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ON THE ULTRA VIOLET COMPONENT IN ARTIFICIAL LIGHT.

By Louis Bell.

With two Plates

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THE ULTRA VIOLET COMPONENT IN ARTIFICIAL LIGHT.

By Louis Bell.

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Purpose of the Investigation. — The fundamental purpose of this study has been definitely to evaluate the amount of energy given by various artificial illuminants in the ultra violet portion of the spectrum. In particular, beside determining the general proportion of ultra violet rays and their actual amount in each lamp investigated, the writer has determined in absolute measure the ultra violet energy delivered by each light source for unit illuminating value. Assuming that each of the artificial lights studied is to be used to produce a certain given illumination, the amount of ultra violet radiation incidental to that illumination has been set down in absolute terms of ergs per second per sq. c. m. This classification of illuminants, which has not hitherto been made, is important in view of the possible harmful effects of radiation of short wave length which have been repeatedly discussed during the past few years. The amount of such possibly injurious radiation given by any particular lamp is a matter of small importance except as it is correlated with the illuminating power of the lamp, so that one may know to what amount of possibly harmful radiations he is exposed in securing a required degree of illumination.

Nature and extent of Radiations under Suspicion as harmful. — There has been much discussion concerning the effects of radiations of different wave lengths upon the eye. Without going extensively into an examination of the literature, which is very scattered and extensive, or of the physiological facts, some of which the writer now has under careful investigation and which will be reported later, it is sufficient here to say that the investigators of this matter may be divided into somewhat divergent schools. All agree that the extreme ultra violet rays, those of wave length less than 300 $\mu\mu$, which are absorbed by the cornea and so do not penetrate to the inner parts of the eye, produce when in sufficient intensity more or less serious damage to the corneal ephithelium, resulting in acute irritation, always accompanied by conjunctivitis, and sometimes by cloudiness of the cornea and other symptoms which go to make up the complex injury which has come to be known as *ophthalmia electrica.* It is in effect a superficial sunburn of the eye and is often accompanied by a similar sunburn in the vicinity of the affected eye. Whether this particular sort of effect is produced also by ultra violet rays of slightly greater wave length, say up to 320 $\mu\mu$ or 330 $\mu\mu$, is a matter of some dispute, but most investigators have held this particular region under suspicion on account of the phenomena of snow blindness, which closely resemble those of the so-called *ophthalmia electrica,* and cannot be produced by the extreme ultra violet rays since the solar spectrum owing to atmospheric absorption is extremely weak at and below 300 $\mu\mu$, very near to which point it is wholly cut off. It is, however, fairly rich at 320 to 330 $\mu\mu$, the cutting off by atmospheric absorption being rather sudden, as shown in *a*, Plate 1.

Now while the cornea cuts off only rays of wave length less than 300 $\mu\mu$ the lens of the human eye ordinarily absorbs the whole ultra violet, it being substantially due to this absorption that we are unable to see beyond the violet. This absorption extends to about wave length 380 $\mu\mu$ and in old persons in whom the lens gets slightly yellow even as far as wave length 420 $\mu\mu$. In early youth there is a very slight transmissibility of the lens in the region 315 to 330 $\mu\mu$ as shown by Hallauer.[1] Now potentially the rays which are absorbed by a medium may produce changes in it and the ultra violet rays up to and including the extreme violet have been reputed by various writers to produce a large variety of lesions, including retinal injury due to the rays which may filter through the lens. The list of reputed dangers is a very long one including erythropsia, color scotomata, cataract and other serious results. The situation from the point of view of the ophthalmologists who seem to be really in fear of ultra violet radiations is well summed up by Schanz and Stockhausen.[2] Other writers like Best[3] and Voege[4] attach relatively little importance to the effect of the ultra violet as such and are inclined to attribute some of the phenomena to over-intense radiation of ordinary light or to causes not connected to radiation at all.

A third group, of which Birch-Hirschfeld[5] is a representative, holds an intermediate view. It should be noted that the permanent

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[1] Klin. Monatsbl. f. Augenheilk., Dec. 1909.
[2] Ztschr. f. Augenheilk., May 1910.
[3] Klin. Monatsbl. f. Augenheilk., May 1909.
[4] Die Ultravioletten Strahlen der modernen kuenstlichen Lichtquellen und ihre augenbliche Gefahr für das Auge. Berl., 1910.
[5] Ztschr. f. Augenheilk., July 1908, and elsewhere.

injuries ascribed to ultra violet rays, like cataract and retinal degeneration, are charged to the radiations running even up to the visible spectrum, while the extreme ultra violet, absorbed by the cornea, produces only superficial lesions generally recovered in a few days.

From the standpoint of the present investigation it did not seem justifiable to attempt to pass without further investigation on the validity of any of the divergent views here noted, but to deal with the radiations of short wave length as a whole, including in the possibly injurious group all those radiations which have been under serious suspicion on clinical evidence by reputable investigators. The line has therefore been drawn between the ordinary lighting radiations and radiations of short wave length in the extreme violet and ultra violet of the spectrum, where the lighting value of the rays is negligible and their actinic value notably high.

Separation of the Ultra Violet from the Visible Spectrum. — Having determined on such a separation of the radiations under grave suspicion of injurious action from the rest of the spectrum, it was next in order to find a suitable screen for making just this division of the spectrum, so that it would be possible to measure the energy in the two portions of the spectrum directly and as a whole, without a resort to the extremely difficult and troublesome measures of the energy in separate spectrum lines, a task of great delicacy when discontinuous have to be compared with continuous spectra. After considerable investigation a suitable medium was found in the so-called Euphos glass. This glass, which has been strongly recommended by Schanz and Stockhausen as eliminating completely all the harmful rays and which was prepared under the direction of one of them, cuts off the ultra violet spectrum with remarkable definiteness while showing relatively little absorption of the general luminous rays.

Plate 1, *b*, *c*, *d*, shows the nature of this absorption very clearly. Spectrogram *b* of this Plate is the spectrum of the mercury quartz arc put on merely for reference, the group at 365 $\mu\mu$ being at the right of the figure and the brilliant green line exactly in the centre of the plate. Spectrogram *c* shows the spectrum of the magnetite arc which is very rich in the ultra violet and *d* shows the same as absorbed by a Euphos glass screen 2 mm. thick. The exposure in each case was one minute with a rather wide slit and a very brilliant grating. The cut off of the shorter wave lengths by the Euphos glass in the ultra violet is very clean and sudden at wave length 390 $\mu\mu$, practically just at the end of the visible spectrum as seen by the average eye. The absorption continues slightly on into the violet, gradually fading away until the transmission becomes nearly complete for the bright blue mercury line (435 $\mu\mu$).

In examining *b, c* and *d* of Plate 1 it must be remembered that the second order ultra violet overlaps the first order so that the group near 365 $\mu\mu$ appears in the first order at the extreme right of the figure and in the second order at the extreme left. In *d* of this Plate the arc spectrum fades off on the left, not from absorption but from the weakening of the photographic action. The Euphos glass is extremely transparent to the radiations throughout all except the extreme violet of the visible spectrum, and well into the infra red, as will hereafter be seen. The results here obtained for its absorption of the ultra violet are altogether parallel with those shown in the paper by Schanz and Stockhausen [6] and also by Hallauer.[7] The Euphos glass thus enables a particularly clean partition of the visible spectrum from the ultra violet and extreme violet to be made.

If it were possible to obtain an equally good absorbent for separating the infra red from the visible spectrum radiometric measurements of efficiency would be greatly facilitated. It should here be noted that Euphos glass appears in various shades and some imitations of it are now upon the market, so that a sample of such glass should be tested in the spectrograph before use for such a purpose as the present, inasmuch as in some of the shades the cut-off of the ultra violet is much less sharp and complete. The sample here used was the original No. 1, 2 mm. thick.

Method of Investigation. — The method taken for the evaluation was the familiar one of measuring the radiation directly by means of a thermopile connected with a sensitive galvanometer in a manner familiar in recent experiments on the efficiency of illuminants in the visible spectrum, e. g., Lux,[8] Féry.[9] The thermopile was chosen as the radiometric instrument merely as a matter of convenience. The instrument actually used was a Rubens linear thermopile, having 20 constantin-iron couples with a total resistance of 4.6 ohms. It was mounted as shown in Figure 1, in a vacuum tube with a quartz window immediately in front of the couples. The inner body of the instrument, containing the couples, was taken out of its original mounting and set up in a tube about 37 mm. in diameter, through the upper end of which was sealed a pair of leading-in wires.

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[6] Zts. f. Augenheilk., May 1910, Table VII, figure 3.
[7] Archiv. of Ophthal., Jan. 1910, Plate II, figure 3.
[8] Zts. f. Beleuchtungswesen, Heft 16, 1 p. 36, 1907.
[9] Bull. Soc. Franc. de Physique, p. 148, 1908.

These were firmly clamped in the binding posts of the instrument by working through the side tube attached for the reception of the quartz window. The thermopile was then pushed up exactly opposite the side tube and wedged in place with cork and cotton wool attached with shellac. The end of the side tube was flanged out and ground flat for the fitting of the quartz window and after the shellac had dried out thoroughly the window was fastened in place and the lower end of the tube drawn out for the attachment of the pump. The tube was pumped to the high vacuum usual in an X-ray tube, and was then sealed. It was mounted as shown in a block of wood to which was secured the disconnecting terminal, reached by a long handled plug,

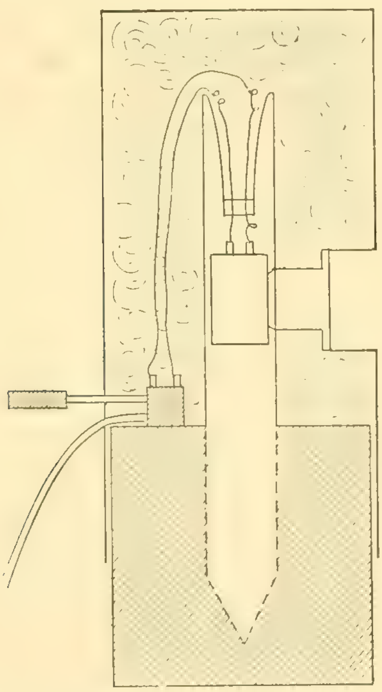

FIGURE 1. Vacuum thermopile.

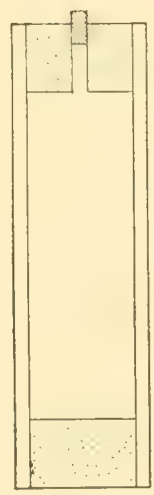

FIGURE 2. Quartz cell.

and the whole was then surrounded by a pasteboard case having a hole just opposite the quartz window, and packed full with loose cotton wool. The galvanometer was of the D'Arsonval type, having a sensibility of 2×10^{-8} ampere per mm. scale deflection. Its period for the attainment of a complete deflection, was, under the ordinary conditions of its use, 1 minute.

The galvanometer deflections were read by a scale and telescope, the scale being a special one bent to 1.5 meters radius. The thermopile indications were calibrated in absolute measure by observations on the radiation of a standard incandescent lamp supplied by the Bureau of Standards. After applying the proper correction for stray thermal losses and spherical reduction factor and reducing the readings as taken to the standard distance of 50 cm. employed throughout this investigation, the constant of the thermopile galvanometer system was found to be 1 mm. = 1 scale division = 35.3 ergs per second per square cm. By this constant the observed deviations were reduced to absolute dynamical measure.

As a matter of convenience and to establish an approximate ratio between the ultra violet radiation from the various sources studied and the radiation in the visible spectrum, an absorption cell which

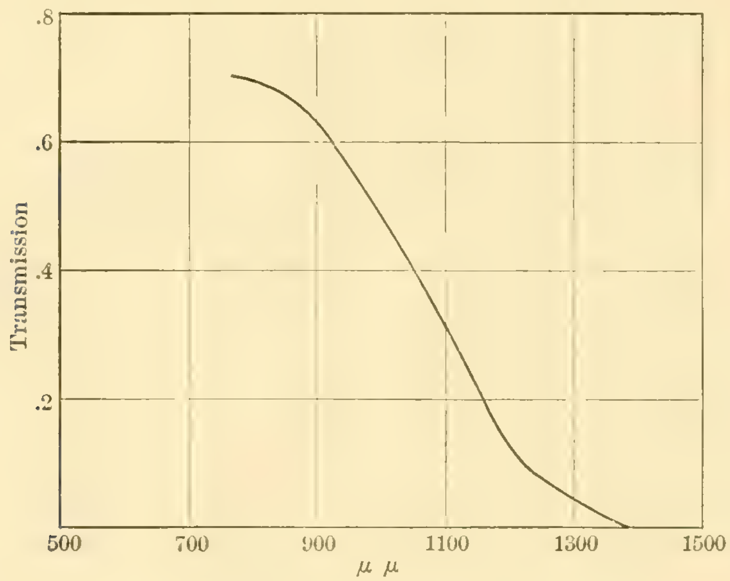

FIGURE 3. Absorption curve of water.

eliminated nearly all the infra red was kept in front of the thermopile window. This cell, Figure 2, was of glass, ground flat and exactly 1 cm. thick, 44 mm. external diameter and 35 mm. internal diameter. The glass ring was provided with a hole for filling and was closed by two quartz plates cut across the axis, each 2.25 mm. thick and 44 mm. diameter. These were fastened with hard shellac to the glass cell, and the cell in use was filled with distilled water. The absorption of a layer of distilled water of this thickness is shown in Figure 3 taken from Nichols's experiments.[10] Quartz has no material absorption in the part of the infra red spectrum transmitted and neither quartz nor

[10] Nichols, Physical Review, Vol. 1, p. 1.

distilled water in this thickness has any material absorption in even the extreme ultra violet up to the limit investigated.

The use of this cell therefore could produce no sensible effect on the accuracy of the ultra violet measurements, while it did serve the extremely useful purpose of limiting the total amount of energy to be measured and of eliminating any difficulties that might arise owing to absorption in the further part of the infra red, all the absorbing media incidentally used being, as compared with water, practically entirely transparent to all the radiations that got through the water cell. It would have been convenient if some substance cutting off the infra red sharply at 750 $\mu\mu$ or 800 $\mu\mu$ had been available. Unfortunately, there is no such substance, so far as has yet been discovered, the very few substances less transparent than water in the region 800 to 1300 $\mu\mu$ being useless for the purpose of this investigation on account of opacity in the ultra violet and generally in the visible spectrum as well. Iron ammonium alum used by Lux (loc. cit.) and the copper salts used by Féry (loc. cit.) are open to this objection and the same is true of all the otherwise useful and promising substances discussed in the very thorough and valuable researches of Coblentz.[11]

In some of the experiments a second similar quartz cell was used, particularly in work on arc lamps. In this case the Euphos glass used to cut out the ultra violet portion of the spectrum was permanently affixed to one of these cells and either the plain quartz cell or the Euphos-quartz cell was thrust into the beam so as quickly to get differential readings. In order to avoid the somewhat large correction due to reflection of energy which would have been produced by the introduction of a plain slip of Euphos glass to cut out the ultra violet the following expedient was adopted.

The Euphos glass was attached to the surface of the quartz cell by spring clips with the addition of a thin capillary film of pure glycerine between the quartz and glass surface. Glycerine is immensely transparent to all radiations, including the extreme ultra violet, to which Canada balsam and gelatine are quite opaque. Its index of refraction, 1.47 for D, is sufficiently near that for quartz and the various glasses to reduce the loss of light at the surfaces to an entirely negligible amount. As the Euphos has a slightly less index of refraction than quartz, there was a minute residual gain in the total transmission of the system when the Euphos glass was added, in the right direction to compensate for the minute losses by absorption in the glycerine film.

[11] Bull. Bureau of Standards, Vol. 2, p. 619.

As a check on the possible magnitude of this virtual absorption by the glycerine film readings were taken on a tungsten lamp through the quartz cell alone, and through the quartz cell plus a disc of optical crown glass 2 mm. thick secured with glycerine in the ordinary manner. The absorption of this crown glass is shown in Plate 1, *e, f, g*, in which *e* is the spectrogram of the quartz arc taken with a wide slit and 2 minutes exposure, *f* the spectrogram through the crown glass in question, and *g* through the Euphos glass. In spite of the fact that there is a slight absorption by the crown glass in the region near 300 $\mu\mu$, the addition of the crown glass and glycerine film reduced the galvanometer deflection by barely 0.5 %, an amount scarcely outside the errors of observation. The energy cut off from the spectrum of a tungsten lamp by the crown glass would be of course very small, but perhaps not negligible, since as Schanz and Stockhausen have shown (loc. cit. table VIII, figure 6) the tungsten lamp spectrum goes quite down to 300 $\mu\mu$ in sufficient strength to give a clear photographic effect. At all events it is evident that the use of the glycerine film involves no material errors.

In the ordinary experimentation in using steady sources, sets of readings were taken alternately with and without the Euphos glass, the glass being either added to the clear cell with the glycerine film, or removed and the film quickly washed away with distilled water. With sources which give trouble from unsteadiness the second quartz cell was brought into play as previously mentioned. Aside from a slight drifting of the zero point, which is generally observable in measurements with a thermopile, the method adopted worked very smoothly. The drift, however, was usually small and slow and satisfactorily taken care of by a time correction. With proper attention to this, the readings, although necessarily slow, were nearly as consistent as would be found in ordinary photometric measurements. The following string of deflections forming a single group of 5 readings is typical of those obtained under ordinary conditions.

Scale readings from bare quartz lamp through quartz cell only.

	cm.
	36.17
	36.10
	36.27
	36.36
	36.16
Av. =	36.21

The mean departure of a single reading from the average here given is slightly less than ¼%, so that the errors of observation, of which this is a fair sample, showed that the thermopile observations are about as reliable as those with a photometer. Some preliminary experiments made on Euphos and other glasses showed that the transmission of the Euphoa glass aside from its absorption in the violet and ultra violet was exceptionally high for such rays as got through the layer of distilled water. In fact the total transmission of energy with Euphos glass was greater than with the ordinary samples of clear glass and was only exceeded by a single sample of optical crown which showed extraordinary transparency to all these radiations, so great that the losses were practically only those chargeable to actual reflection at the surfaces.

Measurements on various Illuminants. — With these preliminaries the apparatus was set up permanently and work begun on commercial illuminants. Readings of current and voltage on the electric lamps were taken by Weston instruments freshly calibrated, and the quantity readings on the gas lamps tested were obtained from a newly adjusted standard meter.

100 Watt Tungsten.— The first source of light investigated was an ordinary 100 watt tungsten lamp, taking actually .951 amperes at 113 volts, i. e. 103.3 watts, and giving 79.4 c. p. in the direction of the thermopile. With this lamp the mean difference of deflection due to energy cut off by the Euphos glass was 1.9 cm. The ultra violet energy cut off, including such losses in the extreme violet as are indicated by Plate **1**, *d*, was 6% of the total energy transmitted by the quartz cell.

100 Watt Gem. — The second source studied was an ordinary 100 watt Gem lamp, taking 100 watts at 114 volts and giving in the marked direction 39.25 c. p. This lamp of course gave a spectrum relatively weak in the ultra violet, but as will be seen from its spectrogram in Plate **2**, *b*, the ultra violet region down to wave length 330 $\mu\mu$ is by no means negligible. The total differential deflection due to the ultra violet was in this case only 0.61 cm., 2.6% of the total deflection. These readings confirm the extraordinarily small absorption of Euphos glass throughout the longer wave lengths, since the transmission observed with the known cut off of a very perceptible amount in the ultra violet, leaves no room for any material selective or general absorption elsewhere.

It should here be noted that while quartz transmits with extraordinary freedom, so far as absorption is concerned, all rays which are allowed to pass by a cm. thickness of distilled water, it still exercises a slight selective action by reflection. The index of refraction of quartz for the longer wave lengths of the visible spectrum is 1.54, while for rays in the further ultra violet this figure rises to about 1.6, hence in accordance with Fresnel's formula $\left(\frac{n-1}{n+1}\right)^2$ there is a small amount of selective stopping of the ultra violet rays by reflection. This occurs both at the quartz water cell and at the quartz window in front of the thermopile so that the total selective effect is proportional to the fourth power of the difference due to the change in the index of refraction for a single surface of transmission. This difference amounts to approximately 2 % as between the red rays and the further part of the ultra violet. The result is to cause a slight under estimation of the ultra violet. No account has been taken in any of these experiments of this very small and troublesome correction, which amounts in ordinary cases to only a small fraction of 1% of the total ultra violet. The existence of the effect should, however, be noted as it has a tendency toward causing a slight under estimate rather than an over estimate of the ultra violet component.

Cooper Hewitt Tube. — The next source investigated was the Cooper Hewitt tube. One of the ordinary commercial 22 inch tubes was used, the particular tube having previously been used in another research and very carefully photometered. A section of this tube, giving 100 c. p., was screened off so that the length might be so reduced that the energy from the whole section taken could fall freely upon the thermopile without causing a material angular error or forcing one to depart widely from the standard distance of 0.5 meter. The horizontal radiation normal to the tube was of course taken, the reflector being removed. The corrected deflection due to the ultra violet amounted to 1.64 cm. which corresponded to 41.7 % of the total energy passing through the quartz cell. The lamp was singularly steady and easy to work with, with the exception of producing an inconveniently small total deflection. The result, however, can be regarded as fairly precise in spite of the small magnitude, the mean deviation of a single reading amounting to barely over .5 % in the total deflection. In this lamp the ultra violet energy is nearly all between 365 $\mu\mu$ and the visible spectrum, the extreme ultra violet being entirely cut off by the glass of the tube and the few lines of wave length between 365 and 300 $\mu\mu$ being reduced by the absorption to very feeble intensity. The total deflection produced by this lamp, of which the portion exposed radiated 100 c. p., was only 17 % of the deflection given by the Gem lamp of the previous experiment, which gave less than 40 c. p.

Quartz Mercury Lamp. — Following the examination of the ordinary glass Copper Hewitt tube, the next source investigated was the quartz mercury lamp. Two tubes were available, each of the ordinary commercial 220 volt type rated at 3.5 amperes. One of these tubes, which is here referred to as the old mercury lamp, was made by the French Cooper-Hewitt Company and had been already used for experimental purposes for about a year and had seen rather hard service, having often been worked above its rated amperage. The second lamp was entirely new, made in the Cooper-Hewitt factory in this country and was not at any time worked above its rating. The spectrum of the quartz lamp is extremely rich in certain portions of the ultra violet, particularly in rays of wave length less than 300 $\mu\mu$. It is well shown in Spectrum *e* of Plate **1**. The brilliant lines in this spectrum, counting from the violet, have wave lengths as follows: —

4077.84	2967.27
4046.55	2925.38
3983.96	2893.60
3906.47	2752.80
3663.27	2698.88
3662.88	2655.14
3654.83	2653.70
3650.14	2652.07
3341.48	2536.52
3131.84	2483.87
3131.56	2482.76
3125.67	2482.07
3027.49	2399.81
3025.61	2399.43
3023.43	2378.39
3021.50	2302.65

The wave lengths here are taken at the value assigned by Stiles[12] in A. u. It will be observed that a number of the lines are associated in close groups which with small dispersion mass into heavy lines. The relative intensity of the lines, as is well known, shifts considerably with the degree of excitation of the tube, so that the relative intensities given by Stiles do not agree with the spectrograms taken from the quartz arc for the same reason that Stiles' arc and spark intensities do not agree. The quartz arc spectrum resembles Stiles' arc spectrum much more closely than it does the spark spectrum.

[12] Astrophysical Journ., Vol. XXX, p. 48.

In particular the quartz arc spectrum displays a very striking gap between wave length 334.14 $\mu\mu$ and the double line at wave length 313.1 $\mu\mu$. Save for the very faint haze of continuous spectrum that characterizes the radiation from the quartz tube this part of the spectrum is blank. Indeed the line 334.14 $\mu\mu$ itself is far from strong relatively to those in the further part of the ultra violet and there is very little effect of radiation between wave length 313.1 $\mu\mu$ and 365.2 $\mu\mu$. This gap is of some significance in interpreting the results of bactericidal experiments, since any failure of bactericidal action in the region between wave length 350 $\mu\mu$ and wave length 313 $\mu\mu$ observed in working with the quartz lamp may be due to the absence of any strong radiation in this region as well as to lack of specific bactericidal power in rays of this particular wave length if they existed.

In the radiometric investigations on the old quartz lamp it was run at 3.7 amperes and about 80 volts, an average of about 260 watts, without an external globe. Under these circumstances the corrected deflection due to the total ultra violet was 16.7 cm. The deflections were not quite so steady as in the case of the ordinary Cooper Hewitt tube, but still the average departure of a single reading was within 1%. After the deflection due to the total ultra violet was determined another set of readings was taken with the bare lamp and quartz cell and then with the Euphos glass replaced by the crown glass of which the absorption spectrum is shown at *f*, Plate 1.

This glass in effect cuts off substantially the whole of the extreme ultra violet spectrum, letting pass in practically undiminished strength only the lines of greater wave length than 300 $\mu\mu$. This separation is of some importance with respect to the bactericidal power of the lamp in water purification and similar work. The result was to show that the transmission of the crown glass was 54.7 % of the transmission found for the Euphos glass. In other words, nearly one half of the total ultra violet energy in this lamp was of wave length below 300 $\mu\mu$. Of the remaining half the spectrum shows, as just indicated, that by all odds the larger part lies between 365 $\mu\mu$ and the visible spectrum.

The new quartz lamp without its globe was then tested, the input in this case being 350 watts. The ultra violet output was greater than in the old tube, the total deflection reduced to the standard distance rising to 32.1 cm. In this case 65.1 % of the energy transmitted by the quartz water cell was cut off by the Euphos glass. Following up the radiometric measurement further, the Euphos glass was replaced by the light crown glass as before with the result of showing that substantially one half, 49.9 %, of the total ultra violet was cut off by the crown glass and hence substantially this proportion was of wave length less than 300 $\mu\mu$.

In running quartz lamps without their globes, as was done in these experiments, the energy output is considerably diminished by the cooling of the tube and the light-giving properties of the lamp are very much reduced. Both the old and the new quartz lamps herein noted were photometered. The lamps were compared against a tungsten secondary standard by means of a Simmance-Abady flicker photometer. The c. p. normal to the length of the tube and in a horizontal direction, was for the old quartz lamp 415, for the new quartz lamp 348, in each case without any enclosing globe. Both lamps were very steady and easy to work with, both on the photometer bar and with the thermopile.

Finally the new quartz lamp was fitted with its regular diffusing globe and tested with the thermopile. In working with the globe the tube operated at a higher temperature and far more intensively, the wattage rising to 460. With the Euphos glass in, the total change in deflection amounted to only 3.7 cm. although the lamp tested on the photometer as in the previous case reached 820 c. p. in the horizontal direction. In percentage the amount of energy cut off by the Euphos glass was 42.5. These figures plainly indicate that the globe absorbed the further ultra violet very strongly, more strongly than the crown glass already referred to. In fact the deflection due to the ultra violet energy which passed through the globe of the lamp was extraordinarily small with respect to the c. p. of the source, very much smaller than in the case of any other illuminant investigated. Without the globe the quartz arc is a very powerful source of radiation in the extreme ultra violet, below wave length 300 $\mu\mu$. With its ordinary globe on, all this energy in the extreme ultra violet is cut off and the small remaining amount, mostly in that part of the ultra violet nearest the visible spectrum, becomes quite insignificant.

The Welsbach Mantle.— At this point study of the radiation from the Welsbach light was taken up. The particular form used was a Graetzin street lamp with a single large inverted mantle fitted with a clear glass globe, which obviously eliminated whatever of extreme ultra violet might be present. This burner took 6.4 feet of gas per hour at 3 inches pressure and gave 75 c. p. in the horizontal direction. Its total deflection was slightly greater than that produced by the quartz lamp with its globe tested immediately before. The addition of the Euphos glass cut down the deflection by .924 c. m., an amount equivalent to the absorption of 8.4 % of the total radiation recorded. The lamp proved fairly easy to work with in point of steadiness and the average variation of a single deflection from the mean was still less than 1 %.

Acetylene Flame. — Following the trial of the Graetzin lamp a series of measurements was made on an acetylene flame fed from a Prestolite tank. This flame gave on the photometer in the direction of measurement 27.35 c. p. and its change in deflection on interposition of the Euphos was .524 cm., corresponding to a cut off of 4.5 % of the total energy. It proved very amenable to measurements and was quite as steady and easy to work with as the mantle burner previously used. The spectrum of the acetylene flame reaches well down into the ultra violet as shown by Schanz and Stockhausen.[13] It reaches, in fact, approximately wave length 310 $\mu\mu$, but the further portion of the spectrum is comparatively weak. The spectrum of the Welsbach mantle with a clear globe, given by the same authorities (loc. cit.), is cut off at about wave length 320 $\mu\mu$, but is notably bright in the part of the ultra violet toward the visible spectrum. These results are fully checked by the spectrograms taken of the particular burners here indicated.

The Carbon Electric Arc. — Next in order the various arc lamps were taken up for investigation, beginning with the arc between carbon electrodes. On account of the relative instability of the arcs the method of experimentation was modified. A second quartz cell similar to the one already in use was constructed and filled with distilled water. The ratio of the absorption between this new cell and the old cell was then determined. From a slight difference in thickness or in polish of the quartz plates the new cell was found to give about 1% more absorption than the original quartz cell and a correction for this difference was introduced in the subsequent measurements. The two quartz cells were mounted in recesses in a sliding screen so that either could be brought quickly in front of the thermopile window. The Euphos glass screen was then mounted with a glycerine film on one of the quartz cells so that the cells with and without the Euphos could be rapidly interchanged in the beam from the lamp under test and the absorption thus determined without having to depend on the constancy of the lamp for any considerable time.

The times of observation were regulated by means of a stop watch so that a time correction for shift of zero could be readily made, and

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[13] Zts. f. Augenheilk., V. XXXIII, plate 8.

by taking several preliminary swings, so as to give the thermopile chance to settle into a steady state, the rate of shift of zero was kept pretty steadily and the corrections were easily applied. It was also necessary to photometer the arcs in the actual condition in which they were under test. To this end the apparatus was set up as shown in Figure 4. Here A is the arc lamp, B the thermopile, C the galvanometer, D the telescope and scale, E an adjustable rotating sector disc just in front of the arc, F the quartz cells in their sliding screen in front of the thermopile window, G a silvered plate glass mirror which could be quickly interposed in the beam between the arc and

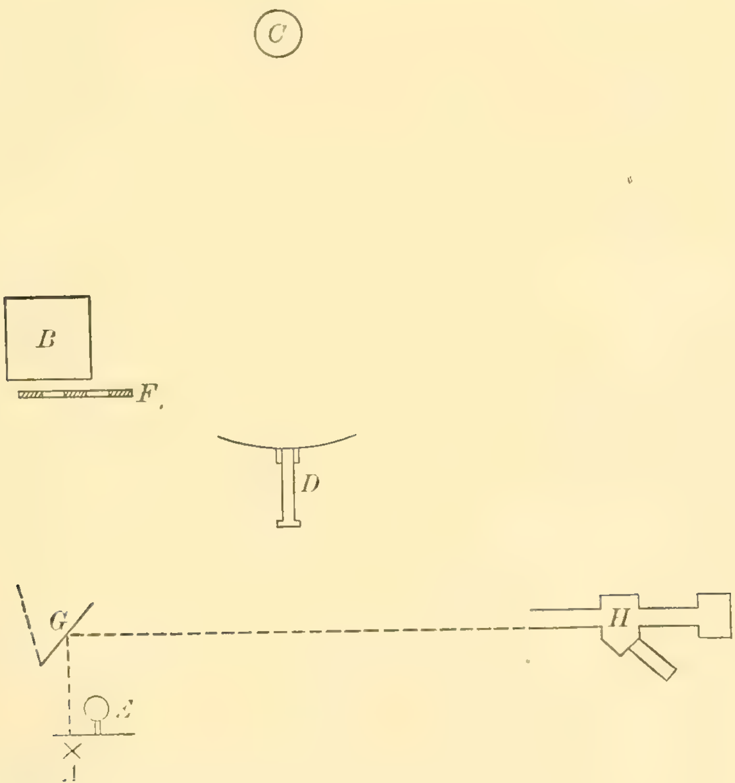

Figure 4. Arrangement of radiometric apparatus.

the thermopile so as to deflect the rays into the portable photometer H, set up on the other side of the photometer room. The coefficient of reflection of the mirror had previously been many times determined as the mirror had been in use for photometric work. The photometer was ready for use at any time simply by closing the switch on the standard lamp. When in course of a series of thermopile measurements it was desired to test the c. p. of the lamp the disc was started, the mirror swung into place and readings were then taken on the portable photometer.

The carbon arc was first attacked and it proved to be a difficult subject for investigation. The particular lamp used was of the enclosed type, having the globe fitted with a short side tube and a quartz window so as to keep the arc as steady as possible without losing the ultra violet. To the same end it was found desirable to adjust a magnet behind the arc so as to keep it burning on the side of the carbons next the thermopile instead of wandering round and round the carbons in the usual manner.

The arc thus operated gave a prodigious amount of ultra violet radiation, showing a continuous spectrum far down into the ultra violet and the three enormously intensive carbon bands usually ascribed to cyanogen, one of them in the extreme violet and the other two near wave lengths 380 $\mu\mu$ and 360 $\mu\mu$ respectively. Reduced to the standard distance the deflection due to the ultra violet cut off by the Euphos glass amounted to 74 cm., being 30 % of the whole energy which passed through the quartz cell. It has, of course, been long known that the naked electric arc gives off very powerful ultra violet radiations and its effect in the production of ophthalmia electrica has been known for more than half a century, but in this case the extent of the ultra violet activity was somewhat unexpected.

It was undoubtedly considerably enhanced by the intensive cyanogen bands as regards that portion of the radiation lying near the visible spectrum, but on the other hand the extreme ultra violet, wave length 300 $\mu\mu$ and less, is unquestionably stronger in the case of an open arc than in the enclosed arc on account of the very intense continuous spectrum emitted from the crater, which is much lessened when the arc is enclosed. No separation between these parts of the ultra violet was attempted with the lamp under consideration since its unsteadiness was a constant source of annoyance and the ordinary variations of independent readings from the mean amounted to 5 or 6 %. It was sufficiently evident, however, that a powerful enclosed arc in a globe which permits all the radiations to pass is an enormously powerful source of ultra violet light. The carbon arc, however, is rapidly passing out of general use so that attention was next directed to the luminous arc.

Magnetite Arcs. — The magnetite arc is one of the commonest and most generally useful outdoor illuminants. It gives a very intense nearly white light due almost wholly to the arc stream itself. The spectrum of this, the active electrode being composed almost wholly of the oxides of iron and titanium, is immensely complicated, containing thousands of bright lines so closely packed as almost to obtain the effect of a continuous spectrum. The actual character of the spectrum photographed with a fairly wide slit, is shown in Plate **2**, *d*. Here, with the quartz arc spectrum for reference at *a* is shown the radiation from the magnetite arc through a quartz window and below it the spectrum of the same arc taken through its ordinary globe. A quartz window was used merely to insure steadiness of the light, which would have been lost by taking off the globe. A glance shows that this spectrum is exceedingly rich in powerful lines all through the ultra violet clear down to wave length 230 $\mu\mu$. The glass globe cuts off the spectrum quite sharply near wave length 300 $\mu\mu$, as in Plate **2**, *e*, but from this region to the visible spectrum lies an almost continuous mass of strong lines, very intense in the region where the quartz mercury arc is conspicuously weak, say from the group at wave length 313 $\mu\mu$ to the group near wave length 365 $\mu\mu$.

For radiometric measurements the magnetite arc, which was operated at 6.6 amperes and about 80 volts, proved much more steady than the carbon arc, showing more small and quick fluctuations, but fewer of the large and relatively slow variations which interfered most with the readings. As a consequence the deflections obtained agreed more closely, the average variations of a single setting running between 3 and 4 %. For the magnetite arc through the quartz window the cut-off of Euphos glass amounted to 29 cm., 28 % of the total deflection. Through the ordinary glass globe the deflection was reduced to 22.4 cm., 22.5 % of the total deflection. The difference between these results shows that while there is a large amount of energy of short wave length produced by the magnetite arc, most of the ultra violet energy is of wave length greater than 300 $\mu\mu$. As compared with the quartz mercury arc used without its globe the magnetite arc gave relatively about 60 % less energy of wave length below 300 $\mu\mu$ and about 40 % more energy in the wave lengths above 300 $\mu\mu$. The candle power in the horizontal direction as measured by the method just described amounted to 760 in the run with the quartz window, and 700 in the run with the ordinary globe.

The Nernst Lamp. — Finally a series of readings was taken on the Nernst lamp. The lamp investigated was of the single glower type for 220 volts, taking 91 watts and giving a downward c. p. of 68. As the spectrum of this source runs to less than wave length 300 $\mu\mu$ and reaches that vicinity with somewhat material strength an attempt was at first made to run the Nernst glower without a globe. It proved so difficult to get steady deflections under these conditions, on account of the effect of air currents, that this measurement was abandoned and the readings taken with the globe on, which proved reasonably easy, the precision being comparable with that obtained with the ordinary incandescent lamps. But even then the lamp proved very sensitive to small changes of voltage and only by very careful regulation of the current could consecutive series of readings be held in reasonably close agreement.

In the average the deflection due to the ultra violet in the Nernst lamp with its globe was 1.81 cm. and the percentage of energy thus cut off was 5.2. This completed the radiometric investigation of ordinary illuminants. Two others which it seemed desirable to investigate, that is the ordinary flame arc, and the arc between iron electrodes as used by Finsen were studied on the spectrograph, since their fluctuations were of a character to make their study by means of a galvanometer of so long period as that used in this investigation quite impracticable. The peculiarities of these sources will be referred to in discussion of the general results.

Sun Light. — Finally it seemed advisable to take some comparative readings on sunlight as a source of ultra violet radiations, particularly with reference to the amount of ultra violet energy with respect to the intensity of the light. Of course the solar radiation in absolute amount has been investigated with great thoroughness, but the ultra violet has received less attention than the rest of the spectrum. In general the sun radiates energy substantially like an incandescent black body at about 6000 degrees C. except in so far as its energy, particularly in the ultra violet, is cut off by the absorption of its own and the terrestrial atmosphere. It behaves then, like an enormously hot incandescent body shining through a medium that cuts off all the ultra violet of less wave length than about 295 $\mu\mu$ and greatly diminishes the shorter radiations even into the violet of the visible spectrum. One would expect therefore to find relatively little total ultra violet per unit of illumination so far as the direct light of the sun is concerned. On the other hand as Schuster [14] and others have shown, much of this cutting off of the ultra violet is due to scattering of the short waves by the molecules of the atmosphere and small bodies suspended in it. In other words, the violet and ultra violet are not wholly lost, but appear in radiation from the blue sky.

Of the energy thus radiated from the sky the maximum lies almost in the edge of the ultra violet. The arrangement of the apparatus for experiments on sunlight is shown in Figure 5. Through the

[14] Nature, XXXI, p. 97.

courtesy of the Director, this part of the work was done in the Rogers Laboratory of Physics where the conditions for getting natural light were good. In Figure 5, A is a *porte lumière* receiving the light from the sun and forming by means of the iris diaphragm B, stopped to 3 mm. diameter, an image of the sun on the thermopile front at C, before which was placed the usual quartz cell D. The thermopile was connected with the galvanometer F, read by the telescope and scale G. By the use of the diaphragm, forming a species of "pin hole" image on the face of the thermopile, at a distance of 3 meters,

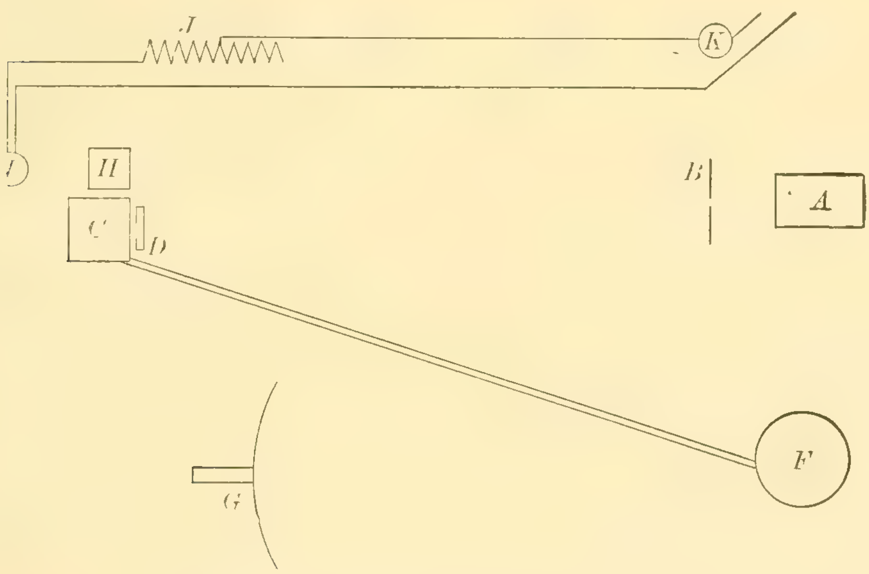

FIGURE 5. Apparatus for solar radiation.

the light and energy were cut down so as to be readable with comparative ease.

To measure the intensity of the illumination a Simmance-Abady flicker photometer H was set up close alongside the thermopile so that the solar image could be quickly moved so as to fall squarely on the photometer disc. On the other side of the photometer at I was an 80 watt tantalum lamp which was previously calibrated, in terms of the current flowing through it, against a standardized Gem lamp. From the source of supply the current was taken to this lamp through an adjustable rheostat J and a mil-amperemeter K. In measuring the light-intensity of the beam which was allowed to fall on the thermopile, it was simply shifted from the face of the thermopile to the face of the photometer and by means of the rheostat J a flicker balance was established. The current read on K and referred to the standardization curve at once gave the c. p. of I, so that the illumination could be computed.

The mirror at A was an electrolytic nickel surface highly polished, inasmuch as nickel gives a considerably higher coefficient of reflection near the end of the solar spectrum than does silver, which is particularly weak at this point. To separate the extreme violet and ultra violet as before and on exactly the same basis, the solar readings were taken with simply the quartz cell and then with the Euphos glass and a glycerine film. The cut off of violet and ultra violet produced by the Euphos glass in the first day's readings was 16.2 % and in a second day's reading 17.9 %, both days being brilliantly clear and cold in late December at noon. The average energy therefore cut off was substantially 17 % uncorrected for the coefficient of reflection of the nickel mirror, or approximately 21 % after the correction for the variation in reflection as between the ultra violet and the visible spectrum.

This figure is somewhat large as compared with the data ordinarily quoted for the ultra violet component of the solar spectrum, but it should be noted that this comparison is not with the spectrum as a whole but with that portion of it transmitted by a quartz cell filled with distilled water which cuts off a large part of the infra red. Also the absorption of the Euphos glass extends into the violet as has been previously noted, and finally the observations were taken in cold winter weather when the aqueous vapor, which is important in the absorption of the atmosphere, is pretty well frozen out.

The observed difference of deflection in these experiments on the sun due to the cut off of the ultra violet was 2.28 cm. and the observed intensity of the illumination was equivalent to 101 foot candles. These readings show precisely what the general theory indicates, that the solar light must be regarded as received from an enormously hot and hence very efficient radiator which has been robbed by atmospheric scattering and absorption of a considerable part of its shorter wave lengths.

Record of General Results.

In these experiments the following artificial sources of light were investigated with respect to the ultra violet component of each as separated from the rest of the spectrum by a disc 2 mm. thick of Euphos glass #1:— G. E. M. lamp; tungsten lamp; Cooper Hewitt tube; quartz lamp of the French Cooper Hewitt Company without globe; quartz lamp, American, without globe; quartz lamp, American, with globe; Graetzin mantle burner; acetylene flame; carbon electric arc through quartz window; magnetite arc through quartz window; magnetite arc with ordinary globe; Nernst glower. In addition, a study was made of sunlight with the thermopile for comparative purposes and spectrographic studies were also made of the ordinary yellow flame arc and of the arc between iron terminals such as is used for therapeutic purposes. The Euphos glass was chosen as the medium for the partition of the ultra violet from the rest of the spectrum for the reason that it cuts out and was intended to cut out by its designers all the rays of any illuminant which are under indictment as having specific harmful action on the eyes.

Broadly, the accusations of short wave lengths as injurious to the eye involve the entire ultra violet from the furthest point reached by natural or artificial illuminants up to and into the chemically active rays of the violet. If on the one hand it is the rays in the extreme ultra violet, wave length 300 $\mu\mu$ and less, which are absorbed by the cornea, that are held responsible for the ordinary phenomena of *ophthalmia electrica*, it is the rays of ultra violet of greater wave length than this, extending clear into the violet, that have been regarded by some recent investigators as producing perhaps serious lesions of the retina and of the lens. Note Schanz and Stockhausen.[15] The former class of injuries which have to do with the radiations absorbed by the cornea are wholly superficial and, according to Van Lint [16] the prognosis is generally good and the recovery rapid. Injuries to the retina and the lens, in-so-far as they take place, involve a far greater danger of permanent injury. Glass-blowers cataract is one of the typical injuries which has been ascribed to ultra violet radiations lying adjacent to the visible spectrum by Schanz and Stockhausen, Birch-Hirschfeld and others. Obviously, the temperature of melting glass (1400° C) is too low to give rise to any material amount of energy in the extreme ultra violet.

The present investigation, therefore, took account of the whole body of radiations of short wave length. So far as possible injury from the ultra violet component in any artificial light source is concerned it is obviously dependent on the amount of actual energy delivered by the source in the ultra violet region and not upon the

[15] Ztschr. f. Augenheilk., Mai, 1910.
[16] Accidents oculaires provoqué's par l'électricité, p. 100.

percentage relation of this energy to the whole input. It is quite clear that in order to do any injury to the eye a certain amount of energy must be spent upon it and must be delivered at a rate in excess of the power of the eye to repair damages. One receives injury from excessive exposure to ultra violet rays just as he receives it by excessive exposure to heat rays. In either case the delivery of energy at a very high rate for a considerable time does damage.

TABLE I.

Source	*Input*	*Total u. v.*	*u. v. per watt*
100 Watt G. E. M.	100	215	2.15×10^{-7}
Glass Mercury Lamp (½ length taken)	96	577	6.02×10^{-7}
Nernst (with globe)	91	640	7.03×10^{-7}
100 Watt Tungsten	103	670	6.50×10^{-7}
New Quartz Lamp (with Alba globe)	460	1305	2.84×10^{-7}
Old Quartz Lamp (without globe)	260	5920	22.8×10^{-7}
Magnetite Arc (with globe)	530	7900	14.9×10^{-7}
Magnetite Arc (no globe)	530	10240	19.7×10^{-7}
New Quartz lamp (without globe)	350	11350	32.5×10^{-7}
Carbon Arc (quartz window)	495	26200	52.9×10^{-7}

At a moderate rate and for a moderate time the constructive forces of the organism are not over balanced by the destructive forces of the radiations. Hence the first application of the data obtained from the sources investigated was to determine the actual rate at which ultra violet energy was delivered by them. Table I shows for all the electric sources of light, of which the input could be readily measured, the gross input in watts at the lamp terminals, the total ultra violet radiation in ergs per second per square cm. at the standard distance of half a meter and finally, this ultra violet output in terms of ergs square cm. per second per watt input. This last column is proportional to the efficiency of the source as a producer of ultra violet radiations in terms of the gross input.

In Table I the highest ultra violet output per watt of input is reached by the carbon arc operated in the manner already described. The next highest figure is given by the quartz lamp operated without its globe, a condition of relatively low luminous efficiency which would only be found in cases where the arc was being used for bactericidal purposes or other special tasks where ultra violet radiations are important. The very high ultra violet output reached by the carbon arc is as has already been pointed out largely due to the very intensive cyanogen bands about in the middle of the ultra violet spectrum and the output of wave length below 300 $\mu\mu$ is materially less than it is in the quartz lamp operated without its globe.

At the other end of the list stand the G. E. M. lamp and the ordinary Cooper-Hewitt tube, the former showing a very low ultra violet output by reason of its relatively low temperature and the latter by reason of the fact that the extreme ultra violet is entirely cut off by the tube, and the middle ultra violet being very weak in the mercury spectrum, the main body of the energy is of wave length greater than 365 $\mu\mu$. In fact since the spectrum of the G. E. M. lamp runs down nearly to wave length 300 $\mu\mu$, and is strong only between say 360 and the visible, the energy distribution of the spectra of these two illuminants is singularly similar, considering their wide difference in character.

The Nernst and tungsten lamps produce rather more total ultra violet than the Cooper-Hewitt tube, most of the output being toward the visible spectrum. The Nernst lamp operated without its globe gives a spectrum relatively stronger in the further ultra violet, reaching wave length 300 $\mu\mu$ with a considerable degree of strength and stretching beyond it. All the lamps running with glass globes show a weak spectrum in that region. For this reason the quartz lamp with its regular diffusing globe shows an ultra violet output per watt almost as low as the G. E. M. lamp, the cut off of the globe in the ultra violet region being very striking. The magnetite arc both with and without its globe gives a considerable ultra violet output. The globe cuts off much less ultra violet than in the case of the quartz lamp, the latter being relatively rich in the rays which the glass most effectively absorbs.

Table II shows the percentage of energy cut off by the Euphos glass in each of the illuminants investigated as compared with the total energy which was transmitted by the quartz water cell, and also the relative horizontal c. p. of the sources dealt with. The percentage ratios of ultra violet are therefore numerically higher than they would be in the case of admitting the whole infra red to the thermopile. The relative composition of the various sources, however, is well expressed by the data.

TABLE II.

Source	*% of energy cut off by euphos*	*Candle power (horizontal)*
100 Watt G. E. M.	2.6	39.25
Acetylene Flame	4.5	27.35
Nernst (with globe)	5.2	68.0
Tungsten (100 wt.)	6.0	79.4
Graetzin Gas Lamp	8.4	75.0
Sunlight	21.0	272.(equivalent)
Magnetite Arc (glass globe)	22.5	700.0
Magnetite Arc (quartz window)	28.0	760.0
Carbon Arc (quartz window)	30.0	720.0
Mercury Arc (glass)	41.7	100.
New Quartz Lamp (with Alba globe)	42.5	820.
Old Quartz Lamp (no globe)	55.7	415.
New Quartz Lamp (no globe)	65.1	348.

It will be noted that the smallest percentage of ultra violet is shown again by the G. E. M. lamp, with the acetylene flame standing second. The Welsbach mantle of the Graetzin lamp runs materially higher than any of the electric incandescent lamps in spite of the fact that this lamp was tested with its globe on. Next higher than the Graetzin lamp, and approximating the arc lamps, comes sunlight, standing between the incandescent sources which give a continuous spectrum and the arcs of various sorts which give highly selective radiation. At the other end of the list is the quartz lamp worked intensively without its globe. These ratings of the various illuminants are instructive as showing the distribution of the energy as between ultra violet and the remainder of the spectrum, but they are not significant as regards the extremely practical matter of illumination. If the ultra violet component of artificial light involves any risk of injury to the eyes the one important thing to find out in comparing sources of light is how much ultra violet they deliver for a given illumination. In other words if one desires to light a room, say to an intensity of five foot candles, with what illuminant can he obtain this intensity while receiving the minimum amount of ultra violet radiation? It is not of the slightest practical consequence from the standpoint of good and safe illumination whether a given light source produces much or little ultra violet per watt, provided it produces an insignificant amount per foot candle, hence the luminous efficiency of the source is in the last resort the thing which determines the presence or absence of ultra violet radiation in material amount. In other words the more efficiently the energy supplied to the illuminant is transformed into light the less important does the ultra violet radiation become in considering the source as a practical illuminant.

TABLE III.

Source	*Deflections due to u. v. in cm.*	*Ultra violet ergs per sec. per cm² per foot candle*
Quartz Arc (Alba globe)	3.70	4.3
Graetzin Gas Lamp	.92	11.7
G. E. M. Lamp	.61	14.8
Cooper-Hewitt (glass)	1.64	15.5
Sunlight (direct)	2.28	16.1
Acetylene Flame	.52	18.4
Tungsten Lamp	1.90	22.7
Nernst Lamp (globe)	1.81	25.5
Magnetite (glass)	22.40	30.3
Magnetite (quartz)	29.00	36.3
Old Quartz Lamp (bare)	16.77	38.3
New Quartz Lamp (bare)	32.10	87.6
Carbon Arc (quartz)	74.00	91.0

Table III assembles the commercial light sources tested, with respect to the ultra violet energy accompanying a given illumination. The first column of the table gives merely for the purpose of record the actual deflections found to be due to the ultra violet energy, and column two the total ultra violet radiation in ergs per second per square cm. per foot-candle of illumination. At the head of the list stands the quartz mercury arc with its diffusing globe. Of the commercial illuminants tested this gives by all odds the smallest proportion of ultra violet per foot candle. As the previous tables show, the ultra violet energy of this source so equipped is small from any point of view. Its unique position, however, is due largely to the fact that the light-giving radiation, which lies practically at the very peak of the luminosity curve for vision, is produced at enormous efficiency, according to Buisson and Fabry [17] not less than 55 candles per watt for the green line at wave length 546 which supplies nearly two thirds

[17] Comptes Rendus, Vol. 153, p. 254.

of the total light and at almost as high efficiency for the pair of yellow lines which supply nearly all the rest. Next in the list, a rather bad second, comes the Graetzin gas lamp, its position again being due to the somewhat selective radiation that gives it a very high luminous efficiency. Third, comes the G. E. M. lamp which, from its relatively low temperature, gives a small absolute amount of ultra violet radiation, although its luminous efficiency is not great.

At the other end of the line comes the special enclosed arc with 91 ergs per second per square cm. per foot candle, and next to it the quartz lamp without its globe. Of course the quartz lamp without its globe is never used for illuminating purposes, but only for such work as sterilization of water and the like in which the ultra violet rays are the things sought. Operated for this purpose it undoubtedly is the most efficient powerful source of extreme ultra violet. To test this feature of the matter energy measurements were taken on the two quartz lamps without their globes and on the magnetite lamp free from its globe while using as a screen instead of the Euphos glass a disc of the very light crown glass previously referred to, which practically effects a separation at wave length 300 $\mu\mu$ absorbing substantially all the energy below this point and transmitting at almost full intensity the rest. The result of this test, measuring the extreme ultra violet and reducing it to the mean spherical output of ultra violet, showed for the extreme ultra violet efficiency of the new quartz lamp 4.07 % and for the efficiency of the old quartz lamp 3.14 %. A similar measurement of the magnetite arc showed an extreme ultra violet efficiency of 1.13 %. These figures may be properly compared with the tests for the ultra violet efficiency of the quartz lamps made by Fabry and Buisson.[18] In this case two mercury lamps showed respectively extreme ultra violet efficiencies of 6.4 and 4.7 %, the ultra violet separation being effected by the screens used by Fabry and Buisson at wave length 320 $\mu\mu$. The values obtained by the French investigators and in this study therefore check each other closely, showing that in the quartz mercury lamp 4 to 5 % of the total input is returned in the form of extreme ultra violet radiation when the lamps are operated, as they are for sterilization purposes, without their globes. The lighting power of the lamp falls off very greatly in this condition.

When operated with the globe the total proportion of ultra violet becomes both absolutely small and extremely small relatively to the

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[18] Comptes Rendus, Vol. 153, p. 93.

light given. In this connection the position of sunlight in Table III is not without importance. On the face of the returns it has a less amount of ultra violet with respect to the illumination than most of the artificial illuminants. This is due to the very high temperature of the source, which insures high luminous efficiency, in connection with strong ultra violet absorption in the atmosphere. Unfortunately one can apply Planck's law to very few practical sources of light. The sun is ruled out by the very erratic and highly selective absorption which produces the Frauenhofer lines and also by an unknown absorption of the extreme ultra violet which may take place in the earth's atmosphere or near the solar surface or in both places. Incandescent lamps involve absorption by their globes and also in the case of more recent ones a certain amount of selective radiation. The whole tribe of arcs which yield in a greater or less degree discontinuous spectra, for which Planck's law does not hold, are also thereby eliminated, so that this otherwise very useful guide to the distribution of radiation ceases to have exact significance.

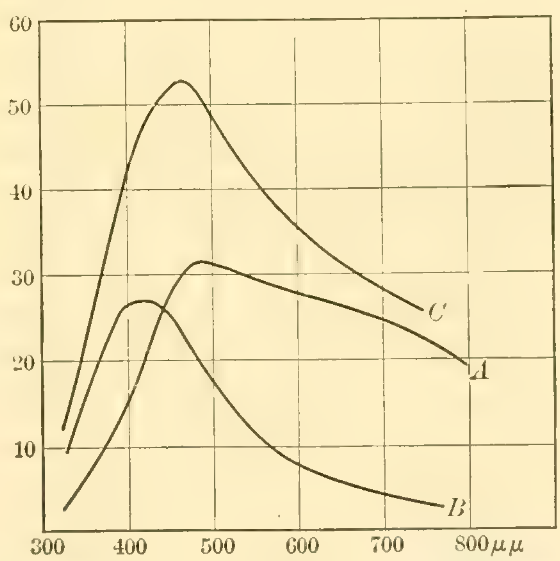


FIGURE 6. Curves of Sun and sky energy.

The ultra violet component of sunlight has been considerably disputed. It has been held by some investigators like Dr. Voege[19] that sunlight contains more ultra violet than the arc light, while Schanz and Stockhausen[20] take the opposite view. In a sense both are right and both wrong. Sunlight undoubtedly contains only a very modest proportion of ultra violet per foot candle of illumination when one considers direct sunlight alone. If, however, one considers the total daylight effect, including skylight under favorable circumstances, the situation takes on a totally different aspect. The light diffused by the blue sky is mainly violet and ultra violet, being substantially that light of which the direct sunbeam is robbed by scattering. Figure 6 shows in curve

[19] The Illuminating Engineer, Lond., Vol. II, p. 205.
[20] The Illuminating Engineer, Lond., Vol. I, p. 1049.

A the distribution of energy in the directly received solar light. Curve B shows the distribution of energy in the diffused light of the blue sky when the total of this diffused energy equals 20 % of the total directly received solar energy, a not uncommon proportion. It will be noted that the maximum for this curve is in the far violet near the edge of the ultra violet. Curve C is the summation of A and B and it will be seen at once that the proportion of ultra violet is something like three times as great as in the case of the direct solar rays. This proportion would raise the ultra violet activity of daylight to a point higher per foot candle than that reached by any ordinary artificial illuminant.

Weisner [21] in photographic observations of light received on horizontal surfaces states for example, "For solar altitudes less than 19 degrees the chemical intensity of the sunlight as compared with diffused daylight is negligible, with increasing solar altitude it gains in comparison with the diffused daylight. * * * Since the intensity of the direct beam may reach twice that of the diffused, the total combined chemical effect may be three-fold that of the diffused light."

Daylight, therefore, varies very greatly in ultra violet energy, ranging from the low value given in Table III for direct sunlight to values which would exceed almost all artificial light sources. The chief claim of sunlight to serious consideration from the standpoint of ultra violet energy, however, lies in the very large amount of energy which the sun delivers. There is considerable doubt as to the exact amount of solar radiation outside of the atmosphere, but that which gets through the atmosphere is pretty well determined and its amount, from the data given by Abbott [22] amounts practically, under favorable conditions, to not less than 1 kw. per square meter, which is 0.1 watt per square cm. If one assumes that only 10 % of this is in the ultra violet region, an amount which may be exceeded at times, the total ultra violet radiation rises to 10^5 ergs per second per square cm., several times that given by the most powerful artificial sources of ultra violet at even a distance of so short as half a meter.

Considering this very large flux of ultra violet energy it is small wonder that troubles from sunburn and snow-blindness are not infrequent. Did we not habitually shield our eyes by interposing the rim of the hat or the brow and by systematically looking away from the direct sunlight ocular troubles would be common and severe.

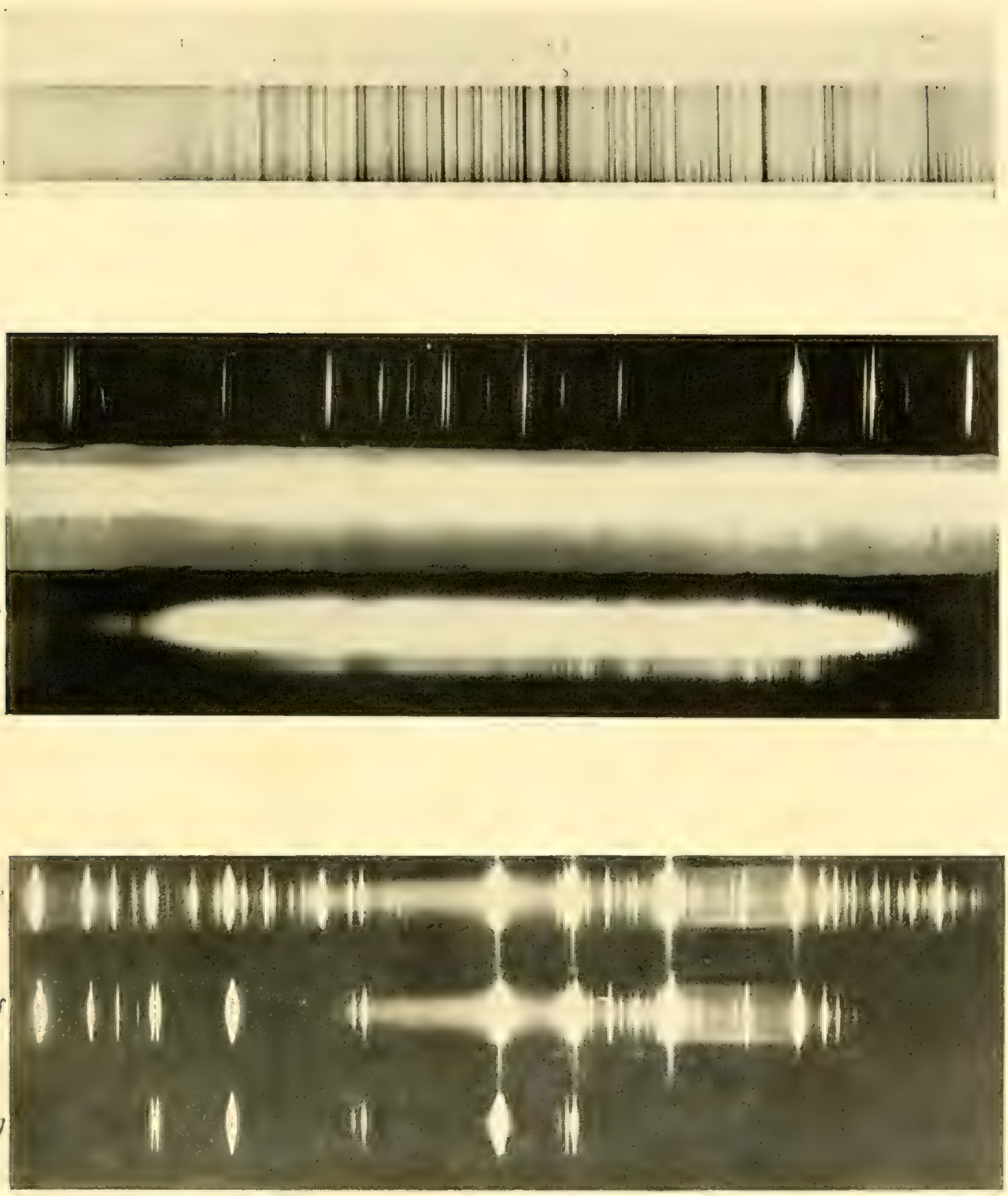
[21] Denkschriften Wien. Akad., Vol. 64, 1897.
[22] "The Sun", Chapter VII.

Snow is a good reflector of ultra violet radiations, at least throughout the limits of the solar spectrum. At two meters distance a square meter of snow surface may reflect to the eye as much as 10^4 ergs per second per square cm. If even one tenth of this is in the ultra violet then a square meter of snow in the field of vision at two meters distance would deliver about 1000 ergs per second of ultra violet per square cm., which is in excess of the greatest amount which would be given at this distance by any of the artificial sources of light here investigated.

Fortunately the sun is weak in the extreme ultra violet, but the very large amount of radiation which can be reflected to the eye from a snow covered surface is quite sufficient to account for all the phenomena observed, even although the ultra violet per foot-candle in the sunbeam is rather exceptionally low.

Two sources of light, not here measured for reasons already stated, should not be forgotten. One of these is the iron arc used for therapeutic purposes, of which the spectrum is shown along side of the mercury spectrum in Plate, **2**, *g*. It will be observed that it is enormously rich in lines, even to the extreme ultra violet, and as the light giving power between iron terminals is not high, this source would stand very near the bottom of Table III. The yellow calcium fluoride arc, of which the spectrum is similarly shown in Plate **2**, *i*, would unquestionably stand near the quartz arc at the head of the list, owing both to its very high luminous efficiency and to the comparatively weak lines in the extreme ultra violet.

In conclusion it may be confidently stated that no commercial illuminant radiates for any ordinary working value of the illumination enough ultra violet energy to be at all harmful, provided one exercises ordinary discretion is keeping unpleasantly bright visible light out of the eyes.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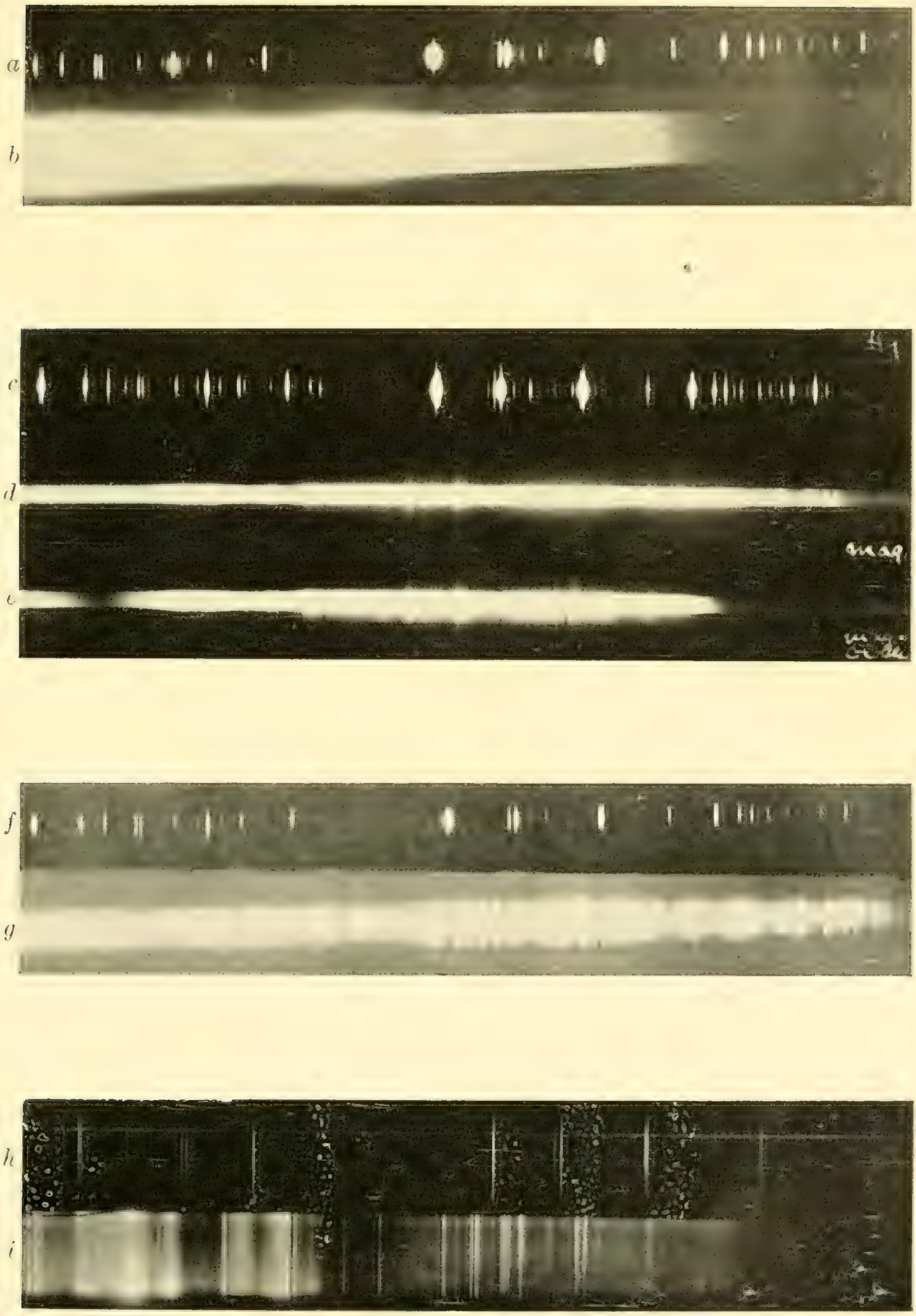
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ALEXANDER AGASSIZ.

By Henry P. Walcott.

whither she had removed from Neuchâtel to the company of her own relatives. Alexander came here into contact with Professor C. T. E. von Siebold, whose character and great scientific attainments did not fail to make a deep impression upon him. Soon after his mother's death in 1848 he came to this country and joined his father at Cambridge. He was prepared for college in the high school of that city and was graduated from Harvard College in the class of 1855.

He received degrees from the Lawrence Scientific School in 1857 and again in 1862, the studies pursued there were Chemistry, Civil Engineering and Zoölogy. This choice of studies shows that at this time he was not yet settled in his mind as to his life work — he had for a short time an interest in a Pennsylvania coal mine, and had thought of taking up the occupation of railroad engineering. He was appointed assistant in the United States Coast Survey in 1859 and was employed in charting the mouth of the Columbia River, Oregon; and in the survey of the northwest boundary, he found time in the intervals of his official duties to study the marine life of San Francisco harbor and to make collections at other points on the Pacific coast for the Museum at Cambridge.

Whatever his own plans may have been, powers beyond his control had been at work to determine his career, he vainly thought it might be in fields remote from those in which his father had labored, but indulgent fates brought him back to the natural sciences and here he remained for that part of his activities in which he found his highest satisfaction. He had lived all his life in an atmosphere of science, he had an inheritance from both father and mother of the mental qualities that promised him successes in these fields.

Louis Agassiz's second marriage, in 1850, to Elizabeth C. Cary, brought into the family a very strong and happy influence in the same direction, and ultimately the valued companionship for Alexander Agassiz which nearly reached the span of his own life.

Another important influence in his preparation for life is to be found in the state of Cambridge social life at this time. The native and unstinted hospitality of the father aided by the gracious manner in which Mrs. Agassiz received his guests brought to this open house every traveler of scientific prominence. The college society of the fifties and the association with the neighboring city could not easily be found elsewhere; some idea may be formed of its quality by reading the lines in which Lowell pictures the scenes, from which his great friend had been recently removed by death. There was no place there for mere wealth, riches were prized only where their possession had contributed to the improvement or happiness of mankind, and the man without a definite occupation in life was practically unknown. It was a very simple life according to the standards of the present day but it yielded results which our larger material resources have not proportionately multiplied.

After a year's absence upon the Pacific coast, he returned to Cambridge in accordance with his father's earnest wishes and definitely entered upon the work of the Museum. His marriage in 1860 to Miss Anna Russell, sister of the wife of Theodore Lyman, his classmate and associate at the Museum, made this place also his home. His methodical habits and financial prudence were of great value to his father in the administration of the business of the establishment and he early became indispensable there. The visitor to the modest quarters of the Museum of those days would probably have failed to discover in the quiet assistant intent only on the work of the laboratories and of the Museum, the power which was destined in a few years to place these collections in halls commensurate with their value and that by resources won by himself in the fierce struggle for the wealth buried in the depths of the earth.

In 1859 was published his first scientific paper which was read before the Natural History Society of Boston, upon the mechanism of the flight of Lepidoptera — a subject hardly to be expected from one who was subsequently to gain his great honors in very different departments of zoölogy. Before the age of thirty he had published more than twenty (20) papers upon scientific subjects, all of which displayed originality and covered a variety of topics. He published in 1865 with his stepmother, Mrs. L. Agassiz, a book of popular character under the title "Seaside Studies in Natural History." He became much interested in 1867 in the dredging operations of his friend Louis F. de Pourtalès, who on the Coast Survey Steamer "Corwin" had successfully brought up material from the then unusual depth of 850 fathoms along the course of the Gulf Stream between the Florida coast and the Bahamas. He assisted in the arrangement and description of the collections. He thus early became interested in the study of the ocean bottom — the problems of which were to occupy so prominent a place in all his work for the rest of his life. The influence of this favorite pupil of his father and his own life long friend is acknowledged in the appreciative notices which were presented to the American Academy and to the National Academy after the death of Pourtalès.

"The Revision of the Echini," which appeared in the years 1872–74, is the best known work of Agassiz and was at once recognized as the performance of a master and made him the leading authority on the subject. The thoroughness of his methods is shown by this extract from a letter to a friend from Leuk, Switzerland, August, 1870, "I have done now with my examination of the Echini collections, having seen them all." It was of this work that Jeffries Wyman spoke when he said that the son had done a piece of work that would live as long as anything accomplished by the father." The manner in which the work was performed by Agassiz is well shown by the quotation from his letter given above — he saw every specimen that was worth seeing before he felt justified in stating his own conclusions.

The activity that marked these early years down to 1873 was a marvel to all — he was intensely busy, and capable of undertaking the most strenuous physical and mental labors, his working day was habitually more than half of the twenty-four hours.

In 1869 came a serious illness from which modern surgery might have brought a more satisfactory cure than that which he obtained. Some of the consequences of this illness affected his mode of life permanently — he avoided thereafter, so far as possible, our New England winters.

The end of the year 1873 was a time of great sadness for Agassiz. His father and his wife died within ten days of each other. He assumed the direction of the Museum and for 37 years labored for its development and administration, a serious task, if it had been his sole occupation.

Louis Agassiz had opened a school for natural history studies on the island of Penikese in Buzzards Bay in the summer of 1873. His immense capacity for teaching, his love for it and success in it carried the school through the first season, but it was the last great effort of his life. In the succeeding year Alexander Agassiz reluctantly took up the burden, he had not shared his father's enthusiastic belief in the possibility of carrying on a school at this remote point. He loyally made the attempt, however, and when it became evident that the necessary financial support could not be obtained, he characteristically did not hesitate to drop the enterprise and pay the deficit from his own pocket.

A few years after the closing of the Penikese school he built in the vicinity of his house at Castle Hill, Newport, an excellent marine laboratory with the required accommodations for about 12 students. Here much valuable work was done by a number of men whose names have become well known throughout the scientific world. During his long service at the head of the Museum and under a variety of titles, he expended from his own resources for collections and the buildings to hold them, more than $1,200,000, not including very considerable sums contributed to other allied interests or to the general purposes of the University. At the end of the year 1874 he set out on the first of the many distant expeditions which were made at intervals through the rest of his life. This journey took him to Chile and Peru, and during the course of it he made the exploration of lake Titicaca, an account of which is given in our proceedings for the year 1876.

His quick eye showed him at Tilibiche in Peru, a fossil coral reef at an elevation of nearly 3000 feet above the sea and 20 miles inland, and he noted with a certain satisfaction the evidence that Darwin's observations had caused on his part an underestimate of the amount of recent elevation of this coast.

He now entered upon that series of deep sea investigations which in some form had always been of exceeding interest to him. He directed three expeditions in the Atlantic on board the U. S. Steamer "Blake" and three in the Pacific on the "Albatross." The vast material collected on these trips was, with combined wisdom and generosity and in obedience to the rule of the Museum laid down in his father's time, distributed for purposes of description and study to those scientific men everywhere who were best qualified for the work.

Sir John Murray says, and no living authority is better able to make the statement, "If we can say that we now know the physical and biological conditions of the great ocean basins in their broad general outlines — and I believe we can do so — the present state of our knowledge is due to the combined work and observations of a great many men belonging to many nationalities, but most probably more to the work and inspiration of Alexander Agassiz than to any other single man."

In these later years he was also much interested in the study of the coral reefs. He organized many expeditions to all parts of the world — to the Maldives, to Australia and to remote portions of the Pacific. He saw, explored, and accurately described every important coral reef region on the globe and having done so he felt that he was ready to give his own views to the world.

Darwin saw but one atoll and upon that founded his theory of coral building. Agassiz was at work in his last days upon the publication which would have given to the world the well considered conclusions acquired by the studies of nearly a lifetime. Though his own final results cannot be surely known, his vast material still exists for some more fortunate investigator. He had written and rewritten his sketch of the book upon this subject and a few days before his death said to his friend, Sir John Murray in London, that it was his intention to practically rewrite the book during the year for the fourth and last time, leaving out all criticism of the work of others and stating exactly what he had himself observed and his own views. It should be understood that Darwin's theory of the coral reefs belonged to his younger years and has no bearing upon his later published theory of natural selection. What Agassiz's views were, upon this and other theories conveniently grouped under the title Darwinism, cannot be accurately stated. It is true that he found much that was objectionable in the opinions maintained by some of Darwin's German followers. No one who knew him, however, can doubt his ability to weigh dispassionately any evidence, which could be produced for this or for any other doctrine, though it might run counter to opinions long entertained by him or by those whom he delighted to honor.

Some intimations of his views upon the position of the Zoölogist of today as compared with that of the great men of an earlier generation may be found in the remarks made by him as representative of his class at the Commencement at Harvard in 1905, that being the 50th anniversary of his graduation. He called attention to the inconveniences and the primitive appliances which hampered the work of the student of natural history in his own student days and added, "The change in scientific thought is most striking — fifty years ago authority was the powerful factor — scientific dictators were not uncommon — now authority as such is no longer recognized beyond the point at which it can be controlled. Successful experiment has taken its place, and while recognizing the value of imagination and of pleasing speculations, men of science no longer accept the dicta of their leaders."

As John Hunter said to his pupil Jenner, who had asked for the explanation of some perplexing phenomenon, "I think your solution is just; but why think, why not try the experiment." So with Agassiz, discussions had little interest for him when it was not possible to put the conclusion to the test of observation or experiment.

The bibliography of his own scientific papers contains 248 titles which cover a great range of subjects and procured for him marked distinctions throughout the world. No man among men of science promoted the interests of zoölogy so generously as he. In 1910 the 54th volume of the Bulletin and the 40th volume of the Memoirs of the Museum of Comparative Zoölogy were coming from the press. These publications began to appear in 1863–64 and in the number of important and finely illustrated papers which are presented there, they have been excelled by few only of the great and most active scientific societies of the world, yet the expense of producing them was largely borne by Agassiz.

Much has been said about the great sums of money spent by him upon the monument he raised in filial piety to the memory of his father, and which he duly commemorated in that characteristically simple inscription upon the walls of the Museum "Alexander, son of Louis Agassiz, to his father." The voice of the public has named it the Agassiz Museum — father and son were both content to call it the Museum of Comparative Zoölogy. Whatever legally that title may be, the memory of these two lives will possess a force greater than the statute, and will preserve for generations to come the name common to the enthusiastic founder and to the wise, patient and munificent builder. Whatever Agassiz's contributions in money may have been and others, not he counted them up to sums exceeding any thus far made to the University, yet he gave a greater still in the devotion of himself to the task of developing and making secure the future activities of the Museum. All the material successes he had won in other fields he pledged to the support of the Museum after he had satisfied the reasonable requirements of his family, but of his own labors he made no reservation. The Museum had all that he could bestow.

On the pages of the quinquennial catalogue of Harvard College are enumerated the distinctions conferred upon him by universities, learned societies and foreign governments, they are a sufficient proof of the esteem in which he was held throughout the world. Such distinctions sometimes reveal a more than passive recipient, but they came to him absolutely unsought. His intimates even had little knowledge of the honors bestowed upon him, and rarely obtained it from himself.

The great gold Victoria Research Medal given to him in 1909, was shown to his friends, but this was more for the exquisite beauty of the workmanship of the Medal, than for the pride in receiving it. He had a keen appreciation of anything that had artistic merit and surrounded himself in his home with many beautiful objects of art collected in his travels from all parts of the world. In addition to the Victoria Research Medal of the Royal Geographical Society, he had received the Walker Grand prize of the Boston Society of Natural History and in 1878 the Serres prize of the French Academy of Sciences, the first foreigner to be so honored.

From 1865 onwards in addition to the scientific work of the Museum he was developing and managing most successfully the largest copper mine in the world. He did not rest content with the development of the mine as a problem in engineering, but always mindful of the just obligation of capital to labor, he employed experts for the purpose of securing good conditions of living, caused careful measures to be taken for the protection of life and limb in this hazardous occupation, and secured the formation of pension and aid funds for the benefit of disabled and aged employees to which the corporation made liberal contributions. No workman was so far removed from the authorities in control that his complaint passed by unheard. The whole conduct of the mine is one of the bright spots in the much beclouded world of such enterprises and must still be reckoned among the more satisfactory attempts to bring the workman and his employer into harmonious relations with each other.

A pleasing instance of his thoughtfulness with regard to the population of this mining community is related by one of his friends, the physician who took care of him through a fever which might have been acquired during one of his visits to the mines at Calumet. The physician was asked one day whether he suspected that the disease could have been brought from that place. If that were so, there was something to do at once and that was to take such measures that his work people should be protected from a like danger. Upon this suspicion, possibly unfounded, a thorough overhauling of water supplies and systems of sewerage was at once undertaken there while Agassiz was still confined to his house.

He was early called to service upon the governing boards of Harvard College, he was elected a member of the Board of Overseers in 1873, became a member of the Corporation in 1878, resigned his place there in 1884, and was promptly elected to the Overseers in 1885, was again transferred to the Corporation in 1886, and definitely gave up his place there in 1890, when he found it necessary to free himself from some of his many occupations. During all the period of his connection with these boards he was an active, much interested and far sighted helper in all the departments of the University. The Jefferson laboratory owed much to him for the friendly coöperation with which he promoted the intentions and plans of the generous founder. He gave valued aid to the Observatory, to the Botanical Museum, the Mineralogical Cabinet and the Peabody Museum of American Archaeology and Ethnology.

He interested himself in the attempt to secure for women a share in the medical instruction offered by the University. He took a generous part in many of the subscriptions for the general purposes of the College. He witnessed with interest the development of the collections of the Arnold Arboretum under auspices not unlike those with which he was himself so familiar. The members of his College class have given expression to their warm feelings of friendship for one who never forgot his college associates and had a genuine pleasure in all his meetings with them.

The secretary of the class closes a feeling notice of Agassiz's death with these words of appreciation, "No one of the class will miss him more than the secretary does who never went to him in vain for aid in the many common undertakings which bound the class together."

He did not forget his early debt to the public schools of Cambridge and willingly accepted service upon the school committee, and while a member of that body devoted all his special knowledge to the service of the city. This appears to be the only public office, subject to election by the people, which he at any time held.

Agassiz was elected a member of the Academy of Arts and Sciences Nov. 12, 1862, he was then in his twenty-eighth year. It was possibly in remembrance of this early election that he suggested in his last note to President Trowbridge the propriety of bringing in to this association a larger number of the younger scientific men than had hitherto been customary. He presented his first paper the next year and made in all thirteen communications, generally upon special subjects in zoölogy. A very interesting account of his work at Lake Titicaca is an exception, and has an added claim to our attention from the fact that it was made at a time when the death of his wife had left him disconsolate, but it is also an evidence of how resolutely he turned again to the occupations which he followed to the end.

The series of publications put forth by the Museum of Comparative Zoölogy received the records of his scientific labors after the date of the last communication made to the Academy. When President Cooke died in the summer of 1894, a feeling soon became manifest that Agassiz was the most fit member for the succession. The Vice-President of that year was Augustus Lowell and he was the prompt and enthusiastic leader in the preliminaries usual to an election. Agassiz as might have been expected was very reluctant to allow the use of his name and probably would not have done so, but for the insistence of Mr. Lowell, whose influence was all the greater from the fact that he was one of the earliest friends acquired by Agassiz when he landed a stranger among people speaking an unknown tongue.

He received a unanimous vote in one of the largest meetings ever held by the Academy; he faithfully performed all the duties of the office interrupted only by the winter vacations which his illness of 1869 made necessary for him. In this place it is a satisfaction to remember that no one of his many and great distinctions gave him a greater pleasure than did this. It was a most unexpected revelation to him of the hold he had upon the respect and good will of his fellows.

It is not possible to escape from some comparison of the two great men of science who have borne this name, and there can be nothing unbecoming in the attempt to make it.

The son was the pupil of the father and different as the two men seemed to be, the son was ever conscious of the debt he owed to his father.

Louis Agassiz came to this country with a great and well deserved reputation fairly earned among the world's great men.

He did more than anyone to encourage the study of the natural sciences here. Endowed with every social attraction — persuasive, a leader and fond of his leadership, great in acquirement, quick in apprehension, rich in imagination, fertile in illustration, a teacher beyond compare. He found listeners in the market place as well as in the halls of the Colleges and of the Legislatures. He laid in magnificent hope the foundation of an establishment so extensive that he had no just right to expect that either he or his son could see its completion.

Alexander Agassiz, patient seeker after truth, skilful organizer of scientific methods, unwearied in researches, prudent, self-denying, pursuing his great ends to a successful issue with silent determination, not eloquent and always reluctant to attempt persuasion by spoken words, he leaves behind him, in the opinion of many competent judges, a more permanent and more important mass of completed work in the study of the natural sciences than fell to his father's lot. He, moreover, by his own exertion completed the structure which his father could only have seen in some prophetic vision.

It is not easy to speak of the personal qualities of Alexander Agassiz. Men expected to find in him the counterpart of his father, and in such intercourse as they may have had with him they met with disappointment. They regarded him as one holding himself somewhat aloof from his fellows, not much interested in their doings and slightly affected by their misfortunes. This conception of his character showed little acquaintance with the real man; beneath the quiet and reserved, certainly not austere demeanor, there lay a nature quick in feeling, sympathetic and tender, not given to verbal expression, but capable of great generosity not in money only, but in the things that money never buys.

They knew him in the serious work of life, wise, fearless and of an indomitable energy, quick and fiery in temper, but harboring no sullen enmities. Many a victim of some sudden expression of a vigorous disapproval had found to his surprise in some future and unmerited trouble no warmer friend or if occasion required more strenuous advocate than Alexander Agassiz.

His emotions were never under his complete control and he steadily avoided the public occasions that might lead to their manifestation. They were always, however, the emotions of a sensitive, generous and strong nature.

His actions often seemed hasty if not premature, but this was in appearance merely, for his whole life long he thought for himself and by himself, and when action came it was true that few, if any, had knowledge of the long and patient thinking that led up to the result.

His intimate friends were comparatively few in number, but to those who had earned his confidence, he showed no reserve, and had a simple charm which made intercourse with him the delight of a lifetime.

The unworthy things in life, or such that seemed so to him, moved him to quick and impetuous judgments and expressions, but if cooler thought led him to believe that he had made a mistake, it was quite certain that any wrong that might have been done would be fully repaired.

His wealth, whatever it may have become, had little effect upon a life simple and free from display. The man who was known all over the world in the assemblies of the great men of science walked unrecognized through the streets of Cambridge, and he would not have had it otherwise. He was modest, somewhat diffident and shy, but he was by no means unconscious of his powers and the recognition of them by his peers was a source of legitimate satisfaction to him. He was courageous, independent and quite ready to fight if need be, for the losing cause. He was not a willing critic of the work of other men, unless it dealt directly with subjects to which he had himself given much attention. He was ever ready to recognize with unselfish praise the results of any honest and thorough investigation. All the resources of the Museum were at the disposal of him who could effectively make use of them.

He suffered without complaint any criticism of his own opinions, but was sure to be roused to instant wrath at any suggestion that he had incorrectly reported observations or experiments. His declaration of scientific faith was his father's adage, that a physical fact was as sacred as a moral principle.

One instance of his fine generosity may well be noted here. Some years since it was announced that a notice of his father was about to be published. Mrs. L. Agassiz and he had reason to believe that the work was not in friendly hands. The printer's proofs of the paper came into Agassiz's possession, together with the intimation that any change he might wish to make would receive serious consideration. He requested a trusted friend to read over the proofs and mark such passages as might appear to him unfitting. The friends met to compare notes, they agreed in substance with the exception of one passage that seemed to the friend mischievous if not malevolent. Agassiz said at once, "As to the spirit in which this statement is made I quite agree with you, but it is a scientific question, and any scientific man has the right to criticise my father's scientific views." The passage remained.

The lessons of the narrow circumstances of his youth and early manhood never left him. He could be apparently reckless in discarding machinery and tools which had served their purposes or were inferior to newer inventions, but it was always with the object of getting a larger return or a better product. For himself he never sought luxuries, but lived without ostentation in the dignified manner that became his station. He cast aside all the lessons of thrift, however, when he turned to the human agencies in his employment. He never discharged an employee who had been long in his service and who was still capable of doing enough work to appear to be doing something.

One of Agassiz's most remarkable characteristics was the systematic and accurate disposal of his time, he might be making a journey to the Maldives or it might be to the barrier reef of Australia. The date of his return was fixed, and punctual to the day he made his appearance at the Museum, and quietly resumed his accustomed occupations there. He made such thorough preparations for these trips, and provided so carefully for any possible mishaps, that the usual uncertainties of ocean voyages for him at least ceased to exist. Many men take measures against the larger accidents, and forget the trifles. Agassiz kept the great emergencies in mind but never neglected the small things of life.

No native born citizen ever carried to Europe a more pronounced spirit of personal independence than he did. His stories of experiences with officials on the other side of the Atlantic were a source of much entertainment for his friends. In the later years of his life his thoughts turned more willingly to the other shores of the Atlantic, he had made warm friends there, and he looked forward with much satisfaction to the few weeks in Paris which generally were the end of his foreign excursions for the winter. Here in the company of kindred spirits — Associates in the Institute of France and others — he spent days of real enjoyment, speaking the language which belonged to his father if not to his mother and which never had become at all unfamiliar to him. The theatres of the better sort attracted him and his distance from the demands of his active life here left him free to indulge in his always temperate pleasures.

Notwithstanding the very serious illness of his early life his originally slender but vigorous frame bore him safely through a life of more than the usual exposures in the varied hardships of a mining camp and journeys which were often perilous. He was spared the usual defects of advancing years and carried to the end a clear head, unimpaired senses and an active body. On Easter morning, March 27th, 1910, on board the Steamer Adriatic in mid ocean he passed from sleep to death without a struggle and the last great mystery was revealed to him who had dealt with the immensities of time and space in all the oceans of the globe.

It was well known to some of Agassiz's friends that he had bestowed much thought upon a plan for giving to this Academy a more satisfactory house than any it had yet had. He had made provision in his will for a bequest to the Academy which would have given it a substantial aid in this direction. He, however, had promised himself a more immediate gratification of this wish and on 16 October, 1909, wrote to President Trowbridge offering to erect upon the land already owned by the Academy and the adjoining lot which he had recently purchased a building which should become, to use his own words, "a scientific and literary Club," while remaining the domicile of the Academy. He had caused plans to be prepared by Mr. S. F. Page, for a building to be erected on this spot — not merely a house for the Academy but a home for its members, a place to which they would gladly at all times come, to which they might bring their friends and associates from other parts of the country or from foreign lands. It was quite clear to those who were most familiar with his plans, that the house was destined to have all the attractive features which he knew so well to give to his own dwellings. His sons in quick response to the father's wishes with a generous piety have carried out his plans. Mr. Page, the architect, had submitted his sketches to Mr. Agassiz and had had frequent conferences with him before he left the country in December, 1909. His death on March 27, 1910, of necessity caused some delay in the progress of the work, but the plans had been so fully developed that there seemed no doubt as to his intentions and the architect under the direction of the sons and of your committee has faithfully and successfully brought the building to completion.

Kings and ambitious noblemen have in other lands and other times been patrons of learned societies and have provided sumptuous accommodations for them. Our house is believed to be the only abode of a scientific society built by a member of the body and devoted to the unrestricted uses of his fellows. If Agassiz had lived to see the completion of this house, it is safe to say that neither his name not his features would have appeared upon these walls. What his singular modesty would have forbidden to him living has been done in the one instance by the authorities of the Academy, and in the other by the loving hands of one of his own family.

In the great Museum at Cambridge is the monument of two great men of science laboring in the service of science alone. Here in this pleasant house and home may their associates and successors for all time remember the gracious spirit of him who asked only of his fellows a kindly remembrance.

May we not speak of him in the words which our own poet used in describing another of our greatest and best loved associates,

> The wisest man could ask no more of fate
> Than to be simple, modest, manly, true,
> Safe from the many, honored by the Few;
> . . . .
> To feel mysterious Nature ever new;
> To touch, if not to grasp, her endless clue,
> And learn by each discovery how to wait.
> He widened knowledge and escaped the praise;
> He wisely taught, because more wise to learn,—
> He toiled for Science not to draw men's gaze,
> But for her lore of self denial stern.

O friend of this house and all who gather here, not of a day but for long years to come may your place still be here to welcome by this visible presence the generations of this Academy, till this solid structure which you have built and all that it contains shall sink in dust.

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A THEORY OF LINEAR DISTANCE AND ANGLE.

By H. B. Phillips and C. L. E. Moore.

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Introduction.

1. In a recent article [1] we developed for the plane a theory of distance and angle such that points equally distant from a fixed point lie on a line and lines making a given angle with a fixed line pass through a point. On account of this property we have called this distance *linear*. In the present paper we extend this theory to higher dimensions. Because of the increased complexity, the synthetic method of the previous discussion cannot be used here and since we know none better we have adopted that of Grassmann. In the first part of the paper we have shown how the extensive quantities of Grassmann can be regarded as matrices and the progressive and regressive multiplication interpreted as simple operations performed upon these matrices. In this way we develop as much of the Grassmann analysis as is needed for our purpose. We then determine for any two spaces R, R' of the same dimension, a distance or angle $\overline{R\,R'}$ having the property that if this invariant is constant and either of the spaces fixed, the other satisfies a linear relation and such that for three spaces R, R', R'' of a pencil

$$\overline{R\,R'} + \overline{R'\,R''} + \overline{R''\,R} = 0.$$

Any distance between points that has these properties is expressible in terms of a hyperplane and a linear line complex. The plane is the locus of infinitely distant points and the complex the locus of minimal lines. If the complex does not degenerate, the hyperplane and line complex in n dimensions determine a point and $n-2$ other complexes forming altogether n elements which we use for a reference system. This system of elements forms a group under outer multiplication

[1] An Algebra of Plane Projective Geometry, Proceedings of the American Academy of Arts and Sciences, Vol. 47, p. 737.

in the sense that any product of these elements is equal to a numerical multiple of a third one. In terms of this fundamental system we define the angle between any two spaces. Each of the complexes of the fundamental system is an infinite locus for spaces complimentary to it. The entire system is invariant under a group of collineations of the same order as the Euclidean group of motions. Degenerate cases are obtained by taking sections having a special relation to the fundamental system.

Matrices in Three Dimensions.

2. **Progressive Matrices.** We represent a point A in three dimensions by a set of four homogeneous coordinates a_i. These coordinates determine a matrix

$$A = \| \, a_1 \; a_2 \; a_3 \; a_4 \, \| = \| \, a_i \, \| \qquad .$$

which may be used to represent the point. Two matrices of this kind will be called equal when their corresponding elements are equal. The matrix is zero if all its elements are zero. If $a_i = k \; b_i$ we shall write

$$A = k \; B.$$

In this case the matrices A and B represent the same point but with different magnitudes. A linear function of A and B is defined by the matrix

$$\lambda \; A + \mu \; B = \| \, \lambda \; a_i + \mu \; b_i \, \|.$$

In a similar manner we define any linear function of points or matrices A, B, C, etc. If the result does not vanish it represents a point in the space determined by A, B, C, etc. If it vanishes and the coefficients are not all zero those points lie in a lower space than a like number of points usually determine.

The coordinates of the line joining A and B are proportional to the two-rowed determinants in the matrix

$$[A \; B] = \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{matrix} \right\| = \left\| \begin{matrix} a_i & a_k \\ b_i & b_k \end{matrix} \right\|.$$

We shall call the elements of this matrix the two-rowed determinants

$$\begin{vmatrix} a_i & a_k \\ b_i & b_k \end{vmatrix}.$$

[This is not in conformity with the usual definition which makes element equivalent to coordinate a_i or b_i but is the only definition which has a value in the present discussion.] The matrix is zero when all its elements are zero. In that case the points A and B have proportional coordinates and hence coincide. If the matrix is not zero it represents the line $A\ B$ in the sense that from the matrix can be obtained the coordinates of the line. Conversely if the line is given a matrix can be formed by taking any two points on the line. Different matrices representing the same line are multiples of any one. For if A, B and P, Q are pairs of distinct points on the line

$$P = \lambda_1 A + \lambda_2 B,$$
$$Q = \mu_1 A + \mu_2 B,$$

and

$$[P\ Q] = \begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{vmatrix} [A\ B].$$

Thus a two-rowed matrix in addition to representing a line, has a definite size.

The matrix $[A\ B]$ is in reality a set of six determinants

$$\begin{vmatrix} a_i & a_k \\ b_i & b_k \end{vmatrix}$$

taken in some definite order. . It can then be considered as a one-rowed matrix of six terms

$$[A\ B] = \|\, a_i\ b_k - a_k\ b_i \,\|.$$

The sum of two matrices $[A\ B]$ and $[C\ D]$ is then a complex matrix, each element of which is the sum of corresponding elements in $[A\ B]$ and $[C\ D]$. In general this sum cannot be represented as a single two-rowed matrix, just as the sum of corresponding Plücker coordinates of two lines are not in general coordinates of a line. For analytical purposes we express this sum by simply writing the two matrices with an addition sign between them. If, however, the lines $A\ B$ and $C\ D$ intersect in a point P, we can find points Q and R on those lines such that

$$[A\ B] = [P\ Q],$$
$$[C\ D] = [P\ R].$$

Then

$$[A\ B] + [C\ D] = [P\ Q] + [P\ R] = \left\| \begin{matrix} p_i & p_k \\ q_i + r_i & q_k + r_k \end{matrix} \right\| = [P(Q + R)].$$

We can consider $[A\ B]$ as a product of A and B. For

$$[A(B + C)] = [A\ B] + [A\ C]$$

as we have just seen in the case of $[P(Q + R)]$. The process of multiplication consists in placing the second matrix under the matrix A

to form a two rowed matrix $[A\ B]$. We shall call this the *progressive* product. From the definition it is evident that

$$[A\ B] = -[B\ A]$$

and

$$[A\ A] = 0.$$

3. If the points A, B, C are not collinear, the coordinates of the plane $A\ B\ C$ are proportional to the three rowed determinants in the matrix

$$[A\ B\ C] = \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix} \right\| .$$

This matrix represents the plane in the sense that from it we can determine the coordinates of the plane. Conversely to represent any plane as a matrix we take three non-collinear points of the plane and form the matrix from them. The elements of such a matrix are the three-rowed determinants belonging to it. In reality we are considering this as a one-rowed matrix of four terms (equal to the three-rowed determinants in $[A\ B\ C]$) arranged in some definite order. Two matrices of this kind will be called equal if corresponding elements are equal and are added by adding corresponding elements.

If P, Q, R are any three points of the plane determined by A, B, C,

$$\begin{aligned} P &= \lambda_1 A + \lambda_2 B + \lambda_3 C, \\ Q &= \mu_1 A + \mu_2 B + \mu_3 C, \\ R &= \nu_1 A + \nu_2 B + \nu_3 C. \end{aligned}$$

Consequently

$$[P\ Q\ R] = \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} [A\ B\ C].$$

Thus a matrix $[P\ Q\ R]$ in addition to representing a plane has a definite size. The vanishing of $[P\ Q\ R]$ signifies that P, Q, R lie on a line.

The matrix $[A\ B\ C]$ can be regarded as a product of $[A\ B]$ and C, A and $[B\ C]$ or of A, B and C, the process of multiplication consisting always of placing the first matrix at the top and the others in order under it to form a single matrix.[2]

[2] In this multiplication each matrix must have four columns. If instead of $[A\ B]$ we have a complex the operation must be performed distributively on each two-rowed matrix of the sum. For purposes of addition we regard our quantities as matrices of one row but for purposes of multiplication as matrices or sums of matrices of four columns.

From this definition it is evident that

$$\begin{aligned}
[A\cdot B\,C] &= [A\,B\cdot C] = [A\,B\,C],\\
[A\,B\,(C+D)] &= [A\,B\,C] + [A\,B\,D],\\
[A\,B\,C] &= -[A\,C\,B] = [C\,A\,B].
\end{aligned}$$

The sum of any number of three-rowed matrices can be expressed as a single three-rowed matrix $[P\,Q\,R]$. In fact let $A_1\,B_1\,C_1$ and $A_2\,B_2\,C_2$ cut in a line $P\,Q$. Then

$$\begin{aligned}
[A_1\,B_1\,C_1] &= [P\,Q\,R_1],\\
[A_2\,B_2\,C_2] &= [P\,Q\,R_2].
\end{aligned}$$

Hence

$$\begin{aligned}
[A_1\,B_1\,C_1] + [A_2\,B_2\,C_2] &= [P\,Q\,R_1] + [P\,Q\,R_2]\\
&= [P\,Q\,(R_1+R_2)] = [P\,Q\,R],
\end{aligned}$$

where

$$R = R_1 + R_2.$$

From four points A, B, C, D we can form a four rowed matrix or determinant

$$(A\,B\,C\,D) = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

This matrix has only one element and hence we write it as a determinant. A matrix of one element is analytically equivalent to a number. We use the parentheses to indicate this fact. Square brackets are used to represent matrices which do not reduce to numbers. The vanishing of $(A\ B\ C\ D)$ is the condition that the four points lie in a plane.

The quantity $(A\ B\ C\ D)$ can be regarded as a product in a number of ways. From the definition it is evident that

$$(A\,B\,C\,D) = (A\cdot B\,C\,D) = (A\,B\cdot C\,D) = -(A\,B\,D\,C).$$

4. **Regressive Matrices.** We can consider space as generated by planes as well as by points. If its coordinates are α_i, a plane α is then represented by a matrix

$$\alpha = \|\,\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\,\|.$$

The same plane may be represented by a matrix $[A\ B\ C]$. Then the

coordinates a_i are proportional to the coefficients of x_i in the determinant $|A\ B\ C\ X|$.[3]
If a_i is equal to the coefficients of x_i in that determinant we shall write

$$\alpha = [A\ B\ C].$$

Thus a three-rowed matrix is for our purpose equivalent to a one-rowed matrix in contragredient variables.

The line of intersection of two planes α and β can be represented by a matrix

$$[\alpha\,\beta] = \left\|\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{matrix}\right\|.$$

The coordinates of the line are here

$$q_{ik} = \begin{vmatrix} \alpha_i & \alpha_k \\ \beta_i & \beta_k \end{vmatrix}.$$

If the same line is the join of two points A and B we know from analytical geometry that the coordinates q_{ik} are proportional to the coefficients of the minors $\begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix}$ in the determinant $|A\ B\ X\ Y|$. If q_{ik} is equal to the coefficient of $|x_i\ y_k|$ in the determinant we shall write

$$[\alpha\,\beta] = [A\ B].$$

This amounts to saying that in the determinant $[A\ B\ \alpha\,\beta]$, each minor in the first two rows is then equal to its algebraic compliment (coefficient in the expansion of the determinant).

Similarly we represent the point of intersection of three planes by a matrix $[\alpha\,\beta\,\gamma]$. The coordinates a_i of this point A are proportional to the coefficients of ξ_i in the determinant $[\xi\ \alpha\,\beta\,\gamma]$. In particular if a_i is equal to the coefficient of ξ_i in that determinant we write

$$A = [\alpha\,\beta\,\gamma].$$

In this case each term of the first row in the determinant $(A\ \alpha\,\beta\,\gamma)$ is equal to its algebraic compliment.

There is a determinant $[\alpha\,\beta\,\gamma\,\delta]$ of four planes just as of four points. These quantities $[\alpha\,\beta]$, $[\alpha\,\beta\,\gamma]$, $(\alpha\,\beta\,\gamma\,\delta)$ can be regarded as products formed according to the same laws as the products of points. These products of matrices expressed in plane coordinates we shall call *regressive*.

[3] It is to be observed that here X is written last. If we take the coefficients of X_i in the determinant $|X\ A\ B\ C|$ they will have different signs from the coefficients used here.

Matrices in Hyperspace.

5. We shall call the order of a space the number of homogeneous coordinates of a point in that space.[4] Thus a point is of order one, a line of order two, etc. A space of order n can be generated either by points or by hyperplanes of order $n-1$. A space R of order $r < n$ can be determined by a set of r points A, giving rise to a matrix.

$$R \equiv \left\| \begin{matrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots & \ldots \\ a_{r1} & a_{r2} \ldots a_{rn} \end{matrix} \right\| = [A_1\, A_2 \ldots A_r].$$

This matrix represents the space R in the sense that from the matrix can be determined the coordinates of the space. The same space R can be determined as the intersection of $n-r$ hyperplanes α_i determining a matrix

$$\left\| \begin{matrix} \alpha_{r+1,\,1} & \alpha_{r+1,\,2} \ldots \alpha_{r+1,\,n} \\ \alpha_{r+2,\,1} & \alpha_{r+2,\,2} \ldots \alpha_{r+2,\,n} \\ \ldots & \ldots \\ \alpha_{n\,1} & \alpha_{n,\,2} \ldots \alpha_{n,\,n} \end{matrix} \right\| = [\alpha_{r+1}\ \alpha_{r+2} \ldots \alpha_n].$$

The condition that these matrices represent the same space is that in the determinant

$$\left| \begin{matrix} a_{11} & a_{12} \ldots \ldots a_{1n} \\ \ldots & \ldots \\ a_{r1} & a_{r2} \ldots \ldots a_{rn} \\ \alpha_{r+1,\,1} & \alpha_{r+1,\,2} \ldots \alpha_{r+1,\,n} \\ \ldots & \ldots \\ \alpha_{n,\,1} & \alpha_{n2} \ldots \ldots \alpha_{nn} \end{matrix} \right|$$

the minors in the first r rows be proportional to their algebraic compliments.[5] If in the determinant each minor of the first r rows is *equal* to its algebraic compliment we shall write

$$[A_1\ A_2 \ldots A_r] = [\alpha_{r+1}\ \alpha_{r+2} \ldots \alpha_n].$$

The r-rowed determinants of a matrix of r rows we call the elements of the matrix. To add such matrices we add corresponding elements. If there exists a matrix of n columns whose elements are the corre-

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[4] Cf. Whitehead, Universal Algebra, page 177.
[5] Cf. Bertini, Geometria Projettiva, p. 33.

sponding sums, it represents the sum of the given matrices. If no such matrix exists the sum is complex. In that case we write the result as an algebraic sum and do not attempt to express it as a single matrix. If some of the matrices are expressed in point coordinates, the others in hyperplane coordinates we replace those of one kind by their dual forms or at least imagine them so replaced. This amounts to adding elements of the former to their algebraic compliments in the latter and considering the result as a term of the first kind.

6. A matrix in which the number of rows is equal to or less than the number of columns can be regarded as a product. If the matrix is expressed in point coordinates we call the product progressive, if in hyperplanar coordinates regressive. To multiply two such matrices (of the same kind) the sum of whose rows is equal to or less than n, we write the second matrix under the first to form a single matrix. If one of the factors is complex we apply the process distributively to the separate matrices of the sum. From the definition it is evident that such products are distributive and associative and that the interchange of two points or hyperplanes (according as the product is progressive or regressive) changes the sign of the result.

If a matrix of r rows vanishes, all the minors of order r in that matrix are zero. There is then a linear relation between the rows of the matrix.[6] If the matrix represents a progressive product of r points there is a linear relation between the points of that product and they therefore lie in a space of order less than r. If the matrix represents a regressive product of r hyperplanes they satisfy a linear relation and therefore intersect in a space of order greater than $n-r$. If the matrix is not zero the progressive form represents the space containing its factors and the regressive the space common to its factors.

The most general product is the result of a succession of operations each consisting of multiplying two factors. If the total number of rows in two matrices of the same kind (progressive or regressive) is less than n, the two are multiplied together according to the rule already given. If the total number of rows is greater than n, the product as previously defined gives a matrix of more rows than columns. For such a matrix we have no interpretation. In that case we replace each factor by its equivalent in contragredient variables. The total number of rows in the new product is less than n and we form the product by the previous method. If the total number of rows is equal to n the result is the same whether the matrices are taken in

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[6] Böcher, Introduction to Higher Algebra, p. 36.

point or hyperplane coordinates. If the matrices are of different kinds we replace one of them by its contragredient form in such a way that the new matrices have a sum of rows equal to or less than n. Thus in every case of a product there is a definite result that has a meaning. We call this the *product* of those factors.

7. **Reduction formulae.** We have just found that in expressing the product of two matrices when the sum of the rows is greater than n, we must change to contragredient forms. We shall now derive certain reduction formulae by which we obtain the same results without that change. For this purpose let

$$[A_1 \;\; A_2 \dots A_p] = [\alpha_{p+1}, \;\alpha_{p+2} \dots \alpha_n]$$
$$[B_1 \;\; B_2 \dots B_q] = [\beta_{q+1}, \;\beta_{q+2} \dots \beta_n].$$

We shall now prove that in the determinant

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & \dots a_{1n} & \pm a_{11} & \pm a_{12} & \dots \pm a_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{p1} & a_{p2} & \dots a_{pn} & \pm a_{p1} & \pm a_{p2} & \dots \pm a_{pn} \\ \alpha_{p+1,1} & \alpha_{p+1,2} & \dots \alpha_{p+1,n} & 0 & 0 & \dots 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{n,1} & \alpha_{n,2} & \dots \alpha_{n,n} & 0 & 0 & \dots 0 \\ 0 & 0 & \dots 0 & b_{11} & b_{12} & \dots b_{1n} \\ 0 & 0 & \dots 0 & b_{q1} & b_{q2} & \dots b_{qn} \\ \mp \beta_{q+1,1} & \mp \beta_{q+1,2} & \dots \mp \beta_{q+1,n} & \beta_{q+1,1} & \beta_{q+1,2} & \dots \beta_{q+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \mp \beta_{n,1} & \mp \beta_{n,2} & \dots \mp \beta_{n,n} & \beta_{n,1} & \beta_{n,2} & \dots \beta_{n,n} \end{vmatrix}$$

each minor from the $n-p$ rows of α's and $n-q$ rows of β's is equal to its algebraic compliment. To prove this we first show that if such a minor M contains a product of a minor A whose order is $n-p$ in the α's by a minor B of order $n-q$ in the β's, then the algebraic complement of M contains a product of minors respectively equal to A and B. Since A is contained in the principal minor $|a_{11} \dots \alpha_{nn}|$, if B is in its complement $|b_{11} \dots \beta_{nn}|$, the result is obvious. For the algebraic complements of A and B respectively in those principal minors are the terms required. If B is not contained entirely in the principal minor, there is a minor B', in $|b_{11} \dots \beta_{nn}|$ containing the same letters as B and in the same order (but having perhaps different signs). In the algebraic complement of a minor M' containing A, B' is then a term $C\,D = A\,B'$ where C is a minor of p rows of a's and D a minor of q rows of b's. If now we permute the columns contained in B' but not in B with their correspondents of the same suffix, this term will pass over into a term of the complement of M. In this process C and D are not changed for each a_{ik} in C is either not changed at all or replaced by the same letter and no letter of D can be in a column so moved. Furthermore the sign is correct for there are as many minus columns introduced as interchanges made. The same argument shows that for every product of minors in the algebraic complement of M there is an equal product in M. Therefore M is equal to its algebraic complement.

Suppose now $p + q \geqq n$. Then

$$n - p + n - q \leqq n.$$

We expand the determinant Δ in terms of minors of the nth order taken from the first n and the last n columns. The *part* of the expansion which contains all the α's and β's in the minor from the first n rows is

$$\Delta_1 = \Sigma \, (A_1 \; A_2 \ldots . A_{n-q} \; B_1 \; B_2 \ldots . B_q) \; (A_{n-q+1} \ldots . A_p \; \alpha_{p+1} \ldots . . \alpha_n \; \beta_{q+1} \; \ldots . \beta_n)$$

the summation being for every combination of $n - q$ A's in the first factor with the remaining $p + q - n$ in the second factor so arranged that the two groups in the order written constitute a positive permutation of A_1 to A_p. The form of this expression is evident since the B's cannot occur in the same factor with α's and β's (the other factor then containing a row of zeros). The sign of the term written is positive since it is obtained as a product of principal minors given by moving q rows of B's past $n - p$ rows of α's and $p + q - n$ rows of A's, interchanging first n and last n columns and changing $n - q$ minus signs. The result should therefore have a sign

$$(- 1)^{q(n - p + p + q - n) + n^2 + n - q} = 1.$$

The signs of the other terms then follow, since any positive rearrangement of A's should not change the sign of the term.

Now in the expression of Δ each minor formed of $n - p$ rows of α's and $n - q$ rows of β's is equal to its coefficient. Furthermore Δ_1 contains all of the terms in Δ given by such minors taken from the first n columns. Therefore in Δ_1 each minor of the matrix $[\alpha_{p+1} \ldots \alpha_n \; \beta_{q+1} \ldots \beta_n]$ is equal to its coefficient. These coefficients constitute the matrix

$$\Sigma \; (A_1 \ldots A_{n-q} \; B_1 \ldots B_q) \; [A_{n-q+1} \ldots . A_p]$$

which is therefore equal to the former. We can write this result in the form [7]

$$[\alpha_{p+1} \ldots \alpha_n\, \beta_{q+1} \ldots \beta_n] = [A_1\, A_2 \ldots A_p\;\, B_1\, B_2 \ldots B_q] = \Sigma\,(C\, B_1 \ldots B_q)\, D \qquad (1)$$

C being the product of any combination of $n - q$ of A's and D the product of the others such that

$$[A_1\;\, A_2 \ldots A_p] = [C\; D]$$

If $p + q < n$ we take the part of Δ which contains all the A's and B's in one n-rowed minor. The result is

$$\Delta_2 = \Sigma(\alpha_{p+1} \ldots \alpha_n\, \beta_{n-p+1} \ldots \beta_n)[A_1 \ldots A_p\;\, B_1 \ldots B_q\;\, \beta_{q+1} \ldots \beta_{n-p}].$$

Hence we have

$$[A_1 \ldots A_p\;\, B_1 \ldots B_q] = [\alpha_{p+1} \ldots \alpha_n\;\, \beta_{q+1} \ldots \beta_n] = \Sigma\,(\alpha_{p+1} \ldots \alpha_n\, \delta)\, \gamma \qquad (2)$$

where δ is any combination of p B's and γ the remaining ones so arranged that

$$[\beta_{q+1} \ldots \beta_n] = [\,\gamma\; \delta\,].$$

If instead of the determinant Δ we use the determinant

$$\Delta' \equiv \begin{vmatrix}
a_{11} & a_{12} \ldots\ldots a_{1n} & 0 & 0 \ldots\ldots 0 \\
\ldots & \ldots\ldots & \ldots & \ldots\ldots \\
a_{p1} & a_{p2} \ldots\ldots a_{pn} & 0 & 0 \ldots\ldots 0 \\
\alpha_{p+1,\,1} & \alpha_{p+1,\,2} \ldots \alpha_{p+1,\,n} & \pm\alpha_{p+1,\,1} & \pm\alpha_{p+1,\,2} \ldots \pm\alpha_{p+1,\,n} \\
\ldots & \ldots\ldots & \ldots & \ldots\ldots \\
\alpha_{n,\,1} & \alpha_{n,\,2} \ldots\ldots \alpha_{n\,n} & \pm\alpha_{n1} & \alpha_{n2} \ldots\ldots \alpha_{nn} \\
\mp b_{11} & \mp b_{12} \ldots\ldots \mp b_{1n} & b_{11} & b_{12} \ldots\ldots b_{1n} \\
\ldots & \ldots\ldots & \ldots & \ldots\ldots \\
\mp b_{q1} & \mp b_{q2} \ldots\ldots \mp b_{q\cdot n} & b_{q1} & b_{q2} \ldots\ldots b_{qn} \\
0 & 0 \ldots\ldots 0 & \beta_{q+1,\,1} & \beta_{q+1,\,2} \ldots \beta_{q+1,\,n} \\
\ldots & \ldots\ldots & \ldots & \ldots\ldots \\
0 & 0 \ldots\ldots 0 & \beta_{n1} & \beta_{n2} \ldots\ldots \beta_{nn}
\end{vmatrix}$$

when $p + q > n$, we obtain the expression in the form

$$[A_1\, A_2 \ldots A_p\, B_1\, B_2 \ldots B_p] = \Sigma\,(A_1\, A_2 \ldots A_p\, D)\, C\,. \qquad (3)$$

where D is any combination of $n-p$ letters B and C the remaining ones so arranged that

$$[B_1\, B_2 \ldots B_q] = [C\; D].$$

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[7] Grassmann, Gesamelte Werke, Vol. I, p. 83; Whitehead, Universal Algebra. p. 188; H. B. Philips, Proceedings of the American Academy of Arts and Sciences, Vol. 46, p. 369. In this last article the formula obtained may have a different sign from the one here given.

When $p + q < n$ this determinant gives

$$[\alpha_{p+1} \ldots \alpha_n \, \beta_{q+1} \ldots \beta_n] = \Sigma \, (\gamma \, \beta_{q+1} \ldots \beta_n) \, \delta \quad . \quad . \qquad (4)$$

where γ is any combination of q α's and δ the remaining ones so arranged that

$$[\alpha_{p+1} \ldots \alpha_n] = [\gamma \, \delta].$$

Symbolic notation.

8. The determinants in the matrix representing a space S are the coordinates s_i of S. Of σ is a space complimentary to S, we consider it as represented by a matrix of the same kind as S. It has then a like number of coordniates σ_i (algebraic compliments of s_i in the determinant $|s\sigma|$). Then

$$(S \, \sigma) = \Sigma \, s_i \, \sigma_i.$$

This is a linear function of the coordinates s_i and by a proper choice of σ (perhaps complex) can be made any linear function of those coordinates. To obtain a bilinear function of the coordinates r_i, s_k of two spaces R and S we take matrices ρ and σ complimentary to R and S. Then

$$(R \, \rho) \;\; (S \, \sigma) = \Sigma \, \rho_i \, \sigma_k \, r_i \, s_k \, . \quad . \quad . \quad . \qquad (5)$$

In order to obtain the most general bilinear function

$$\Sigma \, a_{ik} \, r_i \, s_k$$

we consider the above as a symbolic representation in which $\rho_i \, \sigma_k$ is to be replaced by a_{ik}. Thus $(R \, \rho)$ $(S \, \sigma)$ represents symbolically any bilinear function of the coordinates r_i, S_k. Any linear relation connecting the symbolic quantities $(R \, \rho)$ $(S \, \sigma)$ will be satisfied by the bilinear functions $\Sigma \, a_{ik} \, r_i \, s_k$. This is the symbolic representation so much used by Clebsch.

We can consider $(R \, \rho)$ $(S \, \sigma)$ as resulting from an expression $\rho \, \sigma$ by operating on the first factor with R and on the second with S. This product $\rho \, \sigma$ is the dyadic of Gibbs.[8] It may be considered as a distributive product of ρ and σ. It is called the indeterminate[9] product. In it the order of factors must be preserved. In fact there is no general functional relation between $\rho \, \sigma$ and $\sigma \, \rho$. The dyadic

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8 Vector Analysis, Gibbs-Wilson, page 265.

9 Cf. H. B. Phillips, loc. cit.

$\rho\,\sigma$ represents a transformation which changes a space R complimentary to ρ into a space $(R\,\rho)\,\sigma$ which is given by the locus of S in

$$(R\,\rho)\,(S\,\sigma) = 0.$$

Linear Distance and Angle in Three Dimensions.

9. **Linear distance between two points.** We define the distance between two points A, B as such a function $\overline{A\,B}$ of their coordinates that (1) if one is fixed the other lies in a plane, and (2) for points A, B, C on a line

$$AB + BC + CA = 0\,. \qquad . \qquad . \qquad . \qquad (6)$$

From the first condition the distance must be of the form

$$\overline{A\,B} = \frac{F_1\,(A,\,B)}{F_2\,(A,\,B)} \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (7)$$

where F_1 and F_2 are bilinear functions of A and B. Putting A, B and C equal in the second condition we get

$$\overline{A\,A} = 0.$$

Hence

$$F_1\,(A_1\,A) = 0 \qquad . \qquad .. \qquad . \qquad . \qquad (8)$$

In this last equation replacing A by $A + B$ and cancelling the terms $F_1\,(A,A)$ and $F_1\,(B,B)$ we have

$$F_1\,(A,\,B) + F_1\,(B,\,A) = 0\,. \qquad . \qquad . \qquad . \qquad (9)$$

The numerator of $\overline{A\,B}$ must then change sign when we interchange A and B. In (6) putting $C = B$ we have

$$\overline{A\,B} + \overline{B\,A} = 0.$$

This shows that

$$F_2\,(A,\,B) = F_2\,(B,\,A) \qquad . \qquad . \qquad . \qquad . \qquad (10)$$

or the denominator of $\overline{AB}$ is symmetric in A and B. Let $C = A + B$. Then (6) becomes

$$\frac{F_1\,(A,\,B)}{F_2\,(A,\,B)} + \frac{F_1\,(B,\,A) + F_1\,(B,\,B)}{F_2\,(B,\,A) + F_2\,(B,\,B)} + \frac{F_1\,(A,\,A) + F_1\,(B,\,A)}{F_2\,(A,\,A) + F_2\,(B,\,A)} = 0.$$

Making use of (8), (9) and (10) this becomes

$$F_1\,(A,\,B)\,[F_2\,(A,\,A)\,F_2\,(B,\,B) - F_2\,(A,\,B)^2] = 0.$$

Then either

$$F_1 (A, B) = 0$$

or

$$F_2 (A, B)^2 = F_2 (A, A) F_2 (B, B).$$

This last equation shows that $F_2 (A, B)$ factors into a function of A times a function of B. Calling this function (ϕA), and writing

$$F_1 (A, B) = \tfrac{1}{2} [F_1 (A, B) - F_1 (B, A)] = (\alpha A) (\beta B) - (\alpha B) (\beta A),$$

we have

$$\overline{A B} = \frac{(\alpha A) (\beta B) - (\alpha B) (\beta A)}{(\phi A) (\phi B)}.$$

Using the identity (3) this takes the form [10]

$$\overline{A B} = \frac{(\alpha \beta \cdot A B)}{(\phi A) (\phi B)} = \frac{(q \cdot A B)}{(\phi A) (\phi B)},$$

where we put q in place of the two rowed matrix $(\alpha \beta]$.

10. **Angle between two planes.** We define the angle between two planes as such a function $\overline{\alpha\beta}$ of their coordinates that if the angle is given and one of the planes fixed, the other passes through a point and for three planes of a linear pencil

$$\overline{\alpha\beta} + \overline{\beta\gamma} + \overline{\gamma\alpha} = 0.$$

By the same argument as for the distance between two points we obtain for the angle

$$\overline{\alpha\beta} = \frac{(p \cdot \alpha \beta)}{(F \alpha) (F \beta)}$$

where p is a fixed complex and F a fixed point.

Distance is a relative invariant under the group of collineations that leave the complex q and the plane ϕ fixed. Similarly angle is a relative invariant under the group leaving p and F fixed. In order that fixed relations may exist between distances and angles we wish, if possible, these groups to be the same. We assume that the complex q does not degenerate into a line. Then the only complex and point determined by q and ϕ is the complex q itself and the polar point of ϕ with respect to it. Hence we have

$$p = q,$$
$$F = \lambda [\phi p].$$

10 We consider $(\alpha \beta . A B)$ as a regressive product $(\alpha . \beta . A B)$, in which we expand the product $(\beta . A B)$ and then multiply by α.

We choose the unit angle such that $\lambda = 1$. Then

$$F = [\phi\, p].$$

also [11]

$$[F\, p] = [\phi\, p \,.\, p] = \tfrac{1}{2}\,(p.\, p)\,\phi = \phi$$

if we choose the magnitude of p such that

$$(p\, p) = 2.$$

The relations between ϕ and F are then symmetrical.

Our formulae are now

$$\overline{A\, B} = \frac{(p\, A\, B)}{(\phi\, A)\,(\phi\, B)} \quad . \quad . \quad . \quad . \qquad (11)$$

$$\overline{\alpha\, \beta} = \frac{(p \cdot \alpha\, \beta)}{(F\, \alpha)\,(F\, \beta)} \quad . \quad . \quad . \quad . \qquad (12)$$

with the condition that $F = [\phi\, p]$, $\phi = [F\, p]$ and $(p\, p) = 2$.

The ratio of two distances or of two angles, also the product of a distance and angle are invariant under the seven parameter group of collineations leaving the complex p and the plane ϕ fixed. If one of these transformations leaves a distance or angle unchanged it leaves all distances and angles unchanged. Those quantities then are invariant under a six parameter group. Any tetrahedron can therefore be transformed into an equal tetrahedron (one having equal length of sides) by a collineation leaving distance and angle invariant.

From the formula for the distance between two points, it is seen that distances along a line of the complex p are zero provided neither of the points lies on ϕ. The distance along a line of p to ϕ is indeterminate but along any other line it is infinite. Similarly the angle between two planes intersecting in a line of p but neither passing through F is zero. If one of the planes passes through F, the angle is indeterminate or infinite according as the other plane does or does not cut it in a line of p.

11. The locus of points y at a distance λ from the point x is a plane

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11 The formula $[\phi\, p \cdot p] = \frac{1}{2}\,(p\, p)\,\phi$ can be proved as follows. Let

$$p = \alpha\,\beta + \gamma\,\delta.$$

Then $(p\, p) = 2(\alpha\,\beta\,\gamma\,\delta)$

and
$$\begin{aligned}[\phi\, p \cdot p] &= [\phi(\alpha\,\beta + \gamma\,\delta) \cdot (\alpha\,\beta + \gamma\,\delta)] \\ &= (\phi\,\alpha\,\beta\,\delta)\,\gamma - (\phi\,\alpha\,\beta\,\gamma)\,\delta + (\phi\,\alpha\,\delta\,\beta)\alpha - (\phi\,\gamma\,\delta\,\alpha)\,\beta \\ &= \phi\,(\alpha\,\beta\,\gamma\,\delta) = \tfrac{1}{2}\,(p\, p)\,\phi.\end{aligned}$$

ξ intersecting ϕ on the polar plane of x with respect to p. The correspondence between x and ξ is a correlation. From the equation

$$\overline{x\,y} = \frac{(p\,x\,y)}{(\phi\,x)\,(\phi\,y)} = \lambda,$$

or
$$\lambda\,(\phi\,x)\,(\phi\,y) - (p\,x\,y) = 0,$$

it is seen that the locus of y is

$$\xi = \lambda\,(\phi\,x)\,\phi - [p\,x] \qquad . \quad . \quad . \quad (13)$$

Similarly the locus of planes η making a given angle λ with the plane ξ is a point such that the line connecting it to F passes through the polar point of ξ with respect to p. The locus of planes making with ξ an angle $-\lambda$ is

$$z = -\lambda\,(F\,\xi)\,F - [p\xi].$$

Substituting in this the value of ξ from (13) we get

$$z = \lambda\,(F\,p\,x)\,F - \lambda\,(\phi\,x)\,[p\,\phi] + [p \cdot p\,x]$$

since $(F\,\phi) = 0$, F being a point of ϕ. Using the conditions

$$[F\,p] = \phi, \quad [\phi\,p] = F, \text{ and } \quad [p \cdot p\,x] = \tfrac{1}{2}\,(p\,p)\,x = x$$

we get
$$z = x$$

Hence the correlations determined by a distance λ and by an angle $-\lambda$ are inverse. Now the correlation set up by an angle $-\lambda$ is inverse to that determined by an angle λ. Hence the equations

$$\overline{x\,y} = \lambda,$$
$$\xi\,\overline{n} = \lambda,$$

where x and ξ are given, y and η variable, set up the same correlation.

Through a correlation

$$\overline{x\,y} = \lambda$$

to x_1 and x_2 correspond the planes

$$\lambda\,(\phi\,x_1)\,\phi - p\,x_1,$$
$$\lambda\,(\phi\,x_2)\,\phi - p\,x_2.$$

The angle between these planes is

$$\frac{(p\,[\lambda\,(\phi\,x_1)\,\phi - p\,x_1]\,[\lambda\,(\phi\,x_2)\,\phi - p\,x_2])}{\{F\,[\lambda\,(\phi\,x_1)\,\phi - p\,x_1]\}\,\{F\,[\lambda\,(\phi\,x_2)\,\phi - p\,x_2]\}}.$$

Since $[F\,\phi]$ and $[\phi\,\phi]$ are zero this gives

$$\frac{-\lambda\,[(\phi\,x_1)\,(p\cdot\phi\cdot p\,x_2) + (\phi\,x_2)\,(p\cdot p\,x_1\cdot\phi) + (p\cdot p\,x_1\cdot p\,x_2)]}{(F\,p\,x_1)\,(F\,p\,x_2)}$$

$$= \frac{-\lambda\,\{(\phi\,x_1)\,(\phi\,x_2) + (\phi\,x_2)\,(x_1\,\phi)\} + (p\,x_1\,x_2)}{(F\,p\,x_1)\,(F\,p\,x_2)}$$

$$\doteq \frac{(p\,x_1\,x_2)}{(\phi\,x_1)\,(\phi\,x_2)}.$$

Hence the correlation changes x_1, x_2 into two planes ξ_1, ξ_2 such that

$$\overline{\xi_1\,\xi_2} = \overline{x_1\,x_2}.$$

In particular if $\lambda = 0$, the correlation between x and ξ is the null system determined by the complex p. The distance between any two points is therefore equal to the angle between their polar planes with respect to the complex p.

12. **Angle between two lines.** We define the angle between two lines r, s as such a function $\overline{r\,s}$ of their coordinates that, one of them being fixed and the angle constant, the other satisfies a linear relation (*i. e.* belongs to a linear complex) and for lines r, s, t of a plane pencil

$$\overline{r\,s} + \overline{s\,t} + \overline{t\,r} = 0.$$

By the same argument as for distance between two points we find

$$\overline{r\,s} = \frac{f_1\,(r,\,s)}{f_2\,(r,\,s)},$$

where

$$f_1\,(r,\,s) = -f_1\,(s,\,r)$$

and $f_2\,(r,\,s)$ factors into a linear function of r times the same linear function of s. Hence

$$\overline{r\,s} = \frac{(a\,r)\,(b\,s) - (a\,s)\,(b\,r)}{(c\,r)\,(c\,s)},$$

where a, b, c are matrices of two rows and $a\,b$ a dyadic setting up a correspondence between lines or complexes. The numerator of $\overline{r\,s}$ can be written in a different form. In fact

$$(A\,B\,r)\,(C\,D\,s) - (C\,D\,r)\,(A\,B\,s) =$$
$$\tfrac{1}{2}\{[ABC\cdot r\cdot D\,s] - [ABD\cdot r\cdot Cs] + [BCD\cdot r\cdot As] - [ACD\cdot r\cdot Bs]\},$$

as is seen by expanding the right hand member. The expression in the parentheses may be regarded as gotten by operating on the collineation (dyadic)

$$\tfrac{1}{2}\{[A\,B\,C]\,D - [A\,B\,D]\,C + [B\,C\,D]\,A - [A\,C\,D]\,\mathrm{B}\}$$

with r, s. For this collineation the linear invariant

$$\tfrac{1}{2}\,\{(A\,B\,C\,D) - (A\,B\,D\,C) + (B\,C\,D\,A) - (A\,C\,D\,B)\} = 0.$$

Such a collineation has sometimes been called normal. By summing we get

$$(a\,r)\,(b\,s) - (a\,s)\,(b\,r) = (\alpha\,r\cdot A\,s)$$

where $\alpha\,A$ is a collineation such that

$$(\alpha\,A) = 0.$$

Conversely if $\alpha\,A$ is any normal collineation

$$(\alpha\,r\cdot A\,r) = (\alpha\,A)\,\frac{(r\,r)}{2} = 0$$

r being any line or complex. Replacing r by $r + s$ we have

$$(\alpha\,r\cdot A\,s) + (\alpha\,s\cdot A\,r) = 0,$$

showing that $(\alpha\,r\cdot A\,s)$ changes sign with interchange of r and s and is hence of the type

$$(a\,r)\,(b\,s) - (a\,s)\,(b\,r).$$

We therefore have

$$\overline{r\,s} = \frac{(\alpha\,r\cdot A\,s)}{(c\,r)\,(c\,s)} \qquad \dots \qquad (14)$$

It is to be noticed that this formula determines an angle between two complexes as well as between two lines. In particular the angle is zero if the complexes coincide.

The system of lines s making a zero angle with a line $r = [C\,D]$ may be constructed as follows. Let the correspondents of C and D through the collineation $\alpha\,A$ be

$$\mathrm{C}' = (C\,\alpha)\,A$$
$$D' = (D\,\alpha)\,A.$$

Then s is determined by an equation

$$(\alpha\,r\cdot A\,s) = (\alpha\cdot C\,D\cdot A\,s) = (\alpha\,D)\,(C\,A\,s) - (\alpha\,C)\,(D\,A\,s)$$
$$= (D'\,C\,s) - (C'\,D\,s) = 0.$$

In particular any line of the congruence cutting $D' C$ and $C' D$ has the required property. We may use instead of C, D any two points of the line. If then $C D$ and $C' D'$ do not intersect this gives us an infinite number of congruences generating the complex to which s belongs.

13. For a general collineation αA these lines r, s making with each other zero angles have an interesting geometrical interpretation. It is well known that a general collineation whose linear invariant (αA) vanishes has a system of tetrahedra A, B, C, D such that each point is carried by the collineation into a point of the opposite face. Two opposite edges $A B$ and $C D$ of such a tetrahedron determine a zero angle. For in this case since C', D' are in the planes $A B D$ and $A B C$, the lines $C' D$ and $C D'$ cut $A B$.

Conversely if $A B$ and $C D$ are two non-intersecting lines making with each other a zero angle and those lines are not left entirely invariant by the collineation we construct a tetrahedron upon them as follows. Join A' and C', the correspondents of A, C through the collineation αA, to $C D$ and $A B$ respectively and let these planes determine on $A B$ and $C D$ respectively the points B and D. Then B will pass into a point B' such that $A B'$ cuts $C D$ (i. e. a point of $A C D$). Similarly for D. Thus, with the possible exception of fixed lines, the entire system of non-intersecting lines making with each other a zero angle consists of the opposite edges of these particular tetrahedra associated with the normal collineation αA.

If P, Q, R, S are any four points it is seen on expanding the right side that

$$(\alpha P)\,(A\,Q\,R\,S) = \tfrac{1}{2}\,[\alpha\,P\,Q\cdot A\,R\,S - \alpha\,P\,R\cdot A\,Q\,S + \alpha\,P\,S\cdot A\,Q\,R]$$

Hence if x is any point and ξ any plane $(\alpha\, x)\ (A\,\xi)$ is expressible as a sum of terms of the form $\alpha\, r\cdot A\, s$. Under any collineation leaving all angles invariant this last expression must be covariant. Hence the form $(\alpha\, x)\ (A\,\xi)$ must also be covariant.

Collineations leaving angle invariant must then leave the complex c invariant and the collineation αA fixed. We wish these angles to be invariant under the group of transformations that leave distance fixed. In that case c must coincide with p. There is a transformation of this group changing any distance $\overline{x\,y}$ into any equal distance $\overline{x\,y'}$. Since to x there can correspond through αA only one point y, this point must be fixed under all the collineations. Therefore to each point x corresponds the point F. Hence

$$\alpha\, A = \beta\, F$$

where β is a plane and F a point. Dual considerations show that β is fixed under all the collineations, i. e. coincides with ϕ. Hence by a proper choice of units we have

$$\overline{r\,s} = -\frac{(\phi\, r \cdot F\, s)}{(p\, r)\,(p\, s)} = \frac{(\phi\, s \cdot F\, r)}{(p\, r)\,(p\, s)} \qquad . \quad . \quad (15)$$

The angle between two lines is zero if they cut a line through F in the plane ϕ. The angle is infinite if one of them belongs to the complex p and they are not cut by a line of the plane ϕ passing through the point F.

