that instant.

§ 6. MUTUAL RELATIONS OF THE ELECTROPOTENTIAL AND MAGNETOPOTENTIAL SURFACES.

Two surfaces in four-dimensional hyperspace intersect in general in solitary points (this is to be contrasted with the fact that two surfaces in three-dimensional space intersect in general in a *curve*). Suppose then that (x, y, z, t) is a point of intersection of any electropotential surface with any magnetopotential surface: let (dx_1, dy_1, dz_1, dt_1) be a line-element issuing from this point and lying in the electropotential surface, and let (dx_2, dy_2, dz_2, dt_2) be a line-element issuing from the same point and lying in the magnetopotential surface.

Then from equations (4) and (8) we have

$$\begin{aligned} h_z dy_1 &= h_y dz_1 - d_x dt_1 \\ h_z dx_1 &= h_x dz_1 + d_y dt_1 \\ h_z dt_2 &= d_y dx_2 - d_x dy_2 \\ h_z dz_2 &= -h_x dx_2 + h_y dy_2. \end{aligned}$$

If we substitute from these equations in the expression

$$h_z(dx_1 dx_2 + dy_1 dy_2 + dz_1 dz_2 - dt_1 dt_2)$$

it vanishes identically. Similarly, we see that the same equation is true when any of the other components of the electric or magnetic vectors is put in place of h_z : and therefore we have

$$dx_1 dx_2 + dy_1 dy_2 + dz_1 dz_2 - dt_1 dt_2 = 0.$$

But this is the condition that the line-elements (dx_1, dy_1, dz_1, dt_1) and (dx_2, dy_2, dz_2, dt_2) should be orthogonal: and thus we have the following theorem: *At the point of intersection of any electropotential surface with any magnetopotential surface, every line drawn in the tangent-plane to the one surface is orthogonal to every line drawn in the tangent-plane to the other surface: or, in other words, at the point of intersection of any electropotential surface with any magnetopotential surface, the tangent-planes to the two surfaces are absolutely orthogonal, or the electropotential surfaces and the magnetopotential surfaces are two families of surfaces which are everywhere absolutely orthogonal.*

§ 7. INTRODUCTION OF THE TUBES OF ELECTROMAGNETIC FORCE.

Now let any continuous set of ∞^1 electropotential surfaces be chosen arbitrarily and grouped together. Their aggregate is a three-dimensional region or *hypersurface*, which we shall call G . Moreover, let any continuous set of ∞^1 magnetopotential surfaces be chosen arbitrarily and grouped together. Their aggregate is another hypersurface, which we shall call H . The hypersurfaces G and H will intersect in a surface; let us call this surface Σ . We shall now investigate its properties.

Let P be a point on Σ ; let σ be the electropotential surface through P , and let τ be the magnetopotential surface through P . Since Σ and σ are two surfaces both contained in the same hypersurface G , they intersect in a curve. (This is to be contrasted with the fact that two surfaces in four-dimensional hyperspace intersect in general only in solitary points.) The tangent-plane to Σ at P therefore intersects the tangent-plane to σ at P in a line. Similarly, the tangent-plane to Σ at P intersects the tangent-plane to τ at P in a line.

Now we have seen (§ 6) that any line in the tangent-plane to τ at P is orthogonal to every line drawn in the tangent-plane to σ at P . Therefore the tangent-plane to Σ at P contains one line which is orthogonal to every line drawn in the tangent-plane to σ at P . Hence, at P the tangent-plane to Σ is half-orthogonal to the tangent-plane to σ . Similarly, the tangent-plane to Σ is half-orthogonal to the tangent-plane to τ . Thus we have the theorem that *the surface Σ is, at every one of its points, half-orthogonal to the electropotential surface which passes through the point, and is also half-orthogonal to the magnetopotential surface which passes through the point.*

We may remark that the property here called "half-orthogonality" is really the same as what in ordinary three-dimensional geometry is simply called the "perpendicularity" of two planes: for two planes are said to be perpendicular to each other in ordinary three-dimensional geometry if one of them contains a line which is perpendicular to all the lines of the other.

When the field is purely electrostatic or purely magnetostatic, the "surfaces Σ ," which have been introduced, become the ordinary Faraday "tubes of force." For, taking the electrostatic case, the electropotential surfaces reduce, as we have seen (§ 2), to the ordinary equipotential surfaces. When we take as hypersurface G the aggregate of all the equipotential surfaces which correspond to a fixed value of t , these equipotential surfaces all lie in the same hyperplane (namely, the "instantaneous three-

dimensional space corresponding to this value of t "), and therefore the surface Σ lies wholly in this hyperplane: and the property of half-orthogonality now implies that at every point of intersection of Σ with an equipotential surface, their tangent-planes are perpendicular in the ordinary three-dimensional sense. That is to say, Σ is a tube of force as defined in ordinary electrostatic theory.

Similar reasoning applies to the case when the field is purely magnetostatic.

It appears therefore that the surfaces Σ are a covariant family of surfaces which, when the field is purely electrostatic or purely magnetostatic, reduce to the ordinary Faraday tubes of force. We shall therefore call them the *tubes of force of the electromagnetic field*. It is convenient to introduce a new term, partly for brevity and partly in order to distinguish them from the Faraday tubes, which are limiting cases of them. We shall call them *calamoids*.*

§ 8. THE PARALLEL PROPERTIES OF THE CALAMOIDS.

We have seen (§ 7) that the tangent-plane to a tube of force or calamoid Σ at P intersects the tangent-plane to the electropotential surface σ at P in a line, and therefore these tangent-planes have a point at infinity in common: that is, they are half-parallel. A similar argument applies to the magnetopotential surface τ . Thus we have the theorem that a *calamoid, at every one of its points, is half-parallel and half-orthogonal to the electropotential surface which passes through the point, and is also half-parallel and half-orthogonal to the magnetopotential surface which passes through the point.*

§ 9. THE CONDITIONS FOR HALF-PARALLELISM AND HALF-ORTHOGONALITY.

We shall now find the analytical conditions which must be satisfied when two planes in our space-time hyperspace are half-parallel or half-perpendicular.

First, let us consider half-parallelism.

Let one plane ϖ have the equations

$$\begin{cases} \xi_1 x + \xi_2 y + \xi_3 z + \xi_4 t + \xi_5 = 0 \\ \eta_1 x + \eta_2 y + \eta_3 z + \eta_4 t + \eta_5 = 0, \end{cases}$$

and let the other plane ϖ' have the equations

$$\begin{cases} \xi'_1 x + \xi'_2 y + \xi'_3 z + \xi'_4 t + \xi'_5 = 0 \\ \eta'_1 x + \eta'_2 y + \eta'_3 z + \eta'_4 t + \eta'_5 = 0. \end{cases}$$

* From *κάλamos*, a reed-pipe.

Introducing homogeneous co-ordinates by writing $x = x_1/x_5, y = x_2/x_5, z = x_3/x_5, t = x_4/x_5$, the equations of the plane ϖ become

$$\begin{cases} \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \xi_4 x_4 + \xi_5 x_5 = 0 \\ \eta_1 x_1 + \eta_2 x_2 + \eta_3 x_3 + \eta_4 x_4 + \eta_5 x_5 = 0, \end{cases}$$

and therefore the equations of the line in which the plane ϖ meets the hyperplane at infinity are

$$\begin{cases} \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \xi_4 x_4 = 0 \\ \eta_1 x_1 + \eta_2 x_2 + \eta_3 x_3 + \eta_4 x_4 = 0. \end{cases}$$

Similarly, the equations of the line in which the plane ϖ' meets the hyperplane at infinity are

$$\begin{cases} \xi'_1 x_1 + \xi'_2 x_2 + \xi'_3 x_3 + \xi'_4 x_4 = 0 \\ \eta'_1 x_1 + \eta'_2 x_2 + \eta'_3 x_3 + \eta'_4 x_4 = 0. \end{cases}$$

The condition for half-parallelism is that these lines should meet in a point, which evidently requires

$$\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \xi'_1 & \xi'_2 & \xi'_3 & \xi'_4 \\ \eta'_1 & \eta'_2 & \eta'_3 & \eta'_4 \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Now if a plane in hyperspace is defined as the intersection of two hyperplanes whose equations are

$$\begin{aligned} \xi_1 x + \xi_2 y + \xi_3 z + \xi_4 t + \xi_5 = 0 \\ \eta_1 x + \eta_2 y + \eta_3 z + \eta_4 t + \eta_5 = 0, \end{aligned}$$

then the six quantities

$$\begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix}, \begin{vmatrix} \xi_1 & \xi_3 \\ \eta_1 & \eta_3 \end{vmatrix}, \begin{vmatrix} \xi_1 & \xi_4 \\ \eta_1 & \eta_4 \end{vmatrix}, \begin{vmatrix} \xi_2 & \xi_3 \\ \eta_2 & \eta_3 \end{vmatrix}, \begin{vmatrix} \xi_2 & \xi_4 \\ \eta_2 & \eta_4 \end{vmatrix}, \begin{vmatrix} \xi_3 & \xi_4 \\ \eta_3 & \eta_4 \end{vmatrix}$$

(or any quantities proportional to them) are called the six *direction-ratios* of the plane. They are evidently independent of the choice of the two particular hyperplanes used to define the plane. Denoting the direction-ratios of the plane ϖ by $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, and the direction-ratios of the other plane ϖ' by $\alpha', \beta', \gamma', \delta', \epsilon', \zeta'$, and expanding the determinant in (9) by Laplace's formula in terms of minors selected from the first two and last two rows, we obtain

$$\alpha\zeta' + \alpha'\zeta + \gamma\delta' + \gamma'\delta - \epsilon\beta' - \epsilon'\beta = 0 \quad . \quad . \quad . \quad . \quad (10)$$

This is the condition that the two planes should be half-parallel.

Next, let us find the condition for half-perpendicularity.

Let two straight lines have the equations

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n} = \frac{t - t_0}{q}$$

and

$$\frac{x - x_0'}{l'} = \frac{y - y_0'}{m'} = \frac{z - z_0'}{n'} = \frac{t - t_0'}{q'}$$

The homogeneous co-ordinates of the points in which these lines meet the hyperplane at infinity are evidently $(l, m, n, q, 0)$ and $(l', m', n', q', 0)$: and if these points are conjugate with respect to the absolute whose equation is

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0,$$

we must have

$$ll' + mm' + nn' - qq' = 0,$$

which is therefore the condition that the two lines should be orthogonal.

Now let a plane ϖ through a point (x_0, y_0, z_0, t_0) have the equations

$$\begin{cases} \xi_1(x - x_0) + \xi_2(y - y_0) + \xi_3(z - z_0) + \xi_4(t - t_0) = 0 \\ \eta_1(x - x_0) + \eta_2(y - y_0) + \eta_3(z - z_0) + \eta_4(t - t_0) = 0. \end{cases}$$

If a line

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n} = \frac{t - t_0}{q}$$

lies in this plane, we must have

$$\left. \begin{aligned} \xi_1 l + \xi_2 m + \xi_3 n + \xi_4 q &= 0 \\ \eta_1 l + \eta_2 m + \eta_3 n + \eta_4 q &= 0 \end{aligned} \right\} \dots \dots \dots (11)$$

Let a second plane ϖ' through the same point (x_0, y_0, z_0, t_0) be given by the equations

$$\begin{cases} \xi_1'(x - x_0) + \xi_2'(y - y_0) + \xi_3'(z - z_0) + \xi_4'(t - t_0) = 0 \\ \eta_1'(x - x_0) + \eta_2'(y - y_0) + \eta_3'(z - z_0) + \eta_4'(t - t_0) = 0. \end{cases}$$

If a line

$$\frac{x - x_0}{l'} = \frac{y - y_0}{m'} = \frac{z - z_0}{n'} = \frac{t - t_0}{q'}$$

lies in this plane, we must have

$$\left. \begin{aligned} \xi_1' l' + \xi_2' m' + \xi_3' n' + \xi_4' q' &= 0 \\ \eta_1' l' + \eta_2' m' + \eta_3' n' + \eta_4' q' &= 0 \end{aligned} \right\} \dots \dots \dots (12)$$

If the planes ϖ and ϖ' are half-perpendicular, it must be possible to find values of l, m, n, q , satisfying equations (11), such that the equation

$$ll' + mm' + nn' - qq' = 0 \dots \dots \dots (13)$$

is true for all values of l', m', n', q' , which satisfy equations (12): that is, equation (13) must be a linear combination of equations (12), and therefore we must have

$$\left. \begin{aligned} l &= \lambda \xi_1' + \mu \eta_1' \\ m &= \lambda \xi_2' + \mu \eta_2' \\ n &= \lambda \xi_3' + \mu \eta_3' \\ q &= -\lambda \xi_4' - \mu \eta_4' \end{aligned} \right\} \dots \dots \dots (14)$$

where λ and μ are as yet undetermined.

Substituting from (14) in (11), we have

$$\begin{aligned} \lambda(\xi_1\xi_1' + \xi_2\xi_2' + \xi_3\xi_3' - \xi_4\xi_4') + \mu(\xi_1\eta_1' + \xi_2\eta_2' + \xi_3\eta_3' - \xi_4\eta_4') &= 0 \\ \lambda(\eta_1\xi_1' + \eta_2\xi_2' + \eta_3\xi_3' - \eta_4\xi_4') + \mu(\eta_1\eta_1' + \eta_2\eta_2' + \eta_3\eta_3' - \eta_4\eta_4') &= 0. \end{aligned}$$

Eliminating the ratio $\lambda : \mu$ between these equations, we have

$$\begin{vmatrix} \xi_1\xi_1' + \xi_2\xi_2' + \xi_3\xi_3' - \xi_4\xi_4' & \xi_1\eta_1' + \xi_2\eta_2' + \xi_3\eta_3' - \xi_4\eta_4' \\ \eta_1\xi_1' + \eta_2\xi_2' + \eta_3\xi_3' - \eta_4\xi_4' & \eta_1\eta_1' + \eta_2\eta_2' + \eta_3\eta_3' - \eta_4\eta_4' \end{vmatrix} = 0;$$

that is to say, the product of the arrays

$$\begin{matrix} \xi_1 & \xi_2 & \xi_3 & i\xi_4 \\ \eta_1 & \eta_2 & \eta_3 & i\eta_4 \end{matrix} \quad \text{and} \quad \begin{matrix} \xi_1' & \xi_2' & \xi_3' & i\xi_4' \\ \eta_1' & \eta_2' & \eta_3' & i\eta_4' \end{matrix}$$

vanishes: but this product may be written

$$\begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix} \begin{vmatrix} \xi_1' & \xi_2' \\ \eta_1' & \eta_2' \end{vmatrix} + \begin{vmatrix} \xi_1 & \xi_3 \\ \eta_1 & \eta_3 \end{vmatrix} \begin{vmatrix} \xi_1' & \xi_3' \\ \eta_1' & \eta_3' \end{vmatrix} + \dots + \dots - \begin{vmatrix} \xi_3 & \xi_4 \\ \eta_3 & \eta_4 \end{vmatrix} \begin{vmatrix} \xi_3' & \xi_4' \\ \eta_3' & \eta_4' \end{vmatrix} = 0$$

or

$$aa' + \beta\beta' - \gamma\gamma' + \delta\delta' - \epsilon\epsilon' - \zeta\zeta' = 0 \quad . \quad . \quad . \quad (15)$$

This is therefore the condition that two planes whose direction-ratios are $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$ and $(\alpha', \beta', \gamma', \delta', \epsilon', \zeta')$ should be half-perpendicular in the hyperspace whose absolute has the equations

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0, \quad x_5 = 0.$$

§ 10. THE PARTIAL DIFFERENTIAL EQUATIONS OF THE CALAMOIDS.

We shall now proceed to find the partial differential equations whose solutions are represented by the tubes of force or calamoids. We shall derive them from the properties proved in §§ 7, 8, namely, that the calamoids are half-parallel and half-perpendicular to the electropotential surfaces.

If the equations of a surface are given in the form

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad t = t(u, v),$$

it is easily seen that the direction-ratios of the tangent-plane to this surface at the point (u, v) are given by the proportion

$$\begin{aligned} \alpha : \beta : \gamma : \delta : \epsilon : \zeta \\ = \frac{\partial(z, t)}{\partial(u, v)} : -\frac{\partial(y, t)}{\partial(u, v)} : \frac{\partial(y, z)}{\partial(u, v)} : \frac{\partial(x, t)}{\partial(u, v)} : -\frac{\partial(x, z)}{\partial(u, v)} : \frac{\partial(x, y)}{\partial(u, v)}. \end{aligned}$$

Hence for an electropotential surface, by equation (7), we have

$$\begin{aligned} \alpha : \beta : \gamma : \delta : \epsilon : \zeta \\ = h_z : -h_y : d_x : h_x : d_y : d_z. \end{aligned}$$

Therefore, if $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$ now denote the direction-ratios of the tangent

plane to a calamoid, the conditions that the calamoid is half-parallel and half-perpendicular to the electropotential surface become (by (10) and (15))

$$\begin{aligned} d_z\alpha + h_z\zeta + h_x\gamma + d_x\delta + h_y\epsilon - d_y\beta &= 0 \\ h_z\alpha - h_y\beta - d_x\gamma + h_x\delta - d_y\epsilon - d_z\zeta &= 0, \end{aligned}$$

so the calamoid must satisfy the two equations

$$\left. \begin{aligned} h_z \frac{\partial(x, y)}{\partial(u, v)} - h_y \frac{\partial(x, z)}{\partial(u, v)} + d_x \frac{\partial(x, t)}{\partial(u, v)} + h_x \frac{\partial(y, z)}{\partial(u, v)} + d_y \frac{\partial(y, t)}{\partial(u, v)} + d_z \frac{\partial(z, t)}{\partial(u, v)} &= 0 \\ -d_z \frac{\partial(x, y)}{\partial(u, v)} + d_y \frac{\partial(x, z)}{\partial(u, v)} + h_x \frac{\partial(x, t)}{\partial(u, v)} - d_x \frac{\partial(y, z)}{\partial(u, v)} + h_y \frac{\partial(y, t)}{\partial(u, v)} + h_z \frac{\partial(z, t)}{\partial(u, v)} &= 0 \end{aligned} \right\} \quad (16)$$

If in particular we take u, v to be two of the co-ordinates, say x and y , we obtain

$$\left. \begin{aligned} h_z - h_y \frac{\partial z}{\partial y} + d_x \frac{\partial t}{\partial y} - h_x \frac{\partial z}{\partial x} - d_y \frac{\partial t}{\partial x} + d_z \left(\frac{\partial z}{\partial x} \frac{\partial t}{\partial y} - \frac{\partial z}{\partial y} \frac{\partial t}{\partial x} \right) &= 0 \\ -d_z + d_y \frac{\partial z}{\partial y} + h_x \frac{\partial t}{\partial y} + d_x \frac{\partial z}{\partial x} - h_y \frac{\partial t}{\partial x} + h_z \left(\frac{\partial z}{\partial x} \frac{\partial t}{\partial y} - \frac{\partial z}{\partial y} \frac{\partial t}{\partial x} \right) &= 0 \end{aligned} \right\} \quad (17)$$

These two equations are the partial differential equations whose solutions are represented by the calamoids.

§ 11. THE INTEGRAL-EQUIVALENT OF MAXWELL'S EQUATIONS.

In 1898 M. de Franchis* showed that the theorems regarding the equivalence of surface-integrals and volume-integrals in three-dimensional space, which are known as "Green's theorem" and "Stokes' theorem," can be extended to space of any number of dimensions. If $(h_x, h_y, h_z, d_x, d_y, d_z)$ are any six functions of position in our four-dimensional hyper-space, one of de Franchis' formulæ may be written

$$\begin{aligned} & \iint_S \left\{ h_z \frac{\partial(x, y)}{\partial(u, v)} - h_y \frac{\partial(x, z)}{\partial(u, v)} + d_x \frac{\partial(x, t)}{\partial(u, v)} + h_x \frac{\partial(y, z)}{\partial(u, v)} + d_y \frac{\partial(y, t)}{\partial(u, v)} + d_z \frac{\partial(z, t)}{\partial(u, v)} \right\} du dv \\ &= \iiint_V \left\{ \left(\frac{\partial d_z}{\partial y} - \frac{\partial d_y}{\partial z} + \frac{\partial h_x}{\partial t} \right) \frac{\partial(y, z, t)}{\partial(p, q, r)} + \left(\frac{\partial d_z}{\partial x} - \frac{\partial d_x}{\partial z} - \frac{\partial h_y}{\partial t} \right) \frac{\partial(x, z, t)}{\partial(p, q, r)} \right. \\ & \quad \left. + \left(\frac{\partial d_y}{\partial x} - \frac{\partial d_x}{\partial y} + \frac{\partial h_z}{\partial t} \right) \frac{\partial(x, y, t)}{\partial(p, q, r)} + \left(\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z} \right) \frac{\partial(x, y, z)}{\partial(p, q, r)} \right\} dp dq dr. \end{aligned}$$

Here the integral on the left is taken over any closed surface S, and the integral on the right is taken over any open hypersurface V which is bounded by S. For convenience, position on S is specified by two variables (u, v) , and position on the hypersurface V is specified by three variables (p, q, r) ; but it is evident that the theorem is really independent of the choice of these variables.

* *Palermo Rend.*, xii (1898) p. 163.

Now take $(h_x, h_y, h_z, d_x, d_y, d_z)$ to be the components of the magnetic and electric force at a point in free space: then by Maxwell's equations (2) the triple integral on the right-hand side of de Franchis' equation vanishes; that is to say, *if, in our four-dimensional hyperspace, S be any closed surface to which we can fit an open hypersurface not containing electrons, then the integral*

$$\iiint \left\{ h_x \frac{\partial(x, y)}{\partial(u, v)} - h_y \frac{\partial(x, z)}{\partial(u, v)} + d_x \frac{\partial(x, t)}{\partial(u, v)} + h_x \frac{\partial(y, z)}{\partial(u, v)} + d_y \frac{\partial(y, t)}{\partial(u, v)} + d_z \frac{\partial(z, t)}{\partial(u, v)} \right\} du dv \quad (18)$$

*vanishes when the integration is extended over the surface S .**

Similarly, by use of the other four of Maxwell's equations (namely, equations (1)) we can show that *the integral*

$$\iiint \left\{ -d_z \frac{\partial(x, y)}{\partial(u, v)} + d_y \frac{\partial(x, z)}{\partial(u, v)} + h_x \frac{\partial(x, t)}{\partial(u, v)} - d_x \frac{\partial(y, z)}{\partial(u, v)} + h_y \frac{\partial(y, t)}{\partial(u, v)} + h_z \frac{\partial(z, t)}{\partial(u, v)} \right\} du dv \quad (19)$$

vanishes when the integration is extended over the surface S .

These two formulæ (18) and (19) constitute an integrated equivalent of Maxwell's equations in free space.

§ 12. THE INTEGRAL PROPERTIES OF TUBES OF FORCE.

We shall now apply the formulæ of § 11 to the case when the closed surface S is formed of a portion of a thin tube of force or calamoid, terminated at one end by a portion σ of an equipotential surface, and terminated at the other end by a portion τ of another equipotential surface.

By the equations (16) the integrals vanish over that portion of the surface S which is formed of the calamoid: and therefore we have the result that *the integrals*

$$\iiint \left\{ h_x \frac{\partial(x, y)}{\partial(u, v)} - h_y \frac{\partial(x, z)}{\partial(u, v)} + d_x \frac{\partial(x, t)}{\partial(u, v)} + h_x \frac{\partial(y, z)}{\partial(u, v)} + d_y \frac{\partial(y, t)}{\partial(u, v)} + d_z \frac{\partial(z, t)}{\partial(u, v)} \right\} du dv$$

and

$$\iiint \left\{ -d_z \frac{\partial(x, y)}{\partial(u, v)} + d_y \frac{\partial(x, z)}{\partial(u, v)} + h_x \frac{\partial(x, t)}{\partial(u, v)} - d_x \frac{\partial(y, z)}{\partial(u, v)} + h_y \frac{\partial(y, t)}{\partial(u, v)} + h_z \frac{\partial(z, t)}{\partial(u, v)} \right\} du dv,$$

vanish when the integration is extended over the two small regions σ and τ together.

Now, since σ is part of an equipotential surface, we have by equation (7)

$$\frac{\partial(x, y)}{\partial(u, v)} = - \frac{\partial(x, z)}{d_y} = \frac{\partial(x, t)}{h_x} = \frac{\partial(y, z)}{d_x} = \frac{\partial(y, t)}{h_y} = \frac{\partial(z, t)}{h_z},$$

* This result is not new, and is inserted here only because it is necessary for what follows in § 12.

and each of these fractions is equal to

$$\frac{\left[\left\{ \frac{\partial(x, y)}{\partial(u, v)} \right\}^2 + \left\{ \frac{\partial(x, z)}{\partial(u, v)} \right\}^2 + \left\{ \frac{\partial(y, z)}{\partial(u, v)} \right\}^2 - \left\{ \frac{\partial(x, t)}{\partial(u, v)} \right\}^2 - \left\{ \frac{\partial(y, t)}{\partial(u, v)} \right\}^2 - \left\{ \frac{\partial(z, t)}{\partial(u, v)} \right\}^2 \right]^{\frac{1}{2}}}{(d_x^2 + d_y^2 + d_z^2 - h_x^2 - h_y^2 - h_z^2)^{\frac{1}{2}}}$$

But the area of σ is

$$\left[\left\{ \frac{\partial(x, y)}{\partial(u, v)} \right\}^2 + \left\{ \frac{\partial(x, z)}{\partial(u, v)} \right\}^2 + \left\{ \frac{\partial(y, z)}{\partial(u, v)} \right\}^2 - \left\{ \frac{\partial(x, t)}{\partial(u, v)} \right\}^2 - \left\{ \frac{\partial(y, t)}{\partial(u, v)} \right\}^2 - \left\{ \frac{\partial(z, t)}{\partial(u, v)} \right\}^2 \right]^{\frac{1}{2}} du dv,$$

the symbol of integration being omitted, as σ is a very small area.

Thus we have

$$\frac{\partial(x, y)}{\partial(u, v)} du dv = \frac{d_z}{(d_x^2 + d_y^2 + d_z^2 - h_x^2 - h_y^2 - h_z^2)^{\frac{1}{2}}} \times \text{area of } \sigma,$$

and therefore the first integral-formula becomes nugatory, while the second integral-formula transforms into the theorem that the area of σ multiplied by the value of $(d_x^2 + d_y^2 + d_z^2 - h_x^2 - h_y^2 - h_z^2)^{\frac{1}{2}}$ at σ , is equal to the area of τ , multiplied by the value of $(d_x^2 + d_y^2 + d_z^2 - h_x^2 - h_y^2 - h_z^2)^{\frac{1}{2}}$ at τ . In other words, *the cross-section of a thin calamoid* (measured by the area which it cuts off on the electropotential surfaces which intersect it in curves), *multiplied by the value of $(d_x^2 + d_y^2 + d_z^2 - h_x^2 - h_y^2 - h_z^2)^{\frac{1}{2}}$, is constant along the whole length of the calamoid.*

It will be seen at once that this theorem is the generalisation of the well-known property of Faraday tubes of force in electrostatics, namely, that "the cross-section of a Faraday tube, multiplied by the value of the electric force, is constant along the whole length of the tube." The quantity $(d_x^2 + d_y^2 + d_z^2 - h_x^2 - h_y^2 - h_z^2)^{\frac{1}{2}}$, which occurs in the corresponding property of calamoids, is independent of the velocity of the observer by whom the electric and magnetic forces are measured, and is therefore suited for the expression of an invariant property.

§ 13. THE CHARACTERISTICS.

The Faraday tubes of force in electrostatics and magnetism are constructed in the following way. We first take the system of curves which intersect the equipotential surfaces orthogonally; these are called the *lines* of force: then we take any simple closed curve in the field, and consider the lines of force which intersect this curve: they form a tubular surface, which is called a *tube* of force. A tube of force, which is a *surface*, is therefore formed by the aggregation of lines of force, which are *curves*.

We have now to inquire if there is anything analogous to this in the theory of calamoids.

calamoid along a curve (as contrasted with a solitary point): the curve is then a *characteristic curve*. Let the neighbouring calamoid be specified by $(p_1 + dp_1, q_1 + dq_1, p_2 + dp_2, q_2 + dq_2)$, and suppose that the line-element $(\delta x, \delta y, \delta z, \delta t)$ is in the intersection, so that from (22) we must have

$$\left. \begin{aligned} -dp_1 \cdot \delta x - dq_1 \cdot \delta y &= 0 \\ -dp_2 \cdot \delta x - dq_2 \cdot \delta y &= 0 \end{aligned} \right\} \dots \dots \dots (23)$$

Now by differentiating (20) and (21) we have

$$\left. \begin{aligned} -h_y dq_1 + d_x dq_2 - h_x dp_1 - d_y dp_2 + d_z dr &= 0 \\ d_y dq_1 + h_x dq_2 + d_x dp_1 - h_y dp_2 + h_z dr &= 0 \\ -p_2 dq_1 + p_1 dq_2 + q_2 dp_1 - q_1 dp_2 - dr &= 0 \end{aligned} \right\} \dots \dots \dots (24)$$

Eliminating the ratios of $dp_1, dq_1, dp_2, dq_2, dr$ from equations (23) and (24), we have

$$\begin{vmatrix} \delta x & \delta y & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta x & \delta y \\ -h_x & -h_y & d_z & -d_y & d_x \\ d_x & d_y & h_z & -h_y & h_x \\ q_2 & -p_2 & -1 & -q_1 & p_1 \end{vmatrix} = 0.$$

Performing the operations col. '4 = col. 4 - q_1 col. 3, col. '5 = col. 5 + p_1 col. 3, this gives

$$\begin{vmatrix} \delta x & \delta y & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta x & \delta y \\ -h_x & -h_y & d_z & -d_y - q_1 d_z & d_x + p_1 d_z \\ d_x & d_y & h_z & -h_y - q_1 h_z & h_x + p_1 h_z \\ q_2 & -p_2 & -1 & 0 & 0 \end{vmatrix} = 0.$$

Expanding by Laplace's formula in terms of minors taken from the first three columns and the two last columns, we have

$$\begin{vmatrix} \delta x & \delta y & 0 \\ -h_x & -h_y & d_z \\ q_2 & p_2 & -1 \end{vmatrix} \begin{vmatrix} \delta x & \delta y \\ -h_y - q_1 h_z & h_x - p_1 h_z \end{vmatrix} - \begin{vmatrix} \delta x & \delta y & 0 \\ d_x & d_y & h_z \\ q_2 & p_2 & -1 \end{vmatrix} \begin{vmatrix} \delta x & \delta y \\ -d_y - q_1 d_z & d_x + p_1 d_z \end{vmatrix} = 0 \quad (25)$$

But if in the second of equations (21) we replace $-d_z$ by its value $(h_x d_x + h_y d_y)/h_z$, the equation becomes

$$h_x d_x + h_y d_y + h_z d_y q_1 + h_x h_z q_2 + d_x h_z p_1 - h_y h_z p_2 + h_z^2 p_1 q_2 - h_z^2 q_1 p_2 = 0,$$

or

$$(h_x + p_1 h_z)(d_x + q_2 h_z) + (h_y + q_1 h_z)(d_y - p_2 h_z) = 0 ;$$

so we can write

$$\frac{-d_y + p_2 h_z}{h_x + p_1 h_z} = \frac{d_x + q_2 h_z}{h_y + q_1 h_z} = \text{M say} \dots \dots \dots (26)$$

and similarly

$$\frac{h_y + p_2 d_z}{d_x + p_1 d_z} = \frac{-h_x + q_2 d_z}{d_y + q_1 d_z} = \text{N say} \dots \dots \dots (27)$$

Substituting from (26) and (27) in (25), we have

$$(N - M)\{(h_x + p_1 h_z)\delta x + (h_y + q_1 h_z)\delta y\} \{(d_x + p_1 d_z)\delta x + (d_y + q_1 d_z)\delta y\} = 0$$

and therefore the characteristics satisfy *either* the equation

$$(h_x + p_1 h_z)\delta x + (h_y + q_1 h_z)\delta y = 0 \quad . \quad . \quad . \quad . \quad (28)$$

or *else* the equation

$$(d_x + p_1 d_z)\delta x + (d_y + q_1 d_z)\delta y = 0 \quad . \quad . \quad . \quad . \quad (29)$$

Substituting in these from the first of equations (22), we see that the characteristics satisfy *either* the equation

$$h_x \delta x + h_y \delta y + h_z \delta z = 0 \quad . \quad . \quad . \quad . \quad (30)$$

or *else* the equation

$$d_x \delta x + d_y \delta y + d_z \delta z = 0 \quad . \quad . \quad . \quad . \quad (31)$$

Moreover, from (26) the equation (28) may be written

$$(-d_y + p_2 h_z)\delta x + (d_x + q_2 h_z)\delta y = 0,$$

and substituting in this from the second equation of (22), it becomes

$$-d_y \delta x + d_x \delta y + h_z \delta t = 0 \quad . \quad . \quad . \quad . \quad (32)$$

Similarly, equation (29) leads to the equation

$$h_y \delta x - h_x \delta y + d_z \delta t \quad . \quad . \quad . \quad . \quad (33)$$

Thus the characteristics satisfy *either* the pair of equations (30) and (32), or *else* the pair of equations (31) and (33). But the equations (30) and (32) are the equations which define the magnetopotential surfaces, and the equations (31) and (33) are the equations which define the electropotential surfaces. Thus finally we have the result that *the characteristics, out of which the calamoids are constituted, are all the curves that can be drawn arbitrarily on the electropotential surfaces, together with all the curves that can be drawn arbitrarily on the magnetopotential surfaces.* This result might indeed have been foreseen from the mode in which the calamoids were formed originally: for any curve on an electropotential surface S may be regarded as the intersection of S with a continuous set H of ∞^1 magnetopotential surfaces: and by § 7 this intersection forms part of every one of the calamoids which can be obtained as the intersection of H with a continuous set of ∞^1 electropotential surfaces of which S is one: the curve is therefore an intersection of calamoids, that is, a characteristic.

§ 14. EXTENSION TO GENERAL FIELDS.

Throughout the whole of the preceding discussion we have been subject to one restrictive condition, namely, that the field is one in which the electric and magnetic vectors are everywhere perpendicular to each other. We have now to show what modifications are necessary when this condition is no longer supposed to hold.

It will be remembered that the equipotential surfaces were defined by the covariant set of total differential equations (4), and the magnetopotential surfaces were defined by the covariant set of total differential equations (8): and these equations do not define families of surfaces at all unless the determinants of the coefficients, namely,

$$\begin{vmatrix} 0 & h_z & -h_y & d_x \\ -h_z & 0 & h_x & d_y \\ h_y & -h_x & 0 & d_z \\ -d_x & -d_y & -d_z & 0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 0 & d_z & -d_y & -h_x \\ -d_z & 0 & d_x & -h_y \\ d_y & -d_x & 0 & -h_z \\ h_x & h_y & h_z & 0 \end{vmatrix}$$

are zero: this condition was satisfied by virtue of the relation $d_x h_x + d_y h_y + d_z h_z = 0$, which we assumed to hold, but it is no longer satisfied when the electric and magnetic vectors are not everywhere at right angles. We can, however, form a linear combination* of these two invariant sets of total differential equations, namely,

$$\begin{cases} (\lambda h_z + \mu d_z)dy - (\lambda h_y + \mu d_y)dz + (\lambda d_x - \mu h_x)dt = 0 \\ -(\lambda h_z + \mu d_z)dx + (\lambda h_x + \mu d_x)dz + (\lambda d_y - \mu h_y)dt = 0 \\ (\lambda h_y + \mu d_y)dx - (\lambda h_x + \mu d_x)dy + (\lambda d_z - \mu h_z)dt = 0 \\ -(\lambda d_x - \mu h_x)dx - (\lambda d_y - \mu h_y)dy - (\lambda d_z - \mu h_z)dz = 0 \end{cases}$$

and this set will be equivalent to only two equations provided the ratio $\lambda : \mu$ is such that the determinant of the coefficients of the differentials is zero: that is, provided

$$\begin{vmatrix} 0 & \lambda h_z + \mu d_z & -\lambda h_y - \mu d_y & \lambda d_x - \mu h_x \\ -\lambda h_z - \mu d_z & 0 & \lambda h_x + \mu d_x & \lambda d_y - \mu h_y \\ \lambda h_y + \mu d_y & -\lambda h_x - \mu d_x & 0 & \lambda d_z - \mu h_z \\ -\lambda d_x + \mu h_x & -\lambda d_y + \mu h_y & -\lambda d_z + \mu h_z & 0 \end{vmatrix} = 0$$

or

$$(d_x h_x + d_y h_y + d_z h_z)\lambda^2 + (d_x^2 + d_y^2 + d_z^2 - h_x^2 - h_y^2 - h_z^2)\lambda\mu - (d_x h_x + d_y h_y + d_z h_z)\mu^2 = 0.$$

This is a quadratic equation in the ratio $\lambda : \mu$, and therefore defines in general two ratios $\lambda_1 : \mu_1$ and $\lambda_2 : \mu_2$. We now define one family of geometrical forms in our four-dimensional hyperspace by the set of total differential equations

$$\begin{cases} (\lambda_1 h_z + \mu_1 d_z)dy - (\lambda_1 h_y + \mu_1 d_y)dz + (\lambda_1 d_x - \mu_1 h_x)dt = 0 \\ -(\lambda_1 h_z + \mu_1 d_z)dx + (\lambda_1 h_x + \mu_1 d_x)dz + (\lambda_1 d_y - \mu_1 h_y)dt = 0 \\ (\lambda_1 h_y + \mu_1 d_y)dx - (\lambda_1 h_x + \mu_1 d_x)dy + (\lambda_1 d_z - \mu_1 h_z)dt = 0 \\ -(\lambda_1 d_x - \mu_1 h_x)dx - (\lambda_1 d_y - \mu_1 h_y)dy - (\lambda_1 d_z - \mu_1 h_z)dz = 0 \end{cases}$$

* This device was suggested to me by the recollection of the method published in 1876 by Hamburger for integrating systems of simultaneous partial differential equations.

and another family of geometrical forms by the same equations with λ_2 and μ_2 substituted for λ_1 and μ_1 .

When the two conditions of unconditional integrability are satisfied, these geometrical forms are families of surfaces intersecting each other everywhere at right angles, and resembling the electropotential and magnetopotential surfaces previously introduced; when the conditions of unconditional integrability are not satisfied, the geometrical forms are curves in the four-dimensional hyperspace. In the former case, the tubes of force or calamoids can be introduced in precisely the same way as in § 7; in the latter case, the problem is somewhat more complicated from the geometrical point of view, and will be deferred to a subsequent paper. In all cases, the partial differential equations satisfied by the calamoids are the same as those found in the special case, namely, equations (17): and the integral properties found in § 12 are valid in the general case just as in the special case.

(Issued separately December 27, 1921.)

II.—On the Phenomenon of the “Radiant Spectrum” observed by Sir David Brewster. By C. V. Raman, M.A., Palit Professor of Physics in the Calcutta University. *Communicated by Dr C. G. KNOTT, F.R.S., General Secretary.*

(MS. received July 27, 1921. Read November 7, 1921.)

IN a paper on “The Scattering of Light in the Refractive Media of the Eye” published in the *Philosophical Magazine* for November 1919 (p. 568), I discussed the explanation of the luminous effects observed when a small brilliant source of light is viewed directly by the eye against a dark background, and especially of the marked difference between the cases in which the source emits white light and highly monochromatic light respectively. In both cases the source appears to be surrounded by a diffraction-halo: but the structure of the halo is markedly different in appearance. In the former case, the source appears to shoot out streamers of light radiating from it in all directions, these streamers showing marked colour, and in fact appearing as elongated spectra in the outer parts of the halo. With the monochromatic light-source, on the other hand, the radiant structure of the halo is not observed, and we have instead surrounding the light-source a halo showing dark and bright rings and exhibiting a finely mottled or granular appearance. It was pointed out in the paper that these effects are precisely what might be expected on the hypothesis that the halo seen surrounding the source is due to the diffraction of light by a large number of particles of constant size—presumably the corneal corpuscles—present in the refractive media of the eye. The radiant structure of the halo in white light and its granular structure in highly monochromatic light is, on this view, due to the field of light diffracted by individual particles varying arbitrarily in intensity from point to point as the result of the mutual interference of the effects of the large number of such particles. A closely analogous structure of the luminous field may be observed in diffraction-haloes obtained in other ways, *e.g.* with the aid of a glass plate dusted with lycopodium powder through which a small distant source of light is viewed.

The facts mentioned above provide a very simple explanation of a remarkable observation made long ago by Sir David Brewster, and communicated to the Royal Society of Edinburgh (*Proceedings*, vi, p. 147; * see

* See also a brief note by Tait, *Proc. R.S.E.*, vi, p. 167.

also *Phil. Mag.*, September 1867), which has not up to now been satisfactorily accounted for, and to which my attention has been recently drawn by Dr C. G. Knott while I was on a visit to Edinburgh. Brewster noticed that when a spectrum of a small brilliant source of white light is formed, either by a prism or by diffraction, and viewed directly by the eye, a patch of light is seen lying in the continuation of the spectrum well beyond its violet end and exhibiting streamers radiating from its centre. That this is a diffraction-effect is shown by the fact that a similar and even more striking effect may be observed in the diffraction-halo due to a glass plate dusted with lycopodium held together with a 60° glass prism before the eye, when a small distant source of white light is viewed through the combination. The prism disperses the image of the source into a spectrum. It also disperses the diffraction-halo, and since the diffraction-rings are of different size for the different wave-lengths and are shifted to different extents owing to the dispersive power of the prism, the achromatic centre of the halo is shifted laterally to a considerable extent, its new position generally lying at a point much removed beyond the violet end of the spectrum of the source itself. The elongated spectra which form the radiating streamers are rotated through various angles by the dispersion of the prism, being drawn out laterally on one side and shut up or drawn together on the other side, and they then appear to diverge from the shifted position of the achromatic centre of the halo, which, as remarked above, now lies well beyond the violet end of the spectrum of the source. The analogy between this effect and Brewster's phenomenon is so striking that there can be no doubt that the latter is essentially of the same nature, the diffraction in this case being due to the structures within the eye itself.

(Issued separately December 27, 1921.)

with dry nitrogen, as being the least adsorbible of the chief ingredients of air; if applied to air the results therefore have a "margin of safety."

The apparatus consisted of a glass bulb, holding about 20 grams of charcoal, and provided with a bifurcating neck; one branch of the neck was connected to a mercury manometer and to a large burette holding the gas over strong sulphuric acid. All gas-volume measurements were made at atmospheric pressure. Before being placed in the bulb the charcoal or silica was dried by being heated, in vacuo, as strongly as combustion tubing

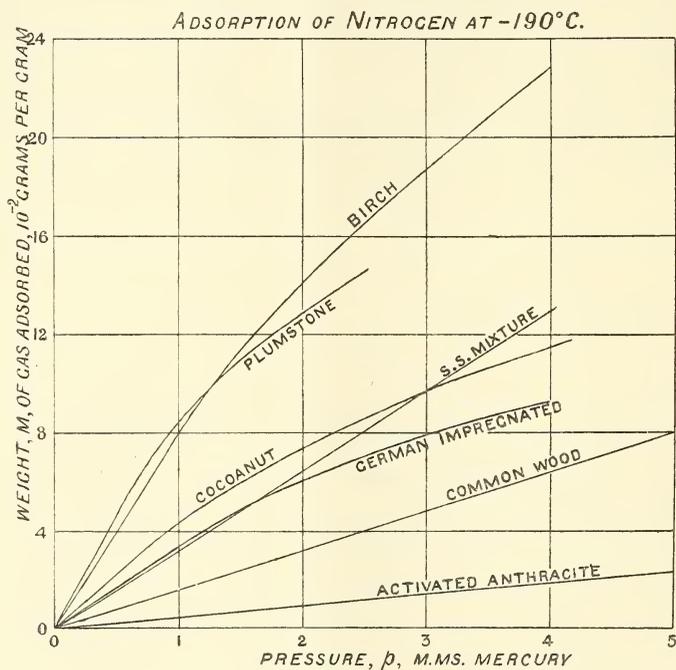


FIG. 2.

would stand; when in the bulb it was subjected to further heating accompanied by intense exhaustion effected by means of coconut charcoal contained in a large bulb, dipped in liquid air, and connected to the second branch of the neck. The evacuating bulb was then shut off, the test-bulb plunged in liquid air, and the manometer reading taken by means of the micrometer of a cathetometer, using a barometer column (standing alongside the manometer) as index. A measured volume of gas was admitted from the burette, and an interval allowed to elapse—one of several hours was sometimes necessary—until the pressure reached stability; the pressure was then taken. More gas was admitted, and the operations repeated until a sufficient number of results had been obtained. These results enabled

curves to be drawn (figs. 2 and 3) in which the abscissæ are pressures and the ordinates the weights of gas (centigrams) adsorbed per gram of the substance.

The materials named on fig. 2 are mostly anti-gas charcoals used in the war. The plumstone and cocoanut charcoals were activated by steaming, and the birch by long-continued heating in the presence of a low proportion of oxygen. The German charcoal is a pine charcoal impregnated with zinc and iron; the common wood charcoal (unactivated) was probably also made from pine. The activated anthracite is an American product. "S.S. mixture" is a briquetted mixture of coal and charcoal dusts prepared by Messrs Sutcliffe, Speakman & Co., Leigh. Silica (A) of fig. 3 was made from the hydrogel in the manner described elsewhere; * it represents the

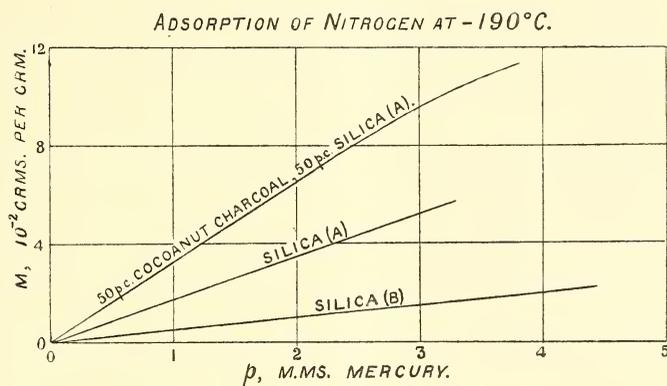


FIG. 3.

highest activation so far achieved with silica. Silica (B) is a colloidal silica prepared on a commercial scale at Baltimore, U.S.A., and dried, as the makers recommend, at about 120°C . The third curve of fig. 3 was obtained with a half-and-half mixture of powdered cocoanut charcoal and silica, the charcoal being mixed in when the silica was in the sol condition.

The slope at the origin of any of the curves of figs. 2 and 3 is the prehensility of the substance concerned under the conditions of the test. The figures of the following table (p. 30) were ascertained from the curves; two values of prehensility on dry hydrogen are also included.

A problem met with in evacuation plant and in Dewar flasks is that of determining the degree of tenuity of the vacuum produced by means of a mass of charcoal cooled (usually) to liquid air temperature, the pressure having been reduced to a known value prior to the charcoal being put

* *Proc. Roy. Soc., A*, 100, 1921, p. 88.

TABLE I.—PREHENSILITY AT -190° C.

Substance.	Nitrogen.	Hydrogen.
Plumstone charcoal	$10^{-2} \times 9.0$	$10^{-3} \times 1.2$
Birch charcoal	$\times 8.6$	
Cocoonut charcoal	$\times 4.5$	$10^{-3} \times 0.6$
German impregnated charcoal	$\times 3.6$	
50 per cent. silica (A) ; 50 per cent. cocoonut charcoal	$\times 3.2$	
S.S. mixture	$\times 3.1$	
Silica (A)	$\times 1.6$	
Common wood charcoal	$\times 1.6$	
Silica (B)	$\times 0.7$	
Activated anthracite	$\times 0.7$	

into use; or again, we may wish to ascertain how far the preliminary evacuation ought to go to enable the charcoal to bring the pressure down to a desired point. When the prehensility is known such problems admit of easy solution.

The vacuum-producing power of a known weight of adsorbent of a known prehensility may be calculated thus:—

L is the volume, in litres, of the space to be exhausted.

w is the weight, in grams per litre at N.T.P., of the gas contained in the space.

A is the weight, in grams, of the adsorbent placed in connection with the space.

σ is the prehensility of the adsorbent at the temperature to which it is cooled.

The space is first roughly pumped out to a pressure P mms., and the final pressure, after the adsorbent has been used, is p mms. mercury. Also let θ_1 be the initial absolute temperature of the space (usually room temperature), and θ_2 its mean absolute temperature when the adsorbent is in use.

When the pressure is P , the weight of gas in the space is $\frac{LPw}{760} \times \frac{273}{\theta_1}$; when the pressure is p , it is $\frac{Lpw}{760} \times \frac{273}{\theta_2}$. The weight of gas adsorbed per gram of substance is therefore

$$0.359 \frac{Lw}{A} \left(\frac{P}{\theta_1} - \frac{p}{\theta_2} \right).$$

As p is very small,

$$\sigma = \frac{\text{weight gas adsorbed per gram}}{p}$$

TABLE II.—PRESSURE-REDUCTION FACTORS (f) FOR THE 50-LBS. CONTAINER, THE GAS IN THE ENVELOPE BEING AIR.

Absorbent Substance.	Relative Apparent Density (Cocoanut Charcoal=100).	Pressure-Reduction Factor at -190° C.
Plumstone charcoal	79	1413
Cocoanut charcoal	100	894
50% cocoanut charcoal; 50% silica (A)	118	750
Birch charcoal	33	564
German impregnated charcoal	73	517
S.S. Mixture	62	382
Silica (A)	105	334
Silica (B)	191	266
Activated anthracite	176	244
Common wood charcoal	46	147

atmospheric pressure. The table shows that a silica can now be made of appreciable evacuating power, though at liquid air temperature over four times as much of silica (A) is required (by volume) as of plumstone charcoal to obtain the same reduction of pressure. Silica, moreover, does not act so rapidly as charcoal in adsorbing gas at low concentration.

Extremely high vacua may be obtained by using a number of charcoal bulbs in succession, the bulbs being severally sealed to the evacuating system and fused off, one at a time, after use. Such a method of exhausting in stages is, for example, useful in evacuating thermionic valves. If we have a mass of dry charcoal which, in accordance with (8), yields a pressure reduction factor of f , we shall get a very different factor if we divide the mass into n equal parts and employ them, in the manner just stated, in separate bulbs. The factor will then be f_2 where

$$f_2 = \left(\frac{f_1}{n}\right)^n \dots \dots \dots (9)$$

Two interesting corollaries follow: first, no advantage can result from dividing the adsorbent and using it in stages if f_1 is equal to or less than 4; secondly, f_2 attains a maximum when $n = \frac{f_1}{e}$. For example, if $f_1 = 8.2$, the best results with a given mass of charcoal would be obtained when it was equally divided between $(8.2/2.72 =)$ 3 bulbs; and the pressure-reduction factors would be:—

With 1 bulb,	8.2
„ 2 bulbs,	16.8
„ 3 bulbs,	20.4
„ 4 bulbs,	17.7

Actually f_1 is usually so large that the optimum value of n is too high to be attainable. Thus, if f_1 be 10^3 , f_2 will become a maximum when the charcoal is divided between 367 bulbs, and the maximal value of f_2 will be e^{367} . For practical purposes, then, it is permissible to state that the evacuating power of a given mass of absorbent can be greatly increased by employing it in stages in the manner indicated, and that the more numerous the stages the higher the vacuum obtained.

SUMMARY.

Prehensility is defined as the slope at the origin of an adsorption isotherm.

A method of measuring prehensility is described, and results given for various adsorbents at liquid air temperature.

It is shown that from the prehensility the evacuating power of a substance may be calculated. In evacuating any given volume, the weight of charcoal required to yield a required reduction of pressure may be computed. The degree of vacuum obtained in a Dewar liquid-air container is discussed.

The plumstone charcoal used by the author had a higher evacuating power than cocoanut charcoal.

Reference is made to a colloidal silica of appreciable evacuating power, though, at -190° C. over four times as much of it is required (by volume) as of plumstone charcoal to attain the same result, and it acts more slowly.

The very high degree of vacuum procurable by using a succession of charcoal bulbs is discussed, and it is shown that with a given weight of charcoal the reduction of pressure obtainable by division of the mass among a number of bulbs does not indefinitely increase with that number, but eventually reaches a maximum.

(Issued separately March 14, 1922.)

IV.—Results obtained at the Physical Test Station, Edinburgh.
By Professor Henry Briggs, D.Sc., Ph.D.

(MS. received November 7, 1921. Read November 7, 1921.)

I. PURPOSE.

It became evident during the War that some men who are organically sound are nevertheless incapable of supporting heavy exertion ; and among the older groups that were called to the Colours in 1918 the proportion of such individuals was considerably more. The effect of forcing these men to carry out the duties falling to the lot of an " A " recruit was in many cases harmful to them, and in extreme instances resulted in permanent disability and sometimes in death. The Army Medical Department therefore sought to obtain a method of physical examination, which would supplement the ordinary medical examination of a recruit, and which could be applied to men who proved to be unserviceable material in the hands of the drill sergeant—a test which would, in fact, enable the malingerers to be distinguished from those who were truly incapable of sustained labour.

At that time, the writer was working on the same problem in its application to the members of mine rescue brigades,* who, like the soldier, ought to be both medically sound and physically efficient, and he succeeded in developing a method of testing which quantitatively assesses the fitness and stamina of the subject. Acting on the advice of Col. Sir William Horrocks, K.C.M.G., the Army Council put the method into service for army purposes by setting up, in Edinburgh University, the Test Station of which this paper gives an account. Arrangements were being made in the autumn of 1918 to establish a second Station under the Southern Command, but the signing of the Armistice in November brought the project to an end, and, shortly after, the Edinburgh Station also ceased to function.

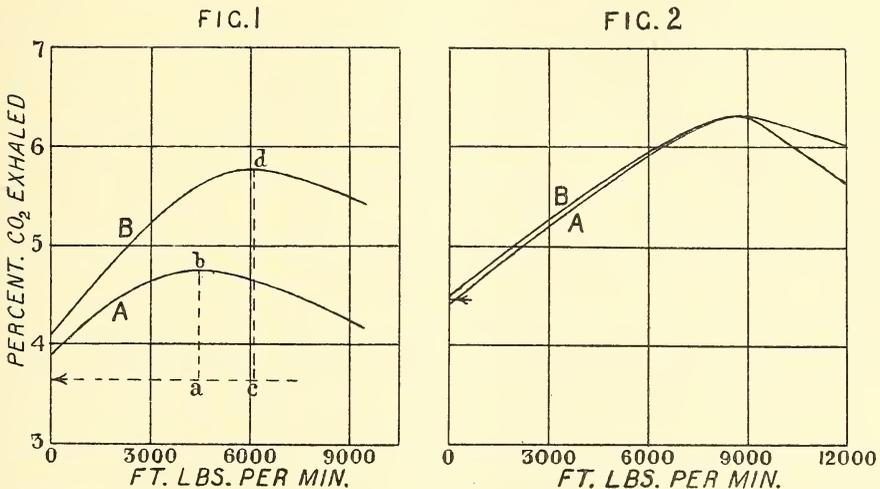
II. PRINCIPLES.

The principle upon which the method is based, and the experimental data bearing upon it have been fully described in the *Journal of*

* The work was carried out under the auspices of the Scientific and Industrial Research Department, which also provided the material for the Test Station.

Physiology; * it will therefore be sufficient to touch briefly upon the physiological aspect of the problem.

It was found that when a man breathing normal air is set to do physical work of gradually increasing amount, as, for example, upon a Martin's ergometer, the percentage of CO₂ in the exhaled air rises from the resting value (average 3.61) to a maximum and then falls again. That is to say, if that percentage be plotted as ordinate against load in foot-pounds per minute as abscissa, a dome-shaped curve (e.g. A, fig. 1) always results. The evidence supports the view that oxygenation of the muscles doing the work is sufficient up to a load corresponding with the apex of the dome (the "crest-load," as it is conveniently called), but is inadequate



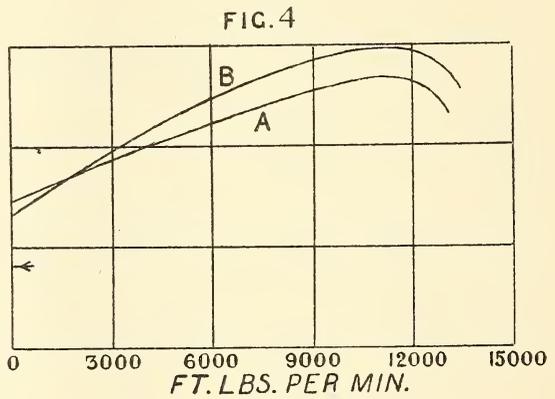
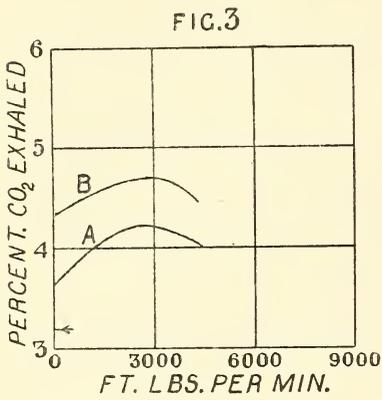
for greater loads. Degrees of exertion which are less than the crest-load are termed "normal loads," and can be supported for a considerable time; while those exceeding the crest-loads are "over-loads," and cannot be kept up for more than a brief period. Fatigue or illness reduced the "crest-load."

When the subject is caused to breathe air containing from 60 to 100 per cent. of oxygen, the effect with most people is to enable them to undertake hard physical work with greater ease and to carry it on longer without fatigue. In other words, oxygenation is improved. The result of breathing enriched air becomes evident when the graph connecting the degree of exertion and the expired-CO₂-percentage is drawn, as at B, fig. 1.

* Henry Briggs, "Physical Exertion, Fitness, and Breathing," *Journ. Physiol.*, liv (1920), p. 292; also *ibid.*, liii (1919): *Proc. Physiol. Soc.*, p. xxxviii; *Journ. R.A.M.C.*, Oct. 1921; and *Trans. Inst. Min. Engs.*, lxi (1921), p. 26.

It is then seen, with the average subject, that the CO₂-proportions are higher, especially at and beyond the crest-load, and that the crest-load itself is greater than when normal air was breathed. Oxygen-want, which is the main factor limiting the duration and intensity of physical exertion, is staved off by breathing enriched air.

By experimenting on a large number of healthy men ranging in type from the athlete in perfect training to the ultra-sedentary person, it was found that the higher the degree of fitness the less the *A* and *B* curves diverged, and, indeed, that when exceptionally fit men were tested, the resulting graphs were similar to those of fig. 2, where the curves for all practical purposes are coincident to the crest and only show divergence at



the over-loads. It was also discovered that if the same man were kept under observation for several months and were tested at different states of health, or at intervals during a course of physical training, the *B* curve would remain constant (within the limits of experimental error), but the *A* curve would vary in form and position according to the state of health or of training. Fitness, which may be defined as the efficiency of oxygenation of brain, heart, and muscles during exercise, is therefore inversely as the extent of divergence of the two curves, and can be evaluated.

By drawing upon the graph the horizontal line *ac* (fig. 1) at the level of the expired-CO₂-percentage at rest, and then measuring the crest ordinates *ab* and *cd*, the fitness factor can be expressed as $ab \div cd$. Thus, fig. 1, which records the data for a sedentary man, gives his fitness as 46 per cent. while the curves of fig. 2, which are those of an athletic sergeant-instructor in physical drill, show his fitness to have been 100 per cent.

If stamina be defined as the power of dealing with sustained exertion,

it is clear that the wider the range of normal load, *i.e.* the higher the crest-load, the higher must be the stamina of the subject. Hence the abscissal position of the crest-load becomes a measure of stamina.

Inasmuch as the *A* curve of a young man in good health rises during physical training until it eventually coincides with the *B* curve up to the crest, the latter curve may be regarded as expressing the subject's performance on air after he has been made quite fit; thus it was preferable to state the stamina as a function of the position of the crest of the *B* curve. The measure of stamina adopted at the Test Station was to take a *B* crest-load of 10,000 ft.-lbs. per minute as indicating 100 per cent. stamina, a *B* crest-load of 5000 ft.-lbs. as 50 per cent. stamina, and so on. This method, though not free from objection, is simple and proved reliable.

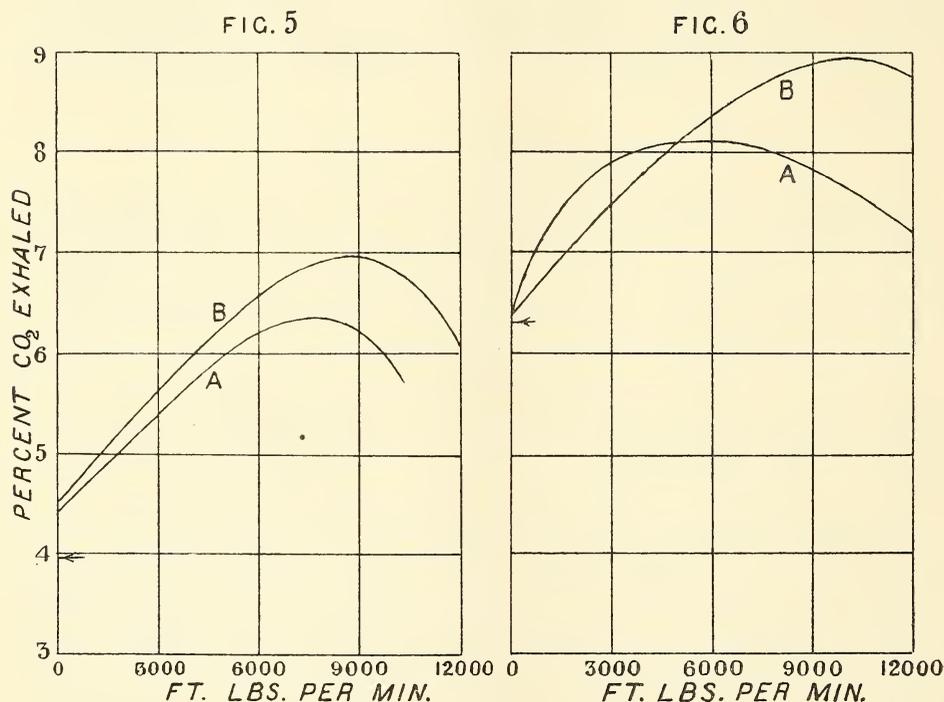
III. APPARATUS AND ROUTINE.

The Station was run by an officer and two N.C.Os. under the writer's superintendence. In the research which preceded the establishment of the Station, apparatus of rather greater complexity had to be employed, since the aim then was to evaluate oxygen consumption during work as well as CO₂-output, and to study other questions such as the composition of alveolar air and the mechanical efficiency of the subject; but at the Station the apparatus was cut down to the minimum and the routine was simplified and standardised for the sake of speed. The whole equipment, with the exception of thirty 100-ft. oxygen cylinders, is shown in fig. 9. The subject, it will be observed, was provided with mouthpiece and nose-clip; he drew air or oxygen (as the case may be) through a dry meter and expired into a Douglas bag. The valves and connecting tubes were large, and their resistance was negligible even when the lung-ventilation was as high as 80 litres per minute. The meter, besides measuring the volume drawn in, served to indicate the rate of breathing; the officer in charge counted the movements of the pointer against a stop-watch. As the dial of the meter was not seen by the man, he was unaware that any notice was being taken of his breathing—a matter of importance with “raw” subjects.

At the start, the empty Douglas bag was connected to the expiratory tube, *A*, the three-way tap being in the “off” position, so that the products of expiration passed directly out into the air of the room. The subject, seated at rest on the saddle of the ergometer, breathed normal air. After he had become accustomed to his position, the three-way tap was turned “on” at the end of an expiration, and the breath passed into the bag.

After an interval of about two minutes the tap was again turned at the end of an expiration. The bag was then placed on the table; kneaded to mix its contents; connected to the supply pipe of the Briggs analysis apparatus,* *B*, and with the tap in the "on" position, squeezed to force a few litres through the burette. The sample so obtained in the burette was then analysed for CO_2 . This procedure avoided the need for sample tubes or bottles. One of the N.C.O.s. made the analysis during the time that the next sample was being obtained in the bag.

After filling the burette, the bag was emptied by pressing it flat, and



was again connected to *A*. The subject was required to pedal at no load, *i.e.* with the belt off, for two minutes; the tap was turned on at the end of an expiration, and expired air allowed to accumulate in the bag for another two minutes of pedalling, when the sample was removed to the analysis apparatus. The belt was put in place; its cords were adjusted to give a difference of 3 lbs. between the balance readings, and the same sequence followed. Similar observations were made at balance-differences of 6, 9, 12 lbs., or even more, if the man could support such heavy exertion.

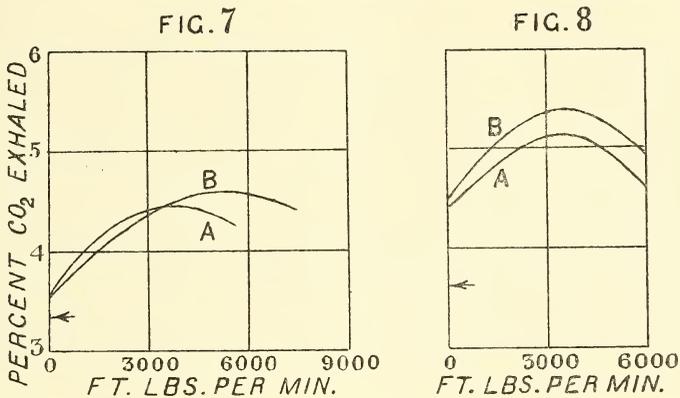
Pedalling was timed to a pendulum which swung 56 to the minute; at

* First Report, "Mine Rescue Apparatus Research Committee," 1918, p. 51.

this rate, and with the gear of the cycle used, the work done was evaluated by multiplying the balance-difference by one thousand.

Longer pauses were permitted between the heavier spells of work to allow the man to recover from the effect of one spell before attempting another. Care was taken that pedalling was kept up two minutes before opening the bag to the exhaled air; at the highest loads, however, *e.g.* 12,000 or 14,000 ft.-lbs., this rule had to be relaxed, as no one in the writer's experience was able to bear them so long. The graphs at these extreme loads are therefore not so reliable as at lower ones.

After the above results had been obtained with the subject breathing normal air, an exactly similar series (excepting that the resting experiment was omitted) was taken with the man breathing oxygenated air from the



reservoir bag, which was kept, in a partially distended state, under the table at *C*, fig. 9. Before the latter series was started he was required to sit still and breathe the enriched air for ten minutes.

The subject did not know that he was breathing oxygenated air. No loophole was allowed for any prejudice against so doing.

The oxygen was supplied to the reservoir bag, *C*, from a 100-ft. cylinder fitted with a reducing valve. On passing the reducing valve the gas flowed through an injector nozzle arranged so that the oxygen drew in, and diluted itself with, a certain proportion of normal air. The air entered through a pipe controlled by a tap which was set by trial (and then soldered in position), so that the mixture passing forward to the reservoir bag was 40 per cent. air. Allowing for the impurity in cylinder oxygen, such a mixture contains about two-thirds oxygen. Besides the mixture being as good from the physiological point of view—even with the least fit subject—as pure oxygen, its use brought about a considerable saving in expense, oxygen being the chief item of cost of the Station.

The officer in charge entered all results as they were obtained and straightway plotted the graphs. A report, based upon his medical history and Test Station performance, was made out for each subject and forwarded to the C.O. of the Unit concerned. The report stated the physical capability of the man and his probable utility when trained. In the numerous instances where the tests showed him to be useless as a fighting

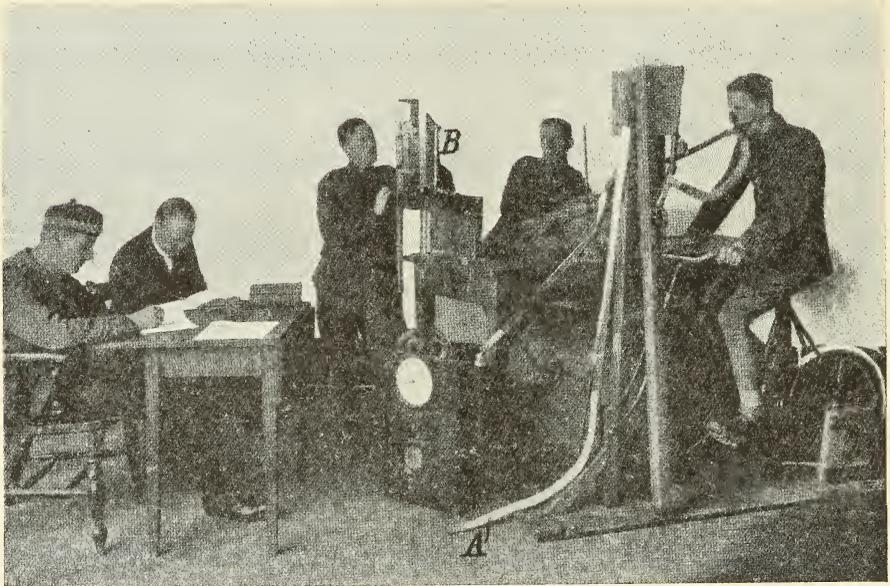


FIG. 9.

unit, a recommendation was added in regard to the purpose (if any) to which he could be put.

A complete test, as described, took about thirty-five minutes. The time taken obviously precluded the use of the method for every recruit, and that was never the intention; the Station was set up to deal with special cases.

IV. RESULTS.

Specimen charts are reproduced in figs. 1 to 8. The arrow-heads indicate the resting value of the CO_2 proportions. The *A* curves show the relation between load and expired- CO_2 -percentage when breathing normal air, and the *B* curves that when breathing oxygenated air.

When the Station was in operation men over forty years of age were being conscripted, and many of these were examined. Only about 20 per cent. of them gave evidence of being worth training. The graphs

indicated very clearly the influence of age, which reduces the "crest-loads" both on normal and on enriched air. In other words, anoxæmia (as might be expected) makes itself felt lower down the scale of exertion as age increases; stamina is reduced and a degree of exercise which would be a normal load to a younger man is an over-load to an older one. The effect in question is shown by figs. 3 and 4, of which the former is the record of a man of forty-four—a painter in civil life—who was classed B 2 owing to kidney trouble, and the latter that of a well-developed and athletic cadet of eighteen. The report sent out in regard to the older man was: "*Stamina*: Very poor. *Condition*: Poor. *Observations*: Not worth training; no use as an infantryman. *Recommendation*: Suggest that he be set to his own trade." And that in regard to the cadet was: "*Stamina*: Excellent. *Condition*: Excellent. *Observations*: First-rate material; fit for anything. Probable increase of fitness from P.D., 10 per cent."

That a man of middle age, who is habituated in civilian life to physical labour, may sometimes preserve the physiological characteristics of youth is illustrated by fig. 5, which gives the curves of a working miner, aged forty-two. Expressed on the system above described, his fitness was 79 per cent. and his stamina 90 per cent.

The physical deterioration brought about by wounds and hard active service is indicated by fig. 7. In this instance the subject was a corporal, aged thirty-two, who had joined the Army in 1914; he had suffered from trench fever and had been wounded three times. His medical category was A1, but the tests showed that, though probably as fit as he was ever likely to be, his stamina had become so impaired that he was of no further use as an infantryman, and it was recommended that he should be re-boarded so that his category might be lowered,

The remaining charts, figs. 6 and 8, are of special interest as representing the extremes of physical capacity. Both subjects were young men of "A" category; but while the former was a highly-intelligent instructor in physical drill, and a first-class footballer, runner, jumper, and all-round athlete, the latter was deficient both physically and mentally.

A complete account of the numerous tests made upon the sergeant-instructor is given in the writer's paper in the *Journal of Physiology*, *loc. cit.*, p. 302. When the curves of fig. 6 were obtained he had a fitness of 70 per cent.; on the lightest loads he respired at the very low rate of 2.5 to 3 breaths per minute, and when dealing with a heavy load, like 12,000 ft.-lbs., he only breathed nine times per minute. As the chart shows, his CO₂ level was very high, and in consequence he used a small

volume of air. For example, on a load of 6000 ft.-lbs., though the heavier man, his lung-ventilation was less than half that of the sedentary subject of fig. 1.

A good deal of trouble was taken with the degenerate subject of fig. 8. He did not know how to pedal, and even after practice could not be induced to pedal in time with a pendulum. The record shows him to be useless in the ranks and not worth training; his stamina was far too low. His response to any form of mental stimulus was unusually tardy. An order, such as "Stop pedalling," would only be obeyed after the lapse of several seconds. Curiously enough, his respiratory centre appeared to operate after a similar lag, with the result that the volume of breathing did not increase at the normal rate upon starting a spell of work; anoxæmia supervened and made him stop the exertion at a load which to a normal healthy man would be easy.

Very few malingerers were met with. They were easily detected, as they would allege a load was more than they could support before the curve had reached its crest.

(Issued separately March 14, 1922.)

V.—On the Connexion between Linear Differential Systems and Integral Equations. By E. L. Ince, D.Sc., Lecturer in Mathematics in the University of Liverpool. *Communicated by Professor E. T. WHITTAKER, F.R.S.*

(MS. received October 29, 1921. Read January 23, 1922.)

§ 1. INTRODUCTION.

THIS paper summarises the results of an attempt to extend the theory upon which the relationship between linear differential equations and integral equations is based.* The case in which the nucleus $K(x, s)$ of the integral equation arises as a Green's function is well known; the nucleus is there characterised by its having discontinuous derivatives when $x=s$. The method here dealt with is virtually an extension of Laplace's and analogous methods for solving linear differential equations by definite integrals, and leads to nuclei which are continuous and have continuous derivatives for $x=s$.

§ 2. THE RELATIONSHIP BETWEEN LINEAR DIFFERENTIAL SYSTEMS OF THE SECOND ORDER AND INTEGRAL EQUATIONS.

Let us, in the first place, consider the homogeneous and self-adjoint linear differential equation of the second order

$$L_x(u) + au \equiv \frac{d}{dx} \left[k(x) \frac{du}{dx} \right] + [a + l(x)]u = 0 \quad (1)$$

in which $k(x)$ and $l(x)$ are defined for $\alpha \leq x \leq \beta$, and are such that $k(\beta) = k(\alpha)$ and $l(\beta) = l(\alpha)$; let us adjoin to (1) the periodic boundary conditions

$$\left. \begin{aligned} u(\beta) &= u(\alpha) \\ u'(\beta) &= u'(\alpha) \end{aligned} \right\} \quad (1')$$

The system (1, 1') is in general incompatible,† *i.e.* admits of no solution not identically zero. There may, however, exist a sequence, finite or infinite, of characteristic numbers a_1, a_2, \dots such that the system becomes

* An abstract of the theory as it stood in 1910 is given by Bateman, "Report on the Theory of Integral Equations," British Association, 1910, § 21. References to later work are given in footnotes in the present paper.

† For the general theory of the compatibility of linear differential systems, see Bôcher, *Leçons sur les méthodes de Sturm* (1917), chap. ii.

compatible when, and only when, a assumes a value in this sequence. In any particular system, the set of characteristic numbers depends, *inter alia*, upon $k(a)$ and $l(a)$.

When the given system is thus rendered compatible, there arises one of two cases. In the more general case, to a characteristic number a_i there corresponds a solution $u_i(x)$, unique apart from a constant factor; in this case the system is said to be of index of compatibility 1. On the other hand, when further relations between a , $k(a)$, and $l(a)$ are introduced, it may occur that two linearly independent solutions of (1), and therefore all solutions of (1), satisfy conditions (1'), in which case the system is said to be of index 2.

Let $K(x, s)$ be a solution of the partial differential equation

$$L_x(K) - L_s(K) \equiv \frac{\partial}{\partial x} \left[k(x) \frac{\partial K}{\partial x} \right] - \frac{\partial}{\partial s} \left[k(s) \frac{\partial K}{\partial s} \right] + [l(x) - l(s)]K = 0 \quad (2)$$

satisfying the periodic conditions

$$\begin{aligned} K(x, \beta) &= K(x, \alpha) \\ K_s(x, \beta) &= K_s(x, \alpha) \end{aligned} \quad (2')$$

for $\alpha \leq x \leq \beta$. Since equation (2) is symmetrical in x and s , it follows that, if $K(x, s)$ exists, it may be so chosen as to be, if not symmetrical, *i.e.* such that $K(s, x) = K(x, s)$, then at least skew-symmetrical, *i.e.* such that $K(s, x) = -K(x, s)$. This being so, the conditions

$$\begin{aligned} K(\beta, s) &= K(\alpha, s) \\ K_x(\beta, s) &= K_x(\alpha, s) \end{aligned} \quad (2'')$$

will also be satisfied for $\alpha \leq s \leq \beta$.

Let $u_i(s)$ be a solution of

$$L_s(u) + a_i u = 0,$$

in which a_i is a characteristic number, and let

$$I(x) \equiv \int_{\alpha}^{\beta} K(x, s) u_i(s) ds.$$

Then *

$$\begin{aligned} L_x(I) &= \int_{\alpha}^{\beta} L_x(K) u_i(s) ds = \int_{\alpha}^{\beta} L_s(K) u_i(s) ds \\ &= \left[R \right]_{\alpha}^{\beta} + \int_{\alpha}^{\beta} K(x, s) L_s(u_i) ds \\ &= -a_i I(x), \end{aligned}$$

since the bilinear concomitant

$$R \equiv k(s) [u_i(s) K_s(x, s) - u_i'(s) K(x, s)] \quad (3)$$

vanishes between the limits of integration for all values of x in (α, β) .

* Forsyth, *Theory of Differential Equations*, vol. iv, p. 252.

It follows that $I(x)$ satisfies the system $(1, 1')$ for the characteristic number a_i . Let us assume for the moment that $I(x)$ is not identically zero, that is to say, that $u_i(s)$ is not orthogonal to $K(x, s)$ in (a, β) . Let us confine our attention to the more general case in which the index of compatibility is 1. The system $(1, 1')$ has then no solutions other than multiples of $u_i(x)$ for the characteristic number a_i , and therefore $I(x)$ is a multiple of $u_i(x)$. In other words, $u_i(x)$ satisfies the integral equation

$$u(x) = \lambda \int_a^\beta K(x, s)u(s)ds \quad . \quad . \quad . \quad . \quad . \quad (4)$$

for a particular value λ_i of λ . This integral equation is therefore satisfied by all solutions of system $(1, 1')$ which are not orthogonal to $K(x, s)$. In particular, if the nucleus $K(x, s)$ is closed, the integral equation is satisfied by all solutions of the system. We may go further and define a conditionally closed nucleus as one not orthogonal to functions of a specified type. If the nucleus satisfies such a condition, the integral equation will be satisfied by all the solutions of $(1, 1')$ which are of that type, *e.g.* by all the even solutions of the system.

§ 3. CONDITIONS THAT A SOLUTION OF THE INTEGRAL EQUATION SHALL SATISFY THE DIFFERENTIAL SYSTEM.

The theorem now to be proved is the converse of the theorem of § 1. The proof depends upon the Hilbert development of the nucleus $K(x, s)$ in terms of the fundamental functions.

It may easily be verified that the iterated nuclei $K_1(x, s), K_2(x, s), \dots$ satisfy the same partial differential equation (2) and the same boundary conditions $(2', 2'')$ as the original nucleus $K(x, s)$. Let us assume that $K(x, s)$ is symmetrical; the treatment of a skew-symmetrical nucleus would follow on similar lines.

Let the development of $K(x, s)$ be

$$K(x, s) = \left. \begin{aligned} &\frac{\phi_1(x)\phi_1(s)}{\lambda_1} + \frac{\phi_2(x)\phi_2(s)}{\lambda_2} + \dots + \frac{\phi_r(x)\phi_r(s)}{\lambda_r} + \dots \\ &\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_r \dots \end{aligned} \right\} \quad (5)$$

then the expansion of the iterated nucleus of rank $p-1$ will be

$$K_{p-1}(x, s) = \left. \begin{aligned} &\frac{\phi_1(x)\phi_1(s)}{\lambda_1^p} + \frac{\phi_2(x)\phi_2(s)}{\lambda_2^p} + \dots + \frac{\phi_r(x)\phi_r(s)}{\lambda_r^p} + \dots \\ &p = 2, 3, 4, \dots \end{aligned} \right\} \quad (5a)$$

Let us assume that the series for $K(x, s)$ and its second derived series are uniformly convergent throughout the square $a \leq x \leq \beta, a \leq s \leq \beta$; the same will be true of the iterated nuclei. [If this condition be not

satisfied by $K(x, s)$, we can replace $K(x, s)$ by $K_1(x, s)$, or by a nucleus of higher rank in which the condition is satisfied.] Then since $K_{p-1}(x, s)$ satisfies (2), we have

$$\left. \begin{aligned} \sum_r \lambda_r^{-p} \{L_x - L_s\} \phi_r(x) \phi_r(s) = 0, \\ p = 1, 2, 3, \dots \end{aligned} \right\} \dots \dots \dots (6)$$

Hence by the classical theory of simultaneous equations, whether finite or infinite in number,*

$$\{L_x - L_s\} \phi_r(x) \phi_r(s) = 0 \dots \dots \dots (7)$$

provided λ_r is not equal to any other number in the λ -sequence. Thus the single product $\phi_r(x) \phi_r(s)$ satisfies (2); it may likewise be shown to satisfy the boundary conditions (2', 2''). Consequently $\phi_r(x)$ is a solution of the system (1, 1') for a particular value a_r of a .

It has thus been proved that provided the nucleus $K(x, s)$ of the integral equation (3) satisfies the partial differential system (2, 2', 2''), provided the expansion of the nucleus in fundamental functions and its second derived series are uniformly convergent in the square (α, β) —a condition which can be ensured by iteration,—and provided that to the characteristic number λ_r there corresponds only one fundamental function $\phi_r(x)$, then $\phi_r(x)$ is a solution of the differential system (1, 1'). It follows immediately that, under the same conditions, solutions of (1, 1') exist which are not orthogonal to the nucleus $K(x, s)$ chosen, thus justifying the assumption made in § 2.

If we suppose two of the characteristic numbers equal, e.g. if $\lambda_2 = \lambda_1$, then we can conclude no more than that $\phi_1(x) \phi_1(s) + \phi_2(x) \phi_2(s)$ satisfies (2, 2', 2''). It does not follow from this that $\phi_1(x)$ and $\phi_2(x)$ satisfy (1, 1'); but if one of these satisfies the system, the other also satisfies it, and hence the linear combination $C_1 \phi_1(x) + C_2 \phi_2(x)$ satisfies the system. This may arise when the index of compatibility is 2, but must be definitely excluded in the more general case of index 1.

§ 4. AN EXAMPLE ILLUSTRATING THE METHOD.

The foregoing theory includes all the known cases of the solution of linear differential equations with periodic coefficients by integral equations.†

* Kowalewski, *Einführung in die Determinantentheorie*, §§ 22, 156.

† Whittaker: (1) *Proceedings International Congress, Cambridge, 1912*, i, p. 367; *Modern Analysis*, § 19·21.

(2) *Proc. R.S.E.*, xxxv (1914), pp. 70–77.

(3) *Proc. Lond. Math. Soc.* (2), xiv (1915), pp. 260–268; *Modern Analysis*, § 23·61.

(4) *Proc. Edin. Math. Soc.*, xxxiii (1915), pp. 14–23.

Reference may also be made to A. Milne's integral equation for the parabolic cylinder functions, *Proc. Edin. Math. Soc.*, xxxii (1914), p. 8.

As a further example, let us consider the determination of $K(x, s)$ so that its fundamental functions may be periodic solutions of the equation

$$\frac{d^2u}{dx^2} + \left(a + k^2 \cos^2 x - \frac{n(n-1)}{\sin^2 x} \right) u = 0 \quad \dots \quad (8)$$

where $0 \leq n \leq 1$. This is a generalisation* of Mathieu's equation

$$\frac{d^2u}{dx^2} + (a + k^2 \cos^2 x) u = 0,$$

whose even periodic solutions were shown by Whittaker† to be the fundamental functions of the integral equation

$$u(x) = \lambda \int_0^{2\pi} e^{k \cos x \cos s} u(s) ds.$$

Let us assume the corresponding nucleus for (8) to be of the form

$$K(x, s) = e^{k \cos x \cos s} \sin^r x \sin^r s;$$

substituting this in the partial differential equation

$$\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial s^2} + \left[k^2 \cos^2 x - k^2 \cos^2 s - \frac{n(n-1)}{\sin^2 x} + \frac{n(n-1)}{\sin^2 s} \right] K = 0,$$

we find that $r = n$ or $1 - n$.

Thus, supposing the conditions enumerated above to be satisfied, we can conclude that equation (8) is satisfied by the fundamental solutions of

$$u(x) = \lambda \int_0^{2\pi} e^{k \cos x \cos s} \sin^n x \sin^n s u(s) ds \quad \dots \quad (9a)$$

and

$$u(x) = \lambda \int_0^{2\pi} e^{k \cos x \cos s} \sin^{1-n} x \sin^{1-n} s u(s) ds \quad \dots \quad (9b)$$

The condition as to the convergence of the development presents no difficulty; the condition as to the inequality of the characteristic numbers is satisfied for sufficiently small values of n .

When $n = 0$, (9a) reduces to Whittaker's integral equation for the even Mathieu functions $ce_m(x)$; and (9b) reduces to

$$u(x) = \lambda \int_0^{2\pi} e^{k \cos x \cos s} \sin x \sin s u(s) ds,$$

an integral equation for the odd Mathieu functions $se_m(x)$.

Since the expansion of $e^{k \cos x \cos s}$ is of the form

$$\sum_0^\infty \mu_m ce_m(x) ce_m(s),$$

and that of $e^{k \cos x \cos s} \sin x \sin s$ is of the form

$$\sum_1^\infty \mu'_m se_m(x) se_m(s),$$

* It is virtually the equation exhaustively treated by M. Abraham, *Math. Ann.*, lii (1901), pp. 81-112.

† *Loc. cit.* (1).

the solutions of (9a) will be expressible in the form

$$\sin^n x \sum_0^\infty C_m e_m(x) + \sin^{n-1} x \sum_1^\infty D_m s e_m(x),$$

where $D_m = 0$ when $n = 0$, and $C_m = 0$ when $n = 1$. The solutions of (9b) will be obtained by writing $1 - n$ for n .

§ 5. SYSTEMS OF ORDER n .

We may readily extend the foregoing results to the general homogeneous linear equation of order n ,* the main theorem being as follows:—

The solutions of the homogeneous integral equation

$$u(x) = \lambda \int_a^\beta K(x, s) u(s) ds$$

satisfy the differential equation

$$L_x(u) + au \equiv l_n(x) \frac{d^n u}{dx^n} + l_{n-1}(x) \frac{d^{n-1} u}{dx^{n-1}} + \dots + [a + l_0(x)]u = 0 \quad (10)$$

and the n periodic boundary conditions

$$\left. \begin{aligned} U_0(u) &\equiv u(\beta) - u(\alpha) = 0 \\ U_i(u) &\equiv u^{(i)}(\beta) - u^{(i)}(\alpha) = 0 \\ & \quad i = 1, 2, \dots, n-1 \end{aligned} \right\} \quad (10')$$

provided the nucleus $K(x, s)$ satisfies the partial differential equation

$$L_x(K) - \bar{L}_s(K) = 0, \quad (11)$$

where \bar{L}_s is the differential operator adjoint to L_s , and satisfies also the periodic conditions

$$\left. \begin{aligned} K(x, \beta) &= K(x, \alpha), & K(\beta, s) &= K(\alpha, s) \\ K_s^{(i)}(x, \beta) &= K_s^{(i)}(x, \alpha) & K_x^{(i)}(\beta, s) &= K_x^{(i)}(\alpha, s) \\ & & & i = 1, 2, \dots, n-1 \end{aligned} \right\} \quad (11')$$

If the index of compatibility of (10, 10') is 1, the characteristic numbers λ in the expansion

$$K(x, s) = \sum_r \frac{\phi_r(x) \psi_r(s)}{\lambda_r} \quad (12)$$

must all be unequal; if the index is i , not more than i of the characteristic numbers may be equal.

Under like conditions, the adjoint differential equation

$$\bar{L}_x(v) + av = 0 \quad (10a)$$

is satisfied by the solutions of

$$v(x) = \lambda \int_a^\beta K(s, x) v(s) ds.$$

* Bateman, *Trans. Camb. Phil. Soc.*, xxi (1909), p. 187.

§ 6. THE FUNCTIONAL RELATIONSHIP BETWEEN SOLUTIONS OF TWO
DISTINCT LINEAR DIFFERENTIAL EQUATIONS.

Let

$$L_x(u) + au = 0 \quad (13)$$

and

$$M_x(v) + av = 0 \quad (13a)$$

be linear differential equations. Let $K(x, s)$ and $k(s, x)$ respectively satisfy the partial differential equations

$$L_x(K) - \bar{M}_s(K) = 0 \quad (14)$$

$$M_x(k) - \bar{L}_s(k) = 0 \quad (14a)$$

We are thus led to study the pair of simultaneous integral equations

$$u(x) = \lambda \int_{\gamma} K(x, s)v(s)ds \quad (15)$$

$$v(x) = \lambda \int_{\gamma'} k(s, x)u(s)ds \quad (15a)$$

where γ is chosen so that the bilinear concomitant of $M_s(v)$ and $K(x, s)$ vanishes over γ , and γ' is similarly chosen with reference to $L_s(u)$ and $k(s, x)$.

Let the development* of $K(x, s)$ be

$$\sum_r \frac{\phi_r(x)\bar{\psi}_r(s)}{\lambda_r},$$

that of $k(s, x)$ will be

$$\sum_r \frac{\bar{\phi}_r(s)\psi_r(x)}{\lambda_r}.$$

The systems $\phi_r(x)$, $\bar{\phi}_r(x)$ and $\psi_r(x)$, $\bar{\psi}_r(x)$ are each biorthogonal and normal.

It is easily demonstrated, as in § 3, that if λ_r is unequal to any other λ , $\phi_r(x)$ and $\psi_r(x)$ satisfy (13) and (13a) respectively. This stipulation may be modified when the index of one or other or of both of the systems is greater than unity.

If we eliminate v or u between (15) and (15a), writing

$$\bar{K}(x, s) = \int_{\gamma} K(x, t)k(s, t)dt$$

and

$$\underline{K}(x, s) = \int_{\gamma'} K(t, s)k(t, x)dt,$$

* This development is due to E. Schmidt, *Inaugural Dissertation*, Göttingen, 1905; see Lalesco, *Théorie des équations intégrales*, p. 96, or Goursat, *Cours d'analyse*, iii, p. 470.

we have respectively the homogeneous integral equations

$$u(x) = \lambda^2 \int_{\gamma} \overline{K}(x, s) u(s) ds \quad . \quad . \quad . \quad . \quad (16)$$

$$v(x) = \lambda^2 \int_{\gamma} \underline{K}(x, s) v(s) ds \quad . \quad . \quad . \quad . \quad (16a)$$

with the same series of characteristic numbers, whose solutions satisfy (13) and (13a). The functional relations (15) and (15a) are thus intermediate to the formation of the ordinary integral equations (16) and (16a).

§ 7. AN IMPORTANT PARTICULAR CASE.

In general it is of advantage to have the equation $M_x(v) + \lambda v = 0$ of as simple and well-known a form as possible, in order that (15) and (15a) may be relations connecting the solutions of (13) with known functions, such as the elementary transcendentals, Bessel or Legendre functions. A particularly simple and interesting case is that in which (13) occurs in the form

$$L_n(u) + n^2 u = 0 \quad . \quad . \quad . \quad . \quad (17)$$

n being an integer, and (13a) reduces to

$$\frac{d^2 v}{dx^2} + n^2 v = 0 \quad . \quad . \quad . \quad . \quad (17a)$$

Then if $K(x, s)$ be a solution, periodic in s , of the partial differential equation

$$L_x(K) - \frac{\partial^2 K}{\partial s^2} = 0 \quad . \quad . \quad . \quad . \quad (18)$$

in which n does not appear, equation (17) will be satisfied by a definite integral of the form

$$\int_0^{2\pi} K(x, s) \frac{\cos ns}{\sin ns} ds \quad . \quad . \quad . \quad . \quad (19)$$

Also, if $k(x, s)$ is a solution, periodic in s , of the equation adjoint to (18), and if $u(x)$ is a solution of (17), it may be possible to find a path of integration γ' such that

$$\frac{\cos nx}{\sin nx} = C \int_{\gamma'} k(s, x) u(s) ds \quad . \quad . \quad . \quad . \quad (20)$$

Thus, taking Bessel's equation

$$x^2 u'' + x u' + (x^2 - n^2) u = 0 \quad . \quad . \quad . \quad . \quad (21)$$

we associate with it the partial differential equation

$$x^2 \frac{\partial^2 K}{\partial x^2} + x \frac{\partial K}{\partial x} + x^2 K + \frac{\partial^2 K}{\partial s^2} = 0 \quad . \quad . \quad . \quad . \quad (22)$$

A solution of this equation periodic in s and having a periodic derivate

with respect to s , and thus fulfilling the limit conditions, is $K(x, s) = e^{ix \sin s}$. Hence Bessel's equation admits of solutions of the form

$$u(x) = \int_0^\pi e^{ix \sin s} \frac{\cos}{\sin} ns \, ds.$$

The real and the imaginary parts may be separated, giving integrals of the form

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \frac{\cos}{\sin} (x \sin s) \frac{\cos}{\sin} ns \, ds \quad \dots \quad (23)$$

such as occur in the classical theory of Bessel functions.

The adjoint partial differential equation is

$$x^2 \frac{\partial^2 K}{\partial x^2} + 3x \frac{\partial K}{\partial x} + (x^2 + 1)K + \frac{\partial^2 K}{\partial s^2} = 0 \quad \dots \quad (22a)$$

and has solutions $\frac{\cos(x \sin s)}{x}$ and $\frac{\sin(x \sin s)}{x}$. These lead to the two equations

$$\left. \begin{aligned} \cos nx &= n \int_0^\infty \cos(s \sin x) \frac{J_n(s)}{s} ds & n \text{ even} \\ \sin nx &= n \int_0^\infty \sin(s \sin x) \frac{J_n(s)}{s} ds & n \text{ odd} \end{aligned} \right\} \quad \dots \quad (23a)$$

§ 8. SOLUTION IN FINITE FORM.

An important case arises when the nucleus $K(x, s)$ can be decomposed by elementary methods into a series of products of the form *

$$K(x, s) = \sum_i C_i g_i(x) h_i(s) \quad \dots \quad (24)$$

the series being either finite or uniformly convergent in the domain considered. Then the relation (15), for example, gives an immediate solution of the corresponding equation (13), thus

$$\begin{aligned} u(s) &= \int_\gamma K(x, s) v(s) ds \\ &= \sum_i \lambda C_i \int_\gamma g_i(x) h_i(s) v(s) ds \\ &= \sum A_i g_i(x) \quad \dots \quad (25) \end{aligned}$$

where

$$A_i = \lambda C_i \int_\gamma h_i(s) v(s) ds.$$

If now $K(x, s)$ is decomposable into a finite number n of products, then

* It is not here a question of the development of the nucleus according to its fundamental functions.

Let us investigate the even periodic solutions of (29) which are expressible in finite form; they must necessarily be included in

$$u(x) = e^{il \sin^2 x} (\cos 3x + c \cos x),$$

where c is a constant to be determined. We have the two equations

$$1 = \frac{\lambda}{4} \int_0^{2\pi} e^{l \sin^2 s} \cos 3s (\cos 3s + c \cos s) ds$$

$$c = \frac{3\lambda}{4} \int_0^{2\pi} e^{l \sin^2 s} \cos s (\cos 3s + c \cos s) ds,$$

which may be written

$$\left. \begin{aligned} 1 &= \frac{\lambda}{8} [I_0 + I_3 + c(I_1 + I_2)] \\ c &= \frac{3\lambda}{8} [I_1 + I_2 + c(I_0 + I_3)] \end{aligned} \right\} \dots \dots \dots (31)$$

in which

$$I_m = \int_0^{2\pi} e^{l \sin^2 s} \cos 2ms \, ds = 2i^m e^{\frac{l}{2}} J_m \left(\frac{il}{2} \right) \dots \dots \dots (31a)$$

Hence

$$c = 3 \frac{I_1 + I_2 + c(I_0 + I_3)}{I_0 + I_3 + c(I_1 + I_2)}.$$

By the use of the recurrence relation

$$4mI_m = l(I_{m+1} - I_{m-1})$$

this equation may be reduced to

$$lc^2 + 2(4-l)c - 3l = 0,$$

whence we conclude that the two even solutions of (29) expressible in finite form are

$$u_1(x) = e^{il \sin^2 x} \left[\cos 3x + \frac{4-l + \{(4-l)^2 + 3l^2\}^{\frac{1}{2}}}{l} \cos x \right] \dots \dots (32)$$

and

$$u_2(x) = e^{il \sin^2 x} \left[\cos 3x + \frac{4-l - \{(4-l)^2 + 3l^2\}^{\frac{1}{2}}}{l} \cos x \right] \dots \dots (32a)$$

VI.—Myriapods collected in Mesopotamia and N.W. Persia by
W. Edgar Evans, B.Sc., late Capt. R.A.M.C. By Henry W.
Brolemann, Pau (France). *Communicated by* WM. EVANS, F.F.A.

(MS. received December 8, 1921. Read March 6, 1922.)

THE Myriapods recorded in this paper were collected in Mesopotamia—mostly at and around Amara on the Tigris—and in the neighbouring uplands of N.W. Persia, by W. E. Evans, B.Sc., Edinburgh, late Captain R.A.M.C.

They are referable to seventeen forms; of these eight are new and three could not be identified owing to want of appropriate material. Amongst the novelties two specimens deserve special mention as belonging to an Iulid group, the known representatives of which are very few. The new genus *Calyptophyllum* has been erected for these.

This is a most gratifying contribution to our knowledge of the fauna of the regions visited, and the interest the study of the material afforded calls for a grateful acknowledgment from the describer.

PAU, November 25, 1921.

CHILOPODA.

Bothriogaster signata georgica, Chalande & Ribaut, 1909.

♀ : length about 89 mm.; breadth of 10th segment 1.70 mm., of 68th segment 3 mm., of penultimate segment 1.90 mm.—109 pairs of legs.

Sternal dimples on segments 38 to 49; thus there are 23 more sternites lacking dimples behind the 49th segment than in front of the 38th—a figure which agrees with that given by the authors. The dimples are closed in front, circular, the second being one-fifth longer than broad, while the following are as long as broad.

A single female from: "Above Amara on Tigris, under clod, — iii. 1918."

Polyporogaster sinuatus, Silvestri, 1919.

♂ : length about 78 mm.; breadth of 3rd segment 1.50 mm., in the middle of the body 2 mm.—89 pairs of legs.

This specimen has met with some accident, probably during its growth,

causing the last three segments to be injured on the right side. Yet it was possible to detect some coxal pores dorsally.

A single male from: "Jebel Hamrin, near Ruz, N.E. of Baghdad, under stone, — xi. 1918."

Mecistocephalus Evansi, n. sp.

♀ : length 38 mm.; breadth 1.20 mm.—51 pairs of legs.

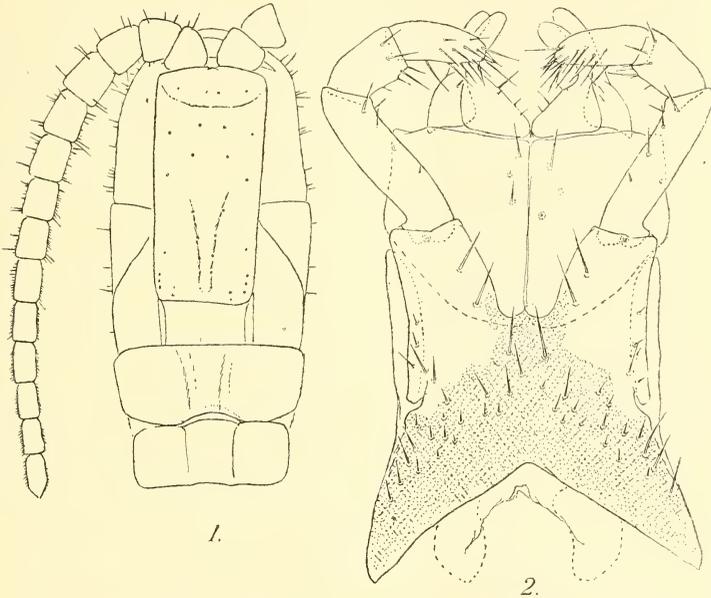


FIG. 1.—*Mecistocephalus Evansi*, n. sp.
Anterior end of body, dorsal
view.

FIG. 2.—*Mecistocephalus Evansi*, n. sp.
Maxillæ, first and second pairs,
ventral view.

Body parallel-sided for four-fifths of its length, tapering gently backwards from about the 40th segment.

Head quadrangular (fig. 1), almost as broad behind as in front, at least twice as long as broad; angles scarcely blunt. Surface with two paramedian sulci and a few deep punctures. Basal shield shorter than broad, parallel-sided, filling the space between the dorsal pleural ridges. The ridges are perfectly straight and parallel, in a line with the lateral margins of the head plate. Maxillipedes widely exposed dorsally, produced in front, the external summit of the femoral joint overreaching the anterior margin of the head plate; their surface is overspread with large and small punctures intermingled. Coxosternum with almost straight anterior margin (at least in the middle); no coxal teeth. Following joints all provided with distinct

apical black teeth pointing forwards, the femoral joint bearing besides a second tooth on its inner edge.

Prelabial region (fig. 3) with the usual strongly chitinised acute angles, without any distinct paler area, but with a transverse row of 4+4 short setæ. Lateral labial plates with their posterior internal angles overlapping each other, the oblique median ridge thick and dark, and the posterior edge thin (not darkened) and destitute of setæ or lashes. Median plate oval, partly concealed behind, exposed towards the front. Mandibula with a dentate (6 to 7-toothed) lamella and seven pectinate lamellæ; the latter formed of 14, 14, 14, 14, 11, 10, and 4 distinct, blunt teeth. Coxosternum of first maxillæ divided, with but few setæ. Coxosternum of second maxillæ

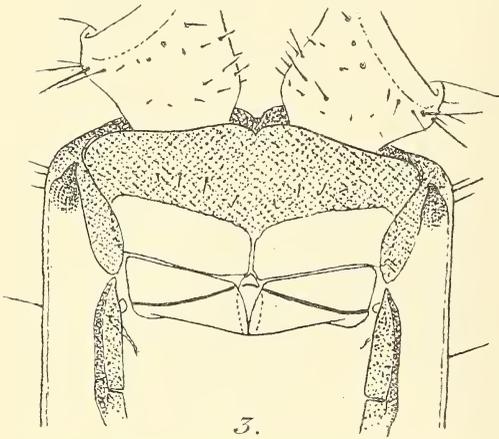


FIG. 3.—*Mecistocephalus Evansi*, n. sp.
Labral region, ventral view.

(fig. 2) long and narrow, deeply excised anteriorly; the structureless lateral (coxal) regions comparatively less developed than the posterior (sternal) reticulate region. Setæ more numerous in the middle; two strong setæ along the inner margin of each of the coxal regions.

First tergite with two wide and shallow, longitudinal impressions; the other tergites smooth, bisulcate from the second to the penultimate. Sternites smooth; the 2nd to about the 23rd with a distinct median sulcus, the anterior end of which is

divided into two short branches, which diverge more and more so as to soon become almost horizontal, the median sulcus growing longer. Towards the 24th segment the sulcus is rather suddenly replaced by a longitudinal shallow impression, more or less distinct on the following sternites. Last segment elongate. Last tergite not quite twice as long as the penultimate, parallel-sided, rounded backwards. Sternite short, not exceeding half the length of the adjoining coxæ of the anal legs, heart-shaped, poorly strewn with short setæ. Coxæ of anal legs long and slender; pores moderately numerous (some 25 to be seen on the ventral surface and perhaps 10 more dorsally). Anal legs wanting.

Male unknown. A single female collected at "Amara on Tigris, — vii. 1918."

The only two species of *Mecistocephalus* known to possess 51 pairs of legs are *M. lifuensis*, Pocock, from the Loyalties (Lifu), and *M. erythroceps*,

Chamberlin, from the Fijis. Little is known of the former, and its geographical distribution makes it highly improbable that the Amara specimens should belong to the same species. It is certainly distinct from Chamberlin's *erythroceps* on account of the shape of its head plate and of its last tergite. The author does not state how many of the sternites bear the median sulcus, which might lead one to believe that it is to be found on most, if not all, of them.

Pachymerium ferrugineum caucasicum, Attems, 1903.

7 specimens from "Amara on Tigris, under clods in garden," as under:—
 ♀ with 57 pairs of legs, ♀ with 55 pairs, ♂ with 53 pairs (13 ii. 1918);
 two ♂s with 53 pairs (17 ii. 1918); ♀ with 55 pairs, ♂ with 53 pairs
 (19 ii. 1918).

One specimen, ♀ with 55 pairs of legs, from Amara (2 iii. 1918).

The difference between *P. ferrugineum* and *P. caucasicum* lies essentially in the presence or absence of longitudinal sulci on the first tergite and in the more or less produced maxillipedes. A dissection of an Amara specimen with 55 pairs of legs, destitute of such sulci, showed that the mouth parts are alike in both forms. Considering, besides, that the sulci of the first tergite may occur or be wanting on specimens collected the same day on the same spot, it has seemed advisable to reduce Attems' species to the rank of a subspecies of *ferrugineum*.

Scolopendra valida, Lucas, 1839.

A specimen from "Jebel Hamrin, near Ruz, N.E. of Baghdad, under rock, — xi. 1918."

Trachycormocephalus mirabilis (Porat, 1876).

"Amara on Tigris, in courtyard"; an immature specimen (19 xii. 1917); and a full-grown specimen (— ii. 1918).

"Amara on Tigris, under clod"; three immature specimens (14 ii. 1918).

Adult specimens show no sulci on the first tergite; but immature ones may occasionally bear traces of it. Such sulci, when found on adult Scolopendrids, might therefore be held as persisting larval structures.

Cryptops, sp. ?.

Two specimens, the legs of which are partly missing, could not be identified.

"Amara on Tigris, under clods in garden" (17 ii. 1918 and 2 iii. 1918).

Lithobius jugorum, Attems, 1904; subspecies *tigriaccola*, n. subsp.

Adult ♂ : length, 17.50 mm.; breadth of 9th tergite, 2.30 mm.

Immat. ♀ : length, 14 mm.; breadth of 9th tergite, 2 mm.

Colour greyish. Body feebly tapering forwards, more conspicuously from the 9th segment backwards.

Angles of 9th, 11th, and 13th tergites not produced. Antennæ 19 to 20 jointed. Coxal teeth of maxillipedes 3+3, the outer pair being smaller and more slender than the others. Ocelli few, 4 to 6; very irregular, some being fused in a large dark patch and more or less distinct. Coxal pores 4, 3, 3, 3, small, circular.

Tarsi 1-5 two-jointed. Spinal armature of legs of adult male as shown below :—

1st pair	o . o . a p. a . a . o . o . mp. amp. am . (♀ : o . o . a p. a p. . o . o . mp. amp. am .)
2nd to 7th pair	o . o . a p. a p. a p. o . o . mp. amp. am .
8th and 9th pairs.	o . o . amp. a p. a p. o . o . mp. amp. am . (♀ with m above on the 2nd joint of 9th pair.)
10th to 12th pair	o . o . amp. a p. a p. o . m . amp. amp. am .
13th pair	o . o . amp. p. p. o . m . amp. am . am .
14th pair	m . o . amp. p. p. o . m . amp. am . . (♀ : m . o . amp. p. p. o . m . amp. am . a .)
15th pair	m . o . amp. p. . o . m . amp. am . a . (Imm. ♂ : m . o . amp. . o . m . amp. am . a .)

15th pair without coxolateral spine and with a single claw.

Fourth joint of 14th and 15th pair of male with a dorsal sulcus.

Almost all the particulars agree with Attems' description of *L. jugorum* from Przewalsk: even the peculiar structure of the eyes is similar. Yet the Amara form is smaller and the spine armature is different (said to be 1, 3, 3, 1 below in the type). The latter detail is in accordance with the presence in *tigriaccola* of the sulci of the hind legs, which, considering the

close resemblance of the two forms, has evidently to be traced back to environmental influences.

“Amara on Tigris, under clod, 13 ii. 1918” (an immature male).

“Garden above Amara, 3 iii. 1918” (an adult male and an immature female).

Lithobius fossipes, n. sp.

Adult male: length, 17 mm.: breadth of the 3rd tergite, 1.40 mm.; of the 8th, 1.60 mm.

Body indistinctly tapering anteriorly. Cephalic plate as long as broad, rounded, with a thick posterior marginal pad. Tergites and sternites smooth and shining; marginal pads of tergites thin, interrupted dorsally on the 5th and the 7th; posterior margin of the large tergites scarcely emarginate, that of the 14th perfectly straight: angles of the 9th, 11th, and 13th tergites not produced.

Antennæ 31-jointed; long setæ are scarce on the ten basal joints and more numerous on the following ones. Ocelli few, disposed in three rows: 1+4, 3, 2. Coxosternal plate of maxillipedes (fig. 4) produced and armed with 2+3 moderate and acute teeth; no lateral spines. Coxal pores small, 3, 4, 4, 3.

Tarsi of all legs two-jointed. Spinal armature of legs thus:—

1st pair	o . o . mp. a p. a .
	o . o . p. amp. .
2nd and 3rd pairs	o . o . mp. a p. a .
	o . o . p. amp. am .
4th to 8th pair	o . o . mp. a p. a p.
	o . o . mp. amp. am .
9th to 11th pair	o . o . amp. a p. a p.
	o . o . mp. amp. am .
12th pair	o . o . amp. p. p.
	o . o . mp. amp. am .
13th and 14th pairs	m . o . amp. p. p.
	o . m . amp. amp. am .
15th pair	m . o . amp. . .
	o . m . amp. amp. .

A coxolateral spine on the 15th pair; none on the 14th. Claw of the anal pair double. The fourth joint of the telopodite of anal legs (fig. 5) is by no means swollen, but its dorsal surface is depressed and furrowed; the corresponding joint of the 14th pair is scarcely flattened and without any special structure.

A single specimen from : "Amara, under clod, 13 ii. 1918."

This species belongs to a group of rather small Lithobiids having none of the tergites produced, and provided with more than four coxosternal teeth and 25 or more antennal joints. The group includes: *devertens* and *magnus*, Trotzina, 1893; *magnus-pleodontus*, Attems, 1904; and *Vinciguerræ*, Silvestri, 1895 (besides the two Italian species, *infossus*, Silv., and *cryptobius*, Silv., which are out of the question).

The male of *magnus-pleodontus* bears a cup-shaped tubercle on the anal legs, and *Vinciguerræ* is a large species (26 mm.), the anal legs of which lack a coxolateral spine; both have thus to be discarded. Trotzina's *devertens* and *magnus*, both from the Alai Mountains, seem to approach very near to each other and to the new species, but neither of them is said to show sulci on the anal legs, and there is no evidence that they should stand as subspecies of the same form.

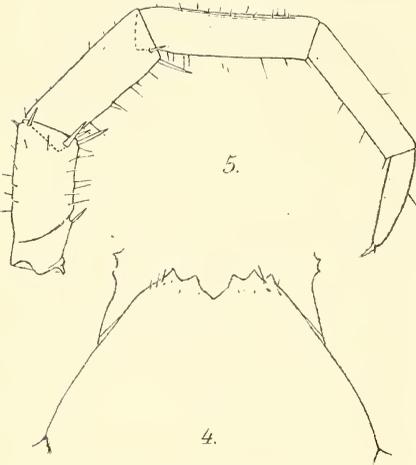


FIG. 4.—*Lithobius fossipes*, n. sp. Anterior margin of maxillipedes.

FIG. 5.—*Lithobius fossipes*, n. sp. Telopodite of right 15th (anal) leg.

Lithobius aeruginosus spinosior,
n. subsp.

This is the species already described in a previous paper (Myriapods collected by Mr P. A. Buxton, *Proc. Bombay Nat. Hist. Soc.*, in the press), under the name *Lithobius ? aeruginosus*, L.

Koch, 1862. Considering the constancy of the characters recorded, it has been found advisable to give it a definite name.

Two female specimens from : "Amara, under clods, 13 ii. 1918."

Lithobiids, ? sp. sp.

Two larvæ and a legless female from : "Jebel Hamrin, near Ruz, N.E. of Baghdad, — xi. 1918."

Thereuonema turkeстана, Verhoeff, 1905; subspecies *hamrinensis*,
n. subsp.

Length, 17 mm. Colour, ochraceous yellow, with scarcely marked dorsal bands.

Frontal region rounded, strongly convex. Dorsal region flattened,

bearing three faint sulci converging anteriorly and crossed before the middle by a short arched sulcus; and six very shallow dimples, one inside the inner angle of each eye, and two pairs (one behind the other) on the posterior half of the shield. First antennal shaft, 55 to 56 jointed; second shaft, 142 to 180 jointed; third shaft, 155 to at least 228 jointed.

Tergites, with coupled spines,* scattered on the surface and along the edges. Spiracles short, not reaching one-tenth of the breadth of the corresponding dorsal sclerite (fifth segment).

Prefemoral spurs of legs 1 to 14:	2 dorsal, 1 ventral.
Femoral	1 to 14: 3 ,, 0 ,,
Tibial	1 to 14: 1 ,, 2 ,,

Legs jointed as under:—

1st pair :	tarsus, 14 jointed; metatarsus, 30 jointed.
2nd ,,	13 ,, ,, 29 ,,
3rd ,,	10 ,, ,, (broken off after 20th joint).
4th ,,	9 ,, ,, (entirely missing).
5th ,,	9 ,, ,, (broken off after 5th joint).
6th ,,	(broken off after the 6th joint).
7th ,,	7 jointed; metatarsus, 25 jointed.
8th ,,	8 ,, ,, 26 ,,
9th ,,	8 ,, ,, 26 ,,
12th ,,	9 ,, ,, 26 ,,
13th ,,	9 ,, ,, 28 ,,
15th ,,	no distinction between tarsus and metatarsus, which is broken off after the 316th joint.

Tarsal spines beginning on the fifth pair of legs, where only one spine is found on the two basal joints. Spines of basal joint increasing backwards from 4 on pair 6 to 24 on pair 13; 7 on pair 15.

Metatarsal pegs (whenever present) very small on all legs, except pair 1 and pair 2, and, to a certain extent, on pair 3. Large pegs were observed on:—

Pair 1, on metatarsal joints	12, 14, 16, 18, 20, 23, 24, 25.
,, 2, ,,	9 to 16, 18, 20, 22, 24.
,, 3, ,,	17 and 19.

Coupled spines to be found on the anterior row of prefemur of the four anterior pairs of legs, *i.e.* 16 (+ 3 apical) on pair 1; 11 (+ 2 apical) on pair 2; 6 (+ 2 apical) on pair 3; 4 (+ 3 apical) on pair 4. Coupled spines

* By "coupled spines" are meant those spines which are found associated with setæ generally twice (or more) their own length.

reappear on the anterior row of pair 8, the number increasing backwards (from 1 to as many as 10).

Seven specimens from: "Jebel Hamrin, near Ruz, N.E. of Baghdad, — xi. 1918."

The differences from the Bukhara original type lie in the metatarsal pegs, the number and, at least, the size of which are considerably reduced, as well as in the coupled spines of the prefemora, which seem to be wanting on pair 4 to pair 14 of the type.

DIPLOPODA.

Calyptophyllum, n. gen.

Gonapods deeply split. Mesomerite much larger than the solenomerite, and fused with its tracheal stalk. Solenomerite with a distinct coxal base, strongly arched forwards, and bearing a slender process near its summit. The presence of a seminal groove is still uncertain. Both the solenomerite and the mesomerite are adorned with veil-like lamellæ.

Peltogonapods proportionally small, without distinct sternal plate, but with a very short flagellum and with short, simple tracheal stalks.

(Body robust; prozonites without circular striæ; metazonites sulcate; pores opening in the suture; last segment angularly produced. Mandibular stipe of male with process. Legs padded in male, and with a supplementary coxal ring in both sexes.)

Type: *Calyptophyllum obvolvatum*, n. sp.

The species hereafter described undeniably bear a close resemblance to the *Pæctophyllidæ*. In *Pæctophyllum Escherichi*, Verh., we find the mesomerite to be very large, while the solenomerite is borne on a coxal base; it appears to be very much the same in *Catamicrophyllum (hamuligerum)*, for instance, the solenomerite of which is said, moreover, to possess a delicate hyaline anterior lamella. It may be added that three out of the four members of the *Pæctophyllidæ* family originate from Western Asia (Anatolian steppe, Jaffa, and Caifa).

But the main character on which the family has been erected lies in the structure of the mesomerite which is not fused with the corresponding tracheal stalk, with which it articulates freely. Therefore, should this character be relied upon, as is done by Verhoeff, *Calyptophyllum* would have to be split off from the bulk of its nearest relatives, and would have to be placed amongst the *Mastigoniulince* of the *Iulidæ* family, with which it has no particular affinities.

This shows that the value of the relation of the mesomerite to the

tracheal stalk has been overestimated, and that one or other of the two structures (stalk fused or free) has to be looked upon as a peculiar case, evidently connected with local environmental conditions. It is consequently proposed to give these structures a secondary rank amongst the family characters, which ought to stand as follows:—

Gonapods deeply split, the mesomerite being by far the larger of the two parts, the solenomerite being provided with a more or less developed coxal base and an apical slender process, and being often “trimmed” with hyaline lamellæ. In four cases out of five the mesomerite articulates freely with its tracheal stalk, while in the remaining case these organs are fused.

Peltogonapods more or less developed, with or without a flagellum; when present, the latter is often short.

The family remains divided into two sub-families, according to the existence (*Pæctophyllinæ*) or the absence (*Catamicrophyllinæ*) of a peltogonopodial flagellum, *Calyptophyllum* taking its place in the former.

In the *Pæctophyllinæ* two genera are already inscribed, *i.e.* *Pæctophyllum* and *Macheiroiulus*; both may be known by the peculiar size and shape of the mesomerite and by the absence of the lamellar expansions of the solenomerite.

Calyptophyllum obvolvatum, n. sp.

♂ : Length about 42 mm.; diameter, 1.95 mm.; 70 segments; 127 pairs of legs; 3 segments apodous. Adult.

Colour uncertain, the only specimen being dry, in which state it is black, with pale brownish limbs.

Head short and broad, smooth, with occipital and transverse sulci particularly thin, though corresponding to well-defined internal apodemas. The transverse sulcus is accompanied by only one (right) dimple, the seta of which is rubbed off; it is not possible to decide whether the animal possesses occipital setæ or not. Four labial setæ. Labial excision extremely shallow. Ocelli distinct, some 52 in number (9, 10, 9, 8, 7, 5, 4), gathered on a large, rounded, dark patch. Antennæ of moderate length, reaching the posterior margin of the third segment. No apical sensory cones (broken off by accident?).

Mandibular stipe provided with a rounded, projecting callus. Four pectinate lamellæ. Gnathochilarium (fig. 6) with a very small promentum, not exceeding one-third of the length of the lingual lamellæ. Two short setæ on each side behind the middle. None of the palpi show sensorial

pegs, these being replaced by minute papillæ. The lateral duplicature is provided with a small but well-formed condyle. Postmentum undivided.

Collum moderately convex. Lateral lobes rather short, evenly rounded,

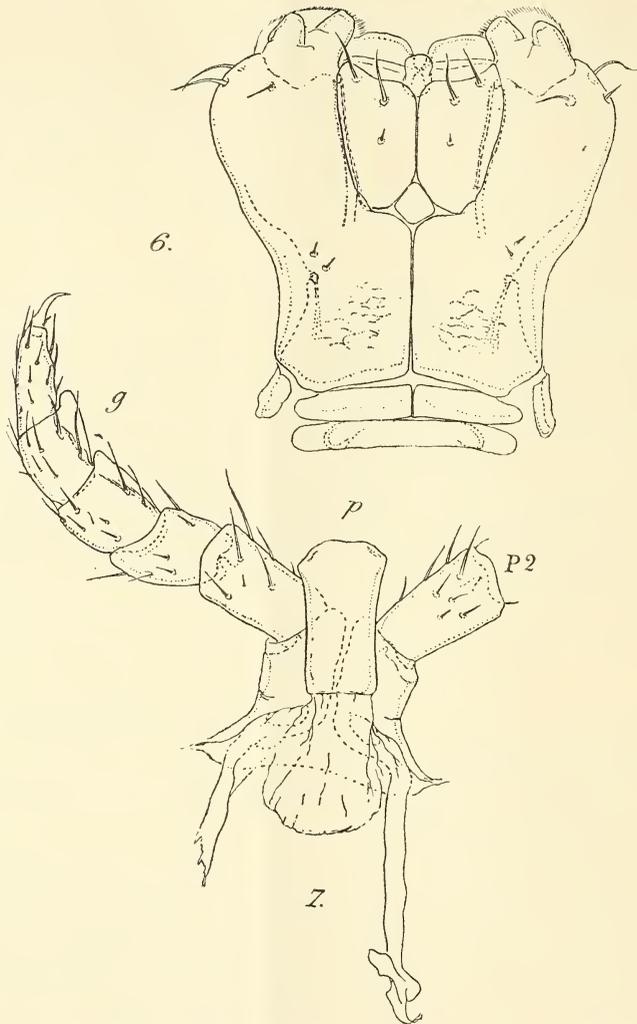


FIG. 6.—*Calyptophyllum obvolvatum*, n. sp. Gnathochilarium of male.

FIG. 7.—*Calyptophyllum obvolvatum*, n. sp. Second pair of legs of the male (*P2*) and penis (*p.*), posterior view. *g.*, Tarsal soles.

and adorned with some more or less distinct sulci along their posterior margin. Following segments dull or with a silky sheen, due to the dense striation; the prozonites bear no traces of transverse striæ, even in their concealed anterior part, but the posterior half is more or less regularly marked with numerous very delicate and long longitudinal striæ. Transverse suture shallow but obvious all round the somite. Metazonites

with strong, regular, and dense sulci running through to the posterior margin, which is neither fringed nor pilose. The pores open in the suture itself; their distance from the posterior margin covers 8 interspaces (on the 10th, 15th, 40th segment) to $9\frac{1}{2}$ (on 62nd).

Seventh segment (of male) with traces of dorso-median suture (*x.*, fig.8) and a thick ventral duplicature (*d.*).

Last segment triangularly produced into a long, straight, thick process, laterally depressed at the base, the apex of which decidedly overlaps the anal valves. No setæ were observed. Anal valves obliquely truncate, not prominent, with rounded margins and fine setæ spread all over their surface. Anal scale large, deeply excised, with a sharp angle on each side of the emargination.

The first pair of legs changed into two non-coalesced, parallel unci. Coxosternal bases evenly arched, broad, projecting laterally, bearing two-jointed limbs. Basal joint globular; apical joint gradually tapering, with rounded apex.

Sternite of second pair of legs a short and broad triangle. The tracheal stalks are wide apart. Coxæ short and thick. The two penultimate joints (as well as the corresponding joints of all the following legs) have large soles (fig. 7, *g.*). The penis (*p.*, fig. 7) is a flattened organ twice as long as broad, with slightly widening summit, in each corner of which opens a seminal duct. Coxæ of following ambulatory legs with a dorsal supplementary ring.

Copulatory organs of a very peculiar structure (figs. 9, 10, 11). Peltogonapods narrow at the base, where they are fused together; gently widened as far up as the middle, then almost parallel-sided. The summit is abruptly expanded, but its sides are folded back and its outline remains

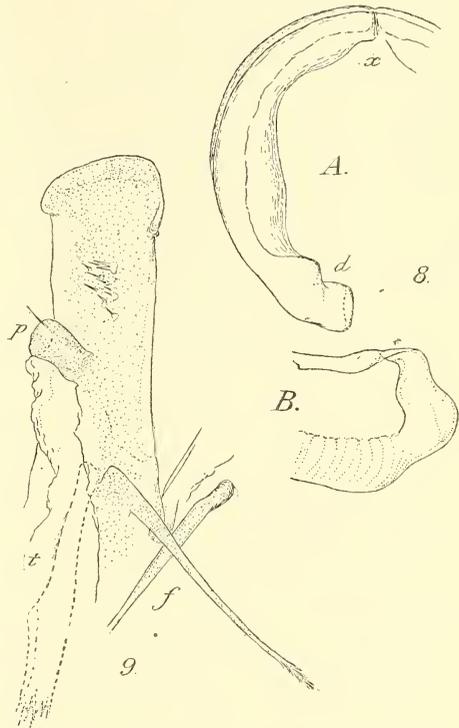


FIG. 8.—*Calyptophyllum obvolvatum*, n. sp. Seventh segment of male. *A.* Anterior section showing traces of dorso-median fusion of the two halves of the somite (*x.*) and the outline of the ventral duplicature (*d.*); and *B.* the ventral duplicature viewed from inside.

FIG. 9.—*Calyptophyllum obvolvatum*, n. sp. Right peltogonapod, posterior view. *f.*, Flagellum; *p.*, process on outer margin; *t.*, tracheal stalk.

rounded; the lateral expansions clearly appear in the side view only. Almost in the middle of the outer margin is a conspicuous process, with its rounded end pointing backwards and tipped with a seta (*p.*, fig. 9). Inwardly, on the posterior surface, are a dozen thick hairs. Below the process the

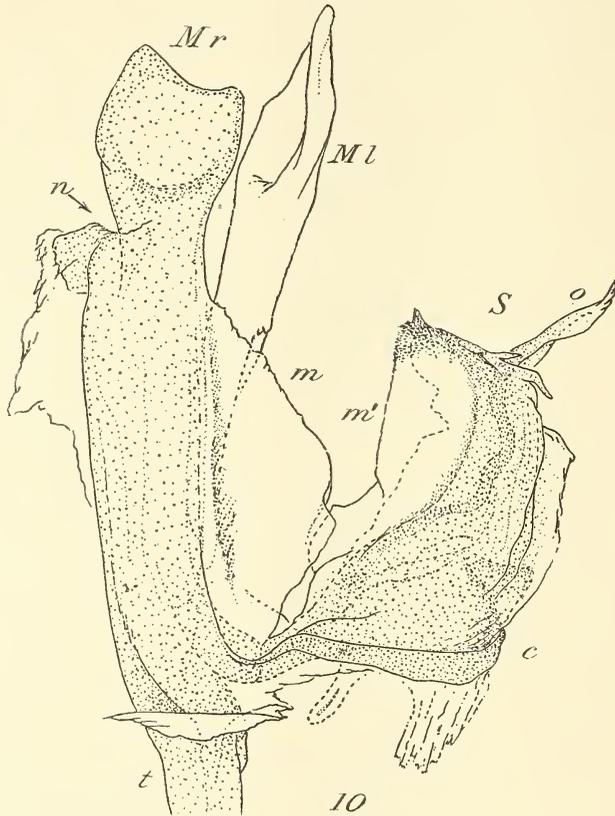


FIG. 10.—*Calyptophyllum obvolvatum*, n. sp. Right gonapod, outer profile. *Mr.*, Mesomerite; *S.*, solenomerite; *t.*, tracheal stalk; *n.*, notch; *m.*, triangular lamella of mesomerite; *m'*, veil-like lamella of solenomerite; *c.*, horse-shoe-shaped coxal base; *o.*, trumpet-like process. (In the background the apex of the left mesomerite is outlined, *ML.*)

outer margin of the peltogonapod is very poorly chitinised, and unites with membranaceous expansions of the mesomerite. From the base of the posterior surface rises a very short flagellum, angularly bent proximally so as to cross the flagellum of the symmetrical organ (*f.*); it ends in a brush of spines. The tracheal stalks (*t.*) are simple, short, and attached some distance above the base of the organs.

The two gonapods are united by hard membranes in which no trace of a sternal plate could be detected. Each gonapod is deeply split into a long

mesomerite and a short solenomerite (fig. 10). The mesomerite (*M.*) is a broad and flattened piece abruptly notched externally before the summit (*n.*); the latter is again slightly expanded, with its apical margin excised, its posterior surface showing a shallow depression. From the posterior surface of the mesomerite rises a triangular hyaline lamella directed backwards (*m.*). In its normal position the end of the mesomerite is pressed against the posterior surface of the peltogonapod, the process of which fits tightly in the notch of the outer margin of the mesomerite. The mesomerite is fused with the corresponding tracheal stalk (*t.*), no other trace of division being present but a deep depression of the outer surface of the stalk, representing the once existing stigmata.

The solenomerite (*S.*) does not exceed two-thirds of the height of the mesomerite. It sits on a horse-shoe-shaped coxal base (*c.*, fig. 10), which is open in front, low outside and behind, but broadens internally into a transverse, squared process (*c.*, fig. 11). The main body of the solenomerite (dotted in both figs. 10 and 11) is strongly chitinised, rapidly tapering towards the end, and conspicuously bent forwards. Both the internal and the external rims of this hook, from top to bottom, are expanded into parallel, delicate, and hyaline, veil-like lamellæ (*m'*), the outer of which meets the triangular lamella of the mesomerite. Noteworthy also are a chitinised dimple at the back of the hook; two backward pointing spines, one behind the other, at a short distance from the apex; and, next to these, a long trumpet-like process (*o.*) turned backwards, ending proximally in a chitinised apodema seen (fig. 10) to wind its way down in the main body of the organ. From the presence of such an apodema might be inferred the existence of a seminal groove—a point which, however, had to be left unascertained.

The female is unknown. A single male from: "Taki Girreh, N.W. Persia, under stone on hillside (limestone), 14 i. 1919."

Calyptophyllum integrum, n. sp.

♀ : length unknown; diameter, 3 mm.; number of segments uncertain, the only specimen being broken (59 segments are preserved); 3 segments apodous. Adult.

All particulars as in *C. obvolvatum*, except that the size is decidedly larger, that the sculpturing of the segments is somewhat less deep, and that the distance from the pore to the posterior margin covers $9\frac{1}{2}$ to $10\frac{1}{2}$ of the sulcal interspaces. Furthermore, the anal scale has its posterior margin completely rounded.

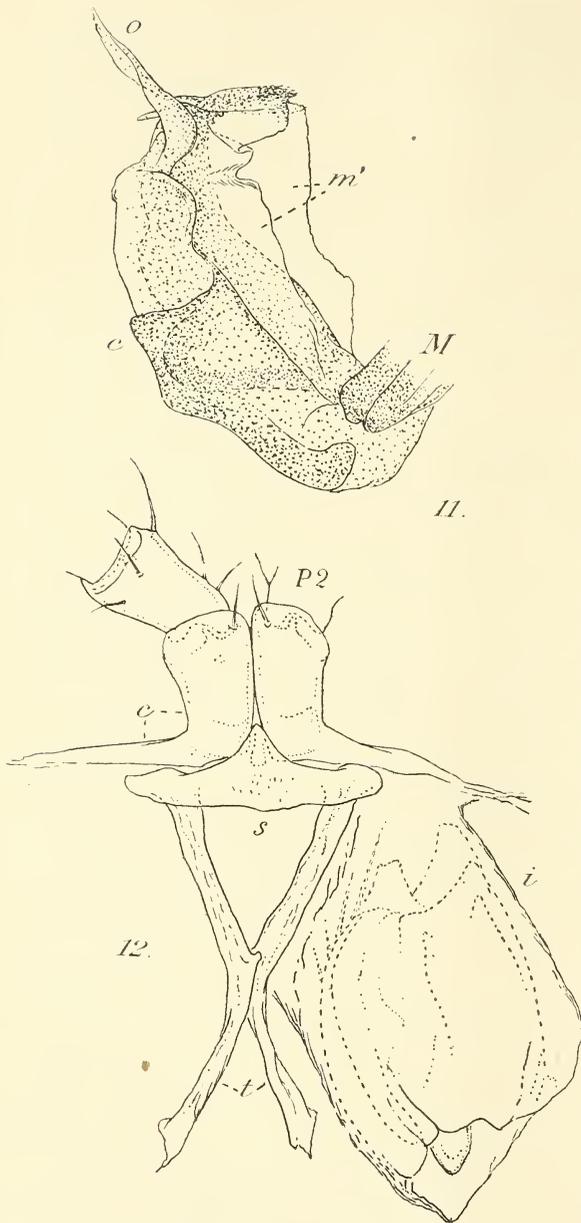


FIG. 11.—*Calyptophyllum obvolvatum*, n. sp. Solenomerite of the right gonapod from the inside. (Same lettering as in fig. 10.)

FIG. 12.—*Calyptophyllum integrum*, n. sp. Base of the 2nd pair of legs (P2), with the right vulvar invagination (*i.*), anterior view. *t.*, Tracheal stalks; *s.*, sternite; *c.*, coxal joint.

Sternite of second pair of legs very much as in the male *obvolvatum*, but not as broad (*s.*, fig. 12); the tracheal stalks (*t.*) do not stand so wide apart, they are long, bent in usual way, with an inconspicuous internal uncus about the middle. Coxæ (*c.*) more slender proximally, thus appearing widened apically.

Vulvar invaginations (*i.*, fig. 12) distant from each other, not deep enough to conceal entirely the vulvæ. Vulvæ strongly chitinised, at least twice as high as long. When at rest, the operculum is turned forwards, the mound facing backwards. The operculum is a subquadrate piece, by one-fourth lower than the mound, with the apical margin shallowly excised and with the outer angle more produced than the inner (fig. 13). The surface shows two parallel, thickened ridges bearing macrochaetæ. When the vulva is closed, the anterior margin of the valves of the mound are pressed tight against these ridges, concealing the lateral margins of the operculum.

The mound (fig. 14) is asymmetrical; the outer half is more developed than the inner half. Valves narrow, not fused with the posterior median plate, tipped with hyaline rounded

lobes slightly bent backwards, each of them bearing a dozen setæ. Posterior-median plate deeply split; the notch is bordered posteriorly by a chitinous horse-shoe-shaped thickening; anteriorly its margins are expanded into lobes (*l.*), the inner of which is much longer than the outer. These lobes are folded in so as to meet in the middle, thus

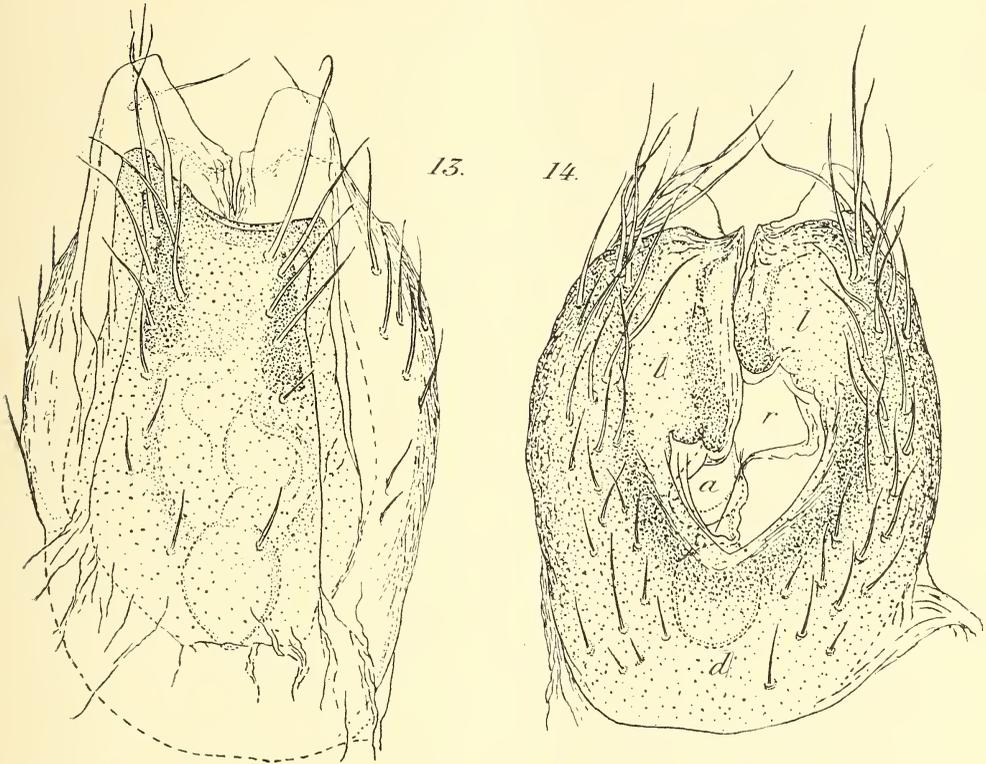


FIG. 13.—*Calyptophyllum integrum*, n. sp. Left vulva, anterior view. (Dotted parts belong to the operculum and to organs of the mound seen through its surface. The more dense the dots the thicker and the darker the chitinous layers.)

FIG. 14.—*Calyptophyllum integrum*, n. sp. Mound of the left vulva, from behind. *l.*, Lobes of the posterior median plate; *r.*, ridge; *a.*, apodema; *d.*, diverticulum. (The horse-shoe-shaped thickening is densely dotted.)

leaving between them a linear slit, which abruptly widens backwards (ridge = *r.*). On both sides the setæ are numerous, thick, and long. The apodema (*a.*) is scarcely as long as the notch, wide at its posterior end, from which rises a single, short, but enormously swollen diverticulum (*d.*) of somewhat irregular shape.

Male unknown. A single female from: "Harunabad, N.W. Persia, under stone, 22 i. 1919."

This specimen could have been identified as the female of *C. obvolvatum* had it not been for the structure of the anal scale. It is difficult to decide

whether this is merely a sexual difference, no similar case being known amongst Iulids. It has, therefore, been deemed justifiable to consider them as specifically distinct.

Mesoiulus Evansi, n. sp.

♂: Length 14 mm.; diameter 0·87 mm.; 44 segments, of which 2 apodous; 77 pairs of legs. Adult.

♂: Length 13 mm.; diameter 0·84 mm.; 44 segments, of which 2 apodous; 77 pairs of legs. Adult.

♂: Length 12 mm.; diameter 0·81 mm.; 41 segments, of which 3 apodous; 69 pairs of legs. Adult.

♀: Length 16·50 mm.; diameter 0·97 mm.; 47 segments, of which 2 apodous; 85 pairs of legs. Adult.

♀: Length 15·50 mm.; diameter 1 mm.; 46 segments, of which 2 apodous; 83 pairs of legs. Adult.

♀: Length ? ; diameter ? ; 44 segments, of which 2 apodous; 79 pairs of legs. Adult.

♀: Length 13 mm.; diameter 0·95 mm.; 43 segments, of which 3 apodous; 75 pairs of legs. Adult.

Colour ochraceous, with dark patches around the repugnatorial pores. In some specimens the ventral region is fulvous. A dark band between the eyes. Legs pale.

♂: Head smooth. Four supralabial setæ. Two occipital setæ. No occipital sulcus. Antennæ short. Eyes composed of somewhat flattened and not very distinct ocelli numbering 25 to 30. Mandibular stipe, though destitute of any characterised lobe, is nevertheless larger than in the female, and shows a subquadrangular outline. Gnathochilarium with groups of few short setæ behind the middle.

Collum rather short, with triangular lobes, the anterior margins of which are broadly excised. No longitudinal sulci on the surface. Prozonites without special structure. Suture thin, yet well marked all around the somite. Longitudinal sulci of metazonites fairly regular, not reaching the posterior margin, wide apart. Pores opening behind the suture on the 6th and 7th segments, gradually more remote backwards so as to stand far behind the suture (on about the first third of the metazonite); their distance from the posterior margin covers $3\frac{3}{4}$ to 4 intersulcal spaces. Posterior margin without setæ, but with an inconspicuously striate, narrow rim. Last segment rounded; the posterior half of its surface bears fairly numerous and long setæ; so do likewise the

anal valves, which are not prominent, strongly convex, and completely destitute of marginal pads.

First pair of legs uncus-like; the telopodite formed of two or three more or less distinct joints. The following pairs with a complementary coxal ring and soles below the fourth and the fifth joint.

Peltogonapods narrowed at the base, abruptly and largely widened before the middle, with rounded apex (fig. 15). The apical tubercles of the posterior surface are quadrangular, subequal in size and shape, and placed side by side on the same level. The reflexed thickenings of the inner margin reach as high as the second third of the total length of the organ.

Gonapods (fig. 16) gradually tapering apically, provided with anterior and posterior basal expansions (the latter wider than the former), and divided almost from the middle of their length. The anterior division, the mesomerite (*M.*), is gradually sharpened and closely laid against the posterior division, or solenomerite (*S.*). The latter is a concave lamella (fig. 17), in which a groove is seen to wind its way down almost to the bottom of the gonapod. Apically the groove widens into a short funnel, the anterior margin of which thins out into a short, sharp spine.

♀: Mandibular stipe narrowed anteriorly. Collum somewhat longer than in the male. Second segment as large as the following segments, much larger than the collum, so that it appears swollen ventrally; its pleural lobes end in sharp angles.

Sternite of second pair of legs short, raised between the insertions of the tracheal stalks (*s.*, fig. 18). Tracheal stalks (*t.*) not branched, slender. Basal expansions of coxæ of second pair very low and poorly developed; the coxa itself moderately widened distally and without complementary ring.

Vulvar invaginations (*i.*) distant from each other, being just deep enough to conceal the vulvæ. When at rest, the vulvæ are disposed so as to have the mound facing backwards and dorsally; consequently the aperture of the invagination is closed by the distal part of the operculum. The latter (*o.*, fig. 19) is much higher than the mound; the median part of its upper margin is bent backwards, partly overlapping the ridge; therefore the summit of the operculum appears excised, with acutely produced angles. On a level with the outer angle are several (6 or 7) macrochaetæ.

The mound (*M.*, fig. 19) is poorly chitinised, rounded backwards, strongly asymmetrical; the outer valve is much lower than the inner; both are tipped with inconspicuous lamellæ and strewn with macrochaetæ along

their upper margin. The posterior median plate seems fused with the valves; its upper notch is deep, but not surrounded by a horse-shoe-shaped

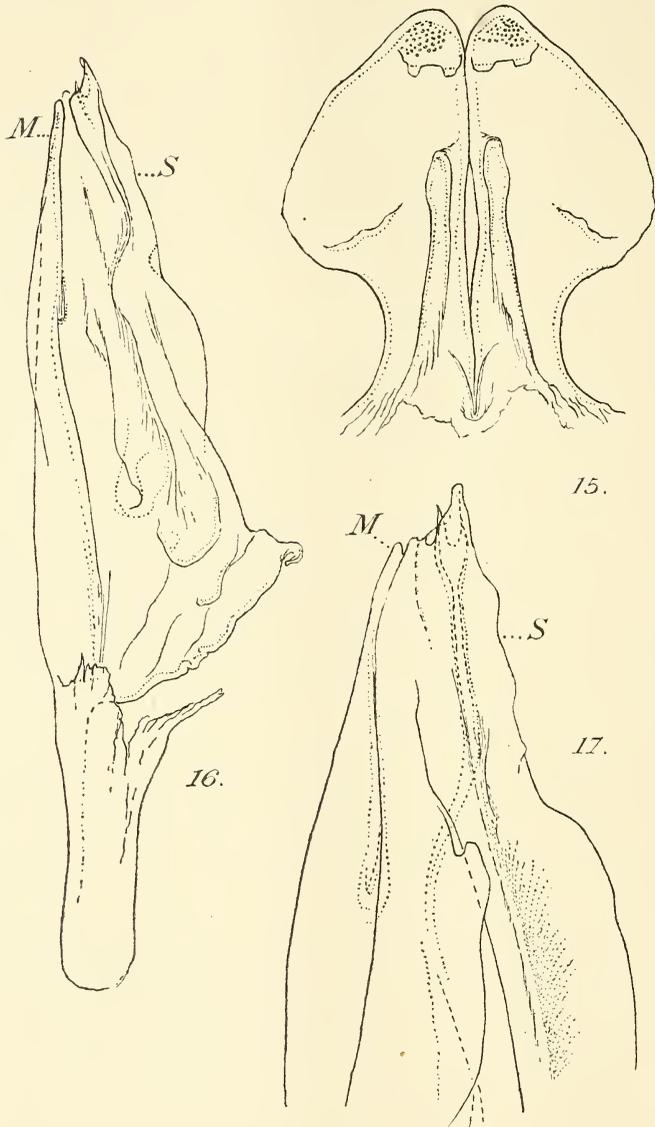


FIG. 15.—*Mesoiulus Evansi*, n. sp. Peltogonapods, posterior view.
 FIG. 16.—*Mesoiulus Evansi*, n. sp. Right gonapod, outer profile.
 FIG. 17.—*Mesoiulus Evansi*, n. sp. Apical half of left gonapod,
 inner profile.

thickening. Ridge (*r.*) triangular, short, wide, depressed. The gutter-like apodema is deep; it ends in a blind diverticulum (cul-de-sac=*a.*), from the posterior surface of which rises a second remarkable appendage. This

is divided into a moderately slender stalk and a much swollen, pear-shaped end (*d.*); it is normally directed outwards at a right angle with and perpendicularly to the main axis of the first diverticulum.

Three males and four females from: "Amara, under clods, 17 ii. 1918."

Had Verhoeff's system been strictly followed, this new species would have fallen under *Micropachyiulus*, owing to the absence of a mandibular process. But it agrees exactly with none of the subgenera proposed by the said author. On the other hand it does not seem justifiable, as was pointed out by Attems in 1902, to extend the generic name *Micropachyiulus* to any other species but the type, *i.e.* *M. paucioculatus*, Verh., as this is the only one the gonapods of which are undivided; all the other forms have a distinct mesomerite, as seen in *M. Evansi*. A new generic name has therefore to be adopted for the bulk of the other *Micropachyiuli*, and Berlese's *Mesoiulus* had to be chosen as being the oldest known (1886).

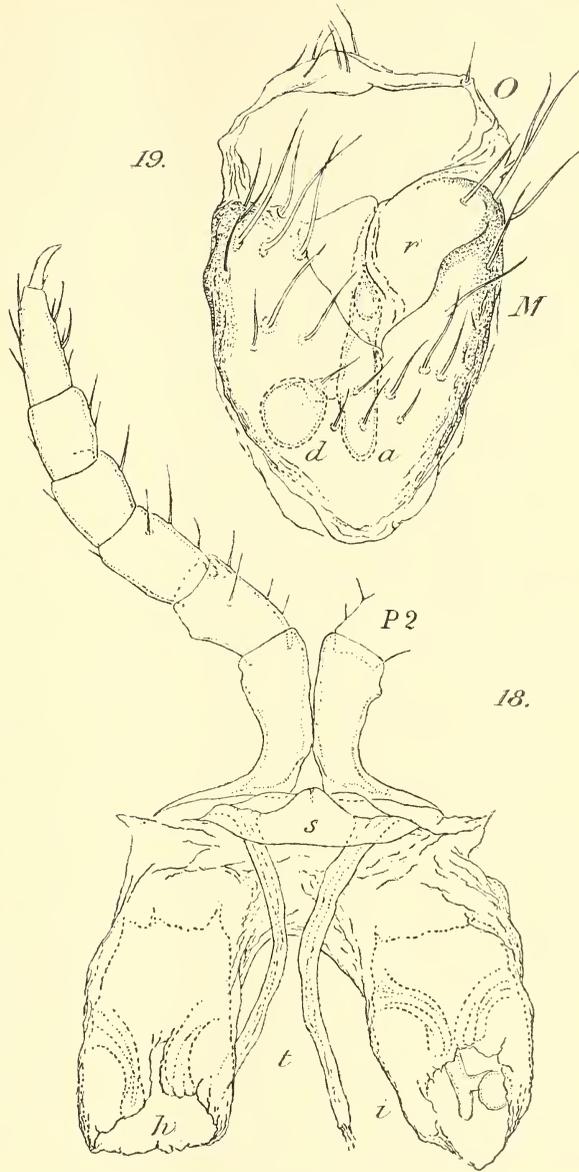


FIG. 18.—*Mesoiulus Evansi*, n. sp. Second pair of legs (*P2*) of female and vulvar invaginations (*i.*) in their normal position, anterior view. *s.*, Sternite; *t.*, tracheal stalks; *h.*, hinges of the vulvæ; *d.*, diverticulum of same.

FIG. 19.—*Mesoiulus Evansi*, n. sp. Right vulva seen from above. *o.*, Operculum; *m.*, mound; *r.*, ridge; *a.*, apodema; *d.*, diverticulum.

Of the species so far recorded having similarly built gonapods the nearest related is *M. aphroditæ*, Attems, from Cyprus. Yet it is said to have a short but well-marked occipital sulcus, few sulci on the lateral lobe of the collum, and a short, broad, and rounded callus at the mandibular stipe in the male, all structures absent in the Mesopotamian species. Furthermore, the apex of the gonapods seems to be different, as it is described (and figured) in *aphroditæ* as a plain, folded lamella, tapering and rounded apically. Moreover, the peltogonapods are more narrow and sharpened in Attems' species, and the two posterior tubercles are very different from each other in size and shape.

Pachyiulus (Dolichoïulus) creticus, Verh., likewise resembles *Evansi*; it is readily distinguished from the latter by the want of occipital setæ and by the more simple structure of the apex of the gonapods.

From other congeneric species of the same eastern region, *M. Evansi* may be known as follows:—From *M. sporadensis*, Verh., in having occipital setæ and a simple (not ramose) mesomerite; from *M. turcicus*, Verh., in having distinct ocelli and the anal segment rounded; from *M. domesticus*, Att., in having no prominent mandibular callus; from *M. cedrophilus*, Att., in having the pores far away from the suture; from *M. obscurus*, Att., in having convex anal valves and tarsal soles; from *M. polyzonus*, Att., *M. sinaimontis*, Verh., and *M. Festa*, Silv., by a much lower number of body segments (41–47, as against 79–81, 63, and 74); etc

Iuloid, sp. ?.

A single female from: "Harunabad, N.W. Persia, 22 i. 1919."

Strongylosoma persicum, Humbert & Saussure, 1869.

Eight specimens (♂ and ♀) from: "Amara, under clods, 13 ii. 1918."
Four immature female specimens from: "Amara, under clods on roadside, 17 ii. 1918." One male from: "Kizil Robot, N.E. of Baghdad, — xii. 1918."

VII.—A New Method of investigating Colour Blindness, with a Description of Twenty-three Cases. By Dr R. A. Houstoun, Lecturer on Physical Optics in the University of Glasgow.

(MS. received December 5, 1921. Read January 9, 1922.)

DURING the past four years I have been conducting surveys of the colour vision of students in the University of Glasgow. The first survey* was made by a colour-perception spectrometer very similar to Dr Edridge-Green's instrument, and embraced 79 observers. The second survey† was made by Dr Edridge-Green's bead test, and embraced 100 observers. The third survey,‡ carried out in collaboration with Miss Margaret A. Dunlop, was made by an original method, called here for short the microscope test, and embraced 1000 observers. At present there are two other surveys under progress. The object of these surveys is to find a numerical method of specifying goodness of colour vision; to see, by the application of statistical methods, whether the colour blind fall naturally into groups or are merely outliers of a homogeneous population; to find whether colour blindness is a Mendelian characteristic for men and merely an extreme case of normal variation for women; and to throw light on the subject of colour vision generally. Consequently, the normal have been investigated with as much care as the colour blind. But in the course of the four years I have made the acquaintance of many trained observers with abnormal colour vision, and have been possessed with an ever-growing desire to know exactly, irrespective of all theory, what was the matter with their colour vision. In spite of the vast literature on the subject, the tests generally have been of a very superficial nature, and unsatisfactory to the man with mathematical instincts. As these abnormal cases were beginning to leave the University, I addressed myself last spring to the problem of finding a method of testing which would describe their condition independent of theory, and, indeed, independent of words. This paper describes how the problem was solved, and gives data for twenty-three cases of colour blindness, four of normal colour vision, and one case of exceptionally good colour vision.

* *Roy. Soc. Proc.*, A, **94**, p. 576, 1918.

† *Proc. Phil. Soc. of Glasgow*, 1920.

‡ *Phil. Mag.*, **41**, p. 186, 1921.

Newton found that almost all colours could be produced by mixing red, green, and blue light. This result is not explicitly stated in his *Opticks*, although it is implied in his diagram. Two centuries later, Helmholtz and Maxwell developed Newton's treatment almost simultaneously; and in a paper read to the Royal Society of Edinburgh sixty-five years ago, Maxwell referred to it as the geographical method of exhibiting the relation of colour. According to this method, red, blue, and green are taken as the corners of an equilateral triangle, and any colour compounded of these three is represented by a point found by conceiving masses proportional to the several components of the colour placed at their respective corners, and taking the centre of gravity of the three masses. I have taken as my standard colours Wratten & Wainwright's standard tricolour gelatine films, now supplied by Kodak Limited, Wratten division.

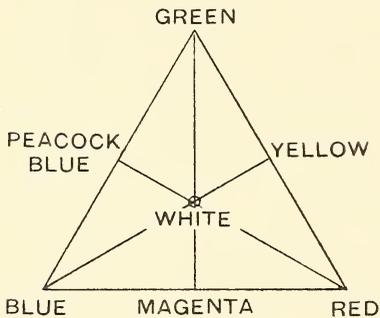


FIG. 1.

These are spectrally very pure indeed, and a long way in advance of the coloured papers and pigments hitherto used in colour-vision work. If we add red and green in equal proportions we get yellow, which is represented by the point midway between red and green (fig. 1). Similarly, red and blue in equal proportions give magenta, and blue and green in equal proportions peacock-blue. The three medians meet in white. It is obvious

from the diagram that peacock-blue, yellow, and magenta are respectively the complementaries of red, blue, and green, and that, for example, twice as much peacock-blue as red is required to make white. The colours of the spectrum lie roughly along the red-green and green-blue sides of the triangle, and the violet end lies below blue outside the triangle. Violet cannot be produced by mixture of the three primaries.

Suppose, now, we have two identical patches of colour on a screen, and gradually alter the tint of one until the difference in colour becomes visible. Then we can represent the gradual alteration of tint by a displacement on the colour diagram. If we take the first colour as a fixed point, and the displacement in different directions, the point representing the second colour traces out a closed curve surrounding it and enclosing an area all the colours within which appear alike. The number of such areas or patches it is possible to draw inside the triangle is the number of different tints which the observer is able to perceive under the conditions of the experiment in question.

Young, Helmholtz, and Maxwell made the assumption that to each of the three primary colours there corresponded a primary sensation. The fact underlying Newton's diagram, namely, that there are three primary colours, is beyond all dispute: it is accepted by all theorists, and is the basis of various systems of colour photography. The additional assumption, that there are three primary colour sensations corresponding to these primary colours, is the fundamental hypothesis of the Young-Helmholtz theory, and has been a subject of debate for years.

Let us suppose that fig. 2 represents the colour-patch system of a normal individual, that the Young-Helmholtz theory is true, and that his red sensation is suddenly completely destroyed. Then his colour-patch system should be given by fig. 3. No matter how much red is added to a colour, he cannot see it; consequently, he should be able to move in a straight line to

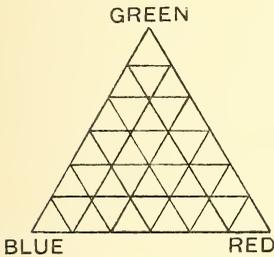


FIG. 2.

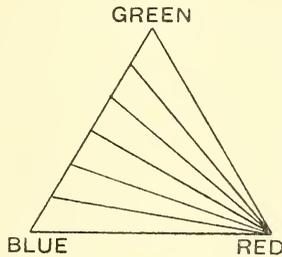


FIG. 3.

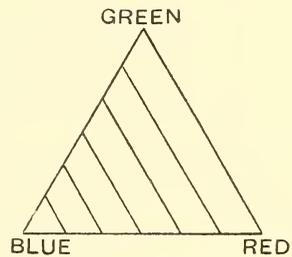


FIG. 4.

the red corner from any point on the green-blue side without crossing a boundary.

In Hering's theory, instead of primary colours there are primary processes, the red-green process and the blue-yellow process. The one acts parallel to the red-green side (fig. 1), and the other along the blue-yellow median. Suppose the red-green process completely destroyed, then the colour-patch system should be given by fig. 4.

Edridge-Green's non-elemental theory does not require that for colour blindness the number of patches should grow less in any particular manner. It merely requires that it should grow less. It is not generally known that Hering's theory and the non-elemental theory explain the facts of colour-mixing quite as well as the Young-Helmholtz theory; this has been shown in the case of Hering's theory by Helmholtz,* and in the case of the non-elemental theory by myself.†

It will be seen from the foregoing that the colour triangle, although

* *Physiologische Optik*, 2nd edition, p. 377.

† *Phil. Mag.*, 38, p. 402, 1919.

ignored for more than half a century, is nevertheless the key to the study of colour vision. I have made it the basis of my new method.

Fig. 5 is a plan of the apparatus employed. C is a collimator mounted horizontally close up to a window; G a piece of ground glass fastened by an elastic band in front of its slit S, which was a very wide one. P is a glass prism cemented to the window pane: the line of vision of the collimator was deflected by this prism 15 degrees up into the western



FIG. 5.

sky. Consequently, it easily cleared the roof of the infirmary at the other side of the recreation field. T is a telescope fitted with a ground-glass screen E instead of an eyepiece. Under ordinary conditions a rectangular white image of the slit would appear on this screen, but in front of the telescope objective was fixed a biprism B, with its refracting edge parallel to the slit, which had the effect of changing the image into two. These images of the slit were about 9 mm. high, $2\frac{1}{2}$ mm. broad, and 8 mm. apart. In front of the biprism was a brass plate with two equal rectangular

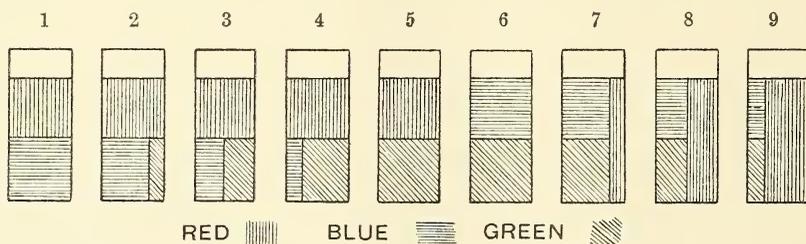


FIG. 6.

openings, one in front of each face of the biprism, and in front of these openings colour filters M and N could be screwed up and down. The positions of these filters, and consequently the proportions of the areas of the openings which they covered, could be read on scales at the side. The object glasses of the telescope and collimator were of 5 cm. aperture, and 57 cm. focal length.

Altogether nine pairs of filters were used, the two filters of each pair being alike. Fig. 6 represents the whole set of nine. Let us suppose that filter 1 is placed at N (fig. 5), with its red half covering the opening in the brass plate. Then the corresponding image at E appears a pure red. If the filter is screwed up, so that its blue half gradually comes on to the

opening, the colour of the image at E changes through pink and magenta to purple and finally blue. The colour of the image always appears uniform all over, and the amounts of the red and blue in it are proportional to the areas of the face of the biprism covered by the red and blue parts of the filter. Filter 1 accordingly enables us to travel along the red-blue side of the triangle. The lower half of filter 2 corresponds to the point A in fig. 7; hence filter 2 enables us to travel along the line red-A. Similarly, filters 3, 4, and 5 enable us to travel along the lines red-B, red-C, and the red-green side. Filter 3 should have produced white one-third of the length of the median from B; as a matter of fact, it produced it at 31 per cent. of the length from B. This is a striking tribute to the accuracy of the Wratten filters. Filters 6, 7, 8, and 9 enable us to travel along the lines blue-green, DE, FG, and HJ respectively. The filters were made specially for the investigation by Kodak Limited. They consist of coloured gelatine cemented between plates of B quality glass. The total coloured area on each filter measures 5.2 cm. by 2 cm.; above the coloured area is a clear part for fitting into the brass holder. One filter can be unscrewed and another fitted in its place in a couple of minutes.

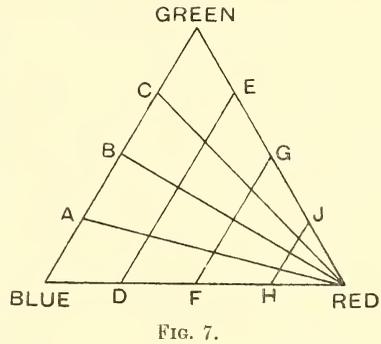


FIG. 7.

The manner in which the tests were conducted is explained best by a full description of one particular case. The observer was seated in front of the screen E in a darkened room, and the two filters 1 were fitted and adjusted, so that the two images on the screen appeared pure red. Then the tint of one of the images was gradually altered, and the observer was requested to say when the difference of tint was perceptible. As soon as he did so, this filter was stopped and the other one altered, the observer being requested to state when the tints were again the same. The proportion of area of aperture covered by the blue of the first filter was then read. It proved to be 28 per cent. This completed the first step. The tint of the first image was then altered again until the difference became perceptible, the other image brought to equality, and the area of aperture covered by the blue of the first filter again read. It proved to be 39 per cent. This completed the second step. The process was then repeated until the images were wholly blue. The ends of the steps are indicated by marks on the red-blue side of fig. 8.

The pair of filters 2 was then substituted for filters 1, the images

adjusted to pure red, and steps taken along the line red-A. Similarly, steps were taken along the lines red-B, red-C, and red-green. The marks in fig. 8 indicate in each case the ends of the steps. These marks were then joined by contour lines; the closer the contour lines, the greater the power of discriminating difference of tint.

Filters 6, 7, 8, and 9 were then fitted, and the steps made on the transverse paths, blue-green side, DE, FG, and HJ, noted. In this case the start was always made from the median, and steps taken down to blue; then the filters were set on the median again, and steps taken up to green. This procedure gave a clearer diagram than when the start was made from a side. The steps made on the transversals are indicated in fig. 9; they were afterwards joined by broken contour lines. The completed

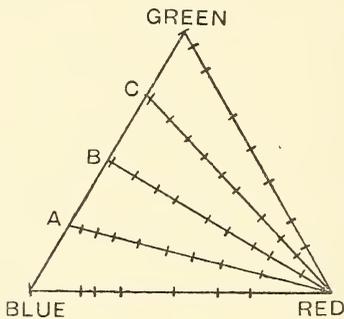


FIG. 8.

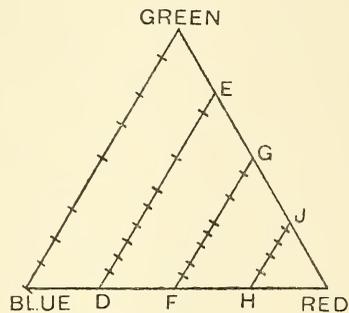


FIG. 9.

diagram of this observer is given in fig. 10. His colour vision was normal. The complete set of observations he made is given below, as reduced in the record of tests; the numbers are in each case percentages:—

Red to blue	28	39	53	70	79	83	100	
„ $\frac{1}{4}$ green	28	42	53	63	73	83	90	96
„ $\frac{1}{2}$ „	13	28	35	47	55	70	78	89
„ $\frac{3}{4}$ „	20	28	39	49	60	69	79	88
„ green	19	29	41	55	68	86	95	100
Median to blue	-20	32	41	50		to green	+14	28
„ $\frac{3}{4}$ „	9	18	27	35	42	„ $\frac{3}{4}$ „	18	43
„ $\frac{1}{2}$ „	9	19	31	41		„ $\frac{1}{2}$ „	21	41
„ $\frac{1}{4}$ „	26					„ $\frac{1}{4}$ „	21	46

A complete test takes anything from thirty to sixty-five minutes. Half the time is occupied in changing and fitting the filters. Observers with

specially good colour vision take longest, as they naturally make more steps than the others. I conducted the test myself in each case; the observer was allowed to repeat the first step, if he desired, but not to repeat any other; and every single observation is entered in the diagrams, as it was taken, without any smoothing or adjustment whatever.

The instrument gives a feeling of what might be called æsthetic satisfaction to the person using it. All the readings check themselves. After producing the least perceptible difference of tint, the observer has next to set to equality. No matter how colour blind he is, he always does this in a confident manner. From my position when turning the screws I could always see what proportion of the area of the aperture was covered by the different colours; in no single case did the observer get the filter into an unreasonable position. It was possible, in producing the difference of tint, to make it appear in the right-hand image instead of the left-hand one, or to slip in a piece of wire gauze while pretending to turn the screw, and so substitute a change of intensity for a change of tint. But the observers always detected the trick at once.

The two images were separated by a dark space. Had they bounded one another sharply, the observer would no doubt have obtained smaller steps. But this would be a disadvantage; the tests would then have taken longer, and it will be observed from the diagrams that the number of steps obtained is quite sufficient to bring out all the characteristic features. The total number of patches is not the absolute number of colours which the observer can see; it is only the number he sees under the conditions given by the apparatus.

The results of the tests are given in the diagrams on pp. 82, 83, and 84. Here follow some notes on the different diagrams:—

1. Normal. 57 patches.
2. Normal. 65 patches.
3. Normal. 98 patches.
4. Normal. My own. No. 4 of perception spectrometer test. Differs from others in two respects: I had some practice in using the apparatus before making the diagram, and I turned the screws and read the scales myself. In all other cases there was no preliminary practice, and I worked the screws and took the readings. 102 patches.
5. Exceptionally good colour vision. This observer got most patches, namely 26, in the perception spectrometer test, and, like Newton, sees indigo as a

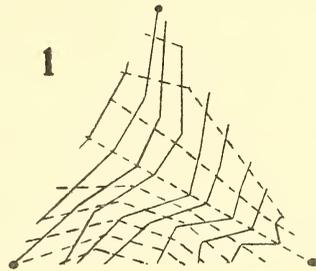
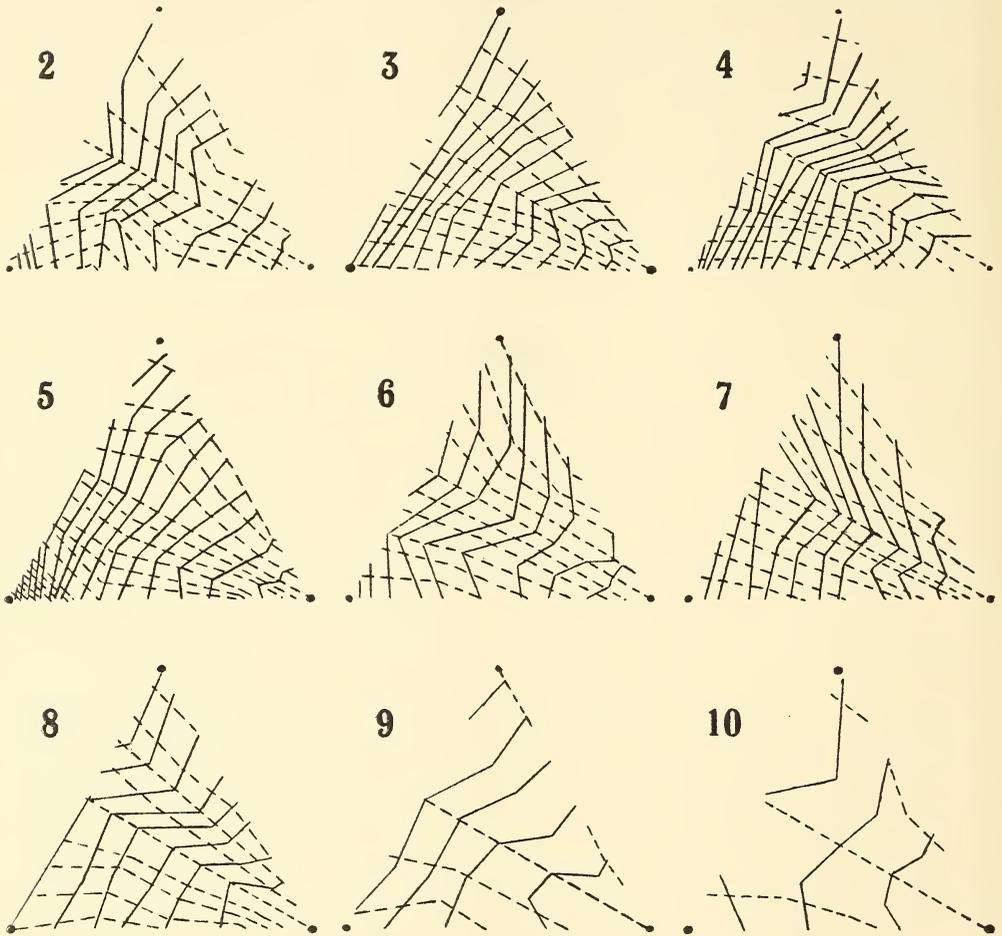


FIG. 10.

separate colour sharply divided off from blue. It will be noticed how good she is at the blue corner of the diagram. 130 patches.

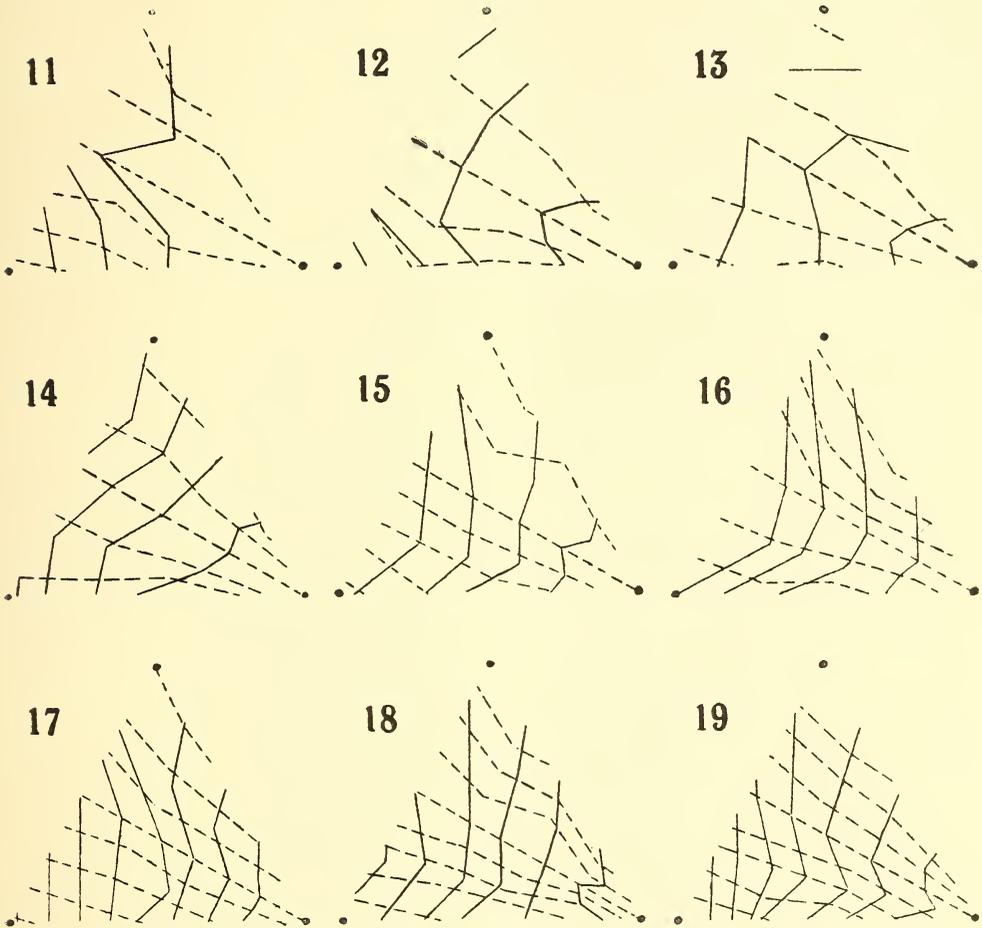
6. I had always regarded this observer as decidedly colour blind. When shown a row of books on a shelf he names their colours fairly accurately, except



2, 3, and 4 are normal; 5 has exceptionally good colour vision, and sees indigo as a separate colour; 6 and 7 are strongly "colour different"; 8 is the woman student with the worst colour vision out of 249 tested; 9 and 10 are slow, but more accurate at naming colours than 6 or 7.

the greens, which appear colourless to him, though he may risk calling them brown or yellow. When examined by the bead test, he put green into red; yellowish green and brown into yellow; pink, peacock-blue, and dark pink into green, in addition to other less important mistakes. When examined with the colour-perception spectrometer he got 15 patches, which is about normal, but he called the range 5500-5200 A.U. "yellow with a slight tint of green." There is no doubt whatever that he would fail decidedly in the Board of Trade examination.

In spite of my earlier tests, however, this observer maintained that his colour vision was good, but admitted that under certain circumstances he was unable to recognise green. The diagram shows that his view is quite a reasonable one ; he gets 64 patches, which is not far from normal. He gets

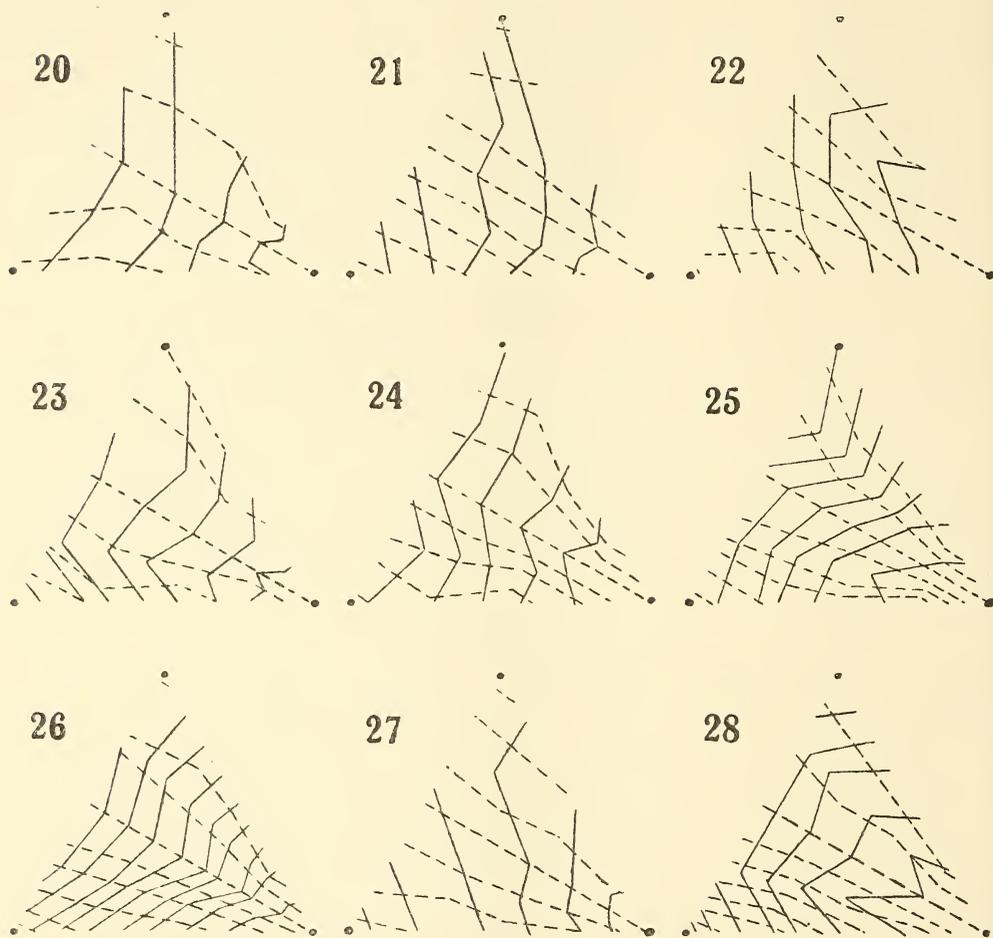


All the observers represented above make very serious errors in naming and matching colours, and would have been classified as totally red or totally green blind by the old Board of Trade test. No. 11 thinks Vandyke-brown a perfect match for vermilion and Prussian blue.

rather a low number of steps, $6\frac{1}{2}$, on the red-green side, and a relatively high number, 4, between peacock-blue and green. The point in the spectrum where the colour changes rapidly is in his case shifted from yellow into green. It is ridiculous to call an observer of this type "colour blind." "Colour different" is a more suitable name. It is owing to the colours in common use not being strictly monochromatic that he makes mistakes ; the greens in common use include wide ranges of spectrum which change slowly

in colour to the normal with wave length, but rapidly to him. Hence his difficulties.

7. This is No. 387 of the microscope test. As he obtained the high value of 12·5 for the red-green reading in that test, he was not regarded as abnormal at the time. But he found difficulty with the colours of the salts in the



The observers represented on this page, though bad enough to fail in the Board of Trade examination, are much better in naming and matching colours in common use than the observers on the previous page.

chemical laboratory, came to me for re-examination, and then, when tested with the bead test, put red in the green hole, and hesitated long with pink over the green hole before putting it into red.

When re-examined again one year later with the microscope test, he got 2·2, 6·4, and 6·2 for the red-green reading, values which are abnormally low; and, when examined on this occasion with the bead test, he put peacock-blue, dark pink, and pale pink in the green hole. When asked to name spectrum

colours, he called green "yellow," red "orange yellow," violet "blue," yellow "yellow with faint tinge of green," red "orange." The number of patches in the diagram, 72, is quite up to normal. There are, however, only four steps on the red-green side, but the very high number of five steps from peacock-blue to green. There is no doubt whatever this observer would fail in the Board of Trade examination, but he cannot be called colour blind. He is a very striking case of colour difference. The rapid change in colour sensation which the normal have in passing along the spectrum at yellow is in his case displaced into green.

8. The woman student who made the lowest mark in the microscope test. Was sure there was nothing abnormal about her colour vision, but, as diagram shows, she is below the average. Passed bead test with ease. 48 patches.
9. No. 585 in microscope test. Did not think there was anything the matter with his colour vision. Differed with a relative about colour of a cloth, but attached no importance to difference. In the bead test put peacock-blue into green, and hesitated extremely long over ultramarine before putting it into blue, but otherwise there was nothing remarkable. His case is a striking one, as the number of patches is so low, only 14; but he was quite certain the steps were not too large. It will be noticed, however, that the diagram here is not distorted in any way, but that the observer is equally weak all over; it is distortion rather than weakness that causes the observer to mistake the colours.
10. No. 38 of perception spectrometer test. The sky was very dark when the test was made—not dark enough, however, to require the use of artificial light indoors. More patches would probably have been obtained under ordinary conditions. Was slow at bead test, but made no errors. 8 patches.
11. No. 200 of the microscope test. Failed altogether to distinguish red from green by that test. Thought Vandyke-brown a perfect match for the red and blue of the Union Jack. My worst case. He made all the readings of the new test in a confident manner, and I found it impossible to confuse him into recognising changes of intensity for changes of colour. 9 patches.
12. This observer knew he was colour blind, but did not consider his case a bad one. In bead test put red into green, and pink into blue. His case brought to my notice through his being unable to do a methyl-orange titration. 11 patches.
13. Had difficulty with methyl-orange titration, and came to see how bad his colour vision was. In bead test put green and peacock-blue into red, and pink into blue. When shown spectrum colours, called orange, yellowish green, and green all "yellow," greenish blue a "bluey white." 12 patches.
14. Confused green and yellow when examined with the lantern for the R.A.F., but was passed. Came as a result of a difficulty with perimeter test in Physiology Department. In bead test put greenish yellow in green, and pale pink and dark pink in blue. 17 patches.
15. No. 554 in microscope test. Knew he was colour blind, as he had been rejected

by Board of Trade examination, and also by R.A.F. Has a brother with defective colour vision. 19 patches.

16. This observer was No. 71 of the colour-perception spectrometer test. He obtained 8 patches by that test instead of the normal 15. Called green a "dirty yellow," and had the other colour names completely mixed up. Could not distinguish the blue and green tramcars by their colour until they were directly opposite. In the four years that have elapsed since his former examination he has improved considerably in using the colour names. 22 patches.
17. No. 211 of the microscope test. In the bead test he had everything wrong except two blues. In the green hole he put nothing except pale pink. When shown spectrum colours he called green "yellow," and yellowish green "reddish yellow." Did not think there was anything wrong with his colour vision, even after the tests were finished. 31 patches.
18. No. 599 in microscope test. Knew he was "partially colour blind," but said he had no difficulty in recognising strong colours. Thought his memory for colours was bad. In bead test put green in red hole, and pink, pale pink, and dark pink in green. 32 patches.
19. No. 579 in microscope test. Put green in red hole, and red, dark pink, and pale pink in green hole in bead test. 39 patches.
20. No. 634 in microscope test. Peacock-blue in green and other slight errors in bead test. 13 patches. It is somewhat surprising that he does so well in the bead test with such a small number of patches.
21. Came as the result of a dispute about the colour of an object. In bead test put gold and greenish yellow into green. 20 patches.
22. No. 246 of the microscope test. Normal in bead test; but when shown a spectrum, called green with a trace of yellow in it "yellow." 19 patches.
23. This observer was a student at a school of navigation. His first Board of Trade examination was inconclusive; the results of his second were referred to London for decision, and he was failed. He appealed, went to London for re-examination, and was failed again. He himself was very positive that his colour vision was good.

In the bead test the very first step he took was to put peacock-blue into green. Everything else was normal. When shown spectrum colours he called orange "red," and both yellowish green and green with a slight tint of yellow "yellow," the latter two in a hesitating manner. He had the other spectrum colours right. His mean red-green reading in the microscope test was 10·9. 21 patches.

24. No. 486 in microscope test. Said he had difficulties with purples and blues. Almost normal in bead test. 28 patches.
25. This was a foreigner, who, I noticed in the laboratory, had the colour names completely wrong, although his knowledge of English was excellent. When examined by the bead test, he put dark pink in blue, left out greenish yellow, and put in the other beads in a hesitating manner. His performance under examination was not bad enough to account for his colour terminology, which doubtless was partly due to English not being his native language. 42 patches.

26. This observer thought he was colour blind, as he had a difficulty in naming shades of pink and peacock-blue. Had been rejected for the R.A.F. by wool test as partially colour blind. In the bead test he left pink and yellowish green in the box. Almost normal. 57 patches.
27. A pronounced green anomaly, requiring only $\cdot 06$ times as much red as the average. In bead test put peacock-blue, greenish yellow, gold, yellowish white, and amber in green, and pink in blue. 18 patches.
28. Bead test correct with a slight effort. When shown a spectrum, called green with a slight trace of yellow "yellow." When examined by Rayleigh test, was found to be a green anomaly, only requiring $\cdot 309$ times as much red as average. 34 patches.

The spectrum colours lie approximately along the red-green and blue-green sides of the triangle. The total number of steps on these sides should therefore correspond roughly to the number of monochromatic regions obtained by the Edridge-Green colour-perception spectrometer; only roughly, however, for the latter instrument works from end to end of the spectrum, whereas the new instrument goes from a red with some orange in it to ultramarine blue. Five of the observers whose diagrams are given in this paper were tested by the perception spectrometer four years ago. The following table gives the results by the two methods:—

No. of observer's diagram in present paper	5	4	6	1	16	10
Number of steps on two sides of triangle	30	$22\frac{1}{2}$	$16\frac{1}{2}$	16	$8\frac{1}{2}$	6
Number of steps by perception spectrometer	26	17	15	18	8	5

The agreement is as good as can be expected. The colour perception spectrometer has the advantage that it works with spectral colours; the new instrument has the advantages that the readings are self-checking, that all possible variations of colour are tested, and that the results are exhibited in their relation to the colour triangle of Helmholtz and Maxwell.

Of the 23 colour-blind observers, 14 were taken from a list of the 20 worst cases, compiled from the surveys of the past few years. The other 6 had left the University. The remaining 9, who are quite as bad as the 14, offered themselves in the course of the investigation. So that the diagrams give a fair idea of the worst cases we should meet with in a random collection of 1000 men.

It is maintained that the colour blind fall naturally into two classes. I am unable to separate the diagrams into two classes, or even to recognise two pronounced types. According to the original view of Helmholtz, described under his name in so many textbooks, these two classes should be the red-blind and the green-blind. The green usually employed was an

imaginary green, but the red was a real colour corresponding closely to the red corner of my triangle. So the diagram of a completely red-blind man should have no full contour lines. All my diagrams have full contour lines. The fundamental characteristic of the system of Helmholtz and Maxwell was that the colour sensations of the normal could be represented on a plane, whereas the sensations of the colour blind required only a line or a point. All my cases require plane diagrams. Thus it follows that the colour blind are trichromatics.

This result I was prepared for, as I have frequently inferred it on previous occasions from other experiments, though I was never able to demonstrate it directly before. But the two cases (6) and (7), whom I have named the colour different, came as a great surprise to me. I had no idea that trained observers with so high a power of discriminating colour could make such serious errors with the colours of everyday objects. The two converse cases (9) and (10), which I have arranged directly below (6) and (7) for purposes of comparison, are also striking: it is remarkable that observers with such a low power of discriminating colour should make so few mistakes. It is difference in the distribution of colour-discriminating power, not lack of it, that causes trouble; also, difference in the relative intensity of the colours has a strong influence.

(Issued separately May 1, 1922.)

VIII.—The Asymptotic Expansion of the Confluent Hypergeometric Function, and a Fourier-Bessel Expansion. By Dr T. M. MacRobert, M.A.

(MS. received February 16, 1922. Read May 1, 1922.)

PART I.

THE ASYMPTOTIC EXPANSION OF THE CONFLUENT HYPERGEOMETRIC FUNCTION.

IN Whittaker and Watson's *Modern Analysis*, chap. xvi, the asymptotic expansion of the confluent hypergeometric function $W_{k,m}(z)$ is established for the region $-\pi < \text{amp } z < \pi, z \neq 0$. The object of the first part of this paper is to show that this expansion is valid in the extended region $-3\pi/2 < \text{amp } z < 3\pi/2, z \neq 0$.

In the equation

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z} z^k}{\Gamma(\frac{1}{2} - k + m)} \int_0^\infty e^{-\xi} \xi^{-k-1+m} \left(1 + \frac{\xi}{z}\right)^{k-1+m} d\xi, \quad (1)$$

which is valid for $\text{R}(m - k + \frac{1}{2}) > 0, -\pi < \text{amp } z < \pi, z \neq 0$, assume for the moment that $\text{amp } z = 0$. Then the path of integration can be deformed into a straight line from the origin to infinity, making an angle ψ with the ξ -axis, where $-\pi/2 < \psi < \pi/2$. Thus

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z} z^k}{\Gamma(\frac{1}{2} - k + m)} \int_0^\infty e^{-\zeta} \zeta^{-k-1+m} \left(1 + \frac{\zeta}{z}\right)^{k-1+m} d\zeta, \quad (2)$$

an equation which is valid for $\text{R}(m - k + \frac{1}{2}) > 0, \psi - \pi < \text{amp } z < \psi + \pi, z \neq 0$.

Now

$$\left(1 + \frac{\zeta}{z}\right)^{k-1+m} = \sum_{r=0}^{s-1} \frac{\Pi(k - \frac{1}{2} + m)}{r! \Pi(k - \frac{1}{2} + m - r)} \left(\frac{\zeta}{z}\right)^r + R_s',$$

where*

$$R_s' = \frac{\Pi(k - \frac{1}{2} + m)}{s! \Pi(k - \frac{1}{2} + m - s)} \left(\frac{\zeta}{z}\right)^s \int_0^1 s(1-t)^{s-1} \left(1 + \frac{\zeta t}{z}\right)^{k-1+m-s} dt.$$

Hence, employing the formula

$$\Gamma(z) = \int_0^\infty e^{-\zeta} \zeta^{z-1} d\zeta,$$

* Cf. Prof. G. A. Gibson, *Proc. Edin. Math. Soc.*, vol. xxxviii.

which is valid, when $R(z) > 0$, for the same path as the integral in (2), we find that

$$W_{k, m}(z) = e^{-\frac{1}{2}z^2} \left[1 + \frac{m^2 - (k - \frac{1}{2})^2}{1! z} + \frac{\{m^2 - (k - \frac{1}{2})^2\} \{m^2 - (k - \frac{3}{2})^2\}}{2! z^2} + \dots + \frac{\{m^2 - (k - \frac{1}{2})^2\} \dots \{m^2 - (k - s + \frac{3}{2})^2\}}{(s-1)! z^{s-1}} + R_s \right],$$

where

$$R_s = \frac{1}{\Gamma(\frac{1}{2} - k + m)} \frac{\Pi(k - \frac{1}{2} + m)}{s! \Pi(k - \frac{1}{2} + m - s) z^s} \times \int_0^1 e^{-\zeta} \zeta^{-k-i+m+s} \int_0^1 s(1-t)^{s-1} \left(1 + \frac{\zeta t}{z}\right)^{k-i+m-s} dt d\zeta.$$

From this formula it can be deduced* that $R_s = \phi(z)/z^s$, where $\phi(z)$ remains finite for all values of z such that $\psi - \pi < \text{amp } z < \psi + \pi$, $z \neq 0$. When $R(m - k + \frac{1}{2}) \neq 0$ we can obtain this result by using the contour integral expression for $W_{k, m}(z)$, as in Whittaker and Watson's *Analysis*, 16·3, and making the contour approach infinity in the direction $\text{amp } \zeta = \psi$. Thus the asymptotic expansion of $W_{k, m}(z)$ is valid for $-3\pi/2 < \text{amp } z < 3\pi/2$, $z \neq 0$.

Since $K_n(z) = \sqrt{\left(\frac{\pi}{2z}\right)} W_{0, n}(2z)$, it follows that the asymptotic expansion of $K_n(z)$ is valid in the same region, while the asymptotic expansions of $G_n(z)$ and $J_n(z)$ hold for $-\pi < \text{amp } z < 2\pi$, $z \neq 0$, and $-\pi < \text{amp } z < \pi$, $z \neq 0$, respectively.

PART II.

A FOURIER-BESSEL EXPANSION.

In a previous paper † the author has given a proof, depending on contour integration, of the validity of the ordinary Fourier-Bessel expansions. A similar proof will now be given of the validity of the expansion ‡

$$f(r) = \sum_s A_s R(\kappa_s),$$

where

$$R(\kappa) = J_n(\kappa r) G_n(\kappa a) - G_n(\kappa r) J_n(\kappa a),$$

and κ_s is a positive zero of

$$S(\kappa) = J_n(\kappa b) G_n(\kappa a) - G_n(\kappa b) J_n(\kappa a).$$

To begin with, let us assume that the expansion is valid for $a < r < b$; then, if

$$Q(\kappa) = J_n(\kappa x) G_n(\kappa a) - G_n(\kappa x) J_n(\kappa a),$$

* Prof. G. A. Gibson, *loc. cit.*

† *Proc. Edin. Math. Soc.*, vol. xxxix.

‡ Cf. Gray and Mathews' *Bessel Functions*, chap. x.

and

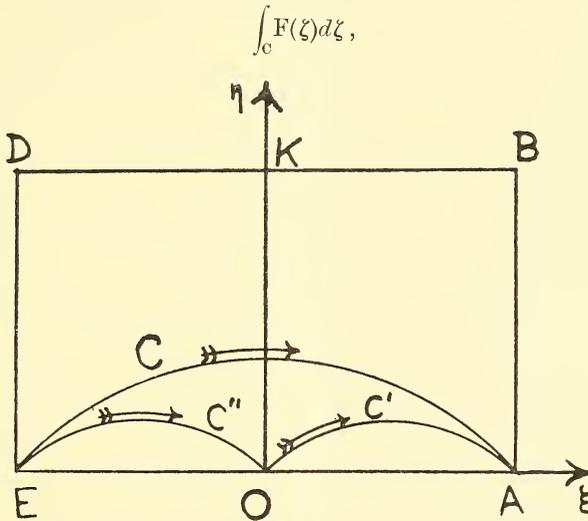
$$T(\kappa) = J_n'(\kappa b)G_n(\kappa a) - G_n'(\kappa b)J_n(\kappa a),$$

$$\int_a^b x f(x) Q(\kappa_s) dx = A_s \frac{b}{2} S'(\kappa_s) T(\kappa_s), \quad \dots \dots \dots (3)$$

so that

$$\sum_{s=1}^{\nu} A_s R(\kappa_s) = \frac{2}{b} \int_a^b x f(x) \sum_{s=1}^{\nu} \frac{Q(\kappa_s) R(\kappa_s)}{S'(\kappa_s) T(\kappa_s)} dx. \quad \dots \dots \dots (4)$$

Validity of the Expansion.—In order to establish the validity of the expansion, consider the integral



where C is a path above the ξ -axis from E to A, A is $\zeta = M$, E is $\zeta = -M$, M lies between κ_ν and $\kappa_{\nu+1}$,

$$F(\zeta) = \frac{\zeta^{2ab} P(\zeta) Q(\zeta) R(\zeta)}{S(\zeta)},$$

and

$$P(\zeta) = J_n(\zeta b) G_n'(\zeta a) - G_n(\zeta b) J_n'(\zeta a).$$

It should be noted that $P(\zeta)$, $Q(\zeta)$, $R(\zeta)$, $S(\zeta)$ are *uniform* functions of ζ , $P(\zeta)$ being odd and the others even. The first terms of their asymptotic expansions are $-\frac{1}{\zeta \sqrt{J(ab)}} \cos \{\zeta(b-a)\}$, $\frac{1}{\zeta \sqrt{(xa)}} \sin \{\zeta(x-a)\}$, $\frac{1}{\zeta \sqrt{(ra)}} \sin \{\zeta(r-a)\}$, $\frac{1}{\zeta \sqrt{(ab)}} \sin \{\zeta(b-a)\}$ respectively, and these are the same for all values of amp ζ . The first term of the asymptotic expansion of $F(\zeta)$ is thus

$$\theta(\zeta) = -\frac{b}{\sqrt{(xr)}} \cot \{\zeta(b-a)\} \sin \{\zeta(x-a)\} \sin \{\zeta(r-a)\}.$$

Now deform C into the real axis indented at the zeros of $S(\zeta)$. Since $F(\zeta)$ is odd, the integrals along the straight parts of this contour cancel. The integral round the semicircle at O tends to zero with the radius, and the sum of the integrals round the other semicircles tends to

$$-2\pi i \sum_{s=1}^{\nu} \frac{\kappa_s^2 ab P(\kappa_s) Q(\kappa_s) R(\kappa_s)}{S'(\kappa_s)} i. \quad (5)$$

But

$$S(\kappa) \{ J_n'(\kappa b) G_n'(\kappa a) - G_n'(\kappa b) J_n'(\kappa a) \} - P(\kappa) T(\kappa) \\ = \{ J_n'(\kappa a) G_n(\kappa a) - J_n(\kappa a) G_n'(\kappa a) \} \{ J_n'(\kappa b) G_n(\kappa b) - J_n(\kappa b) G_n'(\kappa b) \} = \frac{1}{\kappa^2 ab}.$$

Hence (5) is equal to

$$2\pi i \sum_{s=1}^{\nu} \frac{Q(\kappa_s) R(\kappa_s)}{S'(\kappa_s) T(\kappa_s)}. \quad (6)$$

Again, consider the function

$$\phi(\zeta) = \frac{\zeta^2 P(\zeta)}{S(\zeta)} G_n(\lambda \zeta) G_n(\mu \zeta) G_n(\rho \zeta) G_n(\sigma \zeta), \quad (7)$$

where the amplitudes of $\lambda, \mu, \rho,$ and σ are 0 or π , $\lambda + \mu + \rho + \sigma > 0$, and $0 \leq n < \frac{1}{2}$.

The first term in the asymptotic expansion of $\phi(\zeta)$ is

$$\frac{\pi^2}{4\sqrt{(\lambda\mu\rho\sigma)}} \cot \{ \zeta(b-a) \} e^{-2n\pi i + i\zeta(\lambda + \mu + \rho + \sigma)}.$$

Now, if C' is a path from O to A above the ξ -axis,

$$\int_{C'} \phi(\zeta) d\zeta = - \int_{AB} \phi(\zeta) d\zeta + \int_{OK} \phi(\zeta) d\zeta + \int_{KB} \phi(\zeta) d\zeta,$$

where KB is the line $\eta = N$. But, if $N \rightarrow \infty$, $\int_{KB} \phi(\zeta) d\zeta \rightarrow 0$, since $\lambda + \mu + \rho + \sigma > 0$;

hence

$$\int_{C'} \phi(\zeta) d\zeta = - \int_0^\infty \phi(M + i\eta) i d\eta + \int_0^\infty \phi(i\eta) i d\eta.$$

In the two integrals on the right of this equation put $\eta = v/(\lambda + \mu + \rho + \sigma)$; then

$$\int_{C'} \phi(\zeta) d\zeta = \frac{-i}{\lambda + \mu + \rho + \sigma} \int_0^\infty \phi\left(M + \frac{iv}{\lambda + \mu + \rho + \sigma}\right) dv + \frac{i}{\lambda + \mu + \rho + \sigma} \int_0^\infty \phi\left(\frac{iv}{\lambda + \mu + \rho + \sigma}\right) dv.$$

It will now be shown that this is also true in some cases even when $\lambda + \mu + \rho + \sigma < 0$. Assume that

- (i) $\text{amp } \lambda = 0$,
- (ii) $\mu + \rho + \sigma$ is negative and $= -\gamma$,
- (iii) the amplitudes of $\mu, \rho,$ and σ are 0 or π ,
- (iv) $\lambda + \mu + \rho + \sigma = \lambda - \gamma > 0$.

Then

$$\int_{C'} \phi(\xi) d\xi = -\frac{i}{\lambda-\gamma} \int_0^\infty \phi\left(M + \frac{iv}{\lambda-\gamma}\right) dv + \frac{i}{\lambda-\gamma} \int_0^\infty \phi\left(\frac{iv}{\lambda-\gamma}\right) dv. \quad (8)$$

The first term in the asymptotic expansion of $\phi\left\{M + \frac{iv}{\lambda-\gamma}\right\}$ is

$$\frac{\pi^2}{4\sqrt{(\lambda\mu\rho\sigma)}} \cot\left\{\left(M + \frac{iv}{\lambda-\gamma}\right)(b-a)\right\} e^{-2n\pi i + iM(\lambda-\gamma-v)},$$

so that the integrals on the right of (8) are convergent even when $\lambda-\gamma$ is negative, provided that the asymptotic expansions of $\phi\{M + iv/(\lambda-\gamma)\}$ and $\phi\{iv/(\lambda-\gamma)\}$ are still valid.

Now consider the arguments of the factors of $\phi\{M + iv/(\lambda-\gamma)\}$ as given by (7), and let $\lambda-\gamma = de^{i\theta}$, where $d < \gamma$.

I. *The real part of* $\left\{\lambda\left(M + \frac{iv}{\lambda-\gamma}\right)\right\} = M(\gamma + d \cos \theta) + \gamma \frac{v}{d} \sin \theta$, and, as θ

increases from 0 to π , this remains positive or zero for $M \geq 0$: thus the amplitude of the argument varies between $-\pi/2$ and $\pi/2$.

II. *The real part of* $\left\{\mu\left(M + \frac{iv}{\lambda-\gamma}\right)\right\} = \mu\left(M + \frac{v}{d} \sin \theta\right)$. This remains

positive if μ is positive, or negative if μ is negative, for $M \geq 0$: thus the amplitude varies between $-\pi/2$ and $\pi/2$, or $\pi/2$ and $3\pi/2$.

Accordingly, the asymptotic expansions of the integrands on the right of (8) remain valid as θ increases from 0 to π , except for the initial point of the integral along the η -axis. At this point, however, the integral is convergent, so that (8) is still valid. *

Similarly the equation

$$\int_{C''} \phi(\xi) d\xi = \frac{i}{\lambda-\gamma} \int_0^\infty \phi\left(-M + \frac{iv}{\lambda-\gamma}\right) dv - \frac{i}{\lambda-\gamma} \int_0^\infty \phi\left(\frac{iv}{\lambda-\gamma}\right) dv, \quad (9)$$

where C'' is a path from E to O above the real axis, can be shown to hold even when $\lambda-\gamma$ is negative. In this case the amplitudes of μ , ρ , and σ are assumed to be 0 or $-\pi$, and $\text{amp}(\lambda-\gamma)$ is made to *decrease* from 0 to $-\pi$.

Now

$$\int_C F(\xi) d\xi = \int_{C'} F(\xi) d\xi + \int_{C''} F(\xi) d\xi.$$

Along C' and C'' replace $J_n(\xi x)$ by $\frac{1}{i\pi} \{G_n(\xi x) - e^{in\pi} G_n(\xi x e^{i\pi})\}$ and

* *Note.*—The factor $e^{iM(\lambda-\gamma)}$ in the asymptotic expansion of $\phi\left(M + \frac{iv}{\lambda-\gamma}\right)$ is equal to $e^{iM d \cos \theta - M d \sin \theta}$ when $\lambda-\gamma = de^{i\theta}$, so that the integral remains convergent when θ varies from 0 to π .

$-\frac{e^{2in\pi}}{i\pi}\{G_n(\zeta x) - e^{-in\pi}G_n(\zeta x e^{-i\pi})\}$ respectively, and similarly for the other J_n 's in $Q(\zeta)$ and $R(\zeta)$. Then, in the first case,

$$Q(\zeta)R(\zeta) = -\frac{e^{2in\pi}}{\pi^2}\{G_n(\zeta x)G_n(\zeta a e^{i\pi}) - G_n(\zeta a)G_n(\zeta x e^{i\pi})\} \\ \times \{G_n(\zeta r)G_n(\zeta a e^{i\pi}) - G_n(\zeta a)G_n(\zeta r e^{i\pi})\},$$

and in the second case

$$Q(\zeta)R(\zeta) = -\frac{e^{2in\pi}}{\pi^2}\{G_n(\zeta x)G_n(\zeta a e^{-i\pi}) - G_n(\zeta a)G_n(\zeta x e^{-i\pi})\} \\ \times \{G_n(\zeta r)G_n(\zeta a e^{-i\pi}) - G_n(\zeta a)G_n(\zeta r e^{-i\pi})\}.$$

Thus $F(\zeta)$ is replaced in each case by four functions of the same form as $\phi(\zeta)$, to which formulæ (8) and (9) apply.

The integrals along the η -axis will cancel one another. For example, consider the integrals arising from the products

$$G_n(\zeta x)G_n(\zeta a e^{i\pi})G_n(\zeta r)G_n(\zeta a e^{i\pi})$$

and

$$G_n(\zeta a)G_n(\zeta x e^{-i\pi})G_n(\zeta a)G_n(\zeta r e^{-i\pi}).$$

In the first we divide the arguments by $x+r-2a$. In the second we first assume that $2a > x+r$ and divide the arguments by $2a-x-r$. We then make $\text{amp}(2a-x-r)$ decrease from 0 to $-\pi$, so that ultimately, when $2a-x-r < 0$,

$$2a-x-r = (x+r-2a)e^{-i\pi}.$$

Thus

$$\frac{\zeta a}{2a-x-r} = \frac{\zeta a e^{i\pi}}{x+r-2a}, \\ \frac{\zeta x e^{-i\pi}}{2a-x-r} = \frac{\zeta x}{x+r-2a},$$

and so on. The factor $\frac{1}{\lambda-\gamma} \frac{\{\tilde{\zeta}/(\lambda-\gamma)\}^2 \cdot P\{\tilde{\zeta}/(\lambda-\gamma)\}}{S\{\tilde{\zeta}/(\lambda-\gamma)\}}$ being even in $(\lambda-\gamma)$, remains unaltered, and therefore the integrals cancel. Accordingly

$$\int_C F(\zeta) d\zeta = \sum_{r=1}^4 \frac{-i}{\lambda-\gamma} \int_0^\infty \phi_r\left(M + \frac{iv}{\lambda-\gamma}\right) dv + \sum_{r=5}^8 \frac{i}{\lambda-\gamma} \int_0^\infty \phi_r\left(-M + \frac{iv}{\lambda-\gamma}\right) dv,$$

the integrals on the right being all convergent. The restriction $0 \leq n < \frac{1}{2}$ can now be removed.

Again, if $\theta(\zeta)$ is the first term in the asymptotic expansion of $F(\zeta)$, the integral

$$\int_C \theta(\zeta) d\zeta$$

can be discussed in a similar manner. In order to avoid trouble regarding the convergency of the integrals at O, we first of all consider the integral

of the first term in the asymptotic expansion of $\phi(\zeta)$ taken round the rectangle bounded by $\xi = \delta (> 0)$, $\xi = M$, $\eta = 0$, $\eta = N$, and then, when the integrals have been added to give the integral of $\theta(\zeta)$, make $\delta \rightarrow 0$.*

It is thus found that

$$\int_C \theta(\zeta) d\zeta = \sum_{r=1}^4 \frac{-i}{\lambda - \gamma} \int_0^\infty \psi_r \left(M + \frac{iv}{\lambda - \gamma} \right) dv + \sum_{r=5}^8 \frac{i}{\lambda - \gamma} \int_0^\infty \psi_r \left(-M + \frac{iv}{\lambda - \gamma} \right) dv,$$

where ψ_r is the first term in the asymptotic expansion of ϕ_r .

Accordingly

$$\begin{aligned} \int_C F(\zeta) d\zeta - \int_C \theta(\zeta) d\zeta &= \sum_{r=1}^4 \frac{-i}{\lambda - \gamma} \int_0^\infty \left\{ \phi_r \left(M + \frac{iv}{\lambda - \gamma} \right) - \psi_r \left(M + \frac{iv}{\lambda - \gamma} \right) \right\} dv \\ &+ \sum_{r=5}^8 \frac{i}{\lambda - \gamma} \int_0^\infty \left\{ \phi_r \left(-M + \frac{iv}{\lambda - \gamma} \right) - \psi_r \left(-M + \frac{iv}{\lambda - \gamma} \right) \right\} dv. \end{aligned}$$

Now the integrand of each integral on the right is of the form

$$\frac{\chi(v)e^{-v}}{\pm M + \frac{iv}{\lambda - \gamma}},$$

where $\chi(v)$ remains finite throughout the path of integration. Thus, by increasing M , each of these integrals can be made arbitrarily small.

Hence

$$\int_C F(\zeta) d\zeta = \int_C \theta(\zeta) d\zeta + \epsilon, \quad \dots \dots \dots (10)$$

where $\epsilon \rightarrow 0$ as $M \rightarrow \infty$.

Lastly, to evaluate $\int_C \theta(\zeta) d\zeta$, deform C into the ξ -axis indented at the zeros of $\sin \{ \zeta(b-a) \}$. As before, the integrals along the axis cancel, and the integrals round the semicircles give

$$2\pi i \frac{b}{\sqrt{(xr)}} \sum_{s=1}^{\mu} \frac{\sin \left\{ \frac{s\pi(x-a)}{b-a} \right\} \sin \left\{ \frac{s\pi(r-a)}{b-a} \right\}}{b-a},$$

where μ denotes the number of positive zeros of $\sin \{ \zeta(b-a) \}$ between O and M .

This is equal to

$$\begin{aligned} &\pi i \frac{b}{\sqrt{(xr)}} \frac{1}{b-a} \sum_{s=1}^{\mu} \left\{ \cos \left(\frac{s\pi(x-r)}{b-a} \right) - \cos \left(\frac{s\pi(x+r-2a)}{b-a} \right) \right\} \\ &= \frac{ib}{\sqrt{(xr)}} \left[\frac{\sin \left\{ \pi \left(\mu + \frac{1}{2} \right) \frac{(x-r)}{(b-a)} \right\}}{2} \frac{\sin \left\{ \pi \left(\mu + \frac{1}{2} \right) \frac{(x+r-2a)}{(b-a)} \right\}}{2} \right] \\ &\quad \left[\frac{b-a}{\pi} \sin \left\{ \frac{\pi(x-r)}{2(b-a)} \right\} - \frac{b-a}{\pi} \sin \left\{ \frac{\pi(x+r-2a)}{2(b-a)} \right\} \right]. \end{aligned}$$

* This method could also be employed in the discussion of the integral of $\phi(\zeta)$, and then it would be unnecessary to introduce the restriction $0 \leq n < \frac{1}{2}$.

Now multiply by $x f(x)/(\pi i b)$, integrate from a to b , and let μ tend to infinity. Then, from (10), (6), and (4) we see, using the theory of Dirichlet Integrals, that, if $a < r < b$,

$$\sum_{s=1}^{\infty} A_s R(\kappa_s) = \frac{1}{2} \{f(r+0) + f(r-0)\}.$$

When $r = a$ or b , the sum is obviously zero.

(Issued separately May 1, 1922.)

IX.—On Models of Ferromagnetic Induction. By Sir J. Alfred Ewing, K.C.B., F.R.S., Principal of the University of Edinburgh. (With Two Plates and Fifteen Figs. in Text.)

(MS. received February 20, 1922. Read February 20, 1922.)

1. IN 1890 I published a theory of ferromagnetic induction in which it was suggested that the equilibrium of Weber's elementary magnetic particles was due only to magnetic forces.* It was shown that when the elementary magnets are made to turn by applying a magnetising force which is progressively increased, the conditions of equilibrium must be such that there is first a small amount of stable (reversible) deflection, then a break away with irreversible deflection into a new position of stability, and finally a reversible approach to the position of complete parallelism which corresponds to saturation. A model was constructed showing that these conditions could be satisfied by a purely magnetic control. It was made up of little magnets, pivoted on fixed centres which were uniformly spaced. The magnets were all free to turn, but controlled one another by their mutual magnetic forces. They tended to form rows, and when an external field was applied the rows broke up and fresh rows were formed more nearly in the direction of the field. With this simple model the known characteristics of the magnetising process were, qualitatively, well reproduced. A recent study of the stability of such rows of magnets has, however, led me to abandon this model, and to design instead a model in which each atom forms a magnetic system comprising a Weber element capable of turning, but controlled by the magnetic forces exerted on it by other parts of the atom which are taken as fixed.† As in the old model, the control is wholly magnetic. Various forms of the new model will be described in a later part of this paper, but in the first place it may be useful to give some account of the investigations of stability which have convinced me that the old model fails, quantitatively, to represent the process.

2. When a magnetising field is applied to a substance such as soft iron, the amount of magnetism that is taken up during the initial or quasi-elastic stage, before hysteresis begins to show itself, is a very small part,

* "Contributions to the Molecular Theory of Induced Magnetism," *Proc. Roy. Soc.*, xlviii, p. 342; *Phil. Mag.*, Sept. 1890.

† *Proc. Roy. Soc.*, Feb. 1922; *Phil. Mag.*, March 1922.

often only one per cent. or less, of the magnetism of saturation. Hence, whatever be the nature of the control, the range of stable deflection which the Weber element can undergo before instability occurs is very narrow, say about half a degree to either side of its initial position. It follows that in the old model the pivoted magnets had to be thought of as being placed near together with but little clearance between neighbouring poles.* As a result they form rows in which the poles of adjacent magnets in the row are very near together compared with the distance between adjacent rows, and consequently the equilibrium of the row, prior to rupture, is substantially unaffected by the existence of adjacent rows. Under these conditions the control of any one magnet is due almost wholly to the forces exerted by its next neighbours in the same row. In considering the problem of stability it is therefore sufficient to deal only with a single row.

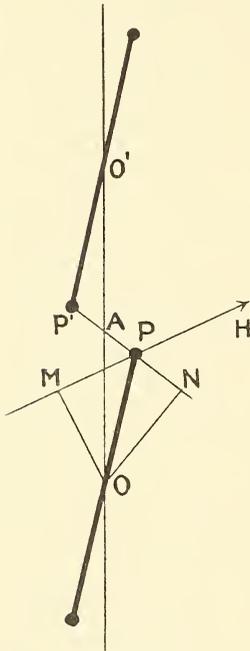


FIG. 1.

3. Imagine an indefinitely extended row of magnets, uniform in length and moment, each free to turn about a fixed centre, and let the centres be spaced at a uniform distance $2a$ in a straight line. Let the length of each magnet be $2r$ and its pole strength m . In the absence of any deflecting field, the magnets will form a straight line with their poles $2(a-r)$ apart. Suppose now a deflecting field H to act, everywhere of the same strength and inclined at a constant angle α to the direction in which the magnets lie when they are pointing along the line of centres OO' (fig. 1). Except when α is not far from 180° —within say 35° of that direction, a case which will be considered later—the effect of H is to deflect

the magnets equally, so that they remain parallel to one another as in the figure. At first, as H is increased, there is stability, and the action is reversible. The angle of deflection, θ , has a single value for each value of H . But unless α is small, as will be shown immediately, instability occurs when θ reaches a certain value θ_r , and the magnets then swing irreversibly towards a new position of equilibrium. The angle of rupture θ_r depends primarily on the ratio a/r which we shall call i . To some extent it also

* In a paper by Honda and Okubo on "Ferromagnetic Substances and Crystals in the light of Ewing's Theory of Molecular Magnetism," *Science Reports of the Tôhoku University*, vol. v, Aug. 1916, also *Phys. Rev.*, vol. x, p. 705, this point seems to have been overlooked. The distance between centres is taken there as probably more than twice the length of each magnet, which gives a wider range of stable deflection than is admissible.

depends on the inclination α of the applied field. For small values of α there is no instability.

4. Consider the equilibrium of any one magnet in the row. The deflecting moment due to H (fig. 1) is $2Hm \cdot OM$. The restoring moment due to adjacent poles is $\frac{2m^2ON}{(PP')^2}$. Since $\alpha - r$ is small compared with r , only adjacent poles need be taken into account. The equation of equilibrium is accordingly

$$H \cdot OM = \frac{m \cdot ON}{(PP')^2}.$$

But $OM = OP \sin OPM = r \sin(\alpha - \theta)$, $(PP')^2 = 4AP^2 = 4(a^2 + r^2 - 2ar \cos \theta)$,

and $ON = OA \sin OAP = \frac{OA \cdot OP \sin AOP}{AP} = \frac{ar \sin \theta}{(a^2 + r^2 - 2ar \cos \theta)^{\frac{1}{2}}}$. Hence the equation of equilibrium becomes

$$H \sin(\alpha - \theta) = \frac{ma \sin \theta}{4(a^2 + r^2 - 2ar \cos \theta)^{\frac{3}{2}}},$$

or, writing i for a/r ,

$$H = \frac{m}{4r^2} \left[\frac{i \sin \theta}{(i^2 + 1 - 2i \cos \theta)^{\frac{3}{2}}} \cdot \frac{1}{\sin(\alpha - \theta)} \right] \quad \dots \quad (1)$$

For brevity we shall write F for the quantity in square brackets. It is a numerical factor, a function of i and α and of the deflection θ , increasing from zero with θ . When values of i and α are assigned, F reaches a maximum, say F_r , for a particular value of θ , say θ_r . At that deflection $\frac{d\theta}{dH}$ is infinite, and consequently θ_r is the angle of rupture and determines the field H_r which will produce instability.

5. For example, if i is 1.1 and α is 20° the values of F calculated from the above expression for various values of θ are as follows:—

VALUES OF F FOR $i=1.1$ AND $\alpha=20^\circ$.

θ .	F .	θ .	F .
1	56.1	10	121.6
2	102.9	12	117.4
3	132.7	15	131.9
4	145.9	17	177.7
5	148.8	18	241.3
6	145.2	19	438.5
8	132.3	20	∞

These are also shown in the curve, fig. 2, from which it will be seen that as H is increased from zero the deflection is at first nearly proportional

to H , and there is no hysteresis. But instability occurs when F reaches a value of 149, the deflection then being about $4^{\circ}7$. At that point (a on the curve) the magnets swing violently into a new position of equilibrium, for the same value of H , the deflection changing to $16^{\circ}2$; and if H is then further increased, they are gradually brought into closer agreement with the direction of the field, taking that direction only when

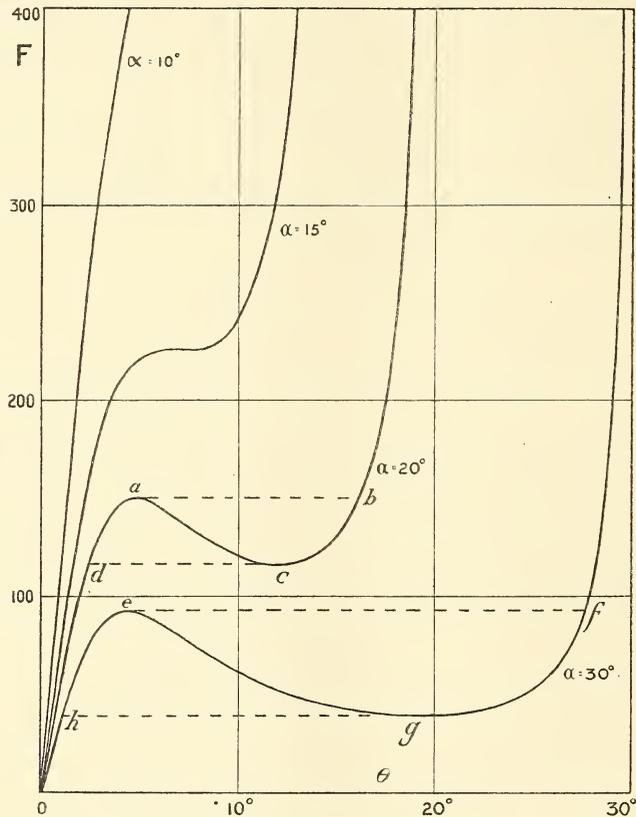


FIG. 2.—Relation of F to θ for various values of α , with $i = 1.1$.

the field is indefinitely increased. Again, if after the field H has been raised to a high value, it is reduced, an unstable stage is reached when θ is about $12^{\circ}5$ and F is 136. The magnets then swing violently back to a position at which θ is $2^{\circ}4$. In the early stage, up to a , and again in the final stage from b onwards, the deflection is reversible with varying H ; but between the early stage and the final stage there is an irreversible step in which the deflection exhibits hysteresis and involves dissipation of energy. In the figure the outward and inward irreversible steps are exhibited by the lines ab and cd respectively.

In the same figure the curve is drawn for $\alpha=30^\circ$ (also with $i=1.1$); there e represents the point of rupture, namely, the point at which F is a maximum. ef is the violent swing outwards during the application of the field, and gh is the violent return swing during the removal of the field. The angle of rupture is $4^\circ.3$.

6. When α is reduced to a certain value α_1 (depending on i), the maximum and minimum of F coalesce, and the curve of F in relation to θ exhibits a point of inflexion at which $\frac{d^2F}{d\theta^2}=0$ as well as $\frac{dF}{d\theta}=0$. This is approximately true of the curve for $\alpha=15^\circ$ in the same figure. For any value of α less than α_1 there is no unstable phase in the deflection of the magnets; as an example the curve is given for $\alpha=10^\circ$ (still with $i=1.1$). Here the whole process of deflection is reversible, though it includes a stage during which $\frac{d\theta}{dH}$ is relatively large.

7. By differentiating equation (1) an expression is found connecting the angle of rupture with α and i , from which the value of H that will produce instability is determined. The condition is that $\frac{dF}{d\theta}=0$, which involves the relation

$$(2 + 3 \sin^2 \theta) \cos \theta - 3 \sin^3 \theta \cot \alpha = i + \frac{1}{i} \quad . \quad . \quad . \quad (2)$$

When α and i are assigned, this gives, for values of α greater than α_1 , two values of θ . The lesser of these is θ_r , the deflection at which instability occurs during the application of the deflecting field; the greater is the value of θ at which instability occurs during the return of the magnets after deflection by a strong field. On substituting these values of θ in equation (1), the corresponding values of H are determined.

Again, to find α_1 , the limiting value of α below which there is no instability, we have the condition that $\frac{d^2F}{d\theta^2}=0$ as well as $\frac{dF}{d\theta}=0$. This requires that

$$\cot \alpha_1 = \frac{\frac{4}{3} - \sin^2 \theta}{\sin \theta \cos \theta} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

At that angle equation (2) must also be satisfied. Hence when $\alpha=\alpha_1$ we obtain from equations (2) and (3) together

$$5 \sec \theta_1 + \cos \theta_1 = 3 \left(i + \frac{1}{i} \right) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

as an expression for the deflection θ_1 at which the point of inflexion occurs for any assigned value of i . From this, along with equation (3), we may

determine α_1 . Thus when $i=1.1$, equation (4) shows that θ_1 is $6^\circ 40'$, making $\alpha_1=14^\circ 59'$, and the value of F at the point of inflexion is 225. These numbers are in agreement with the case illustrated by the curve $\alpha=15^\circ$ in fig. 2.

8. Consider next how the limiting stable deflection θ_r and the corresponding value of F , namely F_r , depend upon the value of i and the inclination α of the applied field. In the particular case where $\alpha=90^\circ$, equation (2) gives

$$(2 + 3 \sin^2 \theta_r) \cos \theta_r = i + \frac{1}{i} \quad (5)$$

Hence when the deflecting field acts at right angles to the line of centres, by giving the angle of rupture θ_r the values shown below, the stated values of i are calculated. The table also gives the corresponding calculated values of F_r , from which H_r may be determined by equation (1).

RUPTURE OF A ROW OF MAGNETS WHEN $\alpha=90^\circ$.

For $\theta_r =$	30'	45'	1°	2°	3°	5°	10°	15°
$i =$	1.0125	1.0187	1.0250	1.0506	1.077	1.130	1.274	1.423
$F_r =$	2500	1110	625	154	67	24	5.8	2.6

Some of these results are plotted in fig. 3. It will be noticed that for small values θ_r is nearly proportional to $i-1$, that is, to $(\alpha-r)/r$. Hence $r\theta_r$ bears a nearly constant ratio to $\alpha-r$; in other words, the line PP' (fig. 1) has nearly the same inclination to the line of centres when rupture occurs, for all small values of θ_r .

When $i-1$ is made indefinitely small, with the result that θ_r also becomes indefinitely small, the quantity F_r , for $\alpha=90^\circ$, tends towards the

limiting value $\frac{2}{3\sqrt{3}(i-1)^2}$. To show this it is convenient to express the

deflection of the magnets in terms of the perpendicular distance of P from the line of centres (fig. 1). Call that distance x , and write c for $\frac{1}{2}PP'$.

Then $x = r \sin \theta$, $ON = \frac{\alpha x}{c}$, and the equation of equilibrium becomes

$$Hr \sin (\alpha - \theta) = \frac{m\alpha x}{4c^3}.$$

Hence for $\alpha=90^\circ$ we have, in the limiting case when θ is indefinitely small,

$$H = \frac{m\alpha x}{4rc^3},$$

and the criterion for rupture is that $\frac{d}{dx} \left(\frac{x}{c^3} \right)$ should be equal to zero, which

makes $x = \frac{a-r}{\sqrt{2}}$, since in the limit $a^2 = x^2 + (a-r)^2$. On substituting these values the field producing rupture is found to have the value

$$H = \frac{m}{4r^2} \left[\frac{2}{3\sqrt{3}(i-1)^2} \right],$$

and rupture occurs when the angle OAP (fig. 1) is $\tan^{-1} \frac{1}{\sqrt{2}}$ or $35^\circ 16'$.

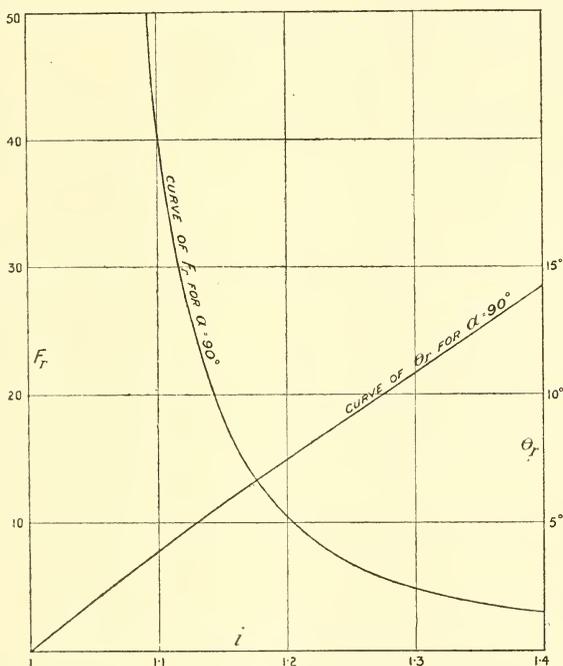


FIG. 3.—Rupture of a row of magnets for various values of i , when $\alpha = 90^\circ$.

When α is made less than 90° (but not less than α_1) the limiting stable deflection for a given value of i is slightly increased in consequence of the term in $\cot \alpha$, in equation (2). The quantity F_r has a minimum when α is approximately $90^\circ + \theta_r$, and rises rapidly as α approaches the value α_1 .

Taking again as an example the case $i = 1.1$, we have the following relation of θ_r and F_r to α :

RUPTURE OF A ROW OF MAGNETS WHEN $i = 1.1$.

For	$\alpha =$	20°	30°	35°	45°	70°	90°	115°	135°	145°
	$\theta_r =$	4.7°	4.4°	4.3°	4.0°	3.8°	3.6°	3.5°	3.5°	3.4°
	$F_r =$	149	92	78	61	44	40.4	43	54	65

These values of F_r are plotted in relation to α in the curve ABC of fig. 4, which also shows by a dotted continuation of the curve (to E) how F_r , if calculated in the same way, would continue to change as the angle α increases towards 180° .

9. But the manner in which the row of magnets becomes ruptured when α approaches 180° is entirely different from that shown in fig. 1, and consequently the values of F_r calculated from the equations that have been given above have no relevancy to that part of the curve. We have then to consider a different type of instability, namely, that in which neighbour-

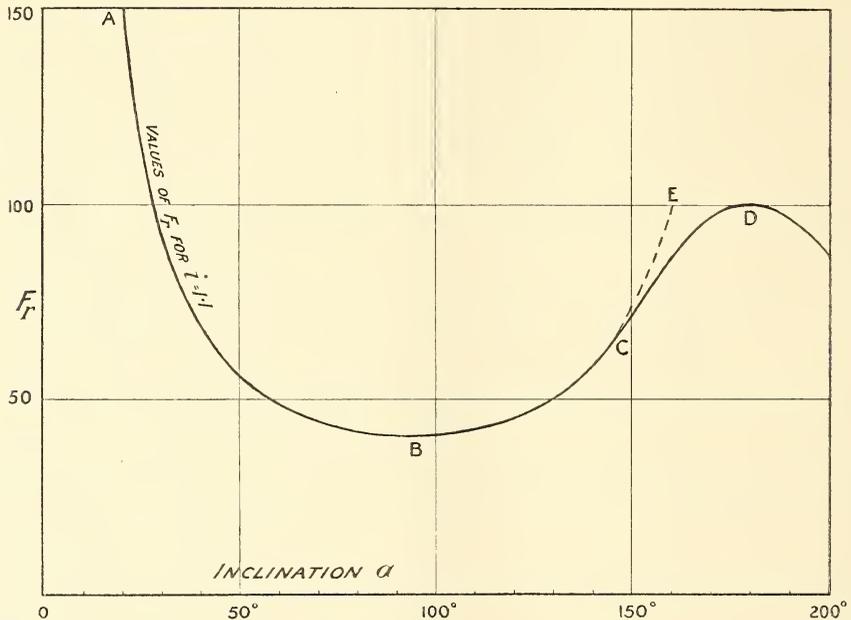


FIG. 4.—Values of F_r calculated for a row of magnets, for $i=1.1$ and for various values of α .

ing poles break away to the same side of the centre line. In that type the row of magnets assumes a serpentine form before instability sets in, with one pair of poles to the right of the centre line, the next pair to the left of it, the next to the right, and so on.

That the manner of rupture is different will be obvious when the condition of instability for $\alpha=180^\circ$ is considered. The externally applied magnetic force is then directly opposed to the magnetic force with which each pole is attracted by its next neighbour, and a state of neutral equilibrium is reached when these oppositely directed forces are equal, namely, when

$$Hm = \frac{m^2}{4(a-r)^2},$$

$$\text{or } H = \frac{m}{4r^2} \left[\frac{1}{(i-1)^2} \right] \dots \dots \dots (6)$$

This therefore (as was pointed out in my paper of 1890) is the criterion which determines the breaking up of the row when $\alpha=180^\circ$. There is then nothing to prevent any one pair of neighbouring poles in the row from deviating—without limit—towards one side, provided the next neighbouring pairs deviate towards the other side. Hence in this mode of rupture

$\frac{1}{(i-1)^2}$ is the factor which corresponds to F_r in the other mode, for the particular case when $\alpha=180^\circ$. Taking the same example as before, in which $i=1.1$, its value is 100. This determines the point D in fig. 4, and shows that the field required to break up a row of magnets when α is 180° is about two and a half times* that which is required to break it up when α is 90° .

10. When the applied field is inclined at an angle not far removed from 180° the magnets will assume, before rupture, an equilibrium position of the kind sketched in fig. 5, where one is deflected through an angle θ and the next through a smaller angle θ' . The line PP' joining the poles tends, as the condition of instability is approached, to become nearly parallel to the applied field. It will be convenient to write ϕ for the inclination of PP' to the line of centres. Draw PL and $P'L'$ perpendicular to that line. Then

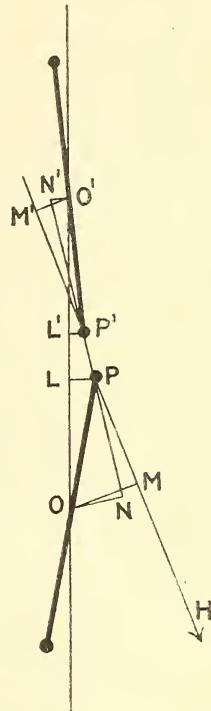


FIG. 5.

$$PP' = \frac{LL'}{\cos \phi} = \frac{2\alpha - r(\cos \theta + \cos \theta')}{\cos \phi} = \frac{r(2i - \cos \theta - \cos \theta')}{\cos \phi},$$

$$\tan \phi = \frac{PL - P'L'}{LL'} = \frac{\sin \theta - \sin \theta'}{2i - \cos \theta - \cos \theta'} \quad (7)$$

$$ON = r \sin(\phi + \theta), \quad ON' = r \sin(\phi - \theta'),$$

$$OM = r \sin(\alpha + \theta), \quad O'M' = r \sin(\alpha - \theta').$$

The condition of equilibrium is that on each of the two magnets the deflecting moment shall be equal to the restoring moment; hence for the alternate magnets of an indefinitely extended row,

$$Hm \cdot OM = \frac{m^2 ON}{(PP')^2}, \quad \text{and} \quad Hm \cdot O'M' = \frac{m^2 ON'}{(PP')^2},$$

which makes

$$\frac{\sin(\phi + \theta)}{\sin(\alpha + \theta)} = \frac{\sin(\phi - \theta')}{\sin(\alpha - \theta')}, \quad (8)$$

* When i is made indefinitely small the ratio is 1 to $\frac{2}{3\sqrt{3}}$ or 2.598. As i increases the ratio is somewhat diminished, becoming 2.54 for $i=1.05$, 2.48 for $i=1.1$, and so on.

and

$$H = \frac{m}{4r^2} \left[\frac{4 \cos^2 \phi}{(2i - \cos \theta - \cos \theta')^2} \cdot \frac{\sin(\phi + \theta)}{\sin(\alpha + \theta)} \right] \quad (9)$$

Here again the quantity in square brackets is a numerical factor, which we shall write F' . The condition for rupture is that $\frac{dF'}{d\theta} = 0$, and this, together with equations (7), (8) and (9), makes F_r' determinate. When θ and θ' at rupture are very small, as will be the case when α is only very little greater than r , ϕ at rupture becomes equal to α and the value of F_r' is $\frac{\cos^2 \alpha}{(i-1)^2}$,

which when $\alpha = 180^\circ$ becomes $\frac{1}{(i-1)^2}$, as it should do to agree with the earlier consideration of that simple case. From the maximum at $\alpha = 180^\circ$ the curve of F_r' falls away to either side proportionally to $\cos^2 \alpha$ when $i-1$ is indefinitely small, and at a rate which is nearly proportional to $\cos^2 \alpha$ for such values of i as we are concerned with, until it meets the curve of F_r in a point of inflexion (as at C in fig. 4), which is the point at which the mode of rupture changes from that of fig. 1 to that of fig. 5. At that point both modes are equally probable. When $i-1$ is very small this change-point is at $\alpha = 145^\circ$, as will be seen from the fact that F_r then approximates to the limiting value $\frac{2}{3\sqrt{3}(i-1)^2} \cdot \frac{1}{\sin \alpha}$, and F_r' to the limiting value $\frac{\cos^2 \alpha}{(i-1)^2}$. To make these equal requires α to be $180^\circ \pm 35^\circ$. With larger values of i the point of inflexion is shifted a little nearer to the 180° position.

11. All these conclusions are applicable with a very simple change to a solitary pair of magnets instead of an indefinitely extended row. In a row the restoring force acts on both poles of each magnet; in a solitary pair the restoring force acts on one pole only, while the deflecting force still acts on both poles. Hence to maintain a given deflection the value of H for the pair is only half its value for the row. This applies in both modes of rupture. Hence for a pair of magnets the field which will produce rupture is $H_r = \frac{m}{8r^2} [F_r]$ from the smallest value of α , namely α_1 up to the value at which the mode of rupture changes; and from that point up to and through the maximum at $\alpha = 180^\circ$ it is $H_r = \frac{m}{8r^2} [F_r']$. The deflection at rupture (for any value of α and of i) is the same for the pair as for the row.

12. In order to confirm and illustrate these conclusions the magnetic

fields which produce instability in pairs and rows of permanent steel magnets have been experimentally measured, under various conditions as to length, proximity of poles, and direction of the deflecting field. All the conclusions stated above for rows and pairs of magnets have in this way been verified. The experiments were made, with the permission of Professor Barkla, in the Physical Laboratory of the University, and I have to thank Dr C. G. Knott for the use of his room there and for the loan of a large pair of Helmholtz coils, with a diameter of 49 cms., set $24\frac{1}{2}$ cms. apart, which gave a nearly uniform field of sufficient extent.* Dr G. A. Carse, lecturer in the same department, kindly volunteered to help me in these experiments, and I am much indebted to him for his skilful co-operation in this part of the work.

The magnets used were of the ball-ended type originally introduced by J. Robison, Professor of Natural Philosophy in the University of Edinburgh from 1774 to 1805, and reintroduced in modern magnetic work by Dr G. F. C. Searle.† Their behaviour, as Searle has pointed out, approximates closely to that of an ideal pair of magnetic poles situated at, or very near to, the centres of the balls. In the experiments now to be described I found that the position as well as the strength of the poles was but little altered by varying the conditions to which the magnets were subjected. When the pivots were so placed as to bring the magnets very near together, the poles shifted a little out; when the pivots were drawn further apart, the poles shifted a little in. These changes of position, on the part of the poles, were to be expected; they were in all cases slight, and their amount was readily inferred from observations of the field which was required to produce instability. Tests by rupture of a pair of pivoted magnets in which the ratio a/r or i is small afford a delicate method of finding the true value of r and so determining the position of the poles. An example is given in § 15 below.

13. The magnets were rods of steel wire 0.21 cm. in diameter, with lengths ranging up to 10 cms. Their ends were screwed into quarter-inch bicycle balls (diameter 0.62 cm.). The rods and balls were softened to have the screws cut, were rehardened before magnetising, and were magnetised by being placed between the conical poles of a powerful electromagnet. They were supported on needle-point centres, as in figs. 6 and 6A. Most of the magnets were straight, as in fig. 6A, which is the better form for exact work, and had gravitational stability given them by a tubular brass weight shown in section at A, which could be slipped sideways along

* A description of these coils will be found in *Proc. R.S.E.*, vol. xiii, 1885, p. 523.

† Searle, *Proc. Camb. Phil. Soc.*, vol. xii, 1902, p. 27.

the rod to adjust the level. A horizontal brass base-plate B carried centres on which the magnets were pivoted, namely, brass uprights with sewing-needle points at the top. The centres could be set at any desired distance apart. The base-plate was mounted on a horizontal brass turntable between the Helmholtz coils, which stood vertically (fig. 7, Pl. I). This allowed the line of centres to be turned into any azimuth, and the turntable was provided with a graduated edge for reading the angle between the line of centres and the direction of the magnetic field. The coils stood so that their field coincided in direction with the horizontal component of the earth's magnetic field; the earth's field could therefore be allowed for by making a constant addition to the actual current.

In tests for rupture two methods were employed. In one, the azimuth of the line of centres was set at a definite value of α , and the current was slowly increased by means of a potentiometer slide until rupture occurred,

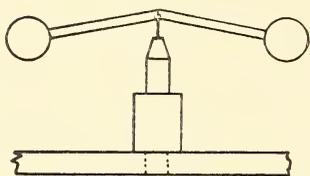


FIG. 6.

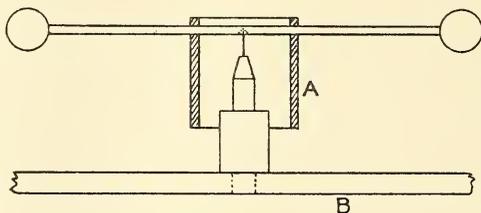


FIG. 6A.

the base-plate being gently tapped to eliminate effects of friction on the pivots. In the other method, the current was kept constant, the turntable was slowly turned into a more favourable direction for rupture, and the angle α was noted at which rupture occurred. The two methods gave extremely concordant results. For values of α approaching 180° the second method was in general the more convenient.

14. By way of illustration the following particulars are quoted for one pair of magnets in which the moment was 104.2 C.G.S. units, and the length, from centre to centre of the balls, was 6.78 cms. The distance between the pivots was, in this test, 7.93 cms., making $\alpha = 3.965$. The magnetic half-length r was found under these conditions to be 3.36 cms. (which was 0.03 cms. shorter than the half-length measured to the centre of the balls), and accordingly i was 1.18. The smallest value of α at which any instability was observed was $22^\circ.5$. The currents which just sufficed to produce instability were measured for a large number of values of α , and (after adding an appropriate constant for the earth's field) are shown in relation to α on the curve marked $i = 1.18$ in fig. 8. In the same figure corresponding curves are given for the same pair of magnets with two other

distances between the centres, namely, distances which made i equal to 1.21 and 1.29 respectively. The curves are symmetrical about the 180° line. It will be seen that they confirm the theory of rupture described above and illustrated in fig. 4. The change from one mode of rupture to the other was observed to occur when α differed from 180° by about 30° or 35°. The ratio of the field that would produce rupture at 180° to the field that would produce rupture in the most favourable position was found to range from about 2.35, when i was 1.18, to 2.1, when i was 1.29.

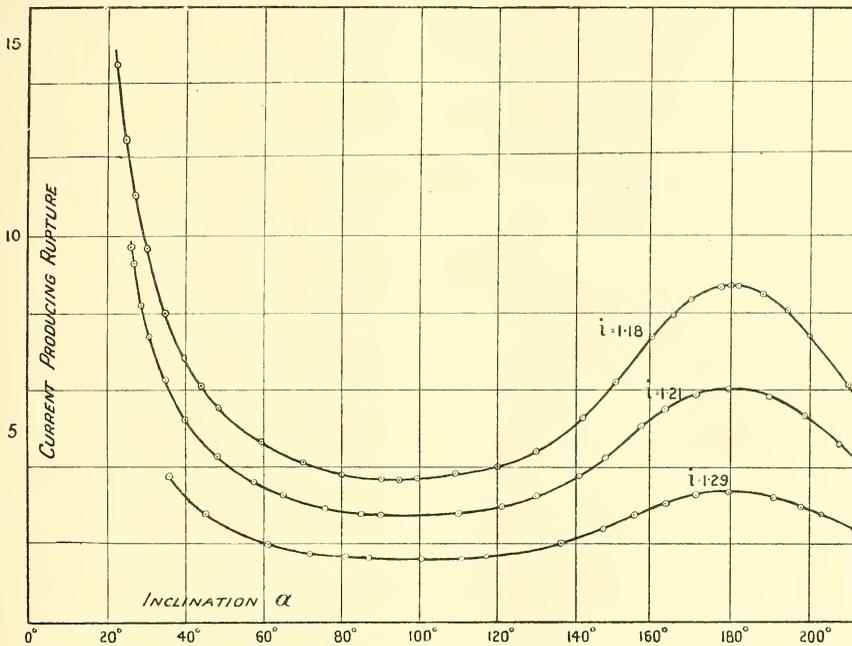


FIG. 8.—Experiments on the rupture of a pair of magnets, with various values of i .

15. From the observed value of the field which produces rupture it is easy to calculate r the true half-length of the magnet.

Applying equation (1) to a single pair of magnets we have

$$H_r = \frac{m}{8r^2} [F_r],$$

which may be written

$$H_r = \frac{M}{16r^3} [F_r]. \quad (10)$$

where M is the moment of each magnet. The value of r which will satisfy this equation is readily found by trial. For example, in the above test a was 3.965 cms., and when $\alpha = 90^\circ$ the current producing rupture was 3.75 amperes, making $H_r = 2.20$ c.g.s. If we try $r = 3.36$ cms. we have $i = 1.18$,

and for that value of i the curve of fig. 3 shows that F_r should be 12·8, consequently $\frac{M}{16r^3} [F_r]$ becomes $\frac{104\cdot2 \times 12\cdot8}{16 \times (3\cdot36)^3}$, which is almost exactly equal to the observed value of H_r . On trying $r=3\cdot35$ or $3\cdot37$, substantial discrepancies are found in excess and defect; hence $3\cdot36$ cms. is accepted as the value of r for that spacing of the magnets.

Similarly, when they were set nearer together r was found to be slightly greater, and when they were further apart it was slightly less. The following are representative figures, got by observing the current which produced rupture for various values of a , for the same pair of magnets, the angle being 90° throughout:—

Half distance a	3·80 cms.	3·965	4·28	4·51
Half length to centre of balls	3·39 "	3·39	3·39	3·39
Field producing rupture	5·10	2·20	0·94	0·62
True half length of magnets, r	3·41 cms.	3·36	3·33	3·32
Value of i	1·115	1·18	1·29	1·36

Results of the same character were got when the observed breaking field for $a=180^\circ$, instead of for $a=90^\circ$, was used as the criterion in determining r .

16. The limiting stable deflection which preceded rupture was examined by attaching a light pointer to each magnet, which allowed the angle θ to be read on a graduated arc. In this way values of θ_r were measured which agreed with those calculated above. When the direction of the field was such that a was near 180° it was observed that as the field was gradually applied the adjacent poles at first deviated to opposite sides of the line joining the centres; then with increasing H one of the two poles became dominant and pulled the other pole over to the same side as itself, and finally rupture took place in the manner described in § 10 and illustrated in fig. 5.

17. Experiments were also made on the rupture of long rows of magnets. The conditions that would apply in an indefinitely extended row were reproduced with a limited number of magnets by the device of using blocks to prevent the two terminal magnets from becoming more deflected than the others during the quasi-elastic stage which precedes rupture. Eight comparatively short magnets were used to form the row. Wooden blocks were set to limit the deflection of the end members; the blocks were adjustable by hand, and it was easy to manipulate them, while the current was being slowly increased, so that the deflection of the magnets at the ends of the row was kept, as nearly as the eye could judge, equal to that of the other magnets. The effect was that the other magnets behaved as

members of an infinite row, and rupture took place somewhere about the middle, with a field which, for every azimuth, was found to be very nearly twice as strong as that which produced the rupture of a single pair. The observed ratio was as exact as could be expected, having regard to the fact that there were unavoidable small differences among the eight magnets, as to length and moment.

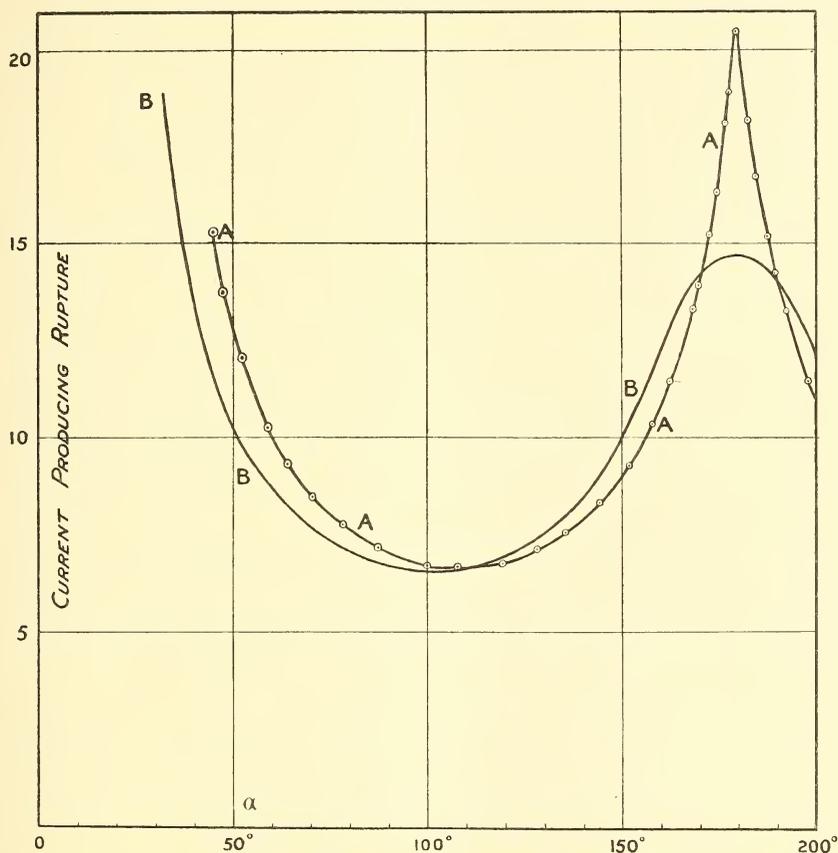


FIG. 9.—AAA, rupture of end members of a long row of magnets. BBB, rupture of a single pair of the same magnets with the same pitch.

18. The same device was used to produce another condition, namely, practical endlessness in one direction only. Taking a row of five uniformly spaced and equal magnets, one end was left free, and at the other a block was placed and adjusted so as to maintain there a state equivalent to indefinite extension, while the deflecting field was increased until rupture took place, the break-up beginning at the free end and travelling towards the other. Fig. 9 shows what was observed when the row was tested to rupture under these conditions. The five magnets used in this experi-

ment were set with a pitch ($2a$) of 4.52 cms. Their mean length from centre to centre of the balls was 3.54 cms.: their true magnetic length between poles was more nearly 3.70, making $i=1.22$. The row of magnets was so set as to bring its free end near the middle of the field of the Helmholtz coils. The curve AAA shows the current required to produce rupture for values of a ranging from 45° , which is the lowest for which instability could be clearly detected, up to 200° . The points of observation are marked by small circles. The curve BBB, which is added for comparison, relates to a single pair of the same magnets with the same half-pitch a . On comparing the two curves it will be seen that there are interesting differences between the behaviour of the isolated pair and that of the end members of a long row. For the end member of a long row the maximum at $a=180^\circ$ is much sharper as well as higher. With values of a between 110° and 170° the isolated pair required a stronger field to upset them than sufficed for the end member of a row. On the other hand, with values of a less than 110° the end member of a row is less easily upset than a single pair, and when a was less than 45° there was no instability, although with the single pair there was clear instability down to $a=32^\circ$ or so. The minimum field was nearly the same in both cases, but it was found at a considerably larger value of a for the end member of a long row.

19. These differences become accentuated when a straight-line group of three or four magnets is tested by itself, without any support at either end. Fig. 10 shows in the curve CCC the result of a test in which three of the same magnets as those of fig. 9, still with the same distance between the centres, formed an isolated group. Here again the curve BBB for a single pair is drawn to facilitate comparison. With three magnets the maximum at 180° becomes so remarkably sharp as to resemble a cusp, and the minimum is further shifted to the right, to about 120° .

20. From the above calculations and observations it follows that a model consisting simply of rows of pivoted magnets fails quantitatively to represent what happens when a ferromagnetic substance is magnetised. This becomes apparent when a quantitative estimate is made of the field that would be required to break up rows of such magnets, assuming them to be so closely spaced as to comply with the essential condition laid down in § 2. Take, for example, the case of iron. From the form of the magnetisation curve, we know that in soft and fairly pure iron the elementary magnets show instability when the applied field reaches a value of the order of 1 c.g.s. unit. A much smaller field, of the order of $\frac{1}{10}$ c.g.s., will suffice to upset them in iron that has been electrolytically deposited

and melted *in vacuo*,* though such iron shows all the characteristics of hysteresis. Compare these values with the calculated field for a row of atomic magnets in the form of rods with point poles. Let h be the side of the unit cube in the iron crystal. Since the space-lattice is the centred cube (see § 24), each unit cube contains two atoms. The volume per atom is

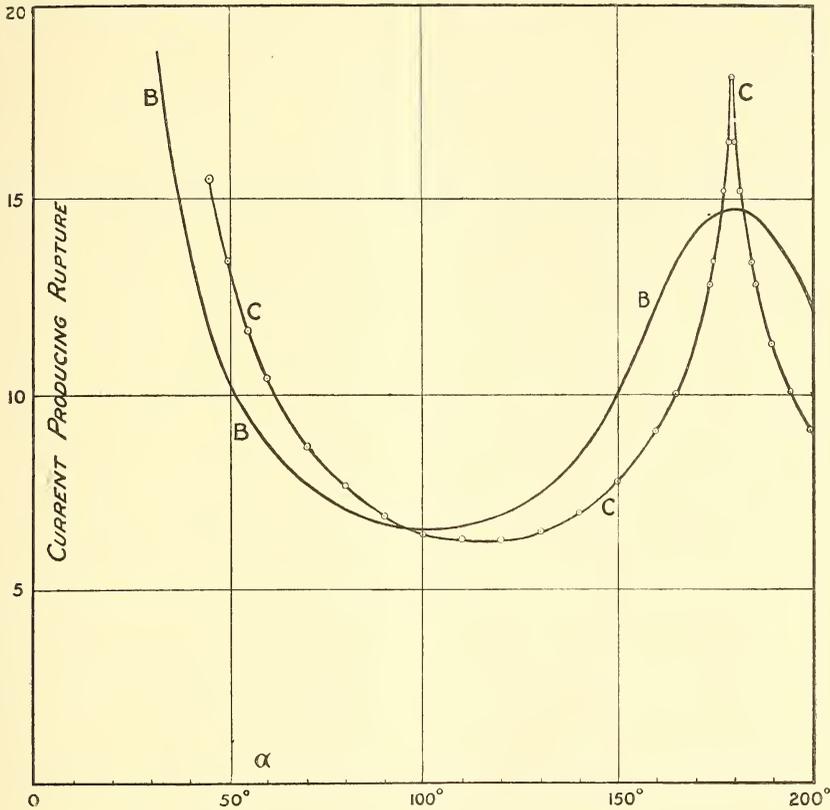


FIG. 10.—CCC, group of three equal magnets. BBB, single pair of the same magnets with the same pitch.

accordingly $\frac{h^3}{2}$, and n , the number of atoms per unit of volume, is $\frac{2}{h^3}$.

Assume that each atom contains a Weber element of moment M , and that when the iron is saturated all these elements have turned into complete alignment with the field. Then the magnetisation of saturation $I_s = nM$, and

$$M = \frac{I_s h^3}{2}$$

If we treat the atoms as spheres in contact, their diameter is one-half the

* Yensen, "Magnetic and other Properties of Electrolytic Iron melted *in vacuo*," *Bulletins of the Engineering Station of the University of Illinois*, Nos. 72, 83, 95.

diagonal of the cube, namely $\frac{h\sqrt{3}}{2}$, and this is the limiting (greatest) length that can be assigned to the atomic magnets. If we treat each atomic magnet as a rod with point poles, its half length r must be less than $\frac{h\sqrt{3}}{4}$, but will approach that value very nearly, since i is small. Hence, taking $r = \frac{h\sqrt{3}}{4}$,

$$\frac{M}{r^3} = \frac{64 M}{h^3 3 \sqrt{3}} = \frac{32 I_s}{3 \sqrt{3}}.$$

Now, the most favourable condition for rupture will occur when a row with unsupported ends lies across the direction of the deflecting field, and in that case, as we saw in § 18, the value of H which initiates rupture is nearly the same as for an isolated pair, namely (§ 15),

$$H = \frac{M}{16r^3} [F_r].$$

On substituting the above value of M/r^3 this becomes

$$H = \frac{2I_s}{3\sqrt{3}} [F_r].$$

In iron I_s is about 1700 c.g.s. units, and accordingly we should have

$$H = 654 [F_r]$$

as the field at which instability begins. Now $[F_r]$ —as was shown in § 8—is of the order of 2500, when the magnetic poles are near enough together to satisfy the condition that the elastic deflection which precedes rupture shall not exceed about half a degree (see § 2). Thus in soft iron the calculated field required to break up the rows is more than a million times as great as the field which actually suffices to make the Weber elements become unstable, when that metal is magnetised. It is obviously necessary to look for an arrangement in which the Weber elements will have a far less stable control.

21. These considerations have led me to devise a new model which, while retaining the idea of magnetic control (a feature that accords well with all the complex phenomena of ferromagnetism), escapes the excessive stability of the old model, and at the same time satisfies the essential condition that the Weber element in each atom may turn only through a very small angle before becoming unstable.

The new model is based on the idea that the system of electrons in each atom comprises two groups: an inner group which constitutes the

Weber element, possessing magnetic moment and capable of turning in response to an applied field; and an outer group or shell which is to be regarded as held more or less completely fixed relative to neighbouring atoms when the atom is part of a solid body—as for instance when it takes its place in the space-lattice of a crystal. The outer group of electrons acts on the inner group or Weber element like a number of fixed directing magnets. The mutual magnetic forces between the Weber element and the outer group determine the range through which the Weber element can turn stably, and when this range is exceeded it turns irreversibly (in a manner involving hysteresis) from one position of stability to another. In each position of stability there are strong magnetic forces acting between the Weber element and separate portions of the outer group; these make the range of stable deflection very narrow, but they are more or less completely balanced, with the result that the stability is feeble.

22. To make this clear we may consider, in the first place, the equilibrium of a magnet *W* (fig. 11) pivoted at its centre *O*, and set between two fixed magnets *A* and *B* which are oppositely directed towards *O*. Thus the fixed poles *P'* and *P''* are of the same name. Let *m'* and *m''* be their pole strengths and let *m* be the pole strength of the pivoted magnet. Assume that the clearance between *W* and the fixed magnets is small, and that it is the same at both ends. Write *r* for *OP*, the half length of *W*, and *a* for *OP'* or *OP''*. Let a field *H* act with a constant inclination *α* to the line of centres, deflecting *W* stably through a very small angle *θ*. Write *x* for *PQ* and *c* for *PP'*. Then for the pole *P* the deflecting moment is *Hm*. *OM* = *Hm r sin (α - θ)*, and the restoring moment due to *P'* is $\frac{mm'ON}{c^2} = \frac{mm'ax}{c^3}$. Taking both poles of *W* into account, but neglecting the effects of other than the nearest fixed poles, the equation of equilibrium is

$$2Hm r \sin (\alpha - \theta) = \frac{m(m' - m'')ax}{c^3},$$

or

$$\frac{2Hr}{a(m' - m'')} = \frac{x}{c^3 \sin (\alpha - \theta)}.$$

As *H* is increased the limit of stable deflection is reached when

$$\frac{d}{dx} \left(\frac{x}{c^3 \sin (\alpha - \theta)} \right) = 0.$$

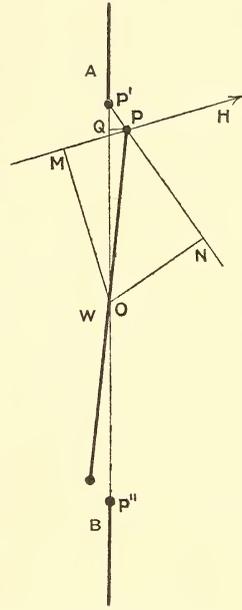


FIG. 11.

We are concerned only with a narrow range of stability; the expression may be simplified when it is applied to cases in which θ is very small and α is not small. Under these conditions $\sin(\alpha - \theta)$ may be treated as sensibly constant, and the criterion for rupture becomes

$$\frac{d}{dx} \left(\frac{x}{c^3} \right) = 0,$$

which gives $x = \frac{\alpha - r}{\sqrt{2}}$ as in § 8. The deflection ceases to be stable when the angle OPP becomes $\tan^{-1} \frac{1}{\sqrt{2}}$ or $35^\circ 16'$. The criterion is a geometrical one; it is independent of the strengths of the poles. Hence a narrow

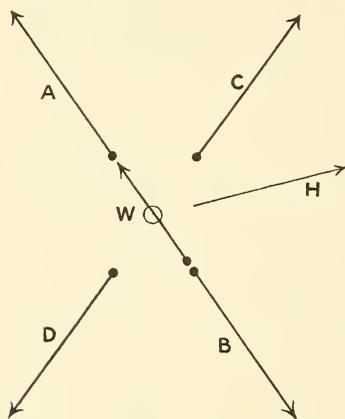


FIG. 12.

range of stable deflection is secured by setting the fixed poles near to those of the pivoted magnet W , though the resultant controlling force due to them may become vanishingly small. It will be evident that the strength of the control may be adjusted to any desired value, say by making the attracting fixed pole rather stronger than the repelling fixed pole, or by bringing it rather nearer. Or again, if P and P' are exactly equal, and are placed equally near, W may have some stability given to it from another source, but the action on it of P and P' will still determine the limit of its stable deflection.

23. When the Weber element breaks away from one position of stability it turns into another which is more favourably inclined to the impressed field H . Accordingly the outer group of electrons furnishes the equivalent not only of one pair of fixed magnets A and B , but of other pairs such as C and D (fig. 12), providing alternative stable positions into which W may be turned. The model photographed in fig. 13 (Pl. I) illustrates this action for movements in one plane. There the four fixed magnets are held in supports which allow their distances from W to be adjusted. When the clearances as well as the pole strengths are made equal, it is found that W has some stability in any one of the four positions in which it may be placed; this is an obvious consequence of the fact that the magnets of the model are not magnetically rigid, but influence one another by induction, with the effect of drawing unlike poles a little closer together and driving like poles further apart. Thus W in turning round from one position to another disturbs the magnetic

distribution in the pair of fixed magnets towards which it happens at any instant to point, in such a manner that in each of the four positions there is a margin of stability. It is possible that there may be an action in the atom corresponding in effect to this feature in the model, the result of which is to make the mutual forces between the attracting poles always a little stronger than the mutual forces between the repelling poles.

24. In a ferromagnetic metal the space-grouping of the parts that are equivalent to fixed magnets in the shell of an atom is presumably subject to the symmetry that characterises crystals of the metal. Ferromagnetic metals are cubic, and according to the X-ray analysis of A. W. Hull the space-lattice of iron is the centred-cube, in which each atom has eight nearest neighbours, set at the corners of a cube of which it is the centre.* It is natural to imagine a corresponding structure for the atom itself, in respect of the grouping of the fixed magnets. This consideration leads to a model (fig. 14, Pl. II) with eight fixed magnets pointing along the diagonals of a cube, namely, the four fixed magnets of fig. 13 and other four in a plane at right angles to the first. In the model shown in fig. 14 they are held by a skeleton cubical structure of brass at the centre of which the pivot of the Weber element may be placed. By loosening a small set-screw each of the fixed magnets can slide in its support, in the direction of its length, so that its distance from the centre may be adjusted. The Weber element and its pivot are omitted in this photograph. If the Weber element is a simple bar magnet, as in fig. 13, it will turn by steps of $70^{\circ} 32'$ in passing from one position of stability to another, and in the absence of an impressed field there are eight possible positions of stability.

25. But the Weber element may itself consist of more than a simple magnet with two poles. It may, for example, be made up of an octet of poles set at the eight corners of a cube, constituting a magnetic system which turns as a whole and possesses magnetic moment, four poles on one face of the cube being of one name, and the opposite four of the other name. In the model atom which is shown in fig. 15 (Pl. II) this structure is realised; the Weber element is there formed by screwing ball-ended steel wires into the eight corners of a small cubic steel boss, and magnetising the whole to give opposite polarities on opposite faces. The least angle through which it turns in passing from one position of stability to another is 90° . When there is no impressed field there are twelve possible positions. In

* *Phys. Rev.*, ix, p. 84, 1917; x, p. 661. This is confirmed by A. Westgren, *Jour. Iron and Steel Inst.*, ciii, p. 303, 1921.

the model as constructed the octet of poles which represents the Weber element is pivoted on a vertical needle, and therefore turns about one axis only, but of course it has to be thought of as having three degrees of rotational freedom. To allow the construction to be more clearly seen, fig. 15A (Pl. II) shows the Weber element removed from its place and standing alongside the frame of fixed magnets.

In another model, shown in fig. 16 (Pl. II), the same outer fixed group is used, but the Weber element consists of a quartet of poles occupying the corners of a regular tetrahedron—two north poles and two south poles. This form is readily derived by removing four of the eight spokes of the Weber element which is shown in figs. 15 and 15A.

A. W. Hull has suggested that the twenty-six electrons of the iron atom are probably grouped so that two of them form a doublet with the nucleus of the atom in the middle, and the remaining twenty-four are spaced further out along the diagonals of a cube, forming three octets. The members of each octet occupy the corners of a cube.* On the basis of this suggestion we may think of the outer octets as supplying the four pairs of fixed magnetic elements shown in the models. The Weber element may be constituted simply by the inner doublet; on the other hand, it may include the innermost of the three octets, leaving the other two octets to form the fixed magnets. In the latter case it may be conjectured that the inner doublet is also included in the part that turns. According to either hypothesis the Weber element—the thing that turns during magnetisation—need not be more than a comparatively small part of the atom. That it is only a very small part has been inferred by Messrs A. H. Compton and O. Rognley from the fact that no change in the X-ray spectrum reflected from the face of an iron crystal could be detected when the crystal was magnetised.†

On this point, however, it should be observed that if in the atoms of a crystal each Weber element is an octet of electrons (or any cubically symmetrical system), its being oriented, by magnetising the substance, into the most favourable of its positions of stability cannot be expected to alter the X-ray spectrum, no matter what proportion its dimensions bear to those of the complete atom. In the model atom there is no difference to the eye between one and another of the positions of stability. It is therefore of doubtful validity to conclude that the dimensions of the Weber element are minute in relation to those of the atom, and it is open to suppose that the central group of electrons, which in my view constitutes the Weber element, extends far enough from the nucleus to

* *Phys. Rev.*, ix, p. 85, 1917.

† *Ibid.*, xvi, p. 464, 1920.

be susceptible to influences of temperature, vibration, strain, and so forth, to which the nucleus itself would be impervious.

26. We have next to consider what circumstances affect the stability of the Weber element when controlled in the manner here suggested. Suppose, as an ideal case, that the opposing actions within each atom (taken separately) are exactly balanced; in other words, that the opposed poles of the fixed magnets are exactly equal, and that the clearances are exactly equal. Suppose also that there is complete magnetic rigidity, so that the presence of the Weber element between the opposed fixed poles does not disturb the symmetry of their action. Imagine atoms in which there is this complete balance of forces to have their centres arranged in rows, as they would be in a crystal. Then there is still some stability on the part of the Weber elements, for they exert magnetic forces on one another from atom to atom as in my old model, and their mutual magnetic forces tend to make them form rows which can be broken up by the application of a feeble external field. Thus in fig. 17 let there be two adjoining atoms with (in one line) the respective fixed poles A_1B_1 and A_2B_2 , and the respective Weber

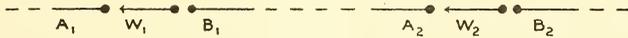


FIG. 17.

elements W_1 and W_2 . The mutual action between W_1 and W_2 will give them some stability even if the action on them of the fixed poles is exactly balanced. When a transverse field is applied there will be, as there should be, a small stable deflection preceding rupture, then a violent swinging over into new rows more favourably oriented to the direction of the field; and when the field is removed they will remain in rows that are so oriented as to give the piece residual magnetism. It is not impossible that in a crystal of pure unstrained iron the resultant controlling force which determines the stability of the Weber element in each atom may be entirely due to the mutual forces which the Weber elements exert on one another from atom to atom. The coercive force in pure annealed iron is so weak as to favour the idea that it may be explained in this way.

On the other hand, the structure of the atom may be such that the attracting poles always act more strongly on one another than the repelling poles, so that in every atom the Weber element has an independent stability. It is at least possible that in the "fixed" parts of the atom there is not complete magnetic rigidity, and that the Weber element is made stable by internal displacements which are due to the forces between it and the shell, namely, by the drawing together of attracting poles, and the pushing apart

of repelling poles, in the manner already suggested in § 23. In either case the Weber element becomes stable in whichever of the possible positions it may assume: in the first case, because it is a member of a row of other Weber elements which have turned into the same direction; in the second case, because the symmetry of the opposed internal forces is disturbed by the Weber element itself, and the disturbance travels round with it as it turns from one to another of the possible positions of stability.

Of the two hypotheses, the second appears to consort better with the idea that the atom is an elastic structure, in which the shell at least is liable to have its form altered by such forces as come into play when the piece is strained. It may be added that it is easy to imagine, and to realise in a model, a Weber element which will have its projecting poles of one name stronger than those of the other name, and this offers another (but less probable) means of accounting for the stability.

27. The effects of stress on magnetic quality are too intricate* to be deduced in detail from the behaviour of so simple a model, which, at the best, cannot claim to do more than represent crudely the mechanism of magnetisation. We should need a fuller knowledge of the distribution of the electrons and of the forces which determine their positions, before speculating about such features as the Villari reversal in iron, or about those differences which have been observed in the magnetic effects of stress in iron, cobalt, and nickel.† But a notable characteristic shared by all ferromagnetic metals is that a simple stress—a pull or a push—produces magnetic æolotropy. This is well reproduced in the model. A pull, for example, applied along one line will increase the clearance for those “fixed” poles which lie more or less in that line, and reduce it for others, with the result that a piece composed of model atoms will show different qualities as to magnetic susceptibility and retentiveness in the longitudinal and transverse directions. The main magnetic effect seen in iron, under fairly strong fields, and more conspicuously in nickel, is that a simple push gives the metal greater susceptibility and very much greater retentiveness in the direction of the push, which is the kind of change the model would lead one to expect.

28. The model also throws light on the influence of impurity in a ferromagnetic metal. When a foreign atom occupies one of the places of the

* For a summary of effects of stress see chapter ix of the author's book on *Magnetic Induction in Iron and other Metals*.

† Hull (*Phys. Rev.*, May 1921) finds that the space-lattice of nickel is the face-centred cube. This gives each atom twelve nearest neighbours. If the shell electrons assume a corresponding grouping, the nickel model should have twelve “fixed” magnets, set in lines inclined to one another at 60°, 120°, and 180°.

space-lattice, the neighbours will be affected in such a way that the balance of forces between the fixed magnets in them will be upset. Each of the neighbours will find itself unsymmetrically surrounded; the symmetry of magnetic forces within it will be disturbed, and the control of the Weber element will consequently be increased. This action will no doubt extend, in diminishing degree, through several layers of atoms near each foreign particle, and it will be irregularly distributed throughout the piece as a whole. Compared with pure metal, the piece will therefore lose some of its susceptibility, and also become less magnetically homogeneous, with the result that the curve of magnetisation will show more rounded outlines, as well as a higher coercive force. This agrees with what is observed.

These remarks apply in general terms, whether the foreign atoms are present in mixture, or in solid solution, or in combination with those of the ferromagnetic metal. In a chemical compound, say of iron, the distortion of the iron atom that comes of its being unsymmetrically surrounded may cause its magnetic properties to differ widely from those of an atom of uncombined iron, and may be associated with the magnetic æolotropy which is a well-known feature of the crystals of certain iron compounds. With the model shown in fig. 15 (Pl. II) we can imitate conditions of extreme æolotropy such, for instance, as Weiss found in crystals of pyrrhotite.* Suppose that in the model one pair of opposite fixed magnets are advanced until they touch one pair of opposite poles in the turning octet which represents the Weber element. One axis of the octet (namely, a trigonal axis) thereby becomes fixed, and it can turn only by rotation about that axis. It turns through 120° in passing from one position of stability to another. This state of things is illustrated in the photograph (fig. 18, Pl. II), where the central pivot is removed and the octet is supported by contact of poles in the axis AB, about which it can turn, two of the ball ends being slightly cupped to keep it from falling out of place. The behaviour of the model now resembles that of pyrrhotite, which takes up magnetic induction readily in one plane but not in a direction perpendicular to that plane. The other fixed magnets may at the same time be adjusted to exhibit differences of susceptibility along different axes in the plane of magnetisation. Such differences were in fact observed in pyrrhotite by Weiss. His experiments showed that when a pyrrhotite crystal was rotated about its non-magnetic axis in a constant field, acting in the plane of magnetisation, there were more or less abrupt magnetic changes at intervals of 60° . Inspection of the model suggests that these may be consequences of the turning of the Weber element about one of its trigonal axes. The project-

* *Jour. de Phys.*, iv, p. 469, 1905.

ing poles of the turning octet lie in planes inclined to one another at 60° round the axis about which it turns, and the successive positions of stability are consequently 120° apart. Hence we have only to think of the crystal as made up of successive layers parallel to the magnetic plane and in twin relation to one another (each turned through 180° with respect to the adjoining layer), to obtain abrupt magnetic changes at intervals of 60° in the rotation of the crystal, though in each iron atom of the compound the electron grouping remains substantially cubic.

29. The forms of the new model that have so far been described may be said to be based on an atom of the Lewis-Langmuir type, in which the electrons are conceived to have small orbits or to be small rings. But the same ideas may be embodied in another form, based on an atom of the

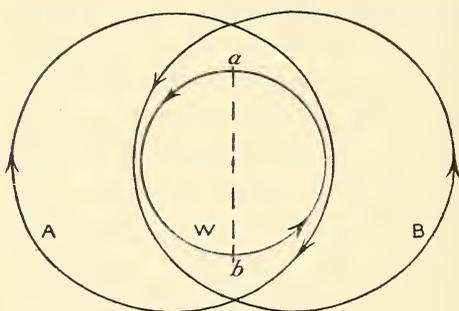


FIG. 19.

Rutherford-Bohr type, with large electron-orbits encircling the nucleus. Thus in fig. 19 the Weber element is represented by a central electron-orbit W, while A and B are a pair of coplanar elliptic orbits with a common focus at the nucleus of the atom, which is also the centre of W. The plane of A and B is to be thought of as fixed; the plane of W may turn.

Under the influence of a deflecting field it turns stably through a small angle and then becomes unstable.

The model photographed in fig. 20 (Pl. I) shows a corresponding arrangement in which the electron-orbits are represented by coils carrying currents which have the directions indicated by the arrows in fig. 19. In the model the controlling elliptical coils are set as nearly as may be in one plane, and the central coil is pivoted to turn about the axis ab , with only a small amount of clearance on each side. A complete model would of course have several pairs of controlling coils forming a symmetrical system in three dimensions, and its Weber element (which might consist of one or more coils) would have freedom to turn in any direction. It would obviously pass from one to another position of stability in such a way as to exhibit hysteresis and the other characteristics of the magnetising process. In any one position the stability would depend on an inequality of controls, just as in the model with magnets, and there would be a correspondingly narrow range of stable deflection. Generally, what has been said about the model with magnets will apply to this one.

If the atomic shell or system of fixed elliptic orbits is to have cubic

symmetry, the least number of orbits constituting it (paired as in fig. 19) would be twelve. These might be made up of three groups of two pairs each, arranged as in fig. 21, in mutually perpendicular planes. But a grouping apparently more appropriate to an atom which is to take its place in a centred-cube space-lattice, would be one in which three pairs of orbits are symmetrically placed (with their major axes inclined at 60°) in each of the four octahedral planes—that is to say, in planes perpendicular to the trigonal axes.

30. Reverting to the model of § 25, it is interesting to notice that the pivoted octet which there constitutes the Weber element may be deprived of magnetic moment (without other change) by reversing, end for end, four of its eight spokes, with the result that all the projecting poles have the same name, the central steel boss then forming a common pole of the opposite name. When this is done, the pivoted element, having as a whole no magnetic moment, is not caused to turn by applying an external field. It sets itself in a stable position with reference to the eight fixed poles of the shell, namely, with its spokes pointing towards them (just as in fig. 15, Pl. II) if its own poles have the opposite name, or in the position of fig. 22 (Pl. II) if its own poles have the same name. It

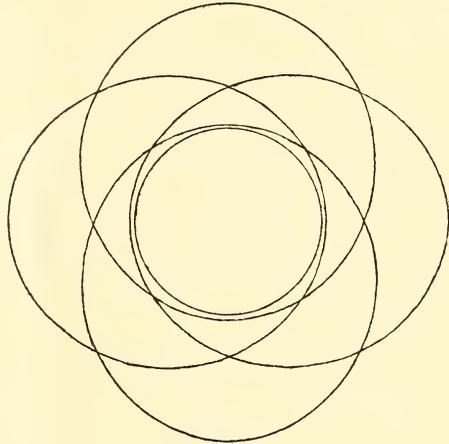


FIG. 21.

is obvious that in either case a substance made up of such atoms has lost its capacity for magnetisation, though each atom is still composed of the same fixed parts, and of a moving part comprising the same magnetic constituents, similarly grouped except that, instead of exposing four north poles and four south poles, they now expose eight poles of the same name. The geometry of the atom is otherwise unchanged. This transformation, which is readily produced in the model by making a simple rearrangement in the magnetised spokes of the central octet, is suggestive of what occurs when iron is heated to the critical temperature at which it loses magnetic quality. As is well known, when iron passes from the magnetisable to the non-magnetisable state there is an arrest in the process of heating—a point at which energy is absorbed—and there is a corresponding arrest when the magnetisable state is recovered in the process of cooling. But as Westgren has shown, X-ray analysis reveals no change at this point in the crystal

structure: the cube-centred space-lattice persists at temperatures above as well as below the magnetic critical point.* The model will serve to illustrate how a complete loss of magnetic quality may occur without apparent change in the structure of the space-lattice or even in that of the atom itself.

31. In conclusion, it may be useful to attempt a summary of chief points, distinguishing those that are well established from those that are conjectural.

Magnetic induction in ferromagnetic solids is characterised not only by its relatively great amount but by the phenomena of saturation and hysteresis. It cannot be doubted that the process occurs through the turning of "Weber elements" which possess magnetic moment. The thing that turns is not the molecule nor the atom, but something within the atom. In a pure magnetic metal—say iron in the normal state—each atom has in it a part possessing magnetic moment,† which turns in response to an impressed magnetic field. Its turning is opposed by controlling forces which give it stability in a number of different angular positions. When it is displaced by an impressed field from any one of these positions, its deflection is at first reversible; but if a narrow range of stable deflection be exceeded, the turning element becomes unstable: it breaks away and falls over, with dissipation of energy, into another position of stable equilibrium.

Thus when a field is first applied to a previously unmagnetised piece, the initial deflections of the Weber elements are reversible: hence within a very narrow range the magnetisation is quasi-elastic, and there will be no residual magnetism if the field be withdrawn. Similarly, at any stage in a magnetising process a change from increasing to decreasing field, or *vice versa*, is marked by a short reversible stage in the turning of the Weber elements. All these points are well established.

In my theory of 1890 the stable and unstable stages were explicitly recognised, and it was pointed out that hysteresis arose from the falling over of the Weber element from one position of stability to another. Under conditions which prevent that falling over there is no hysteresis.

* *Loc. cit.*, p. 315 *et seq.* "The so-called β iron has the same lattice as the α iron. . . . No difference has been found in the structure of iron below and above A2. At A3, however, the atoms of iron are completely rearranged, and the iron passes from one crystal class into another." The magnetic change occurs at the lower arrest point A2.

† From known data as to the space-lattice of the iron crystal and the saturation limit of magnetisation in iron, it is easy to show that the moment of the Weber element in an atom of iron is very nearly 2×10^{-20} c.g.s. units. (See *Proc. Roy. Soc.*, February 1922, vol. 100, A, p. 453.)

This point received a curious development at the hands of Mr James Swinburne, who remarked that according to the theory there should be no dissipation of energy if a cylinder of iron were caused to turn slowly in a constant magnetic field of great strength, for then the Weber elements would remain always parallel to the field with no unstable phase in the course of their turning.* The result was unexpected, but it was experimentally confirmed by Professor F. G. Baily.†

In all probability the Weber element is part of the electron system of the atom, deriving its magnetic moment from the motion of electricity, and the control under which it turns is electromagnetic. My theory of 1890 ascribed that control solely to the mutual magnetic forces between each Weber element and its neighbours. For reasons which have been explained in this paper, though the mutual forces between neighbouring Weber elements must not be overlooked, I now ascribe the control mainly to magnetic forces which are exerted on the Weber element by other portions of its own atom. When this action is taken into account we find the Weber element to be capable of a narrow range of reversible deflection, followed by unstable movement into another position of stability, without unduly strong control. Thus a quantitative correspondence becomes possible between the magnetic field which is required to upset the element and those fields which are known to produce strong magnetisation in iron and other ferromagnetic metals. The modified theory accordingly escapes a fundamental objection to which the original theory was open; at the same time it retains all the features in which that theory was found to agree with the observed facts of magnetic induction.

A basic feature in the modified theory is that it recognises two parts of the atom: a part which turns in response to an applied field, and a part which does not turn—which is fixed in relation to neighbouring atoms, but exerts control on the turning of the other part. This idea harmonises so well with the behaviour of ferromagnetic substances that it is offered for acceptance with considerable confidence. But in suggesting any apportionment of the electron system of the atom into these two parts we enter a region of speculation. The models which are described here claim no more than an illustrative value. They show how, on that basis, one may imagine groupings by which the observed phenomena can be closely reproduced, whether the electron structure is assumed to be that suggested by Rutherford and Bohr or that suggested by Lewis and Langmuir. They imitate the form of the magnetisation curve in a cyclic process: they hint at explanation of the known effects of strain,

* *Industries*, Sept. 19, 1890.

† *Phil. Trans.*, vol. clxxxvii, A, p. 715, 1896.

of temperature, of vibration, and of what is found when magnetic atoms are mixed or compounded with non-magnetic atoms. To the exact form of the models, however, no special importance should be attached: they will have served their purpose if they make the general ideas which they illustrate more intelligible and more widely known.

Note added March 15, 1922.

Since the reading of the paper, Professor Whittaker has made to me the interesting suggestion that these models, apart from their bearing on the theory of ferromagnetic induction, may be applied to elucidate a point in quantum theory. This is because, in the model, a definite amount of energy is absorbed when the part that turns is displaced irreversibly from any one of its various positions of stable equilibrium, and a definite amount of energy is given out when that part settles again into a position of stable equilibrium.

To appreciate the significance of this property of the model, from the point of view of quantum theory, it should be recalled that (as was pointed out in § 30) the part that is capable of turning need not necessarily possess magnetic moment. When used to represent an atom of a substance which is not ferromagnetic, the model may contain an inner part which, although it does not possess magnetic moment, is capable of turning relatively to the outer part, but is held in relation to the outer part by forces between its poles and those of the outer part.* A model satisfying this condition is easily made, for example as in fig. 22 (Pl. II), by having the part W present nothing but like poles to the operative poles of the surrounding portions of the atom. When the atom is placed in a homogeneous magnetic field, the part W will have no tendency to turn. But imagine it to be exposed to a transient deflecting action such as might be caused by an encounter with another atom or with a free electron. Assume, for example, that an electron has an encounter with one side. In that event W will be deflected, and, if the transient deflecting action be sufficiently strong, W will be pulled away from its first position of stability and will fall, after the encounter, into one or another of the possible positions. It will therefore oscillate until the energy of the oscillation is dissipated by the radiation which is thereby given out.

* The words "inner" and "outer" are retained only as a convenient means of fixing the ideas. We are concerned here simply with relative motion between parts of the atom, with elastic relative displacements through a limited range, with the breaking away from a stable configuration, and with the dissipation of energy that attends the settling again into a stable configuration.

In order that this should happen, the transient deflecting force must do an amount of work sufficient to cause W to break away from its original position of stable equilibrium. If W break away, the encounter has the result that a definite amount of energy is communicated to the atom of which W is a part, and this energy is the source of the consequent radiation.

Suppose, on the other hand, that the work done by the transient deflecting force is insufficient to cause W to break away. In that event no energy is communicated in the encounter as a whole. During the first part of the encounter, work will be done in reversibly deflecting W , within the limited range in which it remains stable; but since, by hypothesis, that range is not exceeded, the work done in deflecting W is recovered without loss as it returns to its initial or zero position. The encounter is elastic: there is no dissipation of energy. Oscillations are not set up. The atom does not emit radiation in consequence of the encounter.

Return now to the case of an inelastic encounter in which W breaks away from a position of stable equilibrium and settles down, when its oscillations have subsided, again in a position of stable equilibrium. If we assume the energy to be equal in both positions, it follows that the work done in causing W to break away is entirely spent in producing radiation. That work represents energy which was received in the encounter. Suppose the energy to have come from bombardment by an isolated electron. Then the least kinetic energy which such an electron must possess, in order to make the encounter inelastic and consequently to set up vibrations, is equal to the definite amount of work that is required to make W break away from its position of stability. If the electron possesses just that amount of energy (and encounters the atom in the most effective manner), it will give up all its energy in the encounter. If it possesses more, it will still give up only that amount of energy. If it possesses less, it will give up no energy.

Thus the model serves to illustrate the mechanical possibility of an encounter which involves no scattering of energy except under quantum conditions. It receives energy in quanta and gives out corresponding quanta of radiation.

This action of the model, as Professor Whittaker points out, has a direct bearing on the experiments of Franck and Hertz, M'Lennan, and others, on the emission of radiation by metallic vapours and other gases under the stimulus of a bombardment of electrons.* In these experiments the bom-

* Franck and Hertz, *Verh. d. deutsch. Phys. Gesells.*, 1914, p. 457 and p. 512; M'Lennan (and pupils), *Proc. Roy. Soc.*, vol. 91, A, p. 485; 92, A, p. 305 and p. 574; *Phil. Mag.*, Dec. 1918. See also Davis and Goncher, *Phys. Rev.*, Aug. 1917 and Jan. 1919; Goncher, *Proc.*

barding electrons are set free from a heated body and are caused to acquire a suitable velocity by passing through a space in which there is a measured drop of electrostatic potential. When their velocity is less than a certain limit, there is no visible radiation from the atoms of the gas. When the velocity is increased, by bringing the drop of potential up to a certain value V_0 , radiation of a single definite wave-length begins to be emitted. When the velocity is further increased (by raising the potential drop beyond V_0), the same definite wave-length, and no other, continues to be emitted; until finally, when the velocity is raised above a second and generally much higher limit, ionisation sets in and a many-lined spectrum is produced. The relation of V_0 to the frequency ν of the monochromatic radiation which the atoms continue to emit until they are ionised—that is to say, until the bombardment becomes strong enough to knock electrons out of them—is found to satisfy the quantum condition

$$eV_0 = h\nu,$$

where e is the charge of an electron and h is Planck's constant. Thus with mercury vapour it is found that a potential drop of 4.9 volts is required to give the bombarding electrons velocity enough to make the atoms of mercury emit any radiation. When the potential drop is raised to that value, they are observed to give out monochromatic light with a wave-length of 2536.72 Å.U. When the potential drop is further raised, the same single-line spectrum continues to be emitted until the potential drop is 10.2 volts, when the many-lined spectrum appears. The suggestion made in this note deals, first, with the conditions under which the encounter is elastic, giving rise to no radiation; and, second, with those under which a single-line spectrum is produced, when the bombarding electrons have sufficient, or more than sufficient, kinetic energy to upset a stable grouping of the electrons in the atom, but have not enough energy to deprive the atom of any of its electrons. In the second case, after the encounter has taken place the parts of the atom again assume a stable grouping, but with oscillations that give out a train of electromagnetic waves.

The model can reproduce such effects, and in doing so it goes some way towards reconciling quantum theory with ordinary dynamics.

Phys. Soc., vol. xxxiii, p. 13; Horton and Davies, *Proc. Roy. Soc.*, 95, A, p. 408; *Phil. Mag.*, June 1921; Horton and Bailey, *Phil. Mag.*, Oct. 1920; Mohler, Foote, and Stimson, *Phil. Mag.*, July 1920.

(Issued separately April 27, 1922.)

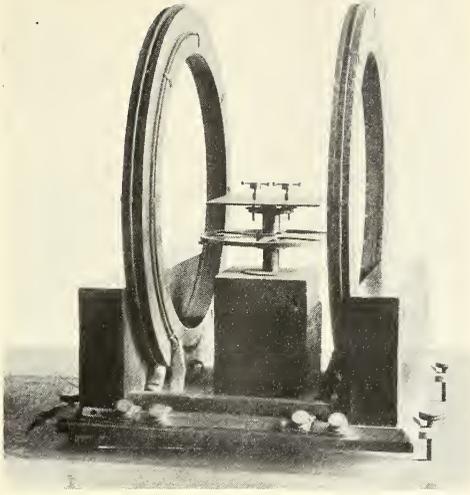


FIG. 7.—Helmholtz coils and turn-table for experiments on the rupture of pairs and rows of magnets.

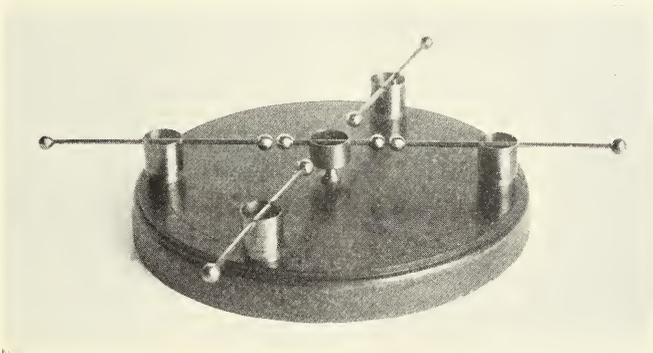


FIG. 13.—Model with magnets grouped in one plane.

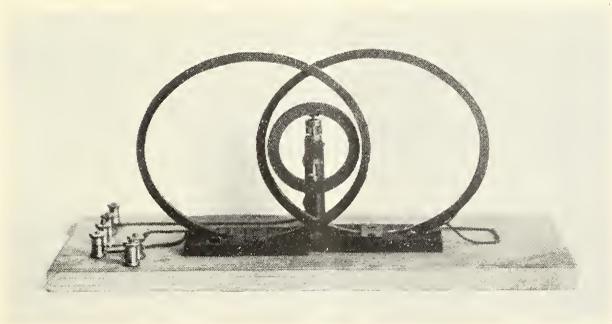


FIG. 20.—One-plane model with elliptical coils.

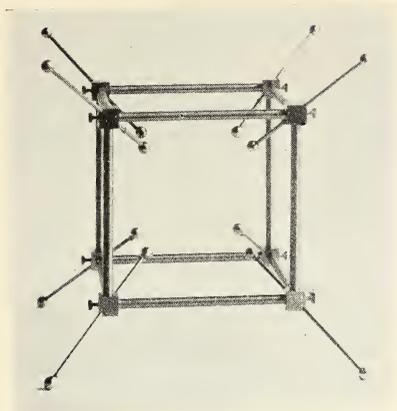


FIG. 14.

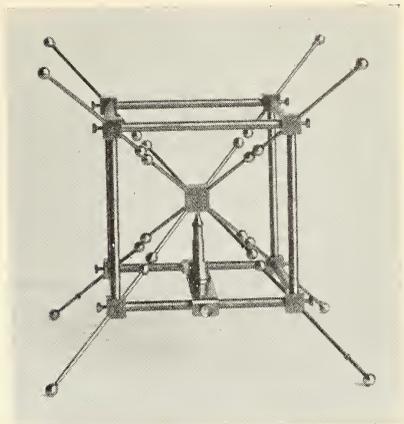


FIG. 15.

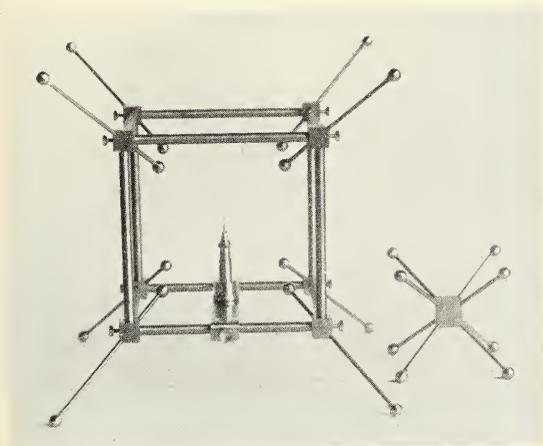


FIG. 15A.

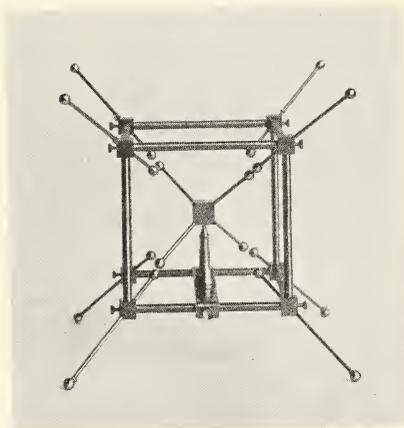


FIG. 16.

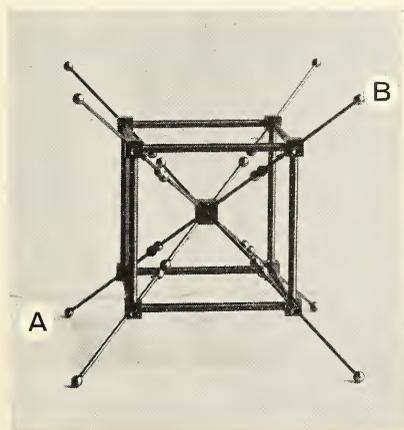


FIG. 18.

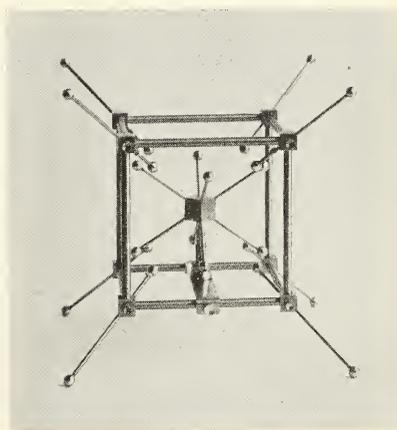


FIG. 22.

Cubic models with eight fixed magnets.

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X.—On the Quantum Mechanism in the Atom.

By Professor E. T. Whittaker, F.R.S.

(MS. received April 12, 1922. Read May 8, 1922.)

§ 1. INTRODUCTION.

It is now well established experimentally that when an atom is caused to emit radiation of frequency ν by collision with an electron, the amount U of the kinetic energy of the electron which is absorbed by the atom is given by the equation

$$U = h\nu,$$

where h denotes Planck's quantum of Action: an electron whose kinetic energy before the encounter is less than $h\nu$ is incapable of stimulating the atom to emit the radiation, and is merely repelled from the atom without any loss of energy. It has long been known that the relation $U = h\nu$ exists between the frequency of X-rays and the kinetic energy of the cathode rays which produce them: and since the discovery of Franck and Hertz in 1914 that when mercury vapour is traversed by electrons possessing the kinetic energy acquired in passing through a potential fall of about 4.9 volts the vapour is stimulated to the emission of the radiation of wave-length $\lambda = 2536.72$ A.U., a great number of experimental investigations have been published dealing with "single-line spectra": the relation $U = h\nu$ is always satisfied.

The aim of the present paper is to investigate the mechanism within the atom which compels all exchanges between the kinetic energy of electrons and radiant energy to conform to the equation $U = h\nu$. In § 2, by a process of deduction from the known results of experiment, it is inferred that the mechanism within the atom must be such that an electron approaching an atom induces in the atom what is here called a "magnetic current": and in § 3 it is shown that this leads directly to the consequence that the atom can absorb energy from the electron only in quanta. In § 4 it is shown that the disturbance in the atom after the collision consists in the displacement of a single electron, and it is further shown that the radiation emitted by the electron during its oscillatory subsidence to its normal state must satisfy the equation $U = h\nu$.

§ 2. THE NATURE OF THE INTERACTION BETWEEN AN ATOM
AND AN APPROACHING ELECTRON.

Let us now consider an electron approaching an atom with a velocity which is not great enough to ionise the atom, but which may be great enough to evoke a "single-line spectrum." We shall speak of the encounter as a "collision," understanding thereby not any actual contact of the two bodies, but simply that when they are close to each other there are forces between them which may involve a considerable exchange of energy.

The experimental results indicate that the electron, as it approaches, experiences a repulsion which is sufficient to turn it back altogether if its kinetic energy is less than $h\nu$. We have first to determine the nature of this repulsion. Now there are two kinds of force which are capable of acting on an electron, namely, electric force and magnetic force, and we have to decide which of these two is operative in the present case. The decision is easy, since we know that motion through a field of electric force affects the kinetic energy of an electron, while motion through a field of magnetic force deflects its direction of motion without altering its energy: and in our case the fate of the electron depends entirely on whether it possesses enough kinetic energy to force its way through the field. The field must therefore be electric: that is to say, *an electron which is approaching an atom experiences, in the vicinity of the atom, a field of electric force.*

We have now to consider whether this field of electric force is always present in the vicinity of the atom, whether the electron is there or not, or whether the field is evoked in some way by the approach of the atom. Here again the decision is not in doubt: the atom cannot maintain a permanent electric field in its vicinity unless it either contains an excess of electricity of one sign, or is permanently polarised electrically: both of which suppositions can be ruled out, since the atoms considered are neutral atoms which do not respond to an ordinary electric field. We thus infer that *the electric field about the atom is not permanent, but is evoked by the approach of the electron.*

We have now to consider what kind of mechanism within the atom provides this responsive electric field when the electron approaches.

A first attempt at an explanation might be made by supposing that there are within the atom elements which behave like conducting surfaces: for if an electric charge is brought into the neighbourhood of a conducting surface, the portions of the surface nearest and furthest from the charge become oppositely electrified, and an electric field is thus set up. Any explanation of this type must, however, be abandoned by reason of the fact

that the fields thus set up are always such as to *attract* the charge towards the conductor : whereas we have to explain the setting up of a field which *repels* the charge away from the atom.

As a second attempt at an explanation, we might suppose that there are in the atom a number of permanent electric dipoles pivoted at their centres, which, when the atom is isolated, take up positions of equilibrium in various directions so as to neutralise each other's fields externally, but which, when an electron approaches the atom, are drawn out in parallelism to each other, and thus set up an electric field. This explanation must, however, be rejected for the same reason as the foregoing, namely, that such an arrangement would cause an attraction instead of a repulsion between the electron and the atom.

Now this latter attempt at an explanation is really an electric analogue of the arrangement of small magnets within the atom by which the facts of induced magnetisation are accounted for in paramagnetic bodies : and this remark leads at once to an observation which supplies the key to the difficulty : namely, that what is wanted in our case is an electric analogue of the mechanism of induced magnetisation in *diamagnetic* bodies.

Induced magnetisation in a diamagnetic body is, however, accounted for by supposing that the inducing magnet, as it approaches the body, induces electric currents in the atoms of the body. We are driven to conclude that a corresponding explanation, or at least a corresponding set of mathematical equations, applies in our case, and thus infer that *the electron, as it approaches the atom, induces within the atom a "magnetic current,"* i.e. *the magnetic analogue of an electric current* : or at any rate induces something which behaves like a magnetic current.

In order to show how this can be done, we shall now describe a model in which the process can be seen at work.

§ 3. THE MODE OF ABSORPTION OF ENERGY BY THE ATOM FROM THE ELECTRON.

Our task is now to describe a mechanism within the atom which will respond to the approach of an electron by setting up a "magnetic current." If the model corresponds to nature, we ought to find that the field thus set up by the mechanism will drive the electron away from the atom, without any loss of energy, unless the kinetic energy of the electron comes up to a certain standard, so that there is a "perfectly elastic impact" : while if the kinetic energy of the electron exceeds this standard, the electron penetrates the structure and loses precisely this standard

amount of kinetic energy. The success or failure of the model must be judged by its ability to act in this way.