14. We have seen that

$$\overline{x\,y} = \lambda$$

sets up a correlation. To x and y correspond planes

$$\lambda(\phi\, x)\,\phi - p\, x,$$
$$\lambda(\phi\, y)\,\phi - p\, y.$$

To $x\,y$ corresponds the intersection which can be written

$$\lambda\,[\phi \cdot x\, y \cdot F] + (p\, x\, y)\, p - \tfrac{1}{2}\,(p\, p)\,[x\, y]$$

Hence to lines r and s correspond lines

$$\lambda\,[\phi\, r \cdot F] + (p\, r)\, p - r,$$
$$\lambda\,[\phi\, s \cdot F] + (p\, s)\, p - s.$$

The angle between these lines is

$$\frac{(\phi\,\{\lambda\,[\phi\, s \cdot F] + (p\, s)\, p - s\}\quad F\,\{\lambda\,[\phi\, r\, F] + (p\, r)\, p - r\})}{(p\,\{\lambda\,[\phi\, S\, F] + (p\, s)\, p - s\})\,(p\,\{\lambda\,[\phi\, r\, F] + (p\, r)\, p - r\})}$$
$$= \frac{(\phi\, s\, .\, F\, r)}{(p\, s)\,(p\, r)}.$$

Hence the angle between two lines is equal to that between the lines corresponding to them through the correlation.

$$\overline{x\,y} = \lambda$$

In particular when $\lambda = 0$ we see that the angle between two lines is equal to that between their polar lines with respect to the complex p.

15. **Distance from point to plane.** We wish to determine a function $\overline{A\,\alpha}$ of the coordinates of a point and plane such that if either is fixed the other satisfies a linear relation and such that

$$\overline{\mathrm{A}\,\alpha} = \overline{\mathrm{A}'\,\alpha'}$$

is a necessary and sufficient condition that A, α be transformable into A', α' by a motion leaving distance invariant. Such a function is

$$\overline{A\,\alpha} = \frac{(A\,\alpha)}{(\phi\,A)\,(F\,\alpha)} \quad . \quad . \quad . \quad . \quad (16)$$

Let α be a plane $B\,C\,D$. Then

$$\overline{A\,\alpha} = \frac{(A\,B\,C\,D)}{(\phi\,A)\,(F\,B\,C\,D)} = \frac{\frac{1}{2}\,(A\,B\,C\,D)\,(p\,p)}{(\phi\,A)\,(\phi\,p\cdot B\,C\,D)}.$$

This expression can be written

$$\frac{(p\,A\,B)\,(p\,C\,D) + (p\,A\,C)\,(p\,D\,B) + (p\,A\,D)\,(p\,B\,C)}{(\phi\,A)\{(\phi\,B)\,(p\,C\,D) + (\phi\,C)\,(p\,D\,B) + (\phi\,D)\,(p\,B\,C)\}}$$

$$= \frac{\overline{A\,B}\cdot\overline{C\,D} + \overline{A\,C}\cdot\overline{D\,B} + \overline{A\,D}\cdot\overline{B\,C}}{\overline{B\,C} + \overline{C\,D} + \overline{D\,B}} \quad (17)$$

That $\overline{A\,\alpha}$ is invariant under the transformations leaving distance unchanged is shown by the last form. Conversely if

$$\overline{A\,\alpha} = \overline{A'\,\alpha'}$$

we take in α a triangle $B\,C\,D$ and in α' a corresponding triangle $B'C'D'$ such that

$$\overline{A'B'} = \overline{A\,B}, \quad \overline{A'C'} = \overline{A\,C}, \quad \overline{B'C'} = \overline{B\,C}$$
$$\overline{A'D'} = \overline{A\,D}, \qquad \overline{C'D'} = \overline{C\,D}.$$

Then the above equation shows that

$$\overline{B'\,D'} = \overline{B\,D}.$$

The two tetrahedra have all their edges equal and hence the one is transformable into the other.

This quantity $\overline{A\,\alpha}$ we call the distance from the point A to the plane α. It has many of the properties of euclidean distance from point to plane. Thus if the point lies in the plane (point not in ϕ and plane not through F) the distance is zero. If the plane is held fixed and the distance kept constant the point lies in a plane cutting α on ϕ. If the point is held fixed the locus of the plane is a point on the line joining the given point to F.

If $$\overline{A\,\alpha} = \lambda$$

the point corresponding to A (enveloped by α) is

$$B = \lambda\,(\phi\,A)\,F - A.$$

The distance from A to B is

$$\overline{A\,B} = \frac{(p\,A\,\{\lambda\,(\phi\,A)\,F - A\})}{-\,(\phi\,A)^2} = \lambda.$$

Thus $\overline{A\,B} = \overline{A\,\alpha}$. This shows that $\overline{A\,\alpha}$ is the distance, measured along $A\,F$, from A to the point of intersection of α with $A\,F$.

16. **Distance from point to line.** We define the distance from a point A to a line r as such a function of their coordinates that one of the quantities being fixed and the distance held constant, the other satisfies a linear relation and such that this distance is invariant under the transformations leaving distance between two points unchanged. Such a function is

$$A\,r = \frac{(F\,A\,r)}{(\phi\,A)\,(p\,r)} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If r joins two points, B, C this can be written

$$\overline{A\,r} = \frac{(A\,B\,C \cdot \phi\,p)}{(A\,\phi)\,(B\,C\,p)} = \frac{(A\,\phi)\,(B\,C\,p) + (B\,\phi)\,(C\,A\,p) + (C\,\phi)\,(A\,B\,p)}{(A\,\phi)\,(p\,B\,C)}$$

Dividing numerator and denominator by $(A\,\phi)\,(B\,\phi)\,(C\,\phi)$, this becomes

$$\overline{A\,r} = \frac{\overline{B\,C} + \overline{C\,A} + \overline{A\,B}}{\overline{B\,C}} \quad . \quad . \quad . \quad (19)$$

This expression shows that $\overline{A\,r}$ is invariant under the distance transformations. Conversely of

$$\overline{A\,r} = \overline{A'\,r'}$$

there is a transformation changing $\overline{A\,r}$ into $\overline{A'\,r'}$. For let B, C be two points of r. Take on r' two points B', C' such that

$$\overline{A\,B} = \overline{A'\,B'}$$

$$\overline{A\,C} = \overline{A'\,C'},$$

then

$$\overline{B\,C} = \overline{B'\,C'},$$

and a transformation of the kind desired can be obtained.

The distance from a point to a line is zero if the point lies on the line or if the plane of the line and the point pass through F (assuming that the point does not lie in ϕ and that the line does not belong to p). Since the order of A and r is immaterial in the formula for $\overline{A\ r}$ we write

$$\overline{A\ r} = \overline{r\ A}.$$

17. **Angle between line and plane.** Dual considerations give for the angle between a line r and a plane α, the expression

$$\overline{\alpha\ r} = \overline{r\ \alpha} = \frac{(\phi\ \alpha\ r)}{(F\ \alpha)\ (p\ r)} \quad . \quad . \quad . \quad (20)$$

Let α be the plane at distance λ from A and r the line all points of which are at distance λ from s. Then

$$\alpha = \lambda\,(\phi\ A)\ \phi - p\ A,$$

$$r = \lambda\,[\phi\ S\ F] + (p\,s)\ p - s.$$

Hence

$$\overline{\alpha\ r} = \frac{(\phi\ \{\lambda\,(\phi\ A)\ \phi - p\ A\}\ \{\lambda[\phi\ s\ F] + (p\ s)\ p - s\})}{(F\ \{\lambda\,(\phi\ A)\ \phi - p\ A\})\ (p\ \{\lambda\ \phi\ s\ F + (p\,s)\ p - s\})}$$

$$= \frac{(-[p\,A]\ \{(ps)\ F - s\phi\})}{(\phi\ A)\ (p\ s)} = \frac{(p\ s)\ (\phi\ A) + (p\ A)\cdot(\phi\ s)}{(\phi\ A)\ (p\ s)}.$$

But $\qquad [p\ A\cdot s\ \phi] = [p\ \{(A\ \phi)\ s - A\cdot s\ \phi\}],$

and $\qquad \{p\ [A\cdot\phi\ s]\} = (A\cdot s\cdot\phi\ p) = (A\ s\ F).$

Hence $\qquad \overline{\alpha\ r} = \dfrac{(F\ A\ s)}{(\phi\ A)\ (p\ s)} = \overline{A\ s}.$

Therefore the angle between a line and plane is equal to the distance between the line and point corresponding to them through the distance correlation. In particular for $\lambda = 0$, we see that the distance between point and line is equal to the angle between their polar plane and point with respect to the complex p.

18. **Line Area of a triangle.** We define the area of a triangle $A\ B\ C$ as a function $\overline{A\ B\ C}$ of three points such that if the vertex is fixed and the base moved along its line, the area is proportional to the base. Hence if A is the vertex of the triangle and s the line on which the base $B\ C$ lies

$$\overline{A\ B\ C} = k\ \overline{B\ C}\cdot\overline{A\ s},$$

where k is a function of A and s. This gives on applying formulae (11) and (18) after replacing s by $[B\,C]$

$$\overline{A\,B\,C} = k\frac{(F\,A\,B\,C)}{(\phi\,A)\,(\phi\,B)\,(\phi\,C)}.$$

The areas of two triangles having the same vertex and base line are then proportional to the quantities

$$\frac{(F\,A\,B\,C)}{(\phi\,A)\,(\,(\phi\,B)\,(\phi\,C)}.$$

By a series of operations consisting of moving one side of the triangle along its line and keeping the opposite vertex fixed we can move the triangle into coincidence with any other having the same area. Under each of these operations the area is proportional to the above quantity. Hence any two areas are to each other as those quantities. Then by a proper choice of unit we have

$$\overline{A\,B\,C} = \frac{(F\,A\,B\,C)}{(\phi\,A)\,(\phi\,B)\,(\phi\,C)} \quad . \quad . \quad . \qquad (21)$$

Writing $F = (\phi\,p)$ we have

$$\overline{ABC} = \frac{(\phi\,p\cdot A\,B\,C)}{(\phi A)\,(\phi B)\,(\phi C)} = \frac{(\phi A)\,(pBC) + (\phi B)\,(p\,CA) + (\phi C)\,(pAB)}{(\phi\,A)\,(\phi\,B)\,(\phi\,C)}$$

$$= \overline{B\,C} + \overline{C\,A} + \overline{A\,B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

Thus the line area of a triangle is equal to its perimeter.

Dually we can find as the trihedral angle between three planes α, β, γ,

$$\overline{\alpha\,\beta\,\gamma} = \frac{(\phi\,\alpha\,\beta\,\gamma)}{(F\,\alpha)\,(F\,\beta)\,(F\,\gamma)}$$

$$= \overline{\alpha\,\gamma} + \overline{\beta\,\gamma} + \overline{\gamma\,\alpha} \quad . \quad . \quad . \qquad (23)$$

19. **Volume of a tetrahedron.** Similarly we define the volume of a tetrahedron $A\,B\,C\,D$ as such a function $\overline{A\,B\,C\,D}$ of the four points that given the vertex and plane of the base, the volume is proportional to the area of the base. From the definition we have

$$\overline{A\,B\,C\,D} = k\,\overline{B\,C\,D}\cdot\overline{A\,\alpha}$$

$$= \frac{k\,(F\,B\,C\,D)\cdot(\,A\,\alpha)}{(\phi\,B)\,(\phi\,C)\,(\phi\,D)\,(\phi\,A)\,(F\,\alpha)} = k\frac{(A\,B\,C\,D)}{(\phi\,A)\,(\phi\,B)\,(\phi\,C)\,(\phi\,D)},$$

where α represents the plane $B\,C\,D$, in which the base lies and k is a function of A and α. By a series of motions consisting of moving one triangle of the tetrahedron in its plane it is seen that the tetrahedron can be moved into any other having equal volume. These motions keep the volume constant and therefore k is an absolute constant. Hence choosing our unit so that $k = 1$, we have

$$\overline{A\,B\,C\,D} = \frac{(A\,B\,C\,D)}{(\phi\,A)\,(\phi\,B)\,(\phi\,C)\,(\phi\,D)} \quad . \quad . \qquad (24)$$

From the definition we have

$$\begin{aligned}\overline{A\,B\,C\,D} &= \overline{B\,C\,D}\cdot\overline{A\,\alpha} = \overline{B\,C\,D}\cdot\overline{A\,(B\,C\,D)} \\ &= (\overline{B\,C}+\overline{C\,D}+\overline{D\,B})\left(\frac{\overline{A\,B}\cdot\overline{C\,D}+\overline{A\,C}\cdot\overline{D\,B}+\overline{A\,D}\cdot\overline{B\,C}}{\overline{B\,C}+\overline{C\,D}+\overline{D\,B}}\right) \\ &= \overline{A\,B}\cdot\overline{C\,D}+\overline{A\,C}\cdot\overline{D\,B}+\overline{A\,D}\cdot\overline{B\,C} \quad . \quad . \quad . \qquad (25)\end{aligned}$$

From (24) we see that if the vertex A lies in the plane $A\,B\,C$ the volume is zero. Hence applying this to (25) we have

$$\overline{A\,B}\cdot\overline{C\,D}+\overline{A\,C}\cdot\overline{D\,B}+\overline{A\,D}\cdot\overline{B\,C} = 0$$

as a relation connecting four points lying in a plane. This relation is seen to be identical with the relation connecting the Plücker coordinates of a line. From this a theory of plane quadrilaterals could be built up.

20. **Summary.** We have defined a bilinear function of any two spaces in three dimensions. In case one of these spaces is a point we call this function a distance otherwise an angle. We have also defined certain areas determined by three elements and volumes determined by four. These functions are all invariant under a six parametered group of collineations projectively equivalent to the group of collineations leaving euclidean volume invariant. Under the correlation

$$\overline{x\,y} = \text{const.}$$

each of these functions is equal to the dual function of the transformed elements. The expressions for these functions are

$$\overline{A\,B} = \frac{(p\,A\,B)}{(\phi\,A)\,(\phi\,B)} \quad . \quad . \quad . \quad . \qquad (11)$$

$$\overline{\alpha\,\beta} = \frac{(p\,\alpha\,\beta)}{(F\,\alpha)\,(F\,\beta)} \quad . \quad . \quad . \quad . \qquad (12)$$

$$\overline{r\,s} = \frac{(\phi\, S \cdot F\, r)}{(p\, s)\,(p\, r)} \quad . \quad . \quad . \quad . \qquad (15)$$

$$\overline{A\,\alpha} = \frac{(A\,\alpha)}{(\phi\, A)\,(F\,\alpha)} \quad . \quad . \quad . \quad . \qquad (16)$$

$$\overline{A\,r} = \frac{(F\,A\,r)}{(\phi\, A)\,(p\,r)} \quad . \quad . \quad . \quad . \qquad (18)$$

$$\overline{r\,\alpha} = \frac{(\phi\, r\,\alpha)}{(p\,r)\,(F\,\alpha)} \quad . \quad . \quad . \quad . \qquad (20)$$

$$\overline{A\,B\,C} = \frac{(F\,A\,B\,C)}{(\phi\, A)\,(\phi\, B)\,(\phi\, C)} \quad . \quad . \quad . \qquad (21)$$

$$\overline{\alpha\,\beta\,\gamma} = \frac{(\phi\,\alpha\,\beta\,\gamma)}{(F\,\alpha)\,(F\,\beta)\,(F\,\gamma)} \quad . \quad . \quad . \qquad (23)$$

$$\overline{A\,B\,C\,D} = \frac{(A\,B\,C\,D)}{(\phi\, A)\,(\phi\, B)\,(\phi\, C)\,(\phi\, D)} \quad . \quad . \qquad (24)$$

$$\overline{\alpha\,\beta\,\gamma\,\delta} = \frac{(\alpha\,\beta\,\gamma\,\delta)}{(F\,\alpha)\,(F\,\beta)\,(F\,\gamma)\,(F\,\delta)} \quad . \quad . \qquad (25)$$

21. **Tetrahedron.** The angles of a triangle will now be expressed in terms of the sides. For the angle $C\,A\,B$ of the triangle $A\,B\,C$ we have

$$\text{Angle } C\,A\,B = \overline{C\,A \cdot A\,B} = \frac{(\phi\, A\,B \cdot F\,C\,A)}{(p\,A\,B)\,(p\,C\,A)}$$

$$= \frac{(\phi\, A)\,(F\,A\,B\,C)}{(p\,A\,B)\,(p\,C\,A)}.$$

Replacing F by $[\phi\, p]$ and applying (3) we have

$$\overline{C\,A,\,A\,B} = \frac{(\phi\, A)\,(\phi\, p \cdot A\,B\,C)}{(p\,A\,B)\,(p\,C\,A)}$$

$$= \frac{(\phi\, A)\,\{(\phi\, A)\,(p\,B\,C) + (\phi\, B)\,(p\,C\,A) + (\phi\, C)\,(p\,A\,B)\}}{(p\,A\,B)\,(p\,C\,A)}.$$

Dividing numerator by $(\phi\, A)^2\,(\phi\, B)\,(\phi\, C)$, this becomes

$$\overline{C\,A,\,A\,B} = \frac{\overline{B\,C} + \overline{C\,A} + \overline{A\,B}}{\overline{A\,B} \cdot \overline{C\,A}}$$

If we use A, B, C for the angles and a, b, c for the sides opposite this becomes

$$A = \frac{a + b + c}{b\,c} \quad . \quad . \quad . \quad . \qquad (26)$$

Similarly we have

$$a = \frac{A + B + C}{B\,C} \quad . \quad . \quad . \quad . \qquad (27)$$

In the tetrahedron if α, β, γ, δ are the planes opposite the vertices A, B, C, D we have for the angle

$$\overline{\alpha\,\beta} = \frac{(p\cdot B\,C\,D\cdot C\,D\,A)}{(F\,B\,C\,D)\;(F\,C\,D\,A)} = \frac{(p\,C\,D)\;(B\,C\,D\,A)}{(F\,B\,C\,D)\;(F\,C\,D\,A)}$$

$$= \frac{\overline{C\,D}\;\overline{B\,C\,D\,A}}{\overline{B\,C\,D}\;\overline{C\,D\,A}}.$$

This gives for the volume

$$\overline{B\,C\,D\,A} = \frac{\overline{B\,C\,D}\;\overline{C\,D\,A}\;\overline{\alpha\,\beta}}{\overline{C\,D}} \quad . \quad . \qquad (28)$$

That is the volume of a tetrahedron is equal to the product of the areas of two faces and the dihedral angle between them divided by the length of the common edge.

The trihedral angle $\overline{\alpha\,\beta\,\gamma}$ is given by

$$\overline{\alpha\,\beta\,\gamma} = \overline{\alpha\,\beta} + \overline{\beta\,\gamma} + \overline{\gamma\,\alpha}$$

$$= \frac{\overline{C\,D}\cdot\overline{B\,C\,D\,A}}{\overline{B\,C\,D}\cdot\overline{C\,D\,A}} + \frac{\overline{D\,A}\cdot\overline{C\,D\,A\,B}}{\overline{C\,D\,A}\cdot\overline{D\,A\,B}} + \frac{\overline{D\,B}\cdot\overline{A\,D\,B\,C}}{\overline{A\,D\,B}\cdot\overline{C\,D\,B}}$$

$$= \overline{B\,C\,D\,A}\left(\frac{\overline{D\,A\,B}\cdot\overline{C\,D} + \overline{B\,C\,D}\cdot\overline{D\,A} + \overline{C\,D\,A}\cdot\overline{D\,B}}{\overline{B\,C\,D}\cdot\overline{C\,D\,A}\cdot\overline{D\,A\,B}}\right). \qquad (29)$$

This formula solved for $\overline{B\,C\,D\,A}$ will also express the volume in terms of the trihedral angle and the three face triangles and three edges which meet at its vertex.

The volume can also be expressed in many other forms.

Linear distance in hyperspace.

22. The argument by which we derived the formula for the distance between two points in three dimensions applies without change to higher dimensions. The formula for distance is then always

$$\overline{A\,B} = \frac{(q\,A\,B)}{(\phi\,A)\,(\phi\,B)},$$

where q is a complex matrix of order $n - 2$ and ϕ a hyperplane. Similarly the angle between two hyperplanes is

$$\overline{\alpha\,\beta} = \frac{(p\,\alpha\,\beta)}{(F\,\alpha)\,(F\,\beta)},$$

where p is a complex matrix of order two and F a point. We wish these quantities to be invariant under the same group of collineations. This will happen if ϕ and q are determined by F and p and conversely. We shall therefore consider the system of complexes determined by a point F and a complex p of the second order. The details of this discussion depend somewhat on whether the space is of even or odd order. We consequently consider these cases separately.

23. **Space of order** $n = 2\,m$. The progressive products of a complex p with itself give a system of complexes $[p\,p]$, $[p\,p\,p]$ etc. we shall denote these by the symbols p^2, p^3 etc. In the present case p^m is represented by a sum of determinants of order n and hence is a scalar. We assume that this quantity is not zero. Such for example is the case if

$$p = A_1\,A_2 + A_3\,A_4 + \,.\,.\,.\,.\,.\, A_{2m-1}\;A_{2m}$$

and the points A_i do not lie in a hyperplane. For then

$$p^m = m!\;(A_1\,A_2\,.\,.\,.\,.\,.\,A_{2m}).$$

Since p^m is not zero none of the lower powers are zero.

We take as a fundamental system the quantities

$$F,\;\; p,\;\; F\,p,\;\; p^2\,.\,.\,.\,.\,F\,p^{m-1},$$

consisting of the powers of p and those powers multiplied by F. We shall find that this system forms a group under progressive and regressive multiplication, in the sense that the product of any two is either zero or a numerical multiple of a third in the system.

To form products it is sufficient to recall that p is a sum of products of two points and hence in *linear* (distributive) operations behaves like a simple product of two points. Furthermore to multiply regressively

R by a product of points S we take from S all combinations D of points such that D is complimentary to R, arrange the others in a product C such that $S = C\,D$, and form the sum $\Sigma\,(R\,D)\,C$. To obtain the product

$$[p^2 \cdot p^{m-1}]$$

by resolving the second factor, we must take the sum of products, of p^2 by all but two letters of any term of the second factor times the product of those two. Those letters will occur in a combination p^2 and this combination may be selected in $\binom{m-1}{2}$ ways. Hence

$$[p^2 \cdot p^{m-1}] = \binom{m-1}{2}[p^{m-1} \cdot p^2] - \left(2\binom{m-1}{2} - (m-1\right) p^m \cdot p,$$

the second term being subtracted because in

$$\binom{m-1}{2}[p^{m-1} \cdot p^2]$$

occur $2\binom{m-1}{2}$ terms of the form $p^m \cdot p$, whereas there should be $m-1$ in the expansion of $[p^2 \cdot p^{m-1}]$. Simplifying the above expression we get, since $[p^2 \cdot p^{m-1}] = [p^{m-1} \cdot p^2]$,

$$[p^{m-1} \cdot p^2] = 2\frac{m-1}{m} p^m \cdot p.$$

Similarly

$$[p^{m-1} \cdot p^r] = \binom{r}{2}[p^{m-1} \cdot p^2 \cdot p^{r-2}] - [2\binom{r}{2} - r]\, p^m\, p^{r-1}$$

$$= r\frac{(m-r+1)}{m} p^m \cdot p^{r-1}.$$

Since p^m is a scalar, we may solve this last equation for p^{r-1}. Changing r into $r+1$ in the result we have,

$$p^r = \frac{m\,[p^{m-1}\, p^{r+1}]}{p^m\,(r+1)\,(m-r)},$$

the equation holding for $r = 0$ if we take $p^0 = 1$. Thus we have an expression for p^r in terms of p^{r+1}. Expressing p^{r+1} in terms of p^{r+2}, etc., we have finally

$$p^r = \left(\frac{m\,p^{m-1}}{p^m}\right)^{m-r} \frac{p^m}{[(r+1)\,(r+2)\,.\,.\,.\,m]\,(m-r)!}.$$

If we choose the magnitude of p such that

$$p^m = m!$$

and let

$$\frac{p^{m-1}}{(m-1)!} = q,$$

this equation may be written

$$\frac{p^r}{r!} = \frac{q^{m-r}}{(m-r)!} \quad . \quad . \quad . \quad . \qquad (30)$$

where $r = 0, 1, \ldots . m$.

Again we have

$$[F\, p^m\,] = m\,[p^{m-1}\, F\cdot p]$$

and

$$[p^{m-1}\, F\cdot p^r] = r\,[(p^{m-1}\, F\cdot p)\, p^{r-1}].$$

Hence

$$[p^{m-1}\cdot F\, p^r] = F\,[p^{m-1}\, p^r] - [p^{m-1}\, F\cdot p^r] = \frac{r\,(m-r)}{m}\, p^m\,[F\, p^{r-1}].$$

Solving this for $[F\, p^{r-1}]$ and changing r into $r+1$, we have

$$[F\, p^r] = \frac{m\,[p^{m-1}\cdot F\, p^{r+1}],}{p^m\,(m-r-1)\,(r+1)},$$

a formula holding for $r = 0$. By continued application of this formula we finally get

$$\frac{[F\, p^r)}{r!} = \frac{[q^{m-r-1}\, F\, p^{m-1}]}{(m-r-1)!\,(m-1)!}.$$

Let

$$\frac{[F\, p^{m-1}]}{(m-1)!} = \phi.$$

Then

$$\frac{[F\, p^r]}{r!} = \frac{\phi\, q^{m-r-1}}{(m-r-1)!} \quad . \quad . \quad . \qquad (31)$$

where $r = 0, 1, 2, \ldots m-1$.

24. **Space of order** $n = 2\,m + 1$. In this case p^m is of order $n - 1$ and hence represents a hyperplane. Since the product p^m is progressive this product must contain p (i. e., p can be expressed as $\Sigma\, \lambda_{ik}\, [A_i\, A_k]$, the points A_i being contained in ϕ). Hence,

$$[p\cdot p^m] = 0.$$

We assume that

$$[F\,p^m] \neq 0.$$

Then p^r and $[F\,p^r]$, $r \overline{\leqq} m$, are not zero.

Since $[p \cdot p^m] = o$, there can be no terms in the expansion of $[p \cdot F\,p^m]$ which have F outside the parenthesis. Hence

$$p\,(F\,p^m) = m\,(F\,p^m)\,p + \binom{m}{2}[F\,p^{m-1} \cdot p^2] - 2\binom{m}{2}(F\,p^m)\,p.$$

Consequently

$$[F\,p^{m-1} \cdot p^2] = 2\,\frac{m-1}{m}\,(F\,p^m)\,p.$$

Similarly

$$[F\,p^{m+1} \cdot p^r] = \binom{r}{2}[F\,p^{m-1} \cdot p^2 \cdot p^{r-2}] - [2\binom{r}{2} - r]\,(F\,p^m)\,p^{r-1}$$

$$= \frac{r\,(m-r+1)}{m}\,(F\,p^m)\,p^{r-1}$$

Solving this for p^{r-1} and changing r into $r+1$, we obtain

$$p^r = \frac{m\,[F\,p^{m-1} \cdot p^{r+1}]}{(F\,p^m)\,(r+1)\,(m-r)}.$$

Repeated use of this formula gives finally

$$\frac{p^r}{r!} = \left(\frac{m\,F\,p^{m-1}}{F\,p^m}\right)^{m-r} \frac{p^m}{m!\,(m-r)!}.$$

If we choose the magnitudes of F and p such that

$$(F\,p^m) = m! \quad . \quad . \quad . \quad . \quad (A)$$

and let

$$\frac{F\,p^{m-1}}{(m-1)} = q. \quad . \quad . \quad . \quad . \quad (B)$$

$$\frac{p^m}{m!} = \phi \quad . \quad . \quad . \quad . \quad (C)$$

we have

$$\frac{p^r}{r!} = \frac{[q^{m-r}\,\phi]}{(m-r)!} \quad . \quad . \quad . \quad . \quad (32)$$

where $r = 0, 1, \ldots m$.

Letting $r = 0$ in (32) we have

$$m! = q^m \cdot \phi \quad . \quad . \quad . \quad . \quad . \quad (A')$$

Hence q^m is not zero. Now q^m is a point, and since q contains F as a factor, must be the point F. Also

$$1 = [F\,\phi].$$

Consequently

$$\frac{q^m}{m!} = F \quad . \quad . \quad . \quad . \quad . \quad (C')$$

again from (32) we have

$$\frac{[\phi\, q^{m-1}]}{(m-1)!} = p \quad . \quad . \quad . \quad . \quad (B')$$

The equations (A), (B), (C) and (A'), (B'), (C') show that F, p are related to ϕ, q in the same way that the latter are to the former. Hence

$$\frac{q^r}{r!} = \frac{[p^{m-r} \cdot F]}{(m-r)!} \quad . \quad . \quad . \quad . \quad (33)$$

where $r = 0, 1, 2 \; . \; . \; . \; m$.

The formulae (30), (31), (32), (33) show that the system of quantities p^r, $F\,p^r$ is generated in the same way from F, p or from ϕ, q. If the product of two of these quantities is progressive the factors can be associated and the result is either zero or equal to a third. If the product is regressive we replace p^r and $F\,p^r$ by their expressions in terms of ϕ and q. The product in this form is represented by a sum of matrices (in hyperplane coordinates) having a smaller number of rows than columns. The factors can therefore be commuted and associated giving a result which is number times a quantity of the form q^r or $\phi\, q^r$. Hence the product of any two quantities of the fundamental system is a numerical multiple of a third.

Let

$$\left.\begin{aligned} S_{2r} &= \frac{p^r}{r!} \\ S_{2r+1} &= \frac{F\,p^r}{r!} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (34)$$

where $r = 0, 1, 2, \; . \; . \; . \; m$. Dually we have the quantities σ_i such that,

$$\left.\begin{aligned} \sigma_{2r} &= \frac{q^r}{r!} \\ \sigma_{2r+1} &= \frac{\phi q^r}{r!} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (35)$$

where $r = 0, 1, 2, \ldots m$. Equations (30), (31), (32), (33) show that

$$\sigma_i = S_{n-i} \quad . \quad . \quad . \quad . \quad . \quad (36).$$

25. **Distance and angle.** The distance between two points A, B is

$$\overline{A\,B} = \frac{(q\,A\,B)}{(\phi\,A)\,(\phi\,B)}.$$

Similarly we define the angle between any two spaces R, T of the same order r by the equation

$$\overline{R\,T} = \frac{(q \cdot \sigma_{r-1}\,R \cdot \sigma_{r-1}\,T)}{(\sigma_r\,R)\,(\sigma_r\,T)}.$$

σ_{r-1} being the complex which multiplied by R and T respectively give points. This expression can be put into two other forms which we shall now obtain. We consider three cases depending on the form of σ_{r+1}.

(1) If
$$\sigma_{r+1} = \frac{p^k}{k!},$$
we have
$$[\sigma_{r+1}\,R] = \frac{[p^k \cdot R]}{k!} = \frac{[p \cdot p^{k+1}\,R]}{(k-1)!} = [p \cdot \sigma_{r-1}\,R].$$
Then
$$(p \cdot \sigma_{r-1}\,R \cdot \sigma_{r-1}\,T) = (\sigma_{r+1}\,R \cdot \sigma_{r-1}\,T).$$

(2). If
$$\sigma_{r+1} = \frac{[F\,p^k]}{k!},$$
then
$$[\sigma_{r+1}\,R] = \frac{(p^k\,R)\,F}{k!} - \frac{[p \cdot F\,p^{k-1}R]}{(k-1)!}.$$

Since in this case σ_{r-1} also contains F, we have

$$(p \cdot \sigma_{r-1}\,R \cdot \sigma_{r-1}\,T) = -(\sigma_{r+1}\,R \cdot \sigma_{r-1}\,T).$$

In both of the preceding cases

$$(p \cdot \sigma_{r-1}\,R \cdot \sigma_{r-1}\,T) = (-1)^{r+1}\,(\sigma_{r+1}\,R \cdot \sigma_{r-1}\,T).$$

(3) If σ_{r+1} is of the form $\frac{q^k}{k!}$ or $\frac{[\phi q^k]}{k!}$ by the dual of the preced-

ing reasoning we have, since the sign must be positive in the first and negative in the second case

$$(\sigma_{r+1} R \cdot \sigma_{r-1} T) = (-1)^{n-r-1} (\sigma_{r+1} R \cdot q \cdot \sigma_{r+1} T)$$
$$= (-1)^{r+1} (q \cdot \sigma_{r+1} R \cdot \sigma_{r+1} T).$$

For every case the following equation holds:

$$\overline{R\,T} = \frac{(p \cdot \sigma_{r-1} R \cdot \sigma_{r-1} T)}{(\sigma_r R)\,(\sigma_r T)} = \frac{(q \cdot \sigma_{r+1} R \cdot \sigma_{r+1} T)}{(\sigma_r R)\,(\sigma_r T)} = (-1)^{r+1} \frac{(\sigma_{r-1} R \cdot \sigma_{r-1} T)}{(\sigma_r R)\,(\sigma_r T)}.$$

It is evident from the definition that

$$R\,T = -\overline{T\,R}.$$

This together with the linearity of the expression, the factored form and symmetry of the denominator, shows that three spaces R, R', R'' of a pencil determine angles such that

$$\overline{R\,R'} + \overline{R'\,R''} + \overline{R''\,R} = 0.$$

To prove this directly it is only necessary to place

$$R'' = \lambda\,R + \mu\,R'$$

in the expression for the above sum and clear of fractions.

26. **Distance and angle in a section of hyperspace.** A space R of our space of order n intersects the complexes S_i of the fundamental system in a set of complexes. For spaces contained in R we can define distance and angle relative to these last complexes. We wish now to show the relation between those invariants and the corresponding invariants relative to the complexes S_i.

First consider the section made by a hyperplane α. This determines with the complex p a point

$$F_1 = [\alpha\,p],$$

and with the complex $[F\,p]$, a complex

$$p_1 = [\alpha \cdot F\,p].$$

We can write this last expression in the form

$$p_1 = (\alpha\,F)\,p - [\alpha \cdot p \cdot F].$$

If we multiply this by itself r times, since the last term is a line, this last term cannot appear more than once as a factor of any term of the result. Hence

$$p_1^r = (\alpha\, F)^r\, p^r - r\,(\alpha\, F)^{r-1}\,[\alpha\cdot p\cdot F\cdot p^{r-1}].$$
$$= (\alpha\, F)^{r-1}\,\{(\alpha\, F)\, p^r - r\,[\alpha\cdot p\cdot F\cdot p^{r-1}]\}.$$
$$= (\alpha\, F)^{r-1}\,[\alpha\cdot F\, p^r],$$

provided that $2\,r + 1 \overline{\gtrless}\, n$. Multiplying the first of these values of p_1^r by $F_1 = [\alpha\, p]$, since $[\alpha\, p]$ is already a factor of the second term, we get

$$[F_1\, p_1^r] = (\alpha\, F)^r\,[\alpha\, p\cdot p^r] = \frac{(\alpha\, F)^r\,[\alpha\cdot p^{r+1}]}{r+1},$$

provided that $2\,r + 2 \overline{\gtrless}\, n$. Dividing the above expressions for p_1^r and $[F_1\, p_1^r]$ by $r!$, we get

$$\left.\begin{aligned} \frac{p_1^r}{r!} &= \frac{(\alpha\, F)^{r-1}\,[\alpha\cdot F\, p^r]}{r!} \\ \frac{[F_1\, p_1^r]}{r!} &= \frac{(\alpha\, F)^r\,[\alpha\cdot p^{r+1}]}{(r+1)!} \end{aligned}\right\} \quad . \quad . \quad . \quad (37)$$

these expressions being valid if the order of the left side is equal to or less than that of a hyperplane.

We next find the intersection of a second hyperplane β with the system of complexes p_1^r, $[F_1\, p_1^r]$. Let

$$F_2 = [\beta\, p_1] = [\beta\,\alpha\cdot F\, p]$$

$$p_2 = \frac{[\beta\cdot F_1\;\, p_1]}{(\alpha\, F)} = \frac{[\beta\;\,\alpha\cdot p^2]}{2!}.$$

By the same argument as before we get

$$\frac{p_2^r}{r!} = \frac{(\beta\, F_1)^{r-1}[\beta\cdot F_1\, p_1^r]}{(\alpha\, F)^r\; r!}$$

$$\frac{[F_2\, p_2^r]}{r!} = \frac{(\beta\, F_1)^r\,[\beta\cdot p_1^{r+1}]}{(\alpha\, F)^r\;(r+1)!}.$$

Using the values of p_1 and $[F_1\, p_1^r]$ in (37) we have

$$\frac{p_2^r}{r!} = \frac{(\beta\,\alpha\, p)^{r-1}\,[\beta\;\alpha\cdot p^{r+1}]}{(r+1)!},$$

$$\frac{[F_2\, p_2^r]}{r!} = (\beta\;\alpha\;\, p)^r\,\frac{[\beta\,\alpha\cdot F\, p^{r+1}]}{r!},$$

these expressions being valid if the order of the left member is equal to or less than that of $[\beta\ \alpha]$. Similarly we obtain the intersection of this system with a hyperplane γ, etc. We thus get finally.

$$F_\lambda = [R\ S_{\lambda+1}],$$
$$p_\lambda = [R\ S_{\lambda+2}].$$

as point and complex in a space R of order $n - \lambda$. For these we have the equations

$$\left.\begin{aligned} R_{2r} &= \frac{p_\lambda{}^r}{r!} = (R\ S_\lambda)^{r-1}\,[R \cdot S_{2r+\lambda}] \\ R_{2r+1} &= \frac{[F^n p_\lambda{}^r]}{r!} = (R\ S_\lambda)^r [R \cdot S_{2r+\lambda+1}] \end{aligned}\right\} \quad . \qquad (38)$$

these expressions being valid for values of r such that the orders of the left members are equal to or less than $n - \lambda$.

If $(R\ S_\lambda) \neq 0$ we choose the magnitude of R such that

$$(R\ S_\lambda) = 1.$$

Then the above equations become

$$R_m = [R \cdot S_{m+\lambda}] \quad . \quad . \quad . \quad . \qquad (39)$$

We consequently have

$$\frac{[R_{n-\lambda-2} \cdot A\ B]\, R}{[R_{n-\lambda-1}\ A]\,[R_{n-\lambda-1}\ B]} = \frac{[R\ S_{n-r} \cdot A\ B]\, R}{[R\ S_{n-1} \cdot A]\ [R\ S_{n-1} \cdot B]}$$
$$= \frac{[A\ B \cdot R\ S_{n-2}]\, R}{[A \cdot R\ S_{n-1}]\ [B \cdot R\ S_{n-1}]} = \frac{R\,(A\ B\ S_{n-2})\, R}{R\,(A\ S_{n-1})\, R\,(B\ S_{n-1})}$$

provided that $A\ B$ is contained in R.[12]

Hence we have

$$\overline{A\ B} = \frac{(S_{n-2} \cdot A\ B)}{(S_{n-1}\ A)\ (S_{n-1}\ B)} = \frac{[R_{n-\lambda-2}\ A\ B]\, R}{[R_{n-\lambda-1}\ A]\ [R_{n-\lambda-1}\ B]}.$$

We may consider R as the unit quantity in the space R. Then the right side of the above equation is the expression for distance relative to the system of complexes in R. Thus whether we take distance in R relative to the fundamental system of complexes S_i or relative to the sections R_i in R, the result is the same. Similar relations of the angles between other spaces in R relative to S_i and R_i can be shown.

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12 Cf. Grassmann, Gesammelte Werke, Vol. I, theil 2, page 91.

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CONTRIBUTIONS FROM THE CRYPTOGAMIC LABORATORIES OF HARVARD UNIVERSITY.—No. LXVIII.

PRELIMINARY DIAGNOSES OF NEW SPECIES OF CHAETOMIUM.

By A. H. Chivers.

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PRELIMINARY DIAGNOSES OF NEW SPECIES OF CHAETOMIUM.

By A. H. Chivers.

Presented by R. Thaxter. Received, June 16, 1912.

For a considerable time the writer has been engaged in the preparation of an illustrated monograph of the genus Chaetomium but owing to unavoidable interruptions, and delay caused by the preparation of plates, he has thus far been obliged to defer a final publication.

At the time when this work was begun, the only comprehensive paper on the subject was the well known monograph of Zopf (Nova Acta Acad. Leop.-Carol. 42. 1881), but after it was well under way a paper by Bainier appeared in the Bull. de la Soc. Myc. de France (Vol. XXV. Fasc. 4. p. 191. 1910) in which a considerable number of new forms were described and illustrated, some of which prove to be American. Up to the present time, however, there has been no further attempt to make a comprehensive review of the genus or to collate the American forms with the exception of the revision of the Chaetomiaceae in volume III of the "Flora of North America" by H. L. Palliser, who enumerates seventeen species including three unpublished names.

In the course of his work upon these widely distributed fungi the writer has been able to examine a very large series of specimens from various herbaria and exsiccati, and to cultivate many species from diverse sources on various media and through many successive generations. As a result of this examination numerous forms have been added to those previously recorded from America, and a number of new species have been recognized of which it seems desirable to publish the following preliminary diagnoses. In this connection it may be mentioned that all of these forms with two exceptions have been extensively cultivated in a pure condition and that it has been possible to determine with accuracy their range of variation as well as their salient, specific characteristics.

Chaetomium subspirale, sp. nov.

Griseum vel roseo-griseum. Peritheciis majoribus, longioribus, $314 \times 213\ \mu$ ($300-337 \times 206-224$), sporidiis irregulariter conglobatis gerentibus; pilis lateralibus numerosis, tenuibus, regulariter et distincte septatis, levibus, basi rectis, apice arcte spiraliter convolutis; pilis terminalibus tenuibus, obscure septatis, pallide-olivaceis, levibus, primum arcte dein laxe spiraliter convolutis; ascis clavatis, octosporis, $45 \times 9.7\ \mu$, p. sporif. $24\ \mu$; sporidiis subdistichis, pallide olivaceis, limoniiformibus, utrinque apiculatis, $6.4 \times 5.2-5.6\ \mu$.

Frequent in cultures of various substrata from New England. Appearing in cultures of dung from Holland and South America.

The species may be distinguished by its characteristic hairs; the lateral ones of which are short, straight, dark below; tightly coiled, hyaline and refractive at the tips; the terminal slender, at first tightly coiled in a delicate spiral, later elongated, twisted rather than coiled and giving the appearance of wooly threads.

Chaetomium sphaerale, sp. nov.

Griseo-flavis, olivaceo-flavis, aetate aureo-flavis. Peritheciis majoribus, subglobosis, basi rotundatis, apice subconstrictis, $312 \times 276\ \mu$ ($300-329 \times 262-300$), sporidiis regulariter conglobatis gerentibus vel cirrhis instructis; pilis lateralibus numerosis, gracilibus, levibus, regulariter et distincte septatis, successive olivaceis, aureoflavis, pallide flavis, hyalinis, apice collabentibus; aliis subrectis, longioribus, 1–2-ramosis, basi $3.7\ \mu$ diam., aliis flexuosis, brevioribus, non ramosis, basi $2.8\ \mu$ diam.; pilis terminalibus longis, gracilibus, pilis lateralibus concoloribus, levibus, irregulariter flexuosis vel subspiraliter convolutis, 1–5-ramosis, basi distincte septatis, apice obscure septatis vel subcontinuis; ascis clavatis, octosporis, $48 \times 13\ \mu$, p. sporif. $26\ \mu$; sporidiis subdistichis, dense olivaceo-brunneis, utrinque umbonatis, limoniiformibus, $7.3-8.1 \times 6.4\ \mu$.

In a culture of caterpillars from Reading, Mass.

The perithecium, globose below, with a tendency to narrow above into a neck, distinguishes this species from all others which the writer has studied. The slender delicate hairs and the entire absence of differentiated rhizoids are also significant characteristics.

Chaetomium quadrangulatum, sp. nov.

Griseum. Peritheciis majoribus, longioribus, 403 × 294 μ (333–456 × 243–350), cirrhis longissimis instructis; pilis lateralibus numerosis, tenuibus, rectis, regulariter et distincte septatis, basi olivaceo-fuscis, asperulis vestitis, apice hyalinis, levibus; pilis terminalibus biformibus, aliis spiraliter convolutis, irregulariter pauciseptatis, asperulis vestitis, basi olivaceo-brunneis vel atris, apice dilute coloratis, aliis subrectis, undulatis vel convolutis, irregulariter pauciseptatis, asperulis vestitis, ramosis, basi olivaceo-brunneis vel atris, apice dilute coloratis; ascis clavatis, octosporis, 39 × 9.7 p. sporif. 21 μ; sporidiis pallide olivaceis, a fronte visis subquadrangulatis, a latere ovatis, 7.3 × 6.3 μ (6.4–8 × 5.6–6.4).

Cultivated on dung from Cambridge, Mass. Appearing also on dung from Chile and from Little Swan Island, Gulf of Mexico (R. Thaxter).

The species may be easily identified by the spores which, when seen in face view, are four sided and four angled but, when seen in profile, are oval. *Chaetomium quadrangulatum* and *Chaetomium trigonosporum* are the only species known to the writer which possess spores with angles, the former having spores clearly quadrangular, the latter clearly triangular.

Chaetomium convolutum, sp. nov.

Cyano-griseum. Peritheciis magnitudine mediis, globosis, 244 × 232 μ (236–254 × 224–240), cirrhis instructis; pilis lateralibus paucis, gracilibus, rectis, regulariter et distincte septatis, basi olivaceo-flavis, asperulis vestitis, apice hyalinis, sparse asperulis vestitis; pilis terminalibus undique asperulis vestitis, olivaceo-atris, subcontinuis vel irregulariter pauciseptatis, 8–10 spiraliter convolutis, ad ipsam apicem convolutionibus terminalibus regulariter successive minoribus; ascis clavatis, octosporis, 56.4 × 10 μ, p. sporif. 27.4 μ; sporidiis pallide olivaceis, ovatis vel limoniiformibus, utrinque obtusis, subapiculatis, 8–8.4 × 6.4 μ.

Cultivated on mouse dung from Germany.

Apparently a rare species having appeared but once. The species may be identified by the distinct blue color of the plant when seen with the naked eye or a hand lens, and by the long spreading terminal hairs whose long series of coils taper abruptly to a blunt point.

Chaetomium spinosum, sp. nov.

Aureo-flavum. Peritheciis magnitudine mediis, subglobosis, 290–224 μ (273–318 × 206–262), cirrhis instructis; pilis lateralibus numerosis rectis, rigidis, acutis, irregulariter et parum distincte septatis, basi atrobadiis, asperulis vestitis, apice hyalinis, levibus; pilis terminalibus rectis, rigidis, acutis, asperis vestitis, ramosis, ramis ramulisque dilute olivaceis; ascis clavatis, octosporis, 41 × 7.5 μ, p. sporif. 22 μ; sporidiis subdistichis, pallide olivaceis, oviformibus, 5.9 × 3.9 μ (5.6–6.4 × 3.2–4).

Growing in cultures of dung from Buenos Ayres (R. Thaxter).

This is, apparently, a rare species having appeared but once. The egg-shaped spores and the branched terminal hairs are peculiar to the species. From the dark, stiff, spine-like shafts or the terminal hairs arise slender, delicate, irregularly swollen and constricted branches, from which secondary branches arise. As the cirrhus of spores forms the branches rise in the form of a column and in this way a support is formed for the spore mass.

Chaetomium ampullare, sp. nov.

Ochraceum. Peritheciis majoribus, longissimis, 489 × 147 μ (456–532 × 137–167), sporidiis irregulariter conglobatis gerentibus; pilis lateralibus paucis, gracilibus, regulariter et distincte septatis, basi rectis, olivaceo-fuscis, asperulis vestitis, apice collabentibus, levibus; pilis terminalibus longis, gracilibus, distincte et regulariter septatis, successive aureo-brunneis, aureo-flavis, hyalinis; levibus, ramosis, in fila hyalina elongatis; ascis clavatis, octosporis, 45 × 9.7 μ, p. sporif. 23 μ; sporidiis subdistichis, laete olivaceo-flavis, utrinque umbonatis, limoniiformibus, 8.1–8.9 × 6.4 μ.

On culture of sail cloth from Lowell, Mass. On dung from North Carolina (R. Thaxter).

The species is clearly characterized by the very much elongated bottle-shaped perithecium, and by the terminal hairs which are drawn out into long, hyaline, tangling, easily collapsing threads.

Chaetomium aureum, sp. nov.

Griseum, pallide-olivaceum, lutescens, demum aureo-flavum. Peritheciis minutis, globosis, 127 × 115 μ (110–140 × 105–123), cirrhis instructis, pilis lateralibus numerosis, tenuibus, rectis vel flexuosis, regulariter et distincte septatis, olivaceo-flavis, asperulis vestitis; pilis terminalibus olivaceo-flavis, regulariter septatis, asperulis vestitis, arcuatis, apice subrectis vel incurvatis; ascis clavatis, octosporis, $42 \times 10\ \mu$, p. sporif. $26\ \mu$; sporidiis olivaceo-brunneis, irregulariter ovatis, utrinque apiculatis, $9.8 \times 5.4\ \mu$ ($9.4–11 \times 4.7–5.6$).

On paper, dung and other materials of various kinds from New England. In cultures of old paper from Java (R. Thaxter).

The small size and characteristic golden yellow color clearly distinguish this species from all others except *Chaetomium trilaterale* and *Chaetomium fusiforme*. From the former of these it differs in that the spores are discharged in long black cirrhi, in the comparative obscurity of the perithecial hairs at maturity, in the incurved tips of the terminal hairs, and in the irregular, oval shape of its spores. From the latter it differs also in producing long black cirrhi, in the incurved extremities of its terminal hairs, and in the size of its spores and their irregular oval shape.

Chaetomium fusiforme, sp. nov.

Griseum vel pallide olivaceum. Peritheciis minutis, subglobosis, $120 \times 102\ \mu$ ($116–123 \times 101–125$), cirrhis carentibus; pilis lateralibus numerosis, tenuibus, flexuosis, regulariter et distincte septatis, olivaceo-flavis, asperulis vestitus; pilis terminalibus crassioribus, asperulis vestitis, olivaceo-brunneis, regulariter et distincte septatis, arcuatis, apice circinantibus vel subconvolutis; ascis clavatis, octosporis, $48 \times 11\ \mu$, p. sporif. $32\ \mu$; sporidiis laete olivaceo-flavis, vel olivaceo-brunneis, longis, angustis, subfusiformibus, apice rotundatis vel apiculatis, $15.8 \times 5.4\ \mu$ ($15–16 \times 4.8–5$).

On paper from Alabama (Herb. R. Thaxter).

The long narrow spores distinguish this form from all other species of Chaetomium. In general characteristics it most nearly resembles *Chaetomium aureum* and *Chaetomium trilaterale*, but differs from both in the long, slender, fusiform spores.

Chaetomium trilaterale, sp. nov.

Olivaceo-flavum. Peritheciis minutis, subglobosis, $106 \times 94\ \mu$ ($100–110 \times 90–97$), cirrhis carentibus; pilis lateralibus numerosis, gracilibus, longioribus, regulariter et distincte septatis, aureo-flavis, basi rectis, asperulis vestitis, apice 1–3 spiraliter convolutis, levibus; pilis terminalibus irregulariter septatis, olivaceo-brunneis, asperulis vestitis, arcuatis, apice 1–3 spiraliter convolutis; ascis clavatis, octosporis, $50 \times 9.5\ \mu$, p. sporif. $26\ \mu$; sporidiis subdistichis, laete olivaceo-flavis, forma sphaerasectoris praeditis, utrinque subapiculatis, $9.5 \times 5.5\ \mu$ (8.9–9.7×5.2–6).

On paper from New England (Herb. R. Thaxter).

This species has certain characteristics in common with *Chaetomium aureum* and *Chaetomium fusiforme.* From the former it differs in the more numerous, stout, 1–3 spirally convolute, terminal hairs; the spirally coiled lateral hairs; the smaller size and unusual shape of the spores. From the latter it differs in the convolute lateral hairs; the shape of its spores and their smaller size.

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A STUDY WITH THE ECHELON SPECTROSCOPE OF CERTAIN LINES IN THE SPECTRA OF THE ZINC ARC AND SPARK AT ATMOSPHERIC PRESSURE.

By Norton A. Kent.

With two Plates.

Investigations on Light and Heat made and published with Aid from the Rumford Fund.

A STUDY WITH THE ECHELON SPECTROSCOPE OF CERTAIN LINES IN THE SPECTRA OF THE ZINC ARC AND SPARK AT ATMOSPHERIC PRESSURE.

By Norton A. Kent.

Presented by Charles R. Cross. Received June 19, 1912.

In November, 1907, the writer published, in collaboration with one of his graduate students, an article [1] attempting to meet certain objections made by Keller [2] to the method of procedure adopted by the writer in certain former work [3] upon the question of the relative wave-lengths of certain lines in the spectrum of titanium and zinc as developed by the arc and spark discharge in air at normal pressure. That displacements of the spark lines to the red from the position of the corresponding arc lines actually existed on the photographic plates obtained, is regarded by the writer as unquestionably proven. It is certain, also, that the displacements were not due to any incorrect experimental procedure.

It appeared to be worth while to study the matter further, seeking to ascertain, if possible, the cause of these displacements. As the echelon spectroscope had revealed structure in the lines of metallic spectra both in Plücker tubes and in the arc in vacuo and at normal atmospheric pressure[4], it seemed advisable to use this instrument to study the spark, noting the change in the form of the image as a function of the constants of the electric circuit. The titanium lines $\lambda\lambda$ 3900 and 3913, formerly studied in detail, presented difficulties because of their short wave-lengths; therefore, it appeared best to concentrate the work upon zinc.

A brief survey of the most important results in the case of this metal recently obtained by various observers is thus in order.

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1 These Proceedings, **43**, No. 11, Nov. (1907).

2 Ueber die angeblicke Verschiebung der Funkenlinien, Inaugural Dissertation, Christian Keller.

3 These Proceedings, **41**, No. 10, July (1905).

4 Janicki, Annalen der Physik, **19**, 36–79, Jan. (1906).
Nutting, Astrophysical Journal, **23**, No. 1, Jan. (1906)
Nutting, Bulletin Bureau of Standards, **2**, No. 3, Dec. (1906).

Historical Survey.

Houston [5] who notes the changes which take place in the reversal system as seen by an echelon when a zinc arc "begins to hiss," speaks of the "striking forms of reversal," the distances between the different components in the line varying in the different parts of the arc. With one of his arcs and a small amount of vapor, he obtained the three blue lines of zinc "without reversals." Under certain conditions the three blue lines were "all doublets with components of equal intensity."

Janicki [6] in his inaugural dissertation (**1905**) states that "an examination by the echelon of the lines of the zinc spectrum developed in a capillary tube of 0.3 mm. diameter with external electrodes at a temperature of about 460° showed them to be single lines."

Nutting,[7] in a paper on line structure, mentions the fact that Plücker tube spectra of rarefied gases moderately excited show narrow lines of the simplest structure, "but with a heavy current or capacity in parallel, if the pressure be greater than 3 or 4 mm. the lines broaden, and finally, with a spark in series with the tube, widen into a continuous spectrum, with the peculiar fluted appearance characteristic of spark lines."

He states further that "sparks between metallic electrodes give lines far too broad for use as monochromatic sources. They are never less than half a tenth-meter broad. The effect appears to depend chiefly upon the amount of capacity used, and is greatly heightened by the use of another spark in series; that is, *it is due to the steepness of the wave-front of the current wave.*[8] Inductance weakens the wings produced by capacity, and sometimes channels them, but never reduces a line to a simple structure. Occasional lines will appear to simply broaden out under the violence of the discharge, but ordinarily it is simply a case of the dark background — between spectra of different order — becoming luminous."

"Using a small current (0.02 amp.) of low voltage (5000) and low frequency (60) and a minimum of capacity, and electrodes of iron and brass, the spark lines were found to be still broad and diffuse. Lines due to impurities (sodium, for example) occasionally appear

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5 Philosophical Magazine, **7**, May (1904).
6 See Annalen der Physik, **19**, 36–79, Jan. (1906).
7 Astrophysical Journal, **23**, No. 1, Jan. (1906).
8 The italics are the writer's.

fairly sharp on but a faint background, but a number of tests indicated that it is impracticable to obtain narrow lines by introducing impurities into the spark."

Further, when discussing arc spectra in general, he writes: "The structure which a line exhibits depends primarily upon its intensity; that is, upon the amount of a substance vaporized and the intensity of its excitation in the arc"; and specifically, in the case of zinc:

"All four zinc lines are rather diffuse, and are usually found double or triple.* * * The blue lines, 4810, 4722, 4680, are broad and diffuse, and show a trace of structure on reversal."

In a general discussion attention is called to the fact that the structure of any one line is very variable, so much so that "we may hardly speak of any line as having a fixed definite structure, even with a minute specification of conditions of production."

Types of lines are classified according to structure and behavior, and the general conclusion drawn that to explain certain types — lines which, when single, under some conditions become double or triple, symmetrically or unsymmetrically, with receding components of various relative intensities — the old absorption theory of reversal is not satisfactory.[9]

In another paper [10] covering the results of a search for intense and yet "pure" light standards, Nutting, sketching the development of the typical normal line in either the open air arc or at pressures less than atmospheric, states:—"with increase of intensity the line broadens, and finally separates into two; * * * with further increase the two components continually broaden and separate"; and of highest "rank as to purity are the composite lines produced in the vacuum tubes measured between extreme components."

In a paper [11] on relative intensities of spectrum lines an attempt is made to show that the changes produced in spectra by varying current, capacity, inductance, temperature and pressure, may be accounted for by a single variable, or at most, two. He writes:—

"Several years ago the writer [12] gave the steepness of the wave-front through a gas as condition for the preponderance of the secondary over the primary spectrum. Crew [13] almost at the same time con-

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9 Nutting advances a theory of broadening, doubling and reversal in the Astrophysical Journal of April (1906).

10 Bulletin Bureau of Standards, **2**, No. 3, Dec. (1906).

11 Nutting, Astrophysical Journal, **28**, 66 (1908).

12 Astrophysical Journal, **20**, 135 (1904).

13 Ibid., **20**, 284 (1904).

cluded that a 'high E. M. F., rapidly changing, is a probable *conditio sine qua non* for the appearance of spark lines in arc spectra.' Both might better have expressed their results in terms of potential gradient." * * * "The lowest gradients are obtained in heavy current arcs and Plücker tubes with wide capillary; in the former case the low gradient is due to the heavy current, in the latter to low gas-pressure. Higher potential gradients are obtained in arcs with very small current, Plücker tubes with fine capillaries and sparks with small capacity and large inductance. The highest potential-gradients are found in sparks and other interrupted arcs, the gradient increasing with the amount of capacity in circuit and with the impressed voltage. Gradients vary from about 20 to 80 volts per cm. in ordinary arcs and tubes up to thousands of volts per cm. in condensed sparks." * * * "Inductance reduces the gradient down to a minimum, beyond which it is inoperative." * * * "In the condensed spark without inductance, the front of the pilot discharge must have a potential-gradient not much below the dielectric strength of the intervening gas. The remainder of the discharge is probably at a very low gradient, approaching that of a direct-current arc. Hence such a spark gives both spark and arc lines. Inductance and resistance lower maximum gradients by smoothing out the current wave. The spectrum of a spark rendered dead beat by series resistance can scarcely be distinguished from that of a low direct-current arc."

In 1909 Janicki [14] writes on the structure of spectrum lines, giving the results of a study made with the Lummer-Gehrcke plate, the source being an arc at low pressure (0.1 mm. or less) in a special form of apparatus having an anode of the desired metal.

The three zinc lines in the blue are described as sharp and simple. They appeared at 0.3 amp., were good at 0.4 amp., and at more than 0.7 amp. were reversed in part.

In certain calcium lines the change of position of their satellites with increase of current is noted, and attention called to an unsymmetrical broadening and reversal. Somewhat later reference is made to the work of Exner and Haschek on the displacement of spark lines.

"They traced these displacements, directed mostly toward longer wave-lengths, to the different density of the metallic vapor. With good reason Eder and Valenta objected that these displacements were only apparent. * * * They photographed arc and spark lines im-

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[14] Annalen der Physik — Band 29 (1909).

mediately above one another with different exposure times. The long exposures seem to give a different center of intensity from the short, if a line is unsymmetrically broadened to one side; whereas on the other hand the real center remains clearly in the same position only in the case of sufficiently short exposures. The long and short exposures play the same rôle, however, as a greater or smaller density of metallic vapor; therefore the shifts observed by Exner and Haschek are to be considered only as apparent. Exner and Haschek then tried to maintain their theory by referring the cause of the shifts to changeable satellites, which cannot be resolved by a Rowland grating and might therefore produce a shift. They studied the arc lines of a series of elements by means of a 15 plate echelon and made the astonishing discovery that a satellite often appeared upon the red side of the line, especially when the arc flickered. With the plane parallel plates at my disposal, which are more efficient than a 15 plate echelon, I have been unable to verify the satellites which they reported." * * * "It is possible that the satellites seen by Exner and Haschek with the flickering of the arc arose from impurities in the carbon and the metal. It is more probable, however, that they must be regarded as ghosts. Ca λ 4527 is supposed to be simple, but with a satellite arising on the side of greater wave-length upon the flickering of the arc; whereas I found no satellite near this strong line. On the contrary, I observed a weak satellite of greater wave-length near Ca λ 4586, while Exner and Haschek did not. Ca λ 5270 is supposedly a triplet, in which with weak current the middle line is the brightest; with strong current the two lines toward the red are the brightest. All my photographs show this very strong line to be single; furthermore, Cu λ 5218 is supposed to have a red companion which grows more rapidly than the head-line as the current is increased; I always found this very strong line to be single. This very line seems to me proof that Exner and Haschek were deceived by ghosts in their echelon. For if the head-line is not very strong, the ghost can scarcely be seen; if the main line becomes stronger, the ghost comes out more strongly; with further increase in intensity, the main line, however, seems to gain less rapidly than the ghost, since the eye (Exner and Haschek make visual observations only) cannot distinguish differences in great intensities so accurately as in the case of small ones. Nutting has also used the ordinary arc for creating spectrum lines and worked with an echelon of 30 plates, of $1\frac{1}{2}$ cm. thickness. The same remarks as above made are valid in case of the use of the carbon arc."

Janicki reviews Nutting's results, characterizes them as extraordinary; states that they should have aroused Nutting's suspicion and regards them as due to ghosts which become visible when the intensity of the source is sufficiently great. He writes: —

"Thus, according to Nutting, the red *Cd* line, the red and the blue *Zn* lines form triplets; whereas, even with the greatest intensity and the most varied sources of development, it is just these very lines that have always been found to be unquestionably single by Michelson, Fabry and Perot, Hamy, Gehrcke and van Baeyer, and myself. * * * Nutting's echelon had about the resolving power of the plane parallel plate C and did not approach that of plate H, so that the objection cannot be raised that he was able to make closer observations by reason of having a finer instrument. According to him all five prominent silver lines are compound, and indeed, both triple and quadruple, while the plate H even with the greatest intensity shows no sign of satellites. * * * The characteristic line-structure remains the same, no matter how the spectrum is produced. This is confirmed by the agreement of the observations of the lines of *Cd* and *Zn*, where it makes absolutely no difference with whatever instrument one observes and no matter how the spectrum is produced. * * * That the designation of the brightness of the satellites sometimes varies, as in *Cd* λ 4800, is immaterial, since the satellites are weak and the differences in their intensity very slight."

Here follows a discussion of unsymmetrical broadening noted with the Rowland grating by Kayser, Rowland and others. The statement is made that "a good Rowland grating would not resolve an unsymmetrical reversal the components of which, like the chromium line, are 0.043 Ångstrom units apart, and the resultant apparent shift about 0.02 Ångstrom units." There follows a reference to the work of the writer who, with Avery, made certain measurements upon two titanium lines. He writes: —

"They found an average shift of 0.019 and 0.018 Ångstrom units for the two titanium lines λλ 3900.7 and 3913.6. In the mean taken from both observers, the minimum and maximum shifts for the line λ 3900.7 are found to be 0.009 and 0.038 Ångstrom units. This very circumstance seems to me to indicate that Kent and Avery were dealing here with unsymmetrical reversals like those of chromium and calcium, reversals which their grating would not resolve and which appeared to them as line-shifts."

General Description and Arrangement of Apparatus.

An echelon spectroscope and a constant deviation spectroscope of the Hilger pattern were ordered of A. B. Porter of the "Scientific Shop," the echelon having 33 plates, a 1 mm. step, 34 mm. height of plate and about 15 mm. thickness, and the lenses of the constant deviation and echelon spectroscopes being of $1\frac{3}{4}''$ and $2''$ diameter and $17''$ and $20\frac{3}{4}''$ focal length, respectively. The constant deviation prism proved to be of insufficient aperture to fill the echelon, and was therefore sent to Hilger for a new prism.[15]

The echelon itself finally appeared to be a poor instrument and wholly unfitted for first-class work; for, upon final adjustment, the green mercury line λ 5471 showed a false pattern and there also appeared in certain zinc spark lines a distinct pattern which the writer, in view of the false satellites in the mercury line, at first deemed likewise spurious, inasmuch as a smaller and less powerful echelon made by Petitdidier, and kindly loaned by Professor Goodwin of the Massachusetts Institute of Technology, did not show it. This was later identified with Nutting's "peculiar fluted appearance, characteristic of spark lines".[16]

The Porter instrument was finally sent to Petitdidier for overhauling. Three plates were taken out and all were adjusted so that the step was more uniform. The instrument again showed both patterns, the mercury line pattern being false. Many months were thus lost with these various difficulties. At length it was decided to continue the work with the borrowed Petitdidier echelon, an excellent instrument, although of only 20 plates, total aperture $27 \times 15\frac{3}{4}$ mm., step $\frac{3}{4}$ mm. and $14\frac{3}{4}$ mm. thickness of plate.

The apparatus generally employed was, then, the Petitdidier echelon and Porter constant deviation spectroscope with a prism fitted by Hilger.

The spark was generated by a Holtzer-Cabot motor-generator set, the alternator of 4.5 K. W. giving 60 complete cycles per second and feeding a 5 K. W. transformer (of ratio of transformation 110 to 30,000) in the secondary of which was a condenser of 0.0226 microfarads, which discharged, at times through various inductances, over a spark gap generally set horizontal.

Two methods of producing the arc were employed, one giving what

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15 Professor Porter died within a short time after the instrument was delivered.

16 Nutting, Astrophysical Journal, **23**, No. 1, Jan. (1906).

we may call the Pfund arc,[17] between two iron rods, the upper, the negative terminal, being 5 mm. in diameter and pointed somewhat, and the lower, the positive, being 16 mm. in diameter, the current varying from about 1 to 9 amp. and the E. M. F. of the circuit being 220 volts; and the other a 110 volt circuit arc between carbon terminals, the lower being positive, and the values of the currents used being within the above limits. In both cases the positive terminal was supplied with small pieces of the necessary metal, ordinary commercial zinc. The echelon image was magnified about $3\frac{3}{4}$ diameters by a Bausch and Lomb microscope.

Two shutters were used, at first a very light wood and wire arrangement, having two sets of openings of three and two openings respectively, placed in the focal plane of the echelon spectroscope; and finally a shutter of cardboard, having two sets of openings of two and one openings respectively, placed over the slit of the constant deviation spectroscope (this method giving good results, as the echelon spectroscope slit was set accurately in the focal plane of the telescope of the constant deviation spectroscope). The echelon was covered with a cotton lined box to prevent temperature changes, which were never more than 0.1° C during any one set of exposures and usually much less. The photographic plates generally used were Seed Gilt Edge #27, in some cases double-coated; the developer generally normal rodinal solution.

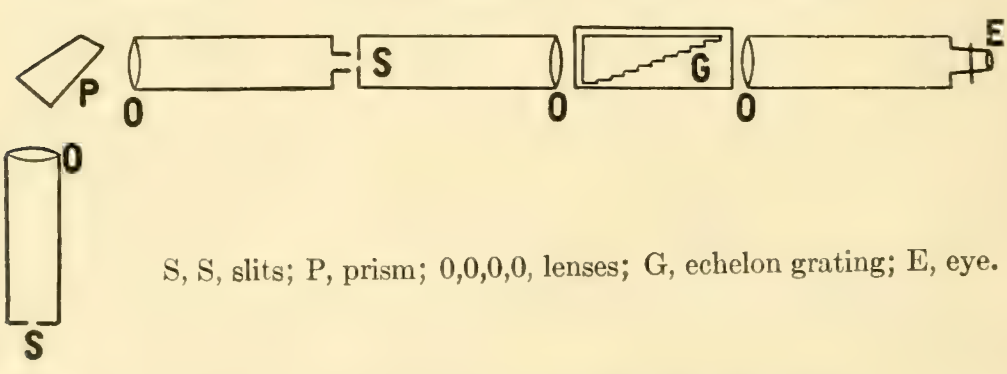
In adjusting and testing the echelons a Cooper-Hewitt mercury lamp, kindly loaned by Mr. William Sawtelle of Harvard, was used.

The two inductance coils used were as follows: —

(a) A coil having three layers as described on page 186 of the *Astrophysical Journal* for October, 1905.

(b) A commercial coil of annunciator wire, weight about 8 lbs., size of wire #18 S. W. G.

The arrangement of the apparatus is shown in the figure.



S, S, slits; P, prism; 0,0,0,0, lenses; G, echelon grating; E, eye.

17 Pfund, Astrophysical Journal, **27**, 296, May (1908).

General Method of Procedure.

Before describing the work in detail it may be stated that the general procedure was to set the echelon at the position of greatest efficiency, such that its axis was parallel to that of the collimator and telescope.

A vertical arc or horizontal spark image was thrown upon the slit and studied visually under numerous and widely different conditions. When a photographic comparison of the two sources was desired, a shutter was used.

Details of the Investigation.

Preliminary comparison of spark and arc. At the outset an attempt was made to compare the position of the image of a highly disruptive spark with that of the arc. This was soon found to be impossible because of the fact that the lines given by a disruptive spark between terminals of the pure metal were not sufficiently monochromatic. Their images given by the Petitdidier instrument cannot be distinguished from those given by the corresponding region of the spectrum of a Nernst lamp (see Plate **1**, 52) and the position of the maximum intensity is a function of the condition of the echelon whether purely of a single or purely of a double order nature at the temperature of the instrument. The only cases in which this method would apply are those in which the spark line is more nearly monochromatic and the condition is absolutely that of a single order. Even then the form of the intensity curve for white or not fully monochromatic light would have to be known.

Visual study of arc lines.[18] As the conditions in the arc and the resulting structure of the lines of the spectrum often change very rapidly, it appeared to be of interest to study these three strong zinc lines visually. A study of this sort was made, an assistant keeping the arc image on the slit and recording the structure of the line as dictated to him. From various sets of observations, many of which are mutually confirmatory, the conclusions given below may be drawn,

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[18] These visual observations were made in a wholly unprejudiced state of mind for, although the papers of Nutting and Janicki referred to had been read when they were first published, the details of the same had been quite forgotten by the writer of this paper.

these being, of course, modified by the condition of the echelon, whether of absolutely single or double order, or part way between the two. However, interpreting the pattern is a simple matter in either case. The echelon was so placed that in the field of the microscope the lower orders lay at the left, the higher at the right.

Zinc 4810. Upon starting the arc after a fresh piece of zinc was put in, the whole field resembled that of a polychromatic source, except that the normal diffuse echelon image was marked by several fine lines similar to the pattern shown in Plate **1**, 12, whether the position of the echelon for monochromatic light be that of single or double order. This structure always accompanied the arc when noisy, and was present in 4722 and 4680, as well as 4810. It is clearly visible in arc lines with the Petitdidier instrument and is similar to the "fluted appearance" of spark lines. At low current and the single order condition, eight components were visible in 4810, the two outermost poorly marked; the two innermost the sharpest of all. As the vapor became less dense, the structure became less extensive. There appeared two lines strongly marked, lying between two other wider, less intense and less sharp satellites. Then the two outer satellites faded, the stronger, inner pair at times receded from each other and then again approached. Finally the reversed region (if, indeed, we are justified in speaking of the phenomena as a "reversal") disappeared and the two lines merged into one, which eventually became a single, narrow line.

The above phenomena were noticed in the Pfund arc at 2.5 amp., with the upper pole (in error) positive. The condition was the double order one, the lower order being slightly stronger. The same phenomena appeared, however, at single order condition, and with lower pole positive.

At 3.4 amp. at another temperature such that the condition was nearly single order, and with the lower terminal positive, the same general phenomena appeared, but when the two line structure was present and the vapor density was decreasing, the *right component became weaker than the left;* whereas when judged by the fact that the adjacent order was stronger, it should have been the stronger of the two. However, *the right component sometimes appeared stronger than the left* and was generally broader. At this point the current was reduced to 1.1 amp. and the right component, although visible near the lower, or positive terminal, disappeared at the center of the arc, the line there being single.

At 5.5 amp. and the lower pole negative, the phenomena of 2.5

amp. were noted, but upon change of polarity and in the single order condition, the fluting changed into three components, the one lying toward the red being the faintest of the three; moreoever, at times six components appeared, the four toward the red being well marked.

At 8.8 amp. the changes were sudden and well defined. The condition was nearly that of the single order, the higher order being slightly stronger. The central component was lacking and the four side components appeared far apart, the two innermost being the strongest. Then at times the central line appeared, attended by two hazy satellites, the left one of which was often the strongest of all three lines. A sudden change here occurred to a very broad image, showing no structure. New zinc was then supplied and there eventually appeared two well-marked lines far apart; these gradually approached each other, a line developed between them, and all three of these lines were at times of the same intensity. Finally, with lessening vapor density, the central component became stronger and the two outer ones shrank toward it.

Zinc 4722. Current 1.3 amp. single order condition. The same general phenomena obtained following the first fluting, which was poorly marked. There appeared two lines, the left a little stronger than the right. These eventually reduced to a single line.

At 2.5 amp. there appeared a single line between two faint satellites. The right component of the three was stronger than the left at certain times; whereas, the arrangement of the orders was such that it would be weaker. With low vapor density the system became a single line.

At 3.5 amp. three lines of nearly equal intensity appeared, the two on the outside somewhat fainter than the central one, which last was the sharpest of all. The outside components often became broader and finally there resulted one single, sharp line with a faint suggestion of side lines at times of greatest brilliancy.

At 5.5 amp. and nearly a double order condition, a central component and two broad satellites were found.

At 8.8 amp. and using carbon terminals, in the single order condition, at first the field showed no structure, then followed the fluting, then there appeared three lines, the right and left strong, the central one weak and diffuse. Finally the left component disappeared, the central one became as intense as its fellow; then the two merged into a single fine line.

Zinc 4680. At 1.2 amp. and a nearly double order condition, after the fluting there followed a condition marked by two components nearly equal in intensity, followed by a single line.

At 2.5 amp. the left component appeared a little weaker and the lines were less sharp than with less current. At low vapor density there appeared a single line, slightly hazy on both sides. With small amount of vapor and in double order condition, the lines appeared hazy, and no reversal was to be seen. This line, 4680, has, however, been observed at this current in single order condition as a single line with two side components which broadened at certain times, all three lines broadening and receding from each other.

At 3.5 amp. the two components were far apart, and equal in intensity. With small amount of vapor the reversal was less well marked.

The results of this visual study of the arc may be summarized as follows: —

(1) All three lines are at times single.

(2) All have been observed "reversed" but,

(3) 4810 and 4680 are generally double or quadruple, while 4722 is generally triple, and,

(4) All are still more complex at times and show asymmetry, but this asymmetry is no more often marked by stronger *red* satellites than *violet*.

Visual study of the inductance spark lines. A visual study of the spark with inductance showed that the conditions could in this case be more easily controlled and were more steady.

4810, in a nearly single order condition, the right order of the three being stronger than the left, with coil (b) as inductance in the condenser discharge circuit, showed two lines, the right component distinctly weaker than the left when the arrangement of the orders would, if the two components were intrinsically of the same intensity, make this right component the stronger.

The central part of the image was under observation. This right component, however, appeared stronger when the end of the spark image was thrown upon the slit. With no inductance the fluting was *faintly* visible (it had not been observed previously with the Petitdidier instrument [19]) for the gap was small so that the spark burned quietly.

4722 showed the fluting more clearly than 4810, and with inductance, the condition being nearly single order, there appeared two bright central lines which were nearly equal at times. Whenever unequal, however, the left line was the stronger. At the end of the image near the terminals the line broadens out and resembles the disruptive spark.

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19 See page 97.

4680 in the double order condition showed the fluting faintly at times, while with inductance the line appeared almost single, the "reversal" being almost invisible.

This visual work with the spark shows: —

(1) That at *high* inductance 4810 and 4680 are generally double; (4810 has been photographed as quadruple, and, *with less inductance,* as a quintuplet and a triplet) and 4722 triple.

(2) That here we have a definite asymmetry, controllable and regular, a relative increase in intensity of the red satellites when the end of the spark gap is used. This is fully confirmed by the photographic study. (See discussion below and Plate **1**, 71 (a), (c), and 111 (b).)

Photographic study of arc and spark lines and a comparison of the two.[20] Among others the following photographs were taken with the Porter echelon:—

The "fluting" not visible in all disruptive spark lines.

12. 4722. Spark without inductance. Center of 4 mm. gap. Shows fluting.

16. *Zn*4924. Without inductance. Center of 4 mm. gap. Shows no fluting. This may be an "air" line, however.

The following photographs were taken with the Petitdidier echelon:

A general comparison of the images of a line as given by different sources.

28. 4810. Pfund arc through two of the openings of the five opening shutter; spark through three openings. Arc current, 0.9 amp. Spark without inductance. Gap 2 mm. and in series an auxiliary gap of 4 mm. Exposures: arc 20 seconds, spark 2 minutes. Compares the pattern and position of arc and spark images, under the double order condition. Confirmed by other similar photographs. However, we cannot tell with the echelon a disruptive spark from an arc burning on a heavy current, as is shown by 36.

32. 4680. Similar to 28. Compares the arc and spark images under single order condition. Confirmed by other similar photographs.

36. 4810. Inside 110 volt arc of about 10 amp. between Pfund terminals and *very dense* vapor. Outside Pfund arc, 1 amp. Compare 28.

52. 4810. Center of 4 mm. gap of disruptive spark, inside opening of three opening shutter; Nernst lamp (in neighborhood of 4810), outside openings. Exposures: Nernst, 1 minute; spark, 15 seconds. Shows that the two cannot be distinguished.

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20 For the photographs which have been reproduced see Plates **1** and **2**.

Inductance spark structure.

70 (a). 4810. Single order condition. Center of 3 mm. gap. Coil (b) in circuit. Exposure: 30 seconds. Notice four components, the two outermost faint.

Inductance spark structure and power in circuit.

81 (b) and (d). 4810. Single order condition. Center of 1.5 mm. gap. Three layers of coil (a) as inductance. Two exposures of 30 seconds each: (b) at 20 amp. and 1 hectowatt; (d) at 50 amp. and 2 hectowatts. These show that approximately doubling the current, and the power in the primary of the transformer has little, if any, effect upon the structure. This is confirmed by another photograph in the case of 4722.

Inductance spark structure as a function of the part of the image viewed.

71 (a). 4722. Single order condition. Electric conditions as in 70 (a). Exposure: 25 seconds. Three components are apparent on the original negative, but the one toward the red is very weak.

71 (c). 4722 Same as (a) but near end of image. Exposure: 35 seconds. Notice new component toward red not due to inequality of exposure, for the main component is just as bright in 71 (a) as in 71 (c). This effect is confirmed for 4810 by another photograph and still further by:—

111 (b). 4810. Center of photographic plate, central part of 4 mm. spark gap; outside of plate, end of gap. Coil (b) as inductance. Notice the new component to red in the exposure of the end of the gap. The original negative shows still another component toward the red. Note further that despite the fact that the photographic image of the central part of the gap is the denser of the two, the components appearing are but two in number. Single order condition with the left of the three central orders slightly *stronger* than the right, giving a condition *unfavorable* for the appearance of components to the right! This is confirmed by three other sets of exposures.

Effect of using an alloy.

88 (a). 4810. Single order condition, center of 1.5 mm. disruptive spark between brass terminals. Shows that the line structure is simple although a continuous pattern is present with it, and that disruptiveness in itself is not the only controlling factor. Other exposures with 5 and 9 mm. gaps gave the same results.

89 (b). 4810. Same as 88 (a) except that three turns of inductance were inserted. The result is a single fine line.

89 (c). 4722. Same as 89 (b). Two fine lines, just separated, appearing on the original negative. Photographs 89 (b) and 89 (c)

thus show that with an alloy and inductance the structure is rendered very simple and the light even more monochromatic than with the lower voltage arc.

Comparison of arc and disruptive spark.

48 (c). 4680. Pfund arc at low current shown by two openings of the five opening shutter: end of image of a disruptive spark of 3 mm. gap with a 4 mm. auxiliary gap in series, shown by three openings. Double-coated Seed, gilt edge 27 plate. Hydrochinone developer. Exposure: arc, 15 seconds, spark 1 minute. Note that there is structure in the spark and that it lies to the right, the region of longer wave-lengths. See especially the middle of the five shutter openings. This is confirmed by two other sets of exposures. The reproduction is poor, owing to the fact that the structure is not strongly marked, and is obscured by a continuous pattern.

Comparison of arc and inductance spark.

85 (a). 4810. A comparison of an inductance spark outside (inductance, three layers of coil (a); exposure, 30 seconds; and center of gap) with Pfund arc inside (low current and exposure 5 seconds). Single order condition but with the stronger of the two adjacent orders toward the violet. Notice that the maximum intensity of the structure lies toward the red in the spark in comparison with the arc. This is confirmed by another set of exposures in which the arc was given relatively greater exposure time. Of course if another part of the arc had by chance been used, the result might possibly have been different. And again, greater arc current might have made some difference in the structure and further, as the rapidly fluctuating conditions in the arc change the structure, the distribution of energy might at another instant have been different. But further exposures are confirmatory with respect to 4810, and show a like phenomenon in the case of 4722; and still others are confirmatory with respect to 4722, and show a like phenomenon in the case of 4680. Further, two other sets of photographs taken some days later, confirm these results for all three lines; and two more using carbon terminals and a 3 amp. current show the same effects in all three lines. And again four other sets taken upon still another day, with a 220 volt, 3.3 amp arc between carbon terminals, give in every case the same results for these three lines.

Such agreement proves that the effect cannot be fortuitous. However, as the inductance spark is steadier and easier to control, it is well to compare sparks having different inductances in circuit: —

Spark line structure as a function of the inductance.

78 (c). 4810. Between single and double order condition. Center of 1.5 mm. gap. Three layers of inductance coil (a). Exposure: 1 minute. Two main components. Note that the right component of the quadruplet is as strong as, or stronger than the left, when the position of orders is such that it would be weaker. This fact is confirmed by other exposures.

79 (e). 4722. Between single and double order condition. Electric conditions as in 78 (c). Exposure: 1 minute. Shows three main components.

80 (a). 4680. Between single and double order condition. The electric conditions are as in 78(c) and 79(e). Two main components. Exposure: 1 minute.

65 (b). 4810. Single order condition. Center of 4 mm. spark gap under different conditions. Outside, no inductance, 5 seconds: inside coil (b) in circuit, 45 seconds. Notice the two side components in the inductance spark image.

68 (a). 4722. Single order condition. Electric conditions, similar to 65. Note inequality of intensity of inductance line components. Exposures: 30 seconds with inductance and 3 seconds without.

94 (c), (d), and (e). 4810. Single order condition. Center of a very small gap — less than 2 mm. Three, two and one layers of coil (b), respectively.

96 (c), (d), and (e). 4680. Double order condition. Same set of operations as in 94. Notice in both plates a continuous increase of intensity of the old components lying toward the red and the development of new ones as the inductance is decreased. Another photographic plate (numbered 95) clearly confirms this for 4722. On all three, 94, 95 and 96, there were also taken shutter comparisons showing the relative positions of the components given with one, two and three turns. These all show that the component coming up with decrease of inductance is the one toward the red: the component toward the violet retains its position while its intensity becomes relatively less. The effect of removal of inductance is similar to that obtained by moving up to the end of a somewhat longer gap leaving the inductance the same. (See 111b).

The conclusions to be drawn from the photographic study are:—

1. That it is impossible by means of the echelon grating to compare the positions of maximum density of any but quite monochromatic sources, whether the condition be either double or single order.

2. That it is impossible in general to distinguish the images given

by a Nernst lamp, an arc of great vapor density, and a highly disruptive spark between terminals of the pure metal.[21] These sources give, in fact, nothing but the so-called "diffraction" as distinguished from the "interference" pattern.

3. That inductance, even in small amounts, is extremely efficient in reducing the intensity of the continuous or diffraction pattern and producing structure in the spark image.

4. That the structure varies with the part of the inductance spark image used whether end or center; the end showing an enhancement of the intensity of the components lying toward the red.

5. That as the value of the inductance is increased, the red components in the structure become less intense.

6. That even a disruptive or non-inductance spark between brass terminals shows structure in the zinc lines studied and that, if in addition inductance be inserted, the resultant lines are as sharp, or even sharper, than those given by a low current arc.

7. That a small amount of vapor in the arc, even with fairly high current (e. g. 8 amp.) produces conditions favorable to structure other than the fluting which occurs when the arc is heavily charged with vapor and is noisy.

8. That on all plates obtained upon which the positions of the components of the spark with small inductance are compared with the positions of the components of the arc at low current (about 3.3 amp.) the center of gravity of the spark structure lies further toward the red than that of the arc.

General Conclusions.

That conflicting results were obtained by Janicki and Nutting is probably due to the fact that different sources of light were employed. The structure Nutting describes is unquestionably real. Certainly echelon gratings may give ghosts. That the Petitdidier instrument used in this investigation is free from such, is shown by the fact that the green line of mercury shows no false lines.

Further, from the visual observations made upon arc lines, it is perfectly clear that the "ghost" argument will not explain the endurance of a satellite or its increase in intensity, when a formerly brighter line grows fainter or disappears entirely, nor, specifically,

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21 This is true of the spark only when the echelon is not powerful enough to resolve the components of the fluting.

a case such as that recorded on page 101 under Zn 4722 at 8.8 amperes. It is impossible for the main line to disappear and the ghost remain; and again, even if ghosts were present, there is no reason why these should appear in the case of any one line with the spark as a source, and not with the arc. The presence of neither a symmetrical nor unsymmetrical ghost structure could produce the enhancement of the red satellites in the spark.

A certain objection may, however, be made: namely, that the presence of the diffraction pattern between the orders when the instrument is in a double order condition, might cause satellites which are of low intensity to appear (when otherwise they would not) in much the same manner as fogging a photographic plate will carry the exposures of "low lights" up along the intensity curve so that they will become visible.[22] In response to this objection, it may be said that the satellites in question are not always of low intensity, either visually or photographically; and they even come up on the *right* side when the diffraction pattern lies to the *left*.

We must conclude, then, that there exists for some unknown reason a fairly progressive increase in the intensity of the red satellites of these three zinc lines with decreasing inductance. There follows at once the unsymmetrical broadening to the red of the images given by instruments of less resolving power, namely, prism or grating spectroscopes.

The unsymmetrical satellite system may be produced by the high potential gradient in the spark; why, the writer, of course, cannot state. Disruptiveness is not a determining factor, for in the same spark we obtain from different parts of the gap different line structure. Vapor density probably does not of itself determine structure, but may influence the potential gradient. In the arc high density seems to produce a tendency toward complexity of structure, but not an asymmetry of a regular or enduring type.

All the writer's observations, both visual and photographic, confirm the results obtained by Nutting, dealing with arc structure. The results of this study also confirm the shifts found by the writer[23] to exist at lower dispersion, shifts,— great at the end of a fairly large gap of a non-inductance spark between terminals of the pure metal, lessened or removed entirely by the addition of inductance, and by the use of the central region of the gap; and lessened also by the use of an alloy. In this former work the standard of reference employed

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22 R. W. Wood actually used this method.

23 Astrophysical Journal, **22**, No. 3, Oct. (1905).

was a carbon arc of somewhat greater current than here used, but the amount of vapor was never great, only small bits of metal being inserted in the arc, and the exposure always being made when it was burning quietly. These two sets of standards were probably much the same. Still, assuming them different, if the potential gradient determine the enhancement of the red satellites and we accept Nuttings classification of gradient, from low to high the order being, (1) heavy current arc, (2) low current arc and inductance spark, (3) high capacity and non-inductance spark, then the assymmetry of satellites (and resultant shift) obtained in this investigation with low current arcs as standards would be even less than that found with the somewhat higher current arcs previously used. However, as stated above, in the arc there seems to be no regular, controllable nor enduring enhancement of either red or violet satellites.

Janicki's suggested explanation of the shifts obtained — namely, as "unsymmetrical reversals like those of chromium and calcium, reversals which their grating would not resolve and which appeared to them as line-shifts" must then be replaced by this enhanced satellite theory.

The distances between the satellites in Plate **2**, 48 (c) are approximately 0.05 Ångstroms. We may then say that the removal of two layers of inductance in coil (a) has shifted the center of gravity of the line at least 0.02 Ångstroms. In the extreme case then, with no inductance in the circuit, the shift might easily be in the neighborhood of 0.032 Ångstroms, as formerly obtained.

The writer wishes to record his appreciation of the kindness of Professor Goodwin of the Massachusetts Institute of Technology in loaning his Petitdidier echelon. To the Rumford and Bache Committees, and a personal friend, Mr. J. DeL. VerPlanck, the writer is indebted for funds which made this investigation possible. In the actual work of obtaining the results he wishes to acknowledge the faithful assistance rendered by various students, especially Messrs. Walter F. Burt, Russell T. Hatch, Charles H. Smith and Carl K. Springfield.

Physics Laboratory, Boston University,
June, 1912.

12

The following negatives, on Plates **1** and **2**, represent approximately a three-fold enlargement of the image as photographed or a twelve-fold enlargement of the echelon image.

The region of longer wave-lengths lies to the right.

12 and 16 were taken with the Porter echelon; 28 to 96e with the Petitdidier instrument.

Much of the detail existing upon the enlarged negatives is not apparent in the reproductions herewith shown.

16

28

32

36

52

70a

81b

81d

71a

71c

111b


88a


89b


89c


48c


85a


78c


79e


80a


65b


68a


94c


94d


94e


96c


96d


96e

THE IMPEDANCE OF TELEPHONE RECEIVERS AS AFFECTED BY THE MOTION OF THEIR DIAPHRAGMS.

By A. E. Kennelly and G. W. Pierce

Received July 16, 1912.

I. Introduction.

The writers have made a series of measurements of the resistance and inductance of several forms of telephone receivers over a wide range of frequency of current. In the course of the measurements some interesting results have been obtained, which form the subject of this paper.

As the period of the e. m. f. used in the measurements approaches the natural period of the diaphragm, the note emitted by the telephone receiver increases markedly in loudness, and the resistance and inductance of the receiver undergo wide deviations from values obtained when the diaphragm is prevented from vibrating by being damped. That is to say, the motion of the diaphragm has an effect upon the resistance and inductance of the receiver, and this effect grows rapidly as the electrical period approaches the mechanical period.

In the tests to be described, the resistance and the inductance of a given receiver were measured, first with the diaphragm free and sounding, and, second, with the diaphragm damped, or arrested. The values when the diaphragm is free may be called *free values;* the values when the diaphragm is damped may be called *damped values*. The difference obtained by subtracting the damped values from the corresponding free values may be called the *motional values* of resistance, inductance, etc.; since such differences are due to the motion of the diaphragm.

It is found that when the impressed frequency differs widely from the natural frequency of the diaphragm, the motional resistance and inductance are very small. In the neighborhood of resonance, which is often very sharply marked, these motional values become relatively large, and one or both pass through a change of sign, in such a manner that, when the motional impedance for different frequencies is drawn vectorially from a fixed point as origin, all the points given by the observations lie upon a circular graph, which may be called the *mo-*

was repeated with the diaphragm of the telephone T at rest and silent. The damping was effected usually by lightly pressing upon the diaphragm with the finger, but in some cases it was affected by inserting a light wedge (a quill) between the diaphragm and pole, when this operation was permitted by an open structure telephone. The balance, when the diaphragm was damped, gave practically complete silence in the head-telephones H, and the settings of resistance and inductance were consistent within about $\frac{1}{2}$ of 1%. The balance, on the other hand, when the diaphragm was in motion, was not so good. In this case, difficulties were introduced by parasitic notes probably due to currents of higher frequency generated by the motion of the telephone diaphragm. It was usually possible, however, to balance out the fundamental tone, with adjustments consistent within 1 or 2 ohms.

III. Particulars of the Telephones Tested.

Several telephones were submitted to measurements. Four of the instruments, for which the results are presented in the present account, were: —

1. A Western Electric Bipolar Bell Telephone, Type 122, here designated "R_b",
2. A Western Electric Bipolar Watch-case Telephone receiver, designated "Watch-case,"
3. An experimental specially-constructed monopolar receiver, here designated "Experimental monopolar," and
4. An experimental bipolar telephone receiver, provided with exploring coils, and here designated "Experimental bipolar."

The following table (Table I) contains some of the mechanical particulars of these instruments.

IV. Experimental Data and Results.

The data obtained by measurements of the resistance and inductance of the first three of the above receivers are contained in Tables II to VI. The data with the "experimental bipolar" receiver are not tabulated, as they were taken for the specific purpose of determining the angle of lag of magnetization of the iron behind the actuating current and this subject is discussed later.

Explanation of Tables. — A brief explanation of Table II, obtained with the bipolar Bell "R_b" with 0.3 effective volts applied at its terminals, will be given as typical of all the tables. The first column contains the frequency in cycles per second. The second column gives the corresponding angular velocity in radians per second. The third column gives R′ the resistance free, at each frequency, as measured on the Rayleigh bridge; while the fourth column gives R the corresponding resistance obtained with the diaphragm damped. The

TABLE I.

MECHANICAL CONSTANTS OF RECEIVERS.

	Bell R_b.	Watch-case.	Exp. Monopolar.	Exp. Bipolar.
Area of each pole in cm. × cm.	1.4 × .225	1.61 × .16	0.53* × 0.53	1.17* × 0.38
Distance separating poles in cm.	0.85	0.80	–	0.73
External diam. of diaphragm in cm.	5.40	5.48	7.22	7.22
Diameter of clamping circle, cm.	4.94	4.84	4.45	4.45
Tchikness of diaphragm, cm.	0.024	0.030	0.032	0.022
Weight of diaphragm, grams	4.0	5.03	5.3	5.3
Direct-current resistance of coils, ohms, at 20° C.	71.0	81.4	89.7	25

* Laminated poles.

fifth column, headed "motional," gives R′ – R; or the difference between the free and damped resistances with a proper sign for the difference. The three remaining columns contain the corresponding reactances, as obtained by multiplying the inductances observed on the Rayleigh bridge by the angular velocity ω in each case. The final column, marked "motional," gives the excess of free reactance over damped reactance, with proper sign.

Tables III to VI contain data similar to those in Table II, but with different applied voltages or different receivers.

An examination of these tables shows that there are two independent phenomena of interest; namely,

First, the effect of the frequency on the resistance, reactance and inductance of the receiver when damped; and

TABLE II.

Resistance and Reactance of Bipolar Bell Receiver R_b, at Different Frequencies.

0.3 volt at Terminals.

Frequency.		Resistance, Ohms.			Reactance, Ohms.		
n. Cycles per Second.	$\omega = 2\pi n$ Radians per Second.	Free R'.	Damped R.	Motional $R' - R$.	Free $X' = L'\omega$.	Damped $X = L\omega$.	Motional $L'\omega - L\omega$.
440	2760	151.8	146.3	5.5	138.1	135.1	3.0
512	3220	163	150.8	12.2	148.7	144.5	4.2
600	3770	166	147.7	18.3	154.5	150.5	4.0
670	4210	188	157.5	30.5	162.4	158.8	3.6
704	4420	206	163.0	43.0	162.0	164.2	− 2.2
720	4520	213	164	49	158	166	− 8.0
744	4660	233	168	65	142	168.8	−26.8
754	4730	237	170	67	133.2	169.5	−36.3
770	4830	240	172	68	115.7	172.3	−56.6
778	4880	212	173	39	78.1	171.4	−93.3
780	4900	173.5	173	0.5	76.5	170.5	−94.0
790	4960	164	172.5	− 8.5	82.3	173.0	−90.7
792	4970	164	172.5	− 8.5	81.9	173.2	−91.3
792	4970	172.5	183	−10.5	78.4	165.9	−87.5
792	4970	178	175	3	78.4	173.6	−95.2
793	4975	169	175	− 6	78.5	172.9	−94.4
794	4980	178	172.5	5.5	75.6	173.5	−97.9
804	5050	142	178.5	−36.5	122	174	−52.0
808	5080	148	177	−29	98.4	174.9	−76.5
822	5160	146.5	180	−33.5	145	175	−30.0
824	5180	142	180	−38	121.6	180.0	−58.4
824	5180	136	176	−40	120.4	177.2	−56.8
826	5190	141	176	−35	117.7	177.9	−60.2
826	5190	141	176	−35	127.2	177.9	−50.7
832	5220	147	179	−32	145.7	172.9	−27.2
840	5280	157	182	−25	162.8	177.5	−14.7
868	5450	160	183	−23	167	181	−14.0
890	5590	166	186	−20	175	184	− 9.0
892	5600	164.5	185	−20.5	175	184	− 9.0
912	5730	180	204	−24	184	192.5	− 8.5
940	5900	178.7	191.5	−12.8	183.5	187.8	− 4.3
1000	6283	205	203	− 8.0	202.5	204	− 1.5
1060	6650	210	216	− 6.0	209.5	211.5	− 2.0
1152	7250	192	202	−10	230.5	234	− 3.5
1248	7850	222	225	− 3	230	228	2.0
1648	10350	264	265	− 1	254	253	1.0
2464	15500	325	325	0.0	291.5	294	− 2.5

Second, the effect of the motion of the diaphragm.' These two effects will be treated in order.

Change of Damped Resistance, Inductance and Reactance with Change of Frequency.— Figure 2 shows the damped resistance, inductance and reactance of the bipolar receiver "R_b" plotted

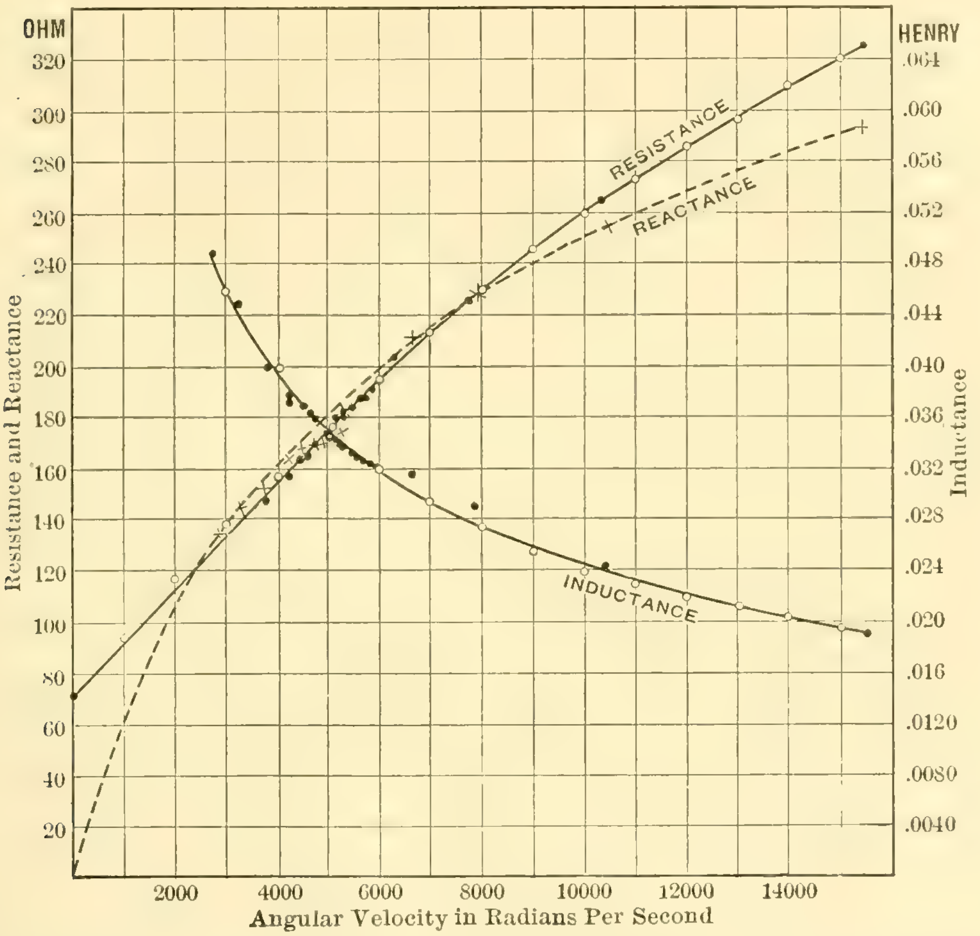

FIGURE 2. Curves of damped resistance, inductance, and reactance plotted against angular velocity, for Bell bipolar receiver, with 0.3 volt at terminals. The dots are observed points; circles calculated; crosses belong to reactance curve.

against the angular velocity of the current used in the measurements. Figure 3 contains the corresponding curves for the bipolar "watch-case" receiver. In each case the resistance and reactance of the telephone when damped increases with increase of frequency, while the damped inductance decreases with increase of frequency. The following empirical relations approximately hold.

For the bipolar Bell "R_b," at 20° C, and with 0.3 volts at its termi-

TABLE III.

Resistance and Reactance of Bell Bipolar Receiver R_b at Different Frequencies.

0.42 Volts at Terminals of Receiver.

Frequency.		Resistance, Ohms.			Reactance, Ohms.		
n. Cycles per Second.	ω Radians per Second.	Free R'.	Damped R.	Motional $R'-R$.	Free $L'\omega$.	Damped $L\omega$.	Motional $L'\omega - L\omega$.
428	2690	138	128.5	9.5	129.5	124.4	5.1
548	3446	162	145.7	16.3	148.8	144.3	4.5
704	4425	211	168	43	164.2	166.4	− 2.2
710	4468	216.5	167.5	49	160.7	166.9	− 6.2
722	4540	224	172	52	155.1	168.7	− 13.6
733	4610	230	170	60	148.7	170.4	− 21.7
744	4680	235	171.5	63.5	144.1	170.4	− 26.3
754	4740	242.5	175	67.5	132.1	171.4	− 39.3
766	4810	237	174	63	120.5	171.6	− 51.1
778	4885	227	175	52	90.8	172.9	− 82.1
792	4976	189	176	13	76.1	177.1	−101.0
805	5060	154	177	−23	94.1	177.5	− 83.4
822	5170	142	181	−39	127.8	179.5	− 51.7
836	5253	143	181	−38	142.9	179.7	− 36.8
853	5354	148	183	−35	156.3	183.1	− 26.8
870	5470	157	185	−28	167.9	184.3	− 16.4
890	5595	161	186	−25	171.5	187.2	− 15.7
2464	15480	306.5	306.5	0.0	291.9	296.5	− 4.6

nals, the damped resistance as a function of the angular velocity is expressible by the equation

$$R = 71 + 0.0234\,\omega - 0.456 \times 10^{-6}\omega^2 \qquad \text{ohms,} \quad (1)$$

in which R is the damped resistance, and ω is the angular velocity in radians per second.

TABLE IV.

RESISTANCE AND REACTANCE OF WATCH-CASE RECEIVER, AT DIFFERENT FREQUENCIES, WITH 0.3 VOLT AT TERMINALS OF RECEIVER.

Frequency.		Resistance, Ohms.			Reactance, Ohms.		
n. Cycles per Second.	ω Radians per Second.	Free R'.	Damped R.	Motional $R' - R$.	Free X'.	Damped X.	Motional $X' - X$.
451	2834	135	136	− 1	115.7	118.5	− 2.8
550	3456	149	149	0	135.8	135.8	0.0
653	4102	163	160	3	150.3	149.5	0.8
702	4410	168	166	2	162.6	160.8	1.8
712	4474	174	171	3	157.4	157.0	0.4
753.5	4738	179	175	4	163.5	163.0	0.5
804	5052	185	181	4	169.5	168.0	1.5
849.5	5340	194	184	10	172.5	174.7	− 2.2
884.3	5554	204	187	17	171.2	178.4	− 7.2
903	5674	212	189	23	160	180.5	−20.5
913	5736	207	190	17	146.1	180.5	−34.4
923	5800	192	191	1	137.8	182.5	−44.7
934	5868	174	192	−18	146	183.5	−37.5
940	5906	169	193	−24	154.4	184.5	−30.1
945	5938	168	193	−25	160.1	184.5	−24.4
957	6014	170	194	−24	172.2	186.0	−13.8
968	6082	173	195	−22	177.6	186.8	− 9.2
980	6158	179	197	−18	182.8	186.5	− 3.7
993	6240	183	198	−15	186.6	188.5	− 1.9
1020	6408	188	200	−12	191.1	191.7	− 0.6
1084	6812	197	206	− 9	197.6	198.3	− 0.7
1157	7270	205	211	− 6	203.1	204.5	1.4
1250	7854	214.5	220	− 5.5	210.6	212.0	1.4
1846	11600	254	256	− 2	261.3	263.0	1.7

For the bipolar "watch-case," at 20° C, and with 0.3 volts applied,

$$R = 81.4 + 0.0214\,\omega - 0.505 \times 10^{-6}\omega^2 \qquad \text{ohms.} \quad (2)$$

In these equations, 71 and 81.4 are the respective resistances of the two instruments to steady currents; the other constants of each equation were determined by using two of the observed points on each of the resistance-frequency curves.

Another more interesting fact, obtainable from the experimental curves, is that the product of the damped resistance by the damped

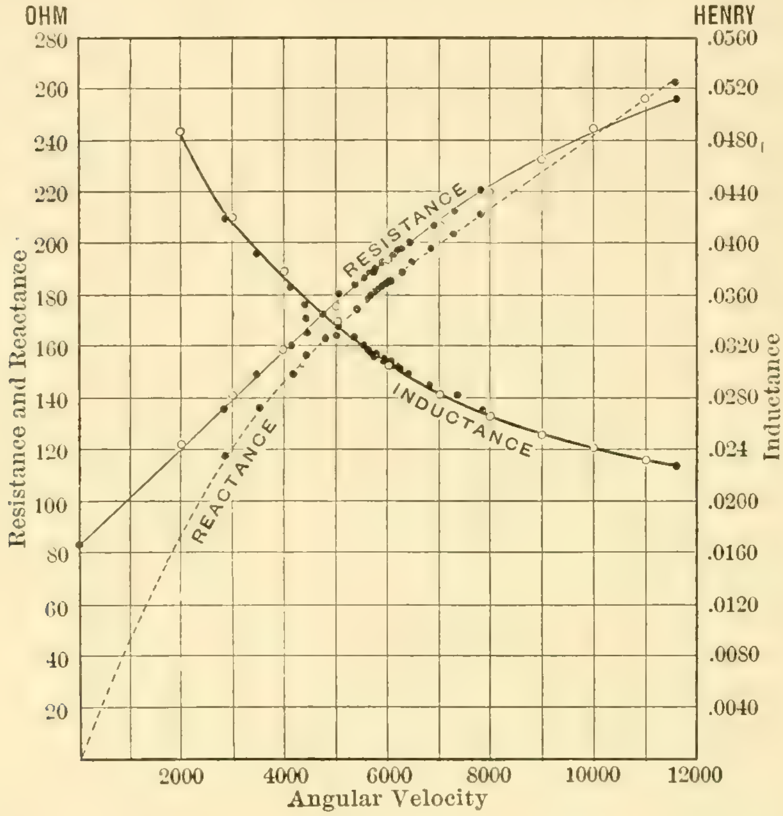

FIGURE 3. Curves of damped resistance, inductance, and reactance *vs.* angular velocity, for watch-case receiver, with 0.3 volt at terminals. Dots, observed; circles calculated.

inductance, for each of the two telephones, is approximately a constant independent of the frequency. That is, for the bipolar "R_b," at 0.3 volt,

$$L = \frac{6.25}{R} \quad \text{henrys} \quad (3)$$

and for the "watch-case" at 0.3 volt

$$L = \frac{5.88}{R} \quad \text{henrys.} \quad (4)$$

The degree of accuracy with which these formulas accord with the observations is shown by the damped resistance and inductance curves of Figures 2 and 3, where the observed points are indicated by black dots, and the points calculated from the formulas are represented by the circles. The agreement between the calculated points and the curve of observations in the case of the watch-case instrument (Figure 3) is within about 1%. In the case of the Bell instrument "R_b," in

TABLE V.

Resistance and Reactance of Experimental Monopolar Receiver, with 0.3 Volt at Terminals of Receiver.

Frequency.		Resistances.			Reactances.		
n.	$\omega = 2\pi n$.	Free R'.	Damped R.	Motional $R' - R$.	Free X'.	Damped X.	Motional $X' - X$.
993	6240	158	140	18	394	365.3	28.7
1012	6360	321	143	178	354	375.8	− 21.8
1016	6390	266	142	124	427	374.5	52.5
1018	6400	161	142	19	408	378.5	29.5
1020	6410	207	142	65	256	375.8	−119.8
1020	6410	195	143	52	429	378.3	50.7
1023	6428	248	144	104	434	378.2	55.8
1023.6	6436	291	143	148	416	380.6	35 4
1024	6454	211	143	68	432	380.5	51.5
1024	6454	175	143	32	421	382.4	38.6
1024	6435	148	144	4	299	380.0	− 81
1024.2	6440	303	143	160	389	380.0	9.0
1024.8	6444	331	143	188	341	380.4	− 39.4
1025.4	6448	321	143	177	333	380.7	− 47.7
1026	6450	291	143	148	273	380.3	−107.3
1029	6460	238	144	94	257	381.5	−124.5
1032	6480	135	144	− 9	320	380.3	− 60.3
1035	6508	175	144	31	271	383.0	−112
1048	6586	146	144	2	307	387.7	− 80.7
1051	6603	134	144	− 10	340	389.5	− 49.5
1054	6624	140	145	− 5	321	388.7	− 67.7

order to have sufficient range of frequency, the writers had to use some of the earliest observations, taken before they had learned the precautions required for accurate results. But in this case also, the values calculated by the formulas (1) and (3) agree closely with the curves that best represent the observed points except in regions where the latter are uncertain.

As a further illustration of the approximate constancy $R \times L$ taken with the telephone damped, reference is made to Table VII, which contains this product at different frequencies for receiver R_b with 0.42

volt at its terminals. At this voltage, the product for this telephone averages 6.21, and within the range of frequencies between 428 and

TABLE VI.

Resistance and Reactance of Experimental Bipolar Receiver Provided with Exploring Coils, With 1 Volt at Terminals.

Frequency.		Resistance, Ohms.			Reactance, Ohms.		
n. Cycles per Second.	ω. Radians per Second.	Free R'.	Damped R.	Motional $R' - R$.	Free X'.	Damped X.	Motional $X' - X$.
1007	6370	39.4	38.3	1.1	153.7	151.8	1.9
1020	6410	42.7	38.6	4.1	156.1	152.9	3.2
1020	6410	41.2	39.2	2.0	155.1	152.5	2.6
1023	6430	44.0	38.7	5.3	156.2	153.0	3.2
1026.5	6450	50.0	38.8	11.2	149.7	153.6	−3.9
1027	6450	42.0	39.0	3.0	156.5	153.6	2.9
1027	6450	48.6	38.8	9.8	153.6	153.9	−0.3
1027	6452	49.0	38.8	10.2	149.7	153.6	−3.9
1027.5	6455	49.0	38.8	10.2	148.0	153.9	−5.9
1028	6458	47.0	39.0	8.0	147.2	153.7	−6.5
1030	6470	43.0	39.0	4.0	157.4	154.1	3.3
1030	6470	48.2	38.8	9.4	148.8	154.0	−5.2
1033	6492	47.2	38.9	8.3	147.4	154.6	−7.2
1033	6492	46.0	39.0	7.0	156.5	154.5	2.0
1033	6492	50.2	38.9	11.3	154.6	154.3	0.3
1036	6510	44.8	39.0	5.8	158.2	154.9	3.3
1036	6510	46.0	39.0	7.0	148.6	155.2	−6.6
1037	6520	46.0	39.0	7.0	150.0	155.3	−5.3
1038	6525	44.0	39.0	5.0	148.4	155.6	−7.2
1039	6530	50.0	39.0	11.0	151.7	154.6	−3.9
1039	6530	44.0	39.0	5.0	149.1	156.0	−6.9
1039.5	6533	43.0	39.0	4.0	148.4	155.6	−7.2
1041	6540	42.0	39.0	3.0	148.5	155.7	−7.2
1042	6546	48.0	39.0	9.0	149.8	155.7	−5.9
1042	6546	40.2	39.1	1.1	149.3	155.9	−6.6
1043	6560	41.0	39.0	2.0	150.0	156.6	−6.6
1045	6570	40.0	39.0	1.0	150.5	156.8	−6.3
1047	6580	39.0	39.0	0.0	150.7	156.7	−6.0
1051	6600	39.9	39.3	0.6	151.3	157.2	−5.9
1051	6604	38.9	39.3	− 0.4	152.5	156.8	−4.3
1054	6624	39.0	39.3	− 0.3	153.0	158.0	−5.0
1059	6660	38.8	39.3	− 0.5	155.2	158.5	−3.3
1064	6686	38.8	39.6	− 0.8	155.2	159.2	−4.0
1084	6812	39.0	39.6	− 0.6	159.0	161.7	−2.7

890 cycles per second, the product does not depart from the average by more than 2%. There is no march of the product within this range.

When, however, the computation is extended to a single observation at 2464 cycles per second, a departure of 5% is obtained.

A third interesting fact shown by the experimental tables is that the damped reactance is approximately equal to the damped resistance for the telephone "R_b" over a wide range of frequency. This may be seen, for this telephone, at 0.3 volt, by a reference to Figure 2 and by a comparison of the observed reactance values, marked by *crosses*, with the observed points on the resistance curve marked by black *dots*. The damped reactances and damped resistances are seen to be nearly the same throughout the range of frequencies between 451 cycles per second ($\omega = 2834$) and 1250 cycles per second ($\omega = 7850$). Within this range the damped resistance and the damped reactance both nearly double and yet remain within a few percent of equality with each other. For the telephone R_b at 0.42 volts, the same approximate equality holds within the range of frequencies between 428 and 2464 cycles per second, as may be seen by a reference to the fourth and seventh columns of Table III. It is to be noted, however, that this same equality cannot persist at low frequencies, for the damped reactance at zero cycles is zero, while the resistance of this instrument at zero cycles is 71 ohms. As a corollary, it may be observed that within the range of equality of damped resistance and damped reactance, the damped angle of lag of current behind impressed e. m. f. is 45°, and the damped impedance is $\sqrt{2}R$. With the other instruments tested, the equivalence of damped reactance and damped resistance was not obtained; but, as may be seen by reference to Figure 3, the curves of damped reactance and damped resistance for the watch-case instrument run nearly parallel and within 10 ohms of each other, for a considerable range of frequencies.

It would be interesting to discuss the relations expressed in equations (1), (2), (3), and (4). Since, however, at this time the primary purpose of the writers is to present an account of the effects of the motion of the diaphragm in modifying the resistance and reactance of the telephone receivers, a further discussion of the relations (1) to (4) will be deferred.

The Effects of Motion of the Diaphragm on the Resistance and Reactance of the Receivers.— As stated in the introduction, the motion of the diaphragm of a telephone receiver has a marked effect on its resistance and reactance. This effect is best shown by subtracting the damped resistance from the free resistance, and the damped reactance from the free reactance and plotting the differences, called respectively *motional resistance* and *motional reactance*, against the frequency in radians per second (angular velocity). This is done in Figures 4, 5, and 6. Taking Figure 4 plotted from Table IV obtained with the watch-case receiver, as typical, it will be seen that the Figure contains curves of motional resistance, motional reactance, motional power, and phase angle of motional impedance, marked respectively *Resistance, Reactance, Power* and *Phase.* These quantities are all plotted against angular velocity. The black dots are observed points, and the circles are computed values, or derived values. Beginning with the resistance curve, and remembering that this curve represents the excess of free resistance over damped resistance, that is to say, the effect of the motion, it will be seen that, starting at a value slightly below zero at 2834 radians per second, the increment of resistance due to motion (motional resistance) increases up to 23 ohms at angular velocity 5674, then descends rapidly to minus 25 ohms at angular velocity 5938 and then increases again toward zero. The motion of the diaphragm markedly increases the resistance at certain frequencies and markedly decreases it at other frequencies. The formulas for computing the motional resistance values are given under heading V below.

Next, let us examine the motional reactance curve. The effect of the motion of the diaphragm is chiefly to decrease the reactance so that the free reactance is less than the damped reactance, giving usually a negative motional reactance, amounting to — 44.7 ohms at angular velocity 5800. The motional reactance is not always negative but shows small positive values in the neighborhood of angular velocities 4500 and 7000.

The resemblance of the motional resistance curves and the motional reactance curve of Figures 4, 5, and 6 to the curves of optical index of refraction and optical absorption plotted against frequency, in the neighboring of an absorption band, will at once strike the attention of the reader familiar with theoretical optics. A difference, however, exists on account of the hysteretic behavior of the iron in the telephone theory, as will be pointed out in the treatment under heading V below.

Effect of Motion of Diaphragm on Draft of Power.— Attention is next directed to the curve marked *Power* in Figure 4. This curve shows the excess of power sent into the telephone when freely vibrating over the power sent into it under the same impressed e. m. f. when damped. The excess of power (i. e. *motional power*) is plotted in microwatts against angular velocity of impressed e. m. f., and is seen to be different for different angular velocities corresponding to different frequencies. The maximum of motional power is in the neighborhood of angular velocity 5820 radians per second, and this is the period of the diaphragm, as is shown later by other methods of analyzing the data. The impressed e. m. f. in this experiment was maintained throughout at 0.3 effective volt.

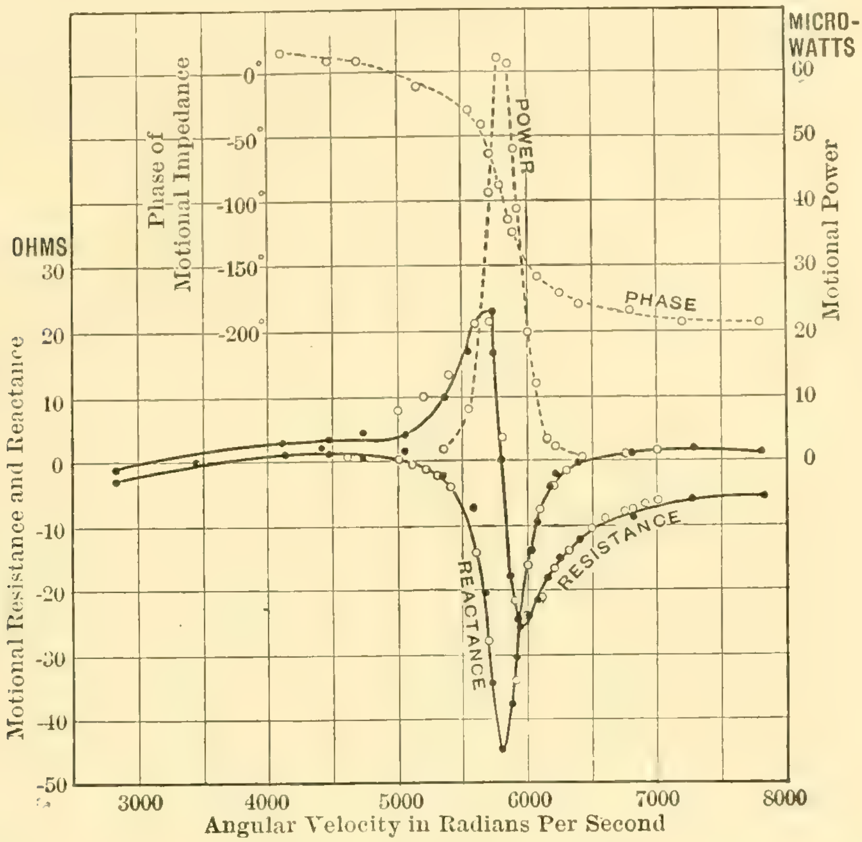

FIGURE 4. Curves of motional resistance, reactance, power, and phase, plotted against angular velocity, for watch-case receiver at 0.3 volt. Dots, observed; circles calculated.

The method of obtaining the motional power curve was as follows: Table III contains measurements of resistance and reactance of this receiver at different frequencies both while free and while damped. The square root of the sum of the squares of resistance and reactance gives directly the impedance. Dividing the impedance into the e. m. f. gives the effective current. The square of the free effective current multiplied by the free resistance gives the free power. Likewise, the square of the damped current multiplied by the damped resistance gives the damped power. The free power minus the damped power gives the *motional power.* These are tabulated for two receivers, for three series of measurements, in Tables VIII, IX and X.

It is not necessarily true that all of the motional power goes into energy of motion. The term means merely the excess of input when

TABLE VII.

Showing Product of R and L for Bipolar Receiver R_b with 0.42 Volt at Terminals.

Frequency Cycles per Second. n.	Damped Resistance R. Ohms.	Damped Inductance L. Henry.	RL.	Departure. Per cent.
766	174	.0357	6.21	.0
778	175	.0354	6.20	.1
792	176	.0356	6.26	.8
805	177	.0351	6.22	.1
822	181	.0347	6.28	1.0
836	181	.0342	6.19	.3
853	183	.0342	6.26	.8
870	185	.0337	6.24	.5
890	186	.0335	6.23	.2
754	175	.0362	6.33	2.0
744	171.5	.0364	6.24	.5
733	170	.0369	6.26	.8
722	172	.0372	6.20	.1
710	167.5	.0373	6.25	.7
704	168	.0376	6.31	1.6
548	145.7	.0419	6.11	1.6
428	128.5	.0463	6.20	.1
2464	306.5	.0192	5.90	5.0
	Average		6.21	

sounding over input when damped. As a matter of fact, the motional power is negative at some frequencies, as is shown in some of the curves

(e. g. Figures 5 and 6). Always, however, at consonance of the impressed e. m. f. with the period of the diaphragm, the motional power

TABLE VIII.

Values of Power. Bell Bipolar *Rb* at 0.3 Volt.

Frequency.		Power in Microwatts.		
n, Cycles per Second.	ω, Radians per Second.	Free.	Damped.	Motiona .
440	2760	325.1	308.7	16.4
512	3220	301.4	303	− 1.6
600	3770	290.5	297.0	− 6.5
670	4210	274.2	283.2	− 9.0
704	4420	270.0	275.0	− 5.0
720	4520	272.7	271.8	.9
744	4660	282.0	266.2	15.8
754	4730	288.8	265.2	23.6
770	4830	304.5	261.0	43.5
778	4880	373.9	262.4	111.5
780	4900	356.9	266.2	90.7
790	4960	438.5	259.2	179.3
792	4970	436.0	259.8	176.2
792	4970	433.0	270.0	163.0
792	4970	423.3	259.2	164.1
793	4975	438.0	260.2	177.8
794	4980	428.0	259.0	169.0
804	5050	364.3	259.2	105.1
808	5080	422.2	257.3	164.9
822	5160	310.4	256.7	53.7
824	5180	371.5	254.0	117.5
824	5180	365.9	250.0	115.9
826	5190	352.0	253.0	99.0
826	5190	376.5	253.0	123.5
832	5220	309.0	260.0	49.0
840	5280	276.7	253.8	22.9
868	5450	269.2	230.6	38.6
890	5590	257.1	241.9	15.2
892	5600	257.1	244.2	12.9
912	5730	244.3	233.3	11.0
940	5900	244.2	239.6	5.6
1000	6283	222.2	220.1	2.1
1060	6650	215.1	213.0	2.1
1152	7250	192.0	190.5	1.5
1248	7850	195.7	197.5	− 1.8
1648	10350	177.4	187.3	− 9.9
2468	15500	153.3	152.1	1.2

has a large positive value, which is no doubt correlative with the large amount of sound produced under this condition. With the receiver giving the curves of Figure 4 (the bipolar watch-case — cf. Table X) the motional power at resonance is 62 microwatts, which is 20% of the total free input (309 microwatts) and 25% of the total input at the same voltage with the diaphragm damped (247 microwatts). In the case of the bipolar Bell receiver R_b at 0.3 volt (see Figure 5 and

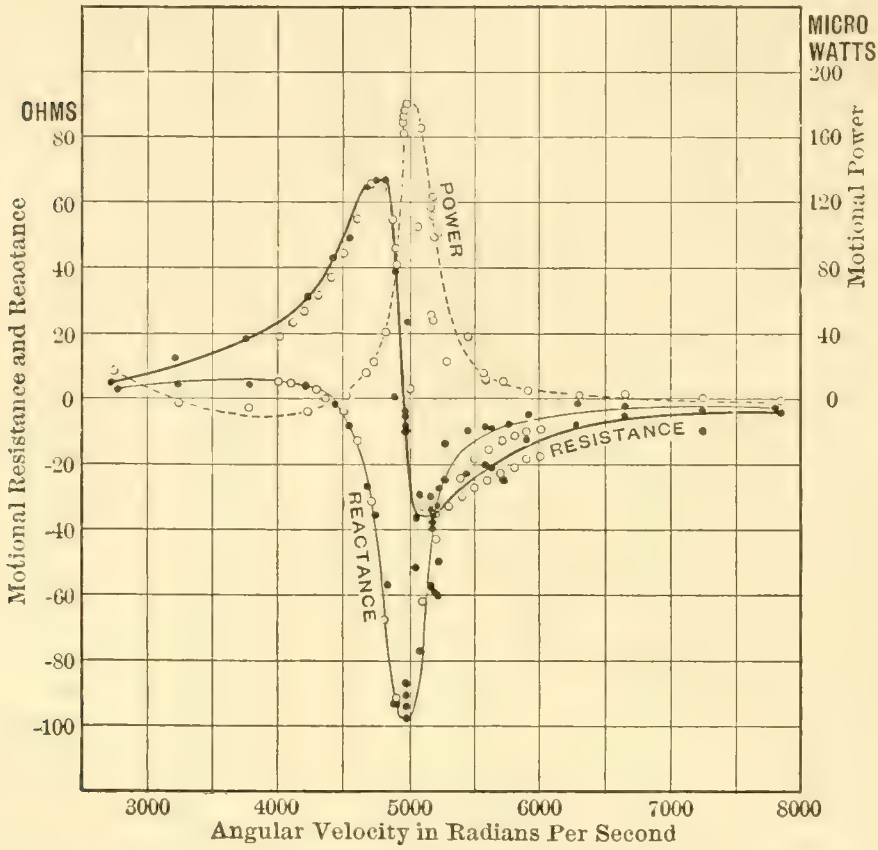

FIGURE 5. Curves of motional resistance, reactance, and power, *vs.* angular velocity, for Bell bipolar, with 0.3 volt. Dots, observed, circles calculated.

Table VIII) the motional power at resonance was 179 microwatts, which is 41% of the free input and 69% of the damped input. At 0.42 volt with receiver R_b (see Figure 6 and Table IX) the motional power was 338 microwatts at resonance, amounting to 40% of the total power input with diaphragm free and to 68% of the power input under the same e. m. f. with the diaphragm damped. That is to say, if one holds his finger on the diaphragm so as to damp it, and measures the power supplied to this receiver at 0.42 volt, the frequency being resonant with the period of the diaphragm, and then takes his finger off, the telephone emits a loud sound and the power input jumps up 68%.

An examination of the curves of Figures 4, 5, and 6 shows how this occurs. The effective resistance of the receiver, with the e. m. f. at resonance, is not very different when it is sounding and when it is damped; that is, the motional resistance is nearly but not quite zero. What causes the large consumption of power at the resonant frequency

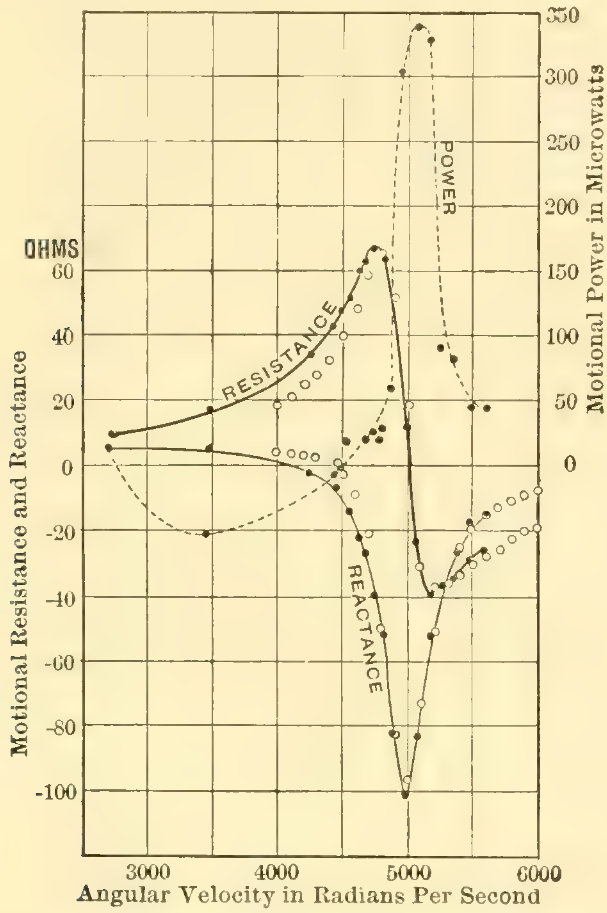

FIGURE 6. Same as Figure 5, but with 0.42 volt at terminals of Bell bipolar. Power points, calculated; other dots, observed; circles, calculated.

is the low value of the effective inductive reactance of the receiver at this frequency and the consequent large draft of current from the source. As we go away from the frequency of e. m. f. resonant with the period of the diaphragm, the motional power consumption may be due either to excess of free resistance over damped resistance or to the excess of free current incidental to a decrease of inductance by the motion.

Effect of Motion on Phase.— The phase angle of the motional impedance for the watch-case receiver is shown in the curve marked

Phase in Figure 4. The angles plotted in this curve were obtained by taking the antitangent of the ratio of motional reactance to motional

TABLE IX.

VALUES OF POWER. BELL BIPOLAR R_b AT 0.42 VOLT.

Frequency.		Power in Microwatts.		
n, Cycles per Second.	ω, Radians per Second.	Free.	Damped.	Motional.
428	2690	677	666	11
548	3446	556	609	− 53
704	4425	520	530	− 10
710	4468	527	528	− 1
722	4540	539	520	19
733	4610	539	518	21
744	4680	546	518	28
754	4740	538	518	20
766	4810	541	512	29
778	4885	570	511	59
792	4976	800	496	304
805	5060	833	495	338
822	5170	825	493	332
836	5253	587	495	92
853	5354	564	481	83
870	5470	524	478	46
890	5595	518	471	47

resistance and plotting the result, which is phase of motional impedance, against angular velocity. The meaning of this phase angle will be made plainer in the discussion of the circular graphs to follow.

Application to Sound Experiments.— It may be noted, in passing, that the effect of the reaction of the motion of the diaphram in modifying the electrical properties of the telephone receiver is of importance in experiments on sound, where an electrically driven tuning fork or telephone is used as the source of sound, because the power consumed in producing the sound may change with the change of the

TABLE X.

VALUES OF POWER. WATCH-CASE RECEIVER AT 0.3 VOLT.

Frequency.		Power in Microwatts.		
n, Cycles per Second.	Radians per Second.	Free.	Damped.	Motional.
451	2834	396	376	20
550	3456	328	330	− 2
653	4102	302	299	3
702	4410	276	280	− 4
712	4474	284	285	− 1
754	4738	274	274	0
804	5052	264	268	− 4
849.5	5340	258	256	2
884	5554	259	251	8
903	5674	271	250	21
913	5736	290	249	41
923	5800	309	247	62
934	5868	305	244	61
940	5906	291	243	48
945	5938	282	243	39
957	6014	261	241	20
968	6082	252	240	12
980	6158	244	241	3
993	6240	241.5	239	2.5
1020	6408	235	234	1
1084	6812	228	227	1
1157	7270	222	219	3
1250	7854	214	212	2
1846	11600	174	172	2

stationary sound-wave system in the room. In our experiments, the sound emitted from the test telephone was reflected from the various walls of the room and formed a stationary system with nodes and loops at various parts of the room. As an assistant walked about the room while the measurements were being made, it was found that the bridge, previously balanced, was thrown successively in and out of balance as the reflection and absorption of the assistant's clothing changed the stationary sound system when he walked through the room. Professor Sabine, in some experiments not yet published, had previously noticed the effect of a shift in the stationary wave system in modifying the draft of power by an electrically-driven tuning fork kept vibrating at constant amplitude. In accordance with the present experiments and as Professor Sabine previously suggested in

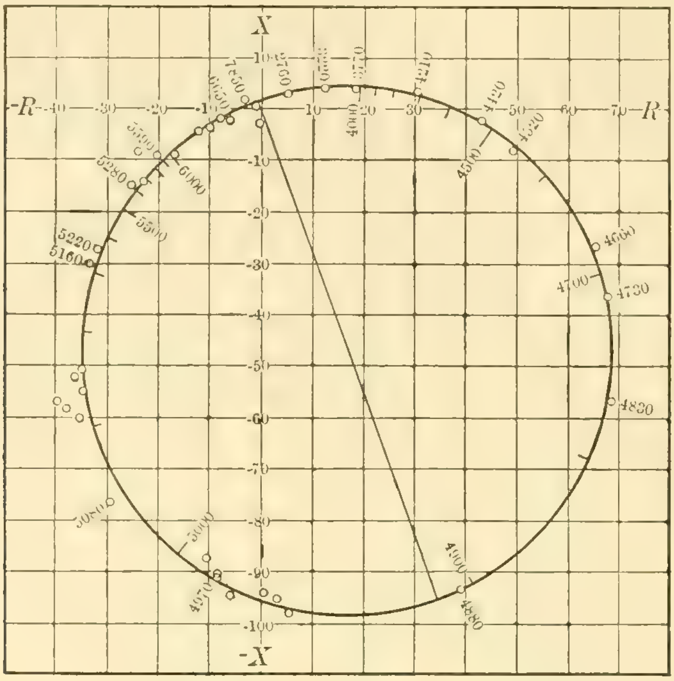

Figure 7. Circular graph for bipolar Bell receiver R_b with 0.3 volt at terminals. Diameter 103 ohms. Depression angle (2β) 70.5°. $\omega_0 = 4885$ radians per sec. $\Delta = 200$. Small circles observed. Internal ring numbers computed.

conversation with one of the writers, the phenomenon is seen to have its explanation in the change of resistance and reactance of the coil of the fork due to the variously affected motion of the fork.