Suppose that a bar magnet, of length a and magnetic moment $\mu\alpha$, is pivoted at the origin (which is one of its magnetic poles) so that the bar is free to rotate in a plane, say the plane of yz . Let ψ be the angle which the bar makes at time t with the axis of y , so that the co-ordinates of the extremity (the other pole) at time t are

$$(0, a \cos \psi, a \sin \psi).$$

Suppose, moreover, that an electron of charge e is free to move along the axis of x , its co-ordinates at time t being $(x, 0, 0)$. When the electron is in motion, it creates a magnetic field, the magnetic force at the movable pole of the magnet due to this cause being of magnitude $\frac{ea\dot{x}}{r^3}$, where r denotes the distance from the electron to the movable pole of the magnet, and being directed at right angles to the magnet in the plane yOz ; so that it tends to set the bar magnet in rotation about the origin in the plane yOz , just as if instead of the moving electron we had an electric current along the axis of x . Thus if the moment of inertia of the bar magnet about the origin is denoted by α , its equation of motion is

$$\alpha\ddot{\psi} - \frac{\mu ea^2\dot{x}}{(\alpha^2 + x^2)^{\frac{3}{2}}} = 0 \quad \dots \dots \dots (1)$$

Moreover, when the bar magnet is rotating, it sets up an electric field, the electric force at the electron along the axis of x due to this cause being $\frac{\mu\alpha\dot{\psi}}{r^3}$: so that if the electron has a mass m , its equation of motion is

$$m\ddot{x} + \frac{\mu ea^2\dot{\psi}}{(\alpha^2 + x^2)^{\frac{3}{2}}} = 0 \quad \dots \dots \dots (2)$$

We can further suppose that there are several such bar magnets rigidly connected like the spokes of a wheel, so that they rotate together in the plane of yz , each magnet having one of its poles at the origin of co-ordinates and having its direction at every instant radial from the origin. We thus obtain a structure essentially similar to one of those which have been proposed by Sir Alfred Ewing* for the purpose of explaining induced magnetisation. Denoting by M the sum of the values of μ corresponding to the different bar magnets in the structure,

* "On Models of Ferromagnetic Induction," *Proc. R.S.E.*, 1922. The structure in question is described in § 30 of his paper, and is depicted in plate ii, fig. 22. It was while I was listening to Sir Alfred Ewing's exposition of his model before the Royal Society of Edinburgh that the ideas which have led to the present paper originated.

and denoting by A the sum of the values of α , equations (1) and (2) become respectively

$$A\ddot{\psi} - \frac{Mea^2\dot{x}}{(a^2 + x^2)^{\frac{3}{2}}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$m\ddot{x} + \frac{Mea^2\dot{\psi}}{(a^2 + x^2)^{\frac{3}{2}}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The magnetic structure now consists essentially of a number of magnetic poles revolving in a circle of radius a , their corresponding poles of contrary sign being at rest at the centre of the circle. We can speak of this arrangement as a *magnetic current*, since it is in some sense a magnetic analogue of an ordinary electric current formed by the motion of electrons along a circular wire: and the equations (3) and (4) express the dynamical interaction between a circular magnetic current and an electron moving along its axis.

We shall take this magnetic structure to be a model of the mechanism within the atom which enables an approaching electron to induce a magnetic current in the atom.

We now proceed to integrate the differential equations (3) and (4) Equation (3) may be integrated as it stands, giving

$$A\dot{\psi} - \frac{Mex}{(a^2 + x^2)^{\frac{3}{2}}} = \text{constant.}$$

We shall suppose that the electron is initially projected from $x = -\infty$ with velocity u , the magnetic structure being initially at rest: so the last equation is

$$A\dot{\psi} - \frac{Mex}{(a^2 + x^2)^{\frac{3}{2}}} = Me \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Moreover, if we multiply equations (3) and (4) by $\dot{\psi}$ and \dot{x} respectively, add, and integrate, we obtain

$$\frac{1}{2}A\dot{\psi}^2 + \frac{1}{2}m\dot{x}^2 = \text{constant,}$$

which is the equation of conservation of energy of the system: since initially $\dot{\psi}$ is zero and \dot{x} is u , the equation is

$$\frac{1}{2}A\dot{\psi}^2 + \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mu^2 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From equations (5) and (6) we see that when the electron, moving with its initial velocity u from $x = -\infty$, comes into the presence of the magnetic structure, its velocity begins to diminish: its kinetic energy is, in fact, being expended in setting the magnetic structure into rotation. It may happen that the electron's initial kinetic energy is entirely used up in this way, in which case the electron gets only as far as a point x determined by the equation

$$-\frac{eMx}{(a^2 + x^2)^{\frac{3}{2}}} = eM - \sqrt{Am} \cdot u; \quad . \quad . \quad . \quad . \quad . \quad (7)$$

depend on the mass m of the electron, but only on its charge e and on quantities depending on the atomic structure with which the electron collides.

The absorbed energy appears in the atom as a magnetic current, specified by the angular velocity $\omega = 2eM/A$, so that between the absorbed energy and this angular velocity there exists the relation (by (8) and (9))

$$U = eM\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

It may, I think, be urged in support of the above explanation of quantum absorption of energy that it postulates no structure in the atom beyond one of a kind which other investigators have introduced in order to account for a totally different class of phenomena, namely, those of induced magnetisation. If on grounds connected with induced magnetisation we have come to accept the existence of a structure within the atom capable of providing what is here called a magnetic current—and it is difficult to see how induced paramagnetism can be explained without some structure which will do this,—then the whole argument of the present section seems to be forced on us inevitably: and, as we have seen, it entails the necessity that exchanges of energy between atoms and electrons must be of quantum type.

At the same time, we must bear in mind that the function of models is merely to suggest the correct differential equations of the phenomenon: when the differential equations have been obtained, the model may be discarded. Instances of this in the history of physics are abundant: to name only the most famous of them, it was a model of rolling particles, idle wheels, and cellular vortices that suggested to Maxwell the correct differential equations of the æther. The model, having served its purpose, soon dropped out of sight: and the æther itself appears to be following it into oblivion. The differential equations alone remain, and in the hands of the relativists have provided the foundation for a complete reconstruction of our ideas of the universe.

We may therefore be well satisfied if the above mathematical equations represent correctly the interaction between an atom and an electron which is approaching it with a velocity not great enough to ionise it, but great enough to evoke a single-line spectrum: * the “magnetic structure” which suggested the equations need not be insisted on.

* The case when the electron approaches with velocity great enough to ionise the atom would correspond in our model to the case when the electron imparts to the magnetic structure so great an angular velocity that the structure explodes under the centrifugal strain.

§ 4. THE TRANSFORMATION OF THE ENERGY INTO THE RADIANT FORM
BY THE MECHANISM WITHIN THE ATOM.

So far we have seen that the energy absorbed by the atom from the moving electron becomes resident in the atom as the energy of a magnetic current: we have next to study the subsequent transformations of this energy.

Now it is well known that a closed electric current is equivalent to a magnetic shell which occupies the gap formed by the circuit, and whose magnetic moment per unit area is proportional to the current-strength. Precisely the same equivalence subsists between a magnetic current and an electric shell—an electric shell being to all intents and purposes what is called a charged condenser in electrostatics: so that we may, if we please, regard our magnetic current as equivalent to a charged condenser in the atom. This will be the preferable way of considering it when we are studying the transformation of the energy into radiant energy: for there is a great body of evidence in favour of the supposition that the element in the atom which performs the function of radiating is something of the nature of a Hertzian oscillator: * and a Hertzian oscillator is essentially a condenser in the act of discharging.

Let us then denote by Q the charge on either plate of the condenser which is equivalent to our magnetic current, by d the distance apart of the plates, and by K the dielectric constant. We have

$$\begin{aligned} \frac{Qd}{K} &= \text{magnetic current} \times \text{area of magnetic circuit,} \\ &= \frac{M\omega}{2\pi} \times \pi a^2, \end{aligned}$$

so

$$Qd = \frac{1}{2} K M a^2 \omega.$$

Introducing the value of ω from (11), we have

$$Qd = \frac{K a^2 U}{2e} \dots \dots \dots (12)$$

But in a condenser whose charge is Q and whose plates are at a distance d apart and have the area πa^2 , the energy is

$$U = \frac{2Q^2 d}{K a^2}, \dots \dots \dots (13)$$

* This seems at any rate to be established as regards the fluorescent radiations of preparations of the sulphides of the alkaline earths. Lenard, Pauli, and others have found that the frequency of the radiations can be altered at will by modifying the dielectric constant of the solvent in a way which conforms exactly to the supposition that the frequency depends on capacity as in a Hertzian oscillator, and the capacity is affected by the dielectric constant just as in a condenser.

and the capacity is

$$C = \frac{Ka^2}{4d} \dots \dots \dots (14)$$

Eliminating d between equations (12) and (13), we have

$$Q = e \dots \dots \dots (15)$$

This equation shows that *the electric separation in the atom, which is caused by the collision with the bombarding electron, is precisely a separation of two electronic charges e and $-e$* , so that the effect of the collision with the bombarding electron is to displace one electron in the atom from its normal position, and the vibrations which generate the emitted radiation are the vibrations of this electron settling down again to its normal position: this agrees with the generally accepted view of experimenters regarding the production of single-line spectra. The acquisition of rotational energy by the magnetic structure, of which we spoke in § 3, must be regarded as a picture of the way in which the atom gets wound up, so to speak, to the stage at which the electron is displaced.

The manner in which the electron, while subsiding to equilibrium, generates radiation is, according to the view here put forward, essentially the same (so far as the differential equations are concerned) as the discharge of a condenser: and we shall therefore continue to speak of a condenser, although it is to be understood throughout that the object which is oscillating is really a single electron.

Eliminating Q and d between equations (12), (13), and (14), we have

$$C = \frac{e^2}{2U}, \dots \dots \dots (16)$$

an equation which expresses the capacity of the condenser in terms of the energy absorbed from the bombarding electron.

Now the frequency of a Hertzian oscillator depends on two factors, namely, its capacity C and its inductance L , the frequency ν being expressed in terms of these by the equation

$$\nu = \frac{1}{2\pi\sqrt{LC}} \dots \dots \dots (17)$$

While it is not to be supposed that there is in the atom anything like an actual coil of wire possessing inductance, it seems difficult to see how oscillations can be generated unless there are two factors which play the same part in the differential equations that L and C play respectively in the differential equation for the oscillatory discharge of a condenser.

We have now to consider how the elements which oscillate—the Hertzian vibrators or discharging condensers—differ from each other in different

Equations (20) and (21) show that the two variables ψ and g oscillate, the energy of the system existing alternately as a magnetic current in the magnetic wheel and an electric current in the wire: the frequency of oscillation is found at once from (20) and (21) to be

$$\nu = \frac{M}{\pi \sqrt{\lambda A}} \quad \dots \quad (22)$$

The quantity $Me^2 \sqrt{\frac{\lambda}{A}}$ now has the same dimensions in every system of units; and by reasoning similar to that in § 4 we obtain the equation

$$Me^2 \sqrt{\frac{\lambda}{A}} = \frac{h}{2\pi} \quad \dots \quad (23)$$

Eliminating λ and A between equations (8), (22), and (23), we have

$$U = h\nu,$$

as was found in equation (19) above.

§ 6. THE EXPLANATION OF PHOTO-ELECTRIC PHENOMENA.

In the discussion on May 8, I gave the following indication of the way in which photo-electric phenomena would be interpreted on the basis of the above theory of quanta.

Suppose that light of frequency ν is incident on a metal. There are grounds for supposing* that in a complex body such as a metal there are systems whose natural frequencies of vibration are distributed almost continuously over a wide range of frequencies. We shall therefore suppose that the incident light provokes resonance in a system within the metal, consisting of a magnetic wheel linked with a wire circuit, as described in § 5 above; so that the system absorbs energy from the incident light and the magnetic wheel is thus set in rotation. This absorption of energy from the incident light proceeds continuously. Suppose now that one of the free electrons of the metal happens to be in the vicinity of the magnetic wheel: the field due to the rotation of the wheel will cause the free electron to be sucked inwards to the wheel, as follows at once from the equations of § 3. If the energy of rotation of the magnetic wheel is less than the amount $U = h\nu$ of §§ 3-5, the free electron, after having been sucked in a certain distance, will be rejected without passing through the wheel or having (in the long run) abstracted any energy from it; but if the energy which the magnetic wheel has absorbed

* Cf. J. J. Thomson, *Phil. Mag.*, **37** (1919), p. 434.

from the incident light happens to be, at the moment, equal to or greater than $h\nu$, then the free electron will be sucked *through* the magnetic wheel and will come out on the other side, having abstracted energy of amount precisely $h\nu$ from the wheel. Thus, when the energy absorbed by an atom of the metal from the incident light amounts to as much as $h\nu$, the first free electron which presents itself is sucked through the atom and relieves the atom of the energy $h\nu$. The electron, then, arrives at the inner surface of the metal on its way out, with energy $h\nu$; it loses energy $h\nu_0$ in passing through the surface, where $h\nu_0$ is equal to Richardson's product ϕe in thermionics; and thus the energy with which an electron escapes from a metal illuminated by radiation of frequency ν is $h(\nu - \nu_0)$.

(Added May 17, 1922.)

§ 7. CONNECTION WITH BOHR'S THEORY OF SERIES-SPECTRA.

The equations (16), (17), (18), which determine radiation according to the present theory, afford an explanation of the most perplexing feature of Bohr's theory of series-spectra, namely, the question as to how the energy, set free by the fall of an electron from one of Bohr's outer orbits to an inner orbit, is transformed into radiation of frequency ν determined by the equation $U = h\nu$. We can, in fact, assimilate this part of Bohr's theory to the theory of the present paper in the following way. In Bohr's theory let a negative electron E fall from an orbit of radius a_1 (position P_1) to an orbit of radius a_0 (position P_0). Now in the initial state of this system, which consists of the electron E at P_1 , let us introduce two coincident electrons E' and E'' at P_0 , one positive and one negative, so that they annul each other; and let us replace Bohr's conception of the fall of the electron from E at P_1 to E' at P_0 , by the conception of the discharge of a condenser whose charges are E and E''; the discharge annihilates E and E'', and so leaves E' surviving alone at the end of the process, and is therefore equivalent to Bohr's notion of a translation of E to the position of E'. Now the energy set free by the fall of the electron in Bohr's theory is

$$U = \frac{e^2}{2} \left(\frac{1}{a_0} - \frac{1}{a_1} \right)$$

which, in conformity with the above idea, may be written

$$U = \frac{e^2}{2C}$$

where C denotes the capacity of a condenser formed of two spherical surfaces of radii a_0 and a_1 respectively. This is identical with equation (16) above,

and thus, by combining it with (17) and (18) as in § 4 of the present paper, we obtain the equations

$$U = h\nu$$

$$\nu = \frac{e^2}{2h} \left(\frac{1}{a_0} - \frac{1}{a_1} \right).$$

Since in Bohr's Hydrogen atom we have $\frac{e^2}{2ha_0} = \frac{N_\infty}{n_0^2}$, where N_∞ is Rydberg's constant and n_0 is the quantum number corresponding to the orbit a_0 , the last equation becomes

$$\nu = N_\infty \left(\frac{1}{n_0^2} - \frac{1}{n_1^2} \right),$$

which is the well-known formula for the frequencies of the series lines of Hydrogen.

It will be seen that, in this view, the underlying reason for Ritz's Principle of Combination in series-spectra is that each spectral line is produced by the discharge of a condenser whose opposite surfaces are concentric spheres; the two terms whose combination determines the frequency of the line depend respectively on the radii of the two spherical surfaces. The determination of the frequency of the oscillatory discharge of this condenser depends on the connection found in the present paper between Planck's quantum constant and the self-induction of the atomic oscillator, namely, the connection expressed by equation (18).

(Issued separately June 2, 1922.)

XI.—Note on Professor Whittaker's Paper "On the Quantum Mechanism in the Atom." By Sir J. Alfred Ewing, K.C.B., F.R.S.

(MS. received May 6, 1922. Read May 8, 1922.)

IN Professor Whittaker's highly interesting and suggestive application of my model the postulated structure is a system of magnets rigidly connected like the spokes of a wheel, having poles of the same name at the circumference, the corresponding poles of contrary name being all at the centre. He shows that when a bombarding electron, moving in the direction of the axis, approaches such a wheel it sets the wheel turning, the rotation of the poles constituting what he calls a "magnetic current." If the initial velocity of the bombarding electron falls short of a certain limiting value, the encounter is perfectly elastic, and the electron is reflected back without loss of energy; the point at which its direction of motion becomes reversed may be on either side of the plane of the wheel, depending on the electron's initial velocity of approach. If the initial velocity exceeds the stated limit, the electron passes clear through the system and escapes, but with a definite reduction of velocity, corresponding to a quantum of energy which it has given up to the wheel. When the encounter is elastic, the angular velocity which the wheel acquires during the approach of the electron is lost during the return of the electron. When the encounter is not elastic, and the electron has passed clear through, the wheel is left with an angular velocity ω ; and the energy which that represents is the source of the consequent radiation. In Professor Whittaker's notation, the energy communicated by the electron to the atom in that case is

$$U = \frac{1}{2}A\omega^2 = \frac{2e^2M^2}{A},$$

where M is the sum of the magnetic poles at the circumference, and A is the moment of inertia of the wheel, e being the charge of an electron. The angular momentum communicated to the wheel is

$$A\omega = 2eM.$$

The electron, whose mass is m , has its speed reduced from u to v , which is such that

$$U = \frac{1}{2}m(u^2 - v^2).$$

This is the quantum of energy, which is related to the frequency of the consequent radiations by the Planck equation

$$U = h\nu.$$

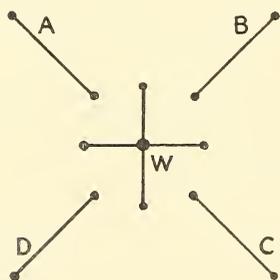
It follows that the acquired angular momentum

$$A\omega = h \cdot \frac{A\nu}{eM},$$

and, since h is a constant of the dimensions of angular momentum, the frequency ν is either equal to eM/A or is a numerical multiple of it. Also, since e is constant, $\nu \propto M/A$.

How is the rotational energy $\frac{1}{2}A\omega^2$, which is communicated to the wheel by the bombarding electron, converted into an oscillatory form? When Professor Whittaker reaches this point in the development of his argument, he leaves the model and compares the system to a Hertzian oscillator or "condenser in the act of discharging." It should, however, be remembered that his "magnetic current"—which is set up by a non-elastic encounter—is a current established in one direction. Such a current is equivalent to a charged condenser simply, not to a condenser undergoing Hertzian oscillations. Obviously a condenser with the charge oscillating from + to - in each plate would be equivalent to a "magnetic current" undergoing reversals of corresponding frequency. To make the comparison to a Hertzian oscillator valid, we must have the wheel execute angular oscillations, and this implies that when it is displaced from its initial position a restoring moment is called into being, so that the angular impulse which it receives as the electron passes through the system sets up oscillations like those of the balance-wheel of a watch. The single-line spectra of Franck and Hertz show that these oscillations are sensibly isochronous.

In accounting for them it seems preferable not to drop the model at this stage. For in the model—as I have described it—there are essentially



two magnetic systems in any atom: a central one forming what is here called the wheel, and another around it, which may for distinction be called the ring. (For simplicity of description we may confine the consideration to one plane.) Thus in the figure we have the ring system ABCD as well as the wheel system W. Each may be taken as rigid, but there is freedom of relative rotation in the plane of the figure, except for the control exerted by magnetic forces

between the two systems. These forces make the systems take up a position of equilibrium as sketched, because the outer poles of the wheel have the same name as the inner poles of the ring. When an electron passes through and escapes, it gives an impulse producing relative angular

displacement. The mutual magnetic forces between wheel and ring tend to restore the whole to its original configuration. Thus oscillations are set up which expend their energy in the emitted radiation.

In this form, therefore, the model furnishes a mechanism not only for extracting a quantum of energy from the passing electron, but for converting that into alternating magnetic currents and thereby into radiation. From the structure of the model it will be apparent that the period is sensibly constant for any moderate amplitude of oscillation.

In the note added under date March 15 to my recent paper "On Models of Ferromagnetic Induction" (*Proc. R.S.E.*, vol. xlii, p. 97, February 20, 1922)—a note which was written after a conversation with Professor Whittaker—it was suggested that the quantum of radiation might be determined by the definite amount of work (say U') which would have to be expended in causing what is here called the wheel to break away from one position of stable equilibrium and fall into another. But in the theory which he has now developed Professor Whittaker finds his quantum in quite another—and as it seems a more convincing—way, namely, as the amount of work U which the electron does in passing clear through the system. Accepting this later view, I conceive U' to be probably greater, and possibly much greater, than U ; with the result that the passing electron simply produces oscillation about the original position of stable equilibrium, without causing the system to pass from one stable configuration to another.

Two other points suggest themselves. The electron, in Professor Whittaker's theory, gives angular momentum to the atom, or part of the atom, and itself loses only linear momentum in the direction of the axis. The principle of conservation would seem to require the assistance of the ether. The electron is in general emitted from some other atom, and in leaving it will give an angular impulse to that atom. The changes of angular momentum, on the part of the two atoms, will be opposite in sign, and may under certain conditions be equal. Taking both atoms together there may be conservation, but even then it appears necessary to think of the angular momentum as stored in the ether during the transit of the electron.

That the ejection of an electron from an atom gives the atom an angular impulse follows from Professor Whittaker's equations. It will be at once apparent if one considers the particular case of an encounter in which the bombarding electron has just enough velocity not to be reflected back. The bombarding electron then stops dead, remaining in the atom, and it gives the atom an angular impulse. The emission of an electron

already in the atom is the converse process. It is associated with an angular impulse in the atom from which it is emitted.

Hence, for the reasons indicated in this note, the emission of an electron is associated with angular oscillations—that is to say, with radiation; from which it may be inferred that the theory has a bearing on the phenomena of photo-electricity. We may imagine the magnetic system of the atom to be thrown into oscillation by resonance as a result of the incidence of light of appropriate wave-length, and when the amplitude reaches a certain limit an electron is ejected. In being ejected it exerts an angular impulse which stops the oscillation and deprives the atom of the quantum of energy which it has absorbed through resonance.

(Issued separately June 6, 1922.)

XII.—The Concomitants (including Differential Invariants) of Quadratic Differential Forms in Four Variables. By Prof. A. R. Forsyth, Sc.D., F.R.S., Hon. F.R.S.E., Imperial College of Science and Technology, S.W.

(MS. received March 20, 1922. Read May 8, 1922.)

STATEMENT OF THE PROBLEM.

1. THE characteristic properties of quadratic differential forms, involving two or more independent variables, have been investigated from the days of Gauss onwards. Initially, the discussion arose for the case of two variables; and, in its most general trend, it was concerned with a form

$$ds^2 = E dp^2 + 2F dpdq + G dq^2,$$

associated with surfaces, E, F, G being integral functions of p and q . But the relation does not, by itself, define a surface completely. When a surface is deformed in any manner, without stretching and without tearing, the quantity ds^2 preserves its measure unchanged; the measure is of fundamental importance. Consequently, the measure must remain unchanged whatever changes of the variables are admitted. Further, changes of the variables, of any kind, allow the existence of covariant concomitants which therefore persist through these changes. In particular, there is one function, of E, F, G and of their derivatives up to the second order inclusive, which persists unaltered; it is the Gauss measure of the curvature of the surface.

The conservation of ds^2 through all changes of variables (as well as that of its concomitant unvarying function) is not sufficient for a complete specification of the surface; but it is a necessary preliminary condition.

Similar remarks apply to ordinary space, when it is referred to triply orthogonal surfaces; the investigations of Lamé, Cayley, and Darboux are well known.

2. In recent years, owing to the various stages in the Einstein theory of relativity, attention has been concentrated upon quadratic differential forms in four variables. Originally, the four variables were the ordinary space-coordinates, the Cartesian coordinates x, y, z or an equivalent set, and the time t . Later, some efforts have been made to use the four variables as specifying a four-dimensional space; and not a little of the

vocabulary of metaphysics has been introduced. My aim is mathematics, not metaphysics; to me, an "event" is sufficiently indicated by the four variables x, y, z, t , subject to all classes of change of variables; and my investigations do not quarrel with the definition of the "interval" between two "events" as given by

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2,$$

with the warning notice that the conservation of this measure of ds^2 through all change does not specify the universe through all time, any more than the conservation of the Gauss surface-element ds^2 would specify all surfaces having that element.

For subsequent purposes, it proves convenient to change the variables by taking x, y, z, t as four independent functions of four independent variables x_1, x_2, x_3, x_4 ; and then ds^2 becomes

$$ds^2 = \sum_{m=1}^4 \sum_{n=1}^4 g_{mn} dx_m dx_n,$$

with the limitation

$$g_{mn} = g_{nm}.$$

But my intention is to discuss these forms without any necessary initial and persistent reference to the symbol measuring an event-interval; and therefore a quadratic form

$$\begin{aligned} & a dx_1^2 + b dx_2^2 + c dx_3^2 + d dx_4^2 \\ & + 2f dx_2 dx_3 + 2g dx_3 dx_1 + 2h dx_1 dx_2 \\ & + 2l dx_1 dx_4 + 2m dx_2 dx_4 + 2n dx_3 dx_4 \end{aligned}$$

is postulated, sometimes to be written in the forms

$$(a, b, c, d, f, g, h, l, m, n \text{ } \delta dx_1, dx_2, dx_3, dx_4)^2, \quad (a \text{ } \delta dx)^2.$$

The ten quantities $a, b, c, d, f, g, h, l, m, n$ are called the *coefficients* of the form: in the general case, they are any functions of the variables; in the particular Einstein case, they must be such functions of the variables as will make the preceding form equivalent to the interval-form.

3. Now, when the variables x_1, x_2, x_3, x_4 are changed, there are consequent changes of distinct types. The four quantities dx_1, dx_2, dx_3, dx_4 become homogeneous linear functions of $dx_1', dx_2', dx_3', dx_4'$, with coefficients that are functions of the variables involving no differential elements dx' . The ten coefficients a give way to ten new coefficients a' , according to the equivalence of the forms, as represented by the relation

$$(a \text{ } \delta dx)^2 = (a' \text{ } \delta dx')^2;$$

that is, the ten new coefficients a' are homogeneous linear functions of

the ten old coefficients, the quantities entering into the linear expressions being functions of the variables.

Two dissimilar considerations show that other magnitudes must be introduced.

4. In the first place, as dx_1, dx_2, dx_3, dx_4 are homogeneous linear functions of $dx'_1, dx'_2, dx'_3, dx'_4$, we write

$$dx_1 = X_1, \quad dx_2 = X_2, \quad dx_3 = X_3, \quad dx_4 = X_4,$$

and regard X_1, X_2, X_3, X_4 as homogeneous coordinates of a point in ordinary space; and we shall then have a quaternary form

$$(a, b, c, d, f, g, h, l, m, n \text{ } \ddot{X}_1, X_2, X_3, X_4)^2,$$

the variables X in which are subject to linear transformation. We are therefore led to consider the concomitants of such a quaternary form; and, in doing so, we are bound to take account of two other sets of variables, viz. plane-variables and line-variables, in addition to point-variables.

The four plane-variables U_1, U_2, U_3, U_4 can be defined in either of two ways: (i) by the relation

$$\sum_{r=1}^4 U_r X_r = \sum_{r=1}^4 U_r' X_r',$$

or (ii) by the equations

$$U_1, U_2, U_3, U_4 = \left\| \begin{array}{cccc} Y_1, & Y_2, & Y_3, & Y_4 \\ Z_1, & Z_2, & Z_3, & Z_4 \\ T_1, & T_2, & T_3, & T_4 \end{array} \right\|,$$

where the sets of variables Y, Z, T are subject to the same transformations as the variables X , that is, are *cogredient* with the variables X . The plane-variables U are *contragredient* with the variables X .

The six line-variables $P_1, P_2, P_3, P_4, P_5, P_6$ can be defined in either of two ways: (i) by the equations

$$\begin{array}{ll} P_1 = Q_2 R_3 - Q_3 R_2, & P_6 = Q_1 R_4 - Q_4 R_1, \\ P_2 = Q_3 R_1 - Q_1 R_3, & P_5 = Q_2 R_4 - Q_4 R_2, \\ P_3 = Q_1 R_2 - Q_2 R_1, & P_4 = Q_3 R_4 - Q_4 R_3, \end{array}$$

where the two sets of variables Q and R are cogredient with the variables X ; or (ii) by the equations

$$\begin{array}{ll} P_1 = V_1 W_4 - V_4 W_1, & P_6 = V_2 W_3 - V_3 W_2, \\ P_2 = V_2 W_4 - V_4 W_2, & P_5 = V_3 W_1 - V_1 W_3, \\ P_3 = V_3 W_4 - V_4 W_3, & P_4 = V_1 W_2 - V_2 W_1, \end{array}$$

where the two sets of variables V and W are cogredient with the variables

U and are contragredient with the variables X. Under either definition, we have a permanent relation

$$P_1P_6 + P_2P_5 + P_3P_4 = 0.$$

But the introduction of these variables has another consequence: it necessitates the introduction of other variables. Thus we must retain the point-variables of the intersection of the line, having P_1, \dots, P_6 for its variables, with the plane, having U_1, \dots, U_4 for its variables; these can be taken in the form

$$\begin{aligned} X_1'' &= U_2P_3 - U_3P_2 + U_4P_6, \\ X_2'' &= -U_1P_3 + U_3P_1 + U_4P_5, \\ X_3'' &= U_1P_2 - U_2P_1 + U_4P_4, \\ X_4'' &= -U_1P_6 - U_2P_5 - U_3P_4, \end{aligned}$$

being, of course, cogredient with the variables X. Next, we must retain the variables of the plane which passes through the point, having X_1, \dots, X_4 for its variables, and through the line, having P_1, \dots, P_6 for its variables; these can be taken in the form

$$\begin{aligned} U_1'' &= X_2P_4 - X_3P_5 + X_4P_1, \\ U_2'' &= -X_1P_4 + X_3P_6 + X_4P_2, \\ U_3'' &= X_1P_5 - X_2P_6 + X_4P_3, \\ U_4'' &= -X_1P_1 - X_2P_2 - X_3P_3, \end{aligned}$$

being, of course, cogredient with the variables U. Next, because we have two points X and X'', we have a line joining these points; its coordinates can be taken to be

$$\begin{aligned} P_1'' &= X_2X_3'' - X_3X_2'', & P_6'' &= X_1X_4'' - X_1''X_4, \\ P_2'' &= X_3X_1'' - X_1X_3'', & P_5'' &= X_2X_4'' - X_2''X_4, \\ P_3'' &= X_1X_2'' - X_2X_1'', & P_4'' &= X_3X_4'' - X_3''X_4, \end{aligned}$$

with the relation

$$P_1''P_6'' + P_2''P_5'' + P_3''P_4'' = 0.$$

Next, we have two planes U and U'', and we therefore have a line which is the intersection of those planes; when its coordinates are taken to be P_1''', \dots, P_6''' , we easily find

$$\frac{P_1''' - P_1''}{P_1} = \dots = \frac{P_6''' - P_6''}{P_6} = U_1X_1 + U_2X_2 + U_3X_3 + U_4X_4.$$

The manner in which these inferred variables are used in the sequel will show that it is not necessary to retain the line P_1''', \dots, P_6''' .

Thus the sets of variables to be retained for the expression of concomitants of the quaternary form are

$$X, U, P, X'', U'', P''.$$

5. The quantities dx_1, dx_2, dx_3, dx_4 may be regarded as defining a direction at the position x_1, x_2, x_3, x_4 in the four-dimensional space; to this direction there corresponds the point X_1, X_2, X_3, X_4 . The line P_1, \dots, P_6 then corresponds to a combination of any two directions at a position. The plane U_1, U_2, U_3, U_4 corresponds to a combination of any three directions at a position. A combination of any four directions at a position gives a four-dimensional volume which is covariantive under the transformations considered.

6. In the second place, and having regard to the occurrence of Gauss's invariant for surfaces, we can expect to have invariants of similar character; also, it may happen—indeed, it does happen—that concomitants of an entirely new character enter. For the purpose, it will be necessary to consider the derivatives of the coefficients a, \dots, n , and to consider their variations when the independent variables are transformed. There is nothing to impose a necessary limit upon the order of the derivatives retained; so, remembering other investigations of a like nature and also the analysis used in Einstein's memoirs, we shall retain derivatives of the coefficients in the differential quadratic form, of the first order, and of the second order.

Thus, among the aggregate of invariantive functions associated with a differential quadratic form, we shall require

- (i) The customary concomitants (point-covariants, line-covariants, plane-covariants, in the different sets of variables) of the quaternary form, arising from the differential form when the differential elements dx_1, dx_2, dx_3, dx_4 are subjected to linear transformation:
- (ii) The concomitants (whether pure differential invariants, or covariants of the various types involving derivatives of the coefficients) of the differential form.

METHOD ADOPTED.

7. In Einstein's work,* and in the exposition of his work as given by others,† the initial analysis is directed towards the construction of the conditions for the equivalence of the general form and the special form.

* For convenience, I refer specially to the general account given by him in his memoir "Die Grundlage der allgemeinen Relativitätstheorie," *Annalen der Physik*, Bd. xlix (1916), pp. 769-822.

† Particular mention should be made of three important and comprehensive accounts of various investigations connected with the general theory of relativity which have recently appeared, viz.: De Donder, *La gravifique einsteinienne* (Gauthier-Villars, Paris, 1921); W. Pauli, jr., "Relativitätstheorie," *Encycl. d. math. Wiss.*, Bd. v, 2, Heft 4 (1921); Marcolongo, *Relatività* (Principato, Messina, 1921). The amplest references are given.

The mathematical process adopted was begun by Christoffel;* and it leads to the quantities known as the Christoffel-Riemann symbols, the second name being added because these quantities occur in a posthumous fragment of Riemann's.† It is proved that, under any transformations of the variables, these quantities are subject to linear transformations; but they are not invariants, they do not belong to any of the three types of covariants, and indeed (except for their connection with the equivalence of the forms) they are not brought into relation with the body of invariantive functions.

8. As my aim is the construction of a complete body of invariantive functions (complete in the sense that they are algebraically independent of one another), I decided to adopt the method of Lie's theory of transformation groups. It is quite general; it provides a system of partial differential equations of the complete Jacobian type, and so furnishes all the tests necessary and sufficient for the construction of an algebraically complete system of covariantive forms; and the only difficulties, which occur in using it in the shape adopted in this paper, are of the kind that always arise in the integration of simultaneous partial differential equations of the first order. The first application of the theory to differential invariants was made by Lie himself‡ to the construction of the invariant which is the Gauss measure of the curvature of a surface; and there have been later extensions and modifications of this application.§

In order that the sequence and the arrangement of the following analysis may become reasoned and clear, a brief statement of the essential elements of the theory as it is used here is prefixed, without any proofs.|| The main results, stated for four variables subject to transformations

$$\begin{aligned}x_1 &= F(x'_1, x'_2, x'_3, x'_4), \\x_2 &= G(x'_1, x'_2, x'_3, x'_4), \\x_3 &= H(x'_1, x'_2, x'_3, x'_4), \\x_4 &= K(x'_1, x'_2, x'_3, x'_4),\end{aligned}$$

* *Crelle*, t. lxx (1869), pp. 46-70, 241-5. An account is also given in Bianchi's *Lezioni di geometria differenziale*, vol. i, chaps. ii, xi.

† *Ges. Werke*, pp. 384 et seq. Riemann died in 1866.

‡ *Math. Ann.*, t. xxiv (1884), pp. 537-578.

§ In particular, reference may be made to a paper by Żorawski, *Acta Math.*, vol. xvi (1892-3), pp. 1-64; and to two papers by myself, *Phil. Trans.*, vol. 201 (1903), pp. 329-402, *ib.* vol. 202 (1903), pp. 277-333. In the present paper, the detailed calculations are systematised in a fashion distinct from the processes in these papers.

|| For the establishment of the various propositions, reference may be made to the following:—

- (1) Lie, (*a*) *Math. Ann.*, Bd. xvi (1880), pp. 441-528; (*b*) the paper quoted in last note but one; (*c*) *Theorie der Transformationsgruppen*, Bd. i, Kap. 25.
- (2) Campbell, *Theory of Continuous Groups*, chaps. iii, iv, v.
- (3) Wright (J. E.), *Invariants of Quadratic Differential Forms*.

where F, G, H, K denote quite general independent functions, are as follows:—

- (i) The general substitutions are adequately secured by the retention of the complete aggregate of infinitesimal substitutions.
- (ii) Each of the fundamental infinitesimal substitutions, obeyed by a function, leads to a linear partial differential equation of the first order which the function must satisfy.
- (iii) When properly arranged, the full system of these linear partial differential equations is a complete Jacobian system.*
- (iv) The original group needs to be "extended" (in Lie's sense of the word), so as to include derivatives of all quantities initially subject to variation.
- (v) A function can only belong to the body of invariantive functions if it satisfies all the equations in the complete Jacobian system.
- (vi) The number of algebraically independent members in the body of invariantive functions is the excess of the number of variable quantities † involved over the numbers of equations in the Jacobian system.
- (vii) A function of the variable quantities involved may satisfy a sub-group, or more than one sub-group, of the equations in the Jacobian system; when it satisfies all the sub-groups except that which involves derivatives with respect to the original coefficients of the differential form, it can be a semi-invariant (such as the leading coefficient in an invariantive form).

After this statement of the method of analysis to be adopted, it is manifest that consideration must, first of all else, be given to the infinitesimal substitutions and to their effect upon all the variable quantities that can occur.

EFFECT OF THE INFINITESIMAL TRANSFORMATIONS.

9. We have taken the most general transformations to be

$$x_1, x_2, x_3, x_4 = F, G, H, K (x_1', x_2', x_3', x_4');$$

and all the quantities concerned are to be subject to them. We denote, temporarily, the modulus of transformation by Ω ; thus

$$\Omega = J \left(\frac{F, G, H, K}{x_1', x_2', x_3', x_4'} \right),$$

* For the nature and properties of a complete Jacobian system, see my *Theory of Differential Equations*, vol. v, ch. iii.

† For the purpose of estimating the number, a coefficient of the quadratic form and all its derivatives are reckoned as independent.

where J denotes the Jacobian of the four functions with respect to the four variables.

A function ϕ , of any of the quantities subject to transformation, is said to be invariantive if, when precisely the same function ϕ' of all the quantities in ϕ after they have been transformed is constructed, a relation

$$\phi' = \phi \Omega^p$$

is satisfied, p being an integer. When p is zero, the function is said to be *absolutely* invariantive or (simply) an invariant; when p is not zero, the function is said to be *relatively* invariantive. Generally, p is called the *index*.

The simplest example of such a function connected with a quadratic differential form

$$(a \delta dx)^2$$

arises when we consider the discriminant Δ , where

$$\Delta = \begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix}.$$

We have

$$a' = \left(a, b, c, d, \dots \left(\frac{\partial x_1}{\partial x_1'}, \frac{\partial x_2}{\partial x_1'}, \frac{\partial x_3}{\partial x_1'}, \frac{\partial x_4}{\partial x_1'} \right)^2, \right.$$

$$h' = \left(a, b, c, d, \dots \left(\frac{\partial x_1}{\partial x_1'}, \frac{\partial x_2}{\partial x_1'}, \frac{\partial x_3}{\partial x_1'}, \frac{\partial x_4}{\partial x_1'} \right) \left(\frac{\partial x_1}{\partial x_2'}, \frac{\partial x_2}{\partial x_2'}, \frac{\partial x_3}{\partial x_2'}, \frac{\partial x_4}{\partial x_2'} \right), \right.$$

and so on; and therefore

$$\Delta' = \begin{vmatrix} a', & h', & g', & l' \\ h', & b', & f', & m' \\ g', & f', & c', & n' \\ l', & m', & n', & d' \end{vmatrix}$$

$$= \begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix} \Omega^2 \left(\frac{x_1, x_2, x_3, x_4}{x_1', x_2', x_3', x_4'} \right).$$

$$= \Delta \Omega^2.$$

This function of the coefficients of the form is of persistent recurrence; it will regularly be denoted by Δ .

10. The infinitesimal transformations, the aggregate of which determines the general transformations, are taken in the form

$$x_1 = x_1' + \epsilon \xi(x_1', x_2', x_3', x_4'),$$

$$x_2 = x_2' + \epsilon \eta(x_1', x_2', x_3', x_4'),$$

$$x_3 = x_3' + \epsilon \zeta(x_1', x_2', x_3', x_4'),$$

$$x_4 = x_4' + \epsilon \theta(x_1', x_2', x_3', x_4'),$$

where ϵ is so small that its square and higher powers are neglected, and where the four functions ξ, η, ζ, θ are perfectly general and entirely independent of one another. It follows that the infinitesimal transformations can be taken also in the form

$$\begin{aligned} x_1' &= x_1 - \epsilon\xi(x_1, x_2, x_3, x_4), \\ x_2' &= x_2 - \epsilon\eta(x_1, x_2, x_3, x_4), \\ x_3' &= x_3 - \epsilon\zeta(x_1, x_2, x_3, x_4), \\ x_4' &= x_4 - \epsilon\theta(x_1, x_2, x_3, x_4). \end{aligned}$$

For the infinitesimal variations of the coefficients of the form and of their derivatives, as well as for the derivatives themselves, it is convenient to define here the symbols that will be used. In the case of any function ψ of the variables x_1, x_2, x_3, x_4 , whether ψ be ξ, η, ζ, θ or any of the coefficients a, \dots, n , we shall write

$$\psi_r = \frac{\partial\psi}{\partial x_r}, \quad \psi_{ar} = \frac{\partial^2\psi}{\partial x_a\partial x_r}, \quad \psi_{pqr} = \frac{\partial^3\psi}{\partial x_p\partial x_q\partial x_r}.$$

11. The infinitesimal variations of the variables x_1, x_2, x_3, x_4 are given by

$$\begin{aligned} dx_1' &= (1 - \epsilon\xi_1)dx_1 - \epsilon\xi_2 dx_2 - \epsilon\xi_3 dx_3 - \epsilon\xi_4 dx_4, \\ dx_2' &= -\epsilon\eta_1 dx_1 + (1 - \epsilon\eta_2)dx_2 - \epsilon\eta_3 dx_3 - \epsilon\eta_4 dx_4, \\ dx_3' &= -\epsilon\zeta_1 dx_1 - \epsilon\zeta_2 dx_2 + (1 - \epsilon\zeta_3)dx_3 - \epsilon\zeta_4 dx_4, \\ dx_4' &= -\epsilon\theta_1 dx_1 - \epsilon\theta_2 dx_2 - \epsilon\theta_3 dx_3 + (1 - \epsilon\theta_4)dx_4. \end{aligned}$$

Hence the modulus of transformation, Ω , is

$$\begin{aligned} \Omega &= \begin{vmatrix} 1 - \epsilon\xi_1, & -\epsilon\xi_2, & -\epsilon\xi_3, & -\epsilon\xi_4 \\ -\epsilon\eta_1, & 1 - \epsilon\eta_2, & -\epsilon\eta_3, & -\epsilon\eta_4 \\ -\epsilon\zeta_1, & -\epsilon\zeta_2, & 1 - \epsilon\zeta_3, & -\epsilon\zeta_4 \\ -\epsilon\theta_1, & -\epsilon\theta_2, & -\epsilon\theta_3, & 1 - \epsilon\theta_4 \end{vmatrix} \\ &= 1 - \epsilon(\xi_1 + \eta_2 + \zeta_3 + \theta_4), \end{aligned}$$

on neglecting squares and higher powers of ϵ .

We at once have the transformations of the variables X_1, X_2, X_3, X_4 in the form

$$(X_1', X_2', X_3', X_4') = (1 - \epsilon\xi_1, -\epsilon\xi_2, -\epsilon\xi_3, -\epsilon\xi_4) \times (X_1, X_2, X_3, X_4) \begin{vmatrix} -\epsilon\eta_1, & 1 - \epsilon\eta_2, & -\epsilon\eta_3, & -\epsilon\eta_4 \\ -\epsilon\zeta_1, & -\epsilon\zeta_2, & 1 - \epsilon\zeta_3, & -\epsilon\zeta_4 \\ -\epsilon\theta_1, & -\epsilon\theta_2, & -\epsilon\theta_3, & 1 - \epsilon\theta_4 \end{vmatrix}$$

The similar transformations of the variables U_1, U_2, U_3, U_4 are

$$(U_1', U_2', U_3', U_4') = (1 + \epsilon\xi_1, \epsilon\xi_2, \epsilon\xi_3, \epsilon\xi_4) \times (U_1, U_2, U_3, U_4) \begin{vmatrix} \epsilon\eta_1, & \epsilon\eta_2, & \epsilon\eta_3, & \epsilon\eta_4 \\ \epsilon\zeta_1, & \epsilon\zeta_2, & \epsilon\zeta_3, & \epsilon\zeta_4 \\ \epsilon\theta_1, & \epsilon\theta_2, & \epsilon\theta_3, & \epsilon\theta_4 \end{vmatrix}$$

and the similar transformations of the variables $P_1, P_2, P_3, P_4, P_5, P_6$ are

$$\begin{aligned} P_1' - P_1 &= \epsilon \{ -(\eta_2 + \zeta_3)P_1 + \eta_1 P_2 + \zeta_1 P_3 + \eta_4 P_4 - \zeta_4 P_5 \}, \\ P_2' - P_2 &= \epsilon \{ \xi_2 P_1 - (\xi_1 + \zeta_3)P_2 + \zeta_2 P_3 - \xi_4 P_4 + \zeta_4 P_6 \}, \\ P_3' - P_3 &= \epsilon \{ \xi_3 P_1 + \eta_3 P_2 - (\xi_1 + \eta_2)P_3 + \xi_4 P_5 - \eta_4 P_6 \}, \\ P_4' - P_4 &= \epsilon \{ \theta_2 P_1 - \theta_1 P_2 - (\zeta_3 + \theta_4)P_4 - \zeta_2 P_5 - \zeta_1 P_6 \}, \\ P_5' - P_5 &= \epsilon \{ -\theta_3 P_1 + \theta_1 P_3 - \eta_3 P_4 - (\eta_2 + \theta_4)P_5 - \eta_1 P_6 \}, \\ P_6' - P_6 &= \epsilon \{ \theta_3 P_2 - \theta_2 P_3 - \xi_3 P_4 - \xi_2 P_5 - (\xi_1 + \theta_4)P_6 \}. \end{aligned}$$

The variables $X_1'', X_2'', X_3'', X_4''$ are cogredient with X_1, X_2, X_3, X_4 . It is, however, unnecessary to take their infinitesimal transformations into consideration; for, if

$$\phi(X_1, X_2, X_3, X_4)$$

is a point-covariant, so also is

$$\phi(X_1 + \lambda X_1'', X_2 + \lambda X_2'', X_3 + \lambda X_3'', X_4 + \lambda X_4'');$$

and therefore

$$\left(X_1'' \frac{\partial}{\partial X_1} + X_2'' \frac{\partial}{\partial X_2} + X_3'' \frac{\partial}{\partial X_3} + X_4'' \frac{\partial}{\partial X_4} \right)^p \phi(X_1, X_2, X_3, X_4)$$

is also a covariant for all powers of p ; that is, we can introduce the variables X'' by means of the polar operator

$$d_x = \sum_{r=1}^4 X_r'' \frac{\partial}{\partial X_r}$$

operating on a covariantive function of X_1, X_2, X_3, X_4 .

Similarly, it is unnecessary to consider the infinitesimal variations of $U_1'', U_2'', U_3'', U_4''$; they can be introduced by means of the polar operator

$$d_u = \sum_{r=1}^4 U_r'' \frac{\partial}{\partial U_r}$$

operating on a covariantive function of U_1, U_2, U_3, U_4 .

Lastly, it is unnecessary to consider the infinitesimal variations of P_1'', \dots, P_6'' ; they likewise can be introduced by means of the polar operator

$$d_p = \sum_{s=1}^6 P_s'' \frac{\partial}{\partial P_s}$$

operating on a covariantive function of P_1, \dots, P_6 .

VARIATIONS OF THE COEFFICIENTS AND THEIR DERIVATIVES.

12. Next, we require the corresponding infinitesimal variations of the coefficients a, \dots, n of the quadratic differential form. We have

$$(a', b', c', d', \dots \text{ \textcircled{X} } dx_1', dx_2', dx_3', dx_4')^2 = (a, b, c, d, \dots \text{ \textcircled{X} } dx_1, dx_2, dx_3, dx_4)^2;$$

so we substitute for dx_1, dx_2, dx_3, dx_4 in terms of $dx'_1, dx'_2, dx'_3, dx'_4$ according to the law

$$(dx_1, dx_2, dx_3, dx_4) = (1 + \epsilon \xi_1, \quad \epsilon \xi_2, \quad \epsilon \xi_3, \quad \epsilon \xi_4) \begin{vmatrix} dx'_1, dx'_2, dx'_3, dx'_4 \\ \epsilon \eta_1, 1 + \epsilon \eta_2, \quad \epsilon \eta_3, \quad \epsilon \eta_4 \\ \epsilon \zeta_1, \quad \epsilon \zeta_2, 1 + \epsilon \zeta_3, \quad \epsilon \zeta_4 \\ \epsilon \theta_1, \quad \epsilon \theta_2, \quad \epsilon \theta_3, 1 + \epsilon \theta_4 \end{vmatrix}$$

we equate the coefficients of the various power-combinations of $dx'_1, dx'_2, dx'_3, dx'_4$, and, neglecting powers of ϵ above the first, we find

$$\begin{aligned} a' - a &= \epsilon(2a\xi_1 + 2h\eta_1 + 2g\zeta_1 + 2l\theta_1), \\ b' - b &= \epsilon(2h\xi_2 + 2b\eta_2 + 2f\zeta_2 + 2m\theta_2), \\ c' - c &= \epsilon(2g\xi_3 + 2f\eta_3 + 2c\zeta_3 + 2n\theta_3), \\ d' - d &= \epsilon(2l\xi_4 + 2m\eta_4 + 2n\zeta_4 + 2d\theta_4), \\ f' - f &= \epsilon(g\xi_2 + f\eta_2 + c\zeta_2 + n\theta_2) + \epsilon(h\xi_3 + b\eta_3 + f\zeta_3 + m\theta_3), \\ g' - g &= \epsilon(a\xi_3 + h\eta_3 + g\zeta_3 + l\theta_3) + \epsilon(g\xi_1 + f\eta_1 + c\zeta_1 + n\theta_1), \\ h' - h &= \epsilon(h\xi_1 + b\eta_1 + f\zeta_1 + m\theta_1) + \epsilon(a\xi_2 + h\eta_2 + g\zeta_2 + l\theta_2), \\ l' - l &= \epsilon(a\xi_4 + h\eta_4 + g\zeta_4 + l\theta_4) + \epsilon(l\xi_1 + m\eta_1 + n\zeta_1 + d\theta_1), \\ m' - m &= \epsilon(h\xi_4 + b\eta_4 + f\zeta_4 + m\theta_4) + \epsilon(l\xi_2 + m\eta_2 + n\zeta_2 + d\theta_2), \\ n' - n &= \epsilon(g\xi_4 + f\eta_4 + c\zeta_4 + n\theta_4) + \epsilon(l\xi_3 + m\eta_3 + n\zeta_3 + d\theta_3). \end{aligned}$$

13. Next, in connection with the differential invariants of the quadratic differential form and with the general body of its invariantive concomitants, we require the infinitesimal variations in the derivatives of the coefficients of the form with respect to x_1, x_2, x_3, x_4 ; as already indicated, derivatives of the first order and the second order (but of no higher order) will be retained. These infinitesimal variations can be obtained as follows.

Let any arbitrary small increments $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be made to x_1, x_2, x_3, x_4 respectively; and let the corresponding small increments of x'_1, x'_2, x'_3, x'_4 be $\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4$ respectively, under the general infinitesimal transformations. We have

$$\begin{aligned} x'_1 &= x_1 - \epsilon \xi(x_1, x_2, x_3, x_4), \\ \alpha'_1 + \alpha_1 &= x_1 + \alpha_1 - \epsilon \xi(x_1 + \alpha_1, x_2 + \alpha_2, x_3 + \alpha_3, x_4 + \alpha_4); \end{aligned}$$

and therefore

$$\alpha'_1 = \alpha_1 - \epsilon \{ \xi(x_1 + \alpha_1, \dots) - \xi(x_1, \dots) \},$$

so that

$$\begin{aligned} \alpha'_1 - \alpha_1 &= -\epsilon \{ \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3 + \alpha_4 \xi_4 \\ &\quad + \frac{1}{2}(\xi_{11}, \xi_{22}, \xi_{33}, \xi_{44}, \dots, \xi_{34}) \alpha_1, \alpha_2, \alpha_3, \alpha_4 \}^2; \end{aligned}$$

and similarly

$$\begin{aligned} \alpha'_2 - \alpha_2 &= -\epsilon \{ \alpha_1 \eta_1 + \alpha_2 \eta_2 + \alpha_3 \eta_3 + \alpha_4 \eta_4 \\ &\quad + \frac{1}{2}(\eta_{11}, \eta_{22}, \eta_{33}, \eta_{44}, \dots, \eta_{34}) \alpha_1, \alpha_2, \alpha_3, \alpha_4 \}^2; \\ \alpha'_3 - \alpha_3 &= -\epsilon \{ \alpha_1 \zeta_1 + \alpha_2 \zeta_2 + \alpha_3 \zeta_3 + \alpha_4 \zeta_4 \\ &\quad + \frac{1}{2}(\zeta_{11}, \zeta_{22}, \zeta_{33}, \zeta_{44}, \dots, \zeta_{34}) \alpha_1, \alpha_2, \alpha_3, \alpha_4 \}^2; \\ \alpha'_4 - \alpha_4 &= -\epsilon \{ \alpha_1 \theta_1 + \alpha_2 \theta_2 + \alpha_3 \theta_3 + \alpha_4 \theta_4 \\ &\quad + \frac{1}{2}(\theta_{11}, \theta_{22}, \theta_{33}, \theta_{44}, \dots, \theta_{34}) \alpha_1, \alpha_2, \alpha_3, \alpha_4 \}^2. \end{aligned}$$

14. For the infinitesimal variations of a , we have

$$\begin{aligned}
 a'(x_1', \dots) &= a(x_1, \dots) + 2\epsilon\{a(x_1, \dots)\xi_1(x_1, \dots) \\
 &\quad + h(x_1, \dots)\eta_1(x_1, \dots) \\
 &\quad + g(x_1, \dots)\zeta_1(x_1, \dots) \\
 &\quad + l(x_1, \dots)\theta_1(x_1, \dots)\}, \\
 a'(x_1' + a_1', \dots) &= a(x_1 + a_1, \dots) + 2\epsilon\{a(x_1 + a_1, \dots)\xi_1(x_1 + a_1, \dots) \\
 &\quad + h(x_1 + a_1, \dots)\eta_1(x_1 + a_1, \dots) \\
 &\quad + g(x_1 + a_1, \dots)\zeta_1(x_1 + a_1, \dots) \\
 &\quad + l(x_1 + a_1, \dots)\theta_1(x_1 + a_1, \dots)\}.
 \end{aligned}$$

We expand the various terms in the latter equation in powers and products of the quantities a and a' , up to the second order inclusive. The terms independent of a and a' balance, owing to the former equation. We then substitute the values of the quantities a' in terms of the quantities a , retaining only the first power of ϵ ; and we compare the coefficients of the various powers and products of the quantities a . The following are the results:—

$$\begin{aligned}
 a_r' - a_r &= \epsilon[2a_r\xi_1 + a_1\xi_r + 2a\xi_{1r} \\
 &\quad + 2h_r\eta_1 + a_2\eta_r + 2h\eta_{1r} \\
 &\quad + 2g_r\zeta_1 + a_3\zeta_r + 2g\zeta_{1r} \\
 &\quad + 2l_r\theta_1 + a_4\theta_r + 2l\theta_{1r}],
 \end{aligned}$$

for $r=1, 2, 3, 4$; and

$$\begin{aligned}
 a_{rs}' - a_{rs} &= \epsilon[2a_{rs}\xi_1 + a_{1s}\xi_r + a_{1r}\xi_s + a_1\xi_{rs} + 2a_s\xi_{1r} + 2a_r\xi_{1s} + 2a\xi_{1rs} \\
 &\quad + 2h_{rs}\eta_1 + a_{2s}\eta_r + a_{2r}\eta_s + a_2\eta_{rs} + 2h_s\eta_{1r} + 2h_r\eta_{1s} + 2h\eta_{1rs} \\
 &\quad + 2g_{rs}\zeta_1 + a_{3s}\zeta_r + a_{3r}\zeta_s + a_3\zeta_{rs} + 2g_s\zeta_{1r} + 2g_r\zeta_{1s} + 2g\zeta_{1rs} \\
 &\quad + 2l_{rs}\theta_1 + a_{4s}\theta_r + a_{4r}\theta_s + a_4\theta_{rs} + 2l_s\theta_{1r} + 2l_r\theta_{1s} + 2l\theta_{1rs}],
 \end{aligned}$$

for $r, s=1, 2, 3, 4$ in all combinations.

Proceeding similarly with the rest of the coefficients of the quadratic differential form, we obtain the results:

$$\begin{aligned}
 b_r' - b_r &= \epsilon[2h_r\xi_2 + b_1\xi_2 + 2h\xi_{2r} \\
 &\quad + 2b_r\eta_2 + b_2\eta_r + 2b\eta_{2r} \\
 &\quad + 2f_r\zeta_2 + b_3\zeta_r + 2f\zeta_{2r} \\
 &\quad + 2m_r\theta_2 + b_4\theta_r + 2m\theta_{2r}], \\
 b_{rs}' - b_{rs} &= \epsilon[2h_{rs}\xi_2 + b_{1s}\xi_r + b_{1r}\xi_s + b_1\xi_{rs} + 2h_s\xi_{2r} + 2h_r\xi_{2s} + 2h\xi_{2rs} \\
 &\quad + 2b_{rs}\eta_2 + b_{2s}\eta_r + b_{2r}\eta_s + b_2\eta_{rs} + 2b_s\eta_{2r} + 2b_r\eta_{2s} + 2b\eta_{2rs} \\
 &\quad + 2f_{rs}\zeta_2 + b_{3s}\zeta_r + b_{3r}\zeta_s + b_3\zeta_{rs} + 2f_s\zeta_{2r} + 2f_r\zeta_{2s} + 2f\zeta_{2rs} \\
 &\quad + 2m_{rs}\theta_2 + b_{4s}\theta_r + b_{4r}\theta_s + b_4\theta_{rs} + 2m_s\theta_{2r} + 2m_r\theta_{2s} + 2m\theta_{2rs}]; \\
 c_r' - c_r &= \epsilon[2g_r\xi_3 + c_1\xi_r + 2g\xi_{3r} \\
 &\quad + 2f_r\eta_3 + c_2\eta_r + 2f\eta_{3r} \\
 &\quad + 2c_r\zeta_3 + c_3\zeta_r + 2c\zeta_{3r} \\
 &\quad + 2n_r\theta_4 + c_4\theta_r + 2n\theta_{3r}],
 \end{aligned}$$

$$c_{rs}' - c_{rs} = \epsilon [2g_{rs}\xi_3 + c_{1r}\xi_s + c_{1s}\xi_r + c_1\xi_{rs} + 2g_s\xi_{3r} + 2g_r\xi_{3s} + 2g\xi_{3rs} \\ + 2f_{rs}\eta_3 + c_{2r}\eta_s + c_{2s}\eta_r + c_2\eta_{rs} + 2f_s\eta_{3r} + 2f_r\eta_{3s} + 2f\eta_{3rs} \\ + 2c_{rs}\zeta_3 + c_{3r}\zeta_s + c_{3s}\zeta_r + c_3\zeta_{rs} + 2c_s\zeta_{3r} + 2c_r\zeta_{3s} + 2c\zeta_{3rs} \\ + 2n_r\theta_3 + c_{4r}\theta_s + c_{3s}\theta_r + c_4\theta_{rs} + 2n_s\theta_{3r} + 2n_r\theta_{3s} + 2n\theta_{3rs}];$$

$$d_r' - d_r = \epsilon [2l_r\xi_4 + d_1\xi_r + 2l\xi_{r4} \\ + 2m_r\eta_4 + d_2\eta_r + 2m\eta_{r4} \\ + 2n_r\zeta_4 + d_3\zeta_r + 2n\zeta_{r4} \\ + 2d_r\theta_4 + d_4\theta_r + 2d\theta_{r4}],$$

$$d_{rs}' - d_{rs} = \epsilon [2l_{rs}\xi_4 + d_{1r}\xi_s + d_{1s}\xi_r + d_1\xi_{rs} + 2l_s\xi_{4r} + 2l_r\xi_{4s} + 2l\xi_{4rs} \\ + 2m_{rs}\eta_4 + d_{2r}\eta_s + d_{2s}\eta_r + d_2\eta_{rs} + 2m_s\eta_{4r} + 2m_r\eta_{4s} + 2m\eta_{4rs} \\ + 2n_{rs}\zeta_4 + d_{3r}\zeta_s + d_{3s}\zeta_r + d_3\zeta_{rs} + 2n_s\zeta_{4r} + 2n_r\zeta_{4s} + 2n\zeta_{4rs} \\ + 2d_{rs}\theta_4 + d_{4r}\theta_s + d_{4s}\theta_r + d_4\theta_{rs} + 2d_s\theta_{4r} + 2d_r\theta_{4s} + 2d\theta_{4rs}];$$

$$f_r' - f_r = \epsilon [h_r\xi_3 + g_r\xi_2 + f_1\xi_r + g\xi_{2r} + h\xi_{3r} \\ + b_r\eta_3 + f_r\eta_2 + f_2\eta_r + f\eta_{2r} + b\eta_{3r} \\ + f_r\zeta_3 + c_r\zeta_2 + f_3\zeta_r + c\xi_{2r} + f\xi_{3r} \\ + m_r\theta_3 + n_r\theta_2 + f_4\theta_r + n\theta_{2r} + m\theta_{3r}],$$

$$f_{rs}' - f_{rs} = \epsilon [g_{rs}\xi_2 + h_{rs}\xi_3 + f_{1r}\xi_s + f_{1s}\xi_r + f_1\xi_{rs} + g_r\xi_{2s} + h_s\xi_{3r} + g_s\xi_{2r} + h_r\xi_{3s} + g\xi_{2rs} + h\xi_{3rs} \\ + f_{rs}\eta_2 + b_{rs}\eta_3 + f_{2r}\eta_s + f_{2s}\eta_r + f_2\eta_{rs} + f_r\eta_{2s} + b_s\eta_{3r} + f_s\eta_{2r} + b_r\eta_{3s} + f\eta_{2rs} + b\eta_{3rs} \\ + c_{rs}\zeta_2 + f_{rs}\zeta_3 + f_{3r}\zeta_s + f_{3s}\zeta_r + f_3\zeta_{rs} + c_r\zeta_{2s} + f_s\zeta_{3r} + c_s\zeta_{2r} + f_r\zeta_{3s} + c\xi_{2rs} + f\xi_{3rs} \\ + n_{rs}\theta_2 + m_{rs}\theta_3 + f_{4r}\theta_s + f_{4s}\theta_r + f_4\theta_{rs} + n_r\theta_{2s} + m_s\theta_{3r} + n_s\theta_{2r} + m_r\theta_{3s} + n\theta_{2rs} + m\theta_{3rs}];$$

$$g_r' - g_r = \epsilon [g_r\xi_1 + a_r\xi_3 + g_1\xi_r + g\xi_{1r} + a\xi_{3r} \\ + f_r\eta_1 + h_r\eta_3 + g_2\eta_r + f\eta_{1r} + h\eta_{3r} \\ + c_r\zeta_1 + g_r\zeta_3 + g_3\zeta_r + c\xi_{1r} + g\xi_{3r} \\ + n_r\theta_1 + l_r\theta_3 + g_4\theta_r + n\theta_{1r} + l\theta_{3r}],$$

$$g_{rs}' - g_{rs} = \epsilon [a_{rs}\xi_3 + g_{rs}\xi_1 + g_{1r}\xi_s + g_{1s}\xi_r + g_1\xi_{rs} + a_s\xi_{3r} + a_r\xi_{3s} + g_s\xi_{1r} + g_r\xi_{1s} + g\xi_{1rs} + a\xi_{3rs} \\ + h_{rs}\eta_3 + f_{rs}\eta_1 + g_{2r}\eta_s + g_{2s}\eta_r + g_2\eta_{rs} + h_s\eta_{3r} + h_r\eta_{3s} + f_s\eta_{1r} + f_r\eta_{1s} + f\eta_{1rs} + h\eta_{3rs} \\ + g_{rs}\zeta_3 + c_{rs}\zeta_1 + g_{3r}\zeta_s + g_{3s}\zeta_r + g_3\zeta_{rs} + g_s\zeta_{3r} + g_r\zeta_{3s} + c_s\xi_{1r} + c_r\xi_{1s} + c\xi_{1rs} + g\xi_{3rs} \\ + l_{rs}\theta_3 + n_{rs}\theta_1 + g_{4r}\theta_s + g_{4s}\theta_r + g_4\theta_{rs} + l_s\theta_{3r} + l_r\theta_{3s} + n_s\theta_{1r} + n_r\theta_{1s} + n\theta_{1rs} + l\theta_{3rs}];$$

$$h_r' - h_r = \epsilon [a_r\xi_2 + h_r\xi_1 + h_1\xi_r + a\xi_{2r} + h\xi_{1r} \\ + h_r\eta_2 + b_r\eta_1 + h_2\eta_r + h\eta_{2r} + b\eta_{1r} \\ + g_r\zeta_2 + f_r\zeta_1 + h_3\zeta_r + g\xi_{2r} + f\xi_{1r} \\ + l_r\theta_2 + m_r\theta_1 + h_4\theta_r + l\theta_{2r} + m\theta_{1r}],$$

$$h_{rs}' - h_{rs} = \epsilon [h_{rs}\xi_1 + a_{rs}\xi_2 + h_{1r}\xi_s + h_{1s}\xi_r + h_1\xi_{rs} + a_s\xi_{2r} + a_r\xi_{2s} + h_s\xi_{1r} + h_r\xi_{1s} + a\xi_{2rs} + h\xi_{1rs} \\ + b_{rs}\eta_1 + h_{rs}\eta_2 + h_{2r}\eta_s + h_{2s}\eta_r + h_2\eta_{rs} + h_s\eta_{2r} + h_r\eta_{2s} + b_s\eta_{1r} + b_r\eta_{1s} + h\eta_{2rs} + b\eta_{1rs} \\ + f_{rs}\zeta_1 + g_{rs}\zeta_2 + h_{3r}\zeta_s + h_{3s}\zeta_r + h_3\zeta_{rs} + g_s\zeta_{2r} + g_r\zeta_{2s} + f_s\zeta_{1r} + f_r\zeta_{1s} + g\xi_{2rs} + f\xi_{1rs} \\ + m_{rs}\theta_1 + l_{rs}\theta_2 + h_{4r}\theta_s + h_{4s}\theta_r + h_4\theta_{rs} + l_s\theta_{2r} + l_r\theta_{2s} + m_s\theta_{1r} + m_r\theta_{1s} + l\theta_{2rs} + m\theta_{1rs}];$$

$$l_r' - l_r = \epsilon [l_r\xi_1 + a_r\xi_4 + l_1\xi_r + l\xi_{1r} + a\xi_{4r} \\ + m_r\eta_1 + h_r\eta_4 + l_2\eta_r + m\eta_{1r} + h\eta_{4r} \\ + n_r\zeta_1 + g_r\zeta_4 + l_3\zeta_r + n\xi_{1r} + g\xi_{4r} \\ + d_r\theta_1 + l_r\theta_4 + l_4\theta_r + d\theta_{1r} + l\theta_{4r}],$$

$$l_{rs}' - l_{rs} = \epsilon[l_{rs}\xi_1 + a_{rs}\xi_4 + l_{1s}\xi_r + l_{1r}\xi_s + a_r\xi_{4s} + a_s\xi_{4r} + l_1\xi_{rs} + l_r\xi_{1s} + l_s\xi_{1r} + l\xi_{1rs} + a\xi_{4rs} \\ + m_{rs}\eta_1 + h_{rs}\eta_4 + l_{2s}\eta_r + l_{2r}\eta_s + h_r\eta_{4s} + h_s\eta_{4r} + l_2\eta_{rs} + m_r\eta_{1s} + m_s\eta_{1r} + m\eta_{1rs} + h\eta_{4rs} \\ + n_{rs}\zeta_1 + g_{rs}\zeta_4 + l_{3s}\zeta_r + l_{3r}\zeta_s + g_r\zeta_{4s} + g_s\zeta_{4r} + l_3\zeta_{rs} + n_r\zeta_{1s} + n_s\zeta_{1r} + n\zeta_{1rs} + g\xi_{4rs} \\ + d_{rs}\theta_1 + l_{r_s}\theta_4 + l_{4s}\theta_r + l_{4r}\theta_s + l_r\theta_{4s} + l_s\theta_{4r} + l_4\theta_{rs} + d_r\theta_{1s} + d_s\theta_{1r} + d\theta_{1rs} + l\theta_{4rs}];$$

$$m_r' - m_r = \epsilon[l_r\xi_2 + h_r\xi_4 + m_1\xi_r + l\xi_{2r} + h\xi_{4r} \\ + m_r\eta_2 + b_r\eta_4 + m_2\eta_r + m\eta_{2r} + b\eta_{4r} \\ + n_r\zeta_2 + f_r\zeta_4 + m_3\zeta_r + n\xi_{2r} + f\xi_{4r} \\ + d_r\theta_2 + m_r\theta_4 + m_r\theta_4 + d\theta_{2r} + m\theta_{4r}],$$

$$m_{rs}' - m_{rs} = \epsilon[l_{rs}\xi_2 + h_{rs}\xi_4 + m_{1r}\xi_s + m_{1s}\xi_r + h_r\xi_{4s} + h_s\xi_{4r} + l_s\xi_{2r} + l_r\xi_{2s} + m_1\xi_{rs} + l\xi_{2rs} + h\xi_{4rs} \\ + m_{rs}\eta_2 + b_{rs}\eta_4 + m_{2r}\eta_s + m_{2s}\eta_r + b_r\eta_{4s} + b_s\eta_{4r} + m_s\eta_{2r} + m_r\eta_{2s} + m_2\eta_{rs} + m\eta_{2rs} + b\eta_{4rs} \\ + n_{rs}\zeta_2 + f_{rs}\zeta_4 + m_{3r}\zeta_s + m_{3s}\zeta_r + f_r\zeta_{4s} + f_s\zeta_{4r} + n_s\zeta_{2r} + n_r\zeta_{2s} + m_3\zeta_{rs} + n\zeta_{2rs} + f\xi_{4rs} \\ + d_{rs}\theta_2 + m_{r_s}\theta_4 + m_{4r}\theta_s + m_{4s}\theta_r + m_r\theta_{4s} + m_s\theta_{4r} + d_s\theta_{2r} + d_r\theta_{2s} + m_4\theta_{rs} + d\theta_{2rs} + m\theta_{4rs}];$$

$$n_r' - n_r = \epsilon[l_r\xi_3 + g_r\xi_4 + n_1\xi_r + l\xi_{3r} + g\xi_{4r} \\ + m_r\eta_3 + f_r\eta_4 + n_2\eta_r + m\eta_{3r} + f\eta_{4r} \\ + n_r\zeta_3 + c_r\zeta_4 + n_3\zeta_r + n\xi_{3r} + c\xi_{4r} \\ + d_r\theta_3 + n_r\theta_4 + n_4\theta_r + d\theta_{3r} + n\theta_{4r}],$$

$$n_{rs}' - n_{rs} = \epsilon[l_{rs}\xi_3 + g_{rs}\xi_4 + n_{1r}\xi_s + n_{1s}\xi_r + g_r\xi_{4s} + g_s\xi_{4r} + l_s\xi_{3r} + l_r\xi_{3s} + n_1\xi_{rs} + l\xi_{3rs} + g\xi_{4rs} \\ + m_{rs}\eta_3 + f_{rs}\eta_4 + n_{2r}\eta_s + n_{2s}\eta_r + f_r\eta_{4s} + f_s\eta_{4r} + m_s\eta_{3r} + m_r\eta_{3s} + n_2\eta_{rs} + m\eta_{3rs} + f\eta_{4rs} \\ + n_{rs}\zeta_3 + c_{rs}\zeta_4 + n_{3r}\zeta_s + n_{3s}\zeta_r + c_r\zeta_{4s} + c_s\zeta_{4r} + n_s\zeta_{3r} + n_r\zeta_{3s} + n_3\zeta_{rs} + n\zeta_{3rs} + c\xi_{4rs} \\ + d_{rs}\theta_3 + n_{r_s}\theta_4 + n_{4r}\theta_s + n_{4s}\theta_r + n_r\theta_{4s} + n_s\theta_{4r} + d_s\theta_{3r} + d_r\theta_{3s} + n_4\theta_{rs} + d\theta_{3rs} + n\theta_{4rs}];$$

for $r, s = 1, 2, 3, 4$ in all combinations.*

15. Thus, within the range of derivation retained, we have, for each of the ten coefficients in the quadratic form, the coefficient itself, its four first derivatives, and its ten second derivatives; consequently there are 150 quantities arising through those coefficients. For the concomitants which are to be associated with the form in the complete system there are the four variables X, the six variables P, and the four variables U; that is, fourteen more. Hence the total number of quantities that occur is 164.

As regards the derivatives of each of the quantities ξ, η, ζ, θ , we have the four of the first order, the ten of the second order, and the twenty of the third order, while the quantities ξ, η, ζ, θ do not themselves occur in the infinitesimal variations of any of the 164 quantities occurring. Thus the total number of derivatives that have to be taken into account is 136. It is convenient to arrange them in sub-groups of 80 (for the third order of derivation), 40 (for the second order), and 16 (for the first order).

* These ten sets of symbols can be grouped into a single set, giving

$$\left(\frac{\partial g_{pq}}{\partial x_r}\right)' - \frac{\partial g_{pq}}{\partial x_r}, \quad \left(\frac{\partial^2 g_{pq}}{\partial x_r \partial x_s}\right)' - \frac{\partial^2 g_{pq}}{\partial x_r \partial x_s},$$

for all values of p, q, r, s ; but the forms are not so easy to manipulate as are the more diffuse expressions in the text.

THE GROUP OF CHARACTERISTIC PARTIAL DIFFERENTIAL EQUATIONS.

16. According to the adopted definition of an invariantive function ϕ , the relation

$$\phi' = \phi \Omega^p$$

must be satisfied where ϕ' is the same function of the transformed quantities that occur in ϕ , as ϕ is of the untransformed quantities; where Ω is the modulus of transformation; and where p is an integer. Further, it is sufficient (after earlier statements) to take only the infinitesimal variations. For these variations,

$$\begin{aligned} \Omega &= J \left(\frac{x'_1, x'_2, x'_3, x'_4}{x_1, x_2, x_3, x_4} \right) \\ &= 1 - \epsilon(\xi_1 + \eta_2 + \zeta_3 + \theta_4); \end{aligned}$$

and so the equation

$$\begin{aligned} &\phi(a'_1, \dots, n', X'_1, \dots, X'_4, P'_1, \dots, P'_6, U'_1, \dots, U'_4) \\ &= \{1 - p\epsilon(\xi_1 + \eta_2 + \zeta_3 + \theta_4)\} \phi(a, \dots, n, X_1, \dots, X_4, P_1, \dots, P_6, U_1, \dots, U_4) \end{aligned}$$

must be satisfied. And it is to be satisfied for all the infinitesimal variations that can be imposed, however arbitrary, arising out of the functions ξ, η, ζ, θ . It follows that, when we substitute the values of $a', \dots, n', X'_1, \dots, X'_4, P'_1, \dots, P'_6, U'_1, \dots, U'_4$ that have been obtained, and expand the left-hand side so as to retain only the first power of ϵ , the coefficients of the various derivatives of ξ, η, ζ, θ on the two sides must be equal. The terms, independent of ϵ on the two sides, cancel one another; the only derivatives, out of the total group of 136 arising from ξ, η, ζ, θ and occurring on the right-hand side, are $\xi_1, \eta_2, \zeta_3, \theta_4$. Thus the coefficients of 132 of these, from the left-hand side, must vanish; and there are four such coefficients arising out of the left-hand side, each of which is equal to $-p\phi$.

THE FIRST SUB-GROUP: EIGHTY EQUATIONS.

17. We consider the sub-groups in turn, dealing first with the sub-group of eighty that arise in connection with the third derivatives of ξ, η, ζ, θ . This sub-group is most conveniently managed by selecting the equations in the sets of four, that are furnished by taking those derivatives of ξ, η, ζ, θ which are exactly similar to one another.

From the coefficients of $\xi_{111}, \eta_{111}, \zeta_{111}, \theta_{111}$ we have

$$\begin{aligned} \left(2a \frac{\partial}{\partial a_{11}} + h \frac{\partial}{\partial h_{11}} + g \frac{\partial}{\partial g_{11}} + l \frac{\partial}{\partial l_{11}} \right) \phi &= 0, \\ \left(2h \frac{\partial}{\partial a_{11}} + b \frac{\partial}{\partial h_{11}} + f \frac{\partial}{\partial g_{11}} + m \frac{\partial}{\partial l_{11}} \right) \phi &= 0, \end{aligned}$$

$$\left(2g \frac{\partial}{\partial a_{11}} + f \frac{\partial}{\partial h_{11}} + c \frac{\partial}{\partial g_{11}} + n \frac{\partial}{\partial l_{11}}\right)\phi = 0,$$

$$\left(2l \frac{\partial}{\partial a_{11}} + m \frac{\partial}{\partial h_{11}} + n \frac{\partial}{\partial g_{11}} + d \frac{\partial}{\partial l_{11}}\right)\phi = 0;$$

and therefore

$$\frac{\partial \phi}{\partial a_{11}} = 0, \quad \frac{\partial \phi}{\partial h_{11}} = 0, \quad \frac{\partial \phi}{\partial g_{11}} = 0, \quad \frac{\partial \phi}{\partial l_{11}} = 0.$$

Similarly, from the coefficients of $\xi_{222}, \eta_{222}, \zeta_{222}, \theta_{222}$ we have

$$\frac{\partial \phi}{\partial h_{22}} = 0, \quad \frac{\partial \phi}{\partial b_{22}} = 0, \quad \frac{\partial \phi}{\partial f_{22}} = 0, \quad \frac{\partial \phi}{\partial m_{22}} = 0;$$

from those of $\xi_{333}, \eta_{333}, \zeta_{333}, \theta_{333}$ we have

$$\frac{\partial \phi}{\partial g_{33}} = 0, \quad \frac{\partial \phi}{\partial f_{33}} = 0, \quad \frac{\partial \phi}{\partial c_{33}} = 0, \quad \frac{\partial \phi}{\partial n_{23}} = 0;$$

from those of $\xi_{444}, \eta_{444}, \zeta_{444}, \theta_{444}$ we have

$$\frac{\partial \phi}{\partial l_{44}} = 0, \quad \frac{\partial \phi}{\partial m_{44}} = 0, \quad \frac{\partial \phi}{\partial u_{44}} = 0, \quad \frac{\partial \phi}{\partial d_{44}} = 0.$$

From the coefficients of $\xi_{112}, \eta_{112}, \zeta_{112}, \theta_{112}$ we similarly have

$$2 \frac{\partial \phi}{\partial a_{12}} + \frac{\partial \phi}{\partial h_{11}} = 0, \quad \frac{\partial \phi}{\partial h_{12}} + 2 \frac{\partial \phi}{\partial b_{11}} = 0, \quad \frac{\partial \phi}{\partial g_{12}} + \frac{\partial \phi}{\partial f_{11}} = 0, \quad \frac{\partial \phi}{\partial l_{12}} + \frac{\partial \phi}{\partial m_{11}} = 0;$$

but, because $\frac{\partial \phi}{\partial h_{11}} = 0$, the first of these leads to

$$\frac{\partial \phi}{\partial a_{12}} = 0.$$

From the coefficients of $\xi_{113}, \eta_{113}, \zeta_{113}, \theta_{113}$ we similarly have

$$2 \frac{\partial \phi}{\partial a_{13}} + \frac{\partial \phi}{\partial g_{11}} = 0, \quad \frac{\partial \phi}{\partial h_{13}} + \frac{\partial \phi}{\partial f_{11}} = 0, \quad \frac{\partial \phi}{\partial g_{13}} + 2 \frac{\partial \phi}{\partial c_{11}} = 0, \quad \frac{\partial \phi}{\partial l_{13}} + \frac{\partial \phi}{\partial n_{11}} = 0;$$

but, because $\frac{\partial \phi}{\partial g_{11}} = 0$, the first of these leads to

$$\frac{\partial \phi}{\partial a_{13}} = 0.$$

From the coefficients of $\xi_{114}, \eta_{114}, \zeta_{114}, \theta_{114}$ we similarly have

$$2 \frac{\partial \phi}{\partial a_{14}} + \frac{\partial \phi}{\partial l_{11}} = 0, \quad \frac{\partial \phi}{\partial h_{14}} + \frac{\partial \phi}{\partial m_{11}} = 0, \quad \frac{\partial \phi}{\partial g_{14}} + \frac{\partial \phi}{\partial n_{11}} = 0, \quad \frac{\partial \phi}{\partial l_{14}} + 2 \frac{\partial \phi}{\partial d_{11}} = 0$$

but, because $\frac{\partial \phi}{\partial l_{11}} = 0$, the first of these leads to

$$\frac{\partial \phi}{\partial a_{14}} = 0.$$

From the coefficients of $\xi_{122}, \eta_{122}, \zeta_{122}, \theta_{122}$ we similarly have

$$2\frac{\partial\phi}{\partial h_{12}} + \frac{\partial\phi}{\partial a_{22}} = 0, \quad 2\frac{\partial\phi}{\partial b_{12}} + \frac{\partial\phi}{\partial h_{22}} = 0, \quad \frac{\partial\phi}{\partial f_{12}} + \frac{\partial\phi}{\partial g_{22}} = 0, \quad \frac{\partial\phi}{\partial m_{12}} + \frac{\partial\phi}{\partial l_{22}} = 0;$$

but, because $\frac{\partial\phi}{\partial h_{22}} = 0$, the second of these leads to

$$\frac{\partial\phi}{\partial b_{12}} = 0.$$

From the coefficients of $\xi_{133}, \eta_{133}, \zeta_{133}, \theta_{133}$ we similarly have

$$2\frac{\partial\phi}{\partial a_{33}} + \frac{\partial\phi}{\partial g_{13}} = 0, \quad \frac{\partial\phi}{\partial h_{33}} + \frac{\partial\phi}{\partial f_{13}} = 0, \quad \frac{\partial\phi}{\partial y_{33}} + 2\frac{\partial\phi}{\partial c_{13}} = 0, \quad \frac{\partial\phi}{\partial l_{33}} + \frac{\partial\phi}{\partial n_{13}} = 0;$$

but, because $\frac{\partial\phi}{\partial g_{33}} = 0$, the third of these leads to

$$\frac{\partial\phi}{\partial c_{13}} = 0.$$

From the coefficients of $\xi_{144}, \eta_{144}, \zeta_{144}, \theta_{144}$ we similarly have

$$2\frac{\partial\phi}{\partial a_{44}} + \frac{\partial\phi}{\partial l_{14}} = 0, \quad \frac{\partial\phi}{\partial h_{44}} + \frac{\partial\phi}{\partial m_{14}} = 0, \quad \frac{\partial\phi}{\partial y_{44}} + \frac{\partial\phi}{\partial n_{14}} = 0, \quad \frac{\partial\phi}{\partial l_{44}} + 2\frac{\partial\phi}{\partial d_{14}} = 0;$$

but, because $\frac{\partial\phi}{\partial l_{44}} = 0$, the last of these leads to

$$\frac{\partial\phi}{\partial d_{14}} = 0.$$

From the coefficients of $\xi_{223}, \eta_{223}, \zeta_{223}, \theta_{223}$ we similarly have

$$\frac{\partial\phi}{\partial h_{23}} + \frac{\partial\phi}{\partial g_{22}} = 0, \quad 2\frac{\partial\phi}{\partial b_{23}} + \frac{\partial\phi}{\partial f_{22}} = 0, \quad \frac{\partial\phi}{\partial f_{23}} + 2\frac{\partial\phi}{\partial c_{22}} = 0, \quad \frac{\partial\phi}{\partial m_{23}} + \frac{\partial\phi}{\partial n_{22}} = 0;$$

but, because $\frac{\partial\phi}{\partial f_{22}} = 0$, the second of these leads to

$$\frac{\partial\phi}{\partial b_{23}} = 0.$$

From the coefficients of $\xi_{224}, \eta_{224}, \zeta_{224}, \theta_{224}$ we similarly have

$$\frac{\partial\phi}{\partial h_{24}} + \frac{\partial\phi}{\partial l_{22}} = 0, \quad 2\frac{\partial\phi}{\partial b_{24}} + \frac{\partial\phi}{\partial m_{22}} = 0, \quad \frac{\partial\phi}{\partial f_{24}} + \frac{\partial\phi}{\partial n_{22}} = 0, \quad \frac{\partial\phi}{\partial m_{24}} + 2\frac{\partial\phi}{\partial d_{22}} = 0;$$

but, because $\frac{\partial\phi}{\partial m_{22}} = 0$, the second of these leads to

$$\frac{\partial\phi}{\partial b_{24}} = 0.$$

From the coefficients of $\xi_{233}, \eta_{233}, \zeta_{233}, \theta_{233}$ we similarly have

$$\frac{\partial\phi}{\partial j_{23}} + \frac{\partial\phi}{\partial h_{33}} = 0, \quad \frac{\partial\phi}{\partial f_{23}} + 2\frac{\partial\phi}{\partial b_{33}} = 0, \quad 2\frac{\partial\phi}{\partial c_{23}} + \frac{\partial\phi}{\partial f_{33}} = 0, \quad \frac{\partial\phi}{\partial n_{23}} + \frac{\partial\phi}{\partial m_{33}} = 0;$$

but, because $\frac{\partial\phi}{\partial f_{33}} = 0$, the third of these leads to

$$\frac{\partial\phi}{\partial c_{23}} = 0.$$

From the coefficients of $\xi_{244}, \eta_{244}, \zeta_{244}, \theta_{244}$ we similarly have

$$\frac{\partial\phi}{\partial h_{44}} + \frac{\partial\phi}{\partial l_{24}} = 0, \quad 2\frac{\partial\phi}{\partial b_{44}} + \frac{\partial\phi}{\partial m_{24}} = 0, \quad \frac{\partial\phi}{\partial f_{44}} + \frac{\partial\phi}{\partial n_{24}} = 0, \quad \frac{\partial\phi}{\partial m_{44}} + 2\frac{\partial\phi}{\partial d_{24}} = 0;$$

but, because $\frac{\partial\phi}{\partial m_{44}} = 0$, the last of these leads to

$$\frac{\partial\phi}{\partial d_{24}} = 0.$$

From the coefficients of $\xi_{334}, \eta_{334}, \zeta_{334}, \theta_{334}$ we similarly have

$$\frac{\partial\phi}{\partial g_{34}} + \frac{\partial\phi}{\partial l_{33}} = 0, \quad \frac{\partial\phi}{\partial f_{34}} + \frac{\partial\phi}{\partial m_{33}} = 0, \quad 2\frac{\partial\phi}{\partial c_{34}} + \frac{\partial\phi}{\partial n_{33}} = 0, \quad \frac{\partial\phi}{\partial n_{34}} + 2\frac{\partial\phi}{\partial d_{33}} = 0;$$

but, because $\frac{\partial\phi}{\partial n_{33}} = 0$, the third of these leads to

$$\frac{\partial\phi}{\partial c_{34}} = 0.$$

From the coefficients of $\xi_{344}, \eta_{344}, \zeta_{344}, \theta_{344}$ we similarly have

$$\frac{\partial\phi}{\partial g_{44}} + \frac{\partial\phi}{\partial l_{34}} = 0, \quad \frac{\partial\phi}{\partial f_{44}} + \frac{\partial\phi}{\partial m_{34}} = 0, \quad 2\frac{\partial\phi}{\partial c_{44}} + \frac{\partial\phi}{\partial n_{34}} = 0, \quad \frac{\partial\phi}{\partial n_{44}} + 2\frac{\partial\phi}{\partial d_{34}} = 0;$$

but, because $\frac{\partial\phi}{\partial n_{44}} = 0$, the last of these leads to

$$\frac{\partial\phi}{\partial d_{34}} = 0.$$

From the coefficients of $\xi_{123}, \eta_{123}, \zeta_{123}, \theta_{123}$ we have

$$\left. \begin{aligned} 2\frac{\partial\phi}{\partial a_{23}} + \frac{\partial\phi}{\partial h_{13}} + \frac{\partial\phi}{\partial y_{12}} &= 0 \\ \frac{\partial\phi}{\partial h_{23}} + 2\frac{\partial\phi}{\partial b_{13}} + \frac{\partial\phi}{\partial f_{12}} &= 0 \\ \frac{\partial\phi}{\partial g_{23}} + \frac{\partial\phi}{\partial f_{13}} + 2\frac{\partial\phi}{\partial c_{12}} &= 0 \\ \frac{\partial\phi}{\partial l_{23}} + \frac{\partial\phi}{\partial m_{13}} + \frac{\partial\phi}{\partial n_{12}} &= 0 \end{aligned} \right\};$$

from those of $\xi_{124}, \eta_{124}, \zeta_{124}, \theta_{124}$ we have

$$\left. \begin{aligned} 2\frac{\partial\phi}{\partial a_{24}} + \frac{\partial\phi}{\partial h_{14}} + \frac{\partial\phi}{\partial l_{12}} &= 0 \\ \frac{\partial\phi}{\partial h_{24}} + 2\frac{\partial\phi}{\partial b_{14}} + \frac{\partial\phi}{\partial m_{12}} &= 0 \\ \frac{\partial\phi}{\partial g_{24}} + \frac{\partial\phi}{\partial f_{14}} + \frac{\partial\phi}{\partial n_{12}} &= 0 \\ \frac{\partial\phi}{\partial l_{24}} + \frac{\partial\phi}{\partial m_{14}} + 2\frac{\partial\phi}{\partial d_{12}} &= 0 \end{aligned} \right\};$$

from those of $\xi_{134}, \eta_{134}, \zeta_{134}, \theta_{134}$ we have

$$\left. \begin{aligned} 2\frac{\partial\phi}{\partial a_{31}} + \frac{\partial\phi}{\partial g_{14}} + \frac{\partial\phi}{\partial l_{13}} &= 0 \\ \frac{\partial\phi}{\partial h_{34}} + \frac{\partial\phi}{\partial f_{14}} + \frac{\partial\phi}{\partial m_{13}} &= 0 \\ \frac{\partial\phi}{\partial g_{34}} + 2\frac{\partial\phi}{\partial c_{14}} + \frac{\partial\phi}{\partial n_{13}} &= 0 \\ \frac{\partial\phi}{\partial l_{34}} + \frac{\partial\phi}{\partial n_{14}} + 2\frac{\partial\phi}{\partial d_{13}} &= 0 \end{aligned} \right\};$$

and from those of $\xi_{234}, \eta_{234}, \zeta_{234}, \theta_{234}$ we have

$$\left. \begin{aligned} \frac{\partial\phi}{\partial h_{34}} + \frac{\partial\phi}{\partial y_{24}} + \frac{\partial\phi}{\partial l_{23}} &= 0 \\ 2\frac{\partial\phi}{\partial b_{34}} + \frac{\partial\phi}{\partial f_{24}} + \frac{\partial\phi}{\partial m_{23}} &= 0 \\ \frac{\partial\phi}{\partial f_{34}} + 2\frac{\partial\phi}{\partial c_{24}} + \frac{\partial\phi}{\partial n_{24}} &= 0 \\ \frac{\partial\phi}{\partial m_{34}} + \frac{\partial\phi}{\partial n_{24}} + 2\frac{\partial\phi}{\partial d_{23}} &= 0 \end{aligned} \right\}.$$

18. Now these eighty partial differential equations constitute a complete Jacobian linear system* by themselves; it is, indeed, a feature of Lie's theory of the infinitesimal transformations of a continuous group that the aggregate of the partial differential equations should be of this nature. The quantities that occur in them are one hundred in number; hence† the system of eighty equations possesses twenty functionally independent integrals.

In the eighty equations, there are twenty-eight which are single-termed; they require that none of the quantities $a_{11}, a_{12}, a_{13}, a_{14}; h_{11}, h_{22}; g_{11}, g_{33}; l_{11}, l_{44}; b_{12}, b_{22}, b_{23}, b_{24}; f_{22}, f_{33}; m_{22}, m_{44}; c_{13}, c_{23}, c_{33}, c_{34}; n_{33}, n_{44}; d_{14}, d_{24}, d_{34}, d_{44}$ can occur in any of the integrals. Thus the integrals must be proper combinations, the simpler the better, of the remaining seventy-two double suffix quantities. There are twelve sets of two-term equations, each set consisting of three members; and there are four sets of three-term equations, each set consisting of four members. Thus there remain fifty-two equations in all, involving seventy-two quantities.

A complete set of twenty functionally independent integrals (or, as

* For the characteristics and the properties of such a system, see my *Theory of Differential Equations*, vol. v, §§ 37-46.

† *Loc. cit.*, p. 86.

proves preferable, a set of twenty-one integrals, subject to a single relation) can be taken as follows:—

$$\begin{aligned}
 a_{44} - 2l_{14} + d_{11} &= \alpha, \\
 b_{44} - 2m_{24} + d_{22} &= \beta, \\
 c_{44} - 2n_{34} + d_{33} &= \gamma, \\
 b_{33} - 2f_{23} + c_{22} &= \delta, \\
 a_{33} - 2g_{13} + c_{11} &= \eta, \\
 a_{22} - 2h_{12} + b_{11} &= \theta, \\
 a_{23} - h_{13} - g_{12} + f_{11} &= \kappa, \\
 a_{24} - h_{14} - l_{12} + m_{11} &= \chi, \\
 a_{34} - g_{14} - l_{13} + n_{11} &= \varpi, \\
 b_{13} - f_{12} - h_{23} + g_{22} &= \lambda, \\
 b_{14} - m_{12} - h_{24} + l_{22} &= \rho, \\
 b_{34} - f_{24} - m_{23} + n_{22} &= \sigma, \\
 c_{12} - f_{13} - g_{23} + h_{33} &= \mu, \\
 c_{14} - n_{13} - g_{34} + l_{33} &= \nu, \\
 c_{24} - n_{23} - f_{34} + m_{33} &= \tau, \\
 d_{12} - l_{24} - m_{14} + h_{44} &= \zeta, \\
 d_{13} - l_{34} - n_{14} + g_{44} &= \iota, \\
 d_{23} - m_{34} - n_{24} + f_{44} &= \xi, \\
 f_{14} - g_{24} - m_{13} + l_{23} &= \psi, \\
 h_{34} - f_{14} - l_{23} + n_{12} &= \omega, \\
 g_{24} - h_{34} - n_{12} + m_{13} &= \nu,
 \end{aligned}$$

with the single relation

$$\nu + \psi + \omega = 0.$$

It should be noted, in passing, that the twenty integrals (except for a factor $\frac{1}{2}$) constitute the terms which involve derivatives of the second order occurring in the twenty independent Riemann-Christoffel symbols.

THE SECOND SUB-GROUP: FORTY EQUATIONS.

19. Next, we proceed to the second sub-group of equations. These arise in a manner similar to the former sub-group, viz. by the consideration of the coefficients of the second derivatives of ξ, η, ζ, θ . In all, there are forty equations in this sub-group.

When the equations are formed, it appears that they involve (among other quantities) second derivatives of the magnitudes $a, b, c, d, f, g, h, l, m, n$. The only admissible combinations of these derivatives are now the preceding twenty integrals in the set just obtained. Consequently, the forty equations must be transformed so as to substitute these combinations as the variables in place of the scattered occurrences of the

second derivatives. After the transformations have been effected, the forty equations run as follows:—

(i) From the coefficients of $\xi_{11}, \eta_{11}, \zeta_{11}, \theta_{11}$,

$$2a \frac{\partial \phi}{\partial a_1} + h \frac{\partial \phi}{\partial h_1} + g \frac{\partial \phi}{\partial g_1} + l \frac{\partial \phi}{\partial l_1} + (l_1 - 2h_2) \frac{\partial \phi}{\partial \theta} + (c_1 - 2g_3) \frac{\partial \phi}{\partial \eta} + (d_1 - 2l_4) \frac{\partial \phi}{\partial \alpha} + (f_1 - h_3 + g_2) \frac{\partial \phi}{\partial \kappa} + (m_1 - h_4 + l_2) \frac{\partial \phi}{\partial \chi} + (n_1 - g_4 - l_3) \frac{\partial \phi}{\partial \omega} = 0,$$

$$2h \frac{\partial \phi}{\partial a_1} + b \frac{\partial \phi}{\partial h_1} + f \frac{\partial \phi}{\partial g_1} + m \frac{\partial \phi}{\partial l_1} - b_2 \frac{\partial \phi}{\partial \theta} - (c_2 - 2f_3) \frac{\partial \phi}{\partial \eta} + (d_2 - 2m_4) \frac{\partial \phi}{\partial \alpha} - b_3 \frac{\partial \phi}{\partial \kappa} - b_4 \frac{\partial \phi}{\partial \chi} + (n_2 - f_4 - m_3) \frac{\partial \phi}{\partial \omega} = 0,$$

$$2g \frac{\partial \phi}{\partial a_1} + f \frac{\partial \phi}{\partial h_1} + c \frac{\partial \phi}{\partial g_1} + n \frac{\partial \phi}{\partial l_1} + (b_3 - 2f_2) \frac{\partial \phi}{\partial \theta} - c_3 \frac{\partial \phi}{\partial \eta} + (d_3 - 2n_4) \frac{\partial \phi}{\partial \alpha} - c_2 \frac{\partial \phi}{\partial \kappa} + (m_3 - f_4 - n_2) \frac{\partial \phi}{\partial \chi} - c_4 \frac{\partial \phi}{\partial \omega} = 0,$$

$$2l \frac{\partial \phi}{\partial a_1} + m \frac{\partial \phi}{\partial h_1} + n \frac{\partial \phi}{\partial g_1} + d \frac{\partial \phi}{\partial l_1} + (b_4 - 2m_2) \frac{\partial \phi}{\partial \theta} + (c_4 - 2n_3) \frac{\partial \phi}{\partial \eta} - d_4 \frac{\partial \phi}{\partial \alpha} + (f_4 - m_3 - n_2) \frac{\partial \phi}{\partial \kappa} - d_2 \frac{\partial \phi}{\partial \chi} - d_3 \frac{\partial \phi}{\partial \omega} = 0.$$

(ii) From the coefficients of $\xi_{12}, \eta_{12}, \zeta_{12}, \theta_{12}$,

$$a \left(2 \frac{\partial \phi}{\partial a_2} + \frac{\partial \phi}{\partial h_1} \right) + h \left(\frac{\partial \phi}{\partial h_2} + 2 \frac{\partial \phi}{\partial b_1} \right) + g \left(\frac{\partial \phi}{\partial g_2} + \frac{\partial \phi}{\partial f_1} \right) + l \left(\frac{\partial \phi}{\partial l_2} + \frac{\partial \phi}{\partial m_1} \right) + 2a_2 \frac{\partial \phi}{\partial \theta} + a_3 \frac{\partial \phi}{\partial \kappa} + a_4 \frac{\partial \phi}{\partial \chi} + (g_2 + h_3 - f_1) \frac{\partial \phi}{\partial \lambda} + (h_4 + l_2 - m_1) \frac{\partial \phi}{\partial \rho} - 2g_3 \frac{\partial \phi}{\partial \mu} - 2l_4 \frac{\partial \phi}{\partial \zeta} + (g_4 + l_3) \left(\frac{\partial \phi}{\partial \nu} - \frac{\partial \phi}{\partial \omega} \right) = 0,$$

$$h \left(2 \frac{\partial \phi}{\partial a_2} + \frac{\partial \phi}{\partial h_1} \right) + b \left(\frac{\partial \phi}{\partial h_2} + 2 \frac{\partial \phi}{\partial b_1} \right) + f \left(\frac{\partial \phi}{\partial g_2} + \frac{\partial \phi}{\partial f_1} \right) + m \left(\frac{\partial \phi}{\partial l_2} + \frac{\partial \phi}{\partial m_1} \right) + 2b_1 \frac{\partial \phi}{\partial \theta} + (h_3 + f_1 - g_2) \frac{\partial \phi}{\partial \kappa} + (h_4 + m_1 - l_2) \frac{\partial \phi}{\partial \chi} + b_3 \frac{\partial \phi}{\partial \lambda} + b_4 \frac{\partial \phi}{\partial \rho} - 2f_3 \frac{\partial \phi}{\partial \mu} - 2m_4 \frac{\partial \phi}{\partial \zeta} - (f_4 + m_3) \left(\frac{\partial \phi}{\partial \nu} - \frac{\partial \phi}{\partial \omega} \right) = 0,$$

$$\begin{aligned}
& g\left(2\frac{\partial\phi}{\partial a_2} + \frac{\partial\phi}{\partial h_1}\right) + f\left(\frac{\partial\phi}{\partial h_2} + 2\frac{\partial\phi}{\partial b_1}\right) + c\left(\frac{\partial\phi}{\partial g_2} + \frac{\partial\phi}{\partial f_1}\right) + n\left(\frac{\partial\phi}{\partial l_2} + \frac{\partial\phi}{\partial m_1}\right) \\
& \quad + 2(g_2 + f_1 - h_3)\frac{\partial\phi}{\partial\theta} + c_1\frac{\partial\phi}{\partial\kappa} + (g_4 - l_3 + n_1)\frac{\partial\phi}{\partial\chi} + c_2\frac{\partial\phi}{\partial\lambda} \\
& \quad + (f_4 + n_2 - m_3)\frac{\partial\phi}{\partial\rho} - 2c_3\frac{\partial\phi}{\partial\mu} - 2n_4\frac{\partial\phi}{\partial\xi} + (c_4 + n_3)\left(\frac{\partial\phi}{\partial\nu} - \frac{\partial\phi}{\partial\omega}\right) = 0, \\
& l\left(2\frac{\partial\phi}{\partial a_2} + \frac{\partial\phi}{\partial h_1}\right) + m\left(\frac{\partial\phi}{\partial h_2} + 2\frac{\partial\phi}{\partial b_1}\right) + n\left(\frac{\partial\phi}{\partial g_2} + \frac{\partial\phi}{\partial f_1}\right) + l\left(\frac{\partial\phi}{\partial l_2} + \frac{\partial\phi}{\partial m_1}\right) \\
& \quad + 2(l_2 + m_1 - h_4)\frac{\partial\phi}{\partial\theta} + (l_3 + n_1 - g_4)\frac{\partial\phi}{\partial\kappa} + d_1\frac{\partial\phi}{\partial\chi} + (m_3 + n_2 - f_4)\frac{\partial\phi}{\partial\lambda} \\
& \quad + d_2\frac{\partial\phi}{\partial\rho} - 2m_3\frac{\partial\phi}{\partial\mu} - 2l_4\frac{\partial\phi}{\partial\xi} + (n_4 + d_3)\left(\frac{\partial\phi}{\partial\nu} - \frac{\partial\phi}{\partial\omega}\right) = 0.
\end{aligned}$$

(iii) From the coefficients of ξ_{13} , η_{13} , ζ_{13} , θ_{13} ,

$$\begin{aligned}
& a\left(2\frac{\partial\phi}{\partial a_3} + \frac{\partial\phi}{\partial g_1}\right) + h\left(\frac{\partial\phi}{\partial h_3} + \frac{\partial\phi}{\partial f_1}\right) + g\left(\frac{\partial\phi}{\partial g_3} + 2\frac{\partial\phi}{\partial c_1}\right) + l\left(\frac{\partial\phi}{\partial l_3} + \frac{\partial\phi}{\partial n_1}\right) \\
& \quad + 2a_3\frac{\partial\phi}{\partial\eta} + a_2\frac{\partial\phi}{\partial\kappa} + a_4\frac{\partial\phi}{\partial\omega} - 2h_2\frac{\partial\phi}{\partial\lambda} \\
& \quad + (h_3 + g_2 - f_1)\frac{\partial\phi}{\partial\mu} + (g_4 + l_3 - n_1)\frac{\partial\phi}{\partial\nu} - 2l_4\frac{\partial\phi}{\partial\iota} + (h_4 + l_2)\left(\frac{\partial\phi}{\partial\psi} - \frac{\partial\phi}{\partial\nu}\right) = 0, \\
& h\left(2\frac{\partial\phi}{\partial a_3} + \frac{\partial\phi}{\partial g_1}\right) + b\left(\frac{\partial\phi}{\partial h_3} + \frac{\partial\phi}{\partial f_1}\right) + f\left(\frac{\partial\phi}{\partial g_3} + 2\frac{\partial\phi}{\partial c_1}\right) + m\left(\frac{\partial\phi}{\partial l_3} + \frac{\partial\phi}{\partial n_1}\right) \\
& \quad + 2(h_3 + f_1 - g_2)\frac{\partial\phi}{\partial\eta} + b_1\frac{\partial\phi}{\partial\kappa} + (h_4 + m_1 - l_2)\frac{\partial\phi}{\partial\omega} - 2b_2\frac{\partial\phi}{\partial\lambda} \\
& \quad + b_3\frac{\partial\phi}{\partial\mu} + (f_4 + m_3 - n_2)\frac{\partial\phi}{\partial\nu} - 2m_4\frac{\partial\phi}{\partial\iota} + (b_4 + m_2)\left(\frac{\partial\phi}{\partial\psi} - \frac{\partial\phi}{\partial\nu}\right) = 0, \\
& g\left(2\frac{\partial\phi}{\partial a_3} + \frac{\partial\phi}{\partial g_1}\right) + f\left(\frac{\partial\phi}{\partial h_3} + \frac{\partial\phi}{\partial f_1}\right) + c\left(\frac{\partial\phi}{\partial g_3} + 2\frac{\partial\phi}{\partial c_1}\right) + n\left(\frac{\partial\phi}{\partial l_3} + \frac{\partial\phi}{\partial n_1}\right) \\
& \quad + 2c_1\frac{\partial\phi}{\partial\eta} + (g_2 + f_1 - h_3)\frac{\partial\phi}{\partial\kappa} + (g_4 + n_1 - l_3)\frac{\partial\phi}{\partial\omega} - 2f_2\frac{\partial\phi}{\partial\lambda} \\
& \quad + c_2\frac{\partial\phi}{\partial\mu} + c_4\frac{\partial\phi}{\partial\nu} - 2n_4\frac{\partial\phi}{\partial\iota} + (f_4 + n_2)\left(\frac{\partial\phi}{\partial\psi} - \frac{\partial\phi}{\partial\nu}\right) = 0, \\
& l\left(2\frac{\partial\phi}{\partial a_3} + \frac{\partial\phi}{\partial g_1}\right) + m\left(\frac{\partial\phi}{\partial h_3} + \frac{\partial\phi}{\partial f_1}\right) + n\left(\frac{\partial\phi}{\partial g_2} + 2\frac{\partial\phi}{\partial c_1}\right) + d\left(\frac{\partial\phi}{\partial l_3} + \frac{\partial\phi}{\partial n_1}\right) \\
& \quad + 2(l_3 + n_1 - g_4)\frac{\partial\phi}{\partial\eta} + (l_2 + m_1 - h_4)\frac{\partial\phi}{\partial\kappa} + d_1\frac{\partial\phi}{\partial\omega} - 2m_2\frac{\partial\phi}{\partial\lambda} \\
& \quad + (m_3 + n_2 - f_4)\frac{\partial\phi}{\partial\mu} + d_3\frac{\partial\phi}{\partial\nu} - 2d_4\frac{\partial\phi}{\partial\iota} + (m_4 + d_2)\left(\frac{\partial\phi}{\partial\psi} - \frac{\partial\phi}{\partial\nu}\right) = 0.
\end{aligned}$$

(iv) From the coefficients of $\xi_{14}, \eta_{14}, \zeta_{14}, \theta_{14}$,

$$\begin{aligned} & a \left(2 \frac{\partial \phi}{\partial \alpha_4} + \frac{\partial \phi}{\partial l_1} \right) + h \left(\frac{\partial \phi}{\partial h_4} + \frac{\partial \phi}{\partial m_1} \right) + g \left(\frac{\partial \phi}{\partial g_4} + \frac{\partial \phi}{\partial n_1} \right) + l \left(\frac{\partial \phi}{\partial l_4} + 2 \frac{\partial \phi}{\partial d_1} \right) \\ & + 2a_4 \frac{\partial \phi}{\partial \alpha} + a_2 \frac{\partial \phi}{\partial \chi} + a_3 \frac{\partial \phi}{\partial \omega} - 2h_2 \frac{\partial \phi}{\partial \rho} \\ & - 2g_3 \frac{\partial \phi}{\partial v} + (l_2 + h_4 - m_1) \frac{\partial \phi}{\partial \zeta} + (l_3 + g_4 - n_1) \frac{\partial \phi}{\partial i} + (l_3 + g_2) \left(\frac{\partial \phi}{\partial \omega} - \frac{\partial \phi}{\partial \psi} \right) = 0, \end{aligned}$$

$$\begin{aligned} & h \left(2 \frac{\partial \phi}{\partial \alpha_4} + \frac{\partial \phi}{\partial c_1} \right) + b \left(\frac{\partial \phi}{\partial h_4} + \frac{\partial \phi}{\partial m_1} \right) + f \left(\frac{\partial \phi}{\partial g_4} + \frac{\partial \phi}{\partial n_1} \right) + m \left(\frac{\partial \phi}{\partial l_4} + 2 \frac{\partial \phi}{\partial d_1} \right) \\ & + 2(h_4 - l_2 + m_1) \frac{\partial \phi}{\partial \alpha} + b_1 \frac{\partial \phi}{\partial \chi} + (f_1 + h_3 - g_2) \frac{\partial \phi}{\partial \omega} - 2b_2 \frac{\partial \phi}{\partial \rho} \\ & - 2f_3 \frac{\partial \phi}{\partial v} + b_4 \frac{\partial \phi}{\partial \zeta} + m_3 \frac{\partial \phi}{\partial i} + (b_3 + f_2) \left(\frac{\partial \phi}{\partial \omega} - \frac{\partial \phi}{\partial \psi} \right) = 0, \end{aligned}$$

$$\begin{aligned} & g \left(2 \frac{\partial \phi}{\partial \alpha_4} + \frac{\partial \phi}{\partial l_1} \right) + f \left(\frac{\partial \phi}{\partial h_4} + \frac{\partial \phi}{\partial m_1} \right) + c \left(\frac{\partial \phi}{\partial g_4} + \frac{\partial \phi}{\partial n_1} \right) + n \left(\frac{\partial \phi}{\partial l_4} + 2 \frac{\partial \phi}{\partial d_1} \right) \\ & + 2(g_4 - l_3 + n_1) \frac{\partial \phi}{\partial \alpha} + (g_2 + f_1 - h_3) \frac{\partial \phi}{\partial \chi} + c_1 \frac{\partial \phi}{\partial \omega} - 2f_2 \frac{\partial \phi}{\partial \rho} \\ & - 2c_3 \frac{\partial \phi}{\partial v} + (f_4 - m_3 + n_2) \frac{\partial \phi}{\partial \zeta} + c_4 \frac{\partial \phi}{\partial i} + (f_3 + c_2) \left(\frac{\partial \phi}{\partial \omega} - \frac{\partial \phi}{\partial \psi} \right) = 0, \end{aligned}$$

$$\begin{aligned} & l \left(2 \frac{\partial \phi}{\partial \alpha_4} + \frac{\partial \phi}{\partial l_1} \right) + m \left(\frac{\partial \phi}{\partial h_4} + \frac{\partial \phi}{\partial m_1} \right) + n \left(\frac{\partial \phi}{\partial g_4} + \frac{\partial \phi}{\partial n_1} \right) + d \left(\frac{\partial \phi}{\partial l_4} + 2 \frac{\partial \phi}{\partial d_1} \right) \\ & + 2d_1 \frac{\partial \phi}{\partial \alpha} + (m_1 + l_2 - h_4) \frac{\partial \phi}{\partial \chi} + (n_1 + l_3 - g_4) \frac{\partial \phi}{\partial \omega} - 2m_2 \frac{\partial \phi}{\partial \rho} \\ & - 2n_3 \frac{\partial \phi}{\partial v} + d_2 \frac{\partial \phi}{\partial \zeta} + d_3 \frac{\partial \phi}{\partial i} + (m_3 + n_2) \left(\frac{\partial \phi}{\partial \omega} - \frac{\partial \phi}{\partial \psi} \right) = 0. \end{aligned}$$

(v) From the coefficients of $\xi_{22}, \eta_{22}, \zeta_{22}, \theta_{22}$,

$$\begin{aligned} & a \frac{\partial \phi}{\partial h_2} + 2h \frac{\partial \phi}{\partial b_2} + g \frac{\partial \phi}{\partial f_2} - l \frac{\partial \phi}{\partial m_2} \\ & - a_1 \frac{\partial \phi}{\partial \theta} + (c_1 - 2g_3) \frac{\partial \phi}{\partial \delta} + (d_1 - 2l_4) \frac{\partial \phi}{\partial \beta} \\ & - a_3 \frac{\partial \phi}{\partial \lambda} + (g_4 + l_3 - n_1) \frac{\partial \phi}{\partial \sigma} + a_4 \left(\frac{\partial \phi}{\partial \omega} - \frac{\partial \phi}{\partial v} \right) = 0, \end{aligned}$$

$$\begin{aligned} & h \frac{\partial \phi}{\partial h_2} + 2b \frac{\partial \phi}{\partial b_2} + f \frac{\partial \phi}{\partial f_2} + m \frac{\partial \phi}{\partial m_2} \\ & + (a_2 - 2h_1) \frac{\partial \phi}{\partial \theta} + (c_2 - 2f_3) \frac{\partial \phi}{\partial \delta} + (d_2 - 2m_4) \frac{\partial \phi}{\partial \beta} + (l_2 - m_1) \frac{\partial \phi}{\partial \rho} \\ & - (h_3 - g_2 + f_1) \frac{\partial \phi}{\partial \lambda} - (f_4 + m_3 - n_2) \frac{\partial \phi}{\partial \sigma} + h_4 \left(\frac{\partial \phi}{\partial \omega} - \frac{\partial \phi}{\partial v} \right) = 0, \end{aligned}$$

$$\begin{aligned}
& g \frac{\partial \phi}{\partial h_2} + 2f \frac{\partial \phi}{\partial b_2} + c \frac{\partial \phi}{\partial f_2} + n \frac{\partial \phi}{\partial m_2} \\
& + (a_3 - 2g_1) \frac{\partial \phi}{\partial \theta} - c_3 \frac{\partial \phi}{\partial \delta} + (d_3 - 2n_4) \frac{\partial \phi}{\partial \beta} + (l_3 - n_1) \frac{\partial \phi}{\partial \rho} \\
& - c_1 \frac{\partial \phi}{\partial \lambda} - c_4 \frac{\partial \phi}{\partial \sigma} + g_4 \left(\frac{\partial \phi}{\partial \omega} - \frac{\partial \phi}{\partial \nu} \right) = 0,
\end{aligned}$$

$$\begin{aligned}
& l \frac{\partial \phi}{\partial a_2} + 2m \frac{\partial \phi}{\partial b_2} + n \frac{\partial \phi}{\partial f_2} + d \frac{\partial \phi}{\partial m_2} \\
& + (a_4 - 2l_1) \frac{\partial \phi}{\partial \theta} + (c_4 - 2n_3) \frac{\partial \phi}{\partial \delta} - d_4 \frac{\partial \phi}{\partial \beta} + (l_4 - d_1) \frac{\partial \phi}{\partial \rho} \\
& - (l_3 + n_1 - g_4) \frac{\partial \phi}{\partial \lambda} - d_3 \frac{\partial \phi}{\partial \sigma} + l_4 \left(\frac{\partial \phi}{\partial \omega} - \frac{\partial \phi}{\partial \nu} \right) = 0.
\end{aligned}$$

(vi) From the coefficients of ξ_{23} , η_{23} , ζ_{23} , θ_{23} ,

$$\begin{aligned}
& a \left(\frac{\partial \phi}{\partial h_3} + \frac{\partial \phi}{\partial g_2} \right) + h \left(2 \frac{\partial \phi}{\partial b_3} + \frac{\partial \phi}{\partial f_2} \right) + g \left(\frac{\partial \phi}{\partial f_3} + 2 \frac{\partial \phi}{\partial c_2} \right) + l \left(\frac{\partial \phi}{\partial m_3} + \frac{\partial \phi}{\partial n_2} \right) \\
& + 2(h_3 + g_2 - f_1) \frac{\partial \phi}{\partial \delta} - a_1 \frac{\partial \phi}{\partial \kappa} + a_2 \frac{\partial \phi}{\partial \lambda} + (h_4 + l_2 - m_1) \frac{\partial \phi}{\partial \sigma} \\
& + a_3 \frac{\partial \phi}{\partial \mu} + (g_4 + l_3 - n_1) \frac{\partial \phi}{\partial \tau} - 2l_4 \frac{\partial \phi}{\partial \xi} + (a_4 + l_4) \left(\frac{\partial \phi}{\partial \omega} - \frac{\partial \phi}{\partial \psi} \right) = 0,
\end{aligned}$$

$$\begin{aligned}
& h \left(\frac{\partial \phi}{\partial h_3} + \frac{\partial \phi}{\partial g_2} \right) + b \left(2 \frac{\partial \phi}{\partial b_3} + \frac{\partial \phi}{\partial f_2} \right) + f \left(\frac{\partial \phi}{\partial f_3} + 2 \frac{\partial \phi}{\partial c_2} \right) + m \left(\frac{\partial \phi}{\partial m_3} + \frac{\partial \phi}{\partial n_2} \right) \\
& + 2b_3 \frac{\partial \phi}{\partial \delta} + (a_2 - 2h_1) \frac{\partial \phi}{\partial \kappa} + b_1 \frac{\partial \phi}{\partial \lambda} + b_4 \frac{\partial \phi}{\partial \sigma} \\
& + (h_3 + f_1 - g_2) \frac{\partial \phi}{\partial \mu} + (f_4 + m_3 - n_2) \frac{\partial \phi}{\partial \tau} - 2m_4 \frac{\partial \phi}{\partial \xi} + (h_4 + m_1) \left(\frac{\partial \phi}{\partial \omega} - \frac{\partial \phi}{\partial \psi} \right) = 0,
\end{aligned}$$

$$\begin{aligned}
& g \left(\frac{\partial \phi}{\partial h_3} + \frac{\partial \phi}{\partial g_2} \right) + f \left(2 \frac{\partial \phi}{\partial b_3} + \frac{\partial \phi}{\partial f_2} \right) + c \left(\frac{\partial \phi}{\partial f_3} + 2 \frac{\partial \phi}{\partial c_2} \right) + n \left(\frac{\partial \phi}{\partial m_3} + \frac{\partial \phi}{\partial n_2} \right) \\
& + 2c_2 \frac{\partial \phi}{\partial \delta} + (a_3 - 2g_1) \frac{\partial \phi}{\partial \kappa} + (g_2 + f_1 - h_3) \frac{\partial \phi}{\partial \lambda} + (f_4 + n_2 - m_3) \frac{\partial \phi}{\partial \sigma} \\
& + c_1 \frac{\partial \phi}{\partial \mu} + c_4 \frac{\partial \phi}{\partial \tau} - 2n_4 \frac{\partial \phi}{\partial \xi} + (g_4 + n_1) \left(\frac{\partial \phi}{\partial \omega} - \frac{\partial \phi}{\partial \psi} \right) = 0,
\end{aligned}$$

$$\begin{aligned}
& l \left(\frac{\partial \phi}{\partial h_3} + \frac{\partial \phi}{\partial g_2} \right) + m \left(2 \frac{\partial \phi}{\partial b_3} + \frac{\partial \phi}{\partial f_2} \right) + n \left(\frac{\partial \phi}{\partial f_3} + 2 \frac{\partial \phi}{\partial c_2} \right) + d \left(\frac{\partial \phi}{\partial m_3} + \frac{\partial \phi}{\partial n_2} \right) \\
& + 2(m_3 + n_2 - f_4) \frac{\partial \phi}{\partial \delta} + (a_4 - 2l_1) \frac{\partial \phi}{\partial \kappa} + (l_2 + m_1 - h_4) \frac{\partial \phi}{\partial \lambda} + d_2 \frac{\partial \phi}{\partial \sigma} \\
& + (l_3 + n_1 - g_2) \frac{\partial \phi}{\partial \mu} + d_3 \frac{\partial \phi}{\partial \tau} - 2d_4 \frac{\partial \phi}{\partial \xi} + (l_2 + d_1) \left(\frac{\partial \phi}{\partial \omega} - \frac{\partial \phi}{\partial \psi} \right) = 0.
\end{aligned}$$

(vii) From the coefficients of $\xi_{24}, \eta_{24}, \zeta_{24}, \theta_{24}$,

$$\begin{aligned}
& a\left(\frac{\partial\phi}{\partial h_4} + \frac{\partial\phi}{\partial l_2}\right) + h\left(2\frac{\partial\phi}{\partial b_4} + \frac{\partial\phi}{\partial m_2}\right) + g\left(\frac{\partial\phi}{\partial f_4} + \frac{\partial\phi}{\partial n_2}\right) + l\left(\frac{\partial\phi}{\partial m_4} + 2\frac{\partial\phi}{\partial d_2}\right) \\
& \quad + 2(h_4 + l_2 - m_1)\frac{\partial\phi}{\partial\beta} - 2a_1\frac{\partial\phi}{\partial\chi} + a_2\frac{\partial\phi}{\partial\rho} + (h_3 + g_2 - f_1)\frac{\partial\phi}{\partial\sigma} \\
& \quad + a_4\frac{\partial\phi}{\partial\xi} + (g_3 + l_3 - n_1)\frac{\partial\phi}{\partial\xi} - 2g_3\frac{\partial\phi}{\partial\tau} + (a_3 + g_1)\left(\frac{\partial\phi}{\partial\psi} - \frac{\partial\phi}{\partial\nu}\right) = 0, \\
& h\left(\frac{\partial\phi}{\partial h_4} + \frac{\partial\phi}{\partial l_2}\right) + b\left(2\frac{\partial\phi}{\partial b_4} + \frac{\partial\phi}{\partial m_2}\right) + f\left(\frac{\partial\phi}{\partial f_4} + \frac{\partial\phi}{\partial n_2}\right) + m\left(\frac{\partial\phi}{\partial m_4} + 2\frac{\partial\phi}{\partial d_2}\right) \\
& \quad + 2b_4\frac{\partial\phi}{\partial\beta} - 2h_1\frac{\partial\phi}{\partial\chi} + b_1\frac{\partial\phi}{\partial\rho} + b_3\frac{\partial\phi}{\partial\sigma} \\
& \quad + (h_4 + m_1 - l_2)\frac{\partial\phi}{\partial\xi} + (f_4 + m_3 - n_2)\frac{\partial\phi}{\partial\xi} - 2f_3\frac{\partial\phi}{\partial\tau} + (h_3 + f_1)\left(\frac{\partial\phi}{\partial\psi} - \frac{\partial\phi}{\partial\nu}\right) = 0, \\
& g\left(\frac{\partial\phi}{\partial h_4} + \frac{\partial\phi}{\partial l_2}\right) + f\left(2\frac{\partial\phi}{\partial b_4} + \frac{\partial\phi}{\partial m_2}\right) + c\left(\frac{\partial\phi}{\partial f_4} + \frac{\partial\phi}{\partial n_2}\right) + n\left(\frac{\partial\phi}{\partial m_4} + 2\frac{\partial\phi}{\partial d_2}\right) \\
& \quad + 2(f_4 + n_2 - m_3)\frac{\partial\phi}{\partial\beta} - 2g_1\frac{\partial\phi}{\partial\chi} + (g_2 + f_1 - h_3)\frac{\partial\phi}{\partial\rho} + c_2\frac{\partial\phi}{\partial\sigma} \\
& \quad + (g_4 + n_1 - l_3)\frac{\partial\phi}{\partial\xi} + c_1\frac{\partial\phi}{\partial\xi} - 2c_3\frac{\partial\phi}{\partial\tau} + (g_3 + c_1)\left(\frac{\partial\phi}{\partial\psi} - \frac{\partial\phi}{\partial\nu}\right) = 0, \\
& l\left(\frac{\partial\phi}{\partial h_4} + \frac{\partial\phi}{\partial l_2}\right) + m\left(2\frac{\partial\phi}{\partial b_4} + \frac{\partial\phi}{\partial m_2}\right) + n\left(\frac{\partial\phi}{\partial f_4} + \frac{\partial\phi}{\partial n_2}\right) + d\left(\frac{\partial\phi}{\partial m_4} + 2\frac{\partial\phi}{\partial d_2}\right) \\
& \quad + 2d_2\frac{\partial\phi}{\partial\beta} - 2l_1\frac{\partial\phi}{\partial\chi} + (l_2 + m_1 - h_4)\frac{\partial\phi}{\partial\rho} + (m_3 + n_2 - f_4)\frac{\partial\phi}{\partial\sigma} \\
& \quad + d_1\frac{\partial\phi}{\partial\xi} + d_3\frac{\partial\phi}{\partial\xi} - 2n_3\frac{\partial\phi}{\partial\tau} + (l_3 + n_1)\left(\frac{\partial\phi}{\partial\psi} - \frac{\partial\phi}{\partial\nu}\right) = 0.
\end{aligned}$$

(viii) From the coefficients of $\xi_{33}, \eta_{33}, \zeta_{33}, \theta_{33}$,

$$\begin{aligned}
& a\frac{\partial\phi}{\partial g_3} + h\frac{\partial\phi}{\partial f_3} + 2g\frac{\partial\phi}{\partial c_3} + l\frac{\partial\phi}{\partial n_3} \\
& \quad - a_1\frac{\partial\phi}{\partial\eta} + (l_1 - 2h_2)\frac{\partial\phi}{\partial\delta} + (d_1 - 2l_4)\frac{\partial\phi}{\partial\gamma} \\
& \quad - a_2\frac{\partial\phi}{\partial\mu} - a_4\frac{\partial\phi}{\partial\nu} - (h_4 + l_2 - m_1)\frac{\partial\phi}{\partial\tau} = 0, \\
& h\frac{\partial\phi}{\partial g_3} + b\frac{\partial\phi}{\partial f_3} + 2f\frac{\partial\phi}{\partial c_3} + m\frac{\partial\phi}{\partial n_3} \\
& \quad + (a_2 - 2h_1)\frac{\partial\phi}{\partial\eta} - b_2\frac{\partial\phi}{\partial\delta} + (d_2 - 2m_4)\frac{\partial\phi}{\partial\gamma} \\
& \quad - f_1\frac{\partial\phi}{\partial\mu} - (h_4 - l_2 + m_1)\frac{\partial\phi}{\partial\nu} - b_4\frac{\partial\phi}{\partial\tau} = 0,
\end{aligned}$$

$$g \frac{\partial \phi}{\partial g_3} + f \frac{\partial \phi}{\partial f_3} + 2c \frac{\partial \phi}{\partial c_3} + n \frac{\partial \phi}{\partial n_3} \\ + (a_3 - 2g_1) \frac{\partial \phi}{\partial \eta} + (b_3 - 2f_2) \frac{\partial \phi}{\partial \delta} + (d_3 - 2n_4) \frac{\partial \phi}{\partial \gamma} \\ - (g_2 + f_1 - h_3) \frac{\partial \phi}{\partial \mu} - (g_4 + n_1 - l_3) \frac{\partial \phi}{\partial v} - (f_4 + n_2 - m_3) \frac{\partial \phi}{\partial \tau} = 0,$$

$$l \frac{\partial \phi}{\partial g_3} + m \frac{\partial \phi}{\partial f_3} + 2n \frac{\partial \phi}{\partial c_3} + d \frac{\partial \phi}{\partial n_3} \\ + (a_4 - 2l_1) \frac{\partial \phi}{\partial \eta} + (b_4 - 2m_2) \frac{\partial \phi}{\partial \delta} - d_4 \frac{\partial \phi}{\partial \gamma} \\ - (l_2 + m_1 - h_4) \frac{\partial \phi}{\partial \mu} - d_1 \frac{\partial \phi}{\partial v} - d_2 \frac{\partial \phi}{\partial \tau} = 0.$$

(ix) From the coefficients of ξ_{34} , η_{34} , ζ_{34} , θ_{34} ,

$$a \left(\frac{\partial \phi}{\partial g_4} + \frac{\partial \phi}{\partial l_3} \right) + h \left(\frac{\partial \phi}{\partial f_4} + \frac{\partial \phi}{\partial m_3} \right) + g \left(2 \frac{\partial \phi}{\partial c_4} + \frac{\partial \phi}{\partial n_3} \right) + l \left(\frac{\partial \phi}{\partial n_4} + 2 \frac{\partial \phi}{\partial d_3} \right), \\ + 2(g_4 + l_3 - n_1) \frac{\partial \phi}{\partial \gamma} - 2a_1 \frac{\partial \phi}{\partial \omega} - 2h_2 \frac{\partial \phi}{\partial \sigma} + a_3 \frac{\partial \phi}{\partial v} \\ + (g_2 + h_3 - f_1) \frac{\partial \phi}{\partial \tau} + a_4 \frac{\partial \phi}{\partial i} + (h_4 + l_2 - m_1) \frac{\partial \phi}{\partial \xi} + (a_2 + h_1) \left(\frac{\partial \phi}{\partial v} - \frac{\partial \phi}{\partial \omega} \right) = 0,$$

$$h \left(\frac{\partial \phi}{\partial g_4} + \frac{\partial \phi}{\partial l_3} \right) + b \left(\frac{\partial \phi}{\partial f_4} + \frac{\partial \phi}{\partial m_3} \right) + f \left(2 \frac{\partial \phi}{\partial c_4} + \frac{\partial \phi}{\partial n_3} \right) + m \left(\frac{\partial \phi}{\partial n_4} + 2 \frac{\partial \phi}{\partial d_3} \right) \\ + 2(f_4 + m_3 - n_2) \frac{\partial \phi}{\partial \gamma} - 2h_1 \frac{\partial \phi}{\partial \omega} - 2b_2 \frac{\partial \phi}{\partial \sigma} + (h_3 + f_1 - g_2) \frac{\partial \phi}{\partial v} \\ + b_3 \frac{\partial \phi}{\partial \tau} + (h_4 + m_1 - l_2) \frac{\partial \phi}{\partial i} + b_4 \frac{\partial \phi}{\partial \xi} + (h_2 + b_1) \left(\frac{\partial \phi}{\partial v} - \frac{\partial \phi}{\partial \omega} \right) = 0,$$

$$g \left(\frac{\partial \phi}{\partial g_4} + \frac{\partial \phi}{\partial l_3} \right) + f \left(\frac{\partial \phi}{\partial f_4} + \frac{\partial \phi}{\partial m_3} \right) + c \left(2 \frac{\partial \phi}{\partial c_4} + \frac{\partial \phi}{\partial n_3} \right) + n \left(\frac{\partial \phi}{\partial n_4} + 2 \frac{\partial \phi}{\partial d_3} \right) \\ + 2c_4 \frac{\partial \phi}{\partial \gamma} + 2g_1 \frac{\partial \phi}{\partial \omega} - 2f_2 \frac{\partial \phi}{\partial \sigma} + c_1 \frac{\partial \phi}{\partial v} \\ + c_2 \frac{\partial \phi}{\partial \tau} + (g_4 + n_1 - l_3) \frac{\partial \phi}{\partial i} + (f_4 + n_2 - m_3) \frac{\partial \phi}{\partial \xi} + (g_2 + f_1) \left(\frac{\partial \phi}{\partial v} - \frac{\partial \phi}{\partial \omega} \right) = 0,$$

$$l \left(\frac{\partial \phi}{\partial g_4} + \frac{\partial \phi}{\partial l_3} \right) + m \left(\frac{\partial \phi}{\partial f_4} + \frac{\partial \phi}{\partial m_3} \right) + n \left(2 \frac{\partial \phi}{\partial c_4} + \frac{\partial \phi}{\partial n_3} \right) + d \left(\frac{\partial \phi}{\partial n_4} + 2 \frac{\partial \phi}{\partial d_3} \right) \\ + 2d_3 \frac{\partial \phi}{\partial \gamma} - 2l_1 \frac{\partial \phi}{\partial \omega} - 2m_2 \frac{\partial \phi}{\partial \sigma} + (l_3 + n_1 - g_4) \frac{\partial \phi}{\partial v} \\ + (m_3 + n_2 - f_4) \frac{\partial \phi}{\partial \tau} + d_1 \frac{\partial \phi}{\partial i} + d_2 \frac{\partial \phi}{\partial \xi} + (l_2 + m_1) \left(\frac{\partial \phi}{\partial v} - \frac{\partial \phi}{\partial \omega} \right) = 0,$$

(x) From the coefficients of $\xi_{44}, \eta_{44}, \zeta_{44}, \theta_{44}$,

$$\begin{aligned} & a \frac{\partial \phi}{\partial l_4} + h \frac{\partial \phi}{\partial m_4} + g \frac{\partial \phi}{\partial n_4} + 2l \frac{\partial \phi}{\partial d_4} \\ & \quad - a_1 \frac{\partial \phi}{\partial \alpha} + (b_1 - 2h_2) \frac{\partial \phi}{\partial \beta} + (c_1 - 2g_3) \frac{\partial \phi}{\partial \gamma} \\ & \quad - a_2 \frac{\partial \phi}{\partial \zeta} - a_3 \frac{\partial \phi}{\partial \iota} - (h_3 + g_2 - f_1) \frac{\partial \phi}{\partial \xi} = 0, \\ & h \frac{\partial \phi}{\partial l_4} + b \frac{\partial \phi}{\partial m_4} + f \frac{\partial \phi}{\partial n_4} + 2m \frac{\partial \phi}{\partial d_4} \\ & \quad + (a_2 - 2h_1) \frac{\partial \phi}{\partial \alpha} + b_2 \frac{\partial \phi}{\partial \beta} + (c_2 - 2f_3) \frac{\partial \phi}{\partial \gamma} \\ & \quad - b_1 \frac{\partial \phi}{\partial \zeta} - (h_3 + f_1 - g_2) \frac{\partial \phi}{\partial \iota} - b_3 \frac{\partial \phi}{\partial \xi} = 0, \\ & g \frac{\partial \phi}{\partial l_4} + f \frac{\partial \phi}{\partial m_4} + c \frac{\partial \phi}{\partial n_4} + 2n \frac{\partial \phi}{\partial d_4} \\ & \quad + (a_3 - 2g_1) \frac{\partial \phi}{\partial \alpha} + (b_3 - 2f_2) \frac{\partial \phi}{\partial \beta} - c_3 \frac{\partial \phi}{\partial \gamma} \\ & \quad - (g_2 + f_1 - h_3) \frac{\partial \phi}{\partial \zeta} - c_1 \frac{\partial \phi}{\partial \iota} - c_2 \frac{\partial \phi}{\partial \xi} = 0, \\ & l \frac{\partial \phi}{\partial l_4} + m \frac{\partial \phi}{\partial m_4} + n \frac{\partial \phi}{\partial n_4} + d \frac{\partial \phi}{\partial d_4} + 2a \frac{\partial \phi}{\partial d_4} \\ & \quad + (a_1 - 2l_1) \frac{\partial \phi}{\partial \alpha} + (b_4 - 2m_2) \frac{\partial \phi}{\partial \beta} + (c_1 - 2n_3) \frac{\partial \phi}{\partial \gamma} \\ & \quad - (l_2 + m_1 - h_4) \frac{\partial \phi}{\partial \zeta} - (l_3 + n_1 - g_4) \frac{\partial \phi}{\partial \iota} - (m_3 + n_2 - f_4) \frac{\partial \phi}{\partial \xi} = 0. \end{aligned}$$

20. This set of equations, forty in number, satisfies all the conditions for a complete linear Jacobian system of the first order.

The number of independent quantities that occur in the system are :—

- 20, being the magnitudes a, β, \dots from the previous sub-group ;
- +40, . . . first derivatives of a, \dots, n ;
- +10, . . . quantities a, \dots, n ;

that is, 70 in all.

Hence under the Jacobian theory the system possesses thirty (70 - 40) independent integrals. As derivatives with regard to a, \dots, n do not occur, it is clear that ten of these integrals are provided by

$$a, b, c, d, f, g, h, l, m, n.$$

We therefore require twenty other integrals, independent of these and independent of one another.

21. Next, there are no other integrals which are independent of the magnitudes $\alpha, \beta, \gamma, \dots$. If such an integral existed not involving these magnitudes, it would satisfy the relations

$$\frac{\partial \phi}{\partial a_1} = 0, \quad \frac{\partial \phi}{\partial h_1} = 0, \quad \frac{\partial \phi}{\partial g_1} = 0, \quad \frac{\partial \phi}{\partial l_1} = 0,$$

because of the $\xi_{11}, \eta_{11}, \zeta_{11}, \theta_{11}$ equations; the relations

$$2\frac{\partial \phi}{\partial a_2} + \frac{\partial \phi}{\partial h_1} = 0, \quad \frac{\partial \phi}{\partial h_2} + 2\frac{\partial \phi}{\partial b_1} = 0, \quad \frac{\partial \phi}{\partial g_2} + \frac{\partial \phi}{\partial f_1} = 0, \quad \frac{\partial \phi}{\partial l_2} + \frac{\partial \phi}{\partial m_1} = 0,$$

because of the $\xi_{12}, \eta_{12}, \zeta_{12}, \theta_{12}$ equations; the relations

$$2\frac{\partial \phi}{\partial a_3} + \frac{\partial \phi}{\partial g_1} = 0, \quad \frac{\partial \phi}{\partial h_3} + \frac{\partial \phi}{\partial f_1} = 0, \quad \frac{\partial \phi}{\partial g_3} + 2\frac{\partial \phi}{\partial c_1} = 0, \quad \frac{\partial \phi}{\partial l_3} + \frac{\partial \phi}{\partial n_1} = 0,$$

because of the $\xi_{13}, \eta_{13}, \zeta_{13}, \theta_{13}$ equations; the relations

$$2\frac{\partial \phi}{\partial a_4} + \frac{\partial \phi}{\partial c_1} = 0, \quad \frac{\partial \phi}{\partial h_4} + \frac{\partial \phi}{\partial m_1} = 0, \quad \frac{\partial \phi}{\partial g_4} + \frac{\partial \phi}{\partial n_1} = 0, \quad \frac{\partial \phi}{\partial l_4} + 2\frac{\partial \phi}{\partial d_1} = 0,$$

because of the $\xi_{14}, \eta_{14}, \zeta_{14}, \theta_{14}$ equations; the relations

$$\frac{\partial \phi}{\partial h_2} = 0, \quad \frac{\partial \phi}{\partial b_2} = 0, \quad \frac{\partial \phi}{\partial f_2} = 0, \quad \frac{\partial \phi}{\partial m_2} = 0,$$

because of the $\xi_{22}, \eta_{22}, \zeta_{22}, \theta_{22}$ equations; the relations

$$\frac{\partial \phi}{\partial h_3} + \frac{\partial \phi}{\partial g_2} = 0, \quad 2\frac{\partial \phi}{\partial b_3} + \frac{\partial \phi}{\partial f_2} = 0, \quad \frac{\partial \phi}{\partial f_3} + 2\frac{\partial \phi}{\partial c_2} = 0, \quad \frac{\partial \phi}{\partial m_3} + \frac{\partial \phi}{\partial n_2} = 0,$$

because of the $\xi_{23}, \eta_{23}, \zeta_{23}, \theta_{23}$ equations; the relations

$$\frac{\partial \phi}{\partial h_4} + \frac{\partial \phi}{\partial l_2} = 0, \quad 2\frac{\partial \phi}{\partial b_4} + \frac{\partial \phi}{\partial m_2} = 0, \quad \frac{\partial \phi}{\partial f_4} + \frac{\partial \phi}{\partial n_2} = 0, \quad \frac{\partial \phi}{\partial m_4} + 2\frac{\partial \phi}{\partial d_2} = 0,$$

because of the $\xi_{24}, \eta_{24}, \zeta_{24}, \theta_{24}$ equations; the relations

$$\frac{\partial \phi}{\partial g_3} = 0, \quad \frac{\partial \phi}{\partial f_3} = 0, \quad \frac{\partial \phi}{\partial c_3} = 0, \quad \frac{\partial \phi}{\partial n_3} = 0,$$

because of the $\xi_{33}, \eta_{33}, \zeta_{33}, \theta_{33}$ equations; the relations

$$\frac{\partial \phi}{\partial g_4} + \frac{\partial \phi}{\partial l_3} = 0, \quad \frac{\partial \phi}{\partial f_4} + \frac{\partial \phi}{\partial m_3} = 0, \quad 2\frac{\partial \phi}{\partial c_4} + \frac{\partial \phi}{\partial n_3} = 0, \quad \frac{\partial \phi}{\partial n_4} + 2\frac{\partial \phi}{\partial d_3} = 0,$$

because of the $\xi_{34}, \eta_{34}, \zeta_{34}, \theta_{34}$ equations; and the relations

$$\frac{\partial \phi}{\partial l_4} = 0, \quad \frac{\partial \phi}{\partial m_4} = 0, \quad \frac{\partial \phi}{\partial n_4} = 0, \quad \frac{\partial \phi}{\partial d_4} = 0.$$

These relations can only be satisfied when each of the differential coefficients of ϕ vanishes; consequently, no function ϕ , involving only the first derivatives, can exist.

22. We therefore require twenty integrals involving the magnitudes $\alpha, \beta, \gamma, \dots$. It has already been pointed out that these magnitudes are

the combinations of the second derivatives of a, b, \dots, n which occur in the symbols usually associated with the names of both Riemann and Christoffel. It should further be noted that very many of the coefficients of the derivatives of ϕ in the preceding forty equations are the combinations often called Christoffel's symbols, which, in the case of our four independent variables, are as follows:—

$$\begin{aligned} \left[\begin{matrix} 11 \\ 1 \end{matrix} \right] &= \frac{1}{2}a_1, & \left[\begin{matrix} 11 \\ 2 \end{matrix} \right] &= \frac{1}{2}(2h_1 - a_2), & \left[\begin{matrix} 11 \\ 3 \end{matrix} \right] &= \frac{1}{2}(2g_1 - a_3), & \left[\begin{matrix} 11 \\ 4 \end{matrix} \right] &= \frac{1}{2}(2l_1 - a_4); \\ \left[\begin{matrix} 12 \\ 1 \end{matrix} \right] &= \frac{1}{2}a_2, & \left[\begin{matrix} 12 \\ 2 \end{matrix} \right] &= \frac{1}{2}b_1, & \left[\begin{matrix} 12 \\ 3 \end{matrix} \right] &= \frac{1}{2}(g_2 + f_1 - h_3), & \left[\begin{matrix} 12 \\ 4 \end{matrix} \right] &= \frac{1}{2}(l_2 + m_1 - h_4); \\ \left[\begin{matrix} 13 \\ 1 \end{matrix} \right] &= \frac{1}{2}a_3, & \left[\begin{matrix} 13 \\ 2 \end{matrix} \right] &= \frac{1}{2}(h_3 + f_1 - g_2), & \left[\begin{matrix} 13 \\ 3 \end{matrix} \right] &= \frac{1}{2}c_1, & \left[\begin{matrix} 13 \\ 4 \end{matrix} \right] &= \frac{1}{2}(n_1 + l_3 - g_4); \\ \left[\begin{matrix} 14 \\ 1 \end{matrix} \right] &= \frac{1}{2}a_4, & \left[\begin{matrix} 14 \\ 2 \end{matrix} \right] &= \frac{1}{2}(h_4 + m_1 - l_2), & \left[\begin{matrix} 14 \\ 3 \end{matrix} \right] &= \frac{1}{2}(g_4 + n_1 - l_3), & \left[\begin{matrix} 14 \\ 4 \end{matrix} \right] &= \frac{1}{2}d_1; \\ \left[\begin{matrix} 22 \\ 1 \end{matrix} \right] &= \frac{1}{2}(2h_2 - b_1), & \left[\begin{matrix} 22 \\ 2 \end{matrix} \right] &= \frac{1}{2}b_2, & \left[\begin{matrix} 22 \\ 3 \end{matrix} \right] &= \frac{1}{2}(2f_2 - b_3), & \left[\begin{matrix} 22 \\ 4 \end{matrix} \right] &= \frac{1}{2}(2m_2 - b_4); \\ \left[\begin{matrix} 23 \\ 1 \end{matrix} \right] &= \frac{1}{2}(h_3 + g_2 - f_1), & \left[\begin{matrix} 23 \\ 2 \end{matrix} \right] &= \frac{1}{2}b_3, & \left[\begin{matrix} 23 \\ 3 \end{matrix} \right] &= \frac{1}{2}c_2, & \left[\begin{matrix} 23 \\ 4 \end{matrix} \right] &= \frac{1}{2}(m_3 + n_2 - f_4); \\ \left[\begin{matrix} 24 \\ 1 \end{matrix} \right] &= \frac{1}{2}(h_4 + l_2 - m_1), & \left[\begin{matrix} 24 \\ 2 \end{matrix} \right] &= \frac{1}{2}b_4, & \left[\begin{matrix} 24 \\ 3 \end{matrix} \right] &= \frac{1}{2}(f_4 + n_2 - m_3), & \left[\begin{matrix} 24 \\ 4 \end{matrix} \right] &= \frac{1}{2}d_2; \\ \left[\begin{matrix} 33 \\ 1 \end{matrix} \right] &= \frac{1}{2}(2g_3 - c_1), & \left[\begin{matrix} 33 \\ 2 \end{matrix} \right] &= \frac{1}{2}(2f_3 - c_2), & \left[\begin{matrix} 33 \\ 3 \end{matrix} \right] &= \frac{1}{2}c_3, & \left[\begin{matrix} 33 \\ 4 \end{matrix} \right] &= \frac{1}{2}(2n_3 - c_4); \\ \left[\begin{matrix} 34 \\ 1 \end{matrix} \right] &= \frac{1}{2}(l_3 + g_4 - n_1), & \left[\begin{matrix} 34 \\ 2 \end{matrix} \right] &= \frac{1}{2}(m_3 + f_4 - n_2), & \left[\begin{matrix} 34 \\ 3 \end{matrix} \right] &= \frac{1}{2}c_4, & \left[\begin{matrix} 34 \\ 4 \end{matrix} \right] &= \frac{1}{2}d_3; \\ \left[\begin{matrix} 44 \\ 1 \end{matrix} \right] &= \frac{1}{2}(2l_4 - d_1), & \left[\begin{matrix} 44 \\ 2 \end{matrix} \right] &= \frac{1}{2}(2m_4 - d_2), & \left[\begin{matrix} 44 \\ 3 \end{matrix} \right] &= \frac{1}{2}(2n_4 - d_3), & \left[\begin{matrix} 44 \\ 4 \end{matrix} \right] &= \frac{1}{2}d_4; \end{aligned}$$

the general term being

$$\left[\begin{matrix} pq \\ r \end{matrix} \right] = \frac{1}{2} \left(\frac{\partial g_{pr}}{\partial x_q} + \frac{\partial g_{qr}}{\partial x_p} - \frac{\partial g_{pq}}{\partial x_r} \right),$$

with $g_{11}=a, g_{12}=h, g_{13}=g, g_{23}=f, g_{22}=b, g_{33}=c, g_{14}=l, g_{24}=m, g_{34}=n, g_{44}=d$.

We also require the first minors of the determinant Δ where

$$\begin{aligned} \Delta &= \begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix} \\ &= abcd \\ &\quad - adf^2 - bdy^2 - cdh^2 - bel^2 - cam^2 - abn^2 \\ &\quad + 2afmn + 2bgnl + 2chlm + 2dfgh \\ &\quad + f^2l^2 + g^2m^2 + h^2n^2 - 2fglm - 2ghmn - 2hfnl. \end{aligned}$$

The first minors of Δ are taken in the form

$$\begin{aligned} A_{11} &= bcd + 2fmn - bn^2 - cm^2 - df^2, \\ A_{22} &= cda + 2gnl - cl^2 - an^2 - dg^2, \\ A_{33} &= dab + 2hlm - am^2 - bl^2 - dh^2, \\ A_{44} &= abc + 2fgh - af^2 - bg^2 - ch^2, \\ A_{12} = A_{21} &= clm + dfy + hn^2 - cdh - gmn - fnl, \\ A_{13} = A_{31} &= bnl + dnm + gm^2 - bdg - flm - hmn, \\ A_{23} = A_{32} &= amn + dgl + fl^2 - adf - hnl - glm, \\ A_{14} = A_{41} &= bgn + chm + lf^2 - bcl - hfn - gfm, \\ A_{24} = A_{42} &= chl + afn + mg^2 - cam - fgl - hgn, \\ A_{34} = A_{43} &= afm + bgl + nh^2 - abn - ghm - fhl. \end{aligned}$$

23. Noting therefore (i) the forms of the coefficients in the sub-group of forty equations, (ii) the fact that there is no integral which involves only derivatives of the first order, and (iii) the fact that the number of independent integrals required is the same as the number of Riemann-Christoffel symbols, we are led to surmise that the integrals can be associated with the symbols, member by member. The surmise proves to be justified in fact. I have found twenty-one integrals in the following forms, the symbols $(ij, i'j')$ in the first line being the actual Riemann-Christoffel symbols* :—

$$\begin{aligned} A &= (41, 14) \\ &= \frac{1}{2}\alpha + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 11 \\ s \end{bmatrix} \begin{bmatrix} 44 \\ t \end{bmatrix} - \begin{bmatrix} 14 \\ s \end{bmatrix} \begin{bmatrix} 14 \\ t \end{bmatrix} \right\}, \dagger \\ B &= (42, 24) \\ &= \frac{1}{2}\beta + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 22 \\ s \end{bmatrix} \begin{bmatrix} 44 \\ t \end{bmatrix} - \begin{bmatrix} 24 \\ s \end{bmatrix} \begin{bmatrix} 24 \\ t \end{bmatrix} \right\}, \\ C &= (43, 34) \\ &= \frac{1}{2}\gamma + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 33 \\ s \end{bmatrix} \begin{bmatrix} 44 \\ t \end{bmatrix} - \begin{bmatrix} 34 \\ s \end{bmatrix} \begin{bmatrix} 34 \\ t \end{bmatrix} \right\}, \\ D &= (32, 23) \\ &= \frac{1}{2}\delta + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 22 \\ s \end{bmatrix} \begin{bmatrix} 33 \\ t \end{bmatrix} - \begin{bmatrix} 23 \\ s \end{bmatrix} \begin{bmatrix} 23 \\ t \end{bmatrix} \right\} \end{aligned}$$

* The reason for the apparently arbitrary notation A, . . . , V for the integrals will appear later (§ 26).