Circular Graphs of Motional Impedance.— A very interesting result is obtained by plotting the motional reactance of a telephone as ordinate against the motional resistance as abscissa. The result is a point on the R X plane, and the point is different for different values of the angular velocity used in the measurement. The locus of this point, as the angular velocity is varied, is a circle passing through the origin; that is, through the point of zero motional resistance and zero motional reactance. Stated otherwise, if the motional impedance, $(R' - R) + j(X' - X)$, is plotted vectorially from a point as origin, the vector for any given frequency is the cord of a circle through the point. As the angular velocity of the impressed e. m. f. increases from zero to infinity, the free end of the vector impedance passes once around the circle. Circular graphs of this character are plotted in Figures 7 to 11 for different instruments or for different values of impressed

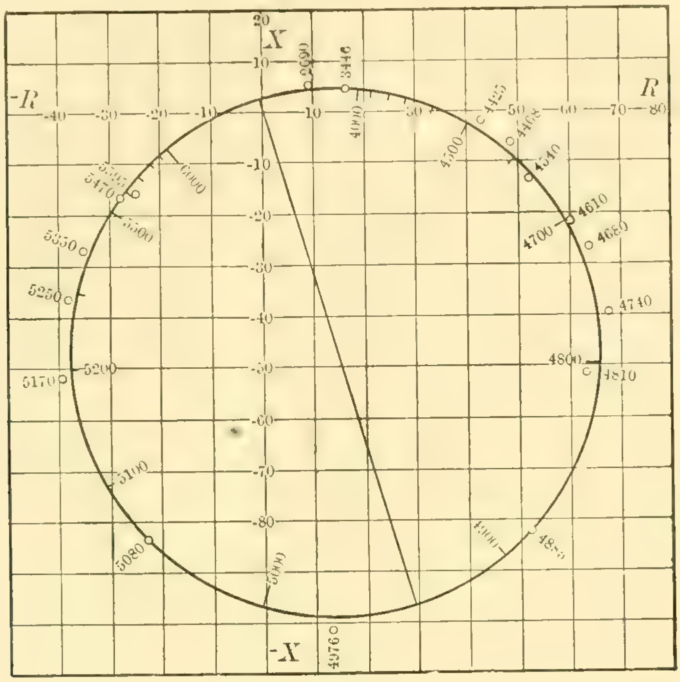

FIGURE 8. Circular graph for bipolar Bell receiver R_b with 0.42 volt at terminals. Diameter 103.5 ohms. Depression angle (2β), 73°; $\omega_0 = 4940$ radians per second; $\Delta = 200$. Small circles observed. Internal ring numbers computed.

e. m. f. These several circular graphs are on different scales, and are summarized to the same scale in Figure 12.

On each of the circular graphs, all of the observed points, for a given receiver with a given impressed e. m. f., are plotted as small circles. The measured angular velocities of the impressed e. m. f. for the observed points are printed at the *outside* of the circular locus. To avoid crowding, not all of the points are designated with angular velocities, and in the selection of the points to bear numerical designation of angular velocity, those points were chosen, for which the measurements were made with especial precaution as to voltage and frequency.

The numbers placed *inside* of each of the circular loci are computed values for the theoretical distribution of angular velocities around

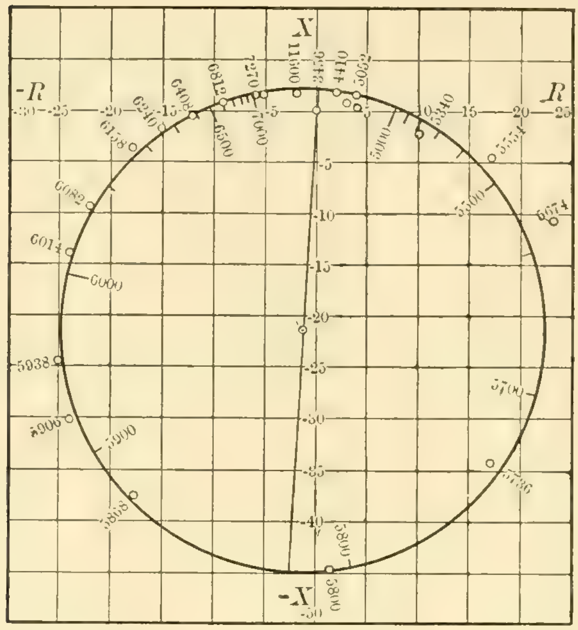

FIGURE 9. Circular graph for watch-case receiver with 0.3 volt at terminals. Diameter 47 ohms. Depression angle (2β) 93°. $\omega_0 = 5820$ radians per second. $\Delta = 150$. Small circles observed. Internal ring numbers computed.

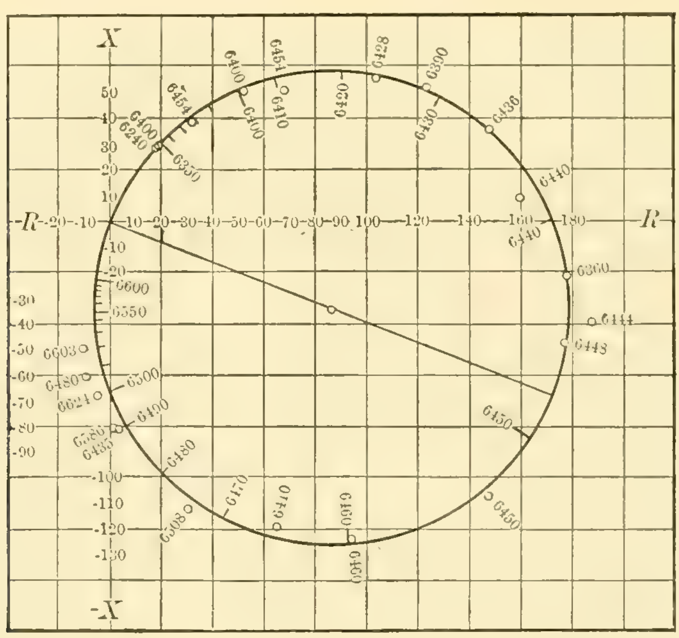

FIGURE 10. Circular graph for experimental monopolar receiver, with 0.3 volt at terminals. Diameter 185.4 ohms. Depression angle (2β) 21.5°. $\omega_0 = 6448$ radians per second. $\Delta = 20$. Small circles observed. Internal ring numbers computed.

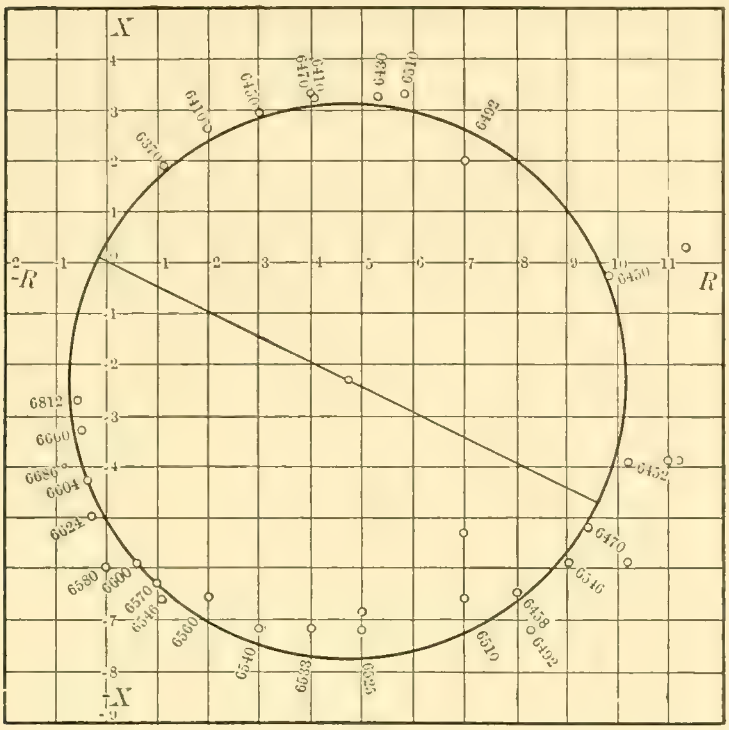

FIGURE 11. Circular graph for experimental bipolar receiver, with one volt at terminals. Diameter 10.9 ohms. Depression angle (2β) 26.5°. $\omega_0 = 6465?$; $\Delta = 20?$.

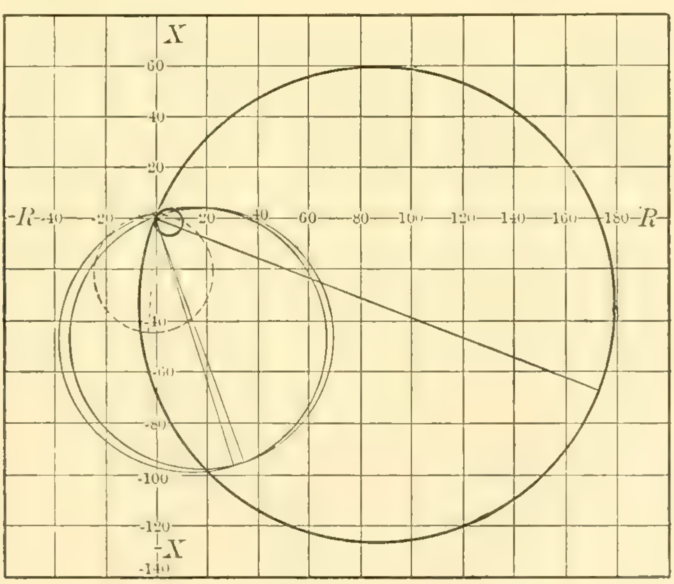

FIGURE 12. Circular graphs collected to one and the same scale. The largest circle and the smallest circle are graphs with experimental receivers. Two light circles near together are graphs with bipolar Bell receiver. Broken circle is graph with watch-case.

the circular graphs. The formulas by which this distribution has been computed are derived under heading V below.

The quantities of theoretical and practical importance in these circular graphs are:

1. The length of diameter of the circular graph for a particular receiver.
2. The dip of the diameter below the axis of resistance.
3. The rate of change of angular velocities around the circle.
4. The angular velocity at the end point of the diameter, remote from the origin, and
5. The impedance at this point.

The significance of these several quantities will appear in connection with the discussion of the theory of the problem, which follows.

V. Theory of the Reactive Effects of Motion of the Diaphragm on the Electrical Constants of the Receiver.

An exact treatment of the electrical properties of a coil containing a magnetic core in proximity to a moving magnetic membrane offers great difficulty. If, however, we confine our attention to terms of the first order, we can obtain a sufficiently close approximation to a solution, to permit an interpretation of the preceding experimental results.

Assumptions Regarding Mechanical Magnitudes.— To this end we shall assume, so far as concerns the fundamental mode of vibration of the diaphragm,

(1) That the elastic restoring force of the diaphragm is all concentrated at the center of the diaphragm, and is proportional to the displacement;

(2) That the motion is opposed by a frictional force proportional to the velocity and also concentrated at the center of the diaphragm; and

(3) That the actual distributed mass of the diaphragm may be replaced by an equivalent mass concentrated at the center of the diaphragm.

Motion of the Center of the Diaphragm under a Pull Maintained Sinusoidal.— As a first step toward the solution, let us assume the diaphragm to be solicited by a force which is maintained sinusoidal; then (*cf.* Figure 13)

$$sx + r\dot{x} + m\ddot{x} = f = F\epsilon^{j\omega t} \qquad \text{dynes } \angle \quad (5)$$ [2]

[2] The sign $\angle$ following the unit indicates that the equation should be interpreted vectorially, or in complex quantities.

in which

x = the displacement of the effective mass of the diaphragm from its position of rest (cm.),

$\dot{x}$ = the displacement velocity (cm/sec),

$\ddot{x}$ = the displacement acceleration (cm/sec^2),

s = the elastic force per unit displacement (dynes/cm),

r = the resisting force per unit velocity (dynes per cm. per sec.)

f = $F\epsilon^{j\omega t}$, the impressed moving force measured in the direction of x toward the poles (dynes),

$\omega = 2\pi n$, the angular velocity of the impressed force (radians per second), and

n = the frequency of the impressed sinusoidal force (cycles per second).

$j = \sqrt{-1}$

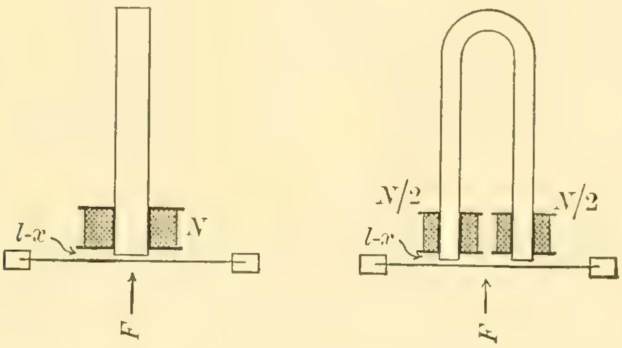

FIGURE 13. Diagram of receivers. N is number of turns. F is force on diaphragm with direction of arrow; l is normal gap length; and x displacement toward poles.

The solution of equation (5) for velocity of displacement, after a steady state has been attained, is well known, and may be written in the form

$$\dot{x} = \frac{f}{r + j\left(m\omega - \frac{s}{\omega}\right)} = \frac{f}{z} = \frac{F\epsilon^{j\omega t}}{z} \quad \text{cm/sec} \angle \qquad (6)$$

in which

$$z = r + j\left(m\omega - \frac{s}{\omega}\right) \qquad \text{dyne sec/cm} \angle \qquad (7)$$

The quantity z may be called "vector mechanical impedance" from its analogy to vector electrical impedance.

We may further write for abbreviation

$$|z| = \sqrt{r^2 + \left(m\omega - \frac{s}{\omega}\right)^2} \quad \text{dyne sec/cm} \qquad (8)$$

and

$$a = tan^{-1}\left\{\frac{m\omega - \frac{s}{\omega}}{r}\right\} \quad \text{radians} \qquad (9)$$

The quantities entering in the above equations, and their analogous electrical quantities, are tabulated below.

Mechanical Quantity.		Electrical Quantity.	
Velocity of displacement	$\dot{x}$	i	Current
Mechanical force	f	e	Electromotive force
Resistance (i. e., force of resistance per unit velocity)	r	R	Resistance (i. e., e. m. f. of resistance per unit current)
Effective mass (i. e., force per unit time-rate of change of velocity)	m	L	Self-inductance (i. e., e. m. f. per unit time-rate of change of current)
Elastic force per unit displacement (i. e., per unit time-integral of velocity)	s	$1/C$	Reciprocal of capacity (i. e., e. m. f. per unit time-integral of current)
Vector mechanical impedance	z	Z	Vector electrical impedance
Mechanical Impedance	$\|z\|$	$\|Z\|$	Electrical impedance
Mechanical phase-angle	a	θ	Electrical phase angle
Mechanical inertia reactance	$m\omega$	$L\omega$	Electrical inductive reactance
Mechanical elastic reactance	s/ω	$(1/C\omega)$	Electrical capacity reactance

Circular Graph of Velocity.— By equation (6) $\dot{x}$ is seen to be sinusoidal, with amplitude $\frac{F}{|z|}$ and lagging by an angle a behind the impressed force. A geometrical representation of the amplitude and phase of $\dot{x}$ is given in Figure 14. In the left hand part of the figure Op is a representation of the vector mechanical impedance z. As ω changes from zero to infinity, the point p moves along the straight line Xp from minus infinity to plus infinity, parallel to OY. In the right hand part of the figure, the circle is the vector graph of F/z, which is given in magnitude and direction by OP. This circle is obtained by taking the reciprocal of the straight line locus of the vector

z and multiplying the reciprocal by F, which gives a circle of diameter F/r symmetrically disposed with reference to the axis of reals.

The use of this circle is as follows. For a given value of ω find the angle α by equation (9) and lay off this angle negatively at O; then the length of the chord OP of the circle gives the amplitude $\frac{F}{|z|}$ of $\dot{x}$,

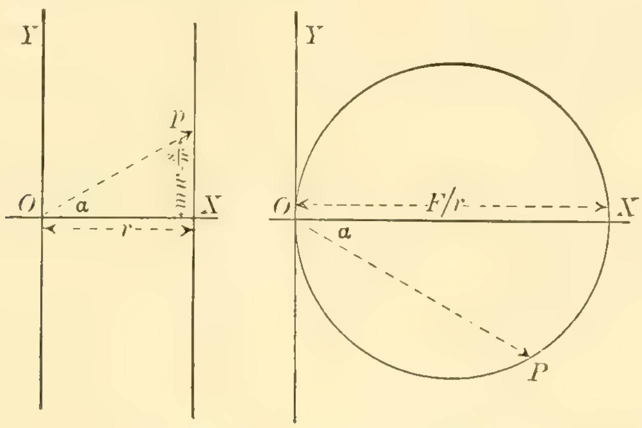

FIGURE 14. Left straight line graph of z. Right, circular graph of F/z.

for the given ω, and the angle is the angle of lag of $\dot{x}$ behind the impressed force. As ω changes from zero to infinity, the point P moves negatively once around the circle from O, through X, back to O.

Magnetic Flux as Dependent on Current and Mechanical Displacement.— In the problem under consideration, the pull f acting on the diaphragm is determined by the magnetic flux through the air gap, or air gaps, of the receiver. If ϕ is the mean flux through the active part of the magnetic circuit, we have, for a bipolar receiver, approximately

$$\phi = \frac{\mathfrak{F}}{\mathfrak{R}} = \frac{\mathfrak{F}_0 + 4\pi Ni}{\mathfrak{R}_0 + \frac{2(l - x)}{S}} = \mathfrak{B} S \quad \text{maxwells} \quad (10)$$

and for a monopolar receiver, approximately

$$\phi = \frac{\mathfrak{F}}{\mathfrak{R}} = \frac{\mathfrak{F}_0 + 4\pi Ni}{\mathfrak{R}_0 + \frac{l - x}{S}} \quad \text{maxwells} \quad (11)$$

in which

$\mathfrak{F}$ = the total m. m. f. due to the permanent magnet and to the current i in the coils (gilberts),

$\mathfrak{F}_0$ = m. m. f. due to permanent magnet alone (gilberts),

$\mathfrak{R}$ = total reluctance of the magnetic circuit (oersteds),

$\mathfrak{R}_0$ = reluctance of circuit exclusive of that of the gaps (oersteds),

N = total number of turns in the receiver coils,

i = instantaneous current in the coils assumed to vary sinusoidally, or according to the real part of $I\epsilon^{j\omega t}$ (absamperes),

l = normal air-gap between poles and diaphragm (cm.),

$\mathfrak{B}$ = mean flux density in the air gap (gausses), and

S = area of one gap (cm^2).

The Equations of Current and Motion.— We can now express the pull on the diaphragm in terms of the flux. It is a well known fact, which may be derived from energy relations, that the pull on the diaphragm is

$$f = \frac{\phi^2}{4\pi S} \text{ for a bipolar receiver} \qquad \text{dynes} \quad (12)$$

and

$$f = \frac{\phi^2}{8\pi S} \text{ for a monopolar receiver} \qquad \text{dynes} \quad (13)$$

If now f_i is used to denote the part of the pull due to the current i, and if this is small in comparison with the pull due to the permanent magnet, we may write

$$f_i = \frac{df}{di}\, i \qquad \text{dynes} \angle \quad (14)$$

which by substitution from equations (12) and (10) becomes

$$f_i = \frac{2\phi}{4\pi S}\frac{d\phi}{di}\, i$$

$$= \frac{2N\phi}{\mathfrak{R}S}\, i = \frac{2N\mathfrak{B}_0}{\mathfrak{R}}\, i \text{ for a bipolar} \qquad \text{dynes} \angle \quad (15)$$

and

$$f_i = \frac{N\mathfrak{B}_0}{\mathfrak{R}}\, i \text{ for a monopolar.} \qquad \text{dynes} \angle \quad (16)$$

In equations (15) and (16) $\mathfrak{B}_0$ has been substituted for $\mathfrak{B}$, since the increment in $\mathfrak{B}_0$ due to i, when multiplied by i, is assumed to be a second order effect.

In order to avoid carrying through separate discussions for the bipolar and for the monopolar receiver, and in order to simplify the equations, let us write

$$A = \frac{2N\mathfrak{B}_0}{\mathfrak{R}} \text{ for the bipolar receiver, and} \qquad \text{dynes/absampere} \quad (17)$$

$$A = \frac{N\mathfrak{B}_0}{\mathfrak{R}} \text{ for the monopolar receiver;} \qquad \text{dynes/absampere} \quad (18)$$

then, for either instrument,

$$f_i = Ai \qquad \text{dynes } \angle \quad (19).$$

Equations (15), (16) and (19) assume that the pull on the diaphragm due to i is in the phase with i; but with hysteresis and eddy currents present, the electromagnetic force will lag[3] behind the current i by an angle β_1; whence the force on the diaphragm due to the current i becomes, by eq. (19),

$$f_i = A\, i \backslash\overline{\beta_1} \qquad \text{dynes } \angle \quad (20)$$

Consequently, by equation (6),

$$\dot{x} = \frac{A\, i \backslash\overline{\beta_1}}{z} \qquad \text{cm/sec } \angle \quad (21)$$

The e. m. f. induced in the coils by the motion of the diaphragm will be, in the absence of hysteresis,

$$e_x = N\frac{d\phi}{dt} = N\frac{\partial\phi}{\partial x}\dot{x} \qquad \text{abvolts, } \angle \quad (22)$$

and by differentiating equation (10) or (11), equation (22) gives to a first approximation

$$e_x = \frac{2N\mathfrak{B}_0\dot{x}}{\mathfrak{R}} = A\dot{x} \qquad \text{abvolts, } \angle \quad (22a)$$

and by substitution from equation (21)

$$e_x = \frac{A^2 i \backslash\overline{\beta_1}}{z}. \qquad \text{abvolts } \angle \quad (23)$$

However, it should be noted that there is also a hysteretic lag of flux with change of gap, and this will cause the induced e. m. f. to lag by a certain angle β_2 behind $\dot{x}$, so that equation (23) should be changed to

$$e_x = \frac{A^2 i \backslash\overline{\beta_1 + \beta_2}}{z} \qquad \text{abvolts } \angle \quad (24)$$

3 On the question of Constancy of β_1 see VI below.

If L and R are the inductance and resistance of the receiver when damped, the impedance of the damped receiver will be

$$Z = R + jL\omega \qquad \text{absohms} \angle \quad (25)$$

and if e is the instantaneous value of the impressed e. m. f. of the type $E\epsilon^{j\omega t}$, we shall have

$$e = iZ \qquad \text{abvolts} \angle \quad (26)$$

But owing to the influence of the e. m. f. of motion, the last equation becomes modified to

$$e - e_x = iZ \qquad \text{abvolts} \angle \quad (27)$$

or

$$e = iZ + e_x \qquad \text{abvolts} \angle \quad (28)$$

That is by equation (24)

$$e = i\left\{ Z + \frac{A^2}{z}\backslash\overline{\beta_1 + \beta_2} \right\} = iZ' \quad \text{abvolts} \angle \quad (29)$$

where Z' is the free impedance of the receiver.

This means that the impedance of the receiver has become increased, through the vibration of the diaphragm, by *a motional impedance:*

$$Z' - Z = \frac{A^2}{z}\backslash\overline{\beta_1 + \beta_2} \qquad \text{absohms} \angle \quad (30)$$

This motional impedance, being the reciprocal of the vector equation of a straight line with ω as variable, is a circle for variable ω, and has a diameter $\frac{A^2}{r}$, depressed below the axis of reals by an angle $\overline{\beta_1 + \beta_2}$.

As to the relative values of β_1 and β_2 it seems reasonable that whether the change of flux of a circuit is caused by a small change of current, changing the m. m. f., or by a small change of gap-length, changing the reluctance, the angle of lag of flux behind the cause is the same; that is $\beta_2 = \beta_1 = \beta$ (Say). This is borne out by one of our experiments to be described below (see VI). With this equivalence substituted in equations above, we obtain,

$$Z' - Z = \frac{A^2}{z}\backslash\overline{2\beta} \qquad \text{absohms} \angle \quad (31)$$

Consequently, if we vary ω from 0 to $+\infty$, keeping the impressed e. m. f. and all other quantities constant, the motional impedance $Z' - Z$ has a circular graph through the origin, with its principal diameter of length $\frac{A^2}{r}$ depressed 2β below the axis of reals. Equation (31) is the theoretical equation to the circular graphs of Figures 7 to 12.

Replacing the vector z of equation (31) by its absolute value $|z|$ and angle α, we have

$$Z' - Z = \frac{A^2}{|z|} \underline{\backslash 2\beta + \alpha} \qquad \text{absohms } \angle \quad (32)$$

Equation (32) may be analysed into

$$R' - R = \frac{A^2}{|z|} \cos (2\beta + \alpha) \qquad \text{absohms} \quad (33)$$

$$X' - X = \frac{-A^2}{|z|} \sin (2\beta + \alpha) \qquad \text{absohms} \quad (34)$$

in which

$$|z| = \sqrt{r^2 + \left(m\omega - \frac{s}{\omega}\right)^2}, \qquad \text{absohms} \quad (35)$$

and

$$\alpha = \tan^{-1} \left[\frac{m\omega - \frac{s}{\omega}}{r} \right] \qquad \text{radians} \quad (36)$$

are functions of ω. The quantity A, involving $\mathfrak{B}_0$ and $\mathfrak{R}$, might be expected to vary with variation of ω, but an examination of the experimental results shows that, with the excitations employed, not much error is introduced by considering A and also β independent of ω.

Equations (32), (33), and (34) are in convenient form for computation, and permit an easy determination of some of the important mechanical constants of the diaphragm.

For example, if we let ω_0 be the angular velocity of impressed mechanical force for which the sustained vibration of the diaphragm is in resonance, we see from equation (6) above that

$$\omega_0 = \sqrt{\frac{s}{m}} \qquad \text{radians/sec.} \quad (37)$$

Now, if ω, the angular velocity of the impressed electromotive force in the telephone circuit, is equal to $\omega_0 = \sqrt{\frac{s}{m}}$, it is seen by equation (36) that α becomes zero; hence the value of ω, which in the experimental circular graphs of Figures 7 to 10 lies at the remote end of the principal diameter is the $\omega = \omega_0$ for which the diaphragm in sustained vibration is resonant. This gives a simple and accurate method of determining ω_0 for a telephone diaphragm.[4]

Again, let Δ be the logarithmic decrement per second of the diaphram, if vibrating under no external force, then by the theory of elasticity,

$$\Delta = \frac{r}{2m}, \qquad \text{numeric/sec.} \quad (38)$$

whence from (36)

$$\omega \tan \alpha = \frac{\omega^2 - \omega_0^2}{2\Delta} \qquad \text{numeric/sec.} \quad (39)$$

Differentiating (39) with respect to α, we obtain

$$\frac{d\omega}{d\alpha} \tan \alpha + \omega \sec^2 \alpha = \frac{\omega}{\Delta} \frac{d\omega}{d\alpha} \qquad \text{numeric/sec.} \quad (40)$$

and if $\alpha = 0$,

$$\left[\frac{d\omega}{d\alpha}\right]_0 = \Delta; \qquad \text{numeric/sec.} \quad (41)$$

That is, in the experimental circular graphs, the rate of change of ω with change of α, at the remote end of the principal diameter, is the logarithmic decrement per second of the diaphragm. This quantity cannot, however, be obtained with the precision with which ω can be obtained.

Another method of obtaining Δ is by taking the values of ω_1 and ω_2 which lie respectively 45° below and 45° above the principle diameter,— these angles being measured at the origin, not at the center. For these points $\tan \alpha$ is respectively $+ 1$ and $- 1$; whence from (39)

$$2\Delta\omega_1 = \omega_1^2 - \omega_0^2$$
$$-2\Delta\omega_2 = \omega_2^2 - \omega_0^2$$

and by subtraction and division by $2(\omega_1 + \omega_2)$,

$$\Delta = \frac{\omega_1 - \omega_2}{2} \qquad \text{numeric/sec.} \quad (42)$$

[4] For another method of finding ω_0 from the humming tone of a telephone receiver, see a paper by A. E. Kennelly and W. L. Upson, Proc. Am. Phil. Soc., 1908, "The Humming Telephone."

Thus we have methods of determining both ω_0 and Δ. The experiments, on the other hand, do not permit a direct determination of the quantities m, r, and s; but it would seem that by adding a known mass, as a small load, to the center of the diaphragm and repeating the series of measurements, these quantities should be capable of determination.

VI. Comparison of Experimental Results with Theory.

An examination of the experimental results with the aid of the theory above developed gives the following results, which may be called the characteristics of the several receivers (Table XI):

TABLE XI.

Summary of Characteristics of Receivers.

	Bipolar Bell R_b at 0.3 Volt.	Bipolar Bell R_b at 0.42 Volt.	Watch-case at 0.3 Volt.	Experimental Monopolar at 0.3 Volt.	Experimental Bipolar at 1 Volt.
Diameter of motional impedance circle in ohms.	103	103.5	47	185.4	10.9
Depression angle (2β) in degrees	70.5	73	93	21.5	26.5
ω_0 in radians per second	4885	4940	5220	6448	6465 ?
Log. dec. per second. (Δ)	200	200	150	20.	20 ?

The method of obtaining these characteristics was as follows: The circular graphs of Figures 7 to 11 were plotted. The diameter of the motional impedance circle and the angle of depression of this diameter below the axis of $R'-R$ could be measured off at once on the diagram. The value of ω at the free end point of the diameter could also be read or obtained by interpolation; this ω is the ω_0 of the diaphragm. The logarithmic decrement per second Δ could have been obtained by either of the two methods derived in the discussion of the theory, equations (41) or (42); but a third method was employed; namely, by the use of the more general equation (39), in which several values of α and the corresponding values of ω from the circular graphs were substituted, and the values of Δ so obtained were averaged.

Having now obtained the constants of Table XI, the theoretical distribution of angular velocities around each of the circular graphs of Figures 7 to 11 were calculated by equation (39), and these theoretical values are designated by numerals on the inside of the circular graphs.

The values of $R'-R$ and of $X'-X$ corresponding to these theoretical values of ω were then plotted as the circles on the rectangular graphs of motional resistance and motional reactance in Figures 4, 5, and 6. It is seen that the agreement of the computed and observed points in these Figures 4 to 6, while not exact throughout the entire range, is yet sufficiently good to show that the theory is essentially correct.

Another significant point in the theory is the interpretation we have given to the depression angle 2β of the circular graphs. We interpreted β to be the angle by which the magnetic flux lags behind the magnetizing current in the telephone receivers. To test this point, this angle of lag of magnetic flux behind magnetizing current was independently measured with the experimental bipolar receiver. This receiver had a separate secondary, or exploring, coil wound around the ends of its poles, near the diaphragm. The e. m. f. generated in this exploring coil is in phase with the time rate of change of flux; and the phase of this e. m. f. was compared with the phase of the alternating current in the exciting coils in two ways (1) by a three-voltmeter method and (2) by an alternating current potentiometer.

In the three-voltmeter method, a known resistance was put in series with the exciting coils, and one end of the exploring coil was connected to the point between the exciting coil and the known resistance. With the frequency and the e. m. f. about the telephone kept the same as that used in the bridge measurements (i. e., the e. m. f. of 1 volt, and the frequency near the resonant frequency of the diaphram) voltages were measured about the known resistance (20 ohms), about the exploring secondary, and about the two in series. These voltages, being small, were measured by a crystal rectifier in series with a galvanometer,[5] — the galvanometer and rectifier having been calibrated immediately before and after the experiment by an a. c. potentiometer operating at the frequency employed.

The readings of voltage were very consistent, and were as follows in a typical case:

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[5] G. W. Pierce: Phys. Review, **25**, p. 31, 1907; ibid., **28**, p. 153, 1909.

Voltage about 20 ohms	= 0.129 volt
" about secondary	= 0.125 "
" about both	= 0.196 "
" about both with secondary reversed	= 0.161 "

Substitution of the first three of these values in the formula for an obtuse-angle oblique triangle gives 79°, as the angle by which the secondary voltage leads the primary current. This is the angle by which the time derivative of the magnetic flux leads the primary current. The flux itself lags its time derivative by 90°, and therefore lags the primary current by 90°—79° = 11°.

Again, a substitution of the first, second, and fourth value of above table in the formula for an acute-angle oblique triangle gives for the flux lag angle the value 11.5°.

This angle of lag of flux behind the magnetizing current was found to be nearly independent of the frequency. To illustrate this, and as a further confirmation of the result obtained by the three-voltmeter method with the crystal rectifier and galvanometer as voltmeter, a second measurement was made by an entirely different method; namely, by the use of a Drysdale alternating-current potentiometer, with a 60-cycle current, and with a vibration galvanometer as indicating instrument. The method employed in this experiment consisted in first measuring the magnitude and phase of the primary current, and then the magnitude and phase of the voltage in the secondary winding. The difference between these two phases, subtracted from 90°, gives the required angle of magnetic flux lag. Balance was in each case indicated by getting a zero deflection of the vibration galvanometer. This method gave 12.5° as the angle by which the flux in the telephone lags behind the magnetising current.

The three values obtained by direct measurement for the flux lag, which should be the angle β according to the theory above proposed, are 11°, 11.5°, and 12.5°; whereas half the depression angle, for this telephone, which, according to the theory, should also be the angle β, is 13.2°. The agreement is not as good as might be desired for a perfect confirmation of the proposed theory; but in view of the difficulty of measuring small angles of lag in circuits containing voltages of the order of 0.1 volt, and in view of the fact that the experimental telephone receiver constructed for this purpose had to be complicated by auxiliary secondary windings and also unfortunately had a diaphragm mounted in such a way as to have a very large temperature coefficient of vibration period, which rendered difficult an accurate determination of the points of the circular graph, the writers believe that the departure of a degree or two in the value of β, as obtained by direct measurement from its value as obtained by the circular graphs, is not unsatisfactory.

VII. Summary of Results.

1. The resistance and inductance of several telephone receivers were measured over a wide range of frequencies with their diaphragms both free and damped.

2. The damped resistance is approximately a quadratic function of the angular velocity of impressed e. m. f. (see equations (1) and (2)).

3. Although the damped resistance and the damped inductance both change with the frequency of e. m. f., their product is approximately constant, independent of the frequency, over a considerable portion of the range of audible frequencies (see eq. (3) and (4) and Table VII).

4. The damped reactance of one form of standard bipolar Bell receiver is approximately equal to its damped resistance, over a considerable range of frequency; so that the current lags the e. m. f. by 45° (see Figure 2).

5. The free resistance and reactance of telephone receivers go through marked changes with changes in frequency of constant e. m. f. in the neighborhood of the natural frequency of their diaphragms (*cf.* Figures 4–6).

6. The motional resistance and motional reactance (by which is meant excess of free resistance of reactance over damped resistance or reactance) conform accurately to certain simple laws as follow:

   I. The motional reactance plotted as ordinates against the motional resistance as abscissas, as the frequency of constant impressed e. m. f. is changed from zero to infinity, gives a circular locus, with various interesting characteristics (*cf.* Figures 7–12).

   II. The rectangular plots of motional reactance and motional resistance against angular velocity of constant impressed e. m. f. give curves somewhat analogous to the curves of index of refraction and absorption of light in an optical medium in the neighborhood of an absorption band (*cf.* Figures 4–6).

7. The power taken by a telephone receiver when sounding at 0.3 volt applied voltage may exceed by 68% the power taken from the

same e. m. f. when the diaphragm is damped (Figures 4–7 and Tables VIII–X).

8. A theoretical explanation of the phenomena is given, and computations are submitted in comparison of experiment and theory (Headings V and VI).

9. The vibration constants of the diaphragms of the several receivers are deduced and collected (Table XI).

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CONTRIBUTIONS FROM THE CRYPTOGAMIC LABORATORIES OF HARVARD UNIVERSITY. — No. LXIX.

NEW OR CRITICAL LABOULBENIALES FROM THE ARGENTINE.

By Roland Thaxter.

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By Roland Thaxter.

Presented April 10, 1912. Received July 28, 1912.

The rapid accumulation during the past six years of Laboulbeniales, which have come to me from various parts of the world and now include some hundreds of new species and genera, has forced me to abandon my intention to figure all new forms as they were published; and it has again become necessary to resort to preliminary diagnoses, a third series of which is entered on with the present paper. It is, however, my purpose to illustrate all the species described in this series as soon as the necessary figures can be prepared and published. The exotic material which is now available is not only very varied, but is in far better condition than that which has formerly been obtained from dried specimens of insects, for the reason that a majority of the hosts have been collected directly into alcohol and the parasites removed before drying.

The examination of large series of forms in good condition has inevitably led to some modification of my views in regard to the limitations of certain genera and species, and while it has in some instances made clearer relationships or differences that were formerly in doubt, it has at the same time served to break down distinctions which were formerly regarded as more or less crucial, so that it has seemed best to modify the treatment of certain genera and species. Thus in the present paper, the limits of *Corethromyces*, for example, are considerably extended to include several genera hitherto kept distinct, and other changes will be noticed applying both to species and genera, which have seemed advisable in the light of a more complete knowledge of numerous forms.

The materials here considered were collected in Argentina, chiefly in the Buenos Aires region, the host insects having been captured for the most part by myself in the Parque 3-de-Febrero, at Palermo, a suburb on the river above Buenos Aires: in the grounds of the Escuela Regional de Santa Catalina near the station of Llavallol, where a small planted wood of various trees affords a good collecting ground already familiar to mycologists, by name at least, from the large number of fungi collected there and described by Prof. Carlos Spegazzini, to whom I am much indebted for guiding me to this locality as well as to the Isla de Santiago near La Plata, where I spent two days collecting. Many hosts were also obtained in the grounds of the Quinta Mackern, at Temperley, a town about ten miles south of Buenos Aires, where I spent several weeks in the spring of 1906.

To Dr. Propile Spegazzini I am greatly indebted for numerous miscellaneous beetles which he kindly collected for me at La Plata and in Tucuman, both during my visit and after my return to the United States: to the Director of the Museo Nacional at Buenos Aires, and to Dr. J. Brèthes I am under obligations for various courtesies and for the privilege of examining the entomological collections of the Museum. For the determination of certain of the hosts I am indebted to Mr. Samuel Henshaw, Dr. Fenyes, Dr. Max Bernhauer, M. Pic, Dr. Malcolm Burr, Dr. Erno Csici and Col. Casey. To all these gentlemen I desire to express my appreciation of their kindness in thus assisting me.

With the exception of perhaps a half dozen species, of which the material is either too scanty or not in condition for description, the following enumeration includes all the forms obtained. As will be seen, a majority of them are hitherto undescribed, but it has seemed desirable also to append a list of the species obtained which are already known, and are listed below in alphabetical order. Of these there are in all forty-nine species, while of the new forms sixty-eight are included, with nine new generic types.

Dimeromyces Anisolabis nov. sp.

Male individual, quite hyaline. Receptacle consisting of four superposed cells obliquely separated, except the upper; the basal subtriangular, larger than the two subequal cells above it, of which the upper always bears an antheridium, while a second may often arise from the cell next below it. The antheridia rather stout and short, the venter and stalk-cells about as long as the abruptly distinguished stout neck, which is bent abruptly outward distally. Appendage consisting of three superposed cells subtended by a more or less conspicuous somewhat prominent red-brown septum; the tip of the appendage hardly extending to the tip of the antheridium. Total length to tip of antheridium, including foot (7 μ), 58 μ. Appendage 20 μ. Receptacle, exclusive of foot, 18–20 μ. Antheridium, including stalk-cell, 31 $\times$ 8 μ.

Female individual, almost hyaline, the perithecium becoming faintly yellowish. Receptacle consisting of five successively smaller cells superposed obliquely, except the uppermost which subtends the primary appendage, and from which it is separated by a red-brown septum, the subterminal cell also bearing a similar somewhat larger, usually five-celled appendage, distinguished from its small subtending cell by a red-brown septum; the subbasal cell of the receptacle bearing a still larger appendage, the somewhat irregular subtending cell of which projects on its inner side and is distally and externally separated from the slightly divergent and inflated portion of the appendage by a narrower deeply blackened isthmus, which includes a portion of the subtending cell, and more than half of the basal cell of the appendage proper. Perithecium usually single, a second rarely developed from the terminal cell of the receptacle, arising between its two appendages; long slender slightly enlarged distally, the tip not clearly distinguished, tapering slightly, inflated at the apex. Perithecia 75–100$\times$14 μ. Receptacle, exclusive of foot, 25–30$\times$20 μ. Primary appendage about 40 μ. Lowest appendage, including its subtending cell, 60–70 μ. Total length to tip of perithecium, including foot, 100–150 μ.

On the inferior surface of the abdomen, near the tip, of *Anisolabis annulipes* Luc., Palermo, No. 1682.

This species is very closely allied to *D. Forficulae*, and may prove only a variety, although the abundant material does not indicate that the form is variable. The male is most readily distinguished by the presence of only one suffused septum in its appendage, as well as by its shorter stouter form and outcurved antheridial necks. The two appendages arising in the female of *D. Forficulae* from the terminal cell of the receptacle, are replaced by only one, and the character of the lower appendage, and the form of the perithecium are also different. A third closely allied form is known to me from the Amazon region.

Dimeromyces Corynitis nov. sp.

Male individual, straw-yellow, the receptacle straight, or but slightly curved, consisting of a single series of from three to eight superposed cells, the basal usually larger; the rest small, broader than long, all bearing antheridia and separated by horizontal, or but slightly oblique, septa; the series terminated by a unicellular mitriform appendage, somewhat variably inflated, symmetrical, broader than the axis which it terminates. Antheridia nearly horizontal, straight, two to seven in number, arising on one side in a single series from all the cells of the receptacle except the basal, their stalk-cells relatively long, sometimes exceeding in length the body of the antheridium, which is short and broad, the discharge tube short, straight and stout. Total length (including foot, 16 μ) about 50–60$\times$9 μ. Appendage-cell 14–20$\times$10–12 μ. Antheridia about 35 μ, the stalk-cell 9–18$\times$6 μ, the venter 10$\times$12 μ.

Female individual, pale straw-yellow. Receptacle similar to that of the male, consisting of four or five superposed cells terminated by a mitriform sterile appendage-cell, the cell immediately below it usually giving rise laterally to an erect, or slightly divergent, appendage of usually five or six successively smaller somewhat inflated cells; the first perithecium usually arising from the cell next below, one or two more perithecia rarely developed from the cells immediately below the first. Perithecium usually solitary, relatively large, its axis nearly at right angles to that of the receptacle or curved upward from it; usually slightly broader distally, the tip not clearly distinguished, the apex blunt, slightly sulcate. Spores (in perithecium) 60$\times$9 μ. Perithecium 150–215$\times$30–40 μ, the sporogenous portion 100–135 μ. Appendage 60–100$\times$8 μ. Receptacle to tip of primary appendage-cell, including foot, 80–100 μ.

On the elytra of *Corynites ruficollis* Fabr., La Plata, No. 1459.

A clearly distinguished species, most readily recognized by its mitriform sterile appendage-cell. Both sexes appear to grow appressed on the elytra, the antheridia and perithecia projecting upward nearly at right angles.

Dimorphomyces Meronevae, nov. sp.

Male individual, relatively large, nearly hyaline, or with faint reddish brown suffusions at the base of the appendage. Basal cell of the receptacle rather large, subtriangular, distally in contact with the outer half of the wedge-like base of the long antheridial stalk-cell; somewhat obliquely separated from the squarish subbasal cell; the appendage relatively short, not extending beyond the base of the neck of the antheridium, its basal cell rounded, somewhat longer than broad, sometimes nearly as large as the whole receptacle and distinguished from it by a marked indentation, distally narrower below the small squarish subbasal cell the terminal cell cylindrical, hyaline. Antheridium large, its slender stalk-cell as long as the inflated venter, the neck somewhat shorter than the stalk and venter combined, and slightly curved. Receptacle including foot, 40×23 μ. Appendage 17 μ. Antheridium 32–35 μ; neck 15 μ, venter 10 μ, stalk-cell 9 μ.

Female individual. Receptacle relatively small, the subbasal cell larger than the basal (without its secondary extension), squarish, distinguished by a deep indentation from the basal cell of the appendage which is subequal, tinged with vinous brown, and rounded in form; the rest of the appendage bent strongly to one side, more deeply suffused, small, blunt or pointed, its two cells not distinguishable. Perithecium relatively large and long, the region below the tip conspicuously suffused with vinous brown, its inner margin concave, the tip hardly distinguished, more faintly suffused, somewhat asymmetrical, as is the hyaline blunt apex; the rest of the perithecium slightly inflated above, more faintly suffused, except the narrow hyaline base. Secondary appendages subcylindrical, somewhat less than half as long as the perithecium, two-celled, the basal cells thick-walled, about half as long as the thin-walled blunt terminal cell. The secondary receptacle narrow, horizontal, or nearly so; its four to eight cells bordered by the narrow extension of the basal cell of the receptacle, the one to three erect perithecia and the appendages rising vertically from it. Perithecia $65–70 \times 12–15$ μ. Spores (in perithecium) 14×1.5 μ. Receptacle, including foot, 18 μ. Secondary receptacle 18–35 μ. Primary appendage 18×9 μ.

On the legs of *Meroneva Sharpi* L. A., Temperley, No. 1503, in company with *Monoicomyces nigrescens.*

A very clearly marked species which was found but once, and is described from four pairs of mature individuals.

Dimorphomyces verticalis nov. sp.

Male individual, relatively small, tinged with blackish brown, the basal cell small, very obliquely separated from the slightly longer narrower subbasal cell which extends downward nearly to its base, and upward to the end of the stalk-cell of the usually single antheridium, which is erect, the venter but slightly inflated; with the short rather stout hardly tapering neck abruptly distinguished. Appendage parallel to the antheridium, or but slightly divergent, consisting of three cells: the basal longer than broad, and distally rounded to the very small much narrower squarish subbasal cell; the terminal cell hyaline elongate slightly inflated below, tapering distally; sometimes extending to or beyond the tip of the antheridium. Total length, including foot, 60 μ. Antheridium, including stalk-cell, 35 μ; its neck 8 μ. Appendage, 20–30 μ.

Female individual, becoming dark blackish brown, the primary appendage erect, consisting of a larger basal cell hardly twice as long as broad, a narrower subbasal cell broader than long, and a terminal cell, hyaline or paler distally, longer than broad, inflated or degenerating. Perithecia usually not exceeding three in number, elongate, straight or curved, blackish brown, very slightly inflated; the tip bluntly rounded, or asymmetrical and snout-like, when viewed laterally: the hyaline apex subtended, on the inner side, by a darker shade. The secondary appendages of two or three superposed cells, hardly a third as long as the perithecia, alternating with them, or somewhat irregular in position, especially above; the series of cells which bears them, and the marginal extension of the basal cell of the receptacle nearly erect, or diverging from the appendage at an angle of not more than 45°. Perithecia 75–100$\times$15–20 μ. Secondary appendages 25–30 μ. Total length to tip of highest perithecium 100–200 μ; to tip of secondary receptacle 75–120 μ.

On *Atheta* sp., Palermo, Nos. 1690, 1965, and 1966.

This species, which was found not infrequently, appears to vary considerably; the older and better developed individuals becoming very dark, and attaining a considerable size. Such individuals, which usually occur on the abdomen, do not appear to be separable from smaller and paler forms which occur, usually, on the legs, antennae and head.

Rickia Lispini nov. sp.

Receptacle short and stout, the basal cell small, hardly longer than broad; the main body consisting of a central cell lying between a pair of marginal cells superposed on either side of it, the two lower united below it and separating it completely from the basal cell; while its extremity lies in oblique contact with the lower half, or less, of the perithecium; the upper marginal cell on one side cutting off one to two small appendiculate cells which subtend the base of the perithecium; the upper marginal cell on the opposite side, bearing two or three to six simple appendages, their origins often lying nearly horizontally, one to five of them arising from single small cells successively separated from above downward (within outward), one, however, always subtended by two minute cells placed not always next the perithecium, and representing the primary and originally terminal appendage. Perithecium short and stout, but slightly longer than its contained spores, subellipsoid to ovoid, the tip hardly distinguished, the apex truncate-papillate. Spores $28 \times 4\ \mu$. Perithecia $40–50 \times 27–31\ \mu$. Receptacle $60–75 \times 28–35\ \mu$. Appendages $20–55\ \mu$. Total length $75–120\ \mu$, average $90–100\ \mu$.

On the abodmen etc. of *Lispinus tenellus* Er., Llavallol No. 1502. Also from Los Amates, Guatemala, No. 1625 (Kellerman).

Were it not for the fact that the genus *Rickia*, as illustrated by the material accumulated from various parts of the world, proves to be a large and very varied one, I should be inclined to separate the present form under a special generic name; and, although it seems best to treat it as a very simple type of *Rickia*, it differs from all others in the fact that all its appendages come from the two distal marginal cells. In a few specimens I have seen a structure associated with the appendages which may be an antheridium; but, in a majority of individuals these organs seem to be quite absent. This appears to be the case also in other and more typically developed species of the genus.

Rickia Melanophthalmae nov. sp.

Hyaline. Perithecium long-ovoid, with a broad truncate apex which may be flat or slightly sulcate, the lumen of the basal wall-cells obliterated, their thick walls forming an ellipsoid cavity in which the spores, which nearly equal it in length, lie somewhat obliquely, and above which the three upper tiers of small subequal wall-cells persist. Receptacle broad and compact, multicellular above the single basal cell; the cells in three vertical series, two lateral and one median; one of the outer consisting of a single somewhat elongated cell, which may rarely be divided into two or three cells, above which lie the three visible basal cells of the perithecium, which are subequal and form an integral part of the receptacle in no way distinguished from it; the marginal series on the opposite side consisting of two to four cells, usually rather narrow, each usually cutting off a small cell obliquely, distally and externally, the uppermost subtending a hardly distinguishable antheridium, the rest developing neither appendages nor antheridia and often becoming wholly obliterated; the series terminated by a small cell which bears the small short stout primary appendage of the usual type; the median series consisting of three superposed cells, the two lower larger, the upper lying beside the base of the perithecium. Perithecium 35–43×23 μ. Spores about 40×2.5 μ. Receptacle 40×27–31 μ. Total length 75–85 μ.

On the elytra of a minute beetle belonging to the genus *Melanophthalma.* Llavallol, No. 1980.

This curious little form is distinguished by the apparent absence of any secondary appendages, the cells which are separated to subtend them in other species, developing nothing more than mere rudiments, and often becoming quite obliterated by the general enlargement of the receptacle, the cells of which may become somewhat displaced. On the perithecial side the usually single marginal cell cuts off no subtending cell even when it becomes divided. Like the preceding species this form is distinctly aberrant.

Monoicomyces Caloderae nov. sp.

Straw-colored, the perithecia and older appendages becoming tinged with amber-brown. Basal cell of the receptacle stout, squarish, the subbasal cell less than half as large, pale straw-colored or nearly hyaline. Primary appendage concolorous with the receptacle, elongate, its tip often extending above the tips of the perithecia; tapering slightly to a blunt extremity, simple or usually producing one or two branches from the third or fourth cells on the inner side. The two primary fertile branches variously complicated by successive proliferation of the secondary branches, the branchlets of which may be of the second or even the fourth order, the perithecia subtending the antheridia. Antheridium of the usual type, its tiers and appendages somewhat variably developed, but resembling in general those of *M. Homalotae.* Perithecia rather short and stout usually symmetrical, inflated below, conical above; the apex small, blunt; the basal cell-region not distinguished from the ascigerous region; the stalk-cell well defined, its basal half usually slightly constricted and suffused with vinous amber-brown. Spores 38×4 μ. Perithecia, including basal cell-region, 80–90×30–35μ; the stalk-cell 25×12 μ. Receptacle about 25×20 μ. Primary appendage 150–175 μ. Appendages 75–100 μ. Antheridium 90 × 35 μ.

Usually on the abdomen of *Calodera* sp. Nos. 1504, 1515, 1691 and 1991, Palermo, Temperley and Llavallol.

Although very common this species is seldom if ever found in good condition, perhaps owing to certain peculiar habits of its host. The appendages are usually broken off entirely and the development of the fertile branches may be very irregular. Although perhaps a dozen perithecia may be formed on a single individual, many are apt to be broken and but few ever mature. The species is most nearly allied to *H. similis* and *H. Homalotae* from both of which it is distinguished by the character of its primary appendage and by the proliferous habit of its fertile branches. The genus of the host has been determined by Dr. Fenyes.

Monoicomyces infuscatus Speg.

Receptacle very small, the basal cell becoming more or less suffused with smoky brown, broader than long, the hyaline subbasal cell hardly distinguishable. Primary appendage stiff rigid upcurved, black externally from its base upward, simple or producing a single branch above its subbasal cell which may be similarly blackened. Fertile branches usually producing a single perithecium and antheridium, more rarely two by proliferation, suffused, especially externally, with blackish olive-brown; the two distal tiers forming a well defined rounded enlargement, terminated by two erect blackened rigid appressed hyaline-tipped appendages. Perithecium hyaline or faintly olivaceous, slightly asymmetrical, subfusiform, the tip hardly distinguished, the apex blunt, the narrower basal cell region not distinguished, the basal cells relatively large, the stalk-cell short and broad, not abruptly distinguished below the basal cell-region. Spores, in perithecium, about $20 \times 2.7\ \mu$. Perithecia $90 \times 26\ \mu$, the stalk-cell $18 \times 12\ \mu$. Antheridium $45–70\ \mu$, its appendages $45–70\ \mu$. Primary appendage with its branches, $110\ \mu$. Receptacle $18 \times 12\ \mu$.

On the abdomen of *Xantholinus Andinus* Fauv., No. 1689, Palermo, No. 1988, Llavallol.

A small and apparently rare species, very closely allied to *M. nigrescens* and distinguished especially by its rigid black primary appendage.

Mimeomyces nov. gen.

Receptacle consisting of two superposed cells, the upper bearing terminally the single appendage and the stalk-cell of the single perithecium. Appendage consisting of a basal cell and several cells superposed above it, the lower bearing single free compound antheridia on the inner side, the upper bearing sterile branches. The antheridia consisting of a group of apparently six similar antheridial cells and originating directly from the slightly swollen extremity of a short stalk-cell and discharging at the same level into the efferent tube. Perithecia stalked and normal.

The characters of this genus correspond exactly to those of *Corethromyces Quedionuchi* which occurs with it on the same host, and in general to that section of *Corethromyces* formerly separated under *Sphaleromyces,* except that the lower branches of the appendages bear conspicuous, typically developed compound antheridia. It seems altogether probable that certain of the species hitherto placed in *Sphaleromyces,* and in which the presence of antheridia has not yet been definitely recognized, may find a place in the present genus when their antheridial characters are known. A careful reexamination of my material of these species has, however, failed to show any indication of the conspicuous antheridia which occur in the present instance.

Mimeomyces decipiens nov. sp.

Perithecium pale translucent yellowish, the basal cells relatively large and clearly distinguished, the ascigerous part usually bent slightly toward the appendages, distally slightly inflated, symmetrical, conical; the tip hardly distinguished, terminating in a small subtruncate apex: stalk-cell short, broader than its length. Basal cell of the receptacle elongate, rather abruptly broader distally, concolorous with the perithecium or more or less deeply and completely suffused with blackish brown, sometimes quite opaque; the subbasal cell small, subtriangular. Appendage consisting of from four to five obliquely superposed cells, subequal in length, the distal ones smaller, the basal without appendages, the subbasal and often the cell above it bearing each a single compound antheridium on a short stalk-cell. Perithecium (sporogenous portion) 55–65 $\times$ 24 μ, including basal and stalk-cells 75–95 μ. Main appendage 50–55 μ, its longest branches 60 μ. Receptacle 50–70 μ, basal cell (longest) 60 μ. Total length to tip of perithecium 125–150 μ. Spores about 30 $\times$ 2.5 (measured in ascus).

On legs and abdomen of *Quedius sorecocephalus* Bernh. (nov. sp.), Llavallol, No. 1520.

The general form and coloration of this species is very similar to that of *Corethromyces Quedionuchi* which may occur with it, but the color and the form of the tip of the perithecium, as well as the conspicuous antheridia distinguish it at a glance. One or more accessory antheridia are sometimes present near the base of the appendage. The host has been determined as a new species of Quedius by Dr. Bernhauer.

Cantharomyces permasculus nov. sp.

Perithecium becoming dark amber-brown with a smoky tinge, subsymmetrical, or usually straighter externally with the inner margin somewhat convex, broadening distally; the short pale rather abruptly subconical tip usually bent outward, the pore subterminal and external, an inner lip-cell forming the small papillate rounded apex: the basal cell region not distinguished, the basal cells extending up about the ascogenic region; the stalk-cell consisting of an upper subtriangular portion, distinguished more or less abruptly by a variably developed constriction from its narrower basal portion, which may equal the distal part in length, and is somewhat obliquely inserted on the receptacle. Receptacle more or less deeply tinged with dirty amber-brown, the basal cell nearly straight and variably elongated, as is the more deeply colored subbasal cell, the base of which is modified by an annular prominence of darker color. Appendage becoming somewhat divergent and curved away from the perithecium, the axis of which coincides in general with that of the receptacle, consisting of five or six superposed cells, the basal one sterile and modified distally by an annular darker ridge similar to that at the base of the subbasal cell of the receptacle, the two to four cells immediately above it becoming compound antheridia, the uppermost or the two uppermost of which may bear a usually simple branch distally, or a pair of such branches arising from opposite sides; the several terminal cells of the appendage bearing distally usually two simple opposite branches which greatly exceed the tip of the perithecium. Perithecia 135–160 $\times$40–50 μ, the stalk-cell 45–60 μ. Spores 70–75$\times$4 μ. Receptacle 100–155$\times$40 μ. Main appendage 200–275 μ, its longer branches 250 μ. Total length to tip of perithecium 275–375 μ.

On a large species of *Parnus,* commonly on the elytra. Palermo, No. 1686.

This species is readily distinguished from the following by the form and color of the perithecium and its short stalk-cell, by the annular prominences of the receptacle and appendage, which are without striations, by its usually more elongate straight receptacle the axis of which coincides with that of the perithecium, not of the appendage as in *C. Bruchi,* and by its much more highly developed appendage, which may produce more antheridial cells than are known in any other of the Laboulbeniales. In its antheridial characters this species, as well as its ally, depart distinctly from the usual type of *Cantharomyces,* which possesses but one antheridium. It should not be separated, however, and is connected with the more normal type by a species, as yet undescribed, which occurs on *Parnus* in north temperate regions. Sufficient material of both species in good condition has been examined and leaves no doubt as to their distinctness.

Cantharomyces Platensis nov. sp.

Perithecium subsymmetrical, more or less tinged with amber-brown, the venter somewhat inflated above its base and more deeply suffused, the distal portion subconical tapering to a rather broad blunt apex, the basal cells small, the outer extending somewhat upward, the region not distinguished from the body of the perithecium; the stalk-cell but slightly suffused, straight erect somewhat divergent from the appendage, the axis of which coincides with that of the receptacle, as long as, or much longer than, the body of the perithecium, the distal end contrasting with and as broad or broader than the darker base of the perithecium; from which it is separated by a horizontal septum more deeply suffused and often abruptly narrower, or distinguished by a pseudo-articulation where it is inserted on the receptacle. The receptacle somewhat darker amber-brown, its basal cell irregularly triangular, geniculate, the subbasal cell usually hardly longer than broad, an annular secondary wall extending around its base and marked by very fine vertical striations. Appendage straight, erect; its basal cell concolorous with the receptacle, its base broad somewhat oblique; the whole cell broader than long, distally modified like the base of the upper cell of the receptacle, and with the same longitudinal striations; usually not more than two of the cells immediately succeeding it, squarish and modified to form antheridia, and succeeded by two or three narrower superposed cells all of which may bear a single erect straight branch; the terminal one often furcate, the branches short or sometimes extending as far as the tip of the perithecium. Perithecia $125–150 \times 32–44\ \mu$, its stalk-cell $135–235 \times 25–35\ \mu$. Spores $60 \times 4\ \mu$. Receptacle $60–75 \times 35–40\ \mu$. Main appendage $110–135\ \mu$, its longest branches $200\ \mu$. Total length to tip of perithecium, about $400\ \mu$ ($350–470\ \mu$).

On the elytra of a smaller species of *Parnus* ?, Palermo, No. 1685.

This species differs from the preceding in its long-stalked more slender perithecium, in its shorter receptacle and appendage, in the smaller number of its antheridia which are never appendiculate, and in the striation and absence of elevation which characterises its peculiar annular secondary walls. Closely allied to *C. Bruchi* Speg. which is half as large and otherwise different.

Amorphomyces Ophioglossae nov. sp.

Male individual, relatively long and slender, nearly straight, the basal and subbasal cells nearly equal, the antheridial cell as long as both combined. The subbasal cell deep reddish brown, contrasting with the hyaline basal cell and the straw-yellow slightly asymmetrical antheridium, the neck of which is about as long as the symmetrically inflated venter. Total length, including foot, 55–65×5 μ, the antheridial cell 28–32×6–7 μ.

Female individual. Basal cell hardly longer than broad, hyaline; its base slightly broader, contrasting with the deep red-brown base of the perithecium above it; the short deeply suffused stalk-cell, and the minute basal cells of the perithecium hardly distinguishable at maturity: the body of the perithecium pale straw-yellow, short, stout; the inner margin straight or concave; the outer strongly convex, tapering from near the middle about equally to the base and apex; the latter broad, flat or somewhat rounded, subtended externally by a reddish brown suffusion, the short tip often slightly bent outward, giving it a snout-like habit. Basal cell 8×8 μ. Total length, including foot (7–11 μ), 100–120×30–35 μ.

On the head and tip of abdomen of *Ophioglossa* sp. Llavallol, No. 1500, and Tucuman, No. 1935, (P. Spegazzini).

A common species at Santa Catalina.

Amorphomyces rubescens nov. sp.

Male individual. Basal cell hyaline, somewhat longer than broad; subbasal cell red-brown, hardly longer than broad; antheridium relatively large, at least twice as long as the two basal cells combined, exclusive of the foot; the venter shorter than the neck, prominent distally on one side, tinged with red-brown below, slightly inflated; the neck erect, clear reddish straw-color. Total length, including foot, 65 μ. The two basal cells 16–18×6 μ. Antheridium 35–37×9 μ, the neck 19–20 μ.

Female individual, relatively slender, the basal cell broader than long, smaller than the foot, hyaline, contrasting. The perithecium tinged throughout with reddish brown, the suffusion deep at and toward the basal and stalk-cells, the latter somewhat shorter than the relatively long narrow basal cells above it. The body of the perithecium straight, relatively narrow, subsymmetrically and slightly inflated, the apex broad, slightly rounded, the tip asymmetrical or bent outward. Basal cell $7 \times 9\ \mu$. Total length, including foot, 140–165$\times$25 μ.

On the abdomen of *Diestota* sp. Temperley, No. 2007, and Llavallol, No. 1498 on *Homalota* sp., the genera doubtfully determined by Dr. Fenyes.

Tetrandromyces nov. gen.

Male individual consisting of four superposed cells the uppermost bearing a crown of four simple antheridia.

Female individual. General structure as in Dioicomyces.

Although the perithecium of the female in this genus is unlike that of any of the species of *Dioicomyces* in external form, it corresponds to this type almost exactly in other respects, so that the genus is based upon the characters of the very peculiar male individual, in which the antheridia are not only grouped, but of a distinctly different type from those of *Dioicomyces,* recalling those of *Synandromyces* or of some species of *Stigmatomyces.*

Tetrandromyces Brachidae nov. sp.

Male individual stout, faintly suffused with brownish olive, basal cell nearly hyaline, longer than broad, the three cells above it subequal or successively smaller, the distal cell triangular or otherwise shaped according to the point of view. Antheridial cells as large as the basal cell, the stout suberect and subsymmetrically arranged brown necks but slightly curved. Antheridia $23 \times 8\ \mu$, the group 16 μ wide. Total length including foot 60 μ.

Female individual. Receptacle faintly yellowish, the basal cell small, about as long as broad; the subbasal cell triangular; minute but clearly distinguished; the subtending cell of the appendage narrow, oblique, the terminal cell stout, distally rounded, deep black-brown. Perithecium relatively very large, the stalk-cell rather short and stout, faintly yellowish; the basal cell and basal wall-cell regions not distinguished externally, and forming an evenly slightly inflated base, or the external basal cell forming a rounded slightly blackened prominence; the second tier or wall-cells marked by a slight inflation distally, not distinguished from the slightly asymmetrical dome-shaped region of the third tier, above which the short and abruptly narrower tip is abruptly distinguished, being subtended by the slightly elevated blackened insertion of the trichogyne; the hyaline apex slightly asymmetrical, bluntly rounded or slightly pointed, subtended by two rounded tooth-like prominences from two adjacent wall-cells of the terminal tier. Spores, in perithecium, male 28–30×4–5 μ, female about 40 μ. Perithecia 200–235×50–65 μ, the subterminal prominence 8 μ long, the stalk-cell 60×20 μ. Sterile appendage-cell 20×12 μ. Total length to tip of perithecium 250–280 μ.

Near the tip on the superior surface of the abdomen of *Brachida Reyi* Shp., Llavallol, No. 1989.

Although in fully matured turgescent individuals the distinction between the basal cell and basal wall-cell regions becomes obliterated, the basal cells, especially the external one, may be distinctly prominent in younger or partly collapsed individuals. The ascogenic cell produces great quantities of asci and spores, unlike the forms of *Dioicomyces*. The general form of the perithecium recalls that of the conventional "fat pig." The host has been determined for me by Dr. Fenyes.

Dioicomyces Formicellae nov. sp.

Male individual rather slender, the foot-cell slightly broadened, blackish or concolorous with the basal cell of the receptacle which is grey brown and usually separated from it by a hyaline line; basal cell a little more than twice as long as broad; the subbasal usually nearly square; the third cell shorter; the antheridial cell somewhat longer than broad; the *neck* terminal at one side, slender slightly bent. Total length including foot and neck 60–70 μ; basal cell 20×8 μ.

Female individual variously curved, sometimes sigmoid, sometimes curved throughout, or the perithecium alone somewhat bent. Basal cell of the receptacle hardly longer than the foot, suffused with brown, the subbasal cell almost obsolete; the sterile appendage-cell short, rounded distally, tinged with brown. Perithecium large, yellowish brown, deeper at the tip and in the middle, strongly curved; the successive wall-cells on the convex side distinguished by slight elevations and depressions, the third wall-cell on the concave side slightly elevated; the venter somewhat inflated; tapering slightly to the coarse bluntly rounded or roughly truncate apex; the basal cell region sometimes abruptly narrower or not distinguished; the stalk-cell elongate, narrower at its base, tinged with yellowish or brownish. Spores, male 35×5 μ, female 40–42×8 μ. Perithecia 145–165×45–50 μ; stalk-cell 140–180×25 μ. Receptacle, including foot and appendage-cell, 40–65 μ.

On the elytra of *Formicella strangulata* Pic, Palermo, Llavallol, and Temperley, No. 1692.

Although its host was very common in the Buenos Aires region, this species was seldom met with. It is the largest species of the genus thus far described, but is otherwise without striking peculiarities.

Dioicomyces malleolaris nov. sp.

Male individual, consisting of three superposed cells and a terminal antheridium, relatively small and stout; the basal cell nearly hyaline, twice as long as broad, the subbasal cell but slightly longer than broad, the third cell much shorter than broad; the antheridium relatively large, slightly suffused, distally somewhat asymmetrical, the well developed neck terminal at the side. Length about $45 \times 7.5\ \mu$.

Female individual, hammer-shaped: the basal cell of the receptacle very small, suffused with blackish brown; the subbasal minute, flat; the appendage-cell blunt-conical, faintly yellowish. Perithecium horizontal, its upper outline straight; the axis of its main body lying at right angles to that of the long, very thick-walled, slightly curved stalk-cell, the lumen of which may be nearly obliterated; the position of the basal cells and basal wall-cells so abnormal that the rounded ascigerous region projects free on one side corresponding to the free tip which projects somewhat further on the other; the whole supported by two cell-series that diverge abruptly from the end of the stalk-cell; on one side, as seen laterally, consisting of two basal cells, on the other of one basal and two squarish wall-cells; the whole including the stalk-cell more or less suffused with pale smoky yellowish brown; the tip tapering slightly to a blunt slightly asymmetrical apex. Spores $28–30 \times 3.5\ \mu$. Perithecia $99–100 \times 26–32\ \mu$. Stalk-cell $65–90 \times 16\ \mu$. Appendage cell $15–16\ \mu$. Receptacle including large foot $28\ \mu$.

On the tip of usually the right elytron of *Anthicus parvus* Pic, Palermo and Llavallol, No. 1513.

This very curious and rather rare species grows more or less appressed, the perithecium lying at right angles to the axis of the elytron. Like all the species of the genus herewith described, the spores begin to germinate normally before discharge and are twice septate when they emerge, with a well developed black foot.

Dioicomyces umbonatus nov. sp.

Male individual almost hyaline or faintly yellowish brown externally, rather slender, straight or slightly curved inward, the basal cell as long as the portion above it; the foot small, the subbasal cell slightly larger than the cell above it, the antheridial neck usually erect, relatively long. Basal cell, including foot, 20 μ. Total length to base of neck 35×5 μ. Neck 12 μ.

Female individual dirty straw-colored with a brownish tinge, the perithecium and the outer margin of the receptacle and appendage becoming somewhat darker. Basal cell of the receptacle larger than the foot, the subbasal flattened, concave below. Basal cell of the appendage hardly distinguishable, the terminal cell blunt pointed, evenly pale yellowish brown. Stalk-cell of the perithecium nearly straight, rather short and stout, deeply constricted just above its origin, about the same diameter throughout; the perithecium short, stout, strongly curved, so that the tip is horizontal, the basal cell region hardly distinguished from the body; one of the basal wall-cells on the convex side forming a conspicuous umbonate projection; the apex broad, slightly sulcate, asymmetrical. Perithecium, from the base to the horizontal edge of the tip, $70\text{–}78 \times 40\text{–}43$ μ (including umbo), the stalk-cell $40\text{–}42 \times 12\text{–}15$ μ. Receptacle to tip of appendage, including foot, 42 μ. Total length 135–145 μ.

At the base of the elytra near the inner margin of several specimens of *Anthicus parvus* Pic; Temperley, No. 1513C.

This species is nearly related to *D. Anthici*, and to the following species from which it is most readily distinguished by the umbonate prominence resulting from the inflation of one of the basal wall-cells. A single specimen was found growing at the base of the anterior leg of one host.

Dioicomyces angularis nov. sp.

Male individual relatively short and stout, rather deeply suffused with olivaceous brown, especially externally; the foot relatively large, the basal cell slightly longer than the rest of the series, the subbasal cell hardly larger than the cell above it; the antheridial cell hardly longer than broad, the antheridial neck slightly divergent. Length, exclusive of neck, 30×6 μ. Basal cell, including foot, 19.5 μ; neck 8 μ.

Female individual much as in the preceding species, the receptacle and appendage more deeply suffused. Stalk-cell of the perithecium elongate, somewhat broader distally, slightly curved distally or near the base. Perithecium rather clear pale straw-yellow, straight or very slightly curved, its axis diverging slightly from that of the stalk-cell, subtriangular, or more or less strongly angular externally owing to a prominence corresponding to the point of separation between the basal and subbasal wall-cells, the perithecium tapering thence to the symmetrically rounded apex of the relatively narrow tip; the basal cell-region distinguished on the inner side only, by a slight indentation marking the base of the lower wall-cell. Perithecium 80–94×35–42 μ; the stalk-cell 98–120×15 μ. Receptacle to tip of appendage 38 μ. Total length 185–125 μ.

On the tips of the elytra and the adjacent free portion of the abdomen of *Anthicus parvus* Pic., Temperley and Llavallol, No. 1513A.

Distinguished from *D. Anthici,* to which it is very closely allied, chiefly by the angular or triangular form of it perithecium.

Autophagomyces nov. gen.

Male individual, attached to the basal cell and foot of the female, consisting of several superposed cells and bearing terminally and laterally from one to several large flask shaped simple antheridia.

Female individual consisting of a single basal cell from which the stalk-cell of the perithecium arises distally. Ascogenic cell single, spores 1-septate.

Although five species of this type have been examined and several individuals destroyed in an attempt to isolate the spores, I have found it impossible to determine whether the male and female spores are more definitely associated than in the other unisexual genera of this type. It is therefore possible that what I have assumed to be a male individual may be an antheridial branch, which arises from the base of the basal cell of the female, although such a condition seems improbable. The relationships of this genus are evidently with *Dioicomyces,* the species of which also occur, for the most part, on Anthicidae, and with *Amorphomyces* which the female very closely resembles, except for its septate spores.

Autophagomyces Platensis nov. sp.

Male individual consisting of three or four superposed cells exclusive of the foot and bearing one to three antheridia. Total length to tip of terminal antheridium 53–60×5 μ. Antheridia 25 μ.

Female individual. Basal cell slightly broader than long, somewhat suffused with brownish below. Stalk of the perithecium short and stout, broader distally, concolorous with the hyaline or faintly yellowish perithecium; which is slightly but distinctly curved throughout; its axis at a slight angle to that of the stalk; its outline somewhat irregular distally, owing to the presence of slight elevations and depressions which correspond to the successive tiers of wall-cells; the tip bluntly rounded, asymmetrical and not well distinguished. Perithecium 106×28–$32\ \mu$, its stalk-cell 14–18×10–$14\ \mu$. Basal cell $9 \times 10.5\ \mu$ exclusive of foot.

On the elytra of *Tomoderus forticornis* Pic, Llavallol, No. 1982.

The base of the stalk-cell is in some specimens slightly constricted or so modified that a very small cell may appear to be separated at its base. There is no indication in this or the succeeding species of any sterile cell which might be formed from the terminal spore-segment. I am indebted to M. Pic for determining the host which he finds to be new.

Autophagomyces nigripes nov. sp.

Male individual, slender, usually consisting of three superposed cells bearing a single terminal, or rarely also a subterminal, antheridium. Total length to tip of antheridium 60–$70 \times 3.5\ \mu$. Antheridium $26\ \mu$.

Female individual. Basal cell relatively large, three to four times as long as broad, slightly broader distally, uniformly suffused with blackish brown, contrasting with the perfectly hyaline stalk of the perithecium; which is slightly longer, broader distally, where it is abruptly bent so as to turn the perithecium at right angles to its axis. Perithecium rather slender, its outline somewhat irregular, bent upward slightly distally; the tip large, broad, well distinguished, blunt-pointed and oblique above; or with the outer, upper lip-cell somewhat prominent. Perithecium $106 \times 26\ \mu$, stalk-cell 26–$28 \times 16\ \mu$. Basal cell exclusive of foot, $26 \times 9\ \mu$.

On the inferior surface of the abdomen of *Tomoderus forticornis* Pic.

Cryptandromyces nov. gen.

Receptacle consisting of two superposed cells, the upper bearing a solitary stalked perithecium, and an appendage formed by a simple series of superposed cells without branches; several consecutive cells of this series at first functioning as antheridial cells, from which sperm-cells appear to be discharged directly through perforations of the wall on the inner side. Perithecia normal, a single ascogenic cell present in the type.

The determination of the characters which distinguish this genus, of which several species are known to me on related hosts, has given much difficulty; since the antheridia appear to be functional only at the moment when the trichogyne is receptive, and the openings through which the sperm cells appear to issue are soon obliterated; the antheridial cells also losing the peculiar densely granular appearance which at first distinguished them. It is only in very few specimens that I have been able to make out these perforations through which there actually seems to be a passage of sperm-cells of the usual type.

Cryptandromyces geniculatus nov. sp.

Wholly hyaline. Receptacle straight, the basal cell becoming broader distally, often longer than its greatest width; the subbasal cell usually angular or subtriangular, slightly larger than the basal cells of the appendage. Appendage slightly divergent, variably developed; sometimes distally elongate and tapering, but often rather short and stout; consisting of usually three to five cells below the antheridial cells, with evenly rounded lumens, the antheridial cells above them, which may be as many as six in number, terminated by a bluntly pointed, slightly incurved cell, or the appendage in some cases becoming long slender and distally attenuated. Stalk-cell of the perithecium slender, two or three times longer than broad, often narrower subterminally; perithecium relatively large short stout, its axis at an angle, sometimes at right angles, to that of the stalk-cell, its inner margin often straight or concave, the outer strongly convex; the tip hardly distinguished, sometimes slightly bent upward, the obtuse apex minutely papillate or slightly sulcate. Spores relatively large $28 \times 3.5\ \mu$. Perithecia $50–70 \times 25–30\ \mu$; stalk-cell $20–26 \times 8\ \mu$. Receptacle $26–35 \times 12–16\ \mu$. Appendage $50 \times 9\ \mu$, the more elongate $130\ \mu$.

On the elytra etc. of *Connophron* nov. sp. Temperley, No. 2001.

The material of this species is sufficiently abundant, and though I at first suspected that it was a unisexual form and that the male had been overlooked, a more careful examination shows that the individuals bearing perithecia are often paired. This host has been kindly determined for me by Col. Casey.

Synandromyces nov. gen.

Receptacle consisting of two cells forming, in conjunction with the basal cell of the appendage, a compact structure in which the subbasal cell of the receptacle occupies a central position, bordered on one side by the subbasal cell, on the other by the basal cell of the appendage, both of which thus tend to become marginal extending to or toward its base. Perithecium relatively large, with a single ascogenic cell, and five wall-cells in each row; the short stalk-cell forming a narrow isthmus between the broad base and the receptacle. The appendage, above its adherent basal cell, forming a compact free structure consisting of a flattened basal cell in some species obliquely divided, which is surmounted by two cells both bearing single simple antheridia; one surmounted by a spine, or bearing also a small cell which subtends a third antheridium, on which the lateral spine is borne; the antheridia arising close together in a characteristic group; their venters closely approximated, their stout necks distally somewhat divergent. Trichogynes bicellular above their insertions, the distal cell elongated at right angles to the basal cell on both sides, and distally beset by numerous vesicular receptive prominences.

The above diagnosis is based upon the examination of several species of this genus which are known to me from various regions, only two having been obtained in the Argentine. It is most nearly related to *Acompsomyces*.

Synandromyces Telephani nov. sp.

Perithecium erect, relatively very large; becoming tinged with amber-brown, straight; the main body, including the basal cell region, symmetrically inflated, subfusiform, but often somewhat more tapering above and rounded at the base; the four cells of the first and second tiers of wall-cells separated by a corresponding number of more or less distinct prominences; a terminal portion rather abruptly distinguished from the main body, and often subtended by slight prominences, straight, narrow, isodiametric above, more deeply suffused, as a rule, than the main body, but nearly hyaline below, slightly inflated distally immediately below symmetrical hyaline truncate or slightly papillate and sulcate apex: the stalk-cell small, constricted to form a short slender isthmus, which is bent sidewise and connects laterally with the basal cells of the perithecium. Receptacle short and compact, its axis straight, the basal cell narrow, clavate above; the subbasal cell extending nearly to the foot, slightly enlarged distally, very narrow below; the basal cell of the appendage extending not quite so low as the subbasal cell, which it closely resembles, though distally more abruptly broadened to form the horizontal insertion of the free appendage. Appendage compact, rounded, subsymmetrical, amber-brown; the flat basal cell undivided, about equalling the pair of cells above it, from which arise two antheridia, and, externally, a small cell bearing laterally a spinose antheridium; the necks of the antheridia lying side by side, erect and parallel, or bent slightly inward and in contact, except distally. Spores $40 \times 6\ \mu$. Perithecia, including basal cells $235–310 \times 45–58\ \mu$, its rostrate termination $80\ \mu$. Receptacle including foot $45–60 \times 35\ \mu$. Appendage, free part including antheridia, $45–50 \times 20\ \mu$.

On the elytra, prothorax and other parts of *Telephanus* sp., Temperley and Llavallol, No. 1992.

Synandromyces geniculatus nov. sp.

Similar in general to the last. Perithecium relatively smaller, the main body tinged with deeper smoky brown, and lying horizontally at right angles to the axis of the receptacle; asymmetrical, the distal portion short, rostrate, tapering more or less to the short hyaline tip; which is often abruptly somewhat narrower, sometimes slightly inflated, irregularly papillate; the base inserted laterally on the short, abruptly bent, constricted stalk-cell. Receptacle as in the previous species, but relatively longer, strongly curved below. The free portion of the appendage relatively smaller, tinged with smoky brown. Spores $30 \times 5\ \mu$. Perithecia $135–155 \times 45–60\ \mu$, rostrate termination $45–50\ \mu$. Appendage including antheridia, free portion, $30 \times 20\ \mu$.

On the superior surface of the tip of the abdomen and less frequently on the adjacent tips of the elytra, often with the last, on the same host, *Telephanus* sp. Temperley and Llavallol; Nos. 1508, 1992.

This species grows, usually somewhat crowded, in the position indicated, and I have not seen it on the elytra except at the very tips, where *S. Telephani* may also occur. It can thus hardly be regarded as a variety due to its position of growth. It may be easily distinguished from *S. Telephani*, even with a hand lens, from its darker color, smaller size, and sigmoid habit.

Stigmatomyces Anoplischii nov. sp.

Faintly yellowish olivaceous with conspicuous brown shades near the base of the appendage on the inner side. Perithecium relatively very large and long, the venter greatly elongated, but slightly inflated;

the neck slightly narrower, squarish or slightly inflated, subtended by a slight elevation; the tip narrower and somewhat shorter than the neck; the apex broader, terminated by four hyaline projections which taper from broad flat bases to blunt, slightly divergent tips, often symmetrical; the two upper basal cells extending upward, and not distinguished from the base of the venter; the stalk-cell very small, often shorter than broad, and bulging externally, separated from the lower basal cell by a marked constriction. Stalk-cell of the appendage narrow, lying in contact with the basal cell of the receptacle; its pointed base reaching nearly to the foot, similar to and symmetrical with the somewhat smaller subbasal cell, which lies beside the narrow enclosed prolongation of the basal cell which reaches nearly to the base of the free appendage. Basal cell of the appendage free, tinged with reddish brown on its inner side, becoming divided into two sometimes subequal cells, the outer sterile or bearing an antheridium, the subbasal cell often as large as the inner division of the basal, its wall red-brown on the inner side, bearing a single antheridium externally, which may or may not be subtended by a small cell; the cell next above smaller, subtriangular, bearing one external and two lateral antheridia, the terminal cell becoming an antheridium, the neck of which is subtended externally by a stout blunt brown spinous process; antheridia tinged with brown, the venters subtriangular, the necks abruptly distinguished, slender, curved, as long as the venters. Spores $60–65 \times 8\,\mu$. Perithecia, including stalk-cell ($8\,\mu$), $280–330 \times 45\,\mu$. Appendage, exclusive of stalk-cell, $50–60 \times 25\,\mu$ (at base): antheridia $25 \times 12\,\mu$. Receptacle, including stalk-cell of appendage, $50–55 \times 26\,\mu$. Total length to tip of perithecium $310–390\,\mu$; to tip of appendage $130\,\mu$.

On the elytra of *Anoplischius* sp., Buenos Aires, No. 2028, La Plata, No. 1518.

A well marked species most nearly related to *S. virescens*, but differing in various essential points. The arrangement of the distal antheridia recalls that seen in *Helminthophana*.

Zeugandromyces, nov. gen.

Receptacle consisting of two superposed cells, the upper bearing a perithecium and antheridial appendage. The appendage consisting of a stalk-cell and a series of superposed cells above it, the lower basal cells clearly distinguished, or not differentiated from those above it and like them, bearing on the inner side a vertical double series of paired antheridia, the terminal cell or cells of the series sterile, or converted directly into antheridia. Perithecium usually solitary, normal, with a well developed stalk-cell; the short trichogyne arising from the base of the prominent free portion of the trichophoric cell.

Were it not that sufficient material is available of two other species of this genus which occur on allied staphylinids, one in Borneo and the other in New England, I should hesitate to separate this type from the very large and varied genus *Stigmatomyces*. The antheridia recall those of *Idiomyces,* in which I have described an arrangement of antheridia in three vertical rows. I have not felt satisfied, however, that this was the actual condition, and a reexamination of fresh material of this curious type may show that here also the antheridia are in two and not in three vertical rows.

The Argentine material is for the most part in poor condition, only one of the dozen or so specimens being fully matured. The perithecia do not greatly resemble those of *Stigmatomyces,* having well developed stalk-cells, while the distinction between venter, neck and tip is not well marked. The apex, in all three species, is rather characteristically shaped, flat-conical, without projections or papillae. There appear to be four ascogenic cells in all cases.

Zeugandromyces australis nov. sp.

Perithecium nearly symmetrical and straight, rather elongate, rich amber-brown, paler distally; the base inflated, tapering thence gradually to the blunt conical apex; the stalk-cell stout, broader distally, faintly yellowish or hyaline, in the type bent abruptly near the base. Receptacle subtriangular, nearly symmetrical, broader distally where the septum is horizontal; subbasal cell somewhat broader, much smaller, irregular. Appendage tinged with brown, the terminal and basal cells darker, the stalk-cell subtriangular, broader externally, the basal cell more or less clearly distinguished from the five to seven cells above it, and like them bearing relatively large antheridia with long appressed upcurved necks; the terminal cell sterile, subtriangular, turned inward, externally spiniferous. Perithecium $155 \times 44\ \mu$; the stalk-cell $16 \times 27\ \mu$ (distally). Appendage, including stalk-cell, 44–54 μ. Antheridia about 20 μ. Total length to tip of appendage 90 μ; to tip of perithecium 250 μ.

On *Scopaeus laevis* Sharp. No. 1695, Palermo.

Found on a single specimen of the three hosts collected.

Corethromyces Th.

A comparison of new material from various parts of the world has led me to the conclusion that the scope of this genus should be considerably extended. Although those forms which, like the type, occur on Cryptobia are all similar and are readily grouped in a section by themselves, owing to the uniform characters of the appendages, there are other closely related forms or groups of forms, like those on *Stilici*, as well as various undescribed species on somewhat varied hosts, that do not seem to be distinguished from the type with sufficient clearness to justify the erection of new genera for their reception. As a result of this extension, it seems desirable, moreover, to discard the genus *Rhadinomyces*, which, though sufficiently well defined in its typical conditions, varies to forms too near *Corethromyces* for proper separation. That this union might prove necessary, I have already mentioned in my second Monograph (p. 317).

A further complication in this connection has been encountered in connection with the species of *Sphaleromyces*, a type in which the antheridial characters are little known. The genus was based on *S. Lathrobii* in which the antheridia appear to be solitary, but in a majority of the species which have been described under this generic name these organs have not been seen at all, or have been but doubtfully recognized: for the reason that the material has all been obtained from dried insects, and was consequently for the most part in poor condition. Among the South American forms are several which would have been placed in this genus had it not been possible to determine from the fresh alcoholic material, that the antheridial characters were those of *Corethromyces*. The striking form for example, described below from material growing on *Pinophilus*, is undoubtedly congeneric with the two species formerly discovered on hosts of this staphyline genus, namely *S. occidentalis* and *S. indicus;* but several of the younger specimens obtained, in which the antheridia still persist, show clearly the intercalary nature of the latter. *S. Quedionuchi* was also obtained both in Chile and in the Argentine, and although the appendages here are densely tufted and small, a seriate disposition of the antheridia seems also to be present. Since, apart from the supposed antheridial distinction, there are no essential differences between *Sphaleromyces* and *Corethromyces*, the former genus must also be abandoned.

The genus *Corethromyces* thus modified, may be considered to include those forms in which a two-celled receptacle gives rise to a free stalked perithecium, normally solitary, and to a single appendage consisting of a main axis of several superposed cells from some of which ramiferous cells are separated on the inner-side, the branches variously developed, the subbasal cell and sometimes the cell above it bearing antheridial branches; the antheridial branchlets themselves, which really form the distinctive feature of the genus, sometimes associated with sterile branchlets and bearing antheridial cells typically arranged in series of two or more superposed members, one or more of which occupy an intercalary position in the series. That even this character may be obscured, or is at least not always recognizable, is evident from an examination of the peculiar series of forms parasitic on species of *Stilicus* of which several additions are herein included. Although in more than one species of this very individual and peculiar group of forms, the seriate arrangement is well marked, instances occur in which it is rarely or perhaps never present. Thus in *Corethromyces Stilicolus,* which I formerly referred provisionally to *Stichomyces,* it is only after the examination of much additional material, that examples have been found in which the characteristic seriate arrangement occurs, the antheridia usually tending to become solitary or at least free, even when grouped: although in the light of further knowledge of this type there can be no question that it is congeneric for example with *C. Stilici* and others of this series, in which one or more of the antheridia may be intercalary.

The conclusion thus seems unavoidable that both *Rhadinomyces* and *Sphaleromyces* should no longer be maintained as distinct genera, but should be merged in *Corethromyces,* which, in addition to the species previously described under this name and the new forms described below, may be regarded as embracing the following species: **Corethromyces cristatus** and **C. pallidus** formerly placed in *Rhadinomyces;* **C. Stilicolus** formerly included in *Stichomyces;* **C. Lathrobii, C. occidentalis, C. Indicus, C. atropurpureus, C. Brachyderi, C. Chiriquensis, C. Latonae, C. obtusus, C. propinquus** and **C. Quedionuchi** formerly placed in *Sphaleromyces.*

That further changes in the disposition of the last mentioned forms may become necessary, when better material of the other species related to *C. Quedionuchi* has been examined, is suggested by the characters of the new genus *Mimeomyces* described above, which are exactly those of the group referred to, except for the presence of well developed *compound* antheridia. *C. atropupureus,* for example, might well belong to the new genus, but in the type material, no signs of compound antheridia can be found.

Owing to the difficulties which are met with in determining the exact nature and association of the antheridia in many forms included in the genus it may be assumed that all those in which a two-celled receptacle bears distally a single perithecium on the one hand and a single main appendage on the other, bearing branches on its inner face and terminally, should be sought under *Corethromyces,* when it possesses no characters which would exclude it from the genus.

Corethromyces Argentinus nov. sp.

Perithecium becoming very large, elongate, asymmetrical; the outer margin more prominent; the region of the subbasal wall-cells greatly elongated, usually distinctly suffused with purple-brown, and more or less inflated; or the whole perithecium of nearly the same diameter to the tip; which is well distinguished, blunt-conical, the apex flat, papillate, subtended by a slight elevation: the basal cell-region relatively short and compact, concolorous with the part above, the stalk-cell hyaline, but externally opaque at its base, short and about twice as long as broad. Receptacle small, the basal cell translucent, reddish, broader above than the opaque subbasal cell. Primary appendage opaque below and externally indistinguishable below from the subbasal cell of the receptacle; consisting of three superposed cells, the two lower translucent along their inner margins, their limits barely indicated externally by a slight elevation, the subbasal cell associated with two unequal cells on its inner side; the lower larger than the subbasal cell itself, inflated, and bearing paired erect branches, which produce branchlets arising near the base only, the two lowest, usually, short, opaque, contrasting, directed obliquely outward; the rest suberect, more or less suffused with purplish or nearly hyaline, coarse, straight or curved toward the perithecium, the tip of which they may exceed when unbroken, the longer branches not numerous (six or more), simple, stout, septate, tapering slightly to blunt tips: the third, terminal cell of the main axis, very small, mostly translucent, bearing distally one or two short branches. Perithecium 160–290$\times$40–55 μ, ascigerous part 165–270 μ, stalk-cell 40–60$\times$20–30 μ. Spores 40$\times$3.5 μ. Primary axis of appendage 50 μ: total length to tip of branches, longest 370 μ; larger branches 8 μ in diameter. Receptacle 40 $\times$ 8 μ.

On legs and abdomen of *Cryptobium* sp. Palermo, Nos. 1703–4.

This species was very common on a dark almost black *Cryptobium* with yellow legs which frequented the low ground in the park. It is well distinguished by its very large and long perithecia, and the stout, erect and elongate simple branchlets of the appendage, certain short oblique branchlets below their origin being alone deeply suffused.

Corethromyces Ophitis nov. sp.

Perithecium rather slender, translucent reddish brown, tapering but slightly to the hyaline blunt papillate tip; the basal cell well developed, hyaline, distinguished above by a slight constriction, the lower large; the stalk-cell relatively small, narrow, hyaline distally, but otherwise rich red-brown, its insertion very oblique, its suffused portion united to the basal cell of the appendage. Basal cell of the receptacle translucent brown, pale, somewhat longer than broad, slightly bent; the subbasal cell somewhat narrower below than the basal, nearly or quite opaque. Basal cell of the appendage opaque like the upper portion of the receptacle, and distinguished from it only by an external well defined rounded prominence; its second and third cells also opaque, both distinguished by a similar rounded prominence: the subbasal separated by an oblique septum from the basal and associated with two cells which occupy its whole inner surface; a lower, subtriangular, nearly equalling it in size, extending from its base for about three fourths of its length and bearing a red-brown ramiferous cell on either side; the upper much smaller and ramiferous; all the branches arising from these cells hyaline, two to four times subdichotomously branched, the ultimate branchlets longer, tapering, erect, the tips often abruptly recurved, some of them extending beyond the tip of the perithecium; the third cell of the main appendage subisodiametric, darker and abruptly constricted externally above its subtending prominence, a crest-like series of branchlets (usually broken) arising from its broad distal surface, the most external opaque or basally suffused. Perithecium $175 \times 28\ \mu$ including basal cell-region ($20\ \mu$). Main appendage $70\ \mu$, to tips of branches $170\ \mu$. Receptacle including foot $50\ \mu$. Total length to tip of perithecium $275\ \mu$.

On *Ophites Fauvelii*, in the Museo Nacional Collection. Collected at Palermo by Dr. J. Brèthes.

Several specimens, only one of which is well matured, have been examined. The species belongs in the section of the genus the members of which occur on *Cryptobia*. It is most nearly allied to *C. purpurascens*, but is readily distinguished by the characters of its appendage.

Corethromyces Platensis nov. sp.

Perithecium becoming translucent amber-brown; usually straight, subconical, tapering more or less from the variably swollen venter to the blunt hyaline apex; the tip more or less clearly distinguished above a slight enlargement; the basal cells rather large; the stalk-cell variably, often greatly, elongated, and tapering somewhat to its insertion. Appendage consisting primarily of three superposed cells; the basal, and sometimes also the others, more or less deeply blackened; the subbasal cell bearing distally from its inner side a pair of antheridial branches, one or both of which often become more or less highly developed through monopodial branching, forming two main axes of obliquely superposed cells; the lowest producing on the inner side fan-like antheridial branches, the ultimate branchlets consisting of two or three superposed antheridial cells; the rest bearing externally simple or branched, sterile, upcurved, appressed branchlets, the lower mostly blackened: the third cell of the primary appendage variably developed; often very small bearing distally and from its inner face, which may become outcurved and recurved, a variable number of simple bristle-like black branches, the lowest external one originally terminal (usually broken off), one of the others often greatly developed by successive monopodial branching, replacing the main appendage and consisting of from three to twelve obliquely superposed cells, each of which bears distally and externally, usually simple branchlets, for the most part short, three-celled, becoming more or less deeply suffused with black or blackish brown, upcurved, more or less closely appressed; the two or three uppermost hyaline, long, multiseptate. Basal and subbasal cells of the receptacle hyaline, small, subequal, or the subbasal larger. Perithecium, including basal cell-region, $118–125 \times 34–40\,\mu$, the sporiferous part $75–100\,\mu$; the stalk-cell $40–60 \times 12–20\,\mu$. Spores $24 \times 2.5\,\mu$. Greatest length of whole appendage $150–360\,\mu$. Receptacle, including foot, $40 \times 20\,\mu$. Total length to tip of perithecium $85–235\,\mu$.

var. **gracilis** nov. var. Perithecium and its stalk-cell longer and more slender than in the type. Appendage divergent, slender, its primary axis consisting of three superposed cells; the basal hyaline below, blackened and slightly constricted above; the subbasal hyaline, rarely externally suffused, nearly twice as long as the basal cell, a small cell separated from its inner distal angle forming a rounded prominence from which arise right and left paired antheridial branches, wholly hyaline, spreading, several times closely branched; antheridial cells single or two to four of these superposed; the third cell bearing distally one to usually not more than three branches; the outer, primary branch, shorter, slender, hyaline; the others, if present, hyaline, stouter, longer, sometimes once furcate above the basal cell. Perithecium 100–156×20–35 μ, including basal cell-region; stalk-cell 175×20 μ. Greatest length of appendage 150–430 μ. Total length to tip of perithecium 180–385 μ.

On *Lathrobium nitidum* Er., Palermo, Temperley and Llavallol, Nos. 1687, 1688, 1998.

The type of this species occurs on various parts of the host and when its appendage is well developed is a very striking form. It is very variable in size and in the development of its appendage, and near the tips of the legs assumes a small, compact stout habit quite unlike the usual form. The variety corresponds exactly to the type formerly distinguished as *Rhadinomyces,* and occurs on the elytra, usually, or at the base of the legs. It differs from the type in its slender form, the absence of sterile branchlets on the antheridial branches, and of the black bristle-like branches of the rest of the appendage. The examination of a sufficient series, however, appears to show that the two are not specifically separable.

Corethromyces Scopaei nov. sp.

Perithecium hyaline becoming faintly tinged with yellowish, relatively rather large, usually slightly asymmetrical owing to an outward curvature, tapering but slightly above the basal portion which is not prominently inflated; the tip short, conical, subsymmetrical; the small rounded papillate apex prominent; the basal cells forming a short compact group not distinguished from the base of the perithecium, the stalk-cell broad hyaline narrower below, set obliquely or sidewise on the small nearly isodiametric hyaline subbasal cell of the receptacle; the basal cell of which is about the same size but of characteristic form, rounded outward, its thick outer wall passing into and not distinguished from the broad undifferentiated hyaline or slightly purplish foot. Appendage wholly hyaline, the basal cell hardly longer than broad, the outer wall greatly thickened and in contact below with the basal cell of the receptacle; the subbasal cell somewhat narrower, the outer wall greatly thickened; the distal portion of the appendage occupied by a more or less crest-like series of hyaline branches derived from the end of the subbasal cell and from one or perhaps more terminal cells which become displaced and appear to be external, their cavities obliterated by their thickened walls, the outer branches short, directed outward and upward, the inner (from the subbasal cell) stouter, longer, once or twice branched near the base and extending not much beyond the middle of the perithecium. Perithecium 65–75: ascigerous portion 55–70; the stalk-cell $28 \times 12\,\mu$. Receptacle $20 \times 16\,\mu$. Total length of appendage including branchlets $60–80\,\mu$. Total length to tip of perithecium $95–120\,\mu$. Spores $18 \times 2\,\mu$.

On superior abdomen of *Scopaeus frater* Lynch. No. 1698 and No. 1702, Palermo.

A small pale species chiefly peculiar from the fact that no foot is distinguished from the peculiar rocker-like basal cell of the receptacle, which is usually quite hyaline. The species bears more resemblance to the Stilicus-inhabiting forms than to the more typical members of the genus.

Corethromyces brunneolus nov. sp.

Perithecium pale reddish brown with a yellowish tinge, usually rather strongly bent inward distally; the basal cells very small not distinguished from the base of the ascigerous portion, which tapers but slightly to the blunt rounded hyaline apex; the tip not distinguished; the small basal cell-region clearly distinguished by a distinct constriction from the stalk-cell, which may be nearly straight, or strongly curved, distally broader or slightly inflated, about twice as long as broad; the stalk-cell and the appendage very asymmetrical in their relation to one another and to the small receptacle; which consists of two subequal cells, concolorous with the perithecium. Basal cell of the appendage relatively large, symmetrically inflated; the subbasal cell, at maturity and through displacement, appearing to bear directly a more or less fan-like series of short, rather stout, somewhat incurved hyaline branches, which may be once or twice branched near the base. Spores $22 \times 2.5\,\mu$. Perithecia $58–62 \times 20\,\mu$; ascigerous portion $54–58\,\mu$; the stalk-cell $23–30 \times 12\,\mu$. Receptacle $24 \times 16\,\mu$ including foot. Appendage, total length including branches, longer, $100\,\mu$; the basal cell $20 \times 16\,\mu$.

On the elytra of *Stilicus* sp., Nos. 1511 and 2012, Temperley.

This pale species appears to be very rare, only a very few specimens having been obtained. It is quite unlike any of the other forms which occur on *Stilicus* and appears to be most nearly allied to the preceding species.

Corethromyces Stilicolus nov. comb.

Stichomyces Stilicolus Thaxter.

This species which, in view of its single free antheridia, I formerly placed provisionally in *Stichomyces,* was found frequently in the vicinity of Buenos Aires on several species of *Stilicus,* and an examination of sufficient material shows that, although the species tends to produce its antheridia singly, or free in groups, the intercalary arrangement also occurs, and there can be no doubt but that the form is congeneric with the other Stilicus-inhabiting species of the genus. The Argentine specimens are similar in all respects to those first obtained on *Stilicus* at Arlington, Mass.

Corethromyces pygmaeus nov. sp.

Perithecium becoming rather deeply suffused with dull reddish amber-brown, asymmetrical; the basal cell-region small and hardly distinguished, one of its cells usually bulging externally to form a distinct prominence; the ascigerous portion, usually rather abruptly inflated externally, the apex of the curvature forming a more or less well distinguished hump, the inner margin usually straight; the tip broad not distinguished, the apex truncate, subtended externally by a rather abrupt rounded prominence: stalk-cell suffused, becoming concolorous with the perithecium, usually strongly curved inward, distally broader below the base of the perithecium, from which it is distinguished by a very slight constriction, and which it nearly equals in length. Axis of foot at right angles to that of the basal cell of the receptacle, which is twice as large as the somewhat flattened subbasal cell; externally strongly concave, its inner margin convex, sometimes distally constricted on its inner side, a deeply suffused outgrowth arising from its outer upper angle; almost uniform in width above its narrower base, extending outward then upward abruptly beside the two basal cells of the appendage, sometimes bent inward near its rounded tip. Basal cell of the appendage large, nearly spherical; the subbasal cell small and surmounted by several hyaline branches, one or two of which may extend nearly to the tip of the perithecium. Perithecium 58–66 $\times$ 24–28 μ: stalk-cell 40–60 $\times$ 20 μ. Spores 26 $\times$ 2.5 μ (measured in perithecium). Receptacle 20 $\times$ 12 μ, its outgrowth 20–30 $\times$ 5 μ. Total length of appendage 30–40 μ. Total length to tip of perithecium 100 μ.

On head and labium of *Stilicus* sp., No. 1963B, Palermo.

This small species was found only once in the park at Palermo but was also obtained on a similar host at Corral, Chile, No. 1902. It is allied to *C. Stilici*, from which it differs in the form of its perithecium and receptacle, as well as in the character of the outgrowth from the latter.

Corethromyces sigmoideus nov. sp.

Axis from tip of perithecium to foot, describing an even sigmoid curve, the lower curvature much shorter. Perithecium strongly curved outward, translucent amber-brown; the basal cell-region concolorous, often slightly distinguished from the ascigerous part, the basal cells well developed and triangular; the apparent apex formed by a blunt outgrowth directly continuous with the ascigerous portion, of which it forms the bluntly rounded slightly asymmetrical termination; the apex proper having its pore lateral in position and hardly distinguishable: stalk-cell but faintly suffused, broader distally, and distinguished from the basal cell-region by a slight constriction; abruptly curved near the base, the axis of which is directly continuous with the subbasal cell of the receptacle. The latter slightly suffused, relatively large, extending on the perithecial side downward nearly to the foot, and obliquely separated from the externally deeply suffused basal cell; which is of about the same diameter throughout, including its upward extension which, lying beside the subbasal cells, extends beyond the base of the first cell of the appendage to which it is adherent, forming a rounded prominence; the upgrowth larger than the basal cell proper, and not distinguished from it. The basal cell of the appendage subelliptical, concolorous with the subbasal cell of the receptacle, its long axis nearly at right angles to that of the rest of the appendage which is curved across the stalk-cell of the perithecium; the subbasal cell small, flattened or rounded, bearing on its inner surface a smaller ramiferous cell, and distally a much larger one, often several times longer than broad, and bearing distally numerous branches; the latter more or less branched, all the branches tapering somewhat, slightly suffused below, hyaline above; the two or three longer ones curved downwards. Perithecia $70–85 \times 23–27$ μ: stalk-cell 60×18 μ. Receptacle including foot 40 μ. Total length to tip of perithecium 135–170 μ. Spores 26×3 μ.

On the superior right lateral margin of the prothorax of *Stilicus elegans* Lynch. Llavallol, No. 1994.

Closely allied to the last species, which grows in a similar position on another species of *Stilicus;* but readily distinguished by its sigmoid habit, and the different structure of its appendage and perithecium.

Corethromyces uncigerus nov. sp.

Perithecium rather bright translucent reddish amber, somewhat concave and more deeply suffused on the inner side, rather strongly convex externally, the basal cells clearly defined, subtriangular in a compact group, the basal cell-region not distinguished from the ascigerous portion, which tapers distally to its peculiarly modified tip, the blackish suffusions of which extend to an opaque, hook-like prolongation which, bending at right angles, forms a lid immediately above and often partly concealing the hyaline apex: the stalk-cell nearly hyaline, variously, often greatly, elongated, curved, or often straight and erect; distally broader than the basal cell-region, from which it is thus separated by a more or less pronounced constriction. Subbasal cell of the receptacle relatively large, hyaline, subtriangular, the basal cell narrow below, smoky, extending obliquely upward to the base of the appendage where it is continued by a deeply suffused broad straight erect upgrowth, which is flattened against the appendage, and extends to or beyond its subbasal cell. Basal and subbasal cells of the appendage subisodiametric and subequal, or the basal larger and longer, the subbasal appearing to bear from its broad distal surface, a small tuft of hyaline, rather short branches and branchlets. Spores $26 \times 2.8\ \mu$. Perithecia $70–85 \times 20–26\ \mu$; its stalk-cell $50–125 \times 15\ \mu$, distally, $20\ \mu$ broad. Appendages, longer, $75\ \mu$. Receptacle, including foot, $30–40\ \mu$, its outgrowth $30–60\ \mu$. Total length to tip of perithecium, $150–250\ \mu$.

On the posterior legs of *Stilicus elegans* Lynch, No. 1994, not uncommon at Llavallol, and easily distinguished by the peculiar tip of its perithecium which recalls that of *Chitonomyces psittacopsis* or of *C. Bullardi.*

Corethromyces armatus nov. sp.

Perithecium nearly uniform dull purplish amber-brown, the basal cell-region not distinguished, or somewhat paler and very slightly narrower than the ascigerous part above; the inner margin slightly convex, the outer strongly so distally, the tip broad undifferentiated; the apex broad, flat, subtended internally by a rounded projection and externally by a prominent conical outgrowth extending obliquely upward and outward and narrower toward its blunt, often slightly contracted, apex: the stalk-cell hyaline, shorter than the perithecium, straight or outcurved, often slightly enlarged on the inner side below the perithecium. Subbasal cell of the receptacle triangular, hyaline, the basal cell abruptly curved at right angles, wholly suffused with blackish, but not opaque; obliquely related to the subbasal cell, and continued below and just beyond the base of the appendage by an external outgrowth which is not free, even at its tip, being adherent to the basal and subbasal cells of the appendage. The basal cell of the appendage nearly hyaline, bent almost at right angles, and thus turning the rest of the appendage across the stalk-cell of the perithecium; the subbasal cell often abruptly narrower, hardly twice as long as broad, bearing distally a few external branches and a large appendiculate cell, from which arise elongate tapering branches, two or three of which may exceed the perithecium and its stalk-cell in length. Spores $32\times3\ \mu$. Perithecium $60–70\times20–23\ \mu$, its terminal projection, upper margin $28\ \mu$, lower $40\ \mu$; stalk-cell $30–45\times12–18\ \mu$. Receptacle $30–40\ \mu$. Longest appendage $175\ \mu$. Total length to tip of perithecium $120–150\ \mu$.

On the upper surface of the prothorax near the right margin of a species of *Stilicus,* Palermo, No. 2012, and Temperley; No. 1932, Tucuman.

This species, which was met with rarely, always occurred in exactly the same position, and is easily distinguished by its appendiculate perithecium, and the peculiar position of its appendage.

Corethromyces rhinoceralis nov. sp.

Perithecium dirty pale brownish amber, a deeper patch of amber-brown involving the subterminal wall-cell on the inner side; subclavate in form, the gradual distal enlargement extending to the subterminal wall-cell; distally curved outward to the subhyaline apex which is slightly cleft, and subtended on the inner side by a long, straight, rather slender unicellular spine-like process which tapers slightly to a blunt apex and projects at right angles; basal cell-region well developed, concolorous, not distinguished from the ascigerous part, narrower below where it connects with the rather slender free, subcylindrical stalk-cell. Receptacle concolorous with the appendage and perithecium, the basal and subbasal cells of about equal length, the subbasal cell half as broad as the basal, except immediately above the latter, and obliquely separated by a curved septum from the basal cell of the appendage which lies beside it and extends but slightly above it: the rest of the appendage rather slender, rigid, its axis of four or five successively smaller superposed cells, each bearing distally, from the inner angle, a short hyaline branch, seldom persistent and producing large bottle shaped antheridia singly or in series of two, one terminal and the other intercalary. Spores (in perithecium) about $45 \times 6\,\mu$. Perithecium, including basal cell-region, $240–250 \times 46\,\mu$: the subterminal spine $80–90\,\mu \times 8–10\,\mu$ near base; the stalk-cell $60 \times 15\,\mu$. Receptacle including foot $70\,\mu$. Free portion of appendage $135\,\mu$.

On the inferior surface of the abdomen of *Pinophilus suffusus* Er., No. 1977, Llavallol.

Closely allied to *C. Indicus*, from which it differs chiefly in the clavate form of the perithecium, and in the highly developed spine which springs from a projection of one of the subterminal wall-cells. The species appears to be very rare, for although very many specimens of its host were obtained it was found in only two instances.

Corethromyces macropus nov. sp.

Nearly hyaline. Perithecium asymmetrical; the outer margin convex, the inner straight below the incurved tip; the basal cell-region not distinguished from the slightly and symmetrically inflated body, which tapers slightly to the undifferentiated tip; the latter slightly suffused with brownish, and rather abruptly bent inward, one of its lateral wall-cells deeply suffused with brown, and forming a free truncate projection immediately beside the flat-conical, hyaline, slightly geniculate apex: stalk-cell small, not distinguished from the basal cells, one of which lies beside it extending nearly to its base. Receptacle relatively large more or less strongly curved, the foot large and long, tapering from a large bulbous portion to its pointed extremity: the basal cell more or less deeply suffused with smoky brown, paler above, rectangular, somewhat longer than broad, distinguished by a horizontal septum from the small subbasal cell, from which the perithecium and appendage arise asymmetrically. The appendage consisting of about five superposed cells; rigid, straight, divergent, nearly hyaline; the basal and subbasal cells not appendiculate, the rest bearing short branches distally on the inner side. Perithecia, including stalk- and basal cells, $100–110 \times 25\,\mu$. Receptacle, including foot, $55 \times 18\,\mu$. Appendage $50–55 \times 8–10\,\mu$. Total length to tip of perithecium $150–180\,\mu$. Spores $30\,\mu$.

On *Heterothops* nov. sp., No. 1987, Llavallol.

This curious form is most clearly distinguished by the peculiar conformation of the tip of the perithecium and its relatively large receptacle and foot; but is included only provisionally in the present genus owing to the fact that the antheridia are not distinguishable in any of the specimens. The host has been determined as a new species by Dr. Bernhauer.

Corethromyces rostratus nov. sp.

Perithecium tinged with pale brownish, long, slender, erect and straight, symmetrical; the basal cell-region distinct from the more or less inflated basal ascigerous part; the mid-region sometimes rather abruptly narrower and elongate; the tip not distinguished, symmetrical; the apex narrow subsymmetrical, hyaline, abruptly papillate: stalk-cell small, concolorous, rather broader than long. Receptacle externally prominent below the insertion of the appendage, the basal cell large, subtriangular, suffused with smoky brown, externally opaque, its broad distal surface obliquely separated from the small flattish subbasal cell. Appendage somewhat divergent, consisting of five or six superposed cells; the basal nearly hyaline; those above it more distinctly suffused, and each bearing a branch from its distal inner angle; the branches once to several times divided, the subbasal cell of the lowest branch, in conjunction with the bases of its two or three branchlets, rather characteristically inflated; the ultimate branchlets slender, hyaline, cylindrical, associated with usually single (?) antheridia. Perithecia, above basal cells, 120–135×20–22 μ: the stalk-cell 6×8 μ. Receptacle 55–58 μ. Spores 30×3 μ. Appendage 95–100×12–14 μ its longest branches 155 μ. Total length to tip of perithecium 200–230 μ.

On various parts, usually the abdomen of *Heterothops* sp., Temperley, No. 2000, Llavallol, Nos. 1985 and 1987.

It seems difficult to obtain this species in very perfect condition, and though I have examined material from a number of different individuals, I have been unable, even in the younger specimens, to determine the exact nature of the antheridia which appear to be solitary near the bases of the lower branches of the appendage. It is possible that I have mistaken short branches for these organs, and in any case the reference of the form to *Corethromyces* as above emended must be considered provisional.

A well marked variety was also found having a hyaline obconical basal cell, separated by a straight horizontal septum from the small triangular cell above, its perithecium and appendage closely approximated.

Stichomyces Catalinae nov. sp.

Perithecium rather stout, nearly hyaline; the basal cell-region well developed, slightly broader than the base of the ascigerous region; the latter becoming gradually and but slightly broader to the broadly conical, symmetrical, or slightly bent, distal region, from which it is distinguished by a slight double corrugation on one or both sides; the apex small, often bent sidewise, rather abruptly distinguished, symmetrical, rounded, hyaline and subtended by dark brown suffusions which often appear like paired rings; the stalk-cell well distinguished, broader than long, distally bent abruptly upward from its insertion which is lateral, from the distal end of the subbasal cell of the receptacle. Receptacle deeply suffused with brown, except its narrow hyaline base just above the small foot; the basal cell broader distally, hardly twice as long as the somewhat broader subbasal cell. The appendage consisting of an axis of four superposed cells not distinguished from the receptacle, and concolorous with it; the subbasal cell bearing from its upper inner angle a group of hyaline branches, which reach to or beyond the tip of the perithecium; the terminal cell smaller, hyaline, and bearing a few hyaline branches. Spores 20×1.5 μ (measured in perithecium). Perithecium 50–60×15–20 μ. Receptacle, including foot, 30–55×9–12 μ. Main axis of appendage 30–35×12 μ; total length to tip of longest branchlets, 75 μ. Total length to tip of perithecium, 90–125 μ.

On *Conosoma testaceum* Lat., No. 1984, Llavallol.

The branches of the appendage in this species are usually badly broken, and even in those which are still intact, are so beset by masses of bacteria, that it has not been possible to make out the antheridia with certainty, although they appear to arise in small groups somewhat as in *S. Conosomae.* The character of the perithecium and of its apex, and the dark continuous axis formed by the receptacle and main appendage, are characteristic of the species, although a few specimens were obtained that are smaller and in which the successive cells of the receptacle and appendage are less evenly continuous.

Laboulbenia Lathropini nov. sp.

Receptacle relatively stout and small, cells I and II faintly suffused, subequal in length; the latter broader, sometimes longer; the rest of the receptacle and the perithecium deeply suffused with dirty olivaceous brown; cells III and IV subequal; the upper angle of cell V free between the perithecium and the slightly oblique insertion-cell, which is thick but rather small. The simple outer appendage enormously elongated, distally hyaline, the cells several times longer than broad, all similar; the first three or four somewhat shorter than the rest; the basal cell of the inner appendage very small, bearing an antheridial branch consisting of one to two small cells, terminated by one to two antheridia, one of which may be replaced by a long simple sterile branch. Perithecium relatively large, not wholly free, slightly and evenly inflated; the wall-cells strongly spiral and marked by fine irregularly parallel lines; the tip deeply suffused, the lip-edges hyaline, subequal, the apex sulcate and turned strongly inward. Spores 75×8 μ. Perithecium 150–175×45–50 μ. Receptacle 120–155 μ. Longest appendage 900×16 μ at base. Total length to tip of perithecium 900×16 μ.

On the upper surface of the abdomen of *Lathropinus fulvipes* Er., No. 1975, Llavallol.

A species of the simpler "polyphaga" type, most nearly allied to *L. Oedodactyli,* and distinguished by its enormously elongated outer appendage and spirally twisted, longitudinally striate wall-cells. The host was found rarely in decaying wood.

Laboulbenia funerea Speg.

This form which is very abundant on species of *Anaedus* in the vicinity of Buenos Aires, especially in the woods at Santa Catalina, is, in my opinion, best regarded as a variety of *L. polyphaga.* It is characterized by its small size, averaging about 175 μ to the tip of the perithecium, the receptacle being usually rather short, about 95–100 μ, although cell II is occasionally considerably enlarged. Cell I is always hyaline, cell II often so, though frequently involved by the characteristic blackish olive-brown suffusion of the rest of the receptacle, which is concolorous with the perithecium except for a small hyaline patch usually present below the insertion-cell. The outer appendage is usually furcate above its subbasal cell, the two branches distally hyaline and tapering; the small basal cell of the inner appendage bearing one or two short branches, the lower cells of which bear a few antheridia. The perithecium is straight, very slightly inflated, the tip clearly distinguished, deeply blackened, the lips hyaline, turned slightly outward, separated by a slight apiculus.

Laboulbenia hemipteralis nov. sp.

Receptacle rather short and stout, the basal and subbasal cells subequal in length; the former hyaline; the rest of the receptacle more or less deeply tinged with olivaceous, especially the relatively broad distal portion; cell VI (stalk-cell) small, triangular, its oblique contact with cell II not extending to the end of the latter; the basal cells of the perithecium obsolete; the ascigerous cavity lying immediately above the stalk-cell. Perithecium olivaceous, tapering, its distal half, only, free; the tip conspicuously blackened and bent slightly inward; the apex subsymmetrically rounded, or slightly pointed, concolorous with the tip; the pore turned inward. Insertion-cell relatively very broad, lying somewhat higher than the middle of the perithecium, the basal cell of the outer appendage bearing a single branch, consisting of a single cell externally suffused at its base, bent inward slightly, producing four or five closely successive branchlets externally, the lowest of which is distinguished by a thin darkened septum and bears about four secondary simple branchlets in a similar fashion, the lowest of which is more slender and suffused especially at its base, usually projecting subhorizontally, the others hyaline; the remaining primary branchlets hyaline, simple or furcate, often spirally curved above: the basal cell of the inner appendage giving rise normally to an outer and an inner and two lateral branches, consisting of single short cells, each bearing a large terminal brown antheridium, which may be replaced by a sterile branch bearing hyaline branchlets like those above the base of the outer appendage. Perithecia 66×20–$23\,\mu$. Spores $22 \times 2.6\,\mu$ (in perithecia). Receptacle $85 \times 23\,\mu$. Appendages to tips of longest branchlets, $105\,\mu$. Total length to tip of perithecium 100–$120\,\mu$.

On the legs and inferior surface of *Velia Platensis* Berg., Palermo, near Belgrano, No. 1951 along the margin of a pool. (Van Duzee det.)

This very clearly distinguished form which was found with the following species. is the first of the genus thus far reported on Hemiptera. The material is abundant and in good condition.

Laboulbenia Veliae nov. sp.

Receptacle dirty olivaceous, concolorous with the perithecium, cells I and II forming a stout elongate stalk about five times as long as the scarcely broader distal portion. The insertion-cell broad and thick, deep reddish, not quite opaque; the outer and inner basal cells of the appendages subequal; the appendages but faintly suffused or subhyaline, once or twice somewhat irregularly branched; the branches divergent, the two or three lowest cells short, slightly inflated, distinguished by dark thin septa. Perithecium not wholly free, narrow, geniculate below the tip, the pore lying laterally on the inner side in the angle formed between the small rounded hyaline prominent inner lips and the greatly enlarged outer lip-cells, which are deeply suffused externally on the side above the pore, above and beyond which they form a characteristic large blunt erect slightly bent process, which terminates the perithecium. Spores $50 \times 7\,\mu$. Perithecia $125–130 \times 24\,\mu$. Receptacle $235–260\,\mu$; cells I and II $200 \times 18\,\mu$. Appendages including longest branchlets, $200\,\mu$. Total length to tip of perithecium, largest, $350\,\mu$.

On the superior surface of the thorax of *Velia Platensis* Berg., No. 1951, Palermo near Belgrano.

A very distinct species, remotely resembling *L. ceratophora* and its allies. A small group of adult specimens was found on the same individual with *L. hemipteralis.*

Laboulbenia Lacticae nov. sp.

Receptacle hyaline, becoming very faintly tinged with brownish yellow; cells I and II subequal, nearly as broad as the much reduced distal portion; cells III, IV and VI not greatly different in size, the insertion-cell occupying but half of the distal surface of cell IV, the rounded outer half of which is free externally. Basal cells of the appendage involved by the opacity of the insertion-cell, and indistinguishable; the outer bearing a compact group of six or eight suberect branches in two radial rows, or more irregularly placed, which bear short branchlets on their inner sides, and consist of two parts; a basal, seated on an almost hyaline cell and composed of rather short cells deeply suffused with blackish brown and constricted at the septa, and a distal portion suffused only at its base, above which it is quite hyaline rigid and tapering: basal cell of the inner appendage bearing one or two short branches on which one or two antheridia may be produced, the latter sometimes occurring on the inner branches of the outer appendage also. Perithecium wholly free, concolorous with the receptacle, narrow, but slightly inflated, the tip nearly as broad as the body, and clearly distinguished by blackish suffusions; the lip-cells large rounded and bent slightly inward. Spores 45×3.5 μ. Perithecium $90–100 \times 24–28$ μ. Receptacle $80 \times 15–155 \times 22$ μ. Longer appendages $135–150$ μ. Total length to tip of perithecium $175–280$ μ.

On the tips of the elytra, wings and abdomen of *Lactica varicornis* Jac. or a closely allied species. Palermo, No. 1462.

Laboulbenia Blechri Spegazzini.

Receptacle slender, hyaline, the basal cell not symmetrically adjusted to the subbasal, which is slightly prominent above it on the posterior side, while the basal bulges below the subbasal on the anterior side; the subbasal somewhat longer than the basal, hardly broader; cells III, IV and VI subequal and subisodiametric, cell V very small. The insertion-cell, black, rather thin, not very broad; the outer appendage erect, simple, its three lower cells rather deeply tinged with olivaceous, especially externally, subequal, each somewhat broader distally and thus rather abruptly distinguished from one another; the rest of the appendage quite hyaline, tapering slightly: basal cell of the inner appendage much smaller than that of the outer, producing the usual branch on either side, each once or twice branched; the whole forming a group of four to six branchlets olivaceous below, which are relatively very stout, short, bent inward or across the perithecium, the longest extending just above its tip, the lower circinate distally. Perithecium colorless, straight, its axis somewhat divergent from that of the slender receptacle, the basal cell-region forming an external rounded prominence, the junction of the basal and subbasal wall-cells also prominent; the tip, rather stout, subtended by a slight external prominence, the apex broad, the hyaline lips outwardly oblique, subtended by an olivaceous patch on the inner side. Spores 35×3 μ. Perithecium $62–70 \times 20–22$ μ. Receptacle $80–100$ μ. Appendages, longer, inner 55 μ, outer 110 μ. Total length to tip of perithecium 140 μ.

On *Blechrus* sp., at the tips of the elytra. Llavallol, No. 1979.

A single specimen of the host was found bearing this species which is most readily distinguished by its relatively very large incurved inner appendages. The perithecium may become suffused with age, but in the specimens examined it is quite hyaline, although they are sufficiently mature to have produced spores.

Laboulbenia Monocrepidii nov. sp.

Cells I and II hyaline or faintly olivaceous, narrow, cell II rather abruptly broader distally, and obliquely separated from cell III by an incurved partition; the distal portion of the receptacle deeply suffused with olive-brown, deeper externally below the very thick dark insertion-cell; cell V paler. Basal cells of the appendage suffused, subequal, each bearing a short single simple rarely once-branched erect similar appendage, the basal cell of which is subhyaline or more faintly suffused, and distinguished above and below by a constriction and by a blackened septum, the rest of the appendage short hyaline, tapering to a blunt point, the inner appendage single short simple, replacing a single small short antheridium found in younger specimens. Perithecium about three quarters free, deeply tinged throughout with olive-brown, slightly inflated; the tip long, not abruptly distinguished, suffused with blackish, the black shades extending downward separated by pale areas; the lips asymmetrical, the edges irregular, outwardly oblique, hyaline. Spores $75 \times 4.5\ \mu$. Perithecia $120–135 \times 40–45\ \mu$. Receptacle $150–225\ \mu$. Longest appendage $80–110\ \mu$. Total length to tip of perithecium $250–325\ \mu$.

On the elytra etc. of *Monocrepidius* sp., Palermo, No. 1683 and also at Llavallol.

A clearly distinguished species, the first as yet recorded on a member of this family (Elateridae).

Laboulbenia fuscata nov. sp.

Receptacle tapering evenly to the small foot, dirty olive brown, cells I and II paler, cell IV externally rounded and prominent below the rather broad insertion-cell which is but little darker than the cells below it. Basal cell of the outer appendage roundish or bell shaped, deep reddish brown, hardly larger than the inner, the appendage externally blackened and curved abruptly outward above it, short, separated by an opaque septum from its deeply suffused reddish brown basal cell, and bearing two to three suberect or incurved short branches; the inner basal cell bearing two deep reddish brown, somewhat bell-shaped cells, terminated by a single short erect usually simple appendage. Perithecium free, except at the very base, dark translucent yellowish olive, subsymmetrical, curved slightly outward, twisted one quarter so that the tip is viewed at right angles to its normal position; the tip large, characteristically and slightly inflated, especially its inner basal half, externally margined with black, the apex nearly opaque, broad, symmetrically bilobed. Spores copious $75 \times 4.5\ \mu$. Perithecium 156×48–$55\ \mu$. Receptacle $200 \times 75\ \mu$. Total length to tip of perithecium 330–350 μ. Longest appendages 120 μ.

On legs of a small species of *Pterostichus* taken on flats outside the docks at Buenos Aires, No. 1968.

A peculiar form, of which four fully developed specimens were obtained, which does not appear to be nearly allied to any of the described species.

Laboulbenia granulosa nov. sp.

Receptacle becoming more or less uniformly tinged with dark olive, the suffused area coarsely granular-punctate, the dark granulation involving the distal portion of the otherwise hyaline basal cell; cell II narrow, very obliquely separated from cell VI which extends nearly to its base, cells III and IV subequal. Insertion-cell broad and thick; cell IV protruding but slightly below it; basal cell of the outer appendage sometimes twice as large as that of the inner, both becoming concolorous with the receptacle; the outer appendage usually furcate above its subbasal cell; the basal cell of the inner appendage producing a branch on either side, usually once branched; the branchlets of both appendages hyaline, eventually curved inward across and beyond the terminal portion of the perithecium. Perithecium evenly olivaceous, a few coarse scattered maculations on the basal third; somewhat inflated in the middle, the tip not abruptly distinguished, rather stout and broad; the apex asymmetrical; the outer lip-cell somewhat more prominent, the inner subtended by a blackish suffusion. Perithecium $110 \times 40\ \mu$. Receptacle $135 \times 40\ \mu$. Total length 215 μ.

On the legs of *Argutor Bonariense* Dej. (thus named in the Museo Nacional) No. 1460, Isla de Santiago, near La Plata.

This species bears a distant resemblance to *L. scelophila,* but is distinguished by its more slender abruptly curved appendages and the blackish powdery granulation of its suffused portions. The host appears to be the same which is called by Spegazzini *Argutoridius oblitus,* which Mr. Henshaw informs me should be placed in *Pterostichus.*

Laboulbenia subinflata nov. sp.

Receptacle rather long but variable, cells III and IV becoming olivaceous, the rest pale dull yellowish, the upper half or more of cell II characteristically swollen, broader than the receptacle above it, from which it is separated by a distinct indentation on one or both sides; cell III relatively large, sometimes twice as large as cell IV, the outer half of which lies external to the insertion-cell, below which it is thus prominent and obliquely rounded outward. The insertion-cell black, rather thick and narrow; the basal cell of the outer appendage several times as large as that of the inner, the subbasal cell similar and subequal, both becoming olivaceous; the latter bearing regularly two parallel branches distally, the outer usually shorter; the whole appendage erect or slightly divergent and reaching a short distance beyond the tip of the perithecium: the small basal cell of the inner appendage bearing a short erect branch on either side, from the base of which arises a unicellular antheridial branchlet terminated by two to three antheridia. Perithecium relatively small, the lower wall-cells and the upper basal cells becoming tinged with olive, distinguished from the part above by a more or less pronounced elevation, later obliterated, from which a darker area of olive-brown extends horizontally across the perithecium, which above it is pale amber-brown; the tip relatively narrow, abruptly distinguished externally above a conspicuous rounded prominence, its concave external margin broadly blackened; the lips outwardly oblique, coarse, the inner more prominnent, rounded, subtended by a blackish patch. Spores $55 \times 5\,\mu$. Perithecium $175–185 \times 45–50\,\mu$. Receptacle $310–415 \times 62–78\,\mu$; largest subbasal cell $187 \times 75\,\mu$. Appendages $200\,\mu$, longest $215\,\mu$. Total length to tip of perithecium $350–585\,\mu$.

On the left margin of the prothorax, superior, of "*Argutor Bonariensis* Dej."; Buenos Aires, Nos. 1512 and 1962; Llavallol, No. 2032.

This species was found on a number of individuals of its host, and always in exactly the same position, sometimes in company with all of the six other species, including *L. polyphaga*, which occur on this host, from which it may be easily distinguished by its perithecium, appendages and inflated subbasal cell.

Laboulbenia Bonariensis nov. sp.

Large, long, slender, and as a rule evenly curved from base to apex. Receptacle becoming more or less evenly suffused with olive brown, the base of cell I hyaline, the distal part more deeply suffused than the rest of the receptacle; cell II somewhat longer anteriorly than cell I, cell IV somewhat obliquely prominent below the insertion-cell, which is relatively narrow and thick: appendages slender, the basal cell of the outer very slightly longer than broad, somewhat larger than that of the inner, becoming deeply suffused with age, bearing a single slightly divergent branch, the slightly smaller basal cell of which bears two to three branchlets distally, its deep external suffusion continuous with that of its short slender outer branchlet, its one or two inner branchlets radially placed, simple hyaline erect, extending to or above the tip of the perithecium: basal cell of the inner appendage bearing one or two branches, sometimes once branched, hyaline, erect, similar to the adjacent branches of the outer appendage. Perithecium bent inward, becoming rich brown with a slight olivaceous tinge when fully mature; the base, above the basal cells, sometimes rather abruptly distinguished and slightly paler; the tip rather long, broad, hardly distinguished, sometimes bent very slightly outward; the apex broad, blunt, often symmetrically rounded; or the lips slightly prominent, subhyaline and subtended by a deeper shade on the inner side. Spores $70 \times 6\,\mu$. Perithecium 135×35 to $210 \times 55\,\mu$, average $175 \times 42\,\mu$. Receptacle $235–335 \times 50–70\,\mu$. Longest appendage $200\,\mu$. Total length to tip of perithecium $300–500\,\mu$.

On "*Argutor Bonariense* Dej." Usually growing in a single group not far from the base of the outer margin of the left elytron, but occurring less frequently on the legs and inferior surface. Llavallol, No. 2032; Temperley, No. 1512; Buenos Aires, No. 1962; La Plata, No. 1460.

A species usually distinguishable with a hand lens from its large size and localized position on the left elytron. In one group of individuals examined there is some variation from the type described, cell I being short, cell II much enlarged and separated from cell VI by a conspicuous indentation, so that the receptacle is subgeniculate; the tip is more prominently distinguished and bent inward, the lips broader and more prominent. The variations in size are considerable and almost straight individuals of the normal type sometimes occur.

Laboulbenia lutescens nov. sp.

"*Laboulbenia fumosa*," Spegazzini, Fungi Chilenses, p. 135.

Receptacle more or less deeply, though not uniformly suffused with clear olive brown, especially along the margin below the appendages, the basal cell small, hyaline below; cell II but slightly longer; cells

II and VI subequal, the latter somewhat shorter; cell IV abruptly prominent externally below the insertion-cell. Insertion-cell deeply suffused, rather thick; the basal cell of the outer appendage somewhat smaller than that of the inner, externally opaque, bearing distally two branches radially placed; the outer branch strongly divergent to horizontal or even slightly recurved, almost wholly opaque, its opacity continuous with that of the basal cell; bearing above several subhyaline branchlets; the inner branch erect, once or twice branched, its basal cell and the outer primary branchlet arising from it, more or less deeply suffused externally: basal cell of the inner appendage slightly longer than that of the outer, bearing two erect slightly olivaceous branches, one on either side, which are usually twice branched; the ultimate branchlets hyaline, rigid, bluntly tipped, the longest scarcely reaching the tip of the perithecium. Body of the perithecium slightly and more or less evenly inflated, broadest in the middle, rich amber yellow, sometimes becoming tinged with olivaceous; usually, but not invariably, twisted one quarter, so that the tip is viewed at right angles to the normal position; the tip more or less deeply suffused with blackish olive, short, rather abruptly distinguished, bent distinctly inward, its outer margin nearly straight, its inner strongly indented, the apex usually broad, horizontal, symmetrically bilobed; the lip-edges hyaline and evenly rounded; if the twist is absent, oblique, or sometimes four-lobed if the twist is one eighth. Spores $78 \times 7\ \mu$, Perithecium $125–145 \times 35–40\ \mu$. Receptacle $100–135\ \mu$. Total length to tip of perithecium $225–275\ \mu$, average $250\ \mu$.

On the outer margin of the left elytron of "*Argutor Bonariense* Dej." Buenos Aires, No. 1962, No. 1431 in Museo Nacional; also at Temperley and Llavallol.

This species does not appear to be nearly allied to *L. fumosa* to which it has been referred by Spegazzini who found it on "Argutoridius" at Santiago, Chile. It was found by me on the same host at the Baños de Apoquindo, near Santiago.

Laboulbenia asperata nov. sp.

Hyaline becoming pale straw- or amber-yellow. Receptacle normal, the subbasal cell variably elongated, rarely minutely corrugated; cell V parallel to cell IV and slightly longer. Appendages hyaline, the insertion-cell transparent, faintly suffused with reddish, the basal cell of the outer appendage usually distinctly larger than the inner, broader than long and forming a more or less prominent rounded or angular external projection variably developed below the usually solitary elongate branch or simple appendage which arises from it and is erect, sometimes divergent or even pendent, especially if it is associated with a second branch within; the basal cell of this appendage, sometimes its subbasal cell, inflated, broader than long, more or less deeply constricted at the very faintly suffused septa: the basal cell of the inner appendage producing two branches which may be simple or once branched at the base, usually slightly exceeding the tip of the perithecium, and sometimes elongate like the outer appendage. Perithecium subhyaline to yellowish, rather narrow, slightly divergent distally, the external basal wall-cell more or less conspicuously roughened by fine transverse ridges; the tip hardly distinguished, tapering very slightly; the apex broad, subtended on the inner side by a small faintly suffused patch, the lips evenly oblique outward, hardly prominent. Perithecia 110×40 μ. Longest appendage 250 μ. Receptacle 100–235 μ. Total length to tip of perithecium, 150–350 μ, average 235 μ.

On the elytra etc. of *Tachys* sp., Palermo, No. 1696.

This species is nearly allied to *L. Tachyis* and to *L. marina* Picard, but differs from both in the characters of its appendages and insertion-cell, as well as by the characteristic external roughening of the outer basal wall-cell of the perithecium.

Laboulbenia australis nov. sp.

Receptacle indistinctly punctate, cells I and II becoming dirty yellowish, often contrasting with the frequently deeply suffused yellow-brown distal portion which often becomes somewhat olivaceous. Insertion-cell horizontal, rather thick; the appendages rather copiously branched the branches subparallel in a rather compact group, usually erect or the whole bent slightly toward the perithecium; the basal cell of the outer appendage twice as long as the inner, not distinguished from the cells above it, the appendage once or twice branched or sometimes simple: the basal cell of the inner appendage producing an erect branch on either side each once or twice branched, the antheridia arising singly or two together even from the third cells of the branches, so that they may lie opposite the tip of the mature perithecium. Perithecium free, except at its very base, usually straight, or concave externally and strongly convex inwardly, especially immediately below the tip, so that the whole perithecium is bent strongly outward distally in a characteristic manner; the tip short, abruptly distinguished, laterally deeply suffused especially externally; the lips rounded, more or less symmetrically, translucent or hyaline. Spores 45×3.5 μ. Perithecia 98×35 μ. Appendages to tips of longest branches 155 μ. Receptacle 125–275 μ. Total length to tip of perithecium average 250–275 μ (150–300 μ).

On all parts of a species of *Apenes*. Tucuman, No. 1940 (P. Spegazzini).

This species of which abundant material is available, is somewhat similar to *L. Oopteri*, but differs in its characteristically and more strongly curved perithecium, and in the absence of dark septa in the outer appendage, the basal cell of which is never as highly developed, in the present species. Individuals growing on the legs are smaller, stouter and darker.

Laboulbenia flexata nov. sp.