† In the double summation for $s, t=1, 2, 3, 4$, all the combinations are to be taken. Thus we must retain

$$A_{12} \left\{ \begin{bmatrix} 11 \\ 1 \end{bmatrix} \begin{bmatrix} 44 \\ 2 \end{bmatrix} - \begin{bmatrix} 14 \\ 1 \end{bmatrix} \begin{bmatrix} 14 \\ 2 \end{bmatrix} \right\}, \quad A_{21} \left\{ \begin{bmatrix} 11 \\ 2 \end{bmatrix} \begin{bmatrix} 44 \\ 1 \end{bmatrix} - \begin{bmatrix} 14 \\ 1 \end{bmatrix} \begin{bmatrix} 14 \\ 2 \end{bmatrix} \right\},$$

while $A_{13}=A_{21}$; and so for other combinations in A and in the succeeding integrals.

$$\begin{aligned}
E &= (31, 13) \\
&= \frac{1}{2}\eta + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 11 \\ s \end{bmatrix} \begin{bmatrix} 33 \\ t \end{bmatrix} - \begin{bmatrix} 13 \\ s \end{bmatrix} \begin{bmatrix} 13 \\ t \end{bmatrix} \right\}, \\
F &= (21, 12) \\
&= \frac{1}{2}\theta + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 11 \\ s \end{bmatrix} \begin{bmatrix} 22 \\ t \end{bmatrix} - \begin{bmatrix} 12 \\ s \end{bmatrix} \begin{bmatrix} 12 \\ t \end{bmatrix} \right\}, \\
G &= (43, 24) \\
&= \frac{1}{2}\xi + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 23 \\ s \end{bmatrix} \begin{bmatrix} 44 \\ t \end{bmatrix} - \begin{bmatrix} 34 \\ s \end{bmatrix} \begin{bmatrix} 24 \\ t \end{bmatrix} \right\}, \\
H &= (43, 14) \\
&= \frac{1}{2}\iota + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 13 \\ s \end{bmatrix} \begin{bmatrix} 44 \\ t \end{bmatrix} - \begin{bmatrix} 34 \\ s \end{bmatrix} \begin{bmatrix} 14 \\ t \end{bmatrix} \right\}, \\
J &= (42, 14) \\
&= \frac{1}{2}\zeta + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 12 \\ s \end{bmatrix} \begin{bmatrix} 44 \\ t \end{bmatrix} - \begin{bmatrix} 24 \\ s \end{bmatrix} \begin{bmatrix} 14 \\ t \end{bmatrix} \right\}, \\
-K &= (13, 21) \\
&= \frac{1}{2}\kappa + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 23 \\ s \end{bmatrix} \begin{bmatrix} 11 \\ t \end{bmatrix} - \begin{bmatrix} 13 \\ s \end{bmatrix} \begin{bmatrix} 12 \\ t \end{bmatrix} \right\}, \\
-L &= (23, 12) \\
&= \frac{1}{2}\lambda + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 13 \\ s \end{bmatrix} \begin{bmatrix} 22 \\ t \end{bmatrix} - \begin{bmatrix} 23 \\ s \end{bmatrix} \begin{bmatrix} 12 \\ t \end{bmatrix} \right\}, \\
-M &= (32, 13) \\
&= \frac{1}{2}\mu + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 12 \\ s \end{bmatrix} \begin{bmatrix} 33 \\ t \end{bmatrix} - \begin{bmatrix} 23 \\ s \end{bmatrix} \begin{bmatrix} 13 \\ t \end{bmatrix} \right\}, \\
-P &= (14, 31) \\
&= \frac{1}{2}\omega + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 34 \\ s \end{bmatrix} \begin{bmatrix} 11 \\ t \end{bmatrix} - \begin{bmatrix} 14 \\ s \end{bmatrix} \begin{bmatrix} 13 \\ t \end{bmatrix} \right\}, \\
Q &= (14, 21) \\
&= \frac{1}{2}\chi + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 24 \\ s \end{bmatrix} \begin{bmatrix} 11 \\ t \end{bmatrix} - \begin{bmatrix} 14 \\ s \end{bmatrix} \begin{bmatrix} 12 \\ t \end{bmatrix} \right\}, \\
-R &= (24, 12) \\
&= \frac{1}{2}\rho + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 14 \\ s \end{bmatrix} \begin{bmatrix} 22 \\ t \end{bmatrix} - \begin{bmatrix} 24 \\ s \end{bmatrix} \begin{bmatrix} 12 \\ t \end{bmatrix} \right\}, \\
S &= (24, 32) \\
&= \frac{1}{2}\sigma + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 34 \\ s \end{bmatrix} \begin{bmatrix} 22 \\ t \end{bmatrix} - \begin{bmatrix} 24 \\ s \end{bmatrix} \begin{bmatrix} 23 \\ t \end{bmatrix} \right\}, \\
-T &= (34, 23) \\
&= \frac{1}{2}\tau + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 24 \\ s \end{bmatrix} \begin{bmatrix} 33 \\ t \end{bmatrix} - \begin{bmatrix} 34 \\ s \end{bmatrix} \begin{bmatrix} 23 \\ t \end{bmatrix} \right\},
\end{aligned}$$

$$\begin{aligned}
 U &= (34, 13) \\
 &= \frac{1}{2}\nu + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 14 \\ s \end{bmatrix} \begin{bmatrix} 33 \\ t \end{bmatrix} - \begin{bmatrix} 34 \\ s \end{bmatrix} \begin{bmatrix} 13 \\ t \end{bmatrix} \right\}, \\
 -N &= (14, 23) \\
 &= \frac{1}{2}\nu + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 24 \\ s \end{bmatrix} \begin{bmatrix} 13 \\ t \end{bmatrix} - \begin{bmatrix} 34 \\ s \end{bmatrix} \begin{bmatrix} 12 \\ t \end{bmatrix} \right\}, \\
 -O &= (13, 42) \\
 &= \frac{1}{2}\omega + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 34 \\ s \end{bmatrix} \begin{bmatrix} 12 \\ t \end{bmatrix} - \begin{bmatrix} 23 \\ s \end{bmatrix} \begin{bmatrix} 14 \\ t \end{bmatrix} \right\}, \\
 -V &= (12, 34) \\
 &= \frac{1}{2}\psi + \frac{1}{\Delta} \sum_s \sum_t A_{st} \left\{ \begin{bmatrix} 23 \\ s \end{bmatrix} \begin{bmatrix} 14 \\ t \end{bmatrix} - \begin{bmatrix} 24 \\ s \end{bmatrix} \begin{bmatrix} 13 \\ t \end{bmatrix} \right\},
 \end{aligned}$$

with the single relation

$$N + O + V = 0.$$

These constitute an algebraically complete set of independent integrals, which simultaneously satisfy all the eighty equations of the first sub-group and all the forty equations of the second sub-group. (The quantity Δ is itself an invariant.)

THE THIRD SUB-GROUP: SIXTEEN EQUATIONS.

24. There remains the sub-group of sixteen equations which arise by equating the coefficients of the first derivatives of ξ , η , ζ , θ on the two sides of the relation

$$\phi' = \{1 - p\epsilon(\xi_1 + \eta_2 + \zeta_3 + \theta_4)\} \phi.$$

The coefficients of $\xi_2, \xi_3, \xi_4; \eta_1, \eta_3, \eta_4; \zeta_1, \zeta_2, \zeta_4; \theta_1, \theta_2, \theta_3$ must all vanish; three equations will be taken in the form

$$\text{co. } \xi_1 = \text{co. } \eta_2 = \text{co. } \zeta_3 = \text{co. } \theta_4,$$

thus making a set of fifteen equations; and a remaining equation

$$\text{co. } \xi_1 = \epsilon p \phi$$

would serve to determine p .

The equations in the set are linear and homogeneous, of the first order; and they are a complete Jacobian set. The quantities that occur in them are

- (i) the fourteen variables $X_1, \dots, X_4; U_1, \dots, U_4; P_1, \dots, P_6;$
- (ii) the ten original coefficients $a, b, \dots, n;$

and, as the first and second derivatives can only occur in the combinations which so far have been called the Riemann-Christoffel symbols,

- (iii) the twenty-one quantities $A, B, \dots, O, V,$ with a single relation between them.

Thus there are forty-five quantities in all. The complete system of fifteen equations will therefore possess thirty independent integrals, which can be taken in a variety of ways; and, among these, we shall have

$$U_1X_1 + U_2X_2 + U_3X_3 + U_4X_4$$

as a mixed concomitant,

$$P_1P_6 + P_2P_5 + P_3P_4$$

as a (vanishing) line-covariant, and

$$N + O + V$$

as a (vanishing) differential invariant.

25. The initial forms of the coefficients of ξ_2, \dots, θ_3 involve derivatives of a, \dots, n of the first and second orders; and the terms must be transformed so as to introduce the necessary combinations. When the rather laborious (but otherwise simple) transformations have been effected, we find the following as the aggregate of the equations, viz. the first twelve, arising out of the coefficients of $\xi_2, \xi_3, \xi_4; \eta_1, \eta_3, \eta_4; \zeta_1, \zeta_2, \zeta_4; \theta_1, \theta_2, \theta_3$, in the form

$$W_1 = X_2 \frac{\partial}{\partial X_1} + P_5 \frac{\partial}{\partial P_6} - P_1 \frac{\partial}{\partial P_2} - U_1 \frac{\partial}{\partial U_2} - \Xi_2 = 0,$$

$$W_2 = X_3 \frac{\partial}{\partial X_1} + P_4 \frac{\partial}{\partial P_6} - P_1 \frac{\partial}{\partial P_3} - U_1 \frac{\partial}{\partial U_3} - \Xi_3 = 0,$$

$$W_3 = X_4 \frac{\partial}{\partial X_1} + P_4 \frac{\partial}{\partial P_2} - P_5 \frac{\partial}{\partial P_3} - U_1 \frac{\partial}{\partial U_4} - \Xi_4 = 0,$$

$$W_4 = X_1 \frac{\partial}{\partial X_2} + P_6 \frac{\partial}{\partial P_5} - P_2 \frac{\partial}{\partial P_1} - U_2 \frac{\partial}{\partial U_1} - H_1 = 0,$$

$$W_5 = X_3 \frac{\partial}{\partial X_2} + P_4 \frac{\partial}{\partial P_5} - P_2 \frac{\partial}{\partial P_3} - U_2 \frac{\partial}{\partial U_3} - H_3 = 0,$$

$$W_6 = X_4 \frac{\partial}{\partial X_2} + P_6 \frac{\partial}{\partial P_3} - P_4 \frac{\partial}{\partial P_1} - U_2 \frac{\partial}{\partial U_4} - H_4 = 0,$$

$$W_7 = X_1 \frac{\partial}{\partial X_3} + P_6 \frac{\partial}{\partial P_4} - P_3 \frac{\partial}{\partial P_1} - U_3 \frac{\partial}{\partial U_1} - Z_1 = 0,$$

$$W_8 = X_2 \frac{\partial}{\partial X_3} + P_5 \frac{\partial}{\partial P_4} - P_3 \frac{\partial}{\partial P_2} - U_3 \frac{\partial}{\partial U_2} - Z_2 = 0,$$

$$W_9 = X_4 \frac{\partial}{\partial X_3} + P_5 \frac{\partial}{\partial P_1} - P_6 \frac{\partial}{\partial P_2} - U_3 \frac{\partial}{\partial U_4} - Z_4 = 0,$$

$$W_{10} = X_1 \frac{\partial}{\partial X_4} + P_2 \frac{\partial}{\partial P_4} - P_3 \frac{\partial}{\partial P_5} - U_4 \frac{\partial}{\partial U_1} - \Theta_1 = 0,$$

$$W_{11} = X_2 \frac{\partial}{\partial X_4} + P_3 \frac{\partial}{\partial P_6} - P_1 \frac{\partial}{\partial P_4} - U_4 \frac{\partial}{\partial U_2} - \Theta_2 = 0,$$

$$W_{12} = X_3 \frac{\partial}{\partial X_4} + P_1 \frac{\partial}{\partial P_5} - P_2 \frac{\partial}{\partial P_6} - U_4 \frac{\partial}{\partial U_3} - \Theta_3 = 0,$$

where the literal operators Ξ_2, \dots, Θ_3 are

$$\begin{aligned} \Xi_2 = & A \frac{\partial}{\partial J} + 2J \frac{\partial}{\partial B} - E \frac{\partial}{\partial M} - 2M \frac{\partial}{\partial D} + Q \frac{\partial}{\partial R} - K \frac{\partial}{\partial L} + H \frac{\partial}{\partial G} - U \frac{\partial}{\partial T} \\ & + (N - O) \frac{\partial}{\partial S} + P \left(\frac{\partial}{\partial O} - \frac{\partial}{\partial N} \right) + a \frac{\partial}{\partial h} + 2h \frac{\partial}{\partial b} + g \frac{\partial}{\partial f} + l \frac{\partial}{\partial m}, \end{aligned}$$

$$\begin{aligned} \Xi_3 = & A \frac{\partial}{\partial H} + 2H \frac{\partial}{\partial C} - F \frac{\partial}{\partial L} - 2L \frac{\partial}{\partial D} + P \frac{\partial}{\partial U} - K \frac{\partial}{\partial M} + J \frac{\partial}{\partial G} - R \frac{\partial}{\partial S} \\ & + (N - V) \frac{\partial}{\partial T} + Q \left(\frac{\partial}{\partial V} - \frac{\partial}{\partial N} \right) + a \frac{\partial}{\partial g} + 2g \frac{\partial}{\partial c} + h \frac{\partial}{\partial f} + l \frac{\partial}{\partial n}, \end{aligned}$$

$$\begin{aligned} \Xi_4 = & E \frac{\partial}{\partial U} + 2U \frac{\partial}{\partial C} - F \frac{\partial}{\partial R} - 2R \frac{\partial}{\partial B} + M \frac{\partial}{\partial T} - Q \frac{\partial}{\partial J} + P \frac{\partial}{\partial H} - L \frac{\partial}{\partial S} \\ & + (O - V) \frac{\partial}{\partial G} + K \left(\frac{\partial}{\partial V} - \frac{\partial}{\partial O} \right) + a \frac{\partial}{\partial l} + 2l \frac{\partial}{\partial d} + h \frac{\partial}{\partial m} + g \frac{\partial}{\partial n}; \end{aligned}$$

$$\begin{aligned} H_1 = & B \frac{\partial}{\partial J} + 2J \frac{\partial}{\partial A} - D \frac{\partial}{\partial M} - 2M \frac{\partial}{\partial E} + R \frac{\partial}{\partial Q} - L \frac{\partial}{\partial K} + G \frac{\partial}{\partial H} - T \frac{\partial}{\partial U} \\ & + (O - N) \frac{\partial}{\partial P} + S \left(\frac{\partial}{\partial N} - \frac{\partial}{\partial O} \right) + b \frac{\partial}{\partial h} + 2h \frac{\partial}{\partial a} + f \frac{\partial}{\partial g} + m \frac{\partial}{\partial l}, \end{aligned}$$

$$\begin{aligned} H_3 = & B \frac{\partial}{\partial G} + 2G \frac{\partial}{\partial C} - F \frac{\partial}{\partial K} - 2K \frac{\partial}{\partial E} + S \frac{\partial}{\partial T} - L \frac{\partial}{\partial M} + J \frac{\partial}{\partial H} - Q \frac{\partial}{\partial P} \\ & + (O - V) \frac{\partial}{\partial U} + R \left(\frac{\partial}{\partial V} - \frac{\partial}{\partial O} \right) + b \frac{\partial}{\partial f} + 2f \frac{\partial}{\partial c} + h \frac{\partial}{\partial g} + m \frac{\partial}{\partial n}, \end{aligned}$$

$$\begin{aligned} H_4 = & F \frac{\partial}{\partial Q} + 2Q \frac{\partial}{\partial A} - D \frac{\partial}{\partial T} - 2T \frac{\partial}{\partial C} + K \frac{\partial}{\partial P} - S \frac{\partial}{\partial G} + R \frac{\partial}{\partial J} - M \frac{\partial}{\partial U} \\ & + (V - N) \frac{\partial}{\partial H} + L \left(\frac{\partial}{\partial N} - \frac{\partial}{\partial V} \right) + b \frac{\partial}{\partial m} + 2m \frac{\partial}{\partial d} + h \frac{\partial}{\partial l} + f \frac{\partial}{\partial n}; \end{aligned}$$

$$\begin{aligned} Z_1 = & C \frac{\partial}{\partial H} + 2H \frac{\partial}{\partial A} - D \frac{\partial}{\partial L} - 2L \frac{\partial}{\partial F} + U \frac{\partial}{\partial P} - M \frac{\partial}{\partial K} + G \frac{\partial}{\partial J} - S \frac{\partial}{\partial R} \\ & + (V - N) \frac{\partial}{\partial Q} + T \left(\frac{\partial}{\partial N} - \frac{\partial}{\partial V} \right) + c \frac{\partial}{\partial g} + 2g \frac{\partial}{\partial a} + f \frac{\partial}{\partial h} + n \frac{\partial}{\partial l}, \end{aligned}$$

$$\begin{aligned} Z_2 = & C \frac{\partial}{\partial G} + 2G \frac{\partial}{\partial B} - E \frac{\partial}{\partial K} - 2K \frac{\partial}{\partial F} + T \frac{\partial}{\partial S} - M \frac{\partial}{\partial L} + H \frac{\partial}{\partial J} - P \frac{\partial}{\partial Q} \\ & + (V - O) \frac{\partial}{\partial R} + U \left(\frac{\partial}{\partial O} - \frac{\partial}{\partial V} \right) + c \frac{\partial}{\partial f} + 2f \frac{\partial}{\partial b} + g \frac{\partial}{\partial h} + n \frac{\partial}{\partial m}, \end{aligned}$$

$$\begin{aligned} Z_4 = & D \frac{\partial}{\partial S} + 2S \frac{\partial}{\partial B} - E \frac{\partial}{\partial P} - 2P \frac{\partial}{\partial A} + L \frac{\partial}{\partial R} - U \frac{\partial}{\partial H} + T \frac{\partial}{\partial G} - K \frac{\partial}{\partial Q} \\ & + (N - O) \frac{\partial}{\partial J} + M \left(\frac{\partial}{\partial O} - \frac{\partial}{\partial N} \right) + c \frac{\partial}{\partial n} + 2n \frac{\partial}{\partial d} + g \frac{\partial}{\partial l} + f \frac{\partial}{\partial m}; \end{aligned}$$

$$\begin{aligned} \Theta_1 &= C \frac{\partial}{\partial U} + 2U \frac{\partial}{\partial E} - B \frac{\partial}{\partial R} - 2R \frac{\partial}{\partial F} + H \frac{\partial}{\partial P} - J \frac{\partial}{\partial Q} + T \frac{\partial}{\partial M} - S \frac{\partial}{\partial L} \\ &\quad + (V - O) \frac{\partial}{\partial K} + G \left(\frac{\partial}{\partial O} - \frac{\partial}{\partial V} \right) + d \frac{\partial}{\partial l} + 2l \frac{\partial}{\partial a} + m \frac{\partial}{\partial h} + n \frac{\partial}{\partial g}, \\ \Theta_2 &= A \frac{\partial}{\partial Q} + 2Q \frac{\partial}{\partial F} - C \frac{\partial}{\partial T} - 2T \frac{\partial}{\partial D} + J \frac{\partial}{\partial R} - G \frac{\partial}{\partial S} + P \frac{\partial}{\partial K} - U \frac{\partial}{\partial M} \\ &\quad + (N - V) \frac{\partial}{\partial L} + H \left(\frac{\partial}{\partial V} - \frac{\partial}{\partial N} \right) + d \frac{\partial}{\partial m} + 2m \frac{\partial}{\partial b} + l \frac{\partial}{\partial n} + n \frac{\partial}{\partial f}, \\ \Theta_3 &= B \frac{\partial}{\partial S} + 2S \frac{\partial}{\partial D} - A \frac{\partial}{\partial P} - 2P \frac{\partial}{\partial E} - G \frac{\partial}{\partial T} - H \frac{\partial}{\partial U} + R \frac{\partial}{\partial L} - Q \frac{\partial}{\partial K} \\ &\quad + (O - N) \frac{\partial}{\partial M} + J \left(\frac{\partial}{\partial N} - \frac{\partial}{\partial O} \right) + d \frac{\partial}{\partial n} + 2n \frac{\partial}{\partial c} + l \frac{\partial}{\partial g} + m \frac{\partial}{\partial f}. \end{aligned}$$

In addition to these twelve equations, $W_1=0, \dots, W_{12}=0$, there are the remaining three equations which arise by equating to one another the four coefficients of $\xi_1, \eta_2, \zeta_3, \theta_4$. And it is in virtue of these three equations that we satisfy the non-evanescent Jacobi conditions, viz.:

$$\begin{aligned} (W_1, W_4) &= 0, & (W_2, W_7) &= 0, & (W_3, W_{10}) &= 0, \\ (W_5, W_8) &= 0, & (W_6, W_{11}) &= 0, & (W_9, W_{12}) &= 0, \end{aligned}$$

in order that the system $W_1=0, \dots, W_{12}=0$ may be a complete Jacobian system. We thus have the fifteen equations which constitute the system; and we have to find a full set of algebraically independent integrals, thirty in number.

26. Now, an inspection of these fifteen equations shows* that the fifteen equations are the equations characteristic of the complete system of concomitants, under linear point-transformations, of the two quantics

$$aX_1^2 + bX_2^2 + cX_3^2 + dX_4^2 + 2fX_2X_3 + 2gX_3X_1 + 2hX_1X_2 + 2lX_1X_4 + 2mX_2X_4 + 2nX_3X_4$$

(which is a point-quantic), and

$$\begin{aligned} DP_1^2 + 2MP_1P_2 + 2LP_1P_3 + 2TP_1P_4 + 2SP_1P_5 + 2NP_1P_6 \\ + EP_2^2 + 2KP_2P_3 + 2UP_2P_4 + 2OP_2P_5 + 2PP_2P_6 \\ + FP_3^2 + 2VP_3P_4 + 2RP_3P_5 + 2QP_3P_6 \\ + CP_4^2 + 2GP_4P_5 + 2HP_4P_6 \\ + BP_5^2 + 2JP_5P_6 \\ + AP_6^2 \end{aligned}$$

* See a memoir of my own, "Systems of Quaternariants that are algebraically complete," *Camb. Phil. Trans.*, vol. xiv (1889), pp. 409-466, specially pp. 430 *et seq.*, 460 *et seq.* The subsequent analysis is modified from the analysis in that memoir.

(which is a line-quantic), being a quadratic line-complex,* in the point-variables X, the line-variables P, and the plane-variables U, together with the variables X'', P'', U'' of § 11.

27. The construction of a complete system of independent concomitants for the two quantics existing simultaneously is facilitated by a knowledge of the complete system of concomitants for each of the quantics taken separately.

28. The complete system for the quantic

$$\Sigma = (a, b, c, d, f, g, h, l, m, n \text{ } \S \text{ } X_1, X_2, X_3, X_4)^2$$

is easily deduced from known results. There are the fifteen partial differential equations to be satisfied; and the number of quantities they involve is twenty-four, viz. fourteen variables and ten constants. Thus the system must contain nine independent integrals.

We write

$$\begin{aligned} \partial_x &= X_1'' \frac{\partial}{\partial X_1} + X_2'' \frac{\partial}{\partial X_2} + X_3'' \frac{\partial}{\partial X_3} + X_4'' \frac{\partial}{\partial X_4}, \\ \partial_u &= U_1'' \frac{\partial}{\partial U_1} + U_2'' \frac{\partial}{\partial U_2} + U_3'' \frac{\partial}{\partial U_3} + U_4'' \frac{\partial}{\partial U_4}, \\ \partial_p &= P_1'' \frac{\partial}{\partial P_1} + P_2'' \frac{\partial}{\partial P_2} + P_3'' \frac{\partial}{\partial P_3} + P_4'' \frac{\partial}{\partial P_4} + P_5'' \frac{\partial}{\partial P_5} + P_6'' \frac{\partial}{\partial P_6}. \end{aligned}$$

Then in addition to Σ we have

$$\Sigma_1 = \frac{1}{2} \partial_x \Sigma, \quad \Sigma_2 = \frac{1}{2} \partial_x^2 \Sigma,$$

* The quadratic line-complex was first considered by Plücker, "New Geometry of Space," *Phil. Trans.* (1865), pp. 725-791, and subsequently in his *Neue Geometrie des Raumes* (1868). I have preserved his notation so far as regards the coefficients of the complex, because it has been used by other writers, and variations of notation tend to be confusing; but a notation which runs

$$\begin{aligned} q_{11} &= D, & q_{12} &= M, & q_{13} &= L, & q_{14} &= T, & q_{15} &= S, & q_{16} &= N, \\ & & q_{22} &= E, & q_{23} &= K, & q_{24} &= U, & q_{25} &= O, & q_{26} &= P, \\ q_{33} &= F, & q_{34} &= V, & q_{35} &= R, & q_{36} &= Q, & q_{44} &= C, & q_{45} &= G, \\ & & & & q_{46} &= H, & q_{55} &= B, & q_{56} &= J, & q_{66} &= A, \end{aligned}$$

would codify the expression of the operators. Thus the operator H_1 becomes

$$\begin{aligned} & q_{15} \frac{\partial}{\partial q_{16}} + q_{25} \frac{\partial}{\partial q_{26}} + q_{35} \frac{\partial}{\partial q_{36}} + q_{45} \frac{\partial}{\partial q_{46}} + q_{55} \frac{\partial}{\partial q_{56}} + 2q_{56} \frac{\partial}{\partial q_{66}} \\ & - \left(q_{11} \frac{\partial}{\partial q_{12}} + 2q_{12} \frac{\partial}{\partial q_{22}} + q_{13} \frac{\partial}{\partial q_{23}} + q_{14} \frac{\partial}{\partial q_{24}} + q_{15} \frac{\partial}{\partial q_{25}} + q_{16} \frac{\partial}{\partial q_{26}} \right); \end{aligned}$$

and so for the others: the relation $N + O + V = 0$ becomes

$$q_{16} + q_{25} + q_{34} = 0,$$

and similarly for other invariants: the notation immediately suggests (or is suggested by) the umbral notation used in my memoir which has just been quoted. As the umbral notation is not used here, I have adhered to the Plücker coefficients.

from taking the function

$$\Sigma(X_1 + \lambda X_1'', X_2 + \lambda X_2'', X_3 + \lambda X_3'', X_4 + \lambda X_4'')$$

for any arbitrary quantity λ and picking out the coefficients of $\lambda^0, \lambda^1, \lambda^2$. Also $\Sigma=0$ is the equation of a quadric surface on which the point X_1, \dots, X_4 lies; $\Sigma_1=0$ is the condition that the points X_1, \dots, X_4 and X_1'', \dots, X_4'' are conjugate with respect to the surface $\Sigma=0$; and $\Sigma_2=0$ is the equation of a quadric surface on which the point X_1'', \dots, X_4'' lies.

We are accustomed to the tangential equation of a quadric; so we have a plane-covariant

$$\Pi = (A_{11}, A_{22}, A_{33}, A_{44}, A_{23}, A_{31}, A_{12}, A_{11}, A_{24}, A_{31}) \text{ } \delta U_1, U_2, U_3, U_4)^2$$

where the coefficients A_{st} have the significance assigned in § 22. And then we have

$$\Pi_1 = \frac{1}{2} \partial_u \Pi, \quad \Pi_2 = \frac{1}{2} \partial_{u^2} \Pi,$$

obtained by selecting the coefficients of μ^1 and μ^2 in the function

$$\Pi(U_1 + \mu U_1'', U_2 + \mu U_2'', U_3 + \mu U_3'', U_4 + \mu U_4'').$$

Also $\Pi=0$ is the condition that the plane U_1, \dots, U_4 touches the quadric; $\Pi_1=0$ is the condition that the planes U_1, \dots, U_4 and U_1'', \dots, U_4'' are conjugate with respect to the quadric; and $\Pi_2=0$ is the condition that the plane U_1'', \dots, U_4'' touches the quadric.

Next, there is a line-covariant Λ . We write

$\mathbf{a} = ad - l^2,$	$\mathbf{g} = df - mn,$	$\mathbf{p} = gl - an,$
$\mathbf{b} = bd - m^2,$	$\mathbf{h} = dg - nl,$	$\mathbf{q} = am - lh,$
$\mathbf{c} = cd - n^2,$	$\mathbf{j} = dh - lm,$	$\mathbf{r} = hm - bl,$
$\mathbf{d} = bc - f^2,$	$\mathbf{k} = gh - af,$	$\mathbf{s} = bn - fm,$
$\mathbf{e} = ca - g^2,$	$\mathbf{l} = hf - bg,$	$\mathbf{t} = fn - cm,$
$\mathbf{f} = ab - h^2,$	$\mathbf{m} = fg - ch,$	$\mathbf{u} = cl - gn,$
$\mathbf{n} = hn - gm,$	$\mathbf{o} = fl - ln,$	$\mathbf{v} = gm - fl,$

with the relation

$$\mathbf{n} + \mathbf{o} + \mathbf{v} = 0,$$

and we have

$$\begin{aligned} \Lambda = & \mathbf{d}P_1^2 + 2\mathbf{m}P_1P_2 + 2\mathbf{l}P_1P_3 + 2\mathbf{t}P_1P_4 + 2\mathbf{s}P_1P_5 + 2\mathbf{n}P_1P_6 \\ & + \mathbf{e}P_2^2 + 2\mathbf{k}P_2P_3 + 2\mathbf{u}P_2P_4 + 2\mathbf{o}P_2P_5 + 2\mathbf{p}P_2P_6 \\ & + \mathbf{f}P_3^2 + 2\mathbf{v}P_3P_4 + 2\mathbf{r}P_3P_5 + 2\mathbf{q}P_3P_6 \\ & + \mathbf{c}P_4^2 + 2\mathbf{g}P_4P_5 + 2\mathbf{h}P_4P_6 \\ & + \mathbf{b}P_5^2 + 2\mathbf{j}P_5P_6 \\ & + \mathbf{a}P_6^2. \end{aligned}$$

And, with Λ , we have

$$\Lambda_1 = \frac{1}{2} \partial_p \Lambda, \quad \Lambda_2 = \frac{1}{2} \partial_{p^2} \Lambda.$$

In addition to these, we have the integrals

$$\begin{aligned} &\Delta, \text{ the discriminant of } \Sigma; \\ &\sum_{r=1}^4 U_r X_r, \text{ a universal concomitant}; \\ &P_1 P_6 + P_2 P_5 + P_3 P_4, \text{ a universal concomitant, which in relations} \\ &\quad \text{is always to be equal to zero;} \end{aligned}$$

so that now we have, as integrals,

$$\Sigma, \Sigma_1, \Sigma_2; \quad \Pi, \Pi_1, \Pi_2; \quad \Lambda, \Lambda_1, \Lambda_2; \quad \Delta; \quad \sum_{r=1}^4 U_r X_r; \quad \sum_{s=1}^3 P_s P_{6-s};$$

twelve in number, while there are only nine functionally independent integrals. There is no difficulty in verifying the three relations

$$\begin{aligned} \Sigma \Sigma_2 - \Sigma_1^2 &= \Lambda_2, \\ \Pi \Pi_2 - \Pi_1^2 &= \Delta \{ \Lambda_2 + 2\Lambda_1 \left(\sum_{r=1}^4 U_r X_r \right) + 2\Lambda \}, \\ \Lambda \Lambda_2 - \Lambda_1^2 &= \Sigma_2 \Pi_2, \end{aligned}$$

which thus reduce the number of independent integrals to nine. The complete system will be regarded as composed of

$$\Sigma, \Sigma_1; \quad \Pi, \Pi_1; \quad \Lambda, \Lambda_1; \quad \Delta; \quad \sum_{r=1}^4 U_r X_r; \quad \sum_{s=1}^3 P_s P_{6-s}.$$

Every concomitant of the quantic can be expressed algebraically in terms of these. Thus there is a concomitant which has $bg^2 - 2fgh + ch^2$ for its leading coefficient,* and is of the second degree in x and in p ; it is easily proved to be

$$\Sigma \Lambda - \Pi_2.$$

The only invariant of the form is Δ . The only point-covariant of the form is the form itself, Σ . The only line-covariant of the form is Λ . The only plane-covariant of the form is Π . The three covariants $\Sigma_1, \Pi_1, \Lambda_1$ are usually called mixed concomitants.

29. We now proceed to obtain a partially complete set of concomitants of the quadratic line complex alone, viz. the line-concomitants, which involve the line-variables only, and its pure invariants. We still have fifteen equations of the type

$$W_1 = P_5 \frac{\partial}{\partial P_6} - P_1 \frac{\partial}{\partial P_2} - \Xi_2 = 0;$$

for the present purpose, these equations involve six variables P and twenty-one constants (with one relation)—that is, twenty-seven quantities in all; and therefore a complete system of concomitants of the specified

* See the memoir quoted in § 26 (footnote), at p. 431.

type must contain twelve members (including the one relation). We shall denote the quadratic line complex by $u(P)$, so that

$$u(P) = DP_1^2 + 2MP_1P_2 + \dots + 2JP_5P_6 + AP_6^2;$$

it is, of course, one of the concomitants of the system.

When the point-variables are subject to a general linear transformation

$$(X_1, X_2, X_3, X_4) = \begin{pmatrix} \lambda_1, & \mu_1, & \nu_1, & \rho_1 \\ \lambda_2, & \mu_2, & \nu_2, & \rho_2 \\ \lambda_3, & \mu_3, & \nu_3, & \rho_3 \\ \lambda_4, & \mu_4, & \nu_4, & \rho_4 \end{pmatrix} \begin{matrix} \text{X}'_1, \text{X}'_2, \text{X}'_3, \text{X}'_4, \\ \end{matrix}$$

where the determinant θ of the coefficients λ, μ, ν, ρ is different from zero, the line-variables become subject to the transformation

$$P_1, \dots, P_6 = \begin{pmatrix} (\mu_2\nu_3), (v_2\lambda_3), (\lambda_2\mu_3), (v_2\rho_3), (\mu_2\rho_3), (\lambda_2\rho_3) \\ (\mu_3\nu_1), (v_3\lambda_1), (\lambda_3\mu_1), (v_3\rho_1), (\mu_3\rho_1), (\lambda_3\rho_1) \\ (\mu_1\nu_2), (v_1\lambda_2), (\lambda_1\mu_2), (v_1\rho_2), (\mu_1\rho_2), (\lambda_1\rho_2) \\ (\mu_3\nu_4), (v_3\lambda_4), (\lambda_3\mu_4), (v_3\rho_4), (\mu_3\rho_4), (\lambda_3\rho_4) \\ (\mu_2\nu_4), (v_2\lambda_4), (\lambda_2\mu_4), (v_2\rho_4), (\mu_2\rho_4), (\lambda_2\rho_4) \\ (\mu_1\nu_4), (v_1\lambda_4), (\lambda_1\mu_4), (v_1\rho_4), (\mu_1\rho_4), (\lambda_1\rho_4) \end{pmatrix} \begin{matrix} \text{P}'_1, \dots, \text{P}'_6, \\ \end{matrix}$$

where $(\mu_2\nu_3)$ denotes $\mu_2\nu_3 - \mu_3\nu_2$, and similarly for the other coefficients; and then differential operators with respect to the quantities P are subject to the transformations

$$\frac{\partial}{\partial P_1}, \dots, \frac{\partial}{\partial P_6} = \frac{1}{\theta} \begin{pmatrix} (\lambda_1\rho_4), (\mu_1\rho_4), (v_1\rho_4), (\lambda_1\mu_4), (v_1\lambda_4), (\mu_1\nu_4) \\ (\lambda_2\rho_4), (\mu_2\rho_4), (v_2\rho_4), (\lambda_2\mu_4), (v_2\lambda_4), (\mu_2\nu_4) \\ (\lambda_3\rho_4), (\mu_3\rho_4), (v_3\rho_4), (\lambda_3\mu_4), (v_3\lambda_4), (\mu_3\nu_4) \\ (\lambda_1\rho_2), (\mu_1\rho_2), (v_1\rho_2), (\lambda_1\mu_2), (v_1\lambda_2), (\mu_1\nu_2) \\ (\lambda_3\rho_1), (\mu_3\rho_1), (v_3\rho_1), (\lambda_3\mu_1), (v_3\lambda_1), (\mu_3\nu_1) \\ (\lambda_2\rho_3), (\mu_2\rho_3), (v_2\rho_3), (\lambda_2\mu_3), (v_2\lambda_3), (\mu_2\nu_3) \end{pmatrix} \begin{matrix} \frac{\partial}{\partial P'_1}, \dots, \frac{\partial}{\partial P'_6}. \\ \end{matrix}$$

It thus appears that

$$\left. \frac{\partial}{\partial P_1} \right\}_{P_6}, \left. \frac{\partial}{\partial P_2} \right\}_{P_5}, \left. \frac{\partial}{\partial P_3} \right\}_{P_4}, \left. \frac{\partial}{\partial P_4} \right\}_{P_3}, \left. \frac{\partial}{\partial P_5} \right\}_{P_2}, \left. \frac{\partial}{\partial P_6} \right\}_{P_1},$$

in bracketed pairs, are subject to the same transformations.

Two useful inferences, among others, can at once be made. The first is that the differential operator

$$\frac{\partial^2}{\partial P_1 \partial P_6} + \frac{\partial^2}{\partial P_2 \partial P_5} + \frac{\partial^2}{\partial P_3 \partial P_4}$$

is covariantive. The second is that, if W and W' be two homogeneous line-covariants, the function

$$\frac{\partial W}{\partial P_1} \frac{\partial W'}{\partial P_6} + \frac{\partial W}{\partial P_2} \frac{\partial W'}{\partial P_5} + \frac{\partial W}{\partial P_3} \frac{\partial W'}{\partial P_4} + \frac{\partial W}{\partial P_4} \frac{\partial W'}{\partial P_3} + \frac{\partial W}{\partial P_5} \frac{\partial W'}{\partial P_2} + \frac{\partial W}{\partial P_6} \frac{\partial W'}{\partial P_1}$$

is also covariantive; and it remains covariantive when W and W' are the same function.

30. On the basis of these two results, an algebraically complete system of concomitants of the required type can be constructed.

Denoting $u(P)$ by u , we have a line-covariant v , where

$$v = \frac{1}{4} \left(\frac{\partial u}{\partial P_1} \frac{\partial u}{\partial P_6} + \frac{\partial u}{\partial P_2} \frac{\partial u}{\partial P_5} + \frac{\partial u}{\partial P_3} \frac{\partial u}{\partial P_4} \right) \\ = (D', M', \dots, A' \text{ \textcircled{ } } P_1, \dots, P_6)^2,$$

where

$$\begin{aligned} D' &= ND + SM + TL, \\ E' &= OE + UK + PM, \\ F' &= VF + QL + RK, \\ C' &= VC + TH + UG, \\ B' &= OB + RG + SJ, \\ A' &= NA + PJ + QH, \\ 2M' &= NM + DP + SE + MO + TK + LU, \\ 2L' &= NL + DQ + SK + MR + TF + LV, \\ 2T' &= NT + DH + SU + MG + TV + LC, \\ 2S' &= NS + DJ + SO + MB + TR + LG, \\ 2K' &= OK + ER + UF + KV + MQ + PL, \\ 2U' &= OU + EG + UV + KC + MH + PT, \\ 2P' &= OP + EJ + UQ + KH + MA + PN, \\ 2R' &= OR + VR + SQ + KB + GF + LJ, \\ 2Q' &= VQ + NQ + RP + KJ + HF + LA, \\ 2G' &= VG + OG + TJ + HS + UB + RC, \\ 2H' &= VH + NH + UJ + GP + QC + TA, \\ 2J' &= OJ + NJ + QG + RH + SA + PB, \\ 2N' &= N^2 + AD + SP + MJ + TQ + LH, \\ 2O' &= O^2 + BE + KG + UR + SP + MJ, \\ 2V' &= V^2 + CF + TQ + LH + KG + UR. \end{aligned}$$

Next, we have a line-covariant w , by forming

$$w = \frac{1}{4} \left(\frac{\partial u}{\partial P_1} \frac{\partial v}{\partial P_6} + \frac{\partial u}{\partial P_2} \frac{\partial v}{\partial P_5} + \frac{\partial u}{\partial P_3} \frac{\partial v}{\partial P_4} + \frac{\partial u}{\partial P_4} \frac{\partial v}{\partial P_3} + \frac{\partial u}{\partial P_5} \frac{\partial v}{\partial P_2} + \frac{\partial u}{\partial P_6} \frac{\partial v}{\partial P_1} \right) \\ = (D'', M'', \dots, A'' \text{ \textcircled{ } } P_1, \dots, P_6)^2,$$

where, if we write

$$\partial = D' \frac{\partial}{\partial D} + M' \frac{\partial}{\partial M} + \dots + A' \frac{\partial}{\partial A},$$

we have

$$D'' = \partial D', \quad M'' = \partial M', \quad \dots$$

for the whole sequence of coefficients.

Next, we have a line-covariant x , obtained by forming either

$$x = \frac{\partial v}{\partial P_1} \frac{\partial v}{\partial P_6} + \frac{\partial v}{\partial P_2} \frac{\partial v}{\partial P_5} + \frac{\partial v}{\partial P_3} \frac{\partial v}{\partial P_4}$$

or (what is the equivalent except as to an arithmetical factor)

$$x = \frac{\partial u}{\partial P_1} \frac{\partial w}{\partial P_6} + \frac{\partial u}{\partial P_2} \frac{\partial w}{\partial P_5} + \frac{\partial u}{\partial P_3} \frac{\partial w}{\partial P_4} + \frac{\partial u}{\partial P_4} \frac{\partial w}{\partial P_3} + \frac{\partial u}{\partial P_5} \frac{\partial w}{\partial P_2} + \frac{\partial u}{\partial P_6} \frac{\partial w}{\partial P_1};$$

it is quadratic in the variables, and of the fourth degree in the coefficients D, M, \dots, A .

Next, we have a line-covariant y , obtained by forming either

$$y = \frac{\partial u}{\partial P_1} \frac{\partial x}{\partial P_6} + \frac{\partial u}{\partial P_2} \frac{\partial x}{\partial P_5} + \frac{\partial u}{\partial P_3} \frac{\partial x}{\partial P_4} + \frac{\partial u}{\partial P_4} \frac{\partial x}{\partial P_3} + \frac{\partial u}{\partial P_5} \frac{\partial x}{\partial P_2} + \frac{\partial u}{\partial P_6} \frac{\partial x}{\partial P_1}$$

or (what is the equivalent except as to an arithmetical factor)

$$y = \frac{\partial v}{\partial P_1} \frac{\partial w}{\partial P_6} + \frac{\partial v}{\partial P_2} \frac{\partial w}{\partial P_5} + \frac{\partial v}{\partial P_3} \frac{\partial w}{\partial P_4} + \frac{\partial v}{\partial P_4} \frac{\partial w}{\partial P_3} + \frac{\partial v}{\partial P_5} \frac{\partial w}{\partial P_2} + \frac{\partial v}{\partial P_6} \frac{\partial w}{\partial P_1};$$

this also is quadratic in the variables, and it is of the fifth degree in the coefficients.

Lastly, as regards line-covariants, there is a line-covariant z , obtained by forming either

$$z = \frac{\partial u}{\partial P_1} \frac{\partial y}{\partial P_6} + \frac{\partial u}{\partial P_2} \frac{\partial y}{\partial P_5} + \frac{\partial u}{\partial P_3} \frac{\partial y}{\partial P_4} + \frac{\partial u}{\partial P_4} \frac{\partial y}{\partial P_3} + \frac{\partial u}{\partial P_5} \frac{\partial y}{\partial P_2} + \frac{\partial u}{\partial P_6} \frac{\partial y}{\partial P_1},$$

or the same combination of v and x , or the combination

$$\frac{\partial w}{\partial P_1} \frac{\partial w}{\partial P_6} + \frac{\partial w}{\partial P_2} \frac{\partial w}{\partial P_5} + \frac{\partial w}{\partial P_3} \frac{\partial w}{\partial P_4};$$

this line-covariant also is quadratic in the variables, and it is of the sixth degree in the coefficients.

31. As regards pure invariants, they now can be obtained simply. The operator

$$\nabla = \frac{\partial^2}{\partial p_1 \partial p_6} + \frac{\partial^2}{\partial p_2 \partial p_5} + \frac{\partial^2}{\partial p_3 \partial p_4}$$

is invariantive; and u, v, w, x, y, z are covariants. Hence

$$\begin{aligned} I_1 &= \nabla u, \\ I_2 &= \nabla v, \\ I_3 &= \nabla w, \\ I_4 &= \nabla x, \\ I_5 &= \nabla y, \\ I_6 &= \nabla z, \end{aligned}$$

are six invariants of degrees one, two, three, four, five, and six respectively.

Some of them can be obtained also in other fashions. Thus (ignoring a mere arithmetical factor) we have

$$\begin{aligned}
 I_2 &= u\left(\frac{\partial}{\partial p_6}, \frac{\partial}{\partial p_5}, \frac{\partial}{\partial p_4}, \frac{\partial}{\partial p_3}, \frac{\partial}{\partial p_2}, \frac{\partial}{\partial p_1}\right)u(p_1, p_2, p_3, p_4, p_5, p_6), \\
 I_3 &= u\left(\frac{\partial}{\partial p_6}, \frac{\partial}{\partial p_5}, \frac{\partial}{\partial p_4}, \frac{\partial}{\partial p_3}, \frac{\partial}{\partial p_2}, \frac{\partial}{\partial p_1}\right)v(p_1, p_2, p_3, p_4, p_5, p_6), \\
 I_4 &= v\left(\frac{\partial}{\partial p_6}, \frac{\partial}{\partial p_5}, \frac{\partial}{\partial p_4}, \frac{\partial}{\partial p_3}, \frac{\partial}{\partial p_2}, \frac{\partial}{\partial p_1}\right)v(p_1, p_2, p_3, p_4, p_5, p_6),
 \end{aligned}$$

and so on.

The actual values of these invariants can be taken as follows:—

$$I_1 = N + O + V,$$

which is permanently equal to zero, being the single relation among the coefficients of the fundamental line-covariant ;

$$\begin{aligned}
 I_2 &= AD + BE + CF + N^2 + O^2 + V^2 \\
 &\quad + 2SP + 2JM + 2TQ + 2LH + 2GK + 2RU,
 \end{aligned}$$

which is the full value of Δv_2 , viz.

$$2N' + 2O' + 2V' ;$$

$$\begin{aligned}
 I_3 &= 2N'' + 2O'' + 2V'' \\
 &= N^3 + O^3 + V^3 \\
 &\quad + 3N(AD + MJ + SP + LH + TQ) \\
 &\quad + 3O(BE + GK + RU + MJ + SP) \\
 &\quad + 3V(CF + LH + TQ + GK + RU) \\
 &\quad + 3A(MS + LT) + 3B(MP + KU) + 3C(LQ + KR) \\
 &\quad + 3D(HQ + JP) + 3E(JS + GR) + 3F(GU + HT) \\
 &\quad + 3J(KT + UL) + 3H(SK + RM) \\
 &\quad + 3Q(GM + US) + 3P(GL + RT) ;
 \end{aligned}$$

$$\begin{aligned}
 I_4 &= A'D' + B'E' + C'F' + N'^2 + O'^2 + V'^2 \\
 &\quad + 2S'P' + 2J'M' + 2T'Q' + 2L'H' + 2G'K' + 2R'U' ;
 \end{aligned}$$

$$\begin{aligned}
 I_5 &= D''A' + 2M''J' + 2L''H' + 2T''Q' + 2S''P' + 2N''N' \\
 &\quad + E''B' + 2K''G' + 2U''R' + 2O''O' + 2P''S' \\
 &\quad + F''C' + 2U''U' + 2R''U' + 2Q''T' \\
 &\quad + C''F' + 2G''K' + 2H''L' \\
 &\quad + B''E' + 2J''M' \\
 &\quad + A''D' ;
 \end{aligned}$$

$$\begin{aligned}
 I_6 &= A''D'' + B''E'' + C''F'' + N''^2 + O''^2 + V''^2 \\
 &\quad + 2S''P'' + 2J''M'' + 2T''Q'' + 2L''H'' + 2G''K'' + 2R''U'' .
 \end{aligned}$$

The full expression of I_5 is long, that of I_6 is very long. Instead of the latter, I prefer to take the discriminant of $u(P)$ which differs from the

foregoing invariant I_6 by combinations of I_1, I_2, I_3, I_4, I_5 ; the expression of this discriminant, say \square , is

$$\square = \begin{vmatrix} D, & M, & L, & T, & S, & N \\ M, & E, & K, & U, & O, & P \\ L, & K, & F, & V, & R, & Q \\ T, & U, & V, & C, & G, & H \\ S, & O, & R, & G, & B, & J \\ N, & P, & Q, & H, & J, & A \end{vmatrix}.$$

In passing it may be remarked that for the line-complex

$$Dp_1^2 + Ep_2^2 + Fp_3^2 + Cp_4^2 + Bp_5^2 + Ap_6^2 + 2Mp_1p_2 + 2Tp_1p_4 + 2Up_2p_4,$$

the other terms being absent owing to vanishing coefficients, we have

$$I_1 = 0, \quad I_3 = 0, \quad I_5 = 0,$$

while

$$\square = ABF \begin{vmatrix} C, & T, & U \\ T, & D, & M \\ U, & M, & E \end{vmatrix},$$

$$I_2 = A^2D^2 + B^2E^2 + C^2F^2 + 2ABM^2 + 2AFT^2 + 2BFU^2, \\ I_2 = AD + BE + CF.$$

32. We thus have a system of twelve members, composed of six line-covariants and six invariants; there might arise a question as to whether these can be declared algebraically independent of one another or (what is the equivalent) can be declared subjected to a single relation satisfied in virtue of the relation

$$p_1p_6 + p_2p_5 + p_3p_4 = 0,$$

the left-hand side of which also is covariantive under our system of equations. The question is most simply resolved by considering the system when the quadratic line-complex has a canonical form, say for the canonical form

$$DP_1^2 + EP_2^2 + FP_3^2 + CP_4^2 + BP_5^2 + AP_6^2 + 2NP_1P_6 + 2OP_2P_5 + 2VP_3P_4,$$

which includes six of the eight canonical forms due to Weiler.* For this form the various expressions are

$$u = DP_1^2 + AP_6^2 + 2NP_1P_6 + \dots$$

(the omitted terms in u and in the succeeding concomitants can be written

* *Math. Ann.*, vol. vii, pp. 145-207.

down, by noticing the symmetry in D, A, N, P₁, P₆; E, B, O, P₂, P₅; F, C, V, P₃, P₄),

$$\begin{aligned}
 v &= N(DP_1^2 + AP_6^2) + (N^2 + AD)P_1P_6 + \dots, \\
 w &= (3N^2 + AD)(DP_1^2 + AP_6^2) + 2N(3AD + N^2)P_1P_6 + \dots, \\
 x &= 2N(N^2 + AD)(DP_1^2 + AP_6^2) + (N^4 + 6N^2AD + A^2D^2)P_1P_6 + \dots, \\
 y &= (A^2D^2 + 10ADN^2 + 5N^4)(DP_1^2 + AP_6^2), \\
 &\quad + 2N(5A^2D^2 + 10ADN^2 + N^4)P_1P_6 + \dots, \\
 z &= N(3A^2D^2 + 10ADN^2 + 3N^4)(DP_1^2 + AP_6^2) \\
 &\quad + (A^3D^3 + 15A^2D^2N^2 + 15ADN^4 + N^6)P_1P_6 + \dots, \\
 I_1 &= N + O + V, \\
 I_2 &= N^2 + AD + \dots, \\
 I_3 &= N(N^2 + 3AD) + \dots, \\
 I_4 &= N^4 + 6ADN^2 + A^2D^2 + \dots, \\
 I_5 &= N^5 + 10ADN^3 + 5A^2D^3N + \dots, \\
 I_6 &= N^6 + 15ADN^4 + 15A^2D^2N^2 + A^3D^3 + \dots
 \end{aligned}$$

It is manifest that the six invariants I₁, I₂, I₃, I₄, I₅, I₆ are independent of one another; they are the sums (save as to a numerical factor 2) of the first, second, third, fourth, fifth, and sixth powers of the six independent quantities

$$k_1, k_2 = N \pm (AD)^{\frac{1}{2}}; k_3, k_4 = O \pm (BE)^{\frac{1}{2}}; k_5, k_6 = V \pm (CF)^{\frac{1}{2}}.$$

(For the form under consideration, the discriminant \square becomes

$$\square = (AD - N^2)(BE - O^2)(CF - V^2);$$

and it is easy to verify that

$$\square = \frac{1}{3}I_6 + \frac{1}{6}I_2^3 - \frac{1}{2}I_2I_4 - \frac{2}{9}I_3^2,$$

when we take I₁=0.)

Again, it is not difficult to see that there is no relation among the line-covariants, which are linear functions of

$$\begin{aligned}
 \sigma_1 &= DP_1^2 + AP_6^2, & \sigma_3 &= EP_2^2 + BP_5^2, & \sigma_5 &= FP_3^2 + CP_4^2, \\
 \sigma_2 &= P_1P_6, & \sigma_4 &= P_2P_5, & \sigma_6 &= P_3P_4,
 \end{aligned}$$

in the forms

$$\begin{aligned}
 u &= \sigma_1 + (k_1 + k_2)\sigma_2 + \dots, \\
 2v &= (k_1 + k_2)\sigma_1 + (k_1^2 + k_2^2)\sigma_2 + \dots, \\
 w &= (k_1^2 + k_1k_2 + k_2^2)\sigma_1 + (k_1^3 + k_2^3)\sigma_2 + \dots, \\
 2x &= (k_1 + k_2)(k_1^2 + k_2^2)\sigma_1 + (k_1^4 + k_2^4)\sigma_2 + \dots, \\
 y &= (k_1^4 + k_1^3k_2 + k_1^2k_2^2 + k_1k_2^3 + k_2^4)\sigma_1 + (k_1^5 + k_2^5)\sigma_2 + \dots, \\
 2z &= (k_1 + k_2)(k_1^4 + k_1^2k_2^2 + k_2^4)\sigma_1 + (k_1^6 + k_2^6)\sigma_2 + \dots
 \end{aligned}$$

These equations can be regarded in either of two ways. We can ignore the relation

$$\sigma_2 + \sigma_4 + \sigma_6 = 0;$$

and then there is no *syzygy* (linear or other) between u, v, w, x, y, z . We could also resolve the equations so as to express $\sigma_2, \sigma_4, \sigma_6$ linearly in terms of u, v, w, x, y, z , and substitute the results in the foregoing relation; owing to the symmetry, the coefficients would be invariants. It is preferable to choose the latter way.

We may therefore summarise the results in a declaration that the required system contains

- (i) six invariants, independent of one another;
- (ii) six line-covariants, connected by a single linear relation;
- (iii) the permanent relation $P_1P_6 + P_2P_5 + P_3P_4 = 0$.

AN ALGEBRAICALLY COMPLETE SYSTEM OF CONCOMITANTS:
DIFFERENTIAL INVARIANTS.

33. It is now possible to construct, with comparative ease, a complete aggregate of independent solutions of the full system of equations in § 25, which are fifteen in number. The quantities that occur in those equations are

- 10, the coefficients in the original quadratic form;
- 21, the coefficients in the associated line-covariant;
- 14, the number of variables (point, line, plane);

that is, 45 in all. Consequently, a complete aggregate of independent solutions of the system of equations must contain thirty members.

We have already constructed a couple of limited aggregates. In the first of them, we had the concomitants which involved none of the twenty-one coefficients of the line-covariant; they were nine in number, and were denoted by

$$\Sigma, \Sigma_1; \quad \Pi, \Pi_1; \quad \Lambda, \Lambda_1; \quad \Delta; \quad \sum_{r=1}^4 U_r X_r; \quad \sum_{s=1}^3 P_s P_{6-s},$$

of which, in particular, Λ is a quadratic line-covariant, also having twenty-one coefficients constructed out of the ten coefficients of the original quadratic form Σ . In the second of them, we had the associated line-covariant $u(P)$, and five other line-covariants $v(P), w(P), x(P), y(P), z(P)$, the six being connected by a single relation in virtue of the equation

$$\sum_{s=1}^3 P_s P_{6-s};$$

and there were also six invariants

$$I_1, I_2, I_3, I_4, I_5, I_6,$$

involving only the coefficients of $u(P)$.

All of these can be included in the complete aggregate required; but they are not sufficient in number to constitute that aggregate. They must therefore be supplemented by others.

34. It is desirable to have a full set of pure invariants, independent of one another; for they involve the coefficients of the original form and the coefficients of $u(P)$, these being combinations of derivatives of the ten original coefficients. Thus the pure invariants are the differential invariants of the original form. As there are fifteen equations and thirty-one coefficients, we shall have sixteen members in the full set of independent quantities; the single relation between the line-coefficients can be introduced later.

These differential invariants can be derived from the six differential invariants already obtained; no further integrations are necessary, because they can be constructed by means of a known theorem in the general theory of concomitants of quantics. We have, on the one hand, the line-covariant of the original quadratic form, represented by

$$\Lambda = (\mathbf{d}, \mathbf{m}, \dots, \mathbf{a} \text{ \textcircled{X} } P_1, \dots, P_6)^2,$$

and the associated line-covariant, involving the Riemann-Christoffel symbols as its coefficient, in the form

$$u = (D, M, \dots, A \text{ \textcircled{X} } P_1, \dots, P_6)^2,$$

both being quadratic in the line-variables. Hence

$$u + \mu\Lambda = (D + \mu\mathbf{d}, M + \mu\mathbf{m}, \dots, A + \mu\mathbf{a} \text{ \textcircled{X} } P_1, \dots, P_6)^2,$$

where μ is any arbitrary parameter, is also a concomitant of the system. The invariants of any concomitant are invariants of the system. We have invariants $I_1, I_2, I_3, I_4, I_5, I_6$ of u alone, as well as the discriminant Δ of the original quadratic form Σ ; consequently, these quantities

$$I_1(u + \mu\Lambda), \quad I_2(u + \mu\Lambda), \quad I_3(u + \mu\Lambda), \quad I_4(u + \mu\Lambda), \quad I_5(u + \mu\Lambda), \quad I_6(u + \mu\Lambda),$$

being the six invariants of $u + \mu\Lambda$, are invariants. This result holds for all values of the parameter μ ; and therefore the coefficient of every power of μ in each of these six modified invariants is itself an invariant.

To express these invariants, we adopt an operator \mathbf{D} , to denote

$$\left(\mathbf{d} \frac{\partial}{\partial D} + \mathbf{m} \frac{\partial}{\partial M} + \dots + \mathbf{a} \frac{\partial}{\partial A} \right);$$

then, except as to an irrelevant numerical factor, each of the quantities

$$\mathbf{D}^s I_r \{u(P)\},$$

for $r=1, 2, \dots, 6$, and $s=0, 1, 2, \dots, r$, is an invariant. There would apparently emerge twenty-seven such invariants ($2+3+4+5+6+7$); but they are not independent of one another, and the independent members must be selected. We take them, in turn, from the successive sets; and shall use the notation

$${}_s I_r = \mathbf{D}^s I_r \{u(P)\}.$$

Out of $I_1(u + \mu\Delta)$, we have

$$\begin{aligned} {}_0 I_1 &= N + O + V, \\ {}_1 I_1 &= \mathbf{n} + \mathbf{o} + \mathbf{v}, \end{aligned}$$

two invariants; they happen, each of them, to be zero; but they occur as invariants of the system.

Out of $I_2(u + \mu\Delta)$, we have

$$\begin{aligned} {}_0 I_2 &= AD + BE + CF + \dots + 2RU \\ &= I_2, \end{aligned}$$

as in § 31,

$$\begin{aligned} {}_1 I_2 &= \mathbf{D} I_2 \\ &= \mathbf{a}D + \mathbf{b}E + \dots + \mathbf{d}\Delta, \end{aligned}$$

and

$$\begin{aligned} {}_2 I_2 &= \frac{1}{2} \mathbf{D}^2 I_2 \\ &= \mathbf{ad} + \mathbf{be} + \mathbf{cf} + \dots + 2\mathbf{ru}. \end{aligned}$$

When the values of $\mathbf{d}, \dots, \mathbf{a}$ are substituted in ${}_2 I_2$, we find that

$${}_2 I_2 = 3\Delta,$$

so that, because Δ has been retained as an invariant, the last gives no new member. Thus out of $I_2(u + \mu\Delta)$, we have the two invariants

$$I_2, \quad {}_1 I_2,$$

in addition to Δ .

Out of $I_3(u + \mu\Delta)$, we have

$$\begin{aligned} {}_0 I_3 &= N^3 + O^3 + V^3 + 3N(AD + \dots) + \dots \\ &= I_3, \\ {}_1 I_3 &= \mathbf{D} I_3 \\ &= 3[\mathbf{n}(AD + \dots) + \mathbf{o}(BE + \dots) + \mathbf{v}(CF + \dots) \\ &\quad + 2(\mathbf{ND} + \mathbf{MS} + \mathbf{LT}) + \dots], \\ {}_2 I_3 &= \mathbf{D}^2 I_3 \\ &= 6\Delta(N + O + V) \\ &= 6\Delta I_1, \\ {}_3 I_3 &= \mathbf{D}^3 I_3 \\ &= 0. \end{aligned}$$

The last but one is not a new invariant; hence, out of $I_3(u + \mu\Lambda)$, we have the two invariants

$$I_3, \quad {}_1I_3.$$

We proceed similarly with the others: the results need only be stated.

Out of $I_4(u + \mu\Lambda)$, the new invariants are

$$\begin{aligned} {}_0I_4 &= I_4, \\ {}_1I_4 &= \mathbf{D}I_4, \\ {}_2I_4 &= \mathbf{D}^2I_4; \end{aligned}$$

the invariant

$${}_3I_4 = \mathbf{D}^3I_4$$

is a numerical multiple of $\Delta \cdot {}_1I_2$, and therefore is not new; and the invariant

$${}_4I_4 = \mathbf{D}^4I_4$$

is a numerical multiple of Δ^2 , and therefore is not new.

Out of $I_5(u + \mu\Lambda)$, the new invariants are

$$\begin{aligned} {}_0I_5 &= I_5, \\ {}_1I_5 &= \mathbf{D}I_5, \\ {}_2I_5 &= \mathbf{D}^2I_5; \end{aligned}$$

the invariant

$${}_3I_5 = \mathbf{D}^3I_5$$

is a numerical multiple of ΔI_3 , and therefore is not new; the invariant

$${}_4I_5 = \mathbf{D}^4I_5$$

is a numerical multiple of $\Delta^2 I_1$, and therefore is not new; and

$$\begin{aligned} {}_5I_5 &= \mathbf{D}^5I_5 \\ &= 0. \end{aligned}$$

For purposes of enumeration, I retain I_6 instead of \square . Out of $I_6(u + \mu\Lambda)$, the new invariants are

$$\begin{aligned} {}_0I_6 &= I_6, \\ {}_1I_6 &= \mathbf{D}I_6, \\ {}_2I_6 &= \mathbf{D}^2I_6, \\ {}_3I_6 &= \mathbf{D}^3I_6, \end{aligned}$$

There is an invariant

$${}_4I_6 = \mathbf{D}^4I_6;$$

it is a linear combination $\Delta \cdot {}_2I_4$ and $\Delta^2 I_2$, so that it is not new. There is an invariant

$${}_5I_6 = \mathbf{D}^5I_6,$$

which (after the last result) is a linear combination of $\Delta \mathbf{D}({}_2I_4)$ and $\Delta^2 \mathbf{D}I_2$; that is, it is a numerical multiple of $\Delta^2 \cdot {}_1I_2$. The invariant

$${}_6I_6 = \mathbf{D}^6I_6$$

is a numerical multiple of Δ^3 .

Thus the tale of invariants is

$$\begin{aligned}
 &I_1(=0), \quad {}_0I_1(=0); \\
 &I_2, \quad {}_1I_2, \quad {}_2I_2(=\Delta); \\
 &\quad \quad I_3, \quad {}_1I_3; \\
 &\quad \quad I_4, \quad {}_1I_4, \quad {}_2I_4; \\
 &\quad \quad I_5, \quad {}_1I_5, \quad {}_2I_5; \\
 &I_6, \quad {}_1I_6, \quad {}_2I_6, \quad {}_3I_6.
 \end{aligned}$$

But owing to the definitions of $\mathbf{n}, \mathbf{o}, \mathbf{v}$, we have

$$\mathbf{n} + \mathbf{o} + \mathbf{v} = 0$$

always, whether the condition $N + O + V = 0$ is retained or is used as an invariantive condition. Excluding both of these, so that we have twenty independent Riemann-Christoffel symbols, together with the ten coefficients of the form, we have thirty quantities in the fifteen equations. There will accordingly be fifteen independent integrals; these can be taken to be the remaining fifteen quantities $I_2, \dots, {}_3I_6$, which constitute the tale of differential invariants of the quadratic differential form

$$(a, \dots, n \delta dx_1, dx_2, dx_3, dx_4)^2$$

up to the second order of derivation of its coefficients.*

35. Down to the present, we have secured, as contributions to the complete aggregate, the nine quantities

$$\Sigma, \Sigma_1; \quad \Pi, \Pi_1; \quad \Lambda, \Lambda_1; \quad \Delta, \sum_{r=1}^4 U_r X_r, \sum_{s=1}^3 P_s P_{6-s};$$

the five independent line-covariants (there are six, subject to one relation)

$$u(P), \quad v(P), \quad w(P), \quad x(P), \quad y(P), \quad z(P);$$

and the fourteen differential invariants other than Δ , which has already been counted. The complete system is to contain thirty members, or, if we insist throughout on the persistence of the relation $N + O + V = 0$, twenty-nine in all. One more constituent member is wanted.

Manifestly the invariantive operator \mathbf{D} can be applied to the line-covariants in the same way as it was applied to the invariants. Now

$$\mathbf{D}u(P)$$

is merely Λ , already retained: and $\mathbf{D}^2u(P)$ is zero; so no new constituent will arise out of $u(P)$. But

$$\mathbf{D}v(P)$$

* This selection of the independent invariants, so as to make up the aggregate, was effected by using the canonical forms of § 32.

is a new line-covariant; it will therefore be retained. (It may be added that

$$\mathbf{D}^2v(P) = 2\Delta(P_1P_6 + P_2P_5 + P_3P_4),$$

because of relations

$$\mathbf{nd} + \mathbf{sm} + \mathbf{tl} = 0,$$

six in number,

$$\mathbf{vk} + \mathbf{ok} + \mathbf{fu} + \mathbf{re} + \mathbf{qm} + \mathbf{pl} = 0,$$

fifteen in number, and

$$\mathbf{ad} + \mathbf{qt} + \mathbf{sp} + \mathbf{lh} + \mathbf{jm} + \mathbf{n}^2 = \Delta,$$

three in number; thus $\mathbf{D}^2v(P)$ would furnish no new constituent.)

Hence by the association of $\mathbf{D}v(P)$ with the preceding concomitants, we have an algebraically complete aggregate of independent concomitants of the quadratic form.

This aggregate can be modified by the substitution of other concomitants for those actually included; the foregoing seem the simplest set to select. Other concomitants can easily be written down; thus we could have

$$\begin{aligned} &\partial_p u(P), \quad \partial_p v(P), \quad \partial_p w(P), \quad \partial_p x(P), \quad \partial_p y(P), \quad \partial_p z(P), \\ &\partial_p^2 u(P), \quad \partial_p^2 v(P), \quad \partial_p^2 w(P), \quad \partial_p^2 x(P), \quad \partial_p^2 y(P), \quad \partial_p^2 z(P), \end{aligned}$$

where ∂_p is the operator obtained in § 28. Again, we could have

$$\begin{aligned} &\mathbf{D}w(P), \quad \mathbf{D}x(P), \quad \mathbf{D}y(P), \quad \mathbf{D}z(P), \\ &\mathbf{D}^2w(P), \quad \mathbf{D}^2x(P), \quad \mathbf{D}^2y(P), \quad \mathbf{D}^2z(P) \\ &\mathbf{D}^3x(P), \quad \mathbf{D}^3y(P), \quad \mathbf{D}^3z(P) \\ &\mathbf{D}^4y(P), \quad \mathbf{D}^4z(P), \quad \mathbf{D}^5z(P), \end{aligned}$$

and so on. But, as already stated, apparently the simplest aggregate has been selected.

FOUR-DIMENSIONAL SPACE: RIEMANN'S MEASURE OF CURVATURE.

36. Consider a five-dimensional space, for which the non-homogeneous coordinates are P, Q, R, S, T. Any four-dimensional amplitude in that space can be represented by a relation

$$f(P, Q, R, S, T) = 0,$$

or by a set of relations

$$\left. \begin{aligned} P &= P(x_1, x_2, x_3, x_4) \\ Q &= Q(x_1, x_2, x_3, x_4) \\ R &= R(x_1, x_2, x_3, x_4) \\ S &= S(x_1, x_2, x_3, x_4) \\ T &= T(x_1, x_2, x_3, x_4) \end{aligned} \right\},$$

where the forms of the functions P, Q, R, S, T are such as to allow one, and only one, eliminant when x_1, x_2, x_3, x_4 are made to disappear. When two neighbouring points are taken, represented by (x_1, \dots, x_4) and $(x_1+dx_1, \dots, x_4+dx_4)$, the "distance" ds between them is given by

$$ds^2 = dP^2 + dQ^2 + dR^2 + dS^2 + dT^2 \\ = (a, b, c, d, f, g, h, l, m, n)(dx_1, dx_2, dx_3, dx_4)^2,$$

with the preceding notation. The simplest case occurs when one of the original coordinates vanishes, say

$$T = 0;$$

the simplest expression in that event occurs by taking

$$P = x_1, \quad Q = x_2, \quad R = x_3, \quad S = x_4,$$

so that

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

In all cases, the variables x_1, x_2, x_3, x_4 are subject, or must be considered subject, to transformations of the most general kind; and we are therefore led to consider what functions of the coefficients a, \dots, n in the differential form are invariantive, and what is the geometrical significance of these invariantive functions.

The analytical investigation of the invariantive forms has been effected in the preceding part of this paper. From the results obtained, especially in connection with the line-covariants, it appears that two of them, viz.

$$u(P), \quad \Lambda(P),$$

are of fundamental importance for the construction alike of line-covariants and of the differential invariants of the quadratic differential form which is the expression of ds^2 . These covariants, however, are only relative: that is to say, when the variables are transformed, the same functions of the new coefficients and the new variables are respectively equal to the functions of the old coefficients and the old variables, save as to a factor which is a power of the modulus of transformation, the power of the modulus depending on the particular covariant.

The consideration of the simplest transformation

$$x_1 = \alpha x'_1, \quad x_2 = \beta x'_2, \quad x_3 = \gamma x'_3, \quad x_4 = \delta x'_4,$$

shows that the covariantive quantity

$$\frac{u(P)}{\Lambda(P)}$$

remains absolutely unaltered: it is therefore an absolute invariant: and, accordingly, it represents some geometrical property of the selected amplitude. We proceed to the interpretation.

37. These variables P_1, \dots, P_6 have been considered as line-variables from the point of view of convenience, just as X_1, X_2, X_3, X_4 have been considered as point-variables; and the adopted definitions are

$$\begin{aligned} P_1 &= Y_2Z_3 - Y_3Z_2, & P_6 &= Y_1Z_4 - Y_4Z_1, \\ P_2 &= Y_3Z_1 - Y_1Z_3, & P_5 &= Y_2Z_4 - Y_4Z_2, \\ P_3 &= Y_1Z_2 - Y_2Z_1, & P_4 &= Y_3Z_4 - Y_4Z_3. \end{aligned}$$

To interpret the covariant quantity in question, imagine a linear transformation of the cogredient sets of variables to be effected such that P_3 does not vanish while P_1, P_2, P_4, P_5, P_6 vanish, thus of course securing the relation

$$P_1P_6 + P_2P_5 + P_3P_4 = 0.$$

In these circumstances, $u(P)$ becomes fP_1^2 and $\Lambda(P)$ becomes fP_1^2 ; and the quotient is an absolute invariant. Hence

$$\frac{u(P)}{\Lambda(P)} = \frac{F}{f}.$$

But in order that the quantities P_1, P_2, P_4, P_5, P_6 may vanish, while P_3 does not vanish, we must have

$$Y_3 = 0, \quad Y_4 = 0; \quad Z_3 = 0, \quad Z_4 = 0.$$

We must now return to the differential form. The point-variables X_1, \dots, X_4 are displacements dx_1, dx_2, dx_3, dx_4 in our amplitude of four-dimensions; and likewise for the sets $Y_1, \dots, Y_4 (= dy_1, dy_2, dy_3, dy_4)$, and $Z_1, \dots, Z_4 (= dz_1, dz_2, dz_3, dz_4)$. Accordingly, for the purposes of the special variables P , we are considering two displacements dy_1, dy_2 when y_3 and y_4 do not vary, and dz_1, dz_2 when z_3 and z_4 do not vary; while for the general variables P , we consider two general displacements without limitations on the variations. Thus the special form of the differential ds^2 , for the immediate choice of coordinates, becomes

$$adx_1^2 + 2hdx_1dx_2 + bdx_2^2,$$

where x_3 and x_4 , in so far as they occur in a, h, b , are not to be subject to variation.

To find the limiting form of F , we can proceed as follows. As f, g, l, m do not occur, and as there is no variation of h through x_3 and x_4 , all the quantities

$$\begin{bmatrix} 11 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 11 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 12 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 12 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 22 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 22 \\ 4 \end{bmatrix}$$

vanish in present circumstances. To obtain the limiting forms of $A_{11} \div \Delta$,

$A_{12} \div \Delta$, $A_{22} \div \Delta$, we momentarily allow c and d of the differential form not to vanish, and make n to vanish; then

$$\begin{aligned} \frac{A_{11}}{\Delta} &= \frac{bcd}{cd(ab-h^2)} = \frac{b}{ab-h^2}, \\ \frac{A_{12}}{\Delta} &= \frac{-cdh}{cd(ab-b^2)} = -\frac{h}{ab-h^2}, \\ \frac{A_{22}}{\Delta} &= \frac{cda}{cd(ab-h^2)} = \frac{a}{ab-h^2}, \end{aligned}$$

in the limit. Hence the limiting form of F is

$$\begin{aligned} F &= \frac{1}{2}(a_{22} - 2h_{12} + b_{11}) \\ &+ \frac{1}{4(ab-h^2)}[-h(a_1b_2 - a_2b_1 - 2a_2h_2 - 2b_1h_1 + 4b_1h_2) \\ &\quad + b\{a_1(2h_2 - b_1) - a_2^2\} + a\{b_2(2h_1 - a_2) - b_1^2\}]. \end{aligned}$$

Also, the value of f is given by

$$f = ab - h^2.$$

Now for the line-element associated with a surface

$$adx_1^2 + 2hdx_1dx_2 + bdx_2^2,$$

the Gauss-measure of curvature of the surface is precisely this quantity

$$-\frac{F}{f}.$$

Accordingly, Riemann takes the quantity

$$\frac{u(P)}{\Lambda(P)},$$

associated with the quantities P , which are compounded out of two directions dy_1, dy_2, dy_3, dy_4 and dz_1, dz_2, dz_3, dz_4 in the four-fold manifold, to be a measure of the curvature of the manifold as associated with two directions. Denoting this measure of curvature by \mathfrak{C} , we have

$$\frac{u(P)}{\Lambda(P)} = -\mathfrak{C},$$

as the Riemann expression for the curvature of the four-fold amplitude, connected with the two directions chosen at the place determined by the values of x_1, x_2, x_3, x_4 .

38. This measure of curvature varies from one pair of directions to another. Where it is a maximum or a minimum or is stationary, we have

$$\frac{\partial \mathfrak{C}}{\partial P_r} = 0, \text{ for } r = 1, 2, 3, 4, 5, 6,$$

and therefore

$$\mathfrak{C} \frac{\partial u(P)}{\partial P_r} + \frac{\partial \Lambda(P)}{\partial P_r} = 0,$$

for $r=1, \dots, 6$. These equations are

$$\begin{aligned} \left(\frac{D}{\mathfrak{C}} + d\right)P_1 + \left(\frac{M}{\mathfrak{C}} + m\right)P_2 + \left(\frac{L}{\mathfrak{C}} + l\right)P_3 + \left(\frac{T}{\mathfrak{C}} + t\right)P_4 + \left(\frac{S}{\mathfrak{C}} + s\right)P_5 + \left(\frac{N}{\mathfrak{C}} + n\right)P_6 &= 0, \\ \left(\frac{M}{\mathfrak{C}} + m\right)P_1 + \left(\frac{E}{\mathfrak{C}} + e\right)P_2 + \left(\frac{K}{\mathfrak{C}} + k\right)P_3 + \left(\frac{U}{\mathfrak{C}} + u\right)P_4 + \left(\frac{O}{\mathfrak{C}} + o\right)P_5 + \left(\frac{P}{\mathfrak{C}} + p\right)P_6 &= 0, \\ \left(\frac{L}{\mathfrak{C}} + l\right)P_1 + \left(\frac{K}{\mathfrak{C}} + k\right)P_2 + \left(\frac{F}{\mathfrak{C}} + f\right)P_3 + \left(\frac{V}{\mathfrak{C}} + v\right)P_4 + \left(\frac{R}{\mathfrak{C}} + r\right)P_5 + \left(\frac{Q}{\mathfrak{C}} + q\right)P_6 &= 0, \\ \left(\frac{T}{\mathfrak{C}} + t\right)P_1 + \left(\frac{U}{\mathfrak{C}} + u\right)P_2 + \left(\frac{V}{\mathfrak{C}} + v\right)P_3 + \left(\frac{C}{\mathfrak{C}} + c\right)P_4 + \left(\frac{G}{\mathfrak{C}} + g\right)P_5 + \left(\frac{H}{\mathfrak{C}} + h\right)P_6 &= 0, \\ \left(\frac{S}{\mathfrak{C}} + s\right)P_1 + \left(\frac{O}{\mathfrak{C}} + o\right)P_2 + \left(\frac{R}{\mathfrak{C}} + r\right)P_3 + \left(\frac{G}{\mathfrak{C}} + g\right)P_4 + \left(\frac{B}{\mathfrak{C}} + b\right)P_5 + \left(\frac{J}{\mathfrak{C}} + j\right)P_6 &= 0, \\ \left(\frac{N}{\mathfrak{C}} + n\right)P_1 + \left(\frac{P}{\mathfrak{C}} + p\right)P_2 + \left(\frac{Q}{\mathfrak{C}} + q\right)P_3 + \left(\frac{H}{\mathfrak{C}} + h\right)P_4 + \left(\frac{J}{\mathfrak{C}} + j\right)P_5 + \left(\frac{A}{\mathfrak{C}} + a\right)P_6 &= 0. \end{aligned}$$

Hence there are six principal values of \mathfrak{C} , being the roots of the sextic equation

$$\begin{vmatrix} \frac{D}{\mathfrak{C}} + d, & \frac{M}{\mathfrak{C}} + m, & \frac{L}{\mathfrak{C}} + l, & \frac{T}{\mathfrak{C}} + t, & \frac{S}{\mathfrak{C}} + s, & \frac{N}{\mathfrak{C}} + n \\ \frac{M}{\mathfrak{C}} + m, & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{L}{\mathfrak{C}} + l, & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{T}{\mathfrak{C}} + t, & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{S}{\mathfrak{C}} + s, & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{N}{\mathfrak{C}} + n, & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0,$$

which, when written in full, is

$$\frac{\square}{\mathfrak{C}^6} + \frac{D\square}{\mathfrak{C}^5} + \frac{\frac{1}{2}D^2\square}{\mathfrak{C}^4} + \frac{\frac{1}{3!}D^3\square}{\mathfrak{C}^3} + \frac{\frac{1}{4!}D^4\square}{\mathfrak{C}^2} + \Delta^2 \frac{I_2}{\mathfrak{C}} + \Delta^3 = 0.$$

All the coefficients in this equation are differential invariants; and thus, if these principal curvatures are $\mathfrak{C}_1, \mathfrak{C}_2, \dots, \mathfrak{C}_6$, we have the significance of six of our differential invariants in terms of symmetric combinations of these principal curvatures. In particular, we have

$$\begin{aligned} \mathfrak{C}_1 + \mathfrak{C}_2 + \mathfrak{C}_3 + \mathfrak{C}_4 + \mathfrak{C}_5 + \mathfrak{C}_6 &= -\frac{DI_2}{\Delta}, \\ \mathfrak{C}_1\mathfrak{C}_2\mathfrak{C}_3\mathfrak{C}_4\mathfrak{C}_5\mathfrak{C}_6 &= -\frac{\square}{\Delta^3}. \end{aligned}$$

Further, there are six principal pairs of directions at a place in the amplitude, respectively associated with the six principal curvatures; each

pair is given by five of the six equations in P_1, \dots, P_6 for the respective values of the principal curvatures.

SPACES OF CONSTANT (AND ZERO) CURVATURE.

39. Two special examples may be noted.

In the first, we take the amplitude of § 36, given by $f(P, Q, R, S, T) = 0$ to be flat, say

$$T = 0;$$

then the element of arc in that amplitude is

$$ds^2 = dP^2 + dQ^2 + dR^2 + dS^2.$$

But now we can take

$$P = x_1, \quad Q = x_2, \quad R = x_3, \quad S = x_4;$$

so that

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

This is a quadratic form for which

$$a = b = c = d = 1, \\ f = g = h = l = m = n = 0.$$

For this form, all the quantities

$$D, \quad M, \quad \dots, \quad J, \quad A$$

vanish, because they involve first and second derivatives of the constant coefficients a, \dots, n .

Hence all the line-covariants vanish except only $\Lambda(P)$, which is

$$\Lambda(P) = P_1^2 + P_2^2 + P_3^2 + P_4^2 + P_5^2 + P_6^2,$$

manifestly not vanishing. Consequently, all the line-covariants of any form, which arises from

$$dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

by transformation of the variables, also vanish. In particular, we must have

$$u(P) = 0$$

for all values of the variables P_1, \dots, P_6 ; and therefore every coefficient in $u(P)$ must vanish; that is, we have the relations

$$D = 0, \quad M = 0, \quad \dots, \quad J = 0, \quad A = 0.$$

Moreover, when these quantities vanish, all the remaining line-covariants vanish. Further, all the invariants will then vanish except only the discriminant Δ .

Now, it is a known proposition* in the theory of homogeneous forms that the necessary and sufficient conditions securing the transformation of one form into another are that the same covariants and invariants vanish for the two sets of concomitants and that the absolute invariants are equal. When we have to deal with a form

$$\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2,$$

there is no absolute invariant under linear transformations. And we have seen that, in order to make all the line-covariants except $\Lambda(P)$ vanish, it is necessary and sufficient that the relations

$$D = 0, \quad M = 0, \quad \dots, \quad J = 0, \quad A = 0$$

should be satisfied.

We therefore infer that, if a form

$$(a, b, c, d, f, g, h, l, m, n)(dx, dy, dz, dt)^2$$

is to be transformably equivalent to a form

$$dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

it is necessary that the twenty-one relations

$$D = 0, \quad M = 0, \quad \dots, \quad J = 0, \quad A = 0$$

(equivalent to only twenty independent relations) shall be satisfied; and these relations are also sufficient, so far as concerns the mere algebraic transformations of dx_1, dx_2, dx_3, dx_4 . Further, under Lie's theory of continuous groups in connection with infinitesimal transformations, we can now infer that the coefficients of all forms equivalent, under any transformations whatever, to the form

$$dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

must satisfy the twenty-one partial differential equations

$$D = 0, \quad M = 0, \quad \dots, \quad J = 0, \quad A = 0,$$

which are connected by a single linear relation.

40. In the second example, we take the amplitude

$$f(P, Q, R, S, T) = 0$$

to be a space of constant measure of curvature W . Then we must have

$$\frac{u(P)}{\Lambda(P)} = W,$$

* Aronhold, *Crelle*, t. lxii (1863), pp. 281-345; Gram, *Math. Ann.*, t. vii (1874), pp. 230-240.

where W is a constant; and this relation must be satisfied for all directions P . Hence we must have the relations

$$\begin{aligned} D &= Wd \\ M &= Wm \\ &\vdots \\ J &= Wj \\ A &= Wa, \end{aligned}$$

necessary and sufficient to secure the constancy of the curvature.* Various forms for the element of arc in such a space have been given, the simplest being Riemann's form †

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2}{\left\{1 + \frac{1}{4}W(x_1^2 + x_2^2 + x_3^2 + x_4^2)\right\}^2}.$$

EINSTEIN'S LAW OF GRAVITATION: INTEGRATION OF THE DEFINING EQUATIONS.

41. It may be convenient, at the beginning of this next stage, to recall the difference between the Riemann-Christoffel symbols (rk, ih) and the symbols $\{rk, ih\}$ due to the same authors. The latter represent the elements of the Riemann-Christoffel tensor in Einstein's theory of relativity. In the notation of § 23, the relation between them is expressed in the form

$$\{\mu\rho, \tau\sigma\} = \frac{1}{\Delta} \sum_k^{1,2,3,4} A_{\rho k}(\mu k, \sigma\nu),$$

while the symbol on the left-hand side is denoted by

$$B_{\mu\nu\sigma}^{\rho}$$

by Einstein.‡

Consequently, for a form $-dx^2 - dy^2 - dz^2 + dt^2$, all the tensors $B_{\mu\nu\sigma}^{\rho}$ vanish; and their identical evanescence for any form

$$(a, b, c, d, f, g, h, l, m, n \text{ \textcircled{X}} dx_1, dx_2, dx_3, dx_4)$$

suffices to secure that it is reducible to the earlier four-square form with constant coefficients.

* The notion of amplitudes with a constant (or variable) measure of curvature originated with Riemann. The literature dealing with amplitudes having a constant measure of curvature is copious, and the developments really belong to the domain of differential geometry. Some account is given by Bianchi, *Lezioni di geometria differenziale*, t. i, cap. xi.

† He obtained it for a space of n dimensions.

‡ Bianchi, *Lezioni di geometria differenziale*, t. i, p. 72; Einstein, *Ann. d. Phys.*, Bd. xlix, p. 800.

In framing the purely mathematical part of his law of gravitation, Einstein introduces a certain tensor, compounded symmetrically from the foregoing quantities $B_{\mu\nu\sigma}^{\rho}$ by the law

$$\begin{aligned}
 B_{\mu\nu} &= \sum_{\rho}^{1, 2, 3, 4} B_{\mu\nu\rho}^{\rho} = \frac{1}{\Delta} \sum_{\rho} \sum_k A_{\rho k}(\mu k, \rho\nu) \\
 &= \frac{1}{\Delta} [A_{11}(\mu 1, 1\nu) + A_{12}(\mu 2, 1\nu) + A_{13}(\mu 3, 1\nu) + A_{14}(\mu 4, 1\nu) \\
 &\quad + A_{21}(\mu 1, 2\nu) + A_{22}(\mu 2, 2\nu) + A_{23}(\mu 3, 2\nu) + A_{24}(\mu 4, 2\nu) \\
 &\quad + A_{31}(\mu 1, 3\nu) + A_{32}(\mu 2, 3\nu) + A_{33}(\mu 3, 3\nu) + A_{34}(\mu 4, 3\nu) \\
 &\quad + A_{41}(\mu 1, 4\nu) + A_{42}(\mu 2, 4\nu) + A_{43}(\mu 3, 4\nu) + A_{44}(\mu 4, 4\nu)];
 \end{aligned}$$

and he introduces* these ten quantities $B_{\mu\nu}$ as the elements of the tensor defining the gravitation field when they vanish. There thus will arise ten differential equations of the second order satisfied by the coefficients of the quadratic differential form which represents the "interval" in the newly defined field of gravitation.

He deals in particular with the form—it may be called a four-square form—

$$a dx_1^2 + b dx_2^2 + c dx_3^2 + d dx_4^2,$$

free from product-terms; and then

$$B_{\mu\nu} = \frac{1}{a}(\mu 1, 1\nu) + \frac{1}{b}(\mu 2, 2\nu) + \frac{1}{c}(\mu 3, 3\nu) + \frac{1}{d}(\mu 4, 4\nu),$$

so that there are ten equations

$$B_{\mu\nu} = 0$$

in all, subsequently found to be connected by relations. For the purpose of investigation, he assumes that

- (i) a, b, d are functions of x_1 only,
- (ii) c is a function of x_1 and x_2 only,

including, of course, the special case when a, b, c, d are pure constants. (The assumption is valid for the field

$$- dx^2 - dy^2 - dz^2 + dt^2$$

or, with spherical polar coordinates,

$$- dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + dt^2).$$

It will be noticed that t , or x_4 , is not supposed to enter into any of the coefficients, particularly not into the coefficient c . With the earlier

* "Deshalb liegt es nahe, für das materiefreie Gravitationsfeld das Verschwinden des Tensors $B_{\mu\nu}$ zu verlangen," *l.c.*, p. 803.

notation of this paper, and having regard to the limiting assumptions, we have

$$B_{11} = \frac{F}{b} + \frac{E}{c} + \frac{A}{d},$$

$$B_{22} = \frac{F}{a} + \frac{D}{c} + \frac{B}{d},$$

$$B_{33} = \frac{E}{a} + \frac{D}{b} + \frac{C}{d},$$

$$B_{44} = \frac{A}{a} + \frac{B}{b} + \frac{C}{c},$$

$$B_{12} = -\frac{M}{c},$$

$$B_{13} = 0, \quad B_{23} = 0, \quad B_{14} = 0, \quad B_{24} = 0, \quad B_{34} = 0,$$

the last five vanishing identically. Thus the differential equations of the limited field of gravitation are

$$B_{11} = 0, \quad B_{22} = 0, \quad B_{33} = 0, \quad B_{44} = 0, \quad B_{12} = 0,$$

which must be satisfied by the coefficients a, b, c, d .

42. I now proceed to the integration of these equations without all the limiting conditions assumed (of course justifiably, in postulating his problem) by Einstein and followed by Schwarzschild. Beyond satisfying these equations, the only conditions that need to be imposed upon the coefficients a, b, c, d of the form

$$a dx_1^2 + b dx_2^2 + c dx_3^2 + d dx_4^2,$$

other than those already imposed upon the occurrence of the variables, are that the values must be such as to allow the quadratic differential form to degenerate into

$$-dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + dt^2,$$

when there is no field of gravitation; that is, when r tends to an infinite value. The supposition as to polar symmetry seems superfluous; and the supposed complete non-occurrence of x_4 (or t) in the coefficients is a very definite limitation.

With our former notation, we have

$$2A = d_{11} - \frac{a_1 d_1}{2a} - \frac{d_1^2}{2d},$$

$$2B = \frac{b_1 d_1}{2a},$$

$$2C = \frac{c_1 d_1}{2a},$$

$$\begin{aligned}
 2D &= c_{22} + \frac{b_1 c_1}{2a} - \frac{c_2^2}{2c}, \\
 2E &= c_{11} - \frac{a_1 c_1}{2a} - \frac{c_1^2}{2c}, \\
 2F &= b_{11} - \frac{a_1 b_1}{2a} - \frac{b_1^2}{2b}, \\
 -2M &= c_{12} - \frac{b_1 c_2}{2b} - \frac{c_1 c_2}{2c}.
 \end{aligned}$$

The expressions can be much simplified by changing the variables. Substitute

$$a = -\alpha^2, \quad b = -\beta^2, \quad c = -\gamma^2, \quad d = \delta^2;$$

and, noting that a is (by hypothesis) a function of x_1 alone, introduce a new variable x_1' such that

$$a dx_1 = dx_1';$$

also let $\beta_1' = \frac{d\beta}{dx_1'}$, $\gamma_1' = \frac{d\gamma}{dx_1'}$, and so on. Then

$$\begin{aligned}
 A &= \alpha^2 \delta \delta_{11}', \\
 B &= \beta \delta \beta_1' \delta_1', \\
 C &= \gamma \delta \gamma_1' \delta_1', \\
 D &= -\gamma(\gamma_{22} + \beta \beta_1' \gamma_1'), \\
 E &= -\alpha^2 \gamma \gamma_{11}', \\
 F &= -\alpha^2 \beta \beta_{11}', \\
 M &= \alpha \gamma \left(\gamma_{12}' - \frac{\gamma_2}{\beta} \beta_1' \right).
 \end{aligned}$$

For the Einstein equations

$$B_{11} = 0, \quad B_{22} = 0, \quad B_{33} = 0, \quad B_{44} = 0, \quad M = 0$$

we can substitute the equivalent set

$$\begin{aligned}
 \frac{A}{a} + \frac{B}{b} + \frac{C}{c} &= 0, \\
 \frac{D}{bc} + \frac{1}{d} \left(\frac{B}{b} + \frac{C}{c} \right) &= 0, \\
 \frac{E}{ca} - \frac{B}{bd} &= 0, \\
 \frac{F}{ab} - \frac{C}{cd} &= 0, \\
 M &= 0,
 \end{aligned}$$

obtained by resolving them linearly so as to express A, D, E, F in terms of B and C .

43. From the first of this equivalent set, we have

$$2\delta\delta_{11}' + 2\frac{\delta}{\beta}\beta_1'\delta_1' + 2\frac{\delta}{\gamma}\gamma_1'\delta_1' = 0,$$

so that

$$\frac{\delta_{11}'}{\delta_1'} + \frac{\beta_1'}{\beta} + \frac{\gamma_1'}{\gamma} = 0,$$

and therefore

$$\delta_1' = \frac{m}{\beta\gamma}\mu,$$

where m is a constant, and μ is independent of x_1' . But while δ_1' and β are functions of x_1' only, γ is a function of x_1' and x_2 ; hence

$$\frac{\mu}{\gamma} = \text{function of } x_1' \text{ only} = \frac{1}{\Gamma},$$

and therefore

$$\left. \begin{aligned} \delta_1' &= \frac{m}{\beta\Gamma} \\ \gamma &= \Gamma\mu \end{aligned} \right\}$$

where Γ is a function of x_1' only and μ a function of x_2 only.

With this value of γ , we have

$$\gamma_1' = \mu\Gamma_1', \quad \gamma_{11}' = \mu\Gamma_{11}', \quad \gamma_{22}' = \mu_{22}'\Gamma;$$

and then the second equation of the equivalent set becomes

$$\frac{\mu_{22}'}{\mu} \frac{1}{\beta^2} + \frac{\beta_1'}{\beta} \frac{\Gamma_1'}{\Gamma} + \frac{\beta_1'}{\beta} \frac{\delta_1'}{\delta} + \frac{\Gamma_1'}{\Gamma} \frac{\delta_1'}{\delta} = 0.$$

Now β, Γ, δ are functions of x_1' only, while μ is a function of x_2 only; hence, in order that this equation may be satisfied, we must have

$$\frac{\mu_{22}'}{\mu} = \text{constant} = -n^2,$$

and therefore

$$\mu = H \sin (nx_2 + K)$$

where H and K are constants. Manifestly, H can be absorbed into Γ ; and so we now have

$$\gamma = \Gamma \sin (nx_2 + K),$$

while

$$\frac{\beta_1'}{\beta} \frac{\Gamma_1'}{\Gamma} + \frac{\beta_1'}{\beta} \frac{\delta_1'}{\delta} + \frac{\Gamma_1'}{\Gamma} \frac{\delta_1'}{\delta} = \frac{n^2}{\beta^2}.$$

The third equation of the equivalent set becomes

$$\frac{\Gamma_{11}'}{\Gamma} = \frac{\beta_1'\delta_1'}{\beta\delta},$$

and the fourth equation of the equivalent set becomes

$$\frac{\beta_{11}'}{\beta} = \frac{\Gamma_1'}{\Gamma} \frac{\delta_1'}{\delta}.$$

Omitting for the moment the equation $M = 0$, we have the four equations

$$\left. \begin{aligned} \frac{\beta_1'}{\beta} \frac{\Gamma_1'}{\Gamma} + \frac{\beta_1'}{\beta} \frac{\delta_1'}{\delta} + \frac{\Gamma_1'}{\Gamma} \frac{\delta_1'}{\delta} &= \frac{n^2}{\beta^2} \\ \frac{\Gamma_{11}'}{\Gamma} &= \frac{\beta_1' \delta_1'}{\beta \delta} \\ \frac{\beta_{11}'}{\beta} &= \frac{\Gamma_1' \delta_1'}{\Gamma \delta} \\ \delta_1' &= \frac{m}{\beta \Gamma} \end{aligned} \right\},$$

while

$$\gamma = \Gamma \sin (n x_2 + k').$$

From the second and the third of these, we have

$$(\beta \Gamma_{11}' - \Gamma \beta_{11}') = \frac{\delta_1'}{\delta} (\Gamma \beta_1' - \beta \Gamma_1'),$$

and therefore

$$(\Gamma \beta_1' - \beta \Gamma_1') \delta = \text{constant} = 2k,$$

so that

$$\frac{\beta_1'}{\beta} - \frac{\Gamma_1'}{\Gamma} = \frac{2km}{\beta \Gamma \delta} = 2k \frac{\delta_1'}{\delta};$$

consequently

$$\frac{\beta}{\Gamma} = l \delta^{2k},$$

where l is a constant. Thus

$$\beta = (lm)^{\frac{1}{2}} \delta^k \delta_1'^{-\frac{1}{2}}, \quad \Gamma = \left(\frac{m}{l}\right)^{\frac{1}{2}} \delta^{-k} \delta_1'^{-\frac{1}{2}}.$$

When these values of β and Γ are substituted in the first of the foregoing equations, it becomes

$$\frac{1}{4} \frac{\delta_{11}'}{\delta_1'^2} - k^2 \frac{\delta_1'^2}{\delta^2} - \frac{\delta_{11}'}{\delta} = \frac{1}{lm} \frac{\delta_1'}{\delta^{2k}};$$

that is, an equation involving δ alone, the independent variable x_1' being given by the definition

$$a dx_1 = dx_1'.$$

Thus far, the second and the third equations have been used only in a single combination; so we must substitute the values of β and Γ in either of them. Now the value of β gives

$$\frac{\beta_1'}{\beta} = k \frac{\delta_1'}{\delta} - \frac{1}{2} \frac{\delta_{11}'}{\delta_1'},$$

and therefore

$$\frac{\beta_{11}'}{\beta} = (k^2 - k) \left(\frac{\delta_1'}{\delta}\right)^2 - \frac{1}{2} \frac{\delta_{111}'}{\delta_1'} + \frac{3}{4} \left(\frac{\delta_{11}'}{\delta_1'}\right)^2.$$

By the third equation,

$$\frac{\beta_{11}'}{\beta} = \frac{\delta_1'}{\delta} \frac{\Gamma_1'}{\Gamma} = -k \left(\frac{\delta_1'}{\delta}\right)^2 - \frac{1}{2} \frac{\delta_{11}'}{\delta};$$

and therefore

$$\frac{\delta_{111}'}{\delta_1'} - \frac{3}{2} \left(\frac{\delta_{11}'}{\delta_1'} \right)^2 = \frac{\delta_{11}'}{\delta_1} + 2k^2 \left(\frac{\delta_1'}{\delta} \right)^2.$$

(This equation for δ emerges from a similar treatment of the expression for Γ and substitution in the second equation.)

We now have two equations for δ alone, viz. the equation just obtained and the former equation, which can be taken in the form

$$\left(\frac{\delta_{11}'}{\delta_1'} \right)^2 - 4 \frac{\delta_{11}'}{\delta} = 4k^2 \left(\frac{\delta_1'}{\delta} \right)^2 + \frac{4}{lm} \frac{\delta_1'}{\delta^{2k}};$$

and they must be consistent with one another. Now the constant l occurs only in the latter equation, which is of the second order; hence the other equation, which is of the third order, must be consistent with

$$\frac{d}{dx_1'} \left\{ \left(\frac{1}{4} \frac{\delta_{11}'}{\delta_1'^3} - \frac{\delta_{11}'}{\delta \delta_1'} - k^2 \frac{\delta_1'}{\delta^2} \right) \delta^{2k} \right\} = 0,$$

that is, with

$$2k \frac{\delta_1'}{\delta} \left(\frac{1}{4} \frac{\delta_{11}'}{\delta_1'^3} - \frac{\delta_{11}'}{\delta \delta_1'} - k^2 \frac{\delta_1'}{\delta^2} \right) + \frac{1}{2} \frac{\delta_{111}'}{\delta_1'^3} - \frac{3}{4} \frac{\delta_{11}'}{\delta_1'^4} - \frac{\delta_{111}'}{\delta \delta_1'} + \frac{\delta_{11}'}{\delta \delta_1'^2} + \frac{\delta_{11}'}{\delta^2} - k^2 \frac{\delta_{11}'}{\delta^2} + 2k^2 \frac{\delta_1'^2}{\delta^3} = 0.$$

In this equation, substitute the former value of δ_{111}' , viz.

$$\frac{\delta_{111}'}{\delta_1'} = \frac{3}{2} \left(\frac{\delta_{11}'}{\delta_1'} \right)^2 + \frac{\delta_{11}'}{\delta} + 2k^2 \left(\frac{\delta_1'}{\delta} \right)^2;$$

after reduction, we find

$$\frac{1}{2} k \frac{\delta_{11}'}{\delta \delta_1'^2} - 2k \frac{\delta_{11}'}{\delta^2} - 2k^3 \frac{\delta_1'^2}{\delta^3} = 0.$$

But we had the relation

$$\frac{1}{4} \frac{\delta_{11}'}{\delta_1'^2} - k^2 \frac{\delta_1'^2}{\delta^2} - \frac{\delta_{11}'}{\delta} = \frac{1}{lm} \frac{\delta_1'}{\delta^{2k}};$$

consequently we must have

$$\frac{k}{lm} \frac{\delta_1'}{\delta^{1+2k}} = 0.$$

We cannot have m infinite, for then both β and Γ would be infinite; we cannot have l infinite, for then β would be infinite and Γ would be zero; and we exclude the case $\delta_1' = 0$, which now is trivial (and it would require that $m = 0$). Hence we must have

$$k = 0.$$

The analysis can be reversed, step by step; therefore, the two equations for δ are consistent with one another, if this condition is satisfied. There then is only a single equation for δ ; it is

$$\frac{\delta_{11}'}{\delta_1'^2} - 4 \frac{\delta_{11}'}{\delta} - 4 \frac{\delta_1'}{lm} = 0,$$

l and m being constants.

With the values of β and Γ that have been obtained, we now have

$$\frac{\beta_1'}{\beta} = \frac{\Gamma_1'}{\Gamma} = -\frac{1}{2} \frac{\delta_{11}'}{\delta_1'};$$

and so the equation

$$\frac{\beta_1'}{\beta} \frac{\Gamma_1'}{\Gamma} + \frac{\beta_1'}{\beta} \frac{\delta_1'}{\delta} + \frac{\Gamma_1'}{\Gamma} \frac{\delta_1'}{\delta} = \frac{n^2}{\beta^2}$$

becomes

$$\frac{1}{4} \frac{\delta_{11}^{\prime 2}}{\delta_1^{\prime 2}} - \frac{\delta_{11}'}{\delta} = \frac{n^2}{\beta^2} = \frac{n^2}{lm} \delta_1'.$$

Hence we have

$$n^2 = 1;$$

and without loss of generality we can take

$$n = 1.$$

Finally, the equation

$$M = 0,$$

being

$$\gamma_{12}' - \frac{\gamma_2'}{\beta} \beta_1' = 0,$$

becomes

$$\frac{\Gamma_1'}{\Gamma} - \frac{\beta_1'}{\beta} = 0,$$

which is satisfied by the values of β and Γ .

Hence all that remains to be done is to integrate

$$\frac{\delta_{11}^{\prime 2}}{\delta_1^{\prime 2}} - 4 \frac{\delta_{11}'}{\delta} - 4 \frac{\delta_1'}{lm} = 0.$$

For the purpose, take a new variable u such that

$$\delta_1' = u\delta^2;$$

then

$$\delta_{11}' = \left(\delta^2 \frac{du}{d\delta} + 2u\delta \right) u\delta^2,$$

and therefore

$$\frac{\delta_{11}'}{\delta_1'} - 2 \frac{\delta_1'}{\delta} = \delta^2 \frac{du}{d\delta}.$$

The equation becomes

$$\delta^2 \left(\frac{du}{d\delta} \right)^2 = 4 \left(u^2 + \frac{u}{lm} \right);$$

and therefore

$$u = \frac{1}{4lm} \left(\frac{\delta}{h} - \frac{h}{\delta} \right)^2,$$

where h is a constant of integration. Hence

$$\delta_1' = u\delta^2 = \frac{h^2}{4lm} \left(\frac{\delta^2}{h^2} - 1 \right)^2.$$

Now take a new variable r , such that

$$\delta^2 = h^2 - 2\frac{mh}{r};$$

then

$$\begin{aligned} adx_1 &= dx_1' = \frac{4lmd\delta}{h^2\left(\frac{\delta^2}{h^2} - 1\right)^2} \\ &= \frac{lh}{\delta} dr, \end{aligned}$$

so that

$$\begin{aligned} a^2 dx_1^2 &= \frac{l^2 h^2}{\delta^2} dr^2 \\ &= \frac{l^2 h^2}{h^2 - 2\frac{mh}{r}} dr^2. \end{aligned}$$

We have

$$\begin{aligned} \beta &= (lm)^{\frac{1}{2}} \delta_1'^{-\frac{1}{2}} = \pm lr, \\ \Gamma &= \left(\frac{m}{l}\right)^{\frac{1}{2}} \delta_1'^{-\frac{1}{2}} = \pm r, \end{aligned}$$

where the doubtful sign is immaterial.

Gathering the results together, we have

$$\begin{aligned} ds^2 &= -a^2 dx_1^2 - \beta^2 dx_2^2 - \gamma^2 dx_3^2 + \delta^2 dx_4^2 \\ &= -\frac{h^2 l^2}{h^2 - 2\frac{mh}{r}} dr^2 - l^2 r^2 dx_2^2 - r^2 \sin^2(x_2 + k') dx_3^2 + \left(h^2 - 2\frac{mh}{r}\right) dx_4^2 \end{aligned}$$

as the expression giving the interval.

44. The expression has been obtained merely by direct integration of the initial differential equations; but, as yet, no conditions have been imposed. Now let the requirement be imposed that, as r tends to become infinite (the equivalent of no gravitation field), the expression for ds^2 should degenerate to

$$-dx^2 - dy^2 - dz^2 + dt^2,$$

or, in polar coordinates,

$$-dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + dt^2.$$

Then we have

$$\begin{aligned} r &= r, \\ x_2 &= \theta, \\ x_3 &= \phi, \\ x_4 &= t, \\ h &= 1, \\ l^2 &= 1, \\ k' &= 0, \end{aligned}$$

the last being in no case a loss of generality. Hence finally, with the single condition that has been imposed, the solution of the initial differential equations gives the form

$$ds^2 = -\frac{1}{1-\frac{2m}{r}}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 + \left(1 - \frac{2m}{r}\right)dt^2.$$

It will be noted that, in deriving the form, there has been no necessity to use the property that the functional determinant should be taken equal to -1 ; it was a property taken by Einstein mainly for convenience.* Further, the condition of comparative symmetry as regards the polar coordinates, imposed by Einstein † and adopted by Schwarzschild, ‡ is superfluous; it is actually, but implicitly, carried by the condition as to the form of ds^2 in the degenerate case. All that has been required is the set of five fundamental equations applied to the type of ds^2 that was assumed, together with the single condition.

45. In passing, it may be pointed out that, in a non-gravitation field, the seven non-evanescent equations

$$A=0, \quad B=0, \quad C=0, \quad D=0, \quad E=0, \quad F=0, \quad M=0$$

are satisfied. Einstein's mathematical definition (§ 41) of the gravitation field rests on the five non-evanescent equations

$$B_{11}=0, \quad B_{22}=0, \quad B_{33}=0, \quad B_{44}=0, \quad B_{12}=0.$$

It is, however, easy to verify that the coefficients of the foregoing form, as deduced from these five equations, actually satisfy the six equations

$$B_{22}=0, \quad B_{44}=0, \quad B_{12}=0, \\ \frac{B}{b} - \frac{C}{c} = 0, \quad \frac{E}{a} - \frac{C}{d} = 0, \quad \frac{F}{a} - \frac{B}{d} = 0;$$

so that it would appear as if the imposition of the limiting condition, under degeneration, has the effect of bringing the number of the equations of the gravitation field one unit nearer the number for the non-gravitation field.

* *Ann. d. Physik*, Bd. xlix, pp. 789, 801.

† *Sitzungsb. Berlin* (1915), p. 801.

‡ *Ib.* (1916), p. 192.

XIII.—The Magnetic Character of the Quantum.*

By H. Stanley Allen, M.A., D.Sc.

(MS. received May 8, 1922. Read May 8, 1922.)

§ 1. Calamoids—the four-dimensional tubes of electromagnetic force introduced by Professor Whittaker in a communication † in November 1921—are not referred to in his work “On the Quantum Mechanism in the Atom” (see pp. 129-142), in which he is content to deal with three dimensions; but there is at least one point of contact between the two papers. The four-dimensional tubes include as particular cases both kinds of Faraday tubes (electric and magnetic), and magnetic forces are treated as on an equality with electric forces. In seeking for a mechanism to elucidate the quantum, a magnetic model is employed; and although Professor Whittaker is careful not to insist on the magnetic structure suggested, careful examination of his paper shows that some form of magnetic element is required both for the absorption of energy by the atom from the electron and for the transformation of the absorbed energy into the radiant form. As I have been for some time a supporter of the view which relates the quantum to a magnetic element, it is gratifying to me to find here such powerful support. In the paper ‡ which I read before the Society in November 1920, I advanced the opinion that the quantum theory involves the existence of discrete tubes of magnetic induction. Applying the same ideas to the calamoids of Professor Whittaker, we are led to the view that the world of events is not a continuum, but is built up of individual tubes of force of this more complicated description.

§ 2. At present, however, we are concerned only with three dimensions, and, more particularly, with the mechanism within the atom which may be regarded as the origin of the quantum magnetic tubes. Such a structure may be called a *magneton*, using that term in its widest sense. The word seems to have been employed first by Dr L. A. Bauer§ in a paper read at a meeting of the Philosophical Society of Washington on May 7, 1910. “The corpuscles in magnetism might be atomic systems

* The substance of the present communication formed a contribution to the discussion on quantum theory and atomic structure following the reading of Professor Whittaker's paper “On the Quantum Mechanism in the Atom” on May 8, 1922, pp. 129-142.

† *Proc. Roy. Soc. Edin.*, vol. xlii, p. 1, 1921.

‡ *Ibid.*, vol. xli, p. 34, 1921.

§ See *Nature*, vol. c, p. 227, 1917, for the history of the word “magneton.”

in which the electron is revolving about an inner nucleus consisting, for example, of a positive ion. . . . Since the system creates an atomic magnetic field the axis of which passes through the centre of rotation of the electron and perpendicular to the plane of rotation, the speaker suggested calling such systems ‘magnetons.’”* The word was used independently by Professor Gans and Professor Weiss, but it should be observed that the magneton of Weiss is an empirical and not a mechanistic deduction, and its existence is not demonstrated with certainty.

In 1913 S. B. M'Laren † discussed the properties of a magneton, regarded as an inner limiting surface of the æther, formed like an anchor ring. Tubes of electric induction terminating on its surface give it an electric charge; tubes of magnetic induction linked through its aperture make it a permanent magnet. This form of magneton seems to be specially adapted to the four-dimensional analysis of Professor Whittaker. In 1915 A. L. Parson ‡ put forward a magneton theory of the structure of the atom, in which the magneton, or ring electron, was looked upon as a thin circular anchor ring of negative electricity rotating about its axis with large velocity. Assuming that the angular momentum of the ring electron (charge e , mass m) has the value $h/2\pi$ appropriate to the quantum theory, its magnetic moment, $\frac{1}{2}(e/m)(h/2\pi)$, is found to be 9.232×10^{-21} E.M.U.§ The structure in the atom postulated by Sir Alfred Ewing || in his recent models of ferromagnetic induction is obviously composed of magnetons, and it is interesting to notice that the magnetic moment of the rotatable Weber element in each atom of iron is found to be nearly 2×10^{-20} E.M.U., or about twice the above value.

The mechanism imagined by Professor Whittaker for the absorption of energy from the electron must also be described as a magneton, if we use that term in the wide sense advocated above. It is true that at the end of § 3 of his paper it is suggested that the model, having served its purpose, may be allowed to drop out of sight, and that the “magnetic structure” which suggested the equations need not be insisted on. But in view of the fact that Professor Whittaker is prepared to retain the electron, and in four dimensions is prepared to place magnetic forces on an equality with electric forces, it must surely be necessary to retain in three dimensions

* *Science*, vol. xxxi, p. 920, June 10, 1910.

† S. B. M'Laren, *British Association Report*, 1913; *Phil. Mag.*, vol. xxvi, p. 800, 1913; *Nature*, vol. xcii, p. 165, 1913.

‡ A. L. Parson, *Smithsonian Misc. Coll.*, vol. lxxv, No. 11, 1915.

§ H. S. Allen, *Phil. Mag.*, vol. xli, p. 119, 1921.

|| Sir J. A. Ewing, *Proc. Roy. Soc.*, vol. c, p. 449, 1922; *Phil. Mag.*, vol. xliii, p. 493, 1922; *Proc. Roy. Soc. Edin.*, vol. xlii, p. 97, 1922.

the conception of the magneton, which in some form or other seems to be essential for the explanation of magnetic phenomena.* Again, the use of inductance in connection with the frequency of the Hertzian oscillator in § 4 implies the existence of tubes of magnetic induction within the atom, and we are consequently led once more to postulate the existence of a magneton.

§ 3. No one has made more brilliant use of models for the elucidation of physical phenomena than Clerk Maxwell; at the same time, no one has understood more clearly their limitations. Most physicists would agree that in the present stage of scientific development it is still necessary for them to work with a model, even if the particular model employed should require modification or should ultimately be discarded.

It is as a rule an easy matter to construct a model to represent a limited number of physical facts. Difficulties are apt to multiply when we endeavour to make our model fit in with other groups of phenomena. One of the chief tests of a satisfactory physical theory is its comprehensiveness. Perhaps the most important fact which requires explanation in connection with atomic structure is the existence of quantum "levels," now firmly established for the outer shell of the atom, and true, if the recent work of Ellis on β and γ rays from radioactive elements be confirmed, for a massive nucleus also. If Professor Whittaker's or any other model can give us a clear mental picture of these levels, and with them explain Bohr's frequency condition

$$h\nu = W_2 - W_1,$$

where W is the energy associated with a particular level, we shall have gone a long way towards establishing an enduring theory of atomic phenomena.

It is well known that Rutherford's theory of the atom, in which there is a positive nucleus of extremely small dimensions, arose in connection with the scattering of α particles, the electrostatic field of the nucleus being responsible for the observed wide-angle scattering. It would be interesting to know whether the mechanism imagined by Professor Whittaker would account for the facts without necessitating such a concentrated nucleus. In a number of papers in the *Philosophical Magazine* † I have discussed the possibility of the core of the atom being of size sufficient to produce appreciable magnetic forces, but the objection to such a model has always

* The difficulty of explaining magnetic effects in terms of the electron theory was clearly pointed out by M'Laren at the British Association meeting of 1913. An abstract of his paper appears in the *Annual Report* for that year.

† H. S. Allen, *Phil. Mag.*, vol. xxix, p. 40, p. 140, p. 714, 1915.

been that it did not yield the inverse square law of electrostatic attraction at extremely small distances. If, as Professor Whittaker infers, the electric field about the atom is not permanent, but is evoked by the approach of an electron, the same should hold with regard to an approaching α particle, and a reconsideration of the prevailing views seems to be demanded.

§ 4. If I am right in attaching physical significance to tubes of magnetic induction, it becomes important to trace the distribution of the tubes in such a case as is presented by Professor Whittaker's magneton. Regarding the model as a wheel with a S-pole at the hub and N-poles distributed round the rim, the lines of force starting from the rim would curve round and enter the wheel at its centre. The number, N , of the tubes is given by $N=4\pi M$, where M is the total pole strength (of one sign) for the different bar magnets which form the spokes of the wheel. Now, in equation (9) Professor Whittaker obtains the expression

$$A\omega = 2eM,$$

that is to say, the angular momentum of the wheel is $2eM$ or $\frac{1}{2\pi}Ne$. But according to the quantum theory angular momentum can be expressed as $nh/2\pi$, where n is an integer. So we find at once $N=n(h/e)$, or *the total number of magnetic tubes associated with the magneton is an integral number of times the fundamental quantum tube defined by h/e* , in exact agreement with the results I have obtained previously in other cases.*

It may be noted that the angular momentum is independent of the size of the magneton.

The general arrangement of the lines of magnetic force in the model is not unlike that which would occur if two thin anchor rings, representing the magnetons either of M'Laren or of Parson, were placed near together with their planes parallel and having a common axis. Thus, if we think of Professor Whittaker's magneton as a bicycle wheel, these rings would be represented by the two beaded edges of the rim. But to obtain the required distribution of the lines of magnetic force the rings must be placed so that the magnetic force between them is one of *repulsion*. It would of course be possible to introduce an electrostatic attraction to balance this repulsion by supposing one ring to be charged positively and the other negatively; and it may be worth noticing that such an arrangement is not unlike that pictured by Professor Whittaker in the latter part of his paper.

[*Added May 13, 1922.*—I am indebted to Professor Whittaker for pointing out in the discussion on May 8 that the two ring electrons,

* H. S. Allen, *Phil. Mag.*, vol. xlii, p. 523, 1921.

with opposed currents, described above, would be equivalent to a magnetic shell forming the curved surface of a cylinder having its edges coincident with the two rings. For suppose the first ring electron is replaced by a magnetic shell in the form of a flat circular disk. The second electron may be regarded as equivalent to a magnetic shell of any chosen form provided only the edge of the shell coincide with the contour of the ring. Let this shell be constructed so as to cover the first shell and also the cylindrical surface between the two rings. Then, as the currents are opposed, the two flat portions of the shells neutralise one another, and we are left with a magnetic shell covering the cylindrical surface between the two rings.

The same result may be obtained very simply by imagining the two rings to be connected by a conductor carrying equal and opposite currents of the same strength as the current in each of the ring electrons.]

If it were found on detailed examination that such a model would absorb energy from an electron in the same fashion as Professor Whittaker's magneton, it would possess certain advantages as compared with the latter. There would be no need to introduce bar magnets, and no difficulty would arise with regard to the passage of the electron through the matter of the bars—there is nothing to prevent free movement through the apertures of the rings. Further, such an arrangement would possess both capacity and inductance; and instead of the charging and discharging of a condenser which is employed as the type of an oscillating system, we could deal with the oscillations of the two rings as they moved nearer together and further apart. Professor Whittaker recognises the difficulty of picturing the inductance of his assumed Hertzian oscillator, but he does not say how the differential equation for such an oscillator can be applied if we retain the conception of an electron as an indivisible unit of electricity, and it is to be noticed that the electric separation in his condenser "is precisely a separation of two electronic charges e and $-e$." If, however, the plates of the condenser are moving with reference to one another, there is no need to consider variations in the value of the charge. This movement of the electrons as such seems indeed to be indicated in his paper in several places.

§ 5. At first sight Professor Whittaker's choice of h/π as the value of the natural constant $e^2 \sqrt{(L/C)}$ seemed somewhat arbitrary, and difficult to justify apart from the fact that it leads to the desired result $h\nu = U$. I find, however, on further examination that it is in complete agreement with the generalised form of the quantum theory proposed by Wilson and Sommerfeld and now commonly accepted. In the present case we have

only one degree of freedom to consider, and the quantum condition for Action is

$$2 \int T dt = nh,$$

where T represents the kinetic energy, n is an integer, and the integration is to be extended over one complete period. We may identify the kinetic energy with the electromagnetic energy $\frac{1}{2}Li^2$, where i is the instantaneous value of the discharging current. If we call q the instantaneous value of the charge on one plate of the condenser, we may write

$$q = e \cos(t/\sqrt{LC}),$$

and

$$i = \frac{dq}{dt} = -\frac{e}{\sqrt{LC}} \sin \frac{t}{\sqrt{LC}}.$$

The Action is

$$L \int i^2 dt = \frac{e^2}{C} \int \sin^2 \frac{t}{\sqrt{LC}} dt = \frac{e^2}{C} \times \pi \sqrt{LC} = e^2 \sqrt{\frac{L}{C}} \times \pi.$$

Thus when the quantum number is unity ($n=1$) we find

$$e^2 \sqrt{\frac{L}{C}} = \frac{h}{\pi},$$

which is precisely Professor's Whittaker's relation.

§ 6. One other point in connection with the relation between L and C may be mentioned. It may be written in the form

$$\frac{2\pi e^2}{hc} = \frac{2}{c} \sqrt{\frac{C}{L}},$$

where c is the velocity of light.

Now, the quantity on the left of the equation is a pure number, which I have discussed in connection with the relation of Lewis and Adams,* and denoted previously by q . It occurs repeatedly in Sommerfeld's theory of the fine structure of spectrum lines, where it is denoted by a . So we may write

$$\sqrt{\frac{C}{L}} = \frac{1}{2} ac,$$

which may suggest an interpretation of this important natural constant a .

§ 7. Finally, I think it is necessary to insist once more on the essential distinction between the new mechanics and the old. When Professor Whittaker introduces a natural constant of Action, he is departing from

* Lewis and Adams, *Phys. Rev.*, vol. iii, p. 92, 1914. H. S. Allen, *Proc. Phys. Soc.*, vol. xxvii, p. 425, 1915.

classical theory and postulating a vibrator which does not lead to the law of equipartition of energy.

Jeans, in his *Report on Radiation and the Quantum Theory*, says (p. 84):—

“The new mechanics must differ from the old even as regards the motion of free electrons. For this reason it seems useless to attempt to explain away the conflict between the radiation laws and the classical mechanics by ingeniously devised special models of atoms, or special detailed mechanisms of emission of radiation, which might seem, while obeying the classical laws, to give something approximating to Planck’s law.” “Any such attempt would first have to surmount the difficulty that any system whatever, if it obeys the classical laws, must also in its state of thermodynamical equilibrium obey the law of equipartition of energy, which is known in turn to lead to Rayleigh’s radiation formula. And if this difficulty could be turned, as, for instance, by postulating a final state which was not one of thermodynamical equilibrium, the question of why all possible mechanisms of radiation lead to Planck’s law, as they certainly appear to do, would remain untouched. And finally, if single electrons do not obey the classical laws, there would seem to be little gain in proving, if it could be proved, that complicated structures might possibly obey them.”

Very similar was the position taken up by Poincaré, who put forward a mathematical argument to show that not merely does Planck’s radiation law involve the hypothesis of discontinuities, but that the existence of quanta is necessitated by any law to which that of Planck might be regarded as a first approximation. Poincaré’s proof of the necessity of Planck’s hypothesis of quanta depends on the use of Fourier’s integral theorem to invert a particular infinite integral. It has been pointed out by Planck,* and also by Fowler,† that there is a gap in Poincaré’s argument due to the fact that the functions actually involved are such that Fourier’s integral theorem does not apply at all. On this ground Planck criticises Poincaré’s conclusion that the second form of the Quantum Theory—discontinuous emission and continuous absorption—is inadmissible, and suggests that it may be due to this gap in the mathematics that Poincaré’s argument distinguishes between the first and second hypotheses, allowing the first to be necessary and the second impossible. Fowler, however, has shown that it is possible to make a simple extension of Fourier’s integral theorem so that the function required is properly catered for, and concludes that, if the groundwork of Poincaré’s argument is admitted to be correct,

* Planck, *Acta Mathematica*, vol. xxxviii, p. 387, 1921.

† R. H. Fowler, *Proc. Roy. Soc.*, vol. xcix, p. 462, 1921.

then it appears to follow inevitably that the energy must change discontinuously both in emission and absorption.

My own position with regard to the Quantum Theory is clearly stated in the paper which I read before the Society in November 1920. I put forward the view that the "quantum" is itself essentially magnetic, being conditioned by the existence of discrete tubes of magnetic induction. "It must not, however, be supposed that the view 'reconciles' the quantum theory and classical dynamics. My object is rather to seek to understand more clearly the nature of the quantum, whilst accepting the conclusion that some modification of the old theories is inevitable."

Professor Whittaker's work is to be welcomed as a valuable contribution towards the attainment of that clearer understanding of the quantum which all physicists must desire.

(Issued separately July 4, 1922.)

XIV.—Note on Professor Whittaker's Quantum Mechanism in the Atom.* By Dr R. A. Houstoun.

(Read May 8, 1922. MS. received May 29, 1922.)

It seems desirable to check Professor Whittaker's model numerically; algebraical expressions when evaluated sometimes have the wrong order of magnitude.

Let us suppose that initially the magnet system is at rest, and that the electron starts from the origin with velocity u . Then, referring to Professor Whittaker's paper, p. 133, we have, instead of (5),

$$\Lambda\dot{\psi} - \frac{Mex}{(a^2 + x^2)^{\frac{3}{2}}} = 0.$$

This gives by substitution in (6)

$$\frac{1}{2\Lambda} \frac{M^2 e^2 x^2}{(a^2 + x^2)} + \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m u^2 \quad \dots \dots \dots (1)$$

The electron consequently oscillates about the origin. The maximum energy of oscillation possible is obtained by putting $\dot{x} = 0$ at $x = \alpha$, and must be identified with the quantum. Thus

$$h\nu = \frac{M^2 e^2}{2\Lambda} \quad \dots \dots \dots (2)$$

This is one quarter of the energy lost by the electron passing through the system from infinity. The oscillations are not isochronous, unless x^2 can be neglected in comparison with a^2 in the denominator in (1). Let us suppose this is so. Then

$$\frac{M^2 e^2 x^2}{2\Lambda a^2} + \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m u^2.$$

This is the energy equation corresponding to vibrations of frequency

$$\nu = \frac{Me}{2\pi a \sqrt{(m\Lambda)}} \quad \dots \dots \dots (3)$$

Squaring (3) and dividing into (2),

$$\frac{h}{\nu} = 2\pi^2 a^2 m,$$

or

$$a^2 = \frac{h}{2\pi^2 m \nu} \quad \dots \dots \dots (4)$$

* Containing the substance of remarks made in the discussion on Professor Whittaker's paper, pp. 129-142.

This enables us to calculate the radius of the magnetic orbit. If we assume, (i) that ν is the frequency of Na yellow, and (ii) that it is the frequency corresponding to 1 Ångström unit in the X-ray region, we obtain respectively $a = 2.7 \times 10^{-8}$ cm. and $a = 3.5 \times 10^{-19}$ cm. As the diameter of a hydrogen molecule is about 2×10^{-8} cm., the first value is on the high side. But it is very satisfactory considering the simple nature of the assumptions made; it will doubtless be possible to devise a more elaborate rotating system of smaller dimensions which will at the same time make the vibrations isochronous throughout a wider range.

It will be noticed that according to equation (4) the radius of the magnetic orbit varies inversely as the root of the frequency. This is in agreement with the view advanced towards the end of the paper, namely, that the oscillators are always of the same type but differ from one another in scale.

If ν be eliminated between (2) and (3),

$$h = \frac{M^2 e^2}{2A} \frac{2\pi a}{Me} \sqrt{(mA)}$$

$$= \mu e \times \text{numerical factor.}$$

This numerical factor becomes π , if the rotating system is a single magnet of moment of inertia ma^2 . Thus Planck's constant comes out as the product of the unit electric charge and a unit magnetic quantity; this has, of course, the correct dimensions. It consequently follows that there must be some truth in the view that there is a unit tube of induction.

Hitherto in dealing with atomic models we have always assumed that the play was between kinetic energy of moving electrons and electrostatic potential energy. Apparently the latter must now be replaced by the kinetic energy of moving magnets; potential energy disappears from the scheme altogether. At the same time the reciprocity which exists between electric and magnetic quantities in the electromagnetic wave must be extended to atomic structure.

The new quantum mechanism is of the highest importance both for the vistas it opens out and the salutary influence it will exert on some of the more speculative theories of radiation.

XV.—The Importance of Professor Whittaker's Atomic Model
and of other Atomic Models.* By Professor W. Peddie.

(Read May 8, 1922. MS. received May 29, 1922.)

THE atom, even in its simplest form, must possess a most complicated structure. The aim of physical science is to find a hypothetical structure, and mode of its functioning, which will give a good account of atomic phenomena. When the phenomena are complicated, and only very partially known, hypothetical investigations are most uncertain and provisional, and make little attempt to work otherwise than by analogy or illustration. In such cases help is often obtained through the discovery of a general law which unifies a wide range of phenomena, and which may be followed by many diverse structures in their activities. Well-known examples are found in the law of conservation of energy, the law of least action, and, recently, the law of quantum action.

Such a law being found, it may be possible to employ a simple mechanism which, in subjection to that law, works according to the manner of nature throughout a wide but limited range. For application in still wider ranges increased complexity in the model is required; but great initial complexity may be avoided in consequence of the discovery of a new general law which, with a little increase in complexity of the model, covers the wider field. Thus, in atomic theory, the smooth, hard, spherical, perfectly elastic atom, with Newton's laws of motion super-added, accounted for much. And the slightly more complicated hydrogen atom of Bohr, consisting of a single electron circulating around a nucleus of positive electricity, when postulated to be subject to the new quantum laws, gives a wonderful revelation of applicability to the extraordinarily complicated phenomena of atomic radiation.

But these additional laws are mere promissory notes which work in lieu of the real valid coinage lying behind them. The quantum postulates are analogous to the motional laws of clock hands which, in co-operation with the simple two-hand structure, enable us to dispense with actual knowledge of the internal mechanical system. But that complicated mechanism exists; and every little advancement in knowledge of the mechanism enables us to pass from the "how" of an unexplained law to

* Containing the substance of remarks made as Chairman of the Meeting at which Professor Whittaker's paper (pp. 129-142) was discussed.

the "why" of it, until, when the final transference is complete, there ceases to be any difference between Why and How.

Until fuller certainty comes, it is well that every hypothetically possible line of advance should be explored. It is in this sense, and not in the sense of singleness of possibility, that the importance of Professor Whittaker's suggestion regarding quantum action, together with the suggestion of Sir Alfred Ewing regarding atomic magnetism, upon which it is based, really lies. Thus about ten years ago I suggested a combined electrostatic and electromagnetic atomic structure which might give quantum action of the type considered by Professor Whittaker and quantum magnetic moment of the type considered by Sir Alfred Ewing as well as radiational quantum action. And about the same time Professor Conway described a different mechanism for radiational quantum emission. I have also shown that known magneto-crystalline and ferromagnetic properties can be deduced from a space-lattice distribution of molecular magnets without the small pole clearance necessitated by Sir Alfred Ewing's scheme. Indeed, the chief importance of his scheme lies in the fact that it constitutes a cubic arrangement of molecular magnets, effectively possessed of variable magnetic moment, and which have their linear dimensions small relatively to their central distances. It is only in the light of many suggestions that the ultimate solution may be found. In no case can a simply constituted atom of the electronic type be sufficient, although it is so wonderfully sufficient when the complexity is eluded by the quantum postulates.

It seems to me that Professor Whittaker's model only furnishes perfectly elastic electron-atom collisions by postulating the prevention of radiation, just as the quantum-dictum of no radiation in steady electronic orbital motion does. In other words, it leaves part of the essential mechanism undescribed. And it seems also that there may be difficulty in connection with the acceleration of an atom containing discrete electric charges, and also discrete distributions of moving magnetism, in consequence of the mutual influence of these charges and magnetic distributions through the medium of the ether; so that Galilei-wise motion may conceivably be impossible.

But such difficulties, should they exist, in no way detract from the value of a suggestion which for the first time exhibits a mechanism capable of giving unreversed action upon an electron in its motion towards, through, and away from an atom.

XVI.—On the Faraday-Tube Theory of Electro-Magnetism. By the late William Gordon Brown. Communicated by Dr C. G. KNOTT, F.R.S., General Secretary, along with a Biographical Note of the Author.

(Read January 9, 1922.)

1. THE method of describing a field of force by means of lines or tubes of induction, which originated with Faraday, was given a quantitative form by Sir J. J. Thomson,* and further discussed by N. Campbell in his book *Modern Electrical Theory*. Since Maxwell himself looked on his work as a mathematical theory of Faraday's lines of force, one is tempted to examine the original physical theory for hints as to the modification of the Maxwellian theory to suit certain modern requirements.

What is attempted in the present paper is a reconstruction of the quantitative theory of Faraday tubes on a dynamical basis from the minimum of hypotheses: partly to enable the electro-magnetic consequences of altering the Principle of Action to be estimated, and partly to suggest plausible directions for modification of the electro-magnetic relations themselves. It will incidentally be shown that the stress which may be supposed to act in the electro-magnetic field requires certain modifications if the theory of lines of force is adopted.

2. The first assumption required is as follows:—A tube of induction, or Faraday tube, may be defined as a continuous line having certain physical properties. Any tube may either be a closed curve, or its ends be connected to a positive and a negative electric particle respectively; the positive direction will then be from the positive to the negative particle. It would be superfluous at present to specify any further properties of the electric particles.

The tubes at any point may be divided into sets distinguished by each set having a common direction and a common velocity of translation.

In what follows the vectorial notation of Heaviside is employed,† and

* *Recent Researches*, chap. i; *Electricity and Matter*, chap. i.

† [Heaviside's vector notation is a modification of Hamilton's quaternion notation, the main difference being that the quaternion product of two vectors \mathbf{AB} is not used in Hamilton's sense but is used to mean the scalar of the complete product—that is, Heaviside's \mathbf{AB} is equivalent to Hamilton's $-\mathbf{SAB}$, and may be defined geometrically as equal to $AB \cos \theta$, where A , B are the lengths of \mathbf{A} , \mathbf{B} , and θ the angle between them. As in other non-associative vector algebras, the square of a vector is equal to the square of its length; in quaternions $\mathbf{A}^2 = -A^2$. The notation introduced by Gordon Brown in equations (9), (10), etc., has been suggested by others but generally discarded. Burali-Forti and Marcolongo, however, make it a feature of their system of vector analysis. As a notation it is misleading; as an operator it is inferior to the quaternion ∇ .—C. G. K.]

electrical quantities are measured in rational units. Let the density of the tubes of the m th set and their direction, at any point, be represented by the magnitude and direction of the vector \mathbf{d}_m ; then the number of tubes of that set passing through unit area normal to the unit vector \mathbf{N} will be $\mathbf{N}\mathbf{d}_m$.

Let

$$\mathbf{D} = \sum \mathbf{d}_m, \quad \dots \quad (1)$$

the summation including all the sets present at the point; then the total of all sets passing through the same unit area is

$$\sum \mathbf{N}\mathbf{d}_m = \mathbf{N}\mathbf{D},$$

where tubes passing through the area in the direction of \mathbf{N} are reckoned positive, and the algebraic total is intended. Thus \mathbf{D} represents vectorially the total flux of tubes; it is to be identified with the \mathbf{D} of Heaviside, and, except for the question of units, with the (f, g, h) of Maxwell.

Let \mathbf{q}_m be the (vector) velocity of the tubes of the m th set at the point in question, and let

$$\mathbf{H} = \sum \mathbf{v}\mathbf{q}_m\mathbf{d}_m \quad \dots \quad (2)$$

The quantity thus defined will be shown to have the properties of magnetic force.

This completes the geometrical and kinematic specification of the properties of the tubes. It is not difficult to see that if we define the charge of an electric particle as the number of tubes *leaving* it, in the sense that the direction of the tubes at a positive particle is outwards, then the density of electric charge will be given by

$$\rho = \text{div } \mathbf{D} \quad \dots \quad (3)$$

If we take the curl of (2) and expand the right member fully, interpreting the terms kinematically, we obtain the equation

$$\begin{aligned} \text{curl } \mathbf{H} &= \dot{\mathbf{D}} + \sum \mathbf{q}_m \text{div } \mathbf{d}_m \\ &= \dot{\mathbf{D}} + \mathbf{u}\rho, \quad \dots \quad (4) \end{aligned}$$

where $\dot{\mathbf{D}}$ is the time rate of change of \mathbf{D} at a *fixed point*, and \mathbf{u} is the mean velocity of translation of the electric particles calculated so as to make $\mathbf{u}\rho$ the convection current.

3. The second assumption made is dynamical. Let us write

$$\mathbf{E} = \frac{\mathbf{D}}{\mathbf{K}} \quad \dots \quad (5)$$

$$\mathbf{B} = \mu\mathbf{H}, \quad \dots \quad (6)$$

where μ and \mathbf{K} are constants, and \mathbf{E} and \mathbf{B} are new vectors, the electric intensity and magnetic induction.

Then we assume that the volume densities of kinetic and potential energy are given by

$$U = \frac{1}{2} \mathbf{E} \mathbf{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$T = \frac{1}{2} \mathbf{H} \mathbf{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The meaning attached to the above quantities is that if we write

$$L = \iiint (T - U) dv,$$

where the volume integral is extended throughout all space, then L may be used as the Lagrangian function in equations of motion of the usual form. For the sake of brevity, vectorial general coordinates will be employed. In order to preserve the form of the equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0,$$

it is sufficient to write, in the case of a vector coordinate \mathbf{r} (equivalent to the three scalar coordinates x, y, z),

$$\begin{aligned} \frac{\partial}{\partial \mathbf{r}} = \nabla &= i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \dot{\mathbf{r}}} &= i \frac{\partial}{\partial \dot{x}} + j \frac{\partial}{\partial \dot{y}} + k \frac{\partial}{\partial \dot{z}} \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This notation in vectorial analysis is of course not generally applicable but is convenient for the purposes of the present paper. The general results of differentiation which will be required are

$$\frac{\partial}{\partial \mathbf{s}} \mathbf{a} \mathbf{s} = \mathbf{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\frac{\partial}{\partial \mathbf{s}} \mathbf{s} \psi \mathbf{s} = 2 \psi \mathbf{s}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where \mathbf{s} is any vector variable, \mathbf{a} is a constant vector, and ψ is a constant self-conjugate linear and vector operator.

4. To define the general coordinates, let all tubes at a given moment be divided into small unit lengths; and let \mathbf{r} be the vector from a fixed origin to the centre of one such unit segment, which forms part of a tube of the m th set, then the Lagrangian equation corresponding to \mathbf{r} will be

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}} - \frac{\partial L}{\partial \mathbf{r}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Now, when a unit length of a tube of the m th set is added to, or removed from, an element of volume, the increase or decrease of the whole Lagrangian function due to this element will be

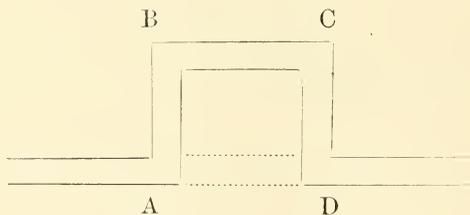
$$\begin{aligned} \delta L &= \frac{\partial L}{\partial \mathbf{d}_m} \delta \mathbf{d}_m \\ &= - \delta \mathbf{d}_m (\mathbf{E} + V \mathbf{q}_m \mathbf{B}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (13) \end{aligned}$$

for

$$\begin{aligned} \frac{\partial}{\partial \mathbf{d}_m}(\mathbf{T} - \mathbf{U}) &= \frac{\partial}{\partial \mathbf{d}_m} \left[\frac{1}{2} \mu (\Sigma \mathbf{V} \mathbf{q}_m \mathbf{d}_m)^2 - \frac{1}{2} \mathbf{D}^2 / \mathbf{K} \right] \\ &= \frac{\partial}{\partial \mathbf{d}_m} \left[\Sigma \Sigma \frac{1}{2} \mu (-\mathbf{d}_m \mathbf{V} \mathbf{q}_m \mathbf{V} \mathbf{q}_s \mathbf{d}_s) - \frac{1}{2} (\Sigma \mathbf{d}_m)^2 / (\mathbf{K}) \right] \\ &= -(\mathbf{E} + \mathbf{V} \mathbf{q}_m \mathbf{B}), \dots \dots \dots (14) \end{aligned}$$

where the summations include all values of the suffixes n, s , the differentiation of terms such as $(-\mathbf{d}_m \mathbf{V} \mathbf{q}_m \mathbf{V} \mathbf{q}_m \mathbf{d}_m)$ being performed by means of (11), since $(-\mathbf{V} \mathbf{q}_m \mathbf{V} \mathbf{q}_m)$ is a self-conjugate operator; and that of cross-products, such as:—

$(-\mathbf{d}_m \mathbf{V} \mathbf{q}_m \mathbf{V} \mathbf{q}_s \mathbf{d}_s)$ by means of (10), writing $\mathbf{a} = -\mathbf{V} \mathbf{q}_m \mathbf{V} \mathbf{q}_s \mathbf{d}_s$.



Thus, if in the figure the unit segment is removed from the position AD (at which (14) has the value $-(\mathbf{E} + \mathbf{V} \mathbf{q}_m \mathbf{B})$) to the parallel position BC (at which (14) has the value $-(1 + \delta \mathbf{r} \nabla) \cdot (\mathbf{E} + \mathbf{V} \mathbf{q}_m \mathbf{B})$, $AB = \delta \mathbf{r}$), then the total increase in L is given by

$$\delta L = -\delta \mathbf{r} \nabla \cdot \delta \mathbf{d}_m (\mathbf{E} + \mathbf{V} \mathbf{q}_m \mathbf{B}).$$

It will now be convenient to suppose (as we may without loss of generality) that the m th set consists of but one tube, so that $\delta \mathbf{d}_m = \mathbf{d}_m$ and is in fact a unit vector.

Then

$$\delta_1 L = -\delta \mathbf{r} \nabla \cdot \mathbf{d}_m (\mathbf{E} + \mathbf{V} \mathbf{q}_m \mathbf{B}), \dots \dots \dots (15)$$

and in applying the axial differentiator $\delta \mathbf{r} \nabla$ we must remember that neither \mathbf{d}_m nor \mathbf{q}_m as they occur explicitly are to be considered variable.

But to preserve the continuity of the tube we require to introduce the segments AB, CD, as shown in the figure, so that, again applying (13), we have the change of L due to this cause.

$$\delta_2 L = \mathbf{d}_m \nabla \cdot \delta \mathbf{r} (\mathbf{E} + \mathbf{V} \mathbf{q}_m \mathbf{B}), \dots \dots \dots (16)$$

in which \mathbf{q}_m is variable (but not \mathbf{d}_m).

Hence

$$\delta L = \delta_1 L + \delta_2 L = \delta \mathbf{r} [\mathbf{d}_m \nabla \cdot (\mathbf{E} + \mathbf{V} \mathbf{q}_m \mathbf{B}) - \nabla \cdot \mathbf{d}_m (\mathbf{E} + \mathbf{V} \mathbf{q}_m \mathbf{B})],$$

\mathbf{q}_m varying in the first term only, and \mathbf{d}_m not at all, and finally

$$\frac{\partial L}{\partial \mathbf{r}} = \mathbf{d}_m \nabla \cdot (\mathbf{E} + \mathbf{V} \mathbf{q}_m \mathbf{B}) - \nabla \cdot \mathbf{d}_m (\mathbf{E} + \mathbf{V} \mathbf{q}_m \mathbf{B}) \quad (17)$$

with the same convention.

In calculating the momentum term $\frac{\partial L}{\partial \dot{\mathbf{r}}}$ we have $\dot{\mathbf{r}} = \mathbf{q}_m$. Then by the method employed above in calculating (14), since T is symmetrical in \mathbf{q}_m and \mathbf{d}_m ,

$$\frac{\partial T}{\partial \mathbf{q}_m} = \mathbf{V} \mathbf{d}_m \mathbf{B} \quad (18)$$

This will be the value of $\frac{\partial L}{\partial \dot{\mathbf{r}}}$ when \mathbf{d}_m is a unit length of tube, but in performing the complete differentiation to time in (7) we must remember that any length of tube will in general be continually varying in direction and magnitude. It is clear that

$$\frac{d}{dt} \mathbf{d}_m = \mathbf{d}_m \nabla \cdot \mathbf{q}_m, \quad (19)$$

since the rate of change of a segment of a straight line, as AD in the figure, will be the relative velocity of its ends (vectorially); while, of course, if \mathbf{q}_m expresses the velocity of any point of the tube, as A, the velocity at D will be $(1 + \overline{AD} \nabla \cdot) \mathbf{q}_m$, where \overline{AD} is the vector element.

Thus

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}} &= \frac{d}{dt} \mathbf{V} \mathbf{d}_m \mathbf{B} \\ &= \mathbf{V} (\mathbf{d}_m \nabla \cdot \mathbf{q}_m) \mathbf{B} + \mathbf{V} \mathbf{d}_m \dot{\mathbf{B}} + \mathbf{V} \mathbf{d}_m (\mathbf{q}_m \nabla \cdot \mathbf{B}), \end{aligned} \quad (20)$$

where $\dot{\mathbf{B}}$ is the rate of change of \mathbf{B} at a fixed point coincident with the moving centre of the segment, $\mathbf{q}_m \nabla \cdot \mathbf{B}$ being of course the term in the rate of change due to motion of the segment with velocity \mathbf{q}_m .

Equation (12) is therefore by (17) and (20),

$$\begin{aligned} &\mathbf{V} (\mathbf{d}_m \nabla \cdot \mathbf{q}_m) \mathbf{B} + \mathbf{V} \mathbf{d}_m \dot{\mathbf{B}} - \mathbf{V} \mathbf{d}_m (\mathbf{q}_m \nabla \cdot \mathbf{B}) \\ &- \mathbf{d}_m \nabla \cdot (\mathbf{E} + \mathbf{V} \mathbf{q}_m \mathbf{B}) + \nabla \cdot \mathbf{d}_m (\mathbf{E} + \mathbf{V} \mathbf{q}_m \mathbf{B}) = 0, \end{aligned} \quad (21)$$

\mathbf{d}_m and \mathbf{q}_m being constant in the last term, and ∇ operating forwards only.

In carrying out the simplifying transformations we may drop for the moment the suffix m .

From the last two terms we have, in part,

$$\begin{aligned} -\mathbf{d} \nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{d} \mathbf{E} &= + \mathbf{V} \mathbf{d} \nabla \nabla \cdot \mathbf{E} \\ &= + \mathbf{V} \mathbf{d} \text{curl } \mathbf{E} \end{aligned} \quad (22)$$

From the remainder we find

$$\begin{aligned}
 & \nabla d\dot{\mathbf{B}} + \nabla(d\nabla \cdot \mathbf{q})\mathbf{B} + \nabla d(\mathbf{q}\nabla \cdot \mathbf{B}) - d\nabla \cdot \nabla \mathbf{q}\mathbf{B} + \nabla_1 \cdot d\nabla \mathbf{q}\mathbf{B}_1 \\
 &= \nabla d\dot{\mathbf{B}} + \nabla(d\nabla \cdot \mathbf{q})\mathbf{B} + \nabla \cdot d(\mathbf{q}\nabla \cdot \mathbf{B}) \\
 &\quad - \nabla(d\nabla \cdot \mathbf{q})\dot{\mathbf{B}} - \nabla \cdot \mathbf{q}(d\nabla \cdot \mathbf{B}) + \nabla_1 \cdot d\nabla \mathbf{q}\mathbf{B}_1 \\
 &= \nabla d\dot{\mathbf{B}} + \nabla \cdot d(\mathbf{q}\nabla \cdot \mathbf{B}) - \nabla \cdot \mathbf{q}(d\nabla \cdot \mathbf{B}) \\
 &\quad - \nabla \cdot d(\mathbf{q}\nabla \cdot \mathbf{B}) + \nabla \cdot \mathbf{q}(d\nabla \cdot \mathbf{B}) + \nabla d\mathbf{q} \cdot \nabla \mathbf{B} \\
 &= \nabla d\dot{\mathbf{B}} + \nabla d\mathbf{q} \operatorname{div} \mathbf{B}, \quad \dots \dots \dots (23)
 \end{aligned}$$

where the suffix restricts the action of ∇ to the vector carrying the same suffix.