Yellowish to hyaline, with variable brown shades; the perithecium becoming uniformly rich translucent brown. Form rather slender, evenly curved throughout, but more or less distinctly geniculate between the basal and subbasal cells of the receptacle which are rather long and about equal in dimensions. Cells IV and V somewhat enlarged and divergent, carrying the very broad and thick black insertion-cell free from the base of the perithecium. Appendage consisting of an outer and an inner branch of the type of *L. Texana;* the outer stout, or curved somewhat away from the inner, and consisting of four to six large subequal cells, each bearing a simple branchlet like those of *L. Texana*, subtended by a small cell from which it is separated by a deeply blackened septum; the small terminal cell of the series bearing two such branchlets: the inner appendage consisting of two branches which spring from a common basal cell; one of them unicellular and terminated by a single antheridium, the other strongly curved across the perithecium, and consisting of five or six small superposed cells, each bearing a simple branchlet similar to those of the outer appendage. Perithecium rather narrow, curved toward the appendage, its middle opposite the insertion-cell; its tip abruptly distinguished, narrow, prominent, opaque, contrasting abruptly with the hyaline symmetrically rounded apex. Perithecium 155–200× 48–55 μ. Receptacle 275–390 μ. Outer appendage 135–155×40 μ at base, longest 200×50 μ: inner appendage 50–60×12 μ; longest branchlets 120–140 μ.

On the inferior left margin of the prothorax of *Brachinus* sp., No.

1457, Isla de Santiago, La Plata; No. 1426 in Museo Nacional, no locality; No. 2030, La Plata (P. Spegazzini).

The present species adds still another form to the well marked series of the *L. Texana* group, all of which occur on the inferior surface or legs of species of *Brachinus*, and which I have hitherto preferred to treat as varieties of *L. Texana*. Sufficient material of several of these forms which is now available, indicates clearly that the members of this series are better regarded as species, which correspond among themselves in a fashion very similar to that which may be seen in the much more numerous species which have developed on the allied host-genus *Galerita* in the Western Hemisphere. Among these forms **Laboulbenia Oaxacana**, alone, has not been found in the Argentine region, although **Laboulbenia pendula** is known only from Montevideo, and but a single specimen of what appears to be the typical *L. Texana* was obtained at the Isla de Santiago.

Of the other members of the group the following were obtained.

Laboulbenia incurvata exactly resembling the types, was found on a large *Brachinus* in the Museo Nacional, No. 1427, labeled "Argentine"; on several specimens of a *Brachinus* taken on the Isla de Santiago, La Plata, and on a *Brachinus* collected in Tucuman by P. Spegazzini.

Laboulbenia retusa, which was first found in Florida, was again obtained on *Brachinus* from the Isla de Santiago near La Plata, No. 1457, as well as from Tucuman No. 1939.

Laboulbenia tibialis, also first obtained in Florida, occurred in good condition on a *Brachinus* collected by P. Spegazzini in Tucuman, No. 1939. All the seven species of this group occupy more or less definite positions on the host, and none of them ever occur, as far as has been observed, on the upper surface; although *L. Brachini*, which is often associated with them, may be found in any position.

Laboulbenia inflecta nov. sp.

Basal cell of the receptacle hyaline or faintly suffused above, much longer than broad, the receptacle above it uniformly dull yellowish olivaceous and compact, the cells not greatly different in size; cell III extending upward sometimes almost to the insertion-cell. Insertion-cell somewhat oblique, thick, deeply suffused; outer and inner basal cells of the appendage subequal, the outer externally rounded and suffused, the axis of the outer appendage consisting of about five obliquely placed cells; those above the basal cell small, their branches stout, relatively short, divergent; the main axis of the inner appendage consisting of five cells, the lower bearing relatively small stalk-cells terminated by single large stout antheridia. Stalk of perithecium hyaline, contrasting, very short, constricted; its axis coincident with that of the perithecium and bent inward at a slight but definite angle to the axis of the receptacle; the body of the perithecium translucent, nearly symmetrical, becoming deeply suffused with clear, slightly reddish olive-brown, subsymmetrically inflated throughout, the tip rather narrow, abruptly distinguished, more deeply suffused; the apex hyaline or becoming suffused, nearly symmetrically rounded or slightly irregular. Perithecium above stalk 110–128×35–38 μ, the stalk 8×15–20 μ. Receptacle 98×40–45 μ, its basal cell 45–50×20 μ. Main appendages 20 μ, their branches 50–75 μ. Antheridia 20 μ, their stalk-cells 10–12 μ.

On the mid left elytron of a black species of *Galerita* (from two specimens), La Plata No. 2021, P. Spegazzini.

This species resembles small forms of *L. punctata*, but differs in the complete absence of maculation, as well as in other minor points.

Laboulbenia marginata nov. sp.

Basal cell of the receptacle hyaline, cells II and III opaque and indistinguishable, forming above a broad black margin extending upward so that the free distal margin is on a level with the insertion-cell; cell IV inwardly yellowish, obliquely elongated, externally dark brown, separated from the upper part of cell III by a clear oblique septum; cell V triangular, similarly suffused externally; both these cells, as well as the rest of the receptacle, transversely punctate. Cell VI and the cells above it subhyaline, soiled with dirty brown: the stalk of the perithecium hyaline, the main body deeply suffused, externally nearly straight and translucent, indistinctly punctate below, inwardly distinctly convex and opaque; the tip abruptly distinguished on both sides, opaque below the asymmetrical sulcate apex; the inner lips prominent, broad, rounded, the outer much smaller, lower, the pore turned obliquely outward. Insertion-cell indistinguishable from the opaque basal cells of the appendages, the blackened portion curved outward and upward and forming a free rounded prominence subtending the first outer branch; this blackened area larger than the hyaline compact main appendages, the cells of which are very narrow; those of the outer seven or eight in number, including the basal cell, somewhat obliquely associated in a but slightly oblique series; the cells of the inner appendage more obliquely superposed, six or seven in number, the three lower bearing antheridial branches consisting of single basal cells terminated by single antheridia; the simple sterile branches of the upper cells extending to about the middle of the perithecium. Perithecium 250–275×52 μ exclusive of the stalk (58×30 μ). Receptacle 190–200×90 μ. Appendages to tips of branches about 175 μ; the antheridia 24 μ, their basal cell 20 μ. Total length to tip of perithecium average 500–510 μ.

On the inferior surface of the abdomen of *Galerita Lacordairii*. Museo Nacional, No. 1428, "Argentina."

Laboulbenia sordida nov. sp.

Resembling *L. perplexa;* rather slender; the basal cell of the receptacle hyaline, the rest becoming irregularly suffused with dirty olive brown; the region below the insertion-cell becoming nearly opaque, the subbasal cell sometimes lighter or hyaline distally; cell IV separated from cell III and V by parallel septa at an angle of 45° to the axis of the receptacle. Insertion-cell broad, thick, horizontal, opaque; the opacity involving the outer basal cell of the appendage which is externally prominent upward. The outer appendage consisting of a series of seven or eight obliquely superposed cells, coherent throughout with the inner appendage, short; all, including the basal cell, bearing erect branches, the two basal cells of which are dark brown, the rest of the branch nearly hyaline and extending to or slightly above the middle of the perithecium: the inner appendage consisting of a series of usually five cells on either side above the basal cell, the distal one bearing a short erect branch, while the four lower bear antheridial branches consisting of a well developed brown basal cell, bearing distally a pair of divergent, brown, somewhat curved antheridia. Stalk of the perithecium clearly distinguished, about as long as broad, hyaline, contrasting; the main body deep olive brown, straight, asymmetrical, very slightly inflated below; the tip slightly darker, short, asymmetrical, more or less well distinguished, its outer margin oblique; the apex translucent, obliquely rounded outward, subtended on the inner side by an opaque suffusion. Perithecium, exclusive of stalk, 215–235×45–47 μ, the stalk 27–31×27 μ. Receptacle 215×66 μ. Appendages, to tips of branches, longest, 160 μ. Antheridia 23–27×6 μ.

On the tips of the elytra of a black *Galerita*, La Plata, No. 2021.

This species is most nearly related to *L. perplexa,* from which it is best distinguished by the short coherent primary appendages, short branches, and numerous paired antheridia.

Laboulbenia Heteroceratis nov. sp.

Uniformly pale straw-yellow, very variable in form. Receptacle usually rather elongate, but sometimes short and stout, the subbasal cells larger than the basal, cells IV and V subequal. Insertion-cell concolorous with the cells below it, the primary outer appendage short, simple, cylindrical, hyaline, becoming distally flaccid; the inner consisting of a few ill defined short flaccid branches; the insertion-cell becoming very variably modified by secondary divisions, which may also involve the basal cells of the appendages so that the primary outer appendage may even become completely surrounded by small cells bearing either branches or curved antheridia, the branches sometimes forming a tuft of some length. Perithecium asymmetrical, the inner margin usually straight or slightly concave, the outer strongly convex; tapering to a snout-like tip so turned (in the Argentine material) that it is viewed sidewise and shows a blunt symmetrically rounded apex, subtended by a purplish shade. Perithecium 110–120×35–40 μ. Receptacle 156–235 μ. Appendages 50–60 μ. Total length to tip of perithecium 220–340 μ.

Growing in various positions on species of *Heteroceros* sent from La Plata by P. Spegazzini in 1907, Nos. 1679–80. Also found on species of *Heteroceros* sent from Kansas by Dr. A. Stewart.

This very peculiar form varies greatly in general habit, and from the secondary divisions of its insertion-cell and the basal cells of its appendages may assume an appearance very similar to that of some of the aquatic forms on Gyrinidae. Its relationships seem to be evidently with the forms found on *Clivina* and its allies; although a similar production of sessile antheridia from proliferous cells such as occurs in the present instance is not seen in other forms. The above description is based in part on material obtained from American species of *Heteroceros* which were found among a small collection of beetles kindly procured for me by Mr. Alban Stewart in Kansas City. The measurements given above are from the Argentine material. The Kansas specimens show the slightly oblique asymmetrical tip of the perithecium from the usual point of view.

Laboulbenia funeralis nov. sp.

Dull blackish olive becoming opaque, except the basal and subbasal cells of the receptacle which are translucent dull olive, subequal, forming a curved or sigmoid stalk not abruptly distinguished from the rest of the receptacle, which is relatively narrow; the basal cell-region of the perithecium bulging externally, and forming a rounded flat, but usually distinct, prominence; above which the narrow perithecium tapers very slightly and evenly to the very broad tip, which is not distinguished; the apex partly hyaline bearing an inner shorter tooth-like appendage, and an outer which is longer and usually irregularly furcate. Appendages not very numerous, erect, septate at the base; the hyaline slender tapering distal portion extending to or beyond the apex of the perithecium. Perithecium 110–155×35–40 μ; the longer terminal appendage (longest) 20 μ. Total length to tip of perithecium 235–350 μ; greatest width 38–66 μ including elevation at base of perithecium.

On the margins of the elytra of a species of *Gyrinus*, No. 1957, in a pond near the railroad station at Palermo.

This species which seems constant in specimens from a considerable number of different individuals, is very closely allied to *L. Gyrinidarum* from which it differs more especially in its smaller size, in the color and conformation of its basal and subbasal cells which have no yellow-brown tint, are similar and subequal; both being much longer than broad; in the marked prominence below the perithecium, the tip of which is not distinguished even on the inner side, as well as by its terminal usually furcate apical appendage.

Rhachomyces Argentinus nov. sp.

Rather slender. Cells of the receptacle tinged with pale brown, small, about as long as broad, ten or twelve of the lower visible; the remainder wholly concealed by the closely appressed, rather slender, copious black appendages; those about the base of the perithecium somewhat stouter with hyaline tips, closely appressed about the perithecium, nearly uniform in length, and extending nearly to its tip, which projects free beyond them. Perithecium straight, symmetrical, brown, the tip nearly black, the apex subhyaline, flat-conical or bluntly pointed. Perithecium 120×40–43 μ. Longest appendages about 95 μ. Total length to tip of perithecium 310–425 μ (longest).

On the legs of a small carabid beetle resembling *Casnonia*. Jujuy, Northern Argentine, No. 1430, Museo Nacional.

This species is most nearly allied to *R. Javanicus*, from which it is distinguished by its more slender, copious and closely appressed appendages, which conceal the axis of the receptacle distally, as well as by the somewhat pointed apex of its perithecium. The material includes two small specimens not more than 200 μ in length.

Scaphidiomyces nov. gen.

Axis consisting of a primary receptacle of two superposed cells, the subbasal bearing a primary branched appendage terminally, and subterminally a secondary receptacle consisting of an indeterminate series of superposed cells, which give rise alternately to stalked perithecia and to branches similar to the primary appendage. Antheridia simple, terminal on short branches. Perithecia normal.

This type, of which two other species are known on scaphidians, from the Argentine and West Africa, appears to be related to the Compsomycetaceae although the number of spores in the asci has not been definitely determined. Some of the branches of the secondary receptacle when young, show the same peculiar oblique septation characteristic of one of the appendages in *Compsomyces;* but this may not be significant, and the perithecium has but a single stalk-cell; the alternate production of branches and perithecia, and their association on the indeterminate secondary axis, have no parallel in any other genus. The characters of this type are nevertheless not clearly defined, and a definite conception of its limitations cannot be arrived at until sufficient material of other species is available.

Scaphidiomyces Baeocerae nov. sp.

Colorless, the perithecia becoming amber-brown at maturity, rather short and stout, somewhat inflated, subsymmetrical, narrowed distally to the broad tip; its apex broad, bluntly rounded or subtruncate; the basal cells similar, rather small, projecting slightly; the region hardly distinguished from the body, and concolorous with it: the stalk-cell hyaline, but slightly longer than broad, narrower below. Basal cell of primary receptacle longer than broad, narrowed and suffused with blackish brown just above the foot. The primary appendage consisting of two to three superposed cells, bearing distally short few-celled branches and branchlets. Secondary receptacle continuous with and not distinguished from the primary, its axis of similar cells of approximately the same size, superposed more or less regularly in a somewhat zigzag fashion, the successive cells bearing with more or less regularity appendages similar to the primary appendage, and stalked perithecia of which there may be from one to four or five in various stages of development produced on the same side or alternating on opposite sides of the axis. Perithecia $75 \times 35\,\mu$, the stalk-cells 15–$18\,\mu$. Appendages to tips of branchlets $70\,\mu$. Total length to tip of primary perithecium 150–$310\,\mu$.

On elytra of an undescribed species of *Baeocera*, a small scaphidian feeding on *Corticia* under moist logs. Llavallol. (Determined by Dr. Csiki.)

Scelophoromyces nov. gen.

Main axis consisting of a basal and subbasal cell forming a primary receptacle, and a series of cells superposed above it; the subbasal cell producing a lateral branch of several superposed cells, terminated by the primary perithecium: the upper cells of the axis, above the subbasal cell, producing more or less copious branches on the inner side and terminally; while one or more secondary perithecia with single stalk-cells may arise from the lower. The lower cells of the primary perithecial branch, and sometimes the subbasal cell of the receptacle, giving rise to slender supporting outgrowths, which curve down toward the substratum. Antheridia (?) simple, and formed terminally from the lower branchlets.

This genus is erected with some reluctance, since the nature of the antheridia is somewhat doubtful. The latter appear to be terminal cells of short lower branchlets from the main branches that arise from the upper cells of the axis above the subbasal cell, and which may be regarded as a primary appendage, or, since it gives rise to perithecia, as a secondary receptacle. Although numerous specimens are available, and the form has also been obtained from the Amazon region, the branches are for the most part not well preserved, even in the youngest individuals. The several-celled stalk of the primary perithecium would suggest that the relationships of the genus might be with the Compsomyceteae, while the production of what may be regarded as a secondary axis suggests *Clematomyces* and *Scaphidiomyces*. The adventitious branches which grow downward from the lower cells toward the substratum undoubtedly act as buffers, like those of *Ceratomyces rhizophorus* described below, and *Hydrophilomyces digitatus*, described recently by Picard to which further reference is made below under *Ecteinomyces.*

Scelophoromyces Osorianus nov. sp.

Pale straw- or amber-yellow, concolorous, becoming dirty amber-brown with age. Perithecium subsymmetrical; main body distinguished from the slightly broader basal cell-region; of nearly equal diameter throughout, or but slightly inflated, the short stout tip abruptly distinguished, bent slightly outward; the apex broad and nearly truncate; the basal cells subequal, large, slightly prominent; two to six cells superposed to form the perithecial branch; the supporting branches simple, septate, tapering throughout to pointed extremities; two to four in number, one of them usually derived from the subbasal cell of the receptacle on the side opposite the perithecial branch. Main appendage, or secondary receptacle, consisting of eight to ten superposed cells, terminated by a more slender portion similar to the branches, which arise distally from cells obliquely separated on one or both sides of the upper cells of the main appendage; the branches more or less copiously branched, the ultimate branchlets forming more or less characteristic tufts, and curved toward the main axis: one to three of the lower cells usually producing a corresponding number of secondary perithecia similar to the primary one. Dimensions very variable. Perithecia, above basal cells, $95–110 \times 30–40\ \mu$, the perithecial branch $25–120\ \mu$, total length, including branch, $130–250\ \mu$; basal cell-region $20–40 \times 25–30\ \mu$. Total length to tip of longest branchlets (largest) $400\ \mu$. Supporting outgrowths $100–275\ \mu$.

On abdomen and elytra of *Osorius sexpunctatus* Bernh., Palermo, No. 1693, and Isla de Santiago, La Plata, No. 1972. Also from the Amazon, (Mann), on a very large *Osorius.*

Ecteinomyces Thaxter.

I have called attention in my second monograph to the uncertain position of this genus, as well as of *Hydrophilomyces;* and also to the similarity between these two and *Misgomyces.* Although the examination of fresh American material of *Misgomyces Dyschirii* from Kansas, recently received in moderately good condition, appears to show that this is a distinct genus more nearly allied to *Laboulbenia,* a further study of forms allied to *Ecteinomyces* and *Hydrophilomyces* has forced me to the conclusion that it is inadvisable to retain both these names, and that all the species are best united under the first. The antheridial characters are doubtful in all the species, and it is still uncertain whether the structures described as simple antheridia in both cases are actually functional as such; since no actual discharge has been observed from them. In these, as in other cases in which the antheridia are not clearly distinguished, either by their position or form, it is often very difficult to distinguish them from young sterile branchlets, unless the material is examined while still fresh, so that the discharge of sperm-cells can be observed. I have therefore concluded to drop the name *Hydrophilomyces*, using *Ecteinomyces* to include the three new forms below described, as well as **E. rhyncophorus** and **E. reflexus**.

Hydrophilomyces digitatus Picard on *Ochtebius marinus* from France described in the Bull. Myc. Soc. de France, Vol. XXV, p. 244, 1910, should also be changed to **Ecteinomyces digitatus** Picard, since it evidently belongs in this group.

Ecteinomyces rhyncophorus was found at Palermo on a small hydrophilid, and has also been obtained from Guatamala; the material in both cases corresponding in all respects to that originally obtained from Florida.

Ecteinomyces filarius nov. sp.

Wholly hyaline. Perithecium rather long and narrow, straight, hardly inflated, the tip rather long-conical with straight margins, subtruncate or rounded, the apex symmetrical and subtended externally by a distinct prominence; the basal cell-region not distinguished, its cells flattened around the ascogenic cells; borne on a distinct short stalk-cell. Receptacle filamentous, slender, elongate, consisting of many (about forty) superposed cells; the distal ones becoming slightly broader, and occasionally cutting off a small cell subterminally or laterally; the axis continuous with an erect primary appendage of similar character, consisting of about six superposed cells, and lying close beside the perithecium and slightly exceeding it in length, bearing distally the remains of one or two branchlets. Spores (in perithecium) 30–35$\times$3 μ. Perithecium 70$\times$14 μ; the stalk-cell 8$\times$10 μ. Receptacle 230–275$\times$7–9 μ. Total length 290–340 μ.

On the elytra of *Coproporus rutilus* Er.; Tucuman, No. 1934, (P. Spegazzini).

The antheridia of this species have not been seen, and the types show only the bases of what appear to have been rather short branches from the end of the appendage. Its hypha-like receptacle is even more striking than that of *E. Trichopterophilus*, from its greater length and more evenly cylindrical form.

Ecteinomyces Thinocharinus nov. sp.

Wholly hyaline. The receptacle usually tapering continuously from above to the minute foot, its axis continuous with that of the perithecium and consisting of from six to twelve more or less flattened cells, which may occasionally be divided longitudinally; the foot-cell of some individuals developing an upcurved appendage, deeply blackened except along its inner margin, of variable length, thicker and bluntly rounded at its tip. Perithecium clearly divided into a nearly symmetrical oval venter and a long, stout, nearly straight, isodiametric neck-portion, the base of which is subtended on the outer margin by a more or less distinct prominence formed by the slightly protruding extremity of the outer basal wall-cell; the tip hardly distinguished, tapering but slightly to the blunt symmetrical apex. Appendage slightly divergent, consisting of six or more superposed cells, the basal larger, angular, in contact on its inner side with the small basal and stalk-cells of the perithecium; the terminal cells bearing a group of rather coarse branches, once or twice branched, the ultimate branchlets not reaching to the tip of the perithecium. Spores, in perithecium, 20×2.5 μ. Perithecia $120–130 \times 23–27$ μ. Receptacle 55–65 μ. Foot-appendage 18 μ. Appendage 35–50 μ, its branches 75–90 μ.

On the abdomen etc. of *Thinocharis exilis* Er., Temperley, No. 2004, and Palermo, No. 1701.

The curious black outgrowth from the foot of this species, occurs in about half the specimens; but while in these it is well developed, there is no trace of it in the others, even when fully matured and growing in the same position.

Ecteinomyces Copropori nov. sp.

Hyaline or faintly tinged with yellowish. Receptacle consisting of from ten to twenty superposed cells some of which may become irregularly divided by one or two longitudinal septa, the cells usually flattened, often irregular, the basal cell subtriangular and deeply suffused with blackish brown above the small foot. Appendage at first not distinguished from the receptacle and continuous with it, slightly divergent when mature, consisting of a variable number (eight to twelve) of superposed cells, the series tapering distally, some or most of the cells cutting off one or two small cells on the inner side, sometimes also on the outer side from which branches arise as well as antheridia (?) which are irregularly flask-shaped, single and sessile or borne one or two together on short branchlets; the sterile branches usually broken and not copiously developed. Perithecium nearly straight, its axis usually continuous with that of the receptacle, a venter neck and tip more or less clearly distinguished, the latter bent very slightly inward, the apex blunt and usually becoming minutely six-papillate; the outer, lower wall-cell slightly prominent below the neck; the two upper basal cells extending upward beside the venter, the stalk-cell short and subtriangular. Perithecium 140–200×38–44 μ, smallest 100×25 μ, stalk-cells and lower basal cells 20 μ. Spores in perithecium 35×3.5 μ. Receptacle average 200 μ. Appendage 60–100 μ. .Total length to tip of perithecium about 325 μ.

On the abdomen of *Coproporus rutilus* Er.; Tucuman, No. 1933, P. Spegazzini. Also from Los Amates, Guatemala, No. 1614 (Kellerman).

The material of this species is not in very good condition and it is difficult to determine the character of the appendages and antheridia from them. The Guatemalan material includes only three specimens in which the perithecia are mature, and in these the papillation of the apex is either indistinct or lacking; but, although the individuals are somewhat larger, the perithecia more divergent, and the cells of the receptacle shorter and broader than the Tucuman material, the two forms seem identical.

Autoicomyces bicornis nov. sp.

Pale yellowish with a smoky tinge, deepest at the base of the perithecium. Basal and subbasal cells of the receptacle rather large, of about equal length. Appendage usually straight, somewhat divergent, comparatively slender; consisting of six or more superposed cells, and bearing a few small branchlets. Perithecium nearly straight externally, its inner margin convex; the tip lying in the fork formed by two outgrowths which arise symmetrically just below it from the wall-cells on either side; the outer shorter, rather closely septate, tapering to a blunt apex, and curved inward; the inner two or three times as long, usually septate only at the base, curved away from the perithecium and tapering to a blunt point. Perithecium 95–110×

40–45 μ, its longer appendage 60–200 μ, the shorter 70–78 μ. Appendage 135 μ. Receptacle $80 \times 35\ \mu$. Total length to tip of perithecium 175–190 μ; to tip of inner appendage 310–370 μ.

On the inferior surface of the abdomen of *Berosus* sp. or a closely allied genus. Palermo near Belgrano, No. 1944.

A species readily distinguished by its paired perithecial appendages, but conforming strictly to the type so clearly marked in this genus.

Ceratomyces rhizophorus nov. sp.

Receptacle small, hyaline, normal; the second and third cells broad and much flattened. The appendage long, of nearly equal diameter throughout, composed of numerous short flattened cells bearing scattered branches. The basal cell, and one or more of the upper cells of the receptacle, developing short rigid curved simple outgrowths, which grow downward to the substratum. Perithecium stout, tapering distally to a well distinguished, abruptly narrower, bluntly rounded tip; each marginal row of wall-cells comprising about twenty cells. Perithecium $100 \times 40\ \mu$. Appendage $135 \times 16\ \mu$ (broken). Receptacle 50 μ, the foot 20 μ. Total length to tip of perithecium 150 μ.

At the tip of the left anterior leg of *Tropisternus* sp. Palermo, near Belgrano, No. 1645.

All but two specimens of this small and peculiar species were unfortunately destroyed by accident, while they were being mounted, so that it has been necessary to base the above description on a single nearly mature, and one younger individual. It is, however, so peculiar, and so well characterized by its supporting outgrowths that it has seemed safe to give it a name. The outgrowths are evidently buffers, similar in function to those described in *Ecteinomyces* (*Hydrophilomyces*) *digitatus* Picard, and of *Scelophoromyces* described above.

Ceratomyces ventriosus nov. sp.

Receptacle relatively long, the subbasal cell and the cell above it deeply blackened laterally, the suffusion extending upward and involving the outer margin or half of the cell which subtends the appendage. Appendage long and relatively slender, bearing a few scattered branches, the lower cells somewhat flattened and becoming divided by a few oblique septa. The receptacle, appendage and base of perithecium pale yellowish, or with a reddish-amber tinge. Perithecium relatively very large and long, about forty-five cells in each row of wall-cells; more or less evenly curved away from the appendage, deeply rich red amber-brown, except at its pale narrower base, of the lower half characterized by a belly-like enlargement; the upper half of nearly the same diameter throughout; the tip subtended externally by a vesicular enlargement of one of the wall-cells, its hyaline apex pointed and bent inward toward the concave base of the long appendage, which is usually abruptly curved at its base, more or less deeply suffused or opaque below, tapering very slightly, consisting of about twelve cells, the lowest of which is comparatively small, and not extending above the apex of the perithecium. Perithecium 550–700×100–110 μ (lower half) and 65–75 μ (upper half), the appendage 250–350×30 μ.

On the inferior surface of the abdomen, near the tip on the left side of *Tropisternus* sp.; Palermo, near Belgrano, No. 1949.

The long appendage of this remarkable species is very similar to that of the last, to which it seems to be most nearly allied, but from which it is easily separated by the form of its receptacle and its enormous pot-bellied perithecium.

Ceratomyces marginalis nov. sp.

Uniform dirty translucent amber-brown. Receptacle small, the foot and basal cell opaque and indistinguishable; the two cells above greatly flattened, the subbasal partly involved below by the suffusion of the cells above. The appendage small, short, consisting of four or five superposed cells, terminated by a few branchlets, erect, appressed against the perithecium or but slightly divergent. Perithecium relatively large, about eight wall-cells in each row, straight, but slightly and rather evenly inflated; the tip not distinguished, but terminated by an erect hyaline nearly cylindrical slender blunt apical prolongation, subtended by a relatively very large sigmoid appendage, which curves toward and beyond it, thence bending and tapering upward, and composed of a series of eight or nine superposed cells of about equal length, sometimes terminated by a few short colorless branchlets. Perithecium 90–110×35–45 μ, the longest appendage 100 μ. The receptacle, including foot, 55–60×30 μ. Appendage 60×7 μ. Total length to tip of perithecium 135–150 μ, to tip of appendage 225 μ.

Beneath the margin of the elytra of a small pale hydrophylid. Palermo, near Belgrano, No. 1952.

In general habit this species is not unlike *C. minisculus* from which it is at once distinguished by its large perithecial appendage.

Ceratomyces intermedius nov. sp.

Receptacle faintly tinged with amber-brown, rather short, externally opaque above the basal cell to the base of the appendage, the blackening involving the outer half or less of the cells concerned; the cell subtending the appendage slightly prominent externally, below the latter. The perithecium and appendage usually divergent at the base of the latter, which is faintly tinged with amber-brown, stout, curved outward; consisting of a series of cells smaller distally, about six of the lowest very broad and flattened, becoming divided more or less irregularly by oblique partitions, and bearing a few scattered branchlets on the inner side. Perithecium large, stout, deeply tinged with dull amber-brown, paler at the base where it is distinctly narrower, the distal two thirds of nearly the same diameter throughout, or the middle third somewhat inflated; the tip short abruptly distinguished externally, being subtended by a rounded prominence in which the series of wall-cells below it ends, its apex hyaline, asymmetrically rounded or outwardly oblique; the simple perithecial appendage becoming deeply suffused or opaque except at its bluntly pointed tip, erect or bent inward, consisting of from about six to eight successively smaller cells, the lower becoming deeply suffused; the basal cell very large, concave within, convex externally, the whole assuming a sigmoid curvature as it matures. Perithecium $310–390 \times 80–105\ \mu$, the base $50–60\ \mu$; the appendage $105–170\ \mu$. Receptacle $74–82 \times 75–78\ \mu$, without foot ($30\ \mu$). Appendage $200 \times 45–48\ \mu$ at base. Total length to tip of perithecial appendage $660\ \mu$.

On the left anterior margin of the thorax of *Tropisternus* sp.; Palermo, near Belgrano, No. 1946.

A large and clearly distinguished species, intermediate between *C. mirabilis*, which it more nearly resembles in its perithecial characters, and *C. cladophorus*, which has a similar though somewhat more highly developed appendage.

Synaptomyces nov. gen.

Receptacle indeterminate, consisting of a series of superposed cells; the uppermost of this series followed by two cells placed side by side, one of which is separated by a single small cell from the basal cell of the appendage, while the other forms the base of the outer series of wall-cells of the perithecium. The appendage consisting of a series of superposed cells bearing scattered branchlets. Perithecium many-celled, indeterminate, without distinction of venter and neck, appendiculate on the inner side below the tip.

This genus, of which two other species are known on *Hydrocharis*, one from North America, and another from Africa, appears to be intermediate between *Ceratomyces*, which it resembles most nearly in the characters of its perithecium, and *Rhyncophoromyces*, which possesses a similar indeterminate receptacle. Although in the present species, which is taken as the type, several appendages develop in a compact group below the apex of the perithecium, in the African form there is only one which is very similar to that seen in species of *Ceratomyces*. The North American form, of which I have only one undeveloped individual, shows that the sperm-cells are developed exogenously exactly as in *Rhyncophoromyces*.

Synaptomyces Argentinus nov. sp.

Receptacle consisting of a series of about twenty superposed, much flattened, cells; surmounted by two somewhat unequal cells separated from one another by an oblique septum; a transversely elongated rounded cell lying obliquely between the anterior of the two and the basal cell of the appendage, which is more or less conspicuously indented externally. The appendage somewhat broken in the types, its basal or subbasal cell giving rise to a simple branch, the main axis of undivided superposed cells proliferating to form several slender branches, which arise from its tip. Perithecium relatively large and stout, hardly inflated above the base, slightly narrower distally, the papillate tip abruptly distinguished; the apex broad and asymmetrically rounded, the perithecial appendages arising in a group just below the tip on the anterior side, usually three being superposed; their extremities free, their bases laterally coherent, some of them proliferating to form slender terminal hyaline branchlets. Perithecium 335×80–390–$105\ \mu$; its appendage without terminal branchlets 110–120 μ. Receptacle 250–275×70–$80\ \mu$ distally. Appendage (broken) 160×15–$18\ \mu$. Total length to tip of perithecium 700–750 μ.

On the left inferior margin of the thorax of *Hydrocharis* sp., No. 943, Palermo, near Belgrano.

In addition to the new forms above described the following species were found, and also a few others that are not determinable.

Acompsomyces brunneolus Th. A species closely allied to the North American form, was obtained at Palermo on a small *Corticaria* (?) The conformation of the tip of the perithecium is very similar, but the latter is shorter and stouter, its broad base abruptly distinguished from the somewhat longer narrower straight stalk-cell. The stalk-cell of the appendage is also quite hyaline. Since the type form has been found only once, its variations are not yet known, and it seems inadvisable to separate the Argentine form until further material of both is available.

Camptomyces melanopus Th. Several well matured and typical specimens of this species were found on the abdomen of *Sunius* sp., No. 2002, at Temperley, but although very many specimens of *Sunii* were examined it was not again met with.

Chaetomyces Pinophili Th. was found very rarely on *Pinophilus suffusus* Er., although its host was very common at Llavallol. The material differs in no respect from that obtained in North America.

Ceratomyces mirabilis Th. was very common on *Tropisterni* at Palermo, near Belgrano, the specimens exactly like those from New England.

Ceratomyces ansatus Th. was also common, and as usual did not occur on the wholly black species of *Tropisternus.*

Ceratomyces filiformis Th. Several typical specimens were obtained growing at the tip of the posterior legs of several *Tropisterni.*

Ceratomyces minisculus Th. was found once on a species allied to *T. lateralis.*

Compsomyces verticillatus Th. was found rather rarely on species of *Sunius* at Temperley and Llavallol, Nos. 1995 and 2002, the individuals differing in no essential respect from the North American type.

Corethromyces purpurascens Th. This species was found very commonly in the vicinity of Buenos Aires on an evenly, rather pale brown species of *Cryptobium,* and appears to be very constant in its characters, varying only in the luxuriance with which the branches of the appendage are developed.

Corethromyces Stilici Th. This species was found in abundance on several species of *Stilicus,* the normal form like that first collected at Interlaken, Switzerland, being sometimes associated with one in which the stalk-cell of the perithecium is enormously developed, the body of the perithecium being at the same time more elongate, its wall-cells more markedly spiral and with the appendage somewhat reduced. Although perhaps worthy of varietal rank, it has not seemed desirable to separate this form specifically.

Dichomyces furciferus Th. was found several times at Palermo and at Temperley on *Philonthus hepaticus* Er., No. 1960.

Dichomyces vulgatus was met with rarely on a large *Philonthus* at Llavallol, No. 1490 and 1936, and occurred on a *Philonthus* collected by Propile Spegazzini in Tucuman.

Dichomyces princeps Th. was found rarely at Palermo on a species of *Philonthus*, No. 1958.

Dichomyces Homalotae Th. Typical material of this species was found several times at Palermo, No. 1964, and at Temperley, No. 2008, on *Atheta sordida* Marsh.

Dichomyces sp., a species apparently unlike the North American form on *Xantholinus*, was found on a small species of this, or a closely allied genus at Llavallol, No. 1497, at Temperley, No. 2003 and at Tucuman, No. 1931 (P. Spegazzini), but the material is too scanty to make a positive determination possible.

Dimeromyces Labiae Th., was found in abundance on *Labia minor*, No. 1974, in the park at Palermo, the specimens corresponding exactly to those obtained at Cambridge.

Ecteinomyces rhyncophorus Th., on a small hydrophilid at Palermo.

Eumonoicomyces Papuanus Th. A form which does not appear to differ essentially from the Papuan material of this species was found occasionally on the legs of a species of *Oxytelus* (?) at Temperley. This appears to be the form described as *E. Argentinensis* Speg.

Herpomyces Paranensis Th. was found in abundance on the antennae of a large roach (*Blabera ?*) inhabiting the roof of the Museo Nacional at Buenos Aires.

Kleidiomyces furcillatus Th. This peculiar species, formerly known only from a single complete specimen, was found in perfect condition and not uncommonly on species of *Aleochara* at Temperley, Llavallol, and the Isla de Santiago. An examination of abundant material shows conclusively that its separation from Monoicomyces is inevitable owing to the quite different character of its antheridium which is furnished with a lateral pore.

Laboulbenia Aspidoglossae Th. on *Aspidoglossa* sp. (?) was common in the park at Palermo and resembled the North American material in all respects.

Laboulbenia bicolor Th. This small species was found abundantly on the elytra and legs of a black *Galerita*, No. 2021, collected at La

Plata by P. Spegazzini, and also on the legs of *G. Lacordairii*, No. 1428, in the Museo Nacional. It resembles the type form from Venezuela in that the basal cell of the outer appendage is similarly modified but lacks the constriction, so characteristic in the type, above the basal cell of the receptacle. In the latter respect it approaches more nearly the distinctly larger Brazilian specimens obtained on *G. carbonaria*, in which, however, the basal cell of the outer appendage is unlike that of the type.

Laboulbenia Brachini Th. was again obtained abundantly from various species of *Brachinus*, and from different regions in the Argentine.

Laboulbenia Clivinae Th. on *Clivina* sp. and entirely typical was found on a specimen in the Museo Nacional, No. 1430, "Argentina."

Laboulbenia compacta Th., was found but twice on *Bembidia* outside the docks at Buenos Aires, No. 1969 and 1967.

Laboulbenia cristata Th. was found but once on *Paederus* sp., No. 2029 La Plata.

Laboulbenia geniculata Th. Several specimens of this species, which correspond exactly to the type, were obtained with several other species on a black *Galerita* collected at La Plata by P. Spegazzini.

Laboulbenia decipiens Th. was found on a black *Galerita*, No. 1439, from Tucuman, in the Museo Nacional.

Laboulbenia Mexicana Th. a pale and variable species, usually found only on the mid-elytra, occurred on two species of *Galerita*, Nos. 2020 and 2021 from La Plata, and Llavallol; also on a species from the Pampa Grenada, No. 1442 and from Jujuy, No. 1445, both in the Museo Nacional.

Laboulbenia Oedodactyli Th. was found repeatedly on *Oedodactylus fuscobrunneus* Fairm. No. 1976, at Llavallol and at Temperley. The material is in good condition and in a majority of individuals the outer appendage is greatly elongated, almost as much so as in *L. Lathropini*, which is its nearest ally, but from which it is distinguished at once by the character of its wall-cells which are neither striate nor spirally twisted.

Laboulbenia pedicillata Th., occurred rather rarely on *Bembidium* at Buenos Aires. No. 2016.

Laboulbenia Philonthi Th. was very common on various species of *Philonthus* throughout the whole Buenos Aires region.

Laboulbenia polyphaga Th. The forms allied to this species and to *L. flagellata* were numerous on many genera of Carabidae. The whole series needs much careful study of abundant material. Nos. 1506, 2019, 2022, 2023, 2024, 2025, 2026, 2027, 1445, 1444, 1970, 1997, 2010, 2014, 2017, 2022.

Laboulbenia Pterostichi Th. was found occasionally on carabids, all allied to *Pterostichus,* near Buenos Aires.

Laboulbenia punctata Th. was found on the head of a large *Galerita* with red prothorax, from Tucuman No. 1441, the individuals for the most part immature and somewhat smaller than the type, but otherwise identical with it.

Laboulbenia Pygmaea Th. was obtained on *Galerita* sp. from Jujuy, northern Argentina, in the Museo Nacional, occurring on the tip of the abdomen. The species seems to vary chiefly in the relative width of its receptacle which may be considerably narrower than it is represented in my Monograph, Part II, Plate LXII, fig. 6.

Laboulbenia sigmoidea Spegazzini. This well marked species which is most nearly allied to *L. elegans* Th., was found on the left inferior margin of the prothorax of a carabid named in the Museo Nacional *Argutor Bonariense,* but referred to by Spegazzini as an *Argutoridius* in his original description, Fungi Chilenses, p. 134 (Buenos Aires, 1910). It was found by me near Santiago, Chile, and in several localities in the vicinity of Buenos Aires, but although the host is common it was rather rare. The host-genus is *Pterostichus.*

Laboulbenia Tachyis Th., or a very closely allied form, was found repeatedly on a *Bradycellus* sp. in the park at Palermo, No. 1697, also at Temperley, No. 1517, and at Llavallol, No. 1996.

Laboulbenia Texana Th. A single immature individual that appears to belong to this species was obtained on a species of *Brachinus* on the Isla de Santiago, La Plata. The other forms heretofore grouped as varieties of this species, are referred to above (p. 56). Among these *L. incurvata, L. retusa* and *L. tibialis* were again found in the Argentine.

Laboulbenia variabilis Th. was common about Buenos Aires, as it appears to be everywhere else in South America, Nos. 1433, 1435, 1443, 1446 etc.

Laboulbenia vulgaris Th., which appears to have been described as *L. Chilensis* by Spegazzini, is everywhere common on *Bembidia* in Chile and the Argentine. There seem to be no characters indicated either by Spegazzini's description or figures which would suggest that *L. Chilensis* should be considered distinct. (Spegazzini, Fungi Chilenses, p. 133.)

Moniocomyces Homalotae Th. A few typical specimens of the smaller form of this species on *Atheta* sp., No. 1510, were found at Palermo. Another species closely allied to *M. Homalotae,* was found on *Ophioglossa* sp., but the material is not sufficient for description.

Monoicomyces nigrescens Th. A form corresponding in all respects to the North American material of this species was found abundantly in the Buenos Aires region on the tip of the abdomen of *Meroneva Sharpi* L. Arrib., No. 1503, Palermo, Temperley and Llavallol.

Rhyncophoromyces rostratus var. similar to that which is figured in my first Monograph, Plate XXIV, fig. 26, was found several times at Palermo on the margins of the elytra of a pale Hydrophilid. This form will probably have to be separated from the type, eventually.

Stigmatomyces virescens Th., which is probably cosmopolitan, having been received from Borneo, as well as Brazil and the West Indies, was obtained on a dull coccinellid collected by P. Spegazzini at La Plata.

Zodiomyces vorticellarius Th. The monstrous Argentine form previously recorded from Rosario, Argentina, was again met with at Palermo, on a large *Hydrophilus*, and the normal type was also found on smaller hydrophilids. A form perhaps specifically distinct was also found on a small hydrophylid, but sufficient material was not obtained.

NOTE. Since the present paper was in type I have received from Professor Spegazzini his "Contribución al Estudio de las Laboulbeniomycetas Argentinas," Buenos Aires, June, 1912, and have made such alterations in my own account as seemed absolutely necessary; reserving further comment on the paper for some more convenient time.

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CONTRIBUTIONS FROM THE CRYPTOGAMIC LABORATORIES
OF HARVARD UNIVERSITY.

No. LXX. — *CULTURE STUDIES OF FUNGI PRODUCING BULBILS AND SIMILAR PROPAGATIVE BODIES.*

By John William Hotson.

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LXX.—CULTURE STUDIES OF FUNGI PRODUCING BULBILS AND SIMILAR PROPAGATIVE BODIES.

By John William Hotson.

Presented by Roland Thaxter. Received June 19, 1912.

Contents.

Introduction.

The term "bulbil" was first employed in connection with Fungi by Eidam in 1883 to designate certain sclerotium-like bodies, somewhat definite in form, and capable of reproducing the plant. They vary greatly in appearance, some consisting of a compact mass of homogeneous cells clearly distinguished from certain others which surround them. The latter form a single layer or in some cases several layers of cells, which may or may not become empty and colorless and which correspond, in a general way, to the pseudospores or accessory spores of certain smuts, while the cells which they surround are functional spores and capable of germination. Bulbils are the predominant type of reproduction in certain fungi, and in some cases the only means at present known. The most typical bodies of this nature are readily distinguished from sclerotia by their smaller size, more definite structure, and peculiar methods of development. There are other types, however, that seem to approach more nearly true sclerotia; while others again resemble very closely the "spore balls" of such forms as Tuburcinia, Urocystis, etc., among the Ustilaginales, or even the compound spores of such forms as Stemphylium, Mystrosporium, etc., among the Hyphomycetes; but from the first they are definitely distinguished by their method of germination, while in general they are readily separated from the last two by their mode of development. They thus seem to possess morphological characters that would place them in an intermediate position between sclerotia, on the one hand, and compound spores of the dictyosporic type on the other, with examples of transitional forms which grade into the former and others that are almost indistinguishable from the latter.

Bulbiferous conditions among the fungi have, in general, been described under the following genera of the so-called "Fungi Imperfecti": Papulospora, Helicosporangium, Baryeidamia and Eidamia;

but in a few instances, in which their association with other and more definite types has been reported, they have been included under the generic name applied to the latter as, for example, Dendryphium or Haplotrichum. There seems to be little or no uniformity or agreement among the writers on this subject, especially among the earlier ones, regarding the morphological significance of bulbils. Preuss, who was the first to describe bodies of this nature in 1851, considered each bulbil a single compound spore and placed the genus Papulospora, which he had created for their reception, in the "Bactridiaceae" of Corda, a family not now recognized, which was established to include fungi like Trichocladium Harz, bearing compound spores and with prostrate fertile hyphae. On the other hand, Karsten ('65) regarded the bulbil-like bodies which were associated with his "Helicosporangium" as an ascus-producing structure, which was included by him among the Erysipheae. Again, Eidam ('83) was of the opinion that the two genera, Papulospora and Helicosporangium, occupied an intermediate position between Ustilagineae and Erysipheae, while E. Fischer is inclined to place them among the Monascaceae. De Bary, in his "Morphology and Biology of Fungi," considers them briefly and includes them in a category which he calls "Doubtful Ascomycetes" and suggests that "the plants should be further investigated." In considering these forms at a later period, Harz ('90) included all structures of this nature then known under a new order, the "Leptoomycetes" and expressed the opinion that they are somewhat closely related to the Oomycetes and coordinate with them and the Zygomycetes.

Inasmuch as these bulbils have received very little attention, our knowledge of their morphology, development, and taxonomy is very meagre. These forms are not as rare as has been generally supposed but are, on the contrary, widely distributed and of common occurrence. Substrata which have produced bulbils have been obtained from various parts of Canada and the United States; from Guatemala, Mexico, and West Indies; from South America and Europe. Their small size, the nature of the substratum on which they grow, and their failure to form a conspicuous fructification in a majority of cases, account to some extent for the fact that they are generally overlooked in the field and in laboratory cultures.

The results of the present investigation emphasize the fact, more recently brought out by several mycologists, that these fungi do not belong to any one of the Natural Orders, nor do they in any sense form a group by themselves, but occur without regularity as imperfect forms among the main groups of Higher Fungi. The forms associated with bulbiferous conditions which are herewith enumerated include among the Discomycetes, a new species of Cubonia; among the Hypocreales, three species of Melanospora; and among the Basidiomycetes at least four types; while nine species of Papulospora as yet unconnected with a perfect form are added to those already known. Among the latter also, *Papulospora candida* Sacc. has been found to be associated with a second and well marked imperfect form, namely *Verticillium agaricinum* var. *clavisedum*. In the life histories that have been worked out, the results have been obtained from pure cultures which, in many cases, have run for a number of years, and care has been taken to avoid any errors resulting from contamination.

In view of the very general occurrence of bulbils, it is somewhat surprising that more attention has not been given to them. The literature on the subject is quite limited and the accounts given often conflicting. Preuss, Karsten and Eidam did their work at a time when Mycology was in a more or less transitional condition, the modern bacteriological methods had not yet been applied to the cultivation of fungi, a fact which may account to a certain extent for the varied and often conflicting opinions of these earlier writers. Certain more recent contributions, however, have given us more accurate information as to certain isolated forms and the investigations of Mattirolo, Berlese, Bainier and Lyman have suggested or demonstrated the actual relationships of certain forms to species among the Ascomycetes and Basidiomycetes, of which they prove to be imperfect conditions. There has been no attempt, however, so far as the present writer is aware, to investigate the general subject of bulbiferous fungi.

The need of further examination of the morphology and development of bulbils was suggested by Professor Roland Thaxter, under whose direction and supervision the work has been conducted. The problem was begun and finished in the Cryptogamic Laboratories of Harvard University, some culture work and collections of material being done in California while the writer was connected with Pomona College.

It is a pleasant duty for the writer to acknowledge, at this point, his indebtedness to those who have rendered him assistance in carrying on this research: especially to Professor Thaxter are grateful acknowledgments due, for suggestions, kindly advice and encouragement, and for placing at the writer's disposal many dried specimens and tube cultures of bulbils which had been collected by him, and for the use of a number of papers belonging to his private library; to Professor Elias J. Durand of the State University of Missouri, for the description and naming of *Cubonia bulbifera;* to Professor W. G. Farlow for material and the use of several articles from his private library.

Review of Literature.

The literature relating to bulbils is, as has been already indicated, by no means extensive, and deals with less than a dozen described forms, some of which do not appear to have been recognized by mycologists since their original publication. In order to give a clearer idea of the present state of our knowledge of the subject, it seems desirable, before proceeding further, to give a brief summary of the more important papers, which may be conveniently considered seriatum under the following heads:

(a) Helicosporangium, (b) Papulospora, (c) Pyrenomycetous Forms, (d) Discomycetous Forms, and (e) Basidiomycetous Forms.

(a) *Helicosporangium.*

The genus Helicosporangium was first described by Karsten ('65) and was based on a form said to be "parasitic" on beet roots, which he named *H. parasiticum.* According to his description the fertile branches of this fungus tend to become erect, and are septate like the rest of the hyphae. In the process of development they coil up spirally at the end to form the bulbil. This character suggested that they might be closely related to such hyphomycetous forms as Helicoma Corda, Helicosporium Nees, Helicomyces Lk., Helicotrichum Nees, etc. In fact, it was this spiral development of the fructification, held in common with these forms, that suggested to Karsten the name, Helicosporangium.

At maturity these bulbils are described as almost spherical, with one large central cell which is surrounded by a single layer of colorless cortical cells which form a complete wall. Karsten believed that one of these cortical cells produced a short protuberance on the inner side, which extended into the large central cell, in which he says a "nucleus" soon appeared and enlarged quite rapidly. He further observed that the contents of the central cell soon became somewhat differentiated and divided into a number of small cells, usually eight in number, but varying from seven to ten, which gradually enlarged to form free, hyaline, elliptical spores; and, after escaping from the central cell, divided, forming compound spores of two cells. On germination each cell produced a germ tube.

Karsten believed that the contents of the cortical cells entered directly, or by diffusion, into the large central cell and that only after the contents intermingled were the spores formed. This suggested the possibility of sexual differentiation of certain cells which made up the coil, the end-cell, in his opinion, acting as an oogonium and the second or even the third or fourth cell acting as an antheridium.

It will thus be seen that in Karsten's opinion the peculiar structures which he described in Helicosporangium were neither bulbils nor homologous with other non-sexual propagative bodies, and although it is possible that he may have been dealing with some form allied to Monascus, in which a sexual process was actually present, it seems not improbable that he was misled by what he saw. Since, however, this subject will be further discussed below in connection with a form which appears to be identical with Karsten's species, it need not be further considered in the present connection.

Eidam ('77, '83) described and figured a bulbil obtained from moist turnips which he referred to *Helicosporangium parasiticum* Karsten, but, as has been pointed out by Karsten himself ('88), Harz ('90), and others, it seems probable that he was dealing with a fungus different from that which Karsten described. Eidam's fungus is said to be saprophytic, producing numerous conidia borne on characteristic bottle-shaped sterigmata and having two kinds of bulbils which do not contain endospores. In these respects it is said to differ from that described by Karsten. This matter, however, will be referred to again below.

De Bary ('87) accepted in general the views expressed by Eidam ('83) regarding *H. parasiticum*, but Karsten ('88) maintained that he did so because he had not read the original article, but formed his opinion on information obtained from "Eidam's unfortunate review of it" ('83), and in conclusion ironically gives the name Baryeidamia to Eidam's fungus, in recognition of what he considered the combined blunders of these two mycologists in dealing with this form.

A third species referred to the genus Helicosporangium was described under the name of *H. coprophilum* by Zukal ('86) and was found by him on horse dung associated with *Stysanus stemonites* Cd. According to Zukal's description, this bulbil consists of two to eight large central cells with thick walls of a dark-red color, which are surrounded by a layer of smaller cortical cells of a lighter color. The form and manner of development of this bulbil are said to vary considerably. "Indeed," he says, "there are hardly two to be found which are exactly alike."

Zukal ('86) also describes a yellowish-brown bulbil under the name of *Dendryphium bulbiferum,* found on birch twigs, the mycelium of which is said to grow up, tree-like, and to branch monopodially, the ultimate branches terminating in rows of small hyaline ellipsoidal cells. At maturity these little cells become brownish and, when they are abstricted, form a dusty mass. The bulbil associated with them is almost spherical and bears a very close resemblance to *Helicosporangium parasiticum* Karsten, both in its mode of development and in its general appearance.

On decayed fruit of *Lycopersicum esculentum* Mill. Zukal ('86) has reported the occurrence of bulbils closely resembling, both in appearance and development, the two types above referred to, but which are said to differ in their greater variations and irregularities, and also in the fact that they are associated with the conidia of *Haplotrichum roseum* Lk. (*Oedocephalum glomerulosum* Bull.). It should be mentioned in this connection, however, that since Zukal did not apparently deal with pure cultures and no such bulbils have been found, as far as the writer is aware, by others who have cultivated this very common Hyphomycete, his statements must be accepted with some reserve. It may be stated at this point that in none of the published accounts of Helicosporangium is there any evidence that pure cultures were used, and thus the possibility of contamination renders these results largely untrustworthy.

(b) *Papulospora.*

Of the several species which have been placed in this genus the first was described by Preuss ('51) from material found growing on decayed pieces of apple and was said to be connected with chlamydospores which resembled those of Sepedonium. He therefore named his species *P. sepedonioides.* These bulbils are described as irregularly arranged on lateral branches, white at first and later becoming rust-colored, with the cortical cells differentiated from the central ones. Preuss regarded this bulbil as a single multicellular spore and not as a cluster of single spores, because they never break up into individual cells, although he thought the cortical layer probably bursts at the time of germination.

Eidam ('83), in the paper already referred to, described a second bulbil found quite abundantly on straw, weeds, dung, etc., which appeared, in his opinion, to be so closely related to the form described by Preuss that he placed it in the same genus; since it was, however, not associated with chlamydospores like those of Sepedonium, but with an Aspergillus-like fructification, he named it *P. aspergilliformis.* Two kinds of bulbils were described as connected with this fungus, which resembled each other in color but differed in their mode of development. Of these two types, one is said to be large, sclerotium-like, without any differentiation into central and cortical cells, while the other is small and consists of several large central cells surrounded by a row of colorless cortical cells resembling those of *Helicosporangium parasiticum*, mentioned in the same paper.

In connection with this fungus Eidam described conidia which, he states, were produced on exceedingly delicate, colorless, conidiophores resembling somewhat those of *Aspergillus albus* Wilhelm, but the sterigmata are usually flask-shaped. These conidia were also borne individually on the sides of ordinary hyphae, being abstricted in chains from flask-shaped sterigmata and resembling those described by Eidam as associated with the form which he referred to *Helicosporangium parasiticum.*

"Chlamydospores" were also described by Eidam in connection with his *P. aspergilliformis.* "This form of reproduction," he says, "seems to be by far the most common one connected with Papulospora and often is the only one. I have found, in great abundance, mycelia with only chlamdospores and no trace of bulbils or conidiophores." On account of the presence of these chlamydospores which resemble the spores of Acremoniella, Lindau ('07) has redescribed this species under the name of *Eidamia acremonioides* Harz. The criticism that was offered as to the reliability of Eidam's investigation of Helicosporangium may equally well be applied here. Bainier ('07) is of the opinion that he mistook the conidia of *Acremoniella atra* Sacc. (*Acremonium atrum* Corda) for chlamydospores belonging to Papulospora, as these two species are often found associated with each other.

Bainier ('07) found a fungus abundantly on straw, paper, cardboard, etc., which he calls *P. aspergilliformis.* His description of the conidia and conidiophores is practically the same as that given by Eidam ('83). His fungus, however, does not produce acremonium-like chlamydospores, as did that of Eidam, but, on the other hand, developed parithecia with long necks, which he refers to the genus Ceratostoma. The asci, which are very transitory, even disappearing before the maturity of the spores, are ovoid with eight simple brownish spores somewhat variable in shape and grouped together, forming a sort of ball. Moreover, he considers that the bulbils of *Helicosporangium parasiticum* described by Eidam are merely abnormal forms of *P. aspergilliformis*, such as are often found among other Mucedineae.

Another Papulospora, which was found in the tubers of Dahlia, has been described under the name of *P. dahliae* by Costantin ('88). The bulbils of this fungus are spherical, brownish-red in color, with two or three large central cells. All the cells are said to contain granular protoplasmic material at first, but the central cells soon become strongly colored violet and more densely filled with granular material and oil globules, and eventually the peripheral cells become empty and transparent. There were found associated with this fungus colorless septate spores which taper at both ends and correspond very closely to those described by Saccardo (Michelia II, p. 20) under the genus Dactylaria. Here again there is little evidence that the investigation was carried on with pure cultures and it is doubtful that the conidia and the bulbils described belong to the same fungus, since they were only found associated and not actually connected. It would thus appear that the only contribution on Papulospora that shows any evidence of work with pure cultures is that of Bainier ('07).

(c) *Pyrenomycetous Forms.*

The first evidence of the definite association of a bulbil with one of the Pyrenomycetes as an imperfect form, is found in the description of *Melanospora Gibelliana*, published by Mattirolo in 1886,— although Zukal ('86) a few months previously had announced that he had found bulbils in connection with *Melanospora fimicola* Hansen, and *M. Zobelii* Corda, but gave no description of them. The fungus studied by Mattirolo was found growing abundantly on decayed chestnuts and was said to produce not only perithecia of Melanospora but also bulbils, conidia and chlamydospores. In appearance and development these bulbils are said to resemble closely those of Baryeidamia, but with more variations. Their color is pale yellow when young, brownish-yellow at maturity, and they are often 100 μ in diameter. Mattirolo considered them immature perithecia, but, although he employed the most varied methods of experimentation, he was unable to make them develop into melanosporous perithecia. The conidia said to be connected with this fungus are described as small, colorless, spherical spores, on bottle-shaped sterigmata, resembling closely those mentioned by Eidam as belonging to Baryeidamia.

The chlamydospores referred to this fungus are said to have very rough, thick walls, resembling somewhat those of Sepedonium. Although Mattirolo is of the opinion that these chlamydospores form a phase of the life history of *M. Gibelliana*, he admits that he has not absolutely proven it. He states he has "cultivated these forms without ever being able to establish unquestionably their origin and relation."

Berlese ('92) described a bulbiferous fungus producing perithecia, which he named *Sphaeroderma bulbilliferum*. This fungus he found growing abundantly on dead leaves of Vitis, Cissus and Ampelopsis. It is said to have several modes of reproduction, such as (a) microconidia, which appear in chains and which resemble those figured by Mattirolo as belonging to *Melanospora Gibelliana* and by Eidam, to *Helicosporangium parasiticum*; (b) chlamydospores, which varied somewhat in size — (these were ovoid, usually smooth, and golden-yellow in color, each with a septum near the base, which divided the chlamydospore into two unequal cells); (c) golden-yellow bulbils, which resembled those described and figured by Mattirolo in *Melanospora Gibelliana* and which seem to be short-lived and, under the most favorable conditions, could not be made to produce mycelia; (d) perithecia, which were represented as almost spherical and when mature measured from 400–500 μ in diameter. They remain without an ostiole almost to maturity and consequently there is no formation of a neck. The color of the young perithecium is yellowish but becomes darker as it grows older, until at maturity it is almost a tan color. The asci are club-shaped with deep smoke-colored spores, ovoid and prolonged at the poles into short obtuse papillae.

Another pyrenomycetous form producing bulbils has been reported by Biffen ('01, '02), and is said to be connected with *Acrospeira mirabilis* Berk., which was originally found on sweet chestnuts (*Castanea vesca*, Gaertn.). By the use of pure cultures, Biffen claims to have succeeded in obtaining not only the chlamydospores, as described by Berkeley and Broome in the Annals and Magazine of Natural History for 1861, but also what he calls "spore-balls" (bulbils) and definite perithecia.

The spore-balls, which he says so closely resemble *Urocystis violae* that he "could not find a single characteristic to separate them by," were obtained by sowing the 'chlamydospores' on a watery extract of chestnuts. Greater difficulty was experienced in producing the perithecia, but finally, by sowing the chlamydospores and bulbils on sterilized chestnuts, he records the following results: — "The 'chlamydospore' infections gave a crop of 'chlamydospores' only; the spore-balls gave spore-balls and small reddish-brown, hard-walled perithecia. The walls of the perithecia were smooth and without bristles and the ostiole was small and flush with the surface, i. e., not raised on a papilla or forming a neck. . . . Berkeley's *A. mirabilis* thus turns out to be one of the stages in the life history of a Sphaeria."

The investigations on the pyrenomycetous forms show more careful work than those under the two preceding headings. In all these there is evidence that pure cultures were used more or less, but in most cases it is uncertain how far the results were thus obtained.

(d) *Discomycetous Forms.*

There have been two fungi described which produce bulbils associated with discomycetous fructifications, one by Zukal ('85, '86) and the other by Morini ('88). Zukal found two kinds of primordia in connection with his fungus; one, he says, consisted of two or three small mycelial branches which wound about each other and eventually produced reddish-brown bulbils with a cortex of small colorless, almost transparent, cells. The other primordium was made up of a number of hyphae massing themselves together and becoming quite large and, under proper conditions of nutrition, developing into apothecia of the Peziza type; but he does not give a name to this form. This fungus produced conidia abundantly on erect, branched conidiophores. The conidia are spoken of as colorless, ellipsoidal, smooth, and they appear in clusters upon the ends of short sterigmata. Zukal's cultures were grown on absorbant paper saturated with Leibig's extract, but there is no evidence in his article that these were pure cultures, or that the life history of the fungus was carefully traced from ascospore to bulbil.

Morini ('88) describes "bulbil-like" bodies associated with *Lachnea theleboloides* (A. & S.) Sacc. in old cultures. Since these occurred only in cultures that had run for a long time, in which the nutrient was probably largely exhausted by the previous growth of the fungus, and since the development was largely the same as that of the apothecium, Morini considers that the bulbils of *L. theleboloides* are abortive apothecia and, further, that they are analagous to the similar structures described by Eidam, Karsten, et al. He apparently has used pure cultures in his investigation, but to what extent his results were obtained from such cultures could not be determined from his paper.

(e) *Basidiomycetous Forms.*

The only account, as far as the writer is aware, of the definite association of bulbils with Basidiomycetes is given by Lyman ('07) in connection with his culture-studies of *Corticium alutaceum* (Schrader) Bresadola, his results having been obtained from pure cultures made of the basidiospores of this fungus. "The bulbils," he says, "are reddish-brown or chocolate-colored clusters of cells, more or less globose in shape, and usually 65–80 μ in diameter, although ranging as high as 220 μ. . . . They are frequently very irregular in shape, due to the unsymmetrical arrangement of the cells, and to the bulging of the free outer walls. There is no distinction between internal and external cells of the cluster." Besides the basidiospores and bulbils this Corticium also produces conidia which are of the Oidium-type. Occasionally whole hyphae break up into chains of spores of this type.

Lyman also mentions two other bulbiferous fungi which were referred to the Basidiomycetes, being recognized as such by the clamp-connections of their hyphae, although the basidiospores were not obtained.

Lastly, it may be well to mention an article by Harz ('90), in which he describes a fungus found growing on material obtained from the reservoir of a factory and which he names *Physomyces heterosporus* (*Monascus heterosporus* (Harz) Schröter). Although this fungus is probably a true Monascus, as Schröter has indicated, yet since it has been associated with bulbils, and since the ascocarps of Monascus in general bear a superficial resemblance to them, it may be well at least to mention it in passing. Harz has associated this form closely with *Helicosporangium parasiticum* Karsten, and created a new family — Physomycetes — for the reception of these two genera. As, however, these two forms will be referred to again in connection with *H. parasiticum* Karsten, a further consideration of them will be deferred until that time.

It will be seen from the foregoing brief review of the literature that much of it is quite vague and untrustworthy. This perhaps is what one would expect from investigations which were carried on during a period prior to the adoption by mycologists of the bacterial methods of handling pure cultures. This is especially true with regard to polymorphic forms, like some of those under consideration, where it is so necessary to adopt these methods in order to be absolutely sure of the different steps in following the life history of the fungus from spore-form to spore-form. The contributions of Lyman and Biffen on this subject show undoubted evidence that their investigations were carried on with pure cultures and that the life history from spore to bulbil was closely traced. It is probable that Bainier, Morini, Berlese, and Mattirolo also used pure cultures more or less, but there is little evidence in their writings that there was careful tracing of the fungus from spore to bulbil.

Sources of Material.

Before recording the results obtained from the study of the various bulbiferous fungi cultivated by the writer, it will be well to refer briefly to the sources of material and the methods used in this investigation.

In 1907, at the suggestion of Dr. Thaxter and with a view to obtaining as much material as possible for examination, the writer began collecting substrata of various kinds from widely different localities. This material was placed in moist chambers in the laboratory and as bulbils appeared pure cultures were made of them. The methods employed in doing this will be referred to later. Most of the material from which bulbils were obtained was collected either in the vicinity of Cambridge, Mass., or Claremont, Calif.; but bulbils were also procured from substrata received from other portions of New England and California, from Kentucky, Canada, Mexico, Guatemala, Cuba, Jamaica, Bermuda Islands, the Argentine Republic, Italy, etc.

The substrata on which these fungi were found were very diverse. The most productive were various kinds of excrement (dog, rat, mouse, rabbit, pig, horse, goose, goat, etc.), dead wood (Acer, Lathyrus, Quercus, Eucalyptus, etc.), decaying vegetables (squash, onions, etc.), straw (wheat, oats, barley, rye, alfalfa, etc.). A number were found on paper and old cardboard, as well as on a variety of other substrata. Of many hundreds of such cultures about two hundred yielded bulbils.

Culture Methods.

The moist chambers used for the cultivation of these materials were usually crystallizing dishes covered with pieces of glass. A large amount of this material was grown in the laboratory and from time to time was carefully examined through the glass top with a hand lens. When bulbils were observed, one of them was picked out by means of fine dissecting-needles under a dissecting microscope, and after thorough washing in sterilized water on a flamed slide, was transferred to a test-tube containing sterilized nutrient material — usually potato agar. In the case of some melanosporous forms the transfer was made by carefully touching the long cirri of ascospores, produced by the perithecia of this genus, with a piece of nutrient agar on the end of a sterilized platinum needle. The ascospores adhering readily to the agar, a pure culture was easily obtained.

Bacteria sometimes gave trouble in some transfers, but as a rule these were gotten rid of either by picking out separate bulbils carefully and washing several times before growing them in acidulated nutrient agar, or by keeping the impure tubes at a temperature of 15–20° C. The growth of the bacteria being retarded either by the cold or acid, the mycelium producing the bulbil soon grew out beyond the affected region, and by gouging out a few of the ends of the hyphae with some of the agar and transferring to another tube, a pure culture was readily obtained.

When these were secured the fungus was cultivated on various kinds of nutrient agar media, some growing better on one medium and some on another. The following were used most frequently: potato, onions, sucrose of different percentages, bran, rice, cornmeal, straw, plums, prunes, grapes, figs, bread, squash, Spanish chestnuts, wood, various kinds of dung, etc. These were usually used with agar, but some materials like wood, dung, straw, nuts, etc., were sterilized in bulk with plenty of water and without using agar while in some instances decoctions were used. In Claremont, California, they were grown in the laboratory at an average temperature of 25–30° C. In Cambridge many were grown in an oven kept at various constant temperatures, 20–25° C. giving the best results.

The vessels used for these cultures were usually medium sized test-tubes, Erlenmeyer flasks of one and two litres, or preserve-jars with cotton plugs. These were filled about one-third full of nutrient agar and usually slanted to give more surface. On this nutrient the fungus would usually grow well for several months, and results were often obtained from pure gross cultures which could not be secured from the smaller ones.

In the germination of the spores and bulbils, Van Tieghem cells were used very freely. For this purpose cover glasses of one inch and two inches in diameter were used and carefully sealed, plenty of sterilized water having previously been put in the cells which corresponded in dimensions with that of the cover glasses. The large Van Tieghem cells afforded an opportunity of using cultures of considerable size which were usually composed of decoctions of different kinds of nutrient material, sometimes with agar to make them solid, while at other times the decoctions were used as hanging drops.

In cases where the transfer of conidia, only, was desired, two methods were employed to avoid getting either bulbils or pieces of mycelium. If the conidia were quite plentiful or were on erect stalks so that they were somewhat separated from the rest of the mycelium, this could be accomplished by means of a piece of nutrient agar on the end of a sterilized platinum needle. By careful manipulation and with the aid of a dissecting microscope, they could be touched with the agar to which they adhered readily, and after examination under a microscope to determine if there were only conidia present, they were immediately transferred to a new tube or a Van Tieghem cell, as the case required. In instances where the above method could not be used, or where cultures from individual conidia were required to verify the relation between a conidial form and the bulbil, Barber's spore-picking apparatus ('07) was employed. Plate-cultures were also used to advantage in some instances for separating the conidia from the bulbils.

Throughout this investigation, as already stated, the results obtained are based upon pure culture methods and every precaution has been taken to avoid error as a result of contamination.

It perhaps should be mentioned at this point that it is the intention of the writer to deposit living cultures of most of the forms described with the Centralstelle für Pilzculturen.

Systematic Consideration of the Forms Studied.

As has already been indicated, "Bulbils" must in all instances be regarded as representing imperfect conditions of the higher fungi; and like the members of other more or less clearly defined "form-genera" may be associated with perfect conditions included in wholly unrelated genera of the Ascomycetes and Basidiomycetes. They may, moreover, not only represent conditions of such perfect forms, but may be further associated with one or more additional imperfect forms. There may thus be present in some instances a succession of three or even four distinct reproductive phases which together make up the individual life-cycle.

It has been the aim of the present investigation, therefore, to endeavor not only to obtain further information as to the occurrence, morphology, and development of these comparatively little known structures, but by means of careful and extended work with pure cultures to make some further contribution to our knowledge of their actual relationship in different cases.

Bulbils, as a rule retain their vitality a long time so that they germinate readily after a year or more. Their maximum longevity has not been precisely determined, but in some instances, as in Grandinia and Corticium, they have been germinated after three years. This fact of the extensive longevity of bulbils is of immense importance to the fungus, enabling it to withstand long periods of unfavorable conditions, the perpetuation of the species being thus comparatively well assured.

In arranging the materials available for systematic consideration it has been found most convenient to group the forms under four main divisions, namely: those which are known or supposed to be connected with perfect forms belonging to the Discomycetes; those thought to be connected with Pyrenomycetes; those which appear to be imperfect conditions of Basidiomycetes, and lastly those the actual relationships of which are still undetermined. It has seemed best to consider the last group under a single form-genus, Papulospora, this name having been the first which was applied to bodies of this nature, and the variations in the morphology and development in the different species being such that a separation into more than one form-genus does not seem advisable.

DISCOMYCETOUS FORMS.

Previous investigations have brought to light but two bulbiferous Discomycetes; an unnamed species of Peziza observed by Zukal ('85, '86), and *Lachnea theleboloides* (A. & S.) Sacc. reported by Morini ('88). To these is added a species of Cubonia now reported for the first time, specimens of which were sent for identification to Professor Elias J. Durand of the University of Missouri, to whom the writer is indebted for the following diagnosis:

Cubonia bulbifera n. sp.

Plate 1, Figures 1–28.

"Plants single or gregarious, often crowded, sessile or narrowed to a stem-like base, turbinate, 3–10 mm. in diameter. Disk cupulate or saucer-shaped, the hymenium pale fawn-color, even when young, but in old specimens wrinkled in a cerebriform manner, externally much darker, becoming almost black with age, smooth or grumous; margin irregularly lacerate-dentate. Consistency subgelatinous, excipulum pseudoparenchymatous throughout, of nearly rounded cells, 20–25 μ

in diameter, the cortical cells blackish, often protruding in groups. Asci clavate, apex rounded, not blue with iodine, 125 × 15 μ. Spores 8, uniseriate, hyaline, smooth, spherical, 12 μ diameter. Paraphyses slender, hyaline, only slightly thickened upward. Mycelium giving rise to numerous rounded, black bulbils, 75–100 μ diameter, composed of rounded cells about 20 μ diameter."

Cultivated on nutrient agar. Found on dog dung from Jamaica, Paestum (Italy), Guatemala and California, and pig dung from Guatemala.

This fungus was first obtained by Dr. Thaxter on dog dung from Jamaica and has been kept growing in pure tube-cultures for twenty years; since then he has found it on the same substratum from Paestum, Italy, and from Guatemala. It was also secured from gross cultures of pig dung and of dead flowers believed to be of the genus Criosanthes from the last named locality, while the writer has found it on gross cultures of dog-dung from Claremont, California, from which a pure culture was obtained in a manner similar to that already described. This was not difficult, since the mycelium grows with great rapidity and the bulbils are produced in abundance. The fungus was grown on a great variety of media until the mature perfect form was obtained. The mycelium grows well on nearly all media, producing numerous dark-colored, almost black, bulbils. The best substratum for producing apothecia is bran, or rat or dog-dung, although they developed quite readily on sweet-potato agar or on Irish potato agar with a little sugar; but it was found that after the fungus had been cultivated for a long time on artificial media, it failed to produce mature apothecia.

On appropriate substrata such as bran, dung, etc. the rate of growth of the mycelium is remarkably rapid. The average of several measurements made of this fungus, grown at the temperature of the laboratory is as follows: 1 cm. in 24 hrs., $2\frac{1}{2}$ cm. in 50 hrs., $3\frac{1}{2}$ cm. in 74 hrs., and 5 cm. in 120 hrs. It is white and somewhat flocculent, and does not grow in a "zonate fashion" like that of the Peziza described by Zukal, but spreads out quite evenly over the surface of the substratum. In older cultures the hyphae become quite large, often over 10 μ in diameter, and densely filled with granular protoplasm, but, as they reach their limit of size, they lose their contents. Frequently when a hypha becomes broken or a portion of it is killed, there seems to be a stimulus for growth at the free end, somewhat similar to that in higher plants which are subjected to wounding. This injury of the hyphae appears to cause a sort of damming up of food material, which is evident from the sprouting out of several small hyphae, not only from the end but also from the sides near the end of the injured part; and these often twine about each other in such numbers, that it gives the appearance of a broom-like structure.

The bulbils.— Often within forty-eight hours, dark bodies, which eventually become black, may be observed with a hand lens, scattered over the substratum or in it; they are most abundant near the point of inoculation, from this point extending out as the peripheral growth of the mycelium increases thus exhibiting a progressive formation. These black bodies are bulbils which soon become very numerous, forming a blackish crust over the substratum and usually giving the whole culture a black aspect. This is especially true when it is grown on such media as potato agar made very hard with about forty grams of agar to the litre. In such cases the mycelium is quite scanty and procumbent, and the bulbils thus become very conspicuous; while on media like rat dung, where there is an abundance of mycelium produced, they are not so readily seen, since they are usually formed on or in the substratum. In the development of these structures which are produced so abundantly, two or three intercalary cells become enlarged and filled with granular nutrient material, as shown in Figures 11–14, Plate 1. From these cells others are produced by budding, or short branches are formed which surround the primordial cells, and which in turn become enlarged so that eventually there is produced an almost spherical bulbil somewhat flattened, 75–100 μ in diameter, the cells in the center, usually considerably larger, but all filled with protoplasm, without any definite differentiation of cell-contents between internal and external cells. Not infrequently, however, the marginal cells of old bulbils lose their contents, although they retain the dark color in the wall, but this is probably due to age. As a result of the unequal production of marginal cells, the bulbils may vary considerably in size and some become quite irregular in outline. Frequently the bulbils or the primordia of imperfect ones, especially as the cultures become old, heap together and form conspicuous dark elevations scattered over the substratum. These structures eventually assume a yellowish color, probably due partly to fading and partly to the immature bulbils that compose them.

The apothecium.— Occasionally there is found a spiral primordium, as shown in Figure 1, Plate 1, produced on short lateral branches which usually divide dichotomously, sometimes of the second or third order, the ultimate branches of which coil up spirally (Figures 1–4,

Plate 1). Ordinarily there are about one and a half to two turns in the spiral, but occasionally there are as many as four. If a lateral branch fails to divide, as it often does, only one primordium is produced (Figure 4, Plate 1). Frequently after the first dichotomy, one of the branches does not divide again, but coils up immediately, while the other may divide once or twice before coiling (Figures 2–3, Plate 1). Thus, according to the number and regularity of these dichotomous divisions, there may appear one, two, or more primordia which are more or less closely related to each other. Usually, however, the pedicels on which they are formed elongate, and thus they may become separated from each other. When this primordium has made about two turns, sometimes as many as four, small branches are produced from the sides of the coils (Figures 5–6, Plate 1), which at this stage often become separated from each other, as shown in Figure 6. It is, however, a very obscure structure, the further details of which are difficult to follow.

Occasionally on media like potato, more frequently on bran, Spanish chestnuts, sweet potato, etc., and quite freely on rat and dog dung, little white patches of hyphae are seen scattered over the substratum. These are the young apothecia. The fine, white, wool-like hyphae become thickly matted together and form a white superficial dome-shaped structure with fine filaments growing out on all sides (Figure 7, Plate 1), and as these become older, they lose their contents and assume a brownish color. Shortly a circular opening appears at the apex (Figure 8, Plate 1), apparently due to the rapid and extensive growth of the inner portion of the apothecium. This opening gradually increases in size, often exhibiting a conical depression in the center which, as the hymenium enlarges, becomes flat and then slightly convex. Microtome sections, made at the time of the opening of the apothecium or shortly before, show the upper region closely crowded with long narrow paraphyses, nearly uniform in thickness, which a little later, slightly enlarge at the ends, forming the somewhat even surface of the hymenium (Figures 9–10, Plate 1).

A short distance below the center of the apothecium, when about the age of that represented in Figure 8, Plate 1, a large cell containing deeply staining material is seen in microtome sections. This appears to be the ascogonium and from it very narrow hyphae, which also stain deeply, grow up between the sterile cells of the apothecium, and eventually produce the asci. At maturity the apothecium is brownish, measuring 3–10 mm. in diameter and 3–5 mm. in height; often in groups and occasionally with a short stem-like base.

When a portion of the hymenium containing some of the large cells below the sub-hymenium was put in a sterilized Van Tieghem cell in an endeavor to induce the ascospores to germinate, it was found that frequently these large cells, which measure 20–25 μ in diameter, sent out germ tubes, or turned brown, secreted thick walls about themselves and resembled considerably chlamydospores (Figures 26, 27).

Germination of the ascospore.— The mature asci are quite uniform, clavate, with the apex rounded, opening by a lid, 125 μ in length and 15 μ in diameter at the widest place. The ascospores are hyaline, spherical, 12 μ in diameter, and arranged in a single row. At maturity all the spores from each ascus are ejected with considerable force blowing off the lid at the apex in a manner somewhat similar to that of Ascobolus, and thus are thrown in a bunch for several centimeters, and, by means of the protoplasmic material which surrounds them, adhere readily to any glass surface with which they may come in contact. These spores were allowed to strike a sterilized cover glass and then supplied with nutrient material and cultivated in a Van Tieghem cell, which had previously been thoroughly sterilized. Not only were the spores alone used as just stated, but frequently a portion of the hymenium with the asci was gouged out with a sterilized platinum needle and hanging drops made of it. In an effort to get these spores to germinate, various kinds of media were used, such as — potato, prunes, bran, horse dung, dog dung, Spanish chestnuts, carrots, etc., either as a decoction, or more often solidified with agar. In spite of these varied efforts, the spores could not be made to germinate. The writer some time ago succeeded in getting the spores of Ascobolus to germinate in Van Tieghem cells by first crushing them lightly between two glass slides, and it occurred to him that the same method might be successful here also. Accordingly hanging drops were made as before, using different media, but the spores were first crushed with a sterilized platinum spatula on the cover-glass. This method proved successful. These spores are composed of a thick brittle episporium and a thin flexible endosporium; the object in crushing was to break the former without injuring the latter. Many of the spores thus crushed were totally destroyed, and broken portions of the episporium were scattered over the culture; but in a few cases, where the pressure was sufficient just to break the episporium without injuring the endosporium, it was found that germination took place in from 24 to 48 hours (Figures 22–24, Plate 1). When this occurs the endospore pushes out, forming a germ tube which is only a little smaller in diameter than that of the spore itself (Figure 22), and frequently when it has grown a short distance, broadens out as much as 14 μ in diameter (Figure 23). Thus the primary hypha from the ascospore is very large (7–14 μ in diameter), well filled with food material, and grows quite rapidly under favorable conditions. The culture of these germinating spores was carried on in Van Tieghem cells until bulbils were produced on the mycelium.

Germination of the Bulbil.— The bulbils, unlike the ascospores, germinate with great readiness within twenty-four hours and any of the cells that contain protoplasmic material may send out a germ tube, which shortly produces other bulbils from intercalary cells, as described above. When the bulbils are crushed, the contents of each of the large cells escapes surrounded by an endosporium (Figure 19) and germinates readily in Van Tieghem cells. Little significance can be attached to this fact, however, as not only are nearly all bulbils similar in this respect, but it is a common occurrence among spores which are surrounded by a thick episporium, such as the ascospores just considered.

In prolonged cultures of this fungus no other spore forms have been observed.

Lachnea theleboloides (A. & S.) Sacc.

The association of this species with bulbil-like bodies is reported by Morini ('88) but it is not clear from his account whether the structures seen were true bulbils, or abortive apothecia, as he believed them to be. The apothecia, which he describes and figures, are very similar to those of *Cubonia bulbifera* but the spherical spores of the latter distinguish it at once.

The bulbil-like structures which he describes were found only in old cultures in which the nutriment was more or less exhausted, and are described as irregularly globose, 160–220 μ, and rather hard. In many cases large cells of somewhat spiral form were visible in these bodies which Morini considered "rudimentary ascogonia." The protoplasm of the external cells, is said to be replaced by an aqueous liquid and the walls become thick and brownish-red in color. A large number of the superficial cells, as in the case of the developing apothecium, give rise to short, often septate setae, which cover nearly the whole surface. When these "bulbils" were transferred to fresh substrata, only those with better developed "ascogonia" continued their development until they formed apothecia identical in character with those produced normally. In all other cases, especially those in which the so called "ascogonium" had completely disappeared, Morini observed no further development, except that in rare cases, a few paraphyses were found.

He is of the opinion that these "bulbil-like" bodies are degenerate apothecia, analogous to the bulbils of Eidam, Karsten, etc., and concludes his article by saying that "the forms heretofore called 'bulbils' or 'spore-bulbils' are to be considered as exactly homologous to apothecia of which they represent forms more or less degenerate or modified during many generations of unfavorable conditions."

Peziza, species; not determined.

A species of "Peziza" found by Zukal growing on a laboratory culture may be here referred to, which according to his account is associated with small bulbils 30–40 μ in diameter, reddish brown in color, and produced by "two or three small hyphal branches which wind about one another like serpents or twist, screw-like." The primordium of the apothecium is somewhat vaguely described. The ascospores are said to be elliptical, hyaline, smooth, about $9 \times 6\ \mu$, obliquely monostichous, germinating readily in from twenty-four to thirty-six hours. Since this form does not appear to have been studied by means of pure cultures its connection with the bulbils described must be regarded as somewhat doubtful.

PYRENOMYCETOUS FORMS.

In the review of the literature a number of pyrenomycetous forms that produce bulbils were mentioned, which have been referred either to the genus Melanospora or to the allied genera Sphaeroderma or Ceratostoma. More than twenty different gross cultures made by the writer of various substrata, such as onions, straw of various kinds, paper, pasteboard, Live Oak chips, rotten planks, tubers of Dahlia, old leather gloves, etc., have produced bulbils which in pure cultures have yielded melanosporous perithecia. In a few cases the perithecial form was found on the original substratum and cultures were made from the cirri of discharged ascospores, which on nutrient agar produced bulbils.

In addition to bulbils, all of these forms also produce ovoid, hyaline conidia borne on characteristic bottled-shaped sterigmata. The ascospores are yellowish brown, becoming black or smoke-colored, asymmetrical, more or less crescent shaped. They vary but little in size, the measurements of *Melanospora papillata* and *M. cervicula* averaging 10 × 25 μ while those of *M. anomala* are slightly larger, 12 × 28 μ. These variations, however, are so small that they could not alone be considered as specific. The size and shape of the ascopores also correspond quite closely with those of *Melanospora Gibelliana* and *Sphaeroderma bulbilliferum*. At maturity the ascospores appear as an irregular black mass in the center of the perithecium. As in all the species of Melanospora the asci are very evanescent. The walls become gelatinous and swell by the absorption of water, which increases the volume to such an extent that the mucilaginous mass protrudes from the ostiole, carrying out with it the embedded spores. If the atmosphere is somewhat humid, this mass of spores, as they are forced out, aggregate in a spherical mass at the mouth of the ostiole; but if the air is dry as they are pushed out, they adhere together into a long, twisted, tendril-like filament, something like the paint as it is squeezed out of an artist's paint-tube. These cirrose structures may measure from 10–18 mm. in length, and twist up into a variety of shapes. The spores not infrequently germinate while still in the cirrus, giving it a white appearance.

Microtome sections show no paraphyses between the asci, but from the walls there grow out more or less conspicuously into the cavity above the asci, numerous hyphal branches, as paraphyses, which converge radially and extend upwards towards the ostiole. These probably aid in the formation of the neck when it is present.

In general the culture methods used were the same for all. Gross cultures of the various substrata were made in crystallizing dishes which were half-filled with sphagnum and covered with white filter paper, on which the substratum was placed. The whole was then well supplied with water and covered with a piece of plain glass and set in a place in the laboratory where it would be protected from the direct sunlight. When bulbils were observed, individual ones were carefully picked out under a dissecting microscope and cultures made from them, until a pure culture was obtained. These were grown on various kinds of media until perithecia with the characteristic long cirri of ascospores, were obtained. Transfers of the ascospores were then made by touching one of the aerial cirri with a piece of nutrient agar on the end of a sterilized needle. In all cases pure cultures of ascopores obtained in this way produced bulbils.

The germination of the ascopores was followed in Van Tieghem cells until bulbils were again produced on the mycelium, thus demonstrating the connection between the ascospore and the bulbil.

In these forms the very young perithecium can be readily distinguished from the bulbil, not only by its mode of development when that is different, but also by the color. The bulbils turn brownish at a very early stage in their development, such as is represented, for example, in Figure 2, Plate 2, while on the other hand, the perithecia frequently remain colorless, or nearly so, until they are beyond the size of the average mature bulbil, and the ascogonium usually can be distinctly seen in the form of one or two large cells lying towards one side of the young perithecium.

The question of sexuality in connection with the formation of the ascogenous primordia has not been worked out. Structures have been observed that might well be taken for antheridial branches, but their attachment was not constantly or certainly observed, so that this phase of the problem will have to be left for future consideration.

Among the twenty bulbil cultures from different sources which have been found by the writer to produce melanosporous perithecia, at least three distinct species appear to be distinguishable. Although these forms possess ascospores that show little if any variation, the differences in their perithecia, bulbils and secondary spore-forms are such that they cannot be included in a single species. Moreover, the characteristics are believed to be sufficiently distinctive to warrant their consideration as separate species. They have therefore been named *Melanospora papillata, M. cervicula,* and *M. anomala.* There thus appear to be several closely related Melanospora-like forms, including *Sphaeroderma bulbilliferum, Melanospora Gibelliana* and *M. globosa* all of which give rise to bulbils.

The differences which distinguish the perithecia of these forms may be summarized as follows:

Melanospora Gibelliana; neck of perithecium long and tapering, with terminal setae, asymmetrical ascospores.

M. globosa; neck of perithecium longer than *M. Gibelliana,* no well-defined terminal setae, symmetrical ascospores.

M. papillata, n. sp.; perithecium with a distinct papilla only with terminal setae, asymmetrical ascospores.

M. cervicula, n. sp.; perithecium with a short neck, terminal and lateral setae, asymmetrical ascospores.

M. anomala, n. sp.; perithecium more or less definitely papillate, with occasional indications of abortive terminal setae, asymmetrical ascospores.

Sphaeroderma bulbilliferum; perithecia without papillae or setae.

The species of "Sphaeria "mentioned by Biffen as associated with *Acrospeira mirabilis* and the species of "Ceratostoma" connected with bulbils by Bainier may also be melanosporous and will be referred to later on.

Melanospora papillata n. sp.

Plate 2, Figures 1–26.

Perithecia scattered or gregarious, superficial, membranous, semi-translucent, straw-colored to light brown, globose to pyriform, 350–450 $\mu \times$ 400–500 μ, papilla surmounted by erect, somewhat divergent, continuous setae, 100–170 μ in length; primordium a group of one or more intercalary cells; ascospores asymmetrical, somewhat crescent-shaped 10×25 μ, yellowish brown becoming black; conidia abundant, hyaline, spherical to ovoid, on flask-shaped sterigmata; bulbils yellowish brown, irregular in outline, 50–60 μ in diameter, sometimes considerably more than this.

On Live Oak bark (*Quercus agrifolia* Née) from Pomona, California.

A pure culture of this species was easily obtained by making a transfer of the ascospores in the manner already described, on rich nutrients, fairly soft, with about 20 gm. of agar to the litre, and both perithecia and bulbils were produced abundantly. On substrata, however, poorly supplied with nutrient material, such as sterilized agar-agar, or even on a medium well supplied with food material if made very hard (about 40–50 gm. of agar to the litre) the bulbils are very sparingly produced if at all, the mycelium is quite inconspicuous and the perithecia appear scattered over the surface more or less abundantly. In its capacity to retain its power of producing perithecia this species resembles *M. cervicula*, while it is in sharp contrast to some other melanosporous forms studied in which, after long artificial cultivation the bulbils tend to become the dominant mode of reproduction and the perithecia are produced sparingly if at all.

The bulbils. The hyphae, which vary in diameter from 4–7 μ, are hyaline, with numerous oil globules and prominent cross walls, and are usually very scantily developed. The bulbils make their appearance as small straw-colored bodies scattered somewhat sparingly and usually in small patches over the surface of the substratum. In the process of development, which was carefully followed in Van Tieghem cells and in pure cultures in test-tubes, hyphae divide up into short intercalary almost isodiametric cells, one or more of which enlarge (Figure 1, Plate 2) while the contents becomes densely granular and filled with oil globules. At this stage these enlarged cells are colorless or opalescent with a comparatively thick wall and look much like chlamydospores. The adjacent cells of the filament on either side of them become stimulated and also enlarge to some extent, but remain colorless longer than the others, although they are eventually incorporated into the bulbil. The primordial cell or cells soon become brownish and produce others by gemmation, which in turn produce still others (Figures 2–5, Plate **2**), so that the mature bulbil finally consists of one or two, occasionally more, large central cells with slightly thickened walls, surrounded by a number of smaller less highly colored ones, with thinner walls. The mature bulbils measure from 50–60 μ in diameter, although they may vary considerably.

Sometimes three or four intercalary cells enlarge and take part in this process, producing an elongated, somewhat irregular bulbil, while at other times there are as many as eight or ten such cells; but in this latter case they seldom go farther than the production of a few lateral cells which soon become empty and colorless, as is shown in Figure 7, Plate **2**.

Not infrequently the terminal cell or series of terminal cells becomes the primordium (Figures 24–25), the further development of which is the same as the one already described. In Van Tieghem cell-cultures, bulbils are sometimes produced with more central cells than ordinarily occur in tube-cultures, and these, which are usually spherical, contain oil globules which give them a peculiar, somewhat opalescent appearance. The cortical cells in such cases are somewhat flattened, as indicated in Figure 22, Plate **2**, a condition which may be due to the pressure exerted by the increased number of the central cells.

The perithecium.— The form of the primordium of the perithecium is essentially the same as that of the bulbil but the former, as has already been stated, can, even in the early stages of its formation, be readily distinguished from the latter by the fact that it is colorless. It can be distinguished also from the primordium of the perithecium of *M. cervicula,* which in many respects it resembles, by the fact that the latter turns brownish at a much earlier stage in its development, producing a large number of radiating hyphae, so that its outline is soon indistinguishable.

Usually one, rarely two, intercalary cells take part in its formation, and from these, two or three large cells are produced laterally by budding (Figure 8, Plate **2**). From the intercalary cells, or, more frequently, from the adjacent ones of the hypha, branches are sent up which eventually enclose this group of large cells. These branches which divide up into short cells, form the wall of the perithecium.

Sometimes, as in the case of the bulbil, a terminal cell may become the primordium, as is evidently the case in Figure 10, Plate **2**, where there are two large cells which have originated from a terminal one.

The mature perithecium is straw-colored, globose or slightly pyriform, measuring 400–500 μ in diameter, but often much smaller than this, the variations in size are largely due to the character of the medium on which it grows. It is surmounted by a crown of setae which surround the ostiole and are colorless, 100–170 μ in length, stiff, erect, straight, and tapering to a point. There are no lateral setae of this nature, but frequently superficial cells near the base of the perithecium may send out filaments which serve as attachments to the substratum. The perithecia often occur grouped in considerable numbers and not infrequently two or three are found which have more or less fused during their development, having no doubt arisen from primordia which were in close contact with each other. Some time after their formation the cirri of ascospores begin to assume a whitish appearance which is due to the presence of numerous germinating spores producing many abnormalities. A very common form in such cases is shown in Figure 14, Plate **2** where, instead of a regular germ tube, a large opalescent, spherical body is formed at the end of the spore, which contains a great deal of granular material and stains deeply. Occasionally a second such body is produced, and from these one or more lateral branches may arise (Figures 18–20, Plate **2**). Not infrequently a series of these swollen cells appears terminating a branch and these become spherical and form a bulbil-like structure (Figure 17) such as is sometimes met with in Van Tieghem cell cultures (Figure 21). One of the most striking features of these germinations is the copious formation on the germ tubes of ovoid conidia which arise from bottle-shaped sterigmata and usually adhere in short chains, although they sometimes cohere at the tips of the sterigmata in a spherical mass. As already mentioned conidia similar to these are also quite frequently met with on the mycelium in all parts of the culture, and when the spores collect in masses the fructification might readily be mistaken for that of Hyalopus.

In some cases the outer cells of the bulbils increase in numbers until the whole structure is about half the size of a perithecium, although very irregular and sclerotium-like. In each case, however, the cells of the original bulbil retain their deep tan-color, while those which have resulted from this secondary growth are distinguished by light colored walls resembling those of the typical perithecium. The occurrence of such abnormal forms, which may be quite frequently produced on media rich in nutriment such as bran-agar for example, and their resemblance to young perithecia, suggested the possibility of a direct development of perithecia from bulbils similar to that suggested by Bainier ('07), and an effort was accordingly made to determine this point. Individual bulbils showing this tendency were isolated and their further development watched in Van Tieghem cells, while others were transferred to different kinds of media, moist cotton, moist filter paper, etc., but in no instance could they be induced to develop into perithecia, although when the moisture was sufficient, they produced numerous germ tubes which grew out forming the typical mycelium.

Melanospora cervicula, n. sp.

Plate 3, Figures 16–24.

Perithecia scattered or gregarious, superficial, membranous, semitransparent, straw-colored to brownish, globose to pyriform, 350–450 × 450–550 μ, with a definite neck 85–140μ in length, terminal setae 100–170 μ in length, erect, somewhat divergent, continuous, sharp, subulate; lateral setae on the neck and upper part of the perithecium; ascospores asymmetrical, somewhat crescent-shaped 10 × 25 μ, yellowish brown becoming black; conidia common in tube cultures, hyaline, spherical to ovoid, on flask-shaped sterigmata; bulbils yellowish brown, irregular, normally 50–60 μ in diameter, sometimes 100 μ, primordium one or more intercalary cells. This form is also said to produce conidia on secondary "Harzia-like" heads, and chlamydospores resembling those of *Acremoniella atra*.

On rabbit dung, Cambridge, Mass.

This melanosporous fungus was obtained from Dr. Thaxter who had grown it for some time as a pure culture. It was originally found on a gross culture of rabbit dung from the vicinity of Cambridge, Mass., and has proved to be of special interest on account of its different methods of reproduction.

In addition to perithecia and bulbils, this fungus seems to have associated with it two other spore forms, chlamydospores resembling those of *Acremoniella atra* and also conidia produced on secondary heads resembling those of the genus Harzia. Alcoholic material furnished by Dr. Thaxter was used for the study of these two modes of reproduction. This material was the result of a series of transfers of the cirri of ascospores and therefore probably pure. The writer has under cultivation transfers of this same fungus but although it has been grown on various kinds of media, both very rich and very poor in nutrient material, and hard and soft, etc., yet thus far he has not succeeded in obtaining either the chlamydospores or the "Harzia-like" fructification. This is probably due to the fact that the production of these structures is secured under certain peculiar conditions not readily controlled.

In general this fungus resembles *M. papillata* in form and habit of growth. The predominant type of reproduction in both is by ascospores the production of bulbils being scanty, while in some cases, as on attenuated agar cultures, they are not produced at all. The perithecium of *M. cervicula* which is usually 400–500 μ in diameter, has a definite neck 85–140 μ in length, while *M. papillata* which is slightly smaller, seldom reaching 500 μ in diameter, has no neck but often a papilla-like structure from which the setae arise. Moreover, the former probably produces conidiophores of the "Harzia type" and also chlamydospores which resemble those of *Acremoniella atra.*

The Bulbils.— The mycelium is colorless, procumbent or only slightly aerial, growing evenly over the surface of the substratum. The hyphae, which are copiously septate, measure 5–7 μ in diameter, but often large swellings occur in them which seem to act as storage organs and from which several branches may grow out as shown in Figure 21, Plate 3. These are found not infrequently on attenuated artificial media such as agar alone without any nutriment, on which the mycelium is very scanty, being barely visible even with the aid of a hand lens. On such media, it should also be noted that as in *M. papillata,* bulbils are not produced. It further resembles the latter in the mode of development of the bulbils, the primordium consisting of a group of intercalary cells. It is, however, subject to considerably greater variation and many irregular, incomplete or imperfect forms appear. Since the mode of development is essentially the same as that described for *M. papillata,* it will be unnecessary to repeat the description here. They are, however, produced very sparingly on most media, and on some, such as that just mentioned, do not occur at all, although on a rich substratum not too hard, such as sugar, chestnut or bran agar they are produced quite abundantly.

The perithecium.— In general the perithecium resembles that of *M. papillata,* but is clearly distinguished by having a definite neck. They, however, vary considerably in size, sometimes reaching 550 μ in diameter, their form often being somewhat contorted, with only a slight difference in size between the neck and body, while at other times several may be grown together. The neck is short, 85–140 μ in length surmounted by a group of terminal setae of about 100–170 μ in length. The mode of development of the perithecia is somewhat variable. Although at times they seem to be produced from intercalary cells, yet more frequently a short lateral branch is produced which may form a close coil of one or two turns, and occasionally even a definite spiral is found as is shown in Figure 19, Plate 3. The young perithecia turn brownish at a much earlier stage of their development than either those of *M. papillata* or *M. anomala.* This fact, together with the large number of radiating hyphae that are produced from the initial cells, a condition not occurring in either species just mentioned, make it very difficult to follow the early development. When the perithecium is young before the neck is produced, filaments with thick brownish walls, apparently stiff and with prominent septa, are seen scattered sparingly over the surface and radiating from it. They are formed by the outgrowth of some of the peripheral cells, and as the perithecium becomes older, as has already been stated, their number increases and some grow down into the substratum and act as hold fasts.

The "Harzia-type" of reproduction.— This mode of reproduction which was studied from material preserved in alcohol appears in small tufts scattered over the surface of the substratum. Short lateral branches become swollen at the end after the fashion of Oedocephalum or Aspergillus, and from this head a number of flask-shaped sterigmata are produced, on the ends of which occur secondary heads crowded with hyaline conidia which are usually spherical and sessile but occasionally more or less ovoid and furnished with short stalks (Figure 24, Plate 3). The secondary heads seem to vary considerably in size, and being so completely covered with conidia it was difficult to determine at all times the exact relation of the different parts of this fructification. In several cases there appeared to be little or no swelling of the secondary head, but with the limited amount of material at hand this could not be determined with certainty. Occasionally the head instead of being spherical is somewhat elongated, and the bottle shaped stalks, on which the secondary heads are formed, are scattered along the margin of this as shown in Figure 23, Plate 3. This fungus also produces numerous spherical conidia on bottle-shaped sterigmata along the margin of the hyphae, similar to those described for the other melanosporous forms.

The chlamydospores.— On the preserved material already referred to, there were also found associated with the "Harzia-like" fructification, chlamydospores which are ovoid, smooth, brownish, thick-walled, and have the distal end rounded. They are produced usually on short lateral branches which taper towards the tips and may be continuous or septate. The mature spores are quite uniform in size, about 17 × 21 μ, although there were some that appeared to be mature, which were slightly smaller than this. These spores resemble both in color and form those of *Acremoniella atra* Sacc. There are certain other fungi that produce imperfect forms of the "Harzia" and "Acremoniella" type which will be further considered below in connection with *P. aspergilliformis.*

Melanospora anomala n. sp.

Plate 2, Figures 27–30; Plate 3, Figures 1–15.

Perithecia scattered or gregarious, superficial membranous, straw-colored or light brown, globose or subglobose, 250–350 μ × 350–450 μ, ostiole formed in connection with a definite but inconspicuous papilla without setae, primordium a spiral of 4 or 5 coils; ascospores asymmetrical, somewhat crescent-shaped 14 × 28 μ, yellowish brown becoming brownish black; conidia, hyaline, spherical to ovoid, on flask-shaped sterigmata: bulbils yellowish brown, variable in size 70–140 μ in diameter, sometimes elongated ones 180 μ in length, primordium a group of intercalary cells.

On Spanish chestnuts in laboratory culture.

Gross cultures of Spanish chestnuts, which were imported probably from Spain obtained by the writer in the Boston market, produced numerous brownish colored bulbils when cultivated in moist chambers. By using the general methods already described, separate bulbils were transferred to sterilized nutrient-agar tubes and, after a few transfers, were obtained pure.

The mycelium of this fungus is white and more or less aerial, varying according to the media in which it is grown. When grown on soft chestnut agar, it becomes quite flocculent, while on chestnut decoction it forms a more or less felted layer over the surface, assuming the brownish color of the liquid; but on potato agar its growth is rather scanty. The diameter of the hyphae varies from 2.5–7 μ.

The bulbils.— Scattered over the aerial hyphae and on the substratum are seen numerous small yellowish-brown bulbils, which, when examined microscopically, are found to vary considerably in size and outline, many of them nearly spherical, others somewhat elongated.

Usually there is no differentiation between the cortical and central cells, but in old bulbils several empty cells, which may or may not be colorless, are often found loosely attached to the periphery. The central cells are often larger than the more superficial ones, but this is not always true, since in many instances they are perfectly uniform throughout. These bulbils are usually developed from a lateral branch which divides up into short cells. These produce short secondary branches (Figures 27, 29, 30, Plate 2) which also divide up into short cells and may produce others by a process of gemmation. Sometimes the primordium consists of a group of intercalary cells (Figures 28, Plate 2, and Figure 14, Plate 3) which may produce other cells by budding in a manner somewhat similar to that of *M. papillata.* At maturity the bulbils are irregularly spherical, about 70–140 μ in diameter, but where several intercalary cells have taken part in its formation, the long axis frequently measures 180 μ. This bulbil may be distinguished from *M. papillata* or *M. cervicula* by the fact that the cells are usually homogeneous throughout, while in the latter two there is a more or less definite cortex. The margin is also often more irregular in the bulbil under consideration as is shown in Figure 15, Plate 3. In the immature bulbils which show this uneven outline more markedly than the mature ones do, there sometimes appear short branches of two or three seriate cells which extend beyond the others.

The perithecium.— In an effort to induce this fungus to produce the perfect form, it was grown on various kinds of media. Decoctions of potato, bran, corn meal, Spanish chestnuts, etc., were hardened with agar-agar, some hard, some soft, but nothing except variations in the size and development of the bulbil could be obtained. Finally, after removing the shells of some fresh, sound chestnuts, the kernels were sliced up and used for cultures. On this medium perithecia were produced in abundance. These are almost spherical in form and vary from 300–400 μ in diameter, no ostiole being developed until they are nearly mature, at which time a few cells about the opening form a definite, though inconspicuous papilla. Terminal setae are wholly absent, and only rarely do the superficial cells produce lateral filaments. Frequently, however, short projections are observed from some of the cells that compose the papilla, as if an attempt were being made to produce setae. The perithecia are light yellowish-brown in color, much lighter than that of the bulbils, and so translucent that the spores can be readily seen grouped together in a black mass in the center (Figure 12, Plate 3).

Development of the perithecium.— The primordium of the perithecium is quite different from that of the bulbil. In this case a short lateral branch coils up spirally, usually making about four or five turns, but in some cases as many as eight. Figures 1 to 8, Plate 3, represent successive stages in the development of the spiral. Usually the second and part of the the third turn become enlarged while branches are given off from the first or from the cells below it. These branches grow up around the spiral and often send secondary branches in between the swollen lower coils so that they are forced apart (Figures 7, Plate 3). The branches continue to grow until they have enveloped the whole spiral, which soon loses its characteristic form. It would appear that the upper portion of the spiral either becomes a disorganized mass of mucilaginous material or not infrequently seems to be pinched off and ejected during the formation of the wall of the young perithecium, as is shown in Figure 7, Plate 3. By the time the wall is completed all that can be recognized of the spiral are two or three large cells which come to lie free in a cavity usually towards one side of the perithecium and which stain deeply (Figures 9–10, Plate 3). Sometimes branches seem to come off from each of the coils, so that one finds the spiral with a number of very short lateral branches produced from its outer surface. Occasionally also the lateral branch that produces the spiral, while making its first coil, divides into short cells and sends off secondary branches from these, as shown in Figure 3, Plate 3. Whether either or both of these develop into perithecia or bulbils, or are to be regarded as abnormalities, could not be determined, since they were of rare occurrence.

Conidia on bottle-shaped sterigmata, similar to those produced by *M. papillata* also occur in this species (Figure 13, Plate 3). Germinating ascospores particularly, produce them abundantly in a dry atmosphere, but they are more sparingly developed on the mycelium.

This fungus resembles somewhat a form described by Berlese ('92) under the name of *Sphaeroderma bulbilliferum*, which is referred to below. The former has, however, a slightly smaller perithecium (300–400 μ in diameter) with a papilla about the ostiole, while the latter is 400–500 μ in diameter, and has no papilla, the ostiole being flush with the surface. The Sphaeroderma moreover is said to have connected with it large two-celled chlamydospores, which have not been found associated with *M. anomala* although the writer has repeatedly searched for them. Berlese does not describe the method of development of the bulbils, but states that "the sporeballs resemble those described by Mattirolo as belonging to *Melanospora Gibelliana*."

The bulbils of the latter are not unlike those of *M. anomala* in size, color and mode of development.

The species of "Sphaeria," referred to by Biffen ('02) in connection with *Acrospeira mirabilis*, also resembles somewhat *M. anomala*. It differs from the latter, however, in several important respects. The perithecium has no papilla about the ostiole, the ascospores are symmetrical and the primordium of the bulbil is a spiral.

Again the mode of development of the perithecium from a spiral primordium resembles somewhat that of *Melanospora stysanophora* described and figured by Mattirolo ('86). The mature perithecia however, are different, *M. stysanophora* having a distinct neck. The latter is also said to be associated with a Stysanus-like fructification.

Melanospora Gibelliana Mattirolo.

This species was found by Mattirolo on a gross culture of decayed chestnuts in moist sand, and besides melanosporous perithecia and bulbils it also produced chlamydospores and conidia on bottle-shaped sterigmata.

The perithecium, which develops from a spiral primordium, is somewhat pyriform with a long neck surmounted by terminal setae. The neck, however, is considerably longer than that described for *M. cervicula*. The ascospores are brownish-black and asymmetrical, somewhat similar to those described for the other melanosporous forms.

The bulbils are said to be nearly spherical, pale yellow to brownish-yellow, and often 100 μ in diameter, with a colorless cortical layer of cells resembling somewhat the appearance of *Papulospora coprophila*. In its development a short lateral branch divides and forms a number of short secondary branches which intertwine forming an irregular spherical body varying considerably in size.

This species also is said to have associated with it chlamydospores somewhat resembling Sepedonium, as well as conidia on bottle-shaped sterigmata.

Melanospora globosa Berl.

In the same article in which he describes *Sphaeroderma bulbilliferum* ('92) Berlese also describes *Melanospora globosa* which he found growing on small pieces of decaying wood and herbaceous material. The perithecium of this species is, as the name indicates, globose, 250–280 μ in diameter and 360–450 μ (rarely 500 μ) long. The neck is well developed, 110–200 μ in length. The ascospores differ from those, already described, in being symmetrical. The other forms have asymmetrical ascospores which are somewhat crescent-shaped.

Besides the perfect form this species is said to have: microconidia which resemble those of Acrostalagmus; chlamydospores that are of the type of *Acremoniella atra;* and bulbils which he considers of the same nature as similar structures described by Mattirolo. Berlese succeeded in obtaining bulbils on the mycelium produced from ascospores but he failed to find any perfectly developed.

SPHAERODERMA BULBILLIFERUM Berl.

This species which is described by Berlese ('92) was found growing on dead leaves of Vitis, Cissus, and Ampelopsis. It is said to have several kinds of reproductive bodies, such as ascospores, bulbils, conidia and chlamydospores.

The perithecium is globose or sub-globose, 400–500 μ in diameter, without any neck, setae or papilla. These characteristics distinguish it from any of the melanosporous forms already referred to. It resembles *M. anomala* but is slightly larger and has no papilla. The ascospores are brownish-black and asymmetrical.

The bulbils are yellowish, nearly spherical, 80–150 μ in diameter, consisting of polyhedral cells and surrounded by a layer of empty cortical cells. They are said to resemble quite closely those described in connection with *Melanospora Gibelliana.*

The conidia occur in chains on bottle-shaped sterigmata resembling those of the melanosporous forms already referred to.

The chlamydospores, which measure $32–40 \times 24–25\ \mu$, are described as yellow, oval, smooth, composed of two unequal cells, and formed terminally on the ends of short lateral branches.

"CERATOSTOMA" sp. indet.

Bainier ('07) has reported that he has determined the connection of a perithecium of the genus Ceratostoma with *Papulospora aspergilliformis.* He is of the opinion that the bulbils in this instance are immature perithecia and that, under proper conditions as regards nutriment and moisture, they may be induced to complete their development.

In this form, the bulbil is produced by a short lateral branch which coils up spirally, the coils becoming quite compact. One or more of the terminal cells enlarge and eventually become filled with conspicuous food material. The cells below the spiral send out branches which divide and may, in turn, produce others. These grow up around the spiral and completely envelop it, thus forming a somewhat spherical mass of cells. In a moist atmosphere these are said to develop into sclerotium-like bodies. By transferring these large bulbils to pieces of moist bread, Bainier succeeded in inducing them to develop into perithecia which he refers to the genus Ceratostoma, although it is not evident why this form should not also be referred to Melanospora. This subject will be further dealt with below under *Papulospora aspergilliformis*.

In connection with pyrenomycetous forms it will be well to consider briefly two additional species which may be regarded as doubtfully pyrenomycetous.

FORMS DOUBTFULLY REFERRED TO PYRENOMYCETES.

Papulospora candida Sacc., parasitic on Geoglossum, has been reported by Dr. Thaxter to be connected with hypocreaceous perithecia found on specimens of the host obtained in South Carolina; but this material was, unfortunately, not available for examination, and since pure cultures of this fungus grown on different media have thus far failed to produce any perfect form, its position must, for the present at least, remain more or less uncertain. The fact, however, that the bulbil is definitely connected with a Verticillium would seem to afford strong evidence of its hypocreaceous nature. A second doubtful form is *Acrospeira mirabilis* (Beck & Br.), with which Biffen ('02) has associated a species of "Sphaeria," but since he was unable to obtain the bulbils or "chlamydospores" as he terms them, of Acrospeira from pure cultures of the ascospores, his conclusions must be accepted with some reserve.

Papulospora candida Sacc.

Plate 4, Figures 1–47.

This fungus was first found by Ellis who collected it in New Jersey and distributed it by N. A. F. No. 3673. The species appears to be common and distributed from N. Carolina to Maine. The material for the present investigation was found growing abundantly as a parasite on *Geoglossum glabrum* in a maple Sphagnum swamp near Walnut Hill, Mass. It was first described (Mich. II, p. 576) as *Papulospora candida*, by Saccardo who also mentions that *Verti-*

cillium agaricinum Link, var. *clavisedum* (Mich. II, p. 577) is associated with it.

A large number of specimens of Geoglossum, with plenty of Sphagnum and leaf mould about each, were collected — some infected, others not — and were grown under bell jars or in a large germinating vessel with a glass top. It was thus kept growing for nearly two months, until it could be determined whether the Papulospora would grow as a saprophyte on artificial media. A number of tube cultures were made of the bulbils on various kinds of media, the most successful of which were the ascoma of Geoglossum itself. About a dozen large specimens of these with long stalks were selected and each put in a test-tube which had previously been supplied with about half an inch agar. These were then sterilized in an autoclave, the object of the agar being simply to hold the specimen in place and thus lessen the chances of contamination in making the transfers, etc. On this medium a pure culture was eventually obtained, which was then transferred to other media such as potato, corn meal, chestnut, horse dung, etc., hardened with agar. This fungus grows fairly well as a saprophyte, better on hard than on soft media such as potato and bran, but very slowly on horse dung, on which, after a month, it had not grown much more than an inch from the point of inoculation. Associated with the Papulospora on the ascoma were found, among other fungi, specimens of *Pleurage anserina* (Rabh) Kuntze and *Verticillium agaricinum* Link, the latter producing in pure cultures very large and conspicuous, brownish sclerotia.

On its natural host *Papulospora candida* forms conspicuous white blotches spreading over the upper portion of the ascoma (Figure 47, Plate 4), and if not too wet, extending down the stem. Although the host is usually found in damp sphagnum swamps, the parasite is largely confined to those specimens that grow tall, so that their tops are comparatively dry. The mycelium is white, procumbent, branching copiously, but soon becoming indistinguishable as such, even with a good hand lens, mainly on account of the large number of bulbils that are formed which give the whole fungus a powdery appearance. When examined under a microscope the mycelium is opalescent, owing to the presence of numerous oil globules (Figures 42, 44, Plate 4) and other colorless material in the cells. The cultures become completely covered with the white powdery bulbils which a little later assume a characteristic cream color.

The bulbils.— During the process of development of the bulbil a short lateral branch divides up into a number of cells and the end one enlarges and usually also the second or third (Figures 29–37, Plate 4). From these, other cells are then produced by budding, the lateral walls of which eventually adhere closely to those adjacent, so that there comes to be from two to six large central cells surrounded by a number of smaller ones, all filled with granular protoplasm, the only apparent difference being in their size. As they mature, however, the inner and outer cells become markedly differentiated. The former, which are large with conspicuously granular contents and with numerous oil globules, secrete a thick hyaline wall, while the latter, which become empty and spherical, adhere to each other loosely, their contents probably being absorbed by the central cells (Figure 41, Plate 4). Although the terminal cell is usually the most prominent in producing the larger central cells, yet one or both of the two adjacent cells may take the lead and, owing to their lateral growth a somewhat crosier-like coil may even occasionally be produced by one or more of these secondary branches.

Germination of the bulbils.— For the purpose of studying the germination, bulbils in different stages of development were placed in Van Tieghem cells. In about twelve or fifteen days the marginal cells of those that were immature — that is, those whose superficial cells still contained protoplasm — began to send out vegetative branches, one or two from each cell (Figure 42, Plate 4); but the central cells were not observed to produce tubes at this stage. After about a month the mature forms begin to germinate, but very sparingly, each of the large central cells usually sends out a single germ tube which readily pushes aside the loosely adhering peripheral cells. The germ tubes or vegetative hyphae, as the case may be, usually divide up into short cells which become swollen with the protoplasmic contents and more or less constricted at the partitions (Figure 42, Plate 4).

The conidia.— The erect septate conidiophores of the so-called *Verticillium agaricinum* (Link) Corda, var. *clavisedum* Sacc., already referred to, are invariably associated with the bulbils in pure cultures, and are thus shown to be not, as Saccardo supposed, accidentally concomitant but a regular phase of the life cycle. Figure 45, Plate 4, shows bulbils and the Verticillium fructification definitely connected on the same erect hypha. This phenomenon is of so frequent occurrence that there is no possibility of error. The conidiophores are simple or branched, with the sterigmata in whorls, varying greatly in number, commonly in threes and frequently clustered at the apex.

The mature conidia are ellipsoidal to oblong and rounded at both ends, varying considerably in size, the average measurements being

14 × 15 μ, although the length may vary from 12 to 15 μ. In this respect it differs from *V. agaricinum* in which the conidia are smaller and ovoid in shape. Both of these forms have been cultivated in pure cultures for some time and seem to be absolutely distinct, the one, *V. agaricinum*, producing ovoid conidia often clustered at the apex of the sterigmata as well as an abundance of large brownish sclerotia not associated with bulbils, while the other has oblong conidia, rounded at both ends, somewhat larger than the former, and on germination the mycelium invariably gives rise to bulbils, without any trace of the sclerotia.

The germination of the conidia of *P. candida* was carefully followed in Van Tieghem cells, using different kinds of nutrient media. In these cultures many interesting variations were observed, as is shown in Figures 1–12 and 15–27, Plate 4, all of which have the same magnification. Figures 1 and 2 show the variation in the size of the conidia. During the first twenty-four hours they enlarge by the absorption of water, becoming almost spherical (Figure 4), in which condition they are ready to germinate, the diameter at this stage varies from 12–18 μ. The germ tube, which may appear at one or both ends (Figures 7, 20) or from one or both sides of the conidium (Figures 6, 8), sometimes grows out to form a mycelium (Figure 10) on which bulbils and the conidial fructifications are produced; but more often, in Van Tiegham cells at least, it rounds up and forms another large cell. Several large cells may be produced in a similar way, which become almost spherical in shape and densely filled with granular protoplasm and oil globules, and from these acting as central cells, other smaller ones are formed laterally by budding, and in about sixteen days a bulbil consisting of two to six large central cells surrounded by a layer of smaller ones, all containing protoplasm, results.

Acrospeira mirabilis (Berk. and Br.).

Plate 5, Figures 18–23.

Acrospeira mirabilis (Berk. and Br.) appeared on a gross culture of Spanish chestnuts obtained from the Boston market. It was from this same material that *Melanospora anomala* was obtained but from other gross cultures. The former was first described by Berkeley and Broome in 1861, a more detailed account being given by Berkeley in his "Introduction to Cryptogamic Botany." Massee ('03) refers to it as a very destructive parasite doing a great deal of damage to chestnuts in Spain, but states that "nothing as to the life history of the parasite is known." Before Biffen ('02) examined this species, the only method of reproduction known was by its so-called "chlamydospores" which at maturity consist usually of one large, thick-walled, chocolate-brown, warty cell and three or more colorless cells adhering closely to it. By the use of pure cultures Biffen claims to have succeeded in obtaining not only the "chlamydospores," as described by Berkeley and Broome, but also what he calls "spore balls" (bulbils) and definite perithecia.

The mycelium of Acrospeira is fine, colorless, procumbent, more or less sparingly developed, and produces large numbers of reproductive bodies, which, in their development and structure, are bulbils rather than "chlamydospores." They are so abundant that the whole surface of a culture, which would otherwise be white, assumes a brownish aspect. The readiness with which these bulbils are produced makes it comparatively easy to trace their development, which, in brief, is as follows: an erect lateral branch usually divides into three secondary branches (Figure 18, Plate 5) each of which coils up much like that of *Papulospora parasitica*, to be considered below. They make about one to one-and-a-half coils and divide into three cells by cross septa. The middle one of these three, as a rule, enlarges rapidly, forming the functional spore (Figure 21, Plate 5) (the central cell of *P. parasitica*), but occasionally the end cell (Figure 20, Plate 5) more rarely the third, is the one that functions in this respect; while the other cells of the coil, ordinarily three or more in number, grow less rapidly and eventually lose their contents, become colorless, and adhere to the side of the large cell. If the marginal cells should increase in number so as to enclose the large cell completely, there would be practically the same condition as exists in *P. parasitica* (Figures 16, 17, Plate 5). In the present form, however, the large cell becomes dark brown in color and develops a thick wall, which eventually becomes warty, and measures 25–30 μ in diameter. Figures 18–23, Plate 5, illustrate the stages in the development of this bulbil. Thus in Acrospeira we have a structure that is only slightly less complex than that seen in *P. parasitica*, a form in which many imperfect bulbils can with difficulty be distinguished from some of those of Acrospeira, their only difference being due to the absence of a warty episporium. These bulbils were grown on various kinds of sterilized nutrient material, and most of the experiments described by Biffen were repeated. The culture conditions were varied with regard to media and other conditions of growth, in many of these experiments, but more bulbils of the same kind were always produced and never, so far as the writer has observed, have any indications been seen of the development of "spore balls," or perithecia such as have been described by Biffen.

BASIDIOMYCETOUS FORMS.

As has already been mentioned (p. 238), bulbils were first reported among the Basidiomycetes by Lyman ('07), who not only definitely connected one form with *Corticium alutaceum* (Schrader) Bresadola, which is dealt with briefly below, but also refers to two other kinds of bulbils, the mycelia of which have well marked clamp-connections; but basidiosporic fructifications were not produced abundantly enough to allow of their identification. Dr. Lyman has kindly supplied the writer with specimens of these forms for the purpose of comparison, which will be referred to under their respective species.

The methods used here were much the same as those already described, except that more gross cultures of wood were used with different amounts of moisture. The best results were obtained from decoctions of bran in one or two litre Erlenmeyer flasks with pieces of rotten wood that extended considerably above the liquid, so that the mycelium could obtain the degree of moisture that best suited it. In order to keep the pieces of wood in place and thus lessen the chances of contamination a quantity of agar was sometimes put in the bottom of the flasks.

GRANDINIA CRUSTOSA (Pers.) Fr.

PLATE 6, FIGURES 1–10.

Bulbils of this species were obtained from at least ten different sources, mostly on substrata such as rotten chips of Live Oak (*Quercus agrifolia* Née), old canvas, paper, cardboard, etc., from Claremont, California. It has been found also by Dr. Thaxter on gross cultures of rabbit dung from Mass. and on rotten wood from Buenos Ayres, and is probably the same as that referred to by Lyman ('07, p. 166), which was obtained by Mr. A. H. Chivers on a gross culture of bits of wood, paper, etc.

The mycelium, which shows quite marked clamp-connections, is colorless, procumbent, producing numerous white fibrous, rope-like strands of hyphae which radiate conspicuously in all directions from the point of inoculation. The white mycelium, however, soon takes on a light straw-colored aspect, owing to the formation of bulbils in large numbers, which gradually become darker as they mature. When grown on nutrient agar in large receptacles like Erlenmeyer flasks, after the mycelium has covered the whole substratum with powdery bulbils, new centers of growth-activity occur at different points on the surface of the culture, and the radiate development of the hyphae and the subsequent formation of bulbils are repeated on the top of those first formed. If the flasks have plenty of nutrient and do not dry up, this process may be repeated two or three times, the amount of mycelium, and consequently the number of bulbils formed, decreasing each time, so that eventually there appears a thick powdery mass with here and there large, white, rope-like strands of hyphae persisting, which is all that can be distinguished of the mycelium.

The bulbils are usually more or less spherical in shape, varying from 52 to 88 μ in diameter, although often exceeding this size, especially when the primordia of two happen to be so close together that their hyphae intertwine, thus forming a large irregular body. The individual cells are large, densely filled with granular material and oil globules, spherical at first; but the central ones soon become angular by pressure, while the marginal ones retain more or less their original form. There is no differentiation of a cortical layer; the cell wall and contents are uniform throughout, except that occasionally some of the peripheral cells which project beyond the others lose their contents, but this is the exception and is probably due to age.

The bulbils.— The hyphae which take part in the formation of the bulbils become enlarged, conspicuous, and more or less contorted on account of the prominence and swollen nature of the clamp-connections, which often occur at short intervals. The lateral branches from these divide up into short cells, so that there comes to be a number of almost spherical hyaline cells with fairly thick walls and filled with granular material and oil globules (Figures 4–9, Plate 6). During the formation of new cells, which are also spherical in shape and produced by budding from the marginal ones, the central cells gradually lose their original form and become angular, as a result of the lateral pressure or resistance offered by the outer cells. When the bulbils are nearly mature, they assume a light straw or "rusty-cinnamon" color. Figure 10, Plate 6, represents a mature bulbil, drawn on the same scale as the other mature forms. This method of development follows very closely that described by Lyman ('07) in connection with *Corticium alutaceum,* considered briefly below.

Formation of basidiospores.— The basidiosporic fructification of

Grandinia has been produced on gross wood cultures of this bulbil and also on test-tube cultures of bran-agar of about 40 gm. of agar to the litre, by three or four of the ten cultures from different sources under cultivation. Preparatory to its formation, the mycelium ceases to produce bulbils and forms a sort of incrustation, chalk-white in color and becoming pustulate by the time the spores are formed, Figure 1, Plate 6. The pustules on examination are found to be made up of more or less thickly interwoven branching hyphae, which have become enlarged and densely filled with granular material and oil globules, the ultimate ramifications of which form the hymenium (Figure 2, Plate 6). The basidia, which form a somewhat loose hymenium, each produce four spores, which are ellipsoidal to oblong in shape, measuring about $4 \times 8\,\mu$. These spores were germinated in Van Tieghem cells and the growth of the mycelium followed until the formation of new bulbils, which were transferred to nutrient agar media, where they produced mycelia and bulbils like the original culture.

On tube cultures this fungus occasionally produces typical sclerotia, which are formed by the massing together of many hyphal branches which remain colorless for some time and thus are easily distinguished from the bulbils. Moreover, they are larger, 400–500 μ in diameter, irregular in shape, somewhat darker in color at maturity, and composed of smaller, compact cells.

Grandinia also produces conidia of the Oidium-type on slender clampless conidiophores, such as are described by Lyman ('07) for *Corticium alutaceum*.

Corticum alutaceum (Schrader) Bresadola.

The bulbils of this species were obtained from Dr. Farlow, who found them on a piece of rotten oak bark collected at Chocorua, N. H. It was comparatively easy to get a pure culture, as the bulbils are produced in large numbers and germinate readily. This form has been carefully compared with specimens of *Corticium alutaceum* obtained from Dr. Lyman and they proved to be the same. The development of the bulbil and the character of the conidia are practically identical with those described for Grandinia and, as these have been well worked out in pure cultures by Lyman ('07), it is not necessary to repeat the results here, a detailed description of which may be obtained by consulting his article, pp. 160 and 196. The mode of development of the bulbils and the character of the conidia, however, have been carefully verified. Lyman obtained his cultures from the basidiospores collected on old rotten oak logs in the field and pure cultures from these produced bulbils. The writer began his cultures with bulbils, also collected in the field, and, after a great number of unsuccessful attempts, finally succeeded in obtaining a basidiosporic fructification similar to that described by Lyman. This was accomplished by using gross cultures of partly decayed wood in two litre Erlenmeyer flasks with sufficient agar to hold them in place. The mycelium, as usual, produced bulbils profusely on the agar and wood, but after six or eight weeks near the top of the pieces of wood conspicuous patches of white mycelium appeared, which eventually produced the hymenium and basidiospores of *C. alutaceum*.

Papulospora anomala n. sp.

Plate 6, Figures 11–19.

This form, which was obtained from four different localities,— three from the vicinity of Claremont, California, found on Live Oak chips, and one on an old paper from Cambridge, Mass.,— has been grown on a variety of substrata in the hope that it would produce its perfect form, but thus far all these efforts have failed. That it belongs to the Basidiomycetes is shown by its clamp-connections, which, however, are not so prominent as those in the two preceding forms, from which it is further distinguished by the dark brown, opaque, almost black color of the bulbils, the compact nature of their cells, and their mode of development. The mycelium is white, procumbent, scanty, slightly aerial on some substrata, with a large number of conspicuous oil globules, and not infrequently contains swollen intercalary cells, which are also densely filled with food material and probably act as storage organs.

The bulbils.— The primary hyphae are small, seldom more than 3 μ in diameter, and do not produce bulbils; but scattered over the secondary hyphae, which vary greatly in width, often reaching 10 μ and under some abnormal conditions 14 μ, are seen slightly swollen, colorless, intercalary cells, quite different from those mentioned above, about 4 or 5 μ in diameter, sometimes projecting considerably and resembling short stunted branches; at other times the base of a short lateral hypha swells slightly and forms the primordium (Figure 12, Plate 6). From the primordial cell or cells branches are sent out in different directions, the basal cells of which become spherical and in turn may produce other similar branches (Figures 13–15, Plate 6).

The lateral walls of these basal cells adhere firmly to each other and the cells become incorporated into the bulbil.

Figures 11–15, Plate 6, illustrate the early stages in the development, and Figures 14 and 15 show the formation of the spherical cells at the center, around the initial cell or cells, while Figure 16 represents a little later stage, which is composed of small hyaline cells with very indistinct walls and forming almost a spherical body with few, if any, cells projecting beyond the others. About this stage, or usually a little later, it would appear that the bulbils cease to form new cells, or, if any, very few, and that the further increase in its size is chiefly due to the enlargement of the individual cells which compose it and which, up to this period, have been small, hyaline, with indistinct walls. As these cells enlarge, there is quite a strong lateral pressure exerted, which tends to make the walls angular, which in the meantime have become more prominent and gradually assumed a brownish tint, that later becomes a dark brown, almost black. As a result of this mode of development, the bulbil at maturity has a clear-cut, even margin, without any appendages or sharp projections, nearly spherical in form, except where some cells in the process of enlargement increased faster than others or in cases where two primordia were formed close together and their early branches became intertwined, forming an elongated, compound structure. The color, which becomes so deep that even the cell walls cannot be distinguished, may be bleached out by placing them in potassium hydroxide for a few hours. The mature bulbils (Figure 17, Plate 6) vary in size, usually measuring from 125 to 175 μ in diameter, although occasionally some are even larger.

Bulbil "No. 200."

This form was obtained from Dr. G. R. Lyman and was originally found by Dr. G. P. Clinton in the vicinity of Cambridge, Massachusetts, on a fragment of an old newspaper in a field. In general this species resembles Grandinia in the mode of development of the bulbils, the presence of conidia and the clamp-connections of the hyphae. The bulbils, however, are much darker and the mycelium does not form the white, fibrous, radiating strands that are so characteristic of Grandinia.

On gross cultures, especially of wood or horse dung agar, the hyphae mass together in conspicuous papilla-like elevations, which are much more prominent than the fructification of Grandinia. These elevations are composed of closely compacted basidia-like structures. Unfortunately thus far the writer has observed only a few scattered basidia with basidiospores so that it has been impossible to obtain a specific determination.

BULBILS NOT YET CONNECTED WITH A PERFECT FORM AND INCLUDED IN THE FORM-GENUS PAPULOSPORA.

Key to the Species of Papulospora.

I. Primordium intercalary.
  A. Bulbils black. . . . . *P. pannosa* n. sp.
  B. Bulbils yellowish to dark brown.
    1. Bulbils, brownish-yellow, central cells 28–55 μ in diameter. *P. immersa* n. sp.
    2. Bulbils straw-color, central cells 10–20 μ in diameter. *P. irregularis* n. sp.
    3. Bulbils dark brown, hyphae with clamp-connections. *P. anomala* n. sp.
II. Primordium one or more lateral branches.
  A. Primordium normally a single lateral branch.
    1. Primordium a spiral.
      a. Cells of bulbil heterogenous, definite cortex.
        i. One central cell.
          α. Cortex complete. . . . . *P. parasitica.*
          β. " incomplete. . . . . *Acrospeira mirabilis.*
        ii. More than one central cell.
          α. Spiral in one plane, cortical cells spinulose. *P. spinulosa* n. sp.
          β. Spiral in more than one plane, 2–6 central cells.
            (a) Bulbils a dark brown. . . . . *P. coprophila.*
            (b) " brick red. . . . . *P. rubida* n. sp.
      b. Cells of bulbil homogenous.
        i. Bulbils brown 21–36 μ in diam. . . *P. sporotrichoides* n. sp.
        ii. " steel gray 21–36 μ in diam. . . . *P. cinerea* n. sp.
    2. Primordium not a spiral.
      a. Bulbils large, 100–750 μ in diam. . . . . *P. aspergilliformis.*
      b. " 30–35 μ in diam. cream color. . . . . *P. candida.*
  B. Primordium two or more lateral branches forming a spherical aggregation of cells at the top. . . . . *P. polyspora* n. sp.

Heretofore fungi producing bulbils have been referred chiefly to the form-genera Papulospora and Helicosporangium, but the characters on which these two have been based are not clearly defined, and as already stated, it does not seem desirable to recognize more than one form-genus. Since Papulospora was the name first employed to represent bodies of this nature, all the fungi that the writer has examined that produce bulbils, the perfect form of which has not been determined, are placed in this form-genus which may be described as follows.

Papulospora.

Mycelium extensive or scanty, flocculent or procumbent, usually white but sometimes dark colored. Reproduction by means of bulbils, i. e., reproductive bodies of more or less definite form, composed of a compact mass of homogeneous or heterogeneous cells which may be few or many, but are always developed from primordia of more than one cell. Other modes of reproduction may be present.

For convenience bulbils may be grouped under three heads: those which form an intercalary primordium of several cells; those which typically originate from a primary spiral; and those that are produced by a perpendicular branch or branches which do not form a spiral.

As has already been pointed out the distinction between simple bulbils and compound spores on the one hand, and the more complex bulbils and sclerotia on the other, is not always definite, and in certain instances it is difficult to determine to which category a given structure belongs. *Compound spores* are reproductive bodies of more than one cell, having a more or less definite form, and are usually the result of a successive or simultaneous division of a single cell. On the other hand, *sclerotia* are compact bodies capable of reproducing the plant and formed rather by the massing together of vegetative filaments, forming a pseudoparenchymatous tissue, but not developed from a group of more or less definitely related cells. Moreover, the individual cells of a sclerotium are not at all spore-like or independent of each other. *Bulbils,* are reproductive bodies, more or less definite in form and mode of development, and normally derived from primordia of more than one cell, rather than the result of successive or simultaneous divisions of a single cell, and their individual cells are more or less independent and spore-like.

Papulospora immersa n. sp.

Plate 10, Figures 17–25.

Mycelium white, septate, scanty, procumbent, growing in or on the substratum; bulbils, light brownish-yellow, irregular, 88–150 μ in diameter, but very variable, sometimes the long axis exceeding 260 μ, often immersed; central cells large 28–55 μ in diameter, angular, with conspicuous oil globules; 50–70 cells in surface view, but in irregular forms 100 cells, no differentiation of internal and external cells. No other mode of reproduction at present known.

On horse and dog dung from Cambridge, Massachusetts, and rabbit dung from Innerkip, Ontario.

Both the bulbils and the mycelium usually grow more or less below the surface of the substratum. The former are often found immersed more than a centimeter. It is easily distinguished from *P. polyspora* by its mode of development and from *P. pannosa* by its color, the latter being black. It resembles most nearly *P. irregularis*, from which it may be distinguished by its darker color, the size and conspicuous contents of the cells of the bulbils and the fact that the latter become more or less imbedded in the substratum.

The mycelium, since it is formed largely in the substratum, is inconspicuous in tube-cultures and is composed of large swollen hyaline cells, densely filled with oil globules and often much contorted (Figure 17, Plate **10**). In older cultures the cells lose their contents.

This fungus was grown on different kinds of media, but could not be induced to develop any other mode of reproduction. It grows well on bran and horse dung agar, the bulbils often becoming very large and numerous just below the surface of the substratum, forming almost a continuous layer, and often producing a more or less hard crust. In contrasts of mycelia in plate cultures, a marked heaping of the hyphae occurs where the two mycelia come together, and the bulbils seem to be somewhat larger, and more irregular in this region, but no other marked difference was observed.