Equation (21) then reduces to

$$\nabla d_m(\operatorname{curl} \mathbf{E} + \dot{\mathbf{B}} + \operatorname{div} \mathbf{B}), \quad \dots \dots \dots (24)$$

Now d_m will have different values according to the different directions of the various sets of tubes; hence (unless all the tubes are parallel) we may write

$$\operatorname{curl} \mathbf{E} + \dot{\mathbf{B}} + \mathbf{q}_m \operatorname{div} \mathbf{B} = 0 \quad \dots \dots \dots (25)$$

From this, since \mathbf{q}_m is the velocity of any set of tubes, unless all the sets have a common velocity, we must have

$$\operatorname{div} \mathbf{B} = 0, \quad \dots \dots \dots (26)$$

and thus

$$-\operatorname{curl} \mathbf{E} = \dot{\mathbf{B}} \quad \dots \dots \dots (27)$$

We have now shown that the first four laws of the ordinary theory of electro-magnetism are consequences of the assumptions which have been made. It may be observed that whereas, in the proof of the first two laws (3) and (4), no departure of importance is made from the method of *Recent Researches*, the proof just given of the laws (26) and (27) is quite different from that adopted in that work. This is rendered necessary by the purpose of the present paper, which is not to deduce the properties of the tubes from the known laws of electro-magnetism, but to show that, given the tubes with the (essential) properties assigned to them by Sir J. J. Thomson, the laws of electro-magnetism follow.

5. It remains to discuss the forces acting on the electric particles. Referring to the figure on p. 228, let B be a particle at the end of the tube B, C, D. Then the change in L due to the displacement of the end of the tube from B to A (introducing a new segment BA), is by (13)

$$\delta L = \delta \mathbf{r}(\mathbf{E} + \nabla \mathbf{q}_m \mathbf{B}), \quad \dots \dots \dots (28)$$

since

$$\delta d_m = \overline{AB} = -\delta \mathbf{r},$$

B being the positive end of the tube, and thus equivalent to a positive unit of electricity. Hence the force acting per unit charge moving with velocity \mathbf{q} is

$$\mathbf{F} = \mathbf{E} + V\mathbf{qB}, \dots \dots \dots (29)$$

the Fifth Law of Electro-magnetism.

6. The definite dynamical assumptions of this theory enable us to examine very thoroughly such questions as the stress in the field and the mechanism of radiation.

Heaviside* has given a general discussion of the problem of stresses from which it is not difficult to deduce the following general result:—

Let ψ_0 be the operator of Maxwell's stress,

$$\psi_0 = \mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B} - \frac{1}{2}(\mathbf{E}\mathbf{D} + \mathbf{H}\mathbf{B}), \dots \dots \dots (30)$$

where any vector operand forms with \mathbf{D} and \mathbf{B} scalar products in the first and second terms. When this operand is a unit vector \mathbf{N} , $\psi_0\mathbf{N}$ is the stress on the plane perpendicular to \mathbf{N} .

Let ψ be the stress derived from ψ_0 by putting for \mathbf{E} , $\mathbf{E} + V\mathbf{qB}$, and for \mathbf{H} , $\mathbf{H} - V\mathbf{qD}$, namely

$$\begin{aligned} \psi &= \psi_0 + V\mathbf{qB} \cdot \mathbf{D} - V\mathbf{qD} \cdot \mathbf{B} - \frac{1}{2}(V\mathbf{qB})\mathbf{D} + \frac{1}{2}(V\mathbf{qD})\mathbf{B} \\ &= \psi_0 + V\mathbf{qB} \cdot \mathbf{D} + V\mathbf{Dq} \cdot \mathbf{B} - \mathbf{DVqB} \\ &= \psi_0 + V\mathbf{DB} \cdot \mathbf{q} \dots \dots \dots (31) \end{aligned}$$

by mere vector transformation.

Then if \mathbf{N} is unit normal to a surface moving with a velocity \mathbf{q} at any point, $\psi\mathbf{N}$ is the flux of momentum through the surface in the direction opposite to the positive direction of \mathbf{N} , per unit surface per unit time.

To see that this is true we have only to apply the theorem of divergence; in the first place we note that since

$$\frac{\partial T}{\partial \mathbf{q}_m} = V\mathbf{d}_m\mathbf{B}, \dots \dots \dots (18)$$

summing for all value of m we have $V\mathbf{DB}$ equal to the momentum per unit volume. But

$$\psi_0\nabla = \frac{\partial}{\partial t}V\mathbf{DB}, \dots \dots \dots (32)$$

a result easily deduced (Heaviside, *loc. cit.*) from the circuital laws, and usually expressed in words by stating that Maxwell's stress gives rise to a translational force per unit volume equal to the rate of change at a fixed point of the momentum per unit volume (the absence of electrification being assumed). We are thus entitled to say that $\psi_0\mathbf{N}$ is the flux of momentum per unit area of a fixed surface. Now it is clear that $V\mathbf{DB} \cdot \mathbf{qN}$

* *Electrical Papers*, vol. ii, pp. 521 *et seq.*; also *Phil. Trans.*, A, 1892.

is the flux per unit area due to the motion of the surface with velocity \mathbf{q} . Hence ψ is the general operator giving the flux of momentum. The equation of rate of change of momentum per unit volume at a point whose velocity is \mathbf{q} is

$$\begin{aligned} \psi \nabla &= \frac{\partial}{\partial t} \mathbf{V} \mathbf{D} \mathbf{B} + \nabla \mathbf{q} \cdot \mathbf{V} \mathbf{D} \mathbf{B} \\ &= \frac{\partial}{\partial t} \mathbf{V} \mathbf{D} \mathbf{B} + \mathbf{q} \nabla \cdot \mathbf{D} \mathbf{B} + \mathbf{V} \mathbf{D} \mathbf{B} \cdot \text{div } \mathbf{q}, \end{aligned} \quad (33)$$

the first two terms giving the rate of change of density of momentum at the moving point, and the last term the rate of change due to expansion at the rate $\text{div } \mathbf{q}$.

This flux of momentum ψ is partly due to convection, and partly to be ascribed to a stress. It is interesting to note that if all the tubes were of one set, we could determine the stress by simply putting \mathbf{q} equal to this velocity. We should then have $\mathbf{H} = \mathbf{V} \mathbf{q} \mathbf{D}$, and the stress would be

$$\begin{aligned} \phi &= (\mathbf{E} + \mathbf{V} \mathbf{q} \mathbf{B}) \cdot \mathbf{D} - \frac{1}{2} (\mathbf{E} + \mathbf{V} \mathbf{q} \mathbf{B}) \mathbf{D} \\ &= \mathbf{F} \cdot \mathbf{D} - \frac{1}{2} \mathbf{F} \mathbf{D} \\ &= \mathbf{F} \cdot \mathbf{D} + \frac{1}{2} (\mathbf{H} \mathbf{B} - \mathbf{E} \mathbf{D}). \end{aligned} \quad (34)$$

In general the stress operator will be obtained by subtracting from ψ_0 the operator $-\Sigma(\mathbf{V} \mathbf{d}_m \mathbf{B} \cdot \mathbf{q}_m)$ which gives the convective flux of momentum relative to a fixed point; thus the stress is

$$\begin{aligned} \phi &= \psi_0 + \Sigma(\mathbf{V} \mathbf{d}_m \mathbf{B} \cdot \mathbf{q}_m) \quad (35') \\ &= \mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B} - \frac{1}{2} \mathbf{E} \mathbf{D} - \frac{1}{2} \mathbf{H} \mathbf{B} + \Sigma \mathbf{V} \mathbf{q}_m \mathbf{B} \cdot \mathbf{d}_m - \Sigma \mathbf{V} \mathbf{q}_m \mathbf{d}_m \cdot \mathbf{B} \\ &\quad + \Sigma(\mathbf{V} \mathbf{q}_m \mathbf{d}_m) \mathbf{B} \\ &= \mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B} - \frac{1}{2} \mathbf{E} \mathbf{D} - \frac{1}{2} \mathbf{H} \mathbf{B} + \Sigma \mathbf{V} \mathbf{q}_m \mathbf{B} \cdot \mathbf{d}_m - \mathbf{H} \cdot \mathbf{B} + \mathbf{H} \mathbf{B} \\ &= \Sigma \{ (\mathbf{E} + \mathbf{V} \mathbf{q}_m \mathbf{B}) \cdot \mathbf{d}_m \} - \frac{1}{2} \mathbf{E} \mathbf{D} + \frac{1}{2} \mathbf{H} \mathbf{B} \quad (35) \end{aligned}$$

From (35') we see that the stress coincides with Maxwell's stress when there is no convection of momentum relative to the (so-called) fixed reference frame; and from (35) that it consists in general of a quasi-tension equal to $\mathbf{E} + \mathbf{V} \mathbf{q}_m \mathbf{B}$ per tube of the m th set together with a hydrostatic pressure $\frac{1}{2}(\mathbf{E} \mathbf{D} - \mathbf{H} \mathbf{B})$. The torque per unit volume is seen to be

$$\begin{aligned} \phi - \phi' = \mathbf{S} &= -\Sigma \mathbf{V} (\mathbf{E} + \mathbf{V} \mathbf{q}_m \mathbf{B}) \mathbf{d}_m \\ &= +\Sigma \mathbf{V} \mathbf{d}_m \mathbf{V} \mathbf{q}_m \mathbf{B} \\ &= +\Sigma \mathbf{V} \mathbf{q}_m \mathbf{V} \mathbf{d}_m \mathbf{B} - \Sigma \mathbf{V} (\mathbf{V} \mathbf{q}_m \mathbf{d}_m) \mathbf{B} \\ &= +\Sigma \mathbf{V} \mathbf{q}_m \mathbf{V} \mathbf{d}_m \mathbf{B}, \end{aligned} \quad (36)$$

the last expression being the rate of change of moment of momentum about a fixed point due to component of velocity perpendicular to the momentum, familiar in the hydrodynamics of the motion of bodies in a fluid.

7. The flux of energy also consists of two parts: the convective flux due to the motion of the tubes, and the flux due to the activity of the stress. To find the convective flux we require to localise the energy in a manner rather difficult to justify. The whole energy per unit volume may be written

$$\frac{1}{2}\mathbf{NB} + \frac{1}{2}\mathbf{ED} \\ = \frac{1}{2}\sum\mathbf{d}_m(\mathbf{E} - \mathbf{V}\mathbf{q}_m\mathbf{B}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Then we may suppose the part $\mathbf{d}_m(\mathbf{E} - \mathbf{V}\mathbf{q}_m\mathbf{B})$ of the energy to be moving with velocity \mathbf{q}_m , and so on. The total convection of energy will therefore be

$$\frac{1}{2}\sum\mathbf{d}_m(\mathbf{E} - \mathbf{V}\mathbf{q}_m\mathbf{B}) \cdot \mathbf{q}_m \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

To find the stress-activity flux from (35), consider first the term $(\mathbf{E} + \mathbf{V}\mathbf{q}_m\mathbf{B}) \cdot \mathbf{d}_m$; the appropriate velocity is clearly \mathbf{q}_m , and the flux (by Heaviside's method)

$$- \mathbf{q}_m(\mathbf{E} + \mathbf{V}\mathbf{q}_m\mathbf{B}) \cdot \mathbf{d}_m = - \mathbf{q}_m\mathbf{E} \cdot \mathbf{d}_m \cdot$$

Again, we may write the second term

$$- \frac{1}{2}\mathbf{ED} + \frac{1}{2}\mathbf{HB} = - \frac{1}{2}\{(\sum\mathbf{d}_m)\mathbf{E} - (\sum\mathbf{V}\mathbf{q}_m\mathbf{d}_m)\mathbf{B}\} \\ = - \frac{1}{2}\sum\mathbf{d}_m(\mathbf{E} + \mathbf{V}\mathbf{q}_m\mathbf{B}),$$

and it seems permissible to write the activity flux due to the term $-\frac{1}{2}\mathbf{d}_m(\mathbf{E} + \mathbf{V}\mathbf{q}_m\mathbf{B})$ as $+\frac{1}{2}\mathbf{q}_m \cdot \mathbf{d}_m(\mathbf{E} + \mathbf{V}\mathbf{q}_m\mathbf{B})$. Hence the total activity flux will be

$$- \sum\{\mathbf{q}_m\mathbf{E} \cdot \mathbf{d}_m - \frac{1}{2}\mathbf{d}_m(\mathbf{E} + \mathbf{V}\mathbf{q}_m\mathbf{B})\}, \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and the whole flux, adding (38) and (39),

$$\mathbf{W} = \frac{1}{2}\sum\mathbf{d}_m(\mathbf{E} - \mathbf{V}\mathbf{q}_m\mathbf{B}) \cdot \mathbf{q}_m - \sum\mathbf{q}_m\mathbf{E} \cdot \mathbf{d}_m + \frac{1}{2}\sum\mathbf{d}_m(\mathbf{E} + \mathbf{V}\mathbf{q}_m\mathbf{B})\mathbf{q}_m \\ = \sum(\mathbf{d}_m\mathbf{E} \cdot \mathbf{q}_m - \mathbf{q}_m\mathbf{E} \cdot \mathbf{d}_m) \\ = \mathbf{VE}\sum\mathbf{V}\mathbf{q}_m\mathbf{d}_m \\ = \mathbf{VEH} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

8. Since we have shown that this theory leads to the ordinary equations of the electro-magnetic field, it is unnecessary to give a separate proof of the uniform propagation of disturbances with velocity $1/\sqrt{\mu\mathbf{K}}$. It is perhaps as well, however, to examine shortly the mechanism of propagation, particularly since the mental picture of electro-magnetic radiation afforded by the theory is in many respects very satisfactory.

N. Campbell gives a short discussion of the question, and shows that a tube at rest may be compared to a flexible cord of linear density $\mu\mathbf{D}$ under a tension \mathbf{D}/\mathbf{K} ; the square of the velocity of propagation of transverse disturbances being then $1/\mu\mathbf{K}$ by the elementary dynamics of cords. To extend this result to the case of a tube having a general velocity v perpendicular to its own direction, we have only to remember that, by

equation (35) above, the stress to which the restoring force is due will now be the quasi-tension $\mathbf{E} + \mathbf{V}\mathbf{q}\mathbf{B}$, where \mathbf{q} is the velocity of the tubes, of which we shall suppose that only one set need be taken into account; and with this last assumption we may drop the suffix m and so write

$$\mathbf{B} = \mu\mathbf{V}\mathbf{q}\mathbf{D}, \quad \mathbf{E} = \frac{\mathbf{d}}{K}.$$

The \mathbf{d} component of $\mathbf{E} + \mathbf{V}\mathbf{q}\mathbf{D}$ is the only effective part of the stress, and its magnitude is given by

$$(\mathbf{E} + \mathbf{V}\mathbf{q}\mathbf{B})\mathbf{d}_1 = \left(\frac{\mathbf{d}}{K} + \mu\mathbf{V}\mathbf{q}\mathbf{V}\mathbf{q}\mathbf{d}\right)\mathbf{d}_1,$$

where \mathbf{d}_1 is the unit vector parallel to \mathbf{d} , or $\mathbf{d} = d\mathbf{d}_1$. This equals

$$\begin{aligned} & \frac{d}{K}(1 + \mu K\mathbf{d}_1\mathbf{V}\mathbf{q}\mathbf{V}\mathbf{q}\mathbf{d}) \\ &= \frac{d}{K}\{1 - \mu K(\mathbf{V}\mathbf{d}_1\mathbf{q})^2\} \\ &= \frac{d}{K}\left\{1 - \frac{v^2}{c^2}\right\}, \quad \dots \dots \dots (41) \end{aligned}$$

where $c^2\mu K = 1$.

The linear density will remain μd , so that the velocity of propagation along the tube will be $\sqrt{c^2 - v^2}$. Since the tube itself is in motion with velocity v in a perpendicular direction, the propagation of the disturbance in space will be with velocity c in a direction making an angle $\sin^{-1} \frac{v}{c}$ with the tube. When $v = c$ the disturbance will not be propagated at all along the tube, which will lie in the wave-front; and the traction $(\mathbf{E} + \mu\mathbf{V}\mathbf{q}\mathbf{V}\mathbf{q}\mathbf{D})$ will vanish.

9. To take into account a general velocity of the tube in the direction of its length, let us restrict ourselves to plane-polarised radiation. We shall take the x -axis in the direction of propagation, and the y -axis in that of the disturbance. Since we are dealing only with transverse vibrations, the velocity of the tubes in the direction of the ray will be constant from point to point along a tube. Let u be this x -component of velocity. Also let (x, y) be the coordinates of a point on some particular tube at time t , so that y is a function of x and t . Then the whole y -component of velocity of the point will be

$$v = \frac{dy}{dt} = \frac{\partial y}{\partial t} + u \frac{\partial y}{\partial x} \quad \dots \dots \dots (42)$$

It is obvious that the shearing motion perpendicular to the x -axis of the tubes in their vibration will not affect the number of tubes per unit area passing through a plane normal to the x -axis. Thus the quantity

d_x , the x -component of electric displacement, will be constant at a point on the tube, or

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)d_x = 0.$$

Also, if d_y is the y -component, we shall have

$$\frac{d_y}{d_x} = \frac{\partial y}{\partial x}.$$

And thus

$$\begin{aligned} d_y &= \frac{\partial y}{\partial x} d_x \\ d^2 &= d_x^2 + d_y^2 \\ &= d_x^2 \left\{ 1 + \left(\frac{\partial y}{\partial x}\right)^2 \right\} \end{aligned} \quad (43)$$

The momentum per unit length along the tube is

$$\begin{aligned} \mathbf{Vd}_1\mathbf{B} &= \mu\mathbf{Vd}_1\mathbf{Vq}\mathbf{d} \\ &= \mu(\mathbf{q} \cdot \mathbf{d} - \mathbf{d} \cdot \mathbf{q}\mathbf{d}_1). \end{aligned}$$

Multiply this by $\frac{d}{d_x}$ to find the value appropriate to unit length along the x -axis, and, taking the y -component, we have

$$\begin{aligned} &\mu\frac{d}{d_x} \left[\left(\frac{\partial y}{\partial t} + u\frac{\partial y}{\partial x}\right)d - d_x\frac{\partial y}{\partial x} \frac{1}{d} \left\{ u d_x + \left(\frac{\partial y}{\partial t} + u\frac{\partial y}{\partial x}\right)d_x\frac{\partial y}{\partial x} \right\} \right] \\ &= \mu d_x \left[\left(\frac{\partial y}{\partial t} + u\frac{\partial y}{\partial x}\right) \frac{d^2}{d_x^2} - \frac{\partial y}{\partial x} \left\{ u + \left(\frac{\partial y}{\partial t} + u\frac{\partial y}{\partial x}\right)\frac{\partial y}{\partial x} \right\} \right] \\ &= \mu d_x \left[\frac{\partial y}{\partial t} \left\{ 1 + \left(\frac{\partial y}{\partial x}\right)^2 \right\} - \left(\frac{\partial y}{\partial x}\right)^2 \right\} + u\frac{\partial y}{\partial x} \left\{ 1 + \left(\frac{\partial y}{\partial x}\right)^2 - 1 - \left(\frac{\partial y}{\partial x}\right)^2 \right\} \right] \\ &= \mu d_x \frac{\partial y}{\partial t} \end{aligned} \quad (44)$$

Hence the rate of change of momentum in the y -direction per unit length along the x -axis is

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\mu d_x \frac{\partial y}{\partial t} = \mu d_x \left[\frac{\partial^2 y}{\partial t^2} + u\frac{\partial^2 y}{\partial x \partial t} \right]. \quad (45)$$

The force to be equated to this arises from the quasi-tension

$$\begin{aligned} \mathbf{E} + \mathbf{Vq}\mathbf{B} &= \frac{\mathbf{d}}{\mathbf{K}} + \mu\mathbf{Vq}\mathbf{Vq}\mathbf{D} \\ &= \frac{\mathbf{d}}{\mathbf{K}} + \mu\mathbf{q} \cdot \mathbf{q}\mathbf{d} - \mu\mathbf{d} \cdot \mathbf{q}^2, \end{aligned}$$

of which the y -component is

$$\begin{aligned} &\frac{1}{\mathbf{K}}d_x\frac{\partial y}{\partial x} + \mu\left(\frac{\partial y}{\partial t} + u\frac{\partial y}{\partial x}\right) \left\{ u d_x + \left(\frac{\partial y}{\partial t} + u\frac{\partial y}{\partial x}\right)d_x\frac{\partial y}{\partial x} \right\} - \mu d_x \frac{\partial y}{\partial x} \left\{ u^2 + \left(\frac{\partial y}{\partial t} + u\frac{\partial y}{\partial x}\right)^2 \right\} \\ &= \frac{1}{\mathbf{K}}d_x\frac{\partial y}{\partial x} + \mu\left(\frac{\partial y}{\partial t} + u\frac{\partial y}{\partial x}\right)u d_x - \mu d_x \frac{\partial y}{\partial x} u^2 \\ &= \frac{1}{\mathbf{K}}d_x\frac{\partial y}{\partial x} + \mu d_x u \frac{\partial y}{\partial t} \end{aligned} \quad (46)$$

Differentiating with respect to x we have the force per unit length

$$d_x \left[\frac{1}{K} \frac{\partial^2 y}{\partial x^2} + \mu u \frac{\partial^2 y}{\partial x \partial t} \right] \dots \dots \dots (47)$$

Equating therefore expressions (45) and (47) and dividing by d_x , we have

$$\frac{1}{K} \frac{\partial^2 y}{\partial x^2} + \mu u \frac{\partial^2 y}{\partial x \partial t} = \mu u \frac{\partial^2 y}{\partial x \partial t} + \mu \frac{\partial^2 y}{\partial t^2}$$

or

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \dots \dots \dots (48)$$

exhibiting the uniform propagation with velocity c independent of the general motion of the tube.

The relation between the electro-magnetic disturbance and the displacement y of the tube is easily seen to be given by

$$E_y = \frac{d_y}{K} = \frac{1}{K} d_x \frac{\partial y}{\partial x} \dots \dots \dots (49)$$

$$H_z = - \frac{\partial y}{\partial t} d_x$$

$$= \frac{1}{\mu c} \mathbf{E} \dots \dots \dots (50)$$

But while in plane-polarised radiation the displacement of the tube from its normal position is thus perpendicular to the plane of polarisation, in circularly polarised rays it is easy to see that the reverse is the case.

10. The intention in presenting the theory of Faraday tubes in the present form was to suggest possibilities of modification which might explain various phenomena of which no entirely satisfactory electrical explanation has been given so far.

In making attempts of this kind we may, for instance, take advantage in various ways of the fact that the electric displacement has been considered as a mean value taken over a small, but not infinitesimal, area. From this point of view the Maxwellian theory is microscopical, and a more microscopical theory may be what is required in various regions of modern physics.

Again, the present theory rests on the localisation of electric and magnetic energy as functions of \mathbf{D} and \mathbf{H} , and on the derivation from these of equations of motion. Hence it would be comparatively simple to estimate the effects either of a modified distribution of energy, or of substituting any different hypothesis for the principle of action.

Lastly, quite a variety of hypotheses are possible as to the exact nature of the electric particles.

11. It will be observed that in describing the properties of the tubes

of force we have so far assumed that two oppositely directed tubes at the same point exactly cancel each other in their effects, if they are moving with the same velocity. Now, just as the electrical theory of matter explains all the phenomena of neutral bodies as due to the existence of the equal mixture of positive and negative electricity, which on the two-fluid theory was supposed to have no recognisable physical properties, so on the lines of force theory we may perhaps speculate with advantage on the possibility of explaining by means of properties of equal mixtures of oppositely directed tubes the phenomenon of gravitation, which seems for many reasons to be on a different level from the ordinary electrical phenomena. Let us consider the potential energy of such a mixture of tubes. So long as we choose an element of area large enough to include many tubes, the density of energy $\frac{1}{2}ED$ must always vanish; but as we take smaller and smaller elements of area, there will be an increasing probability of the number of tubes passing through it in one direction being not quite equal to the number passing through it in the opposite direction: in other words, what to ordinary microscopic electrical measurements is a uniform absence of electric displacement may consist of alternate regions of opposite displacement so small that only the mean field of a considerable number of regions is measured. Such a field would have positive potential energy; but since the more closely the tubes are packed, the smaller is the element of area we can take without considering this effect, it seems reasonable to suppose that the effect will become smaller the more numerous are the tubes of either sign. Not improbably a mathematical form might be given to this hypothesis which would explain and locate the energy of gravitation. Let $de_1, -de_1; de_2, -de_2$, be pairs of opposite charges; r_1, r_2 the (small) distances apart of the components of each pair; and R the distance between the pairs. Then if the hypothesis could be so formulated that the potential energy of the system would include a term of the form

$$\frac{-\gamma de_1^2 de_1^2}{r_1 r_2 R},$$

where γ is a positive constant, the law of gravitation would be completely satisfied, and gravitational mass would be identified exactly with electro-magnetic mass; for

$$\frac{de_1^2}{r_1}$$

is proportional to the element of electro-magnetic mass due to two elements of charge $de_1, -de_1$.

This last question is of some interest in the theory of atomic structure; a number of writers have laid stress on the importance of mutual electro-magnetic mass, and in particular Harkins and E. D. Wilson* have used this phenomenon to explain the departure of atomic *weights* from whole numbers. It appears, however, that such an explanation could alone be valid if mutual mass were ponderable.

12. The theory of Faraday tubes might possibly be employed with advantage in other investigations connected with atom theory. Sir J. J. Thomson † has made several suggestions of this nature; his conception of the electron as possibly simply the end of a single Faraday tube would of course have very important consequences if adhered to in any theory of atomic structure.

Again, if we suppose that electrons and positive nuclei have the property of excluding the tubes of other electrons and nuclei, the attractions between particles of opposite sign would become a repulsion at very small distances. Or we may suppose that some or all of the tubes of an electron in an atom simply end at a nucleus, instead of spreading equally outwards in all directions; and different states of an atom, with different periods of vibration, might arise according to the number of tubes so connected. Suggestions have also been made as to the application of the theory in connection with a possible discrete structure in radiation.‡

CONCLUSION.

13. It has been shown that the general equation of the Maxwell-Lorentz-Heaviside theory of electro-magnetism can be derived as macroscopic consequences of a simple dynamical theory of Faraday tubes.

This theory also gives explicit and non-contradictory expression to the ideas of electro-magnetic stress, momentum, and flux of energy, and an electro-mechanical picture of radiation explaining the law of uniform propagation in spite of the motion of the source.

A number of suggestions are made as to applications to the theory of gravitation and other problems.

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ROYAL NAVAL DIVISION.

* *Phil. Mag.*, Nov. 1915, p. 723.

† *Phil. Mag.* (6), xxvi, p. 792.

‡ Jeans, "Report on Quantum Theories," *Proc. Lond. Phys. Soc.*, 1915.

WILLIAM GORDON BROWN.

Biographical Note by Dr C. G. KNOTT, F.R.S., General Secretary.

William Gordon Brown, the author of the accompanying paper, was born at Edinburgh on August 12, 1895. He was educated at George Watson's College (1901-1914), where he distinguished himself as a pupil of outstanding ability in all lines of study, and exceptionally brilliant in mathematics. In his last year at school he gained the Glasse Bursary and also one of the College bursaries, and he was evidently marked out for a University career. Unfortunately, the outbreak of the war cruelly interrupted the natural development of an intellectual life of the highest order. Along with a group of his schoolmates, William Brown at once enlisted in the 4th Royal Scots. Later, as a private in the Royal Naval Division, he served in Gallipoli, whence he was invalided to Alexandria in 1915, and finally home in 1916. In August of that year he crossed to France, and met his death on November 13, 1916, in the attack on Beaumont-Hamel. He was a general favourite among his comrades, among whom he was recognised as a personality of rare distinction both in character and in intellectual power.

Dr Pinkerton, formerly mathematical master in Watson's College, now Rector of the Glasgow High School, describes William Brown in these words: "Brown was a most infectious pupil. I always called him my Newton. He was far ahead of his fellows at school, and I taught him, in addition to class-work, to explore any region of mathematics or physics that had even the remotest connection with what turned up in my work. He responded to a marvellous extent; and the width of his knowledge as a schoolboy and the methods by which he had arrived at it were probably unique. He had an intellect of the first order; more important still, he was a boy of a simple sincerity of character, of a modest and charming manner, and with an influence on other boys that sprang from a deep respect for self-forgetfulness, courage, directness, and patriotism in a school sense—later all these made him a willing servant of his country, and made him give up everything to be true to himself."

While still at school, William Brown, following a hint from Dr Pinkerton, read with critical appreciation Norman Campbell's book on *Electricity* in the People's Library Series. This led to study of more advanced books, and all through his war experiences he not only kept up an interest in electro-magnetic theory and relativity, but became himself an investigator of important problems. Inspired by J. J. Thomson's *Recent*

Advances in Electricity and Magnetism, he tackled in his own way the many difficult questions suggested by that treatise, and applied his mind to the mastery of vector algebras as an aid in electro-magnetic investigation. From the notes he sent home at intervals it was abundantly evident that, with scanty help from the ordinary sources, he was himself constructing his own vector methods, largely retreading paths which, unknown to him, had been trodden by his predecessors, Hamilton, Tait, Heaviside, M'Aulay, Gibbs, and others. So impressed were his scientific friends with the brilliancy shown by Gordon Brown that an effort was made to get him appointed to a less dangerous post; but he himself would not listen to the proposal.

His correspondence is particularly interesting as throwing light on the way his mind was working. For example, in a letter written to his mother from Haslar R.N. Hospital of date December 5, 1915, we find the following:—

“Up to the present I have written a first scroll of (1) the first part of a paper on ‘The Faraday-tube Theory of Electro-magnetism,’ of which you will find an elementary discussion in Thomson’s *Electricity and Matter* I have in the house as a prize. I think this first part, which deals with the theory in general, and shows that on simple assumptions it leads in general (as it ought) to what are called the Five Equations of Electro-magnetism, of which the first four are the epitome of Maxwell’s theory, and the Fifth Equation is due to Lorentz or possibly (in justice) to Heaviside—I think this part of the paper may be of interest in itself, and is almost certain to be correct and original. The second part I am only just commencing. . . . Its purpose would be to apply the Faraday-tube theory to special problems, such as the law of gravitation, the structure of the atom, and the density of energy in radiation in temperature equilibrium with matter. . . . Besides working the first part of the paper, I have written a rough note on the ‘Energetics of the Electron,’ which is fairly short, and more on the lines of the work I had published in August.” (See *Phil. Mag.*, August 1915, “Note on Reflections from a Moving Mirror.”)

The paper referred to above is the one now being published; and a more particular account of its genesis was given to Dr Pinkerton, in a letter written about the middle of January 1916. This letter brings out the writer’s keen critical faculty, and his determination not to accept anything which was not fully understood by him:—

“DEAR DR PINKERTON,—I am ashamed to trouble you again about my scribbling, but as I am now having typed a paper which you yourself—

indirectly—suggested, I should like to ask your advice about it. You may remember recommending to your classes in Watson's some years ago Dr Norman Campbell's little book on *Electricity* in the People's Library Series. . . . I worked more or less on the lines of the bibliography in Dr Campbell's book, at least at first, but I was always curious to learn more about the theory of Faraday-tubes which he sketches. I eventually read the original investigations in Sir J. J. Thomson's *Recent Advances*, but could not understand his proofs in the light of Dr Campbell's ideas (and also those of Heaviside and Maxwell). When I referred at last to Dr Campbell's *Modern Theory of Electricity* I was disappointed to find him refer readers for proofs to J. J. Thomson's work. The more I began to understand the ideas of the *Recent Advances* proofs, the less inclined was I to admit their justification for Dr Campbell's purposes. I came to the conclusion that the Faraday-tube theory, which perhaps Thomson regarded more as an aid to thought than as a serious physical hypothesis, is too good for such a subordinate place as a mere easy guide to Maxwell; but it had not yet been sufficiently developed in a consistent manner so as to take rank as a physical theory.

"The paper I have written, after struggling with these difficulties for about two years, proposes to supply this development. It is written mostly in Heaviside's vector notation. . . . New proofs directly from dynamical hypotheses are given of three of the five laws of electro-magnetism. There is also a good deal of other matter in the paper . . . for instance, a modified theory of stresses, and an extension of Campbell's discussion of propagation so as to include an explanation of aberration, etc. It was writing this section which suggested my note in the *Phil. Mag.* of last August. . . .

"If you and Professor Gray mean to communicate my note on 'Mass as a Linear and Vector Operator' to some society, I could send you a slightly fuller typewritten copy; but I am afraid it is not up to R.S.E. standard at any rate.

"I am now on convalescent leave after having dysentery at Gallipoli, so have a good deal of time on my hands.

"Hoping you are keeping well during the winter,—Your affectionate pupil,
W. G. BROWN."

The note here referred to on mass as a linear vector operator will be discussed more fully below.

Meanwhile W. G. Brown had been grappling with the Theory of Relativity, as is shown both in his letters to his parents and in the

“By choosing for α and β an appropriate form and by arbitrarily defining the positive direction of the tubes as that of the expressions (2), we could represent moving point charges, and from a discussion of their distribution in the four-dimensional space obtain the modifications in the kinematic scheme (3) due to the presence of electrification.

“The remaining (dynamical) scheme of four equations is of course deducible from the volume density $\frac{1}{2}(\mathbf{H}^2 - \mathbf{E}^2)$ or kinetic potential.

“I noticed particularly that the velocity of the tubes was only defined (by the equations (1)) for directions normal to that of the tubes themselves, that is, only its component in the direction of the momentum is determinate, as in Cunningham’s theory of the stresses; but, while on the simple æther theory only one velocity is considered at any point, when we are dealing with lines or tubes of force each separate set has its peculiar velocity, though the momentum corresponding to a tube is always perpendicular to it.

“With regard to electro-magnetic inertia, I have calculated the mutual electro-magnetic momentum of a pair of point charges moving obliquely to the line joining them, and find it not parallel to the velocity—as indeed is implied in Heaviside’s paper for low velocities (*Phil. Mag.*, 1889). I do not know, however, if this point is of interest. Hoping you will pardon my troubling you so much,—I am, yours truly,
W. G. BROWN.”

On August 1, 1916, Professor Whittaker, writing from Edinburgh, replied as follows:—

“DEAR MR BROWN,—Thank you for sending me an account of your results regarding lines of force.

“Probably you have noticed that your two functions α and β are connected with the old vector-potential and scalar-potential of the electro-magnetic field; thus, if a_x, a_y, a_z are the three components of the vector-potential, and ϕ is the scalar-potential, so that the components of electric force are

$$d_x = -\frac{1}{c} \frac{\partial a_x}{\partial t} - \frac{\partial \phi}{\partial x}, \text{ etc.,}$$

and the components of the magnetic force are

$$h_x = \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}, \text{ etc.,}$$

let us write down the ‘differential form’ or ‘Pfaff’s expression

$$a_x dx + a_y dy + a_z dz - c\phi dt.$$

Then if this differential form is reduced to the form $\alpha d\beta$ (there is a big theory dealing with the reduction of Pfaff's expressions) we have

$$\alpha_x = \alpha \frac{\partial \beta}{\partial x}, \quad \alpha_y = \alpha \frac{\partial \beta}{\partial y}, \quad \alpha_z = \alpha \frac{\partial \beta}{\partial z}, \quad \phi = -\frac{\alpha}{c} \frac{\partial \beta}{\partial t},$$

and therefore

$$d_x = \frac{1}{c} \frac{\partial(\alpha, \beta)}{\partial(x, t)}, \quad h_x = \frac{\partial(\alpha, \beta)}{\partial(y, z)},$$

which are the same as your expressions, except that the electric and magnetic rôles are interchanged.

"I have been so exclusively occupied with pure mathematics for the last four years that I can't say whether any of your work has been anticipated or not.—Yours very truly,
E. T. WHITTAKER."

Before this reply from Professor Whittaker reached him, William Brown was with the Expeditionary Force in France. In later letters to his parents he discussed the significance of the Relativity Theory, and on September 16 asked them to send him, if possible, a copy of Minkowski's *Raum und Zeit*. In acknowledging books received on October 15, he asked for news of the *Relativitäts Prinzip*. Before the lapse of another month this young life had passed from among us, and the world was the poorer by the loss of an intellect brilliant in mathematical power and promise.

W. G. Brown's notes on the investigation described above have not been recovered, but there is no doubt that his mind was taking a firm grip of the mathematical methods associated with the modern theory of Relativity.

To show the extent to which he had mastered the quaternion method, I conclude with giving a brief account of his note on Mass as a Linear Vector Operator. It is an interesting and novel generalisation of Tait's early work on the Rotation of a Rigid Solid (see Tait's *Quaternions*, §§ 406 *et seq.*):—

Let σ be the velocity of the body (more strictly of a definite point in the body), and let the operator μ take the place of the mass factor in ordinary dynamics, being a self-conjugate linear vector function whose axes are fixed in the body. Then the momentum is

$$\eta = \mu\sigma.$$

The force β is assumed to be given by the Newtonian law

$$\text{force} = \frac{d}{dt} (\text{momentum}),$$

or

$$\beta = \dot{\eta} = \frac{d}{dt}(\mu\sigma).$$

Since the quantities β and μ are referred to axes fixed in space, it is necessary to consider the variation $\dot{\mu}$.

Translational motion of the body will not affect the space relations of μ , but rotation will. Any vector a in the body will in virtue of rotation ω change at the rate

$$\dot{a} = V\omega a,$$

and, as is well known in quaternions, the rate of change of $\mu\sigma$ will be

$$\begin{aligned} \frac{d}{dt}(\mu\sigma) &= \dot{\mu}\sigma + \mu\dot{\sigma} \\ &= V\omega\mu\sigma - \mu V\omega\sigma + \mu\dot{\sigma}. \end{aligned}$$

Hence the equation of linear motion is

$$\beta = V\omega\mu\sigma - \mu V\omega\sigma + \mu\dot{\sigma} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The moment of momentum of the body with regard to a fixed origin is of the form

$$\gamma = V\rho\mu\sigma + \phi\omega,$$

where ρ is the vector position of a given point of the body and ϕ is the linear vector function which, operating on the angular velocity ω , gives the angular momentum about the extremity of ρ . (See Tait's *Quaternions*, p. 323.)

If ψ is the torque acting on the body, the equation of rotational motion is

$$\begin{aligned} \psi + V\rho\beta &= \dot{\gamma} \\ &= V\rho\dot{\mu}\sigma + V\rho\frac{d}{dt}(\mu\sigma) + \dot{\phi}\omega + \phi\dot{\omega} \\ &= V\sigma\mu\sigma + V\rho\beta + V\omega\phi\omega + \phi\dot{\omega} \end{aligned}$$

or

$$\psi = V\sigma\mu\sigma + \phi\dot{\omega} + V\omega\phi\omega \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The total activity equation is obtained from (1) and (2) in the form

$$S\sigma\beta + S\omega\psi = \frac{d}{dt}\left\{\frac{1}{2}S\sigma\mu\sigma + \frac{1}{2}S\omega\phi\omega\right\},$$

the integral of which may be represented in the form

$$W = T - T_0,$$

where W is the work done by the forces β and ψ , and $T(=T_1+T_2)$ is the kinetic energy. It will be seen that T is a quadratic function of the components of the velocity and of the angular velocity, the two parts T_1 and T_2 being quite separate, and no products of the components of different type existing. This feature involves or is involved in the self-conjugateness of the operators.

If σ is parallel to one of the axes of μ , so also is $\mu\sigma$, and $V\sigma\mu\sigma$ vanishes.

Equation (2) then reduces to

$$\psi = \phi\dot{\omega} + V\omega\phi\omega,$$

the well-known equivalent of Euler's three equations. Stability or instability will be determined by the usual considerations.

Again, if the momentum $\eta (= \mu\sigma)$ remains constant in amount but changes direction with uniform angular speed, we may write

$$\beta = \dot{\eta} = V\omega\eta = V\omega\mu\sigma.$$

But

$$\beta = V\omega\mu\sigma - \mu V\omega\sigma + \mu\dot{\sigma} \text{ by (1).}$$

Hence

$$\dot{\sigma} = V\omega\sigma,$$

and the body will describe a circular path with uniform speed.

The radial component of force is

$$\begin{aligned} -SU\rho\beta &= -SU\rho\omega\mu\sigma \\ &= +S\mu\sigma V\omega U\rho \\ &= \frac{S\sigma\mu\sigma}{T\rho} = -\frac{2T_1}{r}, \end{aligned}$$

where r is the tensor of ρ , and T_1 the kinetic energy of translation, the same relation as in ordinary dynamics.

The tangential component of the force is

$$-SU\sigma\beta = -SU\sigma V\omega\mu\sigma = \frac{S\omega\sigma\mu\sigma}{T\sigma},$$

vanishing when σ is parallel to a principal axis of μ .

If vibrations are set up about a position of stable equilibrium, any damping action will bring the body to move so that the axis of greatest mass will lie along the tangent to the path, and the motion will be sustained as in ordinary dynamics.

XVII.—The Quantum Theory of the Secondary Spectrum of Hydrogen. By John Marshall, M.A., D.Sc., University College, Swansea.

(MS. received March 18, 1922. Read May 15, 1922.)

INTRODUCTION.

It is proposed in this paper to consider the spectral lines which would arise in the case of an electron describing an orbit about two centres of attracting force. The assumptions of the Quantum Theory as generalised by Wilson and Sommerfeld are made. The centres of attracting force are assumed, as in the simpler applications of the Quantum Theory, to be at rest. Such a spectrum would arise in the case of a hydrogen molecule which had lost one electron and in which the remaining electron described an orbit about the positive nuclei. Novikoff has shown that this configuration would be stable (*vide* Whittaker's *Analytical Dynamics*, 2nd ed., p. 407, ex. 1). It follows that, if at any stage an electron were withdrawn from a molecule, the remaining electron would describe a stable orbit. Much experimental work has been done on the secondary spectrum of hydrogen, but I am not aware that any theoretical explanation has been given of the immense number of lines found by experiment. Dufour in *Ann. de Chimie et de Physique*, 1906, gives a very complete account of the work done on this spectrum and the various controversies which it has evoked. Dufour's conclusion is that the spectrum of the atom is the primary spectrum, and that the secondary spectrum is due to the molecule. H. E. Watson in *Proc. Roy. Soc.*, vol. lxxxii, 1909, states the main points at issue, and gives the most modern measurements of the lines and compares his results with wavelengths found by Hasselberg (*Mém. Acad. Imp. St Petersburg*, vol. xxxi, 1883), Ames (*Phil. Mag.*, vol. xxx, 1890), and Frost (*Astrophysical Journal*, vol. xvi, 1902).

Buisson and Fabry (*Journal de Physique*, vol. ii, p. 442, 1912) have measured with an interferometer the widths of the spectrum lines, and have referred at least a part of the lines of the secondary spectrum to the hydrogen atom. The complications which are introduced into the theoretical investigations, however, seem to indicate that the explanation of the lines is to be sought by the application of the Quantum Theory to molecular orbits.

Fulcher (*Astrophysical Journal*, vol. xxxvii, p. 60, 1913) has discussed the variation in the intensity of the lines when low potential discharges are used. Dr Merton (*Proc. Roy. Soc.*, vol. xevi, p. 382, 1920) has investigated the effect on the spectrum when the hydrogen is at a partial pressure of less than a millimetre in a vacuum tube containing helium at a pressure of about 40 mm. He found that under this condition the secondary spectrum differs from that obtained from pure hydrogen. A large number of lines have their intensities enhanced in the tubes containing helium, and a number of new lines appear. There is another class of lines whose relative intensities are unaffected by the presence of helium; whilst a third class become weaker and disappear. In this paper no attempt is made to discuss the intensities of the lines given, but it is evident that a modification of the physical conditions causing the hydrogen molecule to lose an electron will modify the orbit which the remaining electron describes. Sommerfeld has shown that in the case of helium the number of different ways in which the same λ is obtained gives a measure of the intensity of the line. A change, therefore, in the physical conditions will serve to make certain orbits more likely to be the initial orbits from which the electron jumps to other orbits, and hence a variation in intensity is to be expected.

ANALYSIS.

It is well known that if a curve can be described by a particle under central forces to each of n fixed points, when the forces act separately, it can be described under the action of all the forces, provided the particle is projected properly.

If v_κ is the velocity of the particle at a point when under a central attraction to the κ th centre, then

$$v_\kappa \frac{dv_\kappa}{ds} = -f_\kappa \frac{dr_\kappa}{ds} \quad \text{and} \quad \frac{v_\kappa^2}{\rho} = f_\kappa \frac{p_\kappa}{r_\kappa},$$

where f_κ is the central acceleration, r_κ the distance, and p_κ the perpendicular from the κ th centre on the tangent to the curve of radius of curvature ρ .

If V exists such that $V^2 = \sum v_\kappa^2$, the conditions

$$V \frac{dV}{ds} = \sum v_\kappa \frac{dv_\kappa}{ds} = - \sum f_\kappa \frac{dr_\kappa}{ds}$$

and

$$\frac{V^2}{\rho} = \sum \frac{v_\kappa^2}{\rho} = \sum f_\kappa \frac{p_\kappa}{r_\kappa}$$

are satisfied.

Let us consider an ellipse described under the action of two central forces attracting according to the law of the inverse square of the distance. We have

$$v_1^2 = \frac{\mu_1}{m} \left(\frac{2}{r_1} - \frac{1}{a} \right),$$

$$v_2^2 = \frac{\mu_2}{m} \left(\frac{2}{r_2} - \frac{1}{a} \right),$$

where r_1 and r_2 are the distances of the particle of mass m from the foci of an ellipse of semi-major axis a .

The velocity V required for the description of the ellipse under two centres is given by

$$V^2 = v_1^2 + v_2^2 = \frac{2\mu_1}{mr_1} + \frac{2\mu_2}{mr_2} - \frac{\mu_1 + \mu_2}{ma}.$$

H = sum of kinetic and potential energies

$$= \frac{1}{2}m(v_1^2 + v_2^2) - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}.$$

(1) . . . H = $-\frac{\mu_1 + \mu_2}{2a}$.

In the case of an electron of charge e , attracted by two positive nuclei of charges $\kappa_1 e$ and $\kappa_2 e$, $\mu_1 = \kappa_1 e^2$ and $\mu_2 = \kappa_2 e^2$.

For the ellipse: $x = a \cos \phi$, $y = b \sin \phi$, ϕ being the eccentric anomaly,

$$V^2 = \dot{x}^2 + \dot{y}^2 = (a^2 \sin^2 \phi + b^2 \cos^2 \phi) \dot{\phi}^2$$

$$= a^2(1 - e^2 \cos^2 \phi) \dot{\phi}^2,$$

where e is the eccentricity of the orbit.

$$\text{Potential energy} = \frac{\mu_1}{a(1 - e \cos \phi)} + \frac{\mu_2}{a(1 + e \cos \phi)}$$

$$= \frac{\mu_1 + \mu_2 + e(\mu_1 - \mu_2) \cos \phi}{a(1 - e^2 \cos^2 \phi)}.$$

Hence from (1) we have

$$\frac{1}{2}ma^2(1 - e^2 \cos^2 \phi) \dot{\phi}^2 + \frac{\mu_1 + \mu_2 + e(\mu_1 - \mu_2) \cos \phi}{a(1 - e^2 \cos^2 \phi)} = -\frac{\mu_1 + \mu_2}{2a},$$

$$\dot{\phi}^2 = \frac{\mu_1 + \mu_2}{ma^3} \frac{1 + 2e \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cos \phi + e^2 \cos^2 \phi}{(1 - e^2 \cos^2 \phi)^2},$$

(2) . . . $\dot{\phi}^2 = n^2 \cdot \frac{1 + 2\kappa e \cos \phi + e^2 \cos^2 \phi}{(1 - e^2 \cos^2 \phi)^2},$

where

$$n^2 = (\mu_1 + \mu_2)/ma^3 \quad \text{and} \quad \kappa = (\mu_1 - \mu_2)/(\mu_1 + \mu_2).$$

We have now to “quantise” the variables x and y , using the form of Wilson and Sommerfeld, $\int_C p dq = nh$, where n is an integer, h is Planck’s constant, and the contour C is the ellipse. Here

$$\begin{aligned} p_x &= m\dot{x} = -ma \cos \phi \cdot \dot{\phi}, \\ p_y &= m\dot{y} = mb \sin \phi \dot{\phi}. \end{aligned}$$

For the ellipse, ϕ varies from 0 to 2π .

If n_1 and n_2 are integers,

$$\begin{aligned} n_1 h &= \int_C p_x dx = ma^2 \int_0^{2\pi} \sin^2 \phi \cdot \dot{\phi} d\phi, \\ n_2 h &= \int_C p_y dy = mb^2 \int_0^{2\pi} \cos^2 \phi \cdot \dot{\phi} d\phi. \end{aligned}$$

Substituting from (2) for $\dot{\phi}$, we have

$$(3) \quad \begin{cases} n_1 h = mna^2 \int_0^{2\pi} \sin^2 \phi \frac{\sqrt{(1 + 2\kappa e \cos \phi + e^2 \cos^2 \phi)}}{1 - e^2 \cos^2 \phi} d\phi, \\ n_2 h = mnb^2 \int_0^{2\pi} \cos^2 \phi \frac{\sqrt{(1 + 2\kappa e \cos \phi + e^2 \cos^2 \phi)}}{1 - e^2 \cos^2 \phi} d\phi. \end{cases}$$

The simple case of one centre of force follows at once by putting $\mu_2 = 0$ and $\kappa = 1$.

$$\begin{aligned} n_1 h &= mna^2 \int_0^{2\pi} \frac{\sin^2 \phi}{1 - e \cos \phi} d\phi = \frac{2\pi mna^2}{e^2} [1 - \sqrt{1 - e^2}], \\ n_2 h &= mna^2(1 - e^2) \int_0^{2\pi} \frac{\cos^2 \phi}{1 - e \cos \phi} d\phi = \frac{2\pi mna^2}{e^2} \sqrt{1 - e^2} [1 - \sqrt{1 - e^2}], \end{aligned}$$

which shows that all values of e are not possible, e being restricted to satisfy the equation

$$\sqrt{1 - e^2} = n_2/n_1 \quad \text{or} \quad e^2 = 1 - (n_2/n_1)^2.$$

The elimination of e between the equation for $n_1 h$ and $n_2 h$ gives

$$(n_1 + n_2)h = 2\pi\sqrt{\mu m} \cdot \alpha^{\frac{3}{2}},$$

and substituting for α in

$$E = -H = \frac{\mu}{2a}$$

we obtain

$$E = \frac{2\pi^2 \mu^2 m}{(n_1 + n_2)^2 h^2}.$$

The form given by Sommerfeld follows at once, viz.

$$E - E' = \frac{hc}{\lambda} = \frac{2\pi^2 \mu^2 m}{h^2} \left[\frac{1}{(n_1 + n_2)^2} - \frac{1}{(n_1' + n_2')^2} \right].$$

Reverting to the case of two centres of force, we shall simplify our work by considering the case in which

$$\mu_1 = \mu_2 = \mu \quad \text{and} \quad \kappa = 0.$$

Equations (3) become

$$n_1 h = 4 m n a^2 \int_0^{\pi/2} \frac{s^2 (1 + e^2 c^2)^{\frac{1}{2}}}{(1 - e^2 c^2)} d\phi,$$

$$n_2 h = 4 m n a^2 (1 - e^2) \int_0^{\pi/2} \frac{c^2 (1 + e^2 c^2)^{\frac{1}{2}}}{(1 - e^2 c^2)} d\phi,$$

where $s = \sin \phi$ and $c = \cos \phi$.

Let

$$I = \sqrt{1 + e^2} \int_0^{\pi/2} \frac{s^2 (1 - k^2 s^2)^{\frac{1}{2}}}{1 - e^2 + e^2 s^2} d\phi$$

$$J = \sqrt{1 + e^2} \int_0^{\pi/2} \frac{c^2 (1 - k^2 s^2)^{\frac{1}{2}}}{1 - e^2 + e^2 s^2} d\phi,$$

where

$$k^2 = \frac{e^2}{1 + e^2}, \quad k'^2 = 1 - k^2 = \frac{1}{1 + e^2}, \quad e^2 = k^2 / k'^2.$$

$$I + (1 - e^2)J = \frac{1}{k'} \int_0^{\pi/2} \frac{(1 - k^2 s^2)^{\frac{1}{2}}}{1 - e^2 + e^2 s^2} \{s^2 + (1 - e^2)c^2\} d\phi,$$

$$(4) \quad \dots \quad I + (1 - e^2)J = \frac{1}{k'} \int_0^{\pi/2} (1 - k^2 s^2)^{\frac{1}{2}} d\phi = \frac{1}{k'} E\left(\frac{\pi}{2}, k\right).$$

$$I + J = \frac{1}{k'} \int_0^{\pi/2} \frac{(1 - k^2 s^2)^{\frac{1}{2}}}{(1 - e^2) + e^2 s^2} d\phi$$

$$= \frac{1}{k'(1 - e^2)} \int_0^{\pi/2} \frac{(1 - k^2 s^2)^{\frac{1}{2}}}{1 - k^2 s n^2 v s^2} d\phi, \quad \text{where} \quad k^2 s n^2 v = -\frac{e^2}{1 - e^2},$$

$$= \frac{1}{k'(1 - e^2)} \left[K + \int_0^K \frac{k^2 c n^2 v s n^2 u}{1 - k^2 c n^2 v s n^2 u} du \right]$$

$$= \frac{1}{k'(1 - e^2)} \left[K - \frac{c n v}{s n v d n v} \Pi(K, v) \right].$$

$$(5) \quad I + J = \frac{K}{k'(1 - e^2)} \left[1 - \frac{c n v}{s n v d n v} Z(v) \right],$$

since

$$\Pi(K, v) = K E(v) - v E = K Z(v).$$

Equations (4) and (5) give

$$(6) \quad \dots \quad I = \frac{k'}{k^2} \left[E - K + \frac{c n v}{s n v d n v} K Z(v) \right].$$

$$(7) \quad \dots \quad \begin{cases} (n_1 + n_2)h = 4 m n a^2 [I + (1 - e^2)J] \\ (n_1 + n_2)h = 4 m n a^2 \frac{E}{k'}, \text{ from (4)} \\ n_1 h = 4 m n a^2 I. \end{cases}$$

As I involves functions of an imaginary variable v , we must transform the integrals to functions of a real variable in order to calculate the values of the wave-lengths of the lines which arise.

Since

$$k^2 sn^2(v, k) = -\frac{e^2}{1 - e^2},$$

$$\therefore sn^2(v, k) = -\frac{1 + e^2}{1 - e^2}.$$

Putting $v = iw$ and using the results

$$sn(iw, k) = i \frac{sn(w, k')}{cn(w, k')} = i \sqrt{\frac{1 + e^2}{1 - e^2}},$$

$$cn(iw, k) = \frac{1}{cn(w, k')},$$

$$dn(iw, k) = dn(w, k')/cn(w, k'),$$

we have

$$sn^2(w, k') = \frac{1}{2k'^2},$$

$$cn^2(w, k') = (1 - 2k^2)/2k'^2,$$

$$dn^2(w, k') = 1/2.$$

$$\therefore \frac{cn(v, k)}{sn(v, k)dn(v, k)} = \frac{cn(w, k')}{i sn(w, k')dn(w, k')} = \frac{1}{i} \sqrt{2(1 - 2k^2)}.$$

Also

$$\begin{aligned} iZ(v) &= iZ(iw) \\ &= \frac{sn(w, k')}{cn(w, k')} dn(wk') + \frac{\pi w}{2KK'} + Z(w, k') \\ &= -\frac{1}{\sqrt{2(1 - 2k^2)}} + \frac{\pi w}{2KK'} + Z(w, k'). \end{aligned}$$

$$\therefore \frac{cn v}{sn v dn v} Z(v) = 1 - \sqrt{2(1 - 2k^2)} \left[\frac{\pi w}{2KK'} + Z(w, k') \right].$$

$$\therefore I = \frac{k'}{k^2} \left[E - K + K - \sqrt{2(1 - 2k^2)} \left\{ \frac{\pi w}{2K'} + KZ(w, k') \right\} \right].$$

From (7) we have

$$\begin{aligned} \frac{n_1}{n_1 + n_2} &= \frac{k'^2}{k^2} \left[1 - \sqrt{2(1 - 2k^2)} \frac{K}{E} \left\{ \frac{\pi w}{2KK'} + Z(w, k') \right\} \right] \\ &= \frac{k'^2}{k^2} \left[1 - \sqrt{2(1 - 2k^2)} \left\{ \frac{K}{E} E(w, k') + w \left(1 + \frac{KE'}{EK'} - \frac{K}{E} - \frac{KE'}{K'E} \right) \right\} \right]. \end{aligned}$$

Since

$$\pi/2 = K'E + KE' - KK',$$

we have

$$(8) \quad \delta = \frac{n_1}{n_1 + n_2} = \frac{k'^2}{k^2} \left[1 - \sqrt{2(1 - 2k^2)} \left\{ \frac{K}{E} \left(E(w, k') - F(w, k') \right) + F(w, k') \right\} \right],$$

where

$$F(w, k') = \int_0^{\sin^{-1}/k' \sqrt{2}} \frac{d\phi}{(1 - k'^2 \sin^2 \phi)^{\frac{3}{2}}},$$

and

$$E(w, k') = \int_0^{\sin^{-1}/k' \sqrt{2}} (1 - k'^2 \sin^2 \phi)^{\frac{3}{2}} d\phi.$$

The spectral lines arising from the orbit about two centres of force will be obtained by substituting for a from equation 7 in

$$E_1 = \frac{\mu}{a}.$$

\therefore

$$E_1 = \frac{16E^2 \cdot 2\mu^2 m \pi^2}{\pi^2 k'^2 (n_1 + n_2) \hbar^2} = chN \frac{R}{(n_1 + n_2)^2},$$

where

$$N = \frac{2m\pi^2 e^4}{ch^3} \quad \text{and} \quad R = \frac{16E^2 K^2}{\pi^2 k'^2}.$$

$$(9) \quad \dots \quad \frac{1}{\lambda} = N \left[\frac{R}{(n_1 + n_2)^2} - \frac{R'}{(n_1' + n_2')^2} \right],$$

where n_1' and n_2' are integers, and R' may equal R or correspond to a different value of the modulus k which determines the eccentricity of the orbit. This eccentricity, as in the simple case, cannot assume all possible values, but only those values which satisfy the equation determining k , viz. equation (8).

Substituting $n_1 + n_2 = n_1/\delta$ in (9), we have

$$(10) \quad \dots \quad \frac{1}{\lambda} = N \left[\frac{R\delta^2}{n_1^2} - \frac{R'\delta'^2}{n_1'^2} \right].$$

If then we assign a value for δ , equation (8) fixes the corresponding value of k , and hence R , which is a function of k , is determined. The value of δ also restricts the values possible for n_1 . For, if $\delta = p/q$, where p and q have no common factor, $(q - p)n_1 = pn_2$, i.e., n_1 is a multiple of p .

CALCULATION.

We shall put $k = \sin \theta$, $k' = \cos \theta$. Equation (9) becomes

$$\delta = \cot^2 \theta \left[1 - \sqrt{2} \cos 2\theta \left\{ \frac{K}{E} [E(w, k') - F(w, k')] + F(w, k') \right\} \right].$$

On account of the values assumed by $\cot^2 \theta$ between $\theta = 0^\circ$ and $\theta = 24^\circ$ approx., δ is greater than unity in this range. The upper limit ϕ of the integrals $F(w, k')$ and $E(w, k')$ is determined by $\sin \phi = \frac{1}{k' \sqrt{2}}$. ϕ becomes imaginary when $\sqrt{2} \cos \theta < 1$, i.e., $\theta > 45^\circ$. When $\theta = 45^\circ$, $k'^2 = \frac{1}{2}$ and $e = 1$

corresponding to parabolic orbits. The following table gives the values of R and δ for assigned values of θ .

θ .	R.	δ .
28	4.5568	.19701
29	4.6000	.22001
30	4.6515	.25512
31	4.7094	.30822
32	4.7605	.32059
33	4.8208	.37936
34	4.8900	.39861
35	4.9540	.40890
36	5.0275	.43500
37	5.1020	.46341
38	5.1820	.49339
39	5.2674	.52645
40	5.3560	.55901
41	5.4570	.58670
42	5.5668	.63469
43	5.6655	.68701
44	5.7824	.76310

It is obvious that, on account of the large variation possible in δ which is represented by a rational fraction between 0 and 1, a large number of lines will arise. We confine ourselves to eight simple values of δ obtained by interpolation, together with the appropriate value of $R\delta^2$ and the proper sequence for n_1 .

δ .	$R\delta^2$.	
$\frac{1}{5}$	1.2998	$n_1 = 1, 2, 3 \dots$
$\frac{1}{3}$.53040	$n_1 = 1, 2, 3 \dots$
$\frac{1}{4}$.29025	$n_1 = 1, 2, 3 \dots$
$\frac{1}{5}$.18250	$n_1 = 1, 2, 3 \dots$
$\frac{2}{5}$	2.5009	$n_1 = 2, 4, 6, 8 \dots$
$\frac{3}{5}$	3.2418	$n_1 = 3, 6, 9, 12 \dots$
$\frac{4}{5}$.78370	$n_1 = 2, 4, 6, 8 \dots$
$\frac{5}{5}$	1.9757	$n_1 = 3, 6, 9, 12 \dots$

A value of $1/\lambda$ is obtainable from the subtraction of a value of $NR'\delta'^2/n_1'^2$ from a value of $NR\delta^2/n_1^2$, where the values of n_1 and n_1' are different if the δ 's are the same, and where n_1 may equal n_1' or be different from n_1' if the δ 's are different. We shall use the notation $(n_1, n_2) - (n_1', n_2')$ to denote the values of n taken to obtain the wavelength which corresponds; e.g. $(6, 12) - (4, 2)$ indicates that $n_1 = 6$ is taken in the set for δ given by $\frac{6}{6+12} = \frac{1}{3}$, and that the corresponding value for

$R\delta^2/n_1^2$ is associated with $n_1' = 4$ in the value of $R'\delta'^2/n_1'^2$ corresponding to $\delta = \frac{4}{4+2} = \frac{2}{3}$.

The table which follows gives the first member of each of the first twenty groups of wave-lengths calculated with the foregoing values of δ and $n_1 \geq 12$. 150 wave-lengths were obtained; but as the computation has only been carried out correctly to four significant figures and the experimental values are given to 0.01 A.U., the table of calculated and observed values is intended as an illustration of the kind of agreement obtained. For purposes of calculation, the Rydberg value $N = 109675 \text{ cm.}^{-1}$ has been used. Hasselberg's values reduced to Rowland's standard are given, and Hasselberg's intensities have been ranged from 1 to 10 instead of from 1 to 6. Watson's values have been taken from the paper cited, and 10 has been taken as the maximum intensity. The value 0 for the intensity of a line indicates that the line is just visible.

	$\lambda \times 10^8 \text{ cm.}$ (calculated).	Watson.	Intensity.	Hasselberg.	Intensity.
(2, 2) - (3, 3)	5051	5049.53	2
(1, 3) - (2, 6)	4188	4188.42	0
(1, 4) - (3, 12)	5621	5621.09	0	5620.05	1
(2, 3) - (6, 9)	5232	5231.30	0
(3, 2) - (6, 4)	5538	5537.67	3	5537.40	6
(2, 2) - (1, 2)	4438	4437.77 (Frost)	...
(2, 2) - (2, 6)	3652	3652.88	0
(7, 7) - (1, 4)	5846	5847.84	0
(2, 2) - (6, 2)	3882	3882.19	0
(5, 5) - (2, 3)	6335	6335.53	0
(5, 5) - (3, 2)	5443
(1, 2) - (1, 3)	3797	3796.73	2	3796.8	5
(5, 10) - (1, 4)	5654	5655.98	1	5655.61	4
(6, 12) - (4, 2)	6439	6438.10	0
(2, 4) - (3, 1)	4006	4005.57	2
(4, 8) - (2, 3)	5601	5600.65	0	5599.65	4
(3, 6) - (3, 2)	5678	5676.86	0
(3, 9) - (1, 4)	6068	6067.96	1	6067.92	4
(1, 3) - (6, 3)	4131	4131.59	1
(5, 15) - (4, 2)	6300	6299.60	5	6297.90	5

The kind of agreement in the table given above is of the same order as that between the observed values of λ and the values of λ calculated from Balmer's formula in the case of the primary spectrum of hydrogen with the value of N quoted. Birge, in *Phys. Rev.*, vol. xvii, p. 589, 1921, by calculations based on the fine structure of the Balmer's series and the corresponding lines obtained by the application of relativity mechanics to the atomic model, obtains a value of $N = 109677.7 \pm 0.2$. He states that

“the application of relativity mechanics to Balmer’s formula results in a definite reduction of the discrepancy between observed and computed values, but does not in any way obliterate the discrepancy.” In several cases calculated values of λ seem to have no closely observed correlative, and many values of λ observed by one physicist are not obtained by another. Watson makes the following comment, which is instructive:— “It seems probable that as Hasselberg’s measurements in the yellow-green part of the spectrum were visual, and the eye has a maximum of sensitiveness in this place, these additional lines were too weak to be recorded photographically, and, indeed, indications of several lines can be seen which are too weak to measure.” It is surprising that such a few simple values of δ and n_1 should give rise to such a proportionally large number of values of λ . There is another point of interest which arises, and which deserves mention. In the secondary spectrum a considerable number of lines are found, the wave-lengths of which are less than $\lambda = 3646$, which is the theoretical limit of the primary spectrum according to Balmer’s formula. For a given n , the limit for $\lambda = n^2/N$ according to the Balmer formula, and from equation (10) the limit for $\lambda = n^2/NR\delta^2$. Since $R\delta^2$ can be greater than unity, as is indicated in the table for $R\delta^2$, it follows that equation (10) will give rise to lines of wave-length less than the minimum wave-length to be expected according to the Balmer formula.

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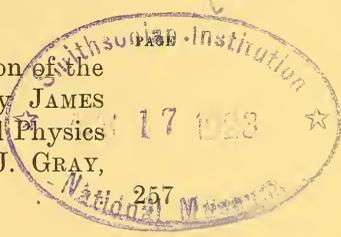
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XVIII.—On the Application of the Gyroscope to the Solution of the "Vertical" Problem on Aircraft. Part I. By James Gordon Gray, D.Sc., Cargill Professor of Applied Physics in the University of Glasgow; and Captain J. Gray, A.M.I.E.E. (late R.A.F.).

(MS. received March 6, 1922. Read March 6, 1922.)

BEFORE the advent of the aeroplane and airship the gyroscope derived its practical importance from its applications to the torpedo and the gyroscopic compass, and neither of these applications originated in this country. With the introduction of aircraft the applications of the gyroscope become very numerous and of great importance. Navigation of such craft must be made precise by the provision of gyroscopic sextants; photography from aeroplanes, to be accurate, must be carried out by means of special cameras stabilised so that the photographs are true vertical productions; war aeroplanes must be steered and controlled in accordance with accurate dynamical principles, and in this respect the claims of the gyroscope cannot be ignored; bombing from aeroplanes must be rendered accurate by designing the bombsight as part of an accurate stabiliser; battleships must be provided with anti-aircraft devices controlled by stabilisers, and the guns on such ships must be controlled both with respect to the vertical and the horizontal, and in azimuth, by means of instruments of the greatest precision.

The inventions described below form part of a series which have resulted, in great part, from long-continued research work carried out in the Natural Philosophy Institute of the University of Glasgow, and relate to apparatus for finding, maintaining, and thus defining, the true vertical and horizontal on aeroplanes and airships. The research was assisted, in its later stages, by means of a grant received from the Royal Society of Edinburgh (in 1915), and first models were made for the Admiralty and the Munitions Inventions Department of the Ministry of Munitions. The invention was adopted for bombing purposes by the R.N.A.S. in the autumn of 1917, but unfortunately, for reasons which cannot be discussed in this paper, the apparatus produced was not used over the German lines.

A bombsight rigidly attached to an aeroplane is of little use, inasmuch as it shares in all the pitching and rolling motions of the machine. In order that a bomb dropped from an aeroplane should hit a target on

the ground it is necessary that at the instant at which the bomb is dropped the aeroplane should be travelling in a line whose trace on the ground passes through the target, and this condition can only be fulfilled with certainty by stabilising the sight employed.

The bombsight used by the R.N.A.S. in 1917 was made up of systems of "line," and "range" or "transit," wires or bars mounted on a framework. The "line" bars consisted of two horizontal wires or bars, one above the other, placed fore and aft with respect to the aeroplane. When bombing "up wind" or "down wind" the aeroplane was steered so that the target was kept as nearly as possible in the plane of these bars. The "range" or "transit" bars consisted of horizontal wires or bars, and by observation of the transit of the target across these the proper instant at which to drop the bomb was determined. To be accurate such a contrivance must not partake of the pitching and rolling of the aeroplane on which it is mounted; the frame with its attachment is said to be stabilised when it is maintained flat (so that the line bars lie in a true vertical plane, a true vertical plane being one containing the thread of a simple pendulum set up on a fixed support, a support which is moving in a straight line at constant speed) no matter how the aeroplane may pitch or roll. If the line bars can be maintained in a truly vertical plane, and the target is on the plane of the bars at the instant of dropping the bomb, the first condition required for accuracy is fulfilled. Obviously it is of equal importance that the "range" bars, or their equivalent, should be stabilised. Hence it is necessary that the sight should be stabilised not only laterally, but longitudinally; errors due to rolling and pitching of the aeroplane must be entirely eliminated if the bombing is to be accurate.

Many inventors have planned apparatus for keeping a platform, mounted on a vehicle horizontal—making use of a gyroscope, or system of gyroscopes, as the means of defining the vertical or horizontal. It is widely supposed that a gyroscope mounted on gimbals so as to be free to precess in all directions will of itself assume the desired position, namely, that in which the axis of the gyroscope is vertical, or horizontal, as the case may be. To be successful a stabiliser must, so to speak, have a sense of the true vertical, and it must not be appreciably disturbed by the forces which it encounters.

Figs. 1 and 2 show, in elevation and plan respectively, a gyroscope mounted on trunnions with its axis of spin vertical. The casing of the gyroscope g is attached by means of pivots p_1p_1 to a gimbal frame f which surrounds the gyroscope casing, with enough clearance to permit

of the gyroscope turning about the pivots p_1p_1 . The frame f is pivoted at p_2p_2 to uprights u , u carried on a base b .

Provided that the lines of the pivots p_1p_1 , p_2p_2 are in the same plane and perpendicular to one another, that the frame f is in neutral equilibrium with respect to the pivots p_2p_2 , and that the intersection of p_1p_1 and p_2p_2 coincides with the centre of gravity of the gyroscope and its casing, the gyroscope is said to be "freely mounted."

In connection with what follows it is necessary to point out some of the properties possessed by a freely mounted gyroscope. If, for example, the gyroscope is turned about the pivots p_2p_2 so that its axis of spin is inclined to the vertical at an angle θ , and the base is then turned in a horizontal plane through 90° , the gyroscope is now inclined at an angle θ to the

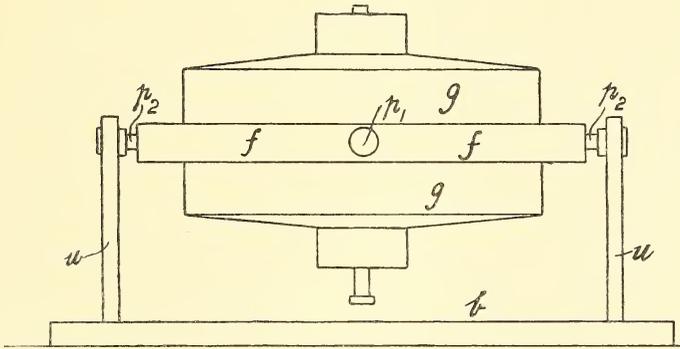


FIG. 1.

vertical, but about the pivots p_1p_1 ; the inclination of the axis does not move round as the base turns in azimuth. If the apparatus is set up on an aeroplane with the pivots p_2p_2 "fore and aft," turning of the plane in azimuth does not result in the carrying round with the machine of any deviation of the apparatus from the vertical which may exist; the deflection from the vertical remains in the same plane as before. Thus, suppose that at an instant the axis of spin is inclined to the vertical about the pivots p_2p_2 . Let the aeroplane now turn in a circle, going once round in time T' . At time $T'/4$ the inclination is about p_1p_1 ; at time $T'/2$ it is about p_2p_2 ; at time $3T'/4$ it is about p_1p_1 ; and at time T' it is about p_2p_2 ; and so on. If at time $t=0$ the apparatus is inclined to the vertical in such manner that the error in range (about the pivots p_1p_1) is θ_r and the error in line (about the pivots p_2p_2) is θ_l , then at time $T'/4$ the amounts of the errors in range and in line are θ_l and θ_r respectively.

The arrangement described is of little use on an aeroplane, inasmuch as it has not itself what may be called a sense of the vertical. It must be

set with respect to the vertical; and on a moving vehicle, subject to accelerations, any ordinary levelling device, such as a spirit-level or damped pendulum, is useless. Efforts have been made to use horizon glasses, or mirrors, for the purpose of setting the gyroscope. Such adjustments are troublesome to make, and are only possible when the horizon is visible. If the gyroscope is to be adjusted about both sets of pivots, the operation of setting the gyroscope becomes extremely difficult, if not impossible, to carry out.

Apparatus, to be effective in war time, must be automatic in action as far as possible. Where practicable, all "personal" errors should be avoided.

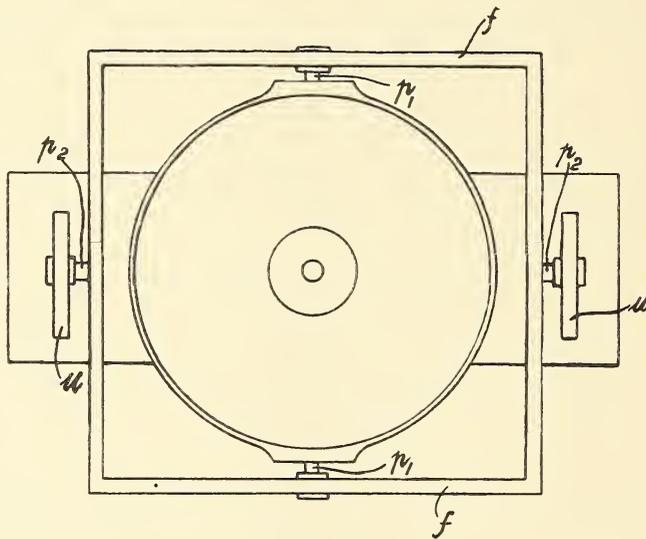


FIG. 2.

It is not fair to require an observer to set up a gyroscope by means of a horizon glass when the aeroplane, or other vehicle, is being shelled, nor is it to be expected that anything like accuracy can be obtained under such conditions.

To endow the gyroscope device above described with a sense of the vertical, many investigators have advocated the use of a gyroscope, or system of gyroscopes, provided with gravity control. In fig. 3 is shown a gyroscope *g* mounted so as to possess gravity control, the arrangement being identical with that illustrated in fig. 1, except that the centre of gravity of the pivoted system is brought below the plane of the pivots by the addition of the mass *m*, which is attached to the casing of the gyroscope *g* by the rod *r*. An identical result would be obtained by mounting the gyroscope within the frame *f* so that its C.G. lies below the line of the

pivots p_1p_1 . The pivoted system hangs from the uprights as shown, and the base b is supposed supported from above in a horizontal position. We suppose the mass m to lie vertically below the intersection of the pivot axes.

The principal properties possessed by this arrangement may be described briefly. In the first place, if the frame f is neutral with respect to the pivots p_2p_2 (that is, if the C.G. of the frame lies in the line p_2p_2) and the axis of spin is upright, the apparatus may be trained (that is, turned in azimuth) without the introduction of errors. If the apparatus is inclined to the vertical on the pivots p_2p_2 , the effect of turning the base through a right angle in azimuth is to transfer the inclination to the pivots p_1p_1 .

If at a given instant the inclination is wholly about the axis p_2p_2 and the base is turned quickly through 90° , the inclination is transferred to the axis p_1p_1 (neglecting the small change which occurs in the interval as a consequence of the existing precessional motion, to be described directly). When the arrangement as described is mounted on an aeroplane with the pivots p_2p_2 fore and aft, the turning of the plane does not, in itself, cause the inclination of the axis of spin to turn with the machine.

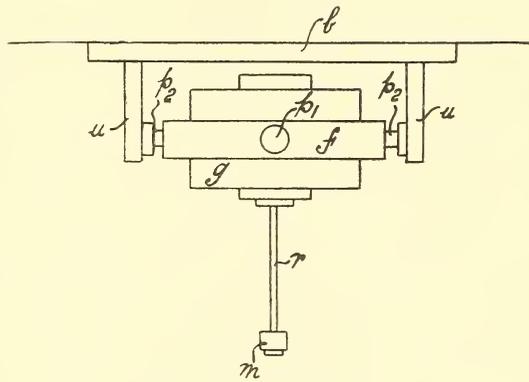


FIG. 3.

Let us suppose the apparatus illustrated in fig. 3 set up in a room. If the axis of spin is tilted from the vertical and the apparatus is left to itself, it assumes after a short time a state of steady precession. The rod r traces out a cone, the vertex of which is at the intersection of the pivot axes, and the semi-vertical angle θ of which is the inclination of the rod to the vertical. If the pivots were frictionless the angle θ would remain constant in amount; in practice θ diminishes very slowly (if the pivots are good) as a consequence of the retardation of the precessional motion by pivot friction. At any instant during steady precessional motion the horizontal component of angular momentum is $Cn \sin \theta$, where C is the moment of inertia of the flywheel of the gyroscope, n is its angular speed, and θ is the inclination of the axis of spin to the vertical. If T is the periodic time of the motion, this horizontal component of angular momentum is turning in azimuth with angular speed $2\pi/T$,

Consequently the rate of growth of angular momentum, in a horizontal direction perpendicular to the instantaneous direction of the horizontal component of spin, is $Cn \sin \theta (2\pi/T)$; and since this must be equal to the resultant couple acting on the gyroscope at the instant, we have, neglecting the couple which results from the centrewards acceleration of the mass m (which couple is very small when T is great),

$$Cn \sin \theta \frac{2\pi}{T} = mgh \sin \theta,$$

or

$$T = 2\pi \frac{Cn}{mgh},$$

where h is the distance of the C.G. of m from the intersection of the pivot axes.

In connection with what follows, the nature of the steady precessional motion should be carefully noted by readers who have not handled gyroscopes. If at a particular instant the pendulum is inclined wholly about the pivots p_2p_2 , then at a later instant it is inclined about p_2p_2 , and also about p_1p_1 . After time $T/4$ it is inclined wholly about p_1p_1 ; and so on. Further, if the pivots are good and Cn large, θ remains sensibly constant in amount over a considerable interval of time.

Let now the gyroscopic pendulum be supposed mounted on an aeroplane with the pivots p_2p_2 lying fore and aft. We suppose that initially the aeroplane is moving in a straight line with uniform speed and that the pendulum is upright. If now the aeroplane moves with constant acceleration a in the direction of motion, which we suppose remains unchanged, it is required to find the inclination of the axis of spin to the true vertical at time t , time being measured from the instant at which the state of uniform velocity is departed from.

Consider a simple pendulum $o'm'$ (fig. 4) carried by the aeroplane. When the velocity is uniform, the thread of the pendulum is vertical. When the aeroplane is accelerated in a constant direction as stated, the thread takes up the direction $o'm''$, where $o'm''$ lies in a vertical fore-and-aft plane. If F is the stretching force in the thread, and β the inclination of the thread to the true vertical, we have

$$F \cos \beta = m'g; \quad F \sin \theta = m'a;$$

$$\tan \beta = \frac{a}{g}; \quad F = m' \sqrt{g^2 + a^2}.$$

The force of gravity is replaced by a force whose amount per unit mass is $\sqrt{g^2 + a^2}$, and whose direction makes with the true vertical an angle given by $\tan \beta = a/g$. From what has been said already it will be clear, on

reflection, that the gyroscopic pendulum will not take up the direction of this apparent gravity, but will precess in such manner that (after a short interval of time) the rod r traces out a cone whose axis is oc and whose semi-vertical angle is $\tan^{-1} a/g$. [In reality, when a state of steady precession has been arrived at, the semi-vertical angle of the cone is slightly less (we suppose T' and Cn very great) than $\tan^{-1}(a/g)$. This follows from the fact that the precessing system possesses kinetic energy, which kinetic energy is obtained at the expense of potential energy. Further, at the start of the motion gyroscopic oscillations of small amplitude are set up, and continue until destroyed by pivot friction.]

The mass m moves in a circular path $abdea$, the plane of which is

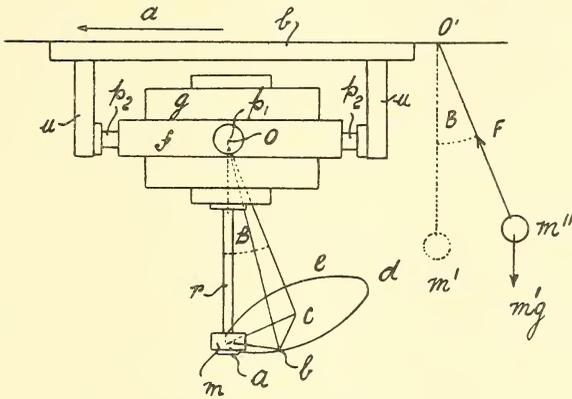


FIG. 4.

inclined to the horizontal at an angle $\tan^{-1} a/g$. Since g is replaced by $\sqrt{g^2 + a^2}$, the precessional period T' of steady precession is given by

$$T' = 2\pi \frac{Cn}{mh \sqrt{g^2 + a^2}}.$$

Let the mass m occupy the position b (fig. 4) at time t . We have

$$\angle acb = \frac{2\pi}{T'}t, \quad \text{and} \quad ab = 2ac \sin \frac{1}{2} \frac{2\pi t}{T'} = 2h \sin \beta \sin \frac{\pi t}{T'}.$$

The angle made by the axis Om of the pivoted system with the true vertical at time t is the angle aob . Hence, denoting this angle by θ , we have

$$\sin \frac{1}{2}\theta = \frac{\frac{1}{2}ab}{h} = \sin \beta \sin \frac{\pi t}{T'};$$

when $t = T'/4$, $\sin \frac{1}{2}\theta = \sin \beta \sin \frac{\pi}{4}$; when $t = T'/2$, $\sin \frac{1}{2}\theta = \sin \beta$, or $\theta = 2\beta$;

when $t = 3T'/4$, $\sin \frac{1}{2}\theta = \sin \beta \sin \frac{3}{4}\pi$; and when $t = T'$, $\theta = 0$.

In point of fact, an aeroplane moving in a straight path at full speed is subject to accelerations and retardations in the direction of motion, but these never last for long. If in the above equation we make θ very small, we obtain

$$\frac{1}{2}\theta = \sin \beta \frac{\pi t}{T'} = \frac{\pi at}{gT'}$$

since $\sin \beta = \frac{a}{\sqrt{g^2 + a^2}}$, $T' = \frac{2\pi Cn}{mh\sqrt{g^2 + a^2}}$, and $T = 2\pi \frac{Cn}{mgh}$.

It is obvious from the figure that when t is small the angle θ is due to turning of the pivoted system about the pivots p_2p_2 , which it is to be remembered are fore and aft.

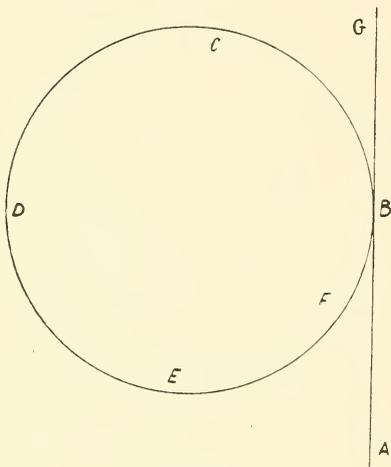


FIG. 5.

The above result may be obtained in a simpler manner as follows. Let the aeroplane be moving with uniform speed, and suppose the gyroscopic pendulum upright. If now the aeroplane is accelerated in the fore-and-aft direction the mass m experiences a force, in the fore-and-aft direction, of amount ma . The torque acting on the gyroscopic pendulum is mah , and this tends to turn the pivoted system about the pivots p_1p_1 . Such a torque would, in fact, turn the pendulum about p_1p_1 in the absence of the gyroscope. But in point

of fact the pendulum turns about the pivots p_2p_2 at angular speed $\dot{\theta}$ given by mah/Cn . Hence if θ is the deviation of the pendulum from the vertical at time t (considered very small), we have

$$\theta = \frac{mah t}{Cn} = \frac{2\pi at}{gT'}$$

as before. It is to be observed that at is the increase of speed attained by the aeroplane in time t .

We now consider the behaviour of the gyroscopic pendulum when it is mounted on an aeroplane which moves in a curved path. Let us suppose that the aeroplane moves in the path ABCDEFG (fig. 5), where the parts AB and BG are straight, and the part BCDEFG is circular. Assuming that at the instant at which the plane is at B the pendulum is upright, it is required to find its inclination to the vertical after the plane has turned through an angle ψ , at which instant the plane is at P_1 . We have the following sufficiently approximate solution.

Let, as before, the periodic time of steady precession be denoted by T , and suppose that the time of the aeroplane in the circle is T' . Let the aeroplane be at P at time t , counting time from the instant corresponding to B . The angle BOP is $\frac{2\pi}{T'}t$ (fig. 6). (This figure is intended to show the pivoted system at the instants corresponding to B , P , and P_1 . It will be readily understood that the scale of the diagram cannot be near the true scale. As a rule the radius of the curve in which a large bombing plane turns is about 200-250 yards.) The centrewards acceleration of the aeroplane is $v\frac{2\pi}{T'}$, where v is the linear speed of the aeroplane. At any instant the force acting on the mass m due to the centrewards acceleration of the plane is $mv\frac{2\pi}{T'}$, and this force turns in azimuth with the aeroplane. The couple experienced by the pivoted system is $mv\frac{2\pi}{T'}h$, and the angular speed of the precessional motion at the instant is this couple divided by the angular momentum of the gyroscope. Now at time t the aeroplane is at P , and the pendulum is turning instantaneously about the pivots p_1p_1 , which at the instant are parallel to OP .

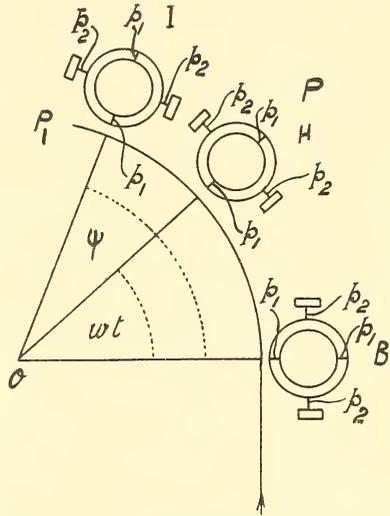


FIG. 6.

When the plane arrives at P_1 the pivots p_1p_1 are parallel, and the pivots p_2p_2 perpendicular, to OP_1 . Denoting the inclinations of the pivoted system to the vertical at time t , about axes parallel and perpendicular to OP_1 , by θ_1 and θ_2 , respectively, we have, with ψ the angle so marked in fig. 6,

$$\frac{d\theta_1}{dt} = \frac{mv\frac{2\pi}{T'}h}{Cn} \cos\left(\psi - \frac{2\pi}{T'}t\right)$$

and

$$\frac{d\theta_2}{dt} = \frac{mv\frac{2\pi}{T'}h}{Cn} \sin\left(\psi - \frac{2\pi}{T'}t\right).$$

The first of these equations gives, since $mh/Cn = 2\pi/(gT)$, where T is the period of the pendulum for steady precessional motion,

$$\theta_1 = -\frac{2\pi v}{gT} \sin\left(\psi - \frac{2\pi}{T'}t\right) + \text{const.},$$

and since $\theta_1 = 0$ when $t = 0$,

$$\theta_1 = \frac{2\pi v}{gT} \left\{ \sin \psi - \sin \left(\psi - \frac{2\pi}{T'} t \right) \right\}.$$

Similarly

$$\theta_2 = \frac{2\pi v}{gT} \left\{ \cos \left(\psi - \frac{2\pi}{T'} t \right) - \cos \psi \right\}.$$

For the instant at which the aeroplane arrives at P_1 we have $2\pi t/T' = \psi$, and then

$$\theta_1 = \frac{2\pi v}{gT} \sin \psi; \quad \theta_2 = \frac{2\pi v}{gT} (1 - \cos \psi).$$

The above results may be obtained more simply as follows. At time 0 the pivoted system is upright and there comes into existence a horizontal accelerating force, due to the turning of the aeroplane. This accelerating force is directed at each instant towards the centre of the path in which the aeroplane is turning. The mass m is constrained to move with the aeroplane, and hence there must be applied to it, at each instant, a force of amount $mv \frac{2\pi}{T'}$, directed towards the centre of the path. This force is applied at the pivots $p_2 p_2$, and so applied is equivalent to an equal force applied to m ,

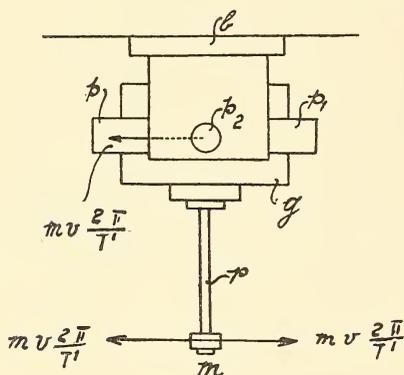


FIG. 7.

together with a couple of moment $mv \frac{2\pi}{T'} h$ applied to the pivoted system about the pivots $p_2 p_2$ (fig. 7). This couple turns with the aeroplane. Considerations of symmetry show that after the aeroplane has turned through an angle ψ there has been applied to the pivoted system an integral couple about an axis which bisects the angle between the initial and final directions of $p_2 p_2$. At time t the pivots $p_2 p_2$ make an angle $\frac{1}{2}\psi - \frac{2\pi}{T'} t$ with the direction of this axis, and the component l of the integral couple is therefore given by

$$l = mv \frac{2\pi}{T'} h \cos \left(\frac{1}{2}\psi - \frac{2\pi}{T'} t \right).$$

And since $l dt = C n d\theta$, where θ is the deflection of the pivoted system, with respect to the vertical, at time t about an axis perpendicular to the

direction of the line which bisects the angle between the initial and final directions of the pivots p_2p_2 , we have

$$\theta = -\frac{mvh}{Cn} \sin\left(\frac{1}{2}\psi - \frac{2\pi}{T}t\right) + \text{const.}$$

When $t=0$, $\theta=0$, so that the constant is $mvh \sin \frac{1}{2}\psi/Cn$.

Hence

$$\theta = \frac{mvh}{Cn} \left\{ \sin \frac{1}{2}\psi - \sin\left(\frac{1}{2}\psi - \frac{2\pi}{T}t\right) \right\}.$$

When the aeroplane has turned through the angle ψ we have $\psi = 2\pi t/T'$, and hence then,

$$\begin{aligned} \theta &= \frac{mvh}{Cn} 2 \sin \frac{1}{2}\psi \\ &= \frac{2\pi v}{gT} 2 \sin \frac{1}{2}\psi. \end{aligned}$$

Thus when the aeroplane has turned through an angle ψ the pendulum is deflected, with respect to the vertical, about a horizontal axis which bisects the angle between the initial and final horizontal projections of the pivots p_1p_1 . Denoting as before the deflections of the pivoted system about the pivots p_1p_1 and p_2p_2 by θ_1 and θ_2 , we have

$$\begin{aligned} \theta_1 &= \frac{2\pi v}{gT} 2 \sin \frac{1}{2}\psi \cos \frac{1}{2}\psi \\ &= \frac{2\pi v}{gT} \sin \psi; \\ \theta_2 &= \frac{2\pi v}{gT} 2 \sin^2 \frac{1}{2}\psi \\ &= \frac{2\pi v}{gT} (1 - \cos \psi). \end{aligned}$$

As an example, we suppose the pivoted system to have a periodic time, for steady motion, of 6 minutes, and the speed of the aeroplane to be 100 feet per second. The aeroplane is supposed to turn in a circular path, and the pivoted system to be initially upright. We have

$$\theta = \frac{2\pi \times 100 \times 57 \cdot 3}{32 \times 360} 2 \sin \frac{1}{2}\psi = 6 \cdot 25 \sin \frac{1}{2}\psi \text{ (in degrees)}$$

$$\theta_1 = 3 \cdot 12 \sin \psi \text{ (in degrees), } \theta_2 = 3 \cdot 12(1 - \cos \psi) \text{ (in degrees).}$$

From the above formulæ the following table has been constructed:—

TABLE I.—EXHIBITING THE PERFORMANCE OF A GYROSCOPIC PENDULUM, OF PRECESSIONAL PERIOD 6 MINUTES, WHEN SITUATED ON AN AEROPLANE EXECUTING A TURN.

Angle turned through by Aeroplane (ψ).	Total Deviation of Pendulum from Vertical (θ).	Error in Range (θ_1).	Error in Line (θ_2).
30°	1·62°	1·57°	0·42°
45	2·36	2·21	0·91
90	4·41	3·12	3·12
135	5·72	2·21	5·3
180	6·25	0	6·25
225	5·72	-2·21	5·30
270	4·41	-3·12	3·12
360	0	0	0

These results are exhibited graphically in fig. 8. If the spin of the gyroscope is clockwise, as seen from above, and the aeroplane moves round in the counter-clockwise direction, as seen from above, the curve (*a*) shows the growth of the deflection of the pivoted system with respect to the vertical. If the aeroplane turns in the clockwise direction the growth of the deflection is as in (*b*).

After the aeroplane has turned through 90° there is an error in line (θ_2), and also an error in range (θ_1); when it has turned through 180° the error is a maximum, and is wholly about the pivots p_2p_2 ; there is no error in range at this instant. After the aeroplane has turned through 270° there are errors in line and in range; and when a complete turn has been executed there is no deflection.

Again, suppose that the pendulum is upright and that the aeroplane yaws, the change in direction of motion amounting, say, to 5°. The resulting deviation from the vertical amounts to $3·12 \sin 5^\circ$, or 0·27 (in degrees).

As a further example we may find the minimum precessional period, for steady precessional motion, which the pivoted system must have if the disturbance due to a half-turn is not to exceed $\frac{1}{5}$ th of a degree. We have, if T is the period,

$$\frac{2\pi \times 100 \times 2}{32T} = \frac{1}{5 \times 57·3},$$

which gives

$$T = 11,260 \text{ seconds, or } 3 \text{ hours } 8 \text{ minutes.}$$

The above results, it is to be observed, hold for the case of an aeroplane which turns at a rate such that a complete circle is executed in a time

small in comparison with the periodic time of the gyroscopic pendulum. It will be clear that if the pendulum is to meet with success it *must have a real periodic time of precessional motion, or virtual periodic time in the presence of curvilinear motion on the part of the aeroplane, which is upwards of 1 hour.* The view is widely held (see various papers published in the *Proceedings of the Advisory Committee for Aeronautics* and the *Proceedings of the Aeronautical Research Committee*) that provided the periodic time of a gyroscopic pendulum exceeds 4 minutes, the aeroplane may manœuvre without serious errors being introduced. *Such pendulums are useless.*

If such a pendulum as that described above is placed on an aeroplane we see that it is deflected from the vertical by motion in a curved path, assuming it to be upright initially. When the aeroplane resumes straight flight, following on a turn, the gyroscopic pendulum precesses so that its axis of spin traces out a cone, the semi-vertical angle of which diminishes

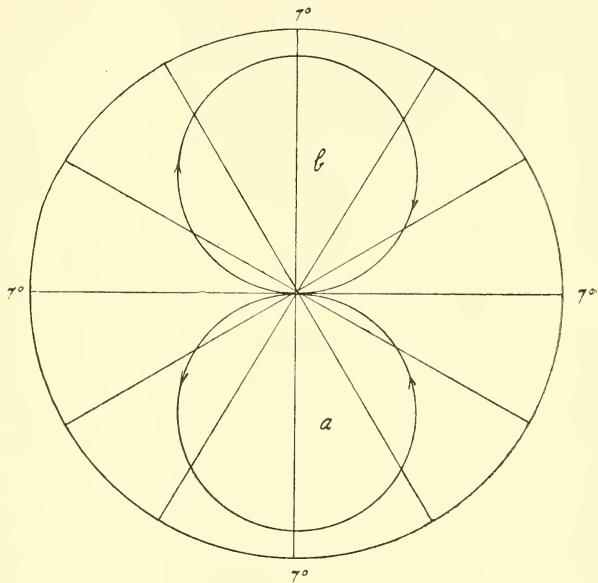


FIG. 8.

gradually as a result of pivot friction. If the pivots are good the axis will approach the vertical very slowly indeed, and consequently much time must elapse before the axis may be regarded as lying even approximately in the true vertical.

To get over this difficulty many exponents of gyroscopics have advocated the provision of a dashpot, or the equivalent. One member of the dashpot is attached to the aeroplane and the other to the pendulum; or a vessel containing oil, or other viscous liquid, is mounted on the pendulum, and the construction is such that, as the pendulum precesses, the potential energy possessed by the pivoted system, when deflected, is dissipated in heat within the oil as a result of the precessional motion. Attached to the casing of the gyroscope *g* (fig. 1) are two pairs of reservoirs, each pair being connected together by means of a tube having a narrow bore. One pair of tubes lies in the line of p_2p_2 and the other in the line of p_1p_1 . The

gyroscope, with its attachments, is mounted within the frame f so that the centre of gravity of the system lies considerably below the plane of the pivot axes.

The action is as follows. When the pivoted system is inclined to the vertical, precession takes place, with the result that, as already explained, the axis of spin moves round so as to sweep out a cone. If, for example, at an instant the inclination is about the pivots p_2p_2 , the system is turning instantaneously about the pivots p_1p_1 . Now consider the reservoirs which lie transverse to the pivots p_2p_2 . One of these is higher than the other and oil moves down the slope, so that it collects in the lower bottle. As time proceeds and the tilt moves round as a result of the precessional motion, it follows that the gyroscopic system is called upon to raise the heavier bottle. Thus work is being continually done by the gyroscopic system, which work is performed at the expense of the potential energy possessed by the system. This energy is of course dissipated in heat within the dashpot.

It will be clear that when the tilt becomes small the action of the dashpot becomes feeble. As the tilt becomes smaller the rate of dissipation of energy in the dashpot diminishes, and, with the construction described, will cease before the true vertical is attained. This follows from the fact that the oil must determine, to some extent, the resting position of the pivoted system, inasmuch as the resting position depends on the distribution of the mass of the entire pivoted system.

Clearly the rate of dissipation of energy in the dashpot, for a given tilt, diminishes as the periodic time of the pivoted system is increased. When the period is great the pivoted system, when disturbed, returns very slowly towards the true vertical; and also the resting position is seriously affected. When the period is small the disturbances introduced by turning motions of the aeroplane are very great. To obtain a quantitative idea of the mechanics of viscous damping, as applied to gyroscopic pendulums contrived with the object of defining the true vertical on aircraft, consider again the device shown in fig. 7. Let it be supposed set up on a table and deflected with respect to the vertical. It precesses, after the manner already described, in periodic time T given by

$$T = 2\pi \frac{Cn}{mgh}.$$

The angular speed of the rod r about the intersection of the pivot axes is $\frac{2\pi}{T} \sin \theta$, where θ is the inclination of r to the vertical. It is characteristic of viscous resistance, for the speeds with which we are here concerned,

that it is proportional to the speed of the resisted body; and if we assume that the pendulum is provided with a dashpot, which results in the resisting couple being proportional to the angular speed of the rod r , we have for the couple $k\frac{2\pi}{T}\sin\theta$, where k is a constant. The rate at which energy is being dissipated in heat within the dashpot, being the product of the couple and the angular speed, is $k\frac{4\pi^2}{T^2}\sin^2\theta$. The potential energy of the system at the instant is $mgh(1-\cos\theta)$, and the rate of diminution of potential energy is $-mgh\sin\theta\dot{\theta}$. Hence

$$k\frac{4\pi^2}{T^2}\sin^2\theta = -mgh\sin\theta\frac{d\theta}{dt}.$$

Since $T = 2\pi\frac{Cn}{mgh}$, this equation may be written

$$dt = -\frac{T Cn}{2\pi k} \frac{d\theta}{\sin\theta},$$

which gives

$$t = -\frac{T Cn}{2\pi k} \log \tan \frac{1}{2}\theta + \text{const.};$$

and denoting the value of θ at time $t=0$ by θ_0 , we have finally

$$t = \frac{T Cn}{2\pi k} \log \frac{\tan \frac{1}{2}\theta_0}{\tan \frac{1}{2}\theta}.$$

As an example we take the following. A gyroscopic pendulum is mounted on an aeroplane, the pendulum being damped by means of a dashpot. The aeroplane executes a half turn and thereafter proceeds in a straight line. It is required to investigate the behaviour of the pendulum.

Let the speed of the aeroplane be 100 feet per second, the precessional period of the pendulum be 6 minutes, and the angular momentum of the gyroscope be 250, in foot, pound, second units. If the pendulum is initially upright, then after the half turn its inclination to the vertical is $6\frac{1}{4}$ degrees, as shown in a previous example. Straight-line motion on the part of the aeroplane is now assumed. Precession takes place, and the value of θ gradually diminishes. We assume a value of k such that when $\theta=30^\circ$ the value of $\dot{\theta}$ amounts to 1 degree in 10 seconds of time, which corresponds to a very powerful dashpot. This gives $k=50$, and

$$t = 280 \log \frac{\tan \frac{1}{2}\theta_0}{\tan \frac{1}{2}\theta},$$

with the values for T and Cn which have been specified.

The behaviour of the pendulum is set out in the following table. Time is counted from the instant at which the aeroplane resumed straight flight following upon the half turn.

TABLE II.—ACTION OF DASHPOT.

Value of θ in degrees.	Time in minutes.
6.25	0
5	1.06
4	2.10
3	3.48
2	5.39
1	9.0
$\frac{1}{2}$	11.9
$\frac{1}{3}$	16.4

These results are shown in fig. 9, which illustrates the vice inherent in the dashpot idea. Assuming that the aeroplane executes the half turn in $\frac{1}{3}$ of

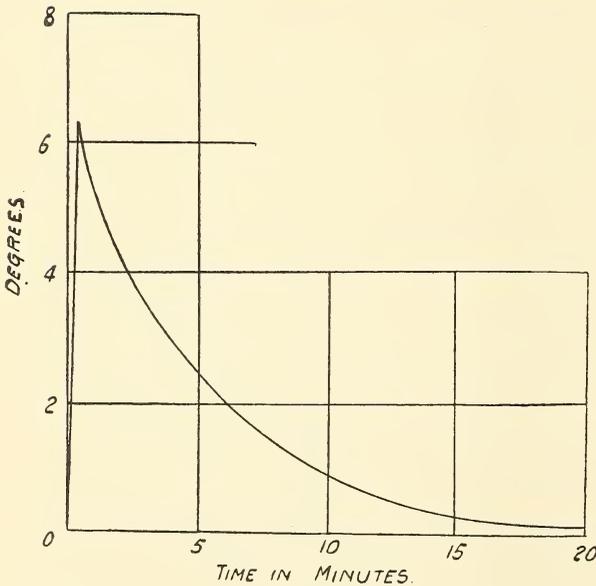


FIG. 9.

a minute, we see that in that time an error of 6.25 degrees is introduced and that it takes the pendulum 16.5 minutes to recover to within $\frac{1}{3}$ of a degree. Under the conditions which prevail on aircraft, gyroscopic pendulums which depend for their action on gravity control and dashpots are of little or no use; *under such conditions these devices leave the vertical quickly and return very slowly; they leave the true vertical with ease and return with difficulty.*

A gyroscopic pendulum of the type described, in which the erecting action depends on the existence of precessional motion, when supported on a steady platform at rest, is subject to an error due to the rotation of the earth. The resting position of the device is one in which the pivoted system is inclined to the vertical at an angle such that the couple which results from the tilt causes the pendulum to precess with the

angular speed of the earth. Thus for a pendulum of the type described set up at the equator, we have approximately

$$\frac{mgh \sin \theta}{Cn \cos \theta} = \frac{2\pi}{\text{Day}},$$

or

$$\tan \theta = \frac{2\pi}{\text{Day}} \frac{Cn}{mgh} = \frac{T}{\text{Day}},$$

where T is the precessional period of the pendulum for steady precessional motion. If the pendulum is set up in latitude λ , we have

$$\tan \theta = \frac{T}{\text{Day}} \cos \lambda.$$

The following table shows the magnitude of this error for various periodic times. The pendulum is supposed set up at a place in latitude 50° .

TABLE III.—EXHIBITING ERROR DUE TO EARTH'S ROTATION FOR THE CASE OF A GYROSCOPIC PENDULUM PROVIDED WITH GRAVITY CONTROL AND VISCOUS DAMPING.

Periodic Time of Pendulum in minutes.	Approximate Error, in Latitude 50° , due to Rotation of Earth.
2	0.045
4	0.09
8	0.18
16	0.36
32	0.72

The above errors relate to a pendulum set up on a steady platform at rest. If the pendulum is carried on a moving vehicle, for example an aeroplane, the errors due to the motion of the vehicle are so large that it is unnecessary to consider those due to the earth's rotation.

COMPLETE SOLUTIONS OF THE PROBLEM. THE GRAY STABILISER.

The following are the requirements which must be fulfilled by a contrivance which completely solves the problem of the true vertical:—

1. It must find and maintain the true vertical with minute accuracy.
2. It must provide for the stabilising of instruments against both rolling and pitching motions of the vehicle on which it is mounted.
3. It must not be sensibly disturbed by such rolling and pitching motions.

4. It must not be sensibly disturbed by azimuthal turning of the vehicle, or by the accelerating forces which result from such turning.

5. It must not be disturbed by the accelerations which result from side-slipping of the vehicle.

6. It should provide a non-revolving platform on which may be mounted instruments of precision, such as sextants, compasses, bomb-sights, and cameras.

7. For use on an aeroplane the construction should be such as to allow of the aeroplane banking up to 90° .

3. There must be no error due to the rotation of the earth.

PRINCIPLE OF THE GRAY STABILISERS.

It has been shown that long-period gyroscopic pendulums, provided with viscous damping, cannot give reliable results when set up on aircraft. As already explained, such devices leave the vertical quickly under the action of the forces which accompany turning movements of such craft, and return to the vertical only very slowly in the absence of such forces. In the Gray stabilisers, about to be described, this defect is entirely removed.

The Gray stabiliser consists of a gyroscopic system, pivoted as in figs. 1 and 2, and having the following properties. Should the pivoted system be inclined to the vertical, during normal flight of the aeroplane or airship, a stabilising couple is applied to the gyroscope and the device is restored to the vertical. The stabilising couple is obtained by means of a special erecting device, and depends in no way on precession of the gyroscope, in the ordinary sense of the term. During curved flight of the aeroplane the erecting device goes automatically out of action. Thus the pivoted system leaves the true vertical very slowly, if at all, in the presence of curved flight, and approaches the vertical relatively quickly during normal flight.

Consider a gyroscope freely mounted as in figs. 1 and 2. The arrangement is shown diagrammatically in plan in fig. 10. A gyroscope g is attached by means of pivots p_1p_1 , to a frame f , and f is in turn attached by pivots p_2p_2 to uprights u, u . The gyroscope is supposed mounted so that its centre of gravity lies at the intersection of the pivot axes. In order to make clear the action of the Gray stabiliser it is necessary to describe further some of the properties possessed by the arrangement. In the first place, suppose that the axis of the gyroscope is vertical. When this is the case the pivots p_1p_1 are horizontal. Let now a couple of moment l be applied to the gyroscope in a vertical plane passing through

p_1p_1 , or in a parallel plane. The mode of applying this couple is of great importance, and will be described immediately. At present suppose the couple produced, no matter how. This couple tends to turn the gyroscope about p_2p_2 , and would in fact do so if the flywheel of the gyroscope were not spinning; but in point of fact the gyroscope turns about the pivots p_1p_1 at the appropriate angular speed. The couple being applied wholly about the pivots p_2p_2 and the system being freely mounted, the gyroscope turns wholly about the pivots p_1p_1 . After time t has elapsed let the inclination of the gyroscope to the vertical be θ . The horizontal component of angular momentum is $Cn \sin \theta$, and the rate at which this is increasing is $Cn \cos \theta \dot{\theta}$. Hence

$$\frac{d\theta}{dt} = \frac{l}{Cn \cos \theta}.$$

If l remains constant in amount and acts always in a vertical plane—that is, if the couple does not depend on the magnitude of θ —then $\dot{\theta}$, or

$\frac{d\theta}{dt}$, is given by the above equation. The couple acts so as to increase θ .

A couple so applied we call a deflecting couple.

Let now the gyroscope be inclined at an angle θ to the true vertical about the pivots p_1p_1 . If there is applied to the gyroscope a couple in a vertical plane passing through p_1p_1 , or in a parallel plane, the gyroscope will, of course, turn about p_1p_1 . For one direction of the applied couple, θ will be increased; for the other, θ will be diminished. If the couple tends to diminish θ , it is called an erecting couple; and if its moment l is independent of the tilt, the axis of the gyroscope moves towards the true vertical in accordance with the equation

$$\frac{d\theta}{dt} = -\frac{l}{Cn \cos \theta}.$$

This equation gives

$$\sin \theta = -\frac{l}{Cn}t + \text{const.};$$

and if we denote the value of θ when $t=0$ by θ_0 , we have

$$t = \frac{Cn}{l}(\sin \theta_0 - \sin \theta).$$

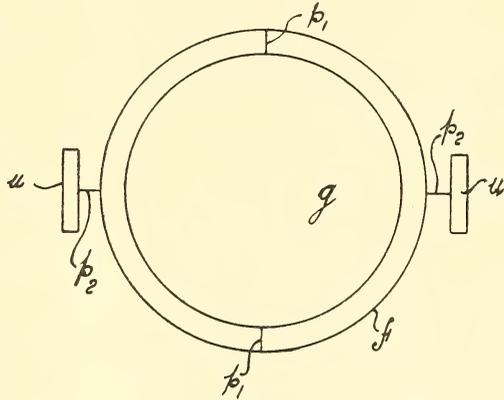


FIG. 10.

Hence the axis of the gyroscope is upright after time $Cn \sin \theta_0 / l$. When θ_0 is small the angular speed of recovery is given by l/Cn , and the time of recovery by $Cn\theta_0/l$.

Again, let the gyroscope be deflected, with respect to the vertical, about the pivots p_2p_2 , and let a couple of moment l , acting in a vertical plane containing p_2p_2 (or in a parallel plane), be applied to the gyroscope. The effective couple about the pivots p_1p_1 is $l \cos \theta$, where θ is the deflection of the gyroscope on the pivots p_2p_2 ; and if the couple is an erecting one, we have

$$\frac{d\theta}{dt} = -\frac{l \cos \theta}{Cn}.$$

This equation may be written

$$\frac{l}{Cn} dt = -\frac{d\theta}{\cos \theta},$$

and its solution is

$$t = -\frac{1}{2} \frac{Cn}{l} \log \frac{1 + \sin \theta}{1 - \sin \theta} + \text{const.};$$

and if $\theta = \theta_0$ when $t = 0$,

$$t = \frac{1}{2} \frac{Cn}{l} \left(\log \frac{1 + \sin \theta_0}{1 - \sin \theta_0} - \log \frac{1 + \sin \theta}{1 - \sin \theta} \right).$$

The time taken by the gyroscope to reach the vertical is thus

$$\frac{1}{2} \frac{Cn}{l} \log \frac{1 + \sin \theta_0}{1 - \sin \theta_0},$$

which reduces to $Cn\theta_0/l$ when θ_0 is small. Thus for small tilts the rates of recovery about the two axes are identical. It is usual in dealing with pivoted gyroscopic systems to assume that the arrangement of pivots shown in fig. 10 may be regarded as an ideal universal joint.

If the constant couple l comes into existence when the deflection amounts to, say, four minutes of angle, then the resting position of the pivoted system will be definite within limits much closer than those required on an aeroplane for purposes of bombing.

Suppose now the pivoted system set up on an aeroplane with the pivots p_2p_2 lying fore and aft. Let the axis of the gyroscope be initially upright, and suppose the aeroplane to move in a circular path.

Let the horizontal acceleration which accompanies the turning motion of the aeroplane result in the establishment of the couple of moment l , the direction of the couple being such as to cause the gyroscope to turn, at any instant, on the pivots p_2p_2 towards the apparent vertical. (It is to be

clearly understood that the gyroscope g is supposed to be in neutral equilibrium with respect to the pivots p_1p_1 and p_2p_2 , and that the turning of the gyroscope on the pivots is due solely to the action of the couple of moment l .) The total deflection of the axis of the gyroscope from the true vertical, after the aeroplane has turned in azimuth through an angle ψ , is easily obtained. In fig. 11, (a) shows the pivoted system in the initial position, and (b) the pivoted system in the final position, after the aero-

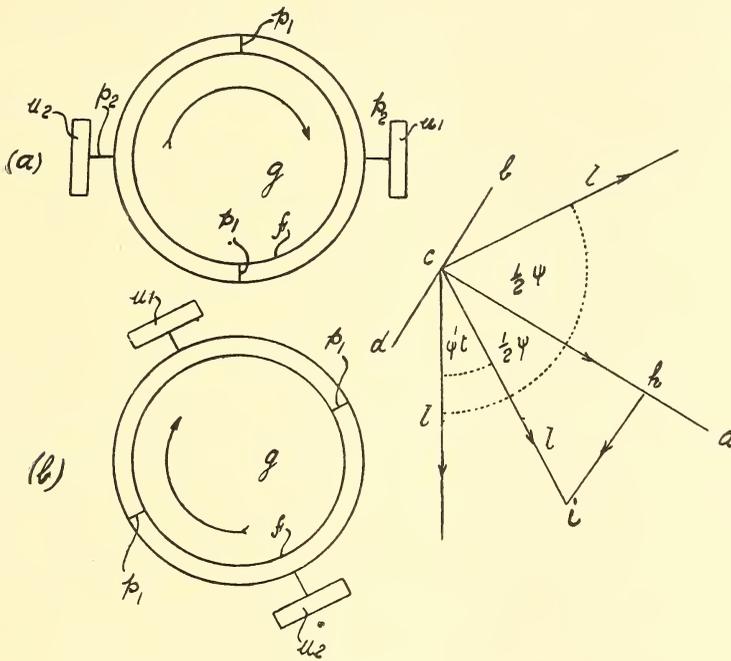


FIG. 11.

plane has turned through the angle ψ . The following is the approximate solution.

The couple acting at any instant (of constant moment l) may be represented by a line drawn perpendicular to the vertical plane containing p_2p_2 at the instant. This line turns round with the aeroplane at angular speed $\dot{\psi}$, and describes a total angle ψ . Let ac (fig. 11) bisect the angle between the initial and final directions of the vector which represents the couple. At time t the couple is represented by ci , and is equivalent to two couples represented by ch and hi respectively. The first of these causes the gyroscope to turn about an axis parallel to db , where db is perpendicular to ca , and the latter causes the gyroscope to turn about an axis perpendicular to db . When ci lies along ca the gyroscope is turning wholly

about the axis parallel to db , and when ci passes to the other side of ca the direction of the second component couple changes sign. During the first half of the interval of time corresponding to ψ the gyroscope is turning about an axis parallel to db and also about an axis perpendicular to db . The same applies to any instant during the second half of the interval; but whereas the turning about the former axis is in the same direction as before, that about the latter axis has been reversed. After the aeroplane has turned through the angle ψ the gyroscope has been deflected wholly about an axis parallel to db . The total deflection θ is given by

$$\theta = \frac{2l}{\dot{\psi}Cn} \sin \frac{1}{2}\psi.$$

If θ_1 and θ_2 denote the inclinations of the gyroscope about the pivots p_1p_1 and p_2p_2 respectively, we have

$$\theta_1 = \frac{l}{\dot{\psi}Cn}(1 - \cos \psi), \quad \theta_2 = \frac{l}{\dot{\psi}Cn} \sin \psi.$$

The aeroplane turns for time $\psi/\dot{\psi}$, and after that time the gyroscope is inclined to the true vertical at an angle $2l \sin \frac{1}{2}\psi/\dot{\psi}Cn$. If now the aeroplane resumes straight flight, the pivoted system approaches the true vertical at angular speed l/Cn , and is upright in time $2 \sin \frac{1}{2}\psi/\dot{\psi}$, or $2t \sin \frac{1}{2}\psi/\psi$, where t is the time taken by the aeroplane to turn through the angle ψ .

For small values of ψ , as for example the turns brought about by yawing motion of the aeroplane, the time required for correction of a deviation from the vertical is equal to the time spent in the turn. If the aeroplane turns through 90° in time t , and thereafter moves in a straight path, the deviation introduced during the turn is corrected in time $2\sqrt{2}t/\pi$. If ψ is 180° —that is, if the aeroplane executes a half turn—the resulting error is corrected in time $2t/\pi$, where t is now the time in the half turn. Thus, by operating the gyroscopic system in the manner described, the disadvantages which are inherent in gyroscopic pendulums depending for their action on the provision of gravity control and viscous damping are completely eliminated.

In calculating the data set out in the following table the angular momentum of the gyroscope is taken as 250, in foot, pound, second units, l as $\frac{1}{2}$ of a poundal at an arm of 1 foot, and the aeroplane is supposed to turn at an angular speed of 9° per second (in 1917 it was usual for a bombing plane, just before running up to a target, to execute a half turn, and the half turn was executed in about 20 seconds). After turning

through the angle ψ the aeroplane is supposed to fly in a straight path. The times given in the fifth column of the table are those which elapse, in each case, before the pivoted system is upright.

TABLE IV.—PERFORMANCE OF EARLY FORM OF GRAY STABILISER WHEN MOUNTED ON AN AEROPLANE EXECUTING A TURNING MOVEMENT.

Angle turned through by Aeroplane (ψ), in degrees.	Total Deviation of Stabiliser from Vertical (θ), in degrees.	Error in Range (θ_1), in degrees.	Error in Line (θ_2), in degrees.	Time required for Correction of existing Error on resumption of Straight Flying, in seconds.
30	0.15	0.04	0.14	3.3
45	0.22	0.07	0.21	4.8
60	0.29	0.14	0.25	6.3
90	0.41	0.29	0.29	8.9
135	0.52	0.50	0.21	11.3
180	0.58	0.58	0	12.6
225	0.52	0.50	-0.21	11.3
270	0.41	0.29	-0.29	8.9
315	0.22	0.07	-0.21	4.8
360	0	0	0	0

If we suppose that a bombsight forms part of the pivoted system, θ_1 gives the error in range, and θ_2 the error in line, after the aeroplane has turned through the angle ψ . When ψ is 90° —that is, after a quarter turn—the errors are equal and amount to $\frac{2}{15}$ of a degree; the time required for correction is 9 seconds. After a half turn (the most important case) there is no error in line, and the error in range amounts to $\frac{2}{3}$ of a degree. As the line bars, or the equivalent, are in a truly vertical plane immediately following upon the half turn, and remain in a truly vertical plane (the correction of the range error takes place wholly about the pivots p_1p_1), no error is introduced in the approach to the target. The error in range is corrected long before the "range" bars are required.

We have seen that if a gyroscopic pendulum, provided with gravity control, is mounted on an aeroplane, the deflection from the true vertical following on a half turn is $4\pi v/gT$, where v is the speed of the aeroplane and T the periodic time of the pendulum for steady precessional motion. If v is 100 feet per second, and the deflection is not to exceed 0.58 degree, the periodic time must be upwards of an hour. The length of a simple pendulum which would have this period is about 2000 miles.

In fig. 12 is shown diagrammatically a form of Gray stabiliser which was used in carrying out early experiments in the air at the R.N.A.S. Experimental Aerodrome at Grain. A gyroscope g is pivoted at p_1p_1

(only one pivot is shown in the diagram) to a frame *f*, which is in turn pivoted at p_2p_2 to uprights *u*, *u* carried by a base *b* which is attached

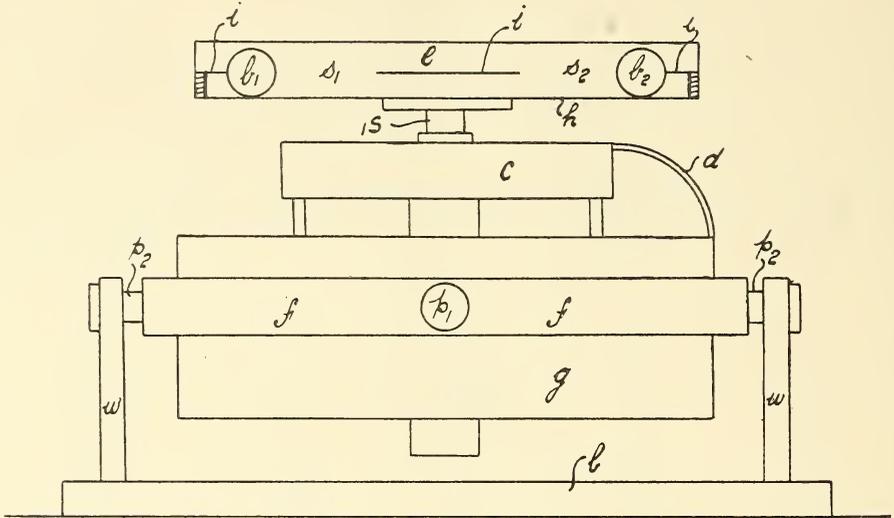


FIG. 12.

rigidly to the floor of the fuselage of the aeroplane, with the pivots p_2p_2 lying fore and aft.

Contained within the case *c* is a small turbine wheel, and this is driven rapidly by means of high-speed air led through the pipe *d* from the outside of the gyroscope casing. This turbine, with reduction gearing, serves to rotate the spindle *S* slowly in the direction of spin of the gyroscope *g*. Mounted on the spindle *S* is an erector *e*, the construction of which is illustrated in figs. 12 and 13.

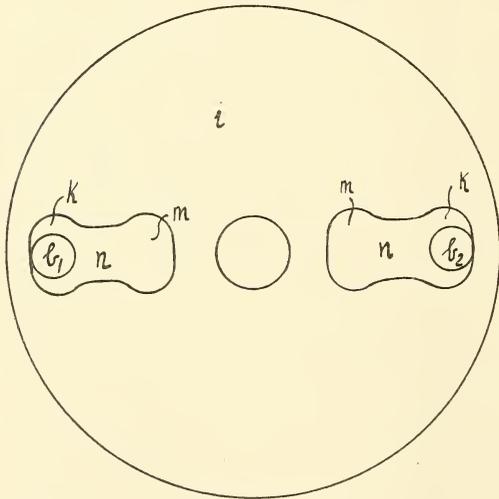


FIG. 13.

peculiar shape formed in the plate *i*. As will be seen from fig. 13, each slot or hole consists of an outer part *k* and an inner part *m* connected together by a narrowed part, or neck, *n*. Each of the slots or holes s_1, s_2

Contained within a shallow cylindrical box are two solid spherical steel balls b_1, b_2 . These rest on a rigid plate *h*, and each is surrounded by a slot of

surrounds a solid spherical steel ball, and their function is to constrain the balls so to move relatively to the plate h , beneath the plate i , when the latter is inclined to the horizontal, that an integral erecting couple is applied to the gyroscope.

The action will be clear from fig. 14. The erector e , it is to be understood, is revolving slowly in the direction of spin of the gyroscope g . We suppose the pivoted system inclined to the vertical about the axis p_2p_2 , the erector being turned towards the observer. The direction of rotation of the gyroscope, and of the erector, is supposed clockwise, as seen from above.

Since the spindle S is inclined to the vertical (about the pivots p_2p_2), each slot moves in a circular path which is inclined to the horizontal.

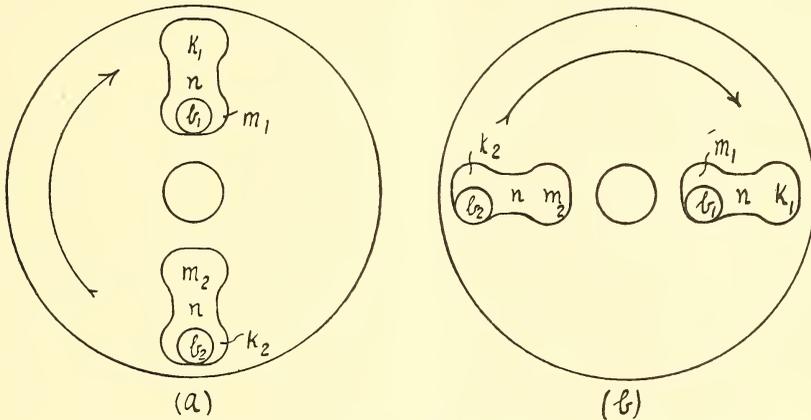


FIG. 14.

Fig. 14 (a) shows the plate i and the balls at an instant at which the slot s_1 is at the crest of the slope. The slot s_2 is now at the lowest position of its path. The ball b_1 has rolled down the slope from k_1 to m_1 , and b_2 has rolled down from m_2 to k_2 . During the ensuing half turn of the erector the balls b_1 and b_2 remain within the inner and outer parts, respectively, of their containing slots. This is a consequence of the peculiar shape of the slots and of the rotation of the erector. The action will be clear from fig. 14 (b), which shows the plate and balls after the erector has advanced through a quarter turn.

After the completion of the half turn of the erector, the ball b_1 moves from the inner to the outer part of s_1 , and similarly b_2 moves from the outer to the inner part of s_2 . Thus, so long as the erector is inclined to the horizontal, each ball when moving uphill is situated at a greater distance from the centre of the plate i than when moving downhill, and, with the direction of the erector that of the gyroscope, this results in the

establishment of an integral erecting couple. The action of the device is illustrated further in fig. 15. As before, the pivoted system (fig. 12) is supposed tilted, with respect to the vertical, about the pivots p_2p_2 , the erector being inclined towards the reader. As the erector rotates, each ball moves so that its centroid describes, relatively to the plate h , the path $abcdef$, where abc and def are semicircles of radii r_1 and r_2 respectively. Consider the instant at which the balls occupy the positions shown. Clearly, if m is the mass of a ball, there is applied, at the instant,

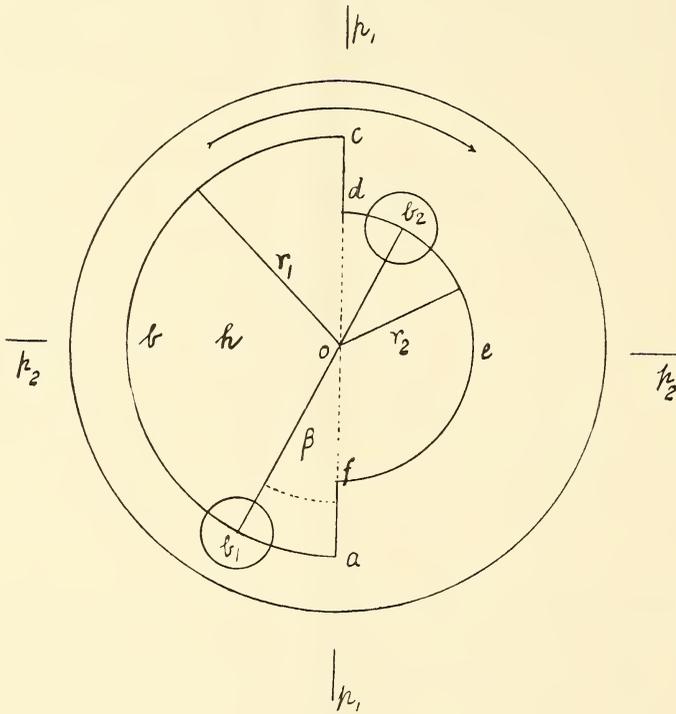


FIG. 15.

to the pivoted system a couple of moment $mg(r_1 - r_2)$, about an axis perpendicular to the line b_1b_2 , and hence at the instant the gyroscope is precessing about an axis parallel to b_1b_2 . As the erector rotates, this couple rotates with it; the moment of the couple remains constant, but the plane in which it acts moves round in azimuth at the speed of the erector. At the start of a half turn the couple is applied wholly about p_2p_2 ; after a quarter turn it is applied wholly about p_1p_1 ; and after the half turn it is again applied about p_2p_2 , but in the opposite direction. So long as b_1 lies between a and b , and b_2 between d and e , the applied couple has components about each of the axes p_1p_1 and p_2p_2 . Call these c_1 and c_2 . At the start of

a half turn, $c_1=0$ and $c_2=mg(r_1-r_2)$. After a quarter turn, $c_1=mg(r_1-r_2)$ and $c_2=0$. After a half turn, $c_2=-mg(r_1-r_2)$ and $c_1=0$. During the half turn executed at uniform angular speed, c_1 increases from 0 to $+mg(r_1-r_2)$ and then diminishes to zero; at the same time c_2 diminishes from $mg(r_1-r_2)$ to 0 and increases, with its direction changed, to $mg(r_1-r_2)$. Thus, after the completion of the half turn the pivoted system has turned wholly about the pivots p_2p_2 . And since the action is continuous the erector applies to the gyroscope an integral couple which causes it to approach the vertical wholly about the pivots p_2p_2 .

With the balls in the positions shown in fig. 15, we have for the couple applied at the instant, about the pivots p_1p_1 , $mg(r_1-r_2)\sin\beta$, and for the angular speed at which the pivoted system is approaching the vertical on the pivots p_2p_2 , $mg(r_1-r_2)\sin\beta/Cn$, where Cn is the angular momentum of the gyroscope. Counting time from the instant at which the ball b_1 was at a , and denoting the angular speed of the erector by ω , we have

$$\frac{d\theta}{dt} = -\frac{mg(r_1-r_2)\sin\omega t}{Cn},$$

where θ is the inclination of the pivoted system (in the pivots p_1p_2) to the vertical at time t . This equation gives

$$\theta = \frac{mg(r_1-r_2)}{Cn\omega} \cos\omega t + \text{const.}$$

Let $\theta=\theta_0$ at time $t=0$, and we have

$$\theta = \theta_0 - \frac{mg(r_1-r_2)}{Cn\omega}(1-\cos\omega t).$$

This equation holds for any instant during the half turn. The ball b_1 arrives at c and the ball b_2 at f , after time π/ω . If the inclination of the pivoted system to the vertical is then θ_1 , we have, since $\cos\omega t$ has then the value -1 ,

$$\theta_0 - \theta_1 = \frac{2mg(r_1-r_2)}{Cn\omega}.$$

Now $\theta_0-\theta_1$ is the angle through which the system has turned towards the vertical in time $T/2$, where T is the periodic time of the erector. The average angular speed of erection is thus $(\theta_0-\theta_1)/(T/2)$, and since $\omega=2\pi/T$,

$$\frac{\theta_0 - \theta_1}{T/2} = \frac{2mg(r_1-r_2)}{\pi Cn}.$$

In the erector as shown in the figures two balls are employed. If the plate i is provided with three slots, each surrounding a ball, the rate of recovery is $3mg(r_1-r_2)/\pi Cn$. If n balls are employed, the rate is $nmg(r_1-r_2)/\pi Cn$.

So long as the balls are constrained to move on the plate h after the manner described, the average speed of recovery is constant, and in no way depends on the inclination of the pivoted system to the vertical. It is important also to notice, in connection with what follows, that the speed of recovery does not depend on the speed of rotation of the erector. That the rate of recovery should not depend on the erector speed may seem puzzling, but the reason will be apparent from consideration of fig. 16. Here, instead of two balls, a series is employed. When the pivoted

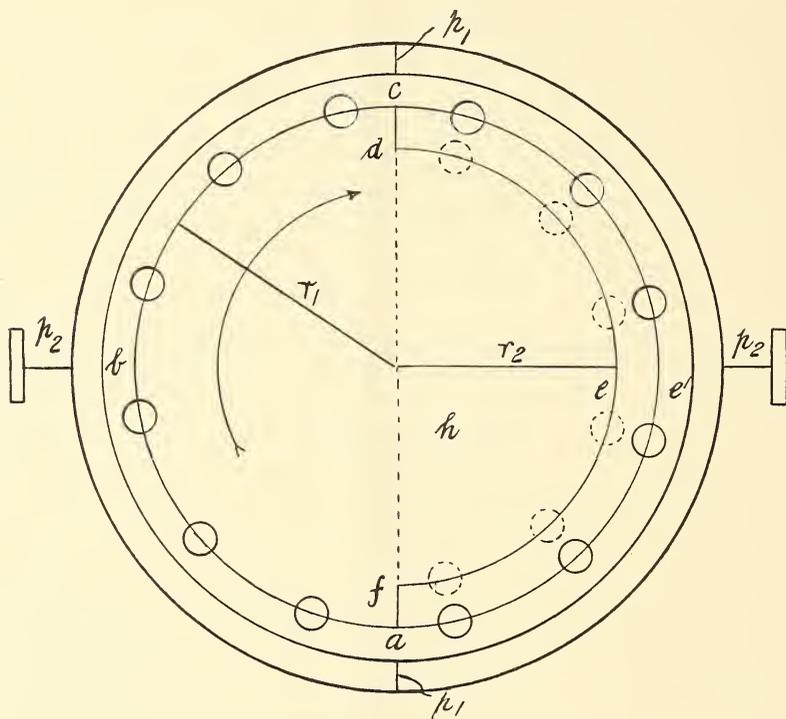


FIG. 16.

system is upright—that is, when the plate h is horizontal—let the balls move round on the plate with uniform speed in the circle $abc'e'a$. If distributed evenly they will apply no couple about the pivots. If the pivoted system is inclined about p_2p_2 , c upwards, a downwards, then with the direction of spin of the gyroscope, and that of the rotation of the balls on the plate, clockwise as seen from above, to cause the pivoted system to approach the vertical on the pivots p_2p_2 , a couple must be applied to the system tending to move b downwards and e' upwards. If the balls can now be constrained to move uniformly in the path $abc'd'f'a$, such a couple is obtained, and obviously the moment of the couple does not depend on

the speed at which the balls travel on the plate. This will be at once apparent if the number of balls employed is supposed very great.

The rate of recovery of the device is proportional to m , the mass of a ball, to the number of balls, and to $r_1 - r_2$; the rate is inversely proportional to Cn , the angular momentum of the gyroscope.

Although the rate of recovery of the device is independent of the erector speed, as has been shown, the action does depend on the speed. In order that the erector may function after the manner described, it is necessary that the balls, when they arrive at c and f respectively (fig. 15), should roll down the slope into the inner and outer compartments of their containing slots. Consider the ball b_1 when at c . It is resting on a plane which is sloped, along co , at an angle θ to the horizontal. Further, it is accelerated towards o , the centre of the path, the amount of the acceleration being $\omega^2 r_1$. The condition that the ball should press against the back of the slot is clearly that $\omega^2 r_1$ should be greater than $g \sin \theta$. Consequently, unless $g \sin \theta$ is greater than $\omega^2 r_1$ the erector will cease to function. Moreover, supposing $g \sin \theta$ to be greater than $\omega^2 r_1$, the acceleration of the ball, relative to the plate h_2 , is $\frac{5}{7}(g \sin \theta - \omega^2 r_1)$, and this acceleration must be sufficiently great to permit of the ball travelling from c to d in a short interval of time.

In practice, of course, the radial motion of the ball starts before the slot arrives at the highest position in its path, and in point of fact the path of each ball on the plate h is not strictly that shown in fig. 16. At an instant at which the line joining the centre of the plate h to the point of contact of the ball with the plate makes an angle ϕ with the line of greatest slope, oc , the acceleration of the ball relative to the erector is $\frac{5}{7}(g \sin \theta \cos \phi - \omega^2 r)$, where r is the distance of the centre of the ball from the centre of the erector. As the ball ascends, the angle ϕ diminishes and the acceleration increases to the value $\frac{5}{7}(g \sin \theta - \omega^2 r)$ and then diminishes. Again, once the radial motion of the ball starts, r diminishes and the acceleration of the ball increases.

The line joining the centre of the plate to the point of contact of the ball with the plate advances from the position in which its angular displacement to the left of oc is ϕ , to one in which its angular displacement to the right of oc is ϕ in time $2\phi/\omega$. Hence it is clear that the ball will arrive at the inner compartment of its surrounding slot provided that

$$r_1 - r_2 = \frac{1}{2} \frac{5}{7} (g \sin \theta \cos \phi - \omega^2 r_1) \frac{4\phi^2}{\omega^2}.$$

For an erector of this type in which $r_1 = 3$ inches, $r_1 - r_2 = 1$ inch, $\omega = 8$ revolutions per minute, the limiting value of θ is about $\frac{1}{8}$ of a degree.

When inclined to the vertical the pivoted system recovers, up to this limiting value of θ , without precessional motion in the ordinary sense of the term. That is, if inclined on the pivots p_1p_1 it approaches the vertical on the pivots p_1p_1 , and similarly for any other axis. The recovery is not accompanied by spiral motion of the axis of the gyroscope.

To render the device delicate the slots are so contrived that the balls possess a certain amount of freedom when situated within the outside compartments of the slots. When the limiting value of θ has been reached, and the radial motion of the balls has ceased, they move to and fro in the outer compartments of the slots, and as a result the pivoted system is erected into the vertical with great exactness. The resting position of the system, with this simple form of erector, is exact to within 3 or 4 minutes of angle.

The dynamical action of the erector during this final stage of recovery is important, and will be returned to later.

An examination of fig. 14 (*b*) will show that the form there given to the slots is defective in that the centres of the balls do not lie on a line which passes through the centre of the plate *i*. When situated as shown in the figure the balls apply a couple about the axis p_2p_2 , and this causes the system to precess about the pivots p_1p_1 . With the slots formed as shown, after a half turn of the erector the effect is to turn the pivoted system about p_1p_1 , and as a result the recovery of the device is not effected wholly about p_2p_2 .

In fig. 17 is shown a later construction given to the plate *i*. In the figure the pivoted system is supposed inclined on the pivots p_2p_2 , the erector towards the reader, so that the erector is functioning. It will be seen that the centres c_1 and c_2 of the balls lie on a line which passes through the centre *o* of the plate. Further, it will be seen that the outside boundary lines of the slots are curved towards the centre of the plate. This has the effect of constraining the balls, as a consequence of their centrewards acceleration, to arrange themselves on the plate *h*, when the pivoted system is upright, so that they rotate as an approximately balanced system. In order that the delicacy of the arrangement should not be diminished, the boundary of each slot, in the neighbourhood of the equilibrium position of the enclosed ball, is cut so as to form a portion of a circle with *o* as centre. The resting positions of the balls are not quite definite, as a result; and as a consequence the pivoted system when upright, in general, performs oscillations of very small amplitude about each of the pivot axes p_1p_1 and p_2p_2 . This is an advantage, in fact, as the pivots are maintained rocking to and fro in their bearings, and pivot friction is thus reduced.

Returning to figs. 13 and 14, let the pivoted system be inclined to the vertical on the pivots p_2p_2 , with the erector displaced in the direction of the reader, and let the erector be rotating in the direction opposed to that in which the gyroscope is spinning. The action is precisely that already described, but the direction of the integral couple applied to the gyroscope by the erector has been reversed, with the result that the pivoted system moves away from the vertical with angular speed $2mg(r_1-r_2)/\pi Cn$. Thus the upright position of the pivoted system is one of stability when the erector is rotating in the direction of spin of the gyroscope, but is one of instability when the erector and gyroscope are rotating in opposite directions. This property possessed by the system of gyroscope and rotating erector is of great importance in connection with what follows.

Now let the stabiliser, as illustrated in fig. 12, be set up on an aeroplane with the pivots p_2p_2 lying fore and aft. Let the direction of rotation of the erector be that in which the gyroscope is spinning. If the aeroplane moves in a straight path with uniform speed it is clear that the pivoted system will set itself in the true vertical.

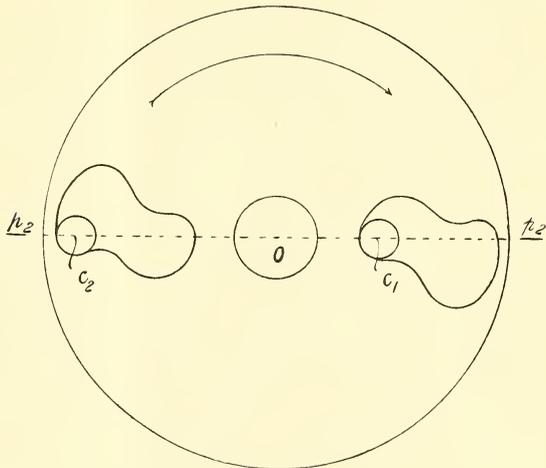


FIG. 17.

Pitching and rolling motions of the aeroplane produce no sensible effect in disturbing the pivoted system. This is not surprising. When fitted up within the cockpit of an aeroplane, the apparatus, it is to be remembered, is situated on, or at all events very near to, the axes about which the aeroplane rolls and pitches, and hence the horizontal accelerations which accompany these motions of the aeroplane are very small. Further, the horizontal accelerations, small in amount, are rhythmic in character; and hence, even if their maximum values were large, the net effects produced on the pivoted system could only be very small. It is a proved fact that the simple apparatus, as described above, when placed on an aeroplane in ordinary flight moving side-on to half a gale of wind, is capable of finding and maintaining the true vertical within an accuracy of $\frac{1}{10}$ of a degree. During such flight the balls are observed to remain, on the whole, in their proper resting positions, with occasional excursions for correcting purposes,

and when the aeroplane yaws. It is to be observed that the device cannot leave the true vertical at an angular speed greater than $2mg(r_1 - r_2)/\pi Cn$, and this speed can be arranged so that, under the conditions which prevail on an aeroplane, very high accuracy is obtained. The forces which accompany bumps are great in amount, but it is obvious that they have no appreciable effect on the instrument.

Let now the aeroplane move in a curved path. The pivoted instrument, it must be clearly understood, is mounted and balanced up so as to be neutral with respect to both the pivot axes $p_1 p_1$ and $p_2 p_2$, so that the disturbance produced by the curved motion will be due solely to the erector. At each instant the entire instrument is being accelerated towards the centre of the path in which the aeroplane is moving, and it will be clear on reflection that the balls function in such manner that the pivoted system moves, at each instant, from the true vertical towards the apparent vertical. The balls are situated in a field of force whose amount per unit of mass is $\sqrt{g^2 + v^2 \dot{\psi}^2}$, where v is the speed of the aeroplane and $\dot{\psi}$ is the azimuthal rate of turning of the aeroplane. The direction of this field is contained in a vertical plane lying athwart the aeroplane, and makes an angle $\tan^{-1}(v\dot{\psi}/g)$ with the true vertical. The direction of the field clearly goes round with the plane; the field is in fact made up of two component fields, one vertical, the other horizontal and rotating with the plane. From what has been said already it will be clear that after the aeroplane has turned through an angle ψ the inclination θ of the pivoted system to the true vertical is given by

$$\theta = \frac{2mg(r_1 - r_2)}{\pi Cn\dot{\psi}} 2 \sin \frac{1}{2}\psi.$$

Let $m = 1$ ounce, $r_1 - r_2 = 1$ inch, $Cn = 250$ (in foot, pound, second units), $\dot{\psi} = 9$ degrees in 1 second of time (a usual value of $\dot{\psi}$ for large bombing aeroplanes in 1917-18). Introducing these values into the equation, we obtain

$$\theta = 18.6 \sin \frac{1}{2}\psi \text{ (in minutes of angle).}$$

With the above values of m , $r_1 - r_2$, etc., the maximum disturbance amounts to less than one-third of a degree, and is introduced by a manoeuvre involving a half turn. As already pointed out, after a half turn of the aeroplane the pivoted system is inclined to the vertical wholly on the pivots $p_1 p_1$, so that the line bars of a bombsight are correct immediately straight flying is resumed.

After a yawed turn amounting to 10° the disturbance introduced is less than 2 minutes of angle. After a complete turn the disturbance is zero.

In the trials carried out by the R.N.A.S. officers of the Gray stabilisers the following manœuvre was frequently performed. Starting with the aeroplane moving in a straight path, it was directed into a circular path and the circular motion maintained for a considerable time. On straightening out the machine the apparatus was found to be upright. The same remarks apply to motion on a banked spiral. In fact, if the apparatus is forcibly deflected from the true vertical when the aeroplane is on a banked

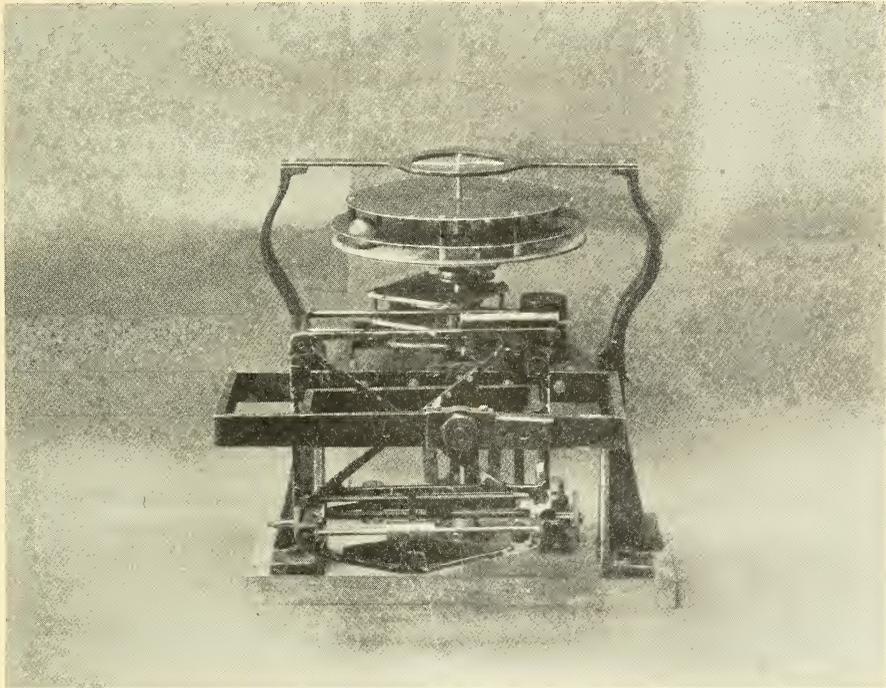


FIG. 18.

curve, the pivoted system, after the required interval of time has elapsed, will be found to be erect.

It was usual, on the conclusion of trials, when approaching the aerodrome, to descend in a continuous spiral just previous to entering the aerodrome. Following on the spiral motion and the bumps consequent on entering the aerodrome, the apparatus was invariably found to be erect.

EARLY MODELS USED. SUN SHADOW TRIALS.

Fig. 18 shows a view of a pioneer instrument constructed by Messrs Elliott Brothers, the well-known instrument-makers. The construction of the instrument is precisely that described above, except that three slots and balls were employed.