The bulbils.— The primordium of the bulbil consists of one or more intercalary cells which become much enlarged. For example, Figure 17, Plate **10**, a later stage of which is seen in Figure 23, shows several such cells, all of which would have taken part in the formation of a somewhat elongated irregular bulbil, such as is shown in Figure 23. On the other hand, Figure 18 represents a primordium which consists of a single cell, and Figures 19–22 are further stages in its development. In the latter case a more or less spherical bulbil is the result (110–148 μ in diameter), while in the former it is more irregular, often exceeding 260 μ through the long axis. The method of enlargement, however, is exactly alike in both cases, that is, short lateral branches are produced from the bases of which are cut off a series of short cells which enlarge, becoming spherical at first and later, as the bulbil increases in size and the cells are subjected to lateral pressure, forming a compact angular mass in the center. Occasionally the branches are replaced by cells which, arising as lateral buds, become spherical and in turn give rise to other buds, the lateral walls of which adhere closely and ultimately form a more or less spherical or elongated bulbil with a fairly even margin, the central cells of which soon become angular. In either case all the cells are filled with conspicuous oil globules. At maturity there is no differentiation of central and cortical cells, but all are uniformly filled with food material, the central ones being larger, 28–35 μ in diameter, and more angular than those nearer the periphery.

Papulospora pannosa n. sp.

Plate 6, Figures 20–25; Plate 8, Figures 28–31; Plate 9, Figures 18–20.

Mycelium white at first, becoming dark smoke-colored, 8–10 μ in diameter, somewhat shaggy; bulbils black, irregular, variable in size and outline, sometimes 350 μ in diameter, but usually considerably less; cells homogeneous throughout, 200–300 cells in surface view; primordium, a group of intercalary or terminal cells. No conidia observed.

On laboratory cultures of rabbit and goat dung, and on corn-cobs from Claremont, California.

Pure cultures of this fungus from about fifteen different sources were obtained and grown on various kinds of media and the mycelium from the different sources contrasted with each other, but thus far it has not developed any other mode of reproduction than the bulbils. This species is easily distinguished from most of the others by the color of its bulbils. The only other black form is that of *Cubonia bulbifera* from which it differs in size and the character of its outline, which is quite even and regular in the latter, as well by the fact that the hyphae are black at maturity.

The bulbils.— The mycelium which grows well on a variety of media in tube-cultures, appears somewhat shaggy, is white at first, gradually becoming dark smoke-colored, with prominent cross walls which remain rigid when the cells collapse (Figure 31, Plate 8). The hyphae which are 3–4 μ in diameter when young and hyaline, gradually increase in size until they are 8–10 μ in diameter, and have already become dark in color at the time the black bulbils are produced. During the formation of the latter, the hyphae become much distorted, and divide into a series of short, somewhat inflated cells which are separated by constriction at the septa (Figure 24, Plate 6), somewhat after the fashion of *Cubonia bulbifera*, but the successive cells of these series are much more irregular and of greater diameter. These enlarged cells send out lateral branches (Figure 18, Plate 9), from which are cut off short basal cells which assume a spherical form, become swollen and may produce other branches similar to the primary ones. This mode of development is illustrated by Figures 20–24, Plate 6, and Figures 18–19, Plate 9. Instead of the enlarged cells producing branches, however, other cells may arise laterally from them by gemmation, become spherical, and may in turn give rise to others in a similar fashion. In either case the lateral walls of adjacent cells eventually adhere firmly, thus forming a compact group, each cell of which is almost spherical at first, but later becomes irregular. The further multiplication of the peripheral cells is subject to considerable variations. Not infrequently the primary or secondary branches, owing to local variation, grow much faster than others and thus produce more cells in that region of the bulbil. If there are several of these points of special activity, the mature bulbils may be quite irregular in outline. Occasionally a bulbil is formed from a single lateral branch (Figures 28–30, Plate 8), new cells being formed by a process of budding or by short branches as in the other cases. Ordinarily, at maturity, they are more or less spherical or somewhat elongated, their margins roughened by projecting cells (Figure 20, Plate 9) and are very variable in size, sometimes as large as 350 μ in diameter. There is no differentiation between the internal and external cells as far as contents are concerned. The central cells are, however, as a rule, larger and more angular.

Papulospora irregularis n. sp.

PLATE 9, FIGURES 11–17.

Mycelium white, more or less procumbent; bulbils hyaline, becoming light straw-color, somewhat spherical (140–170 μ in diam.) to irregular in outline (250–300 μ in diam.), margin very uneven; primordium a group of intercalary cells.

On rat dung, Kittery Point, Maine.

A pure culture of this species was comparatively easy to obtain. In the hyphae, which are hyaline, procumbent and inconspicuous, certain intercalary cells become enlarged and, by a process of budding, these give rise to other cells which in turn may produce still others. Sometimes short lateral branches are produced, the basal cells of which enlarge and take part in the formation of the bulbil (Figure 15, Plate 9). The young bulbils are colorless, covering the substratum, but in older cultures they turn light straw-color. They are usually somewhat spherical in form, measuring 140–170 μ in diameter, but frequently run into irregular sclerotium-like bodies, 250–300 μ in diameter. In old cultures the hyphae often form a felted mass over the substratum. This mode of development is similar to that of *P. pannosa*, from which, however, it is easily distinguished by the color of the mycelium and bulbils, those of the latter species being black. It also resembles *P. immersa*, but it is lighter in color and does not have such large cells with conspicuous oil globules and the bulbils are not immersed in the substratum. Figures 11–17, Plate 9, illustrate the mode of development of this bulbil.

Papulospora spinulosa, n. sp.

Plate 9, Figures 1–10.

Mycelium white, scanty, septate, procumbent, becoming slightly brownish when old, 3.5 μ in diameter, the old hyphae somewhat larger; bulbils hyaline until well developed, at maturity light chocolate-brown, somewhat spherical, 55–88 μ in diameter, 50–60 cells in surface view; primordium a coiled lateral branch which remains prominent throughout the development, becoming empty and showing slight thickenings in the walls. No other means of reproduction known.

On rat dung, Kittery Point, Maine.

This fungus was found on a gross culture of rat dung obtained from Kittery Point, Maine, and has been grown for about three years on various media without producing any reproductive body other than bulbils. The mycelium is white and grows quite sparingly on most media. It has been found that bran agar or rat dung agar is the best nutriment on which this species will grow.

The bulbils.— During their early stages of development the bulbils are hyaline until they are about half grown, at which time they begin to turn a light brown and at maturity assume a chocolate-brown color, often covering the whole substratum with several layers, so that all appearance of hyphae is lost sight of, except around the margin where a white zone about 5 mm. in width indicates the actively growing region of the mycelium and the formation of new bulbils. In the process of development a short lateral branch coils up, usually crosier fashion (Figures 1–4, Plate 9), although occasionally the tip somewhat overlaps, as shown in Figure 3, Plate 9. The primary loop varies greatly in size, as may be seen from a comparison of Figure 1 with the other figures representing the development, all of which are drawn on the same scale, but even these large open primordia form eventually quite close coils. The helix which consists of one to one and one-half turns, divides into cells from which short lateral branches are produced, usually growing towards the center, rarely outward (Figures 5–7, Plate **9**). These branches twine and intertwine, the lateral walls adhering firmly so that eventually a somewhat spherical body is formed which superficially resembles the sporangium of a fern. The cells of the original spiral are more prominent than the others, usually slightly elevated with well marked walls, and correspond to the annulus, as will be seen from Figures 9–10, Plate **9**. Figure 10 is a view of an immature bulbil, looking down on the "annulus," while Figure 9 is a side view of the same. At maturity the bulbil, which is nearly spherical, is 55–88 μ in diameter. The cells of the primary coil usually become empty and lighter colored, showing slight thickenings scattered over their surface, occasionally projecting slightly, thus giving the appearance of minute spines.

Sometimes a lateral hypha divides dichotomously and each branch coils up and produces a bulbil. Similar branches may be produced directly from the superficial cells of a bulbil (Figure 8, Plate **9**). The mode of development in this form resembles that of certain species of Urocystis, such as *U. cepulae*, the common onion smut, in which a lateral branch coils up, making about one turn, and this divides into cells from which secondary branches are given off. Figures 4, 5, 6 and even 7, Plate **9**, might almost equally well illustrate the development of *Urocystis cepulae.*

Papulospora coprophila, nov. comb.

Helicosporangium coprophilum Zukal (**'96**).

PLATE **10**, FIGURES 1–16.

Mycelium white, septate, flocculent, abundant, persistent; bulbils, dark brown, more or less spherical, 30–40 μ (rarely 60 μ) in diameter, with one to four (sometimes as many as 10) large central cells surrounded by a cortex of empty colorless or slightly brownish ones; primordium spiral, of one to four turns, the end cell usually becoming a central cell. Conidia on bottle shaped sterigmata, frequently in white tufts scattered over the surface of the substratum.

On onions, straw, horse dung, etc., Cambridge, Massachusetts, and California.

Onions have proved very productive as a substratum for bulbils. Some onions obtained from the Boston market which had been shipped from New York State, produced several different kinds and among them *P. coprophila* which has been secured from at least ten different sources, not only on onions, but frequently on horse dung and straw. It grows readily on potato and bran agar, but, like many of the other species, after continued artificial cultivation the mycelium becomes scanty and the bulbils few. In such cases it can be rejuvenated by growing on a gross culture of sterilized fresh horse dung, on which the mycelium is developed luxuriantly and becomes flocculent, producing bulbils and conidia abundantly.

This species appears to be the same as that described by Zukal ('86) under the name of *Helicosporangium coprophilum* which he found growing on horse dung. The general appearance of the bulbils of these two forms, their size, color, and at least one phase of their development seem to be identical. The form under consideration, however, differs from the description given by Zukal in producing a copious supply of flocculent hyphae. This may be due to the differences in the conditions of cultivation. *P. coprophila* resembles in mode of development the species referred by Eidam to *Helicosporangium parasiticum* Karsten, but the bulbils of the latter are brick-red, with yellowish cortical cells which, judging from the figures, are much less prominent than in the present form. The only other close allies are *P. parasitica* and *P. spinulosa*, the former easily distinguished by its single large central cell, the latter by its mode of development, and the presence of slight thickenings in the walls of the cortical cells.

This form develops sparingly on very moist substrata. On nutrient potato agar containing sugar, however, or on fresh horse dung, it grows well. Contrast cultures of mycelia from different sources yielded nothing more than additional variations in the filaments and bulbils. The former grew much more luxuriantly at the points of contact of the two sets of mycelia.

The bulbils.— A short lateral branch coils up, making about one or one and a half turns, the end cell enlarges, becomes spherical and frequently turns brownish. As it continues to increase in size its two lateral faces protrude more or less conspicuously and may even become subpendent, as in *P. parasitica* (Figure 4, Plate 5). These projections, however, often behave differently from those of the latter, since they are frequently cut off and thus form other enlarged central cells. Sometimes the second or even the third cell of the coil enlarges and takes part in the formation of the central cells. Those that do not enlarge grow out laterally over the surface of the central cell or cells and eventually completely enclose them. Figures 13–15,

Plate 10, show what appear to be arrested forms of this mode of development, all of which have brownish walls. These conditions resemble somewhat the mode of development figured by Zukal ('86).

About three or four days after inoculation on fresh nutrient agar which contains sugar, there frequently appears a spiral primordium of three or four turns, as shown in Figures 1–6, Plate 10, which divides into cells from which short secondary branches are produced, or other cells are formed by gemmation, so that eventually the spiral is enclosed by them. The cells of the spiral enlarge and usually lose their characteristic form. The lateral walls of the superficial cells adhere firmly together, so that eventually there comes to be one to four (sometimes as many as ten) large central cells, surrounded by a cortical layer of empty and often colorless cells (Figures 10–11, Plate 10). The development of the spiral may be checked at nearly any stage of its formation and thus certain variations in the form and number of the central cells of the bulbil may result. This variability in the formation of the spiral seems to be largely due to the character of the medium which, when favorable, usually produces quite regular primordia with the maximum number of coils, while under less favorable conditions, or after the substratum has been once run over with the hyphae, many variations are found. Some of the spirals are loosely coiled (Figures 1–2, Plate 10), while others are close and compact (Figures 4, 6, Plate 10). Although the primordium usually loses its spiral form early in its development, it is occasionally found surrounded by an irregular layer of cells, as shown in Figure 8, Plate 10. These bulbils resemble somewhat the primordium of a perithecium, like that of Melanospora as shown in Figures 5–6, Plate 3. On account of this resemblance an effort was made to induce them to develop into some perfect form, but although many and varied kinds of experimentation as to media, moisture and temperature, were tried, all efforts proved unsuccessful.

There are also associated with this bulbil spherical or slightly ovoid conidia, on bottle shaped sterigmata, identical with those found in connection with the melanosporous forms. These conidia, which frequently appear on conspicuous white tufts of hyphae scattered over the surface of the substratum, may be formed individually, in chains, or occasionally in a moist atmosphere may cohere at the ends of the sterigmata in a spherical mass. Although, as a rule, the sterigmata occur laterally on the walls of the hyphae, they are often found clustered on irregularly swollen branches and exhibit all the variations referred to below in connection with *P. aspergilliformis*, although the characteristic "Aspergillus-like" fructification illustrated in connection with the latter has never been observed. These conidia were picked out with Barber's apparatus and transferred to nutrient tubes where they germinated and produced mycelium on which bulbils developed. In this respect they differed from those of *P. aspergilliformis*, which, although repeated efforts were made, could not be induced to germinate.

When these bulbils are crushed the contents of the large central cells escape, surrounded by a thick endosporium (Figure 11, Plate 10). These cells germinate readily in Van Tieghem cells (Figure 12, Plate 10).

Papulospora rubida n. sp.

Plate 8, Figures 12–27.

Mycelium white, procumbent or slightly aerial on some media; bulbils more or less spherical, 30–40 μ in diameter, with 2–5 large central cells surrounded by a layer of empty cells which usually retain their yellowish red color, at maturity the whole culture has a brick-red aspect; primordium a spiral, with many modifications; conidia on bottle-shaped sterigmata, but not formed in white tufts.

On dog dung from Buenos Ayres.

This species was obtained from a pure culture received from Dr. Thaxter, which he has had growing for a number of years. It was originally found on dog dung from Buenos Ayres. In general it resembles *P. coprophila* in size, form, and mode of development. It is easily distinguished, however, by the appearance of the culture. The mycelium is more or less procumbent and the bulbils give the whole substratum a brick-red aspect, in old cultures forming a leathery incrustation which often cracks as the medium dries up. The mycelium of *P. coprophila*, on the other hand, is flocculent, filling the whole lower part of the test-tubes in slant cultures, and the bulbils give the culture a dark brown appearance. The cortical layer is colorless and more definitely marked in the latter species.

The hyphae of the form under consideration vary from 3–14 μ in diameter and, especially in old cultures, have well marked cross walls. Large swollen intercalary cells (Figure 24, Plate 8), are often formed, which seem to act as storage cells, as they are densely filled with granular, protoplasmic material and oil globules.

The bulbils.— A short lateral branch coils up spirally usually making one to one and a half turns (Figures 12–15, 21, 22, 27, 25a, Plate 8) and divides up into cells all of which become more or less swollen.

One or more of these cells, as a rule the first or second or both of them, increase in size beyond the rest, becoming densely filled with granular material and oil globules, while the other cells grow out laterally (Figure 16, Plate **8**) and eventually enclose the enlarged cells in a manner similar to that of *P. coprophila* and *P. parasitica.* It sometimes happens that when the end cell enlarges, protuberances are produced from the lateral sides, which may even become subpendent, as in *P. parasitica* (Figure 26, Plate **8**). The development of the cortical cells is shown in Figures 16, 21, 22 and 27, while Figure 25 is a median section and Figure 18 a surface view of the mature bulbil. Thus at maturity the bulbil is more or less spherical, 30–40 μ in diameter with 1–5 (usually 2 or 3) large central cells each of which varies from 10–14 μ in diameter (Figures 16, 25, Plate **8**), surrounded by a cortex consisting of a single layer of empty cells, rarely more, which is often incomplete. The walls of the cells of this cortical layer usually retain their color.

Occasionally the short lateral branch instead of making but one or one and a half turns continues the spiral until from three to five turns are formed (Figures 17, 20, Plate **8**). From the cells of the spiral are produced others laterally by budding, which eventually adhere to each other laterally, thus forming a wall about the spiral. This is similar to the process observed in connection with *P. coprophila.*

This species also produces conidia on bottle-shaped sterigmata similar to those described in *P. coprophila,* but they do not, as far as the writer has observed, occur in white tufts scattered over the substratum as they do in the last named species.

Papulospora sporotrichoides n. sp.

Plate **12**, Figures 1–41.

Mycelium white, procumbent, usually scanty; bulbils dark chocolate colored, somewhat spherical or flattened, 21–36 μ in diameter, primordium a spiral of one to two turns, with conspicuous oil globules, the spiral sometimes not well marked. Conidia and conidiophores of the Sporotrichum type.

On Live Oak chips (*Quercus agrifolia*) and corn cobs from Claremont, California, and Maple chips from Newton, Massachusetts.

The bulbils.— In the development of the bulbil a short lateral or terminal branch coils up, divides into a number of short cells with walls well distinguished, forming a close spiral of two or, rarely, three turns. This process is illustrated by Figures 1–9, Plate **12**. During the very early stages of development, the primordia are colorless, somewhat larger than the ordinary hyphal threads with more granular material. The walls, however, begin to turn brown shortly after division takes place. In Figure 5, for example, the walls are distinctly colored. In the mature bulbil the spiral form can sometimes be recognized (Figure 8, Plate **12**), but more frequently, owing to the unequal enlargement of the cells composing the coils, or some modification in the development which will be spoken of later, all trace of it is lost.

The development of these bulbils was carefully followed in pure Van Tieghem cell cultures, and many interesting modifications were observed. Quite frequently, as illustrated in Figures 12–14, Plate **12**, before the spiral has completed one turn or the walls of the individual cells thickened, one of the cells, usually the third or fourth from the tip, grows out into a vertical branch and coiling divides into cells similar to the first. The second coil may repeat this same process, so that two or three or even four coils like that which is shown in Figure 14, Plate **12**, are formed one above the other, each producing a separate bulbil. These usually continue their development independently of each other, but not infrequently the primordia overlap and a single "compound" bulbil of two or three spirals, as the case may be, is the result. Occasionally this secondary branch is produced on the opposite side of the cell so that it grows into the concave portion of the first coil as shown in Figure 15, Plate **12**. In some instances a single coil only may be formed, the cells of which enlarge as usual (Figures 19–25, Plate **12**) becoming divided during the process, by thin cross partitions which are at first hardly visible without staining. The multicellular bulbil thus produced, does not become dark at once like the normal type but remains hyaline for some time, slowly changing color and only after it has become fully mature does it assume the dark brown tint of the more common type from which, however, it is eventually indistinguishable.

The Conidia.— A conidial form of reproduction, which usually appears on old cultures after a large number of bulbils have been produced, is also connected with this fungus. These conidia are of the Sporotrichum type and were obtained from pure cultures by the transfer of individual bulbils. It seemed desirable, however, to obtain the bulbil-type from germinating conidia in order to eliminate all chance of error; but this was found unexpectedly difficult for the reason that single spores isolated by Barber's apparatus refused to germinate although cultivated in varied media. The conidial form is as a rule scantily developed in older cultures only, but by using a special nutrient composed of a decoction of bran, Spanish chestnuts, horse dung and rotten wood hardened with agar, an abundant production of conidia was obtained after two months, the conidiophores (Figures 35–36, Plate **12**) rising well above the substratum at the margin of the culture, so that large quantities of spores were readily obtained in an absolutely pure condition. Cultures of these yielded about two per cent of germinations after twenty days.

The development of these germinating conidia (Figures 38–41, Plate **12**) was continuously followed in Van Tieghem cells until bulbils were produced on the mycelium derived from them.

The conidiophores (Figures 35–36, Plate **12**) which are colorless at first but become light grayish brown at maturity, are larger (3.5–4 μ in diameter) than the other hyphae from which they arise, with quite irregular walls producing numerous lateral conidia which rest either upon short stalks or upon little projections of the wall of the conidiophore, or are completely sessile. The conidia, which are also colorless at first, but become the same color as the conidiophore, are ovoid, $4 \times 7\ \mu$, with smooth, fairly thick walls. During germination, they swell so as to be almost spherical in shape (Figures 39–41, Plate **12**).

Papulospora cinerea n. sp.

Plate **8**, Figures 1–11.

Mycelium white, septate, procumbent, forming a felted mass over the substratum; bulbils steel-gray or slate-colored, somewhat spherical and flattened, 21–36 μ in diameter, with three or four large angular central cells, and a layer of fairly regular cells forming a cortex, but of the same color as the others; the primordium a spiral of one or two coils. No conidia known.

On gross culture in the laboratory, Cambridge, Mass.

This fungus was found running over a gross culture in the Cryptogamic Laboratories at Harvard University by Dr. Thaxter and has been kept growing as a pure culture for more than ten years. It is easily distinguished from any of the others by the steel gray or slate-color of the bulbils, which are round, somewhat flattened in form, and measure 21–36 μ in diameter, in which respects they resemble those of *Papulospora sporotrichoides*. The mycelium is white, procumbent, forming a felted mass over the substratum, the slate-colored bulbils being scattered among the white hyphal filaments, finally giving the whole culture a bluish gray or steel-gray appearance. When young the hyphae are closely packed with oil globules which escape into the water when the filament is ruptured, and might be mistaken for spores.

The bulbils.— A short lateral branch coils up, usually making one or two turns, rarely more, and frequently less than two, and divides into a number of short cells from which secondary branches are produced, or from which individual cells are formed by budding (Figures 7–8, Plate 8). In either case, spherical cells which gradually increase in size, are developed, and the lateral walls adhere closely to each other. The original coil, the cells of which in the meantime have become much enlarged and filled with granular material and oil globules, is thus eventually completely surrounded. At maturity three or four large central cells may be distinguished which have become angular by pressure, surrounded by a layer of fairly regular cells which are also usually somewhat angular except the outer walls. It often happens that when one turn is made by the primordial coil, the secondary branches begin to form, while at other times two or more turns are formed before this happens. Between these two extremes a number of variations are found. Not infrequently the lateral branch becomes divided into four to eight cells and may or may not be coiled at the end, and from these, secondary branches are produced which coil around each other and around the original branch, dividing and subdividing, the lateral edges eventually adhering closely, and producing a more or less elongated bulbil (Figures 4–6, Plate 8). This process also inhibits the further growth of the coil. An extreme instance of this is shown in Figure 6, Plate 8, where several cells are seen to take part in the formation of lateral branches. Bulbils formed from a primordium of this type are elongated, irregular, and larger than those formed in the usual way.

Although this species was grown on a great variety of nutrient media, it could not be induced to develop any perfect form or even another imperfect type.

Papulospora parasitica nov. comb.

Syn.: *Helicosporangium parasiticum* Karsten. (nec Eidam.)

PLATE 5, FIGURES 1–17.

Mycelium septate, white, flocculent; bulbils light brown, nearly spherical, 15–21 μ in diameter, with a single large central cell surrounded by a single layer of empty colorless cells; primordium a spiral, coiled crosier-fashion.

On bread, Cambridge, Massachusetts; mouse dung, Duarte, California.

This form which appears to be identical with *Helicosporangium parasiticum* Karst. was found by Dr. Thaxter on bread in Cambridge, Massachusetts, and kept as an herbarium specimen, but was too old to be resuscitated. The writer also found it on a gross culture of mouse dung in an old paper bag obtained from Duarte, California. This culture was so overgrown with Penicillium and other foreign material which grew so much more rapidly than the bulbiferous fungus that it was difficult to get it pure. This was finally accomplished by using a gross culture of sterilized peas on which the mycelium of the bulbil grows quite rapidly.

The bulbils.— The development of the bulbils, which are produced in large numbers, agrees in all essential points with the original description and figures of Karsten ('65). Short lateral branches of the hyphae coil up crosier-fashion and, although quite open at first, soon close up, forming a close coil which divides into short cells, all of which increase in size to a certain degree. One of these, usually the end cell, but not infrequently the second, enlarges more rapidly than the others and becomes a "central cell," the remaining members of the coil forming a ring or "annulus" around it and becoming firmly attached to the side of the original lateral branch. As this central cell increases in size more rapidly than those of the coil, considerable lateral pressure is exerted and consequently protuberances usually appear on each side of it which usually becomes subpendent and subsequently may divide into two or three lobes (Figures 4, 5, 9, 10, Plate 5). As this tension is released, probably through the increase in size of the "annulus," the large central cell loses its lobed appearance and assumes a spherical form (Figure 11, Plate 5) and may later become somewhat angular.

In the meantime the cells composing the "annulus" begin to grow out laterally, extending over the surface of the large central cell, and in the mature bulbil completely corticating it, the walls adjacent adhering laterally. Sometimes there is a small pore left at one or both of the centers of the lateral faces of the central cell and through them at germination the germ tube grows, but this is the exception and is probably one of the incomplete stages of development that will be spoken of later.

During the early stages of development and even until they have almost reached their full development these bulbils are colorless, but eventually they become light brown. At maturity they are nearly spherical in form, consisting usually of a single large central cell about 10–14 μ in diameter, densely filled with granular material and oil globules, and surrounded by a single layer of empty colorless cells, the whole bulbil measuring 15–21 μ in diameter. Although the foregoing description of the mode of development of the bulbil is the characteristic one, the process may vary considerably in different cases. Occasionally there appears a tendency to form a helix, at other times a protuberance from the central cell develops only on one side or not at all, and quite frequently the "annulus" is incomplete, or the cortical cells that are derived from it fail to cover the whole central cell. It would thus appear that the development of the bulbil may be arrested at nearly any stage, and these arrested forms, under proper conditions, will germinate almost immediately.

In Van Tieghem cells these bulbils germinate in 24–36 hours and send out one or two germ tubes, as shown in Figures 15–16, Plate 5, which arise from the central cell only. The germ tubes usually proceed from that region where the marginal cells meet or, as sometimes happens fail to meet, leaving two small pores, as already mentioned. In incompletely developed bulbils, the germ tube seems to come out from any point offering the least resistance.

Conidia-like bodies were occasionally found connected with this fungus when grown on straw. A short lateral branch, which not infrequently becomes septate (Figure 17b, Plate 5), enlarges at the end and from it an ovoid cell (4.5×6.5 μ) is abjointed. Unfortunately these were produced so rarely that their germination and further development could not be observed. Figure 17, Plate 5, however, shows a direct connection between these "conidia" and a bulbil.

This form agrees in all respects with the original description and figures of *Helicosporangium parasiticum* (Karsten '65) except that it is saprophytic and that no "endospores" are found in the central cell. As already stated, Karsten was of the opinion that the contents of the cortical cells passed into the central cell, either directly or by diffusion and as a result of the union of these different protoplasmic bodies the spores were formed. If the account given by Karsten is correct, in all its details he was not dealing with a bulbiferous form at all. It would seem, however, that later writers are probably correct in considering them as such, since Karsten may have been misled by the presence of more or less regular oil globules, such as occur in this and other species and which might easily have been mistaken for endospores. On the other hand, it is by no means impossible that he was dealing with a form related to Monascus, which has not been recognized by subsequent investigators. Since, however, the morphology and development of his "Helicosporangium" corresponds so exactly with that of the bulbil under consideration and since also the "parasitism" of his plant on "beets," seems at least very questionable, the writer feels little hesitation in concluding that he was dealing with a bulbil, in all probability identical with the one under consideration.

Harz ('90), in his account of *Physomyces heterosporus* (*Monascus heterosporus* (Harz) Schröter), is of the opinion that this plant is closely related to *Helicosporangium parasiticum* Karsten, and further suggests that *Papulospora sepedonioides* Preuss, belongs near this fungus also, the difference consisting in the fact that the central cell of the latter is said to contain but one or only a few "endospores."

The bulbils described and figured by Zukal ('86), under the name of *Dendryphium bulbiferum*, also resemble this form in appearance and mode of development, except that it does not produce the lateral protuberances from the developing central cell, at least they are not mentioned or figured, and that it is described and illustrated as being intimately connected with hyphae producing spores of the genus Dendryphium.

In this connection it may also be mentioned that the spores of *Stephanoma strigosum* Wallr. (*Asterophora pezizae* Corda, *Synthetospora electa* Morgan, *Asterothecium strigosum* Wallr.) show stages that resemble quite closely certain conditions in the development of *P. parasitica*. Figure 35, Plate 5, for example, is an abnormal spore of Stephanoma and, except for its size and color, might easily be taken for an imperfectly developed bulbil of the form under consideration, such as is represented by Figure 14, Plate 5.

A corresponding resemblance may also be seen between imperfectly developed bulbils of the present species, in which the cortical cells have failed to surround the central cell completely, and the immature bulbils of *Acrospeira mirabilis* described above.

Papulospora aspergilliformis Eidam.

Plate 7, Figures 1–20.

This bulbil was obtained from several different sources, chiefly on onion leaves, wheat chaff, and oat straw from the vicinity of Cambridge, also on straw from Claremont, California. It is not at all rare and can easily be obtained by placing straw in a moist chamber. It is readily distinguished by its relatively large, irregular, sclerotium-like bulbils. Pure cultures from a half-dozen different sources were made by the methods already described, and kept under cultivation on a variety of media.

The septate mycelium grows very slowly on nearly all substrata, producing the best results on bran agar, and on sterilized fresh horse dung on which it becomes somewhat flocculent. The primary mycelium grows on the top of the substratum, or just below the surface, and sends up lateral branches into the air. It is these lateral branches that produce its peculiar Aspergillus-like fructification. The primary mycelium becomes very large, usually somewhat contorted and packed full of granular material and oil globules. The hyphae, which anastomose readily often forming a sort of network, measure as much as 11 μ in diameter, and some of the swollen lateral branches 17 μ (Figure 4, Plate 7). Occasionally, especially in the young hyphae, there occur large swollen intercalary cells containing oil globules and other food material (Figures 17–18, Plate 7). These seem to be cells for the storage of food.

The bulbils.— The mycelium grows out evenly in all directions from the point of inoculation. In about two or three weeks (on horse dung, in about a week), small brownish-red spots appear near the margin of the mycelial growth. These are young bulbils, and on closer examination they are found to develop as follows. A short lateral branch (Figures 2–3, Plate 7) well filled with nutrient material, sends out branches which twine about each other. The former sometimes coils at the tip but this seems to be incidental. These secondary branches may come off near the base of the lateral branch (Figure 3, Plate 7), and by twining about the primary hypha may incorporate it into the bulbil. More often, however, the secondary branches come off a short distance from the hypha (Figures 2, 4, 6, Plate 7), so that, especially in the early stages, it is evident that they are on short pedicels. The secondary branches intertwine with each other, and divide up into short cells, their lateral walls adhering firmly to those of their neighbors and eventually forming a compact mass of uniform cells. At maturity these bodies superficially resemble true sclerotia perhaps more nearly than they do typical bulbils, but they are developed from a group of cells composing the primordia, and not from a mass of interwoven hyphae from different sources. They vary considerably in size and shape, some of them being nearly spherical, about 100 μ in diameter; but most of them are irregular in form, reaching in old cultures $570 \times 750\ \mu$. There is no differentiation between the marginal cells and the central cells. Microtome sections show that the bulbil is uniform throughout (Figure 20, Plate **7**) all the cells containing protoplasm, and under favorable conditions capable of sending out germ tubes. In this respect it differs from the typical sclerotium, which usually has a compact layer of several cells in thickness (the rind) which forms the margin. The primordia are colorless at first (Figures 2–4, Plate **7**), then light-yellow, later ruby-red, and finally reddish brown and opaque.

In this as in most other bulbils the process of development may vary greatly. Figure 1, Plate **7**, shows the primordia of three bulbils, two of which and possibly the third also, would probably have grown together, forming a large, irregular, sclerotium-like body. This phenomenon occurs quite frequently, giving rise to a variety of forms, which vary with the number of the initial primordia taking part in their development, their proximity, and the inequality of their development. In such cases each primordium develops independently, until its lateral branches intertwine with those of one or more that lie adjacent to it, a compound bulbil finally resulting, in which the several origins are indistinguishable.

Aspergillus-like fructification. Conidia are frequently produced both on Aspergillus-like heads and also laterally, on the sides of the hyphae (Figures 10–11, Plate **7**). The latter are usually isolated, sometimes irregularly grouped. The conidiophores arise from erect lateral branches, and are frequently septate; rarely branched. They are very minute, so that one can detect them only with difficulty, even with a good hand lens. The length of the conidiophore varies greatly, some being quite short, others so long that it is difficult to trace them to their origin. The swollen head of the conidiophore is usually spherical, or nearly so, and on it are arranged somewhat irregularly numerous simple sterigmata. These vary slightly in size and shape, but always have a broad base and taper more or less gradually, often to a point, at the distal end. The relative length of the vertical and transverse diameters of the swollen base varies somewhat, so that one may find gradations in shape from almost spherical to napiform. The conidia are nearly spherical, sometimes ovoid, smooth, colorless, minute, occurring in chains, and dropping off very readily; but in moist atmosphere the conidia, instead of being produced in a chain, frequently adhere and form clusters much like those of Hyalopus.

There are many variations in the arrangement of these conidia, which may, for example, arise, as is shown in Figure 9, Plate **7**, terminally and laterally on irregularly clavate extremities of hyphae.

Occasionally a conidiophore may form an intercalary swelling with conidia on it, as if it were a secondary head (Figure 10, Plate 7).

Chlamydospore-like bodies occur quite frequently. They are mostly intercalary but sometimes terminal (Figures 13–16, Plate 7). When young they are colorless, or opalescent, slightly swollen, ovoid cells, filled with granular material. At maturity they are usually more spherical and have thick brown walls (Figures 13, 15, Plate 7). Occasionally more than one cell takes part in the formation of these spore-like bodies. Figure 16, Plate 7, shows two such cells and Figure 5, Plate 7, a large number of "chlamydospores" closely packed together.

There are several forms that have Aspergillus-like fructifications, similar to those just described and which may be considered briefly at this point. As has already been noted, Eidam ('83) describes these structures in his account of *Papulospora aspergilliformis*, and also chlamydospores resembling those of *Acremoniella atra* Sacc. (*Acremonium atrum* Corda.) such as are produced by *Melanospora cervicula*. Eidam, however, described two types of bulbils in *P. aspergilliformis*, a small one that develops in à manner similar to the form examined by the writer, and a large one, the primordium of which is spiral, resembling that described by Bainier ('07). It is quite possible that Eidam has here confused the primordia of two species the larger of which corresponds in all essentials to that studied by the writer. On the other hand his smaller bulbil would correspond more closely with that studied by Bainier.

Bainier ('07), in his article on *Papulospora aspergilliformis* also refers to its "Aspergillus-like" conidial fructification. According to his account the primordium of the bulbil consists of a short lateral branch which coils up spirally and eventually produces a more or less spherical bulbil. Under certain conditions of nutrition and moisture, however, the latter are said to produce large sclerotium-like bodies, which in turn may be induced to develop further and form perithecia, which are referred to the genus Ceratostoma. This form described by Bainier seems to be different from the one under consideration, since the bulbils of the latter do not develop by means of a spiral and are large and sclerotium-like. The present form, moreover, has been grown for nearly three years and during that time it has never been observed to produce any other type of bulbil than the one described. It has, however, produced in abundance conidia on Aspergillus-like conidiophores which sometimes occur in direct connection with the bulbil (Figure 8, Plate 7). This species has been compared with material received from Professor Bainier by Dr. Thaxter, and the two forms have been grown on many and varied kinds of nutrient material for nearly three years during which time, as already mentioned, the American material has never been observed to produce small spherical bulbils; nor has the form received from Bainier developed the large sclerotium-like bodies which he describes, although every effort has been made to obtain them.

There is also a marked difference in the method of growth in these two forms. The mycelium of the American form grows very slowly on bran or corn agar, but fairly rapidly on horse dung, while Bainier's species grows rapidly on a variety of media. There is also a marked difference in the general appearance of the two while growing in cultures; the mycelium of the former being quite inconspicuous at first and often two or three weeks elapse before bulbils are produced. The two forms thus appear to be very probably distinct and there seems little doubt but that Bainier was mistaken in referring his species to *P. aspergilliformis.* Neither of these forms has associated with it Acremoniella-like Chlamydospores, such as Eidam describes and it seems not improbable that Bainier is right in believing that these spores do not belong to *P. aspergilliformis,* but are those of "*Acremonium atrum*" which although frequently associated with it are not a part of its life cycle.

The writer has under cultivation about a dozen pure cultures of *Acremoniella atra* obtained from different sources, some of which were closely associated with bulbils, and these have been grown for nearly three years under varying conditions of temperature, moisture, and nutrient material, the different mycelia having been contrasted on plate-cultures under various conditions. In no instance, however, have bulbils or Aspergillus-like conidiophores been produced.

Harz ('71) has described a form under the name of *Monosporium acremonioides* that produces chlamydospores and conidiophores similar to those of *P. aspergilliformis* Eidam, but not associated with bulbils, and states that the conidia were produced on secondary heads either sessile or short-stalked, like those of *Melanospora cervicula.* This latter character has been used by Costantin ('88) as the basis of a new genus, Harzia, into which he puts the foregoing species under the name of *Harzia acremonioides.* Later, in referring to *Papulospora aspergilliformis* Harz ('90) calls attention to the striking resemblance between the two spore-forms of this fungus and those of *Monosporium acremonioides* Harz, and suggests that, if they are the same, the name should at least be *Papulospora acremonioides,* although he takes exception to the generic name on the ground, as will be seen later, that it does not correspond with the description of the genus by Preuss.

Lindau ('07) apparently is of the opinion that these two forms are the same and he creates a new genus, Eidamia, for their reception under the name *E. acremonioides* (Harz).

The conidial form of *Melanospora cervicula* resembles quite closely *Harzia acremonioides* in having its conidia on secondary heads and in producing Acremoniella-like chlamydospores, but differs in possessing bulbils and melanosporous perithecia. It is quite possible, however, that the two are identical. It is possible also that the so-called "Harzia type" of fructification, as seen in *M. cervicula* and the "Aspergillus-like" type as seen in *P. aspergilliformis*, are modifications of one and the same mode of reproduction: since on several occasions the writer has found in connection with the conidial fructification of *M. cervicula* instances in which secondary heads seemed to be lacking, but, owing to the fact that there was only a limited amount of material available, this point could not be absolutely determined. The perithecium of this form, however, is clearly of the melanosporous type, and can hardly be the same as the Ceratostoma described by Bainier.

The writer has under cultivation the *Mycogone ulmaniae* Potebnia, ('07) (*Chlamydomyces diffusus* Bainier) obtained by Dr. Thaxter from Liberia and kept in cultivation for over fifteen years. In addition to its large two-celled, warty, chlamydospores, this species also produces conidia on "Aspergillus-like" conidiophores similar to those of *P. aspergilliformis*.

Conidial forms similar to those above mentioned are also described by Möller ('93) in connection with the garden fungi of certain species of ants in the tropics.

Again, large chlamydospores, somewhat similar to those of *Melanospora cervicula* except that they are divided into two unequal cells, have been described by Berlese ('92) in connection with *Sphaeroderma bulbilliferum*. They differ from those of *Mycogone ulmaniae*, however, in being smooth.

Papulospora polyspora, n. sp.

Plate 11, Figures 1–13.

Hyphae septate, hyaline, scanty, procumbent, 5–7 μ in diameter (sometimes as much as 9 μ); bulbils dark red-brown usually with a thin mucilaginous film about each, eventually becoming a dry powdery mass, completely concealing the mycelium, more or less spherical, 119–122 μ in diameter, composed of closely compact angular cells, 150–200 cells visible in a surface view; cells homogeneous throughout. Individual cells of the bulbil eventually forming spherical spores, 17–22 μ in diameter loosely held together. No other spore-form known.

On straw, old paper, from California and cotton flowers from Cuba.

This fungus has been obtained from at least three different sources. It was found by Dr. Thaxter running over a gross culture of the flowers of Cuban cotton and also by the writer on gross cultures of barley straw from Claremont, California, and on old paper from Duarte, California.

The usual methods of obtaining a pure culture were employed here, after which the fungus was grown on various kinds of nutrient material, but it could not be made to produce any perfect form. Mycelia from widely different sources were contrasted in Petri dishes but no results were obtained except the production of certain abnormal enlargements and contortions of the hyphae, such as may frequently be observed in contrasting forms of even widely different species.

The mycelium of this fungus is white, inconspicuous, procumbent, the hyphae densely filled with coarse granules or oil globules. At a short distance from the margin of growth small white pustules are seen, which gradually become larger and more frequent as they approach the point of inoculation. These soon turn tan-colored, and are frequently associated with small drops of liquid of nearly the same color, which may often be seen surrounding a bulbil. At maturity these bulbils are almost spherical, 119–122 μ in diameter, composed of closely compacted angular, often irregular cells, uniform throughout, there being no distinction of a definite cortex. They occur in large numbers heaped together, covering the whole substratum and obliterating completely the naturally scanty mycelium. In older cultures they become a dry powdery mass.

The bulbils.— The formation of this bulbil is different from that of any of the others thus far considered, since they result not from the development of a single primordium but from the combined activities of several primary branches. One or more procumbent hyphae send up vertical branches which twine about each other (Figures 1–4, Plate 11). Usually several of these branches arise simultaneously at a given point (Figure 3, Plate 11) and as the bulbil increases in size, more and more of these take part in its formation, their extremities combining to produce the bulbil proper, while just above the substratum there may form a sterile supporting base, often with a diameter nearly equal to that of the bulbil itself and composed of interlacing hyphal strands, which are partly made up of branches from the procumbent hyphae and partly by the branching of the original vertical ones. These supports or "stalk-like" structures vary in length, some being quite long (100 μ), while at other times the bulbils appear to be almost sessile on the horizontal branches. The primordia that are produced later, are hindered in their upward growth by the presence of the first formed bulbils, which, however, are soon broken away from their attachments and pushed up so that eventually several irregular layers of independent spherical bodies are produced, the oldest ones being on the surface. Whether the vertical hyphae first formed fuse at the apex could not be determined. They evidently receive some stimulus, for they begin to send out short branches in different directions, which in turn divide and subdivide, and these intertwine among themselves and, with other hyphae that grow up from the original horizontal branches, form an interlacing weft which becomes more and more compact, producing a hyaline, spherical body in which the walls are very thin and almost indistinguishable except after staining. As they increase in size they assume a brownish tint and finally a rich tan-color, during which time the walls gradually become more definite and eventually are well marked.

Since liquid media appeared to have a peculiar affect on the development of these bulbils, cultures were tried in large flasks on pieces of wood partly immersed in bran decoction, so that the effect of different degrees of moisture might be observed, as the mycelium spread from the liquid medium toward the dryer portions of the wood. Under these conditions it was found that the bulbils formed on the wood about three or four inches above the liquid, began to assume a paler aspect and soon became light straw-colored, instead of the dark tan of the normal bulbil. On examination it was found that the cells composing these pale bulbils, instead of being compact with angular walls as in the normal form, had rounded up and become spherical (17–22 μ in diameter), adhering very loosely by means of a mucilaginous material that had evidently been secreted by them, so that a very slight pressure would separate them into individual spores (Figure 8, Plate 11). The germination of these "spore-masses" was followed carefully in Van Tieghem cells — some crushed, others not — and it was found that nearly all the spores germinated in twenty-four hours, some producing one, others two germ tubes, which were hyaline and septate, becoming much branched (Figures 9–10, Plate 11). When allowed to remain adherent, the spore-mass sent out germ tubes in all directions which shortly forced the individual spores apart. The bulbils were also germinated in Van Tieghem cells, but their germination was much slower and they produced comparatively few germ tubes which seemed to be chiefly from the superficial cells.

In water cultures the hyphae are usually larger and more densely filled with granular material, with numerous large swollen intercalary or terminal cells (Figures 9b–13, Plate **11**). These cells are grouped together irregularly as if attempts were being made to form bulbils but they do not become compact. They often grow very large, as may be seen by a comparison of Figures 9b–13, Plate **11**, all of which have the same magnification.

This development and final fate of the bulbil of *P. polyspora*, suggest a similar condition that is found in Aegerita. In *Aegerita Webberi* Fawcett ('**10**) the "sporodochia" which measure 60–90 μ in diameter, consist of an "aggregation of conidia-like, inflated, spherical, cells, 12–18 μ in diameter," resembling the conditions described for *P. polyspora.* The development of the latter on the other hand recalls also that of the sporodochium of *A. candida* Persoon (*Peniophora candida* Persoon) as described and figured by Lyman ('**07**) and it is possible that the two structures may be similar in nature.

OTHER RECORDED BULBIFEROUS FORMS.

In addition to those above enumerated several other bulbils or bulbiferous forms have been recorded, some of which have already been referred to, but which may here be again mentioned.

Papulospora Dahliae Costantin ('**88**). This species was found by Costantin on roots of Dahlia. Its bulbils appear to be somewhat like those of *P. coprophila,* brownish-red in color, with two or three large central cells surrounded by a layer of empty cortical cells. Conidia belonging to the genus Dactylaria are, however, said to be associated with these bulbils, although it is not evident that the species was cultivated in a pure condition.

Dendryphium bulbiferum Zukal ('**86**) has been mentioned on page 233, and also in connection with *P. parasitica.* The bulbils described and figured by Zukal are said to be directly associated with the conidia of a Dendryphium; but here, as in other forms studied by this author, there is no evidence that pure culture methods were used in studying the fungus.

"*Haplotrichum roseum* Lk." is also stated by the same author ('**86**) to be associated with bulbils said to be very similar to those of the

Dendryphium just mentioned; but here again pure cultures do not appear to have been used. As far as the writer is aware, moreover, this common hyphomycete has never been seen to be thus associated by any other observer.

Papulospora (Stemphylium) **Magnusianum** (Sacc.), (Michelia, I, 132) a form collected by Magnus in the Tyrol, distributed in Vestergren, Micr. Sel., No. 1150, and also figured by Saccardo in Fungi Italici, No. 934, should be mentioned in the present connection, since it is a typical bulbil and by no means a compound spore like that of species of Stemphylium.

Clathrosphaera spirifera Zalewski ('88), is a form which the author, although his observations are concealed in Polish text, appears to regard as bulbiferous, or as producing bodies comparable to bulbils, which are also associated with a species of Helicoon.

The writer has himself observed various other more or less ill defined types of bulbils, which have not been above enumerated, since they do not appear to be sufficiently well marked to warrant a definite name. "No. 170" for example (Figures 24–34, Plate 5), was found in California on straw from Claremont, and on old paper from Duarte. The fungus is characterized by an abundant white mycelium, the hyphae of which produce bulbil-like bodies consisting of a few cells each, as indicated in the figures. Their characters and development, however, are not constant and their exact nature is somewhat doubtful.

COMPOUND SPORES AND OTHER REPRODUCTIVE STRUCTURES WHICH RESEMBLE BULBILS.

Reference has already been made to the close resemblance which exists between the so called "spore-balls" of some of the Ustilaginales, and the structures under consideration; in fact it would be quite impossible to differentiate the spore-balls of Urocystis or Tubercinia from bulbils, as far as concerns their gross structure and method of development which may be exactly similar. They are, however, clearly distinguished in other ways; since in bulbils, spore formation is never preceded by any nuclear fusion, so far as is known; and furthermore the germination of bulbils in no way resembles that of the smuts; and there is never any indication of the formation of anything corresponding to a promycelium.

Attention has also been called to the fact that the compound spores which are associated with the imperfect forms of many of the higher fungi, may bear a close resemblance to bulbils. Although compound spores may in general be distinguished by the fact that they normally arise as the result of the septation of a single cell, while in the production of bulbils two or more cells are primarily involved, to which others are added by a process of budding which may also be combined with secondary septation, it is not always possible to separate them with certainty. Spores like those of Stephanoma, referred to elsewhere, in which the empty superficial cells arise by budding, serve, however, to break down this distinction.

On the other hand, the more complicated types of bulbils are easily comparable to the simpler types of sclerotia, such as occur for example in *Penecillium Italicum, Verticillium agaricinum* and similar forms. Such sclerotia, however, result from the irregular and indefinite massing together of vegetative filaments, the densely compacted cells of which do not partake of the nature of spores, while the functional cells of bulbils are usually spore-like and act independently of one another at the period of germination.

Among the compound spores formed in connection with the imperfect conditions of higher fungi, several may be mentioned which have bulbil-like characteristics.

Stephanoma strigosum Wallr. a parasite on *Peziza hemispherica* which, as Dr. Thaxter informs the writer, occurs also on *Genea hispidula* in this country and is connected with an undescribed hypocreaceous perithecial form, might very well be regarded as a bulbil of a simple type, since not only are its spores similar in their development, but, when mature, are hardly distinguishable from the more simple bulbils which are often produced, for example, by *Papulospora parasitica.*

Stemphylium macrosporoideum Sacc., which has been examined from cultures kept in the Cryptogamic Laboratories, produces a compound spore consisting of one large functional cell to which, at maturity, two or more empty ones are attached. In this condition it resembles very closely the bulbil of *Acrospeira mirabilis;* but in view of the fact that it develops as a result of the successive divisions of a single terminal cell, it must be regarded as a compound spore. Certain other forms also of Stemphylium as well as of Mystrosporium might well be mistaken for bulbils.

Hyalodema Evansii P. Magn., which von Höhnel has referred to *Coniodyctium Chevalieri* H. & Pat., produces a hymenium-like layer bearing compound spores which, except in color, are very like the bulbils of *Papulospora sporotrichoides*. Their development, however, is clearly that of compound spores and not of bulbils.

Eleomyces olei Kirchner ('88) a fungus found growing in poppy oil, produces a compound spore which consists at maturity of a large thick-walled functional cell, surrounded by several empty coherent cells, the whole resembling the bulbil of Acrospeira. If, as suggested by Kirchner, this body results from the coherence of several adjacent cells, it might well be regarded as a bulbil and not a compound spore.

Various other spore-forms might be mentioned which bear more or less resemblance to bulbils, but those above enumerated are sufficient for purposes of illustration. Before leaving bulbil-like forms, however, two or three additional types may be mentioned, the nature of which is not altogether clear, since they are neither compound spores nor typical sclerotia.

Aegerita Webberi Fawcett ('10), a fungus attacking scales on Citrus, produces, under certain conditions, bulbil-like bodies which consist of loosely coherent spore-masses closely comparable to those of the aberrant *Papulospora polyspora*, the development of which, under moist conditions, has been described above.

Sorosporella Agrotidis Sorokin ('88, '89), which attacks the larvae of Agrotis, fills the latter with loosely but definitely coherent cell-groups which might also be compared to those of *P. polyspora*.

Lastly, among structures which bear a striking resemblance to bulbils, the peculiar spore-balls of *Spongospora subterranea* (Wallr.) Johnson should be mentioned; which, although they might readily be taken for a species of Papulospora, have been shown to belong to the life-cycle of one of the Mycetozoa.

THE MORPHOLOGICAL SIGNIFICANCE OF BULBILS.

Opinions concerning the morphological significance of bulbils differ widely. Preuss ('51), Eidam ('83), DeBary ('86), Mattirolo ('86) all regarded them as normal structures which function as auxiliary methods of reproduction; while Karsten ('65), Zukal ('86), Morini ('88), and Baineir ('07) looked upon them as immature ascogenous fructifications of either perithecial or apothecial forms, believing that their arrested growth was due to unfavorable environment, and that, with proper nutriment, they might be able to complete their development.

Although it is possible that the last mentioned view may be correct in some instances, it is quite certain that in many cases, where both bulbils and ascocarps are present, this cannot be the case, since the primordia and development of the two are widely different. Thus in *Cubonia bulbifera*, for example, the bulbil is produced from a group of intercalary cells, while the primordium of the apothecium is a spiral. In like manner *Melanospora anomala* develops bulbils which arise from intercalary cells, somewhat as in Cubonia, while the perithecia arise from free spirals.

It is quite possible, however, that in other cases, as for example in *M. papillata*, where the primordium of the bulbil and that of the perithecium are similar, they may be homologous. But even in such cases, the two primordia are distinguishable so early in their development, that it is more than probable that here, also, they cannot be regarded as immature ascocarps. Various attempts have been made by the writer to induce the bulbils of various species to continue their development and produce ascocarps. Many bulbils of *M. papillata* for example, that had grown larger than the more normal types, were isolated and placed on different media where they were exposed to different degrees of moisture, with this end in view. Similar attempts were also made with the bulbils of *P. coprophila*, in which the spiral bulbil-primordium might be supposed to suggest its ascogonial nature. In no instance, however, was any evidence obtained that would seem to point to the conclusion that they were to be regarded as anything but independent non-sexual propagative bodies, except that, in some instances they increased in size, sometimes becoming approximately half as large as perithecia. This enlargement, however, was unassociated with any structural differentiation such as always characterizes the developing perithecium.

Although Bainier reports that he was successful in inducing the bulbils of *Papulospora aspergilliformis* to develop directly into perithecia which he refers to Ceratostoma, the writer has been as unsuccessful with this species as with others, even when using material derived from a living culture received from Bainier by Dr. Thaxter.

In view of the careful and long continued experiments made by the writer in this connection, and his entire failure to obtain positive results, the assumption seems justified that ordinarily, at least, bulbils are not to be regarded as abortive ascocarps, but rather as an auxiliary method of reproduction that has been interpolated in the life history of certain fungi without definite relation to other forms of reproduction which they may possess; or if they have in reality been derived from some other reproductive body, that this was more probably some type of compound non-sexual spore, rather than the primordium of an ascocarp.

DISTRIBUTION AND OCCURRENCE OF BULBILS.

It is evident from the foregoing account that bulbiferous types are not only widely distributed, but are very readily obtained if sought for, and, like so many other types among the Fungi Imperfecti, have been independently developed by a variety of species wholly unrelated and belonging to widely separated groups among the Pyrenomycetes, the Discomycetes and the Basidiomycetes. Such bulbiferous conditions, therefore, cannot in any sense be regarded as forming anything in the nature of a Natural Group. If one may judge from our actual knowledge of these forms, it would appear, on the contrary, that the bulbiferous condition was a specific one, the habit having been developed by certain species, only, in genera, the other members of which have no such secondary means of propagation: just as the habit of producing sclerotia of a characteristic type, has arisen in a few species, only, of Penecillium, like *P. Italicum.* The same principle is well illustrated in the large genus Corticium many species of which have been tested by means of pure cultures. Here again one finds a single species, only, which possesses the bulbiferous habit, namely *C. alutaceum,* pure cultures of which become completely covered by its dark brown bulbils.

In view of the wide distribution and common occurrence of bulbil-producing forms, it is not a little surprising to find such scanty references to them in mycological literature; and from the experiences of the writer in studying them, it seems certain that further attention to this subject will not only yield numerous other forms, but will show connections with "perfect" conditions even more varied than is at present indicated.

KEY TO THE SPECIES OF BULBILS HEREIN CONSIDERED.

According to their method of development bulbils may be grouped in three more or less well defined categories namely: those which originate from a primary spiral; those which develop from an intercalary primordium of several cells, and those which arise from a group of vertical hyphae. Using these characters as a fundamental basis for separation, the species above enumerated may be distinguished as follows.

Key to the Species of Bulbiferous Fungi.

A. Primordium normally involving more than one cell.
  I. Bulbils black or smoke-colored.
    1. Bulbils 75–100 μ in diam. margin even . . . . *Cubonia bulbifera.*
    2. " 200–300 μ " " " irregular . . *Papulospora pannosa.*
  II. Bulbils yellowish red to dark brown.
    1. Hyphae showing clamp-connections.
      1. Bulbils dark brown or chocolate colored.
        i. Bulbils 65–80 μ in diam. clamps conspicuous. *Corticium alutaceum.*
        ii. " 125–175 μ " " margin even, clamps inconspicuous . . . . *Papulospora anomala.*
      2. Bulbils yellowish or light brown.
        i. Bulbils light yellow, hyphae radiating conspicuously. *Grandinia crustosa.*
        ii. Bulbils brownish yellow, hyphae formed evenly. "No. 200."
    2. Hyphae not showing clamp-connections.
      1. Bulbils scanty, perithecia usually present.
        i. Perithecia with neck, lateral and terminal setae. *Melanospora cervicula.*
        ii. " " papilla and terminal setae. *Melanospora papillata.*
      2. Bulbils abundant, perithecia usually absent.
        i. Primordium intercalary.
          (i). Bulbils brownish-yellow, central cells 28–55 μ in diam. . . . . *Papulospora immersa.*
          (ii). Bulbils straw-colored, central cells 10–20 μ in diam. . . . . *Papulospora irregularis.*
        ii. Primordium one or more lateral branches.
          (i). Primordium normally a single lateral branch.
            a. Primordium a spiral.
              § Cells heterogenous, definite cortex.
                A. One central cell.
                  ⁂ Cortex complete. *Papulospora parasitica.*
                  ⁂ ⁂ Cortex incomplete. *Acrospeira mirabilis.*
                B. More than one central cell.
                  ⁂ Spiral in one plane, cortical cells spinulose *Papulospora spinulosa.*
                  ⁂ ⁂ Spiral normally in more than one plane, 2–6 central cells.
                  α Bulbils dark brown. *Papulospora coprophila.*
                  β Bulbils brick red. *Papulospora rubida.*
              §§ Cells homogenous, bulbils 21–36 μ in diam. brownish producing sporotrichum spores. *Papulospora sporotrichoides.*

b. Primordium not a spiral.
§ Bulbils large, 100–750 μ, irregular. *Papulospora aspergilliformis.*
§§ " 70–150 μ, somewhat spherical, producing perithecia with slight papilla. . . . . *Melanospora anomala.*
(ii). Primordium two or more lateral branches forming a spherical aggregation of cells at the top. *Papulospora polyspora.*
III. Bulbils white to cream colored, 30–35 μ in diam. *Papulospora candida.*
IV. " steel gray, 21–36 μ in diam. . . . . *Papulospora cinerea.*

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EXPLANATION OF PLATES.

The figures of Plates 1–12 were drawn with the aid of a camera lucida using different combinations of the Bausch and Lomb lenses. All the mature bulbils were drawn with the same magnification, namely 4 mm. objective and 3 eye piece, and for the stages of development of the bulbils, 4 mm. objective and 12 eye piece were used. The plates have been reduced in reproduction about three-quarters.

PLATE 1.

Cubonia bulbifera.

Figures 1–6. Different forms of the primordium of the apothecium.
Figures 7, 8. Young apothecia.
Figure 9. Section of the mature apothecium.
Figure 10. Asci and paraphyses.
Figures 11–16. Stages in the development of the bulbil.
Figure 17. Mature bulbil.
Figure 18. Contortions of the hyphae.
Figure 19. Portion of a crushed bulbil with the contents of the cells escaping.
Figure 20. Ascospore.
Figure 21. The endosporium broken off.
Figures 22–24. Germinating Ascospores.
Figures 26, 27. Sprouting vegetative cells from the inner portion of the apothecium.
Figure 28. Germinating bulbil producing spiral primordia directly.

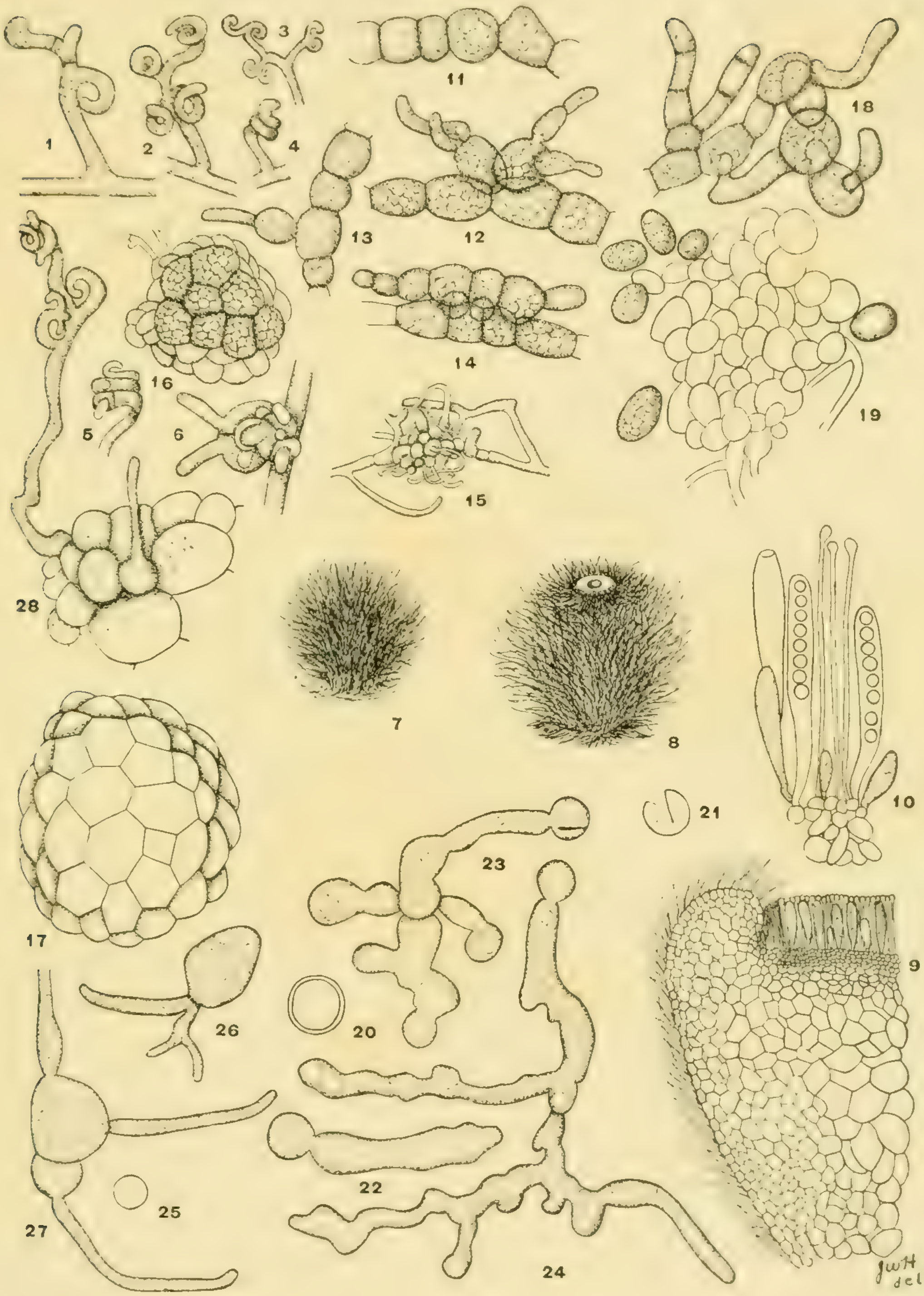
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PLATE 2.

MELANOSPORA PAPILLATA.

FIGURES 1–6. Stages in the development of the bulbil.
FIGURE 7. A group of Chlamydospore-like intercalary cells.
FIGURES 8–10. Stages in the development of the perithecium.
FIGURE 11. Outline of a mature perithecium showing the relative size of the bulbils.
FIGURE 12. A group of asci crushed from a young perithecium.
FIGURES 13–20. Germinating ascospores.
FIGURES 21, 22. Forms produced in Van Tiegham cell cultures.
FIGURE 23. Conidia on flask-shaped sterigmata produced on a hypha.
FIGURES 24, 25. Stages in the development of a terminal bulbil.
FIGURE 26. An intercalary bulbil with three large central cells.

MELANOSPORA ANOMALA.

FIGURES 27–30. Stages in the development of the bulbil.


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PLATE 3.

Melanospora anomala.

Figures 1–12. Stages in the development of the perithecium.
Figure 12. Mature perithecium.
Figure 13. (a) Germinating ascospore showing a bottle-shaped sterigma.
(b) Bottle-shaped sterigma on a hypha.
Figures 14, 15. Other stages in the formation of the bulbil.
Figure 15. A mature bulbil.

Melanospora cervicula.

Figures 16, 17. Primordia of the bulbil.
Figure 18. A bulbil produced from a group of terminal cells.
Figure 19. Primordium of the perithecium and conidia on flask-shaped sterigmata.
Figure 20. Mature perithecium.
Figure 21. Abnormal forms common among the hyphae.
Figure 22. Chlamydospores of the Acremoniella type.
Figures 23, 24. "Harzia-like" fructification.

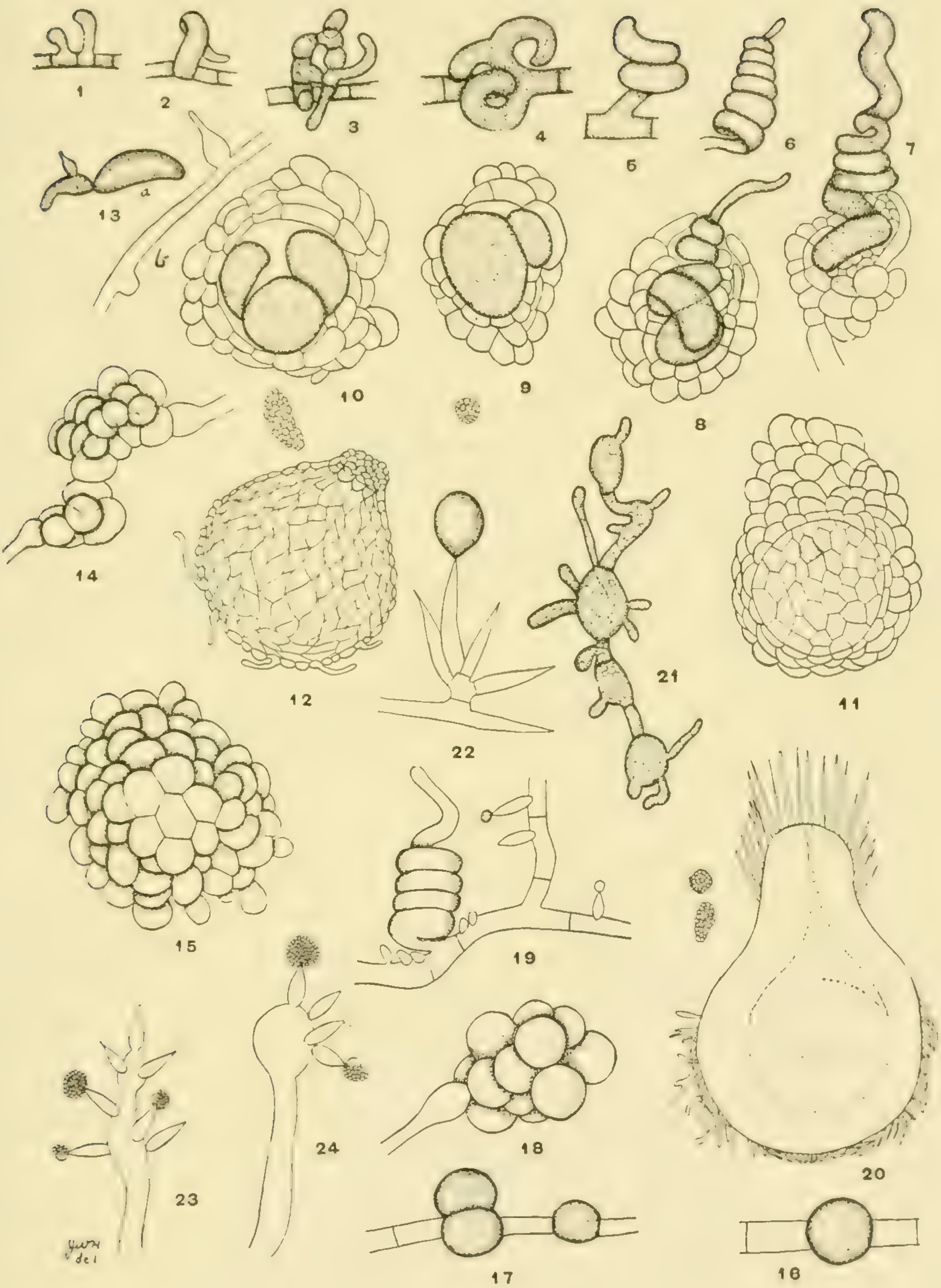
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PLATE 4.

PAPULOSPORA CANDIDA.

FIGURES 1, 2. Variation in the size of the conidia.
FIGURES 3–12, and 15–27. Stages in the germination of the conidia and the development of the bulbil from them.
FIGURES 28–41. Stages in the development of the bulbil from a lateral branch of the hyphae.
FIGURE 42. Germination of the superficial cells of the bulbil.
FIGURE 43. Conidiophores of Verticillium agaricinum var. clavisedum.
FIGURE 44. Portion of the hyphae showing large oil globules.
FIGURE 45. Showing intimate connection between the bulbil and the Verticillium.
FIGURE 46. An irregular primordium of a bulbil.
FIGURE 47. Ascoma of Geoglossum glabrum attacked by the parasite.


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PLATE 5.

Papulospora parasitica.

Figures 1–14. Show various stages in the development of the bulbil.
Figures 4, 5, & 9, 10. Show the protuberance from the lateral surface of the large central cell.
Figures 15, 16. Germinating bulbils.
Figure 17. Conidia-like bodies connected with the bulbil.
Figures 35b, 36. Swollen intercalary cells.

Acrospeira mirabilis.

Figures 18–23. Stages in the development of the bulbil.
Figure 20. The end-cell has enlarged to form the central cell.
Figure 21. The second cell has enlarged to form the central cell.
Figure 22. Several empty cortical cells are shown.

Reproductive Bodies Resembling Bulbils.

Figure 24–34. Irregular forms of a doubtful bulbil (No. 170).
Figure 35. Spore of Stephanoma strigosum Wallr.

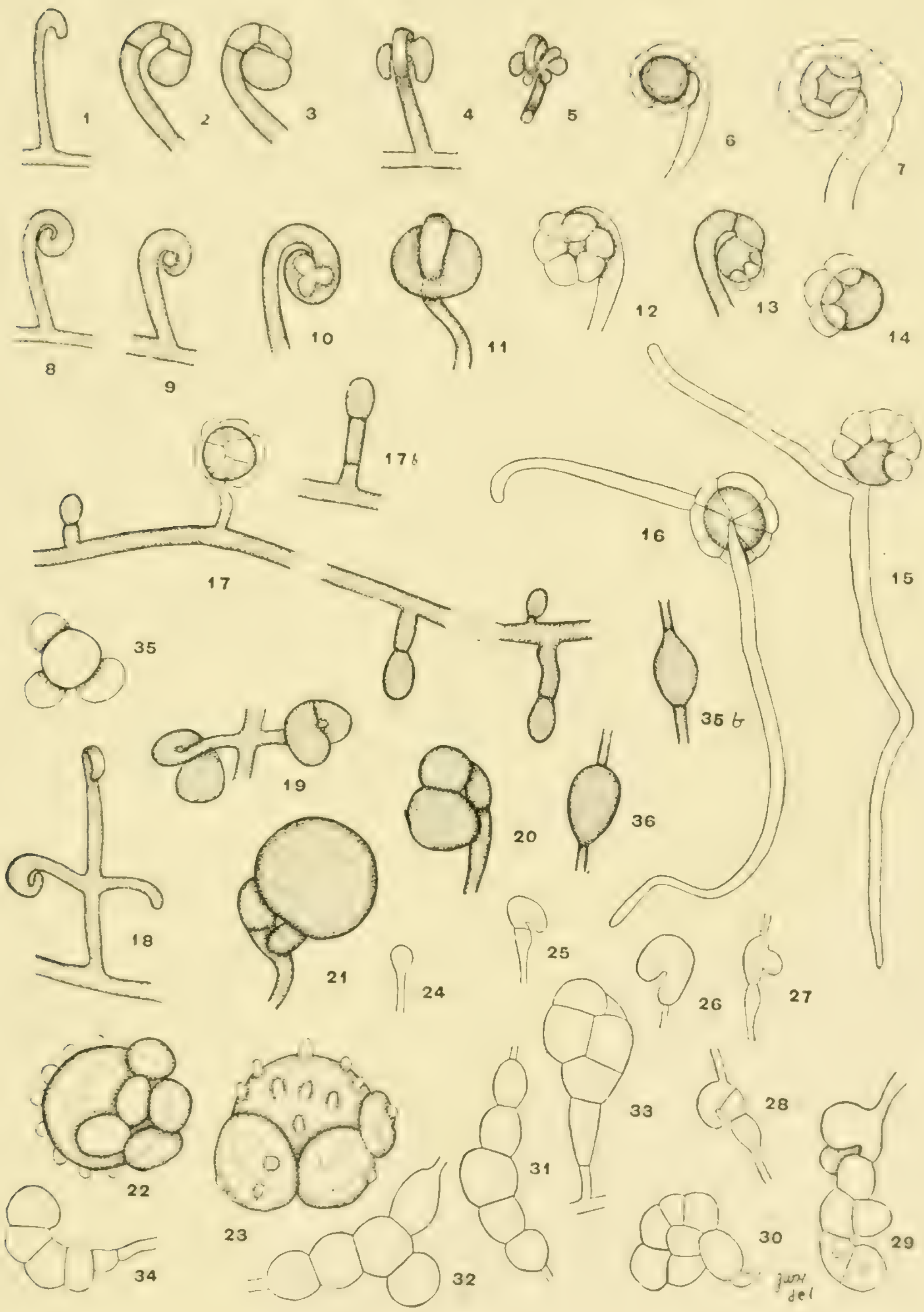
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PLATE 6.

Grandinia crustosa.

Figure 1. Pustulate habit of the fructification.
Figure 2. Hymenium with basidiospores.
Figure 3. Basidiospore.
Figures 4–10. Stages in the development of the bulbil.
Figure 10. Mature bulbil with the same magnification as all the other mature bulbils.

Papulospora anomala.

Figure 11–17. Stages in the development of the bulbil.
Figure 17. Mature bulbil.
Figure 18. Two primordia close together.
Figure 19. Large intercalary cells densely filled with oil globules.

Papulospora pannosa.

Figures 20–24. Stages in the development of the bulbil from intercalary cells.
Figure 25. Occasional mode of formation of intercalary primordia.

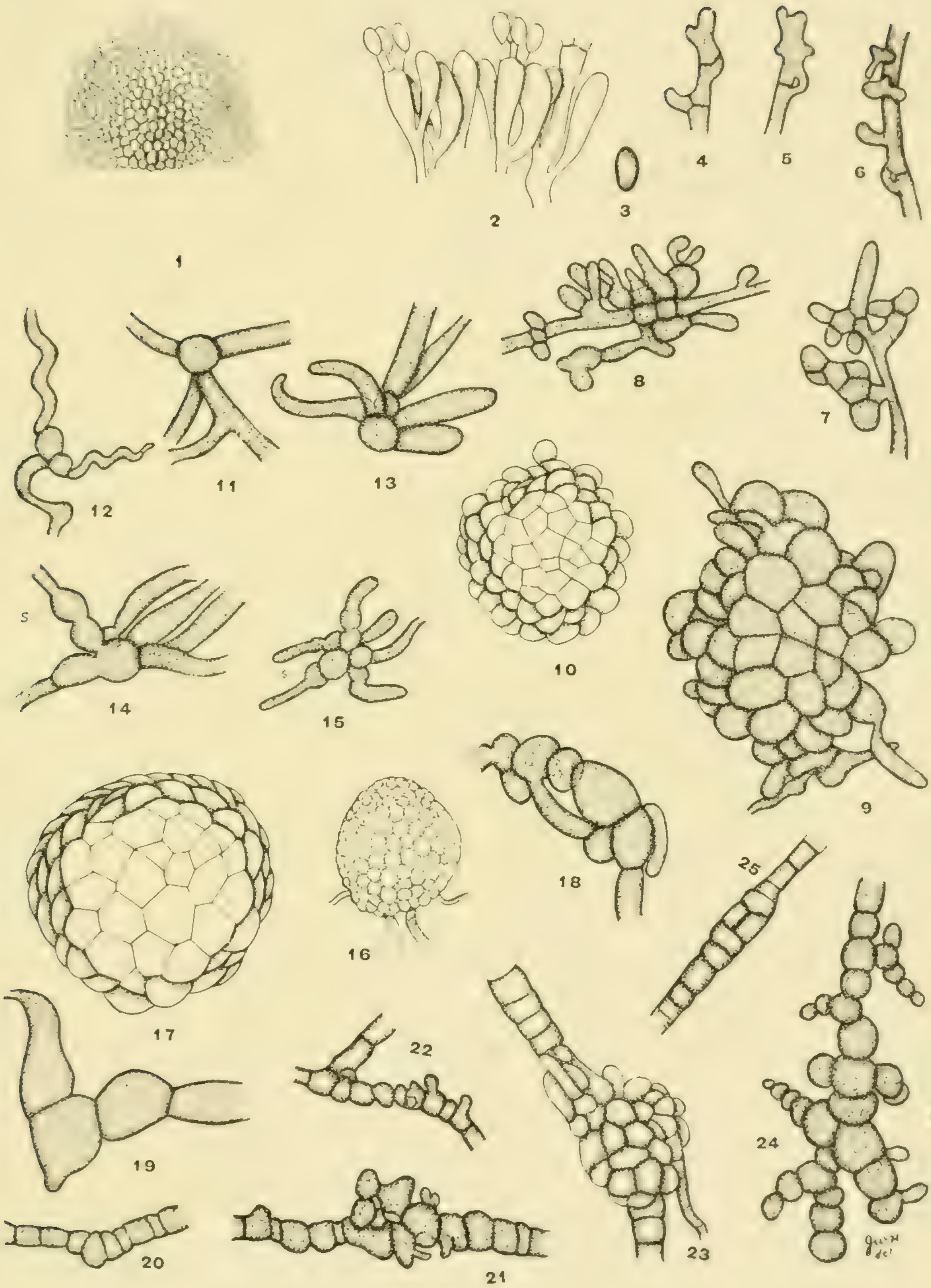
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PLATE 7.

Papulospora aspergilliformis.

Figures 1–4, & 6. Stages in the development of the bulbil.
Figure 5. A group of Chlamydospore-like bodies.
Figure 7. A primordium that produces a very irregular bulbil.
Figure 8. "Aspergillus-like" heads produced directly from the bulbil.
Figures 9–12. Different forms of the "Aspergillus-like" fructification.
Figure 12. Abnormal conditions.
Figures 13–16. Chlamydospores.
Figures 17, 18. Large swollen cells, likely storage cells.
Figure 19. Bulbil forming from terminal cells.
Figure 20. Section of a mature bulbil.

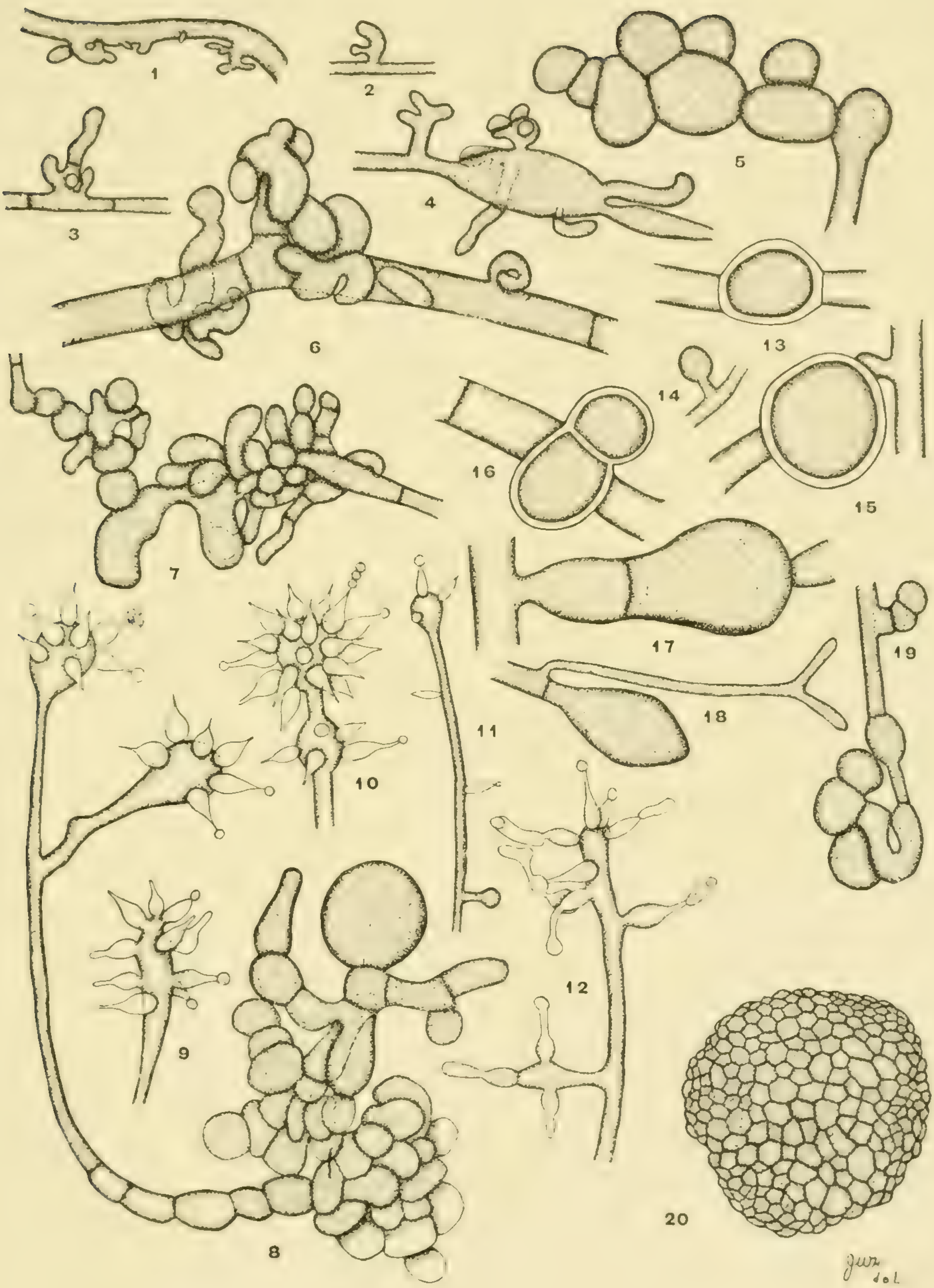
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PLATE 8.

Papulospora cinerea.

Figures 1–10. Stages in the development of the bulbil.
Figures 4, 6, & 9. Modifications of the regular mode of development.
Figures 10, 11. Mature bulbils.

Papulospora rubida.

Figures 12–16. Stages in the development of the bulbil.
Figures 25a–27a, 21, 22. Other stages in the development of the bulbil.
Figures 17, 20. The spiral primordium that sometimes occurs.
Figure 25. Section of a mature bulbil showing five large central cells.
Figure 18. Surface view of a mature bulbil.

Papulospora pannosa.

Figures 28–30. The development of a bulbil from a lateral branch.
Figure 31. A collapsed hypha showing rigid septa.

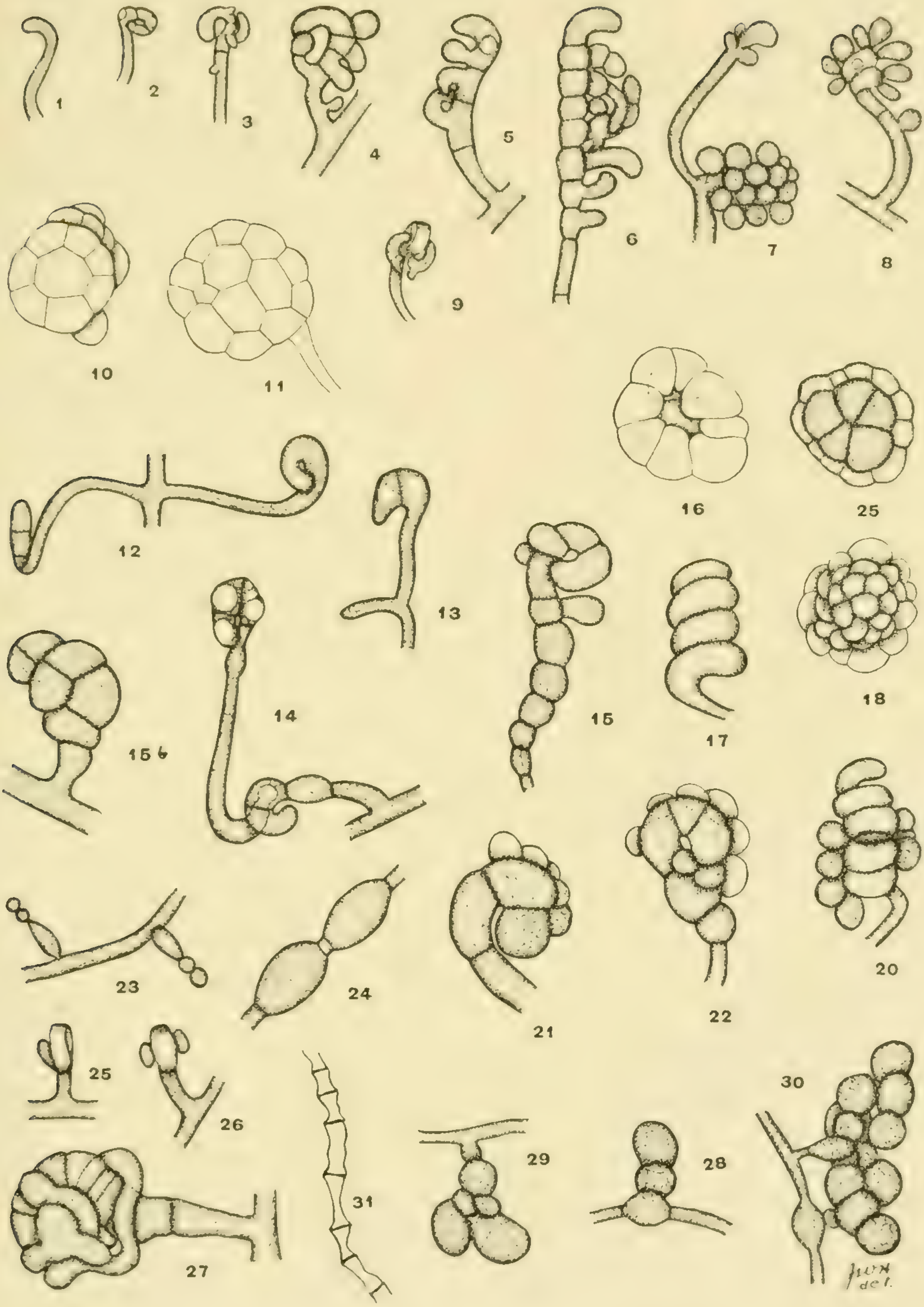
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PLATE 9.

Papulospora spinulosa.

Figures 1–7. Stages in the development of the bulbil.
Figure 8. Primordia produced from a superficial cell of an immature bulbil.
Figure 9. Section of a mature bulbil showing the "Annulus."
Figure 10. A surface view of the same looking down on the "Annulus."

Papulospora irregularis.

Figures 11–17. Stages in the development of the bulbil.
Figure 17. A mature bulbil.

Papulospora pannosa.

Figures 18–20. Stages in the development of the bulbil.
Figure 20. A mature bulbil.

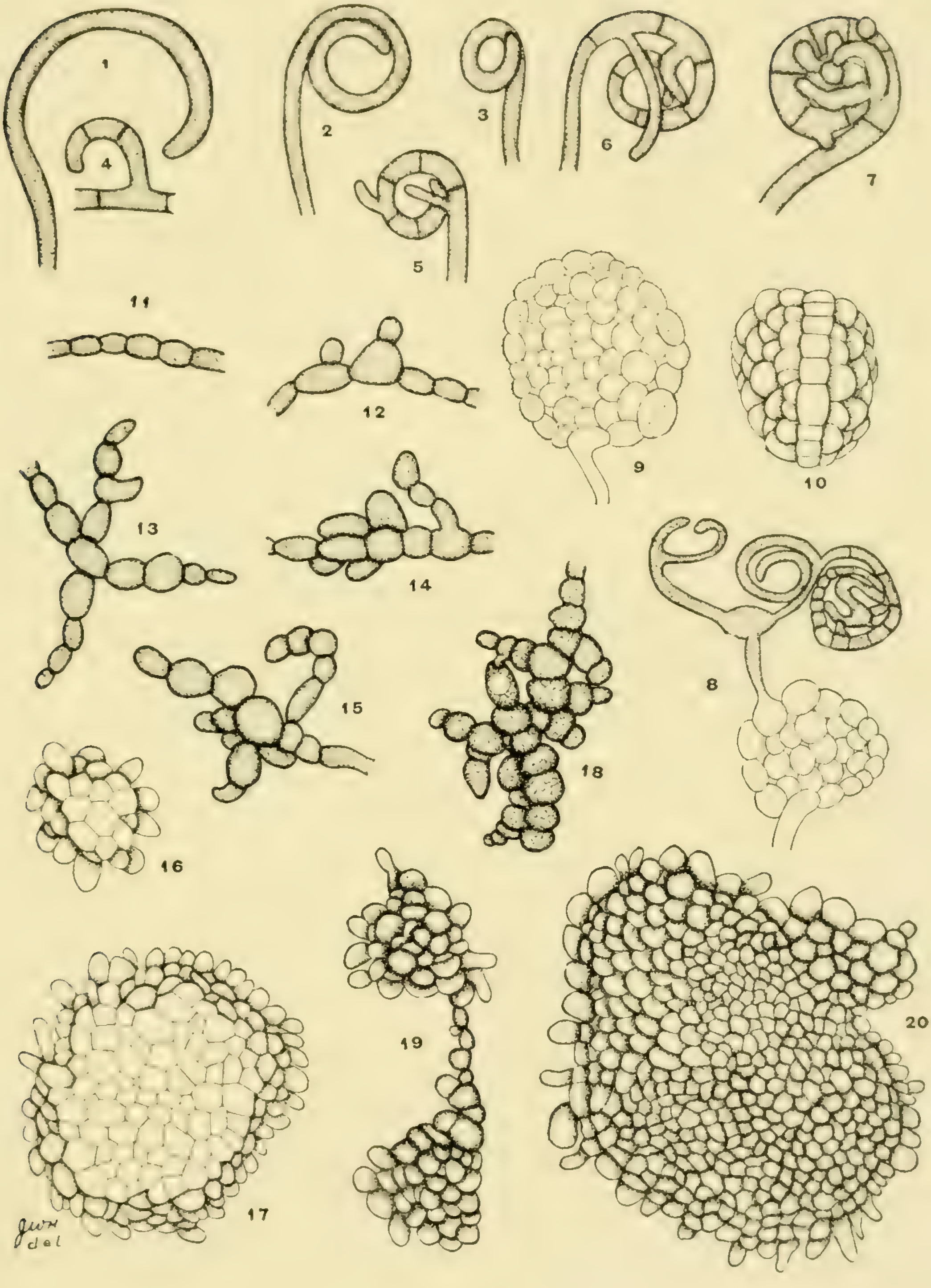
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PLATE 10.

Papulospora coprophila.

Figures 1–8. Stages in the development of a bulbil from a spiral.
Figure 6. An unusual condition, the production of conidia directly from the spiral.
Figure 8. A spiral primordium surrounded by an irregular layer of cells.
Figure 9. Immature bulbil that has developed like Figs. 14 and 15, and also a spiral primordium.
Figure 10. Median section of a mature bulbil with two large central cells.
Figure 11. A mature bulbil with the contents of the large cells crushed out (Fig. 11b).
Figure 12. Germination of one of these cells.
Figures 13–15. Forms arrested in the process of development.
Figures 16. Surface view of the mature bulbil.

Papulospora immersa.

Figure 17. Irregular hypha densely filled with protoplasm. The primordium of the bulbil.
Figure 18. Primordium consisting of a single intercalary cell.
Figure 19–25. Stages in the development of the bulbil.

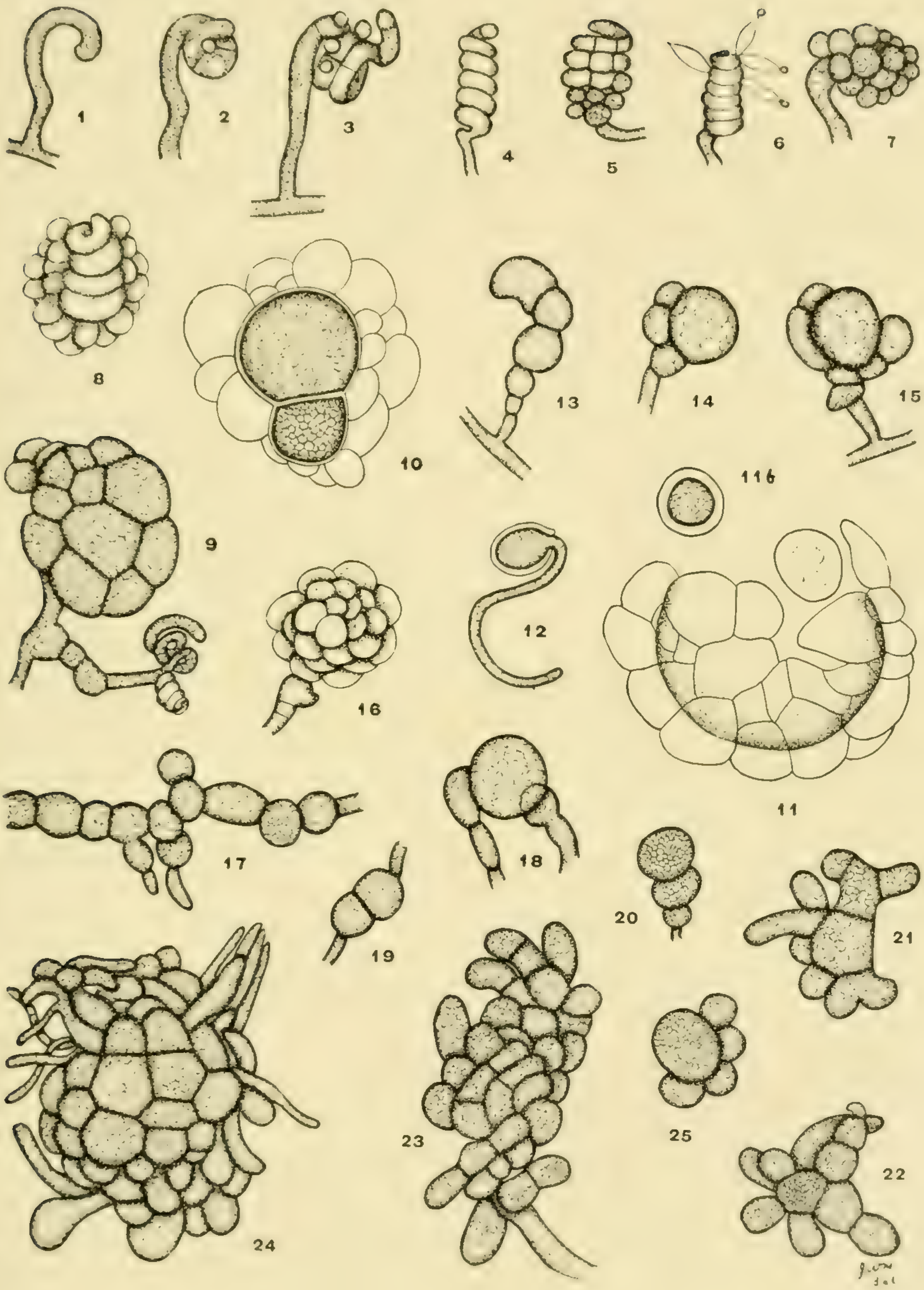
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PLATE 11.

Papulospora polyspora.

Figures 1–7. Stages in the development of the bulbil.
Figure 7. A mature bulbil.
Figure 8. Group of spores adhering loosely together.
Figures 9 & 10. Germinating spores.
Figures 9b, 10b, 11–13. Modifications that occur when grown in liquid media.

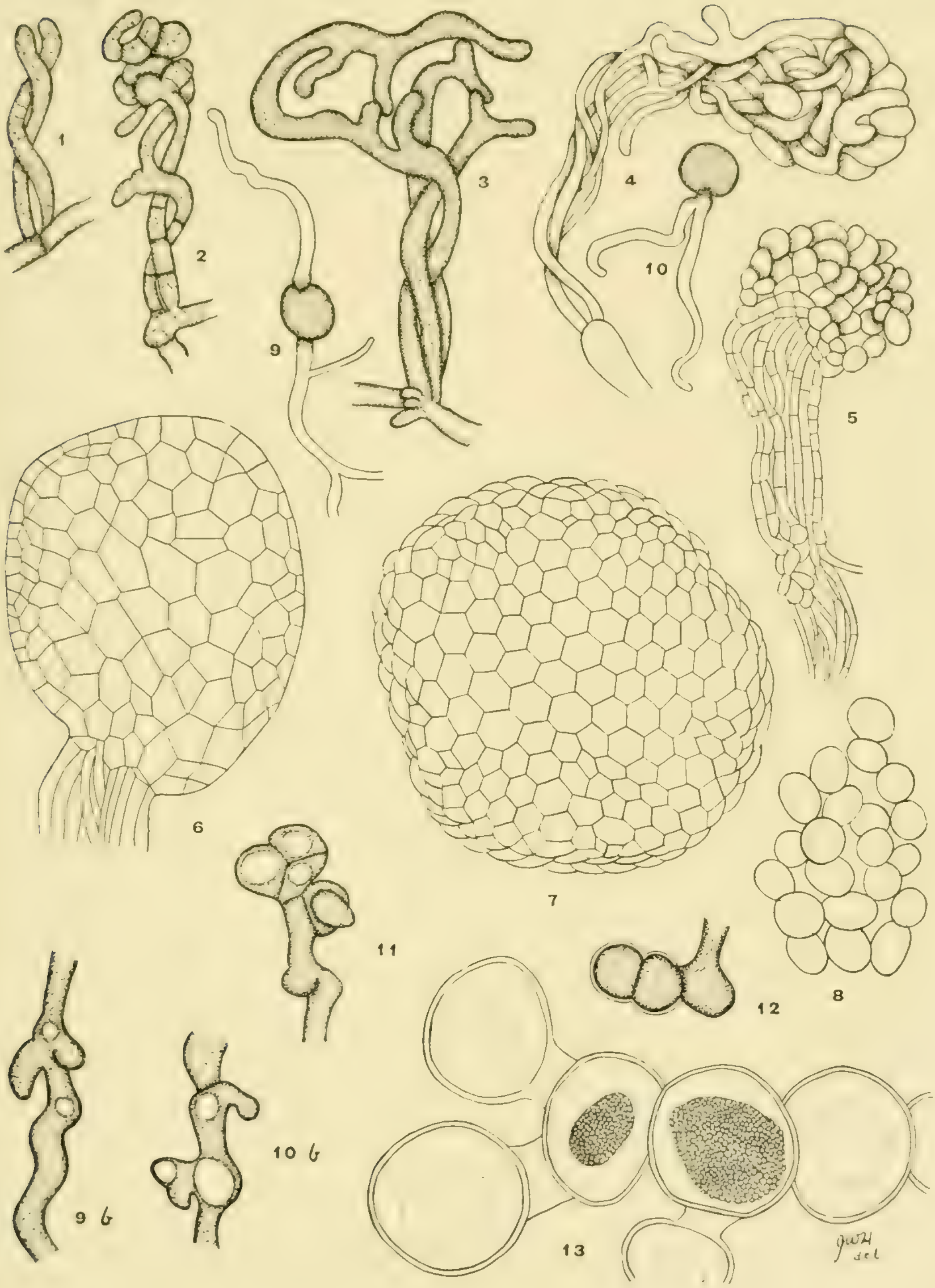
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PLATE 12.

PAPULOSPORA SPOROTRICHOIDES.

FIGURES 1–9. Stages in the development of the bulbil.
FIGURE 8. A mature bulbil.
FIGURE 9. A side view of an immature bulbil.
FIGURES 10, 11. Abortive forms.
FIGURES 12–16. Modifications in the formation of the spiral.
FIGURE 17. An irregular bulbil germinating, magnified more than the others.
FIGURE 18. Branch of the hyphae showing primordia of the bulbils.
FIGURES 19–25. Modifications in the development of the bulbils which are hyaline.
FIGURES 26–28. Semi-diagrammatic representation of the mode of cell formation in the development of the hyaline bulbils.
FIGURE 29. A section of a mature bulbil.
FIGURES 30, 31. Large intercalary and terminal cells found in the hyphae.
FIGURES 32–34. Germinating bulbils.
FIGURES 25–26. Conidiophores with conidia.
FIGURE 37. Conidiophore produced directly from the bulbil in a Van Tieghem cell culture.
FIGURE 38. Conidium.
FIGURE 39. The form the conidia usually assume before germinating.
FIGURES 40, 41. Germinating conidia.


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CONTRIBUTIONS FROM THE JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY.

THERMODYNAMIC PROPERTIES OF LIQUID WATER TO 80° AND 12000 KGM.

By P. W. Bridgman.

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By P. W. Bridgman.

Received June 26, 1912.

Table of Contents.

Introduction.

This paper is in the nature of a supplement to a former paper on the properties of water in the liquid and the solid forms.[1] The solid forms were studied over a range of 20,000 kgm./cm.^{2}, and from $-80°$ to $+76°$, but the study of the liquid reached only from the lowest temperature of its existence to about $+20°$. Above $0°$, measurements were made on the liquid at only $20°$. The two measurements, at $0°$ and $20°$ were sufficient to give the mean dilatation between $0°$ and $20°$, but not the variation of dilatation with temperature. It was assumed in the earlier paper that the variation of dilatation with temperature became negligible at high pressures, since this seemed to be the most plausible assumption in view of all the data then available.

In this present paper the study of the liquid has been continued from $20°$ to $80°$, and to 12000 kgm. The pressure range is greater than that of the preceding paper by about 2,500 kgm. The range is not great enough to entirely cover the region of stability of the liquid, but it is as great as it was convenient to cover with the method used here, which is different from that of the former work. It has the advantage of very much greater rapidity of operation, but since it depends on the complete elastic integrity of the steel pressure cylinders it is not possible to reach so high pressures with it as with the former method. [The former limit of 9500 kgm. was set by the freezing of the liquid and was not due to any limitation of the method.] Nevertheless, it may be hoped that the present temperature and pressure ranges are both wide enough to give a fairly complete idea of the nature of the effects to be expected at high pressures with varying temperature.

Measurements of the dilatation have been made at four temperatures, so that it has been possible to find the variation of dilatation with temperature at any pressure. Perhaps the most unlooked for feature disclosed by the measurements is the fact, contrary to the assumption of the first paper, that the variation of dilatation with temperature does not become vanishingly small at high pressures, but reverses in sign. This means that while at low pressures the volume increases more and more rapidly with rising temperature, at high pressures the expansion becomes more slow at high temperatures.

The data of this paper are sufficient to completely map out the p-v-t surface over the domain in question: Both the first and second

1 Bridgman, These Proceedings, **47**, 439–558 (1912).

derivatives are therefore completely determined, so that we now have all the data at hand for the determination of any one of the thermodynamic properties of the liquid. This means that we are in a position to find such quantities as the specific heats, change of internal energy, adiabatic temperature rise etc., as well as the more easily determined compressibility and thermal dilatation. The latter part of the paper, after the discussion of the method and the presentation of the data in the first part, is occupied with the computation of these various thermodynamic quantities. The accuracy of some of these is probably not very great, because the error in the second derivative of an experimental quantity may be considerable. It has, therefore, seemed best to give the general view of the nature of the quantities which is offered by a graphical representation, rather than to give tables, with the tacit assumption of greater accuracy which usually goes with a set of tables. In spite of the lower order of accuracy of some of these thermodynamic quantities, it has still seemed well worth while to give them, since even the general trend of some of the quantities, such as the specific heats, has not been hitherto known with relation to pressure.

The data presented here are only the beginning of a projected study of the characteristic surface under high pressures for a number of liquids. The measurements have already been carried through for twelve other liquids beside water. The purpose of this study is ultimately the development of a theory of liquids, since it would seem that a much more intimate grasp of the nature of the forces at work in a liquid would be afforded by a study over a wide pressure range, than over the comparatively low pressures hitherto used. It must be admitted, however, that this broader purpose is not particularly furthered by this work on water, because of the well known abnormalities of this substance. In the previous paper several abnormalities had been shown to exist at low pressures. In this paper, new abnormalities are found at higher pressures. Water gives the appearance of becoming completely normal only at the higher temperatures and pressures of the range used here, but of course whether this is really normal or not cannot be told until the behavior of normal liquids has been discovered. The full significance of the present data, in their bearing on such questions as the polymerization of the liquid, for example, cannot appear until after the discovery of the laws for entirely normal liquids. The investigation of water before that of normal liquids was undertaken for two reasons; firstly because of the desire to complete the work for water already begun, and secondly because in this and the following investigation a new method for determining the compressibility was to be used, which had not yet been proved to be reliable, but which could be tested by a comparison of the results obtained by this method with those already obtained by another method at lower temperatures for water.

In addition to the data for liquid water, two other quantities were determined incidentally in the course of the work, and are given at the end of the paper. One of these is the experimental measurement of the compressibility and thermal dilatation of ice VI between 0° and 20° and 6360 and 10,000 kgm. The other is the measurement of the volume of kerosene up to 12,000 kgm. and between 20° and 80°.

The Method.

The method in its fundamental idea is as simple as it would well be possible to devise. The substance, whose compressibility or thermal dilatation is to be measured, is placed in a heavy steel cylinder in which pressure is produced by the advance of a piston of known cross section. The change of volume, given by the distance of advance of the piston, is measured as a function of the pressure. The method is simple, rapid, and above all, applicable to the highest pressures. But there are a number of corrections which must be made, often difficult to determine, which doubtless account for the slight use which has been made hitherto of the method. Apparently, with the exception of the present work, it has been used recently only by Tammann,[2] and by Parsons and Cook.[3] Tammann and Parsons and Cook applied it only to the measurement of compressibility, reaching pressures of about 4000 kgm. The author has previously applied it to the measurement of the thermal dilatation of water at temperatures below 0° C. over a pressure range of about 6500 kgm.

The most serious of the errors which readily occur to one is that of leak. It is almost essential to the success of the method to secure a piston absolutely free from leak, and this has hitherto been a matter of some difficulty at high pressures. Tammann did not entirely secure this freedom from leak, but avoided it in large measure by the use of a very heavy oil, such as castor oil, and still further lessened the error by correcting for the slight amount of leak by measuring the amount of liquid which escaped past the piston in a given time. This method would not be applicable to the highest pressures, however, because

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2 A. D. Cowper and G. Tammann, ZS. Phys. Chem., **68**, 281–288 (1909).
3 Parsons and Cook, Proc. Roy. Soc. A, **85**, 332–349 (1911).

of the freezing of the oil. Parsons and Cook were able to secure entire freedom from leak up to 4000 kgm. by the employment of a cupped leather washer combined with a brass disc of special design. It has been the experience of all those who have worked with high pressures, however, that no leather washer is capable of standing pressures very much in excess of the limit of 4500 kgm., since the leather rapidly disintegrates under the pressure. In the present work the same form of packing was used which was used in the previous work on the freezing of water and mercury under pressure. This has been proved in the previous paper to be absolutely free from leak up to the highest pressures which can be sustained by the steel containing vessels. In the present work this same packing has proved itself to be reliable for the purposes of this method.

The question of the method of measuring pressure is also of considerable importance in using this method, since the usual measuring devices, such as a Bourdon gauge, cannot be applied, for reasons to be discussed later, and attempts to calculate the pressure directly from the force required to produce motion of the piston are likely to be in error because of the friction of the packing. Parsons and Cook did, however, adopt this latter method, and computed the pressure from the known force required to move the piston. The effect of the friction of the packings was allowed for in as large a degree as possible by taking the mean of the readings during increasing and decreasing pressure, assuming that the friction remained constant. The results obtained by Parsons and Cook in this way were surprisingly good. That the friction did remain fairly constant was indicated by the constancy of the results and the fact that the curve nearly always returned to the starting point; but it is doubtful if the method would work at very much higher pressures because of the increase of friction due to the flow of the softer parts of the piston. The brass washers used by Parsons and Cook would almost certainly have upset under two or three thousand more kgm., and it is the experience of the author that it is difficult to obtain even steel washers which will stand much more than 8000 kgm. without taking some set. In fact, at high pressure there must necessarily be some plastic yield, in order to follow the expansion of the cylinder. The result of this set in the washers is that the friction becomes very irregular, and cannot be assumed to be the same during increasing and decreasing pressure. Variations in the amount of friction due to this cause of as much as 200 or 300% have been found at the higher pressures of this work.

The only escape from the difficulty seems to be to measure the pressure directly inside the cylinder. This was done by Tammann by connecting a Bourdon gauge directly to the cylinder. But it is known that the errors of the Bourdon gauge become rapidly more serious at higher pressures,[4] due to the increase of hysteresis, so that this gauge could not be used for the pressures of this experiment. Furthermore, no Bourdon gauge has up to the present been made of sufficient sensitiveness which is capable of standing more than 6500 kgm. In the present work the pressure was measured inside the cylinder by inserting directly into it a coil of manganin wire, which had been already calibrated against an absolute gauge. This method of measuring pressure has been fully described in a previous paper.[5] It was necessary for the purposes of the present work, however, to make a somewhat more careful determination of the temperature coefficient than was done formerly, and this determination will be described in detail later. The method has shown itself perfectly satisfactory and reliable in every respect. One coil of wire has been used almost continuously for over six months, and occasional calibrations have shown no change. These calibrations were made by measuring with the coil certain fixed temperature-pressure points, such as the freezing pressure of mercury or of ice VI, at some fixed temperature.

The apparatus used in the present work is the same in most features as that used in the former work, a detailed account of which has already been given in the papers mentioned. Only the points in which this has been changed will be mentioned here. It was a disadvantage of the former method that the apparatus consisted of two parts; the lower part, a cylinder containing the liquid to be measured, was placed in a thermostat, and the upper part, a cylinder in which pressure was produced, was exposed to the temperature of the room. When temperature was changed in the thermostat below or pressure was changed in the cylinder above, liquid passed from the one cylinder to the other, experiencing in the transition a change of temperature, and so a change of volume also. This change of volume accompanying a known change of temperature varies in an unknown way with the pressure, and to apply the correction it was necessary to make an independent set of experiments. In the present form of apparatus the difficulty was avoided by including everything in one cylinder. This cylinder contained the liquid under investigation, the pressure measuring coil, and the piston by which pressure was produced. It

4 Bridgman, These Proceedings, **44,** 201–217 (1909).
5 Bridgman, These Proceedings, **47,** 319–343 (1911).

was placed in the lower part of the hydraulic press and, together with the lower part of the press, was placed in the thermostat. The dimensions were so small that this could be done without increasing to an unwieldly bulk the size of the apparatus, the four tie rods of the press being 1 1/8″ in diameter and their centers 6″ apart. It is the same form of apparatus which was used for the measurements on ice VI up to 20,500 kgm. The present experiments run to only 12,000 kgm., however, since it is evidently an absolute essential to the success of the method that there should be no permanent distortion of the cylinder. It would be easily possible to reach pressures much higher than those reached in this experiment, but it was felt that the risk and the extra time involved in the probable construction of new apparatus was not justified at present, when it seemed that the most important work was to map out the field, obtain data for as many liquids as possible, and determine the general nature of the significant problems. Later, if there are crucial points which need the use of much higher pressures, it will be a comparatively easy matter to obtain them.

The cylinder used in this experiment was not the same as that used in the previous work on water. This new cylinder is from a piece of chrome-vanadium steel made in the electric furnace by the Halcomb Steel Co., of Syracuse, N. Y. The steel itself is a wonderful product, and without it the present investigation would not have been so easily possible. It shows a tensile strength of 300,000 lbs. per sq. in. when hardened in oil, and an elastic limit of about 250,000 lbs. These figures are considerably in excess of those for the steel used in the previous investigation. The steel furthermore is remarkably homogeneous, because of its production in the electrical furnace. One of these pieces was pierced with a hole 1/8″ diameter and 13″ long, and the drill came through concentrically without any variation from the straight line. The dimensions of the cylinder used in the present work were 4 1/2″ outside diameter, 13″ long, inside diameter 17/32″ for the greater part of its length, with an enlargement to 3/4″ at the lower end for the reception of the manganin coil. The original inside diameter was 7/16″. The cylinder was prepared for use by hardening in oil and then subjecting to a pressure much in excess of that contemplated for the actual experiment. The seasoning pressure was over 30,000 kgm. Even under this high seasoning pressure the cylinder showed very little permanent change of internal dimensions, not stretching as much as 1/32.″ This is less than the amount of stretch which has been found for any other grade of steel. The effectiveness of the treatment is shown furthermore in the fact that in over six months of continual use the inside has not stretched by so much as an additional 1/10000″. The hole was enlarged to a final size of 17/32″, instead of keeping it as small as possible, because of the difficulty of reaming out the hole so as to give a satisfactorily smooth surface after the seasoning process. The difficulty was occasioned by the hardness of the steel, and several attempts were necessary before the desired result was produced.

The pressure measuring coil was the same as that used in the last part of the work on ice VI. The construction of the insulating plug was also the same as that used there. During the course of the work it was necessary to take this plug apart several timess, because water had reached the mica washers, and once or twice the mica washers themselves have given way. These mica washers are the weakest part of the entire apparatus as at present used, since they gradually disintegrate and fail by shear after prolonged use, but it is a matter of only a few hours to replace them. Every time after the insulating plug has been freshly set up it has been tested for insulation resistance, both during application of pressure and after release. The resistance was in all cases as high as several hundred megohms, the limit of the measuring devise. The steel of the insulating plug has also failed once or twice by the "pinching-off effect"[6] after long use. This also is an easy matter to repair. Failure of this type is attended with some danger, however, because of the violence of the explosion with which the ruptured plug is expelled. The surest way of avoiding this danger is to so mount the apparatus that the plug points at the floor or other indestructible object.

The hydraulic press, the method of measuring the displacement of the piston, and the details of the packing of the moving piston, were the same as that used in the former paper.

In the use of the apparatus to determine compressibility there is one serious error which did not enter into its use in the determination of the change of volume during change of state, namely the correction for the distortion of the cylinder in which the piston moves. At low pressure the correction is relatively unimportant, and may be computed from the theory of elasticity, if one is willing to assume that the theory is sufficiently accurate for this type of stress. But at higher pressures the correction becomes more important, increasing in percentage value directly with the pressure, and is almost certainly

6 Bridgman, Phil. Mag., **24**, 63–79 (1912).

not calculable by the theory of elasticity, because of the entrance of such effects as hysteresis. To determine the correction an auxiliary set of experiments is necessary. Evidently if the true value of the compressibility of some one substance were sufficiently well known, then the apparent compressibility as determined by this method would give the correction for the distortion of the cylinder. No such compressibilities are known with any high percentage accuracy, but this is not necessary, provided only that the uncertainty in the standard compressibility is small in comparison with the distortion of the vessel. The substance which most readily suggests itself because of its small compressibility is steel, but this is a solid, whereas the method is applicable directly only to liquids, so that some modification of the procedure is necessary. Such a modification readily suggests itself, and has been used by the author in the previous determinations of the thermal dilatation of water at temperatures below 0°, and has also been used by Parsons and Cook. The modification is to replace part of the liquid under investigation by a steel cylinder, and determine the compressibility of the liquid and the steel together. The difference of two determinations, the one for the liquid alone, the other for the liquid and the steel, gives a value for the difference of compressibility between the liquid and the steel from which the effect of the distortion of the vessel has been almost entirely eliminated. Furthermore, the compressibility of the steel is so small in comparison with that of the liquid that the slight uncertainty in the value for the steel is of no account, so that the compressibility of the liquid is given directly.

The application of this method would demand, then, that the interior of the cylinder be filled first with water and the apparent compressibility determined, and then part of the water replaced by steel and the apparent compressibility determined again. But this demands that the coil of manganin with which the pressure is to be measured come directly in contact with the water, which evidently cannot be allowed because of the short circuiting produced by the water. It seemed to be necessary, then, to devise some sort of protection for the coil, which should not occupy so much volume as to introduce a serious correction, and which should at the same time transmit the pressure readily to the innermost parts of the coil. Considerable time was spent in trying to devise such a protection. The scheme adopted was to surround the coil with a small mass of vaseline enclosed in a flexible sac, formed from the finger of a silk glove, and rendered impervious to water by painting it over with several coats of the collodion of surgeons. This sac was tied with silk thread directly over the end of the insulating plug. It was proved by trial that the vaseline did not become so viscous under pressure as to refuse to transmit the pressure with sufficient freedom, but the arrangement did not prove itself as trustworthy as was to be desired. The collodion might leak after several applications of pressure, which made it necessary to reassemble the insulating plug and redetermine the elastic constants of the apparatus, because the distortion included in the plug itself was sufficient to introduce appreciable error. The device probably could have been made usable with a little more effort, but it would always have been more or less unsatisfactory, and would have been applicable only to those liquids which do not attack the collodion, whereas most of the organic liquids which it was desired to use in the future do so attack the collodion. The attempt to protect the coil was abandoned after a month's work, therefore, and the method replaced by another, which at first sight introduced additional complications, but which is really just as simple as the first, and has the advantage of being applicable with only slight modifications to the investigation of other liquids.

The modified method used two liquids in every determination, one beside the one whose compressibility is to be measured. The water under investigation is placed in a thin shell of steel fitting the inside of the cylinder. This shell, when in position in the cylinder, is surrounded on all sides and above and below by kerosene, which below transmits pressure to the manganin coil, and above reaches to the moving piston with which pressure is produced. In the auxiliary experiment to eliminate the effect of the distortion of the cylinder, the shell with water is replaced by a solid cylinder of steel, and the quantity of kerosene remains the same as before. The motion of the piston due to the change of volume of the kerosene remains the same in the two experiments, therefore, and the difference of readings of the two sets gives directly the difference of compressibility between the water and the steel. The disadvantage of the method is that it is not possible to use so large quantities of water as in the former method, because the steel shell containing the water remains invariable in length under pressure, and enough kerosene must be introduced originally to take up the change of volume of the water in this shell as well as the distortion of the other parts of the apparatus. The reduction in the quantity of water under experiment is not greater than 30%, however, and the other advantages more than outweigh this comparatively small loss of accuracy.

The procedure in using the apparatus in this finally modified form is as follows. The manganin coil is first screwed into the lower part of the cylinder. The rubber washer used to make this plug tight is one cut with a standard set of cutters, so that all the washers used for this purpose are always the same in size. This insures that the distortion due to the compression of the washers shall always be the same. The steel shell with the water in it is next introduced from above. The quantity of water is previously determined by weighing. It is desirable not to fill the shell to closer than 1/4″ of the top, experience having shown that otherwise water is likely to spill out and find its way to the manganin coil. The kerosene is next introduced into the cylinder from above. To ensure entire filling of all parts of the apparatus and the exclusion of air, only part of the kerosene is at first poured in, the air is then exhausted by attaching the mouth of the cylinder to an air pump, or simply by exhausting with the lungs, and then the remainder of the kerosene poured in. The amount of kerosene is determined by weighing the dish from which it is poured before and after filling. Because of the wetting of the dish by the kerosene it is not always possible to obtain exactly the amount of kerosene desired each time, but the variation is seldom over 0.02 gm., and the very slight effect of this discrepancy may be corrected for, as will be described later. Finally the movable plug is introduced into the top of the cylinder, taking particular pains not to allow any of the kerosene to escape in the process. Here again the rubber washer used has been cut with standard cutters, so that the amount of rubber used here is also the same in all the experiments. The cylinder is then placed in the thermostat, and the zero of the manganin coil read at the temperature of the room. The thermostat is then adjusted for the desired temperature and the cylinder seasoned for the run by the application of pressure.

A preliminary seasoning is necessary because of the hysteresis shown by the cylinder, and this hysteresis is shown with respect to both pressure and temperature. Many of the early results were somewhat in error because the necessity of this seasoning for temperature as well as for pressure was not clearly recognized. The method of seasoning to be adopted depends on the kind of data which it is desired to obtain from the run, whether the compressibility at constant temperature or the thermal dilatation at constant pressure. If it is desired to determine the isothermal compressibility, the seasoning consists simply in raising the pressure through the entire range and releasing several times. It was found by experiment that three such preliminary excursions were sufficient; after this the cylinder settles down into a state in which the normal hysteresis cycles are retraced with perfect regularity. Of course it is necessary to make the compressibility determinations immediately after this seasoning, as the effect gradually disappears with time. The time occupied in making the final readings to 12,000 kgm. and back with increasing and decreasing pressure, making in all 20 readings, might vary from two to three hours. After every change of pressure it was necessary to wait for the temperature effect of compression to disappear; this time was from 5 to 7 minutes.

If the thermal dilatation under constant mean pressure is to be determined, the seasoning consists simply in taking the cylinder once through the temperature range contemplated as well as through the pressure range. A word of description as to the general procedure in determining the thermal dilatation at constant mean pressure will not be out of place. The general plan is to change the temperature while the piston is kept invariable in position, and therefore while the volume is also approximately constant. The rise of temperature produces a rise of pressure, so that after the rise of temperature it is necessary to bring the pressure back to the former value by withdrawing the piston if the change of temperature has been an increase, or advancing it if the change of temperature has been a decrease. The amount by which the piston is withdrawn, as also the new final pressure, is noted. The temperature is then changed again, and the same set of readings made again. Thus every observation at any given temperature involves two readings of the position of the piston and the corresponding pressure. The slight change of pressure during the changes of temperature carries with it hysteresis effects, which it is necessary to avoid by previous seasoning, exactly as for pressure changes over a wider range. Two processes of seasoning are necessary for temperature, therefore, one a larger one for the entire temperature range, and another smaller one for the slight changes of pressure incident to the changes of temperature. This second seasoning is made after the first more extensive seasoning simply by running the pressure back and forth several times through the small range of pressure to be met with during the temperature changes. This small range was determined by preliminary experiment.

In the actual calculation of the results there are a number of corrections to be applied. These will now be discussed in detail separately. In the first place the temperature coefficient of the manganin coil has to be determined with particular care. This is because the pressure changes brought about by changes of temperature during the determinations of the thermal dilatation are comparatively slight, so that any change of the pressure coefficient of the coil brought about by the change of temperature appears in the result greatly magnified. Thus for the sake of example, we will suppose that a change of temperature of 20° produces a change of pressure of 400 kgm. at 10,000 kgm. total pressure. This figure is a fair average of the results to be met with in practice. If now the pressure coefficient of the coil is changed by 1% by this same rise of temperature, the pressure will thereby appear to have risen 500 kgm. instead of the actual 400, introducing an error of 25% for a change in the constant of the coil of only 1%. In addition to the effect of the temperature coefficient of the coil, there is an effect due to the change of the zero of the coil with temperature, but this change can be determined by observations of the temperature coefficient of the coil at atmospheric pressure and is easy to measure with the requisite accuracy.

The change in the pressure coefficient of the coil with temperature is more difficult to determine with the desired accuracy. It would not be possible to determine this by a direct calibration against the absolute gauge with which the mean value of the coefficient has been determined, for the reason that the absolute gauge itself is not accurate to better than 1/10%, and this would still leave a possible error in the thermal dilatation of 2.5%. To affect the desired calibration, some standard of pressure must be used which can be relied on to remain absolutely constant. Such a standard pressure is evidently afforded by the transition point of the liquid to the solid form of any convenient substance at some fixed temperature. In previous work the transition points of both water and mercury have been determined at various temperatures with an accuracy in the absolute pressure of 1/10%. To make the calibration it is only necessary to keep the pressure constant automatically at this known value by placing in communication with the chamber in which is the manganin coil to be calibrated another chamber in which are the liquid and solid forms of the substance whose transition temperature and pressure are known. This second chamber is to be kept at constant temperature accurately enough so that slight changes in this temperature will not produce changes of more than the allowed amount in the transition pressure. For this purpose the most convenient fixed temperature seems to be that of melting ice at atmospheric pressure, and the most convenient substance to use mercury, because of the sharpness of the freezing, and the ease with which it can be obtained pure.

The actual arrangements in making this calibration for the temperature coefficient of the pressure coefficient of the coil were as follows. The upper cylinder of the hydraulic press in which pressure was produced contained in addition to the moving plunger a steel shell in which was as large a quantity of mercury as convenient, about 150 gm. This upper cylinder as well as the entire lower part of the press was surrounded by a tank containing ice and water, by which the temperature of the mercury could be kept continuously and accurately at 0°. A heavy nickel steel tube led out of the lower end of the upper cylinder through the bottom of the tank, and connected with the lower cylinder in which was the manganin coil under examination. This lower cylinder was placed in an oil bath with thermostatic regulation, by which the temperature could be set at and retained at any desired value. The experimental procedure was as follows. The temperature of the lower bath was set at any desired value, and the pressure increased until the freezing point of mercury at 0° was slightly passed. The mercury then froze with decrease of volume, thus bringing the pressure back to the known equilibrium value at 0°. After equilibrium had been reached, the resistance of the manganin coil was read. The pressure was then lowered slightly by withdrawing the piston. This was followed by automatic restoration of the equilibrium pressure, brought about by melting of the frozen mercury with increase of volume. The transition point was always so sharp that no difference could be detected in the equilibrium pressure whether approached from above or below. The temperature in the lower cylinder containing the manganin was then changed to another desired value. This change of temperature, if it were an increase, would naturally carry with it a rise of pressure, but the pressure is then automatically lowered by the freezing of the mercury. After a steady state is reached, the new value of the manganin resistance is read, and then the pressure lowered again by slightly withdrawing the piston, and the value of the resistance noted again after the equilibrium conditions have been restored from below. In this way the coil can be calibrated over the entire temperature range contemplated for the experiments. Of course this calibration is good only for one fixed pressure, but in view of the proved linearity of the pressure-resistance relation within 1/10% from 0° to 50°, it seemed safe to let the calibration go at this one determination, particularly since no effect could be found.

The calibration of the manganin was carried out at five temperatures; 25°, 45°, 65°, 85° and 110°. No appreciable change of the coefficient could be found for the four lower temperatures, but between 85° and 110° there is a very perceptible change of 1%. But since the range of temperature of the actual experiment did reach over 80°, no correction was applied to the observations for this effect. It is to be noticed that this result is valid only for this one coil, since previous work, both by Lisell [7] and by the author, have shown that different pieces from the same spool of wire may show slight variations in the temperature coefficient, which is sometimes positive and sometimes negative.

In addition to this special calibration for slight relative changes in the pressure coefficient with temperature, the absolute value of the pressure coefficient has been checked from time to time during the course of the experiments. This could be done conveniently with the apparatus as used for the compressibility determinations by determining the transition point of ice VI, or of mercury at known temperatures. These calibrations have shown no change whatever in the pressure constant of the coil.

It has already been stated that the actual measurements involve two sets of readings, one with the apparatus filled with water, kerosene and a small amount of bessemer steel, and a second set with additional steel replacing the water. By subtracting the piston displacement at any given pressure for these two sets of experiments a value is obtained which gives approximately the piston displacement for the water alone, and from which the effect of the distortion of the vessel has in large measure been eliminated. But a moment's consideration will show that the effect of distortion has not been entirely eliminated, and it is necessary to apply a correction for the slight residual effect. The correction comes because of the fact that the position of the piston at corresponding pressures is not the same in the two sets of experiments, so that the subtraction leaves still uncorrected the distortion due to the part of the cylinder exposed to pressure in the one set of experiments and not so exposed in the other. This correction cannot be determined directly, and the only way seems to be to calculate it by the ordinary theory of elasticity, taking for the constant of the steel the values under ordinary conditions, which are known not to vary much even for the most different varieties of steel. There is undoubtedly some error in the correction as so determined, but the total value of the correction is at best small, and any such error is relatively unimportant.

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7 Lisell, Om Tryckets Inflytande på det Elektriska Ledingsmotståndet hos Metaller samt en ny Metod att Mäta Höga Tryck (Diss. Upsala, 1903).

The compressibility of the steel replacing the water also evidently enters as a correction factor. This compressibility is relatively slight, and it has been previously determined over a range of 10,000 kgm. The value of the compressibility of the steel also changes with the temperature, but this change has also been shown by direct experiment to be slight, so slight that it can be neglected. In the present work the value was assumed to be constant, independent of temperature and pressure, having the value 58×10^{-8} per kgm. per sq. cm.

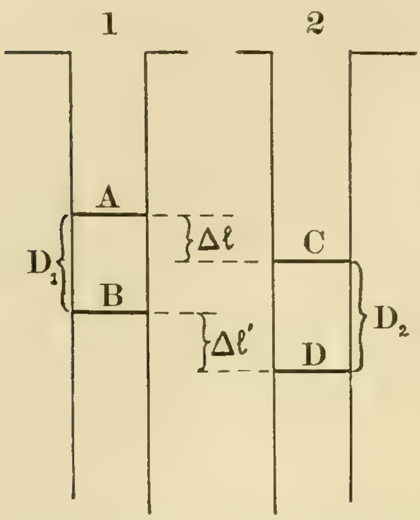

FIGURE 1. Diagram showing the position of the piston. To go with the computations for the corrections to be applied to the compressibility.

There is also a correction to be applied for the compressibility of the kerosene, if the amount does not happen to be the same in the two sets of experiments, and it was seldom that the amount was exactly the same. The variation was very small, however, and the correction is easy to apply if the compressibility of the kerosene itself is known. This was determined with sufficient accuracy for the purpose by an independent set of experiments, exactly the same in principle as those for determining the compressibility of water. The results of these independent experiments are given at the end of the paper.

The following formulas were used in making the corrections, and include all the corrections mentioned qualitatively above. Figure 1 shows the position of the piston at different times in the course of the experiment. The left hand part of the diagram (denoted by the suffix 1) is for the cylinder when it is filled with kerosene and bessemer steel only, and the right hand part (denoted by the suffix 2) is for the cylinder when it contains water, kerosene, and bessemer steel. A and C are the positions of the piston at the arbitrary zero of pressure in these two sets of experiments (this arbitrary zero was usually taken in the neighborhood of 2000 kgm. and will be denoted by p), and B and D indicate the position at some higher pressure, the same in the two sets, which will be denoted by p'. We now write down the expressions for the total volume of the cylinder beneath the piston.

$$\begin{aligned}
&\text{At A, } V_1 = V_{1k} + V_{1s} \\
&\text{At B, } V_1' = V_{1k}' + V_{1s}' \\
&\text{At C, } V_2 = V_{2k} + V_{2\,H_2O} + V_{2s} \\
&\text{At D, } V_2' = V_{2k}' + V_2'\,_{H_2O} + V_{2s}'
\end{aligned}$$

where the suffixes K, H_2O, or S indicate that the volume is for the kerosene, the water, or the steel respectively.

Subtracting the equations above from each other, we obtain

$$\begin{aligned}
(V_1 - V_2) - (V_1' - V_2') = (V_{1k} - V_{1k}') - (V_{2k} - V_{2k}') \\
- (V_{2\,H_2O} - V_2'\,_{H_2O}) + (V_{1s} - V_{1s}') - (V_{2s} - V_{2s}').
\end{aligned}$$

We now denote by Δl the difference of displacements at the two positions A and C, and by $\Delta l'$ the corresponding difference at the positions B and D. We now assume that V_1 and V_2 differ only by the volume of the cylinder of length Δl, and similarly V_1' and V_2' differ only by the cylinder of length $\Delta l'$. This assumption is justified if only the positions of the pistons at A and C are so far removed from the end of the cylinder that the end effects in the distortion of the interior are the same in the two cases. This condition has been shown by the theory to be satisfied when the distance is two or three diameters, as it always was in these experiments. Hence we may write,

$$\begin{aligned}
V_1 - V_2 &= s_0\,(1 + \alpha\,p)\,\Delta l \\
V_1' - V_2' &= s_0\,(1 + \alpha p')\,\Delta l',
\end{aligned}$$

where s_0 is the initial section of the cylinder at atmospheric pressure, and α is the factor of proportionality by which this is changed with pressure. Now if we call the displacement form A to B, D_1 and from C to D, D_2, then.

$$D_2 + \Delta l = D_1 + \Delta l'$$

and the above equation may be thrown into the form

$$V_1 - V_2 - (V_1' - V_2') = -\,s_0(D_2 - D_1)\,(1 + \alpha\,p') + s_0\,\Delta l\alpha\,(p - p')$$

We now make use of the fact that the total change of volume of any substance under pressure is proportional to its mass. If $\Delta\,v$ (positive for a decrease) is taken as the change of volume of 1 gm. between p and p', then,

$$\begin{aligned}
V_{1k} - V_{1k}' - (V_{2k} - V_{2k}') &= \Delta\,v_k\,(m_{1k} - m_{2k}) \\
V_{2\,H_2O} - V_2'\,_{H_2O} &= \Delta\,v_{\,H_2O}\,m_{\,H_2O} \\
(V_{1s} - V_{1s}') - (V_{2s} - V_{2s}') &= \Delta\,v_s\,(m_{1s} - m_{2s})
\end{aligned}$$

This enables us to solve the equations for the compressibility of the water and the kerosene, giving,

$$\Delta v_{H_2O} = \frac{1}{m_{H_2O}} \{ s_0 (D_2 - D_1)(1 + ap') - s_0 \Delta la (p - p') + \Delta v_k (m_{1k} - m_{2k}) + \Delta v_s (m_{1s} - m_{2s}) \}$$

and for the kerosene, when the two runs are both made with kerosene, as in determining the data for kerosene given at the end of the paper,

$$\Delta v_k = \frac{1}{m_{2k} - m_{1k}} \{ s_0 (D_2 - D_1)(1 + ap') - s_0 \Delta la (p - p') - \Delta v_s (m_{2s} - m_{1s}) \}$$

The considerations so far apply only to the measurement of compressibility at constant temperature. The thermal dilatation is determined in the same way as the compressibility from the difference of the thermal dilatation as given by two sets of experiments, one with the water replaced by steel. The piston displacement is not the same at corresponding pressures here, either, and a correction is to be applied for the thermal dilatation of the part of the cylinder which is exposed to pressure in the one experiment and not so exposed in the other. But this portion of the cylinder to which the correction is to be applied was seldom more than 1″ in length, and the correction for this amount of steel is negligible in comparison with the thermal dilatation of the total quantity of water. There is also a correction to be applied for the dilatation of the steel replacing the water, and this correction is small but not negligible. It was assumed that the dilatation of the steel remains independent of the pressure over the pressure range used, and the value for ordinary mild steels at atmospheric pressure was employed. This value is 0.000039 for the cubic expansion per degree Centigrade.

The corrections to the measurements of the thermal dilatation are not so serious or so important as those for the compressibility, since the total effect is much smaller and most of the corrections become negligible. The method of determining the thermal dilatation has already been explained to be that of observing the change of pressure brought about at constant volume by a known change of temperature. From this the change of volume with temperature at constant pressure can be immediately determined if the slope of the p-v curve at that point is known, for $\left(\frac{\partial v}{\partial \tau}\right)_p = -\left(\frac{\partial v}{\partial p}\right)_\tau \left(\frac{\partial p}{\partial \tau}\right)_v$. $\left(\frac{\partial v}{\partial p}\right)_\tau$ is evidently given directly from the curves for compressibility at constant temperature. The slope of this curve changes somewhat with the temperature, so that a correction should be applied for this, but the change is so slight at the higher pressures that for this purpose the compressibility can be assumed constant. At the lower pressures, below 2500 kgm., the change cannot be neglected, and another method of computation must be applied.

The thermal dilatation at low pressures was determined by taking directly the difference between the isothermals traced out at different temperatures. This method is not applicable at high pressures because the irregularities of isothermals traced at different times is sufficient to make their difference an inaccurate measure of the slight change of volume with temperature, but at the low pressures, the errors introduced by hysteresis and other irregular action of the steel cylinder are so slight that the method may be used directly to give the value of the compressibility, and by taking the differences, the value of the thermal dilatation. In fact it would seem that the method would be applicable with slight modifications to the determination of the compressibility of a great variety of substances at low pressures, and it is very much more rapid than the methods hitherto used.

A special setting up of the apparatus was necessary for the experiments at low pressures, because in order to be able to reach low pressure on release of pressure it is necessary that the friction in the movable plug be not too high, and if the pressure has once been run to so high a value as to upset the plug, the friction becomes so great as not to permit release of pressure to much below 1500 kgm. For these experiments, then, the plug was made initially a push fit for the hole, by making it about 0.0015″ smaller than when used for the higher pressures, and in performing the experiment the pressure was never pushed beyond 2500 kgm. In other respects the experiments at low pressures were the same as those at higher pressures. It was not necessary to take quite so elaborate seasoning precautions at these low pressures, however.