In fig. 19 is shown the instrument fitted up for rolling trials. Attached to the frame f of the instrument, with its length perpendicular to the pivots p_2p_2 , is a long rigid bar of wood. Carried at one end of this bar is a white screen, as shown, composed of millimetre paper, and to the other, with its length horizontal and parallel to the screen, is a knife-edge. The apparatus was mounted on the aeroplane with the pivots p_2p_2 lying fore and aft. By flying with the knife-edge and screen in line with the

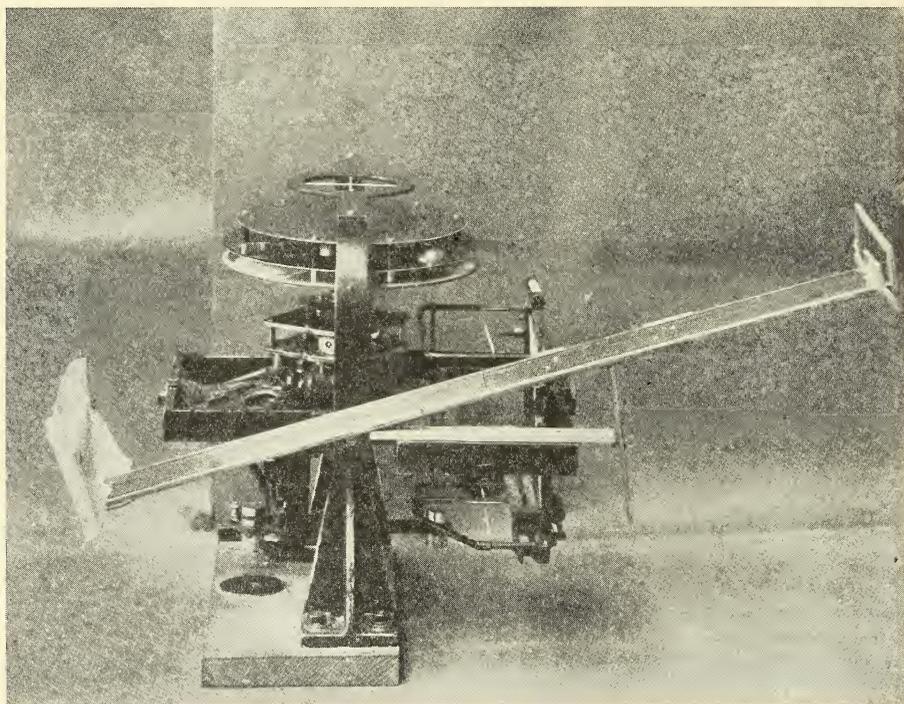


FIG. 19.

sun, a clearly defined shadow of the sharp edge was obtained on the screen, and hence the performance of the instrument could be ascertained. The effects of manœuvres on the instrument were investigated by performing the manœuvre so that at its conclusion the bar was in line with the sun.

A very extensive series of trials was carried out during the spring and summer months of 1917 by officers of the R.N.A.S., and these finished up with the sun-shadow tests above described. It was concluded that the Gray stabiliser furnished an accuracy, for bombing purposes, of from $\frac{1}{8}$ th to $\frac{1}{10}$ th of a degree, or 30 feet on the ground from a height of 15,000 feet. The instrument was adopted by the Naval Air Service, and a great effort was made to construct a very large number of the instruments for use in 1918 over the German lines.

Consider now the device shown in fig. 20. This consists of a large gyroscope g mounted in a frame f and carried by uprights u, u , after the manner already described. Rigidly attached to the pivoted system is a small gyroscope g' , the axis of spin of which is perpendicular to that of the main gyroscope g . When the pivoted system is upright the axis of g' is parallel to p_2p_2 . Let the direction of the spin of g be clockwise, as seen from above, and that of g' be counter-clockwise, as seen from the left-hand side of the apparatus (as viewed by the reader). These directions of rotation are indicated in the figure by the arrows attached to the

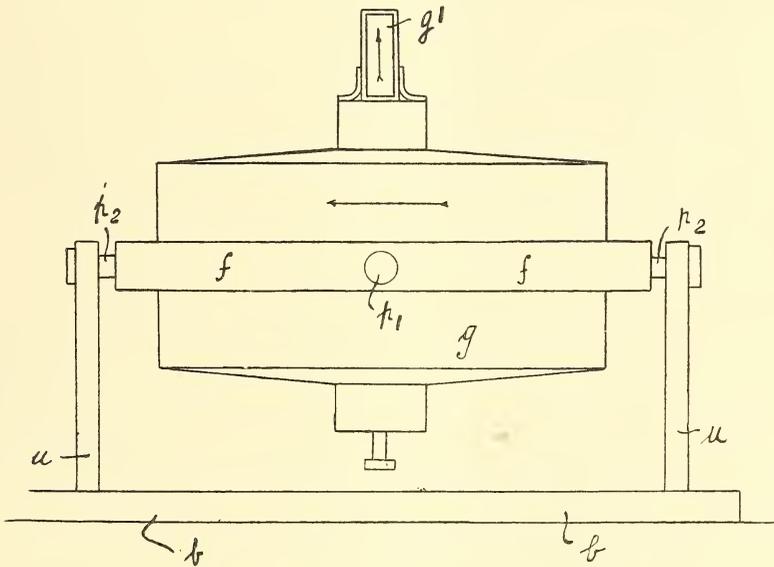


FIG. 20.

diagrams of the gyroscopes. Now suppose this device, as described, to be set up on a table with the axis of g vertical. If the entire apparatus is turned in azimuth at angular speed $\dot{\psi}$ in the clockwise direction, as seen from above, the pivoted system will experience a couple tending to turn it about the pivots p_1p_1 in the counter-clockwise direction, as viewed by the reader. This couple is applied to the gyroscope g , and in consequence the pivoted system turns on the pivots p_2p_2 so that g' moves away from the reader.

The applied couple which brings about this result arises from the turning of the gyroscope g' in azimuth. If $I\omega$ is the angular momentum of g' , the moment of the applied couple is $I\omega\dot{\psi}$. The rate at which the pivoted system turns on the pivots p_2p_2 is $I\omega\dot{\psi}/Cn$, where Cn is the angular momentum of the main gyroscope. If the direction of the

azimuthal turning is reversed, so also is the direction of the applied gyroscopic couple and the direction of turning of the pivoted system on p_2p_2 .

Let now the apparatus be supposed set up on an aeroplane with the pivots p_2p_2 fore and aft, and so that the spin of the gyroscope g' is counter-clockwise as seen from the rear of the cockpit. Starting with the apparatus upright, let the aeroplane turn in azimuth at angular speed $\dot{\psi}$. In consequence of the couple due to the forcible turning of g' , the pivoted system turns on p_2p_2 away from the true vertical, and so as to increase its inclination to the apparent vertical.

Let the pivoted system include an erector of the type illustrated in fig. 12. When the aeroplane changes direction the erector turns the pivoted system on p_2p_2 towards the apparent vertical at angular speed $2mg(r_1 - r_2)/\pi Cn$. At such instant the couple due to the turning of g' in azimuth causes the system to turn on p_2p_2 in the opposite direction, so that the net rate of turning towards the apparent vertical is given by

$$\dot{\theta} = \frac{2mg(r_1 - r_2)}{\pi Cn} - \frac{I\omega\dot{\psi}}{Cn}.$$

The system behaves as a neutral gyroscope if $\dot{\theta} = 0$; that is, if $2mg(r_1 - r_2) = \pi I\omega\dot{\psi}$. If, therefore, the value of $I\omega$ be so chosen that this equation is fulfilled for the value of $\dot{\psi}$ most generally adopted in practice, the apparatus becomes endowed with the property that when the aeroplane is turning the pivoted system can move away from the true vertical towards the apparent vertical only very slowly, but when the curved motion ceases it returns to the true vertical relatively quickly.

Let the speed of recovery due to the erector be, say, 1 degree in 20 seconds of time, and $\dot{\psi}$ be 9 degrees in 1 second of time. If the angular momentum of the gyroscope g is 250, in foot, pound, second units, we have

$$I\omega = \frac{250}{20 \times 9} = 1.4 \text{ (approximately, in foot, pound, second units),}$$

which shows that the small gyroscope may conveniently take the form of a flywheel attached to the spindle of the turbine employed to rotate the erector (fig. 12).

It has been shown that the ball erector as described causes the pivoted system, when the system is inclined, to approach the true vertical at an angular speed which is independent of the speed at which the erector rotates, and that the radial motion of the balls, on which the erecting action depends, ceases when $\omega^2 r_1 > g \sin \theta$. Hence, if, when the device is

used on an aeroplane, means are provided whereby the erector speed becomes very great when the aeroplane is turning, the balls will cease to function, and there will be no disturbance of the pivoted system. Experimental apparatus for bringing about this result has been constructed as shown, diagrammatically, in fig. 21. Mounted in the pivoted system are two electric motors m_1 and m_2 , as shown. When the aeroplane is in straight flight the erector e is driven by the motor m_1 , and revolves slowly in the direction of spin of the gyroscope g ; when the direction of flight is changed the drive is changed automatically from the motor m_1 to m_2 ,

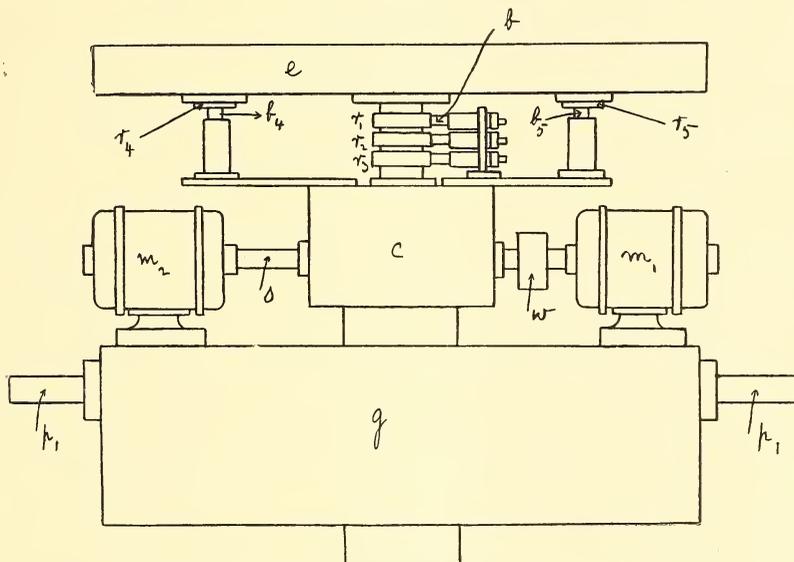


FIG. 21.

and the erector then revolves rapidly in the same direction as before. The motors drive a spindle s , which in turn, by means of reduction gearing contained within the casing c , drives the erector e . With the arrangement of gearing employed, if the spindle s were continuous, the effect of switching over from m_1 to m_2 would be to cause the armature of m_1 to revolve at a very high speed, and to obviate this defect the spindle is fitted with a free-wheel arrangement w . When the gearing is operated by m_2 , m_1 ceases to rotate and is left behind.

Mounted on the erector spindle, as shown, are three slip-rings r_1, r_2, r_3 , and on these press three brushes b , the brush-holders of which are attached to the pivoted system. Mounted below the erector casing are two annular copper segments r_4, r_5 (fig. 22), and on these press two brushes b_4, b_5 (fig. 21). These two latter brushes are parallel to the line of the pivots $p_1 p_1$

which attach the system to the frame f (fig. 12); and as the pivots p_2p_2 lie fore and aft with respect to the aeroplane, the brushes lie athwart the machine.

Fitted to the base plate of the erector are two vertical pivots k_1 and k_2 , and on these are carried two rods g_1g_2 terminating in weights w_1w_2 . The movements of these rods on their supporting pivots k_1k_2 are limited by checks c_1c_2 , and by electric contact pieces d_1 and d_2 . k_1 and k_2 are connected to the ring r_1 , d_1 to r_4 , and d_2 to r_5 . The segments r_4 and r_5 are

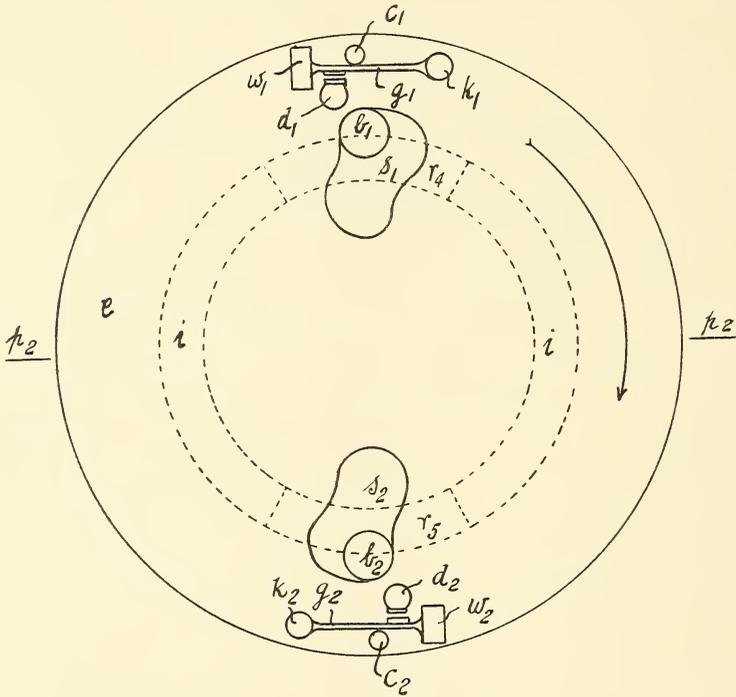


FIG. 22.

separated by insulated segments ii ; and r_4 and r_5 , together with the segments i , form a continuous ring in which press the brushes.

For the particular scheme of operations to be now described one only of the brushes r_1 , r_2 , and r_3 is required (the function of the remaining brushes will be explained presently). One terminal of a battery is connected through a green lamp to the brush-holder of b_4 , and also through a red lamp to the brush-holder of b_5 ; the other terminal of the battery is connected to r_1 .

Now consider the action of the device, as described, during ordinary flight of the aeroplane. The pivoted system is upright, so that the erector is horizontal. The brushes b_4 and b_5 lie, as explained, athwart the aero-

plane. When the masses w_1 and w_2 are in the positions shown—that is, when they are perpendicular to p_2p_2 , or nearly so—the rods remain against the checks c_1, c_2 , provided that the aeroplane is moving in a path which is strictly straight. When the aeroplane is turning slowly the condition that the rods should not move from c_1, c_2 to d_1, d_2 , respectively, is that $\omega^2 a$ should be greater than $v\dot{\psi}$, where ω is the angular speed of the erector, a the distance of one of the masses from the centre of the erector, v the speed of the aeroplane, and $\dot{\psi}$ its speed of turning. If $v\dot{\psi}$ exceeds $\omega^2 a$, one of the rods, say g_1 , will move over from c_1 to d_1 , and the corresponding lamp will light up. When b_4 leaves r_4 the lamp goes out, but is again switched on when b_5 is passing over r_4 , and so on. When the value of $v\dot{\psi}$ exceeds $\omega^2 a$, with the direction of $\dot{\psi}$ reversed, the other lamp will be switched on and off as long as the turning motion of the aeroplane continues. Thus, as the erector rotates, if $v\dot{\psi}$ exceeds $\omega^2 a$ one or other of the lamps is seen to flash in and out, and consequently the device may be utilised as a turn indicator of great delicacy. If $n=10$ revolutions per minute, $a=4$ inches, $v=100$ feet per second, we have for the limiting case $100\dot{\psi}=\pi^2 \times 57.3/27$, where $\dot{\psi}$ is in degrees per second. Hence, with the dimensions and speeds specified, the device is sensitive to all rates of turning exceeding $\frac{1}{3}$ th of a degree per second. This rate of turning corresponds to a banking angle of about $\frac{2}{3}$ rds of a degree; the stabiliser is correct to within $\frac{1}{10}$ th of a degree.

The motors m_1 and m_2 are operated by a commutator, the component parts of which are attached to one of the uprights u (fig. 12) and the frame f respectively. So long as the relative shift of the uprights and the frame does not exceed a few degrees (the amount of the relative shift is, of course, determined by the construction of the commutator), current is supplied to the motor m_1 . When the aeroplane is on a banked turn the current supply is transferred to the motor m_2 . Further, when the plane is banked in one direction a red lamp is lit up, and when the plane is banked in the other direction current is supplied to a green lamp. Thus during night flying the aeroplane may be trimmed by means of the aeroplane. Similarly, by utilising the relative shift of the pivoted system and the frame f to operate lamps, the aeroplane, during night flying, or when in clouds, may be trimmed in the fore-and-aft direction.

A further method of shutting off the erector, in the presence of curved flight of the aeroplane, is shown in fig. 23, which illustrates an erector provided with four slots and four holes. Carried by the plate i (fig. 23) are four small electro-magnets, one for each ball. When the aeroplane is on a banked curve, current is passed through the coils of the electro-magnets, and the attraction of the cores for the steel balls locks the balls in the slots.

This method of shutting off the erector has the advantage that the necessity for using two motors is avoided; the erector may, in fact, be driven by means of a wind turbine after the manner already described.

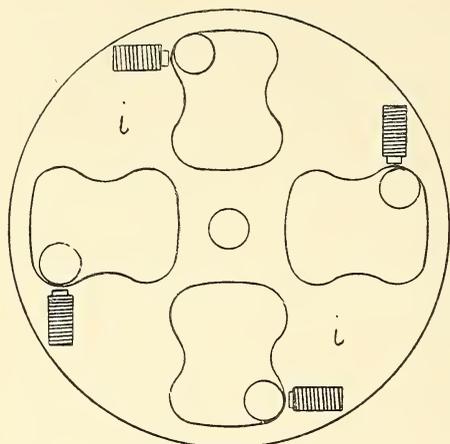


FIG. 23.

With this addition to the erector it is necessary to take in current to the magneto; and if in addition the stabiliser is to be used as a turn-indicator, five brushes in all are necessary. Hence the three rings r_1, r_2, r_3 shown in the experimental model illustrated above.

In fig. 24 is shown a further form of erector. Two rotating ball systems are employed, one of which (*a*) rotates slowly in the direction of spin of the main gyroscope *g* (fig. 12), the other rapidly in the opposite direction. A stabiliser fitted with this arrangement possesses the following properties. When the pivoted system is inclined to the vertical at an angle θ greater than $\sin^{-1}(\omega_2^2 r_1/g)$, where ω_2 is the angular speed of the ball-system (*b*), and r_1 is the distance of the centre of a ball from the centre of (*b*), both systems

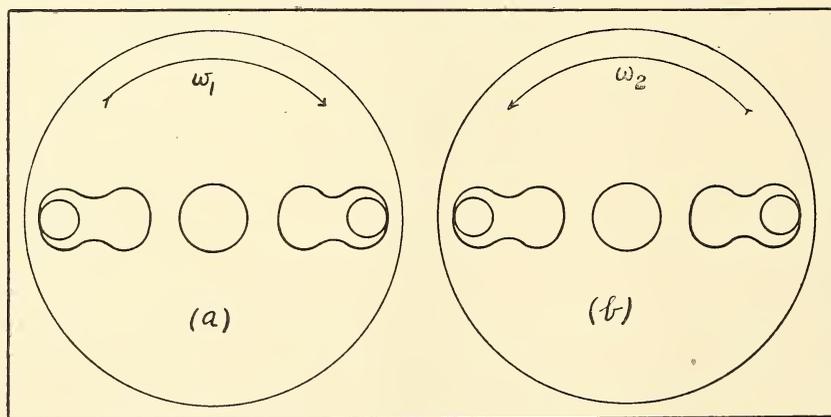


FIG. 24.

function; and if the ball-systems are identical but for their speeds, the pivoted system experiences no erecting couple. This follows from the fact that the magnitude of the integral couple due to the radial motion of the balls in the slots is independent of the erector speed, and that when the direction of the rotation of the erector is reversed the direction of the

integral couple is reversed. When θ is less than $\sin^{-1}(\omega_2^2 r_1/g)$, the ball-system b goes out of action and the pivoted system approaches the vertical at a speed determined solely by a . Further, if the ball-systems are not identical, then when θ is greater than the critical angle the pivoted system experiences a differential couple; and if the system a is more powerful than b , it approaches the vertical at an angular speed which depends on the magnitude of the differential couple and the angular momentum of the gyroscope.

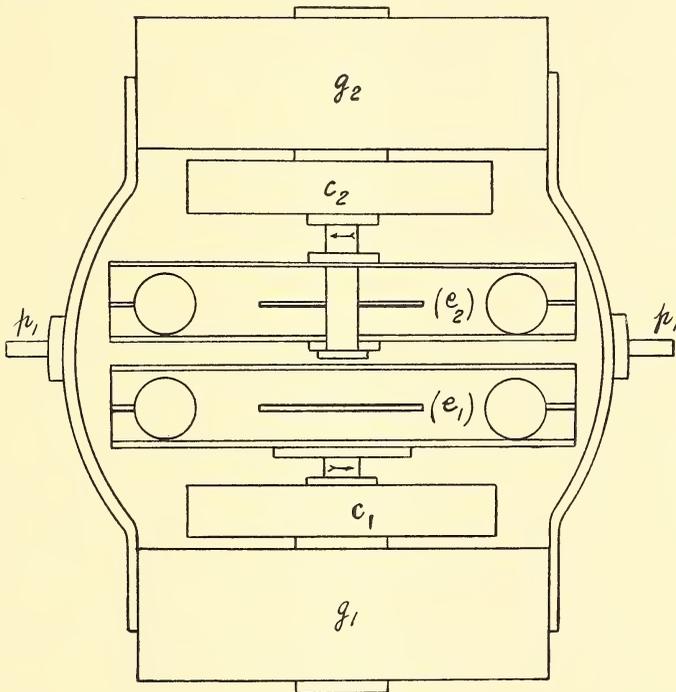


FIG. 25.

This form of stabiliser, when mounted on an aeroplane, has the property that it is blind to the apparent vertical for azimuthal rates of turning of the aeroplane greater than $\omega_2^2 r_1/v$, where v is the speed of the aeroplane, since each ball is being accelerated towards the centre of the path in which the aeroplane moves at the instant. Hence radial motion of the balls takes place when $v\dot{\psi} > \omega_2^2 r_1$. During normal flight the ball-system (a) alone functions; and if any small error has been introduced during the curved flight, correction takes place relatively quickly.

The compound erector described above is peculiarly well suited for use on an aeroplane (or an airship), since the speed v is great, and $\dot{\psi}$ is usually great. There is no difficulty in constructing the compound system so that

b comes into action during yaws, and so employed the apparatus serves to indicate turning motion of the aeroplane. Obviously the radial motion of the balls of the system b may be caused to close electric contacts and light indicating lamps.

Fig. 25 shows a convenient construction of the stabiliser in which two gyroscopes are employed. The cases of these are rigidly connected together, and the axes of spin are parallel, so that they are equivalent to one gyroscope. The erector e_1 is driven round slowly in the direction of spin of the gyroscopes (which is the same for both), and e_2 rapidly in the reverse direction. It is convenient to provide means whereby the speed of the erector e_2 may be varied.

THEORY AND ACTION OF THE FURTHER FORMS OF GRAY STABILISER.

It is necessary at this stage to consider the action of the erector shown, in plan, in fig. 26. Three balls (in this case) are employed. These rotate

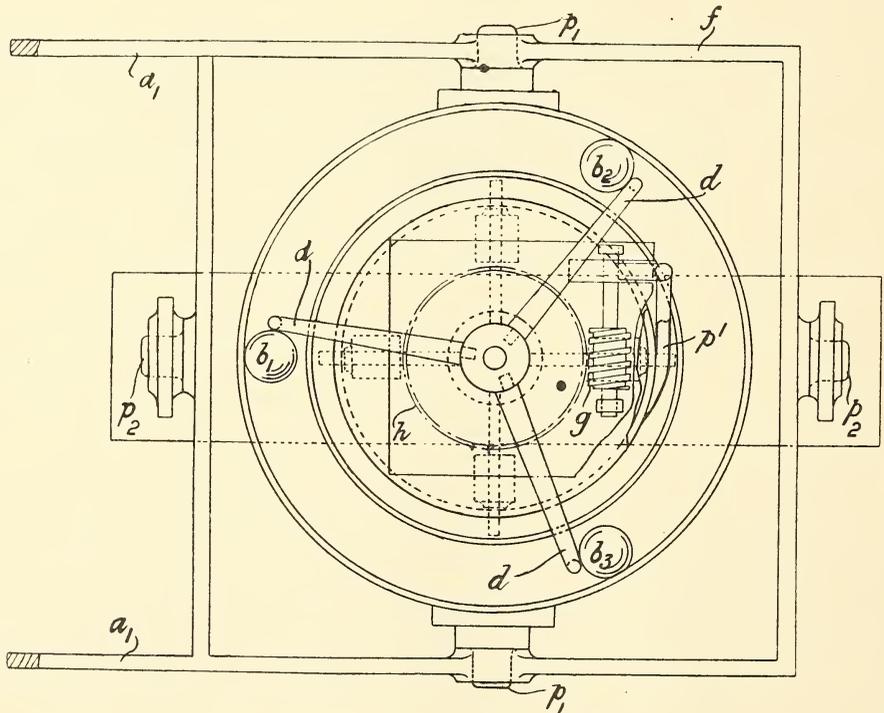


FIG. 26.

as shown on a fixed circular track carried by the pivoted system, each ball being propelled by a pusher d . The pushers d form part of a spider which is driven round slowly in the direction of spin of the gyroscope g .

Otherwise the construction is as before. The pivoted system is attached, as before, at p_1p_1 to a frame f , and this is in turn pivoted at p_2p_2 to uprights carried by a base as shown. When the pivoted system is upright the track in which the balls revolve is horizontal.

Let now the pivoted system be inclined to the vertical. The track is inclined to the horizontal, and each ball in ascending the slope of the track remains against its pusher; but when it crosses the crest of the slope, and descends, it is situated on a slope down which it is accelerated. Thus it runs ahead of its pusher. After passing the lowest point of the track it ascends the slope, its motion is retarded, and it again comes in contact with its pusher, against which it presses until it next crosses the crest of the slope. Hence each ball spends more time in ascending than in descending the slope of the circular track in which it is constrained to move, and the result is the application to the pivoted system of an integral couple which causes the inclination of the system to the vertical to diminish, and finally to become zero, or at all events to arrive at a minimum value.

It is proposed now, on account of the extreme practical value of this type of erector, to investigate the nature and amount of the integral couple obtained, and how the couple depends on the slope of the track to the horizontal, the masses of the balls, the travel of the balls relative to the pushers, and the speed of the erector.

Fig. 27 shows a single ball b which is propelled on a fixed circular track t by a pusher p . Its travel on the track, relative to the pusher, is limited by the check c . The arrangement of ball pusher and check is shown in elevation in fig. 27A. The direction of rotation of the ball in the track is supposed to be clockwise, as viewed from the upper side of the track.

Let m be the mass, and r the radius of the ball, d the distance between the pusher and check, and suppose that the track is inclined at an angle θ to the horizontal about the pivots p_2p_2 (fig. 27) in the clockwise direction as seen from the left-hand side of the diagram, so that d is the crest of the slope. Let the pivoted system be neutral with respect to the pivots p_1p_1, p_2p_2 (fig. 27) when the ball is in contact with its pusher.

The ball, when in the position d , has just ascended the slope, and is in contact with its pusher. At the instant, since it is at the crest of the slope, it is subject to no accelerating force. As it moves round it enters the slope, is accelerated, and comes against its check. It now descends and remains in contact with the check until it is again on the ascending side of the track, when its motion is retarded, with the result that it is

transferred from the check to the pusher. We suppose that when the pusher is at p' , and the check at c' , there is a sudden transference of the ball from the pusher to the check, and that the ball then remains in contact with the check until the latter arrives at c'' , when the ball is suddenly retransferred from the check to the pusher, which is now in the position p'' .

From now on the ball remains in contact with the pusher until it

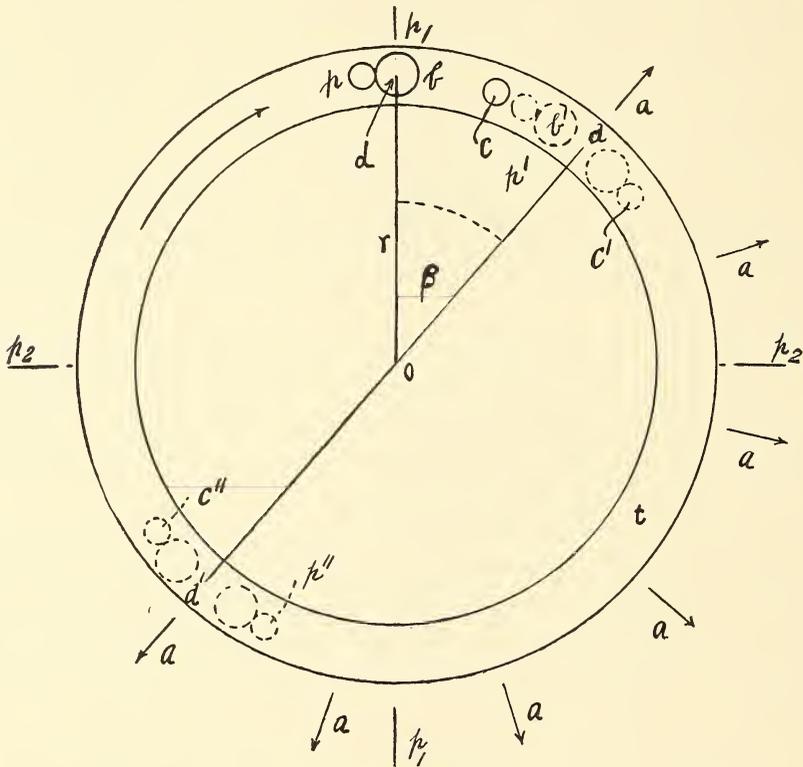


FIG. 27.

crosses the crest of the slope, and again we suppose the ball quickly transferred from the pusher to the check; and so on.

Previous to the first transference of the ball from pusher to check the centre of gravity of the system lies at the intersection of the pivot axes. The shift of the ball brings about a movement of the centre of gravity, and there is applied to the pivoted system a couple of moment $mg(d-2r)$, and so long as the ball remains in contact with the check the amount of this couple does not change. The direction in which the couple is applied goes round with the rotating system. After the second transference of

the ball (from the check to the pusher) the centre of gravity of the system is once more at the intersection of the pivot axes, and it remains so until the pusher and ball arrive once more in the positions of p', b' , when the couple reappears, to advance through an angle π ; it then disappears; and so on.

The couple may be represented completely by a line a , of appropriate length, drawn perpendicular to the plane in which it acts at each instant. This line, which represents the couple, rotates at the angular speed ω of the erector. It appears at the instant corresponding to the angular displacement β (fig. 27), disappears at the instant corresponding to $\beta + \pi$; and so on.

During the first half turn of the couple the integral effect has clearly been to turn the pivoted system about an axis parallel to dod' (fig. 27). From symmetry it follows that the integral couple is completely represented by a line drawn perpendicular to dod' . This couple is applied about an axis perpendicular to dod' , and the gyroscope of course turns about an axis parallel to dod' . When the erector turns in the direction of the spin of the gyroscope, the pivoted system turns, as a consequence of the applied couple, in the direction which results in the inclination of the pivoted system being diminished. Before the application of the integral couple the inclination of the pivoted system was wholly about p_2p_2 . After the application of the couple the inclination has been diminished and the line of greatest slope (fig. 27) has turned in the clockwise direction through a small angle. This follows from the fact that the effect of the integral couple has been to turn the pivoted system on the pivots about an axis parallel to dod' , and not wholly on the pivots p_2p_2 .

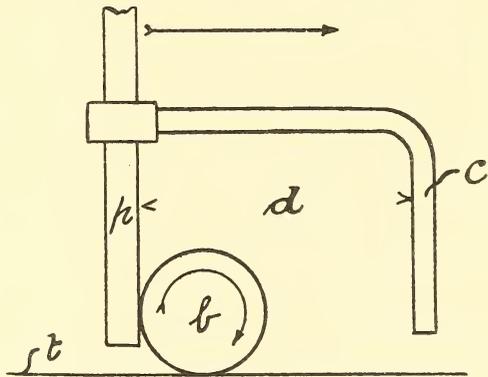


FIG. 27A.

To obtain the angle through which the pivoted system has turned, we count time from the instant at which the first transference took place (the transference is supposed sudden). At time t the couple has turned through an angle ωt , and its component about an axis perpendicular to dod' is $mg(d - 2r) \sin \omega t$ (see fig. 27A). The rate of turning of the pivoted system about the pivots under the action of this component couple is $mg(d - 2r) \sin \omega t / Cn$, where Cn is the angular momentum of the gyroscope.

The angle turned through in the time π/ω , during which the couple acts, is $2mg(d-2r)/Cn\omega$.

This result may also be obtained as follows. The ball applies, at each instant, a couple of amount mgr , about 0. When it is in contact with its pusher the system is balanced. If the ball were removed from the track there would appear a couple of amount mgr , which couple would rotate at angular speed ω . The time taken by the ball to travel a distance $d-2r$ (its maximum shift relative to the pusher and check) is $(d-2r)/r\omega$. When the angular displacement from od (fig. 27) is β the ball is suddenly transferred from the pusher to the check; thus it virtually disappears from the track for the above interval of time. When the angular displacement of the ball is $\beta+\pi$ it spends time $2(d-2r)/r\omega$ on the track; it goes through the distance $d-2r$ in contact with the check, and then, after its retransference to the pusher, over the same distance in contact with the latter. Thus in a complete turn of the erector the integral effect is to take off the ball, when the angular displacement is β for time $(d-2r)/r\omega$, and to place it on the opposite side of the track for the same interval of time. Clearly this results in a couple of amount $2mgr$ acting about an axis perpendicular to dod' for time $(d-2r)/r\omega$. The angle turned through, by the pivoted system about an axis parallel to dod' , is thus $2mgr(d-2r)/r\omega Cn$ or $2mg(d-2r)/\omega Cn$, as before.

In time T' , the periodic time of the spider on which mounted the pusher and check, the system turns, under the action of the integral couple due to the transference of the ball from the pusher to the check, and from the check to the pusher, through an angle $2mg(d-2r)/\omega Cn$, or $2mga/\omega Cn$, where $a=d-2r$. Since the integral couple occurs once in a revolution of the erector, and $\omega=2\pi/T'$, the mean rate at which the pivoted system turns about an axis parallel to dod' is $mga/\pi Cn$. If two balls are employed, each operated by a pusher and check, situated at opposite ends of a diameter of the erector, the average rate of turning is $2mga/\pi Cn$. For three balls, symmetrically distributed, the rate of turning is $3mga/\pi Cn$; and so on.

Let there be two balls operated as described. If each ball is transferred from the pusher to the check when its angular displacement from the line of greatest slope is β , then, as already explained, the pivoted system turns about an axis parallel to dod' at angular speed $2mga/\pi Cn$. Denoting the rate of turning about the axes p_1p_1 , p_2p_2 (fig. 27), the athwart and fore-and-aft pivots, by $\dot{\theta}_1$ and $\dot{\theta}_2$ respectively, we have

$$\dot{\theta}_1 = \frac{2mga \cos \beta}{\pi Cn}; \quad \dot{\theta}_2 = \frac{2mga \sin \beta}{\pi Cn}.$$

In actual practice the behaviour of the balls propelled on a track, and controlled by pushers and checks, after the manner described, approximates closely to that explained above.

Now, when a ball is in the position corresponding to the angular displacement β its acceleration along the plane is $\frac{5}{7}g \sin \theta \sin \beta$ (for the pivoted system is supposed inclined to the vertical at an angle θ on the pivots p_2p_2). If a is small, clearly for large values of θ the transference of the ball from pusher to check will take place before β is great. If θ is small, β will be relatively great. Now if β is small, $\dot{\theta}_1$ is great compared with $\dot{\theta}_2$; when β is great, $\dot{\theta}_2$ is great compared with $\dot{\theta}_1$.

Let now the device as described be mounted on an aeroplane with the pivots p_2p_2 lying fore and aft. When the aeroplane turns, the pivoted system, supposed upright initially, turns on the pivots p_2p_2 towards the apparent vertical, and on the pivots p_1p_1 in a direction depending on the direction of turning of the plane. If v is the speed of the aeroplane, and $\dot{\psi}$ is its azimuthal rate of turning, g is replaced by $\sqrt{g^2 + v^2\dot{\psi}^2}$, and the slope is $\tan^{-1}(v\dot{\psi}/g)$. Since during manœuvres of aeroplanes $\dot{\psi}$ is usually, and v is always, great, it follows that β will be small, and hence that initially $\dot{\theta}_1$ will be great and $\dot{\theta}_2$ small. If a small weight is attached to the pivoted system in line with the axis, so as to provide the stabiliser with a small amount of gravity control, this weight acts in opposition to the couple which is bringing about the turning $\dot{\theta}$, about the pivots p_1p_1 . If the mass of this attached weight is m' and its distance below the intersection of the pivot axes is h , the couple it applies about the pivots p_2p_2 , and which causes turning about the pivots p_1p_1 , is $m'v\dot{\psi}h$. Hence, with this addition, $\dot{\theta}_1$ will be zero if

$$m'v\dot{\psi}h = \frac{2mga \cos \beta}{\pi}$$

As already pointed out, for an aeroplane v is always (and $\dot{\psi}$ is usually) great. Provided that the speed of the erector is small, the transference of a ball from its pusher to its check will be accomplished for a small value of β . Now $\dot{\theta}_2$ is small when β is small. Hence with a proper choice of ω and of m' the device may be rendered blind to the apparent vertical.

When straight flying is resumed, following a turn, any slight deviation of the pivoted system will be corrected about an axis perpendicular to the vertical plane which contains the axis of the pivoted system. This follows from the fact that when β approaches 90° , $\dot{\theta}_1$ is small and $\dot{\theta}_2$ relatively great.

As an example, consider the case of an erector, of the type described, functioning on an aeroplane. Let $m=1$ ounce, $a=1$ inch, $Cn=250$, in

foot, pound, second units. Let the device be supposed initially upright, and suppose that the aeroplane turns at an angular speed of 9° per second. If the speed of the plane is 100 feet per second, the acceleration athwart the aeroplane due to the turning motion amounts to 188 inches per second per second. As a consequence of this acceleration being great, β is small, amounting to less than 5° for an erector speed of 8 revolutions per minute. Taking β as 5° , we obtain $\theta_1 = 1.46$, and $\theta_2 = 0.12$, in minutes of angle per second.

If we attach a mass m' to the pivoted system, as described above, θ_1 becomes zero, and during the turning motion there is no appreciable disturbance. The value of $m'h$ is $2mga \cos \beta / \pi v \dot{\psi}$, and hence, inserting the assumed values of the quantities, we obtain

$$m'h = \frac{2 \times 32 \times 0.97 \times 57.3}{16 \times 12 \times 100 \times 6} \\ = 0.0205.$$

The pivoted system ceases to be neutral for straight flying. Its precessional period T , for steady precessional motion, is given by

$$T = \frac{2\pi \times 250}{32 \times 0.0205} \\ = 41, \text{ (in minutes, approximately).}$$

No disadvantage is brought about by introducing gravity control in the manner here described. In turning rapidly the effect of the mass m' is to counteract the action of the erector. When the aeroplane turns, the pivoted system behaves as a neutral gyroscope—that is, one for which the periodic time is infinite.

The value of m' having been chosen so that θ_1 is zero for a particular value of $\dot{\psi}$, there will come into existence, for other speeds, a differential couple causing the pivoted system to turn, at any instant, about the pivots $p_1 p_1$. If, in the example above, there is a change in $\dot{\psi}$ amounting to 10 per cent., θ_1 will amount to 0.146 minutes of angle per second. There will be no change in the value of θ_2 .

The couple required to cut off the erector, during rapid turns of the aeroplane, may also be brought into existence by causing a small weight, carried by the pivoted system, to move over laterally, to one side or the other, according to the direction of $\dot{\psi}$. The lateral distance through which this weight must be moved does not depend on $\dot{\psi}$, and hence is constant. The weight may be conveniently operated by means of electro-magnets controlled by a commutator the component parts of which are mounted on the frame f (fig. 26) and one of the uprights u .

This simple form of erector is capable of providing an accuracy of 3 or 4 minutes of angle when mounted on an aeroplane. The balls, it is to be remembered, rotate on a hard metal track forming part of a system which includes a high-speed gyroscope. The track is being tapped continuously, as it were, by an automatic hammer which vibrates to and fro with a frequency of about 300. The balls, as a very important consequence, are extremely sensitive to tilt of the track.

Consider now the arrangement of pushers and checks shown in fig. 26. If the device is used as shown in that diagram, each rod acts as a pusher for one ball, and as a check for the ball in the following compartment. The balls are free to move, relatively to their pushers, through a distance on the track subtending at the centre an angle approximating to $\frac{2}{3}\pi$. The value of a is great; if the diameter of the track is 6 inches, the value of a will approximate to 5 inches.

It will be clear, from what has been said, that the device, so used, has the following properties. When set up on a table, deflected through a large angle θ , and left to itself, it will move towards the vertical quickly so long as θ is great. The angle β approximates to 90° , and the device moves up without precession in the ordinary sense of the term. When θ is small the rate of recovery is relatively small, inasmuch as the balls do not move far from their pushers.

This device should not be thus used on an aeroplane. If so used, then when the aeroplane is turning, the device moves rapidly at each instant towards the apparent vertical. The value of β is great for all the values of ψ usually met with on an aeroplane. This follows from the large freedom given to the balls relative to their pushers, and from the fact that the accelerating forces experienced by the balls in the track, when the aeroplane is turning, are usually great. Hence the arrangement, so set up, is of little use.

It is important to note these facts. A novice, viewing the performances of two such instruments when set up on a table in a room, one provided with checks in front of the balls so as to limit their freedom, the other unprovided with checks, as shown in fig. 26, would be apt to conclude that the latter arrangement was preferable to the former. In point of fact, the latter arrangement results in a poor performance on an aeroplane, the former in an excellent one. We shall return to this form of erector.

The complete action of the type of erector described above (figs. 12, 13, 14) may now be explained. So long as the balls move radially in the slots (the first stage of the erecting process) each integral erecting couple results from a couple which comes into existence approximately at A

(fig. 28) (the device is supposed to be inclined to the vertical wholly about the pivots p_2p_2), revolves through a half turn with the erector, its amount remaining constant meanwhile, and goes out of existence at B; simultaneously with its disappearance it reappears at A, revolves as before, to go out of existence at B; and so on. The lines a represent the couple for a few of the directions in which it acts in the course of the half turn which it executes.

When the inclination of the pivoted system to the vertical has reached a limiting value, which value depends, as already explained, on the design and speed of the erector, the radial motion of the ball ceases. But the

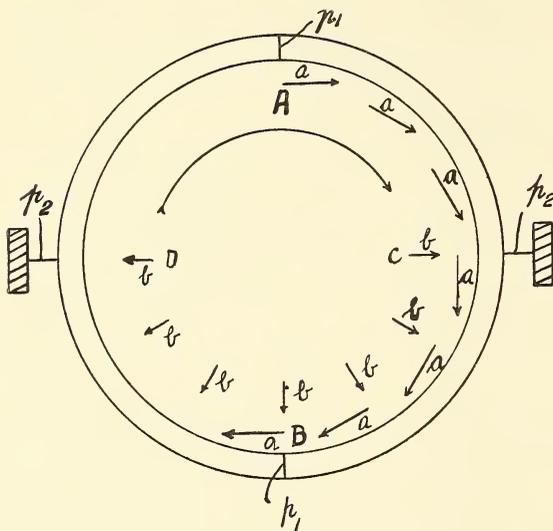


FIG. 28.

balls are allowed a small amount of clearance when situated in the outer compartments of the slots (fig. 17), and the erection of the device continues after the manner described above for the erector illustrated in fig. 26. The inclination of the pivoted system to the vertical being now very small, the transference of a ball from its pusher to its check takes place in the neighbourhood of c , and the retransference from the check to the pusher at D.

As a consequence, a couple (represented by b) comes into existence at c , revolves with the erector to D, and goes out of existence, to be succeeded by a couple of equal moment which appears at c simultaneously with the disappearance of the former couple at D; and so on until the erection of the device is complete. In each case, it will be seen readily from the figure, the direction of the integral couple is such as to erect the pivoted system about the pivots p_2p_2 , and it turns therefore almost wholly about the pivots p_2p_2 . The erection is accompanied by little, if any, turning on the pivots p_1p_1 .

In constructing the above diagram the convention has been adhered to according to which the line which represents a moment is drawn towards the observer when the moment has a counter-clockwise direction as viewed by him.

In the construction of the double erector illustrated in figs. 24 and 25 the slots which limit the freedom of the balls are so constructed that, for

the rapidly revolving ball system, each ball, when in the outer compartment of its slot, possesses very little freedom. In the case of the erector proper, which, it is to be remembered, revolves slowly in the direction of spin of the gyroscope, the balls are allowed sufficient clearance, when in the outside compartments of their slots, to allow of the revolving couple b coming into existence when required.

There is little to choose between the two main forms of erector which have been described, the double-slot erector and the track erector. When the apparatus is properly set

up and balanced, each is capable of supplying an accuracy of $\frac{1}{10}$ of a degree under the conditions which prevail in aircraft.

In figs. 29 and 30 is shown, in elevation and plan respectively, one method of mounting a camera c or sight as part of a Gray stabiliser. The frame f is rectangular, and is pivoted at p_2p_2 to the uprights u, u . Pivoted at p_1p_1 , within the frame f , is the gyroscope g and erector e . The camera c or sight S is pivoted to the frame f at p_3p_3 , and is attached to the gyro-

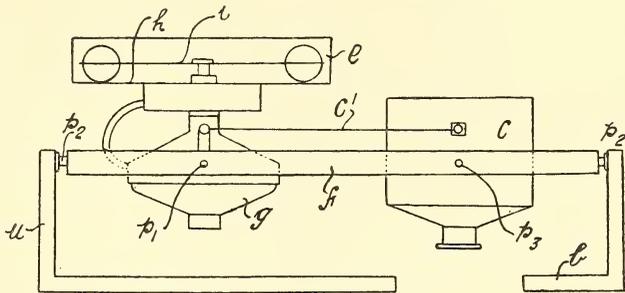


FIG. 29.

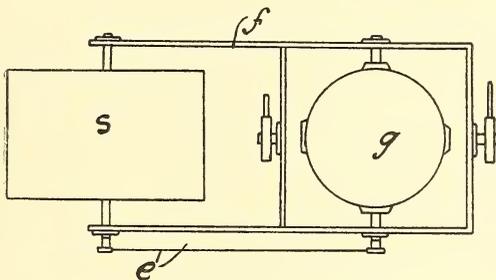


FIG. 30.

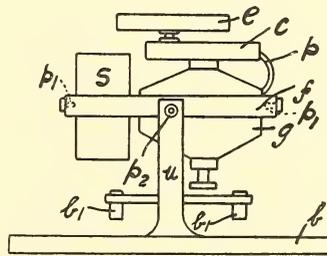


FIG. 31.

scopic system by a pivoted connecting rod c' as shown. This construction was convenient in cases where the observer's cockpit was long and narrow.

In fig. 31 is shown the arrangement which was adopted for use on aeroplanes in the cockpits of which the space in the fore-and-aft direction was limited. The gyroscopic system composed of gyroscope g and erector e was attached rigidly to the sight S , and the entire system pivoted at p_1p_1 to the frame f in such manner that the system was in neutral equilibrium with respect to the axis p_1p_1 . The frame was so pivoted to the

uprights u, u at p_2p_2 that the entire system was neutral with respect to p_2p_2 . When the erector was out of action the system was equivalent to a gyroscope mounted freely. In cases where it was desired to provide gravity control, small weights were attached to the system in positions lying below the plane of the pivot axes. Such weights are shown at b_1b_1 .

GRAY VERTICAL INDICATOR.

For many purposes, notably in the construction of vertical indicators for use in aircraft, or in the construction of stabilised mirrors for use with

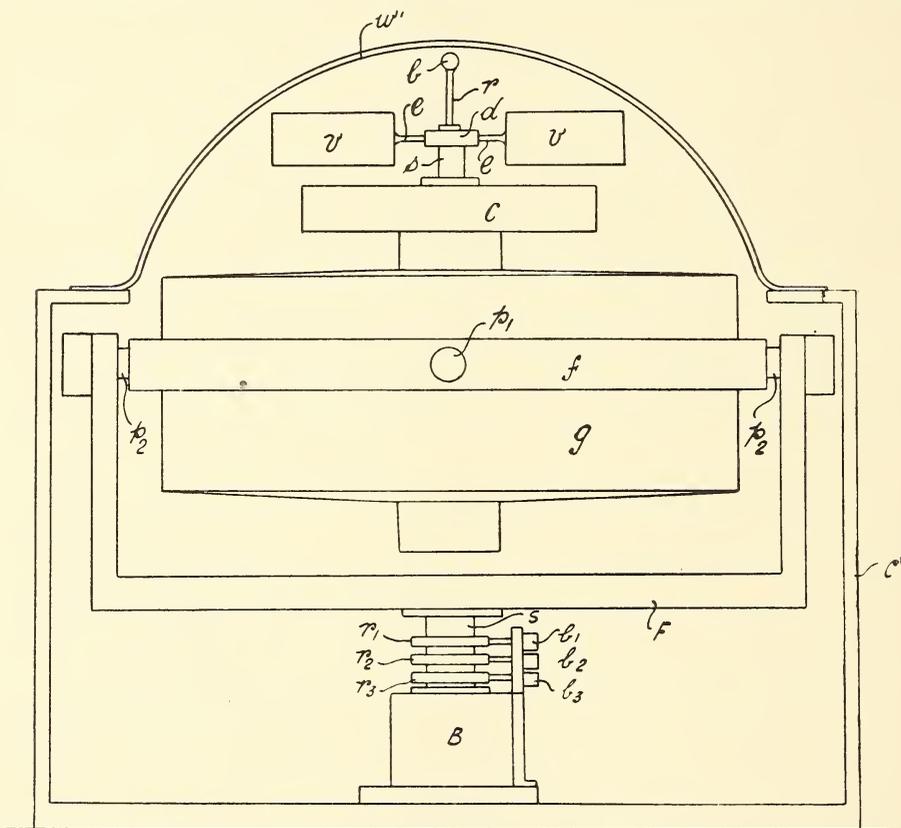


FIG. 32.

sextants, no disadvantage is introduced by rotating the stabilised parts. In fig. 32 is shown a form of vertical indicator in which the entire pivoted system rotates in the direction of spin of the gyroscope g .

The gyroscope g , as in the previous constructions, is pivoted at p_1p_1 (one pivot is indicated in the figure) to a frame f , and this in turn pivoted at p_2p_2 to a fork F . The fork is mounted on a vertical spindle s which

runs in a bearing B. Three slip-rings r_1, r_2, r_3 are carried by s , and these with three brushes b_1, b_2, b_3 serve to take in electric current to the motor of the gyroscope g .

One end of the gyroscope spindle is extended, and is connected, by means of reduction gearing contained in the casing c , to a vertical spindle s , and on this is mounted a boss d from which radiate a number of arms e, e , terminating in plane vanes v, v . The gearing is so arranged that when the gyroscope flywheel is spinning, the spindle s rotates rapidly in the opposite direction, and the vanes v are thus carried round in the direction opposed to that in which the gyroscope is spinning, and owing to the resistance which the air offers to their motion a torque in the opposite direction is applied to the pivoted system. Under the action of this torque the entire system turns on the bearing B in the direction of spin of the gyroscope g .

The apparatus is mounted so that the centre of gravity of the pivoted system lies very slightly to one side of the pivot axes. When the device is in action the equilibrium position is one in which the axis of the gyroscope is vertical. When the device is inclined to the vertical its motion of rotation becomes space-periodic in character, and there is applied to the pivoted system an integral erecting couple, in accordance with the principles already explained.

Attached to the spindle s is a rod r terminating in a bead b . The entire system is enclosed in a case c , and on this is carried a pair of cross-wires w, w (one only is shown in the figure) bent so that they lie on a surface of a sphere. When the device is carried on an aeroplane the relative movements of the wires and head serve to indicate inclinations of the aeroplane with respect to the horizontal.

ADJUSTMENT OF STABILISERS.

Returning now to the ordinary form of stabiliser, that in which the pivoted system does not rotate as a whole, it will be easily seen that the instrument must be carefully adjusted before being set up on an aeroplane. When the pivoted system is upright, with the balls in contact with their pushers, or what corresponds to the pushers, the centre of gravity of the entire pivoted system should lie either at the intersection of the pivot axes or very slightly below the intersection. The pivoted system is made up of parts which do not rotate, and a system of balls which move in a circular track. The centroid of the ball system, when the balls are in contact with their pushers, lies at the centre of the track; but the assumption should not be made that the centre of the track lies

at the intersection of the pivot axes, or on the vertical passing through the intersection. Thus the system should be balanced up with the balls in position, each held in contact with its pusher.

It is of the utmost importance that, before being set up on an aeroplane or other vehicle, the stabiliser should be adjusted by an expert, and that, once installed, the balance of the instrument should not be interfered with.

The final balancing of the stabiliser is accomplished by placing small weights on four pins arranged as shown in fig. 33. These pins lie, as shown, in line with p_1p_1 and p_2p_2 .

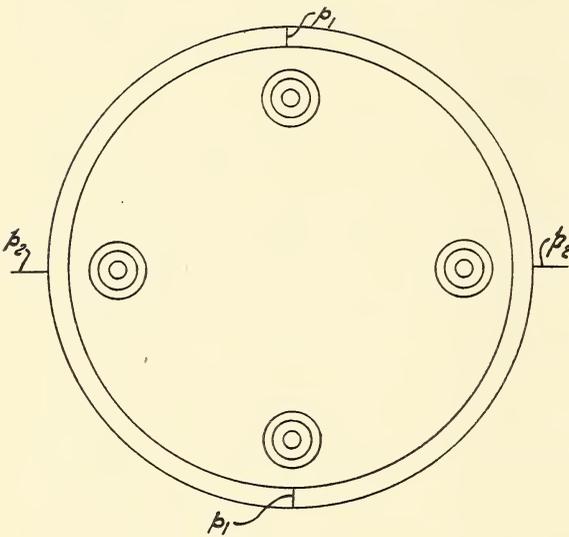


FIG. 33.

may not alter the amount of gravity control, if any, with which the instrument is provided, the pins are placed in the plane of the pivots. Once the adjustments are complete the weights may be fixed in position by passing split-pins through the pins or rods.

The stabiliser should be adjusted as follows. The balls are fixed in their resting positions (so that they are in contact with their pushers) and the gyroscope and erector set in motion. The pivoted system is then adjusted by

hand so that its axis is vertical, and is then watched. In general, of course, the system drifts from the upright position owing to want of balance. Weights are now placed on the pins so as to eliminate drifts; and when the pivoted system, so adjusted, retains the upright position undisturbed for a considerable interval of time, the operation is complete, and the instrument may be installed on the aeroplane.

On no account should the above adjustments be carried out with the gyroscope at rest. Static balancing is of little value. When the gyroscope is spinning the vibration results in the pivot friction being enormously diminished. By arranging that the erector is slightly out of balance the pivoted system may be caused to rock to and fro on the pivots, through a very small angle, in the period of the erector, and this has the result of diminishing the pivot friction. In general, a stabiliser balanced up

with the gyroscope at rest will be found to be, in reality, considerably out of balance.

It has been shown that, under the conditions which prevail on aircraft, a gyroscopic pendulum, if it is to be capable of remaining undisturbed, within the limits required, in the presence of the horizontal accelerations which accompany curved flight, must possess a very great precessional period. For this reason the bombsight, camera, artificial horizon, or other

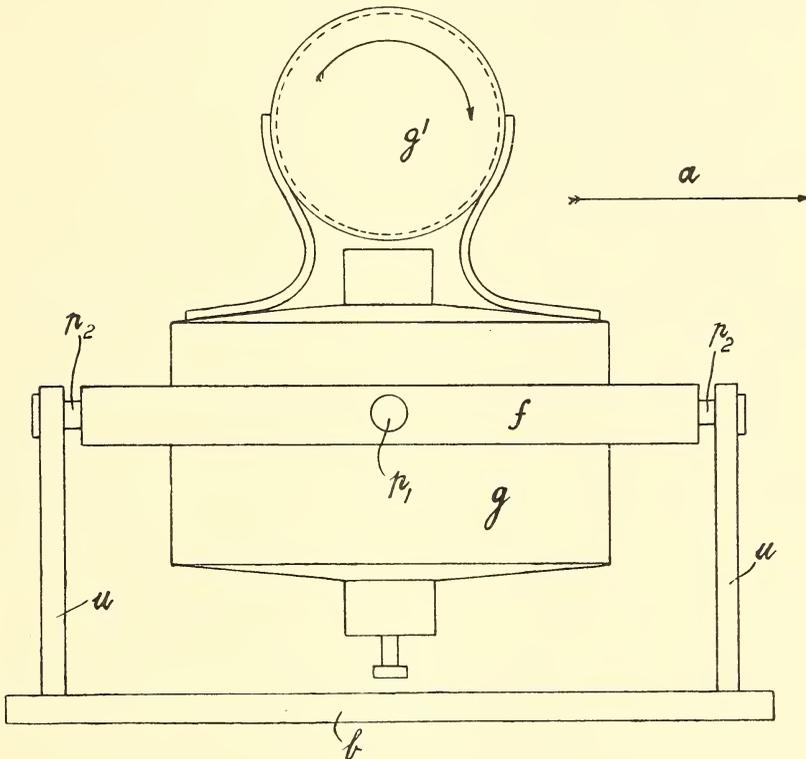


FIG. 34.

instrument stabilised, should be so constructed that any adjustments which have to be made should be capable of being carried out without altering the position of the centre of gravity of the pivoted system. In the case of cameras in which plates are changed, or films rolled from one spool on to another, it is difficult to arrange that this condition shall be precisely fulfilled. In such cases the form of the invention shown in fig. 34 is employed. Rigidly attached to the pivoted system is an auxiliary gyroscope g' with its axis perpendicular to that of the main gyroscope g . Thus when the pivoted system is upright the axis of g' is horizontal. Further, the pivoted system is so mounted on the frame f that

the axis of g' is perpendicular to the pivots p_2p_2 , which lie fore and aft with respect to the aeroplane. Thus the axis g' lies athwart the aeroplane.

Consider the action of this device when mounted on an aeroplane which is moving in the direction indicated by the arrow a . Let the pivoted system be mounted so that its centre of gravity lies below the plane of the pivots p_1p_1 , p_2p_2 , and let the direction of rotation of the gyroscope g' be that indicated by the curved arrow. Suppose the device initially upright, and let the aeroplane move in a curved path.

The couple experienced by the pivoted system as a result of the centrewards acceleration of the aeroplane is $Mv\dot{\psi}h$, where M is the total mass of the pivoted system, h the distance of its centre of gravity below the plane of the pivots, v the linear speed of the aeroplane, and $\dot{\psi}$ its angular speed in azimuth. This couple is applied about the pivots p_2p_2 . But the gyroscope g' is being turned forcibly in azimuth at angular speed $\dot{\psi}$; and if $I\omega$ is the angular momentum of g' , the pivoted system experiences a couple due to this cause of amount $I\omega\dot{\psi}$. With the direction of spin of g' shown this couple is applied about the pivots p_2p_2 in the direction opposed to that of the couple $Mv\dot{\psi}h$. This holds for both directions of turning in azimuth. The resultant couple is thus $(Mvh - I\omega)\dot{\psi}$, and is zero when $Mvh = I\omega$. Since the periodic time of the pivoted system is given by $T = 2\pi Cn/Mgh$, the condition that the resultant couple should be zero becomes

$$\frac{2\pi v}{gT} Cn = I\omega, \quad \text{or} \quad \frac{I\omega}{Cn} = \frac{2\pi v}{gT}.$$

Thus if $v = 100$ feet per second and $T = 6$ minutes, we must have

$$\frac{I\omega}{Cn} = \frac{2\pi \times 100}{32 \times 600} = \frac{1}{19} \quad (\text{about}).$$

If the condition is not exactly fulfilled, we have $(Mvh - I\omega)\dot{\psi}$ for the resultant couple, and for the angular speed at which the system turns about the pivots p_1p_1 , $(Mvh - I\omega)\dot{\psi}/Cn$. Let there be, say, a 5 per cent. error in $I\omega$, so that, say, $I\omega = 19Mvh/20$, then the angular speed at which the pivoted system turns about p_1p_1 is $Mvh\dot{\psi}/20Cn$. In other words, when the aeroplane is in curved flight the virtual precessional period of the system is $20T$, T being the actual precessional period. Thus if the period is 6 minutes, the system behaves during the curved motion of the aeroplane as if the periodic time of the gyroscopic system was 2 hours.

The actual mode of construction is illustrated diagrammatically in fig. 35. Two large gyroscopes are employed, one of which is tilted with respect to the vertical, as shown. The system is attached to the frame f

so that the horizontal component of spin, due to the tilt of the upper gyroscope, lies athwart the aeroplane. Let the gyroscopes be identical, and let the angular momentum of each be Cn . The condition required for complete compensation becomes

$$\frac{Cn \sin \theta}{Cn + Cn \cos \theta} = \frac{2\pi v}{g\Gamma}, \quad \text{or} \quad \tan \frac{1}{2}\theta = \frac{2\pi v}{g\Gamma}.$$

In practice the two gyroscopes are mounted within a frame (fig. 36), one rigidly, the other so that the angle θ may be set in accordance with the above equation. The value of v above is the air speed of the aeroplane,

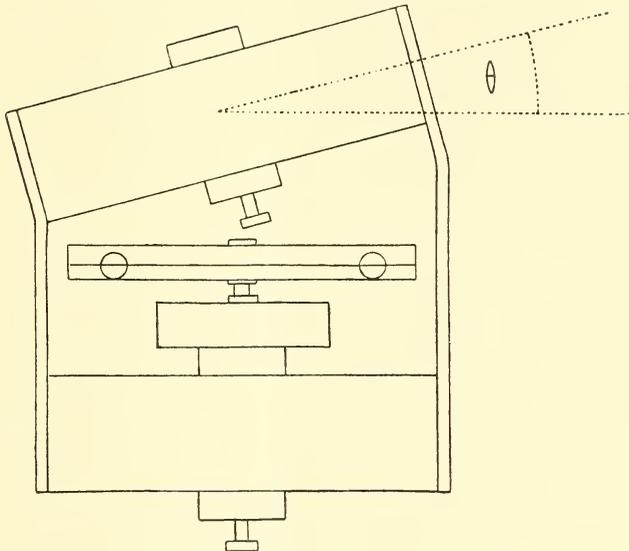


FIG. 35.

or airship. In the case of a ship the value of v is the speed of the ship relative to the water.

The construction will be clear from the figure. The two gyroscopes g_1 and g_2 are mounted on the frame F , g_1 rigidly, g_2 on pivots p, p , as shown. One of the pivots is extended, and carries, at its extremity, a quadrant q . This quadrant may be turned by means of a worm, as shown in the lower figure. The pivots p, p lie fore and aft with respect to the vehicle on which the apparatus is mounted. The erector, or erectors, may conveniently occupy the space between the gyroscopes.

One advantage obtained by using the two gyroscopes in the manner described will be obvious. When the pivoted system is provided with bottom weight, any small change in the position of the centre of gravity of the system due to movement of the film does not appreciably alter the resting position of the system.

SENSITIVENESS OF BALL ERECTORS.

The ball erectors which have been described, when driven round at perfectly uniform speeds (as is the case, for example, when the erector spindle is driven directly, through reduction gearing, by the spindle of the main gyroscope), depend for their action on the balls moving away from their pushers when descending the slope of the track on which they

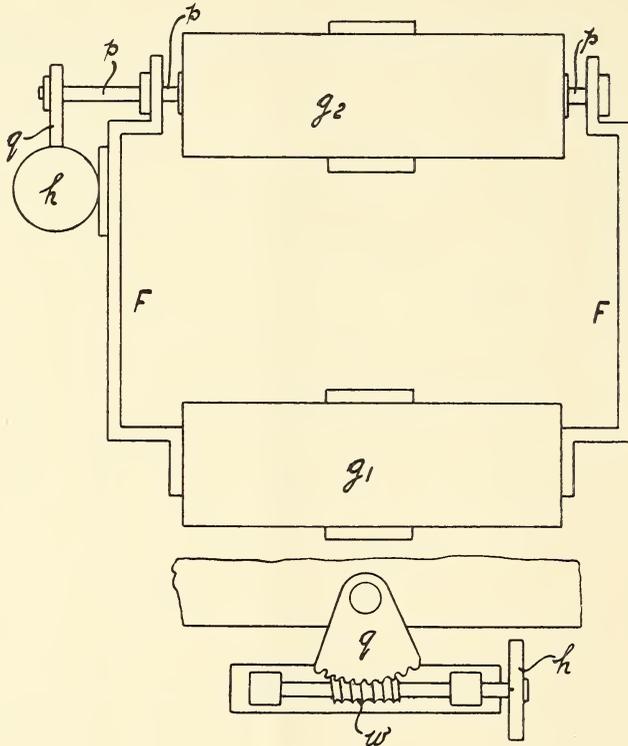


FIG. 36.

move. Such a ball erector, as already stated, is sensitive to within 2 or 3 minutes of angle.

To obtain greater delicacy a number of new erectors have been devised in which the ball, or balls, in ascending the slope, slow down the erector, and thus allow the ball or balls descending the slope to run away from their pushers. In some of the latest erectors the balls themselves do not contribute the stabilising forces directly, but control the application of such forces.

Consider an erector in which two balls, each provided with a pusher and check, are used. Let the balls rotate on a fixed track, being situated always on opposite sides of the track, so that when one ball is ascending

the slope of the track the other is descending. The ball which is ascending is rubbing against its pusher, and if the power available to rotate the erector is not too great the erector will slow down appreciably and the ball which is descending the slope will move up against its check. In fig. 37 is shown an arrangement in which two balls are employed as a brake. The balls lie, as shown, between a pusher p and a check c ; they rotate on the track t . If the track t is inclined to the horizontal, so that the balls are moving uphill, the ball b rubs against the pusher p and also against the ball b' . If the slope of the track is considerable, the ball b slides on the track, and the effect is, of course, to slow down the erector. When these devices are used, the resting position of the pivoted system is definite within half a minute of angle.

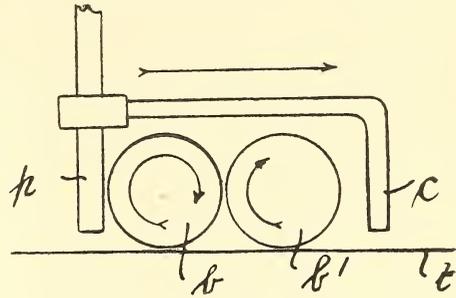


FIG. 37.

In fig. 38 is illustrated a convenient method of cutting off the action of the erector when the vehicle on which the apparatus is mounted is moving

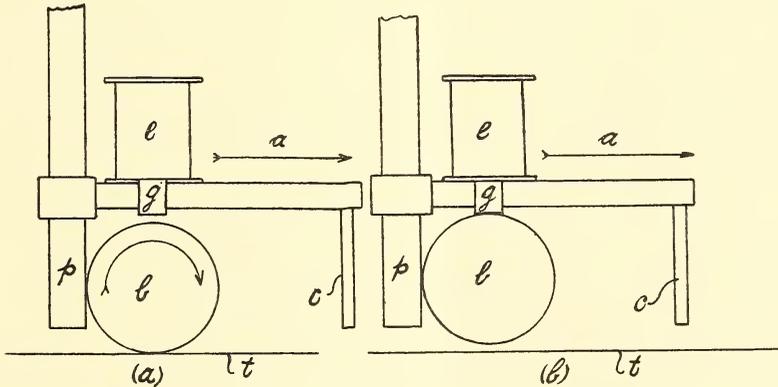


FIG. 38.

in a curved path. The ball b rotates on the track t , its motion limited by the pusher p and the check c . When the ball is against its pusher it lies directly under the core g of the electro-magnet e (fig. 38 (a)). When the vehicle moves in a curved path, current passes through the coils of the electro-magnet, and the ball leaves the track and adheres to g , as shown in fig. 38 (b). An alternative method of bringing about the desired result consists in causing the check c to move close up to the ball when the vehicle is rounding a curve.

The devices which have been described are admirably adapted for use on an aeroplane. A device set up in the cockpit of an aeroplane is situated on, or very near to, the axis of rolling, with the result that the moving parts of the erector experience very little, if any, disturbing force due to rolling. *Experiments carried out by means of the sun-shadow methods described above have shown conclusively that these stabilisers remain absolutely undisturbed by the pitching and rolling motions of an aeroplane. As already pointed out, even were such motions to result in the establishment of horizontal accelerations the integral effects would be very small.*

The writers have repeatedly observed the performances of pioneer instruments during ordinary flights with compass steering, in weather of the most bumpy description. Under these conditions these early instruments maintained the vertical within an accuracy measured by a few minutes of arc.

DESIRABILITY OF USING POWERFUL GYROSCOPES.

Let a gyroscope whose angular momentum is Cn be turning on its pivots towards the true vertical at angular speed θ under the action of a stabilising couple. For the amount l of the couple we have $l = Cn\theta$.

Now, for successful working l should be great, but θ small; hence Cn should be great. Obviously the stabilising couple should be very great in comparison with any frictional couple applied at the pivots. If it is assumed that the frictional couples are proportional to the load carried by the pivots, the advantage gained by employing large swiftly rotating gyroscopes will be at once apparent. The weight of a wheel is proportional to the third power of a linear dimension; its angular momentum, for a given speed of rotation, to the fifth power of the same dimension. Hence, as the diameter of the flywheel of a gyroscope increases, the ratio of the stabilising forces which can be employed, for a given value of θ , to the frictional forces applied at the pivots increases rapidly. With very powerful gyroscopes the frictional forces do not count; with small gyroscopes the frictional forces may be comparable with the stabilising forces.

For use on aeroplanes with their stabilisers the authors recommend the employment of a gyroscope having an angular momentum of 2000, in foot, pound, second units. This is the angular momentum possessed by a wheel of mass 20 pounds, radius of gyration 3 inches, when performing 250 revolutions per second. Complete with its casing and bearings, such a gyroscope would have a mass of about 25 pounds, and the diameter of its casing would lie between 7 and 8 inches.

Where instruments are to be employed on aircraft, consideration has to be paid to the weight of the apparatus. Such considerations do not apply to the application of gyroscopes to problems of naval gunnery, and for stabilising precision instruments on board ships of war in connection with anti-aircraft devices. In such cases it is desirable that the angular momentum of the main gyroscope, or combination of gyroscopes, should be upwards of 5000 in foot, pound, second units. Such gyroscopes are available in America, but not in this country. Given such gyroscopes, it will be found that the authors are in a position to supply apparatus capable of finding and maintaining the vertical, at any part of the ship, within an accuracy of 1 minute of arc.

The above remarks should not be misunderstood. The Gray stabiliser, if properly adjusted and used, is capable of furnishing excellent results when small gyroscopes are employed. But instruments intended for the National Defence should embody the very best practice.

NO ERROR IN GRAY STABILISER DUE TO ROTATION OF EARTH.

It has already been pointed out that the resting position of a gyroscopic pendulum, provided with gravity control and viscous damping, is one in which the axis of the pendulum is inclined to the true vertical at an angle $T \cos \lambda / \text{Day}$, where T is the periodic time of the pendulum for steady precession, and λ the latitude (see Gray's *Gyrostatics*, p. 342). *There is, however, no error in the case of the Gray stabiliser.*

Consider a freely mounted gyroscope placed with its axis vertical. The axis of the gyroscope would leave the vertical in this latitude, in consequence of the rotation of the earth, at an angular speed of about 1 minute of angle in 7 seconds of time. A Gray stabiliser, when deflected, seeks the vertical at angular speed l/Cn , where l is the moment of the erecting couple. *If l/Cn is greater than 1 minute of angle in 7 seconds of time, there is no error.* Critics of the Gray stabiliser have persistently affirmed that the accuracy with which the resting position is attained is affected by the rotation of the earth. Such is not the case. *The stabiliser, properly adjusted, stands erect even when the precessional period for steady motion is infinite; and the reason is obvious. As soon as error begins to grow up, the erector sets to work to annul it, and the vertical is never more than infinitesimally left.* The most persistent supporters of the notion that an error is produced by the rotation of the earth have, however, abandoned it.

XIX.—Elliptic Expansions. By Mr R. T. A. Innes.

(MS. received February 1, 1922. Read February 20, 1922.)

(ABSTRACT.)

IN seeking the expansion of the equation of the centre, and the logarithm of the radius vector in a planetary orbit, in terms of the mean anomaly, the following expressions are obtained:—

$$\begin{aligned} \text{Equation of centre} &= \sum_i \frac{2}{i} (A_i + B_i + C_i) \sin ig, \\ \text{Log } r/a &= \sum_i \frac{2}{i} (B_i - C_i) \cos ig, \end{aligned}$$

where g is the mean anomaly, and

$$\begin{aligned} A_i &= J_i(ie), \quad B_i = \sum_1^\infty \beta^j J_{i-j}(ie), \quad C_i = \sum_1^\infty \beta^j J_{i+j}(ie), \\ \beta &= \{1 - \sqrt{(1 - e^2)}\}/e, \text{ and } J_i(x) \text{ is Bessel's function.} \end{aligned}$$

Mr Innes remarks: "It is shown for the first time, I believe, that the expansion for the true anomaly in terms of the mean (equation of the centre) can be derived quite easily from the addition of the two simple expansions

True in terms of eccentric anomaly,
Mean " " " "

and the two series have most simple recurrence expressions."

The MS. containing the proofs of these expressions, with further developments, is in the hands of the Society.

XX.—On Self Light, Fatigue, Inhibition, and Recurrent Visual Images. By Professor W. Peddie, D.Sc.

(MS. received June 19, 1922. Read June 19, 1922.)

(ABSTRACT.)

IN an earlier note communicated last session the action of self light and fatigue was partially considered. The question of the origin of these effects is one which can only be settled by observation or experiment. The question of the mathematical form of the laws regulating the effects also requires an observational or experimental basis. In the absence of self light, Fechner's well-known law that the change in sensation is proportional to the fractional change in the stimulus forms a close approximation to results of experience throughout a wide range of intensity. Fechner's modification of this law, by the addition of a constant term to the denominator of the fraction, was shown by Helmholtz to account broadly for the action of self light; and he showed also that the introduction of three such constants into three such independent fractions enabled a good description to be given of various phenomena of colour vision in light of different intensities. But, from the phenomena of vision, it is certain that the numerical values of the parameters are influenced by antecedent as well as present illumination. Thus the parameters are really functions which are variable in more than one way, and the form of these functions can only be found by observation. This is the subject of the note referred to above.

Helmholtz extended Fechner's expression for the differential sensation in one colour to a combination of three so as to obtain the curve of differential sensitivity throughout the spectrum. He found remarkable agreement with experimental data, and used the results for the determination of the absolute fundamentals. He also treated the problems of contrast colours and after images, the former being treated from the point of view of psychical considerations, thus avoiding more detailed formal discussion. This was essentially the stage to which the subject was developed at the time of his death, and at which, in the matter of formal development, it has largely remained since. But great experimental development has been made since that time, especially in connection with the questions of fatigue, inhibition, contrast, after images, and

recurrent vision. The evidence of the existence of cross influences amongst stimuli has become conclusive.

The recognition of these makes possible the formal treatment of contrast and other effects. Fatigue is formally taken into account through the employment of the integrated form of Fechner's law. The three integration constants (each of which may expressly contain as a term the corresponding self-light parameter) directly express the extent to which fatigue has developed. They are, of course, constant only under given conditions of illumination and its duration. They are the instantaneous threshold values of the fundamental stimuli. Inhibition occurs whenever the external stimulus falls below the threshold value. In dependence upon the values of the various parameters, including those of the cross connections, a sensation may decay according to an exponential law, simple or complex, or it may have an exponentially decaying oscillatory value. In this way the oscillatory succession of coloured after images receives formal explanation. The determination of the mechanism underlying that formulation is a matter for physiological investigation.

The trichromatic theory, essentially complete to at least a first approximation in its formal development, is as firmly established as the electron theory, or the molecular theory, or the electromagnetic theory of light.

(Issued separately August 10, 1922.)

XXI.—On the Dominance Ratio. By R. A. Fisher, M.A., Fellow of Gonville and Caius College. *Communicated* by Professor J. ARTHUR THOMSON.

(MS. received March 8, 1922. Read June 19, 1922.)

INTRODUCTION.

IN 1918, in a paper published in the *Transactions of the Royal Society of Edinburgh*, the author attempted an examination of the statistical effects in a mixed population of a large number of genetic factors, inheritance in which followed the Mendelian scheme. At that time, two misapprehensions were generally held with regard to this problem. In the first place, it was generally believed that the variety of the assumptions to be made about the individual factors—which allelomorph was dominant; to what extent did dominance occur; what were the relative magnitudes of the effects produced by the different factors; in what proportion did the allelomorphs occur in the general population; were the factors dimorphic or polymorphic; to what extent were they coupled,—besides the more general possibilities of preferential mating (homogamy), preferential survival (selection), and environmental effects, rendered it possible to reproduce any statistical resultant by a suitable specification of the population. It was, therefore, important to prove that when the factors are sufficiently numerous, the most general assumptions as to their individual peculiarities lead to the same statistical results. Although innumerable constants enter into the analysis, the constants necessary to specify the statistical aggregate are relatively few. The total variance of the population in any feature is made up of the elements of variance contributed by the individual factors, increased in a calculable proportion by the effects of homogamy in associating together allelomorphs of like effect. The degree of this association, together with a quantity which we termed the Dominance Ratio, enter into the calculation of the correlation coefficients between husband and wife, and between blood relations. Special causes, such as epistacy, may produce departures, which may in general be expected to be very small from the general simplicity of the results; the whole investigation may be compared to the analytical treatment of the Theory of Gases, in which it is possible to make the most varied assumptions as to the accidental circumstances, and even the essential nature of

the individual molecules, and yet to develop the general laws as to the behaviour of gases, leaving but a few fundamental constants to be determined by experiment.

In the second place, it was widely believed that the results of biometrical investigation ran counter to the general acceptance of the Mendelian scheme of inheritance. This belief was largely due to the narrowly restricted assumptions as to the Mendelian factors, made by Pearson in his paper of 1903 (6). It was there assumed that the factors were all equally important, that the allelomorphs of each occurred in equal numbers, and that all the dominant genes had a like effect. The effect of homogamy was also left out of consideration, and it is to this that must be ascribed the much lower correlations given by calculation, compared to those actually obtained. When the more general system came to be investigated, it was found to show a surprisingly complete agreement with the experimental values, and to indicate with an accuracy which could not otherwise be attained, how great a proportion of the variance of these human measurements is to be ascribed to heritable factors.

At the time when the paper of 1918 was written, it was necessary, therefore, to show that the assumption of multiple, or cumulative, factors afforded a working hypothesis for the inheritance of such apparently continuous variates as human stature. This view is now far more widely accepted: Mendelian research has with increasing frequency encountered characters which are evidently affected by many separate factors. In some fortunate circumstances, as in *Drosophila*, it has been possible to isolate and identify the more important of these factors by experimental breeding on the Mendelian method; more frequently, however, and especially in the case of the economically valuable characters of animals and plants, no such analysis has been achieved. In these cases we can confidently fall back upon statistical methods, and recognise that if a complete analysis is unattainable it is also unnecessary to practical progress.

This fact is meeting with increasing recognition in the United States, and a considerable number of mathematical investigations have been published dealing with the statistical effects of various systems of mating (Wentworth and Remick, 1916; Jennings, 1916, 1917; Robbins, 1917, 1918). A number of the simpler results of my 1918 paper have since been confirmed by independent American investigators (Wright, 1921). The present note is designed to discuss the distribution of the frequency ratio of the allelomorphs of dimorphic factors, and the conditions under which the variance of the population may be maintained. A number of points of general interest are shown to flow from purely statistical premises

Recent work in genetics (East and Jones, 1920) leads unavoidably to the conclusion that inbreeding is not harmful in itself, but is liable to appear harmful only through the emergence of harmful recessive characters. This raises the question as to why recessive factors should tend to be harmful, or why harmful factors should tend to be recessive: unless this association exist we should expect to obtain great improvements by inbreeding ordinarily crossbred species, as often as great deterioration. The statistical reason for this association is clear from the distribution of the ratio of allelomorph frequency which occurs under genotypic selection, for, if we assume that adaptation is the result of selection, the majority of large mutations must be harmful, and these can only be incorporated in the common stock in the sheltered region where the rare recessives accumulate (fig. 4).^{*} Similarly there are many well-attested cases of the crossbred (heterozygous) individual showing surprising vigour; but it is not obvious that there is any biological reason for the heterozygote to be more vigorous than the two homozygotes. From a consideration of the stability of the frequency ratios, however, it appears that there will only be stable equilibrium if the heterozygote is favoured by selection against both the homozygotes: naturally this will occur only in a minority of factors, but when it occurs such a factor will be conserved. In the opposite case it will certainly be eliminated.

Cases in which the heterozygote is favoured by selection in preference to both homozygous forms have an additional interest in that these cases, when the selection is intense, may form the basis upon which is built up a system of balanced lethal factors. Muller (1918) has shown that such systems will tend to be built up when selection strongly favours the heterozygote, and has explained how in the light of such systems the majority of the phenomena, including the "mutations," of *Oenothera*, find a genetic explanation.

The interesting speculation has recently been put forward that random survival is a more important factor in limiting the variability of species than preferential survival (Hagedoorn, 4). The ensuing investigation negatives this suggestion. The decay in the variance of a species breeding at random without selection, and without mutation, is almost inconceivably slow: a moderate supply of fresh mutations will be sufficient to maintain the variability. When selection is at work even to the most trifling extent, the new mutations must be much more numerous in order to

^{*} On the Lamarckian theory of evolution, on the other hand, where most, or all, mutations are assumed to be beneficial, we should expect by inbreeding, which uncovers the accumulated mutations in this region, to make great and immediate progress.

maintain equilibrium. That such is the actual state of the case in mankind may be inferred from the fact that the frequency distribution of the numerical proportion of the allelomorphs, calculated on the assumption of selection maintained in equilibrium by occasional mutation, leads to the value of the Dominance Ratio which is actually observed. In all cases it is worth noting that the rate of mutation required varies as the variance of the species, but diminishes as the number of individuals is increased. Thus a numerous species, with the same frequency of mutation, will maintain a higher variability than will a less numerous species: in connection with this fact we cannot fail to remember the dictum of Charles Darwin, that "wide ranging, much diffused and common species vary most" (1, chap. ii).

1. EQUILIBRIUM UNDER SELECTION.

Let the three phases of a dimorphic factor be born in any generation in the proportion

$$P : 2Q : R,$$

then the proportion of the two allelomorphic genes will be

$$P + Q : Q + R, \quad \text{or} \quad p : q;$$

if by selection those that become parents are in the proportion

$$aP : 2bQ : cR, \quad \text{where} \quad aP + 2bQ + cR = 1,$$

then the proportion born in the next generation will be

$$(aP + bQ)^2 : 2(aP + bQ)(bQ + cR) : (bQ + cR)^2;$$

equilibrium is thus only possible if $Q^2 = PR$, *i.e.* $P = p^2$, $Q = pq$, $R = q^2$, and if $aP + bQ = p$, $bQ + cR = q$.

Hence it follows that, if

$$a = 1 + \alpha, \quad b = 1 + \beta, \quad c = 1 + \gamma,$$

$$\frac{\alpha}{p^2} = -\frac{\beta}{pq} = \frac{\gamma}{q^2}$$

specifies the condition of equilibrium.

If selection favours the homozygotes, no stable equilibrium will be possible, and selection will then tend to eliminate whichever gene is below its equilibrium proportion; such factors will therefore not commonly be found in nature: if, on the other hand, the selection favours the heterozygote, there is a condition of stable equilibrium, and the factor will continue in the stock. Such factors should therefore be commonly found, and may explain instances of heterozygote vigour, and to some extent the deleterious effects sometimes brought about by inbreeding.

If the selective action is sufficiently powerful, it may lead in these cases to the establishment of a balanced lethal system.

2. THE SURVIVAL OF INDIVIDUAL GENES.

If we consider the survival of an individual gene in such an organism as an annual plant, we may suppose that the chance of it appearing in the next generation in 0, 1, 2, 3 individuals to be

$$p_0, p_1, p_2, \dots$$

where

$$p_0 + p_1 + p_2 + \dots = 1.$$

If

$$f(x) = p_0 + p_1x + p_2x^2 + \dots$$

then evidently if there were two such genes in the first generation, the chance of occurrence in r individuals, or more strictly, in r homologous loci, in the second generation, will be the coefficient of x^r in

$$(f(x))^2.$$

It follows that the chance of a single gene occurring in r homologous loci, in the third generation, will be coefficient of x^r in

$$f(f(x)).$$

The form of $f(x)$ will vary from species to species, and in the same species according to the stage of development on which we fix our attention. For simplicity we shall suppose that the successive generations are enumerated at the same stage of development. For the purpose of an evolutionary argument it is indifferent at what stage of development the enumeration is made: in general it will be most convenient to fix our attention on that stage at which the species is least numerous.

In certain important cases the form of $f(x)$ may be calculated. In a field of cross-fertilised grain each mature and ripened plant is the mother of a considerable number of grains, and the father, possibly, of an almost unlimited number. If the number of the species is nearly constant, the average number of its progeny which are destined to become mature is very nearly 2. Or since each gene of a homologous pair occurs in half the gametes, the average number of mature plants in the second generation in which it occurs is 1. Each ovule, therefore, or each pollen grain has individually a very small chance of surviving, and the proportions $p_0, p_1, p_2,$ will be closely given by the Poisson series

$$e^{-1} \left(1, 1, \frac{1}{2!}, \frac{1}{3!}, \dots \right)$$

In the more general case in which the number of the species is not stationary but increases in each generation in the ratio $m : 1$, m being near to unity, the series will be

$$e^{-m} \left(l, m, \frac{m^2}{2!}, \frac{m^3}{3!}, \dots \right)$$

and $f(x) = e^{m(x-1)}$. The chance of extinction of a single gene in one generation is e^{-m} , where m is near to unity. In other species in which an individual may survive for many breeding seasons, or in which the generation is of indeterminate length, the form of the function $f(x)$ will be modified: it is sufficiently clear, however, that if we consider that stage in an animal's or plant's life-history at which reproduction is about to commence, the form of the function will not be very different, and the chance of extinction of a particular gene, thus far established in the species, will be

$$e^{-l},$$

where l is a small number not greatly different from unity.* The arbitrary element thus introduced into the question of the survival of a mutant gene is due to the fact that in the first place its survival depends on that of the individual in which it occurs, and this chance is variable from species to species; once, however, it has reached the point of existing in an adult individual capable of leaving many offspring, the conditions of its survival are closely similar in all cases. While it is rare, its survival will be at the mercy of chance, even if it is well fitted to survive. Using the above expression,

$$f(x) = e^{x-1},$$

it may be seen that only about 2 per cent. will survive 100 generations, while those that do will on the average be represented in some 50 individuals. Only when the number of individuals affected becomes large will the effect of selection predominate over that of random survival, though even then only a very small minority of the population may be affected.

3. FACTORS NOT ACTED ON BY SELECTION.

If p be the proportion of any gene, and q of its allelomorph in a dimorphic factor, then in n individuals of any generation we have $2np$ genes scattered at random. Let

$$\cos \theta = 1 - 2p$$

where θ lies between 0 and π .

* An upper limit can be set to l by the mere fact of segregation, for in the case of the most uniform possible reproduction, when each individual bears 2 offspring the chance of extinction of any gene is $\frac{1}{4}$, so that l cannot exceed 1.4.

Further, if a second generation of n individuals be now formed at random, the standard departure of p from its previous value will be

$$\sigma_p = \sqrt{\frac{pq}{2n}},$$

hence,

$$\sigma_\theta = \sqrt{\frac{pq}{2n}} \frac{d\theta}{dp} = \frac{1}{\sqrt{2n}}.$$

The fact that this is independent of θ makes it easy to calculate the changes in the distribution of θ , in the absence of selection, for let $y(\theta) d\theta$ represent the distribution of θ in any one generation, the distribution in the next will be given by

$$\begin{aligned} y + \Delta y &= \int_0^\pi \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\delta\theta^2}{2\sigma^2}} \left(y + y'\delta\theta + \frac{\delta\theta^2}{2!}y'' + \dots \right) \\ &= y + \frac{\sigma^2}{2}y'' + \dots \end{aligned}$$

Now σ^2 is very small, being $\frac{1}{2n}$, so that measuring time in generations, we have

$$\frac{\partial y}{\partial T} = \frac{1}{4n} \frac{\partial^2 y}{\partial \theta^2}.$$

Since we have drawn no distinction between the gene and its allelomorph, we are only concerned with symmetrical solutions: the stationary case is

$$y = \frac{A}{\pi},$$

where A is the number of factors present.

Besides this, we have when y is increasing

$$y = A_0 e^{kT} \frac{p}{2 \sinh \frac{1}{2} p\pi} \cdot \cosh p \left(\theta - \frac{\pi}{2} \right),$$

and when y is decreasing

$$y = A_0 e^{-kT} \frac{p}{2 \sin \frac{1}{2} p\pi} \cdot \cos p \left(\theta - \frac{\pi}{2} \right),$$

for which

$$k = \frac{p^2}{4n}.$$

4. TERMINAL CONDITIONS.

If we represent by e^{-l} the chance that a particular gene borne by a single individual will not be represented in the next generation, the chance of extinction for a factor of which b genes are in existence will be

$$e^{-bl}.$$

When θ is near to 0, p which is always equal to $\sin^2 \frac{\theta}{2}$, will be very nearly equal to $\frac{1}{4}\theta^2$. Let

$$t = \sin \frac{1}{2}\theta,$$

then the number of genes in existence is $2nt^2$, and the chance of their extinction in one generation is e^{-2nt^2} .

This chance is negligible save when t is very small, and may be equated to $\frac{1}{2}\theta$; hence the number of genes exterminated in any one generation

$$\begin{aligned} & 2 \int_0^{\theta} ye^{-2nt^2} d\theta \\ &= 4 \int_0^{\theta} ye^{-2nt^2} dt. \end{aligned}$$

In the stationary case $y = \frac{\Lambda}{\pi}$, and the number of genes exterminated will be

$$\frac{\Lambda}{\pi} \cdot \frac{2\sqrt{2\pi}}{\sqrt{4ln}} = \Lambda \sqrt{\frac{2}{\pi ln}},$$

if new mutations occur at a rate $n\mu$, then this equilibrium will be possible if

$$\Lambda = \sqrt{\frac{\pi l}{2}} n^{\frac{3}{2}} \mu.$$

For species in this stationary state the variance will vary (1) as the rate of mutation, (2) as the number of the population raised to the power of $\frac{3}{2}$, (3) as \sqrt{l} , a quantity which will seldom differ much from unity.

Using the variate $z = \log_e \frac{p}{q}$, the distribution for this case is shown in fig. 1.

5. THE HAGEDOORN EFFECT.

In the absence of mutation, extinction will still go on, and the number of factors must diminish, hence we may put for this case

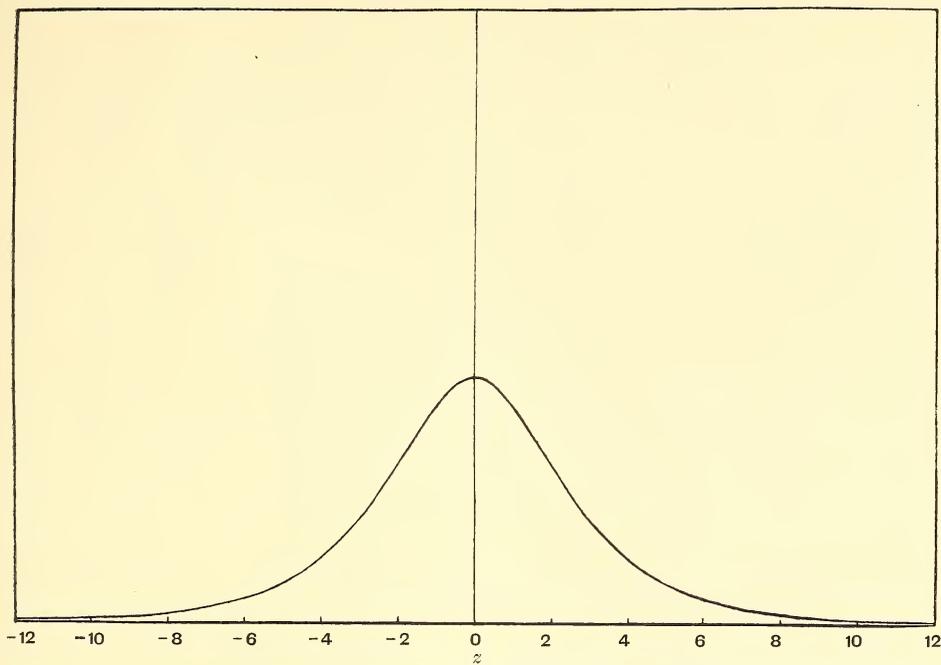
$$y = A_0 e^{-kt} \cdot \frac{p}{2 \sin \frac{1}{2}p\pi} \cdot \cos p\left(\theta - \frac{\pi}{2}\right).$$

If θ is small,

$$\begin{aligned} \cos p\left(\theta - \frac{\pi}{2}\right) &= \cos \frac{1}{2}p\pi + p\theta \sin \frac{1}{2}p\pi - \frac{1}{2}p^2\theta^2 \cos \frac{1}{2}p\pi \dots \\ &= \cos \frac{1}{2}p\pi + 2p \sin \frac{1}{2}p\pi \cdot t - 2p^2 \cos \frac{1}{2}p\pi \cdot t^2 \dots, \end{aligned}$$

so that the rate of extinction is

$$A_0 e^{-kt} \frac{p}{2 \sin \frac{1}{2}p\pi} \cdot \sqrt{\frac{2\pi}{ln}} \left\{ \cos \frac{1}{2}p\pi + 2p \sin \frac{1}{2}p\pi \cdot \sqrt{\frac{2}{4 ln}} \right\}$$



Distribution of the logarithmic frequency ratio $\left(z = \log \frac{p}{q}\right)$ of the allelomorphs of a dimorphic factor.

FIG. 1.— $df = \frac{1}{2\pi} \operatorname{sech} \frac{1}{2}z dz$; †

represents the distribution when, in the absence of selection, fortuitous extinction is counterbalanced by mutation. Dominance Ratio = .2308.

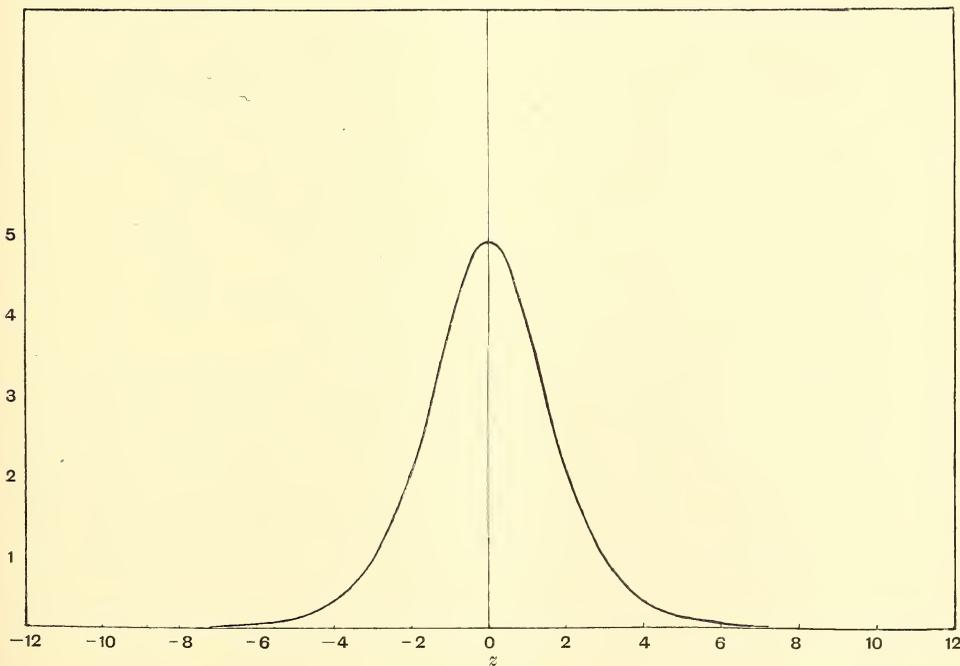


FIG. 2.— $df = \frac{1}{4} \operatorname{sech}^2 \frac{1}{2}z dz$;

represents the distribution when, in the absence of selection and mutation, the variance is steadily decaying owing to fortuitous extinction of genes. Dominance Ratio = .2500. This is the condition emphasised by Hagedoorn.

the third term being evidently negligible compared to the first. For equilibrium, therefore,

$$k = p \sqrt{\frac{2\pi}{ln}} \left\{ \frac{1}{2} \cot \frac{1}{2} p \pi + \frac{p}{2\pi ln} \right\}.$$

Remembering that $k = \frac{p^2}{4n}$, we have

$$\frac{p^2}{n} \left(\frac{1}{4} - \frac{1}{l} \right) = \sqrt{\frac{2\pi}{ln}} \frac{p}{2} \cot \frac{1}{2} p \pi.$$

Hence $\cot \frac{1}{2} p \pi$ is of the order $\frac{1}{\sqrt{n}}$ and is very small, so that p is near to 1.

Then

$$k = \frac{1}{4n}.$$

This is a very slow rate of diminution, a population of n individuals breeding at random would require $4n$ generations to reduce its variance in the ratio 1 to e , or $2.8 n$ generations to halve it. As few specific groups contain less than 10,000 individuals between whom interbreeding takes place, the period required for the action of the Hagedoorn effect, in the entire absence of mutation, is immense. It will be noticed that since l is always less than 1.4 in species stationary in number, the solution above makes p slightly greater than l , which strictly would indicate negative frequencies at the extremes: the value of k is, however, connected with the curvature in the central portion of the curve, and the small distortion at the extremes, where the assumptions, upon which our differential equation is based, break down, will not affect its value. (Fig. 2 shows the distribution of $z = \log \frac{p}{q}$.)

The number by which the number of factors current is reduced in each generation is $\frac{A}{4n}$, and since this number depends on the general form of the distribution curve, it will not be diminished by a number of mutations of the same order. The effect of such very rare mutations would merely be to adjust the terminal of the curve until the rate of extinction is increased sufficiently to counterbalance the additional mutations. It is probable, however, that μ is always far greater than is necessary to make this state of affairs impossible, save in the case of a small colony recently isolated from a very variable species. In this case, with n small and A large, μ might for a time be of the order An^{-2} , rather than of the order $An^{-\frac{1}{2}}$, or An^{-1} .

In the case of a population with A factors, with a supply of fresh

mutations sufficient only to be in equilibrium with a smaller number B factors, we may put

$$B\sqrt{\frac{2}{\pi ln}} = \frac{Ap}{2 \sin \frac{1}{2}p\pi} \cdot \cos \frac{1}{2}p\pi \cdot \sqrt{\frac{2\pi}{ln}},$$

or,

$$\frac{B}{A} = \frac{1}{2}p\pi \cot \frac{1}{2}p\pi,$$

so that if

$$\frac{\alpha}{\tan \alpha} = \frac{B}{A}, \quad 0 < \alpha < \frac{\pi}{2},$$

$$p = \frac{2\alpha}{\pi},$$

and

$$k = \frac{\alpha^2}{\pi n}.$$

Similarly, if $B > A$, the rate of increase in variance may be calculated from the equations

$$\frac{\alpha}{\tanh \alpha} = \frac{B}{A},$$

$$k = \frac{\alpha^2}{\pi n}.$$

The rate of decrease, therefore, cannot, in the absence of selection, exceed the value indicated by $k = \frac{1}{4n}$; no such limit can be assigned to the rate of increase.

6. UNIFORM GENETIC SELECTION.

In section 1 we have seen that the effects of selection on any Mendelian factor may be expressed by the triple ratio $a : b : c$ representing the relative fitness of the three phases. Only when b exceeds both a and c is there a condition of stable equilibrium; when b is less than both a and c there is a condition of unstable equilibrium; and such factors will tend rapidly to disappear from the stock. Generally, however, we may expect that either b will be intermediate, or equal to a , the value for the dominant homozygote. Two hypothetical cases may, therefore, be considered: (1), in which b is the geometric mean of a and c , and the selection merely affects the proportion of the allelomorphous genes; we may call this uniform genetic selection; and (2), in which b is equal to a , which we may call uniform genotypic selection.

In uniform genetic selection the genetic ratio will be altered in a constant ratio r in each generation, so that after n generations of selection we have

$$\frac{p}{q} = r^n \frac{p_0}{q_0},$$

evidently $r = \frac{a}{b} = \frac{b}{c}$ of section 1.

We may suppose that usually r is near to unity, and $\log r$, which may be positive or negative, may be considered to be of the order of 1 per cent. Let $\log r = a$, then for different factors a will have different values, indifferently positive and negative, since we have no reason to suppose that the selection favours either dominant or recessive characters. The mean square value of a for different factors we shall write σ_a^2 .

For any factor

$$\frac{d}{dT} \log \frac{p}{q} = a;$$

therefore

$$\frac{dp}{dT} = pq a,$$

$$\frac{d\theta}{dT} = a\sqrt{pq}.$$

The factors which in one generation are at θ , will in the next be scattered owing to two causes: (1) random survival causing variance, $\frac{1}{2n}$; (2) selection causing variance, $pq \sigma_a^2 (= \frac{1}{4} \sin^2 \theta \cdot \sigma_a^2)$. The total variance at any point will be

$$\frac{1}{2n} + \frac{1}{4} \sigma_a^2 \sin^2 \theta;$$

and so long as σ_a^2 is small as we have supposed, the equilibrium distribution will be

$$y \propto \frac{1}{\sqrt{\sin^2 \theta + \frac{2}{n\sigma_a^2}}},$$

or nearly

$$y = \frac{A}{2 \log (\sigma_a \sqrt{8n})} \cdot \frac{1}{\sqrt{\sin^2 \theta + \frac{2}{n\sigma_a^2}}}.$$

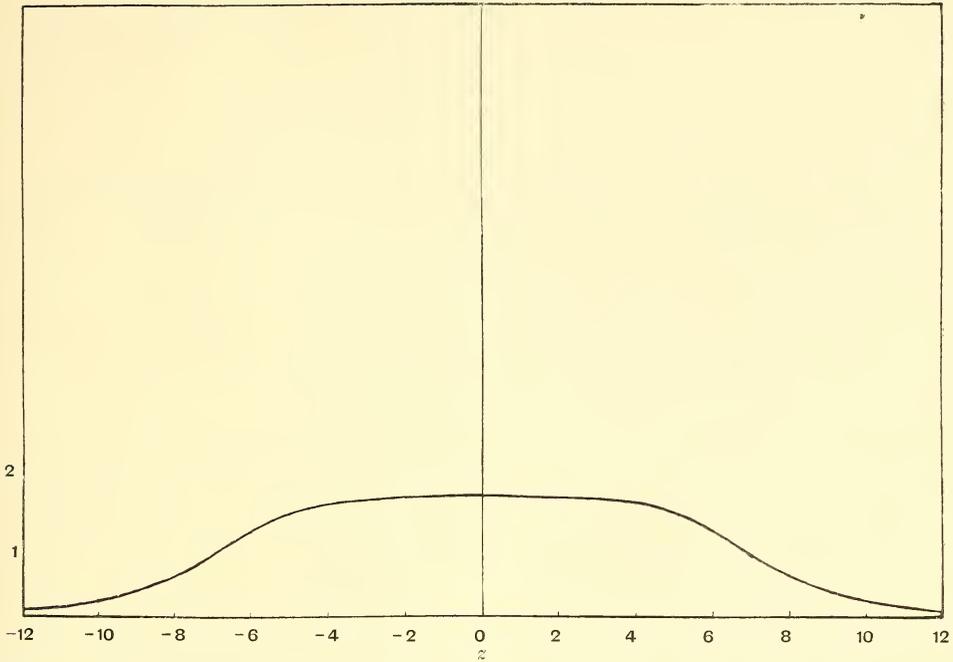
n being large compared with $\frac{1}{\sigma_a^2}$, the effects of selection are, for the more important factors, much more influential than those of random survival. At the extremes, however, for very unequally divided factors the latter is the more important cause of variation. (The distribution of $z = \log \frac{p}{q}$ is shown in fig. 3.)

The amount of mutation needed to maintain the variability with this amount of selection may be calculated from the terminal ordinate

$$\frac{A\sigma_a \sqrt{\frac{n}{2}}}{2 \log (\sigma_a \sqrt{8n})},$$

whence

$$n\mu = \sqrt{\frac{2\pi}{ln}} \cdot \frac{A\sigma_a \sqrt{\frac{n}{2}}}{2 \log (\sigma_a \sqrt{8n})} = \frac{A\sigma_a \sqrt{\frac{\pi}{l}}}{2 \log (\sigma_a \sqrt{8n})}.$$



Distribution of the logarithmic frequency ratio.

FIG. 3.— $df \propto \frac{dz}{\sqrt{1 + k^2 \cosh^2 \frac{1}{2}z}}$; $k = .1$;

genetic selection counterbalanced by mutation. Dominance Ratio, '2000.

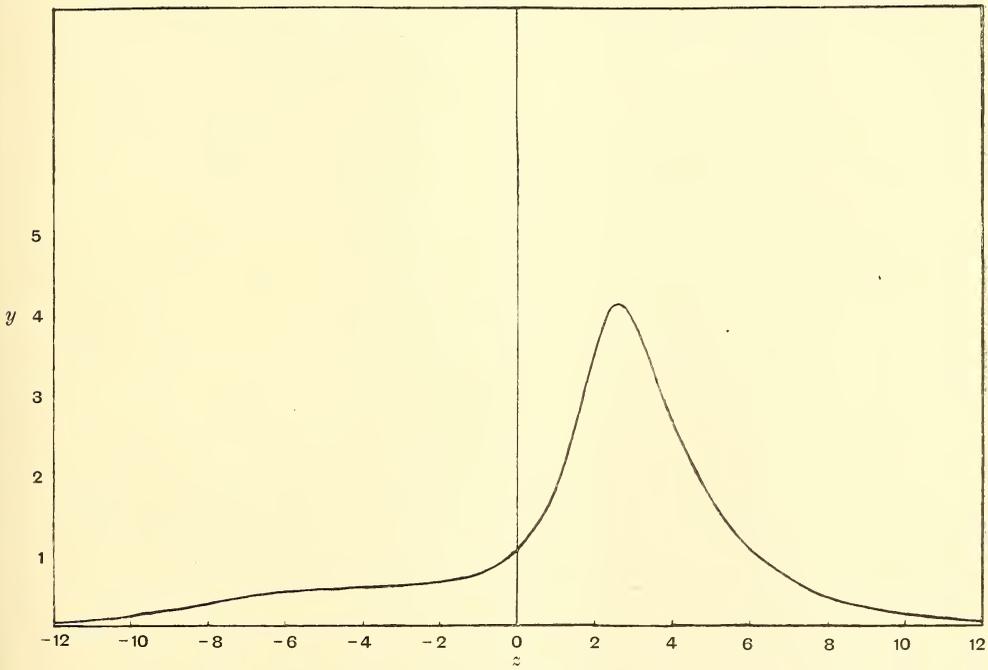


FIG. 4.— $df \propto \frac{dz}{\sqrt{e^{-z} \operatorname{sech}^2 \frac{1}{2}z + k^2 \cosh^2 \frac{1}{2}z}}$; $k = .1$;

genotypic selection, with complete dominance, counterbalanced by mutation. Dominance Ratio, '3333. This is the probable condition of natural species, including man. Note the accumulation of rare recessives.

Since the logarithm does not increase very rapidly, we may say approximately that A is proportional to $\frac{n\mu}{\sigma_a}$.

It will be seen that to maintain the same amount of variability, as in the case of equilibrium in the absence of selection (section 4), the rate of mutation must be increased by a factor of the order $\sigma_a \sqrt{n}$. Even in the low estimate we have made of the intensity of selection on the majority of factors, this quantity will usually be considerable. The existence of even the slightest selection is in large populations of more influence in keeping variability in check than random survival.

A further effect of selection is to remove preferentially those factors for which α is high, and to leave a predominating number in which α is low. In any factor α may be low for one of two reasons: (1) the effect of the factor on development may be very slight, or (2) the factor may effect changes of little adaptive importance. It is therefore to be expected that the large and easily recognised factors in natural organisms will be of little adaptive importance, and that the factors affecting important adaptations will be individually of very slight effect. We should thus expect that variation in organs of adaptive importance should be due to numerous factors, which individually are difficult to detect.

Owing to this preferential removal of important factors the above solution only truly represents an equilibrium of the variability of the species under absolutely uniform conditions of selection when the new mutations which arise have the same frequency distribution of relative importance as those removed by selection. It must be remembered, however, that the change of variability even by selection is a very slow process, and that gradual changes in the physical and biological environment of a species will alter the values of α for each factor, so tending to neutralise the tendency of selection to lower the value of σ_a . Nevertheless, α will be on the whole numerically smaller for factors in the current stock than it is for fresh mutations.

7. UNIFORM GENOTYPIC SELECTION.

If the heterozygote is selected to the same extent as the dominant, or $b = a$, it is easy to see by writing down the first generation, that a genetic ratio $p : q$, becomes in one generation by selection $\frac{p}{q} \cdot \frac{a}{ap + cq}$; or, writing $1 + \beta$ for $\frac{a}{c}$,

$$\frac{p}{q} \frac{1 + \beta}{1 + p\beta};$$

or, when β is small,

$$\frac{p}{q}(1 + q\beta).$$

Such selection is therefore equivalent to a genetic selection

$$a = q\beta.$$

Now

$$\frac{d\theta}{dT} = a \sqrt{pq} = \beta q \sqrt{pq},$$

and for the variance caused by selection, instead of $pq\sigma_a^2$, as in Section 6, we now write $pq^3\sigma_\beta^2$: we have then for the total variance produced in one generation in the value of θ ,

$$\begin{aligned} & \frac{1}{2n} + \frac{1}{16} \sin^2 \theta (1 + \cos \theta)^2 \sigma_\beta^2 \\ &= \frac{1}{2n} + \sin^2 \frac{1}{2} \theta \cos^6 \frac{1}{2} \theta \cdot \sigma_\beta^2, \end{aligned}$$

and the equilibrium distribution will be

$$y \propto \frac{1}{\sqrt{\sin^2 \frac{1}{2} \theta \cos^6 \frac{1}{2} \theta + \frac{1}{2n\sigma_\beta^2}}}.$$

It is important to notice that this distribution, unlike those hitherto considered, is unsymmetrical, factors of which the dominant phase is in excess are in the majority. This has an important influence on the value of the dominance ratio.

If $2n\sigma_\beta^2$ is large, we can write with sufficient accuracy *

$$y = \frac{A}{1.4022(2n\sigma_\beta^2)^{\frac{1}{3}} + \frac{2}{3} \log(8n\sigma_\beta^2) - \frac{2}{3}} \cdot \frac{1}{\sqrt{\sin^2 \frac{1}{2} \theta \cos^6 \frac{1}{2} \theta + \frac{1}{2n\sigma_\beta^2}}}.$$

The terminal ordinate therefore varies nearly as $(2n\sigma_\beta^2)^{\frac{1}{3}}$, and for large populations in equilibrium, μ varies as $n^{-\frac{1}{3}}$ and as $\sigma_\beta^{\frac{1}{3}}$.

Genotypic selection resembles genetic selection in diminishing the amount of variability which a given frequency of mutation can maintain, or *per contra*, increasing the amount of mutation needed to maintain a given amount of variability; it differs, however, in being comparatively inactive in respect of factors in which the dominant allelomorph is in excess, and consequently in allowing a far greater number of factors to exist in this region (see fig. 4).

* I am indebted to Mr E. Gallop, Gonville and Caius College, Cambridge, for the value of the definite integral. Mr Gallop has shown that the three terms given are the heads of three series in descending powers of $n\sigma_\beta^2$, in which the integral may be expanded.

Now when dominance is complete, the dominance ratio from a group of factors having the same ratio $\frac{p}{q}$ is

$$\frac{1}{1 + 2\frac{q}{p}},$$

for in the notation of our previous paper

$$\delta^2 = 4p^2q^2a^2,$$

and

$$a^2 = 4p^2q^2a^2\left(1 + 2\frac{q}{p}\right),$$

where a is half the difference between the two homozygous forms (3, p. 404).

The dominance ratio is therefore raised by an excess of factors in which the dominant gene is the more numerous, such as occurs under genotypic selection.

8. THE DOMINANCE RATIO.

The distribution found for the ratio $\frac{p}{q}$ or for the value of θ , which indicates the same quantity, in sections 3 to 7, enable us to calculate the value attained by the dominance ratio under each of the suppositions there considered.

1. In the Hagedoorn condition, where the variance is steadily decaying by random survival, in the absence of mutations or selection,

$$df = \frac{1}{2}A \sin \theta d\theta,$$

writing $\phi = \frac{1}{2}\theta$, then $p = \sin^2 \phi$, $q = \cos^2 \phi$,

whence

$$\epsilon^2 = S(\delta^2) = 8A\bar{a}^2 \int_0^{\frac{1}{2}\pi} \sin^5 \phi \cos^5 \phi d\phi,$$

$$\sigma^2 = S(a^2) = 8A\bar{a}^2 \int_0^{\frac{1}{2}\pi} (\sin^5 \phi \cos^5 \phi + 2 \sin^3 \phi \cos^7 \phi) d\phi,$$

and

$$\frac{\epsilon^2}{\sigma^2} = \frac{1}{1 + 2 \cdot \frac{3}{2}} = \cdot 2500.$$

2. When in the absence of selection, sufficient mutations take place to counteract the effect of random survival

$$df = \frac{2A}{\pi} d\phi,$$

and we have to consider the ratio of the integrals

$$\int_0^{\frac{1}{2}\pi} \sin^4 \phi \cos^4 \phi d\phi, \quad \int_0^{\frac{1}{2}\pi} \sin^2 \phi \cos^6 \phi d\phi,$$

which are in the ratio 3 : 5.

The dominance ratio is therefore

$$\frac{3}{3 + 2(5)} = .2308;$$

the greater variation in the ratio $\frac{p}{q}$ showing itself in a lower dominance ratio.

3. In the third symmetrical case, when genetic selection is at work, the variation of $\frac{p}{q}$ is even greater (fig. 3); since both δ^2 and a^2 contain the factor p^2q^2 , the factors in which p or q is very small, make no appreciable contribution to these quantities, consequently we only consider the central portion of the distribution, where

$$df \propto \frac{d\phi}{\sin \phi \cos \phi},$$

the intensity of selection appearing only as a constant factor, and therefore influencing the range of variation of the species, but not its dominance ratio. Here we have the integrals

$$\int_0^{\frac{1}{2}\pi} \sin^3 \phi \cos^3 \phi d\phi \quad \text{and} \quad \int_0^{\frac{1}{2}\pi} \sin \phi \cos^5 \phi d\phi,$$

leading to a dominance ratio

$$\frac{1}{1 + 4} = .2000.$$

4. In the case of genotypic selection, which case most nearly reproduces natural conditions, the distribution in the centre of the range is

$$df \propto \frac{d\phi}{\sin \phi \cos^3 \phi},$$

consequently the two integrals with which we are concerned

$$\int_0^{\frac{1}{2}\pi} \sin^3 \phi \cos \phi d\phi, \quad \int_0^{\frac{1}{2}\pi} \sin \phi \cos^3 \phi d\phi$$

are now equal, and the dominance ratio is raised to $\frac{1}{3}$.

In considering the interpretation of the dominance ratio, in our previous inquiry, we found that for symmetrical distribution the value $\frac{1}{3}$ occurred as a limiting value when the standard deviation of z ($= \log \frac{p}{q}$) was made zero. Since the dominance ratio calculated from observed human correlations averaged .32, with a standard error about .03, we were led to consider that either the allelomorphs concerned occurred usually in nearly equal numbers, a supposition for which we saw no

rational explanation, or that the value of the dominance ratio had been raised by the prevalence of epistacy (non-linear interaction of factors), a suggestion for which no direct evidence could be adduced.

In the light of the above discussion in which we have deduced the distribution of allelomorphic ratios from the conditions of equilibrium with selective influences, from which condition it is probable that natural species do not widely depart, we find that the value $\frac{1}{3}$ for the dominance ratio is produced by the asymmetry of the distribution, and in such a manner as to be independent of the activity of the selective agencies, provided that this exceeds a certain very low level. When differential survival to the extent of only about 1 per cent. in a generation affects the different Mendelian factors, in a population of only a million, and far more for more powerful selection, or a larger population, the dominance ratio will be very close to its characteristic value of $\frac{1}{3}$.

The importance of the fact that this ratio is independent of the intensity of selection, lies not only in the fact that the intensity of selection is usually incapable of numerical estimation, but in the fact that factors having effects of different magnitudes on the soma, which are therefore exposed to selection of varying intensity, and contribute very different quota to the variance, are all affected in the same manner; those factors which by their insignificance might be exposed to selective influences which are not large compared to the effects of random survival will be precisely those which have little weight in computing the dominance ratio.

9. ASSORTATIVE MATING.

With assortative mating it has been shown (3, p. 414) that the deviations from the mean of the three phases of any factor have, owing to association with similar factors, mean genotypic values given by the formula

$$I = i + \frac{A}{1-A} \cdot \frac{iP - kR}{p},$$

$$J = j - \frac{A}{1-A} \cdot \frac{p-q}{2pq}(iP - kR),$$

$$K = k - \frac{A}{1-A} \cdot \frac{iP - kR}{q},$$

when i, j, k are the deviations in the absence of association, A measures the degree of association produced by assortative mating: p, q are the gene frequencies, and P, R the corresponding phase frequencies for the homozygous phases.

Writing $j = i$ to represent complete dominance, and $P = p^2$, $R = q^2$, since

$$(p^2 + 2pq)i + q^2k = 0,$$

$$\frac{i}{q^2} = -\frac{k}{p(p+2q)} = \frac{i-k}{1} = \frac{p^2i - q^2k}{2pq^2};$$

and since $i - k = 2a$, we have

$$I = i + \frac{A}{1-A} \cdot 4aq^2,$$

$$J = i - \frac{A}{1-A} \cdot 2aq(p-q),$$

$$K = k - \frac{A}{1-A} \cdot 4apq;$$

or

$$I - J = 2a \cdot \frac{A}{1-A} \cdot q,$$

$$J - K = 2a \left(1 + \frac{A}{1-A} q \right).$$

If now the survival factors of the three phases are a , b , c , the effect of one generation's selection is given by

$$\frac{p_1}{q_1} = \frac{p_0}{q_0} \frac{ap + bq}{bp + cq} = \frac{p_0}{q_0} (1 + p\overline{a-b} + q\overline{b-c}),$$

since a , b , and c are near to 1;

hence

$$a = p(a-b) + q(b-c).$$

Now as $I - J$, $J - K$, the mean differences in any trait due to a single factor, are small compared with the whole variation within the population, we must take $a - b$, $b - c$ proportional to $I - J$ and $J - K$. In other words,

$$a - b = (I - J)\gamma,$$

$$b - c = (J - K)\gamma,$$

where γ measures the intensity of selection per unit change in the trait.

Hence

$$a = \gamma(p\overline{I - J} + q\overline{J - K})$$

$$= \gamma \cdot \frac{2a}{1-A} \cdot q.$$

The general case of uniform genotypic selection when the mean values of the phases are modified by homogamy, therefore, reduces to the case already considered in which homogamy is absent. The total effect of homogamy is to increase the effect of selection by the factor $\frac{1}{1-A}$. The distribution of frequency ratios is unaltered, for although by introducing a difference between I and J the selective effect is made more intense when

p is large, which would tend to make the distribution more symmetrical, this effect is exactly balanced by the increased effect of selection when p is small. The dominance ratio is therefore unaltered by the direct effect of assortative mating.

SUMMARY.

The frequency ratio of the allelomorphs of a Mendelian factor is only stable if selection favours the heterozygote: such factors, though occurring rarely, will accumulate in the stock, while those of opposite tendency will be eliminated.

The survival of a mutant gene although established in a mature and potent individual is to a very large extent a matter of chance; only when a large number of individuals have become affected does selection, dependent on its contribution to the fitness of the organism, become of importance. This is so even for dominant mutants; for recessive mutants selection remains very small so long as the mutant form is an inconsiderable fraction of the interbreeding group.

The distribution of the frequency ratio for different factors may be calculated from the condition that this distribution is stable, as is that of velocities in the Theory of Gases: in the absence of selection the distribution of $\log \frac{p}{q}$ is given in fig. 1. Fig. 2 represents the case of steady decay in variance by the action of random survival (the Hagedoorn effect).

Fig. 3 shows the distribution in the somewhat artificial case of uniform genetic selection: this would be the distribution to be expected in the absence of dominance. Fig. 4 shows the asymmetrical distribution due to uniform genotypic selection with or without homogamy.

Under genotypic selection the dominance ratio for complete dominance comes to be exactly $\frac{1}{3}$, in close agreement with the value obtained from human measurements.

The rate of mutation necessary to maintain the variance of the species may be calculated from these distributions. Very infrequent mutation will serve to counterbalance the effect of random survival; for equilibrium with selective action a much higher level is needed, though still mutation may be individually rare, especially in large populations.

It would seem that the supposition of genotypic selection balanced by occasional mutations fitted the facts deduced from the correlations of relatives in mankind.

REFERENCES.

- (1) DARWIN, CHARLES, *The Origin of Species*.
- (2) EAST, E. A., and JONES, D. F., *Inbreeding and Outbreeding* (1920).
- (3) FISHER, R. A., "The Correlation between Relatives on the Supposition of Mendelian Inheritance," *Trans. Royal Soc. Edin.*, 1918, lii, pp. 399-433.
- (4) HAGEDOORN, A. L. and A. C., *The Relative Value of the Processes causing Evolution* (The Hague : Martinus Nijhoff), p. 294 (1921).
- (5) JENNINGS, H. S., "Numerical Results of Diverse Systems of Breeding," *Genetics*, i, 53-89 (1916).
- JENNINGS, H. S., "The Numerical Results of Diverse Systems of Breeding with respect to two pairs of characters, linked or independent ; with special relation to the effects of linkage," *Genetics*, ii, 97-154 (1917).
- (6) PEARSON, K., "On a Generalised Theory of Alternative Inheritance, with special reference to Mendel's Laws," *Phil. Trans.*, cciii, A, pp. 53-87 (1903).
- (7) MULLER, H. J., "Genetic Variability : twin hybrids and constant hybrids in a case of balanced lethal factors," *Genetics*, iii, 422-499 (1918).
- (8) ROBBINS, R. B., "Some Applications of Mathematics to Breeding Problems," *Genetics*, ii, 489-504 (1917).
- ROBBINS, R. B., "Applications of Mathematics to Breeding Problems, II," *Genetics*, iii, 73-92 (1918).
- ROBBINS, R. B., "Some Applications of Mathematics to Breeding Problems, III," *Genetics*, iii, 375-306 (1918).
- (9) WENTWORTH, E. N., and REMICK, B. L., "Some Breeding Properties of the Generalised Mendelian Population," *Genetics*, i, 608-616 (1916).
- (10) WRIGHT, S., "Systems of Mating," *Genetics*, vi, 111-178 (1921).

(Issued separately October 16, 1922.)

XXII.—Note on a Theorem of Frobenius' connected with Invariant-Factors. By Sir Thomas Muir, F.R.S.

(MS. received June 13, 1922. Read June 19, 1922.)

(1) THE theorem in question appears in his well-known paper of the year 1894, "Ueber die Elementartheiler der Determinanten."* Save that English takes the place of German, the enunciation of it stands exactly as follows:—

If $\xi \leq r \leq s < n$, and D_ξ be any determinant of the n^{th} order in the array

$$a_{\kappa\lambda} \quad (\kappa = 1, \dots, r; \lambda = 1, \dots, s)$$

D'_ξ being its complementary minor of the $(n - \xi)^{\text{th}}$ order in the determinant of the n^{th} order

$$\begin{vmatrix} a_{\kappa\lambda} & a_{\kappa\nu} \\ a_{\mu\lambda} & a_{\mu\nu} \end{vmatrix} \quad \begin{matrix} \kappa = 1, \dots, r; \mu = r + 1, \dots, n \\ \lambda = 1, \dots, s; \nu = s + 1, \dots, n \end{matrix}$$

then

$$\sum_{\xi=0}^r \sum_{D_\xi} (-1)^\xi D_\xi D'_\xi = \begin{vmatrix} 0 & a_{\kappa\nu} \\ a_{\mu\lambda} & a_{\mu\nu} \end{vmatrix}.$$

In regard to it two preliminary remarks are necessary: (1) that in the original there is a very upsetting misprint of r for s in the third line; (2) that the theorem is purely determinantal, any connection with the theory of invariant-factors that may be given to it being solely for the benefit of the latter.

(2) Manifestly the equality predicated in the theorem may be viewed either as a summation of an aggregate of products of pairs of determinants, or as an expansion of a special n -line determinant in terms of products of complementary minors of a more general determinant. In the case of either view it is essential to take pointed note of the fact that the left-hand member is a grossly inflated representative of the member on the right,—a fact readily seen to be inevitable if it is observed that every first factor on the left with one exception is a function of elements not one of which is to be found on the right. This exception, too, is brought in as it were by a side-wind, being the case of D_ξ where ξ is 0 and where D_0 is taken to be 1 and its cofactor D'_0 to be $|a_{1n}|$, thus giving us our only assurance that the right-hand member is represented on the left at all. Indeed, the exact state of matters is that the right-hand member forms a part of the first

* *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), 1894, pp. 31–44.

term on the left, and that the purpose served, as it were, by all the other terms on the left is to nullify the undesired remaining part.

(3) In necessary illustration of the theorem and of what has just been said, let us consider the case where $n=8, r=3, s=4$, the determinant from whose minors all the second factors D'_ζ are taken being

$$| a_1 b_2 c_3 d_4 e_5 f_6 g_7 h_8 |,$$

and the minor array of it that provides all the first factors D_ζ being

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix}.$$

Seeking first the right-hand member of the equality, we see it to be

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & a_5 & a_6 & a_7 & a_8 \\ \cdot & \cdot & \cdot & \cdot & b_5 & b_6 & b_7 & b_8 \\ \cdot & \cdot & \cdot & \cdot & c_5 & c_6 & c_7 & c_8 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 \\ \cdot & \cdot \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 \end{vmatrix} \text{ or } G \text{ say.}$$

Then on the left our first term of the equivalent is the product

$$1 \cdot | a_1 b_2 c_3 d_4 e_5 f_6 g_7 h_8 |,$$

and this is followed by twelve products like

$$- a_1 | b_2 c_3 d_4 e_5 f_6 g_7 h_8 |,$$

eighteen products like

$$+ | a_1 b_2 | \cdot | c_3 d_4 e_5 f_6 g_7 h_8 |,$$

and finally four products like

$$- | a_1 b_2 c_3 | \cdot | d_4 e_5 f_6 g_7 h_8 |.$$

A mere glance at these suffices to make clear that thirty-four out of the thirty-five products contribute no part of the equivalent of the gnomon-like determinant G, and that their net effect is merely to cancel that part of the first product which is not a part of G.

In passing it may be noted that had we taken n equal to 7 or 6, the number of products on the left would not have been altered, and further that in the latter of these two cases G would be 0, and the equality would revert to a very old type in the history of our subject, namely, that of a vanishing aggregate of products of pairs of determinants.

(4) With less symbolism, therefore, Frobenius' theorem may be enunciated as follows:—*If an aggregate of products be formed in which each first factor is a minor of a fixed rectangular array of $|a_{in}|$, and the corresponding second factor is the cofactor of the first in $|a_{in}|$, every minor*

of every order, even the 0th, being taken, and the sign prefixed to any product being + or - according as the first factor of the product is of even or odd order, then the sum of the whole is the determinant got from |a_m| by making all the elements of the fixed array zero.

(5) The character of the theorem being as thus insisted on, it remains to be ascertained whether the knowledge gained may not be utilised as a step towards simplification or generalisation.

Returning to the example of § 3, let us consider the twelve products in the aggregate

$$- \sum a_1 |b_2 c_3 d_4 e_5 f_6 g_7 h_8|.$$

The sum of the first four of the twelve can be expressed as a determinant, namely,

$$- \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \cdot & \cdot & \cdot & \cdot \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \\ \cdot & \cdot \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 \end{vmatrix},$$

and the sum of the second four and the sum of the third four as

$$- \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ b_1 & b_2 & b_3 & b_4 & \cdot & \cdot & \cdot & \cdot \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 \\ \cdot & \cdot \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 \\ c_1 & c_2 & c_3 & c_4 & \cdot & \cdot & \cdot & \cdot \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 \\ \cdot & \cdot \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 \end{vmatrix}$$

Similarly, the sum of the eighteen products

$$+ \sum |a_1 b_2| |c_3 d_4 e_5 f_6 g_7 h_8|$$

can be expressed as the sum of three determinants like

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \cdot & \cdot & \cdot & \cdot \\ b_1 & b_2 & b_3 & b_4 & \cdot & \cdot & \cdot & \cdot \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 \\ \cdot & \cdot \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 \end{vmatrix},$$

and the remaining four products as a single determinant of similar construction. Instead, therefore, of an aggregate of 35 items, we have an aggregate of 8, the theorem reached being:—If Δ be any n-line determinant with its first r rows conceived as separated into two arrays, namely, an r-by-s array (r ≤ s) called P and an r-by-(n-s) array called Q, and if ΣΔ₁ be the sum of all the determinants got from Δ by zero-ising

in it a single row of $Q, \Sigma \Delta_2$ the similar sum got by zero-ising in Δ two rows of Q , and so on, then

$$\Delta - \sum \Delta_1 + \sum \Delta_2 - \dots$$

is equal to the determinant got from Δ by zero-ising P. For example, taking $n=5, r=2, s=3$, the determinant as partitioned being thus

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ \hline c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix},$$

we have

$$\begin{aligned} & |a_1 b_2 c_3 d_4 e_5| \\ - \begin{vmatrix} a_1 & a_2 & a_3 & . & . \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} - \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & . & . \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 & . & . \\ b_1 & b_2 & b_3 & . & . \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} \\ = & \begin{vmatrix} . & . & . & a_4 & a_5 \\ . & . & . & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix}. \end{aligned}$$

This is readily verified by expanding its five determinants in terms of the 2-line minors of their last two columns and the complementary minors.

The number of determinants on the left of the general equality is evidently

$$1 + r + \frac{1}{2}r(r-1) + \dots$$

(6) Our next point is that the principle involved in the theorem of the preceding paragraph extends beyond the sphere of Frobenius' theorem, the places liable to be filled with zeros being not necessarily those of a rectangular array. They may be, for example, the places of the principal diagonal, for we have the theorem that—If Δ be any n -line determinant and Δ_r any determinant got from it by deleting r diagonal elements, then

$$\Delta - \sum \Delta_1 + \sum \Delta_2 - \dots$$

is equal to the diagonal term. Thus

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - \sum \begin{vmatrix} . & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \sum \begin{vmatrix} . & a_2 & a_3 \\ b_1 & . & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - \begin{vmatrix} . & a_2 & a_3 \\ b_1 & . & b_2 \\ c_1 & c_2 & . \end{vmatrix} \\ = a_1 b_2 c_3,$$

where for the purpose of verification we should of course use, not Laplace's expansion-theorem, but Cayley's.

(7) A second point in advance is that the type of theorem which we have thus far been dealing with is not confined to the field of determinants, but finds a place also among Pfaffians, where there are analogues to both the theorems of §§ 5, 6. The latter analogue of the two is remarkably close. For example, taking the Pfaffian

$$|a_1 b_2 c_3 d_4 e_5|,$$

and the places occupiable by zeros to be in the axis of symmetry—that is to say, the places (1, 5), (2, 4), (3, 3)—we have the equality

$$\begin{aligned} & \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ & b_2 & b_3 & b_4 & b_5 \\ & & c_3 & c_4 & c_5 \\ & & & d_4 & d_5 \\ & & & & e_5 \end{vmatrix} - \sum \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \cdot \\ & b_2 & b_3 & b_4 & b_5 \\ & & c_3 & c_4 & c_5 \\ & & & d_4 & d_5 \\ & & & & e_5 \end{vmatrix} \\ & + \sum \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \cdot \\ & b_2 & b_3 & \cdot & b_5 \\ & & c_3 & c_4 & c_5 \\ & & & d_4 & d_5 \\ & & & & e_5 \end{vmatrix} - \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \cdot \\ & b_2 & b_3 & \cdot & b_5 \\ & & \cdot & c_4 & c_5 \\ & & & d_4 & d_5 \\ & & & & e_5 \end{vmatrix} \\ & = a_5 b_4 c_3. \end{aligned}$$

(8) A third step which has been found possible in the direction of generalisation is still more curious. In all the preceding paragraphs zero elements have taken a somewhat conspicuous place: we have now to learn that in the string of determinants forming the left-hand member of our main theorem (§ 5) no zero elements are called for at all, that, in fact, subject to a certain condition we may substitute for the zeros in these determinants any quantities whatever. This condition simply is that the zero occupying a definite fixed place shall always have the same substitute. For example, in the case previously taken where $n=5$, $r=2$, $s=3$, and where there are three determinants not free of zero elements, the substitute for a zero in the place (1,4), if in one instance made x , must in every other instance be made x : in short, the substitution must be of the nature

$$\begin{matrix} (1,4) & (1,5) \\ (2,4) & (2,5) \end{matrix} \Big\} \equiv \begin{matrix} x & y \\ u & v \end{matrix} :$$

and our new result is that, this change having been performed on the left-hand member of the equality, the right-hand member is unaffected. In verification we have only to show that the additions thereby made to the

aggregate on the left cancel each other; and this can without much difficulty be done.

The theorem we have thus reached is of the type already touched on by Le Paige in 1880, by Deruyts in 1881-2, and by myself in 1888, the common feature of the members of the type being *the summation of an aggregate of determinants*.*

(9) When the zeros of the original aggregates are confined to the diagonals (§§ 6, 7), the substitution-theorems thence derived take a different form in the right-hand member. The result reached may be put shortly thus. *If the diagonal elements in the original determinant or Pfaffian be*

$$a_1, b_2, c_3, \dots$$

and the quantities substituted in the aggregates of §§ 6, 7 for zeros occupying the places (1, 1), (2, 2), (3, 3), . . . be

$$x, y, z, \dots$$

then the right-hand member becomes

$$(a_1 - x)(b_2 - y)(c_3 - z) \dots$$

For example,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - \sum \begin{vmatrix} X & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \sum \begin{vmatrix} X & a_2 & a_3 \\ b_1 & Y & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - \begin{vmatrix} X & a_2 & a_3 \\ b_1 & Y & b_3 \\ c_1 & c_2 & Z \end{vmatrix} \\ = (a_1 - X)(b_2 - Y)(c_3 - Z);$$

and

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ & b_2 & b_3 & b_4 & b_5 \\ & & c_3 & c_4 & c_5 \\ & & & d_4 & d_5 \\ & & & & e_5 \end{vmatrix} - \sum \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & X \\ & b_2 & b_3 & b_4 & b_5 \\ & & c_3 & c_4 & c_5 \\ & & & d_4 & d_5 \\ & & & & e_5 \end{vmatrix} \\ + \sum \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & X \\ & b_2 & b_3 & Y & b_5 \\ & & c_3 & c_4 & c_5 \\ & & & d_4 & d_5 \\ & & & & e_5 \end{vmatrix} - \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & X \\ & b_2 & b_3 & Y & b_5 \\ & & Z & c_4 & c_5 \\ & & & d_4 & d_5 \\ & & & & e_5 \end{vmatrix} \\ = (a_5 - X)(b_4 - Y)(c_3 - Z).$$

RONDEBOSCH, S.A.,
24th May 1922.

* *Mém. . . . Soc. R. des Sci.* (Liège), t. ix, x; *Proc. Roy. Soc. Edin.*, xv, pp. 96-105.

XXIII.—Notes on a Correspondence between the French Academy of Sciences and the Royal Society of Edinburgh regarding the Invention of the Pilot Cable (Câble Guide). By The General Secretary.

(Read June 19, 1922.)

DURING 1921 the Academy of Sciences of France awarded a medal and prize to M. W. A. Loth for various important devices having special application to navigation. Among these was mentioned the system of the Câble Guide, which is essentially the system of Pilot Cable invented by Mr C. A. Stevenson, C.E., and described by him in 1893 in the *Proceedings* of the Royal Society of Edinburgh. In a communication to the *Académie des Sciences* the Council of the Royal Society of Edinburgh called attention to this fact and to the further fact that apparently no recognition had been given of the pioneer work of Mr Stevenson.

In a letter of date January 31, 1922, M. Émile Picard, Secrétaire Perpetuel de l'Académie des Sciences, explained that in awarding the prize to M. Loth the Academy of Sciences were guided entirely by the efficacy of M. Loth's ingenious devices, that the report submitted to the Academy by Vice-Admiral Fournier (see *Comptes Rendus*, December 12, 1921, Tome 173, p. 1230) did not touch upon the question of priority of invention, and that the question of principle did not arise.

The Council of the Royal Society of Edinburgh after a careful examination of Vice-Admiral Fournier's report considered that this report did implicitly credit M. Loth with the invention of the Pilot Cable: moreover, in popular accounts of his methods this was explicitly stated.

The Council, feeling it to be a matter of simple justice to emphasise the claim made for Mr C. A. Stevenson as the original inventor of the system which had been so effectively developed by M. Loth in recent years, authorised the General Secretary to send the following statement to the Academy of Sciences, in which the phraseology of the Stevenson Patent of 1893 is compared with one of the Sections in Vice-Admiral Fournier's Report:—

“ Priority Note on the Pilot Cable (Câble Guide).

“ The Council of the Royal Society of Edinburgh desire respectfully to draw attention to a point which has arisen in connection with a Prize recently awarded by the Academy of Sciences.

“In 1921 this Prize (of 6000 francs) was awarded to M. Loth for various important devices having special applications in Navigation. In the ‘Rapport de M. le Vice-amiral Fournier sur les créations scientifiques de M. W. A. Loth et leurs importantes et fécondes applications à la Marine’ (*Comptes Rendus*, t. 173, p. 1230) we read under the fifth section, on p. 1231, these words:—

“ ‘5° Solution du problème de la conduite des navires à leur entrée dans un port et, à leur sortie, en temps de brume et, en temps de guerre, pendant la nuit, les phares éteints, au moyen d’un câble guide.

“ ‘Ce câble élongé sur le fond de la mer est parcouru par un courant alternatif, de fréquence musicale (600 à 1000 périodes par seconde). Ce courant crée dans l’espace environnant un champ magnétique variable, spécial. La forme de ce champ varie avec la fréquence. Ses lignes de force traversent les surfaces de cadres fixes placés à bord et au dessus des navires et disposés, d’après la forme spéciale du champ, de telle façon que tout bâtiment, sans autre moyen de se diriger, puisse connaître à tout instant :

“ ‘1° La direction du câble’;

“ ‘2° Le nombre de degrés d’inclinaison de sa route sur cette direction ;

“ ‘3° Sa distance à ce guide et le côté où il suit.’ ”

In these sentences M. Loth is tacitly credited with the solution of a problem which was solved in essentially the same way by Mr C. A. Stevenson, F.R.S.E., twenty-nine years ago. The use of the pilot cable will be found described in a paper read by him before the Royal Society of Edinburgh on January 30, 1893, and published in that Society’s *Proceedings*, vol. xx, p. 25. In 1891 Mr Stevenson had already exhibited his apparatus at work to the Commissioners of the Northern Lighthouses ; in 1892 he took out a patent for the pilot cable ; and a few years later constructed a working model representing the French coast in the neighbourhood of Ushant, and showing how such a protecting cable could be used to warn ships off that dangerous shore. This model is still exhibited in the Royal Scottish Museum, Edinburgh. The following extract from the complete specification of Mr Stevenson’s patent of 1892 may be compared with the statement given above in the “Rapport” by Vice-Admiral Fournier.

“ My invention consists of an entirely new method of warning vessels electrically of their position or of approach to a coast, shoal, mines, or any other danger. The apparatus consists of a submarine wire or wires laid down in the bed of the sea, river, or estuary, or out to any danger or anchorage or safe channel, through which intermittent currents are made to pass or the electrical state of which is made to alter by means of some form of electrical machine, generator, or battery. These currents or changes of electrical state are detected by a detector of such currents or of the change of magnetic influence or whatever it may be called, which detector may be on the ship or let down by rope or cable, or coiled round the hull of the vessel. Whenever

therefore a vessel comes into the neighbourhood or over the top of the wire, those on board can detect its presence, and in consequence can locate their position. The currents have, if desired, different characteristics just as light-house apparatus has."

In virtue of what has now been stated the Council of the Royal Society of Edinburgh claim that the method of the pilot cable, tacitly credited to M. Loth, is essentially the same as that first described by Mr C. A. Stevenson and published in their *Proceedings* in 1893. It seems to be a simple act of justice that these historical facts should be clearly recognised, and that due credit should be given to Mr Stevenson for his valuable pioneer work in the interests of humanity. The system of the pilot cable is, in fact, Mr Stevenson's system. Compared with M. Loth's beautiful devices made possible in these days by the remarkable developments in methods for detecting electric and magnetic changes, Mr Stevenson's early methods may appear crude: but that does not invalidate his claim as the originator and the first experimenter along these lines. Not only did he invent the pilot cable, but he was the first to demonstrate practically how it could be used in guiding vessels up estuaries and into harbours by means of electric signals from a sunk cable.

C. G. KNOTT,
General Secretary, R.S.E.

A French translation of this statement was sent in the hope that it might find a place in the *Comptes Rendus*. The following reply was received from M. Picard.

"Institut de France,
"Académie des Sciences, Paris,
"Juin 26, 1922.

"M. Emile Picard, Secrétaire Perpetuel de l'Académie des Sciences, à M. C. G. Knott, Secrétaire général de la Société Royale d'Edimbourg.

"Monsieur le Secrétaire général,

"J'ai l'honneur de vous transmettre la lettre que m'a écrite M. l'Amiral Fournier à la suite de votre communication relative au prix attribué à M. Loth par l'Académie des Sciences de Paris. J'y joins deux numéros de la *Revue générale d'Electricité et du Génie civil*, contenant des articles se rapportant à la question du guidage des navires. Il résulte de ces divers documents que le problème pratique de guider un navire restait entier après les intéressantes communications de M. Stevenson donnant seulement un principe général qui paraît d'ailleurs avoir été antérieurement indiqué. On doit reconnaître, avec M. l'Amiral Fournier, que le procédé d'émission et l'appareillage de détection décrit par M. Stevenson dans son brevet et dans la note de *Proceedings* ne permettaient pas de guider avec sécurité les navires entrant dans un port ou en sortant. Or cela seul est intéressant pour la

navigation. Il nous parait donc inutile d'ouvrir dans les *Comptes Rendus* de l'Académie des Sciences une polémique dont la place n'est pas dans cette publication scientifique.

“Veuillez agréer, Monsieur le Secrétaire général, l'assurance de mes sentiments de haute considération,
Sgd. Em. Picard.”

The Council of the Royal Society of Edinburgh note with satisfaction that in the two journals supplied by M. Picard accounts are given of the pioneer work of Mr C. A. Stevenson in this connection, thereby substantiating their claim that the system of the Pilot Cable or Câble Guide is essentially Mr Stevenson's.

(Issued separately October 16, 1922.)

XXIV.—Chemical Combination and Sir Alfred Ewing's Magnetic Atom. By Principal A. P. Laurie, M.A., D.Sc.

(MS. received July 14, 1922. Read June 19, 1922.)

THE outcome of recent investigations into the properties of matter has been to show that the atom, in all probability, consists of a positive nucleus surrounded by electrons. The distribution of the electrons round the positive nucleus is controlled by the attraction of the positive nucleus on the electrons and by the repulsion of the electrons for each other. Sir J. J. Thomson and others have investigated mathematically and experimentally the laws governing such a distribution, on the assumption that the electrons are stationary in relation to each other and to the positive nucleus.

It is also probable from the experimental evidence that atoms of elements containing a considerable number of electrons consist of an inner stable group of electrons round the positive nucleus with an outer shell containing the electrons which take part in chemical change. We also have evidence that when elements combine to form salts which are capable of ionisation one or more electrons pass from one atom to the other. It is evident that the simplest possible conception of the atom is that of a positive element surrounded by stationary electrons, and many chemical phenomena have been explained in terms of this simple atom by Sir J. J. Thomson and others.

It is, however, by no means certain that the electrons are stationary. They are quite possibly moving in orbits with the nucleus as a centre, or they may be moving in tiny orbits of their own, each orbit occupying a fixed position with reference to the other orbits, and the positive nucleus thus endowing the electron with magnetic as well as electrostatic properties.

We may also, instead of assuming the electron to be moving in an orbit small compared with the size of the atom, adopt the assumption made by A. L. Parson that the electron is a continuous negative charge disposed round a ring, and that this ring is rotating with a velocity approaching the velocity of light and so producing a magnetic shell. To this conception of the electron Parson has given the name of the magneton. We shall assume in the subsequent discussion that the atom consists of a positive nucleus surrounded by electrons holding fixed positions in space in relation to the surrounding electron and to the nucleus,

and that in the atoms containing many electrons we have an inner stable group with a few electrons on the outer shell which are active in producing chemical change; and we shall, further, assume that each electron has also magnetic properties in addition to electrostatic properties, either because it is moving in an orbit which is small as compared to the atom, or is itself a rotating ring as supposed by Parson; and we shall also assume that it is able to turn on an imaginary axis under the influence of its magnetic polarity. If we make these assumptions with regard to the structure and properties of the atom, the researches of Sir Alfred Ewing into the hysteresis of iron become of profound significance to the chemist. Sir Alfred Ewing's first researches into the hysteresis of iron were published in 1890, and he has recently further developed his theory of the structure of the iron atom in a paper read to the Royal Society of Edinburgh.* He has demonstrated that if we take a group of tiny magnets, representing Weber's magnetic particles, each able to rotate on a centre, they will arrange themselves into a stable position; and if we then bring them under the influence of an increasing magnetic force, they will at first oscillate about their position of stability, but when the magnetic force becomes sufficiently strong they will break away and will then fall into a new stable position. When the magnetic force is merely sufficient to set up oscillations through angles less than the angle of rupture, the whole system behaves like an elastic body and no work is done on the system; but if the units are rotated until the angle of rupture is reached, they then swing irreversibly into a new stable position and a definite amount of work has been done on the system to induce the break away, and this work is entirely converted into radiation, vibration, or heat when the magnets swing into the new stable position. Moreover, if a still stronger magnetic force is applied to the system of magnetic units than is necessary to produce the break away, the work done and the final dissipation of energy remains the same. We have here, then, a system which when acted upon by an external magnetic force either absorbs no energy at all, or absorbs a definite quantity of energy which is then dissipated. The significance of these results in helping to explain the phenomena associated with chemical combination, and enabling us to get a dynamic statement of what happens when atoms unite, is obvious.

In his recent researches Sir Alfred Ewing has found that the quantitative data of the hysteresis of iron are best explained by assuming a central magnetic unit able to rotate, controlled by fixed magnetic poles with like

* *Proc. Roy. Soc.*, vol. xlviii, p. 342; *Phil. Mag.*, Sept. 1890; *Phil. Mag.*, Sept. 1890; *Proc. Roy. Soc.*, Feb. 1922; *Phil. Mag.*, March 1922; *Proc. Roy. Soc. Edin.*, vol. xlii, p. 97.

poles pointing inwards; and he has pointed out that if, in addition, the rotating unit has like poles pointing outwards—if, for instance, we suppose that the rotating unit has two external south poles, while the fixed magnets have their north poles pointing inwards,—such a system would have no magnetic polarity, and would be unaffected by an external magnetic field, approaching closely to the properties of the atoms of most of the elements which are only very slightly para- or diamagnetic. But if this system is approached by a small magnetic unit, the inner rotating unit might be swung from one position of stability to another with the absorption and dissipation of a definite amount of energy.

Now it is evident, to return to the former discussion of the nature of the atom, that if we assume the atom to consist of electrons with magnetic properties and capable of rotation round a centre, such an atom will have properties corresponding to the atom postulated by Sir Alfred Ewing; and that, while it need not have any magnetic polarity or be affected by a magnetic field, will, if approached sufficiently near by another atom to bring individual magnetic units into play, reproduce the phenomena which he has investigated.

Sir Alfred Ewing in his investigations has assumed the rotating unit to be in the centre and the fixed magnets outside, but in order to bring his conception of the atom into line with the conception of the chemical atom, we shall suppose, in the first instance, the fixed magnets to radiate from the centre and the rotating units to be in the outer ring, and in order that the atom shall have no magnetic polarity, we shall suppose that we have like poles pointing outwards. Let us postulate, to avoid confusion, that the north poles are pointing inwards to the nucleus and the south poles pointing outwards. It is evident that if we assume the atoms of a given element to have the south poles of the fixed magnets pointing outwards, we must make the same assumption for the atoms of all the elements, otherwise we should have those with north poles and those with south poles respectively rushing together.