With regard to the amount of hysteresis or elastic after-effects to be met in the experiments, the difference of the displacment with increasing or decreasing pressure usually amounted at the middle of the range to 0.03 in. This amount was very uniformly consistent, indicating that the cylinder had really settled down to a steady behavior. The piston always returned to the starting point to within the limits of accuracy of reading, indicating that there was no leak or permanent set, or wearing of the packing in appreciable amount.

Of course the experiments at low pressures showed very much less hysteresis, in fact it was so small as to be almost imperceptible. The effect of hysteresis was eliminated as far as possible by using for the displacement at any pressure the mean of the results with increasing and decreasing pressure. The hysteresis was so constant that it would probably have been sufficient to have used consistently the results either at increasing or decreasing pressure. The actual procedure has, therefore, the weight of two independent determinations. In the determinations of thermal dilatation, on the other hand, the hysteresis effects were so much smaller, that except for one run initially to show that there was no effect of this kind, the readings were always made either only with increase or only with decrease of temperature for any mean pressure, never with both increase and decrease.

The Data.

Three independent sets of experiments were performed to give the change of volume with temperature and pressure over the entire range; namely the isothermal compressibility at pressures over 2500 kgm., the isothermal compressibility and the thermal dilatation at pressures below 2500 kgm., and the thermal dilatation at pressures over 2500 kgm. This is the actual order of experiment, but for the purposes of presentation it will be better to use the natural order, proceeding from low to higher pressures.

Compressibility at Low Pressures.

The method with the present form of apparatus is not very sensitive at the low pressures, and not many measurements were made over this range. Two sets of determinations of compressibility were made, the first at 20°, 40°, 60°, and 80°, and the second at only 20° and 80°. Here, just as for the measurements at the higher pressures, there is always sufficient friction in the packing after the pressure has once been applied not to permit of close enough approach to the zero to make an extrapolation back to the zero justifiable. And if the extrapolation to the zero is to be made from the readings during first application of pressure, special effort has to be made to design the washers so as to avoid small initial distortions. For this reason only the second of the above sets could be used by extrapolation back to the zero of pressure. The readings of volume at 20° and 80° were corrected back to 40° from the thermal dilatation as determined by this same set of experiments, so that we have from the above two values for the compressibility at 40° up to 2200 kgm. The first set of readings at five temperatures is consistent with this latter set above 1000 kgm., but at the lower pressures gives values for the compressibility which are doubtless too high. To find the best value for the change of volume at low pressures we now have three sets of data, those of the

TABLE I.

VOLUME OF WATER AT 40° AND LOW PRESSURES BY DIFFERENT METHODS.

Pressure, kgm./cm.^{2}	Change of Volume, cm.^{3}/gm.			
	Steel Piez.	Piston.	Amagat.	Final Mean.
0	.0000	.0000	.0000	.0000
500	.0200	.0202	.0204	.0203
1000	.0368	.0377	.0378	.0376
1500	.0527	.0527	.0534	.0532
000	.0669	.0667	.0676	.0673

present determination, those of the previous work by the method of the steel piezometers, and the results of Amagat. The most probable value for the change of volume has been found by comparing these three sets of values. These values are given in Table I, as also the mean selected from them as the most probable value from the data at present in hand. In taking this mean, the greater weight has been given to the values of Amagat at the lower pressures, since his method of measurement was doubtless more accurate for the low pressures than the present method, which was intended only for high pressures, but at the upper end of the range in the neighborhood of 2000 kgm., more weight has been given to the present determinations. It is to be noticed that the mean value taken as final is lower than that found by Amagat. This divergence is in the same direction as that found by Parsons and Cook, who worked with a method like the present one. The deviation found by them from the results of Amagat is greater than that adopted here.

Dilatation at Low Pressures.

For the thermal dilatation at low pressures, two sets of determinations were made; one was the series of isotherms at four different temperatures already mentioned, and the second was by the method adopted for the higher pressures, namely variation of temperature at constant mean pressure. The method of calculation for this lower

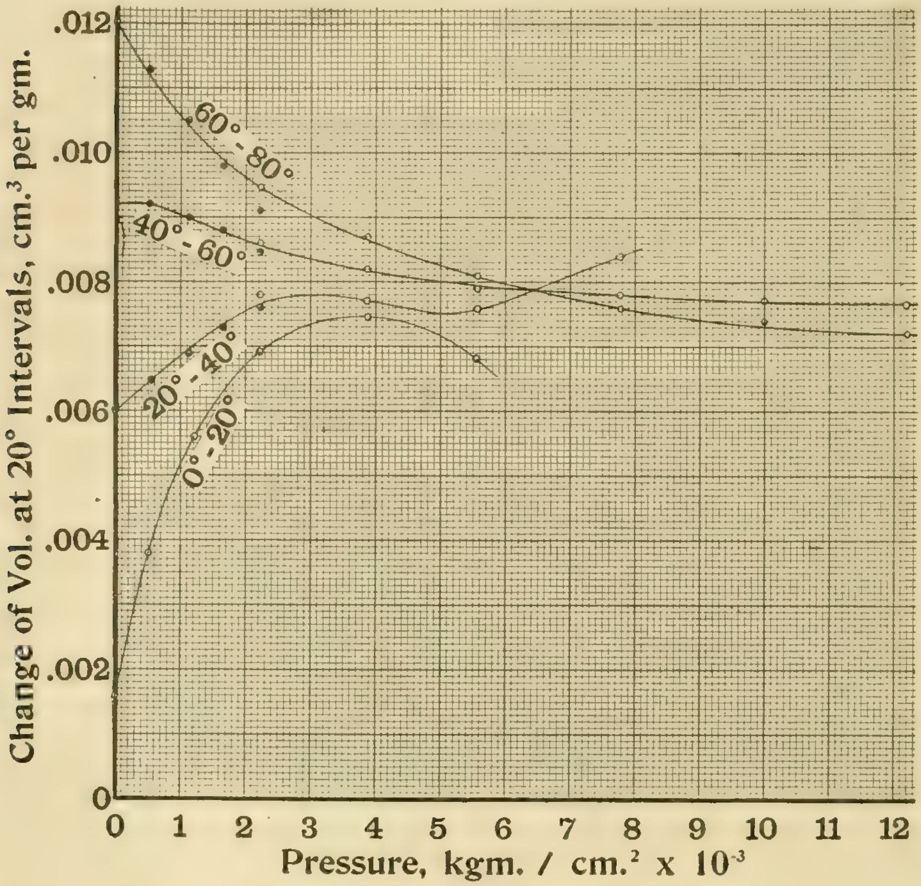

Figure 2. The change of volume of water for intervals of 20° plotted against pressure.

range was not the same as that employed for the higher pressures, as already explained, due to the fact that the slope of the isothermals is not sufficiently independent of temperature at the lower pressures. The method of computation adopted here was a graphical one, by plotting the observed volume and pressure points for the different temperatures and taking the difference between adjacent curves graphically. The temperatures at which the different determinations were made were not exactly the even temperatures desired, namely 20°, 40°, and 60°, and 80°, but they were in all cases within a few tenths of a degree of these temperatures. The results were corrected to these even temperatures by assuming the mean variation with temperature over the whole temperature range to hold for the few tenths of a degree on either side. The final result given by the data is the total change of volume for an interval of 20°; from 20° to 40°, from 40° to 60°, and from 60° to 80°. The mean of the results of the two sets of experiments is shown with satisfactory accuracy in Figure 2, on which are plotted all the values obtained by the different methods. The results for the low pressures are shown in the full black circles. These values are seen to extrapolate, without forcing, to the values already found by other observers for atmospheric pressure, and they also make fairly good connections with the values found by the other method for the higher pressures. In view of this agreement it did not seem to be necessary to make further determinations of this quantity.

Compressibility at High Pressure.

The determinations of the isothermal compressibility at higher pressures extended over a considerable interval of time and are more numerous than any of the other determinations. In all, twelve determinations of this quantity were made, at five different temperatures. These determinations include those made during the early course of the experiment, when the attempt was being made to find the thermal dilatation directly from the difference of compressibility at different temperatures. A little work with the method showed that it was not sufficiently accurate for the purpose, but the results obtained then can be used to give the compressibility at the standard temperature, 40°, by applying the temperature correction found from the later more accurate results. The temperature of 40° was chosen as the standard because this is the lowest of the 20° intervals at which the water is liquid up to 12000 kgm.

The results of these twelve determinations, extending over a period of three months, are shown in Table II. The results as given are reduced to 40°, but the temperature at which the original measurements were made is given also in the table. Two of these sets of determinations differ considerably from the others, and were discarded in taking the mean, although as it happens one of these discarded sets is too high and the other too low, so that it makes very little difference

TABLE II.

Results of Different Determinations of Change of Volume of Water at 40°.

Pressure, kgm./cm.²	Change of Volume, cm.³/gm.											
	Jan 22–27.	Dec. 19–29.	Dec. 4–6.	Nov. 29–Dec. 1.	Nov. 27–28.	Nov. 24–25.	Nov. 9–10.	Nov. 8–9.	Nov. 6–8.	Nov. 3–4.	Nov. 2–3.	Nov. 1–2.	Final Mean.
2220	.0000	.0000	.0000	.0000*	.0000	.0000	.0000	.0000*	.0000	.0000	.0000	.0000	.0000
3330	.0255	.0256	.0266	.0268	.0255	.0262	.0257	.0243	.0260	.0253	.0256	.0250	.0257
4440	.0471	.0476	.0487	.0494	.0473	.0481	.0468	.0452	.0471	.0465	.0472	.0466	.0473
5550	.0656	.0666	.0675	.0687	.0660	.0670	.0652	.0635	.0650	.0648	.0658	.0652	.0659
6670	.0816	.0830	.0836	.0852	.0820	.0830	.0813	.0794	.0806	.0809	.0818	.0810	.0819
7780	.0956	.0974	.0979	.0999	.0962	.0970	.0956	.0937	.0947	.0953	.0961	.0952	.0961
8890	.1079	.1100	.1100	.1126	.1087	.1094	.1086	.1064	.1072	.1083	.1088	.1077	.1087
10000	.1190	.1211	.1213	.1238			.1201	.1178	.1186	.1198	.1201	.1189	.1199
11110	.1291	.1310	.1313	.1337			.1305	.1282	.1291	.1302	.1304	.1289	.1301
12220	.1386	.1398					.1412	.1377			.1413	.1408	.1403

* Discarded in taking mean.

in the final result whether they are included in the mean or not. For convenience in making the computations the pressure was taken in units given conveniently by the changes of the manganin resistance, the intervals of pressure corresponding to a displacement of the slider of the bridge wire of 5 cm.

TABLE III.

Comparison of Results by Two Methods for Change of Volume of Water at 20°.

Pressure, kgm./cm.²	ΔV.		Pressure, kgm./cm.²	ΔV.	
	Piston.	Piezometer.		Piston.	Piezometer.
2220	.0000	.0000	6670	.0814	.0821
3330	.0257	.0259	7780	.0954	.0964
4440	.0472	.0479	8890	.1078	.1105
5550	.0659	.0666	10000	.1190	.1229

These results, reduced to 20° are shown compared with the results of the previous determination in Table III. It is seen that the newer results are lower than the former ones, the difference being about 1%, except at the higher pressures, where the difference is greater. The agreement is perhaps not as close as could be desired, but at present there seems to be no way of choosing between the results. There is no consistent discrepancy, which would indicate a fundamental error in the present method, such as in the correction applied for the distortion of the steel cylinder, for example. If there were any such error it could be eliminated by so choosing the correction as to make the present results agree with the former ones. In the absence of any means of deciding between the two methods therefore, and since the results by the present method reach over a wider temperature and pressure range, and since also the method has been used much more extensively than the former one and with no greater discrepancy in the individual results, these present results have been accepted as the best ones. But it must be remembered that the absolute compressibility given here may be in error by as much as 1% at the higher pressures. This error, however, will not be found to invalidate any of the conclusions drawn from the data.

Dilatation at High Pressures.

The determinations of the thermal dilatation at the higher pressures were made on four occasions. The first two of these were preliminary, during which was discovered the necessity of seasoning for temperature as well as for pressure, and also the necessity for the secondary pressure seasoning over the small range of pressure accompanying the changes of temperature. These first two determinations, while confirming the results of the two later ones, were not given much weight in selecting the final value. The method of computation adopted in finding the thermal expansion from the data requires mention. At first an attempt was made to apply the same graphical method which has been already explained in its application to the determinations at the lower pressures. This method involves the drawing of a curve of the same general slope as the compressibility curve through the two points giving piston displacement against pressure at each temperature. But it was found that even after the seasoning for the small pressure range involved here, the points were too irregular to give good results by this method. The irregularities may be due to residual hysteresis, but are more probably due to slight irregularities brought about by the motion of the piston itself. These irregularities are too minute to have any effect on the compressibility determinations. The best way to avoid them is to utilize in the computations only those readings during which the piston remains stationary. This means that only the change of pressure accompanying a change of temperature is used in making the computations, the second reading at any temperature by which the pressure is brought back to the mean value being ignored. The change of volume at constant pressure for the given change of temperature is then computed from the known change of pressure at constant volume and the previously determined change of volume with pressure at constant temperature. In making this computation it is generally necessary to make two corrections; one to bring the temperature interval to the exact 20° desired for the final results, and the second to correct for the very slight change of measured piston displacement accompanying the change of temperature. This change of displacement is seldom over 0.003″. It is probably not due entirely to actual motion of the piston, but partly to temperature changes in the bars of the press which dip into the thermostat. That this method of computing the results is preferable to the graphical one previously mentioned is shown by the fact that this method gives very much more uniform and consistent results when applied to the same set of data than the graphical method.

The method of computation adopted was first to calculate independently from the individual observations of each set of readings the thermal dilatation at six mean pressures between 2200 and 12,000 kgm. Then smooth curves were drawn through these points for each set of readings, the curves being spaced in the best way so as to give regular variations with both pressure and temperature. The values given by the smooth curves of each set of readings were then combined into the grand mean. In taking this grand mean, as already explained, almost the entire weight was given to the last two sets of readings. The agreement between the different sets was best at the higher temperatures, 60° to 80°, and about equally good between 20° and 40° and 40° and 60°. All four sets of curves, while not agreeing very well as to the numerical value of the coefficient, do agree as to the general character of the results, which are, perhaps, not quite what would be expected. The unexpected feature is the change in the sign of the temperature derivative of the dilatation at the higher pressures. At the low pressures the dilatation is greater at the higher temperatures, but at the higher pressures the thermal dilatation becomes less at the higher temperatures. This essential feature is verified on all four sets of curves. There are indications that it may be an essential characteristic of the behavior of any normal liquid at high pressures, and that it is not peculiar to water alone. This is shown by the work on kerosene, and is also indicated by the work at present being done on still other liquids. This will be taken up in greater detail later. The other feature not to be expected is the increase in the value of the thermal expansion between 20° and 40° at the higher pressures. It is to be distinctly expected that the thermal dilatation will decrease with rise of pressure, as indeed it does for all the other intervals of temperature, but this rise between 20° and 40° is shown by all the sets of determinations and seems to be an undoubted fact. It is probably connected with some new abnormality in the behavior of water at the higher pressures, which may be connected in some way with the appearance of the new variety of ice.

The values finally taken as the best values for the thermal dilatation are the mean of the results of the four determinations, much the greater weight being given, as already explained, to the two latter determinations. Figure 2 gives these results, as also those of the other methods at the lower pressures. The agreement of the two best determinations at the higher pressures is about 5% for the lower temperature interval from 20° to 40°, 3% for the interval 40° to 60°, and 2% from 60° to 80°. The order of accuracy to be expected in these thermal measurements is not so great as that in the compressibility determinations, therefore, but perhaps the accuracy is as great as could be expected when one considers the smallness of the quantities involved, and the difficulty of making such measurements at high pressures. At any rate the absolute value of the coefficient cannot be very much in error. This is made probable by the agreement with the known values at atmospheric pressure. The accuracy is at least high enough to enable us to expect a fairly good quantitative description of the various thermodynamic quantities under high pressure, even those most sensitive to error. The calculation seems to be worth while carrying through in some detail, because such calculations seem never to have been undertaken for any substance, even for the low pressure range up to 3000 kgm., which is the range over which compressibility determinations have been previously made.

Discussion of Results.

The first necessity for a calculation of the various thermodynamic quantities is as accurate as possible a knowledge of the relation between pressure, temperature and volume over the entire pressure-temperature plane. It may be shown that this is sufficient to completely determine the thermodynamic behavior of the substance if in addition the behavior of the specific heat at constant pressure, for example, is known in its dependence on temperature at atmospheric pressure. This may be assumed to be known well enough for the present purpose. The first and the most important outcome of the present data is, therefore, the construction of a table giving pressure, volume, and temperature at sufficiently close intervals. In constructing this table the basis of computation was the compressibility as determined at 40°. This, together with the known value of the volume at 40° and atmospheric pressure, gave the volume as a function of the pressure down a line through the middle of the table at 40°. The values of the volume were tabulated for intervals of the pressure of 500 kgm., the values found graphically from smooth curves through the experimental points being so smoothed as to give smooth second differences. The values of the change of volume for intervals of 20° now were combined directly with these values to give the volume as a function of the pressure at 0°, 20°, 60°, and 80°. To find the intermediate values of the volume, smooth curves were drawn through these five points at every constant pressure, and the intermediate values so chosen as to given smooth values for the second differences over the entire temperature range. The values for the points below zero, which are also given in the table, were taken directly from the previous work, the values for the dilatation found there being kept without modification, but the present value for the compressibility at 0° being used. The differences so introduced may be seen by comparison of the two tables to be only slight.

The table gives the volume to only four significant figures, since this is as many as the variations in the values of the compressibility would entitle one to, but in making the calculations of the thermal expansion it was necessary to keep three significant figures for the expansion, which would mean five figures in the table.

The thermal dilatation per degree rise of temperature was determined from the values used in the construction of the table for the differences of volume at 5° intervals by dividing by 5, and using the result as the thermal expansion at the mean temperature. The values of the total change of volume for five degree intervals had been smoothed so as to give smooth second differences, so that the dilatation as found in this way was smooth also with respect to the second differences, and could be used directly to give the second temperature derivative of the volume at constant pressure.

The difference of thermal dilatation at different temperatures can evidently be combined with the known compressibility at 40° to give the compressibility as a function of the temperature.

These several quantities so determined; the compressibility, the thermal expansion, and the second temperature derivative of the volume, in their dependence on temperature and pressure, are the basis of most of the calculations of the quantities of thermodynamic interest to be given presently. The accuracy of most of these quantites is not so high but that they can be shown as well in figures as in tables, and this manner of presenting them has been chosen as giving the most ready general survey of the facts.

The tables and figures follow. The results are given simply for themselves, without much comment, except to call attention to the unexpected features, or those properties which seem to be peculiarly characteristic of high pressures. It would not be safe to generalize from the behavior of this one liquid, abnormal at low pressures, to the general behavior to be expected for any liquid for high pressures and the bearing on a possible theory of liquids. Such a general treatment must be reserved for another paper, when the data for more liquids are in hand.

TABLE IV.

Volume of Water as a Function of Pressure and Temperature.

Pressure. kgm./cm.	—20°,	—15°,	—10°,	—5°,	0°,	5°,	10°,	15°,	20°,	25°,	30°,	35°,	40°,	45°,	50°,	55°,	60°,	65°,	70°,	75°,	80°.
0			1.0017	1.0006	1.0000	0.9999	1.0001	1.0007	1.0016	1.0028	0.0041	1.0057	1.0076	1.0096	1.0118	1.0143	1.0168	1.0195	1.0224	1.0255	1.0287
500			.9795	.9778	.9771	.9778	.9786	.9796	.9808	.9821	.9837	.9854	.9873	.9894	.9916	.9940	.9965	.9992	1.0020	1.0049	1.0075
1000			.9598	.9584	.9578	.9589	.9602	.9616	.9630	.9646	.9663	.9681	.9700	.9721	.9743	.9766	.9791	.9816	.9842	.9869	.9896
1500		.9404	.9416	.9407	.9410	.9424	.9439	.9454	.9471	.9488	.9506	.9525	.9544	.9564	.9586	.9599	.9632	.9657	.9682	.9707	.9732
2000	.9228	.9235	.9243	.9252	.9260	.9276	.9293	.9310	.9327	.9345	.9364	.9383	.9403	.9423	.9445	.9467	.9489	.9513	.9537	.9561	.9585
2500	.9085	.9094	.9104	.9117	.9133	.9150	.9167	.9185	.9203	.9221	.9220	.9259	.9279	.9299	.9320	.9341	.9363	.9386	.9409	.9433	.9457
3000	.8963	.8972	.8984	.8997	.9015	.9032	.9050	.9068	.9087	.9106	.9105	.9144	.9164	.9184	.9205	.9226	.9247	.9269	.9292	.9314	.9337
3500		.8864	.8876	.8888	.8907	.8924	.8943	.8961	.8979	.8998	.8997	.9036	.9056	.9076	.9096	.9117	.9138	.9160	.9182	.9204	.9226
4000		.8766	.8774	.8786	.8807	.8825	.8843	.8861	.8880	.8889	.8897	.8936	.8956	.8976	.8996	.9016	.9037	.9058	.9080	.9101	.9123
4500			.8684	.8695	.8717	.8734	.8751	.8770	.8788	.8807	.8805	.8844	.8864	.8884	.8904	.8924	.8945	.8965	.8966	.9008	.9028
5000			.8599	.8610	.8632	.8649	.8666	.8684	.8702	.8721	.8719	.8758	.8778	.8798	.8818	.8838	.8858	.8879	.8899	.8920	.8940
5500				.8537	.8554	.8569	.8585	.8603	.8621	.8640	.8639	.8678	.8698	.8718	.8737	.8757	.8777	.8798	.8818	.8838	.8858
6000				.8464	.8480	.8494	.8509	.8527	.8545	.8564	.8654	.8603	.8623	.8643	.8662	.8682	.8702	.8722	.8742	.8762	.8781
6500					.8409	.8423	.8438	.8454	.8473	.8492	.8493	.8532	.8552	.8572	.8591	.8611	.8631	.8650	.8670	.8689	.8709
7000							.8370	.8386	.8404	.8424	.8425	.8465	.8485	.8505	.8524	.8544	.8564	.8583	.8602	.8621	.8640
7500							.8305	.8321	.8338	.8360	.8361	.8401	.8421	.8441	.8460	.8480	.8499	.8519	.8538	.8557	.8575
8000								.8259	.8275	.8298	.8300	.8340	.8360	.8380	.8399	.8419	.8438	.8457	.8477	.8495	.8513
8500								.8200	.8216	.8240	.8262	.8283	.8303	.8323	.8342	.8361	.8381	.8400	.8419	.8437	.8455
9000									.8160	.8185	.8208	.8229	.8249	.8269	.8288	.8308	.8327	.8346	.8364	.8383	.8401
9500										.8133	.8156	.8178	.8198	.8218	.8237	.8256	.8275	.8294	.8313	.8331	.8349
0000										.8083	.8107	.8129	.8149	.8169	.8188	.8207	.8226	.8245	.8264	.8282	.8300
10500											.8060	.8082	.8102	.8122	.8141	.8160	.8179	.8198	.8216	.8235	.8252
11000												.8036	.8056	.8076	.8095	.8114	.8133	.8152	.8170	.8188	.8206
11500												.7991	.8011	.8031	.8050	.8069	.8088	.8107	.8125	.8143	.8160
12000													.7966	.7986	.8005	.8024	.8043	.8062	.8080	.8098	.8115
12500													.7922	.7942	.7961	.7980	.7999	.8017	.8036	.8054	.8071

In presenting the results, the quantities have been arranged in order of simplicity of the thermodynamic formulae, which is also the order of directness with which they are derived from the experimental data.

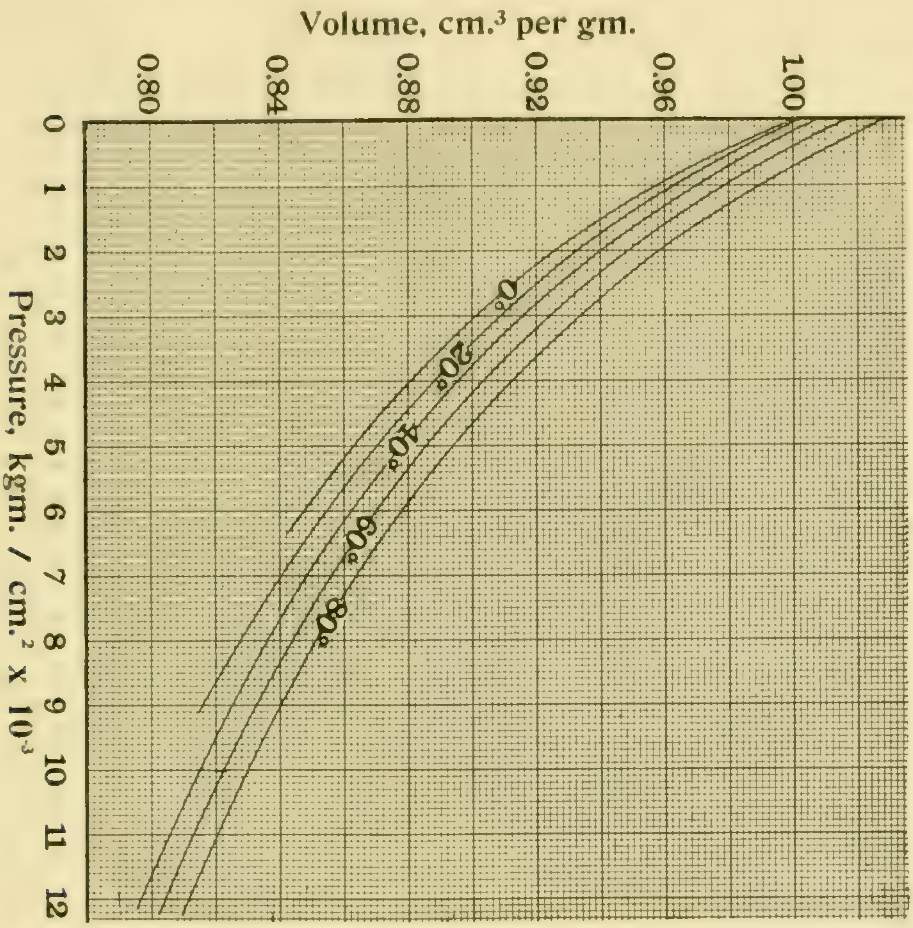

FIGURE 3. Isothermal lines for water, showing volume against pressure.

In Table IV are given the values of the volume for intervals of pressure of 500 kgm., and intervals of temperature of 5°. The table does not require comment. It was computed in the way already described. The values of the volume at intervals of temperature of 20° are shown as a function of the pressure in Fig. 3. The figure does not show the results as accurately as the table, but enables one to form a clearer mental picture of the nature of the results. The curves, on the scale of the figure, do not show any abnormalities to the eye, except in the neighborhood of the origin, where the well known negative expansion at 0° results in the curves drawing together.

There are various abnormalities besides those in the neighborhood of 0°, however, as will be shown by the other figures.

With regard to the compressibility there seems to be some variance of usage, so that it will be well to call attention to the fact that the quantity used throughout this paper in the sense of compressibility is

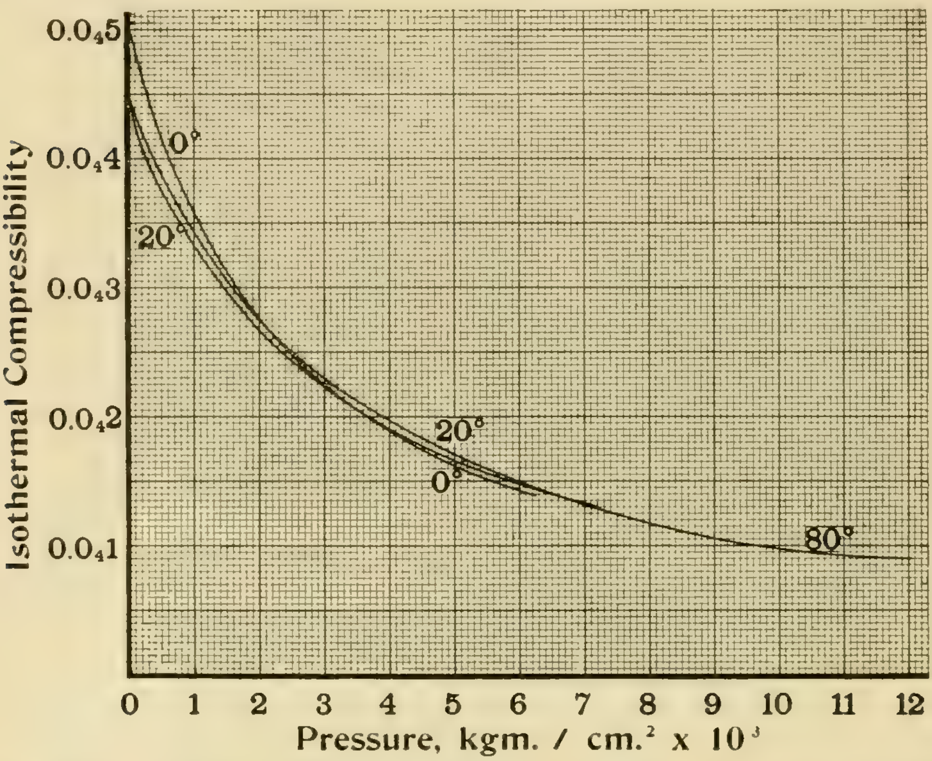

FIGURE 4 The isothermal compressibility of water, $\left(\frac{\partial v}{\partial p}\right)_t$ against pressure.

the derivative $\left(\frac{\partial v}{\partial p}\right)_t$. Sometimes the expression $\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_t$ is used in the same sense. Figure 4 shows the compressibility, that is, the analytic expression $\left(\frac{\partial v}{\partial p}\right)_t$, as a function of the pressure at 0°, 20°, and 80°. It would have made the figure too crowded to have tried to show the values for 40° and 60° also. The complete values for the five standard temperatures are shown in Table V separately, however. The figure shows the well known abnormality in the compressibility at the low pressures, namely a higher compressibility at the lower than at the higher temperatures. This abnormality disappears above 50°, and from here on the compressibility increases with rising temperature. The figure shows that at 80° the initial compressibility is higher than at 20°, although it has not yet risen to the value at 0°. In addition to the abnormality at low pressures, the curve shows also a slight

TABLE V.

Compressibility of H_2O.

Pressure, kgm./cm.²	$\left(\frac{\partial v}{\partial p}\right)_t$, cm.³/gm.				
	0°.	20°.	40°.	60°.	80°.
	.0000	.0000	.0000	.0000	.0000
0	504	453	440	438	450
500	417	381	371	373	388
1000	360	336	328	332	346
1500	313	298	292	296	306
2000	276	267	263	267	274
3000	225	223	222	224	229
4000	189	191	191	193	198
5000	162	166	165	166	170
6000	143	148	146	147	149
7000		133	130	130	132
8000		121	117	117	119
9000			104	104	105
10000			096	096	097
11000			091	091	092
12000			089	089	090

abnormality at the higher pressures in the neighborhood of 6500 kgm. Here the compressibility at 20° rises and at the melting point of ice VI, it has become higher than the compressibility at 80°. The thermal dilatation shows abnormality in the same locality; it would seem to be connected in some way with the appearance of the new variety of ice, but the exact connection cannot at present be stated.

The large change in the value of the compressibility brought about by pressure should be noticed, amounting at 12,000 kgm. to a decrease of five fold. Furthermore the rapid flattening of the curve at the higher pressures also should be commented on. The curve gives the appearance, for the pressure ranges used here, of becoming asymptotic to some value greater than zero. Of course this cannot really be the case for infinite pressures, for otherwise we should have the volume completely disappearing for some finite value of the pressure, but it may indicate the entrance of another effect at the higher pressures, which may persist in comparative constancy for a greater range of pressure than will ever be open to direct experiment, such an effect as the compressibility of the atom, for example. This possibility has been already mentioned and made plausible from the data of the preceding paper.

If instead of the compressibility as defined above, the quantity $\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_t$, which in this paper will be called the relative compressibility, is plotted, a curve of the same general character as that shown will be obtained.

The compressibility may also be plotted against a different argument than the pressure. For many purposes the pressure is perhaps not the most significant independent variable that might be chosen. This is because the external pressure is not a measure of what is happening inside of the liquid. We conceive a liquid as composed of molecules in a state of constant motion and of collision with each other, acted on also by attractive forces between each other. The effect of these attractive forces is to produce at the interior points a pressure which may be much higher than the external pressure. The external pressure is equal to the interior pressure diminished by the amount of the attractive pressure drawing the molecules to the interior at the exterior surface, where the attraction is an unbalanced action in one direction. The amount of the unbalanced pressure at the outside depends in a complicated way on the law of attraction between the molecules, on their mean distance apart in this surface layer, and on the distribution of velocities in this layer. The external pressure required to hold the liquid in equilibrium is, therefore, largely a surface phenomonon, and is connected in a complicated way with the state of affairs at inside points. A more significant independent variable, therefore, would be one involving only the condition of the molecules on the average throughout the mass, and not one dependent on the surface layer. There are only a few such quantities depending on the state of the liquid at interior points. Any quantities involving in any way the constancy of pressure or of entropy, for example, do depend on the complicated action of the surface layer. One of the quantities which is independent of this surface layer, however, is the volume. In many theoretical considerations the use of the volume as an independent variable is known to produce simplifications.

If the volume, instead of the pressure is taken as the independent variable for the compressibility, curves are obtained of the same general appearance as when the pressure is used for the variable. The compressibility falls with decreasing volume, and the curvature is in the same direction as when the pressure is the independent variable. The same general characteristics are also shown if the relative compressibility instead of the compressibility is plotted against the volume. The two sets of curves, for the compressibility and the relative compressibility, do show one feature in common, however, different from the curves when the pressure is used as the variable. This is the fact that the compressibility is always lower for the same volume at the higher temperature. This is true throughout the entire range of volume used; there is no crossing of the curves indicating abnormalities, such as is the case when the pressure is used as the variable. This is what one would expect on the kinetic theory. A liquid, at two different temperatures but at invariable volume, differs only in the violence of the motion of its molecules. At the higher temperature, the kinetic pressure due to the motion is greater, and so the resistance offered to change of volume under a given increase of external pressure is greater when the temperature is higher.

Fig. 5 shows the thermal dilatation as a function of pressure at various temperatures. The thermal dilatation plotted in the figure is the expression $\left(\frac{\partial v}{\partial t}\right)_p$ instead of the expression $\frac{1}{v}\left(\frac{\partial v}{\partial t}\right)_p$, which is sometimes used as the dilatation. The usage adopted here for the dilatation is analagous to that explained above for the compressibility. The values listed in the figure were obtained from the table of volumes in the manner already described. The curve at 0° was obtained from the data of the previous paper for the low temperatures, but in that paper the mean value of the thermal expansion for the range 0°–20° was given, whereas here the instantaneous value at 0° is given instead. The substitution of the instantaneous for the mean dilatation produces no change in the general character of the curves, however.

The points at the higher temperatures were obtained from the data of this paper alone. There are two striking features that call for special comment. The first of these is the abnormal behavior of the curve for 20°. In the initial stages, the dilatation rises with increasing pressure, unlike normal liquids, but this merely indicates the return of water to the normal behavior to be expected at high pressures. At about 3500 kgm. the curve at 20° has reached a maximum and begins to descend with increasing pressure, as it does for the curve at 0°. But the descent continues for only a little way, and at 5500 kgm. the curve begins to rise again, indicating the entrance of a new abnor-

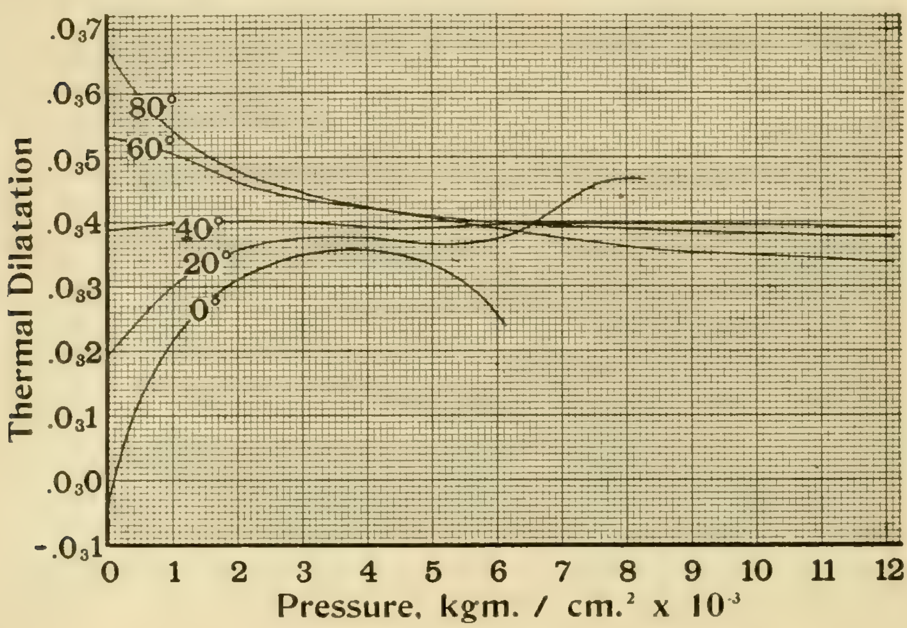

FIGURE 5. The dilatation of water, $\left(\frac{\partial v}{\partial t}\right)_p$, against pressure.

mality. The abnormality is not so striking or so great in amount as that in the neighborhood of 0° and atmospheric pressure. The abnormality at 20° continues for about 2500 kgm., up to 8000, where the curve is terminated by the entrance of the solid phase, but the direction of the curve has already begun to change, indicating that if it could be continued, this abnormality also would probably disappear at higher pressures. As to the question of experimental error here, there would seem to be no room for doubt as to the actual existence of this new abnormality, for it was shown by all four of the dilatation curves, even those taken before the method was got to running satisfactorily, and in which the accuracy was not very high. The curves at the higher temperatures behave as one would be prepared to expect in the region of low pressures. The curve for 40° shows vestiges of the abnormal behavior at the low pressures, namely slight initial rise of dilatation with rising pressure, followed by a fall, but the curves at the higher temperatures, 60° and 80°, show the regular initial decrease with rising pressure shown by all normal liquids. But at higher pressures, the behavior of all three of these curves, for 40°, 60° and 80° is different from what might be expected. The unexpected feature consists in the crossing of the curves, all in the vicinity of the same pressure, 5500 kgm., so that at higher pressures the thermal dilatation at the higher temperatures is lower than it is at the lower temperatures. It has been already remarked that there are indications, both from the present work and from that of Amagat, that this may be the behavior for any normal liquid at sufficiently high pressures. The comparative constancy of the thermal dilatation at the higher pressures, is also a matter perhaps not to be expected. Thus the expansion at 40° remains nearly constant over the entire range of pressure, while the compressibility has in the same range dropped from 44 to 9. It was distinctly expected, before these measurements were taken, that the dilatation would show the greater variation with pressure, so that the effect of temperature on the volume would tend to disappear at the higher pressures, but such is not the case.

The relative thermal dilatation may be plotted against pressure, as was the relative compressibility. The curve shows no striking features. The curve plotting relative dilatation against volume has also been plotted, and this is the same in general character as the others. The slight differences consist in an accentuation of the abnormalities in the neighborhood of 5500 kgm., and the fact that at the lower volumes, that is at the higher pressures, the dilatation against volume increases with decreasing volume for 40° and 60°, but decreases for 80°.

These figures for the thermal dilatation and the compressibility complete those which are obtainable directly from the table. Other quantities of thermodynamic interest may be obtained by combining these, however. Perhaps the simplest of these quantities are those connected with the absorption of energy when the pressure is changed at constant temperature. The first of these is the actual mechanical work done by the external pressure in compressing the liquid at constant temperature. This of course is simply the expression

$W = \int p \left(\frac{\partial v}{\partial p}\right)_t dp$. It was obtained by a mechanical integration from curves similar to the volume curves of Figure 3, drawn on a larger scale. For this purpose the integrating machine owned by the mathematical Department of Harvard University was used. The

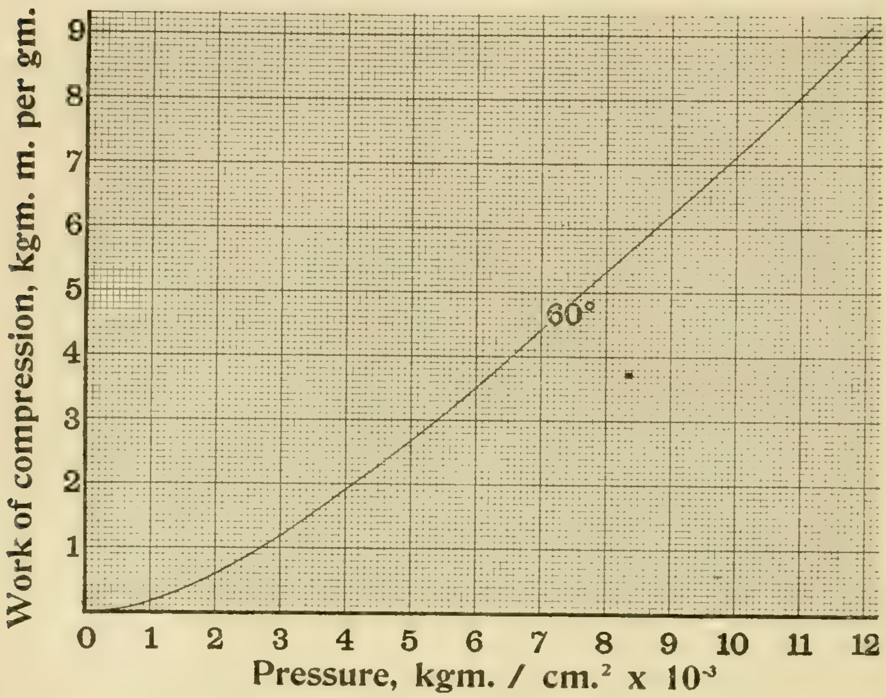

FIGURE 6. The mechanical work of compression at 60°.

actual value of the mechanical work at any pressure is of course dependent on the temperature, but since the variation is so slight that it would have been impossible to show it in the figure (see Figure 6), the work of compression is plotted for only the one temperature, 60°. Although the change of external work with temperature was too slight to show in the diagram, the change with temperature was nevertheless taken account of in making the calculations of the quantities depending on it to be described immediately. After the first 4000 kgm. it is seen that the curve becomes very approximately linear. The curve for a substance which retains the same compressibility unchanged over a wide pressure range, as steel for example, is a parabola, the work increasing directly as the square of the pressure. That this curve for water becomes linear, means that the compressibility decreases so fast with increasing pressure that the decrease in the yield of the liquid for a given increment of pressure decreases almost at the same rate that the pressure itself increases.

The total heat given out during an isothermal compression may be derived from the formula $\left(\frac{\partial Q}{\partial p}\right)_\tau = -\tau\left(\frac{\partial v}{\partial \tau}\right)_p$. This quantity is shown in Figure 7. The figure does not call for especial comment. The

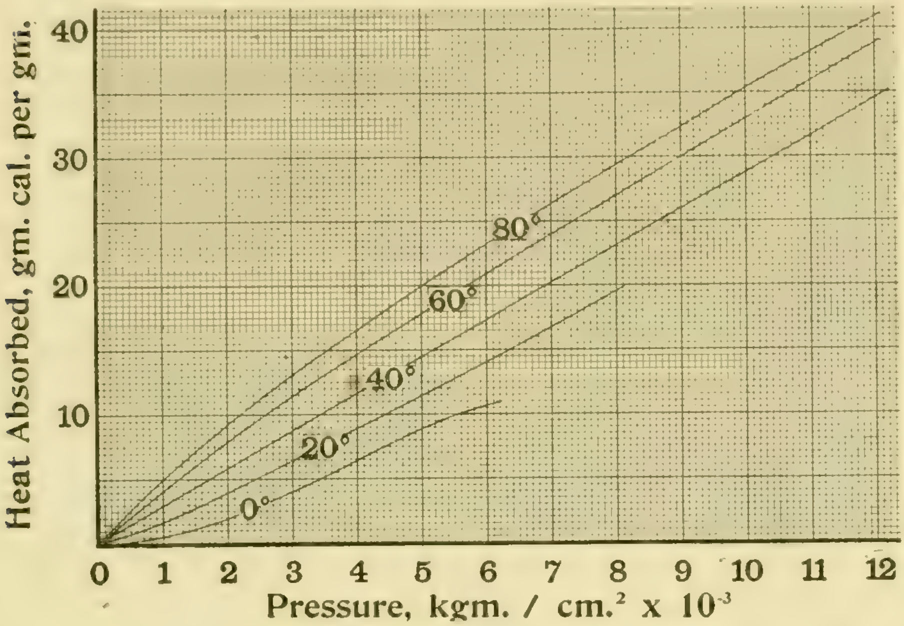

FIGURE 7. The heat given out by water during an isothermal compression.

rapid change in the direction of the isothermal lines in the vicinity of the origin due to the abnormal behavior at low temperatures and pressures is manifest from the figure, as also the slight abnormalities at the upper ends of the 0° and the 20° curves, already commented upon in other connections. Beyond 5000 kgm. the curves for all temperatures tend to become linear and parallel to each other.

These two quantities, the heat liberated in compression and the mechanical work, combine to give the change of internal energy along an isothermal, this change of energy being equal to the difference of the heat and the mechanical work. The change of energy so calculated is shown in Figure 8. The change is a decrease, which continues at all temperatures up to the highest pressures. In the previous paper a value of this quantity was given, confessedly inaccurate, since in the computation the mean thermal dilatation between 0° and 20° had been used instead of the actual dilatation at 0° or 20°. The curve so obtained had the characteristics of the curve now given for 0°, but the maximum at the top was much more strongly accentuated than in the present figure. It was surmised in the previous paper that at high enough pressures the internal energy of all liquids would probably increase instead of decrease along an isothermal. This surmise seemed

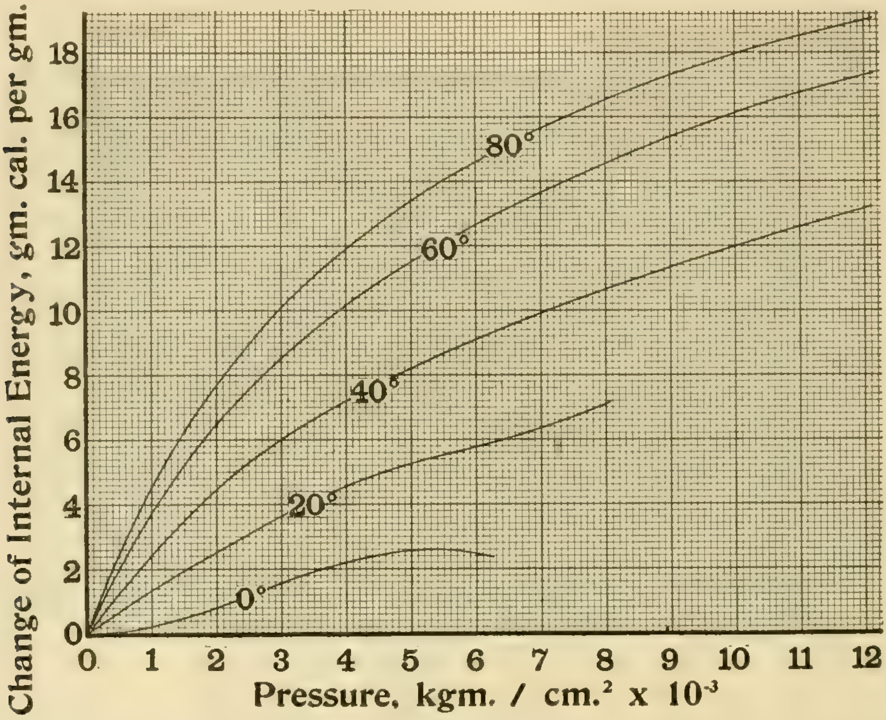

FIGURE 8. The decrease of internal energy of water during an isothermal compression.

plausible because one would expect that at high enough pressures the energy stored up as strain in the interior of the molecules in virtue of the extremely high pressures would more than counterbalance the work done by the attractive forces of the molecules themselves as they were brought closer together by the action of the pressure. This present figure shows that this is not the case, however, for the range of pressure reached here. The lower temperature, 0°, is the only one at which this reversal of the direction of the change of internal energy manifests itself, and this change, in comparison with the other curves, is now seen probably to be an effect of the other abnormalities shown at low pressures and temperatures. Nevertheless it would still seem as if at very high pressures the energy must increase instead of decrease along an isothermal, but the only indication of it from the present curves is in the direction of curvature, which is in the direction to indicate the possibility of a maximum and a reversal of direction at higher pressures. The pressure for a maximum, however, if there is one, is much beyond the reach of any at present attainable. Within the pressure range of these measurements, the attraction between the molecules still remains the dominant feature, so that the work done by the attractive forces and liberated as heat much more than suffices to overbalance the mechanical work of compression.

The internal energy of a substance is one of those quantities which depend only on the properties of the mass of the substance at interior points and do not involve the action of the surface layer. Change of energy plotted against volume shows in the first place that the change of internal energy is much more nearly a linear function of the volume than it is of the pressure. The average slope of the isothermal lines of energy increases rapidly with rising temperature for the lower temperatures, but the two curves for 60° and 80° run nearly parallel to each other for their length. Abnormalities are shown at the upper ends of the 0°, 20° and the 40° curves, and the 0° curve shows the same maximum as it does when plotted against pressure. The origin, of course, for the curves at different temperatures does not coincide as it does for the same quantities when plotted against pressure.

One other quantity may be simply determined in terms of the compressibility and the thermal dilatation alone, the so-called pressure coefficient, that is, the change of pressure following a rise of temperature when the temperature is raised by 1° at constant volume. This quantity is given immediately in terms of the compressibility and the thermal dilatation by the well known formula,

$$\left(\frac{\partial p}{\partial \tau}\right)_v = -\left(\frac{\partial v}{\partial \tau}\right)_p \Big/ \left(\frac{\partial v}{\partial p}\right)_\tau .$$

It is shown plotted in Figure 9. The curves for 0° and 20° show anomalies, as is indicated by the unexpected direction of curvature. The other curves for the higher temperatures seem to be regular enough, though of course it cannot be told whether the course of these curves is the same as that which would be shown by a normal liquid or not. At the upper ends of the high temperature curves, the curvature is in such a direction that if they were continued far enough the pressure coefficient would decrease instead of increasing with rising pressure.

This quantity, the pressure coefficient, has been made the basis of theoretical speculation. It has been enunciated as a law, approximately true, by Ramsay and Shields, that the pressure coefficient is a function of the volume only. This means that if the coefficient were plotted against volume instead of pressure the curves for all five temperatures would fall together. That this is not the case for water at high pressures is shown very distinctly in Figure 10. At the lower pressures and the larger volumes, the curves for the different tempera-

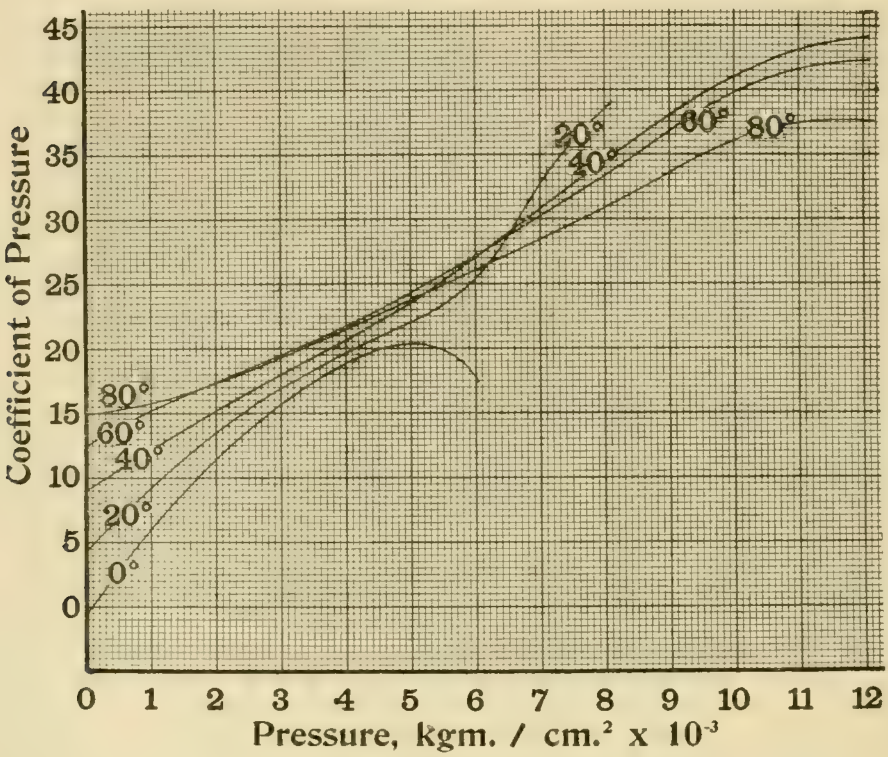

FIGURE 9. The pressure coefficient, that is the change of pressure accompanying a rise of temperature of one degree, as a function of the pressure.

tures are very widely separated. The abnormality on the curve at 0° in the neighborhood of the locality where the new variety of ice makes its appearance is very striking. At the higher pressures the curves do draw together, but they are not approaching coincidence, for they cross in the neighborhood of a volume of about 0.85. It does not seem likely that the entire failure of coincidence throughout the whole range of pressure can be due to abnormalities, since even at low pressures water is nearly normal at the higher temperatures, and certainly at the higher pressures and temperatures we have every reason to expect that its behavior is quite like that of other liquids.

This completes the list of quantities which can be deduced directly from the compressibility and the thermal dilatation. Other quantities of thermodynamic interest involve the specific heats, and these in turn involve the second temperature derivative of the volume.

The first of these quantities is the specific heat at constant pressure. This is given by the thermodynamic equation $\left(\frac{\partial C_p}{\partial p}\right)_\tau = -\tau\left(\frac{\partial^2 v}{\partial t^2}\right)_p$. It will be seen that only the derivative of the specific heat is given by the data as directly determined. In order to obtain the specific heat itself, the derivative, obtained from the tables in a manner already described, must be integrated. This integration was performed mechanically, in the same manner as the integration for the mechanical work of compression. The results are shown in Figure 11. The values for the specific heat as a function of temperature at atmospheric pressure were taken from the steam tables of Marks and Davis.[8] These values seem to be open to some slight question at the present time due to experimental work done by Bousfield [9] since the publication of the tables, but in any event the possible error is slight, too slight to be visible on the scale of the figure. The curves show the now expected abnormalities at 0° and 20°. The striking feature about the curves for the higher temperatures is the very rapid increase of the specific heat with rising temperature at the higher pressures. The specific heat at first decreases on all the curves except at 0°, but passes through a minimum, and then increases. The pressure of the minimum rapidly becomes less with rising temperature, and is situated at 6500 kgm. for 40°, 5500 kgm. for 60°, and at 1100 kgm. for 80°. At 80° the specific heat rises rapidly beyond the minimum, reaching the value 1.17 at 12000 kgm.

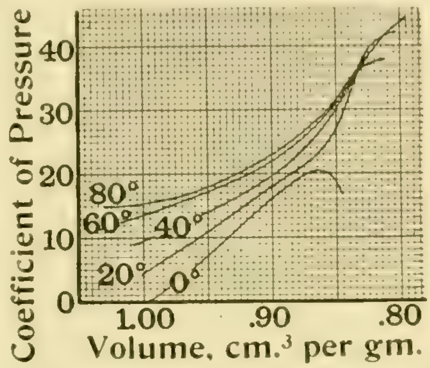

FIGURE 10. The pressure coefficient of water as a function of the volume.

Any valid characteristic equation should predict the behavior of the specific heat at high pressures as well as giving the volume in terms of pressure and temperature, since from the equation the second temperature derivative of the volume may be found. The equation of Tumlirz [10] has been mentioned in the preceding paper as giving perhaps as good agreement as any with the previously known facts over

8 Marks and Davis, Steam Tables. (Longmans, Green, and Co.)
9 W. R. and W. E. Bousfield, Trans. Roy. Soc. (A), **211**, 199–251 (1911).
10 Tumlirz, Sitzber. Wien, Bd. **68**, Abt. IIa (Feb., 1909), pp. 39.

a pressure range of 3000 kgm. This equation would predict a continuous diminution in the specific heat up to infinite pressures, the limiting value being very approximately 0.5. It was shown in the preceding paper that there is some new effect introduced at the high pressures which does not make itself felt at the low pressures, with the

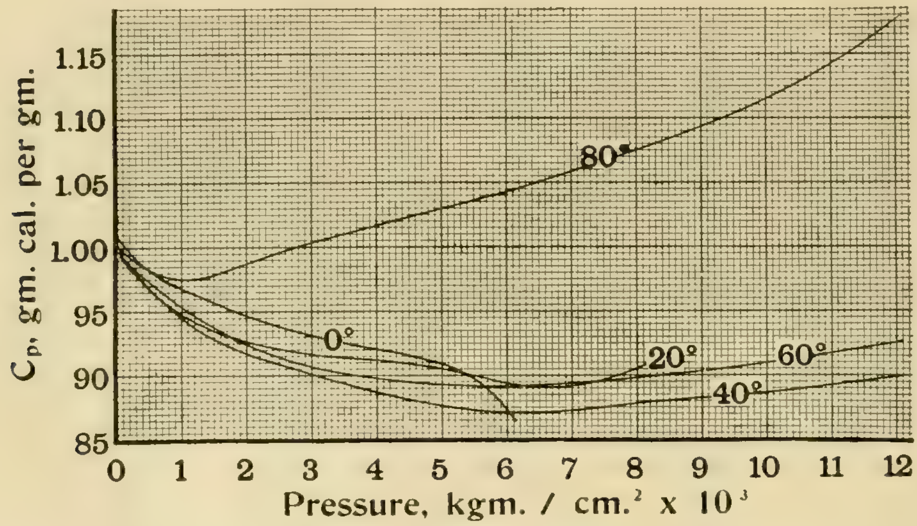

FIGURE 11. The specific heat at constant pressure of water as a function of the pressure.

result that an extrapolation to infinite pressures from the behavior for the first 3000 kgm. is not safe. This was shown in that paper by the behavior of the volume, which tended to decrease more rapidly at the high pressures than was predicted by the formula. The present data also show that there is a new effect at the high pressures, and indicate that the effect, whatever it is, is such as to have a much greater influence on the specific heats than on the volume itself.

The specific heat at constant volume may be found from the specific heat at constant pressure by means of the formula, $C_p - C_v = -\tau \frac{\left(\frac{\partial v}{\partial \tau}\right)_p^2}{\left(\frac{\partial v}{\partial p}\right)_\tau}$. This quantity, so calculated, is shown in Figure 12. The same abnormalities are shown at 0° and 20° as were shown in the curves for C_p. The curves for 40° and 60° decrease for nearly their entire lengths, although they are just beginning to rise at the very highest pressures, but the curve for 80° shows the same sharp turning point and the same rise through the greater part of its length as the curve for C_p. This quantity, the specific heat at constant volume, has more theoretical significance than the other specific heat, since this represents the heat going into the rise of internal energy of the liquid when the temperature rises, and does not involve the work done against external pressure in expanding the liquid. The external work in-

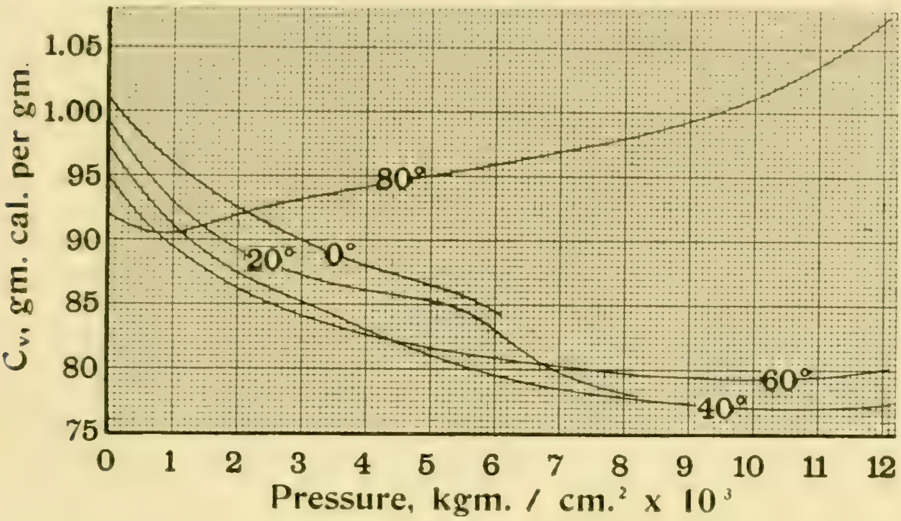

FIGURE 12. The specific heat at constant volume of water as a function of the pressure.

volves in a complicated and at present unknown way the action of the surface layer, while the specific heat at constant volume does not contain this surface effect. This specific heat is therefore one of the quantities mentioned in the beginning as having significance because it does not involve the unknown attractive forces between the molecules as displayed in the surface layer. In order to show this independence of the surface layer, of course C_v should be plotted against a variable not itself involving the action of the layer. It is evidently not adequate, therefore, to plot C_v against the pressure as as been done in Figure 12. C_v plotted against volume may be expected to show this independence of the action of the surface layer. It is shown so plotted in Figure 13. The figure is of the same general character as that in which it is plotted against pressure, but the separation of the curves for the different temperatures is greater, partly because the curves do not start from a common origin. The minimum on the curves for 40° and 60° comes at a lower pressure than it does in the former figure, and the upper end of the 80° curve is perhaps a trifle steeper at the upper end, but there are no essential differences. The entire behavior of the curves is not what one would expect from the ordinary theoretical considerations, however. It is usually considered that when the volume of a substance is kept invariable all, or else a fixed fraction, of the heat put in during a rise of temperature goes toward increasing the kinetic energy of the molecules. This is because the temperature is supposed to be proportional

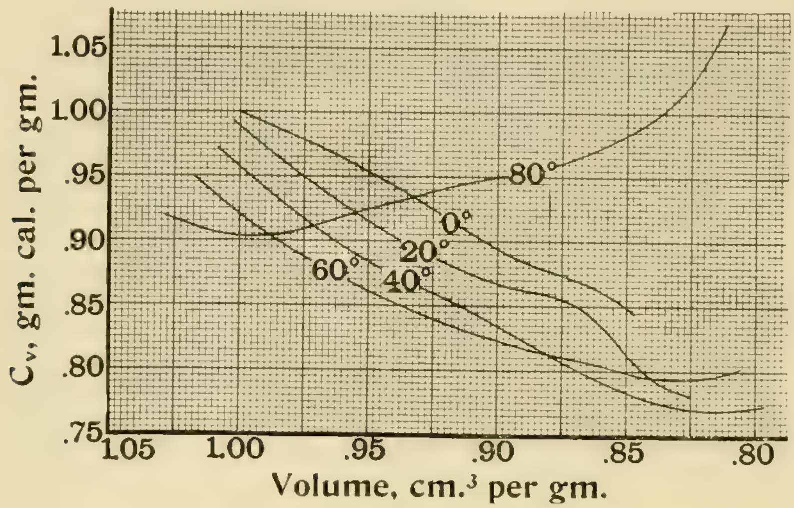

FIGURE 13. The specific heat at constant volume of water as a function of the volume.

to the energy of translation of the molecules, and therefore, because of the law of the equipartition of energy, to the total energy of the molecules. We should expect, therefore, that the input of energy required to raise the temperature by a specified amount would involve only the interval of temperature, and would be independent of the absolute value of the temperature and of the volume. The curves show most convincingly that this is not the case. This suggests that in formulating a theory of liquids it would be well to scrutinize pretty carefully several assumptions that underlie the above considerations, namely that the temperature is proportional to the kinetic energy, that a fixed fraction of the total energy of the molecules is kinetic, and that the law of the distribution of velocities is independent of temperature.

Another quantity of thermodynamic interest which may be found in terms of the specific heats is the thermal effect of compression, that is the rise of temperature in degrees accompanying a change of pressure adiabatically of one kgm. per sq. cm. This may be computed by the thermodynamic formula $\left(\frac{\partial\tau}{\partial p}\right)_\varphi = \frac{\tau\left(\frac{\partial v}{\partial\tau}\right)_p}{C_p}$. The results so calculated are shown in Figure 14 for the five standard temperature intervals. The character of the curves is the same as that shown so many times before, namely a rise to a maximum and then a fall at 0°, the abnormal behavior at the upper end of the 20° curve, and the more or less regular behavior of the three curves for the higher

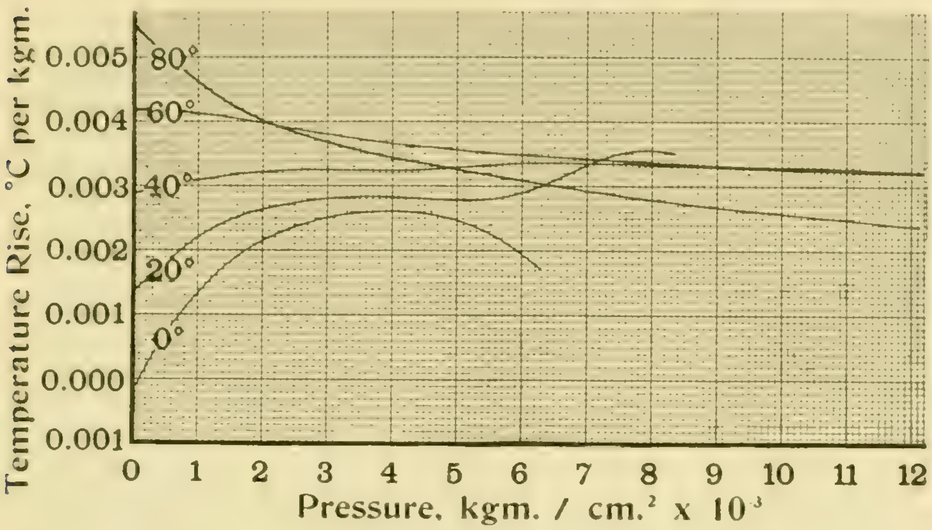

FIGURE 14. The adiabatic rise of temperature of water against pressure.

temperatures, with the crossing of the high temperature curves below the low temperature curves at the higher pressures. In the preceding paper only the approximate values for the very lowest temperature interval could be found. The calculation was based on the mean value of the dilatation between 0° and 20°. The general character of the curve was the same as that found here for 0°, namely a rise to a maximum and then a fall.

Finally it is possible to compute from the quantities in hand the difference between the isothermal and the adiabatic compressibilities. This is found from the formula $\left(\frac{\partial v}{\partial p}\right)_\varphi - \left(\frac{\partial v}{\partial p}\right)_\tau = \frac{\tau}{C_p}\left(\frac{\partial v}{\partial \tau}\right)_p^2$. The results are shown in Figure 15. The general character of the results is exactly the same as those previously given for the temperature effect of compression. Here again, the results at the lowest temperature agree with those of the previous paper which were based on a mean value for the dilatation.

Properties of Kerosene under Pressure.

In the course of the experiment other data were gathered incidentally which are of interest for themselves, and which will now be given. First of these is the compressibility and the thermal dilatation of kerosene. It was not necessary to determine this quantity in

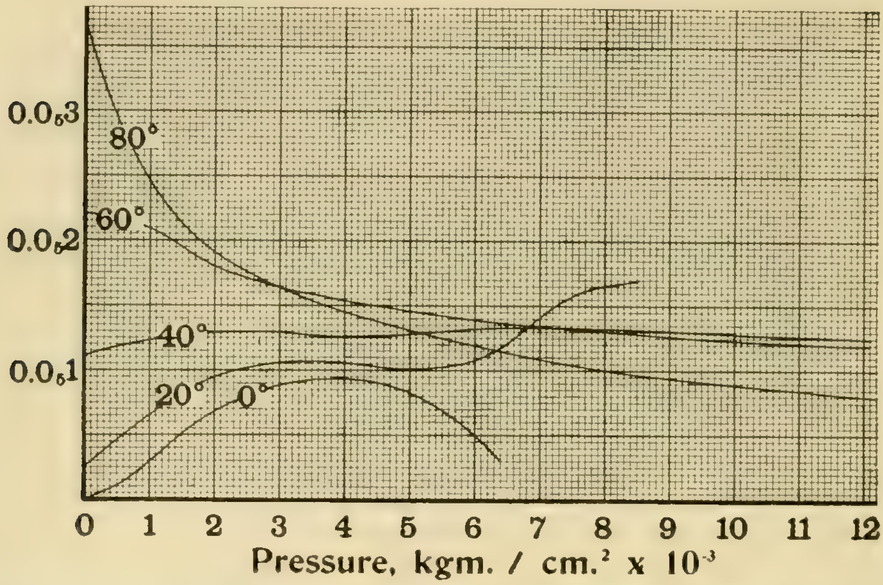

Figure 15. The difference between the adiabatic and the isothermal compressibilities of water.

order to find the corrections to be made for the distortion of the vessel, but since half the work was already done in determining the effect with the cylinder partly filled with kerosene and the other part filled with bessemer steel, it seemed worth while to make the additional run necessary to determine the pressure and temperature effects on the kerosene. Not so many determinations were made of these quantities for kerosene as were made for the water. The results are given in Table VI. The curves showing the total thermal change of volume for 20° intervals are shown in Figure 16. This figure is the analog of Figure 2 for water. The results are very different. At the lower pressures the dilatation is greater at the higher temperatures, as it is for all normal substances, but with rising pressure the effect is reversed, the dilatation becoming greater for the lower temperatures. This is the same behavior which takes place for water at higher temperatures after it has regained normality. But above 5000 kgm. the kerosene shows other abnormalities quite different in their character from those of water. This is shown plainly in the figure as a separation and then a drawing together again of the curves. The curve for 20°–40° between 6000 and 8000 and the curve for 60°–80° beyond 9000 accomplish this separation and drawing together again

TABLE VI.

Volume of Kerosene as a Function of Temperature and Pressure.
(The volume at 0° and atmos. pressure is taken as unity.)

Pressure, kgm. cm.^{2}	Volume.			
	20°.	40°.	60°.	80°.
0	1.0221	1.0412	1.0611	1.0819
1000	.9643	.9763	.9885	1.0010
2000	.9274	.9376	.9468	.9553
3000	.8995	.9086	.9165	.9239
4000	.8781	.8861	.8931	.8997
5000	.8606	.8681	.8747	.8807
6000	.8456	.8529	.8592	.8647
7000	.8323	.8396	.8456	.8508
8000	.8201	.8275	.8334	.8384
9000	.8090	.8161	.8220	.8269
10000	.7989	.8057	.8115	.8164
11000	.7897	.7960	.8017	.8069
12000	.7815	.7872	.7928	7982

by rising with rising pressure, exactly as do some of the curves for water. The abnormality is doubtless due to an entirely different cause, however. In this case the effect is to be explained by the delayed freezing of the kerosene. Kerosene is not a simple pure substance, but is a mixture of several components with different melting points. Freezing under these conditions is not sharp, but is spread out over a considerable interval of temperature or pressure as the case may be. Neither is there any necessity that the freezing should ever be perfectly complete, as indeed it is probably not. This may be shown at atmospheric pressure by plunging the kerosene into solid CO_2. The effect is to change the kerosene to a white pasty mass, like white vaseline. The pressure at which this transition occurs will rise with increasing pressure. The existence of a transi-

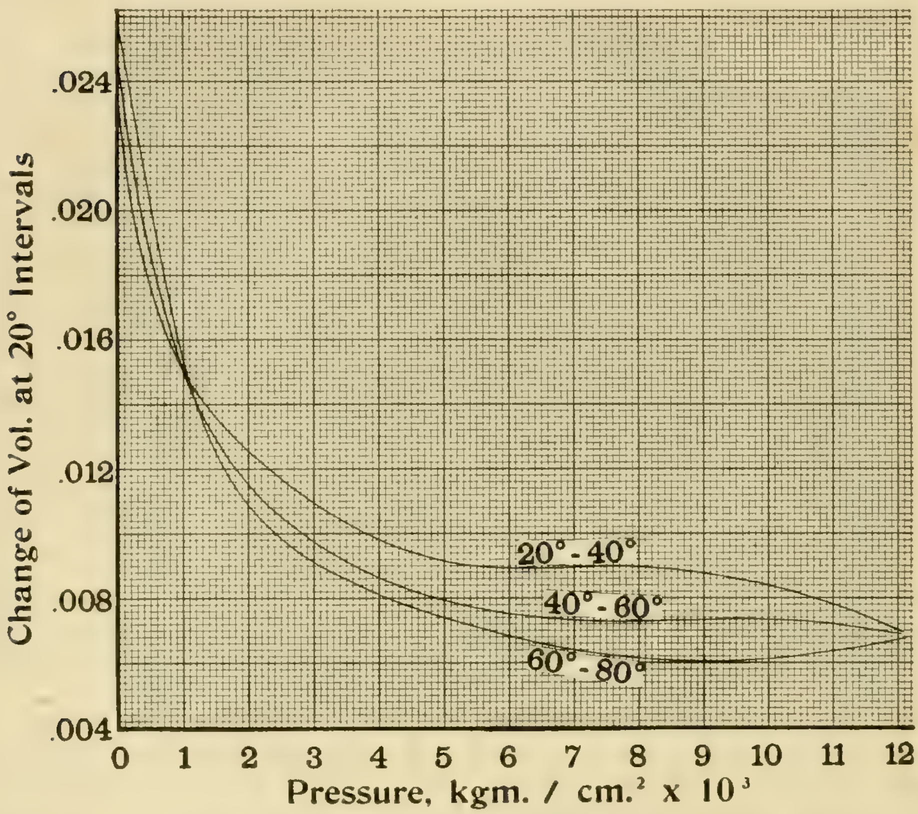

FIGURE 16. The change of volume of kerosene at constant pressure for a rise of temperature of 20°.

tion point, if there were one perfectly sharp, would be shown by an abrupt rise of the curve by an amount corresponding to the change of volume on freezing. But with the delayed freezing which takes place here due to the separation out of the separate components from a solution of varying strength, this abrupt rise becomes converted into a gradual rise extending over a fairly wide pressure range. Furthermore, the mean pressure at which this rise takes place increases with rising pressure, just as the ordinary freezing point is raised by increasing pressure. These features are all clearly shown in the diagram. At the extreme right of the diagram, at pressures over 12,000 kgm., there is evident the beginning of the reversal of the effect, that is, the curves are going to cross again, and the thermal dilation become greater at the higher temperatures. This may possibly indicate a reversal of the reversal of the effect mentioned above for liquids, but more probably the meaning is simply that at pressure above 12,000 the substance is practically a solid, and that for solids the reversal of the effect found in liquids at high pressures does not occur.

There is one bearing which these observations have on the previous data which should perhaps be mentioned. This is in connection with the delayed freezing. Whenever freezing takes place there is usually the possibility of subcooling before separation to the solid form takes place. The amount of subcooling usually taking place depends on the nature of the liquid. In some it is very considerable, while in others it is negligible. If such subcooling took place here, it would produce irregular results, because the change of volume in the kerosene transmitting pressure to the water would not always be the same under the same pressure. The only answer to be made to this objection is that in this experiment the subcooling was not great enough to produce sensible irregularity. No discrepancies were found in the data suggesting that they were due to this effect. It was feared in the beginning of the work that the effect might be very troublesome, but such did not turn out to be the case.

Also with respect to the solidification of the kerosene, the experiments showed that the solidification could not be complete, but the kerosene, even at the highest pressures, must remain a pasty mass like vaseline in nature, always capable of transmitting pressure nearly hydrostatically. But that on the other hand the kerosene does undoubtedly become pretty stiff under pressure has been already shown in the course of some measurements on the linear compressibility of steel rods.

The second bit of data collected incidentally in the course of the work was a measurement of the expansion and the thermal dilatation of the high temperature variety of ice.

The Compressibility and Thermal Dilatation of Ice VI.

Although these data are not directly concerned with the properties of liquid water, which forms the subject matter of this paper, still it was so easy to obtain them without any modification in the arrangement of the apparatus, that it was thought worth while to measure them. In the previous paper on the properties of water and the several varieties of ice, a very rough experimental value for the compressibility was given, as also a computation of the approximate compressibility, neglecting the thermal dilatation of the ice, for which no experimental value was found at that time. These measurements here include a direct measurement of the thermal dilatation, and two different determinations of the compressibility by two different methods. The value for the dilatation may be combined with the already determined values for the volume of the liquid and the change of volume when ice VI separates out, to give a third independent value for the compressibility.

The determinations of the dilatation will first be described. This was found in the same manner as the dilatation of the liquid, by changing the temperature at constant mean pressure, and measuring the change of pressure brought about thereby. Three determinations of this were made for the combination of ice and kerosene, and two for the combination of kerosene and bessemer. The agreement of the different determinations was within 2% of the mean. The dilatation was found between 0° and 20° at a mean pressure of 10,000 kgm. The correction introduced by the thermal dilatation of the bessemer cylinder in the control experiment is fairly large here, being about 25% of the entire effect. The value assumed for the cubic dilatation was 0.000036, which is the value for atmospheric pressure. The effect of pressure is to decrease this number slightly, which would result in a larger value for dilatation of the ice. The effect of pressure on this quantity is, however, very small, and the error so introduced is probably negligible. The mean dilatation found in this way for the 20° above 0° at 10,000 kgm. was 0.00241 $cm.^3/gm.$, or 0.000120 $cm.^3/gm.$ per degree. This is considerably less than the dilatation of the liquid in this neighborhood, for which the value 0.00040 has been found previously.

This value for the dilatation may now be combined with the other data for the liquid and the solid to give the compressibility of the solid along the equilibrium curve. For this we have the following data: vol. of 1 gm. of water at 0° and 6360 kgm., 0.8428 $cm.^3$, and at 20° and 9000 kgm. (these are the equilibrium pressures at these temperatures) 0.8160 $cm.^3$. For the change of volume when the liquid freezes to the solid we have at 0°, 0.0916, and at 20°, 0.0751. This gives for the volume of ice at the equilibrium pressures at 0° and 20° the values 0.7512 and 0.7409 respectively. The decrease of volume of the ice along the equilibrium curve is 0.0103. Part of this is an increase due to rise of temperature, which according to the above data is 0.0024. This leaves a decrease of 0.0127 to be accounted for by the increase of pressure of 2640 kgm. which gives a mean compressibility over this range of 0.0000048, a little more than one third of the compressibility of the liquid over the same range.

The direct determination of the compressibility of the ice was made by two different methods. One of these was the same as that used roughly in the preceding paper, that is by finding the difference of the slope of the curves plotting piston displacement against pressure above and below the transition point to the solid. The values obtained in the preceding paper for this were very rough. In these determinations the cylinder was very much more carefully seasoned, and the readings were made with all the precautions which had been suggested by all the experience of this paper. Two determinations of this quantity were made at 0° and also two determinations at 20°. The two values for the difference of compressibility differed by 2.5% at 0° and by 0.7% at 20°. The value found for the difference was 0.0000087 at 0° and 0.0000067 at 20°. Combining with the values given already for the compressibility of the liquid, this gives for the compressibility of ice VI 0.0_549 at 0° and 6360 kgm., and 0.0_543 at 20° and 9000 kgm. Mean 0.0_546.

The second method for determining the compressibility was exactly the same as that for finding the same quantity for the liquid, comparing the displacements when the apparatus was filled with ice and kerosene with those when the ice was replaced by bessemer steel. This determination was made over a wider pressure range, to find if possible the variation of compressibility with pressure. No variation with pressure could be found over a range of 4500 kgm. at 0° and 3300 kgm. at 20°. The absolute values do not agree with those found by the two other methods, however, the figures being 0.0_531 at 0° and 0.0_535 at 20°. The cause of the discrepancy is not clear, but is probably connected in some way with the hysteresis of the cylinder. The hysteresis was not regular for these small pressure ranges, being at times almost negligible, and again being as large as for almost the entire pressure range from atmospheric pressure to the maximum. There seems little question but that the greater weight is to be attached to the values found by the first two methods. This third determination does show, however, that the variation of the compressibility with pressure and temperature over this range is so small as to be beyond the accuracy of these measurements. In selecting the best probable value for the compressibility the only weight that will be assigned to this third determination is in slightly lowering the mean of the other two.

The final most probable values for Ice VI are as follows: for the compressibility 0.0_545, and for the thermal dilatation 0.000120 cm.^{3}/ gm. over the range 6360–10,000 kgm. and 0° to 20°.

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CONTRIBUTIONS FROM THE CRYPTOGAMIC LABORATORY OF HARVARD UNIVERSITY.

LXXI. — *PRELIMINARY DESCRIPTIONS OF NEW SPECIES OF RICKIA AND TRENOMYCES.*

By Roland Thaxter.

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LXXI.—PRELIMINARY DESCRIPTIONS OF NEW SPECIES OF RICKIA AND TRENOMYCES.

By Roland Thaxter.

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Rickia.

The genus Rickia has proved to be a large and varied one, and although I have enumerated below only those forms parasitic on Acari which have come under my notice, many others are known to me on a variety of hosts, an account of which I have reserved for a future paper. The general habit appears to be very variable, including in addition to the condition seen in the type form, others in which the median cell-series is undeveloped, as well as various species with a more or less complicated system of branches. The antheridial characters, moreover, appear to be equally variable. Not only do the antheridia which are extraordinarily abundant in some species seem wholly lacking in others, but their character may vary in different cases. In some there may be a single antheridium, only, similar to that of *Peyritschiella*, definitely placed at the base of the perithecium; or an antheridium of this type may be associated with others of the normal habit variously disposed. Again even in forms having the three characteristic cell series, antheridia may be present like those of the genus formerly separated as *Distichomyces*, each antheridial cell becoming more or less free in a compact group. Since both the antheridial characters and those of the receptacle thus appear to be so variable, it has not seemed desirable to limit the genus to the type form as illustrated by *Rickia Wasmanni*, and I have therefore given it a more liberal interpretation; including under it forms with two or with three cell-series, whether they be simple or branched, and whether their antheridia be of the Rickia or the Distichomyces type. The latter genus is, therefore, abandoned, one species only, **Rickia Leptochiri**, being involved in this change.

The only American form, *R. minuta*, thus far recorded on Acari, has been described by Paoli ("Redia," Vol. VII, fasc. 2, 1911, republished in Malpighia, Vol. XXIV, 1912) from immature specimens with undeveloped perithecia, a practice which it is surely most desirable to avoid in the systematic study of a group which presents such great difficulties as do the Laboulbeniales. I have been fortunate, however, in obtaining abundant material of this species, fully matured, from the Amazon region, for which as well as for other hosts, I am indebted to the kindness of Mr. W. H. Mann who has allowed me to look over his collections made on the Leland Stanford Expedition in 1911. I am further greatly indebted to the kindness of Messrs. T. Petch, Geo. Schwab and J. B. Rorer who have most generously collected or caused to be collected for me numerous insects, in Ceylon, Kamerun and Trinidad respectively, from among which a majority of the following hosts were obtained. I am also indebted for two species of Acari collected in Grenada to Mr. C. T. Brues and kindly placed at my disposal; while lastly I am much indebted to Mr. Nathan Banks for his determinations of the host-genera.

In the following diagnoses I have assumed that the side bearing the perithecium is "anterior." The spore measurements are for the most part made within the perithecium.

Rickia furcata nov. sp.

Furcate, sometimes irregularly branched. Basal cell short and rather stout, the receptacle above it dividing in two straight divergent branches; an anterior, bearing a perithecium, and a posterior. Anterior branch consisting of a series of usually eleven cells, the lower superposed horizontally, the upper obliquely; all cutting off appendiculate cells externally; the series extending nearly to the apex of the perithecium, to which it is united throughout its length; the second cell of the series extending inward below the base of the latter, the outline of which is symmetrically subfusiform, the inner lip-cell protruding as a finger-like process. Posterior branch indeterminate, formed by a double series of cells which are more or less regularly paired above the second cell of the outer row, the third cell bearing the primary appendage on its narrow subtending and long cylindrical basal cell; many, but not all of the cells above in both rows cutting off distally and externally small cells which bear well-developed appressed appendages or antheridia (?). Appendages subcylindrical, 8–16 × 2.5 μ. Perithecium 30–40 × 8–10 μ, including terminal projection

(2.5–3 μ). Spores about 25 × 2.5 μ. Total length to tip of perithecium 40–70 μ, to tip of posterior branch 50–175 μ.

On *Euzercon* spp. No. 2431, Trinidad; No. 2236, Manaos, Amazon; No. 2058, Grenada, W. I.

This species, and to a more marked degree the following, depart greatly from the normal type, and would be placed in a new genus with little hesitation were it not for the structure which characterizes various others of the many undescribed species known to me. It is evident that the "posterior branch" is an indeterminate proliferation beyond the primary appendage, which appears to involve both the "median" and the "posterior" marginal series of the more normal forms. The receptacle, especially when a primary perithecium fails to develop, may become variously branched and more than one secondary perithecium may be produced. Antheridia of a type like that of *Distichomyces* appear to be developed externally on the posterior branch nearer the base. The specimens from Brazil and Trinidad seem to be identical, although those from Grenada, though otherwise similar, are constantly somewhat smaller.

Rickia arachnoidea nov. sp.

Basal cell rather short and stout, the receptacle above it dividing into two usually furcate arachnoïd branches; an anterior on which a perithecium is produced, and a posterior. Anterior branch indeterminate, consisting of two parallel series of cells usually not opposite, irregularly appendiculate, furcate at a variable distance from its base; one of the branchlets sterile, often greatly elongated; the other short but variable, bearing a perithecium which on one side is usually united to the upper six cells, some of them appendiculate, which continue one of the two series forming the perithecial branchlet which thus extends to the apex of the perithecium, beside which it terminates in a short brown appendage: the perithecium long, slightly and nearly symmetrically inflated, the tip bent distally abruptly sidewise; the other row of the perithecial branchlet ending horizontally or obliquely below the base of the perithecium and consisting of from three to eight cells, some of which are appendiculate. Posterior branch indeterminate, furcate, usually, just above its first to fifth pair of cells, the cells of the two indeterminate branchlets not paired, irregularly appendiculate, indeterminate, usually greatly elongated: the second cell of the main receptacle below its furcation bearing the large long nearly cylindrical basal and subtending cells of the primary appendage, which may be on either side. Appendages suffused with brownish, mostly rather short and stout, 7–18 × 4 μ. Spores 30 × 3 μ. Perithecia 70 × 18–20 μ. Diameter of branches 8–10 μ, greatest length 460–520 μ, in largest specimens. Basal and subtending cell of primary appendage 18–20 × 4 μ, the former rarely divided.

On *Discopoma* sp. Trinidad, No. 2433; on *Trachyuropoda* sp. Trinidad, No. 2429; also an immature specimen from the Amazon on same host; on *Euzercon* sp., Trinidad, No. 2432.

When normally developed this curious form appears to be more or less regular in its structure, as above described, but especially when injured or when the first perithecium aborts, secondary branching takes place, and more than one perithecium may be formed. That there is no significance in "anterior" and "posterior" as applied to the main branches of this form, is indicated by the variable position of the primary appendage beyond which they proliferate. The plant has a characteristic sprawling habit, its branches resting on the upper surface of its host, which is its usual position of growth. Unless it is viewed sidewise, the cell-series bordering the perithecium is not visible, and may thus be wholly overlooked. The appendages, as in all the species, are borne from small subtending cells. Among described species it is most nearly allied to *R. furcata*.

Rickia anomala nov. sp.

Hyaline, rather strongly curved throughout above the basal cell. Median cell-series wanting. Basal cell wholly free, longer than broad, of nearly the same diameter throughout. Anterior series consisting of three or rarely four cells, subisodiametric, externally convex, subequal, without appendages. Posterior series of usually nine cells, the two or three lower larger, rounded; the rest smaller, subequal, irregularly rounded; the first, third, fifth, and seventh cells separating distally small cells which subtend appendages, the second cell subtending the basal cell of the primary appendage, which is relatively very large, wholly free, constricted at the base, terminated by a small cell which subtends the appendage proper; the latter somewhat smaller than the others, but otherwise similar, faintly brownish, bladder-like, roundish, or somewhat longer than broad. Perithecium directly continuous with the anterior series, externally wholly free, rather long and narrow, the tip well distinguished, narrowed, its lower half united on the inner side to the distal cell of the posterior series, which ends in a minute suffused roundish hardly distinguishable cell; the inner lip-cell forming a finger-like straight free process. Spores about $25 \times 3\,\mu$ (in perithecium). Perithecia 30–35 $\times$ 8–10.5 μ. Basal cell 14–18 $\times$ 5–6.5 μ. Basal and subtending cell of primary appendage 16–17 $\times$ 7 μ. Appendages 9×4.5–$7 \times 6\,\mu$. Total length 48–56 $\times$ 14–16 μ.

On a minute mite belonging to a new genus, near *Iphiopsis*. Trinidad, No. 2440.

Although there are fourteen specimens of this peculiar species in various stages of development, none of them show any indication of the presence of an antheridium.

Rickia Discopomae nov. sp.

Hyaline, becoming slightly soiled with dirty brownish throughout. Basal cell large, twice as long as broad. Main body of the receptacle of about the same diameter throughout, broadening slightly below the perithecium, usually rather strongly curved. Cells of the three cell-series small, subequal, squarish or subisodiametric, arranged in tiers of three cells each with some regularity; the middle series extending half way along the tip of the perithecium, its two or three terminal cells free beyond the base of the primary appendage, which terminates the posterior marginal row. Cells of the median row fifty to sixty in number, sometimes less; those of the anterior marginal row thirty to fifty; of the posterior marginal row fifty to sixty, the cells of both marginal rows cutting off appendiculate cells irregularly, except those of the posterior row opposite the perithecium which produce them uninterruptedly; the appendages and antheridia thus irregularly and rather sparingly distributed along the margins. Appendages short and usually inflated. Perithecium rather short and stout, the tip often somewhat bent outward, the apex blunt. Spores $30 \times 5\,\mu$. Perithecium 48–52 $\times$ 18–25 μ. Total length 250–350 $\times$ 18–32 μ, measured below the perithecium. Appendages 7–10 $\times$ 3–4 μ.

On superior surface of *Discopoma* sp. Peradenyia, Ceylon, No. 2111.

The antheridia of this species are not certainly recognized, but appear to be of the type seen in "*Distichomyces.*" The appendages appear to branch occasionally, becoming furcate, a condition possibly resulting from the proliferation of old antheridia.

Rickia elegans nov. sp.

Basal cell hyaline; cells of median row small, rounded; those of marginal rows horizontally elongated or their axes directed upward somewhat obliquely, more than fifty cells in the posterior row, about twenty-five in the anterior; the cells at maturity in all the rows becoming deeply suffused with rich blackish brown and quite indistinguishable; all the cells of the marginal rows cutting off small cells which remain almost wholly hyaline and bear short appendages, their cup like bases rich brown, the distal portion hyaline. Perithecium wholly united on its inner side to the median row, the last two or three free cells of which reach to the middle of the short stout deeply suffused rather broad tip, which is bent rather abruptly outward; the apex hyaline, or translucent; the body nearly straight, about the same diameter throughout, rather narrow, rich brown, not as deep as the tip, the outer margin somewhat irregular, continuous with that of the receptacle below. The whole plant straight or curved, tapering gradually from apex to base. Perithecium 65–85 $\times$ 20 μ. Appendages about 15 $\times$ 4 μ. Total length 200–220 $\times$ 35–40 μ.

On legs and margin of body of *Discopoma* sp. Peradeniya, Ceylon, No. 2110.

This species is very closely allied to *R. Berlesiana* Paoli (Bac.), differing chiefly in its much more numerous cells, which are smaller and differently arranged and the total suffusion of the receptacle. In fully mature specimens, the perithecium is concolorous with the receptacle, and not distinguishable from it.

Rickia cristata nov. sp.

Basal cell three times as long as broad, its upper half or less included between the lower cells of the marginal rows. Posterior row crest-like, the cells radially elongated, each separating several appendiculate cells, the pointed bases of which are intruded between them nearly to their bases, the appendiculate cells becoming so multiplied, where the series curves over against the tip of the perithecium, that the primary cells are obliterated; the primary cells of this series about eighteen, the appendiculate cells thirty-six to forty: the anterior series extending slightly beyond the middle of the perithecium, the base of which it incloses, consisting of three or four cells from which a number of appendiculate cells are cut off, as in the posterior series, one or two of the uppermost bearing pointed antheridia: the appendages six to eight: the middle series of six flattened cells lying along the inner margin of the perithecium for a little more than two thirds of its length. Perithecium rather short and stout, slightly curved, the apex blunt and opposite the bases of the distal appendages of the posterior series, the tip well distinguished externally. Spores $30 \times 4\,\mu$. Perithecium $45 \times 18\,\mu$. Free portion of the basal cell about $18\,\mu$; the rest of the plant $60–75 \times 48–52\,\mu$. Appendages $16–25 \times 4\,\mu$, becoming brownish and subtended by the usual dark cup-like base.

On the inferior surface of a mite parasitic on *Prioscelis* sp. (?) and belonging to a new genus near *Cilliba*. Kamerun, No. 2438.

A species closely allied to *R. Coleopterophagi* Paoli and *R. minuta* Paoli, differing in the form of its appendages and the arrangement of its cell-series. The single type of *R. Coleopterophagi* as well as those of *R. minuta*, are immature, so that it is not possible to judge of the perithecial characters in these species. The latter, however, has been received from Brazil (Mann) on various mites parasitic on Scarabeidae, and an abundance of well matured individuals are available for comparison. The species though very variable is quite well distinguished from the one above described. The tip of its perithecium is wholly free; the cells of the middle series vary considerably in number and extend as far as those of the posterior series, which is more nearly vertical, the general habit of the plant being more slender; the basal cell is not intruded between the lower cells of the anterior and posterior series and there are other differences.

Rickia pulchra nov. sp.

Basâl cell variably developed, more often short, the upper half enclosed by the lower cells of the marginal series; or long and very stout distally. Posterior marginal series consisting usually of four cells, the lower opaque blackish brown bearing distally a very minute rounded appendage, the next above somewhat rounded and cutting off a small cell which subtends an antheridium, the third large triangular, its pointed end directed upward, and cutting off three to five appendiculate cells which lie external to it; the uppermost small, flattened, distally pointed, separating a single minute cell which lies external to it and subtends a small short brownish spine-like appendage: the anterior series consisting of three cells, similar to and symmetrical with the corresponding cells of the posterior series, and bearing an antheridium and appendages in a similar fashion so that the individual is bilaterally subsymmetrical: the middle series consisting of but two flattened cells, the upper, its broader extremity free beyond the distal cell of the posterior series, nearly twice as long as the lower, which is opaque below and forms with the two lower cells of the two other series a suffused area in which cell-divisions are not visible and which extends upward so as to involve the lower half of the perithecium; the tip of which is nearly free, usually bent slightly toward the anterior series, and subtended anteriorly by a straight appendage about $15 \times 3\ \mu$, suffused towards the base, and apparently the indurated base of the trichogyne. Appendages nearly symmetrical on either side, long and slender, hyaline, becoming deeply suffused at and towards the base, cylindrical, tapering slightly at base and apex. Antheridia normally solitary, borne distally from the subbasal cells of the two marginal series, hyaline, the necks purplish, curved outward. Spores, in perithecium about $22 \times 3.5\ \mu$. Perithecia $35–40 \times 15\ \mu$. Basal cell $18–50 \times 6–15\ \mu$. Appendages $35–60 \times 4–6\ \mu$. Total length exclusive of stalk $48–56 \times 35–38\ \mu$.

On the inferior surface and legs of *Macrocheles* sp. and *Celaenopsis* sp. Kamerun, Nos. 2438, 2439.

A very beautiful species, quite unlike any other known form. The specimens on *Celaenopsis* are somewhat smaller.

Rickia obcordata nov. sp.

Hyaline. Basal cell bent, its pointed upper half filling the sinus of the slightly asymmetrical obcordate body. The marginal series consisting of typically six cells each and subsymmetrical with one another, the posterior shorter, terminated by the slender basal cell of the primary appendage which, like all the appendages and the antheridia, projects radially in a more or less regular fashion: basal cells of the marginal series radially extended, broad and rounded externally, separating a small triangular cell above, which subtends an appendage symmetrically placed on either side of the body, the second and third cells of both series separating externally three to four small cells which subtend each an antheridium, the necks quite hyaline projecting more or less radially, usually straight, the third cell on the posterior side usually bearing an appendage distally: the fourth and fifth an antheridium and an appendage, or an appendage only in both series, except in cases where there are but five cells in the posterior series, the uppermost of which always subtends the primary appendage; the sixth cell of the anterior series producing neither appendage nor antheridium. Appendages subcylindrical, several times as long as broad, faintly suffused above the conspicuous blackened slightly constricted base. Median series consisting of five cells successively smaller from below upward, the three lower rounded, the uppermost triangular, its upper face free below the slightly projecting truncate or bluntly rounded free tip of the perithecium. The latter otherwise completely enclosed, vertical or slightly oblique, and lying almost wholly anterior to the median axis. Perithecium $60 \times 25\,\mu$. Body $90–100 \times 78–85\,\mu$. Basal cell including foot $28–35 \times 15–18\,\mu$. Appendages $24–35 \times 5\,\mu$. Projecting antheridia $12\,\mu$.

On a minute mite. Kamerun, No. 2441.

A very minute form characteristic from its obcordate almost symmetrical form and radiating antheridia and appendages.

Rickia elliptica nov. sp.

Hyaline, elliptical to nearly circular in outline. Basal cell very short, sometimes entirely included in the angle between the inner surfaces of the basal cells of the marginal rows, the foot, only, projecting beyond the general outline of the main body. Anterior marginal row consisting of from five to eight cells subradially elongated, the two uppermost extending downward to sharp points, all or nearly all cutting off distally a small triangular appendiculate cell; the appendage which terminates the distal cell appressed against the free anterior face of the tip of the perithecium: posterior marginal row consisting of from seven to nine cells, similar to the anterior series except that the upper cells are smaller, the uppermost much smaller, bearing distally the basal cell of the primary appendage which is small, narrow, free, not greatly longer than the subtending cell of the very small appendage; other appendages stouter, short, irregular with slightly suffused bases. Median series of six to eight cells, one to three of the terminal ones externally free above the basal cell of the primary appendage, the successive cells subisodiametric, somewhat irregular in outline, and not greatly differing in size. Perithecium almost wholly inclosed, the tip free externally, slightly bent outward below the apex which is subtended on its inner side by an erect finger-like upgrowth, geniculate at its base; body of the perithecium rather long and narrow, subsymmetrical. Spores (in perithecium) $22 \times 2.5\,\mu$. Perithecium $30–40 \times 10–12\,\mu$, not including the projection which is $7 \times 2\,\mu$. Basal cell, including foot, 8–16 μ. Total length of body 50–66 $\times$ 35–40 μ.

On legs of *Discopoma* sp. Trinidad, No. 2433.

Although seven specimens in perfect condition and of various ages have been examined, I have seen no indication of an antheridium. The base of the trichogyne persists as a minute dark rounded body below the base of the upgrowth from the inner terminal wall-cell.

Rickia inclinata nov. sp.

Minute, hyaline, of irregularly rounded form. Basal cell forming a well defined slender stalk, the upper third or half inserted in the angle between the two basal cells of the marginal rows. Anterior marginal row not extending above the base of the perithecium, consisting of two radially elongated cells which are subequal and cut off distally and externally two to three appendiculate cells: posterior marginal row consisting of seven cells like those of the anterior, externally convex, the second to the fourth more radially elongate than those above, which are successively smaller; the basal usually separating one, the rest two appendiculate cells distally and externally; the terminal cell much flattened followed by the broad basal cell of the primary appendage, which appears to be a member of the series, its inner margin in contact with the fourth cell of the median series: median series of four subequal irregularly rounded cells. Perithecium stout, its axis straight and characteristically tilted inward at a slight angle to that of the receptacle, its base in contact with the distal cell of the anterior series, externally wholly free; the tip quite free, bent very slightly outward, the apex broad, flat, each lip-cell projecting very slightly and somewhat irregularly. Spores $25 \times 3\,\mu$ (in perithecium). Perithecium 38–40 $\times$ 11 μ. Basal cell, including foot, $25 \times 8\,\mu$. Total length of body to tip of perithecium 50 $\times$ 41–44 μ. Appendages hyaline, tapering very slightly, $16 \times 3\,\mu$, with clearly defined dark basal septa.

On a minute mite, as yet undetermined. Trinidad, No. 2307.

A characteristic and minute species, distinguished by its tilted perithecium, which is externally free. It is closely allied to *R. Celaenopsis,* from which it differs in the number and arrangement of its cells, etc. I have been unable to dete mine the presence of an antheridium in either of the two adult types.

Rickia Celaenopsis nov. sp.

Hyaline, minute, somewhat angular in outline. Basal cell forming a well developed stalk, the upper third or less inserted in the angle between the two basal cells of the marginal rows. Anterior series consisting of two cells, the lower characteristically triangular in form, its outer margin straight and evenly continuous with that of the upper cell, which is radially elongated and cuts off distally an appendiculate cell which is relatively very long, its distal half or two thirds projecting free beyond the margin and subtending a relatively very large and long antheridium which projects above it just at the base of the perithecium: posterior series consisting of typically six cells, the basal like that of the anterior series, triangular, but cutting off distally a slightly prominent appendiculate cell; the four cells above obliquely elongated, lying subparallel, and separating distally a conspicuously protruding upturned appendiculate cell; the terminal cell triangular, subtending the wholly enclosed sublenticular basal cell of the primary appendage, the subtending cell of which is free, bell- or dome-shaped, bearing a rather stout appendage. The appendages subcylindrical, several times longer than broad, rarely furcate, with the usual dark subtending base: median series consisting of usually six cells, the basal and distal somewhat larger, the rest squarish or slightly compressed, subequal, the upper margin of the distal cell free, its oblique wall very thick and directly continuous with the margin of the tip and the distal portion of the venter of the perithecium which rise erect beyond it. Perithecium thick walled, somewhat inflated, quite free and convex externally, erect or nearly so, the tip symmetrical, truncate conical, the apex flattened or slightly rounded. Spores $20 \times 3\,\mu$ (in perithecium). Perithecium $40 \times 20\,\mu$. Basal cell including foot $25 \times 8\,\mu$. Total length of body to tip of perithecium, $50 \times 38\,\mu$, largest. Antheridium about $12\,\mu$ long.

On legs of *Celaenopsis* sp. Trinidad, No. 2426.

Closely allied to *R. inclinata*, but differing in many details of structure, the triangular form of the two basal cells of the lateral series giving it a characteristic appearance.

Rickia discreta nov. sp.

Hyaline, rather elongate. Basal cell relatively large and long, distally symmetrical, but slightly intruded between the lower cells of the marginal series. Anterior marginal series consisting of three to four subequal obliquely separated cells, the lowest cutting off an appendiculate cell distally and externally, the upper an antheridium of the Peyritschiella-type, which subtends the base of the perithecium from which its hyaline sharply pointed stout extremity projects obliquely upward: posterior marginal series consisting of usually seven obliquely separated cells, usually the first, third and fifth, only, separating a rather large appendiculate cell; the uppermost cell triangular, its upper margin continuous with that of the distal cell of the median series, subtending the basal and large subtending cell of the primary appendage, the two latter subequal, the basal somewhat broader: median series consisting of normally six successively smaller, vertically slightly elongated cells. Perithecium erect, slightly curved outward distally, the tip free, the apex symmetrical, truncate, slightly papillate. Appendages relatively long and stout, yellowish, subcylindrical, the basal ring black and conspicuous; 15–25 $\times$ 3.5 μ, the primary one 30–45 μ, its basal and subtending cells 10 $\times$ 4 μ. Perithecium 25 $\times$ 9 μ. Basal cell including foot 20 $\times$ 7 μ. Total length to tip of perithecium 55–65 $\times$ 18–22 μ.

On an undetermined gamasid mite. Trinidad, No. 2308.

This species is well distinguished by its large discrete yellowish appendages, somewhat elongate form, and large single antheridium. In one of the nine specimens examined a second antheridium is developed just below the first.

Rickia spathulata nov. sp.

General form spathulate except for the projecting tip of the perithecium. Basal cell rather stout, its upper half or less inserted in the sharp angle between the lower cells of the marginal series. Anterior series consisting of six to eight cells, the lowest irregularly triangular, externally slightly concave, and without appendage, the rest usually but not always appendiculate, radially elongated, and slightly oblique upward; the subterminal cell bearing also an antheridium, the basal cell of which penetrates three fourths of its length; the terminal cell sometimes separating a second antheridium, its inner margin in contact with the lower two thirds of the perithecium, narrow, its extremity broader and convex: posterior series consisting of ten to thirteen cells, usually eleven, the lowest externally convex like the rest, the other members of the series each usually cutting off an appendiculate cell about half their length and lying between them; the upper ones successively narrower and more elongated radially;

the cells above the second or third curved inward in a somewhat crest-like series which lies parallel to the median series and the inner margin of the perithecium, the terminal cell of the series small, triangular, bearing the large basal cell of the primary appendage which, with the small subtending cell, forms a free straight projection, its axis bent inward at an angle of about 45° to that of the receptacle: median series consisting of eight to ten cells, the two or three lowest enclosed by the marginal series, the rest lying against the strongly convex inner margin of the perithecium, the free slightly convex margin of the uppermost reaching almost to the base of the free tip. Perithecium rather stout, its outer margin nearly straight, its inner convex, the outcurved tip, and externally a small portion of the body, free; the apex flat, protruding slightly externally. Spores $28 \times 3\ \mu$, in perithecium. Perithecium $40–46 \times 16–20\ \mu$. Basal cell, including foot, $28–33 \times 9–11\ \mu$. Total length, not including primary appendage base, $12–16 \times 6–8\ \mu$. Appendages $6 \times 2\ \mu$ or smaller, wholly smoky brown, usually broken off, the dark base not conspicuous.

On legs of *Celaenopsis* sp. No. 2229, Amazon, "M. & M." (Mann No. 41).

A very well marked species peculiar for its more or less regularly spathulate outline, which is broken only by the projecting tip of the perithecium and the primary appendage. It is not nearly allied to other known acarine species, but is perhaps most nearly related to *R. minuta*.

Rickia excavata nov. sp.

General form roughly triangular, distally concave. Basal cell three or four times as long as broad, its distal fourth included in the angle between the two lower cells of the marginal series. Anterior series consisting of four cells, the lower three subequal, usually all appendiculate, the uppermost vertically elongated, externally convex, extending to the middle of the venter of the perithecium: posterior series consisting of usually seven cells, the four lower similar to those of the anterior series, usually all appendiculate, the subtending cells hardly intruded between adjacent members of the series, the three terminal cells successively smaller, flattened, their septa at right angles to the axis of the series which they form, and which is continuous with that of the primary appendage and its basal and subtending cells, which, together with the two terminal cells of the posterior series form a free subtriangular projection directed at an angle of somewhat over 45° to the axis of the body of the perithecium: the median series consisting of usually five cells, the lowest larger, longer than broad and lying mostly below the base of the perithecium; the three upper successively narrower, extending to the base of the tip of the perithecium, forming a series almost symmetrical with that of the three terminal cells of the posterior series and the primary appendage, the axes of the two series nearly at right angles. Tip of the perithecium wholly free, bent strongly inward, the apex abruptly distinguished, the lip-cells rather prominent, the inner more so, rounded; the body nearly vertical or inclined very slightly outward, rather long and narrow and symmetrically rounded basally and distally. Spores $18 \times 3\,\mu$. Perithecium $30 \times 10\,\mu$. Appendages subcylindrical, small, about $6 \times 2.5\,\mu$. Basal cell $20 \times 6\,\mu$. Total length to tip of perithecium $75 \times 34\,\mu$, not including basal cell of primary appendage.

On *Celaenopsis* sp. Trinidad, No. 2427.

Clearly distinguished from other known species by its general form and excavated superior margin.

Rickia Euzerconalis nov. sp.

General form short-spathulate, hyaline. Basal cell very small and short, separating an appendiculate cell distally on the anterior side. Posterior marginal row consisting of usually eight, often nine cells, radially and obliquely but slightly elongated; all usually cutting off an appendiculate cell, except the distal one, which is small, triangular and subtends the large usually outcurved basal cell of the primary appendage which is free above it, two to three times as long as broad, and about the same diameter throughout: anterior marginal series consisting of usually five cells, more rarely four or six, the lowest separating an appendiculate cell below, which lies between it and the basal cell of the receptacle; the remaining cells large, each, except sometimes the lowest, separating an appendiculate cell distally; the uppermost extending to or beyond the middle of the perithecium with which its appendiculate cell with the appendage is in contact: median series consisting of almost invariably six, rarely five or seven, cells, not differing greatly in size, extending from just below the base of the perithecium nearly to its apex. Perithecium narrow, erect, its tip externally free, the inner lip-cell projecting as a short finger-like process. Appendages stout, yellowish-brown, $7 \times 3.5\,\mu$. Spores $25 \times 2.5\,\mu$. Perithecia $22–24 \times 8\,\mu$. Basal cell including foot, 14–16 × 6–8 μ. Total length to tip perithecium 50–70 × 24–32 μ. Basal and subtending cell of primary appendage 12–15 × 5 μ.

On *Euzercon* spp. Trinidad, Nos. 2432 and 2430; Kamerun, No. 2443.

This species is most nearly related to *R. Megisthani* from which it differs in its more complicated receptacle, larger size and more or less evenly spathulate outline. In this, as well as in the following species (*R. Megisthani*) the lowest appendage on the anterior side is subtended by a cell which lies external and inferior in relation to the lowest cell of this series, instead of distal, and has the appearance often of having been separated, not from this cell, but from the basal cell of the receptacle below and it is possible that this is its actual relation.

Rickia Megisthani nov. sp.

Hyaline. Basal cell rather short and stout, obliquely separated from the basal cell of the anterior series, which is angular, subisodiametric and lies immediately below the base of the perithecium, cutting off an appendiculate cell which sometimes covers its whole outer margin, or more often lies external and inferior in relation to it; the series consisting of two other cells which are subequal, elongate; the base of the upper lying obliquely over the distal end of the lower, which may or may not cut off an appendiculate cell distally; the cell above it, sometimes lacking, with or without an appendiculate cell which lies in contact with the outer margin of the perithecium reaching to its upper third or half: the posterior series consisting of normally four cells, the lowest more often not appendiculate; the second and third equal and appendiculate; the fourth vertically elongated, its upper third or half quite free, straight or distally slightly geniculate and continued by the long free finger-like slightly curved basal cell of the primary appendage. Median series of three subequal cells, vertically placed and extending almost to the apex of the perithecium. Perithecium rather stout, its inner margin straight, its outer convex and one half to one third free; the tip very slightly bent inward; the outer lip-cell forming a small, short, finger-like projection. Appendages very short and small, 5 × 2.5 μ. Spores 20 × 2 μ. Perithecia 30–32 × 8–117 μ. Basal cell, including foot, 16 × 7 μ. Total length to tip of perithecium 50–60 × 20–30 μ. The free termination of the posterior series, including basal and subtending cell of primary appendage 25–40 × 5 μ.

On *Megisthanus* sp. Trinidad, No. 2435.

No antheridia have been seen in the numerous specimens examined. The species is very closely allied to *R. Euzerconalis* from which it differs in its smaller size, simpler structure and more irregular outline.

Var. **Trachyuropodae** nov. var. Similar in general to the type. Somewhat smaller, the distal cell of the anterior series extending cushion-like usually to the tip of the perithecium; the posterior series consisting of five cells, the distal one wholly enclosed or hardly protruding, directed slightly inward, bearing the more slender base of the primary appendage which is erect or curved very slightly outward. Appendages stouter.

On the thin anterior and lateral margins of *Trachyuropoda* spp. Itacoatiara, Amazon, No. 2206, and Trinidad, No. 2429.

Abundant material of both type and variety have been examined and the differences noted seem constant, though not sufficient for specific separation.

Rickia Kameruna nov. sp.

Hyaline asymmetrical. Basal cell small and short, abruptly distinguished from the receptacle and from its broad pointed end, which is but slightly intruded between the two basal cells of the lateral series. Anterior series consisting of two cells without appendages, the upper partly overlapping the base of the perithecium which it subtends, and which is otherwise wholly free externally, rather long, its upper half bent slightly inward, the apex, only, free on the inner side, the outer and especially the inner lip-cell slightly prominent: the median series erect, consisting of five cells, the lowest not extending to the base of the perithecium: posterior series consisting of seven to eight cells, all except the upper one or two cutting off a relatively large appendiculate cell, the two lower slightly elongated radially, the rest very similar to those of the median series beside which they lie; the terminal one bearing terminally and externally the basal cell of the primary appendage which projects outward obliquely, its axis parallel to that of the free upper oblique margin of the terminal cell of the median series. Appendages rather stout, $10 \times 3\ \mu$. Spores $18–20 \times 2\ \mu$. Perithecium $30–34 \times 6–8\ \mu$. Basal cell exclusive of foot $8\ \mu$. Total length to tip of perithecium $40 \times 20\ \mu$. Basal cell of primary appendage, with subtending cell, $8\ \mu$.

On *Euzercon* sp. Kamerun, No. 2437.

Although the posterior series in this species is not extended to form an appendage, it seems as nearly related to *R. filifera* as to any of the other species, owing to the small development of its posterior series, which leaves the external margin of the perithecium wholly free as in *R. Celaenopsis.* There appear to be two cells in the anterior series, the upper of which is almost concealed by the base of the perithecium. I have seen no indication of an antheridium in either of the three specimens from which the description has been drawn.

Rickia filifera nov. sp.

Small and slender. Basal cell obliquely separated from the lower cell of the anterior marginal series which consists of two subequal cells; the upper extending a short distance upward external to the base of the perithecium: posterior series consisting of a variable number of cells (eight to fifteen) the basal extending above the base of the perithecium, the subbasal lying opposite to it; the third extending beyond its tip; the rest superposed to form a long, slender, erect, or slightly outcurved appendage, terminated by the undifferentiated basal cell of the primary appendage: the basal cell of the series, and many of the others, cutting off a small appendiculate cell distally and externally: median series consisting of two cells, the lower lying opposite the upper half or less of the perithecium, the upper in contact with the third and fourth cells of the posterior marginal series, its inner margin wholly free. Perithecium slender, the tip well distinguished externally and bent slightly outward, the inner lip-cell forming a short projection. Appendages slender, cylindrical, hyaline, $10 \times 2\,\mu$. Spores $24 \times 2.5\,\mu$. Perithecia $35–45 \times 8–12\,\mu$. Basal cell including foot $12 \times 4–5\,\mu$. Total length to tip of perithecium $35–45 \times 8–12\,\mu$. Longest free flagellum, including primary appendage, $175\,\mu$.

On a very large mite allied to *Megisthanus,* on *Passali.* Kamerun, No. 2442.

This species varies considerably in size and in the length of the extension of its posterior marginal row. No antheridia have been recognized, although material of various ages is available. It is perhaps most nearly related to *R. Megisthani* but resembles it only remotely, and cannot be confused with it on account of its free "flagellum."

Trenomyces.

This very curious genus was first discovered by Chatton in France on Mallophaga infesting domestic fowls, and had been received by me from Dr. Müller who collected it at Elbing, Prussia, and from Dr.

Trinchieri who found it at Naples, before the appearance of the preliminary paper by Chatton & Picard in Comptes Rendus (CXLVI, p. 208, 1908) was published. It was thus discovered almost simultaneously in Italy, Germany and France, and has since been found in New England and received from various other parts of North America.

Having been interested to learn something further as to the distribution and characteristics of the species in this genus, I have made a special effort to accumulate material, and am especially indebted for an opportunity to do so to the kindness of Prof. V. L. Kellogg, who has allowed me to go over his very large accumulations of duplicates in alcohol, and of Mr. M. A. Carriker who put his valuable collection at my service. Mr. Kirkpatrick has also sent me Mallophaga from turkeys and pigeons collected for me at the Rhode Island Experiment Station, for which I am greatly indebted to him, and I have also obtained material from Guatemala collected by the late Professor W. A. Kellerman; from the Bahamas, (W. W. Worthington), as well as from other sources.

The results of my examination of some thousands of Mallophaga have been somewhat disappointing, since their parasites are generally rare, and, if the data obtained may be assumed to indicate the actual conditions, have not found this aberrant group of insects a very favorable substratum for the development of numerous or characteristic species. As will be seen the following enumeration includes only six additional forms, none of them, with the possible exception of *T. gibbus*, departing very far from the characters of the type-species. In all a more or less complicated rhizoidal apparatus is developed, simple in one species, which penetrates the host. The receptacle consists of two cells terminated by a bicellular apiculate appendage resembling a spore of Puccinia, the upper giving rise to fertile branches which grow downward and corticate the lower, the corticating cells producing perithecia or antheridia according to the sex of the individual; although in some instances the corticating cells of the male are hardly developed, the antheridia arising directly from single cells obliquely separated from the lower margin of the subbasal cell of the receptacle. As in *Dimeromyces* and *Dimorphomyces*, to which the genus is most nearly related, the basal cells of the perithecium break down, and the cavity of the latter and that of the stalk-cell become continuous.

TRENOMYCES HISTOPHORUS Chat. & Picard.

This species, which appears to vary greatly in size, has been received from Dr. Müller, from Elbing, Prussia; from Prof. Trinchieri from Naples, Italy, and I have examined type material kindly sent me by Professor Chatton. In this country it has been obtained on species of *Menopon* and *Goniocotes* from Kittery Point, Maine, and from Newton, Mass. (on hosts kindly sent me by Mr. Walter Deane), on *Menopon* sp. from Gundlach's mockingbird, Bahamas; on *Menopon* from hen, Jamaica, W. I., and Guatemala: in the Kellogg collection on *M. mesoleucum* (crow), Palo Alto, California; *M. tridens*, Iowa; *Menopon* sp., No. 256b; on *Goniocotes*, Guatemala.

A species has been examined from various species of *Nirmus*, *N. punctatus* (Calif.), *N. maritimus* (N. E. and Cal.), *N. olivaceus* (Elbing, Prussia, Dr. Müller), which seems hardly separable from the many variations of *T. histophorus*. A variety, which may possibly prove a distinct species has also been found on *Menopon numerosum* (Kellogg, No. 24b), *Menopon* spp. (Kellogg, Nos. 80b, 256b, 74b), *Docophorus* sp. (Kellogg, No. 997). In this form the basal cell and the upper enlarged portion of the rhizoid are more or less conspicuously suffused with smoky brown in all cases. The ascogenic cell is usually near the base of the short stalk, and the distal cell of the appendage is somewhat more compressed than in the type but there are otherwise no distinctive characters.

Trenomyces Lipeuri nov. sp.

Male individual. Rhizoid more or less abruptly enlarged immediately below the integument, the swollen portion variably elongated and passing below into a rather stout simple, cylindrical prolongation of variable length. Basal cell of the receptacle bent at right angles to the rhizoid, horizontally elongated and corticated on the upper surface by an irregularly double series of small cells, which give rise to a corresponding series of erect or slightly divergent antheridia. Stalk-cell of the antheridium very slender, broadened below the basal cells; the body rather short and stout, subfusiform, the efferent tube short and slender. Appendage lying horizontally; the distal cell twice as long as the basal. Length from tip of appendage to last corticating cell, largest specimen, 42 μ. Appendage $15 \times 9\ \mu$. Total length of antheridium including stalk 35 μ; efferent tube 4 μ long; rest of body about $18 \times 10\ \mu$.

Female individual. General structure like that of the male; the base of the rhizoid shorter and relatively broader with very thick walls, the rhizoid proper, simple. Corticating cells of the basal cell vertically elongated, closely associated in a double crest-like series, bearing two or three to fifteen perithecia. The latter yellowish more or less distinctly tinged with brown, the stalk rather slender and clearly distinguished, about one third as long as the body of the perithecium which is rather short and stout, subfusiform; the apex blunt and relatively broad, crowned by four more or less clearly defined prominences which surround the short rounded or slightly sulcate apex. Perithecium, including stalk, 90–110 μ. The main body 60–80 $\times$ 20–28 μ. Total length of rhizoid about 90–100 μ the slender portion about 7 μ in diameter.

On various parts of *Lipeurus* sp. on Buzzard, Los Amates, Guatemala, No. 1547. On *L. celer*, Nos. 1564–67, California (Kellogg, Nos. 20a, 684c, 39a).

This species is clearly distinguished by the horizontal arrangement of its perithecigerous cells and by its simple rhizoid. It is somewhat variable in size, the specimens from Guatemala producing a greater number of smaller perithecia than those from California. The appendage which also lies horizontally is usually quite hidden or broken off, and appears to be rather narrow, the distal cell larger.

Trenomyces Laemobothrii nov. sp.

Male individual. Corticating cells extending but slightly below the subbasal cell, the lower two thirds of the basal cell quite free, the latter thick-walled, somewhat broader distally, about as long as broad. Antheridia of the usual form suberect in a compact group, six or more in number, the stalk-cells rather long, broader distally and not abruptly distinguished from the body. Appendage relatively very large, the cells subequal, broadly rounded, the apiculus hardly distinguishable. Basal cell 18 $\times$ 18 μ. Appendage 28 $\times$ 18 μ. Antheridia including stalk 45–50 $\times$, the body 12 $\times$ 25 μ, including efferent tube.

Female individual. Basal cell rather large and rounded, more or less completely corticated, except at the base where the ends of the corticating branches may be clearly visible. Perithecia about six in number, rather slender, subfusiform, the stalk relatively short, not distinguished from the body, the tip large, its margins slightly convex, but otherwise not distinguished from the main body; the rather prominent sulcate apex subtended by four somewhat spreading bisulcate prominences. Appendage relatively very large, the subequal cells rounded as in the male. Perithecium, including stalk 140–160 × 20–25 μ. Appendage 30 × 20 μ.

On *Laemobothrium atrum* from Coot, New England. M. C. Z., No. 1537.

This species is most easily distinguished by its unusually large appendage, which resembles a stout spore of Puccinia. It seems most nearly related to *T. Lipeuri*, the perithecia being very similar. The mode of growth is however, quite different. The rhizoids are entirely broken off in all the specimens.

Trenomyces circinans nov. sp.

Male individual. Corticating cells few and irregular, producing usually not more than two to four antheridia. Antheridia of the usual form, the body bent often at a right angle to the slender stalk-cell or sometimes recurved, the stalk 18 × 4 μ, the body 18 × 14 μ. Appendage relatively small, the cells about equal, 18 × 11 μ, the distal cell blunt pointed.

Female individual. Swollen portion of the rhizoid bearing several horizontal or upcurved lobes from which arise usually furcate smaller lobes running to slender threads of no great length. Perithecia two to four, usually strongly circinate when young, at maturity typically bent or even recurved, rarely straight, the stalk relatively slender, the body often rather abruptly distinguished, broader distally below the tip, which may be subtended by a distinct elevation on one side and is well distinguished, its margin usually slightly convex, separated by a slight constriction from the crown formed by four symmetrically placed somewhat spreading lobes which surround the hardly prominent apex, the whole surface of the stalk and body more or less distinctly roughened or granular, the walls much thickened. Appendage relatively small like that of the male. Perithecium including stalk 225–280 × 28–35 μ; the stalk 70–125 × 10 μ or broader. Appendage 20 × 10–14 μ.

On various parts, especially the head of *Lipeurus* sp., on pigeons, Kingston, R. I., No. 1549; on *L. baculus*, Elbing, Prussia (Dr. Müller); on *Docophorus Californicus*, California, No. 1555 (Kellogg No. 666); on *D. Montereyi*, No. 1554 (Kellogg No. 264c).

The Californian forms on Docophorus are not quite so well marked as those from Prussia and Rhode Island which, by their abruptly curved habit, slender stalks, and roughened surface, are clearly distinguished from other species of the genus. The tip of the perithecium in well developed specimens is not unlike that of *Arthrorhynchus Eucampsipodae,* but the conformation varies considerably and comparatively few specimens have a well defined subterminal hunch. Several specimens on *Docophorus colymbinus,* Nos. 1556–7 (Kellogg, Nos. 14a and 12a), differ distinctly in that the tip is unmodified and hardly distinguished, the stalks stouter and less well distinguished. Further material may indicate that this form is distinct.

Trenomyces gibbus nov. sp.

Male individual unknown.

Female individual. General structure like that of *T. histophorus.* Swollen portion of the rhizoid producing several, horizontal lobes. Corticating cells very irregular, completely concealing the somewhat irregular basal cell, giving rise to numerous perithecia. Perithecia faintly tinged with yellowish, stout elongate, the stalk not distinguished from the body, the whole indistinctly roughened, and having the appearance of a goose's neck and head owing to a subterminal protrusion which causes the tip and apex to be bent to one side at an angle 45° or more; the tip nearly symmetrical above the protrusion, broadly conical, the apex rather narrow, subtruncate, slightly indented. Total length of perithecium 300 μ, including stalk, which may be 30 μ broad just above its origin; the tip above the hunch, 32 μ long, the base 28 to 30 μ broad, the apex about 7 μ. Appendage 25 $\times$ 10 μ.

Described from a single female on *Lipeurus longipilus.* No. 1563 (Kellogg, No. 128d), California.

This form is so peculiar that I have not hesitated to describe it from a single well developed female in good condition. There are a dozen or more perithecia on the specimen in various stages of development, the four which are mature suggesting the heads and necks of a flock of geese. The distal cell of the appendage is somewhat longer than the basal, tapering from base to apex.

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THE SPACE-TIME MANIFOLD OF RELATIVITY. THE NON-EUCLIDEAN GEOMETRY OF MECHANICS AND ELECTROMAGNETICS.

By Edwin B. Wilson and Gilbert N. Lewis.

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Introduction.

1. The concept of space has different meanings to different persons according to their experience in abstract reasoning. On the one hand is the common space, which for the educated person has been formulated in the three dimensional geometry of Euclid. On the other hand the mathematician has become accustomed to extend the concept of space to any manifold of which the properties are completely determined, as in Euclidean geometry, by a system of self-consistent postulates. Most of these highly ingenious geometries cannot be expected to be of service in the discussion of physical phenomena.

Until recently the physicist has found the three dimensional space of Euclid entirely adequate to his needs, and has therefore been inclined to attribute to it a certain reality. It is, however, inconsistent with the philosophic spirit of our time to draw a sharp distinction between that which is real and that which is convenient,[1] and it would be dogmatic to assert that no discoveries of physics might render so convenient as to be almost imperative the modification or extension of our present system of geometry. Indeed it seemed to Minkowski that such a change was already necessitated by the facts which led to the formulation of the Principle of Relativity.

2. The possibility of associating three dimensional space and one dimensional time to form a four dimensional manifold has doubtless occurred to many; but as long as space and time were assumed to be wholly independent, such a union seemed purely artificial. The idea of abandoning once for all this assumption of independence, although fore-shadowed in Lorentz's use of local time, was first clearly stated by

[1] See, for example, H. Poincaré, La Science et l'Hypothèse.

Einstein. The theorems of the principle of relativity which correlate space and time appeared, however, far less bizarre and unnatural when Minkowski showed that they were merely theorems in a four dimensional geometry.

Suppose that a student of ordinary space, habituated to the interpretation of geometry with the aid of a definite horizontal plane and vertical axis, should suddenly discover that all the essential geometrical properties of interest to him could be expressed by reference to a new plane, inclined to the horizontal, and a new axis inclined to the vertical. Whereas formerly he had attributed special significance to heights on the one hand and to horizontal extension on the other, he would now recognize that these were purely conventional and that the fundamental properties were those such as distance and angle, which remain invariant in the change to a new system of reference.

Let us now consider a four dimensional manifold formed by adjoining to the familiar x, y, z axes of space a t axis of time. Any point in this manifold will represent a definite place at a definite time. Space then appears as a sort of cross section through this manifold, comprising all points of a given time. For convenience we may temporarily ignore one of the dimensions of space, say z, and discuss the three dimensional manifold of x, y, t. This means that we will consider only positions and motions in a plane. The locus in time of a particle which does not change its position in space, that is, of a particle at rest, will be a straight line parallel to the t axis. Uniform rectilinear motion of a particle will then be represented by a straight line inclined to the t axis.

3. If we adopt the view that uniform motion is only relative, we may with equal right consider the second particle at rest and the first particle in motion. In this case the locus of the second particle must be taken as a new time axis. What corresponding change this will necessitate in our spacial system of reference will depend entirely upon the kind of geometry that we are led to adopt in order to make the geometrical invariants of the transformation correspond to the fundamental physical invariants whose occurrence in mechanics and electromagnetics has led to the principle of relativity.

It is immediately evident that if uniform motion is to be represented by straight lines, the statement that all motion is relative shows that the transformation must be of such a character as to carry straight lines into straight lines. In other words, the transformation must be linear. Further we must assume that the origin of our space and time axes is entirely arbitrary.

The further characteristics of this transformation must be determined by a study of the important physical invariants. Fundamental among these invariants is the velocity of light, which by the second postulate of the principle of relativity must be the same to all observers. Hence any line in our four dimensional manifold which represents motion with the velocity of light must bear the same relation to every set of reference axes. This is a condition which certainly cannot be fulfilled by any transformation of axes to which we are accustomed in real Euclidean space. It is indeed a condition sufficient to determine the properties of that non-Euclidean geometry which we are to investigate.

Minkowski, in his two papers on relativity,[2] used two different methods. In his first and elaborate treatment of the subject he introduced the imaginary unit $\sqrt{-1}$ in such a way that the lines which represent motion with the velocity of light become the imaginary invariant lines familiar to mathematicians who discuss the real and imaginary geometry of Euclidean space. In this way, however, the points of the manifold which represent a particle in position and time become imaginary; the transformations are imaginary; the whole method becomes chiefly analytical. In his second, a brief paper, Minkowski makes use of certain geometrical constructions which have their simplest interpretation only in a non-Euclidean geometry.

4. It is the purpose of the present work to develop the four dimensional non-Euclidean geometry which is demanded by the principle of relativity, and to show that the laws of electromagnetics and mechanics not only can be simply interpreted in this way but also are for the most part mere theorems in this geometry.

In the first sections we shall develop in some detail the non-Euclidean geometry in two dimensions. For it is only by a thorough comprehension of this simpler case that it is possible to proceed into the more difficult domains involving three and four dimensions. This part of the paper will be continued by a discussion of vectors and the vector notation that will be employed. At this point it is possible in a few simple cases to show the applications of the non-Euclidean geometry to problems in kinematics and mechanics.

The sections devoted to three dimensions will be occupied largely with numerous analytical developments of the vector algebra, many of which are directly applicable not only in space of higher dimensions

[2] Gesammelte Abhandlungen von Hermann Minkowski, Vol. 2, pp. 352–404 and pp. 431–444.

but also in Euclidean space. We are led further to a consideration of certain vectors of singular character. The study of the singular plane leads to the brief consideration of another interesting and important non-Euclidean plane geometry.

Passing to the general case of four dimensions we shall meet further new types of vectors, and shall attempt even here to facilitate as far as is possible the visualization of the geometrical results. We shall continue further the analytical development, and in particular consider the properties of the differential operator *quad*. In this connection a very general and important equation for the transformation of integrals is obtained. The idea of the geometric vector field will then be introduced, and the properties of these fields will be taken up in detail.

The subject of electromagnetics and mechanics is prefaced with a short discussion of the possibility of replacing conceptually continuous and discontinuous distributions by one another, and we shall point out that in one important case such a transformation is impossible. The science of electromagnetics is treated both from the point of view of the point charge and from that of the continuous distribution. In both cases it is shown that the field of potential and the field of force are merely the geometrical fields previously mentioned, except for a constant multiplier. Particular attention is given to the field of an accelerated electron,[3] and in this field we find that the vectors of singular properties play an important rôle. With the aid of these vectors the problem of electromagnetic energy is discussed. The science of mechanics, which is treated in a fragmentary way in some preceding sections, is now given a more general treatment, and the conservation laws of momentum, mass and energy are shown to be special deductions from a single general law stating the constancy of a certain four dimensional vector, which we have called the vector of extended momentum. Finally it is pointed out that this last vector gives rise to geometric vector fields which can be identified with the

3 There seems to be a widespread impression that the principle of relativity is inadequate to deal with problems involving acceleration. But the essential idea of relativity can be expressed by the statement that there are certain vectors in the geometry of four dimensions which are independent of any arbitrary choice of the axes of space and time. Those problems which involve acceleration will be shown to possess no greater inherent difficulties than those that involve only uniform motion. It is, moreover, especially to be emphasized that the methods which are to be employed in this paper necessitate none of the approximations that are commonly employed in electromagnetic theory. Such terms as "quasi-stationary," for example, will not be used.

fields of gravitational potential and gravitational force. Moreover, it is shown that these fields are identical in mathematical form with the electromagnetic fields, and that all the equations of the electromagnetic field must be directly applicable to the gravitational.

In an appendix a few rules for the use of Gibbs's dyadics, which have occasionally been employed in the text, are stated. And a brief discussion of some of the mathematical aspects of our plane non-Euclidean geometry is given.

The Non-Euclidean Geometry in Two Dimensions.

Translation or the Parallel Transformation.

5. In discussing a non-Euclidean geometry various methods of procedure are available; a set of postulates may be laid down, or the differential method of Riemann may be followed, or the theory of groups may be used as by Lie, or (if the geometry falls under the general projective type, as is here the case) the projective measure of length and angle may be made the basis. For our present purpose we need not restrict ourselves to any one of these; but since the first is familiar to all, we shall employ it as far as convenience permits. Some of the other methods will, however, be briefly discussed in the appendix, §§ 64, 65.

With a view to simplicity we shall at first limit the discussion to the case of a plane. Points and lines will be taken as undefined, and most of the relations connecting them will be the same as in Euclidean plane geometry. Thus: [4]

1°. Through two points one and only one line can be drawn.

2°. Two lines intersect in one and only one point, except that

3°. Through any point not on a given line one and only one parallel (non-intersecting) line can be drawn.

4°. The line shall be regarded as a continuous array of points in open order.

6. In regard to congruence or "free mobility" it is important to proceed more circumspectly than did Euclid. The transformations of Euclidean geometry may be divided into translations and rotations, of which the former alone are the same for our geometry. It seems desirable, therefore, to discuss first and in some detail the postulates

[4] We make no claim of completeness or independence for these postulates, which are designed primarily to show the points of similarity or dissimilarity between our geometry and the Euclidean. A like remark may be made with respect to proofs of theorems.

and propositions relating to this type of transformation, and common to the two geometries. We therefore postulate for translation:

5°. Any point P can be carried into any point P', and any two translations which carry P into P' are identical.

6°. Any line is carried into a parallel line.

7°. Any line parallel to PP' remains unchanged.

8°. The succession of two translations is a translation.

These postulates determine the characteristics of a group of geometries of which the two most important are Euclidean geometry and that non-Euclidean geometry with which we are here concerned. Another non-Euclidean geometry belonging to this same group will be discussed briefly in § 31. This group excludes such geometries as the Lobachewskian and the Riemannian in which a parallel to a given line at a given point is not uniquely defined. We shall first proceed to develop some of those general theorems which are true in this whole group of geometries.

I. If two intersecting lines are parallel respectively to two other intersecting lines, the corresponding angles [5] are congruent.

For by translation the points of intersection may be made to coincide, and the lines of the first pair, remaining parallel with the lines of the other pair (6°), must come into coincidence with them, by postulate 3°.

II. The opposite sides of a parallelogram are congruent.

For if $ABCD$ is a parallelogram and if A be translated to B, the line of DC remains unchanged, by 7°, and the line of AD falls along the line of BC by I. Hence D falls on C by 2°.

Cor. If two points P, P' are carried by a translation into Q, Q', the figure $PP'\,Q'\,Q$ is a parallelogram.

7. We may now set up a system of measurement along any line and hence along the whole set of parallel lines. Consider the segment PP'. By the translation which carries P into P', the point P' is carried into a point P'' of the same line. The measure of the separation of P and P' we will call the *interval* [6] PP'. And since the segment PP' is congruent to the segment $P'\,P''$, the intervals PP' and $P'\,P''$ are said to be equal. We may thus mark off any number of equal intervals along the line. We shall assume further the Archimedean postulate.

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[5] The word angle here refers to a geometrical figure only, and does not as yet imply any measure of angle.

[6] We use the word interval to avoid all ambiguity. The notion of distance will be separately considered in Appendix, § 65.

9°. If a sufficient number of equal intervals be laid off on a line, any point of the line may be surpassed.

Now the whole theory of commensurability or incommensurability of two intervals along the same line or parallel lines may be treated by the usual methods. Thus the intervals along a line, starting from any origin upon the line, may be brought into one-to-one correspondence with the series of real numbers. It is, however, to be especially emphasized that we have not established, and cannot establish by the translation alone, any comparison between intervals on non-parallel lines.

III. The diagonals of a parallelogram bisect each other.[7]

For let (Figure 1) the parallelogram $ABCD$, of which the diagonals intersect at E, be translated into the position $BB'\ C'\ C$ (by translating A to B), in which the diagonals intersect at E'. Now BE' is parallel to EC, and EB to CE'. Hence BE' which is congruent to AE, is congruent to EC by II. Consequently AE is congruent to EC by 8°.

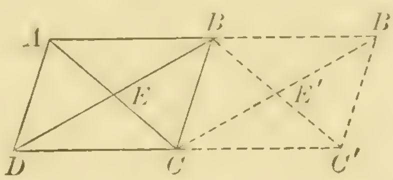

FIGURE 1.

IV. If two triangles have the sides of one respectively parallel to the sides of the other, and if one side of one is congruent to one side of the other, then the remaining sides of the one are respectively congruent to the remaining sides of the other.

For if the two congruent sides are brought into coincidence by translation, the two triangles will either coincide throughout, or will together (Figure 2) form a parallelogram (II).

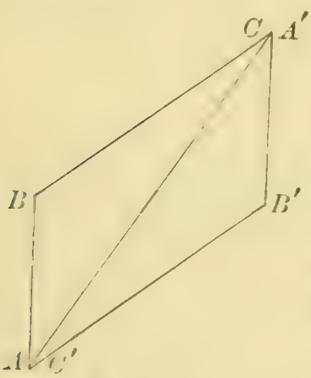

FIGURE 2.

Two triangles with the sides of one respectively parallel to the sides of the other will be called similar.

V. In two similar triangles the sides of the one are respectively proportional to the sides of the other.

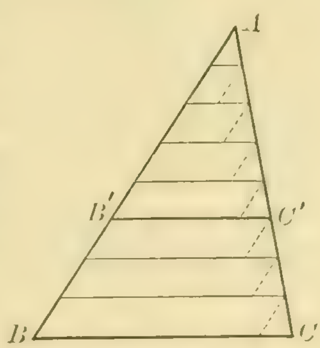

FIGURE 3.

For if ABC and $A'B'C'$ are the triangles, the vertex A' may be made to coincide with A by a translation (Figure 3). Suppose, now,

[7] Theorems like this and the preceding and some which are to follow are proved in elementary geometries by the aid of propositions (on congruence of triangles) not deducible from translations alone.

that AB' falls along AB, and AC' along AC. Assume that AC and AC' are commensurable. Apply the common measure to the side AC, and through the points of division draw lines parallel to BC and to AB. In the small triangles thus formed the parallel sides will be equal by IV, and therefore the intervals cut off on AB must be equal by II. In case of incommensurability the method of limits may be applied.[8] The case in which the two triangles fall on opposite sides of the common vertex may be treated in a similar manner by the aid of IV.

8. For our future needs, the conception and the measure of area are fundamental, and it is important to show that this subject may be satisfactorily treated with the aid of the parallel-transformation (that is, the translation) alone. Indeed, any arbitrarily chosen unit intervals along any selected pair of intersecting lines determine a parallelogram which may be taken as having a unit area. By ruling the parallelogram into equal parallelograms by lines parallel to its sides, an arbitrarily small element of area may be obtained. The area enclosed by any curve may be divided into like elements by similar rulings, and thus by the method of limits the enclosed area may be compared with the assumed unit area.[9] In particular some simple propositions on areas will now be deduced.

VI. Any parallelogram with sides parallel to those of the unit parallelogram has an area equal to the product of the intervals along two intersecting sides.

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[8] It may be observed at this point that if two intersecting lines be taken as axes of reference, if systems of measurement (as yet necessarily independent) be set up along the two lines with the point of intersection as common origin, and if to each point P of the plane are assigned coordinates (x, y) equal to the intercepts cut off from the axes by lines through P parallel to the axes, then straight lines are represented by linear equations, and conversely. For the deduction of the equation of a line depends merely upon the properties of triangles similar in our sense. The transformation from any such set of axis to any other such set will clearly be linear.

[9] If axes be introduced as above, the area of a triangle and the area of any closed curve are expressed analytically by the usual formulas.

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad \text{and} \qquad \iint dxdy = \int_{\circ} xdy = -\int_{\circ} ydx,$$

in terms of our assumed unit parallelogram. The theorems on areas could then be proved analytically, but the elementary geometric demonstrations seem preferable. It is important to observe further that in a transformation to new axes, such that

$$x = ax' + by' + c, \qquad y = a'x' + b'y' + c',$$

VII. The diagonal of a parallelogram divides it into two equal areas.

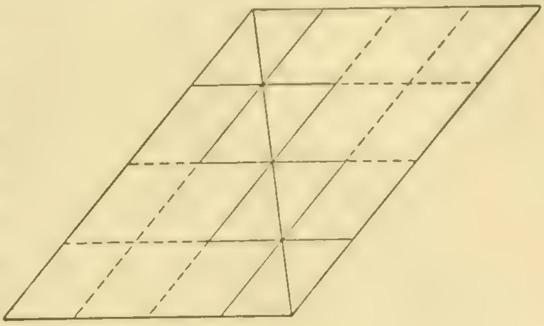

FIGURE 4.

For if the sides of the parallelogram be divided by repeated bisection into 2^n parts, there will be an equal number of equal parallelograms on each side of the diagonal (Figure 4), and in the limit the total area of these parallelograms approaches the area of the triangles.

VIII. If from any point in the diagonal of a parallelogram lines be drawn parallel to the sides, the two parallelograms formed on either side of the diagonal are equal in area (Figure 5).

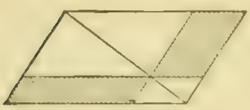

FIGURE 5.

IX. Two parallelograms between the same parallel lines and with congruent bases are equal in area.

Cor. Two triangles having congruent bases on one line and vertices on a parallel line have equal areas.

Cor. The diagonals divide a parallelogram into four equal triangular areas.

Proofs may be given by obvious and familiar methods.

X. Of all parallelograms having two sides common to two sides of a given triangle and a vertex on the third side of the triangle, that one has the greatest area whose vertex bisects that third side.

For in the figure (Figure 6), where ABC is the triangle and E is the middle point of the third side, the difference of the two parallelograms is

$$\begin{aligned} HBFE - IBGD &= MGFE - IHMD = KMEL - IHMD \\ &= KMEL - KDNL = DMEN. \end{aligned}$$

Propositions IV and VIII are used in the proof.

the value of the area, in terms of the area measured with reference to the new axes, is

$$dxdy = \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} dx'dy.'$$

Hence if the measure of area is to be the same, that is, if the unit parallelogram on the new axes is to have a unit area referred to the old axes, the determinant of the transformation must be unity. This implies a relation between the choice of unit intervals on the new axes. Indeed when the unit interval on one of the new axes has been arbitrarily chosen, the unit interval on the other is determined. In other words the unit intervals on the new axes must each vary inversely as the other.

As an extension of the idea of similarity for triangles, we may say that any two polygons which have their corresponding sides parallel and in proportion are similar. It follows that if any two corresponding lines are drawn in the polygons, these lines must be parallel.

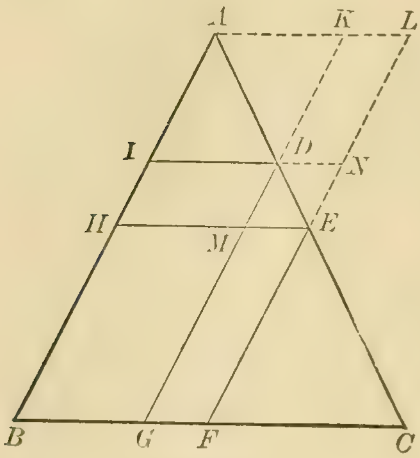

FIGURE 6.

XI. If on two sides of a triangle similar parallelograms be constructed, and on the third side a parallelogram with diagonals parallel to the diagonals of the other parallelograms, the area of this parallelogram will be equal to the difference of the areas of the other two.

The areas (Figure 7) of the parallelograms on AB, CA, BC are respectively four times the areas of the triangles ABF, CAE, BCD. If we take the unit parallelogram with sides parallel to the diagonals, it will suffice to prove that

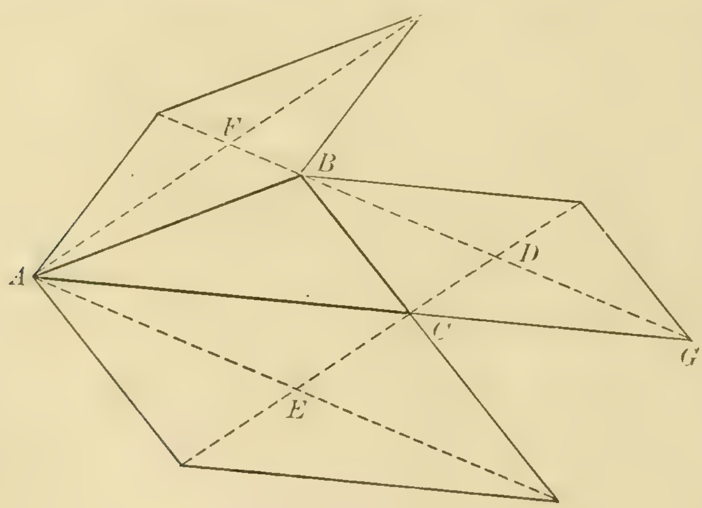

FIGURE 7.

$$FB \times AF = AE \times EC - BD \times CD,$$

for each of these areas is twice the area of the corresponding triangle. In the similar triangles ACE and GCD,

$$EC : CD :: AE : DG.$$

But by III, BD is equal to DG. And writing $AE = FB + BD$, we have

$$EC \times BD = CD \times FB + CD \times BD.$$

Add to each side the product $FB \times EC$. Then

$$EC(BD + FB) = CD \times BD + FB(CD + EC).$$

Hence

$$EC \times AE - CD \times BD = FB \times AF.$$

Non-Euclidean Rotation.

9. The group of parallel geometries determined by Postulates 1°–9°, which, notwithstanding its generality, gives rise, as we have seen, to some interesting and important theorems, may be subdivided by adding a set of postulates belonging to a second transformation which by analogy may be called rotation. It is this set of postulates which will differentiate our non-Euclidean geometry from the Euclidean.

The difference between our non-Euclidean rotation and the ordinary kind is that in addition to a fixed point, two *real* lines through the point remain unchanged. We may postulate for rotation:

10°. Any one point and only that one remains fixed.

This point may be called the center of rotation.

11°. Two lines through this point remain unchanged.

These lines may be called the fixed lines of the rotation.

12°. Any half-line (or ray) from the center, and lying in one of the angles determined by the fixed lines, may be turned into any other ray in the same angle, and this uniquely determines the rotation.

13°. The succession of two rotations about the same point is a rotation.

14°. The result of a rotation about O and a translation from O to O' is independent of the order in which the rotation and translation are carried out.

It follows immediately from 14° that the fixed lines in a rotation about any point O are parallel to the fixed lines in a rotation about any other point O'. All lines in the plane may now be divided into classes in such manner that neither translation nor rotation can change the classification. Namely,

(α) lines parallel to one of the fixed directions,

(β) lines parallel to the other of the fixed directions,

(γ) lines which lie in one of the pairs of vertical angles determined by the fixed directions,

(δ) lines which lie in the other pair of vertical angles determined by the fixed directions.

The lines of fixed direction, namely, the (α)-lines and (β)-lines, will be called *singular* lines.

A system of measurement may be set up for angles between rays [10] which issue from a point into one of the angles determined by the fixed lines through the point. For a succession of rotations may be used (in the same manner as the succession of translations was used to establish the measure of interval along a line). Thus if a line a is carried into a line a' and at the same time the line a' is carried into the line a'', the angles between a and a' and between a' and a'' are congruent and the measures of the angles are said to be equal. Now as the rotation may be repeated any number of times without reaching the fixed line, it is possible to find an angle $aa^{(n)}$ which shall be n times the angle aa'. We shall assume the postulate, analogous to the Archimedean:

15°. If a sufficient number of equal angles be laid off about a point from any initial ray, any ray of that class may be surpassed.

It thus appears that the angles between any given line and other lines of the same class may be placed into one-to-one correspondence with all positive and negative real numbers, just as the intervals from a point on a line may be thus correlated.[11] This constitutes a very great difference between our geometry and the Euclidean.

It is impossible to show from the preceding statements that any given figure maintains a constant area during rotation.[12] We shall therefore lay down the additional postulate:

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10 The relations of order of all lines of a given class, (γ) or (δ), are the same as those of points on a line, as in 4°.

11 The angle between two singular lines (α) and (β) can obviously not be measured. Such an angle, and also the angle between any line and a line of fixed direction, must be regarded as infinite.

12 This matter may readily be discussed analytically. As axes of reference choose the fixed lines, and let u, v denote coordinates. As rotation is a linear transformation, the point P (u, v) and the transformed point P' (u', v') are connected by the equations

$$u' = au + bv + c, \qquad v' = du + ev + f.$$

As the lines $u = 0$ and $v = 0$ are fixed, these equations reduce to $u' = au$, $v' = ev$; and as rotation depends on only one parameter, we may write $e = \phi(a)$. The succession of two rotations is then expressed by

$$\begin{cases} u' = au \\ v' = \phi(a)v, \end{cases} \qquad \begin{cases} u'' = bu' \\ v'' = \phi(b)v', \end{cases} \qquad \begin{cases} u'' = abu \\ v'' = \phi(a)\phi(b)v, \end{cases}$$

16°. In rotation an area becomes an equal area.[13]

10. We are now prepared to discuss in some detail the general characteristics of our rotation. Consider (Figure 8) a series of rotations about O, whereby the point P assumes the positions P', P'', ... Let the parallelograms on OP, OP', OP'', ... as diagonals and with sides along the fixed lines be constructed. Then by 16° the areas of these parallelograms are equal, and in terms of the intervals on the fixed lines

$$OA \times OB = OA' \times OB' = OA'' \times OB''.$$

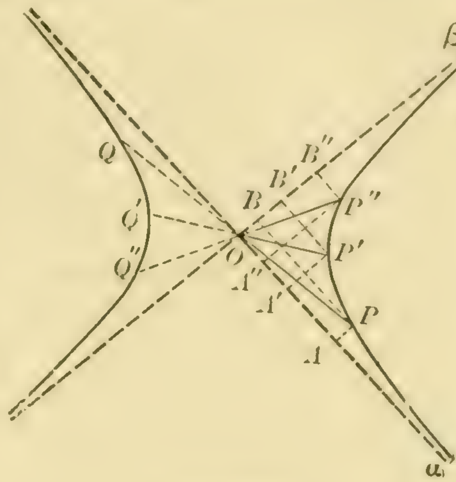

FIGURE 8.

The point P thus traces a curve which in ordinary geometry would be

with the condition

$$\phi(a)\phi(b) = \phi(ab)$$

necessitated by 13°. This is a functional equation of which the only (continuous) solution is $\phi(a) = a^r$. Hence rotation must be of the form

$$u' = au, \; v' = a^r v.$$

The unit parallelogram on the axes of u and v is hereby transformed into a parallelogram on these same axes with intervals a and a^r along u and v. By VI the area of the new parallelogram is therefore a^{r+1}. If this is to be unity, $r = -1$. The transformation equations for rotation are therefore

$$u' = au, \quad v' = v/a,$$

where a is necessarily positive because points do not change from one side of the axes to another.

The intrinsic significance of these equations should not be overlooked. A rotation may be represented as a multiplication of all intervals along one of the fixed lines by a constant factor and a division of all intervals along the other fixed line by the same factor. Or, increasing the unit interval along one fixed line and decreasing it in the same ratio along the other is equivalent to a rotation. (This process effected along any other axes than the fixed lines would leave the area unchanged, but would not be a rotation). As the unit interval along one fixed line cannot be compared either by translation or by rotation with the unit along the other, and as one of these units is arbitrary, we have additional evidence that there is no natural zero of angle.

13 Such a postulate is unnecessary in Euclidean geometry owing to the periodic nature of the Euclidean rotation. Postulate 16° could be replaced by one involving only the notion of symmetry between rotations in opposite directions.

considered a branch of a hyperbola.[14] Since, however, this curve is here generated by the rotation of a line OP about its terminus Q, we shall call this locus (taken with the other branch $Q\ Q'\ Q''$ symmetrically situated with respect to O) the *pseudo-circle.*

By means of such a rotation we are able to compare intervals upon any line with intervals upon any other line of the same class. For the intervals of the congruent radii OP, OP', OP'' will be called equal.

When we consider the fixed lines we observe that the effect of rotation is to carry the segment OA into OA' or OA''. It is therefore evident that segments are congruent by rotation which are incongruent by translation. This source of ambiguity exists only in the case of singular lines, for in no other case is it possible to compare two segments both by rotation and by translation. We may remove this ambiguity at once by stating that intervals along singular lines, although metrically comparable with intervals on other singular lines of the same class by translation, are all of zero magnitude when compared with intervals on any non-singular line. This will become more evident later.

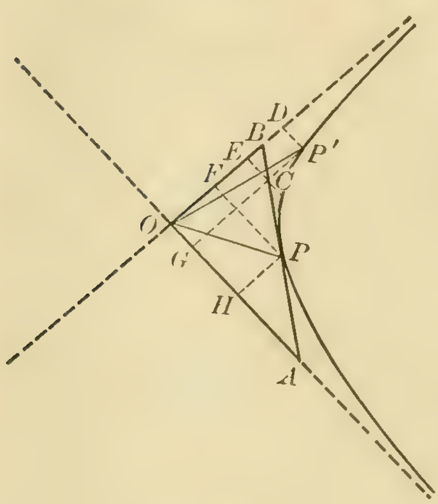

FIGURE 9.

Consider next (Figure 9) the intercept AB terminating on the fixed lines corresponding to a rotation with center at O. Let P be the middle point of the line, and C any other point. Through C draw a line parallel to OB, and on this line mark the point P' such that the area $ODP'G$ equals the area $OFPH$. The area $OECG$ is less than each of these by X. Hence P' lies on the further side of AB from O. But P' is a point on the pseudo-circle through P concentric with O, as we have just seen. Since C was any point of AB, it follows that P' may be any point of the pseudo-circle. Hence as the line AB meets the pseudo-circle at P and only at P, it is tangent to the curve. As a species of converse, we may state the theorem:

14 There is no special significance in the fact that a rectangular hyperbola is drawn in the figure and that the fixed lines α, β are perpendicular in the Euclidean sense; in subsequent figures the singular lines are often oblique. From the non-Euclidean viewpoint the question of perpendicularity or obliquity of the singular lines is of course meaningless.

XII. The tangent to a pseudo-circle lies between the curve and its center, and the portion of the tangent intercepted between the two fixed lines is bisected at the point of tangency.

11. In a pseudo-circle the radius and the tangent at its extremity are said to be perpendicular. Or in virtue of XII we may say that the perpendicular from any point O to any non-singular line is the line from O to the middle point of that segment of the line which is intercepted by the fixed lines through O. The construction of a perpendicular to any line of class (γ) or (δ) at a point of the line is equally simple.

By the aid of propositions concerning similar triangles, the following theorems concerning perpendiculars are readily proved.

XIII. If a line a is perpendicular to a line b, then b is perpendicular to a.

XIV. Through any point one and only one perpendicular can be drawn to any line.

XV. All lines perpendicular to the same line are parallel.

XVI. The singular line of one class which is drawn through the intersection of any two perpendicular lines will bisect the segment intercepted by these lines upon any singular line of the other class (Figure 10).[15]

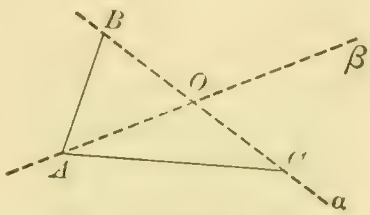

FIGURE 10.

XVII. The perpendicular to a (γ)-line is a (δ)-line, and vice versa.

Intervals along lines of class (δ) cannot be compared by congruence with intervals along lines of the (γ) class. We may, therefore, arbitrarily define equality of intervals between the two classes. *If two mutually perpendicular lines are drawn from any point and terminate on a singular line, the intervals of these lines will be said to be equal.*[16] The consistency of this definition is readily proved.

The definition of perpendicularity is such that if two lines are perpendicular they must remain perpendicular after a translation or rotation. The former case is obvious, and the latter becomes so when the lines are considered as radius and tangent in a pseudo-circle generated by the rotation; the more general case in which neither of the perpendicular lines passes through the center of rotation then follows with the aid of XV. It is important to observe one peculiar

15 In the figure BO and OC are equal, and AB and AC are perpendicular.

16 In Figure 10, the intervals AC and AB are therefore equal by this definition.

characteristic of our rotation, namely that two perpendicular lines approach each other and the fixed line between them scissor-wise, as may be seen, in Figure 11, where OC and OD become respectively OC' and OD', OC'' and OD'', $\cdots$ The pseudo-circles traced by OC and OD may be called conjugate pseudo-circles, since the interval OC equals the interval OD, the lines CD, $C'D'$, $\cdots$, being singular, and bisected by a fixed line.

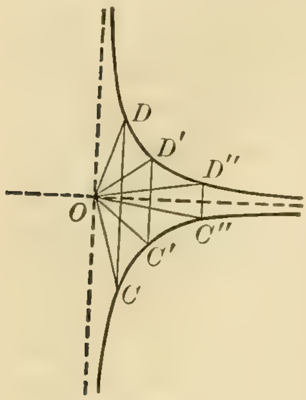

FIGURE 11.

Since two mutually perpendicular lines approach, during rotation about their point of intersection, the same fixed line, we may extend our definition of perpendicularity by regarding every singular line as perpendicular to itself. This extension is also suggested by the fact that the fixed line may be considered an asymptote of a pseudo-circle. Special caution must be given against the idea that a singular line of one class is perpendicular to a singular line in the other class. The peculiarities of singular lines will become clearer in the work on vector analysis.

12. A triangle of which two sides are perpendicular will be called a right triangle, and the third side will be called the hypotenuse. A parallelogram of which the two adjacent sides are perpendicular and of equal interval will be called a square. The following theorem is obvious:

XVIII. One diagonal of every square is a singular line and the other diagonal is a singular line of the other class.

XIX. *Pythagorean Theorem.* The area of the square on the hypotenuse of a right triangle is equal to the difference of the areas of the squares on the other two sides.

For by XVIII the diagonals of the squares are lines of fixed direction, and hence parallel each to each. The squares on the two legs are similar. And the proposition is evidently a special case of XI. (In Figure 7 if the dotted lines are singular lines, the lines AC and BC are so drawn as to be approximately perpendicular.)

XX. Any two squares whose sides are of unit interval are equal in area.

For by suitable translation and rotation one may be brought into coincidence with the other. The unit of area will henceforth be taken as the area of a square whose sides are of unit interval. Hence follows:

Cor. The area of any rectangle is the product of the intervals of two adjoining sides.

We may therefore obtain from XIX the theorem

XXI. The square of the interval of the hypotenuse of a right triangle is equal to the difference in the squares of the intervals of the other two sides.

Cor. The perpendicular from a point to a line has a *greater* interval than any other line of the same class drawn from the given point to the given line.

Having now given a final definition of the measure of area, we may define the unit of angle. The radius of the pseudo-circle, in advancing by rotation over equal angles, necessarily sweeps out equal areas (by 16°). Hence by the familiar argument sectorial areas in any pseudo-circle are proportional to the angles at the center. The unit angle will be taken as that angle which, in a pseudo-circle of unit radius, encloses a sectorial area of one-half the unit area.

Vectors and Vector Algebra.

13. Translation or the parallel-transformation leads at once to the consideration of vectors. We have shown that when a translation carries A into B and A' into B' the directed segments AB and $A'B'$ are parallel and congruent (Cor. to II). Hence a translation may be represented by a vector, that is, by any directed segment laid of from any origin and having the same interval and direction as AB. The succession of two translations is represented by the sum of their corresponding vectors. The addition and subtraction of vectors and their multiplication by scalars follows the usual laws (by §§ 5–7).

If two vectors **a** and **b** are laid off from a common origin, the parallelogram constructed on the vectors is called their outer product $\mathbf{a} \times \mathbf{b}$, and the magnitude of this product will be taken numerically equal to the area of the parallelogram.[17] We must bear in mind that not this magnitude (nor yet a vector perpendicular to the plane), but the parallelogram itself is the outer product. We may, however, represent the outer product by any other closed figure of equal area, provided that it is taken with the same sign. The sign attributed to an

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[17] Our vector notation will be based upon that of Gibbs, and is identical with that employed by Lewis (Four dimensional Vector Analysis, These Proceedings, **46**, 163–181) except in the designation of the inner product which we shall define as in that paper, but represent by $\mathbf{a} \cdot \mathbf{b}$ instead of $\mathbf{ab}$; the latter form will be reserved to denote the dyad. The scalar magnitude of a vector will be represented by the same letter in italic type.

area does not arise from any positive or negative geometric characteristics of the area itself, but from an interpretation or convention concerning the way in which one area is considered as generated relative to another, and is required for analytic work. We shall make the convention that $\mathbf{a}\times\mathbf{b}$ and $(-\mathbf{a})\times\mathbf{b}$ or $\mathbf{a}\times(-\mathbf{b})$ have opposite signs.

The outer product of a vector by itself or by any parallel vector is zero, because the parallelogram determined by these vectors has zero area; thus $\mathbf{a}\times\mathbf{a} = 0$. The associative law for a scalar factor is valid, because multiplying one side of a parallelogram by a number multiplies the area by that number; thus

$$(n\mathbf{a})\times\mathbf{b} = n\,\mathbf{a}\times\mathbf{b} = \mathbf{a}\times(n\mathbf{b}).$$

The distributive laws,

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}, \qquad (\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c},$$

also hold; for inspection shows that the parallelogram $\mathbf{a}\times(\mathbf{b}+\mathbf{c})$ is equal to $\mathbf{a}\times\mathbf{b}$ plus $\mathbf{a}\times\mathbf{c}$. The anti-commutative law,

$$\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a},$$

holds; for

$$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = \mathbf{a}\times\mathbf{a} + \mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{a} + \mathbf{b}\times\mathbf{b} = 0.$$

Hence

$$\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a}.$$

14. Thus far we have proceeded by means of the parallel-transformation alone. It is evident that this much of vector algebra is common to all geometries, including the Euclidean and our non-Euclidean geometry, in which there is such a parallel-transformation. The other type of product, the inner product, cannot be defined without some concept of rotation or perpendicularity, or its equivalent.

We shall so define this inner product $\mathbf{a}\cdot\mathbf{b}$ that it obeys the associative law for a scalar factor and the distributive and commutative laws, namely,

$$(n\mathbf{a})\cdot\mathbf{b} = n\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot(n\mathbf{b}),$$

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c},$$

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a},$$

and furthermore remains invariant during rotation.

As the fixed lines are fundamental in rotation it is sometimes expedient to resolve vectors into components along these directions. Let $\mathbf{p}$ and $\mathbf{q}$ be definite vectors in the two fixed lines; any vector in the plane may be written as $\mathbf{r} = x\mathbf{p} + y\mathbf{q}$. By the postulated formal laws,

$$\mathbf{r}\cdot\mathbf{r} = x^2\,\mathbf{p}\cdot\mathbf{p} + y^2\,\mathbf{q}\cdot\mathbf{q} + 2xy\,\mathbf{p}\cdot\mathbf{q}.$$

We may now note that by rotation a vector along a fixed line is converted into a multiple of that vector. If $\mathbf{p}$ becomes $n\mathbf{p}$, and the inner product $\mathbf{p}\cdot\mathbf{p}$ remains invariant, then $\mathbf{p}\cdot\mathbf{p} = n^2\,\mathbf{p}\cdot\mathbf{p}$; whence it is obvious that $\mathbf{p}\cdot\mathbf{p} = 0$. In general: The inner product of any singular vector by itself is zero, and this suffices to characterize a singular vector. Hence $\mathbf{r}\cdot\mathbf{r}$ reduces to

$$\mathbf{r}\cdot\mathbf{r} = 2xy\,\mathbf{p}\cdot\mathbf{q}.$$

Before proceeding further with the definition of the inner product, we may observe that the signs of x and y are determined by that one of the four angles (made by the fixed lines) in which $\mathbf{r}$ lies. According, then, as x and y have the same sign or different signs, the vector $\mathbf{r}$ belongs to one or the other of the classes (γ) or (δ), and the product $\mathbf{r}\cdot\mathbf{r}$ will have one sign or the other. These considerations suffice to show that if $\mathbf{r}$ and $\mathbf{r}'$ are two vectors, and if $\mathbf{r}\cdot\mathbf{r}$ and $\mathbf{r}'\cdot\mathbf{r}'$ have the same sign, the vectors are of the same class, but if $\mathbf{r}\cdot\mathbf{r}$ and $\mathbf{r}'\cdot\mathbf{r}'$ are of opposite sign, $\mathbf{r}$ and $\mathbf{r}'$ are of different classes. We have here a marked departure from Euclidean geometry, in which the inner product of a real vector by itself is always positive.

We are now in a position to complete the definition of the inner product by stating that the product is a scalar, and that the product of a vector by itself is equal to the square of the interval of the vector, taken positively if the vector is of class (γ), negatively if of class (δ). This does not imply any dissymmetry between the classes (γ) and (δ), but is only such a convention as is often made with respect to sign.

The equation $\mathbf{r}\cdot\mathbf{r} = 2xy\,\mathbf{p}\cdot\mathbf{q}$ shows that the inner product of any singular vector and any singular vector of the other class is equal to one-half the inner product by itself of the diagonal of their parallelogram.

The inner product of any vector and a perpendicular vector is zero.[17] For by XVI it is evident that if $\mathbf{p}$ and $\mathbf{q}$ be the components along the fixed directions of any vector $\mathbf{r}$, so that $\mathbf{r} = \mathbf{p} + \mathbf{q}$, then $\mathbf{p} - \mathbf{q}$ is a perpendicular vector, and in general any perpendicular vector $\mathbf{r}'$ has the form $n(\mathbf{p} - \mathbf{q})$. Hence

$$\mathbf{r}'\cdot\mathbf{r} = n(\mathbf{p} - \mathbf{q})\cdot(\mathbf{p} + \mathbf{q}) = n(\mathbf{p}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{q} - \mathbf{q}\cdot\mathbf{p} - \mathbf{q}\cdot\mathbf{q}) = 0.$$

17 The fact that the inner product of a singular vector by itself vanishes justifies our convention that a singular line is perpendicular to itself.

The inner product of any two vectors is equal to the inner product of either one by the projection of the other along it. For either vector may be resolved into two vectors one of which is parallel and the other perpendicular to the other vector. Thus $\mathbf{b}$ may be written as $n\mathbf{a} + \mathbf{a}'$, where $n\mathbf{a}$ is the projection of $\mathbf{b}$ on $\mathbf{a}$, and $\mathbf{a}'$ is perpendicular to $\mathbf{a}$. Therefore

$$\mathbf{b}\cdot\mathbf{a} = n\mathbf{a}\cdot\mathbf{a} + \mathbf{a}'\cdot\mathbf{a} = n\mathbf{a}\cdot\mathbf{a},$$

which was to be proved. Geometrically the only puzzling case is that in which the vectors are of different classes. Let OA (Figure 12) be a vector of class (γ) and OB of class (δ). The projections of OA on OB and of OB on OA are respectively OB' and OA'. Note that whereas OB' extends in the same direction as OB, the vector OA' extends along the opposite direction to OA. Thus OB' is a positive multiple of OB, whereas OA' is a negative multiple of OA. But the inner product of OB by itself is negative, since the vector is of class (δ), while the inner product of OA by itself is positive, since the vector is of class (γ). Hence the inner product of OA and OB has the same sign, whichever way the projection is taken.

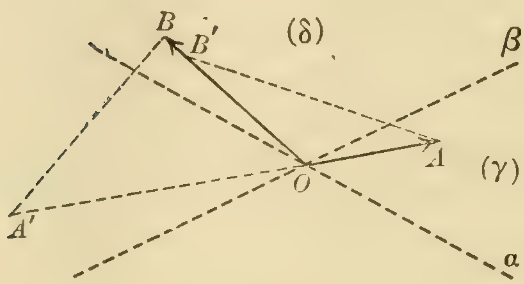

FIGURE 12.

In obtaining the inner product of a singular and a non-singular vector by projecting one upon the other, it is necessary to project the singular vector upon the non-singular vector; for it is impossible to make a perpendicular projection upon a singular vector. In case both vectors are singular the method of perpendicular projection fails entirely, and we must use analytical methods (or have recourse to parallel projection).

15. It will often be convenient to select two mutually perpendicular lines as axes of reference. We will denote[18] by $\mathbf{k}_1$ and $\mathbf{k}_4$ unit vectors along such axes, $\mathbf{k}_1$ being the vector of the (γ)-class, and $\mathbf{k}_4$ of class (δ). For these vectors we have the rules of multiplication

$$\mathbf{k}_1\cdot\mathbf{k}_1 = 1, \qquad \mathbf{k}_4\cdot\mathbf{k}_4 = -1, \qquad \mathbf{k}_1\cdot\mathbf{k}_4 = \mathbf{k}_4\cdot\mathbf{k}_1 = 0.$$

18 We reserve the symbols $\mathbf{k}_2$ and $\mathbf{k}_3$ for other unit vectors of class (γ) in space of higher dimensions.

Any two vectors **a** and **b** may be written in the form

$$\mathbf{a} = a_1\mathbf{k}_1 + a_4\mathbf{k}_4, \qquad \mathbf{b} = b_1\mathbf{k}_1 + b_4\mathbf{k}_4,$$

and the inner product is then, by the distributive law,

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 - a_4b_4.$$

In terms of these unit vectors we may also express outer products. If we write, for brevity, $\mathbf{k}_{14} = \mathbf{k}_1\times\mathbf{k}_4$, the rules for outer multiplication are

$$\mathbf{k}_{14} = -\mathbf{k}_{41}, \qquad \mathbf{k}_{11} = \mathbf{k}_{44} = 0.$$

The outer product of the vectors **a** and **b** is therefore

$$\mathbf{a}\times\mathbf{b} = (a_1b_4 - a_4b_1)\,\mathbf{k}_{14}.$$

Since $\mathbf{k}_{14}$ represents a parallelogram of unit area, the question arises as to why we write $\mathbf{k}_1\times\mathbf{k}_4$ as $\mathbf{k}_{14}$ and not simply $\mathbf{k}_1\times\mathbf{k}_4 = 1$. The answer is that the outer product $\mathbf{a}\times\mathbf{b}$ possesses a certain dimensionality, which, it is true, is not exhibited in a marked degree until we proceed into a space of higher dimensions, but which renders it undesirable to regard the outer product as merely a scalar. We may call it a pseudo-scalar, and later extend this designation to n-dimensional figures in a manifold of n dimensions.

Every vector in two dimensional space uniquely determines, except for sign, another vector, namely, the one equal in interval and perpendicular to the first. This vector will be called the complement of the given vector. To specify this sign, the complement $\mathbf{a}^*$ of the vector **a** may be defined as the inner product of **a** and the unit pseudo-scalar $\mathbf{k}_{14}$, namely, $\mathbf{a}^* = \mathbf{a}\cdot\mathbf{k}_{14}$, where the laws of this inner product are

$$\mathbf{k}_1\cdot\mathbf{k}_{14} = -\mathbf{k}_4, \qquad \mathbf{k}_4\cdot\mathbf{k}_{14} = -\mathbf{k}_1.$$

Thus if $\mathbf{a} = a_1\mathbf{k}_1 + a_4\mathbf{k}_4$, then for the complement

$$\mathbf{a}^* = (a_1\mathbf{k}_1 + a_4\mathbf{k}_4)^* = (a_1\mathbf{k}_1 + a_4\mathbf{k}_4)\cdot\mathbf{k}_{14} = -a_4\mathbf{k}_1 - a_1\mathbf{k}_4.$$

This type of multiplication, as will be seen later, obeys all the general laws of inner products (§§ 27, 29).

Referred to a set of perpendicular unit vectors, the singular vectors take the form $n(\pm\mathbf{k}_1 \pm \mathbf{k}_4)$. The complement of a singular vector is

$$n(\pm\mathbf{k}_1 \pm \mathbf{k}_4)^* = n(\pm\mathbf{k}_1 \pm \mathbf{k}_4)\cdot\mathbf{k}_{14} = n(\mp\mathbf{k}_4 \mp \mathbf{k}_1),$$

that is, the complement of a singular vector is its own negative.

We may extend the idea of complements to scalars and pseudo-scalars. The complement of the scalar n will be defined as the pseudo-scalar $n\mathbf{k}_{14}$; the complement of the pseudo-scalar $n\mathbf{k}_{14}$ will be defined as the scalar $-n$. This may be written

$$(n\mathbf{k}_{14})^* = n\mathbf{k}_{14}\cdot\mathbf{k}_{14} = -n,$$

thus establishing the convention $\mathbf{k}_{14}\cdot\mathbf{k}_{14} = -1$. It may readily be shown that, for any two singular vectors $\mathbf{p}$ and $\mathbf{q}$ of different class, the outer product is the complement of the inner product, that is,

$$\mathbf{p}\times\mathbf{q} = (\mathbf{p}\cdot\mathbf{q})\mathbf{k}_{14}.$$

In other words the inner and outer products of singular vectors are numerically equal.

Some Differential Relations.

16. As the inner product $\mathbf{r}\cdot\mathbf{r}$ of a vector by itself is numerically equal to the square of the interval of the vector $\mathbf{r}$, the equation of the unit pseudo-circle of which the radii are all (γ)-lines is $\mathbf{r}\cdot\mathbf{r} = 1$; and the equation of the conjugate unit pseudo-circle of which the radii are (δ)-lines is $\mathbf{r}\cdot\mathbf{r} = -1$. As the tangents to a pseudo-circle are perpendicular to the radii, they must be of opposite class. A pseudo-circle of which any tangent is a (δ)-line (the radii being (γ)-lines) is called a (δ)-pseudo-circle; and a pseudo-circle of which any tangent is a (γ)-line (the radii being (δ)-lines) is called a (γ)-pseudo-circle. In general if a curve has tangents which are all of the same class (δ) or (γ), the curve may be designated as a (δ)- or a (γ)-curve; the normals to the curve will then be respectively of the opposite class (γ) or (δ). The interval of the arc of any such curve will be the limit of the sum of the intervals of the infinitesimal chords along the arc. We shall not be obliged to consider any curve which is not altogether of one class as here defined.

As $d\mathbf{r}$ is the infinitesimal chord as a vector quantity, the formula for the scalar arc is

$$s = \int \sqrt{d\mathbf{r}\cdot d\mathbf{r}} \qquad \text{or} \qquad s = \int \sqrt{-\,d\mathbf{r}\cdot d\mathbf{r}}$$

according as the curve is a (γ)- or a (δ)-curve.

The sectorial area in a unit pseudo-circle may be regarded as the sum of infinitesimal right triangles, of which the area is numerically equal to $\frac{1}{2}\mathbf{r}\times d\mathbf{r}$ if $\mathbf{r}$ is drawn from the center. The numerical value of the area is therefore one-half the numerical value of $d\mathbf{r}$, that is, one-half the infinitesimal interval of arc. From our definition of unit angle (§ 12), it is evident that an angle is equal to the arc subtended upon a unit pseudo-circle centered at the vertex of the angle. This might, in fact, have been made the definition of the measure of angle. It is evident from these considerations that a rotation turns all non-singular lines through the same angle.

Angles may be classified according to the classes of their sides. If the two sides are (γ)-lines, the angle will be designated as of class $(\gamma\gamma)$; if they are (δ)-lines, the angle is of class $(\delta\delta)$. Consideration of angles $(\gamma\delta)$, which have one side a (γ)-line and the other a (δ)-line, and which cannot be generated by rotation, need not detain us here. (See Appendix.)

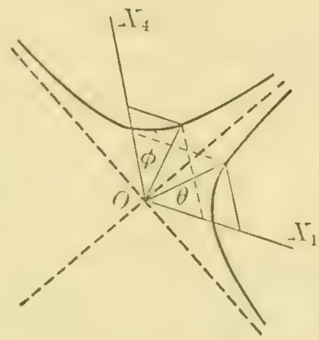

FIGURE 13.

If any line (Figure 13) through the center be taken from which to measure angle, position upon the unit pseudo-circle may be expressed parametrically in terms of the angle as follows. Let the given line be a line of class (γ) (the pseudo-circle then being of class (δ)), and construct the perpendicular line of class (δ). These two lines may be taken respectively as axes of x_1 and x_4 with the unit vectors $\mathbf{k}_1$ and $\mathbf{k}_4$ along them. The equation of the unit pseudo-circle is then

$$\mathbf{r}\cdot\mathbf{r} = (x_1\mathbf{k}_1 + x_4\mathbf{k}_4)\cdot(x_1\mathbf{k}_1 + x_4\mathbf{k}_4) = x_1^2 - x_4^2 = 1.$$

The differential of angle or arc is in this case

$$d\theta = ds = \sqrt{-d\mathbf{r}\cdot d\mathbf{r}} = \sqrt{(\mathbf{k}_1 dx_1 + \mathbf{k}_4 dx_4)\cdot(\mathbf{k}_1 dx_1 + \mathbf{k}_4 dx_4)} = \sqrt{dx_4^2 - dx_1^2}$$

Whence, by differentiation of $x_1^2 - x_4^2 = 1$,

$$\int d\theta = \int ds = \int \frac{dx_4}{\sqrt{1 + x_4^2}} = \int \frac{dx_1}{\sqrt{x_1^2 - 1}},$$

and $$x_1 = \cosh\theta, \qquad x_4 = \sinh\theta,$$

where θ is the angle between the x_1-axis and the radius vector, and therefore of the class $(\gamma\gamma)$. If the given line had been of class (δ) (the pseudo-circle of class (γ)), and if the angle ϕ had been of class $(\delta\delta)$ measured from the x_4-axis to the radius vector, the results would have been

$$x_1 = \sinh \phi, \qquad x_4 = \cosh \phi,$$

with $x_1^2 - x_4^2 = -1$ as the equation of the pseudo-circle.

If now in general $\mathbf{r}$ be the radius of any pseudo-circle, the foregoing results may readily be generalized, and we obtain the following pair of equations.

$$\begin{array}{lll} x_1 = r \cosh \theta, & x_4 = r \sinh \theta, & x_4 = x_1 \tanh \theta; \\ x_1 = r \sinh \phi, & x_4 = r \cosh \phi, & x_1 = x_4 \tanh \phi. \end{array} \tag{1}$$

In the first case $\mathbf{r}$ is a (γ)-vector and θ is a $(\gamma\gamma)$-angle; in the second, $\mathbf{r}$ is a (δ)-vector and ϕ is a $(\delta\delta)$-angle. We thus have equations which express the relations between the hypotenuse and the sides of any right triangle in terms of one angle. The inclination of the vector $\mathbf{r}$ to the axes $\mathbf{k}_1$ or $\mathbf{k}_4$ in the respective cases is the angle

$$\theta = \tanh^{-1} \frac{x_4}{x_1} \qquad \text{or} \qquad \phi = \tanh^{-1} \frac{x_1}{x_4};$$

and the slope of $\mathbf{r}$ relative to the axes is the hyperbolic tangent of the angle, not the trigonometric tangent.

17. Consider next any curve of class (δ). Let

$$s = \int \sqrt{dx_4^2 - dx_1^2}$$

denote scalar arc along the curve, and let $\mathbf{r}$ be the radius vector from a fixed origin to any point of the curve. Then the derivative

$$\mathbf{w} = \frac{d\mathbf{r}}{ds} = \mathbf{k}_1 \frac{dx_1}{ds} + \mathbf{k}_4 \frac{dx_4}{ds} \tag{2}$$

is a unit vector tangent to the curve. If this vector makes the angle ϕ with the axis $\mathbf{k}_4$, so that the slope of the curve is

$$v = \tanh \phi = \frac{dx_1}{dx_4}, \tag{3}$$

the components of the vector are

$$\frac{dx_1}{ds} = \sinh \phi = \frac{v}{\sqrt{1 - v^2}}, \qquad \frac{dx_4}{ds} = \cosh \phi = \frac{1}{\sqrt{1 - v^2}}, \tag{4}$$

and

$$\mathbf{w} = \frac{1}{\sqrt{1 - v^2}} (v\mathbf{k}_1 + \mathbf{k}_4). \tag{5}$$

If we had chosen a different set of perpendicular axes $\mathbf{k}_1'$, $\mathbf{k}_4'$, where $\mathbf{k}_4'$ makes an angle $\psi = \tanh^{-1} u$ with $\mathbf{k}_4$, so that the inclination of $\mathbf{w}$ to $\mathbf{k}_4'$ is $\phi' = \phi - \psi$, the new components of $\mathbf{w}$ would be

$$\frac{dx_1'}{ds} = \sinh \phi' = \cosh \phi \cosh \psi - \sinh \phi \sinh \psi = \frac{v'}{\sqrt{1 - v'^2}}$$

$$= \frac{v - u}{\sqrt{1 - v^2}\sqrt{1 - u^2}},$$

$$\frac{dx_4'}{ds} = \cosh \psi' = \cosh \phi \cosh \psi - \sinh \phi \sinh \psi = \frac{1}{\sqrt{1 - v'^2}}$$

$$= \frac{1 - vu}{\sqrt{1 - v^2}\sqrt{1 - u^2}},$$

where

$$v' = \frac{dx_1'}{dx_4'} = \tanh \phi' = \frac{\tanh \phi - \tanh \psi}{1 - \tanh \phi \tanh \psi} = \frac{v - u}{1 - vu}. \tag{6}$$

It will be convenient to have a general equation for the components of a vector upon one set of axes in terms of its components on another set. Let $\mathbf{k}_1$, $\mathbf{k}_4$ be one set of perpendicular unit vectors, and $\mathbf{k}_1'$, $\mathbf{k}_4'$ another set. If the angle from $\mathbf{k}_1$ to $\mathbf{k}_1'$ be ψ, the angle from $\mathbf{k}_4$ to $\mathbf{k}_4'$ is also ψ by § 16. The products

$$\mathbf{k}_1 \cdot \mathbf{k}_1' = \cosh \psi, \qquad \mathbf{k}_4 \cdot \mathbf{k}_4' = -\cosh \psi,$$

$$\mathbf{k}_1 \cdot \mathbf{k}_4' = \sinh \psi, \qquad \mathbf{k}_1' \cdot \mathbf{k}_4 = -\sinh \psi,$$

follow from (1). To obtain the transformation equations we write

$$\mathbf{r} = x_1 \mathbf{k}_1 + x_4 \mathbf{k}_4 = x_1' \mathbf{k}_1' + x_4' \mathbf{k}_4',$$

and multiply by $\mathbf{k}_1$, $\mathbf{k}_4$, $\mathbf{k}_1'$, $\mathbf{k}_4'$;

$$\begin{aligned} \mathbf{r} \cdot \mathbf{k}_1 &= x_1 = x_1' \cosh \psi + x_4' \sinh \psi, \\ -\mathbf{r} \cdot \mathbf{k}_4 &= x_4 = x_1' \sinh \psi + x_4' \cosh \psi, \\ \mathbf{r} \cdot \mathbf{k}_1' &= x_1' = x_1 \cosh \psi - x_4 \sinh \psi, \\ -\mathbf{r} \cdot \mathbf{k}_4' &= x_4' = -x_1 \sinh \psi + x_4 \cosh \psi. \end{aligned} \tag{7}$$

Curvature in our non-Euclidean geometry is defined, as is ordinary geometry, as the rate of turning of the tangent relative to the arc. As $\mathbf{w}$ is a unit tangent, $d\mathbf{w}$ is perpendicular to $\mathbf{w}$ and in magnitude is equal to the differential angle through which $\mathbf{w}$ turns. Hence

$$\mathbf{c} = \frac{d\mathbf{w}}{ds} = \frac{d^2\mathbf{r}}{ds^2} \tag{8}$$

is the curvature, taken as a vector normal to the curve. Hence

$$\mathbf{c} = \left[\frac{\mathbf{k}_1}{(1-v^2)^2} + \frac{v\mathbf{k}_4}{(1-v^2)^2}\right]\frac{dv}{dx_4}. \tag{9}$$

In magnitude the curvature is

$$\sqrt{\mathbf{c}\cdot\mathbf{c}} = \frac{\dfrac{dv'}{dx_4}}{(1-v^2)^{\frac{3}{2}}} = \frac{\dfrac{d^2x_1}{dx_4^2}}{\left[1-\left(\dfrac{dx_1}{dx_4}\right)^2\right]^{\frac{3}{2}}}.$$

Relative to axes $\mathbf{k}_1'$, $\mathbf{k}_4'$, the result is

$$\mathbf{c} = \left[\frac{\mathbf{k}_1'}{(1-v'^2)^2} + \frac{v'\mathbf{k}_4'}{(1-v'^2)^2}\right]\frac{dv'}{dx_4'}$$
$$= \left[\frac{(1-uv)\,\mathbf{k}_1'}{(1-v^2)^2\sqrt{1-u^2}} + \frac{(v-u)\,\mathbf{k}_4'}{(1-v^2)^2\sqrt{1-u^2}}\right].$$

In complete analogy with the circle in Euclidean geometry the pseudo-circle in our non-Euclidean geometry has a curvature of constant magnitude throughout. The curvature of any other curve may always be represented as the curvature of the osculating pseudo-circle, and in magnitude is inversely proportional to the radius of that pseudo-circle.

Kinematics in a Single Straight Line.

18. Before proceeding to the discussion of the non-Euclidean geometry of more than two dimensions we may consider some simple but fundamental problems of physics which may be treated with the aid of the results which we have already obtained.

The science of kinematics involves a four dimensional manifold, of which three of the dimensions are those of space, and one that of time. By neglecting two of the spacial dimensions, in other words by restricting our considerations to the motion of a particle [19] in a single straight line, kinematics becomes merely a two dimensional science. The theorems of kinematics, not in the classical form, but in the form given to them by the principle of relativity, are simply theorems in our non-Euclidean geometry.

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19 By particle we do not as yet mean a material particle but merely an identifiable point in motion.

The units of distance and time, namely the centimeter and second, were chosen without reference to each other. Retaining the centimeter as the unit of distance, we may take as the unit of time one which had been frequently suggested as the rational unit long before the principle of relativity was enunciated, namely, the second divided by 3×10^{10}, or the time required by light in free space to travel one centimeter. The velocity of light then becomes unity.

Let us consider in our geometry two perpendicular lines, and measure along the (γ)-line extension in space, along the (δ)-line extension in time. Then any point in the plane will represent a given position at a given time. We are considering the motion of a particle along a specified straight line in space. If x denotes distance along the line from a chosen origin, then in terms of our previous nomenclature, we shall take $x = x_1$ and $t = x_4$. The $\mathbf{k}_4$- or t-axis, or any line in the xt-plane parallel to this axis, represents the locus in time of a particle which does not change its position in space, in other words, of a stationary particle. Any straight line of the (δ)-class making a non-Euclidean angle ψ with $\mathbf{k}_4$, represents the locus in space and time of a particle moving with a constant velocity

$$u = \frac{dx}{dt} = \tanh \psi$$

A singular line in our plane represents a velocity $u = 1$, and is the locus of a particle moving with the velocity of light.

We have seen that in our plane no pair of perpendicular lines is better suited to serve as coordinate axes than any other pair. If then we consider (Figure 14) two (δ)-lines, marked t and t', and the respectively perpendicular (γ)-lines, marked x and x', and if we regard the first (δ)-line as the locus of a stationary particle and the second as the locus of a moving particle, we might expect to find that we could equally well regard the second (δ)-line as the locus of a particle at rest and the first as the locus of a moving particle. And this is, in fact, the first postulate of the principle of relativity. The one relation between the two lines, which is independent of any assumption as to which line is the locus of a stationary point, is

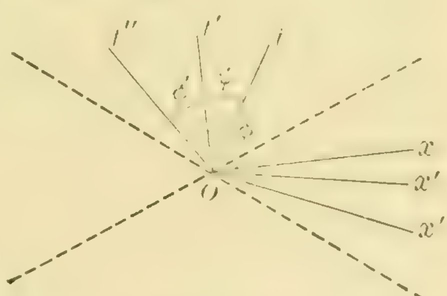

FIGURE 14.

the angle ψ whose hyperbolic tangent is the *relative* velocity which is the same by either of the assumptions.

If now we have a third (δ)-line t'' making an angle ϕ with the first (δ)-line, and ϕ' with the second, where $\phi' = \phi - \psi$, and if we call the relative velocities corresponding to these angles

$$v = \tanh \phi, \qquad v' = \tanh \phi', \qquad u = \tanh \psi,$$

then it is not true that $v' = v - u$, but since $\phi' = \phi - \psi$,

$$v' = \frac{v - u}{1 - vu}$$

by (6). This is the theorem regarding the addition of velocities obtained by Einstein.[20] The true significance of this result cannot be emphasized too strongly, namely, that the velocity as such can only be determined after a set of axes have been arbitrarily chosen; relative velocity, however, has a meaning independent of any coordinate system. Furthermore it is not the relative velocities, but the non-Euclidean angles, which are their hyperbolic anti-tangents, which are simply additive. If we were constructing a new system of kinematics uninfluenced by the historical development of the science, it might be preferable to make these angles fundamental rather than the velocities.

Suppose that from a given (δ)-line we lay off successively equal angles, so that each line determines with the preceding line the same relative velocity, then the angle measured from the given line increases without limit, but its hyperbolic tangent, which is the velocity relative to this line, approaches unity, that is, the velocity of light. The relative velocity, therefore, determined by any two (δ)-lines whatever, is less than the velocity of light. The velocity of light itself appears the same regardless of the choice of coordinate axes. This is the second postulate of the principle of relativity. Indeed if angle, instead of relative velocity, had been made fundamental, the motion of light, as compared with all other motions, would have been characterized by an infinite value of the angle.

19. Let us return to our figure and consider once more the lines that have been marked t, t', and x, x'. If we take the t-line as the locus of a stationary particle, then all points along the line x or along any parallel line are said to be simultaneous, for along any line perpendicular to the t-axis the value of t is constant. In like manner if we con-

20 Einstein, Jahrb. d. Radioak, **4**, 423.

sider the t'-line as the locus of a particle at rest, then simultaneous points are those along x' or along lines parallel to x'. Hence points which are simultaneous from one point of view, are not simultaneous from the other. In fact any two points through which a line of class (γ) can be drawn may be regarded as simultaneous by choosing this (γ)-line as the axis x, and the perpendicular line as the axis t. Similarly any two points through which a (δ)-line can be drawn may be regarded as having the same spacial position; in other words any point may be taken as a point at rest.

It thus appears that the measurements of time and space are determined only relative to some selected set of axes. Further to exhibit this fact, and to determine the relations which exist between the measures of time and space when different sets of axes are chosen, let us consider (Figure 15) two parallel (δ)-lines in our non-Euclidean plane. These lines represent the loci of two particles which have no relative velocity. Let any set of axes of time and space be drawn. The constant intervals cut off by the two parallel (δ)-lines from the x-axis and all lines parallel to this axis represent the constant distance, as measured by these axes, between the two particles at any time. The constant intervals cut off by the two parallel (δ)-lines on the t-axis and all lines parallel thereto represent the constant interval of time as measured by these axes, which must elapse between the instant when one of the particles has a certain position (upon the line in which we are considering rectilinear motion as taking place) and the instant when the other of the particles has this same position.

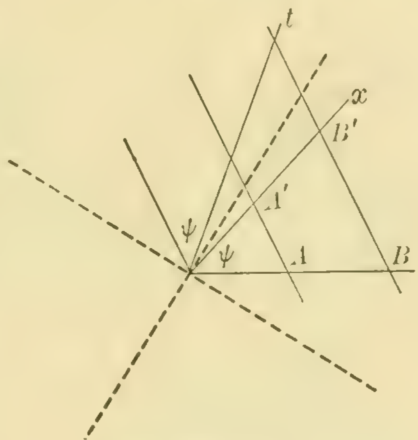

FIGURE 15.

One particular choice of axes is especially simple, namely, that in which the t-axis is parallel to the two (δ)-lines, and the x-axis is perpendicular. Relative to this assumption of axes the particles are at rest. The distance between them is AB. If another set of axes is drawn, the particles appear to be in motion, and the distance between them is taken as $A'\,B'$. If ψ denotes the angle between the axes, the projection of $A'B'$ on AB is equal to AB,

$$AB = A'B' \cosh \psi = \frac{A'B'}{\sqrt{1 - u^2}}.$$

where u is the relative velocity determined by ψ. Or,

$$A'B' = AB \operatorname{sech} \psi = AB \sqrt{1 - u^2}.$$

That is to say, the distance $A'B'$ between the particles when considered in motion with the velocity u is to the distance AB between the particles when considered at rest as $\sqrt{1 - u^2} : 1$. This statement embodies Lorentz's theory of the shortening of distances in the direction of motion.

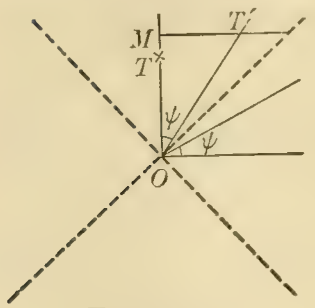

FIGURE 16.

Consider now (Figure 16) two intersecting (δ)-lines along which equal (unit) intervals OT and OT' are marked. If OT is taken as the time-axis, the point M, obtained by dropping from T' the perpendicular $T'M$ to OT, is simultaneous with T'. But the interval OM is greater than OT in the ratio $1 : \sqrt{1 - u}$ where $u = \tanh \psi$ is the relative velocity determined by the two lines. Hence a unit time OT' as measured along OT' appears greater with reference to OT than the unit OT itself. This is another statement of Einstein's theorem that unit time, measured in a moving system, is longer than unit time measured in a stationary system.

All of these special thorems follow directly from the general transformation equations (7). We have

$$x_1' = x_1 \cosh \psi - x_4 \sinh \psi,$$
$$x_4' = - x_1 \sinh \psi + x_4 \cosh \psi.$$

Now substituting

$$u = \tanh \psi, \qquad \sinh \psi = u / \sqrt{1 - u^2}, \qquad \cosh \psi = 1 / \sqrt{1 - u^2},$$

$$x_1' = \frac{1}{\sqrt{1 - u^2}} (x_1 - u x_4).$$

$$x_4' = \frac{1}{\sqrt{1 - u^2}} (x_4 - u x_1).$$

Or, replacing x_4 by t and x_1 by x, we have the fundamental transformation equations of Einstein for the change from stationary to moving coordinates.

20. Let us next consider instead of a (δ)-line any (δ)-curve. This will represent the space-time locus of a particle undergoing accelerated rectilinear motion. As the distinction between curved and straight lines is independent of any reference to axes, it follows that accelerated motion must remain accelerated motion regardless of the axes chosen. Moreover, the curvature (§ 17) of a curve is also independent of any choice of axes. Hence, although it is impossible, as we have seen, to define absolute velocity (that is, all velocity is relative to some assumed set of axes), we may define absolute acceleration if we are willing to define it as the curvature or as any function of the curvature alone. If, however, we wish to use the ordinary measure of acceleration, we must consider the projection of the curvature upon a chosen x-axis, namely,

$$c_x = \frac{1}{(1-v^2)^2}\frac{dv}{dt}, \quad \text{or} \quad \frac{dv}{dt} = (1-v^2)^2 c_x.$$

It is evident that curvature of constant magnitude does not mean uniform acceleration. Indeed if the numerical value of the curvature is constant the point in the xt-plane must move upon a pseudo-circle. Since the tangent to this curve approaches, but never reaches, the asymptotic fixed direction, it is clear that the velocity of the particle approaches as its limit the velocity of light. For such a motion, the relation between x and t is easily seen to be

$$(1-v^2)^{-\frac{3}{2}}\frac{dv}{dt} = \frac{1}{R}, \quad \text{or} \quad (x-c_2)^2 - (t-c_1)^2 = R^2,$$

where R is the radius of curvature, and c_1, c_2 are constants of integration depending on the choice of origin for x and t.

The interval of arc along any (δ)-curve is that which was called by Minkowski the Eigenzeit. This quantity is of course invariant in any change of axes. Thus

$$\int ds = \int \sqrt{dt^2 - dx^2} = \int \sqrt{dt'^2 - dx'^2}.$$

Mechanics of a Material Particle and of Radiant Energy.

21. Hitherto we have not assigned to our moving particles any distinguishing characteristics. Let us now consider what follows if we attribute to each particle a mass. It is true, as we shall later see, that the phenomena which must be discussed in connection with the dynamics of a material particle, even in the case where that particle moves only in a straight line, cannot be adequately represented in our two dimensional diagram. Nevertheless those results which can

be discussed are so much more readily visualized in this simple case that we shall consider a few important theorems before entering upon the treatment of three and four dimensional manifolds.

The meaning of the mass of a particle, when that mass is determined by a person at rest relative to the particle, will be taken as understood. We shall call that value of the mass m_0. Let us consider a (δ)-curve which represents the locus in time and space of this material particle, and at any point of the locus a tangent of unit interval (or unit tangent) $\mathbf{w}$. By multiplying $\mathbf{w}$ by the scalar m_0, we make a new vector which we shall call the *extended momentum*. If now we choose any pair of axes x and t, the slope of the locus with respect to these axes, that is, the velocity of the particle, we have called v. The momentum vector may then be written, by (5),

$$m_0\mathbf{w} = \frac{m_0 v}{\sqrt{1 - v^2}}\mathbf{k}_1 + \frac{m_0}{\sqrt{1 - v^2}}\mathbf{k}_4. \tag{10}$$

If the t-axis were chosen parallel to the tangent $\mathbf{w}$, the coefficient of $\mathbf{k}_4$, that is, the component of the extended momentum $m_0\mathbf{w}$ along the time axis, would be simply m_0, the stationary mass. If, as we have assumed, the particle is regarded as moving with the velocity v, we shall take the component of $m_0\mathbf{w}$ along the t-axis as the mass m. In other words, the mass of a body appears to increase with its velocity in the familiar ratio

$$m = \frac{m_0}{\sqrt{1 - v^2}}. \tag{11}$$

The component along the x-axis is then mv, the momentum. We may therefore write the vector of extended momentum as

$$m_0\mathbf{w} = mv\mathbf{k}_1 + m\mathbf{k}_4. \tag{12}$$

22. From our equation for the curvature we may write

$$m_0\mathbf{c} = \frac{d\,m_0\mathbf{w}}{ds} = \frac{d\,mv}{ds}\mathbf{k}_1 + \frac{dm}{ds}\mathbf{k}_4 = \frac{1}{\sqrt{1 - v^2}}\left(\frac{d\,mv}{dt}\mathbf{k}_1 + \frac{dm}{dt}\mathbf{k}_4\right). \tag{13}$$

The vector $m_0\mathbf{c}$ we shall call the extended force. Since our ordinary definition of force is time-rate of change of momentum, it is evident that the x-component of the extended force multiplied by $\sqrt{1 - v^2}$ is ordinary force. That is,

$$f = \sqrt{1 - v^2}\, m_0 c_x = \frac{d\,mv}{dt}. \tag{14}$$

By comparison with equation (9), or by substituting for m from (11) and differentiating, we obtain the results[21]

$$f = \frac{dmv}{dt} = \frac{m_0 v}{(1-v^2)^{\frac{3}{2}}}\frac{dv}{dt}, \tag{15}$$

$$\frac{dm}{dt} = \frac{m_0 v}{(1-v^2)^{\frac{3}{2}}}\frac{dv}{dt} = fv = \frac{dE}{dt}, \tag{16}$$

where dE/dt represents the rate at which energy is acquired by the particle when acted upon by the force f. Since dE/dt and dm/dt are equal, we may, except possibly for a constant of integration, write $E = m$. This is a special statement which falls under the more general law, that the mass of a body, in the units which we employ, is equal to the energy of the body. We may therefore use the terms mass and energy interchangeably.

The type of motion which, from the viewpoint of the principle of relativity, corresponds most closely to motion under uniform acceleration in Newtonian mechanics, is motion under a constant force f. The equation of motion may readily be integrated.

$$f = \frac{dmv}{dt} = m_0 \frac{d}{dt}\frac{v}{\sqrt{(1-v^2)}} = K,$$

$$\frac{v}{\sqrt{(1-v^2)}} = \frac{K}{m_0}(t-t_0), \qquad \frac{dx}{dt} = \frac{Kt}{\sqrt{m_0^2 + K^2(t-t_0)^2}},$$

and
$$\left(x - x_0 + \frac{m_0}{K}\right)^2 - (t-t_0)^2 = \frac{m_0^2}{K^2}.$$

The representative point in the xt-plane therefore describes a pseudo-circle of which the curvature is the constant force acting on the particle divided by m_0. The mass of the particle at any time is

$$m = \frac{m_0}{\sqrt{1-v^2}} = \frac{K(t-t_0)}{v} = K\left(x - x_0 + \frac{m_0}{K}\right),$$

which shows that the increase in mass is equal to the product of the force by the distance traversed, as it should be from the principle of energy above stated.

23. Let us consider the problem of the impact of two particles A and B of which the vectors of extended momentum ($m_0\mathbf{w}$) are respec-

[21] See later discussion (§36) of the so-called longitudinal mass.

tively **a** and **b** before collision, and **a**′ and **b**′ after collision. Several important laws are subsumed under a law which we may call the law of conservation of extended momentum, namely,

$$\mathbf{a} + \mathbf{b} = \mathbf{a}' + \mathbf{b}'. \tag{17}$$

Assume any set of space-time axes, and write

$$\mathbf{a} = a_1\mathbf{k}_1 + a_4\mathbf{k}_4, \qquad \mathbf{b} = b_1\mathbf{k}_1 + b_4\mathbf{k}_4,$$

$$\mathbf{a}' = a_1'\mathbf{k}_1 + a_4'\mathbf{k}_4, \qquad \mathbf{b}' = b_1'\mathbf{k}_1 + b_4'\mathbf{k}_4.$$

Then the law states that

$$(a_1 + b_1)\,\mathbf{k}_1 + (a_4 + b_4)\,\mathbf{k}_4 = (a_1' + b_1')\,\mathbf{k}_1 + (a_4' + b_4')\,\mathbf{k}_4,$$

or

$$a_1 + b_1 = a_1' + b_1', \tag{18}$$

$$a_4 + b_4 = a_4' + b_4'. \tag{19}$$

Now (by § 21) a_4 and b_4 are the masses of the two particles before collision, a_4', b_4' the masses after collision, and equation (19) expresses the law of conservation of mass or energy. The components a_1, b_1, a_1', b_1', are the respective momenta (in the ordinary sense), and equation (18) is the law of conservation of momentum.

To assume that the impact is elastic is equivalent to assuming that the value of m_0 for each particle is unchanged by the collision; and since each value of m_0 is the magnitude of the corresponding vector of extended momentum, the assumption may be expressed in the equations

$$a = a', \qquad b = b'.$$

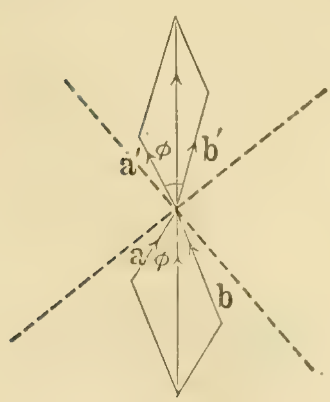

Figure 17.

The condition that the extended momentum is unchanged gives

$$(\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} + \mathbf{b}) = (\mathbf{a}' + \mathbf{b}')\cdot(\mathbf{a}' + \mathbf{b}'),$$

or

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{a}'\cdot\mathbf{b}'$$

by the above relations. Hence it follows (Figure 17) that

$$\cosh\phi = \cosh\phi', \qquad \text{or} \qquad \phi = \phi',$$

as is evident from the rules of projection previously deduced. It is thus seen that the relative velocity is the same before and after collision, and thereby a rule which has been found very useful in the discussion of simple problems in Newtonian mechanics proves equally applicable in the new mechanics.

If the impact, instead of being perfectly elastic, were such that the particles remained together after the collision, the two vectors **a** and **b** would merely be merged into a single vector **a** + **b**. The sum of the m_0's would not in this case remain constant, but would be increased by the heat (or mass) produced by the impact and obtained from the "kinetic energy" of the relative motion. This is all equivalent to the simple geometrical theorem that the (δ)-diagonal of a parallelogram whose sides are (δ)-lines is greater than the sum of the two sides.

24. The concepts of momentum and energy (mass) are ordinarily extended from the primitive mechanical phenomena to those involving so-called radiant energy. We shall see that the ascription of mass and momentum to light or other radiation is in consonance with the geometrical representation which we have adopted.

Let us consider a ray of light emitted in a single line for a definite interval of time. Such a ray alone can be considered in our two dimensional system. If the interval of time is very short, so that the front and the rear of the ray are very near together, we may regard the ray as a particle of light. The motion of such a light-particle can only be represented in our geometry by a singular vector, and to any observer its velocity is unity. Although the interval of any singular vector is zero as compared with the interval of any (γ)- or (δ)-vector, intervals along a given singular vector are, as we have pointed out, comparable with one another.

Supposing now that a given light-particle is represented by a definite singular vector, let us see whether such a vector can be regarded as an extended momentum. If so, its projection on any chosen space-axis must represent momentum, and its projection on the corresponding time-axis mass or energy. These two projections must, moreover, be of equal magnitude in this case, since the velocity of light is unity. It is immediately obvious that this latter condition is fulfilled, since the vector is singular (§ 11). If **a** is the vector, then in terms of two sets of axes

$$\mathbf{a} = m\mathbf{k}_1 + m\mathbf{k}_4 = m'\mathbf{k}_1' + m'\mathbf{k}_4'.$$

If then **a** represents extended momentum, m must represent the mass of the light to an observer stationary with respect to the first system of axes, and m' the mass as it appears to an observer stationary with respect to the other system.

If ϕ is the angle from $\mathbf{k}_1$ to $\mathbf{k}_1'$ or from $\mathbf{k}_4$ to $\mathbf{k}_4'$, we have from (7)

$$m' = m \cosh \phi - m \sinh \phi = m \frac{1 - v}{\sqrt{1 - v^2}}, \tag{20}$$

where $v = \tanh \phi$ is the relative velocity of the two sets of axes. But this is in fact the very relation between the energy of a given particle of light as measured by two different observers whose relative velocity is v. It is therefore, as far as the energy relations are concerned, proper to consider $\mathbf{a}$ as a vector of extended momentum.

The final proof of the desirability of considering the vector $\mathbf{a}$ as extended momentum comes when we consider the interaction of a light-particle with a particle of the ordinary sort. We shall see that the law of the constancy of extended momentum is true, and is only true, when we include the momentum of radiant energy as well as that of so-called material particles.

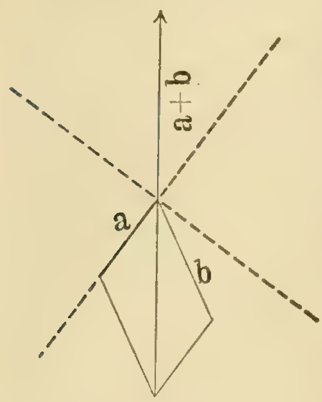

FIGURE 18.

Let the vector $\mathbf{a}$ (Figure 18) be the vector due to a light-particle, and $\mathbf{b}$ that due to a material particle which has the power of absorbing light. Then if our law of extended momentum applies to $\mathbf{a}$ and $\mathbf{b}$, there will be a single vector after impact equal to $\mathbf{a} + \mathbf{b}$ which will represent the extended momentum of the material particle after it has absorbed the light. Let us choose any set of axes. Then

$$\mathbf{a} = a_1 \mathbf{k}_1 + a_4 \mathbf{k}_4, \qquad \mathbf{b} = b_1 \mathbf{k}_1 + b_4 \mathbf{k}_4,$$

where $a_4 = a_1$ is the mass of the light-particle, and b_4 is the mass of the material particle before impact, while a_1 and $b_1 = b_4 v$ are the respective momenta. The momentum after impact is

$$a_1 + b_1 = a_4 + b_4 v.$$

Hence the change in momentum of the material particle is equal in our units to the energy of the light absorbed, which gives at once the well known formula of Maxwell and Boltzmann for the pressure of light.

While it is evident, therefore, that such a vector $\mathbf{a}$ satisfies fully all the conditions of an extended momentum, it must as a singular vector have properties quite distinct from those of a momentum vector which can be written in the form of $m_0\mathbf{w}$. Since a singular vector has zero magnitude we can ascribe to the light no finite value of m_0 or $\mathbf{w}$. In this case, as in the case of inelastic impact between material particles, the total values of m_0 does not remain constant, but is larger after impact. In all cases we obtain the same results from the law of the constancy of extended momentum as those obtained by the application of the ordinary laws for the conservation of energy, mass, and momentum, whatever axes be arbitrarily chosen.

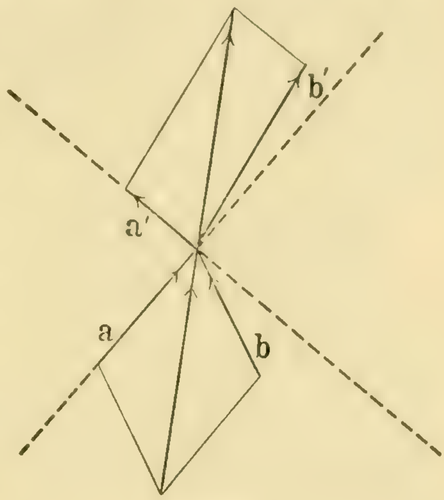

FIGURE 19.

Another simple illustration of these laws is furnished (Figure 19) in the case where the material particle does not absorb the light, but acts as a perfect reflector, which corresponds closely to elastic impact between particles. Here $\mathbf{a}'$ and $\mathbf{b}'$ are the vectors of the light-particle and the material particle after impact; and these vectors are readily shown to be determined either by the condition that the magnitude of $\mathbf{b}$ is equal to the magnitude of $\mathbf{b}'$, that is that the value of m_0 for the material particle undergoes no change, or from the condition that the angle between $\mathbf{b}$ and $\mathbf{a} + \mathbf{b}$ is the same as the angle between $\mathbf{b}'$ and $\mathbf{a}' + \mathbf{b}'$. This latter condition may in fact be regarded as necessary *à priori*, since it is the only construction which can be, in the nature of the case, uniquely determined.

Let us now consider light traveling back and forth in a single line between two mirrors whose positions are fixed relative to one another. If the mirrors are very close to one another, we may as before consider the whole system as concentrated at a point. This gives us a new kind of particle, an infinitesimal one-dimensional *Hohlraum*. Since however the energy contained within the particle is in part moving with the velocity of light in one direction and in part with the velocity of light in the other direction, we may draw two singular vectors (Figure 20) to represent the extended momenta in the two directions. Now these vectors added together give a (δ)-vector which will behave in every way like the extended momentum $m_0\mathbf{w}$ of

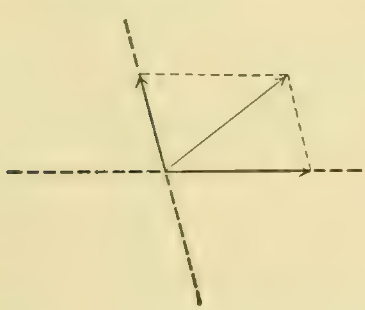

FIGURE 20.

a material particle, and m_0 represents the mass or energy of the *Hohlraum* as it appears to any observer at rest with respect to it. To such an observer the amount of energy traveling in one direction appears equal to that traveling in the opposite direction, and the resultant momentum is zero. To any observer moving with the velocity v relative to the particle, the momentum is the difference between the momenta which he observes in the two directions, and the mass of the particle is increased in the ratio $1/\sqrt{1-v^2}$. These results are all evident geometrically, and follow analytically from (20).

The Non-Euclidean Geometry in Three Dimensions.

Geometry, Outer and Inner Products.

25. We shall now consider a three-dimensional space in which the meaning of points, lines, planes, parallelism, and parallel-transformation or translation are precisely as in ordinary Euclidean geometry. In such a space, in addition to directed segments of lines or one-dimensional vectors, we have directed portions of planes or two-dimensional vectors. Any two portions of the same or parallel planes having the same area and the same sign will be considered identical two-dimensional vectors, briefly designated as 2-vectors. The ordinary one-dimensional vectors may be called 1-vectors for definiteness. It is evident that the outer product $\mathbf{a}\times\mathbf{b}$ of two 1-vectors in space is no longer a pseudo-scalar but a 2-vector lying in the plane determined by the two vectors and having a magnitude equal to the area of their parallelogram.

The addition of two 2-vectors may be accomplished geometrically in the following way. Take a definite segment of the line of intersection of the planes of the 2-vectors. In each plane construct on this segment as one side parallelograms equal respectively to the given 2-vectors. Complete the parallelepiped of which these two parallelograms are adjacent faces. The diagonal parallelogram of the parallelepiped, passing through the chosen segment, is the vector sum; the diagonal parallelogram parallel to the chosen segment is the vector difference.

Let us consider the outer product of a 1-vector and a 2-vector,[22] $\mathbf{a}\times\mathbf{A}$. Let $\mathbf{A}$ be represented as a parallelogram, and $\mathbf{a}$ as a vector through one vertex; the product $\mathbf{a}\times\mathbf{A}$ is the parallelepiped thus

22 In general 2-vectors will be designated by Clarendon capitals (except in the case of the unit coordinate 2-vectors).

determined. This outer product $\mathbf{a}\times\mathbf{A}$, being three-dimensional in a three-dimensional space, is a pseudo-scalar; and different pseudo-scalars are distinguished only by magnitude and sign.

If in $\mathbf{a}\times\mathbf{A}$ we regard $\mathbf{A}$ as itself an outer product $\mathbf{b}\times\mathbf{c}$, the parallelepiped is written as $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. This same parallelepiped can be regarded, with the possible exception of sign, as $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$. We shall in fact consider the sign as the same, and write

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{b}\times\mathbf{c},$$

so that the associative law holds for the three factors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. As $\mathbf{b}\times\mathbf{c} = -\mathbf{c}\times\mathbf{b}$, we shall write $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = -\mathbf{a}\times(\mathbf{c}\times\mathbf{b})$, in order that we may keep the law of association for the scalar factor. By successive steps we may write

$$\mathbf{a}\times\mathbf{b}\times\mathbf{c} = -\mathbf{b}\times\mathbf{a}\times\mathbf{c} = \mathbf{b}\times\mathbf{c}\times\mathbf{a};$$

and hence the outer product of a 1-vector and a 2-vector is not anti-commutative but commutative, namely,

$$\mathbf{a}\times\mathbf{A} = \mathbf{A}\times\mathbf{a}.$$

All of these statements are valid in any geometry of the group characterized by the parallel transformation.

26. In the three-dimensional non-Euclidean space, rotation about a fixed point is characterized by the existence of a fixed cone through the point, corresponding to the fixed lines in our plane geometry. An element of this cone always remains an element; points within the cone remain within, and points without remain outside. Besides the lines which are elements of this cone, or parallel to them, there are two classes, namely,

(δ)-lines through the vertex and lying within the cone, and all lines parallel to them,

(γ)-lines through the vertex and lying outside the cone, and all lines parallel to them.

In like manner planes may be separated into classes. Besides the planes of singular properties which are tangent to the cone along an element, or planes parallel to these, there are

(δ)-planes through the vertex cutting the cone in two elements, and all planes parallel thereto,

(γ)-planes through the vertex and not otherwise cutting the cone, and all parallel planes. The former set, the (δ)-planes, contain (δ)-

lines and also (γ)-lines; the latter set, the (γ)-planes, contain only (γ)-lines.

Any plane passed through a given (δ)-line cuts the cone in two elements and is therefore a (δ)-plane. The geometry of such a plane is the non-Euclidean plane geometry above described, and the elements of the cone are the fixed directions. The perpendicular in this plane to the given (δ)-line is a (γ)-line. The locus of the lines perpendicular to the given (δ)-line in all the planes through the line is a (γ)-plane. This (γ)-plane will be called perpendicular to the (δ)-line. Such a plane possesses no elements of the cone, that is, no lines which are fixed in rotation; hence the geometry of a (γ)-plane is ordinary Euclidean geometry. In the plane any line may be rotated into any other line, and the locus of the extremity of a given segment issuing from the center of rotation is a closed curve which is the circle in that plane. Moreover, the idea of angle, and of perpendicularity between lines in the (γ)-plane, being the same as in ordinary Euclidean geometry, need not be further defined.

A plane passed through a (γ)-line may cut the cone in two elements and be a (δ)-plane, or may fail to cut the cone and will then be a (γ)-plane.[23] The perpendiculars to a (γ)-line will therefore be in part (δ)-lines and in part (γ)-lines, and the plane perpendicular to a (γ)-line will therefore be a (δ)-plane. Thus a plane perpendicular to a (δ)-line is a (γ)-plane, and a plane perpendicular to a (γ)-line is a (δ)-plane.

In any three dimensional rotation one line, the axis of rotation, remains fixed, and points in any plane perpendicular to the axis remain in that plane. If the axis is a (δ)-line, the rotation is Euclidean; if a (γ)-line, non-Euclidean.

When all possible rotations, Euclidean and non-Euclidean, about axes through a given point are considered, the locus of the termini of a (γ)-vector of fixed interval, and a (δ)-vector of equal interval, issuing from the common center of the rotations, is a surface which from a completely Euclidean point of view appears to be the two conjugate hyperboloids of revolution asymptotic to the fixed cone, but which from our non-Euclidean viewpoint is really analogous to the sphere. The (δ)-lines cuts the two-parted hyperboloid; the (γ)-lines, the one-parted.

27. If we construct at a point three mutually perpendicular axes, two will be (γ)-lines, and one a (δ)-line. The unit vectors along these

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23 Planes tangent to the cone will be discussed later.

axes will be denoted respectively by $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_4$. The outer products $\mathbf{k}_1\times\mathbf{k}_2$, $\mathbf{k}_1\times\mathbf{k}_4$ $\mathbf{k}_2\times\mathbf{k}_4$ will be denoted for brevity by $\mathbf{k}_{12}$, $\mathbf{k}_{14}$, $\mathbf{k}_{24}$.

In terms of these arbitrarily chosen axes a 1-vector may be represented as

$$\mathbf{a} = a_1\mathbf{k}_1 + a_2\mathbf{k}_2 + a_4\mathbf{k}_4.$$

Similarly a 2-vector may be represented by the sum of its projections on the coordinate planes as

$$\mathbf{A} = A_{12}\mathbf{k}_{12} + A_{14}\mathbf{k}_{14} + A_{24}\mathbf{k}_{24}.$$

If we had chosen $\mathbf{k}_{21}$ in place of $\mathbf{k}_{12}$ as one of our unit coordinate 2-vectors, we should have written

$$\mathbf{A} = A_{21}\mathbf{k}_{21} + A_{14}\mathbf{k}_{14} + A_{24}\mathbf{k}_{24}.$$

Since $A_{12}\,\mathbf{k}_{12} = A_{21}\,\mathbf{k}_{21}$ and $\mathbf{k}_{12} = -\,\mathbf{k}_{21}$, we have $A_{12} = -\,A_{21}$.

If we denote by $\mathbf{k}_{124}$ the outer product $\mathbf{k}_1\times\mathbf{k}_2\times\mathbf{k}_4$, then

$$\mathbf{k}_{124} = -\,\mathbf{k}_{142} = \mathbf{k}_{412} = -\,\mathbf{k}_{421} = \mathbf{k}_{241} = -\,\mathbf{k}_{214},$$

by the rules of outer products given above. In three-dimensional space these products are unit pseudo-scalars.

In terms of their components we may now expand the two types of outer product which occur in three-dimensional space. In this expansion we employ the distributive law and the law of association for scalar factors. Then

$$\mathbf{a}\times\mathbf{b} = (a_1b_2 - a_2b_1)\,\mathbf{k}_{12} + (a_1b_4 - a_4b_1)\,\mathbf{k}_{14} + (a_2b_4 - a_4b_2)\,\mathbf{k}_{24},$$

$$\mathbf{a}\times\mathbf{A} = (a_1A_{24} + a_2A_{41} + a_4A_{12})\mathbf{k}_{124}.$$

At this point we may discuss the general characteristics of inner and outer products of vectors of various geometric dimensionalities in an n-dimensional space. In such a space we have vectors of 0, 1, 2, . . ., n–1, n-dimensions, designated as 0-vectors (or scalars), 1-vectors, 2-vectors, . . ., (n–1)-vectors, and n-vectors (or pseudo-scalars). The outer product of a p-vector and a q-vector is a $(p+q)$-vector; the product vanishes if by translation the p-vector and q-vector can be made to lie in space of less than $p+q$ dimensions The inner product of a p-vector and a q-vector, where $p \geqq q$, will always be defined as a $(p-q)$-vector. Thus whereas the inner product of a 1-vector by a 1-vector is a scalar, the inner product of a 1-vector and a 2-vector is a 1-vector.

Both the inner and outer products will obey the distributive law, and the associative law as far as regards multiplication by a scalar factor. Furthermore the outer product will always obey the associative law, and the inner product the commutative law.

28. The inner product of any 1-vector into itself may, by an immediate generalization of the definition in plane geometry (§ 14), be defined as equal to the square of its interval, taken positively for (γ)-vectors, negatively for (δ)-vectors. The inner product of two 1-vectors is equal to the inner product of either one and the projection of the other upon it. The rules for the unit coordinate vectors are therefore

$$\mathbf{k}_1 \cdot \mathbf{k}_1 = \mathbf{k}_2 \cdot \mathbf{k}_2 = 1, \qquad \mathbf{k}_4 \cdot \mathbf{k}_4 = -1, \qquad \mathbf{k}_1 \cdot \mathbf{k}_2 = \mathbf{k}_1 \cdot \mathbf{k}_4 = \mathbf{k}_2 \cdot \mathbf{k}_4 = 0.$$

The product of two vectors

$$\mathbf{a} = a_1 \mathbf{k}_1 + a_2 \mathbf{k}_2 + a_4 \mathbf{k}_4, \qquad \mathbf{b} = b_1 \mathbf{k}_1 + b_2 \mathbf{k}_2 + b_4 \mathbf{k}_4,$$

is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 - a_4 b_4.$$

The inner product $\mathbf{a} \cdot \mathbf{A}$ of a 1-vector and a 2-vector will be a 1-vector in the plane $\mathbf{A}$ and perpendicular to $\mathbf{a}$ (that is, perpendicular to the projection of $\mathbf{a}$ on $\mathbf{A}$); its magnitude will be equal to the product of the magnitude of $\mathbf{A}$ and the magnitude of the projection of $\mathbf{a}$ on $\mathbf{A}$; its sign is best determined analytically. If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular 1-vectors we may make the convention

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \mathbf{a} (\mathbf{b} \cdot \mathbf{b}), \qquad \text{or} \qquad (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = -\mathbf{b} (\mathbf{a} \cdot \mathbf{a}). \tag{21}$$

Thence follow the rules for the unit vectors,

$$\begin{array}{lll}
\mathbf{k}_1 \cdot \mathbf{k}_{12} = -\mathbf{k}_2, & \mathbf{k}_1 \cdot \mathbf{k}_{14} = -\mathbf{k}_4, & \mathbf{k}_1 \cdot \mathbf{k}_{24} = 0, \\
\mathbf{k}_2 \cdot \mathbf{k}_{12} = \mathbf{k}_1, & \mathbf{k}_2 \cdot \mathbf{k}_{14} = 0, & \mathbf{k}_2 \cdot \mathbf{k}_{24} = -\mathbf{k}_4, \\
\mathbf{k}_4 \cdot \mathbf{k}_{12} = 0, & \mathbf{k}_4 \cdot \mathbf{k}_{14} = -\mathbf{k}_1, & \mathbf{k}_4 \cdot \mathbf{k}_{24} = -\mathbf{k}_2.
\end{array}$$

Hence [24]

$$\mathbf{a} \cdot \mathbf{A} = (a_2 A_{12} - a_4 A_{14}) \mathbf{k}_1 + (-a_1 A_{12} - a_4 A_{24}) \mathbf{k}_2 + (-a_1 A_{14} - a_2 A_{24}) \mathbf{k}_4.$$

24 We may show that these rules do give an inner product which in all cases agrees with the geometric definition above stated.

The condition that $\mathbf{a} \cdot \mathbf{A}$ lies in the plane $\mathbf{A}$ is that the outer product of it and $\mathbf{A}$ shall vanish, that is, $(\mathbf{a} \cdot \mathbf{A}) \times \mathbf{A} = 0$; the condition that it is perpendicular to $\mathbf{a}$ is that the inner product of it and $\mathbf{a}$ shall vanish, that is, $(\mathbf{a} \cdot \mathbf{A}) \cdot \mathbf{a} = 0$. These two products are

$$(\mathbf{a} \cdot \mathbf{A}) \times \mathbf{A} = [(a_2 A_{12} - a_4 A_{14}) A_{24} + (a_1 A_{12} + a_4 A_{24}) A_{14} \\ - (a_1 A_{14} + a_2 A_{24}) A_{12}] \mathbf{k}_{124} = 0,$$

$$(\mathbf{a} \cdot \mathbf{A}) \cdot \mathbf{a} = a_1 (a_2 A_{12} - a_4 A_{14}) - a_2 (a_1 A_{12} + a_4 A_{24}) + a_4 (a_1 A_{14} + a_2 A_{24}) = 0,$$

as required. It is also necessary to show that the component of $\mathbf{a}$ perpendicular to $\mathbf{A}$ contributes nothing to the product $\mathbf{a} \cdot \mathbf{A}$, so that the component in

The inner product of two 2-vectors is a scalar which is equal to the inner product of either vector by the projection of the other upon it. The inner product of two perpendicular 2-vectors is zero. The inner product of a 2-vector by itself is numerically equal to the square of its magnitude, and is positive in sign if the vector is of class (γ), negative if of class (δ). Hence we have as rules of inner multiplication for 2-vectors

$$\mathbf{k}_{12}\cdot\mathbf{k}_{12} = 1, \qquad \mathbf{k}_{14}\cdot\mathbf{k}_{14} = \mathbf{k}_{24}\cdot\mathbf{k}_{24} = -1,$$

$$\mathbf{k}_{12}\cdot\mathbf{k}_{14} = \mathbf{k}_{12}\cdot\mathbf{k}_{24} = \mathbf{k}_{14}\cdot\mathbf{k}_{24} = 0,$$

$$\mathbf{A}\cdot\mathbf{A} = A_{12}^2 - A_{14}^2 - A_{24}^2, \qquad \mathbf{A}\cdot\mathbf{B} = A_{12}B_{12} - A_{14}B_{14} - A_{24}B_{24}.$$

29. Every 1-vector $\mathbf{a}$, or 2-vector $\mathbf{A}$ in a three-dimensional space uniquely determines, except for sign, another vector (respectively a 2-vector or 1-vector) perpendicular to it and of equal magnitude. This vector will be called the complement of the given vector, and designated as $\mathbf{a}^*$ or $\mathbf{A}^*$ respectively. To specify the sign, the complement may be defined as the inner product of the vector $\mathbf{a}$ or $\mathbf{A}$ and the unit 3-vector or pseudo-scalar $\mathbf{k}_{124}$, where the laws of this inner product are

$$\mathbf{k}_1\cdot\mathbf{k}_{124} = \mathbf{k}_{24}, \qquad \mathbf{k}_2\cdot\mathbf{k}_{124} = -\mathbf{k}_{14}, \qquad \mathbf{k}_4\cdot\mathbf{k}_{124} = -\mathbf{k}_{12},$$

$$\mathbf{k}_{12}\cdot\mathbf{k}_{124} = \mathbf{k}_4, \qquad \mathbf{k}_{14}\cdot\mathbf{k}_{124} = \mathbf{k}_2, \qquad \mathbf{k}_{24}\cdot\mathbf{k}_{124} = -\mathbf{k}_1.$$

Thus

$$\mathbf{a}^* = (a_1\mathbf{k}_1 + a_2\mathbf{k}_2 + a_4\mathbf{k}_4)\cdot\mathbf{k}_{124} = -a_4\mathbf{k}_{12} - a_2\mathbf{k}_{14} + a_1\mathbf{k}_{24},$$

$$\mathbf{A}^* = (A_{12}\mathbf{k}_{12} + A_{14}\mathbf{k}_{14} + A_{24}\mathbf{k}_{24})\cdot\mathbf{k}_{124} = -A_{24}\mathbf{k}_1 + A_{14}\mathbf{k}_2 + A_{12}\mathbf{k}_4.$$

These complements satisfy the condition of perpendicularity previously derived (footnote 24), and the inner products

$$\mathbf{a}^*\cdot\mathbf{a}^* = a_4^2 - a_2^2 - a_1^2, \qquad \mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 - a_4^2,$$

$$\mathbf{A}^*\cdot\mathbf{A}^* = A_{24}^2 + A_{14}^2 - A_{12}^2, \qquad \mathbf{A}\cdot\mathbf{A} = A_{12}^2 - A_{14}^2 - A_{24}^2$$

the plane is alone of importance. We shall do this by deriving the expression for a vector perpendicular to the plane $\mathbf{A}$. Let

$$\mathbf{c} = c_1\mathbf{k}_1 + c_2\mathbf{k}_2 + c_4\mathbf{k}_4, \quad \mathbf{n} = n_1\mathbf{k}_1 + n_2\mathbf{k}_2 + n_4\mathbf{k}_4$$

be respectively any vector in the plane $\mathbf{A}$ and a vector perpendicular to the plane. Then the products

$$\mathbf{c}\times\mathbf{A} = (c_1A_{24} - c_2A_{14} + c_4A_{12})\mathbf{k}_{124} = 0, \qquad \mathbf{c}\cdot\mathbf{n} = c_1n_1 + c_2n_2 - c_4n_4 = 0$$

vanish. Hence it follows that the condition of perpendicularity for the vectors $\mathbf{n}$ and $\mathbf{A}$ is

$$n_1 : n_2 : n_4 = A_{24} : -A_{14} : -A_{12},$$

and that $\mathbf{n}$ must be some multiple of $A_{24}\mathbf{k}_1 - A_{14}\mathbf{k}_2 - A_{12}\mathbf{k}_4$. By the rules, the inner product of this vector and $\mathbf{A}$ vanishes.

show that the magnitudes are equal. The reversal of sign is to be expected from the fact that the complement of a vector (whether 1– or 2–) of class (γ) is a (δ)-vector (whether 2– or 1–), and vice versa.

The use of the term complement in connection with scalars and pseudo-scalars is sometimes convenient. Since, by the rule of inner multiplication, we have $\mathbf{k}_{124} \cdot \mathbf{k}_{124} = -1$, the complement of any pseudo-scalar is a scalar of the same magnitude and of opposite sign. We may define the complement of a scalar a as the product of the scalar and the unit pseudo-scalar; thus $a^* = a\mathbf{k}_{124}$.

All the special rules for the inner products of unit vectors (and pseudo-scalars) are comprised in the following general rule, which will also be applied in space of four dimensions: If either of two unit vectors has a subscript which the other lacks, the inner product is zero; in all other cases the inner product may be found by so transposing the subscripts that all the common subscripts occur in each factor at the end, and in the same order, by then canceling the common subscripts, and by taking as the product the unit vector which has the remaining subscripts (in the order in which they stand), provided that if the subscript 4 has been canceled, the sign is changed.[25] Thus

$$\mathbf{k}_{124} \cdot \mathbf{k}_{34} = 0, \qquad \mathbf{k}_{124} \cdot \mathbf{k}_{12} = \mathbf{k}_{412} \cdot \mathbf{k}_{12} = \mathbf{k}_4, \qquad \mathbf{k}_{12} \cdot \mathbf{k}_1 = -\mathbf{k}_{21} \cdot \mathbf{k}_1 = -\mathbf{k}_2,$$
$$\mathbf{k}_{124} \cdot \mathbf{k}_4 = -\mathbf{k}_{12}, \qquad \mathbf{k}_{134} \cdot \mathbf{k}_{14} = -\mathbf{k}_{314} \cdot \mathbf{k}_{14} = \mathbf{k}_3.$$

30. Hitherto we have given little attention to the singular vectors of our geometry, namely, the lines which are elements of a singular cone and the planes which are tangent to a singular cone. We have seen (§ 14) that the inner product of a singular 1-vector by itself is zero, and have expressed that fact by stating that a singular line is perpendicular to itself. Analytically expressed, the condition that a 1-vector $\mathbf{a}$ shall be singular is that

$$\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 - a_4^2 = 0.$$

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25 Instead of regarding the common subscripts as canceled, it is possible to regard their corresponding unit 1-vectors as multiplied by inner multiplication,— and in this case the change of sign takes care of itself. Thus

$$\mathbf{k}_{pqr} \cdot \mathbf{k}_{qr} = \mathbf{k}_p (\mathbf{k}_q \cdot \mathbf{k}_q)(\mathbf{k}_r \cdot \mathbf{k}_r).$$

Indeed if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are mutually perpendicular 1-vectors, then all the rules given above may be expressed in the equations

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}), \qquad (\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{c}),$$
$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \mathbf{a}(\mathbf{b} \cdot \mathbf{b}), \qquad (\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \cdot \mathbf{c} = \mathbf{a} \times \mathbf{b}(\mathbf{c} \cdot \mathbf{c}),$$
$$(\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a}(\mathbf{b} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{c}).$$

$$\mathbf{a} = a_1\mathbf{k}_1 + a_2\mathbf{k}_2 \pm \sqrt{a_1^2 + a_2^2}\,\mathbf{k}_4.$$

The complement of a singular vector is

$$\mathbf{A} = \mathbf{a}^* = \mathbf{a}\cdot\mathbf{k}_{124} = a_1\mathbf{k}_{24} - a_2\mathbf{k}_{14} \mp \sqrt{a_1^2 + a_2^2}\,\mathbf{k}_{12}.$$

This 2-vector **A** must be itself a singular plane vector; for we have seen that the complement of any (δ)-plane is a (γ)-line and of any (γ)-plane a (δ)-line, and vice versa. The inner product of **A** by itself is obviously zero,[26] for,

$$\mathbf{A}\cdot\mathbf{A} = -a_1^2 - a_2^2 + (a_1^2 + a_2^2) = 0.$$

Conversely if we consider any 2-vector

$$\mathbf{A} = A_{12}\mathbf{k}_{12} + A_{14}\mathbf{k}_{14} + A_{24}\mathbf{k}_{24},$$

such that

$$\mathbf{A}\cdot\mathbf{A} = A_{12}^2 - A_{14}^2 - A_{24}^2 = 0,$$

its complement is a singular line, and it is itself a singular 2-vector. The standard form may be taken as

$$\mathbf{A} = \pm\sqrt{A_{14}^2 + A_{24}^2}\,\mathbf{k}_{12} + A_{14}\mathbf{k}_{14} + A_{24}\mathbf{k}_{24}.$$

The outer product of a singular vector by its complement, whether a 1-vector or a 2-vector, vanishes, as may be seen by multiplying out. Thus the singular vector and its complement lie in the same plane, that is, an element of the cone and the tangent plane through that element are mutually complementary.

When we have to consider the inner product of any singular vector with any other vector, singular or not, the geometrical method dependent on projection often fails to be applicable; for it is impossible to project a vector upon a singular vector. We may in such cases employ the analytical method, which is universally applicable, or replace the inner product with an outer product by a method introduced in a following section (§ 32).

We have seen that an element of the cone is complementary to the tangent plane to the cone through that element, that is, the element is perpendicular to the plane. Hence the element is perpendicular to every line in the plane (including itself).

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26 A singular vector, or vector of zero magnitude, has, like any other vector, a real geometrical existence and is not to be confused with a zero vector, that is, a non-existent vector.

31. We have seen that rotation in a (γ)-plane about the perpendicular (δ)-line is Euclidean, whereas rotation in a (δ)-plane about the normal (γ)-line is non-Euclidean. In this latter case not only do the (δ)-planes normal to the axis remain fixed during the rotation, but the *two singular planes* through the axis and tangent to the cone also are fixed; for the axis remains fixed and the lines in which the planes are tangent to the cone are respectively the two fixed lines in the (δ)-plane. As every point in the axis of rotation is fixed, the whole set of lines parallel to the elements of tangency is fixed. The effect in the two singular planes of a rotation is therefore to leave one line, the axis, fixed point for point, to leave a set of lines fixed, and to move the points on these lines either toward the axis or away from it by an amount which is proportional to the interval from the point to the axis.

Since a rotation in a (δ)-plane multiplies all intervals along one of the fixed directions in a certain ratio, and divides all intervals along the other fixed direction in the same ratio, the effect upon areas in the two singular planes is to multiply all areas in one of the planes in that same ratio, and to divide areas in the other in that ratio. This however is not inconsistent with our condition that areas should remain invariant; for it is evident that, when compared with areas in other planes, areas in singular planes are all of zero magnitude. This is also shown by the fact that the inner product of any singular vector by itself vanishes. That areas in a singular plane have a zero magnitude does not prevent our comparing two areas in the same singular plane or in parallel singular planes, just as the fact that intervals along singular lines had zero magnitude did not prevent our comparing intervals along any such line.

A limiting case of rotation occurs when the axis of rotation is itself an element of the cone, that is, a singular line. Here the infinity of fixed planes perpendicular to the axis, and the two singular planes through it, have all coalesced into the one singular plane through this line and tangent to the cone. In this plane the rotation consists in a sort of shear. Every point moves along a straight line parallel to the axis. In this case areas are rotated into areas which are from every point of view equal. For if a parallelogram whose base is on the axis, which is fixed point for point, is subjected to this rotation, its base remains fixed and the parallelogram remains enclosed between the same two parallel lines (Theorem IX).

The geometry in this plane, depending upon translation and upon such a rotation as has just been described, is interesting as affording a third geometry intermediate between the Euclidean and the non-Euclidean which we have discussed. In Euclidean plane geometry there is no line fixed in rotation, in our non-Euclidean plane geometry there are two fixed directions, in this new case there is just one. If we were to investigate this geometry, we should find one set of (parallel) singular lines and one set of non-singular lines. Every non-singular line may be rotated into any other. Angles about any point range from $-\infty$ to $+\infty$ on each side of the singular line through that point. The interval along any line intercepted between two singular lines is equal to the interval along any other line thus intercepted. Every non-singular line is perpendicular to the singular lines, as the singular line is complementary to the singular plane through it.

Some Algebraic Rules.

32. We shall develop here a number of important relations between outer products, inner products, and complements which will be of frequent use later. Many of these relations hold in any number of dimensions. We shall consider primarily a non-Euclidean space in which one of a set of mutually perpendicular lines is a (δ)-line, the rest being (γ)-lines. But except for occasional differences of sign, the results are valid in a Euclidean space.

In a space of n dimensions, the complement of a vector of dimensionality p is itself of dimensionality $n-p$. If a is a scalar and α is a vector of any dimensionality, then from the associative law for scalar factors, we have

$$(a\alpha)^* = (a\alpha)\cdot\mathbf{k}_{12\ldots n} = (a\mathbf{k}_{12\ldots n})\cdot\alpha = a(\alpha\cdot\mathbf{k}_{12\ldots n}) = a^*\cdot\alpha = a\alpha^*. \quad (22)$$

Let α, β, . . . be vectors of the respective dimensionalities p, q . . . . Then

$$\beta\times\alpha = (-1)^{pq}\,\alpha\times\beta. \quad (23)$$

Owing to the availability of the distributive laws it is sufficient to prove such relations as this for the simpler case where the constituent vectors α, β are unit vectors $\mathbf{k}_{g\ldots}$, $\mathbf{k}_{h\ldots}$ of dimensionality p, q. In the permutation of α and β, there are involved pq simple transpositions of subscripts; for each subscript in $\mathbf{k}_{h\ldots}$ has to be carried past all the subscripts of $\mathbf{k}_{g\ldots}$ Hence there are pq changes of sign. Hence the outer product is commutative if either of the factors is even, but is anti-commutative if both factors are odd in dimensionality.

We may next show that

$$(\alpha\times\beta)^* = \alpha\cdot\beta^*. \tag{24}$$

Suppose again that α, β are unit vectors $\mathbf{k}_g\ldots$, $\mathbf{k}_h\ldots$. We have to show

$$(\mathbf{k}_g\ldots\times\mathbf{k}_h\ldots)\cdot\mathbf{k}_1\ldots = \mathbf{k}_g\ldots\cdot(\mathbf{k}_h\ldots\cdot\mathbf{k}_1\ldots),$$

where $\mathbf{k}_1\ldots$ denotes the unit pseudo-scalar. Without changing this equation, it is possible on both sides to arrange at the end, the subscripts of the pseudo-scalar $\mathbf{k}_1\ldots$ in the same order as in the factors $\mathbf{k}_g\ldots$, $\mathbf{k}_h\ldots$ Thus we have to show that

$$(\mathbf{k}_g\ldots\times\mathbf{k}_h\ldots)\cdot\mathbf{k}_{j\ldots g\ldots h\ldots} = \mathbf{k}_g\ldots\cdot(\mathbf{k}_h\ldots\cdot\mathbf{k}_{j\ldots g\ldots h\ldots}).$$

But now the products on the right are found by canceling successively the common subscripts $h \ldots$ and $g \ldots$; whereas the product on the left is found by canceling simultaneously the subscripts of $\mathbf{k}_{g\ldots h\ldots}$. The identity is therefore proved.

As a corollary of the two preceding results we may write the formula

$$(\alpha\times\beta)^* = (-1)^{pq}(\beta\times\alpha)^* = \alpha\cdot\beta^* = (-1)^{pq}\beta\cdot\alpha^*. \tag{25}$$

All these rules are true for any space, Euclidean or non-Euclidean.

The complement of the complement of a vector α is the vector itself, except for sign. If α is of dimensionality p in a space of n dimensions, the exact relation is

$$(\alpha^*)^* = -(-1)^{p(n-p)}\alpha. \tag{26}$$

The complement of the complement of a vector will therefore be the negative of the vector except when $p\,(n-p)$ is odd, that is, when the dimensionalities of the vector and of the space are respectively odd and even.[27] For the proof, the consideration may be restricted to the case where α is a unit vector $\mathbf{k}_g\ldots$. Then

$$\begin{aligned}(\alpha^*)^* = (\mathbf{k}_g\ldots\cdot\mathbf{k}_i\ldots)\cdot\mathbf{k}_i\ldots &= (\mathbf{k}_g\ldots\cdot\mathbf{k}_{j\ldots g\ldots})\cdot\mathbf{k}_{j\ldots g\ldots}\\ &= (-1)^{p(n-p)}(\mathbf{k}_g\ldots\cdot\mathbf{k}_{j\ldots g\ldots})\cdot\mathbf{k}_{g\ldots j\ldots}\end{aligned}$$

Here again the subscripts in the pseudo-scalar $\mathbf{k}_i\ldots$ have been rearranged so as to bring $g \ldots$ to the end. Then as $g \ldots$ denotes p subscripts and $j \ldots$ denotes $n-p$, the permutation involves $p\,(n-p)$

27 In Euclidean space $(\alpha^*)^* = (-1)^{p(n-p)}\alpha$. Some writers who have identified vectors with their complements have perhaps overlooked this relation which would, upon their assumption, make a vector sometimes identical with its own negative.

changes of sign. In the final form thus found the subscripts $g \ldots$ and $j \ldots$ have successively to be canceled. But one of these is necessarily the subscript 4 (corresponding to the (δ)-vector), which requires a change of sign. Hence

$$(\mathbf{k}_{g\ldots} \cdot \mathbf{k}_{i\ldots}) \cdot \mathbf{k}_{i\ldots} = -(-1)^{p(n-p)} \mathbf{k}_{g\ldots},$$

and the desired result is proved.

Consider the product $\alpha^* \cdot \beta^*$. We have by (24) either

$$\alpha^* \cdot \beta^* = (\alpha^* \times \beta)^* \qquad \text{or} \qquad \beta^* \cdot \alpha^* = (\beta^* \times \alpha)^*. \tag{27}$$

Now, although $\alpha^* \cdot \beta^*$ and $\beta^* \cdot \alpha^*$ are equal, the two expansions obtained are usually different. In fact, as the total dimensionality of an outer product cannot exceed n, the first formula holds only when $p \geqq q$, and the second only when $q \geqq p$. Let us assume $q \geqq p$. Then

$$\begin{aligned} \alpha^* \cdot \beta^* = \beta^* \cdot \alpha^* = (\beta^* \times \alpha)^* &= (-1)^{p(n-q)} (\alpha \times \beta^*)^* \\ &= (-1)^{p(n-q)} \alpha \cdot \beta^{**} = -(-1)^{(q-p)(n-q)} \alpha \cdot \beta. \end{aligned} \tag{28}$$

As a corollary

$$\alpha^* \cdot \alpha^* = -\alpha \cdot \alpha. \tag{29}$$

The complement of an inner product may likewise be proved to be

$$(\alpha \cdot \beta)^* = (-1)^{p(n-p)} \alpha \times \beta^*, \tag{30}$$

where it is assumed that the product $\alpha \cdot \beta$ has been so arranged that the second factor is of dimensionality q greater than the dimensionality p of the first. We have furthermore

$$\alpha^* \times \alpha = (\alpha \cdot \alpha)^*; \tag{31}$$

and also if β is a pseudo-scalar

$$(\alpha \cdot \beta)^* = (-1)^{p(n-p)} \beta^* \alpha = \beta \cdot \alpha^*. \tag{32}$$

It is important to observe that by means of these rules it is possible to replace any outer product by an inner product, and vice versa.

33. We are now able to obtain rules for the expansion of the various products in which three vectors occur. The simplest type, and one which needs no further comment, is

$$(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma), \tag{33}$$

which follows from the associative law.

Consider next the product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ of three 1-vectors. Here

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}. \tag{34}$$

Perhaps the simplest proof is obtained from the relation [28]

$$\mathbf{b} = \frac{(\mathbf{b}\cdot\mathbf{c})\,\mathbf{c}}{\mathbf{c}\cdot\mathbf{c}} + \frac{\mathbf{c}\cdot(\mathbf{b}\times\mathbf{c})}{\mathbf{c}\cdot\mathbf{c}},$$

which states that a vector is equal to the sum of its components. By clearing and transposing, and by permuting the letters, we have

$$\mathbf{c}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\,\mathbf{c},$$

$$\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{b}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{b})\,\mathbf{c}.$$

If now $\mathbf{d}$ is any vector perpendicular to $\mathbf{b}$ and $\mathbf{c}$, we have identically

$$\mathbf{d}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{d}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{d}\cdot\mathbf{b})\,\mathbf{c} = 0.$$

If these equations be multiplied by x, y, z and added, we have

$$(x\mathbf{c} + y\mathbf{b} + z\mathbf{d})\cdot(\mathbf{b}\times\mathbf{c}) = [(x\mathbf{c} + y\mathbf{b} + z\mathbf{d})\cdot\mathbf{c}]\mathbf{b} - [(x\mathbf{c} + y\mathbf{b} + z\mathbf{d})\cdot\mathbf{b}]\mathbf{c},$$

and any vector $\mathbf{a}$ may be represented in the form $x\mathbf{c} + y\mathbf{b} + z\mathbf{d}$.

From the rules (33), (34) combined with the rules (22)–(32) we may obtain a number of other reduction formulas by simply taking complements of both sides of the equation.

Thus

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{C} = \mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{C}) = -\,\mathbf{b}\cdot(\mathbf{a}\cdot\mathbf{C}). \tag{35}$$

[28] With the aid of inner and outer products we may write down expressions for the components of a 1-vector $\mathbf{a}$ along and perpendicular to another 1-vector $\mathbf{b}$ or a 2-vector $\mathbf{A}$. The components of $\mathbf{a}$ along $\mathbf{b}$ and perpendicular to $\mathbf{b}$ are

$$\frac{(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}} \quad \text{and} \quad \frac{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}.$$

The components of $\mathbf{a}$ along $\mathbf{A}$ and perpendicular to $\mathbf{A}$ are

$$-\frac{(\mathbf{a}\cdot\mathbf{A})\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}} \quad \text{and} \quad \frac{(\mathbf{a}\times\mathbf{A})\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}.$$

The component of the plane $\mathbf{A}$ on the plane $\mathbf{B}$ is

$$\frac{(\mathbf{A}\cdot\mathbf{B})\,\mathbf{B}}{\mathbf{B}\cdot\mathbf{B}},$$

and a vector in the line of intersection of the two planes is

$$\mathbf{A}^*\cdot\mathbf{B} \quad \text{or} \quad \mathbf{A}\cdot\mathbf{B}^*.$$

For by (33) and (24),

$$[(\mathbf{a}\times\mathbf{b})\times\mathbf{c}]^* = [\mathbf{a}\times(\mathbf{b}\times\mathbf{c})]^*,$$

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}^* = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})^* = \mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c}^*).$$

But since $\mathbf{c}$ is any 1-vector, its complement $\mathbf{C}$ is any 2-vector.

Again,

$$\mathbf{a}\times(\mathbf{b}\cdot\mathbf{C}) = (\mathbf{a}\times\mathbf{C})\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{C}. \tag{36}$$

For by (34), (22), and (30),

$$[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]^* = [(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}]^* - [(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}]^*,$$

$$(-1)^{1(3-1)}\,\mathbf{a}\times(\mathbf{b}\times\mathbf{c})^* = (-1)^{1(3-1)}\,(\mathbf{a}\times\mathbf{c}^*)\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}^*,$$

$$\mathbf{a}\times(\mathbf{b}\cdot\mathbf{C}) = (\mathbf{a}\times\mathbf{C})\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{C}.$$

Again,

$$(\mathbf{b}\cdot\mathbf{C})\times\mathbf{A} = -\,\mathbf{C}\times(\mathbf{b}\cdot\mathbf{A}). \tag{37}$$

For from (35), (30), and (24),

$$[\mathbf{C}\cdot(\mathbf{a}\times\mathbf{b})]^* = [(\mathbf{b}\cdot\mathbf{C})\cdot\mathbf{a}]^*,$$

$$(-1)^{2(3-2)}\,\mathbf{C}\times(\mathbf{a}\times\mathbf{b})^* = (-1)^{1(3-1)}\,(\mathbf{b}\cdot\mathbf{C})\times\mathbf{a}^*,$$

$$-\,\mathbf{C}\times(\mathbf{b}\times\mathbf{a})^* = -\,\mathbf{C}\times(\mathbf{b}\cdot\mathbf{A}) = (\mathbf{b}\cdot\mathbf{C})\times\mathbf{A}.$$

Again

$$(\mathbf{b}\cdot\mathbf{C})\cdot\mathbf{A} = -\,\mathbf{b}\,(\mathbf{C}\cdot\mathbf{A}) + \mathbf{C}\cdot(\mathbf{b}\times\mathbf{A}). \tag{38}$$

For from (36), (24), (32), (22), and (30),

$$[(\mathbf{b}\cdot\mathbf{C})\times\mathbf{a}]^* = -\,[\mathbf{b}\cdot(\mathbf{C}\times\mathbf{a})]^* + [\mathbf{C}\,(\mathbf{b}\cdot\mathbf{a})]^*,$$

$$(\mathbf{b}\cdot\mathbf{C})\cdot\mathbf{A} = -\,\mathbf{b}\,(\mathbf{C}\times\mathbf{a})^* + \mathbf{C}\cdot(\mathbf{b}\cdot\mathbf{a})^*,$$

$$= -\,\mathbf{b}(\mathbf{C}\cdot\mathbf{A}) + (-1)^{1(3-1)}\,\mathbf{C}\cdot(\mathbf{b}\times\mathbf{A}).$$

These rules (33) to (38) involve every possible combination of three vectors in three dimensional space. Since the formulas which we have used in deriving them, have the same form in Euclidean space, the rules will be true in Euclidean space. The particular use of the complement has implied a three dimensional space, and a similar use of the complenent in a four dimensional space would obtain analogous but different formulas; it should be observed, however, that the rules here obtained (with the exception of (37)) must hold in space of four dimensions, even when the three vectors in question do not lie wholly in a three dimensional space. For consider (36) as a typical case. Let $\mathbf{b}$ be a 1-vector which does not lie in the space of $\mathbf{a}$ and $\mathbf{C}$; we may write $\mathbf{b} = \mathbf{b}' + \mathbf{b}''$, where $\mathbf{b}'$ is in the space of $\mathbf{a}$ and $\mathbf{C}$ and $\mathbf{b}''$ is perpendicular to $\mathbf{a}$ and $\mathbf{C}$. Then by (36)

$$\mathbf{a}\times(\mathbf{b}'\cdot\mathbf{C}) = (\mathbf{a}\times\mathbf{C})\cdot\mathbf{b}' - (\mathbf{a}\cdot\mathbf{b}')\,\mathbf{C},$$

and

$$\mathbf{a}\times(\mathbf{b}''\cdot\mathbf{C}) = (\mathbf{a}\times\mathbf{C})\cdot\mathbf{b}'' - (\mathbf{a}\cdot\mathbf{b}'')\mathbf{C}$$

holds identically, since each of its terms vanishes. Hence by addition (36) is seen also to hold in general.

Some products involving more than three 1-vectors are of frequent occurrence. By (35) and (34) we may write immediately

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\cdot\mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}. \quad (39)$$

In a similar manner we may prove

$$(\mathbf{a}\times\mathbf{b}\times\mathbf{c})\cdot(\mathbf{d}\times\mathbf{e}\times\mathbf{f}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{d} & \mathbf{a}\cdot\mathbf{e} & \mathbf{a}\cdot\mathbf{f} \\ \mathbf{b}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{e} & \mathbf{b}\cdot\mathbf{f} \\ \mathbf{c}\cdot\mathbf{d} & \mathbf{c}\cdot\mathbf{e} & \mathbf{c}\cdot\mathbf{f} \end{vmatrix},$$

and

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}\times\mathbf{e}) = (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{d}\times\mathbf{e})\,\mathbf{c} + (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{e}\times\mathbf{c})\,\mathbf{d} + (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})\,\mathbf{e}.$$

These formulas are all valid in space of any dimensions.

The Differentiating Operator ∇.

34. In discussing the differential calculus of scalar and vector functions of position in space, the vector differentiating operator ∇ is fundamental. The definition of this operator may be most simply obtained as follows. Consider a scalar function F of position in space. Let $d\mathbf{r}$ denote any infinitesimal vector change of position, and let dF denote the corresponding differential change in F. Then let ∇ be defined by the equation

$$dF = d\mathbf{r}\cdot\nabla F.$$

Now ∇F is a vector. If $d\mathbf{r}$ is a vector in the tangent plane to the surface $F =$ const., dF is 0, and as $d\mathbf{r}\cdot\nabla F$ then vanishes, the vector $d\mathbf{r}$ and ∇F are perpendicular. Hence ∇F is a vector perpendicular to the surface $F =$ const. Now ∇F may be a vector of the (δ)-class or of the (γ)-class, and the tangent plane is then respectively a (γ)-plane or a (δ)-plane.[29]

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29 In our non-Euclidean geometry ∇F will not be a vector in the line of the greatest change of F. If $d\mathbf{r}$ be written as $\mathbf{u}\,ds$, where $\mathbf{u}$ is a unit vector in the

If we select three mutually perpendicular axes $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_4$, and denote by x_1, x_2, x_4 the coordinates (intervals) along these axes, then

$$dF = dx_1 \frac{\partial F}{\partial x_1} + dx_2 \frac{\partial F}{\partial x_2} + dx_4 \frac{\partial F}{\partial x_4} = (dx_1\mathbf{k}_1 + dx_2\mathbf{k}_2 + dx_4\mathbf{k}_4)\cdot \nabla F.$$

From this ∇ may be determined as

$$\nabla = \mathbf{k}_1 \frac{\partial}{\partial x_1} + \mathbf{k}_2 \frac{\partial}{\partial x_2} - \mathbf{k}_4 \frac{\partial}{\partial x_4}\cdot \tag{40}$$

Thus ∇ appears formally as a 1-vector, and may be treated formally as such.[30]

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direction of $d\mathbf{r}$ and where ds is the interval or magnitude of $d\mathbf{r}$, we may write

$$dF = ds\mathbf{u}\cdot\nabla F \qquad \text{or} \qquad \mathbf{u}\cdot\nabla F = \frac{dF}{ds}.$$

Hence the component of ∇F along the direction $d\mathbf{r}$ is the directional derivative of F in that direction. Consider now two neighboring surfaces of constant F. Suppose first that the (approximately parallel) tangent planes to the surfaces are of class (γ), so that the perpendicular ∇F is a (δ)-vector. Then, in our geometry, the perpendicular from a point on one surface to a point of the other is, of all lines of its class, the line of greatest interval ds (§12). The directional derivative along the normal is therefore numerically a minimum (instead of a maximum) relative to neighboring directions. In fact, the derivative along a line of fixed direction would be infinite, because along the fixed cone $ds = 0$. Along the (γ)-lines the directional derivative varies between 0 and ∞. Suppose next that the tangent planes are of class (δ), so that the perpendicular ∇F is a (γ)-line. Then the interval ds along the perpendicular from a point on one surface to a point on the other is neither a maximum nor a minimum, but a minimax. For it is less than along any neighboring direction (of the same class) which with the perpendicular determines a (γ)-plane, but greater than along any neighboring direction (of the same class) which with the perpendicular determines a (δ)-plane.

30 The above definition of ∇F depends on inner multiplication, and hence upon the notion of perpendicularity or rotation. It is, however, interesting to observe that we may define a differential operator ∇' dependent upon the outer product, and hence upon the idea of translation alone. The definition would then read

$$\mathbf{a}\times\mathbf{b}\times\mathbf{c}dF = d\mathbf{r}\times\nabla' F = (\mathbf{a}dx_1 + \mathbf{b}dx_2 + \mathbf{c}dx_3)\times\nabla' F,$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are any three independent vectors, and where x_1, x_2, x_3 are coordinates referred to a set of axes along $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Then

$$\nabla' = \mathbf{b}\times\mathbf{c}\frac{\partial}{\partial x_1} + \mathbf{c}\times\mathbf{a}\frac{\partial}{\partial x_2} + \mathbf{a}\times\mathbf{b}\frac{\partial}{\partial x_3}\cdot \tag{41}$$

Now ∇' may be regarded as a 2-vector operator in the same sense as ∇ is regarded as a 1-vector. To show the relation of ∇' to ∇, when the ideas of perpendicularity are assumed, we may take $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_4$ and x_3 as x_4. Then

$$\nabla' = \mathbf{k}_{21}\frac{\partial}{\partial x_1} + \mathbf{k}_{41}\frac{\partial}{\partial x_2} + \mathbf{k}_{12}\frac{\partial}{\partial x_4} = \left(\mathbf{k}_1\frac{\partial}{\partial x_1} + \mathbf{k}_2\frac{\partial}{\partial x_2} - \mathbf{k}_4\frac{\partial}{\partial x_4}\right)^*.$$

Thus ∇' is the complement ∇^* of ∇. In fact if

$$(dF)^* = d\mathbf{r}\times\nabla' F \qquad \text{and} \qquad dF = d\mathbf{r}\cdot\nabla F,$$

our rule of operation (30) shows that $\nabla' = \nabla^*$.

If we consider a field of 1-vectors, that is, a 1-vector function **f** of position in space, we are naturally led to enquire what meaning, if any, should be associated with the formal combinations

$$\nabla \cdot \mathbf{f} \quad \text{and} \quad \nabla \times \mathbf{f}$$

obtained by operating with the 1-vector ∇. Let

$$\mathbf{f}(x_1, x_2, x_4) = f_1 \mathbf{k}_1 + f_2 \mathbf{k}_2 + f_4 \mathbf{k}_4.$$

Then

$$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_4}{\partial x_4},$$

$$\nabla \times \mathbf{f} = \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) \mathbf{k}_{12} + \left(\frac{\partial f_4}{\partial x_1} + \frac{\partial f_1}{\partial x_4}\right) \mathbf{k}_{14} + \left(\frac{\partial f_4}{\partial x_2} + \frac{\partial f_2}{\partial x_4}\right) \mathbf{k}_{24}.$$

Of these the first, $\nabla \cdot \mathbf{f}$, is a scalar function of position, and the second, $\nabla \times \mathbf{f}$, is a 2-vector function of position. They correspond respectively to the divergence and curl in Euclidean three dimensional space. The first, $\nabla \cdot \mathbf{f}$, has indeed the same form as usual. And this was to be expected: for physically or geometrically the idea of divergence depends on translation alone and not on rotation, and it would also have appeared analytically evident if we had used in the definition of divergence the operator ∇^* instead of ∇. The second, $\nabla \times \mathbf{f}$, differs from the ordinary curl not only in that we have retained it as a 2-vector (instead of replacing it by the 1-vector, its complement, as is usually done in Euclidean geometry of three dimensions), but also in that it represents non-Euclidean rotation in the vector field in the same sense that the curl represents ordinary rotation.

If F is a scalar function of position, then ∇F is a 1-vector function. We may then form

$$\nabla \cdot \nabla F, \quad \nabla \times \nabla F.$$

Of these the second, $\nabla \times \nabla F$, vanishes identically, as may be seen by its expansions or by regarding it as an outer product in which one vector is repeated. The first, $\nabla \cdot \nabla F$, may be expanded as

$$\nabla \cdot \nabla F = \frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial x_2^2} - \frac{\partial^2 F}{\partial x_4^2},$$

and $\nabla \cdot \nabla$ corresponds to Laplace's operator in Euclidean geometry.

If **f** is a 1-vector function, there are four different expressions which involve the operator ∇ twice, namely

$$\nabla \nabla \cdot \mathbf{f}, \qquad \nabla \cdot \nabla \mathbf{f}, \qquad \nabla \cdot \nabla \times \mathbf{f}, \qquad \nabla \times \nabla \times \mathbf{f}.$$

Of these the last is a 3-vector function, which clearly vanishes identically. The first three are 1-vector functions, and are connected by the relation

$$\nabla \cdot \nabla \times \mathbf{f} = \nabla (\nabla \cdot \mathbf{f}) - \nabla \cdot \nabla \mathbf{f},$$

as may be seen by expansion or by the application of (34).

Kinematics and Dynamics in a Plane.

35. The three dimensional non-Euclidean geometry which we have developed is adapted to the discussion of the kinematics and dynamics of a particle constrained to move in a plane. The two dimensions of space and the one of time constitute the three dimensions of our manifold. Any (γ)-plane in this manifold may be called space, and extension along the complementary (δ)-line may be called time. As in the simpler case, any (δ)-line represents the locus in time and space of an unaccelerated particle, and any (δ)-curve the locus of an accelerated particle. If we choose any two perpendicular axes x_1, x_2 of space, and the perpendicular time axis x_4, then if the locus of any particle is inclined at the non-Euclidean angle ϕ to the chosen time axis, the particle is said to be in motion with the velocity $\mathbf{v}$ of which the magnitude is $v = \tanh \phi$.

For the locus of a particle let

$$s = \int \sqrt{dx_4^2 - dx_1^2 - dx_2^2}$$

be the arc measured along the (δ)-curve, and let $\mathbf{r}$ be the radius vector from any origin to a point of the curve. Then the derivative of $\mathbf{r}$ by s is the unit tangent $\mathbf{w}$ to the curve. We have

$$\mathbf{w} = \mathbf{k}_1 \frac{dx_1}{ds} + \mathbf{k}_2 \frac{dx_2}{ds} + \mathbf{k}_4 \frac{dx_4}{ds}$$

If the velocity $\mathbf{v}$ is

$$\mathbf{v} = \mathbf{k}_1 \frac{dx_1}{dx_4} + \mathbf{k}_2 \frac{dx_2}{dx_4},$$

then since

$$\frac{dx_4}{ds} = \cosh \phi = \frac{1}{\sqrt{1 - v^2}},$$

we write [31]

$$\mathbf{w} = \frac{1}{\sqrt{1 - v^2}} \left(\mathbf{k}_1 \frac{dx_1}{dx_4} + \mathbf{k}_2 \frac{dx_2}{dx_4} + \mathbf{k}_4 \right) = \frac{\mathbf{v} + \mathbf{k}_4}{\sqrt{1 - v^2}}. \tag{42}$$

31 By a transformation to a new set of axes we may derive at once the general form of Einstein's equation for the addition of velocities.

To obtain the vector curvature of the locus we write

$$\mathbf{c} = \frac{d\mathbf{w}}{ds} = \frac{dx_4}{ds}\frac{d\mathbf{w}}{dx_4} = \frac{1}{1-v^2}\frac{d\mathbf{v}}{dx_4} + \frac{\mathbf{v}+\mathbf{k}_4}{(1-v^2)^2}v\frac{dv}{dx_4},$$

or

$$\mathbf{c} = \frac{1}{1-v^2}\frac{d\mathbf{v}}{dt} + \frac{\mathbf{v}+\mathbf{k}_4}{(1-v^2)^2}v\frac{dv}{dt}. \tag{43}$$

If $\mathbf{v}$ be written as $\mathbf{v} = v\mathbf{u}$, where $\mathbf{u}$ is a unit vector, the resolution of $\mathbf{c}$ into three mutually perpendicular components along $\mathbf{u}$, $\mathbf{k}_4$, and $d\mathbf{u}$ follows immediately:

$$\mathbf{c} = \frac{\mathbf{u}\dfrac{dv}{dt}}{(1-v^2)^2} + \frac{v\dfrac{d\mathbf{u}}{dt}}{1-v^2} + \frac{v\mathbf{k}_4\dfrac{dv}{dt}}{(1-v^2)^2}. \tag{44}$$

The magnitude of $\mathbf{c}$ is

$$\begin{aligned}\sqrt{\mathbf{c}\cdot\mathbf{c}} &= \left[\frac{\left(\dfrac{dv}{dt}\right)^2}{(1-v^2)^3} + \frac{v^2\dfrac{d\mathbf{u}}{dt}\cdot\dfrac{d\mathbf{u}}{dt}}{(1-v^2)^2}\right]^{\frac{1}{2}} \\ &= \frac{1}{1-v^2}\left[\dot{\mathbf{v}}\cdot\dot{\mathbf{v}} + \frac{1}{1-v^2}(\mathbf{v}\cdot\dot{\mathbf{v}})^2\right]^{\frac{1}{2}} \\ &= \frac{1}{(1-v^2)^{\frac{3}{2}}}[\dot{\mathbf{v}}\cdot\dot{\mathbf{v}} - (\mathbf{v}\times\dot{\mathbf{v}})\cdot(\mathbf{v}\times\dot{\mathbf{v}})]^{\frac{1}{2}}.\end{aligned} \tag{45}$$

In case the acceleration is along the line of motion, these expressions reduce to those previously found; the additional term is due to the acceleration normal to the line of motion.

36. Mass may now be introduced just as in the simpler case already discussed, and here likewise we are led to the equation

$$m = \frac{m_0}{\sqrt{1-v^2}}.$$

The extended momentum in this case is also $m_0\mathbf{w}$, that is,

$$m_0\mathbf{w} = m\mathbf{v} + m\mathbf{k}_4. \tag{46}$$

We may speak of $\mathbf{w}$ as the extended velocity, of $\mathbf{c}$ as the extended acceleration, and of $m_0\mathbf{c}$ as the extended force. It is to be noted that while ordinary momentum is the space component of extended momentum, ordinary velocity, acceleration, and force are not the space components of the corresponding extended vectors. Indeed the space component of the extended velocity is $\mathbf{v}/\sqrt{1-v^2}$. The ordinary force, measured as rate of change of momentum, is

$$\mathbf{f} = \frac{dm\mathbf{v}}{dt} = m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt} = \frac{m_0\mathbf{u}\dfrac{dv}{dt}}{(1-v^2)^{\frac{3}{2}}} + \frac{m_0 v\dfrac{d\mathbf{u}}{dt}}{(1-v^2)^{\frac{1}{2}}}, \qquad (47)$$

which is the space component of $m_0\mathbf{c}$ multiplied by $\sqrt{1-v^2}$.

It is evident that in our mechanics the equations

$$\mathbf{f} = \frac{dm\mathbf{v}}{dt} \qquad \text{and} \qquad \mathbf{f} = m\mathbf{a},$$

where $\mathbf{a} = d\mathbf{v}/dt$, are not equivalent, and it is the first of these which we have chosen as fundamental. This makes the mass a definite scalar property of the system. Those who have used the second of the equations have been led to the idea of a mass which is different in different directions, and indeed have introduced as the "longitudinal" and the "transverse" mass the coefficients

$$\frac{m_0}{(1-v^2)^{\frac{3}{2}}}, \qquad \frac{m_0}{(1-v^2)^{\frac{1}{2}}}$$

of the components of acceleration along the path and perpendicular to it, that is, of the longitudinal and transverse accelerations, which are respectively

$$\mathbf{u}\frac{dv}{dt}, \qquad v\frac{d\mathbf{u}}{dt}.$$

The disadvantages of this latter system are obvious.

An interesting case of planar motion is that under a force constant in magnitude and in direction, say $f_x = 0$, $f_y = -k$. The momentum in the x-direction is constant, that in the y-direction is equal to its initial value less kt. From these two equations the integration may be completed. Or, in place of the second, the fact that the increase in mass (that is, energy) is equal to the work done by the force, may be used to give a second equation. The trajectory of the particle is not a parabola, but a curve of the form $y + a = -b \cosh (cx - d)$, resembling a catenary.

The space-time locus of uniform circular motion is a helix

$$\mathbf{r} = a(\mathbf{k}_1 \cos nt + \mathbf{k}_2 \sin nt) + \mathbf{k}_4 t.$$

Then

$$m\mathbf{v} = man(-\,\mathbf{k}_1 \sin nt + \mathbf{k}_2 \cos nt) + m\mathbf{k}_4,$$

$$\mathbf{f} = \frac{dm\mathbf{v}}{dt} = -\,man^2(\mathbf{k}_1 \cos nt + \mathbf{k}_2 \sin nt) = -\,mn^2\mathbf{r}_s,$$

where $\mathbf{r}_s$ is the component of $\mathbf{r}$ on the two-dimensional "space." The force is directed toward the center, as usual. It may be observed that if in general the force is central, the moment of momentum is constant. For if

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{f}, \qquad \mathbf{r}_s\times\frac{d}{dt}(m\mathbf{v}) = \frac{d}{dt}(\mathbf{r}_s\times m\mathbf{v}) = \mathbf{r}_s\times\mathbf{f} = 0.$$

That the rate of change of moment of momentum is equal to the moment of the force is therefore a principle which holds in non-Newtonian as in ordinary mechanics.

The Non-Euclidean Geometry in Four Dimensions.

Geometry and Vector Algebra.