We shall then picture the central stable group of electrons as consisting of magnets radiating from the centre, which can be diagrammatically illustrated in one plane, as shown in fig. 1. A monovalent element like potassium could then be represented diagrammatically with one magnetic electron in the outer shell, which we shall, for our present purpose, represent as a little magnet able to rotate on a central pivot.

Let us now suppose a second atom to approach the first atom, and let us suppose, for simplicity, that the second atom consists of fixed magnets alone (fig. 2). As it approached it would be repelled by the south

poles of the first atom, and also, as the two atoms came very near, by the electrostatic repulsion of the electrons. It would cause the outer magnetic electron to oscillate about its centre, but if the repulsive forces were sufficiently strong it would not approach near enough to cause the little magnet to break away and swing into a new position of stability. If, however, it was moving with sufficient momentum to approach sufficiently near to the little magnet, it would cause it to break away and then swing round into a new position, with the north pole to the south pole of the approaching atom. This operation would involve the doing of a definite amount of work to swing the little magnet into an unstable position, and the conversion of that amount of work into heat, when it falls into its new position.

If we suppose the chemical reaction represented above to be the combination of two elements like potassium and chlorine to form a salt which

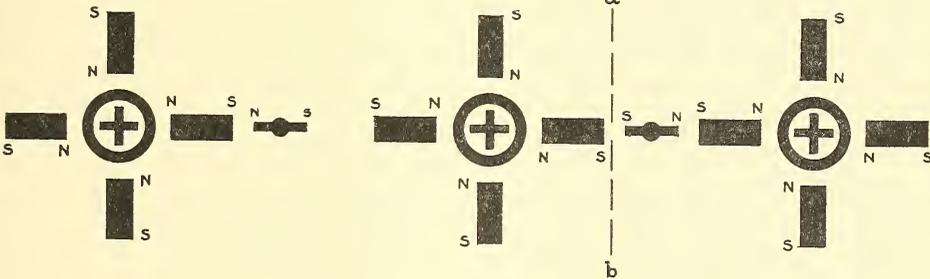


FIG. 1.

FIG. 2.

ionises, then if the two atoms separate again we can suppose the second atom as carrying off the little magnet attached to it, thus depriving the first atom of an electron which will now be electropositive, while the second atom departing with an extra electron will be electronegative.

This view of the nature of chemical combination also suggests a possible explanation of catalytic action; for suppose, as before, one atom approaches another, and that while it can approach near enough to set one of the magnetic units swinging, it cannot approach near enough to swing it to the angle of instability, and while this is going on another atom approaches at some angle to the first two atoms, it is evident that its approach may give the necessary additional magnetic force to swing the little magnet to the angle of instability, when it will then fall into a new position; and if, while in the new position, it is drawn nearer to the second atom under the magnetic attraction, it will remain permanently in the new position, while the catalysing atom will move away unaltered.

Up to the present we have considered only one possible arrangement of

the magnetic units within the atom, viz. fixed radiating magnets with rotating units at the external poles. There is an objection to this arrangement, which is evident when we examine critically what is supposed to happen during chemical combination. On the near enough approach of the second atom the energy which is transformed during combination is entirely due to the swinging of the little magnet in the first atom, which is controlled by the magnetic force of the fixed magnets of the first atom alone, and therefore, in every case, no matter what the nature of the second atom was, or the momentum with which it approaches, or the strength of its magnetic field, the amount of energy transformed during the combination would be the same. Even if we supposed that the action is between two little magnets on the outer shell of the two atoms, the same is true, for on approaching each other both magnetic units will swing outwards, but only one, the one most weakly controlled, will rotate through 180° , while the other will swing back to its old position. This would require us to suppose, to take an example, that if in the formation of potassium chloride and potassium bromide we imagine in each case that it is the little magnet on the potassium atom which swings through 180° , the heat of formation of the two salts would be the same. It is evident then that we must suppose some arrangement of our magnetic units which requires a magnetic unit in each atom to be swung permanently into a new position. If we imagine small magnets to be arranged in concentric circles round the nucleus, as shown in fig. 3, we would have a very stable arrangement which also would have no external magnetic polarity. If we now imagine two such atoms approaching, at that portion of the circles of each where they approach most nearly to each other two magnetic units could swing outwards, as shown in fig. 4. It is a matter of indifference whether they arrange themselves south to north or north to south, but in both atoms the magnetic units will swing through about 90° , and in each atom work would be done and energy dissipated as heat. It is difficult, however, on this assumption to explain valency. There seems no reason why atoms should not attach themselves all round the first atom until the outer shell of magnetic units is completely broken up. If, however, we adopt the view that comparatively few magnetic units are in the outer shell, and those are the ones that take part in chemical change, then only the magnetic units in the outer shell could approach near enough to each other to swing into a new position of stability. We can suppose that a magnetic unit in the outer shell, attracted radially by the nucleus, and electrostatically repelled by two neighbouring magnetic electrons in the inner shell, and magnetically attracted by their north and south poles

respectively, will place itself as shown in fig. 3. On the approach of another atom on which a similarly placed magnetic electron existed in the outer shell, they would both swing through 90° to take up a new stable position, thus requiring work to be done and energy dissipated on both atoms. Both these electrons are already held each to their own atom by the attraction of the positive nucleus, and, in addition, they are now attracted to each other magnetically, so that we seem to have here all the conditions required for the chemical combination of two elements. In the first place, a definite amount of work is done by each atom on their close approach to each other in swinging the two magnetic electrons to the

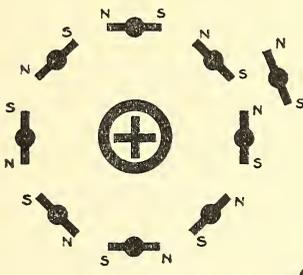


FIG. 3.

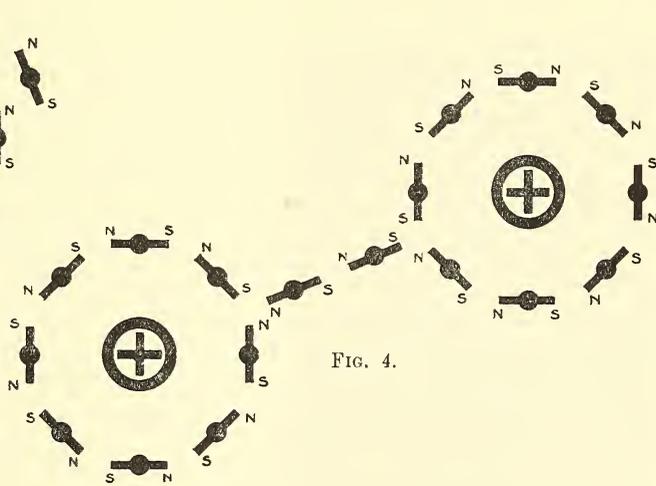


FIG. 4.

angle of instability. In the second place, that work is dissipated as heat as the two magnetic electrons swing irreversibly into their new stable position. In the third place, the two atoms may be supposed to remain attracted and held together by the combination of electrostatic and magnetic attraction which is brought into play.

So far we have only dealt with purely diagrammatic arrangements of magnets in one plane, and it is necessary to see whether we can arrive at similar results if we assume groups of magnetic electrons moving in small orbits, and arranged on a series of shells round the positive nucleus.

If the atom is purely electrostatic, Sir J. J. Thomson has shown that there cannot be more than eight electrons in each shell if the positive nucleus has just a sufficient charge to neutralise the negative charges on the electrons, but if the electrons have also got magnetic properties they will attract each other, and, consequently, the numbers in each shell and the charge on the positive nucleus can be increased.

Mr Langmuir, adopting the Parson hypothesis, has designed an atom made of ring electrons round a positive nucleus in which the number of electrons in each shell is twice the square of the radius, giving 2 in the first shell, 8 in the second, 18 in the third, 32 in the fourth, and so on, and he has also supposed the ring electrons to lie with their orbits in radial planes passing through the centre of the atom. It is evident that, in order to conceive of such an atom, we must assume a certain rigidity in the electron ring; and if we imagine these electron rings to place themselves equally spaced in a series of shells owing to the electrostatic attraction and repulsions, each ring will turn within its given shell so as to obtain the maximum number of lines of magnetic

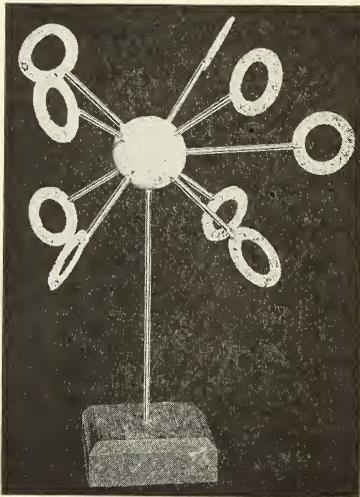


FIG. 5.

force passing through the ring. Such a radial arrangement of the rings, therefore, seems reasonable, as, at any rate, a first approximation to their position, though doubtless it will be modified to a certain extent by the layer of rings below. If imagined to occupy such positions, they will correspond to the circle of little magnets which we have already considered, head to tail round the ring.

In the model shown in fig. 5 the second shell of the Langmuir atom is shown, though, with a view to simplicity, the first shell of two electrons and the double number of electrons which he supposed to be contained in each cell have been left out. A single electron is also shown in the third shell, thus corresponding to a monovalent element like potassium or sodium. If the model is inspected, it is evident that this ring in the outer shell is in a stable position, having set itself with reference to the rings in the inner shell so as to have the maximum number of lines of force passing through it.

Diagram 6 shows two models, the second model being also supposed to have one outer electron and these two electrons to have approached each other and swung into their new position. This also is in agreement with Mr Langmuir's hypothesis, who showed models with the attracting electrons arranged in this way at the British Association Meeting in Edinburgh in 1921. We must suppose, then, that each of these outer electrons is held in its shell by the electrostatic attraction of the

nucleus, and that as they approach each other they each swing through approximately 90° to take up their new position under their mutual polar attractions. This will involve, just as in the case of our magnets, of the doing of a certain amount of work on each electron until it has swung to its angle of instability, when it will then rotate into its new position with vibration and dissipation of energy as radiation or heat.

Evidently these electrons in their new position are no longer magnetically held by their respective atoms, but they are still held electrostatically to the nucleus, and therefore we have the two atoms bound together by the electrostatic attraction of each electron for its own nucleus, and by the magnetic attraction between them.

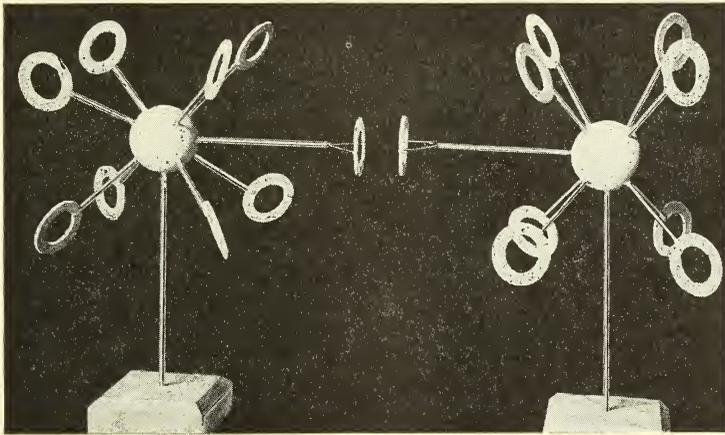


FIG. 6.

All this, therefore, is in harmony with the conclusions as to the construction of the atom come to by Mr Langmuir, but to his description a dynamic explanation of chemical combination has been added, the application of the Ewing effect showing how a definite amount of work would be done and a definite amount of heat produced.

We must next consider whether the two electrons will remain in this position or will swing together on a common axis so as to lie again in a radial plane. Now if we assume the electrons to be placed into the shells of *both* atoms with their north poles facing the same way, it is evident that the poles of the electrons of two atoms approaching each other would be related, as shown in fig. 7, and that consequently, if the two external electrons, after swinging into the tangential plane, begin to swing together on a common centre in either direction, they will be repelled by the magnetic poles on one atom or on the other, as the case may be, and

therefore may remain lying tangentially to both atoms. At the same time the electrons in the two inner shells will be magnetically attracted to each other. It is, however, conceivable that the electrons in one atom may be placed so that their north and south poles are reversed, as shown in fig. 8. In this case it is evident that the two external electrons are no longer stable in the tangential plane, and will tend to swing round a common centre into the radial plane of one atom or the other.

We thus seem to have a possible explanation of why in the case of some atoms combination is followed by ionisation, while in the case of other atoms no ionisation takes place.

Let us suppose two atoms, as shown in fig. 8, to have approached

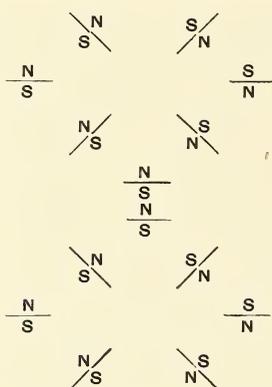


FIG. 7.—Combination without ionisation.

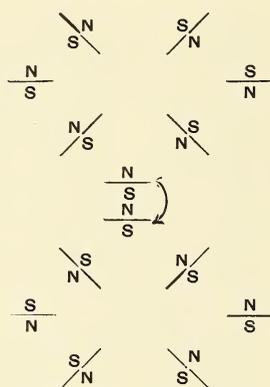


FIG. 8.—Combination to be followed by ionisation.

each other, and the two electrons to have swung into a tangential plane; they will now tend to swing on a common centre into a radial plane, with a lowering of the magnetic potential energy of the system. They are now subjected both to the electrostatic attraction of the nuclei of the two atoms and to the magnetic attraction of the two atoms tending to draw them into their respective magnetic fields of force. Probably, therefore, the question as to which atom will depart with both electrons will depend upon the algebraical sum of these electrostatic and magnetic attractions. There will be a further lowering of magnetic potential energy as the electrons are drawn into the magnetic field of one of the atoms (fig. 9).

We can thus understand how it is that while chlorine in combining with potassium ionises, chlorine in combining with oxygen will produce a compound and will not ionise; as we may suppose that in the case of the non-metallic elements the electrons are all inserted into the shell with

their north and south poles placed in the same way, while in the case of the metallic elements they are placed in the opposite way and thus ionisation will result.

We also have a simple explanation of why it is that if we arrange a chemical reaction so as to produce a voltaic current, the changes of energy which take place do not result in the local production of heat; for, to take a simple example, suppose we introduce a zinc plate into a solution of iodine and zinc iodide, and suppose we imagine the solution to be sufficiently diluted to produce zinc and iodine ions, thus avoiding any controversy on this subject, we shall have the electrons on the outer shells of the two atoms swinging into their new position with the production of heat; but if we arrange the whole thing as a voltaic cell with the zinc plate in a solution of zinc iodide and with a platinum plate surrounded by iodine, dissolved in zinc iodide, then the

iodine atoms can be supposed to attract free electrons from the platinum plate which will be drawn into the outer shell, while at the same time at the other pole electrons are withdrawn from the zinc atom passing as free electrons into the zinc plate, while the zinc ion passes into solution.

As in both cases the electrons are free, this process will take place without vibration and local production of heat, while at the same time the experiment can be so arranged as to make the total transformation of energy the same in both cases, thus supplying a source of energy for the external circuit.

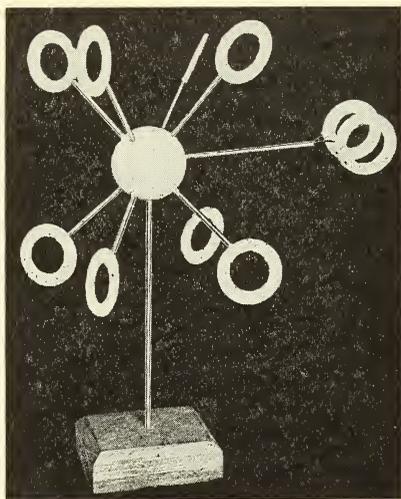


FIG. 9.

OBITUARY NOTICES.

William Spiers Bruce, LL.D. By R. N. Rudmose Brown, D.Sc.,
The University, Sheffield, and James Ritchie, M.A., D.Sc., Royal
Scottish Museum.

(MS. received March 8, 1922. Read March 20, 1922.)

I.

IN William Spiers Bruce, whose death occurred at Edinburgh on 28th October 1921, the Royal Society of Edinburgh has lost a Fellow of many years' standing, and science the foremost authority on Polar regions and an experienced and successful explorer. Born on 1st August 1867, the son of Dr S. N. Bruce, Bruce studied medicine at the University of Edinburgh, but before completing his course sailed for the Antarctic with the *Balena* in 1892. In 1895 he took charge of the Meteorological Observatory on Ben Nevis, and during 1896-97 he was in Franz-Josef Land with the Jackson-Harmsworth expedition. In 1898 he sailed with Major Andrew Coats in the *Blencathra* to Novaya Zemlya and the Barents Sea, and the same summer accompanied the Prince of Monaco to Spitsbergen in the *Princesse Alice*. In 1899 he was again with the Prince of Monaco in Spitsbergen. In 1902 Bruce organised and led the Scottish National Antarctic Expedition in the *Scotia* to the Weddell Sea, returning home in 1904. Later expeditions to Spitsbergen under his leadership were in 1906, 1907, 1909, 1912, 1914, and 1919. His last visit was in 1920. In 1910 he announced plans for a second Scottish Antarctic Expedition, but did not succeed in raising sufficient funds to start. In 1914-15 Bruce was in the Seychelles in charge of a sperm-whaling venture which closed down on account of the war. Bruce was one of the founders of the Scottish Zoological Park, a scheme of which he had long been an advocate. By his own efforts he equipped and maintained the Scottish Oceanographical Laboratory as a centre of Polar research, until failing health compelled him to disband it a year ago.

Bruce never sought reward for his work, and shrank from any form of publicity, but he was a gold medallist of the Royal Scottish Geographical

Society, the Royal Geographical Society, and the American Geographical Society. The Royal Society of Edinburgh awarded him the Neill prize and gold medal, 1911-13. He was an honorary LL.D. of Aberdeen, and a Membre de Comité de perfectionnement de l'Institut Océanographique de Paris.

Enthusiasm, modesty, and single-minded devotion to science were characteristics of Bruce, and endeared him to a wide circle of friends. Indomitable, unselfish, and ever thoughtful of others, he made an ideal leader in the field, and succeeded in accomplishing an immense amount of work in a relatively short life. He has left an imperishable mark on the annals of Scottish scientific endeavour.

R. N. R. B.

II. GEOGRAPHICAL AND OCEANOGRAPHICAL WORK.

The expedition of the Dundee whalers in 1892, which Bruce accompanied in the *Balena*, marked the first expedition to Antarctic regions for over half a century. Sailing nominally as surgeon, and actually engaged principally in sealing, Bruce managed, nevertheless, to make many important observations. A series of two-hourly meteorological observations, taken between lat. 60° and 65° S. and long. 51° and 57° W. during three months, was more complete than any previous set of observations, and gave the first strong evidence in favour of an Antarctic anticyclone. Many soundings and sea temperatures around the north-east of Graham Land were also taken. This expedition did much to reawaken interest in the Antarctic, and paved the way for the great effort of the opening years of this century. In the *Challenger* Office, and as meteorologist in charge of the summit observatory on Ben Nevis, Bruce found congenial work for a year. In Franz-Josef Land his work was largely zoological, but he took part in the survey of the western islands of the archipelago.

In 1898 began a long series of visits to Spitsbergen and the Barents Sea which gave Bruce a wider knowledge of Spitsbergen lands and waters than any other explorer of his time. With the Prince of Monaco he helped in the detailed hydrographical charting of Red Bay and the survey of the surrounding land. Bruce Point marks the proximity of the rock where the *Princesse Alice* ran aground, and where Bruce organised a shore camp when the loss of the vessel seemed imminent. Many of the deep-sea soundings off western Spitsbergen were also made by Bruce in company with the Prince of Monaco and Mr J. Y. Buchanan. In Storfjord he also took soundings on several occasions, and removed from the chart the mythical "flat island" which for long was reported to lie in the fjord. Bruce specialised in the exploration of Prince Charles Foreland, which

before his first visit in 1906 was practically unknown. He spent the greater part of three summers in surveying and exploring the island. The map, on a scale of 1 to 140,000, was published in 1913 by the Prince of Monaco. Expeditions under Bruce's command also completed a map of Bünsow (Garwood) Land at the head of Icefjord, and took many hydrographical observations in Foreland Sound, Sassen Bay, Klaas Billen Bay, and elsewhere. He made one of the first landings on Hope Island, and on another occasion made a new crossing of Spitsbergen from the Sassendal to Mohn Bay.

Bruce took a leading part in the economic development of Spitsbergen, and as long ago as 1899, before any claims to mining estates had been made, brought home samples of coal for analysis. Only those who sailed with Bruce to Spitsbergen could appreciate his marvellously detailed knowledge of its coasts and anchorages, the localities for camping, the distribution of bird rookeries and of driftwood, and the routes for land travel; while his acquaintance with the course and nature of sea ice round the coasts was seldom, if ever, at fault.

It was in the Antarctic, however, that Bruce's most important geographical work was done. The expedition of the *Scotia*, financed in Scotland, largely by Mr James Coats of Paisley and Major Andrew Coats, was designed principally for research in oceanography (including zoology) and meteorology. Bruce was too earnest in the advancement of science to put the attainment of a high latitude in the forefront of his plans. The results can only be briefly summarised. During two summers the *Scotia*, ably handled by Capt. T. Robertson, penetrated the dangerous Weddell Sea without serious mishap, and at the end of the second season discovered a new part of the coast-line of Antarctica in lat. 74° S. Coats Land, as Bruce named it, was traced for 150 miles to the south-west. No landing on its ice-cliffs was possible, but high land could be discerned in the far interior. This discovery, together with a long series of soundings in the uncharted Weddell Sea and South Atlantic, especially the re-sounding in 2660 fathoms of Ross' 4000 fathoms no bottom (lat. 68° 32' S., long. 12° 49' W.), entirely revised ideas of the extent and conformation of Antarctica on the Atlantic side. Other discoveries included the southern extension of the mid-Atlantic ridge, and strong evidence in favour of submarine connection between Graham Land and South America *via* the South Sandwich group and the South Orkneys. Wintering at the South Orkneys, the expedition explored and mapped Laurie Island, and founded, in Scotia Bay, a meteorological observatory which has since been maintained by the Argentine Government—the only Antarctic observatory in existence.

Gough Island (lat. 40° 20' S., long. 9° 56' W.) was visited for the first time by a scientific expedition.

The scientific results of the *Scotia* expedition were published in a series of volumes, many of the papers previously appearing in the *Transactions* of this Society. Some six volumes have appeared, but there is material for several more if funds were available. A Government grant towards the cost, after years of effort on Bruce's part, was on the eve of materialising when the outbreak of war checked all such schemes. Other papers of Bruce's have appeared in the *Scottish Geographical Magazine*, the *Geographical Journal*, etc. He wrote a volume on *Polar Exploration* (1911), and contributed a section on "The Falkland Islands and Dependencies" to *The Oxford Survey of the British Empire* (1914). His researches into the early history and exploration of Graham Land are partly incorporated in the new *Antarctic Pilot*; and his views on the structure of Antarctica appeared in "Über die fortsetzung des antarktischen Festlandes," *Schweiz. natur. Gesell. zu Basel*, 1910.

R. N. R. B.

III.—BIOLOGICAL WORK.

Deeply implanted in Dr Bruce's temperament was an intense love of nature, and this passion was directed into definite channels during his attendance at the University of Edinburgh. There the influence especially of Professor Cossar Ewart and Sir William Turner, of whose teaching he always spoke with enthusiasm, laid a broad foundation of zoological knowledge which stood Bruce in good stead during his wanderings. Two characteristics stand out prominently in his biological work—a scientific opportunism and an universal interest. No chance that offered of adding to the raw material of zoological science was allowed to slip by unused: when meteorology took him to the summit of Ben Nevis, he spent his spare hours making the first extensive collection of insects gathered at a high altitude in this country; when commercial whaling took him to the Seychelles, he seized every opportunity of collecting zoological material, and confirmed a recent observation, first communicated to this Society, regarding the functioning of the so-called embryonic upper teeth of the sperm whale. The wideness of his biological interests could not be more concisely illustrated than by the materials he gathered, in his spare moments and often at great inconvenience to himself, during the Seychelles enterprise; for these range from portions of whales, sucker-marks of giant squids upon whale-skin, marine polychaet worms and other invertebrates, to odds and ends of drift refuse picked up on the shore, by the identifica-

tion of which and of their original provenance he hoped to determine the prevalent oceanic currents in the area.

It was fortunate for biological science that Bruce preserved his wide sympathies throughout life, and refused to be drawn into the narrow path of the specialist. The expeditions in which he took part and which he planned afforded unique opportunities of observation and of collecting on broad lines, and these he used to such advantage that he stands in the first rank of the great naturalist travellers. No man in recent years has done more to enrich, in variety as well as in numbers, the accumulated stores on which the science of systematic zoology is based.

How much Bruce might have accomplished had he been free to follow the lines of special investigation of animal life he had planned for himself it is difficult to say, for the task of gathering specimens was perhaps the least onerous part of his labours, and, with a generosity that was characteristic of him, he spent many years and grudged no pains in the sorting and grouping of his enormous wealth of material, so that each specialist might enter upon the last stage of detailed classification free from the drudgery of the initial unravelling.

To glance more closely at some of the results of Bruce's biological labours. Botany, geology, and zoology have one and all been enriched through his energies. The recent floras of Gough Island, the isolated volcanic islet midway between South America and South Africa, and of the South Orkney Islands, were described for the first time by Dr R. N. Rudmose Brown—the former as the result of the first scientific exploration of the island, made during the return voyage of the *Scotia*; and the plants obtained there, together with representatives of the cryptogamic floras of the Antarctic islands and seas he visited, form, according to Professor Sir Isaac Bailey Balfour, late Keeper of the Royal Botanic Garden in Edinburgh, one of the most valuable collections deposited there in recent years. His careful botanical collecting in Prince Charles Foreland added much to the knowledge of the flora of Spitsbergen.

In spite of the fact that numerous geological surveys of Spitsbergen had preceded Bruce's visits, it was he who first discovered that Prince Charles Foreland was not wholly of Lower Palæozoic formation, as had been asserted; and Dr G. W. Lee's examination of his fossil collections proved that Bruce was right, and that over the Lower Palæozoic formations there lay in places on the east coast deposits representing Tertiary strata.

But it was on zoology that Bruce's main efforts were concentrated, and scarcely a branch of the science, especially in its systematic and biological aspects, but is the richer for his travels.

His first voyage to the Antarctic, on the *Balena* in 1892 and 1893, brought news of an undreamed-of abundance of finner whales in high southern latitudes; and although, with a misfortune that seemed to dog his steps, his attempt to start a commercial venture failed, others have since reaped where he had sown. His subsequent voyages added materially to our knowledge of the mammal life of Spitsbergen, and of both northern and southern Polar seas.

Ornithology owes much to his keen observation. At Spitsbergen in 1906 he discovered for the first time the chicks of the Sanderling, as well as the first European breeding station of this wader. But already he had added several birds to the island faunas of Northern Europe. In 1896-7 he found in Franz-Josef Land the Lapp Bunting, the Shore Lark, the Turnstone, and the Purple and Bonaparte's Sandpipers, all previously unrecorded, and in 1898 added the Grey Phalarope to the fauna of Novaya Zemlya. As was to be expected, even greater gains resulted from so well conceived and executed an expedition as that of the *Scotia*. Not only were many new facts regarding the distribution of sea-birds discovered (for example, the range of several species was extended to within the Antarctic circle), but much information regarding the life-histories and habits of southern birds was accumulated, including the discovery of the eggs of the Cape Petrel, of the eggs and young of the Snowy Petrel, and of the eggs and two stages of down plumage in the chicks of the Ringed Penguin. The bird-life of the South Orkneys was all but unknown until the visit of the *Scotia*; the known avifauna of Gough Island was doubled; and the series of bird skins is one of the most important ever made in the regions of the far South.

Perhaps even more important, from the freshness of the knowledge they added to zoology, were the collections of fishes made by Dr Bruce. The *Scotia* collections alone contained seven new genera and more than two dozen new species, with many interesting representatives of the distinctively Antarctic Notothenidæ, and several abyssal forms from depths of two miles and over.

The sea and life in the seas ever stood in the forefront of Bruce's plans; and in spite of the difficulties of marine collecting, owing to the need of complicated apparatus, of more discrimination in selecting from captures, and of greater expenditure of time and labour, his series of marine invertebrates form the most important of all his contributions to the raw material of zoology. His earliest Arctic voyages extended many a known range of distribution, and added some seven new species of Entomostracan crustaceans to the fauna of the Arctic seas; and of his

extraordinary *Scotia* collections need more be said than that, in the three volumes of the *Scientific Reports* which deal with invertebrates, the colossal total of some 1100 species is recorded, of which 212 are made known to science for the first time?

To the advance of our knowledge of some of the general problems of zoology Bruce's investigations contributed not a little. His unique experience of both Arctic and Antarctic seas and their inhabitants aroused again prolonged discussion of the bi-polar theory of the distribution of marine life, the chief result of which was to place a new emphasis upon the similarity of habit and of structural adaptation induced by similar conditions of environment and livelihood.

Till recently the Arctic and Antarctic collections made by Dr Bruce were housed in the Scottish Oceanographical Laboratory at Surgeon's Hall, and the arrangements he adopted showed how far his interest extended beyond the identification of specimens. Nothing could have been more instructive than the cases in which, layer above layer, he showed the characteristic inhabitants of different depths in the sea; or those where the marine faunas of the islands he visited were placed side by side in comparative series. Some months before his death he presented these great collections, as well as his meteorological and physical apparatus, to the Royal Scottish Museum, and there a series of specimens representative of his most striking discoveries has been arranged for exhibition.

I cannot close these remarks on Dr Bruce's services to zoology without adding that no earnest student was turned away empty from his vast stores of knowledge. Close contact with him and his work for many years impressed me more and more with his striking personality, with his loveliness, generosity, and enthusiasm, which throughout sacrificed self for science.

J. R.

Jacobus Cornelius Kapteyn. By Professor R. A. Sampson, F.R.S.

(Read January 8, 1923.)

JACOBUS CORNELIUS KAPTEYN occupied a unique place in astronomy. Born on January 19, 1851, he was appointed in 1878 to the professorship of astronomy at the University of Groningen, a post which he held till his retirement in 1921, and which he made one of the most famous chairs in the world. There was no observatory. Under parallel circumstances, he might have occupied himself with mathematical developments, as many celebrated astronomers have done. In place of that he conceived and gradually perfected a new branch of discussion. The first phase of it was the measurement and reduction of the photographic survey of the southern heavens. The plates had been taken by Gill at the Cape Observatory, but they lay unusable because unreduced. The work of reduction was new, it required a man of first-rate ability, and was immensely laborious. Kapteyn volunteered to undertake it, and it occupied him for thirteen years. The *Cape Photographic Durchmusterung*, which resulted, contains measures of place and magnitude of 450,000 stars. Laborious as the task was, it gave Kapteyn a sureness of judgment in dealing with voluminous stellar material which could not perhaps have been otherwise acquired. It secured him permanently against unreality in discussing the documents of astronomy. He saw, perhaps the first, that the time was now ripe for a real attack upon the question, What is the Stellar Universe? The Groningen publications are devoted to this question—or since, when Kapteyn began to write, we were very far from being in a position to give any answer, he set himself to collect such contributory particulars as were available, to criticise them, to point out what more was required, to devise means of evading some obstacle that could not be conquered, to select regions for intensive study that should give an intelligent statistical view of the whole, to define practicable problems, the solution of which could throw a light on dark places; and so forth. His zeal and judgment made him eminently successful in this, and he gradually acquired an unquestioned and magisterial authority in the astronomical world. His contributions were not, however, confined to pointing out the path. In his searches through the material, he made many discoveries. The most famous of these, as it well deserves to be, is the discovery of the two Star Streams. From Herschel's time it had

been recognised that the Sun's motion among the stars could be seen reflected in a systematic spreading or closing of the fields towards or away from which the Sun was moving. Kapteyn discovered that, apart from this, there was a polarity in the proper motion which might be understood as dividing the stars considered, into two great clusters, one moving across the other. This wholly unexpected revelation, which has since been amply confirmed, gave a new orientation to research, showing, as it did, the possibility of separating the stellar problem into manageable portions, and of thinking in terms of groups and clusters of stars, the common properties of which were discoverable.

Kapteyn's writings are often mathematical, but with mathematics severely subordinated to his actual problem, which is usually of a statistical or numerical character. Though not himself an observer, he kept touch with observation by annual visits, for many years, to the great observatory of the Carnegie Institution at Mount Wilson, California, of which he was a Research Fellow. His simple-hearted enthusiasm endeared him to many friends. He was well known to British astronomers, and was an Associate of the Royal Astronomical Society and a Foreign Member of the Royal Society of London, as well as of our own Society. He visited Edinburgh on the occasion of the meeting of the British Association in 1921, when the University conferred upon him the Honorary Degree of LL.D. He died on 18th June 1922.

Sir John Rankine, K.C., M.A., LL.D. By **The Hon. Lord Johnston,**

(Read January 8, 1923.)

THE Royal Society have lost, since their last session concluded, a member who deserves to be held in special remembrance.

The late Sir John Rankine, who died in the beginning of August 1922, though himself laying no claim to scientific attainment, always followed the proceedings of the Society with the sympathetic interest which he showed in all which made for the advancement of his fellow-countrymen.

But his work, unremitting and effective, in other directions calls for more than a passing notice. He was not widely known to the general public, but no man of his day and generation was better known and more appreciated by those for and among whom his work was done, or has done more for the good of the community in which his lot was cast.

John Rankine was born in 1846 in the manse of Sorn, of which parish his father, the Rev. John Rankine, was minister. His father was a member of an Ayrshire family belonging to the district of Maybole, and his mother of the family of Simson, well known and long established in Lauderdale. On both sides he was connected with the land, and back to the land he came with his brother Charles on the death of their maternal uncle, Charles Simson, proprietor of the small estate of Threepwood in Lower Lauderdale. This family connection with the land largely affected his future professional career.

As son of Dr Rankine, a Moderator of the General Assembly, and, which is perhaps unique in the history of the Church, having as brothers-in-law two Moderators and a Moderator-designate, John Rankine, was as might be expected, a staunch supporter of the Church of Scotland. But, like many another son of the manse, he looked for his own career in the Parliament House. He studied in Edinburgh, where he took his degree in 1865. He attended the Law classes there, and afterwards followed an old Scottish practice, now obsolete, by spending a year at Heidelberg studying Roman Law.

He was called to the Bar in 1869. He had but few opportunities as a junior of appearing in court; and though, after publication of his first venture in legal literature, he was much relied upon as an adviser in chambers, he never as a senior counsel sought to enter the arena of the court, to which his urbane and placid disposition was not suited.

But Rankine was not long in finding his true rôle to be that of legal scholar, writer, and professor. It is not too much to say, reading his life backwards, that he had not been a couple of years at the Bar when he fixed his mind's eye on a University Chair of Scottish Law as the goal of his ambition, and that he had in ten years from his call to the Bar fairly written himself into such a position in the estimation of the legal profession in Scotland, and more particularly of his brethren in the Faculty of Advocates, as ensured him the first vacancy.

In 1879 he published the first edition of his *Law of Land-ownership*, a marvellous feat for a man who had been but ten years at the Bar. But they had been ten years of close study of the Principles and Case Law of Scotland. When John Rankine entered on his selected task, there was no room for a further work on the Principles of Scots Law, nor even for one which might affect the development of that law in any particular direction, as did the *Commentaries* of George Joseph Bell. Rankine wisely confined himself to the systematic ordering of the results of the Case Law of Scotland upon the subject in which he was, by heredity, most interested—the Law of Scotland, not as it affects the title to land, but as it affects the ownership and occupancy of land. He was a rapid and most methodical worker, and it was always an enigma to his contemporaries how and when, in so brief a period, he collected such a mass of authoritative material. But the leading feature of his published work was the perfect marshalling of the material so collected, and the terseness and perspicacity of the statement. It is not too much to say that Rankine's *Law of Land-ownership* stands unrivalled as a masterly compendium of the law on a very wide and important subject. The *Law of Land-ownership* was followed in May 1887 by his *Treatise on Leases*, the two together covering the whole field which he had appropriated as his own.

In the second decade of his career at the Bar, Rankine had thus established his claim to be an unrivalled exponent of the Law of his country. It was in 1888 that an opening occurred to the position for which he felt himself most fitted, and for which he had been, consciously or unconsciously, preparing himself; and on the resignation of Professor Norman Macpherson, Rankine was appointed to the Chair of Scots Law in the University of Edinburgh, which he held to within two months of his death.

The new Professor was not long in bringing the results of his close study of the Law of Scotland to bear upon the preparation of his lectures. Taking as his textbook Erskine's well-known *Principles of the Law of Scotland*, a work which he edited, and re-edited repeatedly during his tenure of the chair, for the benefit of his students, he was ready to meet

his class in the winter session of the year 1888-89. Scottish students have a sure instinct for estimating the character and capacity of their professors, and in neither did they find John Rankine wanting. He quickly established the best relations with his large class, and continued to the end respected, nay, indeed revered, by them. Their relations were not confined to the bare walls of the lecture-room, but extended to much personal intercourse and kindly interest in their prospects. In the course of the thirty-four years during which he held the chair, most lawyers now in practice in the south-east of Scotland, and many others, passed through his hands; and few, if any, do not look back upon his lectures as their first illumination of the great fabric of Scots Law.

The establishment of John Rankine in the Chair of Scots Law practically opened the way to a wholly new sphere of activity. His interests were not so much transferred from the Parliament House to the University as that the transfer to the University opened new interests on parallel lines. The new Professor at once entered heart and soul into the general work of the University, and with it into the care and support of other institutions in Edinburgh to which his position in the University gave him access. In 1889, as the representative of the Senatus of the University of Edinburgh, he was sent to the Board of Management of Edinburgh Royal Infirmary, and either as representing the Senatus or the Faculty of Advocates he remained till the end an active member of this Board. He took his work on that Board as no sinecure, but gave unremitting attention to all that concerned it, acting frequently as convener of the finance and other important committees. The value of the work which he did for this and kindred institutions was greatly enhanced by the experience which he obtained as a director of the Commercial Bank of Scotland, a position which he held from 1887 till the day of his death.

He was also the representative of the Senatus on the Board of Management of the Students' Union, an institution in which he took the greatest interest.

He represented the University Court on the Board of Management of the Dick Veterinary College.

But the piece of public work with which his name will always be most associated did not directly emanate from his position in the University, for his co-option to the Board of Management of the Royal Edinburgh Asylum occurred in 1887, the year before he became Professor. It was at that date that the remodelling and extension of the Asylum was becoming a pressing question. It was evident that, if it was to fulfil its functions, a very large work of extension must be undertaken. But

the Board was strongly divided on the subject of financial possibility, and after his election the voice of the new member was at once raised in favour of a bold policy, and his views soon became those of the Board, who with unexampled courage launched on the scheme which has given to the Asylum Craig House and the adjoining buildings on Craiglockhart Hill. That they faced the expenditure of £150,000, and that the venture had proved an absolute success, financially as well as practically, was an assured fact before Sir John's death, and it was in the year 1921 that, at the request of his colleagues, he consented to sit for his portrait, as being the only survivor of the 1887 Board. This portrait now hangs in the hall of Craig House. It was in the same year that the honour of knighthood was conferred upon him.

With his work on *Leases* Rankine's more important contributions to legal literature ended. But shortly before his death he published a briefer work on the more technical subject of *Personal Bar or Estoppel*, which was well received on both sides of the Border. It will, however, be of interest to the Society to know that during the last three or four years of his life he occupied some of his leisure at Threepwood in collecting references to Scottish law in the Waverley Novels, which it is believed he contemplated making the foundation of an article on Sir Walter Scott's obligations to the law of Scotland in the creation of many of his characters.

It would not do justice to the subject of this brief reference were something not to be said of John Rankine at Threepwood, familiar to him from boyhood, and the country home of his later years. There was to be found, not the student of Scots Law, the writer or professor, but the landowner, farming his own land, ready and able to talk of crops and stock with grieve or farm hand. But there especially did he play most perfectly the rôle of the genial host, and many a friend has enjoyed a visit to his semi-moorland domain through which, as he was always careful to point out, ran the ancient Girthgate.

There was much indeed in Threepwood, simply as it stands on a hillside as high as the top of Arthur Seat, to attract the visitor, for the scene of much of Border history and Border fiction is spread out to view. It is within three or four miles of the spot where Angus "belled the Cat." Near the site of the old house was the White Lady's well of Sir Walter's *Monastery*. On Threepwood rises the stream which in a couple of miles passes between the Border keeps, once the strongholds of monastery tenants, one of which, Glendearg, was the home of Halbert Glendinning. Down the banks of its lower course and through the Faery Dean recklessly rode Sir Percy Shafton, until he was brought up by the Tweed and the old

bridge with its surly porter and closed gate, the site of which is still known as "Bridge End." From one point to another in the surrounding fields can be seen the route from Glendearg to Avenel Castle (Smailholm), passing between Cowdenknowes and the Black Hill of Earlston on the one hand and Drygrange and Bemersyde on the other, with the Eildons and Cheviots forming middle distance and background to a fair prospect. Thus there was much to attract, and here John Rankine rejoiced as host, and delighted to gather round him the members of his own family in its different branches and his many friends. And it was to Threepwood that he went in July 1922, after the resignation of his chair, and fully expected to spend the summer there as usual. But in a very few days he was taken seriously ill, and died on the 8th of August.

Sir James Ormiston Affleck. By Dr Alexander James.

(Read January 8, 1923.)

To those of us medicals who have lived long enough in Edinburgh, and who, during our life's span have had the privilege and good fortune to acquire the intimacy and friendship of Sir James Affleck, his loss must appear little short of irreplaceable, for the simple reason that men of the type of which he was such a unique example are now no longer being bred.

As a physician, Affleck was self-made, in the truest and in the highest sense of the word. In the truest, because the position which he attained, the esteem in which he was held, and the honours which were accorded to him, were gained, not only by his own unaided industry and effort, but by industry and effort in the performance of which all ideas of future recompense or reward were as completely absent from his mind as is humanly possible. In the highest sense of the word, because esteem, honour, and position, as they severally all came to his lot, made absolutely no difference to him. To all who knew him, Affleck was the same simple, upright, kind, and self-sacrificing man in the second half of his life as he had been in the first.

Born in Edinburgh in 1840, and having made up his mind early to become a doctor, Affleck qualified in 1867, taking then the M.B.,C.M. at the University, and the Licence at the Royal College of Surgeons, Edinburgh. He was thus later than most of his fellows in entering the profession, but this was simply because he had first to earn for himself the means required to carry him through the curriculum. For the same reason, after qualifying, he settled down at once to general practice in the Stockbridge district of the city, and there he worked conscientiously and untiringly for the first ten or twelve years of his medical life. From the beginning, however, it was evident that he was destined for more responsible work than that of general practice, for, combining evening study with his daily toil, he wrote his thesis on "Functional Disorders of the Heart," and took the M.D. degree in 1869. About the same time he obtained the Fellowship of the Royal College of Surgeons, Edinburgh, and, having thus qualified himself for the post of a dispensary physician, he obtained this, and indeed, for some years, combined with it the duties of public vaccinator at the New Town Dispensary.

But in these early days of his medical life a piece of real and deserved

good fortune came to him. As he had distinguished himself in the class work of his student days, and as at his graduation his thesis had obtained commendation, it goes without saying that he had been earning for himself the recognition and approbation of all his teachers. But among these, the late Professor Sir Douglas Maclagan had so clearly discerned his character and talents that he selected him as his Assistant in the Public Health and Medical Jurisprudence Department of the University. This opened out for Affleck the opportunity to acquire what everyone desiring to attain to any eminence in medical circles in Edinburgh must possess, viz. capacity to perform scientific work and practical experience in teaching. In both of these lines he soon gained for himself a solid reputation; and as the Professor of Medical Jurisprudence was in those days also one of the clinical teachers in the Royal Infirmary, Affleck, as his Assistant, soon found himself becoming as well known and respected in the wards of the Infirmary as in the Medical Jurisprudence Laboratory and Classroom. About this time, also, his acknowledged literary leanings found expression in the contribution by him of the medical articles required for the Ninth Edition of the *Encyclopædia Britannica*.

Bearing all this in mind, we can easily understand that the amount of work which Affleck managed to get through in the twenty-four hours of each day for the first ten or twelve years of his medical life must have been enormous. Yet he never flinched; and though his rather slight frame might suggest fragility, he hardly ever showed fatigue, and he never was really ill.

Having obtained the Fellowship of the Royal College of Physicians, Edinburgh, in 1875, Affleck was in 1877 appointed Assistant Physician to the Royal Infirmary; and inasmuch as this meant to him a future in which his energies would be concentrated upon purely medical work and teaching, he in the early eighties changed his domicile to Heriot Row, and gradually relinquished his general practice and other appointments. About this time also he started as Lecturer on the Practice of Medicine at Surgeons' Hall; so that when, in 1885, he became one of the Ordinary Physicians to the Infirmary, he was able to prove himself almost at once one of the foremost teachers and clinical physicians which the Edinburgh Medical School possessed. On the expiry of his term of office in the Royal Infirmary in 1900 he was appointed Consulting Physician to the City Fever Hospital, thus continuing his active teaching work till 1908. But there were other institutions which were indebted to him for most valuable service, and in which he took keen interest. Foremost amongst them was the Longmore Hospital for Incurables, to which he acted as a Physician and as a

Director from its foundation, and to the welfare of which he ministered with a lifelong devotion. He also acted as a member of the Board of Management of the Royal Infirmary, of the Royal Asylum for the Insane, and of the Sick Children's Hospital. He was elected a Fellow of the Royal Society of Edinburgh in 1896; in 1905 he was President of the Medico-Chirurgical Society of Edinburgh; in 1908 he received the Honorary Degree of LL.D. from the University; and in 1911 he was knighted by the King at Holyrood.

As a consulting physician Affleck was pre-eminent alike as regards perception of the significant points in any case of disease and as regards direction towards, and resourcefulness in, treatment. For this he was undoubtedly largely indebted to the medical experience of his earlier years. In the practice of medicine, as in every other walk of life, the things that matter most must be got at by personal experience alone, and during the first ten or twelve years of his medical life, when he was daily and hourly meeting with diseases of all kinds—medical, surgical, and obstetric—and when, with his keen sympathy and intense conscientiousness he was, by study and thought, straining every nerve to discover how and when he could best treat them, he had built up on a solid foundation a truly solid framework of medical knowledge and insight. Affleck was one of those who well grasped the truth of the old saying, "There is no curing of diseases by art, without first knowing how they are being cured by Nature." And so, in the plethora of new methods and processes for the diagnosis, prevention, and treatment of disease, which the last few decades have brought into notice, he was invariably recognised as the man who might be trusted to discern among them those which would stand the test of time and those which would not.

As a teacher he was very highly appreciated; he was lucid and clear in exposition, and the thoughtfulness which was evident in everything which he said or did brought out the best in the minds of the students who listened to him. He was a fluent writer, and during his long life contributed largely to the various medical journals. At the meetings of the Medico-Chirurgical Society of Edinburgh his frequent communications and demonstrations were always welcomed.

Emphatically not a diner-out, and shunning society functions of all kinds, when he found himself in congenial surroundings Affleck showed himself a most interesting and well-informed conversationalist. All his life he read largely, and in the latter half of it he assiduously added to his intellectual and artistic endowments the fruits of Continental travel. Though not a performer, he was artistic and musical through and through.

Quiet, earnest, and perhaps even solemn in manner, Affleck yet had a very keen sense of humour; and stories of little mishaps, misadventures, or foibles displayed in lecture-hall, laboratory, classroom, or consulting-room he could retail or listen to with real and intense enjoyment: they only required to be good-natured.

A lifelong abstainer and non-smoker, Affleck was neither a bigot nor a pussyfoot. He was no bigot, because he knew human nature and frailty, and he knew how much individual circumstances and surroundings can alter cases. He was no pussyfoot, because he knew that, humanity being what it is, stealth must play a part in it, and to his heart stealth was as repugnant as to his head it was impotent as a factor for any real good. In olden times I was often with him in Ward X. (the D.T. ward of the Old Infirmary), and with others I have been present when he was severely laying before a delinquent the inherent wrongness of his conduct, and the inevitably evil consequences of its continuance. What impressed us all was that though it frequently appeared that his love for the sinner made him almost forget the sin, yet his advice and admonitions lost none of their efficacy thereby. This of course was due to Affleck's personality: his boundless human kindness and sympathy were irresistible; indeed, it may truthfully be said of him that only one thing could make him rage, and that was cruelty.

For over a year before his death indications of heart weakness had been making themselves known to him, but he never let them interfere with his work to any appreciable extent. He died quietly in his sleep in the morning of Sunday, 24th September 1922.

The Rt. Hon. Charles Scott Dickson, The Lord Justice-Clerk.
By The Rt. Hon. Lord Salvesen.

(Read January 8, 1923.)

CHARLES SCOTT DICKSON was born in Glasgow on 13th September 1850, his father being a well-known family doctor in that city. He was educated at the High School of Glasgow, from which he passed to Glasgow University, where he took his degree as M.A. His academic education was completed at Edinburgh University, where he attended the law classes necessary to fit him for his future career. He gave early promise of the distinction to which he afterwards attained, for he took his degree with honours in Mathematics and Mental Philosophy—an unusual combination,—which testified alike to his industry and his aptitude for acquiring knowledge in wholly divergent fields of mental activity. In neither did he pursue his studies in after life, and it cannot be said that either had much attraction for him when his qualifying course of study came to an end. A genius for mathematics is a rare attribute in the successful lawyer, and the same is generally true of metaphysics, although there are distinguished exceptions, such as the late Lord Moulton in mathematical science and Lord Haldane in philosophy.

It was wholly otherwise with the science to which Dickson intended to devote his chief energies. He was a born lawyer, although never a philosophical jurist, and the lectures on law which he attended were followed by him with absorbing interest. In 1871-2 he carried off the second prize in Scots Law against keen and able competitors, and he also gained the first of the prizes given by the Faculty of Procurators in Glasgow for eminence in a special written examination on the whole course of study. In the following session he was second prizeman in the Conveyancing Class.

In order to gain a practical knowledge of his profession he became apprenticed to a firm of writers in Glasgow, and qualified as a law agent in 1875. For a short time he practised as such, and quickly satisfied the shrewd writers who practised in the Sheriff Court that he had exceptional gifts as a pleader. All this was, however, merely by way of preparation for the career on which he had set his heart. In 1877 he was admitted to the Faculty of Advocates, and from that time till his death he was continuously resident in Edinburgh.

For most men who join the Bar recognition comes slowly if it comes at all. Some who have afterwards reached the highest pinnacle of distinction in their profession have been practically briefless for eight or ten years after they offered themselves for practice. It was otherwise with Dickson. His Glasgow friends who had known him as a pleader in the Sheriff Court showered briefs upon him, and before the year was out he found it necessary to resign a lectureship on Constitutional Law and History to which he had been appointed in Glasgow University. From that time onwards he had no lack of work, and for many years ranked as the busiest junior counsel at the Scottish Bar. For a short time he acted as an Advocate-Depute, a good preparation for the office of Solicitor-General, to which he was appointed in 1896. He held this position till 1903, when he succeeded Graham-Murray as Lord Advocate on the latter undertaking the duties of Secretary for Scotland. He held office as Lord Advocate till the winter of 1905-6, when a Liberal Government came into power. In 1908 he was appointed Dean of Faculty by the unanimous vote of his brethren at the Bar. He thus successively held all the highest honours that fall to the lot of the successful advocate. In 1915, on the retirement of Lord Kingsburgh, he was appointed Lord Justice-Clerk under the title of Lord Scott Dickson, and presided in the Second Division of the Court of Session until his death on 5th August 1922.

His political career was more chequered. From his student days his politics were pronouncedly Conservative, and as President of the Glasgow University Students' Association he was largely instrumental in securing the return of Disraeli as Lord Rector. He never changed his politics, although in Scotland there were then few seats where a Conservative had much chance of success. In 1892 he contested the Kilmarnock Burghs, in 1895 and again in 1896 the Bridgeton Division of Glasgow. In all three contests he was unsuccessful. In 1900 he was at last returned for Bridgeton, lost it again in 1906, but was returned at an election in 1909, and continued to represent it till he renounced politics for the serener atmosphere of the Bench. Few men who were not professional politicians worked so hard and sacrificed so much for the party to which they belonged.

In the House of Commons he had few opportunities of joining in debate, and then only in connection with Scottish Bills, but in the real business which is often done in Committee he was a most useful member. He was popular with all parties, and by the members of his own party was affectionately known as "Scotty." His freedom from bias and all trace of bitterness, combined with his sound judgment, accounted largely for his success in handling Scottish Bills.

Of the Church of Scotland he was a loyal son. He was an elder of St George's Parish Church, Edinburgh, and a frequent member of Assembly, where he was a vigorous advocate of the union of the Churches. His life was the best testimony of the faith that was in him. No one ever heard him utter an unkind or uncharitable word; to all who deserved it (and to many who didn't) he was generously and unobtrusively helpful. He was conspicuously sincere, sympathetic, open-handed, and tender-hearted. It is not surprising, therefore, that he inspired affection amongst those with whom he came in contact, to a degree that is indeed rare. At the Bar he was the most popular man of his time throughout his long career; on the Bench he was equally beloved by all his colleagues. When he died, spontaneous tributes of esteem and affection were published by such outstanding men as Viscount Cave, Lord Dunedin, and Lord Strathclyde.

From his boyhood his whole life was one of strenuous, unremitting work. Probably there is no profession that makes such calls on a man's energies as that of the successful advocate. When to that are added the constant claims of party politics, the burden is one that few can long sustain. Dickson bore it longer than most, but the strain had told upon him before he reached the Bench, and some of his energy had been sapped. He did not spare himself even then. During the last years of his life (apart from the War work which he undertook connected with recruiting, Red Cross, Child Welfare, and the like) he exhausted himself in taking long criminal trials which, consistently with even his high standard of duty, he might well have delegated to younger colleagues. When the summer session of 1922 closed and the Bench and Bar fled to the country to enjoy the long vacation, Dickson sat continuously as a member of the Judicial Committee of the Privy Council until the day before his death. He had just reached the country house which he had rented for the summer, when the overwrought system gave out, and he died during the night.

Outside of his own profession Dickson received full recognition of his many eminent qualities. Thus both the Universities at which he had studied conferred upon him the degree of LL.D.

As early as 1884 he was elected a Fellow of the Royal Society of Edinburgh. At that time he was in the full swing of his practice as a junior counsel, and had no time for other occupations. As the years went on, his practice, along with his pursuit of politics, became more and more engrossing. It is not, therefore, surprising that he took little or no advantage of his opportunities as a Fellow, and it must also be confessed that science in the ordinary acceptation of the term had little attraction

for him, although he had no difficulty in mastering such parts of a scientific subject as formed the subject of a litigation in which he was engaged. Such knowledge, however, being quickly acquired and for a special purpose only, is as quickly forgotten.

Although endowed with a splendid constitution, he was one of the least athletic of the Faculty of Advocates. He indulged in no outdoor recreations, and found his sole recreation in the companionship of his friends, with whom he loved to exchange his views. For many years before his death he never even took a "constitutional," and seemed to require no physical exercise outside of his work. When he became a judge and ceased to make the same vigorous use of his lungs as when conducting a lawsuit or addressing political gatherings, which stood him in good stead as a substitute for the exercise of his limbs, his bodily health gradually declined and his gait became feebler and slower on his short walk from the Parliament House to his club or home. It is not improbable that this want of attention to the physical side of his nature may have contributed to his premature death. I say premature, for, although he reached the age of seventy-two, so far as one could judge from his intellectual powers he appeared to have years of usefulness as a judge before him.

The enormous practice which he enjoyed for so many years is the most conclusive proof of his capacity as a counsel. Even the best backing will not enable an advocate to maintain a practice for long unless he possesses the qualities which satisfy those who instruct him as well as the clients for whom they act. A pleader in the law courts has the most competent and discriminating critics in those able men of the other branch of the profession who sit behind him and listen day after day to his conduct of a great case. They have been responsible for its preparation, and if he fails to make the most of every point of which the facts are capable, or displays any lack of mastery of the law applicable, or does not come up to the standard of some brilliant opponent, a mental note is made which may affect future employment. The special gifts which an advocate displays come under the same keen scrutiny. Some who are admittedly in the front line in conducting proofs are voted useless for jury trials. Others shine more in the region of debate than in cross-examination, and their practice becomes restricted accordingly. Others, again, are admitted to be experts in one branch of the law and in no other. No such limitations applied to Dickson. He was a first-rate all-round counsel, to whom the intricacies of feudal conveyancing or of patent law presented no special difficulties. He was just as much at home in addressing juries as Courts of Appeal, and his cross-examination of

witnesses was always vigorous and effective, if it sometimes lacked finish. The same may be said of his style of argument. The sentences were sometimes disjointed—the periods were seldom rounded—the words were not carefully chosen; but the substance was there, and received expression in simple, terse, and direct language that could not be misunderstood. No wonder, then, that Dickson enjoyed the confidence of his clients and their immediate advisers, for whether he won or lost they always felt that no one could have identified himself more thoroughly with their point of view.

As a senior counsel he did not come quite up to the level of expectation derived from his success as a junior. This was noticeable only when he appeared in the Divisions or before other appellate Courts. Some of his contemporaries outshone him in well-ordered, close, and consecutive reasoning such as the finished presentment of a purely legal argument demands if it is to conform to the highest standard.

He came too late to the Bench and occupied it for too short a time to play any conspicuous part in the development of the common law of Scotland. His conservative instincts and his deference to authority militated against his taking a bold or independent view, however much he might be satisfied that it was more in accordance with the underlying principles of jurisprudence than previous decisions. No better illustration of this tendency can be given than his being a party to a decision which affirmed that it was the law of Scotland that no woman, however old, can be considered as past child-bearing. There was no prior binding decision to this effect, although it had some support from one or two judges of former generations who were ignorant of the facts of medical science. The truth was that Dickson's unrivalled acquaintance with case law tended to fetter the free use of his intellect when questions of legal principle called for decision. The common law of Scotland is based on the experience of the race, and is supposed to represent the highest embodiment of the commonsense of the community for the time being. It is therefore capable of development as human knowledge broadens the outlook, and is thus unlike statute law, which, so long as the statute remains unrepealed or not in desuetude, must be interpreted strictly in the light of the language in which it is embodied. Judged by this standard the law as to the age of child-bearing (as it has now been provisionally settled) is in accordance neither with human experience nor medical science.

So far as the public, including especially the pleaders, were concerned, Dickson was an ideal judge. He was patient, courteous, attentive in his attitude to the Bar, absolutely impartial and painstaking in his judgments.

No one who pleaded before him could ever say that he had not been fully heard or understood. In his conduct of criminal trials he leaned more to the accused than to the prosecution, and his sentences, if they erred at all, erred on the side of leniency.

For the reasons I have already indicated, Dickson cannot be ranked among the few who can be justly called great judges. But if he was not a great judge, he was at all events a great personality, and, what is still better, a delightful personality. As Lord Strathclyde wrote in a masterly appreciation that appeared in the *Juridical Review*: "His genuine humanity will continue to live for many a day in the haunts which he brightened by his sunny presence. . . . It is by nothing that Dickson said or wrote, but by his own fine nature made manifest by what he did and was, that his memory will long remain green among us."

Henry Newton Dickson, M.A., D.Sc. (Oxon.).
By Dr Hugh Robert Mill.

(Read March 5, 1923.)

HENRY NEWTON DICKSON was born in Edinburgh on June 24, 1866, being the younger son of Mr William Dickson, F.R.S.E. After an early education at the Edinburgh Collegiate School under Dr A. H. Bryce, he entered the Arts classes of Edinburgh University in 1882. The time was one of extraordinary progress in research on the air and the oceans, thanks to the labours of Dr Alexander Buchan and Sir John Murray, who had made Edinburgh the world-centre of meteorology and oceanography. Many students were pressed into the service of Professor P. G. Tait in his experiments for the *Challenger* Reports, and so it was that Dickson served his apprenticeship in research by taking part in determining the pressure-corrections of deep-sea thermometers, and the coefficient of compressibility of sea-water in the physical laboratory of the University. He also assisted me on several oceanographical trips in connection with the newly founded Scottish Marine Station, and volunteered as an assistant to Mr R. T. Omond in the High-Level Meteorological Observatory on Ben Nevis. The newly-established Scottish Geographical Society also flourished in the invigorating atmosphere of research in physical geography, which inspired many students of the period to enthusiasm; but his native city offered no scope to Dickson's ambition.

In 1891 he was working on the salinity and temperature of the water in the English Channel at the Marine Biological Association's laboratory at Plymouth. Two years later he moved to Oxford, where he pursued his oceanographical studies in the University Chemical Laboratory. Dickson came more and more into touch with practical work in geography, and in 1899 he became Lecturer on Physical Geography and teacher of Surveying in the School of Geography in the University of Oxford. He also undertook the study of the water-level in the chalk formation, which depends equally on meteorological and geological conditions, and has a great practical bearing on water-supply. He took part in the work of the scientific societies of London, especially the Royal Geographical Society on the Council of which he sat for some time, and the Royal Meteorological Society which he served in many capacities and acted as President in 1911-12. He was diligent in attendance at the meetings of the British

Association, serving successively as Secretary, Recorder, and, in 1913, President of Section E, Geography. At the meeting of the Association at Birmingham in that year he delivered a notable Presidential Address to the Section on "The World's Resources and the Distribution of Mankind," which was the subject of much public interest.

For fourteen years from 1906 he was Professor of Geography in University College, Reading, where he was an inspiring teacher and a strong promoter of the practical application of the various branches of physical geography to agriculture, engineering, and other objects of vital interest in public life.

When the war broke out his services were utilised in organising a department of geographical information in association with the Intelligence Division of the Admiralty. This soon grew to considerable dimensions and involved the production of a series of geographical handbooks on various parts of the world, some of which have since been published. Dickson received the third class of the Order of the British Empire (C.B.E.) in recognition of the value of his labours.

He had already received the degree of M.A. by decree, from the University of Oxford, on joining the School of Geography, and he took the degree of D.Sc. from the same University.

On resigning his professorship, Dickson devoted himself to literary work in London, where for some time he threw his whole energy into the preparation of the supplementary volumes to complete the Twelfth Edition of the *Encyclopædia Britannica*, of which he was principal assistant editor under Mr Hugh Chisholm.

In the autumn of 1921 his health began to give way, and in the following January he underwent a serious operation. He died at his brother's house in Edinburgh on April 2, 1922. He married in 1891 Margaret, daughter of Richard Stephenson of Chapel, Duns, Berwickshire. He is survived by his wife and by a son, Lieutenant T. H. Dickson, R.N., and a daughter.

Dickson was a hard worker, devoting himself with single-minded intensity to whatever he had in hand. In the early days I often had the opportunity of working with him, and always found him acute, conscientious, and painstaking, with a degree of accuracy in computation that is not often attained. He was the best of good company at all times, and was famous in Oxford and London for his quaint dry humour and curiously apposite anecdotes.

Apart from his work in general geography, which was largely educational or literary, Dickson made solid additions to knowledge in the

departments of meteorology and oceanography, to which, in my opinion, less than justice was done by his contemporaries. His first publication was in the *Proceedings of the Royal Society of Edinburgh* for 1885, on observations which he made when working with me at Granton on the measurement of air temperature by means of Dr John Aitken's experimental thermometer screens. Soon afterwards, he discussed the hygrometry of Ben Nevis, and published an interesting collection of the weather folklore of Scottish fishermen, picked up while working for the Fishery Board. In 1901 he read to the British Association meeting at Glasgow a suggestive paper on the possible cause of glacial and interglacial periods by a comparatively slight fall in atmospheric temperature at the poles with no change in the equatorial belt. He showed how the quickening of the planetary atmospheric circulation and the resultant acceleration of oceanic currents might well account for most of the facts.

His two little books on meteorology have an importance quite out of proportion to their size. In his *Elementary Meteorology*, published as a University Extension Manual in 1893, he gave for the first time a system of meteorology in which the physical facts of wind motion were made the basis of the weather map, the arrangement of the isobars being introduced as the result of the winds, rather than as their cause. This little book did much to vivify interest in meteorology. His *Climate and Weather*, published in 1912, showed equal freshness and originality of view-point, and illustrates the importance, in treating climatology, of a wide and deep knowledge of other departments of physiography.

Dickson's chief contributions to science were the results of his oceanographical researches, chief amongst them that carried out at Oxford in conjunction with the Marine Department of the Meteorological Office. For several years he obtained regularly samples of surface water, collected by Atlantic liners at numerous points on their voyages between Europe and America, and a much more extensive series of temperature observations made as part of the ship's routine. He analysed the samples, and constructed an elaborate set of maps of salinity and temperature from which he deduced the movements of the surface waters for each month in a consecutive period of two years, January 1896 to December 1897. The accuracy of the work was of a higher order than had previously been found possible, and the duration was greater than that of any previous attempt to keep continuous track of seasonal changes. The result was to show that the seasonal variations in the surface water were due partly to changes *in situ* and partly to the translational swing of currents at various seasons to the right or left of their normal lines of flow.

From the two years which he studied in detail, Dickson was able to deduce the general order of seasonal displacement of the so-called Gulf Stream and all its associated currents, and so to correct the vague and contradictory speculations which had figured in popular descriptions and even in textbooks for generations. It is to be regretted that the work did not attract more attention at the time, but it remains, richly illustrated by maps produced by Bartholomew, in the *Philosophical Transactions of the Royal Society of London* for 1901 (Series A, vol. cxcvi, pp. 61-203).

APPLEYARD, JAMES R., F.I.C., was born at Bradford in 1870, received his education at the Bradford Ryan Street School, and subsequently at the Bradford Technical College. He left Bradford in order to take up a position as assistant to Professor Frankland at Dundee, and was for several years lecturer there on dyeing. From there he came as chief lecturer on dyeing to the Royal Technical Institute, Salford. While at Bradford Mr Appleyard carried out some original work on textile chemistry and on the theory of dyeing, and did further work on the latter subject in collaboration with Professor Walker at Dundee. He has since contributed largely on his special subject, most of his work having appeared in the *Dyer and Calico Printer*.

He joined this Society in 1899, and died on 26th November 1921.

BARCLAY, GEORGE WALTER WOODFALL, M.A., held the post of Manager to the North British & Mercantile Insurance Co., Ltd., at Aberdeen.

He was elected a Fellow of this Society in 1883, and published in the *Proceedings*, vol. xiii, 1884–86, a paper, "On some Algoid Lake-Balls found in South Uist." He died at his residence, Raeden House, Aberdeen, on 15th May 1922, aged 74 years.

CLEGHORN, ALEXANDER, M.Inst.C.E., was until recently Engineering Director of the Fairfield Shipbuilding & Engineering Co., Govan, and one of the foremost marine engineers in the country. He had a distinguished University career, and obtained honours at St Andrews, Edinburgh, and Glasgow. After serving his apprenticeship as an engineer with Messrs Robert Napier & Sons, he became assistant manager of the firm, and represented them in Russia in connection with naval work. He was engineering manager with Messrs Barclay, Curle & Co. from 1897–1905, and in the latter year entered the service of the Fairfield Co. as engineering manager, subsequently becoming engineering director. During the war he had charge of much important naval work carried through by his firm.

Mr Cleghorn was a past President of the Institution of Engineers and Shipbuilders in Scotland. He was elected a Fellow of this Society in 1913, and died in Glasgow on 4th May 1922.

CRAIG, WILLIAM, M.D., F.R.C.S.E., Edinburgh's oldest Medical Practitioner, was born near Strathaven, Lanarkshire, on 28th March 1832. On leaving the Parish School he studied Arts at the University of Glasgow. After completing his studies there, he proceeded to the

University of Edinburgh, where he graduated M.B., C.M., in 1868, M.D. in 1870. For over forty years Dr Craig practised in Edinburgh, but during that period his inclinations turned more towards the teaching and examining branches of his profession, and it is in this connection that he was most widely known. He was elected a Fellow of the Royal College of Surgeons in 1878; acted for a number of years as a member of the President's Council; was an examiner in *Materia Medica*, Public Health, and for the Dental Diploma. For several years he acted as an examiner in *Materia Medica* for Edinburgh University.

Dr Craig was the author of the following works:—*Manual of Materia Medica* and *Posological Tables*, and of papers in the *Edinburgh Medical Journal*, etc. He edited *Milne's Materia Medica* (3rd edition) and *Milne's Posological Tables*. From 1875 to 1910 he was Treasurer of the Edinburgh Obstetrical Society, and acted as editor of the *Transactions* of the Medico-Chirurgical Society from 1882-1909. It was due to Dr Craig's initiative in 1881 that this Society first commenced to publish annual *Transactions*. He was a prominent member of the Botanical Society of Edinburgh, and was, for a period, its President. He was one of the oldest members of the Scottish Alpine Botanical Club, a Director of the Edinburgh Dental Hospital, and a Manager of the Edinburgh Savings Bank.

Dr Craig was elected to the Fellowship of this Society in 1875, and died at his residence, 71 Bruntsfield Place, on 3rd February 1922.

HARRISON, JOHN, C.B.E., D.L., J.P., LL.D., was the son of the late Sir George Harrison, M.P., a former Lord Provost of Edinburgh. Born in 1847, he was educated at the High School, and attended the classes of English literature and political economy under Professors Masson and Hodgson at Edinburgh University. At the age of 15 he commenced a business training in his father's establishment in the South Bridge. Afterwards he became the head of the firm of Harrison & Son, George Street, one of the leading tailoring establishments in Edinburgh. He entered the Town Council, and for three years was City Treasurer. Various important improvements in the City can be traced to his initiative. In education he was specially interested. He was prominently identified with the administration and development of the Public Library, serving as a member of Committee from 1886 and latterly as Chairman. Dr Harrison was closely associated with the establishment of the College of Art, the scheme taking form while he was City Treasurer. In the Town Council, at a later date, he acted as Convener of the Tramway Committee during the period of transmission and reorganisation of the undertaking;

and there again his wise guidance was of great service to the community. In 1919 the University conferred on Mr Harrison the honorary degree of LL.D.

Dr Harrison occupied some years ago the office of Master of the Edinburgh Merchant Company, and served as Chairman of the Chamber of Commerce and the Edinburgh Savings Bank. He took an important part in the reorganisation of the Training Colleges, and specially interested himself in the Merchant Company's Schools. During the war he threw himself into emergency work with great zeal, and perhaps overtaxed his strength. His war services were recognised in 1918, when he was made a C.B.E.

He was a writer of repute, and was the author of a spirited sketch of the founding and early struggles of *Oure Tounis Colledge*, and of *The Scot in Ulster*, which displayed an extensive knowledge of the subject. His last important work was *The History of the Monastery of Holyrood, and of the Palace of Holyrood House*.

Dr Harrison was elected to the Fellowship of the Society in 1917, and died on 10th July 1922.

HILL, ALFRED, M.D., M.R.C.S., F.I.C., held the following positions:—Medical Officer of Health and Analyst of Birmingham; Examiner in Public Health, Chemistry, and Medical Jurisprudence, University of Aberdeen; and Professor of Chemistry and Toxicology, Queen's College, Birmingham. He was the author of several works on Sanitation, and of papers in the *Transactions of the Society of Medical Officers of Health*, *Transactions of the Congress of Sanitary Institutes*, and *Public Health*.

Dr Hill became a Fellow of this Society in 1894.

LIPPMAN, GABRIEL C., past President of the Paris Academy of Sciences (1912), Nobel Laureate in Physics, 1908, Professor of Physics in the University of Paris, 1878, and Director of the Laboratory for Physical Research at the Sorbonne (1886), was born in 1845. He graduated Ph.D. (Heid.) and D.Sc. (Paris), was a Commander of the French Legion of Honour, and a Foreign Member of the Royal Society of London. His published works include:—*L'Etude des phénomènes électrocapillaires*, *La Photographie des couleurs*, *Cours de Thermodynamique*, *Unités électriques absolues*, and many papers in various scientific journals.

Professor Lippman was elected a Foreign Honorary Fellow of this Society in 1897, and died at sea on board the liner *La France* on 12th July 1921, while returning from Canada, where he had taken part in the mission of Marshal Foyolle.

NATHORST, ALFRED GABRIEL, was born in 1850 at Wäderbrunn, was educated at Malmö, and at the Universities of Lund and Upsala. The following degrees were conferred on him during his career:—Ph.D. (Lund), 1874; Hon. Ph.D. (Greifs.), 1906 (Christiania, 1911); Hon. Sc.D. (Cantab.), 1907; and Hon. LL.D. (St Andrews), 1911. He was a Member of the Academies of Science of Christiania, Copenhagen, Berlin, Vienna, and St Petersburg; a Foreign Member of the Linnean and Geological Societies, London; a Corresponding Member of the Geological Societies of Edinburgh and Glasgow and the Royal Physical Society, Edinburgh; and received many awards for distinguished work. He conducted explorations in Spitsbergen, 1870 and 1882; in Greenland, 1883; Bear Island, Spitsbergen, and King Charles Land, 1898; and E. Greenland, 1899. Professor Nathorst's published works include:—*History of the Earth, Geology of Sweden, Two Summers in Arctic Regions, Geology of Bear Island, Spitsbergen, and King Charles Land, Swedenborg as Geologist, and Linnæus as Geologist*. He also contributed over three hundred papers to various journals on Geology, Botany, Palæobotany, Geography, etc.

Professor Nathorst was elected a Foreign Honorary Fellow of the Royal Society of Edinburgh in June 1920, and died at Stockholm on 20th January 1921, aged 70 years.

NORRIS, RICHARD HILL, M.D., L.R.C.S. (Edin.), and L.M., was a J.P. of County Warwick and Birmingham City, a member of the British Medical Association, and Assistant Demonstrator in Physiology in Queen's College, Birmingham, from 1874-75. He was elected a Fellow of this Society in 1878.

WALKER, JAMES, C.A., LL.D., Lord Rector's Assessor of the University of Edinburgh, was born in Edinburgh in 1864. He was educated at George Watson's College, the Edinburgh Collegiate School, and continued his education at Leipzig and Lausanne. He served his apprenticeship with the late firm of Messrs Dall & Miller, C.A., and was admitted a Member of the Edinburgh C.A. Society in 1886. Dr Walker had also technical training at the University and Heriot Watt College, which stood him in good stead in capacities connected with the development of several businesses of an engineering nature—in particular the Scottish Central Electric Power Co. In the C.A. Society he took a prominent part, serving on the Council of the Edinburgh Society from 1911-12 to 1914-15. One of the pioneers of the institution of the Edinburgh C.A. Students' Society in 1886, he was President for two terms.

In University circles Dr Walker was a prominent figure for many years, serving on numerous committees. He was a member of the University Court and of the Faculty of Music. His services to the latter, especially in connection with the Reid Concerts, and to the Students' Union, were as unstinted as they were unobtrusive, and will long be remembered. In 1919, in virtue of his great services to the University, the degree of LL.D. was conferred on him.

Dr Walker was elected a Fellow of the Society in March 1922, and died at his residence in Edinburgh on 27th July 1922.

WOODHEAD, SIR GERMAN SIMS, K.B.E., was born in 1855 at Huddersfield. He was educated at Huddersfield College, from which he entered the Medical Faculty of the University of Edinburgh, graduating in 1878. He also studied in Berlin and Vienna. For three years (1887-90) Professor Woodhead was superintendent of the Laboratory of the Royal College of Physicians, Edinburgh, resigning this post in 1890 on his appointment as director of the Conjoint Laboratories of the Royal Colleges of Physicians and Surgeons in London. This appointment he held until 1899, when he was elected to the Chair of Pathology in the University of Cambridge, where it was largely due to his initiative and energy that the New Medical School buildings were erected, including the Memorial Museum to Sir George Humphry.

Professor Woodhead was an Hon. LL.D. of Birmingham and Toronto Universities, a Fellow of Trinity Hall, Cambridge, Hon. Fellow of the Henry Phipps Institute, Philadelphia, a member of the Executive Committee of the Imperial Cancer Research Fund, and a member of the Scottish Universities Committee. He was President of the Royal Medical Society (1878), President of the Royal Microscopical Society (1913-16), and founder and conductor for many years of the *Journal of Pathology and Bacteriology*. In 1895 he drew up a report to the Royal Commission on Tuberculosis, and was a member of that commission of 1902.

During the war Professor Woodhead was appointed Inspector of Government Laboratories in the Military Hospitals in the United Kingdom, a post which involved much travelling and discomfort. Within this period he devised a method for the chlorination of drinking water for the troops. In 1919, in recognition of much valuable work, he was created K.B.E.

He contributed papers to several medical journals, and was the author of the following works:—

1883. *Practical Pathology*, which reached its fourth edition in 1910.

1885. *Pathological Mycology* (with A. W. Hare).

1891. *Bacteria and their Products.*

1894. Published with Dr Cartwright Wood *An Investigation on the Efficiency of Domestic Water Filters.*

1894. *Village Settlements for the Tuberculous* (with P. C. Jones Varrier).

1895. *Report on Diphtheria to the Metropolitan Asylums Board.*

Professor Woodhead was elected a Fellow of this Society in 1886, served on the Council from 1887-90, and published several papers in the *Proceedings* between the years 1884 and 1889. He died on 29th December 1921.

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PROCEEDINGS OF THE STATUTORY GENERAL MEETING

Beginning the 139th Session, 1921-1922.

At the Statutory Meeting of the Royal Society of Edinburgh, held in the Society's Lecture Room, 24 George Street, on Monday, October 24, 1921, at 4.30 p.m.,

Professor F. O. BOWER, F.R.S., President, in the Chair,

the Minutes of the last Statutory Meeting of October 25, 1920, were read, approved, and signed.

The CHAIRMAN nominated as Scrutineers Mr C. H. MILNE and Dr J. B. RITCHIE.

The Ballot for the Election of Office-Bearers and Members of Council was then taken.

The SECRETARY submitted the following Report:—

The number of papers read at our meetings during the last Session was 35, as compared with 27 the previous Session. Of these, 3 were addresses: the first by the PRESIDENT on "Size a Neglected Factor in the Internal Morphology of Plants"; the second by Lt.-Col. LISTON on "Plague and Rats"; and the third by Mr C. T. R. WILSON on "Some Recent Work on Lightning and Thunderstorms." Of the others, 5 were in mathematics, 1 in statistics, 9 in physics, 2 in chemistry, 5 in zoology, 2 in botany, 5 in geology, 2 in oceanography, and 1 in engineering. Twenty-eight of these have been, or are being, published—11 in the *Transactions* and 17 in the *Proceedings*.

Last Session the Society elected 25 new Ordinary Fellows, and 9 Honorary Fellows—4 British and 5 Foreign. Three Fellows have resigned, and we have lost by death 4 Honorary Fellows and 22 Ordinary Fellows.

The Makdougall-Brisbane Prize was awarded to Professor J. H. MACLAGAN WEDDERBURN of Princeton, U.S.A.; and the Gunning Victoria Jubilee Prize to Mr C. T. R. WILSON of Cambridge.

The vacancy in the post of Assistant Librarian, occasioned by the resignation of Miss LE HARIVEL, has been filled up by the appointment of Mr ERNST M. STEWART.

The high cost of printing and publication still continues to hamper the work of the Society. To meet to some extent the increased expense, the Council asked the Society to amend Law VI and raise the annual subscription by one guinea. This was agreed to, and the amendment of Law VI was duly carried out at the Meeting of February 7, 1921.

The Grant of £3000 from the Carnegie United Kingdom Trust for binding the journals in the Library is now nearly all expended; and again the Society takes the opportunity of expressing its thanks to the Trust for their generous help.

The Conjoint Board of Scientific Societies has prepared an important report on the question of aid to Scientific Societies for publication of papers, and our representatives have been instructed to look specially to our interests in any further developments which may occur.

One of the great events of the year was the Meeting of the British Association for the Advancement of Science. This our Society helped by a donation of £100 towards the local expenses and by inviting the Members to a free use of the Library and Reading-room. The Secretaries also organised, by request of the Council, a Royal Societies dinner, at which our Society entertained Fellows of the Royal Society of London, Members of the Royal Irish Academy, and distinguished Foreign visitors who were members of their National Academies. The dinner was held in the Freemasons' Hall on Monday, September 12, when our President, with the Maharaj Rana of Jhalawar on his right hand, presided over a gathering of one hundred and eighty-one scientific men from many lands.

During the summer, and especially in view of the British Association Meeting, the Council arranged for cleaning and painting the west staircase and lecture room, and laying new matting on the stairs.

As will be seen from the Treasurer's Report, the financial condition of the Society is fairly satisfactory, notwithstanding these exceptional but necessary outlays during the Session.

The TREASURER in submitting his Report for the year compared the Income and Expenditure with those of the previous year.

Dr R. KIDSTON, F.R.S., moved the adoption of the Reports and the reappointment of Messrs LINDSAY, JAMIESON & HALDANE, C.A., as auditors of the accounts for the ensuing Session.

This was unanimously agreed to.

The Scrutineers reported that the Ballot Papers were in order, and that the following Office-Bearers and Members of Council had been elected :—

Professor FREDERICK O. BOWER, M.A., D.Sc., LL.D., F.R.S., F.L.S., President.	
Sir GEORGE A. BERRY, M.B., C.M., LL.D., F.R.C.S.E.,	} Vice-Presidents.
Professor WILLIAM PEDDIE, D.Sc.,	
Principal Sir JAMES ALFRED EWING, K.C.B., M.A., D.Sc.,	
LL.D., M.Inst.C.E., F.R.S.,	
Professor JOHN WALTER GREGORY, D.Sc., F.R.S.,	
Major-General W. B. BANNERMAN, C.S.I., I.M.S., M.D.,	} Secretaries to Ordinary Meetings.
D.Sc.,	
W. A. TAIT, D.Sc., M.Inst.C.E.,	
CARGILL G. KNOTT, D.Sc., LL.D., F.R.S., General Secretary.	
Professor E. T. WHITTAKER, Sc.D., F.R.S.,	
Professor J. H. ASHWORTH, D.Sc., F.R.S.,	
JAMES CURRIE, M.A., LL.D., Treasurer.	
A. CRICHTON MITCHELL, D.Sc., Hon. D.Sc. (Geneva), Curator of Library and Museum.	

ORDINARY MEMBERS OF COUNCIL.

HENRY MOUBRAY CADELL, of Grange, B.Sc.	THE HON. LORD SALVESEN.
Professor ARTHUR ROBERTSON CUSHNY, M.A.,	Professor J. ARTHUR THOMSON, M.A., LL.D.
M.D., LL.D., F.R.S.	HERBERT STANLEY ALLEN, M.A., D.Sc.
Professor FRANCIS GIBSON BAILY, M.A.,	Sir ROBERT BLYTH GREIG, M.A., LL.D.,
M.Inst.E.E.	F.Z.S.
GEORGE JAMES LIDSTONE, F.F.A., F.I.A.	JAMES RITCHIE, M.A., D.Sc.
ROBERT CAMPBELL, M.A., D.Sc., F.G.S.	ERNEST MACLAGAN WEDDERBURN, M.A., LL.B.,
Principal JAMES COLQUHOUN IRVINE, C.B.E.,	W.S., D.Sc.
Ph.D., D.Sc., LL.D., F.R.S.	

SOCIETY'S REPRESENTATIVE ON GEORGE HERIOT'S TRUST.

W. A. TAIT, D.Sc., M.Inst.C.E.

The CHAIRMAN, in the name of the Society, thanked the Scrutineers for their services.

PUBLIC BUSINESS.

An Obituary Notice of Robert Munro, M.A., M.D., LL.D., by Dr GEORGE MACDONALD, C.B., was read by Dr C. G. KNOTT, General Secretary. *Proc.*, vol. xli, pp. 158-169.

The PRESIDENT summarised the position of the Society on the working of the year just closed, as regards the volume of work presented for publication and the showings of the Balance Sheet. He pointed out that the volume of work was at present considerably below the average of pre-war years, and forecasted the strong probability that in the near future, when the immediate effects of the war had been worked off, an even larger volume of work would be offered than the average of the years before the war.

He drew attention to the importance of keeping in close touch with the younger men, who are at the outset of their productive period: and he suggested as being worthy of consideration by the Society, the possible constitution of a junior grade of membership to which the younger men might belong. It might stand to the full Fellowship in somewhat the same relation as in the Royal Academy the A. R. A. stands to the rank of full Academician.

**PROCEEDINGS OF THE ORDINARY MEETINGS,
Session 1921-1922.**

FIRST ORDINARY MEETING.

Monday, November 7, 1921.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

The following Communications were submitted :—

1. Obituary Notice of Dr J. G. Bartholomew By GEORGE G. CHISHOLM, M.A., B.Sc. *Proc.*, vol. xli, pp. 170-176.
2. On the Phenomenon of the "Radiant Spectrum" observed by Sir David Brewster. By Professor C. V. RAMAN, M.A. Communicated by the GENERAL SECRETARY. *Proc.*, vol. xlii, pp. 24-25.
3. Prehensility, a Factor of Gaseous Adsorption. By Professor HENRY BRIGGS, D.Sc. *Proc.*, vol. xlii, pp. 26-33.
4. The Military Physical Test Station, Edinburgh. By Professor HENRY BRIGGS, D.Sc. *Proc.*, vol. xlii, pp. 34-42.
5. Tubes of Electromagnetic Force. By Professor E. T. WHITTAKER, F.R.S. *Proc.*, vol. xlii, pp. 1-23.

SECOND ORDINARY MEETING.

Monday, December 5, 1921.

Professor J. W. Gregory, D.Sc., F.R.S., Vice-President, in the Chair.

The following Communications were submitted :—

1. Observations on the Geology of Iona. By Professor T. J. JERU, M.A., M.D., F.G.S. *Trans.*, vol. liii, pp. 165-187.
2. The Geology of the Blackness District as disclosed by recent Borings. By H. M. CADELL, Esq., B.Sc. *Trans.*, vol. liii, pp. 189-207.

THIRD ORDINARY MEETING.

Monday, January 9, 1922.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

The following Communications were submitted :—

1. Obituary Notices of Fellows. Communicated by the GENERAL SECRETARY. *Proc.*, vol. xli, pp. 197-207.
2. On a New Method of investigating Colour Blindness, with a Description of Twenty-three Cases. By Dr R. A. HOUSTOUN, M.A. *Proc.*, vol. xlii, pp. 75-88.
3. On the Faraday Tube Theory of Electromagnetism, and other Notes. By the late W. GORDON BROWN. With Introductory Note by Dr C. G. KNOTT, F.R.S. *Proc.*, vol. xlii, pp. 225-246.
4. Some Simple Experiments on the Colloidal Content of Soils. By T. BEDFORD FRANKLIN, B.A.

FOURTH ORDINARY MEETING.

Monday, January 23, 1922.

Sir George A. Berry, M.B., C.M., LL.D., F.R.C.S.E., Vice-President, in the Chair.

The following Communications were submitted :—

1. Obituary Notice of Sir Thomas R. Fraser, M.D., LL.D., Sc.D., F.R.S. By HARRY RAINY, M.A., M.D., F.R.C.P.Ed. *Proc.*, vol. xli, pp. 186-192.
2. On the Connexion between Linear Differential Systems and Integral Equations. By E. L. INCE, D.Sc. Communicated by Professor E. T. WHITTAKER, F.R.S. *Proc.* vol. xlii, pp. 43-53.

FIFTH ORDINARY MEETING.

Monday, February 6, 1922.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

The following Communications were submitted :—

1. The Floral Structure of *Napoleona imperialis*. By Professor J. M'LEAN THOMPSON, M.A., D.Sc., F.L.S. *Trans.*, vol. liii, pp. 265-275.
2. The Pre-Devonian Basement Complex of Central Spitsbergen. By G. W. TYRRELL, A.R.C.Sc., F.G.S. *Trans.*, vol. liii, pp. 209-229.

SIXTH ORDINARY MEETING.

Monday, February 20, 1922.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

The following Communications were submitted :—

1. On Models of Ferromagnetic Induction. By Principal Sir J. A. EWING, K.C.B., M.A., LL.D., D.Sc., F.R.S. *Proc.*, vol. xlii, pp. 97-128.
2. Elliptic Expansions. By Mr R. T. A. INNES. *Proc.*, vol. xlii, p. 318.

SEVENTH ORDINARY MEETING.

Monday, March 6, 1922.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

The Ballot for the election of Fellows was taken. Professor HUDSON BEARE and Mr J. CAMPBELL DEVAR being nominated as Scrutineers. The following Candidates, who had been recommended for election by the Council, were duly elected :—CHARLES LAWRENCE ABERNETHY, GEORGE BARGER, Sir DUGALD CLERK, FRANCIS ALBERT ELEY CREW, WILLIAM OSBORNE GREENWOOD, WILLIAM ALEXANDER GUTHRIE, ROBERT KERR HANNAY, EDWARD HINDLE, CHARLES FREDERICK JURIUZ, JONATHAN CAMPBELL MEAKINS, MURRAY MACGREGOR, BIJALI BEHARI SARKAR, HERBERT WESTREN TURNBULL, JAMES WALKER, JOHN WILSON, and JAMES MANN WORDIE.

The following Communications were submitted :—

1. Solutions of the Problem of the Vertical on Moving Vehicles, with special reference to Aircraft. The Gray Gyroscopic Stabilisers. By Professor J. G. GRAY, D.Sc., and Captain J. GRAY, M.I.E.E. (late R.A.F.). *Proc.*, vol. xlii, pp. 257-317.
2. Myriapods collected in Mesopotamia and N.W. Persia by W. Edgar Evans, B.Sc., late Capt. R.A.M.C. By M. HENRI W. BROLEMANN. Communicated by WM. EVANS, F.F.A. *Proc.*, vol. xlii, pp. 54-74.

EIGHTH ORDINARY MEETING.

Monday, March 20, 1922.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

The following recently elected Fellows were admitted and signed the Roll :—Professor G. BARGER, Dr F. A. E. CREW, Professor R. K. HANNAY, Professor J. MEAKINS, Professor H. W. TURNBULL.

The following Communication was submitted :—

Obituary Notice of Dr William Spiers Bruce, Arctic and Antarctic Explorer. By Dr R. N. RUDMOSE BROWN and Dr JAMES RITCHIE. *Proc.*, vol. xlii, pp. 362-368.

By request of the Council, Professor Sir CHARLES SCOTT SHERRINGTON, G.B.E., M.A., M.D., Sc.D., LL.D., President of the Royal Society of London, gave an Address on "Some Points Regarding Present-day Views of Reflex Action."

NINTH ORDINARY MEETING.

Monday, May 8, 1922.

Professor W. Peddie, D.Sc., Vice-President, in the Chair.

The following Communications were submitted:—

1. On the Quantum Mechanism in the Atom. By Professor E. T. WHITTAKER, F.R.S., which was followed by a "Discussion on Quantum Theory and Atomic Structure," in which Sir Alfred Ewing, Dr H. S. Allen, Dr R. A. Houstoun, Professor Peddie, and others took part. *Proc.*, vol. xlii, pp. 129-142.

1a. Note on Professor Whittaker's paper "On the Quantum Mechanism in the Atom." By Sir J. ALFRED EWING, K.C.B., F.R.S. *Proc.*, vol. xlii, pp. 143-146.

1b. The Magnetic Characters of the Quantum. By H. STANLEY ALLEN, M.A., D.Sc. *Proc.*, xlii, pp. 213-220.

1c. Note on Professor Whittaker's "Quantum Mechanism in the Atom." By Dr R. A. HOUSTOUN. *Proc.*, vol. xlii, pp. 221-222.

1d. The Importance of Professor Whittaker's Atomic Model and other Atomic Models. By Professor W. PEDDIE, D.Sc. *Proc.*, vol. xlii, pp. 223-224.

2. The Concomitants (including Differential Invariants) of Quadratic Differential Forms in Four Variables. By Professor A. R. FORSYTH, F.R.S. *Proc.*, vol. xlii, pp. 147-212.

3. The Asymptotic Expansion of the Confluent Hypergeometric Function, and a Fourier-Bessel Expansion. By Dr T. M. MACROBERT. *Proc.*, vol. xlii, pp. 89-96.

The Chairman intimated the awards of the Keith, Neill, and James Scott Prizes as under:—

The Council have awarded the Keith Prize (1919-1921) to Professor R. A. SAMPSON, F.R.S., for his Astronomical Researches, including the papers, "Studies in Clocks and Time-keeping: No. 1, Theory of the Maintenance of Motion; No. 2, Tables of the Circular Equation," published in the *Proceedings* of the Society within the period of the award.

The Neill Prize (1919-1921) has been awarded to Sir EDWARD SHARPEY SCHAFER, F.R.S., for his recent contributions to our knowledge of Physiology, and in recognition of his published work, extending over a period of fifty years.

The James Scott Prize (first award) "for a lecture or essay on the Fundamental Concepts of Natural Philosophy" will be presented to Professor A. N. WHITEHEAD, F.R.S., at the Ordinary Meeting on 5th June, when he will deliver a lecture on the above subject.

TENTH ORDINARY MEETING.

Monday, May 15, 1922.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

By request of the Council, Professor JONATHAN CAMPBELL MEAKINS, M.D., F.R.C.P.Ed., gave an Address on "The Royal Society (London) Physiological Expedition to the Andes."

A vote of thanks to Professor MEAKINS was proposed by Professor LORRAIN SMITH, supported by the President, and carried unanimously.

The following Communication was presented:—

The Quantum Theory of Secondary Spectra. By JOHN MARSHALL, M.A., D.Sc. *Proc.*, vol. xlii, pp. 247-256.

ELEVENTH ORDINARY MEETING.

Monday, June 5, 1922.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

Mr CHARLES L. ABERNETHY signed the Roll, and was admitted a Fellow of the Society.

By request of the Council, Professor ALFRED NORTH WHITEHEAD, Sc.D., F.R.S., gave an Address on "The Relatedness of Nature."

The James Scott Prize (first award) "for a lecture or essay on the Fundamental Concepts of Natural Philosophy" was presented to Professor WHITEHEAD.

A vote of thanks to Professor WHITEHEAD was proposed by Professor SAMPSON, seconded by Professor KEMP SMITH, and carried unanimously.

TWELFTH ORDINARY MEETING.

Monday, June 19, 1922.

Professor Frederick O. Bower, M.A., D.Sc., LL.D., F.R.S., F.L.S., President, in the Chair.

The KEITH and NEILL PRIZES for the period 1919-1921 were presented. On making the presentation the Chairman read the following notes on the work of the recipients.

The Council has awarded the KEITH PRIZE for the period 1919-1921 to Professor RALPH ALLEN SAMPSON, F.R.S., Astronomer Royal for Scotland, for his Astronomical Researches, including two papers, "Studies in Clocks and Time-keeping: No. 1, Theory of the Maintenance of Motion; No. 2, Tables of the Circular Equation," published in the *Proceedings* of the Society within the period of award.

These two papers belong to a long series in which Professor SAMPSON has from time to time effected improvements in Instrumental Astronomy and the Science of Astronomical Observation. More than twenty years ago his work on the Durham Almucantar was recognised as a marked advance, and since then his papers in Astronomical Optics and on the Measurement of Time have greatly developed this side of his subject.

Perhaps Professor SAMPSON's best known work is the "Tables of the Four Great Satellites of Jupiter," published in 1909, on which all ephemerides are now based. The precision of these tables, which corresponds to one-tenth of a second of time in the phenomena, is much greater than that of any previous tables. In recognition of their value Professor SAMPSON has been elected "Membre correspondant du Bureau des Longitudes," and the Royal Society of Edinburgh is glad to take this opportunity afforded by Professor SAMPSON's absence from the Council this year in order to express its sense of the merits of work which has long been appreciated abroad.

The Council has awarded the NEILL PRIZE to Sir EDWARD SHARPEY SCHAFER, F.R.S., in recognition of his contributions to the science of Physiology. Sir EDWARD's work, extending over a period of fifty years, constitutes a remarkable record of his many-sided investigations, and he continues as actively as ever, by skilfully devised experiment and penetrative interpretation, to advance our knowledge of the structure and functions of the animal body.

Beginning with studies on the coagulation of the blood, the first results of which were printed as a note in 1872, he passed on to investigate the structure of protoplasm, muscle, bone, liver, and blood corpuscles, the absorption of fat in the intestine, the proteins of the blood, ciliary movement, milk secretion, and other physiological problems.

Sir EDWARD is known everywhere for his masterly work on the structure and physiology of the nervous system. But above all, his name will ever be associated with his discovery of the nature and functions of the ductless glands, which opened a new vista in Physiology and in Medicine.

By his original memoirs and his admirable systematic works, his teaching, and his stimulating influence on younger workers, Sir EDWARD has added greatly to the prestige of British physiology, and the Council is glad to have the opportunity of marking its warm recognition of his distinguished services to Science.

The following Communications were submitted:—

1. On the Septal and Pharyngeal Glands of the Microdrili (Oligochæta). By Dr JOHN STEPHENSON. *Trans.*, vol. liii, pp. 277-295.
2. On Self Light, Fatigue, Inhibition, and Recurrent Visual Images. By Professor W. PEDDIE, D.Sc. *Proc.*, vol. xlii, pp. 319-320.
3. Chemical Combination and Sir Alfred Ewing's Magnetic Atom. By Principal A. P. LAURIE, D.Sc. *Proc.*, vol. xlii, pp. 352-361.
4. On the Dominance Ratio. By Mr R. A. FISHER. Communicated by Professor J. ARTHUR THOMSON. *Proc.*, vol. xlii, pp. 321-341.
5. On the Question of Priority of Invention of the Leader System for Ships. By the GENERAL SECRETARY. *Proc.*, vol. xlii, pp. 348-351.
6. Note on a Theorem of Frobenius' connected with Invariant-Factors. By Sir THOMAS MUIR, LL.D., F.R.S. *Proc.*, vol. xlii, pp. 342-347.

PROCEEDINGS OF THE STATUTORY GENERAL MEETING
Ending the 139th Session, 1921-1922.

At the Statutory Meeting of the Royal Society of Edinburgh, held in the Society's Lecture Room, 24 George Street, on Monday, October 23, 1922, at 4.30 p.m.,

Professor Frederick O. Bower, F.R.S., President, in the Chair,

the Minutes of the last Statutory Meeting, October 24, 1921, were read, approved, and signed.

The CHAIRMAN nominated as Scrutineers Dr LAUDER and Dr CARSE.

The Ballot for the Election of Office-Bearers and Members of Council was then taken.

The SECRETARY submitted the following Report:—

The number of papers read at our Meetings during the last Session was 33, as compared with 35 the previous Session. Of the number, 3 were addresses: one by Professor Sir C. S. SHERRINGTON, C.B.E., F.R.S., President of the Royal Society of London, on "Some Points regarding Present-day Views of Reflex Action," the second by Professor J. C. MEAKINS on "The Royal Society (London) Physiological Expedition to the Andes," and the third by Professor A. N. WHITEHEAD, F.R.S., on "The Relatedness of Nature." Of the other papers, there were 5 in pure mathematics, 4 in applied mathematics, 1 in statistics, 9 in physics, 3 in chemistry, 2 in zoology, 1 in botany, 3 in geology, and 1 in physiology. The remaining paper was an account of a correspondence with the French Academy of Sciences regarding the origination of the Leader System for Ships. Twenty-nine of these papers have been, or are being, published—24, in the *Proceedings*, and 5 in the *Transactions*.

The recognition by the scientific world of the great value of some of these papers is indicated by the increase in our sales of *Proceedings* and *Transactions*, as shown in the Treasurer's Statement of Accounts.

Last Session the Society elected 16 new Fellows. One Fellow has resigned, and we have lost by death 3 Honorary Fellows and 17 Ordinary Fellows.

The Keith Prize was awarded to Professor R. A. SAMPSON, F.R.S., the Neill Prize to Sir EDWARD SHARPEY SCHAFFER, F.R.S., and the James Scott Prize (first award) to Professor A. N. WHITEHEAD, F.R.S.

It is satisfactory to know that the cost of publication is steadily going down, having decreased by about 25 per cent. during the year. It is still, however, higher than before the War; and the Council have found it impossible to keep expenses within the limits of the Society's normal income. Fortunately there still remained a large balance of the Special Subscription Fund so generously contributed to by our Fellows a few years ago; and the deficit on the year's working has been balanced by a transfer from this special fund. In this connection, the Society desires to express its grateful thanks to the Carnegie Trust of the Scottish Universities, to the Moray Fund Trustees, to the Royal Society of London, and to Professor J. W. Gregory for their generous grants towards our expenses of publication.

The grant of £3000 from the Carnegie United Kingdom Trust to aid in binding the loose journals in our Library has now been expended; and the Society again takes the opportunity of expressing its thanks for this welcome and timely help.

During the six months from February to July the Council arranged to have the Society's House open till 6.30 instead of 5 in the afternoon on Mondays, Wednesdays, and Fridays. The number of Fellows who took advantage of this extension was exceedingly small, and the Council does not feel encouraged to repeat the experiment.

The question of the institution of a lower grade of membership was discussed by a committee specially appointed for the purpose and by the Council as a whole. It was resolved to take no action in the meantime.

By means of chosen representatives our Society took part in various Celebrations and Congresses during the Session. Sir GEORGE BERRY acted as our delegate to the celebration of the 150th anniversary of the foundation of the Academy of Sciences, Literature, and Art of Belgium, the Latin Address which was presented having been kindly drawn up by Professor MAIR of the University of Edinburgh. Also, the PRESIDENT represented our Society at the Centenary Celebration of the Yorkshire Philosophical Society; and Professor JEHU was our delegate to the International Geological Congress at Brussels.

The sum of £300 placed in the hands of the Council during the War to expend on scientific experiments for War purposes has now been fully expended, the balance of fully £75 (including interest) having been with the donor's consent used to meet the expenses of publication of the paper by Professor J. C. GRAY and Captain J. GRAY on Gyrostatic Stabilisers in Aeroplanes.

The attention of the Fellows is drawn to the rearrangement of the portraits in the Reception Room and the East Staircase. The portraits of Sir Walter Scott and Sir David Brewster have been placed in the Reception Room along with the others which were previously there.

The TREASURER, in submitting his Report for the year, compared the Income and Expenditure with those of the previous year, and stated that the deficit on the year's working (£408, 11s. 1½d.), largely due to an increase in the amount published, was met from the Special Subscription Fund.

Sir EDWARD SHARPEY SCHAFER moved the adoption of the Reports, and the reappointment of Messrs LINDSAY, JAMIESON & HALDANE, C.A., as auditors of the accounts for the ensuing Session. This was unanimously agreed to.

The Scrutineers reported that the Ballot Papers were in order, and that the following Office-Bearers and Members of Council had been elected:—

Professor FREDERICK O. BOWER, M.A., D.Sc., LL.D., F.R.S., F.L.S., President.	
Principal Sir JAMES ALFRED EWING, K.C.B., M.A., D.Sc., LL.D., M.Inst.C.E., F.R.S.,	} Vice-Presidents.
Professor JOHN WALTER GREGORY, D.Sc., F.R.S.,	
Major-General W. B. BANNERMAN, C.S.I., I.M.S., M.D., D.Sc.,	
W. A. TAIT, D.Sc., M.Inst.C.E.,	
Principal J. C. IRVINE, C.B.E., Ph.D., D.Sc., LL.D., F.R.S.,	
The Rt. Hon. LORD SALVESEN, P.C., K.C.,	
CARGILL G. KNOTT, D.Sc., LL.D., F.R.S., General Secretary.	
Professor J. H. ASHWORTH, D.Sc., F.R.S.,	} Secretaries to Ordinary Meetings.
Professor R. A. SAMPSON, M.A., D.Sc., F.R.S.	
JAMES CURRIÉ, M.A., LL.D., Treasurer.	
A. CRICHTON MITCHELL, D.Sc., Hon. D.Sc. (Geneva), Curator of Library and Museum.	

ORDINARY MEMBERS OF COUNCIL.

Professor FRANCIS GIBSON BAILY, M.A., M.Inst.E.E.	Professor THOMAS H. BRYCE, M.A., M.D., F.R.S.
ROBERT CAMPBELL, M.A., D.Sc., F.G.S.	Professor JAMES YOUNG SIMPSON, M.A., D.Sc.
Professor J. ARTHUR THOMSON, M.A., LL.D.	Professor D'ARCY W. THOMPSON, C.B., D.Litt., F.R.S.
HERBERT STANLEY ALLEN, M.A., D.Sc.	Professor Sir JAMES WALKER, D.Sc., Ph.D., LL.D., F.R.S.
Sir ROBERT BLYTH GREIG, M.C., LL.D., F.Z.S.	Professor EDMUND TAYLOR WHITTAKER, Sc.D., F.R.S.
JAMES RITCHIE, M.A., D.Sc.	
Professor ERNEST MACLAGAN WEDDERBURN, M.A., LL.B., W.S., D.Sc.	

SOCIETY'S REPRESENTATIVE ON GEORGE HERIOT'S TRUST.

W. A. TAIT, D.Sc., M.Inst.C.E.

The CHAIRMAN, in the name of the Society, thanked the Scrutineers for their services.

The PRESIDENT then delivered an Address "On the Primitive Spindle as a Fundamental Feature in the Embryology of Plants."

THE KEITH, MAKDOUGALL-BRISBANE, NEILL, GUNNING
VICTORIA JUBILEE, AND JAMES SCOTT PRIZES.

The above Prizes will be awarded by the Council in the following manner:—

I. KEITH PRIZE.

The KEITH PRIZE, consisting of a Gold Medal and from £40 to £50 in Money, will be awarded in the Session 1923–1924 for the “best communication on a scientific subject, communicated,* in the first instance, to the Royal Society of Edinburgh during the Sessions 1921–1922 and 1922–1923.” Preference will be given to a paper containing a discovery.

II. MAKDOUGALL-BRISBANE PRIZE.

This Prize is to be awarded biennially by the Council of the Royal Society of Edinburgh to such person, for such purposes, for such objects, and in such manner as shall appear to them the most conducive to the promotion of the interests of science; with the *proviso* that the Council shall not be compelled to award the Prize unless there shall be some individual engaged in scientific pursuit, or some paper written on a scientific subject, or some discovery in science made during the biennial period, of sufficient merit or importance in the opinion of the Council to be entitled to the Prize.

1. The Prize, consisting of a Gold Medal and a sum of Money, will be awarded before the close of the Session 1924–1925, for an Essay or Paper having reference to any branch of scientific inquiry, whether Material or Mental.

2. Competing Essays to be addressed to the Secretary of the Society, and transmitted not later than 8th July 1924.

3. The Competition is open to all men of science.

4. The Essays may be either anonymous or otherwise. In the former case, they must be distinguished by mottoes, with corresponding sealed billets, superscribed with the same motto, and containing the name of the Author.

5. The Council impose no restriction as to the length of the Essays, which may be, at the discretion of the Council, read at the Ordinary Meetings of the Society. They wish also to leave the property and free disposal of the manuscripts to the Authors; a copy, however, being deposited in the Archives of the Society, unless the paper shall be published in the Transactions.

* For the purposes of this award the word “communicated” shall be understood to mean the date on which the manuscript of a paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

6. In awarding the Prize, the Council will also take into consideration any scientific papers presented * to the Society during the Sessions 1920-21, 1921-22, whether they may have been given in with a view to the prize or not.

III. NEILL PRIZE.

The Council of the Royal Society of Edinburgh having received the bequest of the late Dr PATRICK NEILL of the sum of £500, for the purpose of "the interest thereof being applied in furnishing a Medal or other reward every second or third year to any distinguished Scottish Naturalist, according as such Medal or reward shall be voted by the Council of the said Society," hereby intimate :

1. The NEILL PRIZE, consisting of a Gold Medal and a sum of Money, will be awarded during the Session 1923-1924.

2. The Prize will be given for a Paper of distinguished merit, on a subject of Natural History, by a Scottish Naturalist, which shall have been presented * to the Society during the two years preceding the fourth Monday in October 1923,—or failing presentation of a paper sufficiently meritorious, it will be awarded for a work or publication by some distinguished Scottish Naturalist, on some branch of Natural History, bearing date within five years of the time of award.

IV. GUNNING VICTORIA JUBILEE PRIZE.

This Prize, founded in the year 1887 by Dr R. H. GUNNING, is to be awarded quadrennially by the Council of the Royal Society of Edinburgh, in recognition of original work in Physics, Chemistry, or Pure or Applied Mathematics.

Evidence of such work may be afforded either by a Paper presented to the Society, or by a Paper on one of the above subjects, or some discovery in them elsewhere communicated or made, which the Council may consider to be deserving of the Prize.

The Prize consists of a sum of money, and is open to men of science resident in or connected with Scotland. The first award was made in the year 1887. The next award will be made in 1924-1925.

In accordance with the wish of the Donor, the Council of the Society may on fit occasions award the Prize for work of a definite kind to be undertaken during the three succeeding years by a scientific man of recognised ability.

V. JAMES SCOTT PRIZE.

This Prize, founded in the year 1918 by the Trustees of the JAMES SCOTT Bequest, is to be awarded triennially, or at such intervals as the Council of the Royal Society of Edinburgh may decide, "for a lecture or essay on the fundamental concepts of Natural Philosophy."

* For the purposes of this award the word "presented" shall be understood to mean the date on which the manuscript of a paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

RESOLUTIONS OF COUNCIL IN REGARD TO THE MODE
OF AWARDING PRIZES.

(See *Minutes of Meeting of January 18, 1915.*)

I. With regard to the Keith and Makdougall-Brisbane Prizes, which are open to all Sciences, the mode of award will be as follows :—

1. Papers or essays to be considered shall be arranged in two groups, A and B, —Group A to include Astronomy, Chemistry, Mathematics, Metallurgy, Meteorology and Physics; Group B to include Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, and Zoology.
2. These two Prizes shall be awarded to each group in alternate biennial periods, provided papers worthy of recommendation have been communicated to the Society.
3. Prior to the adjudication the Council shall appoint, in the first instance, a Committee composed of representatives of the group of Sciences which did not receive the award in the immediately preceding period. The Committee shall consider the Papers which come within their group of Sciences, and report in due course to the Council.
4. In the event of the aforesaid Committee reporting that within their group of subjects there is, in their opinion, no paper worthy of being recommended for the award, the Council, on accepting this report, shall appoint a Committee representative of the alternate group to consider papers coming within their group and to report accordingly.
5. Papers to be considered by the Committees shall fall within the period dating from the last award in groups A and B respectively.

II. With regard to the Neill Prize, the term “Naturalist” shall be understood to include any student in the Sciences composing group B, namely, Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, Zoology.

**AWARDS OF THE KEITH, MAKDOUGALL - BRISBANE,
NEILL, GUNNING, AND JAMES SCOTT PRIZES.**

I. KEITH PRIZE.

- 1ST BIENNIAL PERIOD, 1827-29.—Dr BREWSTER, for his papers “on his Discovery of Two New Immiscible Fluids in the Cavities of certain Minerals,” published in the Transactions of the Society.
- 2ND BIENNIAL PERIOD, 1829-31.—Dr BREWSTER, for his paper “on a New Analysis of Solar Light,” published in the Transactions of the Society.
- 3RD BIENNIAL PERIOD, 1831-33.—THOMAS GRAHAM, Esq., for his paper “on the Law of the Diffusion of Gases,” published in the Transactions of the Society.
- 4TH BIENNIAL PERIOD, 1833-35.—Professor J. D. FORBES, for his paper “on the Refraction and Polarization of Heat,” published in the Transactions of the Society.
- 5TH BIENNIAL PERIOD, 1835-37.—JOHN SCOTT RUSSELL, Esq., for his researches “on Hydrodynamics,” published in the Transactions of the Society.
- 6TH BIENNIAL PERIOD, 1837-39.—Mr JOHN SHAW, for his experiments “on the Development and Growth of the Salmon,” published in the Transactions of the Society.
- 7TH BIENNIAL PERIOD, 1839-41.—Not awarded.
- 8TH BIENNIAL PERIOD, 1841-43.—Professor JAMES DAVID FORBES, for his papers “on Glaciers,” published in the Proceedings of the Society.
- 9TH BIENNIAL PERIOD, 1843-45.—Not awarded.
- 10TH BIENNIAL PERIOD, 1845-47.—General Sir THOMAS BRISBANE, Bart., for the Makerstoun Observations on Magnetic Phenomena, made at his expense, and published in the Transactions of the Society.
- 11TH BIENNIAL PERIOD, 1847-49.—Not awarded.
- 12TH BIENNIAL PERIOD, 1849-51.—Professor KELLAND, for his papers “on General Differentiation, including his more recent Communication on a process of the Differential Calculus, and its application to the solution of certain Differential Equations,” published in the Transactions of the Society.
- 13TH BIENNIAL PERIOD, 1851-53.—W. J. MACQUORN RANKINE, Esq., for his series of papers “on the Mechanical Action of Heat,” published in the Transactions of the Society.
- 14TH BIENNIAL PERIOD, 1853-55.—Dr THOMAS ANDERSON, for his papers “on the Crystalline Constituents of Opium, and on the Products of the Destructive Distillation of Animal Substances,” published in the Transactions of the Society.
- 15TH BIENNIAL PERIOD, 1855-57.—Professor BOOLE, for his Memoir “on the Application of the Theory of Probabilities to Questions of the Combination of Testimonies and Judgments,” published in the Transactions of the Society.
- 16TH BIENNIAL PERIOD, 1857-59.—Not awarded.
- 17TH BIENNIAL PERIOD, 1859-61.—JOHN ALLAN BROWN, Esq., F.R.S., Director of the Trevandrum Observatory, for his papers “on the Horizontal Force of the Earth’s Magnetism, on the Correction of the Bifilar Magnetometer, and on Terrestrial Magnetism generally,” published in the Transactions of the Society.
- 18TH BIENNIAL PERIOD, 1861-63.—Professor WILLIAM THOMSON, of the University of Glasgow, for his Communication “on some Kinematical and Dynamical Theorems.”
- 19TH BIENNIAL PERIOD, 1863-65.—Principal FORBES, St Andrews, for his “Experimental Inquiry into the Laws of Conduction of Heat in Iron Bars,” published in the Transactions of the Society.
- 20TH BIENNIAL PERIOD, 1865-67.—Professor C. PIAZZI SMYTH, for his paper “on Recent Measures at the Great Pyramid,” published in the Transactions of the Society.
- 21ST BIENNIAL PERIOD, 1867-69.—Professor P. G. TAIT, for his paper “on the Rotation of a Rigid Body about a Fixed Point,” published in the Transactions of the Society.
- 22ND BIENNIAL PERIOD, 1869-71.—Professor CLERK MAXWELL, for his paper “on Figures, Frames, and Diagrams of Forces,” published in the Transactions of the Society.

- 23RD BIENNIAL PERIOD, 1871-73.—Professor P. G. TAIT, for his paper entitled “First Approximation to a Thermo-electric Diagram,” published in the Transactions of the Society.
- 24TH BIENNIAL PERIOD, 1873-1875.—Professor CRUM BROWN, for his Researches “on the Sense of Rotation, and on the Anatomical Relations of the Semicircular Canals of the Internal Ear.”
- 25TH BIENNIAL PERIOD, 1875-77.—Professor M. FORSTER HEDDLE, for his papers “on the Rhombohedral Carbonates,” and “on the Felspars of Scotland,” published in the Transactions of the Society.
- 26TH BIENNIAL PERIOD, 1877-79.—Professor H. C. FLEMING JENKIN, for his paper “on the Application of Graphic Methods to the Determination of the Efficiency of Machinery,” published in the Transactions of the Society; Part II having appeared in the volume for 1877-78.
- 27TH BIENNIAL PERIOD, 1879-81.—Professor GEORGE CHRYSAL, for his paper “on the Differential Telephone,” published in the Transactions of the Society.
- 28TH BIENNIAL PERIOD, 1881-83.—THOMAS MUIR, Esq., LL.D., for his “Researches into the Theory of Determinants and Continued Fractions,” published in the Proceedings of the Society.
- 29TH BIENNIAL PERIOD, 1883-85.—JOHN AITKEN, Esq., for his paper “on the Formation of Small Clear Spaces in Dusty Air,” and for previous papers on Atmospheric Phenomena, published in the Transactions of the Society.
- 30TH BIENNIAL PERIOD, 1885-87.—JOHN YOUNG BUCHANAN, Esq., for a series of communications, extending over several years, on subjects connected with Ocean Circulation, Compressibility of Glass, etc.; two of which, viz., “On Ice and Brines,” and “On the Distribution of Temperature in the Antarctic Ocean,” have been published in the Proceedings of the Society.
- 31ST BIENNIAL PERIOD, 1887-89.—Professor E. A. LETTS, for his papers on the Organic Compounds of Phosphorus, published in the Transactions of the Society.
- 32ND BIENNIAL PERIOD, 1889-91.—R. T. OMOND, Esq., for his contributions to Meteorological Science, many of which are contained in vol. xxxiv of the Society’s Transactions.
- 33RD BIENNIAL PERIOD, 1891-93.—Professor THOMAS R. FRASER, F.R.S., for his papers on *Strophanthus hispidus*, Strophanthin, and Strophanthidin, read to the Society in February and June 1889 and in December 1891, and printed in vols. xxxv, xxxvi, and xxxvii of the Society’s Transactions.
- 34TH BIENNIAL PERIOD, 1893-95.—Dr CARGILL G. KNOTT, for his papers on the Strains produced by Magnetism in Iron and in Nickel, which have appeared in the Transactions and Proceedings of the Society.
- 35TH BIENNIAL PERIOD, 1895-97.—Dr THOMAS MUIR, for his continued communications on Determinants and Allied Questions.
- 36TH BIENNIAL PERIOD, 1897-99.—Dr JAMES BURGESS, for his paper “on the Definite Integral $\frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt$, with extended Tables of Values,” printed in vol. xxxix of the Transactions of the Society.
- 37TH BIENNIAL PERIOD, 1899-1901.—Dr HUGH MARSHALL, for his discovery of the Persulphates, and for his Communications on the Properties and Reactions of these Salts, published in the Proceedings of the Society.
- 38TH BIENNIAL PERIOD, 1901-03.—Sir WILLIAM TURNER, K.C.B., LL.D., F.R.S., etc., for his memoirs entitled “A Contribution to the Craniology of the People of Scotland,” published in the Transactions of the Society, and for his “Contributions to the Craniology of the People of the Empire of India,” Parts I, II, likewise published in the Transactions of the Society.
- 39TH BIENNIAL PERIOD, 1903-05.—THOMAS H. BRYCE, M.A., M.D., for his two papers on “The Histology of the Blood of the Larva of *Lepidosiren paradoxa*,” published in the Transactions of the Society within the period.
- 40TH BIENNIAL PERIOD, 1905-07.—ALEXANDER BRUCE, M.A., M.D., F.R.C.P.E., for his paper entitled “Distribution of the Cells in the Intermedio-Lateral Tract of the Spinal Cord,” published in the Transactions of the Society within the period.
- 41ST BIENNIAL PERIOD, 1907-09.—WHEELTON HIND, M.D., B.S., F.R.C.S., F.G.S., for a paper published in the Transactions of the Society, “On the Lamellibranch and Gasteropod Fauna found in the Millstone Grit of Scotland.”
- 42ND BIENNIAL PERIOD, 1909-11.—Professor ALEXANDER SMITH, B.Sc., Ph.D., of New York, for his researches upon “Sulphur” and upon “Vapour Pressure,” appearing in the Proceedings of the Society.

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- 43RD BIENNIAL PERIOD, 1911-1913.—JAMES RUSSELL, Esq., for his series of investigations relating to magnetic phenomena in metals and the molecular theory of magnetism, the results of which have been published in the Proceedings and Transactions of the Society, the last paper having been issued within the period.
- 44TH BIENNIAL PERIOD, 1913-15.—JAMES HARTLEY ASHWORTH, D.Sc., for his papers on "Larvæ of *Lingula* and *Pelagodiscus*," and on "*Sclerocheilus*," published in the Transactions of the Society, and for other papers on the Morphology and Histology of Polychæta.
- 45TH BIENNIAL PERIOD, 1915-17.—ROBERT C. MOSSMAN, for his work on the Meteorology of the Antarctic Regions, which originated with the important series of observations made by him during the voyage of the "*Scotia*" (1902-1904), and includes his paper "On a Sea-Saw of Barometric Pressure, Temperature, and Wind Velocity between the Weddell Sea and the Ross Sea," published in the Proceedings of the Society.
- 46TH BIENNIAL PERIOD, 1917-19.—JOHN STEPHENSON, Lt.-Col. I.M.S., for his series of papers on the Oligochæta and other Annelida, several of which have been published in the Transactions of the Society.
- 47TH BIENNIAL PERIOD, 1919-21.—RALPH ALLEN SAMPSON, F.R.S., for his Astronomical Researches, including the papers "Studies in Clocks and Time-keeping: No. 1, Theory of the Maintenance of Motion; No. 2, Tables of the Circular Equation," published in the Proceedings of the Society within the period of the award.

II. MAKDOUGALL-BRISBANE PRIZE.

- 1ST BIENNIAL PERIOD, 1859.—Sir RODERICK IMPEY MURCHISON, on account of his Contributions to the Geology of Scotland.
- 2ND BIENNIAL PERIOD, 1860-62.—WILLIAM SELLER, M.D., F.R.C.P.E., for his "Memoir of the Life and Writings of Dr Robert Whytt," published in the Transactions of the Society.
- 3RD BIENNIAL PERIOD, 1862-64.—JOHN DENIS MACDONALD, Esq., R.N., F.R.S., Surgeon of H.M.S. "*Icarus*," for his paper "on the Representative Relationships of the Fixed and Free Tunicata, regarded as Two Sub-classes of equivalent value; with some General Remarks on their Morphology," published in the Transactions of the Society.
- 4TH BIENNIAL PERIOD, 1864-66.—Not awarded.
- 5TH BIENNIAL PERIOD, 1866-68.—Dr ALEXANDER CRUM BROWN and Dr THOMAS RICHARD FRASER, for their conjoint paper "on the Connection between Chemical Constitution and Physiological Action," published in the Transactions of the Society.
- 6TH BIENNIAL PERIOD, 1868-70.—Not awarded.
- 7TH BIENNIAL PERIOD, 1870-72.—GEORGE JAMES ALLMAN, M.D., F.R.S., Emeritus Professor of Natural History, for his paper "on the Homological Relations of the Coelenterata," published in the Transactions, which forms a leading chapter of his Monograph of Gymnoblatic or Tubularian Hydroids—since published.
- 8TH BIENNIAL PERIOD, 1872-74.—Professor LISTER, for his paper "on the Germ Theory of Putrefaction and the Fermentive Changes," communicated to the Society, 7th April 1873.
- 9TH BIENNIAL PERIOD, 1874-76.—ALEXANDER BUCHAN, A.M., for his paper "on the Diurnal Oscillation of the Barometer," published in the Transactions of the Society.
- 10TH BIENNIAL PERIOD, 1876-78.—Professor ARCHIBALD GEIKIE, for his paper "on the Old Red Sandstone of Western Europe," published in the Transactions of the Society.
- 11TH BIENNIAL PERIOD, 1878-80.—Professor PIAZZI SMYTH, Astronomer-Royal for Scotland, for his paper "on the Solar Spectrum in 1877-78, with some Practical Idea of its probable Temperature of Origination," published in the Transactions of the Society.
- 12TH BIENNIAL PERIOD, 1880-82.—Professor JAMES GEIKIE, for his "Contributions to the Geology of the North-West of Europe," including his paper "on the Geology of the Faroes," published in the Transactions of the Society.
- 13TH BIENNIAL PERIOD, 1882-84.—EDWARD SANG, Esq., LL.D., for his paper "on the Need of Decimal Subdivisions in Astronomy and Navigation, and on Tables requisite therefor," and generally for his Recalculations of Logarithms both of Numbers and Trigonometrical Ratios,—the former communication being published in the Proceedings of the Society.
- 14TH BIENNIAL PERIOD, 1884-86.—JOHN MURRAY, Esq., LL.D., for his papers "On the Drainage Areas of Continents, and Ocean Deposits," "The Rainfall of the Globe, and Discharge of Rivers," "The Height of the Land and Depth of the Ocean," and "The Distribution of Temperature in the Scottish Lochs as affected by the Wind."

- 15TH BIENNIAL PERIOD, 1886-88.—ARCHIBALD GEIKIE, Esq., LL.D., for numerous Communications, especially that entitled "History of Volcanic Action during the Tertiary Period in the British Isles," published in the Transactions of the Society.
- 16TH BIENNIAL PERIOD, 1889-90.—Dr LUDWIG BECKER, for his paper on "The Solar Spectrum at Medium and Low Altitudes," printed in vol. xxxvi, Part I, of the Society's Transactions.
- 17TH BIENNIAL PERIOD, 1890-92.—HUGH ROBERT MILL, Esq., D.Sc., for his papers on "The Physical Conditions of the Clyde Sea Area," Part I being already published in vol. xxxvi of the Society's Transactions.
- 18TH BIENNIAL PERIOD, 1892-94.—Professor JAMES WALKER, D.Sc., Ph.D., for his work on Physical Chemistry, part of which has been published in the Proceedings of the Society, vol. xx, pp. 255-263. In making this award, the Council took into consideration the work done by Professor Walker along with Professor Crum Brown on the Electrolytic Synthesis of Dibasic Acids, published in the Transactions of the Society.
- 19TH BIENNIAL PERIOD, 1894-96.—Professor JOHN G. M'KENDRICK, for numerous Physiological papers, especially in connection with Sound, many of which have appeared in the Society's publications.
- 20TH BIENNIAL PERIOD, 1896-98.—Dr WILLIAM PEDDIE, for his papers on the Torsional Rigidity of Wires.
- 21ST BIENNIAL PERIOD, 1898-1900.—Dr RAMSAY H. TRAQUAIR, for his paper entitled "Report on Fossil Fishes collected by the Geological Survey in the Upper Silurian Rocks of Scotland," printed in vol. xxxix of the Transactions of the Society.
- 22ND BIENNIAL PERIOD, 1900-02.—Dr ARTHUR T. MASTERMAN, for his paper entitled "The Early Development of *Cribrella oculata* (Forbes), with remarks on Echinoderm Development," printed in vol. xl of the Transactions of the Society.
- 23RD BIENNIAL PERIOD, 1902-04.—Mr JOHN DOUGALL, M.A., for his paper on "An Analytical Theory of the Equilibrium of an Isotropic Elastic Plate," published in vol. xli of the Transactions of the Society.
- 24TH BIENNIAL PERIOD, 1904-06.—JACOB E. HALM, Ph.D., for his two papers entitled "Spectroscopic Observations of the Rotation of the Sun," and "Some Further Results obtained with the Spectroheliometer," and for other astronomical and mathematical papers published in the Transactions and Proceedings of the Society within the period.
- 25TH BIENNIAL PERIOD, 1906-08.—D. T. GWYNNE-VAUGHAN, M.A., F.L.S., for his papers, 1st, "On the Fossil Osmundaceæ," and 2nd, "On the Origin of the Adaxially-curved Leaf-trace in the Filicales," communicated by him conjointly with Dr R. Kidston.
- 26TH BIENNIAL PERIOD, 1908-10.—ERNEST MACLAGAN WEDDERBURN, M.A., LL.B., for his series of papers bearing upon "The Temperature Distribution in Fresh-water Lochs," and especially upon "The Temperature Seiche."
- 27TH BIENNIAL PERIOD, 1910-12.—JOHN BROWNLEE, M.A., M.D., D.Sc., for his contributions to the Theory of Mendelian Distributions and cognate subjects, published in the Proceedings of the Society within and prior to the prescribed period.
- 28TH BIENNIAL PERIOD, 1912-14.—Professor C. R. MARSHALL, M.D., M.A., for his studies "On the Pharmacological Action of Tetra-alkyl-ammonium Compounds."
- 29TH BIENNIAL PERIOD, 1914-16.—ROBERT ALEXANDER HOUSTOUN, Ph.D., D.Sc., for his series of papers on "The Absorption of Light by Inorganic Salts," published in the Proceedings of the Society.
- 30TH BIENNIAL PERIOD, 1916-18.—Professor A. ANSTRUTHER LAWSON for his Memoirs on "The Prothalli of *Tmesipteris Tannensis* and of *Psilotum*," published in the Transactions of the Society, together with previous papers on Cytology and on The Gametophytes of various Gymnosperms.
- 31ST BIENNIAL PERIOD, 1918-20.—Professor J. H. MACLAGAN WEDDERBURN of Princeton University for his Memoirs in Universal Algebra, etc., published in the Transactions and Proceedings of the Society, and elsewhere.

III. THE NEILL PRIZE.

- 1ST TRIENNIAL PERIOD, 1856-59.—Dr W. LAUDER LINDSAY, for his paper "on the Spermogones and Pycnides of Filamentous, Fruticulose, and Foliaceous Lichens," published in the Transactions of the Society.
- 2ND TRIENNIAL PERIOD, 1859-61.—ROBERT KAYE GREVILLE, LL.D., for his contributions to Scottish Natural History, more especially in the department of Cryptogamic Botany, including his recent papers on Diatomaceæ.

- 3RD TRIENNIAL PERIOD, 1862-65.—ANDREW CROMBIE RAMSAY, F.R.S., Professor of Geology in the Government School of Mines, and Local Director of the Geological Survey of Great Britain, for his various works and memoirs published during the last five years, in which he has applied the large experience acquired by him in the Direction of the arduous work of the Geological Survey of Great Britain to the elucidation of important questions bearing on Geological Science.
- 4TH TRIENNIAL PERIOD, 1865-68.—Dr WILLIAM CARMICHAEL M'INTOSH, for his paper "on the Structure of the British Nemerteans, and on some New British Annelids," published in the Transactions of the Society.
- 5TH TRIENNIAL PERIOD, 1868-71.—Professor WILLIAM TURNER, for his papers "on the Great Finner Whale; and on the Gravid Uterus, and the Arrangement of the Fœtal Membranes in the Cetacea," published in the Transactions of the Society.
- 6TH TRIENNIAL PERIOD, 1871-74.—CHARLES WILLIAM PEACH, Esq., for his Contributions to Scottish Zoology and Geology, and for his recent contributions to Fossil Botany.
- 7TH TRIENNIAL PERIOD, 1874-77.—Dr RAMSAY H. TRAQUAIR, for his paper "on the Structure and Affinities of *Tristichopterus alatus* (Egerton)," published in the Transactions of the Society, and also for his contributions to the Knowledge of the Structure of Recent and Fossil Fishes.
- 8TH TRIENNIAL PERIOD, 1877-80.—JOHN MURRAY, Esq., for his paper "on the Structure and Origin of Coral Reefs and Islands," published (in abstract) in the Proceedings of the Society.
- 9TH TRIENNIAL PERIOD, 1880-83.—Professor HERDMAN, for his papers "on the Tunicata," published in the Proceedings and Transactions of the Society.
- 10TH TRIENNIAL PERIOD, 1883-86.—B. N. PEACH, Esq., for his Contributions to the Geology and Palæontology of Scotland, published in the Transactions of the Society.
- 11TH TRIENNIAL PERIOD, 1886-89.—ROBERT KIDSTON, Esq., for his Researches in Fossil Botany, published in the Transactions of the Society.
- 12TH TRIENNIAL PERIOD, 1889-92.—JOHN HORNE, Esq., F.G.S., for his Investigations into the Geological Structure and Petrology of the North-West Highlands.
- 13TH TRIENNIAL PERIOD, 1892-95.—ROBERT IRVINE, Esq., for his papers on the Action of Organisms in the Secretion of Carbonate of Lime and Silica, and on the solution of these substances in Organic Juices. These are printed in the Society's Transactions and Proceedings.
- 14TH TRIENNIAL PERIOD, 1895-98.—Professor COSSAR EWART, for his recent Investigations connected with Telegony.
- 15TH TRIENNIAL PERIOD, 1898-1901.—Dr JOHN S. FLETT, for his papers entitled "The Old Red Sandstone of the Orkneys" and "The Trap Dykes of the Orkneys," printed in vol. xxxix of the Transactions of the Society.
- 16TH TRIENNIAL PERIOD, 1901-04.—Professor J. GRAHAM KERR, M.A., for his Researches on *Lepidosiren paradoxa*, published in the Philosophical Transactions of the Royal Society, London.
- 17TH TRIENNIAL PERIOD, 1904-07.—FRANK J. COLE, B.Sc., for his paper entitled "A Monograph on the General Morphology of the Myxinoid Fishes, based on a Study of Myxine," published in the Transactions of the Society, regard being also paid to Mr Cole's other valuable contributions to the Anatomy and Morphology of Fishes.
- 1ST BIENNIAL PERIOD, 1907-09.—FRANCIS J. LEWIS, M.Sc., F.L.S., for his papers in the Society's Transactions "On the Plant Remains of the Scottish Peat Mosses."
- 2ND BIENNIAL PERIOD, 1909-11.—JAMES MURRAY, Esq., for his paper on "Scottish Rotifers collected by the Lake Survey (Supplement)," and other papers on the "Rotifera" and "Tardigrada," which appeared in the Transactions of the Society—(this Prize was awarded after consideration of the papers received within the five years prior to the time of award: see Neill Prize Regulations).
- 3RD BIENNIAL PERIOD, 1911-13.—Dr W. S. BRUCE, in recognition of the scientific results of his Arctic and Antarctic explorations.
- 4TH BIENNIAL PERIOD, 1913-15.—ROBERT CAMPBELL, D.Sc., for his paper on "The Upper Cambrian Rocks at Craigeven Bay, Stonehaven," and "Downtonian and Old Red Sandstone Rocks of Kincardineshire," published in the Transactions of the Society.
- 5TH BIENNIAL PERIOD, 1915-17.—W. H. LANG, F.R.S., M.B., D.Sc., for his paper in conjunction with Dr R. KIDSTON, F.R.S., on *Rhynia Gwynne-Vaughani*, Kidston and Lang, published in the Transactions of the Society, and for his previous investigations on Pteridophytes and Cycads.

6TH BIENNIAL PERIOD, 1917-19.—JOHN TAIT, D.Sc., M.D., for his work on Crustacea, published in the Proceedings of the Society, and for his papers on the blood of Crustacea.

7TH BIENNIAL PERIOD, 1919-21.—SIR EDWARD A. SHARPEY SCHAFFER, F.R.S., for his recent contributions to our knowledge of Physiology, and in recognition of his published work extending over a period of fifty years.

IV. GUNNING VICTORIA JUBILEE PRIZE.

1ST TRIENNIAL PERIOD, 1884-87.—SIR WILLIAM THOMSON, Pres. R.S.E., F.R.S., for a remarkable series of papers "on Hydrokinetics," especially on Waves and Vortices, which have been communicated to the Society.

2ND TRIENNIAL PERIOD, 1887-90.—Professor P. G. TAIT, Sec. R.S.E., for his work in connection with the "Challenger" Expedition, and his other Researches in Physical Science.

3RD TRIENNIAL PERIOD, 1890-93.—ALEXANDER BUCHAN, Esq., LL.D., for his varied, extensive, and extremely important Contributions to Meteorology, many of which have appeared in the Society's publications.

4TH TRIENNIAL PERIOD, 1893-96.—JOHN AITKEN, Esq., for his brilliant Investigations in Physics, especially in connection with the Formation and Condensation of Aqueous Vapour.

1ST QUADRENNIAL PERIOD, 1896-1900.—Dr T. D. ANDERSON, for his discoveries of New and Variable Stars.

2ND QUADRENNIAL PERIOD, 1900-04.—SIR JAMES DEWAR, LL.D., D.C.L., F.R.S., etc., for his researches on the Liquefaction of Gases, extending over the last quarter of a century, and on the Chemical and Physical Properties of Substances at Low Temperatures: his earliest papers being published in the Transactions and Proceedings of the Society.

3RD QUADRENNIAL PERIOD, 1904-08.—Professor GEORGE CHRYSTAL, M.A., LL.D., for a series of papers on "Seiches," including "The Hydrodynamical Theory and Experimental Investigations of the Seiche Phenomena of Certain Scottish Lakes."

4TH QUADRENNIAL PERIOD, 1908-12.—Professor J. NORMAN COLLIE, Ph.D., F.R.S., for his distinguished contributions to Chemistry, Organic and Inorganic, during twenty-seven years, including his work upon Neon and other rare gases. Professor Collie's early papers were contributed to the Transactions of the Society.

5TH QUADRENNIAL PERIOD, 1912-16.—SIR THOS. MUIR, C.M.G., LL.D., F.R.S., for his series of Memoirs upon "The Theory and History of Determinants and Allied Forms," published in the Transactions and Proceedings of the Society between the years 1872 and 1915.

6TH QUADRENNIAL PERIOD, 1916-20.—C. T. R. WILSON, Esq., F.R.S., in recognition of his important discoveries in relation to Condensation Nuclei, Ionisation of Gases, and Atmospheric Electricity.

V. JAMES SCOTT PRIZE.

1ST AWARD, 1918-22.—Professor A. N. WHITEHEAD, F.R.S., for his lecture delivered on June 5, 1922, "On the Relatedness of Nature."

ABSTRACT

OF

THE ACCOUNTS OF JAMES CURRIE, ESQ., LL.D.

As Treasurer of the Royal Society of Edinburgh.

SESSION 1921-1922.

I. GENERAL FUND.

CHARGE.

1. Arrears of Contributions at 30th September 1921	£86	2	0
<i>Less</i> —Commutation of one Fellow included in above		2	2 0
			£84 0 0
2. Contributions for present Session:—			
1. 253 Fellows at £3, 3s. each	£796	19	0
40 Fellows at £4, 4s. each		168	0 0
			£964 19 0
<i>Deduct</i> —			
Commutation of Contributions of two Fellows—proportion thereof included in above		6	6 0
			£958 13 0
2. Fees of Admission and Contributions of sixteen new Fellows at £6, 6s. each		100	16 0
3. Commutation Fee in lieu of future Contributions of three Fellows		86	2 0
			1145 11 0
3. Extra Contributions for 1921-1922 under Amended Law, No. 6—			
Voluntary Contributions—34 Fellows at £1, 1s. each	£35	14	0
Additional Contributions—5 Fellows at £1, 1s. each		5	5 0
Commutations—4 Fellows at £10, 10s. each		42	0 0
Subscription for 1922-23 in advance		3	3 0
Donation received		5	5 0
			91 7 0
4. Interest received—			
Interest on £7830 five per cent. War Loan, 1929-47, Untaxed	£391	10	0
Annuity from Edinburgh and District Water Trust, less Tax, £13, 19s. 6d.		32	14 6
Interest on Deposit Receipts		39	15 4
Interest on £2100 2½ per cent. Consols, less Tax, £3, 5s. 7d.		9	16 11
			473 16 9
5. Transactions and Proceedings		161	13 8
6. Annual Grant from Government		600	0 0
7. Income Tax repaid for year to 5th April 1922		21	17 11
8. Receipts from Sale of Napier Tercentenary Memorial Volume		2	16 5
9. British Association Dinner Tickets sold		4	6 0
			£2585 8 9
Amount of the Charge			

DISCHARGE.

1. TAXES, INSURANCE, COAL AND LIGHTING :—

Inhabited House Duty	£0 6 3
Insurance	30 2 10
Coal, etc., to 18th March 1922	56 8 0
Gas to 10th May 1922	5 5 5
Electric Light to 15th September 1922	8 4 4
Water, 1921-22	4 4 0

£104 10 10

2. SALARIES :—

General Secretary, 1921-22	£100 0 0
Librarian and Assistant Secretary	252 0 0
Assistant Librarian	78 0 0
Office Keeper	127 12 0
Treasurer's Clerk	35 0 0
Extra Duty Money	15 0 0

607 12 0

3. EXPENSES OF TRANSACTIONS :—

Neill & Co., Ltd., Printers	£440 16 6
Hislop & Day, Ltd., Engravers	57 10 0
Orrock & Son	35 11 3
The Zinco-ColloTYPE Co.	173 1 6
Macfarlane & Erskine	29 15 0

Less Receipts—

Grants by Carnegie Trustees towards Messrs Spath's, Kidston and Lang's, Rennie's, Ferguson's, Gregory's, Stephenson's, Jehu's, and Thompson's Papers	£262 7 0
Moray Grant towards Miss Lamont's Paper	12 0 0
Professor Gregory Grant towards Spath's Paper	50 0 0

£736 14 3

4. EXPENSES OF PROCEEDINGS :—

Neill & Co., Ltd., Printers	£869 19 0
Hislop & Day, Ltd., Engravers	36 17 1
D. Stevenson & Co.	12 17 8

412 7 3

Less—

Carnegie Trustees—Grant towards Miss Mann's and Professor Bower's Papers	£10 1 0
Grant by Anonymous Donor, per Messrs Mackenzie & Kermack, W.S., towards Dr Gray's Paper	67 0 8

77 1 8

842 12 1

5. BOOKS, PERIODICALS, NEWSPAPERS, ETC. :—

James Thin, Bookseller	£256 17 3
R. Grant & Son, Booksellers	6 19 6
W. Green & Son, Ltd., Booksellers	2 1 0
M. A. T. Thomson	1 0 3
Robertson & Scott, News Agents	8 0 0
Ray Society, Subscription	1 1 0
Berwickshire Naturalists' Club, Do.	1 0 0
Palaeontographical Society, Do.	1 1 0
Board of Scientific Societies, London, Donation	10 0 0
Wm. Blackwood & Sons	3 14 0
Williams & Norgate	2 0 0
Alex. Cowan	16 14 6
Edinburgh and Leith Post Office Directory, Ltd.	0 15 0
Thos. Peck & Son	0 8 6
Hay, Foster & Morley	1 16 10

313 8 10

6. ORROCK & SON, BOOKBINDERS

£500 0 0

*Less—*Received from Carnegie Trust

500 0 0

Carry forward £2280 11 0

27

7. OTHER PAYMENTS :—	Brought forward	£2280 11 0
Neill & Co., Ltd., Printers	£120 2 2	
J. & T. Scott	4 5 0	
S. Duncan & Sons, Tailors (uniforms)	9 9 0	
M. A. T. Thomson	1 16 6	
Andrew H. Baird	11 17 0	
Lindsay, Jamieson & Haldane, C.A., Auditors	10 10 0	
A. Muirhead & Son	174 17 0	
R. Maule & Son	4 8 7	
Orrock & Son, Bookbinders	26 2 8	
Gillies & Wright, Joiners	24 17 5	
Edward & Co.	6 13 5	
Burn Brothers, Plumbers	8 0 10	
Allan & Son	6 5 0	
Aitken & Dott	1 0 0	
G. Waterston & Sons, Ltd.	5 6 8	
Travelling Expenses of Delegate to Brussels	12 4 8	
A. Black & Co., Brushmakers	4 15 6	
T. & A. Constable, Printers	2 13 6	
Petty Expenses, Postages, Carriage, etc.	186 16 11½	
		622 1 10½
8. ARREARS of CONTRIBUTIONS written off		1 1 0
9. ARREARS of CONTRIBUTIONS outstanding at 30th September 1922 :—		
Present Session	£57 15 0	
Previous Sessions	32 11 0	
		90 6 0
Amount of the Discharge		£2993 19 10½
Amount of the Discharge		£2993 19 10½
Amount of the Charge		2585 8 9
Excess of Payments over Receipts for 1921-1922 transferred out of Special Subscription Fund		£408 11 1½

SPECIAL SUBSCRIPTION FUND

To 30th September 1922.

CHARGE.

1. Balance at 30th September 1921 :—		
Due by Union Bank of Scotland, Ltd., on Deposit Receipt	£350 0 0	
Due by Union Bank of Scotland, Ltd., on Account Current	629 18 8	
Balance of Loan to Makerstoun Magnetic Meteorological Observation Fund	51 7 1	
Due by Treasurer	6 5 5	
		<u>£1037 11 2</u>

DISCHARGE.

1. Amount transferred to General Fund, being deficiency for year to 30th September 1922		£408 11 1½
2. Balance at 30th September 1922 :—		
Due by Union Bank of Scotland, Ltd., on Deposit Receipt	£358 12 1	
Due by Union Bank of Scotland, Ltd., on Account Current	223 8 5	
Balance of Loan to Makerstoun Magnetic Meteorological Observation Fund	38 17 1	
Amount advanced to Dr Aitken's Fund	2 9 9	
Due by Treasurer	5 12 8½	
		629 0 0½
		<u>£1037 11 2</u>

Note :—

Total Subscriptions received towards Special Subscription Fund	£1128 17 9
Sums transferred to General Fund to meet Deficiency on Accounts—	
To 30th September 1919	£301 16 11
To 30th September 1920	34 9 5
To 30th September 1922	408 11 1½
	<hr/>
Amount transferred from General Fund, being Surplus	£744 17 5½
for year to 30th September 1921	244 19 9
	<hr/>
Net Amount transferred to General Fund out of Special Subscription Fund as at 30th September 1922	£499 17 8½
	<hr/>
	499 17 8½
Amount of Special Subscription Fund at 30th September 1922, as above	<u>£629 0 0½</u>

II. KEITH FUND*To 30th September 1922.*

CHARGE.

1. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1921	£72 4 10
2. INTEREST RECEIVED :—	
On £650 five per cent. War Loan, 1929-47, Untaxed	32 10 0
On Deposit Receipt of £72, 4s. 10d., Uplifted	1 6 10
	<hr/>
	£106 1 8

DISCHARGE.

1. Paid Alex. Kirkwood for Keith Medal	£ 21 0 0
Paid Professor R. A. Sampson Money Portion of Prize, 1919-21	45 12 2
	<hr/>
	66 12 2
2. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1922	39 9 6
	<hr/>
	<u>£106 1 8</u>

III. NEILL FUND*To 30th September 1922.*

CHARGE.

1. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1921	£30 10 6
2. INTEREST RECEIVED :—	
On £300 five per cent. War Loan, 1929-47, Untaxed	£15 0 0
On Deposit Receipt	0 1 1
	<hr/>
	15 1 1
	<hr/>
	£45 11 7

DISCHARGE.

1. Paid Alex. Kirkwood for Neill Medal	£21 0 0
Paid Sir E. Sharpey Schafer Money Portion of Prize, 1919-21	9 13 5
	<hr/>
	30 13 5
2. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1922	14 18 2
	<hr/>
	<u>£45 11 7</u>

IV. MAKDOUGALL-BRISBANE FUND

To 30th September 1922.

CHARGE.

1. BALANCE due by Union Bank of Scotland, Ltd., at 30th September 1921 :—		
On Deposit Receipt	£38	13 11
On Account Current	2	10 0
		<hr/>
	£41	3 11
2. INTEREST RECEIVED :—		
On £400 five per cent. War Loan, 1929-47, Untaxed	£20	0 0
On Deposit Receipt—Union Bank of Scotland, Ltd.	0	18 6
		<hr/>
	20	18 6
		<hr/>
	£62	2 5

DISCHARGE.

1. BALANCE due by Union Bank of Scotland, Ltd., at 30th September 1922 :—		
On Deposit Receipt	£39	12 5
On Account Current	22	10 0
		<hr/>
	£62	2 5

V. MAKERSTOUN MAGNETIC METEOROLOGICAL OBSERVATION FUND

To 30th September 1922.

CHARGE.

1. INTEREST RECEIVED :—		
On £250 five per cent. War Loan, 1929-47, Untaxed	£12	10 0
2. BORROWED from General Fund £100, less repaid £61, 2s. 11d.	£38	17 1
		<hr/>
3. BALANCE due to General Fund at 30th September 1922	38	17 1
		<hr/>
	£51	7 1

DISCHARGE.

1. Balance due to General Fund at 30th September 1921		£51	7 1
2. Repaid on Account of Loan	£12	10 0	
		<hr/>	
		£51	7 1

VI. GUNNING VICTORIA JUBILEE PRIZE FUND

To 30th September 1922.

CHARGE.

1. BALANCE due by Union Bank of Scotland, Ltd., at 30th September 1921 :—		
On Deposit Receipt	£57	4 1
2. INTEREST RECEIVED :—		
On £570 five per cent. War Loan, 1929-47, Untaxed	£28	10 0
On Deposit Receipt—Union Bank of Scotland, Ltd.	1	7 10
		<hr/>
	29	17 10
		<hr/>
	£87	1 11

DISCHARGE.

1. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt, at 30th September 1922	£58	11 11
On Account Current	28	10 0
		<hr/>
	£87	1 11

VII. JAMES SCOTT PRIZE FUND*To 30th September 1922.***CHARGE.**

1. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1921	£286 14 6
2. INTEREST RECEIVED :—	
On Deposit Receipt, Union Bank of Scotland, Ltd.	6 16 10
From General Fund	0 0 6
	<u>£293 11 10</u>

DISCHARGE.

1. Paid Professor Whitehead Money Prize	£36 15 0
2. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1922	256 16 10
	<u>£293 11 10</u>

VIII. DR JOHN AITKEN FUND*To 30th September 1922.***CHARGE.**

1. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1921	£1041 8 10
2. INTEREST RECEIVED :—	
On Deposit Receipts, Union Bank of Scotland, Ltd.	25 12 1
3. Received from General Fund	2 9 9
	<u>£1069 10 8</u>

DISCHARGE.

1. ACCOUNTS PAID :—	
Macfarlane & Erskine	£2 1 6
Harrison & Sons	0 8 3
	<u>£2 9 9</u>
2. BALANCE due by Union Bank of Scotland, Ltd., on Deposit Receipt at 30th September 1922	1067 0 11
	<u>£1069 10 8</u>

STATE OF THE FUNDS BELONGING TO THE ROYAL SOCIETY OF EDINBURGH*As at 30th September 1922.***1. GENERAL FUND—**

1. £7830 five per cent. War Loan, 1929-47, at 99½ per cent.	£7800 12 9
2. £2100 2½ per cent. Consolidated Stock at 57 per cent.	1197 0 0
3. Deposit Receipt Union Bank of Scotland, Ltd., being balance of Legacy received during 1917-18, from the Trustees of the late Mr Robert Mackay Smith, £500 less legacy duty £50	450 0 0
4. Arrears of Contributions, as per preceding Abstract of Accounts	90 6 0
5. Balance of Special Subscription Fund	629 0 0½
6. Balance of Loan to Makerstoun Magnetic Meteorological Observation Fund	38 17 1
7. Balance due by Dr John Aitken's Fund	2 9 9

AMOUNT	<u>£10,208 5 7½</u>
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Exclusive of Library, Museum, Pictures, etc., and Furniture in the Society's Rooms at George Street, Edinburgh.

2. KEITH FUND—

1. £650 five per cent. War Loan, 1929-47, at 99 $\frac{5}{8}$ per cent.	£647 11 3
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	39 9 6

AMOUNT £687 0 9

3. NEILL FUND—

1. £300 five per cent. War Loan, 1929-47, at 99 $\frac{5}{8}$ per cent.	£298 17 6
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	14 18 2

AMOUNT £313 15 8

4. MAKDOUGALL-BRISBANE FUND—

1. £400 five per cent. War Loan, 1929-47, at 99 $\frac{5}{8}$ per cent.	£398 10 0
2. Balance due by Union Bank of Scotland, Ltd. :—	
On Deposit Receipt	£39 12 5
On Account Current	22 10 0
	<u>62 2 5</u>

AMOUNT £460 12 5

5. MAKERSTOUN MAGNETIC METEOROLOGICAL OBSERVATION FUND—

1. £250 five per cent. War Loan, 1929-47, at 99 $\frac{5}{8}$ per cent.	£249 1 3
Less—Balance of Loan from General Fund	38 17 1

AMOUNT £210 4 2

6. GUNNING VICTORIA JUBILEE PRIZE FUND—Instituted by Dr Gunning of Edinburgh and Rio de Janeiro—

1. £570 five per cent. War Loan, 1929-47, at 99 $\frac{5}{8}$ per cent.	£567 17 3
2. Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	58 11 11
3. Balance due do. do. on Account Current	28 10 0

AMOUNT £654 19 2

7. JAMES SCOTT PRIZE FUND—

Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	£256 16 10
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Whereof—Applicable to Capital	£250 0 0
Do. Revenue	6 16 10

£256 16 10

8. TAIT MEMORIAL FUND—

This Fund consists of War Loan, and is to mature for a period of about ten years from 1918, when it is expected to yield about £75 per annum.

9. DR JOHN AITKEN FUND—

Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt	£1067 0 11
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Whereof—Applicable to Capital	£1000 0 0
Do. Revenue	67 0 11

£1067 0 11

Less—Due to General Fund	2 9 9
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£1064 11 2

EDINBURGH, 16th October 1922.—We have examined the preceding Accounts of the Treasurer of the Royal Society of Edinburgh for the Session 1921-1922, and have found them to be correct. The securities of the various Investments at 30th September 1922, as noted in the foregoing Statement of Funds (with the exception of No. 8), have been exhibited to us.

LINDSAY, JAMIESON & HALDANE, C.A.

Auditors.

LIST OF VOLUNTARY CONTRIBUTORS OF TEN GUINEAS
(Single Payment) under Law VI (end of para. 3) up to 30th
September 1922.

Sir J. O. AFFLECK, M.D., LL.D.
Sir J. J. BURNET, R.S.A., LL.D.

Dr T. W. DEWAR.
Lt.-Col. W. GLEN LISTON.

LIST OF VOLUNTARY CONTRIBUTORS OF ONE GUINEA
under Law VI (end of para. 3) up to 30th September 1922.

Dr F. J. ALLAN.
Maj.-Gen. W. B. BANNERMAN.
Dr W. L. BELL.
Sir G. A. BERRY, M.B., C.M., LL.D.
Dr W. B. BLAIKIE.
Dr J. T. BOTTOMLEY, F.R.S.
Prof. F. O. BOWER, F.R.S.
Dr J. J. GRAHAM BROWN.
Sir ARCH. DENNY, Bt., LL.D.
D. B. DOTT.
JOHN S. FORD.
Prof. A. GRAY, F.R.S.
Sir R. B. GREIG.
Em.-Prof. Sir W. A. HERDMAN, F.R.S.
Dr J. HORNE, F.R.S.

Dr W. F. HUME.
Sir J. KEMNAL.
Rev. JAMES LINDSAY, D.D., 1920-21,
1921-22.
GEO. M. LOW.
Dr D. F. LOWE.
Dr P. M'BRIDE.
R. C. MOSSMAN, 1920-21, 1921-22.
FRANK SPENCE.
CHARLES A. STEVENSON, M.Inst.C.E.
Prof. J. ARTHUR THOMSON, 1920-21,
1921-22.
Prof. Sir JAMES WALKER, F.R.S.
A. C. WILSON, F.C.S.
J. C. WRIGHT.

THE COUNCIL OF THE SOCIETY.

October 1922.

PRESIDENT.

PROFESSOR FREDERICK O. BOWER, M.A., D.Sc., LL.D., F.R.S., F.L.S.

VICE-PRESIDENTS.

PRINCIPAL SIR JAMES ALFRED EWING, K.C.B., M.A., D.Sc., LL.D., M.Inst.C.E., F.R.S.,
Principal, University of Edinburgh.

PROFESSOR JOHN WALTER GREGORY, D.Sc., F.R.S., Professor of Geology, University of
Glasgow.

MAJOR-GENERAL W. B. BANNERMAN, C.S.I., I.M.S., M.D., D.Sc.

W. A. TAIT, D.Sc., M.Inst.C.E.

PRINCIPAL J. C. IRVINE, C.B.E., Ph.D., D.Sc., LL.D., F.R.S.

THE RIGHT HON. LORD SALVESEN, P.C., K.C.

GENERAL SECRETARY.

CARGILL G. KNOTT, D.Sc., LL.D., F.R.S.

SECRETARIES TO ORDINARY MEETINGS.

PROFESSOR J. H. ASHWORTH, D.Sc., F.R.S., Professor of Zoology, University, Edinburgh.

PROFESSOR R. A. SAMPSON, M.A., D.Sc., F.R.S., Astronomer Royal for Scotland.

TREASURER.

JAMES CURRIE, M.A., LL.D.

CURATOR OF LIBRARY AND MUSEUM.

A. CRICHTON MITCHELL, D.Sc., Hon. D.Sc. (Geneva).

COUNCILLORS.

PROFESSOR FRANCIS GIBSON BAILY,
M.A., M.Inst.E.E.

ROBERT CAMPBELL, M.A., D.Sc., F.G.S.

PROFESSOR J. ARTHUR THOMSON, M.A.,
LL.D.

HERBERT STANLEY ALLEN, M.A., D.Sc.

SIR ROBERT BLYTH GREIG, M.A., LL.D.,
F.Z.S.

JAMES RITCHIE, M.A., D.Sc.

ERNEST MACLAGAN WEDDERBURN,
M.A., LL.B., W.S., D.Sc.

PROFESSOR THOMAS H. BRYCE, M.A., M.D.,
F.R.S.

PROFESSOR JAMES YOUNG SIMPSON, M.A.,
D.Sc.

PROFESSOR D'ARCY W. THOMPSON, C.B.,
D.Litt., F.R.S.

PROFESSOR SIR JAMES WALKER, D.Sc.,
Ph.D., LL.D., F.R.S.

PROFESSOR EDMUND TAYLOR WHIT-
TAKER, Sc.D., F.R.S.

OFFICE STAFF.

Librarian and Assistant Secretary, G. A. STEWART.

Assistant Librarian, ERNST. M. STEWART.

Housekeeper, SAMUEL HEDDLE.

ALPHABETICAL LIST OF THE ORDINARY FELLOWS OF THE SOCIETY,

Corrected to January 31, 1923.

N. B.—Those marked * are Annual Contributors.
 ,, ,, † have commuted Voluntary Contribution (see 3rd Paragraph, Law VI).

E. prefixed to a name indicates that the Fellow has received a Makdougall-Brisbane Medal.
 K. " " " " Keith Medal.
 N. " " " " Neill Medal.
 V. J. " " " " the Gunning Victoria Jubilee Prize.
 C. " " " " contributed one or more Communications to the Society's TRANSACTIONS or PROCEEDINGS.

Date of Election.		Service on Council, etc
1898	C. Abercromby, the Rt. Hon. Lord, LL.D., 62 Palmerston Place, Edinburgh	
1922	* Abernethy, Charles Lawrence, M.A. (Hons.), B.Sc., Research Physicist, 3 Marchmont Crescent, Edinburgh	
1898	Adami, J. G., C.B.E., M.A., M.D. (Camb., M'Gill, and Belfast), LL.D., F.R.S., Vice-Chancellor of the University of Liverpool.	
1889	† Alison, John, M.A., LL.D., Head Master, George Watson's College, Edinburgh	
1894	Allan, Francis John, M.D., C.M., Edin., M.O.H. City of Westminster, Westminster City Hall, Charing Cross Road, London 5	
1888	C. Allardice, R. E., M.A., Professor of Mathematics in Stanford University, Palo Alto, Santa Clara Co., California	
1920	C. * Allen, Herbert Stanley, M.A. (Cambridge), D.Sc. (London), Professor of Natural Philosophy in the University of St Andrews	1921-
1906	Anderson, Daniel E., M.D., B.A., B.Sc., Green Bank, Merton Lane, Highgate, London, N.	
1920	C. * Anderson, Ernest Masson, M.A., B.Sc., F.G.S., Geologist, H. M. Geological Survey of Scotland, 5 Riselaw Road, Edinburgh	
1905	* Anderson, William, M.A., Head Science Master, George Watson's College, Edinburgh, 6 Lockharton Crescent, Edinburgh 10	
1903	Anderson-Berry, David, M.D., LL.D., F.R.S.L., M.R.A.S., F.S.A. (Scot.), Versailles, Highgate, London, N.	
1905	* Andrew, George, M.A., B.A., H.M.I.S., Royal Technical College, George Street, Glasgow	
1881	C. Anglin, A. H., M.A., LL.D., M.R.I.A., Professor of Mathematics, Queen's College, Cork	
1921	C. Annandale, Nelson, B.A. (Oxon.), D.Sc. (Edin.), Director of the Zoological Survey of India, and Vice-Chairman of the Trustees of the Indian Museum, Calcutta	
1915	Anthony, Charles, M.Inst.C.E., M. Am. Soc. C.E., F.R.San.I., F.R.Met.S., F.R.A.S., F.C.S., General Manager, Water Works Company, Vieytes y Gorriti, Bahia Blanca, Argentina 15	
1906	Appleton, Colonel Arthur Frederick, F.R.C.V.S., 19 Cumberland Road, Bromley, Kent	
1910	C. Archibald, E. H., B.Sc., Professor of Chemistry, University of British Columbia, Vancouver, Canada	
1907	* Archibald, James, M.A., 31 Leamington Terrace, Edinburgh	
1921	* Arthur, William, M.A., Assistant to the Professor of Mathematics in the University of Glasgow. 149 Stanmore Road, Mount Florida, Glasgow	
1911	C. K. * Ashworth, James Hartley, D.Sc., F.R.S., Professor of Zoology, University of Edinburgh (SECRETARY), 69 Braid Avenue, Edinburgh 20	1912-14, 1915-18. Sec. 1918-

Date of Election.		Service on Council, etc.
1907	* Badre, Muhammad, Ph.D., Almuneerah, Cairo, Egypt	
1920	* Bagnall, Richard Siddoway, Member of the Entomological and other Scientific Societies of Britain, France, Italy, Belgium, and Spain, 39 Eslington Terrace, Newcastle-on-Tyne	
1920	C. * Bailey, Edward Battersby, M.C., B.A., F.G.S., District Geologist, H.M. Geological Survey of Scotland, 23 Pentland Terrace, Edinburgh	
1896	C. Baily, Francis Gibson, M.A., M.Inst.E.E., Professor of Electrical Engineering, Heriot-Watt College, Edinburgh, Newbury, Colinton, Midlothian	1909-12. 1920-
1921	* Baker, Bevan Braithwaite, M.A., B.Sc. (Lond.), Lecturer in Mathematics in the University of Edinburgh. 30 Murrayfield Gardens, Edinburgh 25	
1905	C. Balfour-Browne, William Alexander Francis, M.A., F.Z.S., F.E.S., Barrister-at-Law, Lecturer in Zoology (Entomology) in the University of Cambridge. Oaklands, Fenstanton, near St Ives, Hunts	
1918	* Balsillie, David, B.Sc., F.G.S., Department of Mineralogy and Geology, Royal Scottish Museum, Edinburgh	
1902	C. Bannerman, W. B., C.S.I., I.M.S., M.D., D.Sc., Maj.-General, Indian Medical Service (Retd.), 11 Strathearn Place, Edinburgh (Vice-President)	1919-21. V-P 1921-
1889	Barbour, A. H. F., M.A., M.D., LL.D., F.R.C.P.E., 4 Charlotte Square, Edinburgh	
1886	Barclay, A. J. Gunion, M.A., 3 Chandos Avenue, Oakleigh Park, London, N. 30	
1903	Bardswell, Noël Dean, M.D., M.R.C.P. Ed. and Lond., King Edward VII Sanatorium, Midhurst	
1922	* Barger, George, M.A., D.Sc., F.R.S., Professor of Chemistry (Medical) in the University of Edinburgh, 48 St Alban's Road, Edinburgh	
1914	C. * Barkla, Charles Glover, M.A., D.Sc., F.R.S., Professor of Natural Philosophy in the University of Edinburgh, Nobel Laureate, Physics, 1917, The Hermitage of Braid, Edinburgh	1915-18.
1921	* Barr, Archibald, D.Sc., LL.D. (Glasgow and Birmingham), Em.-Professor of Engineering in the University of Glasgow. Westerton of Mugdock, Milngavie	
1904	Barr, Sir James, C.B.E., M.D., LL.D., F.R.C.P. Lond., 72 Rodney Street, Liverpool 35	
1874	Barrett, Sir William F., Kt., F.R.S., M.R.I.A., formerly Professor of Physics, Royal College of Science, Dublin, 31 Devonshire Place, London, W. 1	
1921	* Bartholomew, John, M.C. M.A., F.R.G.S., Geographical Institute, Edinburgh	
1895	C. Barton, Edwin H., D.Sc., F.R.S., F.P.S.L., F.Inst.P., Professor of Physics, University College, Nottingham	
1913	Beard, Joseph, F.R.C.S. (Edin.), M.R.C.S. (Eng.), L.R.C.P. (Lond.), D.P.H. (Camb.), Medical Officer of Health and School Medical Officer, City of Carlisle, 8 Carlton Gardens, Carlisle	
1888	Beare, Thomas Hudson, B.A., B.Sc., M.Inst.C.E., J.P., D.L., Professor of Engineering in the University of Edinburgh 40	1907-09. V-P 1909-15.
1897	C. Beattie, Sir John Carruthers, K.B., D.Sc., Vice-Chancellor and Principal, The University, Cape Town	
1893	C. B. Becker, Ludwig, Ph.D., Regius Professor of Astronomy in the University of Glasgow, The Observatory, Glasgow, Millbank Terrace, Crieff	
1882	C. Beddard, Frank E., M.A. Oxon., D.Sc., F.R.S., formerly Prosector to the Zoological Society of London, 20 Sherriff Road, Kilburn, London, N.W. 6	
1887	Begg, Ferdinand Faithfull, 46 Saint Aubyns, Hove, Sussex	
1906	Bell, John Patrick Fair, F.Z.S., Springbank, Ayton, Berwickshire 45	
1916	* Bell, Robert John Tainsh, M.A., D.Sc., Professor of Mathematics in the University of Otago, New Zealand	
1915	Bell, Walter Leonard, M.D. Edin., F.S.A. Scot., 123 London Road, North Lowestoft, Suffolk	
1893	C. Berry, Sir George A., M.B., C.M., LL.D., F.R.C.S.E., M.P., 31 Drumsheugh Gardens, Edinburgh	1916-19. V-P 1919-22.
1897	C. Berry, Richard J. A., M.D., F.R.C.S.E., Professor of Anatomy in the University of Melbourne, Victoria, Australia	
1880	C. Birch, De Burgh, C.B., M.D., Emeritus Professor of Physiology in the University of Leeds 50	
1907	* Black, Frederick Alexander, Solicitor, 59 Academy Street, Inverness	
1884	C. Black, John S., M.A., LL.D., 125 St James' Court, London, S.W. 1	1891-94, 1916-18. Cur. 1906-16.

Alphabetical List of the Ordinary Fellows of the Society. 427

Date of Election.		Service on Council, etc.
1897	C. Blaikie, Walter Biggar, LL.D., Firbank, Colinton	
1904	C. * Bles, Edward J., M.A., D.Sc., Elterholm, Cambridge	1914-17.
1918	* Blight, Francis James, Chairman and Managing Director of Charles Griffin & Co., Ltd., Publishers, Tregenna, Wembley, Middlesex	55
1894	Bolton, Herbert, D.Sc., F.G.S., F.Z.S., Director of the Bristol Museum and Art Gallery, Bristol, 58 Coldharbour Road, Redland, Bristol	
1915	* Boon, Alfred Archibald, D.Sc., F.I.C., B.A., Professor of Chemistry, Heriot-Watt College, Edinburgh	
1872	C. Bottomley, J. Thomson, M.A., D.Sc., LL.D., F.R.S., F.C.S., 13 University Gardens, Glasgow	
1886	C. Bower, Frederick O., M.A., D.Sc., LL.D., F.R.S., F.L.S. (PRESIDENT), Regius Professor of Botany in the University of Glasgow, 1 St John's Terrace, Hillhead, Glasgow	1887-90, 1893-96, 1907-09, 1917-19 V-P 1910-16. P 1919-
1884	C. Bowman, Frederick Hungerford, D.Sc., F.C.S. (Lond. and Berl.), F.I.C., A.Inst.C.E., A.Inst.M.E., M.Inst.E.E., etc., 76 Acomb Street, Whitworth Park, Manchester	60
1901	Bradbury, J. B., M.D., Downing Professor of Medicine, University of Cambridge	
1916	Bradley, His Honour Judge (Francis Ernest), M.A., M.Com., LL.D., Barrister-at-Law, Examiner to the Council of Legal Education, 8 Balmoral Road, St Annes-on-the-Sea	
1903	C. * Bradley, O. Charnock, M.D., D.Sc., Principal, Royal (Dick) Veterinary College, Edinburgh	1907-10, 1915-17.
1886	Bramwell, Byrom, M.D., F.R.C.P.E., LL.D., 23 Drumsheugh Gardens, Edinburgh	1890-93.
1907	* Bramwell, Edwin, Professor of Clinical Medicine in the University of Edinburgh, M.D., F.R.C.P.E., F.R.C.P. Lond., 23 Drumsheugh Gardens, Edinburgh	65
1918	* Bremner, Alexander, M.A., D.Sc., Headmaster, Demonstration School, Training Centre, Aberdeen, 13 Belgrave Terrace, Aberdeen	
1916	C. * Briggs, Henry, D.Sc., A.R.S.M., Professor of Mining, Heriot-Watt College, Allermuir, Liberton, Midlothian	
1895	Bright, Sir Charles, M.Inst.C.E., M.Inst.E.E., F.R.Aë.S., F.Inst.Radio.E., F.R.A.S., F.R.G.S., Leigh Grange, Kent, and Athenæum Club, Pall Mall, London, S.W.	
1893	Brock, G. Sandison, M.D., 6 Corso d'Italia, Rome, Italy	
1901	C. * Brodie, W. Brodie, M.B., Camden House, Bletchingley, Surrey	70
1907	Brown, Alexander, M.A., B.Sc., Professor of Applied Mathematics, The University, Cape Town	
1883	C. Brown, J. J. Graham, M.D., F.R.C.P.E., 3 Chester Street, Edinburgh	
1885	C. Brown, J. Macdonald, M.D., F.R.C.S., 64 Upper Berkeley Street, Portman Square, London, W.	
1921	* Bruce, Alexander, B.Sc. (Edin.), Government Agricultural Chemist and City Analyst, Colombo, Ceylon	
1912	* Bruce, Alexander Ninian, D.Sc., M.D., 8 Ainslie Place, Edinburgh	75
1898	C. K. Bryce, T. H., M.A., M.D. (Edin.), F.R.S., Professor of Anatomy in the University of Glasgow, 2 The University, Glasgow	1911-14, 1922-
1870	C. K. Buchanan, John Young, M.A., F.R.S., Athenæum Club, Pall Mall, London, S.W.	1878-81, 1884-86.
1905	Bunting, Thomas Lowe, M.D., 27 Denton Road, Scotswood, Newcastle-on-Tyne	
1902	* Burgess, A. G., M.A., Rector of The Academy, Rothesay, Blythswood, Rothesay	
1887	† Burnet, Sir John James, A.R.A., R.S.A., LL.D., Architect, 1 Montague Place, Bedford Square, London, W.C. 1.	80
1888	Burns, Rev. T., D.D., J.P., F.S.A. Scot., Minister of Lady Glenorchy's Parish Church, Croston Lodge, Chalmers Crescent, Edinburgh	
1917	* Burnside, George Barnhill, M.I. Mech.E., 104 Beechwood Drive, Glasgow, W.	
1915	* Butchart, Raymond Keiler, B.Sc., Ph.D., University College, Dundee, 5 Briarwood Terrace, West Park Road, Dundee	
1896	Butters, J. W., M.A., B.Sc., Rector of Ardrrossan Academy	
1887	C. Cadell, Henry Moubray, of Grange, B.Sc., D.L., Linlithgow	85
1910	* Calderwood, Rev. Robert Sibbald, Minister of Cambuslang, The Manse, Cambuslang, Lanarkshire	1919-22.

Date of Election.			Service on Council, etc.
1893	C.	Calderwood, W. L., Inspector of Salmon Fisheries of Scotland, South Bank, Canaan Lane, Edinburgh	
1894		Cameron, James Angus, M.D., Medical Officer of Health, Firhall, Nairn	
1905	C.	Cameron, John, M.D., D.Sc., M.R.C.S. Eng., Professor of Anatomy, Dalhousie University, Halifax, Nova Scotia	
1921		* Campbell, Andrew, Advisory Chemist, Burmah Oil Co., Ltd., and Anglo-Persian Oil Co., Ltd. Lindum, 9 Foxgrove Road, Beckenham, Kent 90	
1904		* Campbell, Charles Duff, Scottish Liberal Club, Princes Street, Edinburgh	
1918		* Campbell, John Menzies, L.D.S. (Glas.), D.D.S. (Toronto), L.D.S. (Ontario), 14 Buckingham Terrace, Glasgow, W.	
1915	C. N.	* Campbell, Robert, M.A., D.Sc., F.G.S., Lecturer in Petrology, University of Edinburgh, 2 Woodhall Road, Colinton	1920-
1899	C.	* Carlier, Edmund W. W., M.D., M.Sc., F.E.S., Professor of Physiology, University, Birmingham	
1910		Carnegie, Col. David, C.B.E., M.Inst.C.E., M.Inst.Mech.E., M.I.S.Inst., Haven, Seasalter, Whitstable 95	
1920	C.	* Carruthers, R. G., F.G.S., District Geologist, H. M. Geological Survey, High Barn, Stocksfield-on-Tyne	
1905	C.	* Carse, George Alexander M.A., D.Sc., Reader in Natural Philosophy, University of Edinburgh, 3 Middleby Street, Edinburgh	
1901		Carshaw, H. S., M.A., D.Sc., Professor of Mathematics in the University of Sydney, New South Wales	
1905		Carter, Joseph Henry, F.R.C.V.S., Avalon, Western Road, Henley-on-Thames	
1898		Carus-Wilson, Cecil, F.R.G.S., F.G.S., Waldegrave Park, Strawberry Hill, Middlesex, and Sandacres Lodge, Parkstone-on-Sea, Dorset 100	
1882		Cay, W. Dyce, M.Inst.C.E., Junior Carlton Club, Pall Mall, London, S.W. 1	
1899		Chatham, James, Actuary, c/o Robert Murrie, Esq., 28 St Andrew Square, Edinburgh	
1912		Chaudhuri, Banawari Lal, B.A. (Cal.), B.Sc. (Edin.), Assistant Superintendent, Natural History Section, Indian Museum, 120 Lower Circular Road, Calcutta, India	
1874		Chiene, John, C.B., M.D., LL.D., F.R.C.S.E., Emeritus Professor of Surgery in the University of Edinburgh, Barnton Avenue, Davidson's Mains	1884-86, 1904-0
1891		Clark, John B., M.A., Head Master of Heriot's Hospital School, Lauriston, Garleffin, 146 Craiglea Drive, Edinburgh 105	
1911		* Clark, William Inglis, D.Sc., 22 Buckingham Terrace, Edinburgh	
1903		Clarke, William Eagle, I.S.O., LL.D., F.L.S., Honorary Supervisor of the Bird Collection and formerly Keeper of the Natural History Collections in the Royal Scottish Museum, Edinburgh. 53 Castle Street, Edinburgh	
1909		Clayton, Thomas Morrison, M.D., D.Hy., B.Sc., D.P.H., Medical Officer of Health, Gateshead, 13 The Crescent, Gateshead-on-Tyne	
1922		* Clerk, Sir Dugald, K.B.E., D.Sc., F.R.S., M.Inst.C.E., etc., Lukyns, Ewhurst, Surrey	
1904	C.	Coker, Ernest George, M.A., D.Sc., Hon. D.Sc. (Sydney), F.R.S., M.Inst.C.E., M.Inst.E.E., Professor of Civil and Mechanical Engineering, University of London, University College, Gower Street, London, W.C. 110	
1904		Coles, Alfred Charles, M.D., D.Sc., York House, Poole Road, Bournemouth, W.	
1888	V. J. C.	Collie, John Norman, Ph.D., D.Sc., LL.D., F.R.S., F.C.S., F.I.C., F.R.G.S., Professor of Organic Chemistry in the University College, Gower Street, London	
1904		* Colquhoun, Walter, M.A., M.B., 18 Walmer Crescent, Ibrox, Glasgow	
1909	C.	* Comrie, Peter, M.A., B.Sc., Head Master of Leith Academy, 19 Craighouse Terrace, Edinburgh	
1886		Connan, Daniel M., M.A. 115	
1905		* Corrie, David, F.C.S., 159 Lauderdale Mansions, Maida Vale, London, W. 9.	
1914		* Coutts, William Barron, M.A., B.Sc., Senior Lecturer in Range Finding and Optics, Artillery College, Red Barracks, Woolwich, S.E. 18. 74 Vanbrugh Park, Blackheath, S.E. 3.	
1911		* Cowan, Alexander, Papermaker, Valleyfield, Penicuik, Midlothian	
1920		Craib, William Grant, M.A. (Aberdeen), Regius Professor of Botany in the University of Aberdeen	
1916	C.	Craig, E. H. Cunningham, B.A. (Cambridge), Geologist and Mining Engineer, The Dutch House, Beaconsfield 120	
1903		Craig, James Ireland, M.A., B.A., Woolwich House, The Drive, Sydenham, London, S.E. 26	
1903		Crawford, Lawrence, M.A., D.Sc., Professor of Pure Mathematics, The University, Cape Town	

Alphabetical List of the Ordinary Fellows of the Society. 429

Date of Election.		Service on Council, etc.
1922	* Crew, Francis Albert Eley, M.B., Ch.B., D.Sc., Lecturer in Genetics in the University of Edinburgh, and Director of the Animal Breeding Research Station. Castleview, Hawthornden, Midlothian	
1870	Crichton-Browne, Sir Jas., Kt., M.D., LL.D., D.Sc., F.R.S., Lord Chancellor's Visitor and Vice-President and Treasurer of the Royal Institution of Great Britain, 45 Hans Place, S.W., and Royal Courts of Justice, Strand, London	
1916	* Crombie, James Edward, M.A., LL.D., Millowner, Parkhill House, Dyce, Aberdeenshire 125	
1886	Croom, Sir John Halliday, Kt., M.D., LL.D., F.R.C.P.E., formerly Professor of Midwifery in the University of Edinburgh, late President, Royal College of Surgeons, Edinburgh. 8 Morningside Place, Edinburgh	
1914	* Cumming, Alexander Charles, O.B.E., D.Sc., Roselands, Crescent Road, Blundell Sands, Liverpool	
1917	* Cunningham, Brysson, D.Sc., B.E., M.Inst.C.E., Lecturer on Waterways, Harbours, and Docks, University College, London, 16 Beechwood Road, Sandstead, Surrey	
1898	Currie, James, M.A. Cantab., LL.D. (TREASURER), Larkfield, Goldenacre, Edinburgh	Treas. 1906-1919-22.
1919	* Cushny, Arthur Robertson, M.A., M.D., LL.D., F.R.S., Professor of Materia Medica and Pharmacology, University, Edinburgh 130	
1904	* Cuthbertson, John, Secretary, West of Scotland Agricultural College, 6 Charles Street, Kilmarnock	
1885	Daniell, Alfred, M.A., LL.B., D.Sc., Advocate, The Athenæum Club, Pall Mall, London	
1921	* Datta, Rasik Lal, D.Sc., Industrial Chemist to the Government of Bengal Department of Industries. 78 Manicktola Street, Calcutta, India	
1884	Davy, R., F.R.C.S. Eng., Consulting Surgeon to Westminster Hospital, Burstone Manor, Bow, North Devon	
1917	* Day, T. Cuthbert, Partner of the firm of Hislop & Day, 36 Hillside Crescent, Edinburgh 135	
1894	Denny, Sir Archibald, Bart., LL.D., Spencer House, Park Side, Wimbledon, S.W. 19	
1869	C. † Dewar, Sir James, Kt., M.A., LL.D., D.C.L., D.Sc., F.R.S., F.C.S., Jacksonian Professor of Natural and Experimental Philosophy in the University of Cambridge, and Fullerian Professor of Chemistry at the Royal Institution of Great Britain, London	1872-74.
1905	* Dewar, James Campbell, C.A., 27 Douglas Crescent, Edinburgh	
1906	† Dewar, Thomas William, M.D., F.R.C.P., Kincairn, Dunblane	
1876	C. Dickson, J. D. Hamilton, M.A., Senior Fellow and formerly Tutor, St Peter's College, Cambridge 140	
1885	C. Dixon, James Main, M.A., Litt. Hum. Doctor, Professor of English, University of Southern California, University Avenue, Los Angeles, California, U.S.A.	
1897	Dobbie, James Bell, F.Z.S., 12 South Inverleith Avenue, Edinburgh	
1904	* Dobbie, Sir James Johnston, Kt., M.A., D.Sc., LL.D., F.R.S., formerly Principal of the Government Laboratories, London. Fairlie Cottage, Fairlie, Ayrshire	1905-08.
1881	C. Dobbin, Leonard, Ph.D., Lecturer in Chemistry in the University of Edinburgh, 6 Wilton Road, Edinburgh	1904-07, 1913-16.
1918	* Dodd, Alexander Scott, B.Sc., F.I.C., F.C.S., City Analyst for Edinburgh, 20 Stafford Street, Edinburgh 145	
1905	* Donaldson, Rev. Wm. Galloway, J.P., F.R.G.S., F.E.I.S., The Manse, Forfar	
1882	C. Dott, David B., F.I.C., Memb. Pharm. Soc., Ravenslea, Musselburgh	
1921	B. C. * Dougall, John, M.A., D.Sc., Publisher's Reader, 26 Underwood Street, Langside, Glasgow	
1901 & 1918	* Douglas, Carstairs Cumming, M.D., D.Sc., Professor of Medical Jurisprudence and Hygiene, Anderson's College, Glasgow, 110 South Brae Drive, Jordanhill, Glasgow	
1910	* Douglas, Loudon MacQueen, Author and Lecturer, 29 W. Saville Terrace, Newington, Edinburgh 150	
1908	C. Drinkwater, Harry, M.D., M.R.C.S. (Eng.), F.L.S., Lister House, Wrexham, North Wales	
1901	* Drinkwater, Thomas W., L.R.C.P.E., L.R.C.S.E., Chemical Laboratory, Surgeons' Hall, Edinburgh	
1917	* Dron, Robert W., A.M.Inst.C.E., M.Inst.M.E., Professor of Mining in the University of Glasgow	
1921	* Drysdale, Charles Vickery, D.Sc. (Lond.), M.I.E.E., F.Inst.P., O.B.E., Superintendent of the Admiralty Research Laboratory, Teddington, Middlesex	
1904	* Dunlop, William Brown, M.A., 4A St Andrew Square, Edinburgh 155	

Date of Election.			Service on Council, etc.
1903		Dunstan, John, M.R.C.V.S., Inversnaid, Liskeard, Cornwall	
1892	C.	Dunstan, M. J. R., M.A., F.I.C., F.C.S., Principal, South-Eastern Agricultural College, Wye, Kent	
1906	C.	Dyson, Sir Frank Watson, Kt., M.A., D.Sc., LL.D., F.R.S., Astronomer Royal, Royal Observatory, Greenwich	1907-10.
1893		Edington, Alexander, M.D., Howick, Natal	
1904		* Edwards, John, LL.D., 4 Great Western Terrace, Kelvinside, Glasgow	160
1904		* Elder, William, M.D., F.R.C.P.E., 4 John's Place, Leith	
1913 & 1921		* Elliot, George Francis Scott, M.A. (Cantab.), B.Sc., F.R.G.S., F.L.S., Drumwhill, Mossdale	
1906	C.	* Ellis, David, D.Sc., Ph.D., Lecturer in Botany and Bacteriology, Royal Technical College, Glasgow	
1897	C.	Erskine-Murray, James Robert, D.Sc., H.M. Signal School, Portsmouth	
1879	C. N.	Ewart, James Cossar, M.D., F.R.C.S.E., F.R.S., F.Z.S., Regius Professor of Natural History, University of Edinburgh, Craigyfield, Penicuik, Midlothian	1882-85, 1904-07. V-P 1907-12.
1902		* Ewen, John Taylor O.B.E., B.Sc., M.I.Mech.E., J.P., H.M. Inspector of Schools, Clairmont, 54 Albert Drive, Pollokshields, Glasgow	
1878	C.	Ewing, Sir James Alfred, K.C.B., M.A., D.Sc., LL.D., M.Inst.C.E., F.R.S., J.P. (VICE-PRESIDENT), Principal of the University of Edinburgh, formerly Director of Naval Education, Admiralty, 16 Moray Place, Edinburgh	1888-91, 1919-20. V-P 1920-
1900	C.	Eyre, John W. H., M.D., M.S. (Dunelm), D.P.H. (Camb.), Professor of Bacteriology, Guy's Hospital, London	
1910	C.	* Fairgrieve, Mungo M'Callum, M.A. (Glasg.), M.A. (Cambridge), Master at the Edinburgh Academy, 37 Queen's Crescent, Edinburgh	
1907	C.	Falconer, John Downie, M.A., D.Sc., F.G.S., Lecturer on Geography, The University, Glasgow	170
1888	C.	Fawsitt, Charles A., Glenfruin, Milton Road, Harpenden, Herts	
1883	C.	Felkin, Robert W., M.D., F.R.G.S., Whare Ra, Havelock North, Hawkes Bay, New Zealand	
1899		* Fergus, Andrew Freeland, M.D., LL.D., 14 Newton Place, Chaing Cross, Glasgow	
1907		* Fergus, Edward Oswald, Aineroft, Bridge of Weir, Renfrewshire	
1904		* Ferguson, James Haig, M.D., F.R.C.P.E., F.R.C.S.E., 7 Coates Crescent, Edinburgh	175
1898		Findlay, Sir John R., K.B.E., M.A. Oxon., LL.D., 3 Rothesay Terrace, Edinburgh	
1899		* Finlay, David W., B.A., M.D., LL.D., F.R.C.P., D.P.H., Emeritus Professor of Medicine in the University of Aberdeen, Honorary Physician to His Majesty in Scotland, Balgownie, Helensburgh	
1911		Fleming, John Arnold, F.C.S., etc., Pottery Manufacturer, Locksley, Helensburgh, Dumbartonshire	
1906		* Fleming, Robert Alexander, M.A., M.D., F.R.C.P.E., Physician, Royal Infirmary, 10 Chester Street, Edinburgh	
1900	C. N.	* Flett, John S., M.A., D.Sc., LL.D., F.R.S., O.B.E., Director of the Geological Survey of Great Britain and of the Museum of Practical Geology, London, 28 Jermyn Street, S.W. 1	180 1916-19.
1872		Forbes, George, M.A., M.Inst.C.E., M.Inst.E.E., F.R.S., F.R.A.S., formerly Professor of Natural Philosophy in Anderson's College, Glasgow. 11 Little College Street, Westminster, S.W.	
1892		Ford, John Simpson, F.C.S., 7 Corrennie Drive, Edinburgh	
1921		* Forrest, George Topham, Architect to the London County Council, and Superintending Architect of Metropolitan Buildings, New County Hall, Westminster Bridge, London, S.W.	
1920	C.	* Franklin, Thomas Bedford, B.A. (Hons. Mathematics), Cambridge, Stancliffe Hall, near Matlock, Derbyshire	
1910		* Fraser, Alexander, Actuary, 15 S. Learmonth Gardens, Edinburgh	185
1896		Fraser, John, M.B., F.R.C.P.E., formerly one of H.M. Commissioners in Lunacy for Scotland, 54 Great King Street, Edinburgh	
1915		* Fraser, Rev. Joseph Robert, U.F. Manse, Kinneff, Bervie	
1914		* Fraser, William, Managing Director, Neill & Co., Ltd., Printers, 212 Causeway-side, Edinburgh	
1891		Fulton, T. Wemyss, M.D., Scientific Superintendent, Scottish Fishery Board, 41 Queen's Road, Aberdeen	
1907		* Galbraith, Alexander, "Ravenswood," Dalmuir, Dumbartonshire	190

Date of Election.		Service on Council, etc.
1888	C. Galt, Alexander, D.Sc., late Keeper of the Department of Technology, Royal Scottish Museum, Edinburgh, c/o Clydesdale Bank, 1 Melville Pl., Edinburgh	
1901	Ganguli, Sanjiban, M.A., Principal, Maharaja's College, and Director of Public Instruction, Jaipur State, Jaipur, India	
1909	C. * Geddes, Rt. Hon. Sir Auckland C., K.C.B., G.C.M.G., M.D., D.C.L., British Ambassador to the U.S.A., The British Embassy, Washington	
1880	C. Geddes, Patrick, Professor of Sociology and Civics, University of Bombay, India.	
1861	C. B. Geikie, Sir Archibald, O.M., K.C.B., D.C.L. Oxf., D.Sc., LL.D., Ph.D., Past-Pres. R.S., Foreign Member of the Reale Accad. Lincei, Rome, of the National Acad. of the United States, of the Academies of Stockholm, Christiania, Göttingen, Corresponding Member of the Institute of France and of the Academies of Berlin, Vienna, Munich, Turin, Belgium, Philadelphia, New York, etc., Shepherd's Down, Haslemere, Surrey 195	1869-72, 1874-76, 1879-82.
1914	Gemmell, John Edward, M.B., C.M., Hon. Surgeon, Hospital for Women and Maternity Hospital; Hon. Gynaecologist, Victoria Central Hospital, Liscard, 28 Rodney Street, Liverpool	
1909	* Gentle, William, B.Sc., 12 Mayfield Road, Edinburgh	
1920	C. * Ghosh, Sudhamoy, M.Sc. (Cal.), D.Sc. (Edin.), F.C.S., Professor of Chemistry, School of Tropical Medicine and Hygiene, Central Avenue, Calcutta, India	
1914	* Gibb, Sir Alexander, G.B.E., C.B., formerly Director-General of Civil Engineering, Ministry of Transport. Queen Anne's Lodge, Westminster, London, S.W. 1.	
1916	* Gibb, A. W., M.A., D.Sc., Professor of Geology in the University of Aberdeen, 1 Belvidere Street, Aberdeen 200	
1910	C. * Gibb, David, M.A., B.Sc., Lecturer in Mathematics, Edinburgh University, 15 South Lauder Road, Edinburgh	
1917	C. * Gibson, Alexander, M.B., Ch.B., F.R.C.S. (Eng.), 661 Broadway, Winnipeg, Canada	
1910	* Gibson, Charles Robert, Lynton, Mansewood, by Pollokshaws	
1890	Gibson, George A., M.A., LL.D. Professor of Mathematics in the University of Glasgow, 10 The University, Glasgow	1905-08, 1912-13. V-P 1917-20.
1921	* Gibson, Walcot, D.Sc., F.G.S., Assistant Director, H.M. Geological Survey (Scotland), 33 George Square, Edinburgh 205	
1911	Gidney, Lt.-Col. Henry A. J., F.R.C.S., D.O., D.P.H., J.P., M.L.A., I.M.S. (retired), Army Specialist Public Health, c/o Thomas Cook & Sons, Ludgate Circus, London	
1900	Gilechrist, Douglas A., B.Sc., Professor of Agriculture and Rural Economy, Armstrong College, Newcastle-upon-Tyne	
1907	Gilruth, John Anderson, M.R.C.V.S., D.V.Sc. (Melb.), Administrator, Government House, Darwin Northern Territory, Australia	
1909	* Gladstone, Hugh Steuart, M.A., M.B.O.U., F.Z.S., Capenoch, Thornhill, Dumfriesshire	
1911	Gladstone, Reginald John, M.D., F.R.C.S. (Eng.), Lecturer and Senior Demonstrator of Anatomy, King's College, University of London, 22 Regent's Park Terrace, London, N.W. 210	
1898	Glaister, John, M.D., F.R.F.P.S. Glasgow, D.P.H. Camb., Regius Professor of Forensic Medicine and Public Health in the University of Glasgow, 3 Newton Place, Glasgow	
1910	Goodall, Joseph Strickland, M.B. (Lond.), M.R.C.P., F.R.C.S. Ed., M.S.A. (Eng.), Professor of Physiology and Biology, City of London Hospital, London. 136 Harley Street, London, W. 1.	
1901	Goodwillie, James, M.A., B.Sc., Liberton, Edinburgh	
1920	C. * Gordon, William, B.Sc., A.M.I.Mech.E., Lecturer in Engineering in the University of Edinburgh, 3 Wellington Street, Edinburgh	
1913	C. * Gordon, William Thomas, M.A., D.Sc. (Edin.), M.A. (Cantab.), Professor of Geology, University of London, King's College, Strand, W.C. 215	
1897	Gordon-Munn, John Gordon, M.D., Heigham Hall, Norwich	
1898	C. Gray, Albert A., M.D., 4 Clairmont Gardens, Glasgow	
1883	C. Gray, Andrew, M.A., LL.D., F.R.S., Professor of Natural Philosophy in the University of Glasgow	1903-06. V-P 1906-09.
1910	Gray, Bruce M'Gregor, C.E., A.M.Inst.C.E., Westbourne Grove, Selby, Yorkshire	
1909	C. * Gray, James Gordon, D.Sc., Professor of Applied Physics in the University of Glasgow 11 The University, Glasgow 220	1913-15.

Date of Election.		Service on Council, etc.
1918	* Gray, Wm. Forbes, F.S.A. (Scot.), Editor and Author, 8 Mansionhouse Road, Edinburgh	
1897	Greenlees, Thomas Duncan, M.D. Edin., 5 Warrior Square Terrace, St Leonards-on-Sea, Sussex	
1922	* Greenwood, William Osborne, M.D. (Leeds), B.S. (Lond.), L.S.A., Obstetric Surgeon and Physician, Woodroyd, 17 Ripon Road, York	
1905	C. * Gregory, John Walter, D.Sc., F.R.S. (VICE-PRESIDENT), Professor of Geology in the University of Glasgow, 4 Park Quadrant, Glasgow	1908-11 V-P 1920-
1906	Greig, Edward David Wilson, C.I.E., M.D., D.Sc., Lt.-Col., H.M. Indian Medical Service, Pasteur Institute, Kasauli, India	225
1905	Greig, Sir Robert Blyth, M.C., LL.D., F.Z.S., Chairman of the Board of Agriculture for Scotland, 29 St Andrew Square, Edinburgh	1921-
1910	* Grimshaw, Percy Hall, Assistant Keeper, Natural History Department, The Royal Scottish Museum, 49 Comiston Drive, Edinburgh	
1899	* Guest, Edward Graham, M.A., B.Sc., 5 Newbattle Terrace, Edinburgh	
1907	C. * Gulliver, Gilbert Henry, D.Sc., A.M.I.Mech.E., 99 Southwark Street, London, S.E.	
1911	C. * Gunn, James Andrew, M.A., M.D., D.Sc., Professor of Pharmacology in the University of Oxford	230
1888	C. Guppy, Henry Brougham, M.B., Rosario, Salcombe, Devon	
1922	* Guthrie, William Alexander, M.B.E., F.I.C., Memb. Inst. Petroleum Technologists, Oil Refining Expert; at present Petroleum Research Chemist with the Egyptian Government. Turf Club, Cairo	
1911	* Guy, William, F.R.C.S., L.R.C.P., L.D.S.Ed., Consulting Dental Surgeon, Edinburgh Royal Infirmary; Dean, Edinburgh Dental Hospital and School; Lecturer on Human and Comparative Dental Anatomy and Physiology, 11 Wemyss Place, Edinburgh	
1911	Hall-Edwards, John Francis, L.R.C.P. (Edin.), Hon. F.R.P.S., Senior Medical Officer in charge of X-ray Department, General Hospital, Birmingham, 141A and 141B Great Charles Street (Newhall Street), Birmingham	
1922	* Hannay, Robert Kerr, M.A., Professor of Ancient History and Palæography in the University of Edinburgh. 14 Inverleith Terrace, Edinburgh	235
1918	* Hardie, P. S., M.A., B.Sc., Physics Department, School of Medicine, Cairo, Egypt	
1896	C. Harris, David Fraser, B.Sc. (Lond.), D.Sc. (Birm.), M.D., F.S.A. Scot., Professor of Physiology in the Dalhousie University, Halifax, Nova Scotia	
1914	Harrison, Edward Philip, Ph.D., Professor of Physics, Presidency College, University of Calcutta, The Observatory, Alipore, Calcutta	
1921	* Harrison, John William Heslop, D.Sc. (Durham), Lecturer in Genetics, Armstrong College, Newcastle. The Avenue, Birtley, Co. Durham	
1914	C. Harvey-Gibson, Robert John, C.B.E., D.L., J.P., M.A., D.Sc., Mem. Roy. Dub. Soc., Emer. Professor of Botany, University of Liverpool. "Beckallans," Grasmere, Westmorland.	240
1892	C. Heath, Thomas, B.A., formerly Assistant Astronomer, Royal Observatory, Edinburgh, 7 Comiston Road, Edinburgh	
1893	Hehir, Sir Patrick, K.C.I.E., C.B., C.M.G., F.R.C.P.E., F.R.C.S.E., D.P.H. (Camb.), D.T.M. (Liv. Univ.), Maj.-General I.M.S. (retired), Ennisfarne, Westward Ho! Devon	
1900	Henderson, John, D.Sc., A.Inst.E.E., Kinnoull. Gregory's Road, Beaconsfield, Bucks	
1908	* Henderson, William Dawson, M.A., B.Sc., Ph.D., Lecturer, Zoological Laboratories, University, Bristol	
1890	C. Hepburn, David, C.M.G., M.D., Professor of Anatomy in the University College of South Wales and Monmouthshire, Cardiff	245
1881	C. N. Herdman, Sir W. A., Kt., C.B.E., D.Sc., LL.D., F.R.S., Past Pres. L.S., Pres. Brit. Assoc., Emeritus Professor of Natural History in the University of Liverpool. Croxteth Lodge, Ullet Road, Liverpool, and Rowany, Port Erin, I.O.M.	
1916	* Herring, Percy Theodore, M.D., F.R.C.P.Ed., Professor of Physiology, University of St Andrews, Hepburn Gardens, St Andrews	1917-20.
1922	Hindle, Edward, M.A. (Cantab.), Ph.D., A.R.C.Sc., Professor of Biology, School of Medicine, Cairo, Egypt	
1902	Hinxman, Lionel W., B.A., formerly of the Geological Survey of Scotland. 4 Morant Gardens, Ringwood, Hants	
1904	Hobday, Major Frederick T. G., C.M.G., F.R.C.V.S., Officier du Merite Agricole, Cavaliere dei S.S. Maurizio e Lazzaro, Hon. Veterinary Surgeon to H.M. the King, Editor of the Veterinary Journal, 165 Church Street, Kensington, London, W.	250

Alphabetical List of the Ordinary Fellows of the Society. 433

Date of Election.			Service on Council, etc.
1885		Hodgkinson, W. R., C.B.E., M.A., Ph.D., F.I.C., F.C.S., Professor of Chemistry and Physics at the Ordnance College, Woolwich, 89 Shooter's Hill Road, Blackheath, Kent	
1911		Holland, William Jacob, LL.D. St Andrews, etc., Director Carnegie Institute, Pittsburg, Pa., 5545 Forbes Street, Pittsburg, Pa., U.S.A.	
1920	C.	* Horne, Alexander Robert, O.B.E., B.Sc., M.I.Mech.E., A.M.I.C.E., Head of the School of Engineering, The Polytechnic, Regent Street, London. 31 Methuen Park, Muswell Hill, London, N. 10	
1881	C. N.	Horne, John, LL.D., F.R.S., F.G.S., formerly Director of the Geological Survey of Scotland, 20 Merchiston Gardens, Edinburgh	1902-05, 1906-07, 1914-15. V.P 1907-13. P 1915-19.
1896		Horne, J. Fletcher, M.D., F.R.C.S.E., The Poplars, Barnsley 255	
1904	C.	* Horsburgh, Ellice Martin, M.A., D.Sc., Reader in Technical Mathematics, University of Edinburgh, 11 Granville Terrace, Edinburgh	
1897		Houston, Sir Alex. Cruikshanks, K.B.E., C.V.O., M.B., C.M., D.Sc., 19 Fairhazel Gardens, South Hampstead, London, N.W.	
1912	C. B.	* Houstoun, Robert Alexander, M.A., Ph.D., D.Sc., Lecturer in Physical Optics, University, Glasgow, 45 Kirklee Road, Glasgow	
1893		Howden, Robert, M.A., M.B., C.M., D.Sc., Professor of Anatomy in the University of Durham, 14 Burdon Terrace, Newcastle-upon-Tyne	
1883	C.	Hoyle, William Evans, M.A., D.Sc., M.R.C.S., Director of the Welsh National Museum : Crowland, Llandaff, Wales 260	
1910		Hume, William Fraser, D.Sc. (Lond.), Director, Geological Survey of Egypt, Helwán, Egypt. 4 Ma'adi, near Cairo, Egypt	
1916		* Hunter, Charles Stewart, L.R.C.P.E., L.R.C.S.E., D.P.H., Walden, Anerley Road, London, S.E. 20	
1911		Hunter, Gilbert Macintyre, M.Inst.C.E., M.Inst.E.S., M.Inst.M.E., Resident Engineer, Nitrate Railways, Iquique, Chile, and Maybole, Ayrshire	
1887	C.	Hunter, William, M.D., M.R.C.P.L. and E., M.R.C.S., 103 Harley Street, London	
1908		Hyslop, Theophilus Bulkeley, M.D., M.R.C.P.E., 5 Portland Place, London, W. 265	
1920		* Inglis, James Gall, Publisher and Editor of Educational Works, Edinburgh, 36 Blacket Place, Edinburgh	
1912		* Inglis, Robert John Mathieson, M.Inst.C.E., District Engineer, W. Division, Lond. & N.E. Railway, S. Section. Tintah, Peebles	
1904	C.	Innes, R. T. A., Director, Government Observatory, Johannesburg, Transvaal	
1917		* Irvine, James Colquhoun, C.B.E., Ph.D., D.Sc., LL.D., F.R.S. (VICE-PRESIDENT), Principal of the University of St Andrews	1920-22. V.-P. 1922-
1914		Jack, John Noble 270	
1875		Jack, William, M.A., LL.D., D.Sc., Emeritus Professor of Mathematics in the University of Glasgow	1888-91.
1889	C.	James, Alexander, M.D., F.R.C.P.E., 9 Randolph Crescent, Edinburgh	
1901		* Jardine, Robert, M.D., M.R.C.S., F.R.F.P.S. Glas., 20 Royal Crescent, Glasgow	
1912	C.	* Jeffrey, George Rutherford, M.D. (Glasg.), F.R.C.P. (Edin.), etc., Bootham Park Private Mental Hospital, York	
1906	C.	* Jehu, Thomas John, M.A., M.D., F.G.S., Professor of Geology in the University of Edinburgh : 35 Great King Street, Edinburgh 275	1917-20.
1900		* Jerdan, David Smiles, M.A., D.Sc., Ph.D., 26 Avenue du Château d'Eau, Saventhem, Belgium	
1895		Johnston, Col. Henry Halero, C.B., D.Sc., M.D., F.L.S., late A.M.S., Orphir House, Kirkwall, Orkney	
1903	C.	* Johnston, Thomas Nicol, M.B., C.M., Pogbie, Humble, East Lothian	
1874		Jones, Francis, M.Sc., Lecturer in Chemistry, 17 Whalley Road, Whalley Range, Manchester	
1888		Jones, John Alfred, M.Inst.C.E., Fellow of the University of Madras, Sanitary Engineer to the Government of Madras, c/o Messrs Parry & Co., 70 Gracechurch Street, London 280	
1922		* Juritz, Charles Frederick, M.A., D.Sc., F.I.C., Chief of the Union Department of Chemistry, Villa Marina, Three Anchor Bay, Cape Town, South Africa	
1915		Kemnal, Sir James Hermann Rosenthal, Managing Director and Engineer-in-Chief of Babcock & Wilcox, Ltd., Kemnal Manor, Chislehurst, Kent	

Date of Election.			Service on Council, etc.
1912		Kennedy, Robert Foster, M.D. (Queen's Univ., Belfast), M.B., B.Ch. (R.U.I.), Assistant Professor of Neurology, Cornell University, New York, 20 West 50th Street, New York, U.S.A.	
1909		Kenwood, Henry Richard, C.M.G., M.B., C.M., Chadwick Professor of Hygiene in the University of London, "Wadhurst," Queen's Road, Finsbury Park, London, N.	
1908		* Kerr, Andrew William, F.S.A.Scot., 81 Great King Street, Edinburgh	285
1903 & 1923	C. N.	* Kerr, John Graham, M.A., F.R.S., F.L.S., F.Z.S., Regius Professor of Zoology, University of Glasgow, 9 The University, Glasgow	1904-07, 1913-16.
1891		Kerr, Joshua Law, M.D., 16 High Street, Swindon, Wilts	
1913		* Kerr, Walter Hume, M.A., B.Sc., Lecturer on Engineering Drawing and Structural Design in the University of Edinburgh	
1908		Kidd, Walter Aubrey, M.D., 2 Suffolk Square, Cheltenham	
1886	C. N.	Kidston, Robert, LL.D., D.Sc., F.R.S., F.G.S., 12 Clarendon Place, Stirling	290 1891-94, 1903-06. Sec. 1909-16. V-P 1917-20.
1907		* King, Archibald, M.A., B.Sc., formerly Rector of the Academy, Castle Douglas; H.M. Inspector of Schools, Inverspey, Fochabers, Morayshire	
1880		† King, W. F., Lonend, Russell Place, Trinity, Leith	
1918		* Kingon, Rev. John Robert Lewis, M.A. (Edin. and Cape of Good Hope), D.Sc. (Ghent), F.L.S., Observatory Congregational Church, Lower Main Road, Observatory, Cape Town, C.P., South Africa	
1878		Kintore, The Right Hon. the Earl of, P.C., G.C.M.G., M.A. Cantab., LL.D. Cambridge, Aberdeen, and Adelaide, Keith Hall, Inverurie, Aberdeenshire	
1901		* Knight, Rev. G. A. Frank, M.A., 5 Granby Terrace, Hillhead, Glasgow	295
1907		* Knight, James, M.A., D.Sc., F.C.S., F.G.S., Head Master, Queen's Park High School, Enterkin, Douglas Gardens, Uddingston, by Glasgow	
1921		* Lamb, James Alexander George, Banker, 11 Braid Crescent, Edinburgh	
1920		* Lamont, John Charles, Lient.-Col., I.M.S. (retired), C.I.E., M.B., C.M. (Edin.), M.R.C.S. (Eng.), 7 Merchiston Park, Edinburgh	
1878	C.	Lang, Sir P. R. Scott, Kt., M.A., LL.D., B.Sc., Emeritus Professor of Mathematics, University of St Andrews	
1910	C.	* Lauder, Alexander, D.Sc., Lecturer in Agricultural Chemistry, Edinburgh and East of Scotland College of Agriculture, 13 George Square, Edinburgh	300 1917-20.
1885	C.	Laurie, A. P., M.A., D.Sc., J.P., Principal of the Heriot-Watt College, Edinburgh	1908-11, 1913-16.
1894	C.	Laurie, Malcolm, B.A., D.Sc., F.L.S., 4 Wordsworth Road, Harpenden, Herts	
1921		* Laurie, The Rev. Albert Ernest, M.C., C.F., Rector of Old St Paul's, Edinburgh, and Canon of St Mary's Cathedral, Edinburgh. Lauder House, Jeffrey Street, Edinburgh	
1910	C. B.	* Lawson, A. Anstruther, B.Sc., Ph.D., D.Sc., F.L.S., Professor of Botany, University of Sydney, New South Wales, Australia	
1905		* Lawson, David, M.A., M.D., L.R.C.P. and S.E., Druimdarroch, Banchory, Kincardineshire	305
1910	C.	* Lee, Gabriel W., D.Sc., Palæontologist, Geological Survey of Scotland, 33 George Square, Edinburgh	
1903		* Leighton, Gerald Rowley, O.B.E., M.D., Medical Officer, Scottish Board of Health, 125 George Street, Edinburgh	
1910		Levie, Alexander, F.R.C.V.S., D.V.S.M., Rannock, Carlton Road, Derby	
1916	C.	* Levy, Hyman, M.A., D.Sc., Assistant Professor of Mathematics, Imperial College of Science and Technology, London, S.W. 7, "Eskbank," 105 Cambridge Road, Teddington, Middlesex	
1914	C. N.	Lewis, Francis John, D.Sc., F.L.S., Professor of Biology, University of Alberta, Edmonton South, Alberta, Canada	310 1919-22
1918		* Lidstone, George James, F.F.A., F.I.A., Manager and Actuary of the Scottish Widows' Fund Life Assurance Society, 8 Eglinton Crescent, Edinburgh	
1905		* Lightbody, Forrest Hay, 53 Queen Street, Edinburgh	
1889		Lindsay, Rev. James, M.A., D.D., B.Sc., F.R.S.L., F.G.S., M.R.A.S., Corresponding Member of the Royal Academy of Sciences, Letters and Arts, of Padua, Associate of the Philosophical Society of Louvain, Annick Lodge, Irvine	
1912		* Lindsay, John George, M.A., B.Sc. (Edin.), Rector of Dunfermline High School	
1920	C.	* Lindsay, Thomas A., M.A. (Hons.), B.Sc., Head Master, Higher Grade School, Bucksburn, Aberdeenshire	315

Alphabetical List of the Ordinary Fellows of the Society. 435

Date of Election.		Service on Council, etc.
1912	* Linlithgow, The Most Honourable the Marquis of, Hopetoun House, South Queensferry	
1903	† Liston, William Glen, M.D., Lt.-Col. Indian Medical Service, 11 Learmonth Gardens, Edinburgh	
1903	* Littlejohn, Henry Harvey, M.A., M.B., B.Sc., F.R.C.S.E., Professor of Forensic Medicine, and late Dean of the Faculty of Medicine in the University of Edinburgh, 11 Rutland Street, Edinburgh	
1898	Lothian, Alexander Veitch, M.A., B.Sc., Training College, Jordanhill, Glasgow	
1888	Lowe, D. F., M.A., LL.D., J.P., formerly Headmaster of Heriot's Hospital School, Lauriston, 19 George Square, Edinburgh 320	
1900	† Lusk, Graham, Ph.D., M.A., Professor of Physiology, Cornell University Medical College, New York, N. Y., U.S.A.	
1894	Mabbott, Walter John, M.A., Rector of County High School, Duns, Berwickshire	
1887	M'Aldowie, Alexander M., M.D., 8 Holland Road, Cheltenham	
1917	* Macalister, Sir Donald, K.C.B., M.D., M.A., B.Sc., Principal of the University of Glasgow, The University, Glasgow	
1907	MacAlister, Donald Alexander, A.R.S.M., F.G.S., 10 St Alban's Road, Kensington, London, W. 8 325	
1921	* M'Arthur, Neil, M.A., B.Sc., Lecturer in Mathematics, Glasgow University, 31 Windsor Terrace, Glasgow	
1883	M'Bride, P., M.D., F.R.C.P.E., 20 South Drive, Harrogate	
1903	M'Cormick, Sir W. S., M.A., LL.D., Chairman of the Advisory Council, Department of Scientific and Industrial Research, 16-18 Old Queen Street, Westminster, S. W. 1 1910-13.	
1918	* M'Culloch, Rev. James David, 43 Brougham Street, Greenock	
1905	* Macdonald, Hector Munro, M.A., F.R.S., Professor of Mathematics, University of Aberdeen, 52 College Bounds, Aberdeen 330 1908-11.	
1897	C. Macdonald, James A., M.A., B.Sc., H.M. Inspector of Schools, Meldon, Southside Road, Inverness	
1904	* Macdonald, John A., M.A., B.Sc., King Edward VII School, Johannesburg, Transvaal	
1920	* M'Donald, Stuart, M.A., M.D., F.R.C.P.E., Professor of Pathology, School of Medicine, Newcastle-on-Tyne	
1904	Macdonald, William, M.S. Agr., Sc.D., Ph.D., D.Sc., Editor, <i>Agricultural Journal</i> of South Africa, Rand Club, Johannesburg, Transvaal	
1886	Macdonald, William J., M.A., LL.D., 15 Comiston Drive, Edinburgh 335	
1901	C. * MacDougall, R. Stewart, M.A., D.Sc., Professor of Biology, Royal Veterinary College, Edinburgh, 9 Dryden Place, Edinburgh 1914-17.	
1910	Macewen, Hugh Allen, M.B., Ch.B., D.P.H. (Lond. and Camb.), Local Government Board, Ministry of Health, Whitehall, London, S. W.	
1888	C. M'Fadyean, Sir John, Kt., M.B., B.Sc., LL.D., Principal, and Professor of Comparative Pathology in the Royal Veterinary College, Camden Town, London	
1885	C. Macfarlane, J. M., D.Sc., LL.D., Emeritus Professor of Botany, 4320 Osage Avenue, Philadelphia, Pennsylvania, U.S.A.	
1897	MacGillivray, Angus, C.M., M.D., D.Sc., F.S.A. (Scot.), 23 South Tay Street, Dundee 340	
1878	M'Gowan, George, F.I.C., Ph.D., 21 Montpelier Road, Ealing, London, W. 5	
1922	* Macgregor, Murray, M.A., B.Sc., District Geologist (Scotland), H.M. Geological Survey, 33 George Square, Edinburgh	
1903	* M'Intosh, Donald C., M.A., D.Sc., Education Offices, Elgin	
1911	M'Intosh, John William, A.R.C.V.S., Dollis Hill Farm, Cricklewood, London, N. W. 2	
1869	C. N. M'Intosh, William Carmichael, M.D., LL.D., F.R.S., F.L.S., Emeritus Professor of Natural History in the University of St Andrews, Pres. Ray Society, 2 Abbotsford Crescent, St Andrews 345 1885-88.	
1895	C. Macintyre, John, M.D., LL.D., 179 Bath Street, Glasgow	
1912	C. M'Kendrick, Anderson Gray, M.B., Major, Indian Medical Service, Superintendent, Research Laboratory, Royal College of Physicians, Edinburgh	
1914	* M'Kendrick, Archibald, F.R.C.S.E., D.P.H., L.D.S., 12 Rothesay Place, Edinburgh	
1873	C. B. M'Kendrick, John G., M.D., F.R.C.P.E., LL.D., F.R.S., Emeritus Professor of Physiology in the University of Glasgow, Maxieburn, Stonehaven } 1875-78, 1885-88, 1893-94, 1900-02, V-P 1894-1900.	
1900	C. * M'Kendrick, John Souttar, M.D., F.R.F.P.S.G., 2 Buckingham Terrace, Hillhead, Glasgow 350	

Date of Election.		Service on Council, etc.
1910	C. * Mackenzie, Alistair, M.A., M.D., D.P.H., Principal, College of Hygiene and Physical Training, Dunfermline	
1916	C. * Mackenzie, John E., D.Sc., Lecturer in Chemistry, University of Edinburgh, Major-Adjutant, O.T.C., 2A Ramsay Garden, Edinburgh	
1894	Mackenzie, Robert, M.D., Napier, Nairn	
1904	C. * Mackenzie, Sir W. Leslie, M.A., M.D., D.P.H., LL.D., Medical Member of the Scottish Board of Health, 14 Belgrave Place, Edinburgh	
1918	* Mackie, Wm., M.A., M.D., D.P.H., 13 North Street, Elgin	355
1910	* MacKinnon, James, M.A., Ph.D., Professor of Ecclesiastical History, Edinburgh University, 12 Lygon Road, Edinburgh	
1904	* Mackintosh, Donald James, C.B., M.V.O., M.B., C.M., LL.D., Supt. Western Infirmary, Glasgow	
1899	Maclean, Ewan John, M.D., M.R.C.P. Lond., J.P., Professor of Obstetrics and Gynæcology, Welsh National School of Medicine, 12 Park Place, Cardiff	
1888	C. Maclean, Magnus, M.A., D.Sc., LL.D., M.Inst.C.E., M.I.E.E., Professor of Electrical Engineering in the Royal Technical College, 51 Kerrsland Terrace, Hillhead, Glasgow	1916-19.
1913	* M'Lellan, Dugald, M.Inst.C.E., Divisional Engineer, L.M. and S. Railway, Buchanan Street Station, Glasgow	360
1916	C. * M'Lintock, W. F. P., D.Sc. (Edin.), Museum of Practical Geology, 28 Jermyn Street, London, S.W. 1	
1907	C. * Macnair, Professor Peter, Curator of the Natural History Collections in the Glasgow Museums, Kelvingrove Museum, Glasgow	
1917	* Macpherson, Rev. Hector Copland, M.A., F.R.A.S., Guthrie Memorial U.F. Church, 30 Pilrig Street, Edinburgh	
1921	* M'Quistan, Dougal Black, M.A., B.Sc., Head Mathematical Master in Allan Glen's School, Glasgow, 29 Viewpark Drive, Rutherglen, near Glasgow	
1921	C. * MacRobert, Thomas Murray, M.A., D.Sc., Lecturer in Mathematics in the University of Glasgow, 6 Lothian Gardens, Kelvinside, N., Glasgow	365
1921	C. * M'Whan, John, M.A. (Glasgow), Ph.D. (Gött.), Lecturer in Mathematics in the University of Glasgow, 59 Greenlees Road, Cambuslang	
1898	C. Mahalanobis, S. C., B.Sc., Professor of Physiology, Presidency College, Calcutta, India	
1913	Majumdar, Tarak Nath, D.P.H. (Cal.), L.M.S., F.C.S., Health Officer, Calcutta, IV, 37 Lower Chitpore Road, Calcutta, India	
1917	* Malcolm, L. W. Gordon, M.Sc. (Cantab.), F.R.G.S., F.R.S.G.S., Capt. R.G.A. (R.O.), Department of Archaeology and Ethnology, Bristol Museum and Art Gallery	
1908	Mallik, Devendranath, Sc.D., B.A., Professor of Mathematics, Astronomical Observatory, Presidential College, Calcutta, India	370
1912	Maloney, William Joseph, M.D. (Edin.), Professor of Neurology at Fordham University, New York City, N.Y., U.S.A.	
1913	Marchant, Rev. Sir James, K.B.E., LL.D., F.R.A.S., F.L.S., Director, National Council for Promotion of Race-Regeneration, Rhondda House, 60 Gower Street, London, W.C. 1	
1909	C. B. * Marshall, C. R., M.D., M.A., LL.D., Professor of Materia Medica, Marischal College, Aberdeen	1915-18.
1882	C. Marshall, D. H., M.A., Em.-Professor of Physics, Queen's University, Elmtree House, Union Street, W., Kingston, Ontario, Canada	
1901	C. Marshall, F. H. A., Sc.D., F.R.S., Lecturer on Agricultural Physiology in the University of Cambridge, Christ's College, Cambridge	375
1920	C. * Marshall, John, M.A., D.Sc. (St Andrews), B.A. (Cantab.), Senior Lecturer in Mathematics, University College, Swansea	
1913	Masson, George Henry, M.D., D.Sc., M.R.C.P.E., Port of Spain, Trinidad, British West Indies	
1885	C. Masson, Sir David Orme, K.B.E., M.A., D.Sc., F.R.S., Professor of Chemistry in the University of Melbourne	
1898	C. B. Masterman, Arthur Thomas, M.A., D.Sc., F.R.S., formerly Superintending Inspector, H.M. Board of Agriculture and Fisheries, Mill House, Wooburn Green, Bucks	1902-04.
1911	Mathews, Gregory Macalister, F.L.S., F.Z.S., Foulis Court, Fair Oaks, Hants	380
1921	* Mathieson, John, F.R.S.G.S., late Division Superintendent, Ordnance Survey (retired), 42 East Claremont Street, Edinburgh	
1906	* Mathieson, Robert, F.C.S., St Serf's, Innerleithen	
1902	Matthews, Ernest Romney, A.M.Inst.C.E., F.G.S., Chadwick Professor of Municipal Engineering in the University of London, University College, Gower Street, London, W.C.	

Date of Election.			Service on Council, etc.
1917		* Maylard, A. Ernest, M.B., B.Sc. (Lond.), F.R.F.P.S. (Glasgow), 1 Windsor Terrace, W., Great Western Road, Glasgow	
1922		* Meakins, Jonathan Campbell, M.D., F.R.C.P.Ed., Professor of Therapeutics in the University of Edinburgh, Physician to Royal Infirmary. 12 Oxford Terrace, Edinburgh 385	
1901	C.	Menzies, Alan W. C., M.A., B.Sc., Ph.D., F.C.S., Professor of Chemistry, Princeton University, Princeton, New Jersey, U.S.A.	
1917		* Merson, George Fowlie, Manufacturing Technical Chemist, 9 Hampton Terrace, Edinburgh	
1888		Methven, Cathcart W., M.Inst.C.E., F.R.I.B.A., Durban, Natal, S. Africa	
1902	C.	Metzler, William H., A.B., Ph.D., Corresponding Fellow of the Royal Society of Canada, Professor of Mathematics Syracuse University, Syracuse, N.Y., U.S.A.	
1885	C. B.	Mill, Hugh Robert, D.Sc., LL.D., Hill Crest, Dormans Park, E. Grinstead 390	
1908		* Miller, Alexander Cameron, M.D., F.S.A. Scot., Craig Linnhe, Fort-William, Inverness-shire	
1910		* Miller, John, M.A., D.Sc., Professor of Mathematics, Royal Technical College, 2 Northbank Terrace, North Kelvinside, Glasgow	
1909		Mills, Bernard Langley, M.D., F.R.C.S.E., M.R.C.S., D.P.H., Lt.-Col. R.A.M.C., formerly Army Specialist in Hygiene, c/o National Provincial Bank, Fargate, Sheffield	
1905		* Milne, Archibald, M.A., D.Sc., Lecturer on Mathematics and Science, Edinburgh Provincial Training College, 108 Comiston Drive, Edinburgh	
1905		* Milne, C. H., M.A., Head Master, Daniel Stewart's College, 19 Merchiston Gardens, Edinburgh 395	
1904	C.	* Milne, James Robert, D.Sc., Lecturer in Natural Philosophy, University of Edinburgh, 17 Manor Place, Edinburgh	
1886		Milne, William, M.A., B.Sc., 70 Beechgrove Terrace, Aberdeen	
1899		* Milroy, T. H., M.D., B.Sc., Professor of Physiology in Queen's College, Belfast	
1889	C.	Mitchell, A. Crichton, D.Sc., Hon. Doc. Sc. (Genève), formerly Director of Public Instruction in Travancore, India (CURATOR OF LIBRARY AND MUSEUM), 246 Ferry Road, Edinburgh	1915-16. Cur. 1916-
1897		† Mitchell, George Arthur, M.A., 9 Lowther Terrace, Kelvinside, Glasgow 400	
1900		* Mitchell, James, M.A., B.Sc., Monydrain, Lochgilphead	
1911		Modi, Edalji Manekji, D.Sc., LL.D., Litt.D., F.C.S., etc., Proprietor and Director of Arthur Road Chemical Works, Meher Buildings, Tardeo, Bombay, India	
1906	C.	Moffat, Rev. Alexander, M.A., B.Sc., Professor of Physical Science, Christian College, Madras, India	
1890	C.	Mond, R. L., M.A. Cantab., F.C.S., Combe Bank, near Sevenoaks, Kent	
1887	C.	Moos, N. A. F., D.Sc., L.C.E., J.P., Director of Bombay and Alibag Observatories (retired), Gowalia, Tank Road, Bombay, India 405	
1896		Morgan, Alexander, M.A., D.Sc., Principal, Edinburgh Provincial Training College, 1 Midmar Gardens, Edinburgh	
1919		* Morris, Robert Owen, M.A., M.D., C.M. (Edin.), D.P.H. (Liverpool), Tuberculosis Institute, Newtown, N. Wales	
1892	C.	Morrison, J. T., M.A., B.Sc., Professor of Mathematical Physics, University, Stellenbosch, Cape Colony	
1914		Mort, Spencer, M.D., Ch. B., F.R.C.S.E., Lieut.-Col., R.A.M.C., North Middlesex Hospital, Upper Edmonton, London, N. 18.	
1901		Moses, O. St John, I.M.S., M.D., D.Sc., F.R.C.S., Lt.-Col. I.M.S., Professor of Medical Jurisprudence, c/o Messrs King, Hamilton & Co., 4 and 5 Koila Ghat Street, Calcutta, India 410	
1892	C. K.	Mossman, R. C., Oficina Meteorologica Argentina, Paseo Colon 974, Buenos Aires	
1916		* Muir, Robert, M.A., M.D., Sc.D., F.R.S., Professor of Pathology, University of Glasgow, 16 Victoria Crescent, Dowanhill, Glasgow	
1874	C. K. V. J.	Muir, Sir Thomas, C.M.G., M.A., D.Sc., LL.D., F.R.S., lately Superintendent-General of Education for Cape Colony, Elmcote, Sandown Road, Rondebosch, South Africa	1885-88. V-P 1888-91.
1888	C.	Muirhead, George, Commissioner to His Grace the Duke of Richmond and Gordon, K.G., Speybank, Fochabers	
1907		Muirhead, James M. P., J.P., F.R.S.L., F.S.S., c/o Royal Societies Club, St James's Street, London, S. W. 415	
1887		Mukhopādhyay, Asitosh, M.A., LL.D., F.R.A.S., M.R.I.A., Professor of Mathematics at the Indian Association for the Cultivation of Science, 77 Russa Road North, Bhowanipore, Calcutta, India	
1894 & 1921		* Munro, J. M. M., M.Inst.E.E., Assoc.M.Inst.C.E., Consulting Electrician and Engineer, 11 Randolph Place, Edinburgh	

Date of Election.		Service on Council, etc.
1896	Murray, Alfred A., M.A., LL.B., 20 Warriston Crescent, Edinburgh	
1907	Musgrove, James, M.D., F.R.C.S. Edin. and Eng., LL.D., Emeritus-Professor of Anatomy, University of St Andrews, The Swallowgate, St Andrews	
1888	Napier, A. D. Leith, M.D., C.M., M.R.C.P., 4 Kent Street, Hawthorn, Unley, S. Australia 420	
1897	Nash, Hon. Alfred George, M.L.C., B.Sc., F.R.G.S., C.E., Belretiro, Mandeville, Jamaica, W.I.	
1898	Newman, Sir George, K.C.B., M.D., D.C.L., F.R.C.P., Chief Medical Officer of the Ministry of Health and the Board of Education, Whitehall, S.W. 1	
1884	Nicholson, J. Shield, M.A., D.Sc., Professor of Political Economy in the University of Edinburgh, 3 Belford Park, Edinburgh	1885-87, 1892-95, 1897-1900.
1880	C. Nicol, W. W. J., M.A., D.Sc., 15 Blasket Place, Edinburgh	
1888	† Ogilvie, Sir F. Grant, Kt., C.B., M.A., B.Sc., LL.D., Principal Assistant Secretary, Department of Scientific and Industrial Research, 15 Evelyn Gardens, London, S.W. 425	1901-03.
1886	Oliver, James, M.D., F.L.S., Physician to the London Hospital for Women, 123 Harley Street, London, W.	
1895	C. Oliver, Sir Thomas, Kt., M.D., LL.D., F.R.C.P., Professor of Physiology in the University of Durham, 7 Ellison Place, Newcastle-upon-Tyne	
1915	* Orr, Lewis P., F.F.A., Manager of Scottish Life Assurance Co., 19 St Andrew Square, Edinburgh, 3 Belgrave Place, Edinburgh	
1914	* Oswald, Alfred, Lecturer in German, Glasgow Provincial Training College, 11 Nelson Terrace, Hillhead, Glasgow	
1908	Page, William Davidge, F.C.S., F.G.S., M.Inst.M.E., 10 Clifton Dale, York 430	
1905	Pallin, Lt.-Col. William Alfred, C.B.E., D.S.O., F.R.C.V.S., Headquarters, Eastern Command, Nainital, India	
1914	Pare, John William, M.D., C.M., L.D.S., Lecturer in Dental Anatomy, National Dental Hospital, 9A Cavendish Square, London, W.	
1901	* Paterson, David, F.C.S., Lea Bank, Roslin, Midlothian	
1918	* Paterson, Rev. William Paterson, D.D., LL.D., Professor of Divinity, University, Edinburgh, 3 Royal Terrace, Edinburgh	
1886	C. Paton, D. Noël, M.D., B.Sc., LL.D., F.R.C.P.E., F.R.S., Professor of Physiology in the University of Glasgow 435	1894-97, 1904-06, 1909-12. V-P 1918-21
1919	C. * Patterson, Thomas Stewart, D.Sc. (London and Glasgow), Ph.D. (Heidelberg), Professor of Organic Chemistry in the University of Glasgow, 10 Oakfield Terrace, Hillhead, Glasgow	
1892	Paulin, Sir David, Actuary, 6 Forres Street, Edinburgh	
1881	C. N. Peach, Benjamin N., LL.D., F.R.S., F.G.S., formerly District Superintendent and Acting Palæontologist of the Geological Survey of Scotland, c/o Wm. Marshall, Esq., 33 Comiston Drive, Edinburgh	1905-08, 1911-12. V-P 1912-17.
1907	* Pearce, John Thomson, B.A., B.Sc., School House, Tranent	
1914	Pearson, Joseph, D.Sc., F.L.S., Director of the Colombo Museum, and Marine Biologist to the Ceylon Government, Colombo Museum, Ceylon 440	
1904	* Peck, James Wallace, C.B., M.A., Scottish Board of Health. 10 South Learmonth Gardens, Edinburgh	
1889	† Peck, Sir William, Kt., F.R.A.S., Town's Astronomer, City Observatory, Calton Hill, Edinburgh	
1887	C. B. Peddie, Wm., D.Sc., Professor of Natural Philosophy in University College, Dundee, The Weisha, Ninewells, Dundee	1904-07, 1908-11. V-P 1919-22
1893	Perkin, Arthur George, F.R.S., F.I.C., Grosvenor Lodge, Grosvenor Road, Leeds	
1913	† Philip, Alexander, M.A., LL.B., Writer, The Mary Acre, Brechin 445	
1889	Philip, Sir R. W., Kt., M.A., M.D., LL.D., F.R.C.P.E., Professor of Tuberculosis, University of Edinburgh, 45 Charlotte Square, Edinburgh	
1907	C. Phillips, Major Charles E. S., O.B.E., Castle House, Shooters Hill, Woolwich, S.E. 18	
1914	* Pilkington, Basil Alexander, "Kambla," Davidson's Mains	
1905	* Pinkerton, Peter, M.A., D.Sc., Rector, High School, Glasgow, 7 Park Quadrant, Glasgow, W.	

Date of Election.

Service on Council, etc.

1908	C.	* Pirie, James Hunter Harvey, B.Sc., M.D., F.R.C.P.E., Superintendent of the Routine Division of The South African Institute for Medical Research, P.O. Box 1038, Johannesburg, South Africa 450	
1911		* Pirie, James Simpson, M.Inst.C.E., 28 Scotland Street, Edinburgh	
1906		Pitchford, Herbert Watkins, C.M.G., F.R.C.V.S., Lt.-Col., Oaklands Drive, Weybridge, Surrey	
1921		* Pollard, Sir George Herbert, K.B., M.D., C.M. (Edin.), Barrister-at-Law, Inner Temple. 79 Albert Road, Southport	
1919		* Porritt, B. D., M.Sc. (Lond.), F.I.C., Research Association of British Rubber and Tyre Manufacturers, 105-7 Lansdowne Road, Croydon, London	
1888		Prain, Sir David, Lt.-Col., Indian Medical Service (retired), Kt., C.M.G., C.I.E., M.A., M.B., LL.D., F.L.S., F.R.S., For. Memb. K. Svensk. Vetensk. Akad.; Hon. Memb. Soc. Lett. ed Arti d. Zelanti, Acireale; Pharm. Soc. Gt. Britain; Corr. Memb. Bayer Akad. Wiss., etc.; formerly Director, Royal Botanic Gardens, Kew, Surrey 12 Heathview Gardens, Putney Heath, London, S.W. 15 455	
1902		* Preller, Charles du Riche, M.A., Ph.D., A.M.Inst.C.E., M.I.E.E., F.G.S., 61 Melville Street, Edinburgh	
1892	C.	Pressland, Arthur J., M.A. Camb., Edinburgh Academy	
1915		Price, Frederick William, M.D., M.R.C.P. Edin., Physician to the Great Northern Hospital, London, 133 Harley Street, London, W.	
1903		* Pullar, Laurence, Dunbarney, Bridge of Earn, Perthshire	
1911		Purdy, John Smith, D.S.O., M.D., C.M. (Aberd.), D.P.H. (Camb.), F.R.G.S., Town Hall, Sydney, N.S.W., Australia 460	
1920		* Purser, George Leslie, M.A. (Cantab.), Assistant in the Natural History Department of the University of Edinburgh, c/o Muir, 21 Buccleuch Place, Edinburgh	
1898		Purves, John Archibald, D.Sc., 52 Queen Street, Exeter	
1899	C.	* Ramage, Alexander G., Marchfield, Davidson's Mains, Midlothian	
1914		* Ramsay, Peter, M.A., B.Sc., Head Mathematical Master, George Watson's College, 63 Comiston Drive, Edinburgh	
1911		* Rankin, Adam A., British Astronomical Association, West of Scotland Branch, 24 Woodend Drive, Jordanhill, Glasgow 465	
1904		Ratcliffe, Joseph Riley, M.B., C.M., c/o The Librarian, The University, Birmingham	
1900		Raw, Nathan, C.M.G., M.D., M.P., 45 Weymouth Street, Harley Street, London, W. 1	
1883	C.	Readman, J. B., D.Sc., F.C.S., Frankleigh House, Bradford-on-Avon, Wilts	
1902		Rees-Roberts, John Vernon, M.D., D.Sc., D.P.H., 11 Oak Hill Park, Hampstead, London, N.W. 3	
1902		Reid, George Archdall O'Brien, M.B., C.M., 9 Victoria Road South, Southsea, Hants 470	
1913		Reid, Harry Avery, O.B.E., F.R.C.V.S., D.V.H., Bacteriologist and Pathologist, Department of Agriculture, Wellington, New Zealand	
1908	C.	* Rennie, John, D.Sc., Lecturer on Parasitology and Experimental Zoology, University of Aberdeen, 60 Desswood Place, Aberdeen	
1914		Renshaw, Graham, M.D., M.R.C.S., L.R.C.P., L.S.A., Editor of the <i>Avicultural Magazine</i> , Sale Bridge House, Sale, Manchester	
1913		* Richardson, Major Harry, M.Inst.E.E., M.Inst.M.E., 16 Stratford Place, London, W. 1	
1908		Richardson, Linsdall, F.G.S., 10 Oxford Parade, Cheltenham, Glos. 475	
1875		Richardson, Ralph, W.S., 29 Eglinton Crescent, Edinburgh	
1916	C.	* Ritchie, James, M.A., D.Sc., Keeper of the Natural History Department in the Royal Scottish Museum, 20 Upper Gray Street, Edinburgh 1921-	
1914	C.	* Ritchie, James Bonnyman, D.Sc., Headmaster, Academy, Forres	
1906	C.	* Ritchie, William Thomas, M.D., F.R.C.P.E., 14 Rothesay Place, Edinburgh	
1898	C.	Roberts, Alexander William, D.Sc., F.R.A.S., Lovedale, South Africa 480	
1919		* Roberts, Alfred Henry, O.B.E., M.Inst.C.E., Superintendent and Engineer, Leith Docks, 2 Cargil Terrace, Trinity, Edinburgh	
1900		* Robertson, Joseph M'Gregor, M.B., C.M., 26 Buckingham Terrace, Glasgow	
1902	C.	* Robertson, Robert A., M.A., B.Sc., Reader in Botany in the University of St Andrews	
1919		* Robertson, William Alexander, F.F.A., Century Insurance Co., Ltd., 18 Charlotte Square. 12 Lonsdale Terrace, Edinburgh	
1896	C.	Roberts, W. G. Aitchison, D.Sc., M.D., F.R.C.P.E., The Grange, Ashford, Middlesex 485	

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Date of Election.		Service on Council, etc.
1910	C. *Robinson, Arthur, M.D., M.R.C.S., Professor of Anatomy, University of Edinburgh, 35 Coates Gardens, Edinburgh	1910-12. Sec. 1912-18. V-P 1918-21.
1916	*Ronald, David, M.Inst.C.E., Chief Engineer, Scottish Board of Health, 125 George Street, Edinburgh	
1881	Rosebery, The Right Hon. the Earl of, K.G., K.T., LL.D., D.C.L., F.R.S., Dalmeny Park, Edinburgh	
1909	C. *Ross, Alex. David, M.A., D.Sc., F.R.A.S., Professor of Mathematics and Physics, University of Western Australia, Perth, Western Australia	
1921	*Ross, Edward Burns, M.A. (Edin. and Camb.), Professor of Mathematics in the Madras Christian College, Madras 490	
1906	*Russell, Alexander Durie, B.Sc., Mathematical Master, Falkirk High School, 14 Heugh Street, Falkirk	
1902	C. K. *Russell, James, 22 Glenorchy Terrace, Edinburgh	
1906	Saleeby, Caleb William, M.D., 10 Campden Mansions, Kensington, London, W. 8	
1916	C. *Salvesen, The Rt. Hon. Lord, P.C., K.C. (VICE-PRESIDENT), Judge of the Court of Session (Retired), Dean Park House, Edinburgh	1920-22. V.-P. 1922-
1914	*Salvesen, Theodore Emile, 37 Inverleith Place, Edinburgh 495	
1912	C. K. *Sampson, Ralph Allen, M.A., D.Sc., F.R.S. (SECRETARY), Astronomer Royal for Scotland, Professor of Astronomy, University, Edinburgh, Royal Observatory, Edinburgh	1912-15, 1919-21. V-P 1915-18. Sec. 1922-
1903	*Samuel, Sir John S., K.B.E., D.L., J.P., F.S.A. (Scot.), 13 Park Circus, Glasgow, W.	
1922	*Sarkar, Bijali Behari, M.Sc., D.Sc. (Edin.), Assistant to the Professor of Physiology, University, Calcutta. 33/1c Lansdowne Road, Calcutta	
1903	*Sarolea, Charles, Ph.D., D.Litt., Professor of French, University of Edinburgh, 21 Royal Terrace, Edinburgh	
1900	C. N. *Schafer, Sir Edward Albert Sharpey, M.D., LL.D., D.Sc., F.R.S., Professor of Physiology in the University of Edinburgh 500	1900-03, 1906-09, 1918-19. V-P 1913-17.
1919	*Scott, Alexander, M.A., D.Sc., Carnegie Scholar, 1912-14; 1851 Exhibition Scholar, 1914-16; lectured (temp.) on Petrology, Oxford, 1914-15, and at Glasgow University, 1917-18; Physical Chemist in charge of Radiometric Laboratory, Glasgow University, 1916-18; Chief Assistant to Principal, Pottery Laboratory, Stoke-on-Trent	
1885	C. Scott, Alexander, M.A., D.Sc., F.R.S., 34 Upper Hamilton Terrace, London, N.W.	
1919	*Scott, Alexander Ritchie, B.Sc. (Edin.), D.Sc. (Lond.), Principal London County Council, Beaufoy Institute, Prince's Road, Vauxhall Street, London, S.E. 11	
1917	*Scott, Henry Harold, M.D., M.R.C.P., L.R.C.P. (London), M.R.C.S. (Eng.), D.P.H., Bacteriologist and Pathologist to the Government of Hong Kong	
1908	*Simpson, George Freeland Barbour, M.D., F.R.C.P.E., F.R.C.S.E., 43 Manor Place, Edinburgh 505	
1900	C. *Simpson, James Young, M.A., D.Sc., Professor of Natural Science in the New College, Edinburgh. 25 Chester Street, Edinburgh	1922-
1911	C. Simpson, Sutherland, M.D., D.Sc. (Edin.), Professor of Physiology, Medical College, Cornell University, Ithaca, N.Y., U.S.A., 118 Eddy Street, Ithaca, N.Y., U.S.A.	
1900	Sinhjee, Sir Bhagvat, G.C.I.E., M.D., LL.D. Edin., H.H. the Thakur Sahib of Gondal, Gondal, Kathiawar, Bombay, India	
1903	*Skinner, Robert Taylor, M.A., J.P., Head Master, Donaldson's Hospital, Edinburgh	
1901	*Smart, Edward, B.A., B.Sc., Tillylross, Tullylumb Terrace, Perth 510	
1920	*Smellie, William Robert, M.A., D.Sc., Geologist on the Staff of the Anglo-Persian Oil Company, Ardene, Mossend, Lanarkshire	
1915	*Smith, James Lorrain, M.A., M.D., LL.D., F.R.S., Professor of Pathology, University of Edinburgh, 9 Carlton Terrace, Edinburgh	1918-21
1921	*Smith, The Right Hon. James Parker, P.C., M.A. (Camb.), 41 Drumsheugh Gardens, Edinburgh	

Alphabetical List of the Ordinary Fellows of the Society. 441

Date of Election.

Service on Council, etc.

1921		* Smith, Norman Kemp, M.A., D.Phil., Professor of Logic and Metaphysics, University of Edinburgh	
1911		* Smith, Stephen, B.Sc., Engineer, 31 Grange Loan, Edinburgh	515
1907	C.	Smith, William Ramsay, D.Sc., M.D., C.M., Permanent Head of the Health Department, South Australia, Belair, South Australia	
1880		Smith, Sir William (Robert), M.D., D.Sc., LL.D., Principal of The Royal Institute of Public Health, Em.-Professor of Forensic Medicine and Toxicology in King's College, University of London, 36 Russell Square, London, W.C. 1	
1919		* Smith, William Wright, M.A. (Edin.), Regius Professor of Botany, University of Edinburgh, Regius Keeper of the Royal Botanic Garden, and King's Botanist in Scotland. Inverleith House, Edinburgh	
1899		Snell, Ernest Hugh, M.D., B.Sc., D.P.H. Camb., Medical Officer of Health, Coventry	
1880		Sollas, W. J., M.A., D.Sc., LL.D., F.R.S., Fellow of University College, Oxford, and Professor of Geology and Palæontology in the University of Oxford	520
1910		* Somerville, Robert, B.Sc., 31 Cameron Street, Dunfermline	
1889		Somerville, Wm., M.A., D.Sc., D.Oec., Sibthorpien Professor of Rural Economy and Fellow of St John's College in the University of Oxford, 121 Banbury Road, Oxford	
1911	C.	* Sommerville, Duncan M'Laren Young, M.A., D.Sc., Professor of Pure and Applied Mathematics, Victoria College, Wellington, New Zealand	
1882		Sorley, James, 73 Onslow Square, London, S.W. 7	
1896		Spence, Frank, M.A., B.Sc., 25 Craiglea Drive, Edinburgh	525
1906		Squance, Major Thomas Coke, M.D., M.S., F.R.M.S., F.S.A.Scot., Physician and Pathologist in the Sunderland Infirmary, President Sunderland Antiquarian Society, Sunderland Naturalists' Association, The Cottage, Newbiggin, Aysgarth, Yorks	
1891		Stanfield, Richard, Professor of Mechanics and Engineering in the Heriot-Watt College, Edinburgh	
1885 & 1915		* Steggall, John Edward Aloysius, M.A., Professor of Mathematics at University College, Dundee (St Andrews University), Woodend, Perth Road, Dundee	
1912	C. K.	† Stephenson, John, M.B., D.Sc. (Lond.), Lt.-Col. I.M.S., Zoological Department, University, Edinburgh	
1910		* Stephenson, Thomas, D.Sc., F.C.S., Editor of the <i>Prescriber</i> , 6 South Charlotte Street, Edinburgh	530
1916		* Steuart, D. R., F.I.C., Formerly Chemist to the Broxburn Oil Company, 20 Hillview, Blackhall, Midlothian	
1886	C.	Stevenson, Charles A., B.Sc., M.Inst.C.E., 28 Douglas Crescent, Edinburgh	
1884		Stevenson, David Alan, B.Sc., M.Inst.C.E., 84 George Street, Edinburgh	
1919		* Stevenson, David Alan, B.Sc., A.M.Inst.C.E., 28 Douglas Crescent, Edinburgh	
1888	C.	Stewart, Charles Hunter, D.Sc., M.B., C.M., Professor of Public Health in the University of Edinburgh, Usher Institute of Public Health, Warrender Park Road, Edinburgh	535
1921		* Stewart, Ian Struthers, M.D. (Edin.), Nordrach-on-Dee, Banchory	
1902		* Stockdale, Herbert Fitton, LL.D., Director of the Royal Technical College, Glasgow, Clairinch, Upper Helensburgh, Dumbartonshire	
1889	C.	Stockman, Ralph, M.D., F.R.C.P.E. F.F.P.S.G., Professor of Materia Medica and Therapeutics in the University of Glasgow	1903-05.
1906		Story, Fraser, formerly Professor of Forestry, University College, Bangor, North Wales. 4K Artillery Mansions, Victoria Street, London, S.W. 1	
1907		* Strong, John, C.B.E., M.A., LL.D., Professor of Education in the University of Leeds	540
1903		Sutherland, David W., M.D., M.R.C.P., C.I.E., Lt.-Col. I.M.S., Principal and Professor of Medicine, Medical College, Lahore, India	
1905		Swithinbank, Commander Harold William, Crag Head, Bournemouth, Hants	
1912		* Syme, William Smith, M.D. (Edin.), 11 Lynedoch Crescent, Glasgow	
1885	C.	Symington, Johnson, M.D., LL.D., F.R.C.S., F.R.S., formerly Professor of Anatomy in the Queen's University, Belfast. Palace Hotel, Princes Street, Edinburgh	1892-93.
1917	C. N.	* Tait, John, D.Sc., M.D., Professor of Physiology, McGill University, Montreal, Canada	545
1904		* Tait, John W., B.Sc., formerly Rector of Leith Academy, 18 Netherby Road, Leith	

Date of Election.			Service on Council, etc.
1898	C.	Tait, William Archer, D.Sc., M. Inst. C.E. (VICE-PRESIDENT), 72A George Street, Edinburgh (Society's Representative on George Heriot's Trust)	1914-17, 1918-21. V-P 1921-
1895		† Talmage, James Edward, D.Sc., Ph.D., F.R.M.S., F.G.S., formerly Professor of Geology, University of Utah, 47 East S. Temple Street, Salt Lake City, Utah, U.S.A.	
1890	C.	Tanakadate, Aikitu, Professor of Natural Philosophy in the Imperial University of Japan, Koisikawa, Zōsigayamati, 144, Tokyo, Japan	
1870		Tatlock, Robert R., F.C.S., City Analyst's Office, 156 Bath Street, Glasgow 550	
1899		* Taylor, James, M.A., Mathematical Master in the Edinburgh Academy	
1917	C.	* Taylor, William White, M.A., D.Sc., Lecturer on Chemical Physiology, University, Edinburgh, Park Villa, Liberton, Edinburgh	
1892		Thackwell, J. B., M.B., C.M., D.P.H., Carlton House, 1 Prince of Wales Road, Battersea Park, London, S.W.	
1885	C.	Thompson, D'Arcy W., C.B., D.Litt., F.R.S., Professor of Natural History, University, St Andrews, 44 South Street, St Andrews	1892-95, 1896-99, 1907-10, 1912-15, 1922- V-P 1916-19.
1917	C.	* Thompson, John M'Lean, M.A., D.Sc., F.L.S., Professor of Botany, University of Liverpool 555	
1905		* Thoms, Alexander, 7 Playfair Terrace, St Andrews	
1887		Thomson, Andrew, M.A., D.Sc., F.I.C., 145 Bruntsfield Place, Edinburgh	
1896		Thomson, George Ritchie, C.M.G., M.B., C.M., Professor of Surgery, University College, Johannesburg, Transvaal	
1903		Thomson, George S., F.C.S., Ferma Albion, Marculesci, Roumania	
1906		* Thomson, Gilbert, M. Inst. C.E., 164 Bath Street, Glasgow 560	
1887	C.	Thomson, J. Arthur, M.A., LL.D., Regius Professor of Natural History in the University of Aberdeen	1906-08, 1920-
1906	C.	Thomson, James Stuart, M.Sc., Ph.D., Zoological Department, University, Manchester	
1880		Thomson, John Millar, LL.D., F.R.S., Em.-Professor of Chemistry in King's College, London. Athenæum Club, Pall Mall, London, S.W.	
1899		* Thomson, R. Tatlock, F.C.S., 156 Bath Street, Glasgow	
1912	C.	Thomson, Robert Black, M.B. Edin., Professor of Anatomy, The University, Cape Town 565	
1870		Thomson, Spencer C., Actuary, 10 Eglinton Crescent, Edinburgh	
1882		Thomson, Sir Wm., K.B., M.A., B.Sc., LL.D., formerly Registrar, University of South Africa, Somerset House, Vermeulen Street, Pretoria	
1876	C.	Thomson, William, Royal Institution, Manchester	
1917		* Thorneycroft, Wallace, J.P., Coal and Iron Master, Plean House, Plean, Stirling	
1920		* Todd, John Barber, B.Sc., A.M.I.Mech.E., Lecturer in Engineering in the University of Edinburgh. 38 Upper Gray Street, Edinburgh 570	
1917		* Tovey, Donald Francis, B.A., Professor of Music, University, Edinburgh	
1914		† Tredgold, Alfred Frank, M.D. (Durham), M.R.C.P., Lecturer on Mental Deficiency at London University, and Bethlem Royal Hospital, "St Martins," Guildford	
1915		* Trotter, George Clark, M.D., Ch.B. Edin., D.P.H. (Aberdeen), Medical Officer of Health, Metropolitan Borough, Islington. Braemar, 17 Haslemere Road, Crouch End, London, N. 8	
1922	C.	* Turnbull, Herbert Westren, M.A., Professor of Mathematics in the University of St. Andrews. Fereday Fellow of St. John's College, Oxford. 2 Queens Terrace, St Andrews	
1905		* Turner, Arthur Logan, M.D., F.R.C.S.E., 27 Walker Street, Edinburgh 575	
1906	C.	* Turner, Dawson F. D., B.A., M.D., F.R.C.P.E., M.R.C.P., Lecturer on Medical Physics, Surgeons' Hall, Physician in charge of Radium Treatment, Royal Infirmary, Edinburgh, 37 George Square, Edinburgh	
1895		Turton, Albert H., M.I.M.M., 233 George Road, Erdington, Birmingham	
1898	C.	Tweedie, Charles, M.A., B.Sc., formerly Lecturer on Mathematics in the University of Edinburgh. Marine View, Belhaven, Dunbar	
1918	C.	* Tyrrell, G. W., A.R.C.Sc., F.G.S., Chief Assistant and Lecturer in Petrology, Geological Department, University, Glasgow	

Alphabetical List of the Ordinary Fellows of the Society. 443

Date of Election.		Service on Council, etc.
1910	Vincent, Swale, M.D. Lond., D.Sc. Edin., etc., Professor of Physiology in the University of London, Physiological Laboratory, Middlesex Hospital Medical School, Berners Street, London, W. 1 580	
1891	C. B. Walker, Sir James, Kt., D.Sc., Ph.D., LL.D., F.R.S., Professor of Chemistry in the University of Edinburgh, 5 Wester Coates Road, Edinburgh	1903-05, 1910-13, 1922- V-P 1916-19.
1902	* Wallace, Alexander G., M.A., 56 Fonthill Road, Aberdeen	
1886	C. Wallace, R., M.A., F.L.S., Em.-Professor of Agriculture and Rural Economy in the University of Edinburgh. 45 E. Claremont Street, Edinburgh	
1898	Wallace, Wm., M.A., Belvedere, Alberta, Canada	
1891	Walmsley, R. Mullineux, D.Sc., Principal of the Northampton Institute, Clerkenwell, London 585	
1920	* Wahnsley, Thomas, M.D. (Glasgow), Professor of Anatomy in the University of Belfast, 59 South Street, Greenock	
1901	C. * Waterston, David, M.A., M.D., F.R.C.S.E., Professor of Anatomy, University, St Andrews	1916-19.
1911	* Watson, James A. S., M.C., B.Sc., etc., Professor of Agriculture in the University of Edinburgh. 30 Mayfield Terrace, Edinburgh	
1900	* Watson, Thomas P., M.A., B.Sc., Principal, Keighley Institute, Keighley	
1907	* Watt, Andrew, M.A., Meteorological Office, 10 Rothesay Place. 6 Woodburn Terrace, Edinburgh 590	1912-14.
1911	† Watt, James, W.S., F.F.A., Craiglockhart House, Slateford, Edinburgh	
1911	* Watt, Rev. Lauchlan Maclean, D.D., Minister of St Stephen's Parish, 7 Royal Circus, Edinburgh	
1896	Webster, John Clarence, B.A., M.D., F.R.C.P.E., Professor of Obstetrics and Gynæcology, Rush Medical College, Shediac. N.B., Canada	
1907	B. C. * Wedderburn, Ernest Maclagan, M.A., LL.B., W.S., D.Sc., Professor of Conveyancing in the University of Edinburgh. 6 Succoth Gardens, Edinburgh	1913-16. 1921-
1903	B. C. * Wedderburn, J. H. Maclagan, M.A., D.Sc., P.O. Box 53, Princeton, N.J., U.S.A. 595	
1904	Wedderspoon, William Gibson, M.A., LL.D., Indian Educational Service, Senior Inspector of Schools, Burma, The Education Office, Rangoon, Burma	
1896	Wenley, Robert Mark, M.A., D.Sc., D.Phil., Litt.D., LL.D., D.C.L., Professor of Philosophy in the University of Michigan, Ann Arbor, U.S.A.	
1916	* White, J. Martin, Esq., of Balruddery, Balruddery, near Dundee	
1896	C. White, Philip J., M.B., Professor of Zoology in University College, Bangor, North Wales	
1911	* Whittaker, Charles Richard, F.R.C.S. (Edin.), F.S.A. (Scot.), Lynwood, Hatton Place, Edinburgh 600	
1912	C. * Whittaker, Edmund Taylor, Sc.D., F.R.S., Foreign Member of the R. Accademia dei Lincei, Rome, Professor of Mathematics in the University of Edinburgh. 35 George Square, Edinburgh	1912-15. 1922- Sec. 1916-22.
1918	* Whyte, Rev. Charles, M.A., LL.D., F.R.A.S., U.F. Church Manse, Kingswells, Aberdeen	
1918	* Wight, John Thomas, M.I.Mech.E., General Manager, Messrs MacTaggart, Scott & Co., Ltd., Station Iron Works, Loanhead. Calderwood Villa, Lasswade	
1908	* Williamson, Henry Charles, M.A., D.Sc., Naturalist to the Fishery Board for Scotland, Marine Laboratory, Aberdeen	
1910	C. * Williamson, William, F.L.S., 79 Morningside Drive, Edinburgh 605	
1900	Wilson, Alfred C., F.C.S., Voewood Croft, Stockton-on-Tees	
1911	* Wilson, Andrew, M.Inst.C.E., 66 Netherby Road, Trinity, Edinburgh	
1902	V. J. * Wilson, Charles T. R., M.A., F.R.S., 14 Cranmer Road, Cambridge, Sidney Sussex College, Cambridge	
1922	* Wilson, John, F.R.I.B.A., Fellow of the Inst. of Scottish Architects. Principal Architect, Scottish Board of Health. 59 Murrayfield Gardens, Edinburgh	
1895	Wilson-Barker, Sir David, Kt., R.N.R., F.R.G.S., late Captain-Superintendent Thames Nautical Training College, H.M.S. "Worcester," off Greenhithe, Kent, Flimwell Grange, near Hawkhurst, Kent 610	
1882	Wilson, George, M.A., M.B., LL.D.	
1920	* Wilson, Malcolm, D.Sc. (London), Lecturer in Mycology and Bacteriology in the University of Edinburgh. Royal Botanic Garden, Edinburgh	

Date of
Election.Service on
Council, etc.

1922	C.	* Wordie, James Mann, M.A. (Camb.), B.Sc. (Glasg.), Lecturer in Oceanography, University of Cambridge. 52 Montgomery Drive, Glasgow, and St John's College, Cambridge	
1911		* Wrigley, Ruric Whitehead, B.A. (Cantab.), Assistant Astronomer, Royal Observatory, Edinburgh	
1890		Wright, Johnstone Christie, Conservative Club, Edinburgh	615
1896		Wright, Sir Robert Patrick, LL.D., formerly Chairman of the Board of Agriculture for Scotland. Wenallt House, Crosswood, Cardiganshire	
1882		Young, Frank W., F.C.S., Scottish Education Department (Ex-Service Student's Branch), 25 Drumsheugh Gardens, Edinburgh	
1892		Young, George, Ph.D., "Bradda," Church Crescent, Church End, Finchley, London, N.	
1896	C.	Young, James Buchanan, M.B., D.Sc., Dalveen, Braeside, Liberton	
1904		Young, R. B., M.A., D.Sc., F.G.S., Professor of Geology and Mineralogy in the South African School of Mines and Technology, Johannesburg, Transvaal	620

LIST OF HONORARY FELLOWS OF THE SOCIETY

At January 31, 1923.

HIS MOST EXCELLENT MAJESTY THE KING.
HIS ROYAL HIGHNESS THE PRINCE OF WALES.

FOREIGNERS (LIMITED TO THIRTY-SIX BY LAW I).

Elected

- 1916 Charles Barrois, Professor of Geology and Mineralogy, Université, Lille, France: 37 rue Pascal, Lille.
- 1905 Waldemar Christofer Brøgger, Professor of Mineralogy and Geology, K. Frederiks Universitet, Christiania, Norway.
- 1916 Douglas Houghton Campbell, Professor of Botany, Leland Stanford Junior University, California, U.S.A.
- 1920 William Wallace Campbell, Director of the Lick Observatory, Mt. Hamilton, California, U.S.A.
- 1921 Reginald Aldworth Daly, Professor of Geology, Harvard University, Cambridge, Mass.
- 1910 Hugo de Vries, Professor of Plant Anatomy and Physiology, Lunteren, Holland.
- 1916 Marcel Eugène Emile Gley, Professor of Biology, Collège de France, Paris, Membre de l'Académie de Médecine, Paris: 14, rue Monsieur le Prince, Paris.
- 1910 Karl F. von Goebel, Professor of Botany, Universität, München, Germany.
- 1916 Camillo Golgi, Professor of Pathology, Università, Pavia, Italy.
- 1916 Gio. Battista Grassi, Professor of Comparative Anatomy, Regia Università, Roma, Italy: Via Agostino Depretis N. 91, Rome.
- 1905 Paul Heinrich von Groth, Professor of Mineralogy, Universität, München, Germany.
- 1913 George Ellery Hale, Director of Mount Wilson Solar Observatory (Carnegie Institution of Washington), Pasadena, California, U.S.A.
- 1921 Johan Hjort, Director of Norwegian Fisheries, Bergen.
- 1920 Hendrik Antoon Lorentz, Nobel Laureate, Physics, 1902, Professor of Physics, Leiden University.
- 1910 Albert Abraham Michelson, Nobel Laureate, Physics, 1907, Professor of Physics, University, Chicago, U.S.A.
- 1897 Fridtjof Nansen, Professor of Oceanography, K. Frederiks Universitet, Christiania, Norway.
- 1921 Heike Kamerlingh Onnes, Nobel Laureate, Physics, 1913, Universiteit, Leiden, Holland.
- 1908 Henry Fairfield Osborn, Professor of Zoology, Columbia University and American Museum of Natural History, New York, N. Y., U.S.A.
- 1908 Ivan Petrovitch Pavlov, Emeritus Professor of Physiology, Kais. Inst. Exper. Med., Petrograd: Wedenskaja Strasse 4, Petrograd, Russia.
- 1920 Ch. Emile Picard, Perpetual Secretary, Academy of Sciences, Paris.
- 1921 Salvatore Pincherle, Professor of Mathematics in the University of Bologna.
- 1889 Georg Hermann Quincke, Emeritus Professor of Physics, Bergstrasse 41, Heidelberg, Germany.
- 1913 Santiago Ramón y Cajal, Nobel Laureate, Medicine, 1906, formerly Professor of Histology and Pathological Anatomy, Universidad, Madrid, Spain.
- 1920 Charles Richet, Professor of Physiology, Faculty of Medicine, Paris.
- 1920 Georg Ossian Sars, formerly Professor of Zoology, Christiania, and Director of Norwegian Fisheries.
- 1913 Vito Volterra, Professor of Mathematical Physics, Regia Università, Rome, Italy.
- 1916 Eugenius Warming, Emeritus Professor of Botany at the University of Copenhagen and Director of the Botanical Garden: Bjerregaardsvej 5, Copenhagen, Valby.

Total, 27.

BRITISH SUBJECTS (LIMITED TO TWENTY BY LAW I).

- 1916 Sir Francis Darwin, Kt., D.Sc., M.B., LL.D., F.R.S., Hon. Fellow, Christ's College, Cambridge, 10 Madingley Road, Cambridge.
- 1900 Sir David Ferrier, Kt., M.A., M.D., LL.D., F.R.S., Emer.-Professor of Neuro-Pathology, King's College, London, Athenæum Club, London.

446 Proceedings of the Royal Society of Edinburgh.

Elected

- 1900 Andrew Russell Forsyth, M.A., Sc.D., LL.D., Hon. Math.D., F.R.S., Chief Professor of Mathematics in the Imperial College of Science and Technology, London, formerly Sadleirian Professor of Pure Mathematics in the University of Cambridge, Imperial College of Science and Technology, London, S.W. 7.
- 1910 Sir James George Frazer, D.C.L., LL.D., Litt.D., F.R.S., Fellow of Trinity College, Cambridge.
- 1916 James Whitbread Lee Glaisher, M.A., Sc.D., F.R.S., Fellow of Trinity College, Cambridge.
- 1908 Sir Alexander B. W. Kennedy, Kt., LL.D., F.R.S., Past Pres. Inst. C.E., A7, Albany, Piccadilly, London, W.
- 1913 Horace Lamb, M.A., Sc.D., D.Sc., LL.D., F.R.S., lately Professor of Mathematics in the University of Manchester. 65 Grange Road, Cambridge.
- 1916 John Newport Langley, M.A., Sc.D., M.D., LL.D., F.R.S., Fellow of Trinity College, Professor of Physiology in the University of Cambridge, Hedgerley Lodge, Madingley Road, Cambridge.
- 1908 Sir Edwin Ray Lankester, K.C.B., M.A., D.Sc., LL.D., F.R.S., 44 Oakley Street, Chelsea, London, S.W. 3.
- 1910 Sir Joseph Larmor, Kt., M.A., D.Sc., LL.D., D.C.L., F.R.S., Lucasian Professor of Mathematics in the University of Cambridge, St John's College, Cambridge.
- 1900 Archibald Liversidge, M.A., LL.D., F.R.S., Emer.-Professor of Chemistry in the University of Sydney, Fieldhead, Coombe Warren, Kingston, Surrey.
- 1921 William Henry Perkin, M.A., Ph.D., Sc.D., LL.D., F.R.S., Waynflete Professor of Chemistry in the University of Oxford.
- 1921 Sir Ronald Ross, K.C.B., K.C.M.G., F.R.S., Hon. Consultant in Malaria, Ministry of Pensions, London, 41 Buckingham Palace Mansions, S.W. 1.
- 1921 Sir Ernest Rutherford, Kt., M.A., D.Sc., B.A., LL.D., F.R.S., Nobel Laureate, Chemistry, 1908, Cavendish Professor of Experimental Physics in the University of Cambridge.
- 1916 Sir Arthur Schuster, Kt., Ph.D., D.Sc., LL.D., D. ès Sc. Geneva, Foreign Secretary and V.P. of the Royal Society, Honorary Professor of Physics in the University of Manchester, Yeldall, Twyford, Berks.
- 1908 Sir Charles Scott Sherrington, G.B.E., M.A., D.Sc., M.D., LL.D., P.R.S., Waynflete Professor of Physiology in the University of Oxford, Physiological Laboratory, Oxford.
- 1921 Sir Jethro J. H. Teall, Kt., M.A., Sc.D., LL.D., F.G.S., F.R.S., lately Director of the Geological Survey of Great Britain and of the Museum of Practical Geology.
- 1913 Sir William Turner Thiselton-Dyer, K.C.M.G., C.I.E., M.A., LL.D., F.R.S., formerly Director of the Royal Botanic Gardens, Kew: The Ferns, Witcombe, Gloucester.
- 1905 Sir Joseph John Thomson, Kt., O.M., D.Sc., LL.D., Past Pres. R.S., Nobel Laureate, Physics, 1906, lately Cavendish Professor of Experimental Physics, University of Cambridge, Trinity College, Cambridge.
- 1900 Sir Thomas Edward Thorpe, Kt., C.B., D.Sc., LL.D., F.R.S., formerly Principal of the Government Laboratories, Emeritus-Professor of Chemistry, Imperial College of Science and Technology, South Kensington, London, S.W. Whinfield, Salcombe, South Devon.

Total, 20.

CHANGES IN FELLOWSHIP DURING SESSION 1921--1922.

ORDINARY FELLOWS OF THE SOCIETY ELECTED.

CHARLES LAWRENCE ABERNETHY.
GEORGE BARGER.
SIR DUGALD CLERK.
FRANCIS ALBERT ELEY CREW.
WILLIAM OSBORNE GREENWOOD.
WILLIAM ALEXANDER GUTHRIE.
ROBERT KERR HANNAY.
EDWARD HINDLE.

CHARLES FREDERICK JURITZ.
MURRAY MACGREGOR.
JONATHAN CAMPBELL MEAKINS.
BIJALI BEHARI SARKAR.
HERBERT WESTREN TURNBULL.
JAMES WALKER.
JOHN WILSON.
JAMES MANN WORDIE.

ORDINARY FELLOWS DECEASED.

SIR J. O. AFFLECK.
J. B. APPELYARD.
G. W. W. BARCLAY.
WM. S. BRUCE.
ALEXANDER CLEGHORN.
WILLIAM CRAIG.
HENRY N. DICKSON.
RT. HON. LORD SCOTT DICKSON.
D. G. ELLIOT.

JOHN HARRISON.
ALFRED HILL.
RICHARD NORRIS.
SIR JOHN RANKINE.
ALEXANDER SMITH.
C. MICHIE SMITH.
JAMES WALKER.
SIR G. SIMS WOODHEAD.

HONORARY FELLOWS DECEASED.

JACOBUS CORNELIUS KAPTEYN.

| CHARLES LAVERAN.

ORDINARY FELLOWS RESIGNED.

WM. M. BAXTER.

| JOHN BROWNLEE.

Additions to the Library—Presentations, etc.—1921–1922.

- Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität herausgegeben von W. Blaschke, E. Hecke, J. Radon. 8vo. Hamburg, 1921.
- L'Academie Royale de Belgique depuis sa Fondation (1772–1922). 8vo. Bruxelles, 1922.
- Algebra, a Professor's Text-book of. Late 18th or early 19th Century. (*In manuscript.*) (*Presented by Mr Charles Tweedie.*)
- Archivos de la Asociación Peruana para el Progreso de la Ciencia. Año 1921. 8vo. Lima, 1921.
- Astrographic Catalogue, 1900·0. Hyderabad Section. Dec. -16° to -21° . Vol. IV. 4to. Edinburgh, 1921. (*Presented.*)
- Bellows, Max. Dictionary of German-English, English-German. 2nd edition. 8vo. London, 1915. (*Purchased.*)
- Bolton, Herbert. A Monograph of the Fossil Insects of the British Coal Measures. (Palæontographical Society.) 4to. London, 1921–1922. (*Presented by the Author.*)
- Bombay Magnetical, Meteorological, and Seismographic Observations, 1911–1915. 4to. Bombay, 1921. (*Presented.*)
- British Antarctic ("Terra Nova") Expedition, 1910. Natural History Report. 4to. London, 1921–22.
- Zoology. Vol. II, No. 11. Hirudinea. By W. A. Harding.
- Vol. IV, No. 4. Rhabdopleura. By J. R. Norman.
- Vol. VI, No. 2. Foraminifera. By Edward Heron-Allen and Arthur Earland.
- Geology. Vol. I, No. 2. Fish Remains of the Upper Old Red Sandstone of Granite Harbour, Antarctica. By Arthur Smith Woodward.
- Vol. I, No. 3. Recent and Local Deposits of McMurdo Sound Region. By F. Debenham.
- Vol. I, No. 4. The Sedimentary Rocks of South Victoria Land. By F. Debenham, R. H. Rastall, and R. E. Priestley.
- Vol. I, No. 5. The Metamorphic Rocks of South Victoria Land. By W. Campbell Smith and R. E. Priestley.
- British Association for the Advancement of Science: A Retrospect, 1831–1921. By O. J. R. Howarth, O.B.E. 8vo. London, 1922. (*Presented by the B.A.*)
- Report, 1921. Edinburgh. 8vo. London, 1922. (*Presented by Dr James Currie.*)
- Campbell, James. Treatise of Algebra (18th Century.) (*In manuscript.*) (*Presented by Mr Charles Tweedie.*)
- Carnegie Institution of Washington Publications. Year Book. No. 20, 1921. La. 8vo. Washington, 1922.
85. Index of Economic Material in the Documents of the States of the United States. Pennsylvania, 1790–1904. Part II—F—Railroads. By Adelaide R. Hasse. 4to. Washington, 1921.

175. Researches of the Department of Terrestrial Magnetism. Vol. IV.—Land Magnetic Observations, 1914-1920. By L. Bauer, J. A. Fleming, H. W. Fisk, and W. J. Peters. 4to. Washington, 1921.
258. The Federal System of the Argentine Republic. By L. S. Rowe. 8vo. Washington, 1921.
268. Olympic Victor Monuments and Greek Athletic Art. By Wm. W. Hyde. La. 8vo. Washington, 1921.
- 273, 275, and 276. Contributions to Embryology. Vol. X, Nos. 47 and 48; Vol. XII, No. 56; Vol. XIII, 57-64. 4to. Washington, 1921.
284. The Distribution of Vegetation in the United States, as related to Climatic Conditions. By Burton E. Livingston and Forrest Shreve. 8vo. Washington, 1921.
293. Leodicidæ of the West Indian Region. Vol. XV. By Aaron L. Treadwell. 4to. Washington, 1921.
294. Letters of Members of the Continental Congress. Vol. I. Edited by Edmund C. Burnett. 8vo. Washington, 1921.
303. Tables, Factors, and Formulas for computing Respiratory Exchange and Biological Transformations of Energy. Prepared by Thorne M. Carpenter. 8vo. Washington, 1921.
305. Selection in Cladocera on the Basis of a Physiological Character. By Arthur M. Banta. 8vo. Washington, 1921.
307. Growth in Trees. By D. T. MacDougal. 8vo. Washington, 1921.
306. Fossil Echini of the West Indies. By Robert T. Jackson and T. W. Vaughan. 8vo. Washington, 1922.
308. Plant Habits and Habitats in the Arid Portions of South Australia. By Wm. A. Cannon. 8vo. Washington, 1921.
309. Gaseous Exchange and Physiological Requirements for Level and Grade Walking. By H. M. Smith. 8vo. 1922.
310. Displacement Interferometry applied to Acoustics and to Gravitation. By Carl Barus. 8vo. Washington, 1921.
311. Shallow-Water Foraminifera of the Tortugas Region. By J. A. Cushman. 8vo. Washington, 1922.
313. Rubber Content of North American Plants. By Harvey M. Hall and Frances L. Long. 8vo. Washington, 1921.
314. The Behaviour of Stomata. By J. V. G. Lotfield. 8vo. Washington, 1921.
315. Aeration and Air-Content: The Rôle of Oxygen in Root Activity. By F. E. Clements. 8vo. Washington, 1921.
316. Development and Activities of Roots of Crop Plants: A Study in Crop Ecology. By J. E. Weaver, F. C. Jean, J. W. Crist. 8vo. Washington, 1922.
320. Genetic Studies of Rabbits and Rats. By W. E. Castle. 8vo. Washington, 1922.

- 450 Proceedings of the Royal Society of Edinburgh. [Sess.
- Catalogue of the Library of the Royal Statistical Society. 8vo. London, 1921. (*By exchange.*)
- Printed Books in Edinburgh University Library. Vol. II, G–O. 4to. Edinburgh, 1921. (*Presented.*)
- Census of Scotland, 1921. Report on the Thirteenth Decennial Census of Scotland. Vol. I, Parts 1 to 17. Fol. Edinburgh, 1922. (*Presented by the Registrar-General.*)
- Comptes Rendus du Congrès International des Mathématiciens (Strasbourg, 22–30 Septembre 1920). Publiés par Henri Villat. 4to. Toulouse, 1921. (*Purchased.*)
- Contribution to Indiana Palæontology. Vol. I. 8vo. New Albany, Indiana, 1898. (*Presented by the American Museum of Natural History.*)
- Copy Register of Members of the General Council of the University of Edinburgh for the Year commencing 1st January 1922. Fol. Edinburgh, 1922. (*Presented by the University of Edinburgh.*)
- Danish Ingolf Expedition. Vol. V, part 9. Actiniaria. Part 1, by Oskar Carlgren. 4to. Copenhagen, 1921. (*Presented, Zool. Museum.*)
- De Moivre. Miscellanea Analytica de Seriebus et Quadraturis. 4to. London, 1730. (*Presented by Mr Charles Tweedie.*)
- Dictionary of Applied Physics. Edited by Sir Richard Glazebrook, K.C.B., D.Sc., F.R.S. Vol. I.—Mechanics—Engineering—Heat. Vol. II.—Electricity. 8vo. London, 1922. (*Purchased.*)
- Dominion Museum Monograph. No. 1. 8vo. Wellington, 1922. (*Presented by Dominion Museum, Wellington.*)
- Dunk, John L. Hyperacoustics. Division II: Successive Tonality. 8vo. London, 1921. (*Presented by J. M. Dent & Sons, Ltd.*)
- Encyclopædia Britannica. Vols. XXX, XXXI, and XXXII. (New Series.) 4to. London and New York, 1922.
- Euclidis Elementorum. Book 15: Græcè and Latinè. 12mo. Paris, 1573. (*Presented by Mr Charles Tweedie.*)
- Die Fauna Südwest-Australien Ergebnisse der Hamburger südwest-australischen Forschungsreise, 1905. Bds. 1–4, Bd. 5. Liefs 1 und 2. La. 8vo. Jena, 1907–1914. (*Presented by Mr Wm. Williamson.*)
- Fowler, A. Report on Series in Line Spectra. La. 8vo. London, 1922. (*Presented by the Physical Society of London.*)
- Fulton, Robert V. Medical Practice in Otago and Southland in the Early Days. 8vo. Dunedin, N.Z., 1922. (*Presented.*)
- Garcæi, Johannis. Astrologiæ Methodus. Fol. Basle, 1570. (*Presented by Mr Charles Tweedie.*)
- Gregory, D. Astronomiæ Physicæ & Geometricæ Elementa. Fol. Oxford, 1702. (*Presented by Mr Charles Tweedie.*)
- Herdman, Sir William. Spolia Runiana V. Some Results of Plankton Investigation in the Irish Sea. 8vo. London, 1922. (*Presented by the Author.*)

Howarth, O. J. R. See *British Association*.

Janet, Charles. *Considérations sur l'Être Vivant*. 1^{re} partie, 1920; 2^{de} partie, 1921. 8vo. Beauvais.

Jeanbernat, Capt. J. Barthélemy de Ferrari Doria *Lettres de Guerre (1914-1918)*. 8vo. Paris, 1920. (*Presented by M. Jeanbernat, Villa Doria, Marseille.*)

Johnston, Sir Harry H. *A Comparative Study of the Bantu and Semi-Bantu Languages*. Vol. II. 4to. Oxford, 1922. (*Presented by Mr Percy A. Molteno.*)

Journal Russe de Zoologie. Sous redaction de MM. les prof. V. Schimkévitch et V. Dogiel. Tome 1. 8vo. Petrograd, 1916. (*Presented.*)

Kenneth, John H. *Osmics, the Science of Smell*. 8vo. Edinburgh, 1922. (*Presented by the Author.*)

Kramer, John B. *Radiations from Slow-Radium; with a note on their Therapeutic Value by John Hall Edwards*. 8vo. London, 1921. (*Presented by the Author.*)

Loria, Gino. *Storia d. Geometria Descrittiva*. 12mo. Milan, 1921. (*Presented by the Author.*)

MANCHESTER UNIVERSITY PUBLICATIONS:

Classical Series—

England, E. B. (Editor). *The Laws of Plato*. (Books 1-12.) Vols. I and II. 8vo. 1921.

Economic Series—

Forrester, R. B. *The Cotton Industry in France*. 8vo. 1921.

Higgins, S. H. *Bleaching*. 8vo. 1921.

Mascart, Jean. *Impressions et Observations dans un Voyage à Tenerife*. 8vo. Paris, 1921. (*Presented.*)

Maseres, F. *Scriptores Logarithmici*. Vols. V and VI. 4to. London, 1804 and 1807. (*Presented by Mr Charles Tweedie.*)

Mémoires de la Société Vaudoise des Sciences Naturelles. Nos. 1 and 2. 8vo. Lausanne, 1922.

National Research Council of Japan:

Proceedings. No. 1.

Japanese Journal of Astronomy and Geophysics. No. 1.

Japanese Journal of Geology and Geography. No. 1.

Japanese Journal of Botany. No. 1.

8vo. Tokyo, 1922. (*Presented by the National Research Council of Japan.*)

New York State Museum. *Memoir No. 9: Early Devonian History of N. York and Eastern N. America*. By John M. Clark. Parts 1 and 2. Two vols. 4to. Albany, 1908-1909. (*Presented.*)

Pardies, P. I. Gaston. *Elementa Geometriæ*. 12mo. Oxford, 1694.
(Presented by Mr Charles Tweedie.)

Picard, Émile. *Discours et Mélanges*. 8vo. Paris, 1922. (Presented.)

Population of Scotland, 1755. Webster's Account. Fol. Edinburgh, 1921. (Presented by J. C. Dunlop, Registrar-General, Scotland.)

Prescriber, The. Vol. XVI. Jan. 1922. (Presented by Mr Thomas Stephenson.)

Proceedings of the Benares Mathematical Society. Vol. I. 8vo. Benares, 1919.

Publicazioni della Specola Vaticana, Roma :

Serie II. Astronomica. No. 1.

Miscellanea Astronomica. Parte I.

Serie III. Astrofotografica.

1. Catalogo Astrografica Stellare. Vol. I.

Redwood, Sir Boverton, Bt. *A Treatise on Petroleum*. With a Foreword by Sir Frederick W. Black, K.C.B. Vols. I-III. 4th edition. 8vo. London, 1922. (Presented by Charles Griffin & Co., Ltd.)

Report of British Association for the Advancement of Science, 1921. Edinburgh. 8vo. London, 1922. (Presented by Dr James Currie.)

——— the Canadian Arctic Expedition, 1913-18. Vol. XII. *The Life of the Copper Eskimos*. By J. Jenness. 8vo. Ottawa, 1922. (Presented by Dept. of Mines, Canada.)

——— the Proceedings of the Fourth Entomological Meeting. (Pusa, February 1921.) 8vo. Calcutta, 1921. (Presented.)

Revue de Geologie (Liège), 1920. 8vo. Liège. (Purchased.)

Royal Society of London. *Catalogue of Scientific Papers. 1800-1900. Subject Index. Vol. III—Physics; Part 2—Electricity and Magnetism*. 8vo. Cambridge, 1914. (Purchased.)

Scientific Results of the Zoological Expedition to British East Africa and Uganda made by Prof. V. Dogiel and I. Sokolow in the year 1914. Vol. I. 8vo. Petrograd. (Presented by V. Dogiel.)

Seismological Notes, No. 1. Imperial Earthquake Investigation Committee. 8vo. Tokyo, 1921.

——— *Bulletin, No. 1*. 4to. Tokyo, 1921. (Central Meteorol. Observatory of Japan.)

Service des Grandes Forces Hydraulique (Région du Sud-Ouest). Résultats obtenus pour le bassin de l'Adour. T. I, bis fasc. A et T. VII, fasc. B.

——— *Bassin de la Garonne. T. II, bis fasc. G, T. VII, fasc. G*. (Presented by Dr Brysson Cunningham.)

Small Science Manual of 17th Century. (In Manuscript.) (Presented by Mr Charles Tweedie.)

Spencer, Sir Baldwin. *Guide to the Australian Ethnological Collection exhibited in the National Museum of Victoria*. 3rd edition. 8vo. Melbourne, 1922. (Presented by Trustees of the National Museum.)

- Stensiö, Erik A : son. Triassic Fishes from Spitzbergen. Part 1. 4to. Vienna, 1921. (*Upsala University.*)
- Tables Annuelles de Constantes et Données Numeriques de Chemie, de Physique et de Technologie. Vol. IV. Années 1913-16. 4to. Paris, 1921-1922. (*Purchased.*)
- Tcherkinsky, M. The Landschaften and their Mortgage Credit Operations in Germany [1770-1920]. 8vo. Rome, 1922. (*From Internat. Inst. of Agric. Rome.*)
- Trabajos del Laboratorio de Investigaciones Biologicas de la Universidad de Madrid. Publicados por S. Ramón Cajal. Tomo I (1901-02). (*Presented by Prof. S. Ramón y Cajal.*)
- Tweedie, Charles. James Stirling: A Sketch of his Life and Works along with his Scientific Correspondence. 8vo. Oxford, 1922. (*Purchased.*)
- University of Edinburgh. Roll of Honour, 1914-1919. 4to. Edinburgh, 1921. (*Presented by the University Court.*)
- University of Wisconsin :
- Studies in Science. No. 1-3.
 - in the Social Sciences and History. Nos. 1-3.
 - in Language and Literature. No. 1-13.
- Verhandelingen van Dr P. Zeeman over Magneto-Optische Verschijnselen. 8vo. Leiden, 1921. (*Presented.*)
- Veröffentlichungen des Magnetischen Observatoriums der Finnischen Akademie der Wissenschaften zu Sodankylä. No. 1. 4to. Helsinki. (*Presented.*)
- Wall, Frank. Snakes of Ceylon. 8vo. Colombo, 1921. (*Presented by Colombo Museum.*)
- Waters, H. H. Astronomical Photography for Amateurs. 8vo. London, 1921. (*Presented by Mr James Gall Inglis.*)
- Williamson, H. Chas. Crustacea Decapoda. Larven. (Nordisches Plankton.) 4to. Kiel and Leipzig, 1915. (*Presented by Author.*)

LAWS OF THE SOCIETY.

Adopted July 3, 1916 ; amended December 18, 1916.

(Laws VIII, IX, and XIII amended May 3, 1920. Law VI amended February 7, 1921.)

I.

THE ROYAL SOCIETY OF EDINBURGH, which was instituted by Royal Charter in 1783 for the promotion of Science and Literature, shall consist of Ordinary Fellows (hereinafter to be termed Fellows) and Honorary Fellows. The number of Honorary Fellows shall not exceed fifty-six, of whom not more than twenty may be British subjects, and not more than thirty-six subjects of Foreign States.

Fellows only shall be eligible to hold office or to vote at any Meeting of the Society.

ELECTION OF FELLOWS.

II.

Each Candidate for admission as a Fellow shall be proposed by at least four Fellows, two of whom must certify from personal knowledge. The Official Certificate shall specify the name, rank, profession, place of residence, and the qualifications of the Candidate. The Certificate shall be delivered to the General Secretary before the 30th of November, and, subject to the approval of the Council, shall be exhibited in the Society's House during the month of January following. All Certificates so exhibited shall be considered by the Council at its first meeting in February, and a list of the Candidates approved by the Council for election shall be issued to the Fellows not later than the 21st of February.

III.

The election of Fellows shall be by Ballot, and shall take place at the first Ordinary Meeting in March. Only Candidates approved by the Council shall be eligible for election. A Candidate shall be held not elected, unless he is supported by a majority of two-thirds of the Fellows present and voting.

IV.

On the day of election of Fellows two scrutineers, nominated by the President, shall examine the votes and hand their report to the President, who shall declare the result.

V.

Each Fellow, after his election, is expected to attend an Ordinary Meeting, and sign the Roll of Fellows, he having first made the payments required by Law VI. He shall be introduced to the President, who shall address him in these words :

In the name and by the authority of THE ROYAL SOCIETY OF EDINBURGH, I admit you a Fellow thereof.

PAYMENTS BY FELLOWS.

VI.

Each Fellow shall, before he is admitted to the privileges of Fellowship, pay an admission fee of three guineas, and a subscription of three guineas for the year of election. He shall continue to pay a subscription of three guineas at the beginning of each session so long as he remains a Fellow.

Each Fellow who was elected subsequent to December 1916 and previous to December 1920 shall also pay a subscription of three guineas at the beginning of each session so long as he remains a Fellow.

Each Fellow who was elected previous to December 1916, and who has not completed his twenty-five annual payments,* shall, at the beginning of each session, pay three guineas or four guineas according as he has or has not made ten annual payments as reckoned from the year of election. Each Fellow who has completed or shall complete his payments shall be invited to contribute one guinea per annum or to pay a single sum of ten guineas.

A Fellow may compound for the annual subscriptions by a single payment of fifty guineas, or on such other terms as the Council may from time to time fix.

VII.

A Fellow who, after application made by the Treasurer, fails to pay any contribution due by him, shall be reported to the Council, and, if the Council see fit, shall be declared no longer a Fellow. Notwithstanding such declaration all arrears of contributions shall remain exigible.

* The following is an extract from the previous law affecting Annual Subscribers elected prior to December 1916:—"Every Ordinary Fellow, within three months after his election, shall pay Two Guineas as the fee of admission, and Three Guineas as his contribution for the Session in which he has been elected; and annually at the commencement of every Session, Three Guineas into the hands of the Treasurer. This annual contribution shall continue for ten years after his admission, and it shall be limited to Two Guineas for fifteen years thereafter."

ELECTION OF HONORARY FELLOWS.

VIII.

Honorary Fellows shall be persons eminently distinguished in Science or Literature. They shall not be liable to contribute to the Society's Funds. Personages of the Blood Royal may be elected Honorary Fellows at any time on the nomination of the Council, and without regard to the limitation of numbers specified in Law I.

IX.

Honorary Fellows shall be proposed by the Council. The nominations shall be announced from the Chair at the First Ordinary Meeting after their selection. The names shall be printed in the circular for the last Ordinary Meeting of the Session, when the election shall be by Ballot, after the manner prescribed in Laws III and IV for the Election of Fellows.

EXPULSION OF FELLOWS.

X.

If, in the opinion of the Council, the conduct of any Fellow is injurious to the character or interests of the Society, the Council may, by registered letter, request him to resign. If he fail to do so within one month of such request, the Council shall call a Special Meeting of the Society to consider the matter. If a majority consisting of not less than two-thirds of the Fellows present and voting decide for expulsion, he shall be expelled by declaration from the Chair, his name shall be erased from the Roll, and he shall forfeit all right or claim in or to the property of the Society.

XI.

It shall be competent for the Council to remove any person from the Roll of Honorary Fellows if, in their opinion, his remaining on the Roll would be injurious to the character or interests of the Society. Reasonable notice of such proposal shall be given to each member of the Council, and, if possible, to the Honorary Fellow himself. Thereafter the decision on the question shall not be taken until the matter has been discussed at two Meetings of Council, separated by an interval of not less than fourteen days. A majority of two-thirds of the members present and voting shall be required for such removal.

MEETINGS OF THE SOCIETY.

XII.

A Statutory Meeting for the election of Council and Office-Bearers, for the presentation of the Annual Reports, and for such other business as may be arranged by the Council, shall be held on the fourth Monday of October. Each Session of the Society shall begin at the date of the Statutory Meeting.

XIII.

Meetings for reading and discussing communications and for general business, herein termed Ordinary Meetings, shall be held, when convenient, on the first and third Mondays of each month from November to July inclusive, with the exception that in January the meetings shall be held on the second and fourth Mondays.

XIV.

A Special Meeting of the Society may be called at any time by direction of the Council, or on a requisition to the Council signed by not fewer than six Fellows. The date and hour of such Meeting shall be determined by the Council, who shall give not less than seven days' notice of such Meeting. The notice shall state the purpose for which the Special Meeting is summoned ; no other business shall be transacted.

PUBLICATION OF PAPERS.

XV.

The Society shall publish Transactions and Proceedings. The consideration of the acceptance, reading, and publication of papers is vested in the Council, whose decision shall be final. Acceptance for reading shall not necessarily imply acceptance for publication.

DISTRIBUTION OF PUBLICATIONS.

XVI.

Fellows who are not in arrear with their Annual Subscriptions and all Honorary Fellows shall be entitled gratis to copies of the Parts of the Transactions and the Proceedings published subsequently to their admission.

Copies of the Parts of the Proceedings shall be distributed by post or otherwise, as soon as may be convenient after publication ; copies of the Transactions or Parts thereof shall be obtainable upon application, either personally or

by an authorised agent, to the Librarian, provided the application is made within five years after the date of publication.

CONSTITUTION OF COUNCIL.

XVII.

The Council shall consist of a President, six Vice-Presidents, a Treasurer, a General Secretary, two Secretaries to the Ordinary Meetings (the one representing the Biological group and the other the Physical group of Sciences),* a Curator of the Library and Museum, and twelve ordinary members of Council.

ELECTION OF COUNCIL.

XVIII.

The election of the Council and Office-Bearers for the ensuing Session shall be held at the Statutory Meeting on the fourth Monday of October. The list of the names recommended by the Council shall be issued to the Fellows not less than one week before the Meeting. The election shall be by Ballot, and shall be determined by a majority of the Fellows present and voting. Scrutineers shall be nominated as in Law IV.

XIX.

The President may hold office for a period not exceeding five consecutive years; the Vice-Presidents, not exceeding three; the Secretaries to the Ordinary Meetings, not exceeding five; the General Secretary, the Treasurer, and the Curator of the Library and Museum, not exceeding ten; and ordinary members of Council, not exceeding three consecutive years.

XX.

In the event of a vacancy arising in the Council or in any of the offices enumerated in Law XVII, the Council shall proceed, as soon as convenient, to elect a Fellow to fill such vacancy for the period up to the next Statutory Meeting.

* The Biological group includes Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, Zoology; the Physical group includes Astronomy, Chemistry, Mathematics, Metallurgy, Meteorology, Physics.

POWERS OF THE COUNCIL.

XXI.

The Council shall have the following powers :—(1) To manage all business concerning the affairs of the Society. (2) To decide what papers shall be accepted for communication to the Society, and what papers shall be printed in whole or in part in the Transactions and Proceedings. (3) To appoint Committees. (4) To appoint employees and determine their remuneration. (5) To award the various prizes vested in the Society, in accordance with the terms of the respective deeds of gift, provided that no member of the existing Council shall be eligible for any such award. (6) To make from time to time Standing Orders for the regulation of the affairs of the Society. (7) To control the investment or expenditure of the Funds of the Society.

At Meetings of the Council the President or Chairman shall have a casting as well as a deliberative vote.

DUTIES OF PRESIDENT AND VICE-PRESIDENTS.

XXII.

The President shall take the Chair at Meetings of Council and of the Fellows. It shall be his duty to see that the business is conducted in accordance with the Charter and Laws of the Society. When unable to be present at any Meetings or attend to current business, he shall give notice to the General Secretary, in order that his place may be supplied. In the absence of the President his duties shall be discharged by one of the Vice-Presidents.

DUTIES OF THE TREASURER.

XXIII.

The Treasurer shall receive the monies due to the Society and shall make payments authorised by the Council. He shall lay before the Council a list of arrears in accordance with Rule VII. He shall keep accounts of all receipts and payments, and at the Statutory Meeting shall present the accounts for the preceding Session, balanced to the 30th of September, and audited by a professional accountant appointed annually by the Society.

DUTIES OF THE GENERAL SECRETARY.

XXIV.

The General Secretary shall be responsible to the Council for the conduct of the Society's correspondence, publications, and all other business except that which relates to finance. He shall keep Minutes of the Statutory and Special

Meetings of the Society and Minutes of the Meetings of Council. He shall superintend, with the aid of the Assistant Secretary, the publication of the Transactions and Proceedings. He shall supervise the employees in the discharge of their duties.

DUTIES OF SECRETARIES TO ORDINARY MEETINGS.

XXV.

The Secretaries to Ordinary Meetings shall keep Minutes of the Ordinary Meetings. They shall assist the General Secretary, when necessary, in superintending the publication of the Transactions and Proceedings. In his absence, one of them shall perform his duties.

DUTIES OF CURATOR OF LIBRARY AND MUSEUM.

XXVI.

The Curator of the Library and Museum shall have charge of the Books, Manuscripts, Maps, and other articles belonging to the Society. He shall keep the Card Catalogue up to date. He shall purchase Books sanctioned by the Council.

ASSISTANT-SECRETARY AND LIBRARIAN.

XXVII.

The Council shall appoint an Assistant-Secretary and Librarian, who shall hold office during the pleasure of the Council. He shall give all his time, during prescribed hours, to the work of the Society, and shall be paid according to the determination of the Council. When necessary he shall act under the Treasurer in receiving subscriptions, giving out receipts, and paying employees.

ALTERATION OF LAWS.

XXVIII.

Any proposed alteration in the Laws shall be considered by the Council, due notice having been given to each member of Council. Such alteration, if approved by the Council, shall be proposed from the Chair at the next Ordinary Meeting of the Society, and, in accordance with the Charter, shall be considered and voted upon at a Meeting held at least one month after that at which the motion for alteration shall have been proposed.

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- . The Pre-Devonian Basement Complex of Central Spitsbergen, vol. liii, No. 10, 1922, pp. 209-229.
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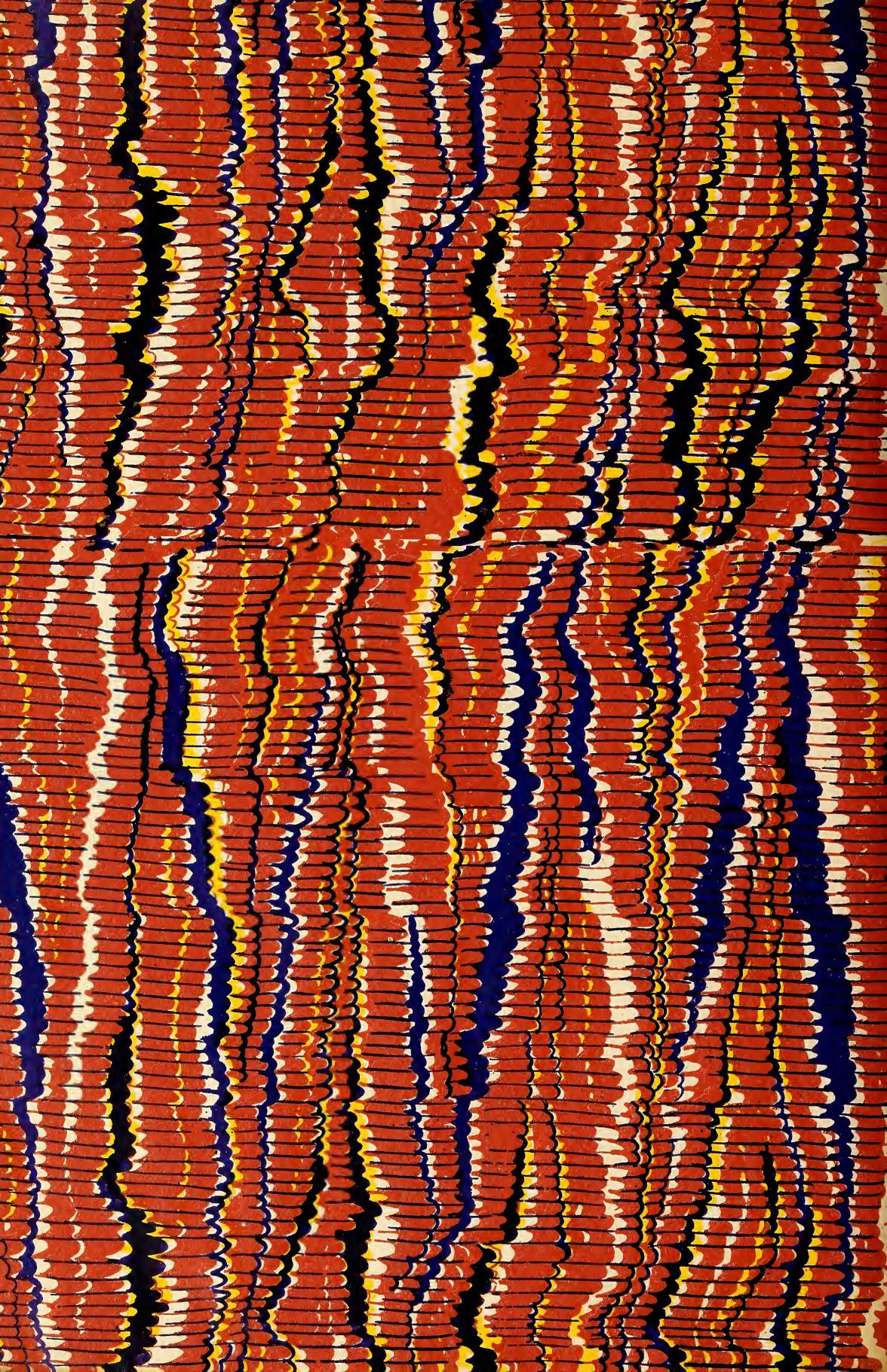
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