37. Consider now a space of four dimensions in which the elements are points, lines, planes, flat 3-spaces or planoids, and which is subject to the same rules of translation or parallel-transformation as two or three dimensional space. If $\mathbf{a}$ and $\mathbf{b}$ are two 1-vectors, the product $\mathbf{a}\times\mathbf{b}$ is a 2-vector, that is, the parallelogram determined by the vectors. The parallelograms $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$ will be taken as of opposite sign, but otherwise equal. The equation $\mathbf{a}\times\mathbf{b} = 0$ expresses the condition that $\mathbf{a}$ and $\mathbf{b}$ are parallel. If $\mathbf{c}$ is any third 1-vector, not lying in the plane of $\mathbf{a}$ and $\mathbf{b}$, the product $\mathbf{a}\times\mathbf{b}\times\mathbf{c}$, which is now itself a vector will represent the parallelepiped determined by the three vectors. The condition $\mathbf{a}\times\mathbf{b}\times\mathbf{c} = 0$ therefore states that the three 1-vectors lie in a plane. If now $\mathbf{d}$ is a fourth 1-vector, not lying in the 3-space or planoid determined by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, the product $\mathbf{a}\times\mathbf{b}\times\mathbf{c}\times\mathbf{d}$ will represent the four dimensional parallel figure determined by the vectors. This product is a pseudo-scalar of which the magnitude is the four dimensional content of the parallel figure. The condition $\mathbf{a}\times\mathbf{b}\times\mathbf{c}\times\mathbf{d} = 0$ shows that the four vectors lie in some planoid. In all these outer products the sign is changed by the interchange of two adjacent factors, as in the case of lower dimensions. Moreover, the associative law, the distributive law, and the law of association for scalar factors will also hold, as is evident from their geometrical interpretation.

Two 1-vectors are added in the ordinary way by the parallelogram law. The same is true of two 2-vectors if they intersect in a line, that is, if they lie in the same 3-space (§ 25). It is, however, clear that in four dimensional space it is possible to have two parallelograms which have a common vertex but which do not lie in any planoid, that is, do not intersect in a line. For two such 2-vectors the construction previously given for the sum is not applicable, and it is indeed impossible to replace the sum of the two 2-vectors by a single plane vector. The sum may, however, be replaced in an infinite variety of ways by the sum of two other 2-vectors. For if **A** and **B** are any two 2-vectors, and if **a** and **b** be two 1-vectors drawn respectively in the planes of **A** and **B**, then the 2-vector $\mathbf{a}\cdot\mathbf{b} = \mathbf{C}$ may be added or subtracted from **A** and **B** so that

$$\mathbf{A} + \mathbf{B} = (\mathbf{A} + \mathbf{C}) + (\mathbf{B} - \mathbf{C}) = \mathbf{A}' + \mathbf{B}'.$$

The sum of more than two 2-vectors can, however, always be reduced to a sum of two. For if three planes in four dimensional space pass through a point, at least two must intersect in a line. A sum of 2-vectors, which is not reducible to a single uniplanar or simple 2-vector will be called a biplanar or double 2-vector whenever it is important to emphasize the difference. Since the analytical treatment of these two kinds of 2-vectors is not materially different, they will be designated by the same type of letters (clarendon capitals).

A vector of the type $\mathbf{a}\times\mathbf{b}\times\mathbf{c}$ will be called a 3-vector. As two planoids which have a point in common, intersect in a plane, a geometric construction for the sum of two 3-vectors may be given in a manner which is the immediate extension of the rule for 2-vectors in three dimensional space. The sum of two 3-vectors is always a simple 3-vector.

In respect to rotation and to the classification of lines, planes, and planoids, our four dimensional geometry will be non-Euclidean in such a manner as to be the natural extension of the non-Euclidean geometries of two and three dimensions which have been already discussed. As in two dimensions there were two fixed lines through a point, and in three dimensions a fixed cone, so in four dimensions there will be a fixed conical spread of three dimensions, or hypercone, which separates lines within the hypercone and called (δ)-lines, from lines outside the hypercone, which are called (γ)-lines. Besides the singular planes which are tangent to the hypercone, there are two classes of planes, namely, (δ)-planes which contain a (δ)-line, and (γ)-planes which contain no (δ)-line. Besides the singular planoids which are tangent to the hypercone, there are two classes of planoids, namely, (δ)-planoids which contain a (δ)-line, and (γ)-planoids which contain no (δ)-line. In the (γ)-planoids the geometry is the ordinary three dimensional Euclidean geometry; in the (δ)-planoids the geometry is that three dimensional non-Euclidean geometry which we have discussed at length.

Every (δ)-line determines a perpendicular planoid of class (γ), and every (γ)-line determines a perpendicular planoid of class (δ). Thus if we construct four mutually perpendicular lines, one will be a (δ)-line, and three will be (γ)-lines. A plane determined by one pair of these four mutually perpendicular lines is *completely* perpendicular to the plane determined by the other pair, in the sense that every line of one plane is perpendicular to every line of the other, and the planes therefore have no line in common. In general every plane determines uniquely a completely perpendicular plane. One of the planes is a (γ)-plane and the other is a (δ)-plane.

As in our previous geometries, perpendiculars remain perpendicular during rotation. If then in a rotation any plane remains fixed, its completely perpendicular plane will also remain fixed; and a general rotation may be regarded as the combination of a certain ordinary Euclidean rotation in a certain (γ)-plane, combined with a certain non-Euclidean rotation in the completely perpendicular (δ)-plane.

38. Let $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, $\mathbf{k}_4$ be four mutually perpendicular unit vectors of which the last is a (δ)-vector. The six coordinate 2-vectors may then be designated [32] as $\mathbf{k}_{14}$, $\mathbf{k}_{24}$, $\mathbf{k}_{34}$, $\mathbf{k}_{23}$, $\mathbf{k}_{31}$, $\mathbf{k}_{12}$. There are furthermore four coordinate unit 3-vectors $\mathbf{k}_{234}$, $\mathbf{k}_{314}$, $\mathbf{k}_{124}$, $\mathbf{k}_{123}$; and a unit pseudoscalar $\mathbf{k}_{1234}$. We may represent 1-vectors, 2-vectors and 3-vectors, as the sum of their projections on the coordinate axes, coordinate planes, and coordinate planoids. Thus

$$\mathbf{a} = a_1\mathbf{k}_1 + a_2\mathbf{k}_2 + a_3\mathbf{k}_3 + a_4\mathbf{k}_4,$$

$$\mathbf{A} = A_{14}\mathbf{k}_{14} + A_{24}\mathbf{k}_{24} + A_{34}\mathbf{k}_{34} + A_{23}\mathbf{k}_{23} + A_{31}\mathbf{k}_{31} + A_{12}\mathbf{k}_{12},$$

$$\boldsymbol{\mathfrak{A}} = \mathfrak{A}_{234}\mathbf{k}_{234} + \mathfrak{A}_{314}\mathbf{k}_{314} + \mathfrak{A}_{124}\mathbf{k}_{124} + \mathfrak{A}_{123}\mathbf{k}_{123}.$$

The outer product of any two vectors is defined geometrically and expressed analytically in a manner entirely analogous to that of the simpler cases already discussed. We thus obtain the following equations for the different types of products.

32 The particular order of subscripts is chosen for convenience only.

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} = (a_1b_4 - a_4b_1)\mathbf{k}_{14} + (a_2b_4 - a_4b_2)\mathbf{k}_{24} + (a_3b_4 - a_4b_3)\mathbf{k}_{34}$$
$$+ (a_2b_3 - a_3b_2)\mathbf{k}_{23} + (a_3b_1 - a_1b_3)\mathbf{k}_{31} + (a_1b_2 - a_2b_1)\mathbf{k}_{12},$$

$$\mathbf{a}\times\mathbf{A} = (a_2A_{34} - a_3A_{24} + a_4A_{23})\mathbf{k}_{234} + (a_3A_{14} - a_1A_{34} + a_4A_{31})\,\mathbf{k}_{314}$$
$$+ (a_1A_{24} - a_2A_{14} + a_4A_{12})\,\mathbf{k}_{124} + (a_1A_{23} + a_2A_{31} + a_3A_{12})\,\mathbf{k}_{123},$$

$$\mathbf{a}\times\mathfrak{A} = -\mathfrak{A}\times\mathbf{a} = (a_1\mathfrak{A}_{234} + a_2\mathfrak{A}_{314} + a_3\mathfrak{A}_{124} - a_4\mathfrak{A}_{123})\mathbf{k}_{1234},$$

$$\mathbf{A}\times\mathbf{B} = (A_{14}B_{23} + A_{24}B_{31} + A_{34}B_{12} + A_{23}B_{14} + A_{31}B_{24} + A_{12}B_{34})\,\mathbf{k}_{1234}.$$

The outer product of two vectors the sum of whose dimensions is greater than four vanishes. The outer product of a vector by itself vanishes except in the case of the biplanar or double 2-vector where the product becomes

$$\mathbf{A}\times\mathbf{A} = 2(A_{14}A_{23} + A_{24}A_{31} + A_{34}A_{12})\,\mathbf{k}_{1234}.$$

If the biplanar vector be written as $\mathbf{A} = \mathbf{B} + \mathbf{C}$, where $\mathbf{B}$ and $\mathbf{C}$ are two simple plane vectors, the product may be written

$$\mathbf{A}\times\mathbf{A} = (\mathbf{B} + \mathbf{C})\times(\mathbf{B} + \mathbf{C}) = 2\mathbf{B}\times\mathbf{C}.$$

It thus appears that $\mathbf{A}\times\mathbf{A}$ is twice the four dimensional parallelepiped constructed upon any pair of planes into which the double vector may be resolved. The vanishing of the outer product, $\mathbf{A}\times\mathbf{A} = 0$, is the necessary and sufficient condition that $\mathbf{A}$ be uniplanar.

The general rule for all cases of inner product has been stated (§ 29). We may tabulate the following cases.

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4,$$

$$\mathbf{a}\cdot\mathbf{A} = (a_2A_{12} - a_3A_{31} - a_4A_{14})\,\mathbf{k}_1 + (-a_1A_{12} + a_3A_{23} - a_4A_{24})\,\mathbf{k}_2$$
$$+ (a_1A_{31} - a_2A_{23} - a_4A_{34})\,\mathbf{k}_3 + (-a_1A_{14} - a_2A_{24} - a_3A_{34})\,\mathbf{k}_4,$$

$$\mathbf{a}\cdot\mathfrak{A} = (a_3\mathfrak{A}_{314} - a_2\mathfrak{A}_{124})\,\mathbf{k}_{14} + (a_1\mathfrak{A}_{124} - a_3\mathfrak{A}_{234})\,\mathbf{k}_{24}$$
$$+ (a_2\mathfrak{A}_{234} - a_1\mathfrak{A}_{314})\,\mathbf{k}_{34} + (a_1\mathfrak{A}_{123} - a_4\mathfrak{A}_{234})\,\mathbf{k}_{23}$$
$$+ (a_2\mathfrak{A}_{123} - a_4\mathfrak{A}_{314})\,\mathbf{k}_{31} + (a_3\mathfrak{A}_{123} - a_4\mathfrak{A}_{124})\,\mathbf{k}_{12},$$

$$\mathbf{A}\cdot\mathbf{B} = -A_{14}B_{14} - A_{24}B_{24} - A_{34}B_{34} + A_{23}B_{23} + A_{31}B_{31} + A_{12}B_{12},$$

$$\mathbf{A}\cdot\mathfrak{A} = (-A_{24}\mathfrak{A}_{124} + A_{34}\mathfrak{A}_{314} + A_{23}\mathfrak{A}_{123})\,\mathbf{k}_1 + (A_{14}\mathfrak{A}_{124} - A_{34}\mathfrak{A}_{234}$$
$$+ A_{31}\mathfrak{A}_{123})\,\mathbf{k}_2 + (-A_{14}\mathfrak{A}_{314} + A_{24}\mathfrak{A}_{234} + A_{12}\mathfrak{A}_{123})\,\mathbf{k}_3$$
$$+ (A_{23}\mathfrak{A}_{234} + A_{31}\mathfrak{A}_{314} + A_{12}\mathfrak{A}_{124})\,\mathbf{k}_4,$$

$$\mathfrak{A}\cdot\mathfrak{B} = -\mathfrak{A}_{234}\mathfrak{B}_{234} - \mathfrak{A}_{314}\mathfrak{B}_{314} - \mathfrak{A}_{124}\mathfrak{B}_{124} + \mathfrak{A}_{123}\mathfrak{B}_{123}.$$

The geometrical interpretation of these inner products follows the same lines as before. The inner product of a vector into a vector of equal dimensions is a scalar, and is the product of either into the projection of the other upon it. In the case where a biplanar 2-vector is projected, or is projected upon, each simple plane has to be treated, and the results compounded. That this may be done follows at once from the distributive law. The product of two vectors of different dimensionality is a vector of which the dimension is the difference of the dimensions of the factors; this vector lies in the factor of larger dimensions and is perpendicular to the factor of smaller dimensions. However, the product $\mathbf{a} \cdot \mathbf{A}$, if $\mathbf{A}$ is biplanar, is compounded of two 1-vectors lying in the two component planes.

The complement of a vector is again defined as its inner product with the unit pseudo-scalar $\mathbf{k}_{1234}$. The complement of a 1-vector is a perpendicular 3-vector, and vice-versa; that of a simple 2-vector is the completely perpendicular 2-vector. We may tabulate the results for the unit vectors.

$$\mathbf{k}_1{}^* = -\mathbf{k}_{234}, \qquad \mathbf{k}_2{}^* = -\mathbf{k}_{314}, \qquad \mathbf{k}_3{}^* = -\mathbf{k}_{124}, \qquad \mathbf{k}_4{}^* = -\mathbf{k}_{123},$$
$$\mathbf{k}_{14}{}^* = -\mathbf{k}_{23}, \qquad \mathbf{k}_{24}{}^* = -\mathbf{k}_{31}, \qquad \mathbf{k}_{34}{}^* = -\mathbf{k}_{12},$$
$$\mathbf{k}_{23}{}^* = \mathbf{k}_{14}, \qquad \mathbf{k}_{31}{}^* = \mathbf{k}_{24}, \qquad \mathbf{k}_{12}{}^* = \mathbf{k}_{34},$$
$$\mathbf{k}_{234}{}^* = -\mathbf{k}_1, \qquad \mathbf{k}_{314}{}^* = -\mathbf{k}_2, \qquad \mathbf{k}_{124}{}^* = -\mathbf{k}_3, \qquad \mathbf{k}_{123}{}^* = -\mathbf{k}_4.$$

With the aid of complements a unique resolution of a given 2-vector into two completely perpendicular parts may be accomplished. Suppose the resolution effected as

$$\mathbf{A} = m\mathbf{M} + n\mathbf{N}$$

where $\mathbf{M}$ is a unit vector of class (γ) and $\mathbf{N}$ one of class (δ) so chosen that $\mathbf{M} \times \mathbf{N}$ is a positive unit pseudo-scalar. Then

$$\mathbf{A}^* = -n\mathbf{M} + m\mathbf{N},$$

and $$\mathbf{M} = \frac{m\mathbf{A} - n\mathbf{A}^*}{m^2 + n^2}, \qquad \mathbf{N} = \frac{n\mathbf{A} + m\mathbf{A}^*}{m^2 + n^2}.$$

Hence $$\mathbf{A} = \frac{m^2\mathbf{A} - mn\mathbf{A}^*}{m^2 + n^2} + \frac{n^2\mathbf{A} + nm\mathbf{A}^*}{m^2 + n^2}.$$

Let $$p = \mathbf{A} \cdot \mathbf{A} = m^2 - n^2, \qquad q = \mathbf{A} \cdot \mathbf{A}^* = -2mn.$$

The quantities m, n may then be expressed in terms of p, q, that is, in terms of $\mathbf{A} \cdot \mathbf{A}$, $\mathbf{A} \cdot \mathbf{A}^*$. The result is

$$\mathbf{A} = \tfrac{1}{2} \frac{(\sqrt{p^2 + q^2} + p)\,\mathbf{A} + q\mathbf{A}^*}{\sqrt{p^2 + q^2}} + \tfrac{1}{2} \frac{(\sqrt{p^2 + q^2} - p)\,\mathbf{A} - q\mathbf{A}^*}{\sqrt{p^2 + q^2}}.$$

The general relationships between products of vectors and their complements have been developed in a previous section for a space of any dimensions. It was there shown that (except 37) formulas (34)-(39) for the expansion of all types of products involving 1-vectors and 2-vectors would be true in higher dimensions, and this is true even if the 2-vectors involved happen to be biplanar, because any such vectors is the sum of two uniplanar vectors and the equations are linear or bilinear in the vectors. Similar equations may, if occasion requires, be developed for products involving 3-vectors.

39. We have not yet considered those vectors whose inner products with themselves are zero. The case of the 1-vector, which is an element of the hypercone, need not be treated again in detail. For such a vector

$$\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 - a_4^2 = 0.$$

A uniplanar 2-vector such that $\mathbf{A} \cdot \mathbf{A} = 0$ satisfies the conditions

$$\mathbf{A} \times \mathbf{A} = 2\,(A_{14}A_{23} + A_{24}A_{31} + A_{34}A_{12})\,\mathbf{k}_{1234} = 0,$$

$$\mathbf{A} \cdot \mathbf{A} = -A_{14}^2 - A_{24}^2 - A_{34}^2 + A_{23}^2 + A_{31}^2 + A_{12}^2 = 0.$$

Such a vector is obviously a plane tangent to the hypercone; for it can be neither a (γ)- nor a (δ)-plane. The singular plane has the same properties as in three dimensional space. The element of tangency may be found as follows. If $\mathbf{a}$ is any vector, $\mathbf{a} \cdot \mathbf{A}$ is a line in the plane $\mathbf{A}$, and $(\mathbf{a} \cdot \mathbf{A}) \cdot \mathbf{A}$ is a perpendicular line of the plane. But the only line which is perpendicular to another line in this peculiar two dimensional space is the singular line, that is, the element of tangency with the hypercone. If $\mathbf{k}_4$ be taken as $\mathbf{a}$, the element may be written as

$$(\mathbf{k}_4 \cdot \mathbf{A}) \cdot \mathbf{A} = \mathbf{k}_1\,(A_{34}A_{31} - A_{24}A_{12}) + \mathbf{k}_2\,(A_{14}A_{12} - A_{34}A_{23})$$
$$+ \mathbf{k}_3\,(A_{24}A_{23} - A_{14}A_{31}) + \mathbf{k}_4\,(A_{14}^2 + A_{24}^2 + A_{34}^2),$$

an equation which we shall find serviceable.

The complement of a uniplanar singular 2-vector is itself such a vector, and it may readily be shown to pass through the same element of tangency. Indeed through every element of the hypercone is a whole single infinity of tangent planes which are mutually complementary in pairs.

If a 2-vector be biplanar, that is, if $\mathbf{A} \times \mathbf{A}$ is not zero, the condition $\mathbf{A} \cdot \mathbf{A} = 0$ is satisfied when, if the vector be resolved into the two complementary (γ)- and (δ)-vectors, these have the same magnitude. For if

$$\mathbf{A} = m\mathbf{M} + n\mathbf{N}, \qquad \mathbf{A} \cdot \mathbf{A} = m^2 - n^2.$$

Such a vector is singular only in an analytical sense.

The complement of a singular 1-vector is a 3-vector which itself is evidently singular. It is the planoid tangent to the hypercone through the given element.[33] It contains, besides the pencil of singular planes through the element of tangency, only (γ)-planes.

We may take this opportunity of summarizing the properties of singular vectors in general. The inner product of any singular vector by itself is 0. Every singular vector is perpendicular to itself and to every singular vector lying within it. The magnitude of a singular vector is zero. This does not imply that such a vector is not a definite geometric object, but only that the interval of a singular 1-vector, the area of a singular 2-vector, and the volume of a singular 3-vector are zero when compared with non-singular intervals, areas, and volumes.

The visualization of the geometrical properties of a four dimensional and especially of a non-Euclidean four dimensional geometry is extremely difficult. It is of course possible to rely wholly on the analytic relations, and thus avoid the difficulty. But we believe that it is of the greatest importance to realize that we are dealing with perfectly definite geometrical objects which are independent of any arbitrary axes of reference, and that it is therefore advisable to make every effort toward the visualization. It seems probable that Minkowski, although he employed chiefly the analytical point of view in his great memoir, must himself have largely employed the geometrical method in his thinking.

The Differentiating Operator $\diamond$.

40. By analogy we may in four dimensions define the operator $\diamond$, called quad, by the equation

$$d(\ ) = d\mathbf{r}\cdot\diamond(\ ). \tag{48}$$

When referred to a set of perpendicular axes, quad takes the form

$$\diamond = \mathbf{k}_1 \frac{\partial}{\partial x_1} + \mathbf{k}_2 \frac{\partial}{\partial x_2} + \mathbf{k}_3 \frac{\partial}{\partial x_3} - \mathbf{k}_4 \frac{\partial}{\partial x_4}, \tag{49}$$

and like ∇ it may be regarded formally as a 1-vector.

[33] The geometry in a singular planoid is analogous to that in a singular plane (§ 31). In this 3-space there are two classes of lines, singular lines, all of which are parallel to each other, and non-singular lines, (γ)-lines, all of which are perpendicular to the singular lines. Similarly there are two classes of planes, singular planes, all of which are parallel to the singular lines, and non-singular (γ)-planes, which are perpendicular to every singular plane. Volumes are comparable with one another but are all of zero magnitude as compared with a volume in any non-singular planoid.

We may therefore write the following equations. The result of applying $\diamondsuit$ to a scalar function F is a 1-vector $\diamondsuit F$, which might be called the gradient of F.

$$\diamondsuit F = \mathbf{k}_1 \frac{\partial F}{\partial x_1} + \mathbf{k}_2 \frac{\partial F}{\partial x_2} + \mathbf{k}_3 \frac{\partial F}{\partial x_3} - \mathbf{k}_4 \frac{\partial F}{\partial x_4}\cdot$$

The application of $\diamondsuit$ to a 1-vector function $\mathbf{f}$ by inner multiplication is a scalar, which might be called the divergence of $\mathbf{f}$.

$$\diamondsuit \cdot \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} + \frac{\partial f_4}{\partial x_4}\cdot$$

The application of $\diamondsuit$, by outer multiplication, to the 1-vector $\mathbf{f}$ is a 2-vector function, which might be called the curl of $\mathbf{f}$.

$$\begin{aligned}\diamondsuit \times \mathbf{f} = {} & \left(\frac{\partial f_4}{\partial x_1} + \frac{\partial f_1}{\partial x_4}\right)\mathbf{k}_{14} + \left(\frac{\partial f_4}{\partial x_2} + \frac{\partial f_2}{\partial x_4}\right)\mathbf{k}_{24} + \left(\frac{\partial f_4}{\partial x_3} + \frac{\partial f_3}{\partial x_4}\right)\mathbf{k}_{34} \\ & + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)\mathbf{k}_{23} + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)\mathbf{k}_{31} + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\mathbf{k}_{12}.\end{aligned}$$

The expression $\diamondsuit \cdot \mathbf{F}$ is a 1-vector.

$$\begin{aligned}\diamondsuit \cdot \mathbf{F} = {} & \left(\frac{\partial f_{12}}{\partial x_2} - \frac{\partial f_{31}}{\partial x_3} + \frac{\partial f_{14}}{\partial x_4}\right)\mathbf{k}_1 + \left(\frac{\partial f_{23}}{\partial x_3} - \frac{\partial f_{12}}{\partial x_1} + \frac{\partial f_{24}}{\partial x_4}\right)\mathbf{k}_2 \\ & + \left(\frac{\partial f_{31}}{\partial x_1} - \frac{\partial f_{23}}{\partial x_2} + \frac{\partial f_{34}}{\partial x_4}\right)\mathbf{k}_3 - \left(\frac{\partial f_{14}}{\partial x_1} + \frac{\partial f_{24}}{\partial x_2} + \frac{\partial f_{34}}{\partial x_3}\right)\mathbf{k}_4.\end{aligned}$$

The product $\diamondsuit \times \mathbf{F}$ is a 3-vector.

$$\begin{aligned}\diamondsuit \times \mathbf{F} = {} & \left(\frac{\partial f_{34}}{\partial x_2} - \frac{\partial f_{24}}{\partial x_3} - \frac{\partial f_{23}}{\partial x_4}\right)\mathbf{k}_{234} + \left(\frac{\partial f_{14}}{\partial x_3} - \frac{\partial f_{34}}{\partial x_1} - \frac{\partial f_{31}}{\partial x_4}\right)\mathbf{k}_{314} \\ & + \left(\frac{\partial f_{24}}{\partial x_1} - \frac{\partial f_{14}}{\partial x_2} - \frac{\partial f_{12}}{\partial x_4}\right)\mathbf{k}_{124} + \left(\frac{\partial f_{23}}{\partial x_1} + \frac{\partial f_{31}}{\partial x_2} + \frac{\partial f_{12}}{\partial x_3}\right)\mathbf{k}_{123}.\end{aligned}$$

We might likewise expand $\diamondsuit \cdot \mathfrak{F}$ and $\diamondsuit \times \mathfrak{F}$.

The rules (30) and (24) for operation with the complement enable us to write

$$(\diamondsuit \cdot a)^* = -\diamondsuit \times a^*, \qquad (\diamondsuit \times a)^* = \diamondsuit \cdot a^*,$$

when a is a vector function of any dimensionality in four dimensional space.

It is important to note in all these equations that while quad operates as a 1-vector, it is not a 1-vector in any geometrical sense.

Thus we find, for example, that $\diamondsuit\times\mathbf{f}$ is not always a plane passing through **f**, and in fact will usually be a biplanar vector. Also $\diamondsuit\cdot\mathbf{F}$ is not necessarily in the plane of **F**.

We have used the same symbol $\diamondsuit$ for our differential operator as was used by Lewis in his discussion of the vector analysis of four dimensional Euclidean space, and which corresponded to the "lor" of Minkowski. There seems no danger of confusion, since it will never be desirable to work simultaneously in Euclidean and non-Euclidean geometry. Sommerfeld[34] has also developed a vector analysis of essentially Euclidean four dimensional space, and his notation is an extension of that current in Germany for the three dimensional case. For the sake of reference we will compare the two notations, as far as the differential operator is concerned, in the following table.

$$\diamondsuit F \backsim \mathrm{Grad}\, F,$$
$$\diamondsuit\cdot\mathbf{f} \backsim \mathrm{Div}\,\mathbf{f},$$
$$\diamondsuit\times\mathbf{f} \backsim \mathrm{Rot}\,\mathbf{f},$$
$$\diamondsuit\cdot\mathbf{F} \backsim \mathfrak{Div}\,\mathbf{F}.$$

Operations involving $\diamondsuit$ twice are of frequent use in a number of important equations. These may be obtained by rules already given if $\diamondsuit$ be regarded as a 1-vector.

$$\diamondsuit\times(\diamondsuit F) = 0, \qquad (50) \qquad\qquad \diamondsuit\times(\diamondsuit\times\mathbf{f}) = 0, \qquad (51)$$
$$\diamondsuit\cdot(\diamondsuit\cdot\mathbf{F}) = 0, \qquad (52) \qquad\qquad \diamondsuit\times(\diamondsuit\times\mathbf{F}) = 0, \qquad (53)$$
$$\diamondsuit\cdot(\diamondsuit\cdot\mathfrak{F}) = 0, \qquad (54)$$
$$\diamondsuit\cdot(\diamondsuit\times\mathbf{f}) = \diamondsuit(\diamondsuit\cdot\mathbf{f}) - (\diamondsuit\cdot\diamondsuit)\,\mathbf{f}, \qquad (55)$$
$$\diamondsuit\cdot(\diamondsuit\times\mathbf{F}) = \diamondsuit\times(\diamondsuit\cdot\mathbf{F}) + (\diamondsuit\cdot\diamondsuit)\,\mathbf{F},\text{[35]} \qquad (56)$$
$$\diamondsuit\cdot(\diamondsuit\times\mathfrak{F}) = \diamondsuit\times(\diamondsuit\cdot\mathfrak{F}) - (\diamondsuit\cdot\diamondsuit)\,\mathfrak{F}. \qquad (57)$$

The important operator $\diamondsuit\cdot\diamondsuit$ or $\diamondsuit^2$ has sometimes been called the D'Alembertian. In the expanded form it is

$$\diamondsuit^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2} = \nabla^2 - \frac{\partial^2}{\partial x_4^2}, \qquad (58)$$

where ∇ now denotes the Euclidean differentiating operator in the $\mathbf{k}_{123}$ space.

34 Sommerfeld, Ann. d. Physik [4] **33**, 649.
35 Kraft (Bull. Acad. Cracovie A, 1911, p. 538) devotes a paper to the proof and application of this formula.

41. In the ordinary integral calculus of vectors the theorems due to Gauss and Stokes play an important rôle. In our notation we may express these laws with great simplicity and generalize them to a space of any dimensions. Let us consider first the form of these theorems in the case of two dimensions, beginning with the more familiar Euclidean case.

Stokes's theorem states that the line integral of a vector function $\mathbf{f}$ around a closed path is equal to the integral of the curl of $\mathbf{f}$ over the area bounded by the curve. The analytic statement is

$$\int d\mathbf{s}\cdot\mathbf{f} = \int\int dS \, \mathrm{curl}\, \mathbf{f},$$

where $d\mathbf{s}$ is the vector element of arc, and dS the scalar element of area. In our notation [36] this becomes

$$\int d\mathbf{s}\cdot\mathbf{f} = \int\int d\mathbf{S}\cdot \nabla\times\mathbf{f},$$

where $d\mathbf{S}$ is now the 2-vector element of area (a pseudo-scalar) and $\nabla\times\mathbf{f}$ is a pseudo-scalar (the complement of curl $\mathbf{f}$, which itself is a scalar in the two dimensional case). Transforming by (35), we may also write

$$\int d\mathbf{s}\cdot\mathbf{f} = -\int\int (d\mathbf{S}\cdot \nabla)\cdot\mathbf{f}. \tag{59}$$

Gauss's theorem states that the integral of the flux of a vector through a closed curve is equal to the integral of the divergence of the vector $\mathbf{f}$ over the area bounded by the curve. The analytic statement is

$$\int f_n ds = \int\int dS \, \mathrm{div}\, \mathbf{f},$$

where f_n is the component of $\mathbf{f}$ normal to the curve. In our notation this becomes

$$-\int (d\mathbf{s}\times\mathbf{f})^* = \int\int dS \;\nabla\cdot\mathbf{f} = \int\int d\mathbf{S}^*\nabla\cdot\mathbf{f},$$

or, by taking the complement of both sides,

$$-\int d\mathbf{s}\times\mathbf{f} = \int\int d\mathbf{S}\nabla\cdot\mathbf{f};$$

36 One of the advantages of our system of notation is that if one term in an equation is a vector of p dimensions, every other term is a vector of p dimensions. This furnishes at once a check on the correctness of any equation.

and transforming by (36), where in two dimensions $\mathbf{f}\times d\mathbf{S}$ vanishes, we obtain the form

$$\int d\mathbf{s}\times\mathbf{f} = -\int\int (d\mathbf{S}\cdot\nabla)\times\mathbf{f}. \tag{60}$$

Equations (59) and (60) can be combined into the operational equation

$$\int d\mathbf{s}\,(\ ) = -\int\int (d\mathbf{S}\cdot\nabla)\,(\ ), \tag{61}$$

where the operators may be applied to $\mathbf{f}$ in either inner or outer multiplication.

In three dimensions Stokes's theorem states that the line integral of a vector around a curve is equal to the surface integral of the normal component of the curl of the vector over any surface spanning the curve, with proper regard to sign. The ordinary statement is

$$\int d\mathbf{s}\cdot\mathbf{f} = \int\int dS\,(\text{curl }\mathbf{f})_n,$$

which in our notation becomes

$$\int d\mathbf{s}\cdot\mathbf{f} = \int\int d\mathbf{S}\cdot(\nabla\cdot\mathbf{f});$$

and may be transformed by (35) into

$$\int d\mathbf{s}\cdot\mathbf{f} = -\int\int (d\mathbf{S}\cdot\nabla)\cdot\mathbf{f}. \tag{62}$$

In like manner Gauss's theorem states that the integral of the flux of a vector through a closed surface is equal to the integral of the divergence of the vector over the volume inclosed by the surface. Thus, if $d\mathfrak{S}$ is the scalar element of volume,

$$\int\int \mathbf{f}_n dS = \int\int\int \text{div}\,\mathbf{f}\,d\mathfrak{S}.$$

In our notation, if $d\boldsymbol{\mathfrak{S}}$ denotes vector element of volume, this becomes

$$\int\int (d\mathbf{S}\times\mathbf{f})^* = \int\int\int d\mathfrak{S}\nabla\cdot\mathbf{f} = \int\int\int d\boldsymbol{\mathfrak{S}}^*\nabla\cdot\mathbf{f},$$

which, by transformation by (24) and (32), becomes

$$\int\int d\mathbf{S}\times\mathbf{f} = \int\int\int (d\boldsymbol{\mathfrak{S}}\cdot\nabla)\times\mathbf{f}. \tag{63}$$

As an example of a similar formula involving a scalar function f, we may take the familiar theorem of hydro lynamics that the surface integral of the pressure is equal to the volume integral of the gradient of the pressure f. This is usually written as

$$\int\int f\mathbf{n}dS = \int\int\int \operatorname{grad} f\, d\mathfrak{S},$$

but in our notation becomes

$$\int\int d\mathbf{s} f = \int\int\int d\mathbf{\mathfrak{S}}\cdot(\nabla f) = \int\int\int (d\mathbf{\mathfrak{S}}\cdot\nabla) f.$$

42. All these formulas lead us to suspect the existence of a single operational equation which is valid when applied to scalar functions and to any vector functions whether with the symbol $(\cdot)$ or $(\times)$. This would have the form

$$\int_{(p)} d\sigma_p(\,) = (-1)^p \int_{(p+1)} (d\sigma_{(p+1)}\cdot\Diamond)\,(\,), \tag{64}$$

where $d\sigma_p$ is the p-vector element of a closed spread bounding a spread of $p+1$ dimensions. We may extend this equation to four (or more) dimensions, and demonstrate its validity as follows.

It will perhaps be sufficient to give the proof of the formula in case the $(p+1)$-spread is a rectangular parallelepiped with $p+1$ pairs of opposite faces. For let

$$d\sigma_{(p+1)} = \mathbf{k}_{123\ldots p+1}\, dx_1 dx_2 dx_3 \ldots dx_{p+1}.$$

Then, by the rules for multiplication,

$$\int_{(p+1)} d\sigma_{(p+1)}\cdot\Diamond = (-1)^p \int_{(p+1)} \Big[dx_2 dx_3 \ldots dx_{p+1}\, \mathbf{k}_{23\ldots p+1} dx_1 \frac{\partial}{\partial x_1}$$
$$- dx_1 dx_3 \ldots dx_{p+1}\, \mathbf{k}_{13\ldots p+1}\, dx_2 \frac{\partial}{\partial x_2} + \ldots \Big].$$

The partial integrations may now be effected upon the right, and leave

$$\int_{(p+1)} d\sigma_{(p+1)}\cdot\Diamond = (-1)^p \int_{(p)} d\sigma_{(p)},$$

if it be remembered that $\mathbf{k}_{23\ldots p+1}$, $-\mathbf{k}_{13\ldots p+1}$, . . . are the positive faces perpendicular to $\mathbf{k}_1$, $\mathbf{k}_2$, . . .

It will be evident from this mode of proof that (64) is valid both for Euclidean and for our non-Euclidean geometry. The equation may be put in another form by the aid of rules previously given.[37]

$$\int_{(p)} d\sigma_p{}^*() = \int_{(p+1)} d\sigma_{(p+1)}{}^* \times \Diamond\,(). \tag{65}$$

In four dimensions a large number of special formulas may be obtained by applying our operational equation to scalars and to vectors of any denomination with either symbol of multiplication. As examples we may write the formulas corresponding to Stokes's and Gauss's theorems. Let $p = 1$ and apply the operator by inner multiplication to a 1-vector function. Then

$$\int d\mathbf{s}\cdot\mathbf{f} = -\int\!\!\int (d\mathbf{S}\cdot\Diamond)\cdot\mathbf{f} = \int\!\!\int d\mathbf{S}\cdot(\Diamond\times\mathbf{f}).$$

This is the extended Stokes's theorem. Again let $p = 3$ and apply the operator by outer multiplication to a 1-vector function. Then

$$\int\!\!\int\!\!\int d\mathfrak{S}\times\mathbf{f} = -\int\!\!\int\!\!\int\!\!\int (d\Sigma\cdot\Diamond)\times\mathbf{f} = -\int\!\!\int\!\!\int\!\!\int d\Sigma(\Diamond\cdot\mathbf{f}).$$

This is the extended Gauss's theorem, where $d\Sigma$ represents a differential (pseudo-scalar) element of four dimensional volume.

In these cases also the same equations apply in Euclidean and in our non-Euclidean space. If, however, we write these two equations in non-vectorial form, they become in the non-Euclidean case

$$\begin{aligned}\int (f_1dx_1 + f_2dx_2 + f_3dx_3 - f_4dx_4) \\ = \int\!\!\int \Big[\left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2dx_3 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) dx_3dx_1 \\ + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1dx_2 - \left(\frac{\partial f_4}{\partial x_1} + \frac{\partial f_1}{\partial x_4}\right) dx_1dx_4 \\ - \left(\frac{\partial f_4}{\partial x_2} + \frac{\partial f_2}{\partial x_4}\right) dx_2dx_4 - \left(\frac{\partial f_4}{\partial x_3} + \frac{\partial f_3}{\partial x_4}\right) dx_3dx_4 \Big]\end{aligned}$$

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37 This equation embraces both of the operational equations given by Gibbs in §§ 164–5 of his pamphlet Vector Analysis (1884) reprinted in his Scientific Papers, **2**. In case $p + 1$ is equal to n, the number of dimensions of space, then $d\sigma_{(p+1)}{}^*$ is a scalar and the equation has no meaning unless we adopt the convention $m\times\alpha = m\alpha$, where m is a scalar and α any vector. This convention would lead to no contradiction, and might occasionally be useful.

and

$$\int\int\int\int (f_1 dx_2 dx_3 dx_4 + f_2\, dx_3 dx_1 dx_4 + f_3 dx_1 dx_2 dx_4 - f_4 dx_1 dx_2 dx_3)$$
$$= \int\int\int\int \left[\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} + \frac{\partial f_4}{\partial x_4}\right] dx_1 dx_2 dx_3 dx_4.$$

The theorems may be used to demonstrate in a vectorial manner such an equation as (52), $\diamond\cdot(\diamond\cdot\mathbf{F}) = 0$. For

$$\int\int\int\int d\Sigma \diamond\cdot(\diamond\cdot\mathbf{F}) = -\int\int\int d\mathfrak{S}\times(\diamond\cdot\mathbf{F})$$
$$= \int\int\int (d\mathfrak{S}\cdot\diamond)\times\mathbf{F} = \int\int d\mathbf{S}\times\mathbf{F}.$$

As the final integral extends over the *boundary* of the *closed* three dimensional spread which bounds the given region of four dimensions, the final integral vanishes, since the closed spread has no boundary.

Geometric Vector Fields.

43. The idea of a vector field is ordinarily associated with concepts such as those of force or momentum, which are not wholly geometrical in character; but it is perfectly possible to construct vector fields which are purely geometrical. Thus in ordinary geometry we may derive a vector field, when a single point is given, by constructing at every other point the vector from that point to the given point, or that vector multiplied by any function of the distance.

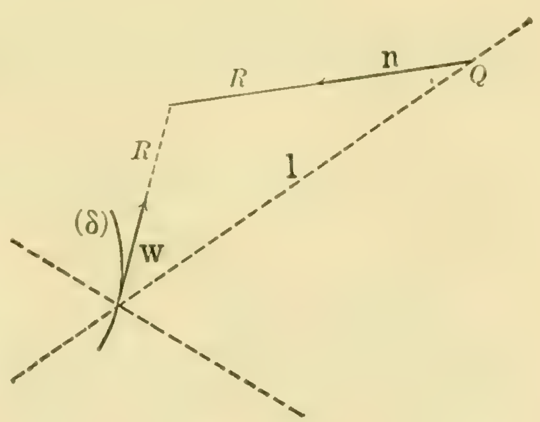

FIGURE 21.

In our non-Euclidean four dimensional space we may associate with any (δ)-curve a vector field derived from that curve in the following way. At each point of the (δ)-curve construct the forward unit tangent $\mathbf{w}$, and the forward hypercone.[38] At each point Q of these hypercones construct the vector $\mathbf{w}/R$, parallel to the vector

[38] That half of the hypercone lying above the origin, enclosing points which will represent later times than the time of the origin, will be called the forward hypercone.

w at the vertex, and equal in magnitude to the reciprocal of the interval R along the perpendicular drawn from the point Q to that tangent produced (Figure 21). On account of analogies which will soon become apparent we shall call this vector function the extended vector potential of the given (δ)-curve.[39] We shall write

$$\mathbf{p} = \frac{\mathbf{w}}{R}. \tag{66}$$

We shall next consider the 2-vector field

$$\mathbf{P} = \diamondsuit\times\mathbf{p} = \left(\diamondsuit\frac{1}{R}\right)\times\mathbf{w} + \frac{1}{R}(\diamondsuit\times\mathbf{w}). \tag{67}$$

We shall consider the evaluation of $\diamondsuit\times\mathbf{p}$ in two steps. First we shall assume that the original (δ)-curve is a straight line. In this case **w** is constant and $\diamondsuit\times\mathbf{w} = 0$. If we arbitrarily take $\mathbf{k}_4$ along **w**, we may write

$$\diamondsuit\frac{1}{R} = \nabla\frac{1}{R} - \mathbf{k}_4\frac{\partial}{\partial x_4}\frac{1}{R} = \nabla\frac{1}{R};$$

for it is clear that a displacement parallel to **w** does not change R. It is evident that R becomes a radius vector in the 3-space perpendicular to **w**. If **n** represents a unit vector from the point Q normal to **w**, that is, in the direction in which R was measured, then by the well known formula, $\nabla R^{-1} = \mathbf{n}/R^2$. Hence

$$\diamondsuit\frac{1}{R} = \frac{\mathbf{n}}{R^2}.$$

And hence

$$\mathbf{P} = \diamondsuit\times\mathbf{p} = \frac{\mathbf{n}\times\mathbf{w}}{R^2}. \tag{68}$$

The determination of $\diamondsuit\cdot\mathbf{p}$ follows in precisely the same way; in each of the above formulas the symbol of inner multiplication will replace that of outer multiplication, and we find that

$$\diamondsuit\cdot\mathbf{p} = \frac{\mathbf{n}\cdot\mathbf{w}}{R^2} = 0, \tag{69}$$

for **n** is perpendicular to **w**.

Of all the geometrical vector fields which might have been constructed from a given (δ)-curve, we shall show later that those which we have just derived are the most fundamental (footnote § 44). The

39 The vector fields produced at a point by two or more (δ)-curves may be regarded as additive. The locus of all possible singular lines **1** drawn (as in Fig. 21) from (δ)-curves to a given point is the *backward* hypercone of which that point is the apex.

2-vector $\diamondsuit \times \mathbf{p}$ is a simple plane vector in the plane of the point Q and of $\mathbf{w}$. The 1-vector $\mathbf{p}$ has everywhere the direction of the fundamental vector $\mathbf{w}$; if $\mathbf{l}$ be the singular vector from the vertex of the cone to the point Q, the scalar product $\mathbf{l} \cdot \mathbf{p}$ is constant. In fact

$$\mathbf{p} = -\frac{\mathbf{w}}{\mathbf{l} \cdot \mathbf{w}}, \qquad \mathbf{P} = \frac{\mathbf{l} \times \mathbf{w}}{(\mathbf{l} \cdot \mathbf{w})^3} \tag{70}$$

are the expressions for the fields in terms of $\mathbf{l}$ and $\mathbf{w}$.

Let us now choose arbitrarily a time-axis $\mathbf{k}_4$, and then the perpendicular planoid is our three dimensional space. We may resolve our 1-vector and 2-vector fields as follows.

$$\begin{aligned} \mathbf{p} &= -\frac{\mathbf{w}}{\mathbf{l} \cdot \mathbf{w}} = -\frac{\mathbf{v} + \mathbf{k}_4}{(\mathbf{l}_s + l_4 \mathbf{k}_4) \cdot (\mathbf{v} + \mathbf{k}_4)}, \\ \mathbf{p} &= \mathbf{p}_s + p_4 \mathbf{k}_4 = \frac{\mathbf{v}}{l_4 - \mathbf{l}_s \cdot \mathbf{v}} + \frac{\mathbf{k}_4}{l_4 - \mathbf{l}_s \cdot \mathbf{v}}, \end{aligned} \tag{71}$$

where $\mathbf{l}_s$ and $\mathbf{p}_s$ are the space components of $\mathbf{l}$ and $\mathbf{p}$. As $\mathbf{l}$ is a singular vector, l_4 is equal to the magnitude of $\mathbf{l}_s$.

$$\begin{aligned} \mathbf{P} &= \frac{\mathbf{l} \times \mathbf{w}}{(\mathbf{l} \cdot \mathbf{w})^3} = -(1 - v^2) \frac{(\mathbf{l}_s + l_4 \mathbf{k}_4) \times (\mathbf{v} + \mathbf{k}_4)}{(l_4 - \mathbf{l}_s \cdot \mathbf{v})^3} \\ \mathbf{P} &= -\frac{(1 - v^2)\, \mathbf{l}_s \times \mathbf{v}}{(l_4 - \mathbf{l}_s \cdot \mathbf{v})^3} - \frac{(1 - v^2)\, (\mathbf{l}_s - l_4 \mathbf{v}) \times \mathbf{k}_4}{(l_4 - \mathbf{l}_s \cdot \mathbf{v})^3}. \end{aligned} \tag{72}$$

Of these two planes into which $\mathbf{P}$ is now resolved, the first lies in "space" and the second passes through the time axis and is perpendicular to "space."

We shall attempt to show with the aid of a diagram (Figure 22) the geometrical significance of the various terms which we have employed in the above formulas. The origin, that is, the vertex of the hypercone, is any chosen point O on the given (δ)-line $\mathbf{w}$. A point upon the forward hypercone is Q, and $\mathbf{l}$ is the element OQ. The unit vector $\mathbf{n}$ is drawn along QJ from Q towards and perpendicular to the vector $\mathbf{w}$. The intervals OJ and QJ are equal, and equal to $R = -\mathbf{l} \cdot \mathbf{w}$. The vector $\mathbf{p}$ drawn at Q parallel to $\mathbf{w}$ and of magnitude $1/R$ is the extended vector potential at Q due to $\mathbf{w}$. The 2-vector $\mathbf{P}$ lies in the plane OJQ, and is equal in magnitude to $1/R^2$. The arbitrarily chosen time-axis is $\mathbf{k}_4$, and on the planoid perpendicular to $\mathbf{k}_4$ (that is, on "space") the vector $\mathbf{l}$ projects into $\mathbf{l}_s = O'Q$. The intersection of the line of $\mathbf{w}$ with the planoid is G (the point of the line $\mathbf{w}$ which is simultaneous with Q). Similarly O' is the intersection of $\mathbf{k}_4$ with the planoid. The line $OO' = l_4$ represents the lapse of time between O and O'; and this is equal in magnitude to $O'Q$ or $\mathbf{l}_s$, the space component of $\mathbf{l}$. The interval $OG = l_4 \sqrt{1 - v^2}$ and the interval $O'G = l_4 v = l_s v$. The direction $\mathbf{w}$ projects into the direction $\mathbf{v}$. Hence as a vector, $O'G$ is equal to $l_s \mathbf{v}$. The quantity $\mathbf{l}_s \cdot \mathbf{v} = O'F$ may be obtained by

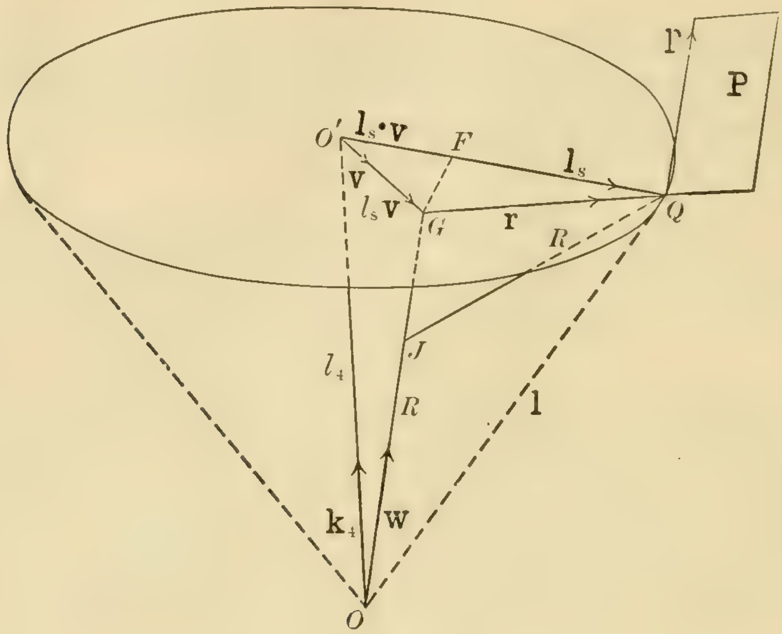

FIGURE 22.

dropping a perpendicular from G to $O'Q$. The interval FQ is then $l_s - \mathbf{l}_s \cdot \mathbf{v}$ or $l_4 - \mathbf{l}_s \cdot \mathbf{v}$, the expression which occurs in the denominators. The vector $GQ = \mathbf{r}$ is clearly $\mathbf{l}_s - l_s \mathbf{v}$ or $\mathbf{l}_s - l_4 \mathbf{v}$.

44. We shall now remove the restriction that the (δ)-curve which gives rise to the potential $\mathbf{p} = \mathbf{w}/R = -\mathbf{w}/(\mathbf{l} \cdot \mathbf{w})$ is rectilinear, and consider the general case of any (δ)-curve. For the sake of simplicity in this complex problem we shall use dyadic notation (see appendix § 61, ff.). The results, however, might all be obtained by means of the more elementary geometric and vector methods.

We may write

$$\Diamond \mathbf{p} = \Diamond \frac{\mathbf{w}}{R} = \left(\Diamond \frac{1}{R}\right)\mathbf{w} + \frac{1}{R} \Diamond \mathbf{w} = -\frac{1}{R^2} (\Diamond R)\, \mathbf{w} + \frac{1}{R} \Diamond \mathbf{w}.$$

Now $\Diamond \mathbf{w}$ is defined so as to satisfy the relation $d\mathbf{r} \cdot \Diamond \mathbf{w} = d\mathbf{w}$. A displacement (Figure 23) $d\mathbf{r} = \mathbf{w}\, ds$ parallel to $\mathbf{w}$, makes a change $d\mathbf{w} = \mathbf{c}\, ds$. A displacement $d\mathbf{r}$ along the vector $\mathbf{l}$ (Figure 24) introduces no change in **w**, and in like manner a displacement $d\mathbf{r}$ in the plane perpendicular to that of **w** and **1** does not affect **w**. Hence we may write

$$\Diamond\mathbf{w} = \frac{1}{\mathbf{1}\cdot\mathbf{w}}\mathbf{c} = -\frac{1}{R}\mathbf{1c}. \tag{73}$$

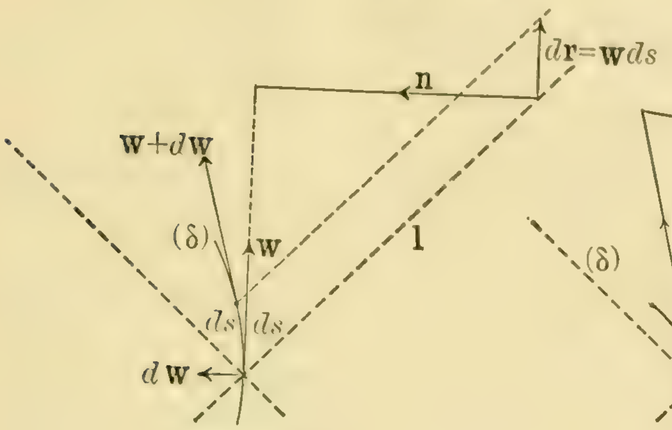

FIGURE 23.

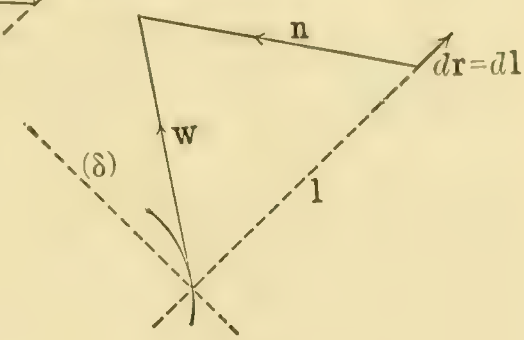

FIGURE 24.

To compute $\Diamond R = -\Diamond(\mathbf{1}\cdot\mathbf{w})$, we may write

$$\Diamond(\mathbf{1}\cdot\mathbf{w}) = (\Diamond\mathbf{1})\cdot\mathbf{w} + (\Diamond\mathbf{w})\cdot\mathbf{1}.$$

Here $\Diamond\mathbf{w}$ is already known. To find $\Diamond\mathbf{1}$ observe that $d\mathbf{1} = d\mathbf{r}\cdot\Diamond\mathbf{1}$ is equal to $d\mathbf{r}$ when $d\mathbf{r}$ is along **1** (Figure 24). Further if $d\mathbf{r}$ is elsewhere in the hypercone, for instance, in the plane perpendicular to that of **1** and **w** then also $d\mathbf{1} = d\mathbf{r}$. But when $d\mathbf{r} = \mathbf{w}ds$ is along **w** the differential $d\mathbf{1}$ vanishes. Hence we may write

$$\Diamond\mathbf{1} = \mathrm{I} - \frac{1}{\mathbf{1}\cdot\mathbf{w}}\mathbf{w} = \mathrm{I} + \frac{1}{R}\mathbf{1w}, \tag{74}$$

where I is the idemfactor. Thus we have

$$\Diamond(\mathbf{1}\cdot\mathbf{w}) = \left(\mathrm{I} + \frac{1}{R}\mathbf{1w}\right)\cdot\mathbf{w} - \frac{1}{R}\mathbf{1c}\cdot\mathbf{1},$$

or, performing the multiplication by **w**,

$$\Diamond R = -\Diamond(\mathbf{1}\cdot\mathbf{w}) = -\mathbf{w} + \frac{1 + \mathbf{1}\cdot\mathbf{c}}{R}\mathbf{1}. \tag{75}$$

From this it follows at once that

$$\begin{aligned}\Diamond\mathbf{p} &= -\frac{1}{R^2}(\Diamond R)\,\mathbf{w} + \frac{1}{R}\Diamond\mathbf{w} \\ &= -\frac{1}{R^2}\left(\mathbf{1c} + \frac{1 + \mathbf{1}\cdot\mathbf{c}}{R}\mathbf{1w} - \mathbf{ww}\right).\end{aligned} \tag{76}$$

The two expressions $\diamondsuit\times\mathbf{p}$ and $\diamondsuit\cdot\mathbf{p}$ may now be obtained by inserting the cross and dot in $\diamondsuit\mathbf{p}$. Hence

$$\diamondsuit\times\mathbf{p} = -\frac{1}{R^2}\left(\mathbf{l}\times\mathbf{c} + \frac{1+\mathbf{l}\cdot\mathbf{c}}{R}\,\mathbf{l}\times\mathbf{w}\right), \tag{77}$$

$$\diamondsuit\cdot\mathbf{p} = -\frac{1}{R^2}\left(\mathbf{l}\cdot\mathbf{c} + \frac{1+\mathbf{l}\cdot\mathbf{c}}{R}\,\mathbf{l}\cdot\mathbf{w} + 1\right) = 0. \tag{78}$$

Here also $\diamondsuit\cdot\mathbf{p}$ vanishes, since $\mathbf{l}\cdot\mathbf{w} = -R$.

As $\mathbf{l}$ varies with R, the parts of $\diamondsuit\times\mathbf{p}$ may be separated into one which varies as R^{-1} and one which varies as R^{-2}, namely.

$$\mathbf{P} = \diamondsuit\times\mathbf{p} = -\frac{1}{R^2}\left(\mathbf{l}\times\mathbf{c} + \frac{\mathbf{l}\cdot\mathbf{c}}{R}\,\mathbf{l}\times\mathbf{w}\right) - \frac{1}{R^3}\,\mathbf{l}\times\mathbf{w}. \tag{79}$$

This may be brought out most clearly by expressing $\mathbf{l}$ as

$$\mathbf{l} = R\,(\mathbf{w} - \mathbf{n}), \tag{80}$$

where $\mathbf{n}$ is a unit vector from Q perpendicular to $\mathbf{w}$.

$$\mathbf{P} = -\frac{1}{R}\,[\mathbf{w}\times\mathbf{c} - \mathbf{n}\times\mathbf{c} + \mathbf{n}\cdot\mathbf{c}\;\;\mathbf{n}\times\mathbf{w}] + \frac{1}{R^2}\,\mathbf{n}\times\mathbf{w}. \tag{81}$$

Another manner of expressing $\mathbf{P}$ is

$$\mathbf{P} = -\frac{1}{R^3}\;\mathbf{l}\times[\mathbf{l}\cdot(\mathbf{w}\times\mathbf{c})] - \frac{1}{R^3}\,\mathbf{l}\times\mathbf{w} \tag{82}$$

or

$$\mathbf{P} = -\frac{1}{R^3}\;(\mathbf{l}\times\mathbf{w}\times\mathbf{c})\cdot\mathbf{l} - \frac{1}{R^3}\,\mathbf{l}\times\mathbf{w}. \tag{83}$$

Any of these forms of $\mathbf{P}$ shows, what perhaps appears clearest from (82), that the part of $\mathbf{P}$ which varies inversely as R is a singular plane, through the element $\mathbf{l}$ and cutting the plane of $\mathbf{w}\times\mathbf{c}$; for $\mathbf{l}\times[\mathbf{l}\cdot(\mathbf{w}\times\mathbf{c})]$ is a plane through $\mathbf{l}$ and the vector $\mathbf{l}\cdot(\mathbf{w}\times\mathbf{c})$ (in $\mathbf{w}\times\mathbf{c}$), and the inner product of the plane by itself is readily shown to be zero.

In a similar manner we may calculate $\diamondsuit\mathbf{P}$, a dyadic with its first vectors 1-vectors and its second vectors 2-vectors. The differentiation requires nothing new except $\diamondsuit\mathbf{c}$. And by the same reasoning applied to find $\diamondsuit\mathbf{w}$, it appears that

$$\diamondsuit\,\mathbf{c} = \frac{1}{\mathbf{l}\cdot\mathbf{w}}\frac{d\mathbf{c}}{ds} = -\frac{1}{R}\frac{d\mathbf{c}}{ds}. \tag{84}$$

Hence $\diamond \mathbf{c}$ brings in, as might be expected, the rate of change of curvature, just as $\diamond \mathbf{w}$ brought in the curvature. We have

$$\diamond \mathbf{P} = \diamond (\diamond \times \mathbf{p}) = \diamond \left(-\frac{\mathbf{l} \times \mathbf{c}}{R^2} - \frac{1 + \mathbf{l} \cdot \mathbf{c}}{R^3}\, \mathbf{l} \times \mathbf{w} \right)$$

$$= \frac{2}{R^3}\left(-\mathbf{w} + \frac{1 + \mathbf{l} \cdot \mathbf{c}}{R}\, \mathbf{l} \right) \mathbf{l} \times \mathbf{c} - \frac{1}{R^2}\left(\mathrm{I} + \frac{1}{R}\, \mathbf{lw} \right) \times \mathbf{c} + \frac{1}{R^2}\left(-\frac{1}{R}\frac{d\mathbf{c}}{ds} \right) \times \mathbf{l}$$

$$+ \frac{3}{R^4}(1 + \mathbf{l} \cdot \mathbf{c}) \left(-\mathbf{w} + \frac{1 + \mathbf{l} \cdot \mathbf{c}}{R}\, \mathbf{l} \right) \mathbf{l} \times \mathbf{w} - \frac{1}{R^3}\left(\mathbf{c} - \frac{1}{R}\frac{d\mathbf{c}}{ds} \cdot \mathbf{l} \right) \mathbf{l} \times \mathbf{w}$$

$$- \frac{1 + \mathbf{l} \cdot \mathbf{c}}{R^3}\left(\mathrm{I} + \frac{1}{R}\, \mathbf{lw} \right) \times \mathbf{w} - \frac{1 + \mathbf{l} \cdot \mathbf{c}}{R^3}\frac{1}{R}\, \mathbf{c} \times \mathbf{l}.$$

In this expression the product indicated by the cross is always performed first, regardless of the parentheses. If now the cross be inserted to find $\diamond \times \diamond \times \mathbf{p}$, the result $\diamond \times \diamond \times \mathbf{p} = 0$ is obtained, as required by equation (51). Moreover, if the dot be inserted so as to find $\diamond \cdot (\diamond \times \mathbf{p})$, the result is also

$$\diamond \cdot \diamond \times \mathbf{p} = 0. \tag{85}$$

We have, of course, proved this theorem only for points lying off the given (δ)-curve.

We have the mathematical relation (55), namely,

$$\diamond \cdot \diamond \times \mathbf{p} = \diamond (\diamond \cdot \mathbf{p}) - (\diamond \cdot \diamond)\, \mathbf{p}.$$

But we have seen that $\diamond \cdot \mathbf{p} = 0$, and therefore

$$\diamond \cdot \diamond\, \mathbf{p} = \diamond^2 \mathbf{p} = 0. \tag{86}$$

The existence of this extended Laplacian equation justifies the use of the term potential [40] for $\mathbf{p}$.

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40 It is interesting to enquire what form the potential $\mathbf{p}$ might be given other than $\mathbf{w}/R$. Suppose that $\mathbf{p}$ should be independent of the curvature of the (δ)-curve. The only vectors then entering into the determination of $\mathbf{p}$ at any point Q would be $\mathbf{w}$ and $\mathbf{l}$. The only possible form of a 1-vector potential would therefore be

$$\mathbf{p} = \varphi(R)\, \mathbf{w} + f(R)\, \mathbf{l},$$

where $R = -\mathbf{l} \cdot \mathbf{w}$. The expression for $\diamond \mathbf{p}$ becomes

$$\diamond \mathbf{p} = \varphi'(R)\left(-\mathbf{w} + \frac{1 + \mathbf{l} \cdot \mathbf{c}}{R}\, \mathbf{l} \right) \mathbf{w} - \varphi(R)\frac{1}{R}\, \mathbf{lc}$$

$$+ f'(R)\left(-\mathbf{w} + \frac{1 + \mathbf{l} \cdot \mathbf{c}}{R}\, \mathbf{l} \right) \mathbf{l} + f(R)\left(\mathrm{I} + \frac{1}{R}\, \mathbf{lw} \right).$$

Electromagnetics and Mechanics.

The Continuous and Discontinuous in Physics.

45. It has been customary in physics to regard a fluid as composed of discrete particles (as in the kinetic theory) or as a continuum (as in hydrodynamics) according to the nature of the problem under investigation; it has been assumed that even if a fluid were made up of discrete particles, it could be treated as a continuum for the sake of convenience in applying the laws of mathematical analysis. For example we introduce the concept of density which may have no real exact physical significance, but which by the method of averages yields apparently correct results. Provided that the particles in a discontinuous assemblage are sufficiently small, numerous, and regularly distributed, it is assumed that any assemblage of discrete particles can be replaced without loss of mathematical rigor by a continuum.

However, when we investigate problems of this character in the light of our four dimensional geometry, we are led to the striking conclusion that in some cases it is impossible, except by methods which are unwarrantably arbitrary, to replace a discontinuous by a continuous distribution and vice versa. Especially we shall see that this is the case with radiant energy, a conclusion which is particularly

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Hence

$$\Diamond\cdot\mathbf{p} = -\,\mathbf{l}\cdot\mathbf{c}\left(\varphi'(R) + \frac{1}{R}\varphi(R)\right) + \left(Rf'(R) + 3f(R)\right)\cdot$$

If $\Diamond\cdot\mathbf{p}$ is to vanish regardless of the curvature of the (δ)-curve, then

$$\varphi'(R) + \frac{1}{R}\varphi(R) = 0, \qquad Rf'(R) + 3f(R) = 0.$$

The integration of these equations determines φ and f as

$$\varphi = \frac{A}{R}, \qquad f = \frac{B}{R^3},$$

where A and B are constants. The expression for $\Diamond\times\mathbf{p}$ is

$$\Diamond\times\mathbf{p} = -\,\frac{A}{R^2}\left(\mathbf{l}\times\mathbf{c} + \frac{1 + \mathbf{l}\cdot\mathbf{c}}{R}\,\mathbf{l}\times\mathbf{w}\right) - \frac{2B}{R^4}\,\mathbf{l}\times\mathbf{w}.$$

The calculation of $\Diamond\cdot\Diamond\times\mathbf{p} = -\,\Diamond\cdot\Diamond\mathbf{p}$ gives

$$\Diamond\cdot\Diamond\mathbf{p} = 2B\left(\frac{\mathbf{w}}{R^4} + 3\frac{\mathbf{l}\cdot\mathbf{c}}{R^5}\mathbf{l}\right)\cdot$$

It therefore appears impossible to satisfy $\Diamond\cdot\mathbf{p} = 0$ and $\Diamond\cdot\Diamond\mathbf{p} = 0$ with any other form of potential, dependent only on $\mathbf{l}$ and $\mathbf{w}$, than the one chosen.

notable when taken in connection with the recent theories regarding the constitution of light, embodied in the quantum hypothesis.

Let us for simplicity first consider such cases as arise in our two dimensional geometry. Consider a material rod of infinitesimal cross section moving uniformly in its own direction. Suppose now that we regard this rod as made up of discrete particles. Then in our geometrical representation each particle will give rise to a vector of extended momentum $m_0\mathbf{w}$, and these vectors will all be parallel. The whole space-time locus of the rod will be a set of parallel (δ)-lines. The rod as a spacial object possessing length has no meaning until a definite set of space-time axes have been chosen, and this choice is arbitrary. There is, however, one such choice which is unique, and that is the selection of the time-axis along $\mathbf{w}$, and the space-axis perpendicular thereto. In this system the mass of each particle is its m_0, and the sum of the m_0's of any segment of the rod divided by the length of the segment is the average density. If the particles are sufficiently numerous, we may regard the rod as continuous and replace conceptually the locus of the rod as a set of discrete (δ)-lines by a vector field continuous between the two (δ)-lines which mark the termini of the rod, and represented at each point by a vector parallel to $\mathbf{w}$ and equal in magnitude to the density at that point. This is the density as it appears to an observer at rest with respect to the rod, and may be called μ_0. The vector $\mu_0\mathbf{w}$ has therefore a definite four dimensional significance. Its projections on any arbitrarily chosen space and time axes are, however, not respectively the density of momentum and mass in that system. For

$$\mu_0\mathbf{w} = \frac{\mu_0}{\sqrt{1 - v^2}}(\mathbf{v} + \mathbf{k}_4). \tag{87}$$

But μ, the density in this system, is not equal to $\mu_0/\sqrt{1 - v^2}$, but

$$\mu = \frac{\mu_0}{1 - v^2} \tag{88}$$

as the units of mass and length both change with a change of axes.

Conversely we may replace a continuous by a discrete distribution. Let us consider a continuous vector field $\mathbf{f}$ of (δ)-lines. Then any region of this field, embraced between two (δ)-lines sufficiently near together, may be replaced by one or several parallel (δ)-vectors, of which the sum is equal to $\mathbf{f}$ multiplied by the length of the line drawn between and perpendicular to the boundary (δ)-lines. We may also use another construction which is essentially identical with this. Let $d\mathbf{r}$ be any vector drawn from one boundary line to the other. Then $(d\mathbf{r}\times\mathbf{f})^*\mathbf{f}/f$ is the same vector as the one just obtained. Although the method of obtaining this vector may seem somewhat artificial, the vector is, however, a definite vector obtainable from the field without any choice of axes.

46. These methods fail completely when the vector field is composed of singular vectors. Let us consider instead of a material rod, a segment of a uniform ray of light. If this can be represented by a continuous vector field bounded by two lines representing the loci of the termini of the segment then all these vectors must be singular. Let $\mathbf{1}$ be (Figure 25) the value of the vector throughout the field. It is evident that we cannot, as in the former case, draw any line across the field perpendicular to $\mathbf{1}$. The second method likewise fails because it would involve the magnitude of $\mathbf{1}$ which is zero. Moreover it can be stated that there is no method whatever, independent of any choice of axes, which will enable us to change from this continuous distribution of the light to a set of light particles. Conversely it is equally true that given a system of light particles moving in a single ray it is quite impossible to replace them by means of any continuous distribution, and this is true no matter how small and numerous and close together these particles are. This statement regarding singular vectors will be seen to hold also in space of higher dimensions,[41] and is of fundamental importance.

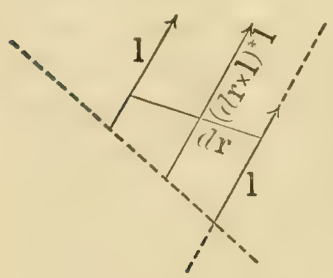

FIGURE 25.

While it is impossible, therefore, to find continuous and discontinuous distributions of singular vectors which are equivalent to one another, it is possible to obtain by four dimensional methods out of a specified region of a singular vector field a single vector or group of discrete vectors uniquely determined by that vector field but quadratic instead of linear in the vectors of the field. Consider any portion of the field bounded by two singular vectors sufficiently near together. Let $\mathbf{1}$ be the vector of the field, and then if $d\mathbf{r}$ is any vector drawn from

[41] In the case of the peculiar geometry of a singular plane (§ 31), the interval dr from one singular line to another is independent of the direction of $d\mathbf{r}$. It is therefore possible to replace the field $\mathbf{1}$ between two boundary lines by the single vector $\mathbf{1}dr$ linear in $\mathbf{1}$. Thus there are exceptional singular fields in higher dimensions for which the passage from continuous to discrete and vice versa may be accomplished.

one boundary to the other (Figure 25), the 2-vector $d\mathbf{r}\,\mathbf{l}$ is independent of the way in which $d\mathbf{r}$ was drawn and the 1-vector $(d\mathbf{r}\times\mathbf{l})^*\mathbf{l}$ is determined, and is in a certain sense representative of the region of the field chosen.

It may be of interest to obtain the projection of $\mathbf{l}$ and $(d\mathbf{r}\times\mathbf{l})^*\mathbf{l}$ upon two sets of axes $\mathbf{k}_1$, $\mathbf{k}_4$ and $\mathbf{k}_1'$, $\mathbf{k}_4'$ where the angle from $\mathbf{k}_4$ to $\mathbf{k}_4'$ is $\phi = \tanh^{-1} v$. Let the vector $\mathbf{l}$ be written as

$$\mathbf{l} = a\,(\mathbf{k}_1 + \mathbf{k}_4) = a'\,(\mathbf{k}_1' + \mathbf{k}_4').$$

Now by the transformation equations (7) we have

$$a' = a\,(\cosh\psi - \sinh\psi) = a\frac{1 - v}{\sqrt{1 - v^2}} = a\,\sqrt{\frac{1 - v}{1 + v}}.$$

Hence the ratio of the components of $\mathbf{l}$ along the new axes to the components along the old axes is $\sqrt{1 - v}/\sqrt{1 + v}$. But $(d\mathbf{r}\times\mathbf{l})^*$ is a member independent of any system of axis. Hence the ratio for $(d\mathbf{r}\times\mathbf{l})^*\,\mathbf{l}$ is the same as that for $\mathbf{l}$.

Now while it is impossible by any four dimensional methods to redistribute the vector $(\mathrm{d}\mathbf{r}\times\mathbf{l})^*\mathbf{l}$ as a continuous vector field, it is always possible after arbitrary axes of space and time have been chosen to make such a distribution. Thus if between the two boundary lines $d\mathbf{r}$ be taken parallel to $\mathbf{k}_1$ and $d\mathbf{r}'$ parallel to $\mathbf{k}'_1$, then as before $d\mathbf{r}\times\mathbf{l} = d\mathbf{r}'\times\mathbf{l}$. By taking the complement of both sides and applying (24), then, since $\mathbf{l}$ is its own complement, we find $d\mathbf{r}\cdot\mathbf{l} = d\mathbf{r}'\cdot\mathbf{l}$. But $d\mathbf{r}\cdot\mathbf{l}$ is equal to $a d\mathbf{r}\cdot\mathbf{k}_1 = a dr$, and $d\mathbf{r}'\cdot\mathbf{l} = a' dr'$. Hence $dr/dr' = a'/a$. Thus the *density* of the components of the vector $(d\mathbf{r}\times\mathbf{l})^*\mathbf{l}$ in the one case is to the density of the components in the other case as a^2 is to a'^2, equal to $(1 - v)/(1 + v)$. Thus while we have seen that the energy and momentum of a light-particle (§ 24) appear different in the ratio $\sqrt{1 - v}/\sqrt{1 + v}$ to two observers, if the energy and momentum are regarded as distributed their densities will appear different to the two observers in the ratio $(1-v)/(1 + v)$.

Let us proceed at once to the discussion of similar problems arising in space of four dimensions. Here also it is possible to pass at will from a consideration of continuous 1-vector fields to a consideration of equivalent discontinuous distributions of 1-vectors in the case of all non-singular vectors, by an extension of either of the methods which we have used in two dimensional space. Thus if a region of the field is cut out by a (hyper-) tube of lines parallel to the vector of the field, then the original vector multiplied by the volume of intersection of a perpendicular planoid is a single vector (or the sum of a group of vectors) which may replace the original field within the tube. Or if **f** represents the vector field and $d\mathfrak{S}$ the 3-vector cut off on any planoid by the tube, then the same result as before may be obtained by the operation $(d\mathfrak{S}\times\mathbf{f})^*\mathbf{f}/f$.

In the case of singular vectors we encounter the same difficulties as in two dimensions. Let us consider a field of singular 1-vectors **l**, and a portion of this field cut off by a small tube of lines parallel to **l**. A little consideration shows that it is impossible by any means whatever to replace this portion of the field by a single equivalent vector along **l**. It is possible, however, as before to obtain a single vector quadratic in **l** and determined by the given portion of the field. Let $d\mathfrak{S}$ be the 3-vector volume cut off on any planoid by the tube. Then $(d\mathfrak{S}\times\mathbf{l})$ is independent of the planoid chosen, and $(d\mathfrak{S}\times\mathbf{l})^*\,\mathbf{l}= d\mathbf{g}$ is the vector thus determined.

47. Now it is impossible to distribute the vector just obtained over that portion of the four dimensional spread which has given rise to it. But there is, nevertheless, in one case another kind of distribution which is possible and which possesses considerable interest. In order to introduce the somewhat difficult construction which is necessary in this case let us investigate first a particular type of singular vector field in three dimensions. Let $d\mathbf{s}$ be a small vector segment of a (δ)-curve. Each point of this segment determines a forward cone. The field which we wish to consider is such that at each point the vector **l** is along an element of the cone and of any interval which is a continuous function of position. This construction gives a limited field bounded by the two forward cones from the termini of the segment $d\mathbf{s}$. Let a plane cut across the two cones. The region of this plane intercepted between the two boundary cones is the surface lying between two nearly concentric circles. Let $d\mathbf{S}$ be an element of this surface. Now just as before the vector $(d\mathbf{S}\times\mathbf{l})^*\mathbf{l} = d\mathbf{g}$ may be formed and is different for each element $d\mathbf{S}$. The singular lines drawn from all the points bounding $d\mathbf{S}$ to the corresponding points of the segment $d\mathbf{s}$ determine a sort of tube of nearly parallel singular lines. The value of $d\mathbf{g}$ for each tube is at each point independent of the particular position of the plane through that point whose intersection with the tube is $d\mathbf{S}$. If therefore the whole field is divided up into an infinite number of such tubes, the infinitesimal vectors of the second order in **l** obtained for the several tubes are at each point independent of the plane which was used in constructing them.

Now it is impossible to redistribute the discrete vectors $d\mathbf{g}$ over the three dimensional field from which they were derived, but it is possible to replace them by a continuous distribution over a two dimensional spread in one of the cones. Let us assume that the infinitesimal tubes are so chosen that the elements of surface $d\mathbf{S} = d\mathbf{q}\times d\mathbf{r}$ are four-sided figures approximately rectangular and that the outer cone is divided into small regions lying between the elements of the cone, a, a', a'', . . . (Figure 26). In each of these small two dimensional regions we may place the corresponding vector $d\mathbf{g}$. Now any two neighboring lines drawn from a to a' are of equal interval because they lie in a singular plane between two singular lines (see preceding footnote and § 31). The vector $d\mathbf{g}/dr$ is therefore determined at each point of the cone independent of the direction of $d\mathbf{r}$. It is a vector representing a kind of density and when all the vectors $d\mathbf{g}$ are similarly treated, it is continuously distributed over the whole cone.

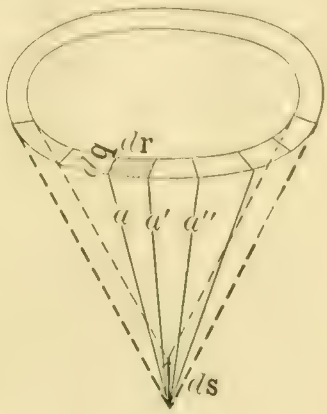

FIGURE 26.

The vector $d\mathbf{g}/dr$ is a function of the interval ds. Let us determine this relation analytically. Since $d\mathbf{S}= d\mathbf{q}\times d\mathbf{r}$ we may write

$$d\mathbf{g} = (d\mathbf{q}\times d\mathbf{r}\times\mathbf{l})^*\mathbf{l} = [(d\mathbf{q}\times d\mathbf{r})^*\cdot\mathbf{l}]\,\mathbf{l} = l_4\mathbf{l}\; dqdr,$$

where l_4 is the component of $\mathbf{l}$ perpendicular to $d\mathbf{q}\times d\mathbf{r}$; for since $d\mathbf{q}$ is perpendicular to $d\mathbf{r}$, $(d\mathbf{q}\times d\mathbf{r})^*$ is a 1-vector perpendicular to $d\mathbf{q}\times d\mathbf{r}$ and of magnitude $dqdr$. We therefore find $d\mathbf{g}/dr = l_4\mathbf{l}dq$. It remains to determine dq in terms of ds.

The plane of intersection having been chosen, the two circles are in general eccentric and the distance dc between their centers is the projection of the segment ds upon their plane (Figure 27). If the normal to this plane makes an angle with ds whose hyperbolic tangent is v, then $dc = vds/\sqrt{1 - v^2}$. The two segments cut off by the two circles on dc produced are found as follows. Pass a plane through dc and ds. Then AB is readily shown to be

$$ds\sqrt{1 - v}/\sqrt{1 + v}, \qquad \text{and} \qquad CD = ds\sqrt{1 + v}/\sqrt{1 - v}\cdot$$

Then the value of dq is readily proved by Euclidean methods to be

$(1-v\cos\phi)\,ds/\sqrt{1-v^2}$, where ϕ is the angle between dq and AD. Hence

$$\frac{d\mathbf{g}}{dr} = l_4 \frac{1 - v\cos\phi}{\sqrt{1-v^2}} \mathbf{1}\,ds. \tag{89}$$

We have gone through this somewhat complicated calculation for the three dimensional case because of the greater ease of visualisation

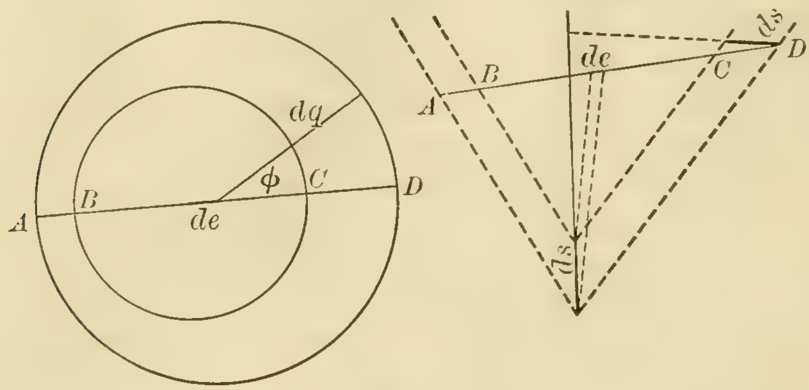

FIGURE 27.

and because the results obtained are applicable without essential change to four dimensions. Again let $d\mathbf{s}$ be a segment of any (δ)-curve each point of which determines a forward hypercone. Let us consider the four dimensional vector field $\mathbf{1}$ bounded by the two limiting forward hypercones, $\mathbf{1}$ at every point lying along an element of one of the hypercones whose apex is on $d\mathbf{s}$. Any (γ)-planoid will intersect the limited vector field in a three dimensional volume bounded by the intersections of the two limiting hypercones with the planoid; these surfaces of intersection appear in the planoid as two nearly concentric spherical surfaces.

If as before the vector field is divided into infinitesimal portions, so that the volume of intersection is divided into the infinitesimal volumes $d\mathfrak{S}$, each of which is approximately a rectangular parallelepiped, and one of the surfaces of intersection is thus divided into the infinitesimal portions dS such that $d\mathbf{q}\times d\mathbf{S} = d\mathfrak{S}$, then for each infinitesimal portion of the field we may at any point obtain as above the vector $d\mathbf{g} = (d\mathfrak{S}\times\mathbf{1})^*\mathbf{1}$. Then precisely as in the previous case [42]

[42] In the peculiar three dimensional geometry of a tangent (singular) planoid there is one set of parallel singular lines, and every plane in the planoid is perpendicular to these lines. Every cross-section of a given tube of singular lines has the same area.

$$d\mathbf{g} = (d\mathbf{q}{\times}d\mathbf{S}{\times}\mathbf{l})^*\mathbf{l} = l_4\mathbf{l}\,dq\,dS, \qquad \text{and} \quad d\mathbf{g}/dS = l_4\mathbf{l}\,dq.$$

This vector is distributed uniformly over one of the hypercones and is independent of the particular planoid used in obtaining it. Then also just as before

$$\frac{d\mathbf{g}}{dS} = l_4 \frac{1 - v\cos\phi}{\sqrt{1 - v^2}} \mathbf{l}\,ds, \tag{90}$$

where ϕ is the angle between $\mathbf{v}$, which passes through the centers of the two spheres, and the line, from either center, to the chosen point upon the surface.

The Field of a Point Charge.

48. Much of recent progress in the science of electricity has been due to the introduction of the electron theory, in which electricity is regarded not as a continuum but as an assemblage of discrete particles. In Lorentz's development of this theory he has deemed it necessary, however, to regard the electron itself as distributed over a minute region of space known as the volume of the electron. This deprives the theory of some of that simplicity which it would possess if the charge of an electron could be regarded as in fact concentrated at a single point. Whether the theory of the point charge can be brought into accord with observed facts and with the laws of energy cannot at present be decided. It seems, however, highly desirable to develop this theory as far as possible. In our application of our four dimensional geometry to electricity we shall therefore consider first an electric charge as a collection of discrete charges or electrons, each of which is concentrated at a single point.

The locus of a point electron in time and space must be a (δ)-curve. If $\mathbf{w}$ is a unit tangent to such a curve, then we may consider at every point the vector $\epsilon\mathbf{w}$, where ϵ is the magnitude of the charge, negative for a negative electron, and positive for a positive electron (if such there be). It is explicitly assumed that ϵ is a constant. We shall show that the geometric fields obtained from this vector by the methods of § 43 give precisely the equations which are of importance in electromagnetic theory.

The vector $\mathbf{w}$ determines at every point of our time-space manifold the vector $\mathbf{p} = \mathbf{w}/R$. Similarly the vector $\epsilon\mathbf{w}$ determines the vector field

$$\mathbf{m} = \epsilon\mathbf{p} = \frac{\epsilon\mathbf{w}}{R} = \frac{\epsilon\mathbf{v}}{l_4 - \mathbf{l}_s\cdot\mathbf{v}} + \frac{\epsilon\mathbf{k}_4}{l_4 - \mathbf{l}_s\cdot\mathbf{v}}. \tag{91}$$

The last equality is obtained when any $\mathbf{k}_4$ axis has been arbitrarily chosen. Then $\mathbf{v}$ is the velocity of the electron and $l_4 - \mathbf{l}_s \cdot \mathbf{v}$ is the distance FQ in Figure 22, that is, the projection of the distance from the point of observation to the contemporaneous position of the electron (if assumed to be moving uniformly) upon the line $\mathbf{l}_s$ joining the "retarded" position of the electron to the point of observation.

We may call $\mathbf{m}$ the extended electromagnetic vector potential. Its projections on space and on the time-axis are respectively the vector potential $\mathbf{a}$ and the scalar potential ϕ,

$$\mathbf{a} = \frac{\epsilon \mathbf{v}}{l_4 - \mathbf{l}_s \cdot \mathbf{v}}, \qquad \phi = \frac{\epsilon}{l_4 - \mathbf{l}_s \cdot \mathbf{v}}, \tag{92}$$

precisely in the form first obtained by Liénard.[43] From (69) we have

$$\Diamond \cdot \mathbf{m} = \left(\nabla - \mathbf{k}_4 \frac{\partial}{\partial t}\right) \cdot (\mathbf{a} + \phi \mathbf{k}_4) = 0.$$

Hence

$$\nabla \cdot \mathbf{a} + \frac{\partial \phi}{\partial t} = 0.$$

We see therefore that the Liénard potentials are connected by the same familiar equation as connects the ordinary vector and scalar potentials. Assuming that vector fields produced by two or more electrons are additive, these equations are true for the general case.

The 2-vector field produced by an electron, whether in uniform or accelerated motion, is obtained immediately from (81)–(83).

$$\mathbf{M} = \Diamond \times \mathbf{m} = \epsilon \Diamond \times \mathbf{p} = -\frac{\epsilon}{R} [\mathbf{w} \times \mathbf{c} - \mathbf{n} \times \mathbf{c} + \mathbf{n} \cdot \mathbf{c}\ \mathbf{n} \times \mathbf{w}] + \frac{\epsilon}{R^2} \mathbf{n} \times \mathbf{w}. \tag{93}$$

Or

$$\mathbf{M} = -\frac{\epsilon}{R^3} \mathbf{l} \times [\mathbf{l} \cdot (\mathbf{w} \times \mathbf{c})] - \frac{\epsilon}{R^3} \mathbf{l} \times \mathbf{w} = -\frac{\epsilon}{R^3} (\mathbf{l} \times \mathbf{w} \times \mathbf{c}) \cdot \mathbf{l} - \frac{\epsilon}{R^3} \mathbf{l} \times \mathbf{w}. \tag{94}$$

The first term in this expression vanishes when the curvature is zero. The fact that this term is a *singular* vector has already been pointed out, and the great importance of this fact in electromagnetic theory will be pointed out later. In the second term $\mathbf{n} \times \mathbf{w}$ is the unit 2-vector determined by the line $\mathbf{w}$ and the point Q where the field is being discussed.

49. In case the electron is unaccelerated the equation assumes the simple form

$$\mathbf{M} = \frac{\epsilon}{R^2} \mathbf{n} \times \mathbf{w}. \tag{95}$$

[43] Eclairage électrique, **16**, 5 (1898).

This may be expanded according to (72) when an axis of time has been chosen. Then, noting that $\mathbf{l}_s\times\mathbf{v} = (\mathbf{l}_s - l_4\mathbf{v})\times\mathbf{v}$,

$$\mathbf{M} = -\epsilon\frac{1-v^2}{r'^3}\mathbf{r}\times\mathbf{v} - \epsilon\frac{1-v^2}{r'^3}\mathbf{r}\times\mathbf{k}_4. \tag{96}$$

Where $\mathbf{r}$ is the vector $\mathbf{r} = \mathbf{l}_s - l_4\mathbf{v}$ from the contemporaneous position of the charge to the point Q in the field, and $r' = l_4 - \mathbf{l}_s\cdot\mathbf{v}$. The 2-vector $\mathbf{M}$ is thus split automatically into two 2-vectors, of which one passes through the time-axis $\mathbf{k}_4$, and the other lies in the planoid $\mathbf{k}_{123}$ which constitutes ordinary space. These will be designated respectively by the letters $\mathbf{E}$ and $\mathbf{H}$. Thus

$$\mathbf{M} = \mathbf{H} + \mathbf{E}. \tag{97}$$

This separation may in all cases be made whether the field is caused by one or more electrons in constant or accelerated motion. We shall thus see that the 2-vector $\mathbf{M}$ is precisely the "Vektor zweiter Art" which Minkowski introduced to express the electric and magnetic forces.

Out of $\mathbf{H}$ and $\mathbf{E}$ spacial 1-vectors $\mathbf{h}$ and $\mathbf{e}$ may be obtained by the equations

$$\mathbf{h} = \mathbf{H}\cdot\mathbf{k}_{123}, \qquad \mathbf{e} = \mathbf{E}\cdot\mathbf{k}_4. \tag{98}$$

Then $\mathbf{h}$ is the three-dimensional complement of $\mathbf{H}$, and $\mathbf{e}$ the intersection of $\mathbf{E}$ with three-dimensional space. Evidently

$$\begin{aligned} h_1 &= H_{23}, & h_2 &= H_{31}, & h_3 &= H_{12}, \\ e_1 &= -E_{14}, & e_2 &= -E_{24}, & e_3 &= -E_{34}. \end{aligned} \tag{99}$$

Referring now to (96) we see that in the case of a uniformly moving electron

$$\mathbf{e} = \epsilon\frac{1-v^2}{r'^3}\mathbf{r}, \qquad \mathbf{h} = -\epsilon\frac{1-v^2}{r'^3}(\mathbf{r}\times\mathbf{v})\cdot\mathbf{k}_{123}, \tag{100}$$

or

$$\mathbf{v}\times\mathbf{e} = \mathbf{h}\cdot\mathbf{k}_{123} = \mathbf{H}.$$

Noting that $(\mathbf{r}\times\mathbf{v})\cdot\mathbf{k}_{123}$ is that which in ordinary vector analysis is known as the vector product of $\mathbf{r}$ and $\mathbf{v}$, we see that these equations are precisely the equations for the electric and magnetic forces.[44]

It may seem surprising to one who is not fully convinced of the very fundamental relationship between the four dimensional geometry of relativity and the science of mechanics that we should thus be led

44 See Abraham, Theorie der Elektrizität, **2**, p. 88.

from simple geometrical premises to conclusions of so purely physical a character. Of course it is to be noted that while our values of **e** and **h** are identical in mathematical form with equations for electric and magnetic force, we should need some additional assumptions before actually identifying these quantities.

50. Our next step will be to show that the values of **e** and **h** derived from the 2-vector $\diamond\times\mathbf{m} = \mathbf{M}$ are identical with the expressions for electric and magnetic force in the general case in which the electron is no longer restricted to uniform motion. We have from (94)

$$\mathbf{M} = \epsilon\mathbf{P} = -\frac{\epsilon}{R^3}(\mathbf{l}\times\mathbf{w}\times\mathbf{c})\cdot\mathbf{l} - \frac{\epsilon}{R^3}\mathbf{l}\times\mathbf{w}. \tag{101}$$

Thus, assuming some time-axis, we see from (43) that

$$\mathbf{w}\times\mathbf{c} = \mathbf{w}\times\dot{\mathbf{v}}/(1-v^2).$$

Then

$$\mathbf{M} = -\frac{\epsilon}{R^3}\frac{(\mathbf{l}\times\mathbf{w}\times\dot{\mathbf{v}})\cdot\mathbf{l}}{1-v^2} - \frac{\epsilon}{R^3}\mathbf{l}\times\mathbf{w}. \tag{102}$$

Hence

$$\mathbf{M} = -\frac{\epsilon}{R^3}\frac{\mathbf{l}\cdot\dot{\mathbf{v}}\,\mathbf{l}\times\mathbf{w} + R\,\mathbf{l}\times\dot{\mathbf{v}}}{1-v^2} - \frac{\epsilon}{R^3}\mathbf{l}\times\mathbf{w}, \tag{103}$$

$$\begin{aligned}\mathbf{M} &= -\frac{\epsilon}{R^3}\left[\frac{\mathbf{l}_s\cdot\dot{\mathbf{v}}\,\mathbf{l}_s\times\mathbf{v}}{(1-v^2)^{\frac{3}{2}}} + \frac{R\,\mathbf{l}_s\times\dot{\mathbf{v}}}{1-v^2}\right] - \frac{\epsilon}{R^3}\frac{\mathbf{l}_s\times\mathbf{v}}{(1-v^2)^{\frac{1}{2}}} \\ &\quad -\frac{\epsilon}{R^3}\left[\frac{\mathbf{l}_s\cdot\dot{\mathbf{v}}\,(\mathbf{l}_s - l_4\mathbf{v})\times\mathbf{k}_4}{(1-v^2)^{\frac{3}{2}}} - \frac{Rl_4\dot{\mathbf{v}}\times\mathbf{k}_4}{1-v^2}\right] - \frac{\epsilon}{R^3}\frac{(\mathbf{l}_s - l_4\mathbf{v})\times\mathbf{k}_4}{(1-v^2)^{\frac{1}{2}}}.\end{aligned} \tag{104}$$

Hence, if again we use $\mathbf{r} = \mathbf{l}_s - l_4\mathbf{v}$ and $r' = l_4 - \mathbf{l}_s\cdot\mathbf{v} = R(1-v^2)^{\frac{1}{2}}$, we have

$$\begin{aligned}\mathbf{e} = \mathbf{E}\cdot\mathbf{k}_4 &= \epsilon\left[\frac{\mathbf{l}_s\cdot\dot{\mathbf{v}}}{r'^3}\mathbf{r} - \frac{l_4\dot{\mathbf{v}}}{r'^2} + \frac{1-v^2}{r'^3}\mathbf{r}\right] \\ \mathbf{h} = \mathbf{H}\cdot\mathbf{k}_{123} &= -\epsilon\left[\frac{\mathbf{l}_s\cdot\dot{\mathbf{v}}\;\mathbf{r}\times\mathbf{v}}{r'^3} + \frac{\mathbf{l}_s\times\dot{\mathbf{v}}}{r'^2} + \frac{(1-v^2)\mathbf{r}\times\mathbf{v}}{r'^3}\right]\cdot\mathbf{k}_{123}.\end{aligned} \tag{105}$$

If we look at the form in $\mathbf{l}_s - l_4\mathbf{v}$ (104) we observe that

$$\mathbf{H} = \frac{1}{l_4}\mathbf{l}_s\times\mathbf{e}, \qquad \mathbf{E} = -\mathbf{e}\times\mathbf{k}_4. \tag{106}$$

Hence

$$\mathbf{M} = \left(\frac{1}{l_4}\mathbf{l}_s + \mathbf{k}_4\right)\times\mathbf{e} = \frac{1}{l_4}\mathbf{l}\times\mathbf{e}. \tag{107}$$

These are the equations for the field of an accelerated electron which were obtained by Abraham and Schwarzschild.[45] It will be convenient to divide the field $\mathbf{M}$ into that part $\mathbf{M}'$ which is due to acceleration alone and that $\mathbf{M}''$ which is independent of acceleration. The former, which is the first term in any of the above expressions for $\mathbf{M}$, (101)-(103), is a singular vector field, and is the only one which is important at great distances from the electron, for it varies as $1/R$ (since $\mathbf{l}$ varies with R) whereas $\mathbf{M}''$ varies as $1/R^2$. If we divide the field $\mathbf{M}'$ into its two parts $\mathbf{M}' = \mathbf{E}' + \mathbf{H}'$, we see here also that

$$\mathbf{H}' = \frac{1}{l_4}\mathbf{l}_s \times \mathbf{e}', \qquad \mathbf{E}' = -\mathbf{e}' \times \mathbf{k}_4; \tag{108}$$

and since, in this case, $\mathbf{l}_s \cdot \mathbf{e}' = 0$ (as may be seen by performing the multiplication) and $\mathbf{l}_s$ is perpendicular to $\mathbf{e}'$, we find that $\mathbf{E}'$, $\mathbf{H}'$ are equal in magnitude. Moreover $\mathbf{e}'$, $\mathbf{h}'$ are equal in magnitude and perpendicular to each other and to $\mathbf{l}_s$. In other words in a radiation field the electric and magnetic forces are equal in magnitude, perpendicular to each other, and perpendicular to the "direction of propagation." All these results are geometric consequences of the fact that the 2-vector $\mathbf{M}'$ is singular.

51. In four dimensional space every singular 2-vector determines a singular 1-vector, namely, a vector pointing outward along the element of tangency of the 2-vector with a forward hypercone. This 1-vector is the complement of the 2-vector in the tangent planoid. If $\mathbf{l}'$ is the 1-vector thus determined by the 2-vector $\mathbf{M}'$, then we may write

$$\mathbf{M}' = \mathbf{u} \times \mathbf{l}',$$

where $\mathbf{u}$ is any unit vector in the plane of $\mathbf{M}'$, provided the sign of $\mathbf{u}$ be properly chosen.[46] In the case of the singular vector $\mathbf{M}'$ which we have obtained in the previous section we may write, from (94),

$$\mathbf{M}' = -\frac{\epsilon}{R^3}\mathbf{l} \times [\mathbf{l} \cdot (\mathbf{w} \times \mathbf{c})] = -\frac{\epsilon}{R^3} a \mathbf{l} \times \frac{\mathbf{l} \cdot (\mathbf{w} \times \mathbf{c})}{a}, \tag{109}$$

where a is the magnitude of $\mathbf{l} \cdot (\mathbf{w} \times \mathbf{c})$ and therefore the last vector is a unit vector. Hence we may write at once for the 1-vector determined by $\mathbf{M}'$,

$$\mathbf{l}' = \frac{\epsilon}{R^3} a \mathbf{l}. \tag{110}$$

45 See Abraham, Theorie der Elektrizität, **2**, p. 95.

46 Owing to the nature of the geometry in a singular plane, the unit vector $\mathbf{u}$ drawn from a given point always terminates on a definite singular line and thus determines the same 2-vector $\mathbf{u} \times \mathbf{l}'$ for all values of $\mathbf{u}$. (§ 31)

The value of α is, from (80),

$$\alpha = \sqrt{[\mathbf{l}\cdot(\mathbf{w}\times\mathbf{c})]\cdot[\mathbf{l}\cdot(\mathbf{w}\times\mathbf{c})]} = R\sqrt{(\mathbf{n}\times\mathbf{c})\cdot(\mathbf{n}\times\mathbf{c})}. \qquad (111)$$

Now the vector $\mathbf{l}'$, being a singular vector continuously distributed, can be treated by the method of § 47 to give at any point a discrete vector of the second order in $\mathbf{l}'$, namely,[47]

$$d\mathbf{g} = (d\mathfrak{S}\times\mathbf{l}')^*\mathbf{l}' \qquad (112)$$

where $d\mathfrak{S}$ is the vector volume cut off on any planoid by an infinitesimal tube of singular lines parallel to $\mathbf{l}'$. If $d\mathbf{s}$ is an infinitesimal portion of the locus of the electron which gives rise to the fields $\mathbf{M}'$ and $\mathbf{l}'$, and if we consider the region of the $\mathbf{l}'$ field bounded by the two forward hypercones from the termini of $d\mathbf{s}$, then all the vectors $d\mathbf{g}$ belonging to this region can be redistributed continuously on one of the hypercones, and just as in § 47 we obtain the vector

$$\frac{d\mathbf{g}}{dS} = l_4'\frac{1 - v\cos\phi}{\sqrt{1 - v^2}}\mathbf{l}'ds.$$

Now we may substitute the value of $\mathbf{l}'$ and obtain

$$d\mathbf{g} = \frac{\epsilon^2}{R^6}\alpha^2(d\mathfrak{S}\times\mathbf{l})^*\mathbf{l}, \qquad (113)$$

$$\frac{d\mathbf{g}}{dS} = \frac{\epsilon^2}{R^6}\alpha^2 l_4\frac{1 - v\cos\phi}{\sqrt{1 - v^2}}\mathbf{l}ds. \qquad (114)$$

Before proceeding further with the second of these equations, let us obtain $d\mathbf{g}$ in another form. We may first show that

$$d\mathbf{g} = (d\mathfrak{S}\times\mathbf{l}')^*\mathbf{l}' = (d\mathfrak{S}^*\cdot\mathbf{M}')\cdot\mathbf{M}'. \qquad (115)$$

For $\mathbf{M}' = \mathbf{u}\times\mathbf{l}'$ where $\mathbf{u}$ is a unit vector perpendicular to $\mathbf{l}'$. Hence

$$(d\mathfrak{S}^*\cdot\mathbf{M}')\cdot\mathbf{M}' = [d\mathfrak{S}^*\cdot(\mathbf{u}\times\mathbf{l}')]\cdot(\mathbf{u}\times\mathbf{l}') = [(d\mathfrak{S}^*\cdot\mathbf{l}')\,\mathbf{u} - (d\mathfrak{S}^*\cdot\mathbf{u})\,\mathbf{l}']\cdot(\mathbf{u}\times\mathbf{l}')$$

by (34). Applying this rule again and noting that $\mathbf{u}\cdot\mathbf{u} = 1$ and $\mathbf{u}\cdot\mathbf{l}' = 0$,

$$(d\mathfrak{S}^*\cdot\mathbf{M}')\cdot\mathbf{M}' = -(d\mathfrak{S}^*\cdot\mathbf{l}')\,\mathbf{l}'.$$

From this, (115) follows by (24). Now we have written $\mathbf{M}'$ as

$$\mathbf{M}' = \mathbf{H}' + \mathbf{E}' = \frac{1}{l_4}\mathbf{l}_s\times\mathbf{e}' - \mathbf{e}'\times\mathbf{k}_4.$$

47 Since $\mathbf{l}'$ involves α and therefore $\mathbf{n}\times\mathbf{c}$, the vector $d\mathbf{g}$ is zero for all points in the line of $\mathbf{c}$, and is a maximum when $\mathbf{n}$ is perpendicular to $\mathbf{c}$.

Now we may choose $d\mathfrak{S}$ perpendicular to $\mathbf{k}_4$ and with proper sign, then $d\mathfrak{S}^* = \mathbf{k}_4\, d\mathfrak{S}$. Hence, performing the multiplication,

$$dg = \left(e'^2 \frac{\mathbf{l}_s}{l_4} + e'^2 \mathbf{k}_4\right) d\mathfrak{S}. \tag{116}$$

Now if $\mathbf{e}'$ is interpreted as electric force in a radiation field, then we are accustomed to regard e'^2 $(= h'^2)$ as the *density of electromagnetic energy*, and the vector $e'^2\, \mathbf{l}_s/l_4$, where $\mathbf{l}_s/l_4$ is a unit vector perpendicular to $\mathbf{e}'$ and $\mathbf{h}'$, as the *Poynting vector*. Therefore $d\mathbf{g}$ becomes a vector of extended momentum of which the components are the total energy and the total momentum in the chosen volume $d\mathfrak{S}$. The vector $d\mathbf{g}$ is moreover independent of any choice of axes and is representative at any point of the tube whose cross section with any chosen space is the volume $d\mathfrak{S}$. But the vector $d\mathbf{g}/d\mathfrak{S}$ obtained by combining the Poynting vector and a vector along the $\mathbf{k}_4$ axis representing the density of energy is by no means independent of the choice of axes. In fact we may state that no way can be found of representing the density of energy by a strictly four dimensional vector. Thus we have a vector of extended momentum for energy-quanta, but not for energy density — an observation which is not without significance in view of certain modern theories of light.

52. It is interesting to note that the same energy vector $d\mathbf{g}$ may be obtained from different 2-vectors $\mathbf{M}'$. For any two singular 2-vectors of the same magnitude and passing through the same element of the hypercone determine the same vector $\mathbf{l}'$ as above defined. If we regard any singular 2-vector $\mathbf{M}'$ produced by an accelerated electron as the extended electromagnetic field of the radiant energy which is moving out along the space projection of the element $\mathbf{l}$ with the velocity of light, then it is evident that, since there is an infinite number of such 2-vectors to which the element $\mathbf{l}$ is common, there is something else necessary to characterize the light besides its energy. In fact a 1-vector such as $\mathbf{l}'$ or $d\mathbf{g}$ upon which the condition is imposed that it shall be singular has three degrees of freedom; a 2-vector such as $\mathbf{M}'$ subject to the two conditions that it shall be singular and uniplanar has four degrees of freedom. It is this additional degree of freedom in $\mathbf{M}'$ which gives rise to such phenomena as polarisation which show a dissymmetry of light with respect to the direction of propagation.

If the vector $d\mathbf{g}$ represents radiant energy (moving out along the hypercone with unity velocity), then the integration of equation (114) around the whole hypercone should give a vector representing the extended momentum of all the energy emitted by the electron, between the ends of the segment $d\mathbf{s}$ of its locus. We wish to evaluate the integral

$$\int \frac{d\mathbf{g}}{dS} dS = ds \int \frac{\epsilon^2}{R^6} \alpha^2 l_4 \frac{1 - v\cos\phi}{\sqrt{1 - v^2}} \mathbf{l} dS. \tag{117}$$

This integration may be simplified by the observation that the vector $d\mathbf{g}$ is not only independent of the direction of the planoid which cuts the boundary of the elementary tube in the surface dS, as has already been shown in general, but is also in this case independent of the position of the planoid, for $d\mathbf{g}/dS$ varies as $1/R^2$ and dS varies as R^2. The integral therefore is the same for any planoid whatsoever, and we may therefore choose for simplicity a planoid perpendicular to the locus $d\mathbf{s}$, and cutting the hypercone in a spherical surface of unit radius, that is $R = l_4 = 1$. Substituting the value of α from (111) gives, since $v = 0$ and $\mathbf{l} = R(\mathbf{w} - \mathbf{n})$,

$$\int \frac{d\mathbf{g}}{dS} dS = ds \int \epsilon^2 (\mathbf{n} \times \mathbf{c})^2 (\mathbf{w} - \mathbf{n})\, d\omega,$$

where $d\omega$ is a solid angle at the center of the sphere subtended by dS. The vector $\mathbf{c}$, normal to $\mathbf{w}$, is then along some diameter of the sphere; and $\mathbf{n}$ is directed from the various points of the surface toward the center. For diametrically opposite points the terms $(\mathbf{c} \times \mathbf{n})^2\, \mathbf{n}$ cancel. We need only integrate the terms $(\mathbf{c} \times \mathbf{n})^2\, \mathbf{w}$. If the diameter determined by $\mathbf{c}$ be taken as polar axis, these terms may be expressed as $c^2 \sin^2\theta\, \mathbf{w}$; and the element of surface is $\sin\theta\, d\theta\, d\phi$. The integral is therefore

$$\int d\mathbf{g} = \frac{8\pi}{3} \epsilon^2 c^2 \mathbf{w} ds. \tag{118}$$

This integral should be the vector of extended momentum for all the energy emitted by the electron between the two points considered, and its projections on any chosen time and space should be the corresponding energy and momentum. If the $\mathbf{k}_4$ axis is chosen parallel to $d\mathbf{s}$, that is if the electron is considered momentarily at rest, we obtain a simple expression; for then $\mathbf{w} = \mathbf{k}_4$, $c^2 = \dot{\mathbf{v}} \cdot \dot{\mathbf{v}}$, and $ds = dt$. The momentum altogether is zero, and the energy is

$$\frac{8\pi}{3} \epsilon^2 (\dot{\mathbf{v}} \cdot \dot{\mathbf{v}})\, dt. \tag{119}$$

When some other $\mathbf{k}_4$ axis is chosen, such that the electron is assumed to have the velocity $\mathbf{v}$, the expression becomes more complicated. Since $\mathbf{w} = (\mathbf{v} + \mathbf{k}_4)/\sqrt{1 - v^2}$ and $ds = \sqrt{1 - v^2}\, dt$, we have from (45),

$$\int^{\bullet} d\mathbf{g} = \frac{8\pi}{3} \frac{\epsilon^2}{(1 - v^2)^3} [\dot{\mathbf{v}} \cdot \dot{\mathbf{v}} - (\mathbf{v} \times \dot{\mathbf{v}}) \cdot (\mathbf{v} \times \dot{\mathbf{v}})] (\mathbf{v} + \mathbf{k}_4)\, dt. \qquad (120)$$

The two parts of this expression are precisely in the form obtained by Heaviside and Abraham [48] for the momentum and energy radiated from an accelerated electron.

53. When a singular vector field such as $d\mathbf{g}/dS$ is distributed continuously over a hypercone and is of such a character that its magnitude falls off along any element inversely as the square of the interval of that element measured from the apex (that is, inversely as R^2), or in other words, if it is of such character that the integral of the vector over the surface of intersection of the hypercone with any three dimensional spread is constant, then we may call such a field a simple radiation field. (In three dimensional space the magnitude would fall off inversely with R, and in two dimensional space would be constant.) The fact that the integral of $d\mathbf{g}/dS$ over the intersection of the hypercone with any two parallel planoids is constant may be regarded as equivalent to the law of conservation of radiant energy.

While the discussion which we have given of the vector $d\mathbf{g}$ is in complete accord with current theories of electromagnetic energy, there is another singular 1-vector which is suggested by the geometry and which may be of importance in case it is necessary to revise our ideas of radiant energy. This vector also gives a simple radiation field, in the sense just defined, and is likewise of the second order in $\mathbf{M}'$; but unlike the vectors $d\mathbf{g}$ and $d\mathbf{g}/dS$ it is continuously distributed over a four dimensional field. This is the vector [49] $(\mathbf{w} \cdot \mathbf{M}') \cdot \mathbf{M}' = \mathbf{b}$. The vector $\mathbf{b}$ is along the element of tangency $\mathbf{l}$ by § 39. Indeed if we take $\mathbf{M}'$ from (93) we have

$$\mathbf{b} = (\mathbf{w} \cdot \mathbf{M}') \cdot \mathbf{M}' = \frac{\epsilon^2}{R^2} [\mathbf{c} \cdot \mathbf{c} - (\mathbf{n} \cdot \mathbf{c})^2] (\mathbf{w} - \mathbf{n}) = \frac{\epsilon^2}{R^3} (\mathbf{c} \times \mathbf{n})^2 \mathbf{l}. \qquad (121)$$

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48 Abraham, Theorie der Elektrizität, **2**, 116.

49 To obtain a vector, of the second degree in $\mathbf{M}'$, out of $\mathbf{M}'$ itself is out of the question; for the only two products of the second degree in $\mathbf{M}'$ which are geometrically significant, namely $\mathbf{M}' \cdot \mathbf{M}'$ and $\mathbf{M}' \cdot \mathbf{M}'$, both vanish, since $\mathbf{M}'$ is singular and uniplanar. The vector $\mathbf{b}$ involves not only $\mathbf{M}'$, the field of the electron, but also $\mathbf{w}$ which expresses the state of motion of the electron itself.

If a $\mathbf{k}_4$ axis has been chosen, $\mathbf{b}$ may be obtained in terms of $\mathbf{e}'$, or of $\mathbf{e}'$ and $\mathbf{h}'$. For instance with $\mathbf{M}'$ taken from (108),

$$\mathbf{b} = \left(\frac{\mathbf{v} + \mathbf{k}_4}{\sqrt{1 - v^2}} \cdot \frac{\mathbf{l}_s \times \mathbf{e}' + l_4 \mathbf{k}_4 \times \mathbf{e}'}{l_4}\right) \cdot \frac{\mathbf{l}_s \times \mathbf{e}' + l_4 \mathbf{k}_4 \times \mathbf{e}'}{l_4}.$$

When we perform the reductions, remembering that $\mathbf{l}_s \cdot \mathbf{e}' = 0$, we find simply

$$\mathbf{b} = \frac{e'^2}{\sqrt{1 - v^2}}\left(1 - \frac{\mathbf{l}_s \cdot \mathbf{v}}{l_4}\right)\left(\frac{\mathbf{l}_s}{l_4} + \mathbf{k}_4\right). \tag{122}$$

If we use $\mathbf{M}'$ in the form $\mathbf{M}' = \mathbf{E}' + \mathbf{H}'$, we find [50]

$$\mathbf{b} = \frac{1}{\sqrt{1 - v^2}}[\overline{\mathbf{e}' \times \mathbf{h}'} + \mathbf{v} \cdot \mathbf{e}'\mathbf{e}' + \mathbf{v} \cdot \mathbf{h}'\mathbf{h}' - h'^2 \mathbf{v} + (e'^2 - \mathbf{v} \cdot \overline{\mathbf{e}' \times \mathbf{h}'})\mathbf{k}_4], \tag{123}$$

where $\overline{\mathbf{e}' \times \mathbf{h}'}$ has been used to denote the 1-vector $(\mathbf{e}' \times \mathbf{h}') \cdot \mathbf{k}_{123}$, which is the three dimensional complement of the 2-vector $\mathbf{e}' \times \mathbf{h}'$. Another equivalent form is

$$\mathbf{b} = \frac{l_4 - \mathbf{l}_s \cdot \mathbf{v}}{\sqrt{1 - v^2}\, l_4}(\overline{\mathbf{e}' \times \mathbf{h}'}^{s} + e'^2 \mathbf{k}_4). \tag{124}$$

The coefficient $(l_4 - \mathbf{l}_s \cdot \mathbf{v})/l_4 \sqrt{1 - v^2}$ is unity when v is negligible compared with the velocity of light, and therefore in all such cases $\mathbf{b}$ is the sum of two vectors one of which is the Poynting vector and the other along $\mathbf{k}_4$ equal in magnitude to the density of energy. Since the vector $\mathbf{b}$ comes so near to being the extended vector of energy density, the possibility is suggested that the energy of an electromagnetic field may not depend solely upon the field itself but to some

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50 For rapid calculation a rule for obtaining the three dimensional form of some products is useful. The most important of these rules is that if

$$\mathbf{A} = \mathbf{a} \cdot \mathbf{k}_{123} - \mathbf{b} \times \mathbf{k}_4 \quad \text{and} \quad \mathbf{c} = \mathbf{c}_s + c_4 \mathbf{k}_4,$$

where $\mathbf{a}$, $\mathbf{b}$ are three dimensional vectors, then

$$\mathbf{c} \cdot \mathbf{A} = \overline{\mathbf{c}_s \times \mathbf{a}} + c_4 \mathbf{b} + (\mathbf{c}_s \cdot \mathbf{b})\,\mathbf{k}_4.$$

Thus we have here

$$\begin{aligned}\mathbf{b} = (\mathbf{w} \cdot \mathbf{M}') \cdot \mathbf{M}' &= \frac{1}{\sqrt{1 - v^2}}[(\mathbf{v} + \mathbf{k}_4) \cdot (\mathbf{h}' \cdot \mathbf{k}_{123} - \mathbf{e}' \times \mathbf{k}_4)] \cdot (\mathbf{h}' \cdot \mathbf{k}_{123} - \mathbf{e}' \times \mathbf{k}_4) \\ &= \frac{1}{\sqrt{1 - v^2}}[\overline{\mathbf{v} \times \mathbf{h}'} + \mathbf{e}' + (\mathbf{v} \cdot \mathbf{e}')\mathbf{k}_4] \cdot (\mathbf{h}' \cdot \mathbf{k}_{123} - \mathbf{e}' \times \mathbf{k}_4) \\ &= \frac{1}{\sqrt{1 - v^2}}[\overline{\overline{\mathbf{v} \times \mathbf{h}'} \times \mathbf{h}'} + \overline{\mathbf{e}' \times \mathbf{h}'} + (\mathbf{v} \cdot \mathbf{e}')\mathbf{e}' + (\overline{\mathbf{v} \times \mathbf{h}'} \cdot \mathbf{e}' + \mathbf{e}' \cdot \mathbf{e}')\mathbf{k}_4],\end{aligned}$$

which is identical with the form given.

extent upon the velocity of the emitting electron. It is interesting further to note that by the application of rules already given we may evaluate $\diamond \cdot \mathbf{b}$ and show that it vanishes. Hence

$$\diamond \cdot \mathbf{b} = \nabla \cdot \mathbf{b}_s + \frac{\partial b_4}{\partial t} = 0, \tag{125}$$

where $\mathbf{b}_s$ is the vector which we have just shown to be approximately equal to the Poynting vector, and b_4 is approximately equal to the density of energy. This equation is therefore entirely analogous to the familiar theorem of Poynting. If we integrate over a three-dimensional volume,

$$\int\int\int \nabla \cdot \mathbf{b}_s \, dx_1 dx_2 dx_3 = -\frac{\partial}{\partial t}\int\int\int b_4 dx_1 dx_2 dx_3,$$

or

$$\int\int b_{sn} dS = -\frac{\partial}{\partial t}\int\int\int b_4 dx_1 dx_2 dx_3. \tag{126}$$

Thus the induction of $\mathbf{b}_s$ through any closed surface is equal to the rate of loss of b_4 in the enclosed volume.[51]

If in the vector field $\mathbf{b}$ we cut the hypercone by any planoid, it will be evident that the integral of $\mathbf{b}dS$ over the surface of intersection will be independent of the position and direction of the planoid; for the surface dS always lies in a tangent plane and $\mathbf{b}$ varies inversely as R^2 and hence as dS. The vector $\mathbf{b}dS$ bears a simple relation to $d\mathbf{g}$ which we have studied. For $d\mathbf{g} = (d\mathfrak{S}^* \cdot \mathbf{M}') \cdot \mathbf{M}'$, where $d\mathfrak{S}$ is determined by any planoid. We may therefore choose $d\mathfrak{S}$ perpendicular to $d\mathbf{s}$, that is, to $\mathbf{w}$. Then $d\mathfrak{S}^*$ is $\mathbf{w}\,d\mathfrak{S}$ and $d\mathfrak{S} = dSds$, and since $\mathbf{b}$ by definition is $(\mathbf{w} \cdot \mathbf{M}') \cdot \mathbf{M}'$, the integral of $d\mathbf{g}$ is the product of ds and the integral of $\mathbf{b}dS$. We might therefore by a consideration of $\mathbf{b}$ alone have obtained the same vector of extended momentum for the total energy emitted by an electron in the interval $d\mathbf{s}$.

We shall not pursue further the study of this interesting vector $\mathbf{b}$, but it may be well to point out that the two fields $\mathbf{M}'$ and $\mathbf{b}$ cannot both be additive. For since $\mathbf{b}$ is quadratic in $\mathbf{M}'$, we obtain a differ-

51 In general if a 1-vector field in four dimensions is of such a character that its four-dimensional divergence vanishes, we may obtain in three dimensions an equation of the type just found, wherein the surface integral over a closed surface of the space component of the vector is equal to the negative time derivative of the integral of the time-component of the vector over the enclosed volume. Such an equation may be interpreted as a continuity or conservation equation whenever the space component appears as a velocity multiplied by the quantity defined by the time-component.

ent result when we obtain **b** from a resultant **M** (no longer necessarily a singular vector) and when we add the **b**'s obtained from the original **M**''s. All the classic ideas of electromagnetic energy assume that it is the vectors **M** that are additive at a point.

The Field of Continuous Distributions of Electricity.

54. Since the locus of an electric charge is not a singular line, we may regard the charge as distributed continuously over a given region or regions rather than as concentrated at one or more discrete points. Thus instead of a single vector representing the locus of an electron, we may consider a vector field. Let a small (δ)-tube be parallel to and comprise n electron-loci each of charge ϵ. Then we may replace these on the one hand by a single vector $n\epsilon\mathbf{w}$, and on the other hand by a vector field $\mathbf{q}$ such that, if $d\mathfrak{S}$ is the volume of any portion of the tube cut off by a planoid perpendicular to $\mathbf{w}$,

$$\int \mathbf{q}\,d\mathfrak{S} = n\epsilon\mathbf{w}.$$

Or if $d\boldsymbol{\mathfrak{S}}$ is the vector volume cut off by any planoid whatever, then as in § 45,

$$\int (d\boldsymbol{\mathfrak{S}}\times\mathbf{q})^*\mathbf{w} = n\epsilon\mathbf{w}. \tag{127}$$

If now we write

$$\mathbf{q} = \rho_0\mathbf{w}, \tag{128}$$

ρ_0 evidently represents the density of electricity as it appears to an observer stationary with respect to the charge. To an observer with respect to whom the charge appears to be moving with the velocity $\mathbf{v}$ the density appears to be different. For we may write (127) in the form

$$\int - (d\boldsymbol{\mathfrak{S}}^*\cdot\mathbf{q})\,\mathbf{w};$$

and if $d\boldsymbol{\mathfrak{S}}$ is the volume cut off by the planoid perpendicular to the chosen time-axis $\mathbf{k}_4$, $d\boldsymbol{\mathfrak{S}}^* = d\mathfrak{S}\mathbf{k}_4$; then writing

$$\mathbf{w} = (\mathbf{v} + \mathbf{k}_4)/\sqrt{1 - v^2},$$

we have

$$\int \frac{\rho_0\mathbf{w}}{\sqrt{1 - v^2}}\,d\mathfrak{S} = n\epsilon\mathbf{w}. \tag{129}$$

If then ρ is the density of the moving charge, we must write

$$\rho = \frac{\rho_0}{\sqrt{1 - v^2}}. \tag{130}$$

When we compare the two vectors

$$\epsilon \mathbf{w} = \frac{\epsilon}{\sqrt{1 - v^2}} (\mathbf{v} + \mathbf{k}_4) \qquad \text{and} \qquad \rho_0 \mathbf{w} = \rho\, (\mathbf{v} + \mathbf{k}_4)$$

with the two vectors which we have obtained for a material system

$$m_0 \mathbf{w} = m\, (\mathbf{v} + \mathbf{k}_4) \qquad \text{and} \qquad \mu_0 \mathbf{w} = \frac{\mu_0}{\sqrt{1 - v^2}} (\mathbf{v} + \mathbf{k}_4)$$

we see that they are identical in mathematical form. But the components of $\epsilon \mathbf{w}$ are not quantities which are commonly used in physics, while the components of $\rho_0 \mathbf{w}$ are the density of electricity and of electric current. On the other hand the components of $m_0 \mathbf{w}$ are the fundamental quantities known as mass and momentum, while the components of $\mu_0 \mathbf{w}$ are not commonly used. This is probably due to the fact that the fundamental conservation law for electricity is $\Sigma\epsilon$ = const., whereas the fundamental conservation law for mass is not Σm_0 = const., but Σm = const.

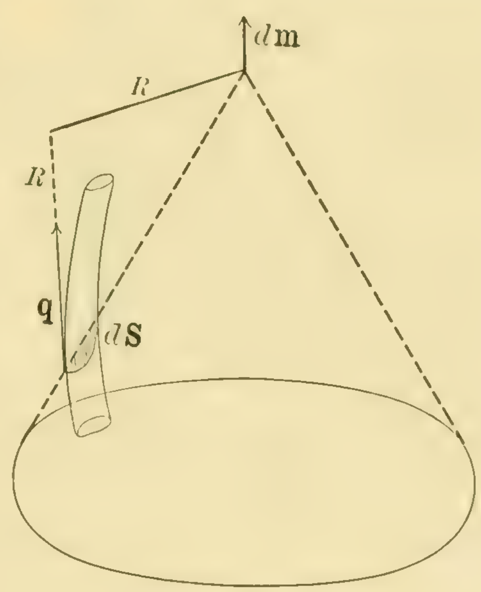

FIGURE 28.

55. We may now construct the potential at a point due to a continuous distribution of electricity, directly from (91) and (127).

$$\mathbf{m} = \int (d\mathfrak{S} \times \mathbf{q})^* \frac{\mathbf{w}}{R}. \tag{131}$$

The interpretation of this equation will be evident from an examination of a diagram which is an immediate extension of the one previously used in discussing potentials. And we may then show that, when a particular space and time have been assumed, the components of the vector $\mathbf{m}$ on the chosen space and time are the ordinary "retarded" potentials.

If (Figure 28)[52] we draw the backward hypercone from the point at which the potential $\mathbf{m}$ is to be determined, and if this backward hypercone cuts an elementary tube of the field $\mathbf{q}$ in the vector volume $d\mathfrak{S}$, then R is the perpendicular interval from the point in question to $\mathbf{w}$ or $\mathbf{w}$ produced (where $\mathbf{w}$ is the direction of $\mathbf{q}$ at the point where the tube cuts the cone). That part of the additive potential vector $\mathbf{m}$ which is due to this particular tube is

$$d\mathbf{m} = (d\mathfrak{S}\times\mathbf{q})^*\frac{\mathbf{w}}{R} = -\,d\mathfrak{S}^*\cdot\mathbf{q}\,\frac{\mathbf{w}}{R}. \tag{132}$$

Evidently the integration of $d\mathbf{m}$ is to be taken over the whole three dimensional spread produced by the intersection of the backward hypercone with the whole assemblage of infinitesimal tubes.

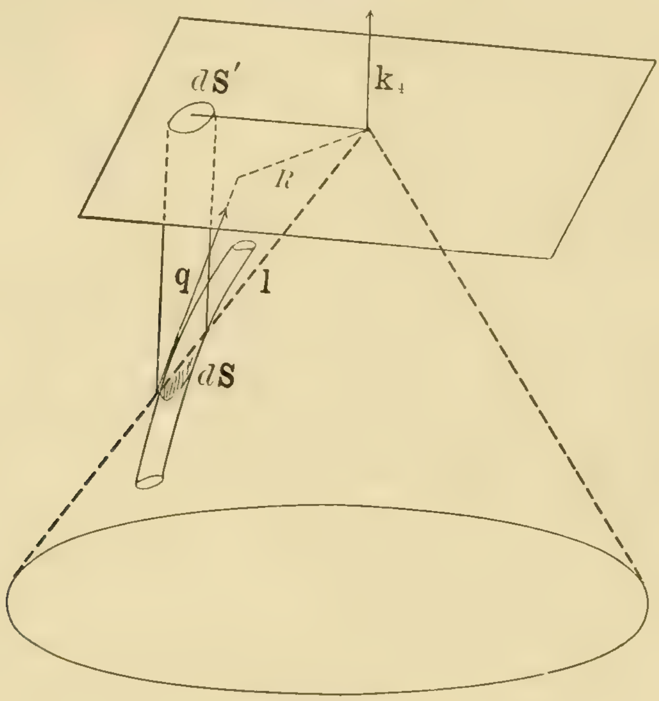

FIGURE 29.

Now if (Figure 29) we construct any planoid through the point in question, the retarded potentials are calculated as follows. This planoid, which we may regard as our space, is divided into elements of volume $d\mathfrak{S}'$ (corresponding to dS' in the figure). We consider the

[52] Figure 28 and Figure 29 are drawn and lettered for one dimension lower.

values $[\rho]$ and $[\rho\mathbf{v}]$ of the density and the current density which were in the element $d\mathfrak{S}'$ at a time previous by the length of time required for light to pass from $d\mathfrak{S}'$ to the point in question. From the four dimensional point of view this means that we project the element $d\mathfrak{S}'$ parallel to the time-axis upon the hypercone, and take as $[\rho]$ and $[\rho\mathbf{v}]$ the projections on time and space of the vector $\mathbf{q}$ at this point of the hypercone. We then form the integrals

$$\int \frac{[\rho]}{r}\, d\mathfrak{S}' \quad \text{and} \quad \int \frac{[\rho\mathbf{v}]}{r}\, d\mathfrak{S}', \tag{133}$$

where r is the distance from $d\mathfrak{S}'$ to the point at which the potential is wanted.

Let us now consider the element $d\mathbf{m}$ of our potential. The vector $d\mathbf{\mathfrak{S}}$ (corresponding to $d\mathbf{S}$ of the figure), being cut out of the hypercone, is a singular 3-vector, and its complement $d\mathbf{\mathfrak{S}}^*$ is therefore a singular 1-vector. Hence $d\mathfrak{S}'$ is numerically the projection of $d\mathbf{\mathfrak{S}}^*$ upon $\mathbf{k}_4$, and it is readily seen that

$$\frac{d\mathbf{\mathfrak{S}}^*}{d\mathfrak{S}'} = \frac{\mathbf{1}}{l_4}.$$

Substituting in (132),

$$d\mathbf{m} = -\frac{\mathbf{1}\cdot\mathbf{q}}{l_4}\frac{\mathbf{w}}{R}\, d\mathfrak{S}' = \frac{\mathbf{1}\cdot\rho_0\mathbf{w}}{l_4}\frac{\mathbf{w}}{R}\, d\mathfrak{S}'.$$

But $\mathbf{1}\cdot\mathbf{w} = -R$ by (80) and l_4 is equal to l_s, that is, to the r in (133). Hence

$$\mathbf{m} = \int \frac{[\rho\mathbf{v}] + [\rho]\,\mathbf{k}_4}{r}\, d\mathfrak{S}'. \tag{134}$$

If we designate the vector and scalar potentials as $\mathbf{a}$ and ϕ respectively, then

$$\mathbf{m} = \mathbf{a} + \phi\mathbf{k}_4. \tag{135}$$

We may show as before [53] that

$$\Diamond\cdot\mathbf{m} = 0 \quad \text{or} \quad \nabla\cdot\mathbf{a} + \frac{\partial\phi}{\partial t} = 0. \tag{136}$$

We have seen (§ 44) that $\Diamond\cdot\Diamond\,\mathbf{p} = 0$, or $\Diamond^2\mathbf{p} = 0$, and consequently $\Diamond^2\mathbf{m} = 0$ in the case of a point electron for all points not upon the

[53] A single differentiation under the sign of integration is permissible if ρ_0 remains finite; but a second differentiation is not permissible, as is well known in the theory of the potential.

locus of the electron. In the case of a continuous distribution of electricity we have [54]

$$\Diamond^2\mathbf{m} = -4\pi\mathbf{q}, \tag{137}$$

which might be proved directly; but this is unnecessary since it has frequently been shown by familiar methods that

$$\Diamond^2\mathbf{a} = -4\pi\rho\mathbf{v} \quad \text{and} \quad \Diamond^2\phi = -4\pi\rho. \tag{138}$$

Furthermore it is unnecessary to evaluate once more in detail the 2-vector

$$\mathbf{M} = \Diamond\times\mathbf{m} = \nabla\times\mathbf{a} + \left(\nabla\phi + \frac{\partial\mathbf{a}}{\partial t}\right)\times\mathbf{k}_4. \tag{139}$$

For $\nabla\times\mathbf{a}$ is the three dimensional complement of what is ordinarily known as curl $\mathbf{a}$ or $\mathbf{h}$, and $\nabla\phi + \dot{\mathbf{a}} = -\mathbf{e}$. Hence

$$\mathbf{M} = \mathbf{H} + \mathbf{E},$$

where the components of $\mathbf{H}$ and $\mathbf{E}$ are once more the components of magnetic and electric force.

56. Whether the 2-vector $\mathbf{M}$ of extended electric and magnetic force be derived from a number of point charges or from a charge continuously distributed, it is in general a complex or biplanar 2-vector.[55] The two invariants of $\mathbf{M}$ are $\mathbf{M}\cdot\mathbf{M}$ and $\mathbf{M}\cdot\mathbf{M}^* = (\mathbf{M}\times\mathbf{M})^*$. If, after choosing space and time axes, we write

$$\begin{aligned} \mathbf{M} &= h_1\mathbf{k}_{23} + h_2\mathbf{k}_{31} + h_3\mathbf{k}_{12} - e_1\mathbf{k}_{14} - e_2\mathbf{k}_{24} - e_3\mathbf{k}_{34}, \\ \mathbf{M}^* &= e_1\mathbf{k}_{23} + e_2\mathbf{k}_{31} + e_3\mathbf{k}_{12} + h_1\mathbf{k}_{14} + h_2\mathbf{k}_{24} + h_3\mathbf{k}_{34}, \end{aligned} \tag{140}$$

54 The vector $4\pi\mathbf{q}$ which we use is identical with the vector $\mathbf{q}$ used by Lewis, owing to a different choice of units of electrical quantity.

55 Since it is customary to divide a complex 2-vector into the two completely perpendicular uniplanar vectors which are uniquely determined, one being a (γ)-vector, the other a (δ)-vector, we might expect that the two lines of intersection of the (δ)-plane with the hypercone, and their projections upon a chosen space, might prove important. This is, however, not the case, although indeed from an analytic point of view the four directions, two of them imaginary, in which the hypercone is cut by the completely perpendicular (δ)-vector and (γ)-vector form a set of four independent directions possessing some advantages over the system $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, $\mathbf{k}_4$. In fact four vectors $\mathbf{j}_1$, $\mathbf{j}_2$, $\mathbf{j}_3$, $\mathbf{j}_4$ can be selected along these directions such that

$$\mathbf{j}_1\cdot\mathbf{j}_1 = \mathbf{j}_2\cdot\mathbf{j}_2 = \mathbf{j}_3\cdot\mathbf{j}_3 = \mathbf{j}_4\cdot\mathbf{j}_4 = 0, \qquad \mathbf{j}_1\cdot\mathbf{j}_2 = \mathbf{j}_3\cdot\mathbf{j}_4 = 1,$$
$$\mathbf{j}_1\cdot\mathbf{j}_3 = \mathbf{j}_1\cdot\mathbf{j}_4 = \mathbf{j}_2\cdot\mathbf{j}_3 = \mathbf{j}_2\cdot\mathbf{j}_4 = 0.$$

In terms of such a set of vectors the differential of arc is given by the equation

$$d\mathbf{r}\cdot d\mathbf{r} = dx^2 + dy^2 + dz^2 - dt^2 = Adudv + Bdwds.$$

(See Bateman, Proc. Lond. Math. Soc. [2] **10**, 107).

Other vectors which might be thought important would be the two lines in which the completely perpendicular planes cut the planoid which is taken then $\mathbf{M}\cdot\mathbf{M} = h^2 - e^2 = 2L$, where L is known as the Lagrangian function, and $\mathbf{M}\cdot\mathbf{M}^* = 2\,\mathbf{e}\cdot\mathbf{h}$. It is not surprising that the Lagrangian function should prove to be one of the fundamental invariants, but it is strange that the other invariant should be a quantity which has not been regarded as of fundamental importance in electromagnetic theory.

Since we have obtained our 2-vector from the equation

$$\mathbf{M} = \diamondsuit\times\mathbf{m},$$

we may readily evaluate $\diamondsuit\times\mathbf{M}$ and $\diamondsuit\cdot\mathbf{M}$. By (51) as a mathematical identity we have

$$\diamondsuit\times\mathbf{M} = \diamondsuit\times\diamondsuit\times\mathbf{m} = 0. \tag{141}$$

By (55)

$$\diamondsuit\cdot\mathbf{M} = \diamondsuit\cdot(\diamondsuit\times\mathbf{m}) = \diamondsuit(\diamondsuit\cdot\mathbf{m}) - (\diamondsuit\cdot\diamondsuit)\mathbf{m};$$

and since we have seen that in general $\diamondsuit\cdot\mathbf{m} = 0$, and substituting for $\diamondsuit\cdot\diamondsuit\mathbf{m}$ or $\diamondsuit^2\mathbf{m}$ from the preceding section,[56] we find

$$\diamondsuit\cdot\mathbf{M} = 4\pi\mathbf{q}. \tag{142}$$

By (52) as a mathematical identity,

$$\diamondsuit\cdot(\diamondsuit\cdot\mathbf{M}) = 0. \tag{143}$$

By the expansion of these equations we obtain directly the familiar equations of the electromagnetic field and the continuity equation

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as space. Following the method of § 38 we may write $\mathbf{M}$ as the sum of its two completely perpendicular parts in the form

$$\mathbf{M} = \tfrac{1}{2}\frac{\left(\sqrt{(\mathbf{M}\cdot\mathbf{M})^2 + (\mathbf{M}\cdot\mathbf{M}^*)^2} + \mathbf{M}\cdot\mathbf{M}\right)\mathbf{M} + (\mathbf{M}\cdot\mathbf{M}^*)\mathbf{M}^*}{\sqrt{(\mathbf{M}\cdot\mathbf{M})^2 + (\mathbf{M}\cdot\mathbf{M}^*)^2}}$$
$$+ \tfrac{1}{2}\frac{\left(\sqrt{(\mathbf{M}\cdot\mathbf{M})^2 + (\mathbf{M}\cdot\mathbf{M}^*)^2} - \mathbf{M}\cdot\mathbf{M}\right)\mathbf{M} - (\mathbf{M}\cdot\mathbf{M}^*)\mathbf{M}^*}{\sqrt{(\mathbf{M}\cdot\mathbf{M})^2 + (\mathbf{M}\cdot\mathbf{M}^*)^2}}.$$

Now the lines in which these two completely perpendicular planes cut the space $\mathbf{k}_{123}$ may be found by multiplying the planes by $\mathbf{k}_4$ by inner multiplication. As $\mathbf{k}_4\cdot\mathbf{M} = \mathbf{e}$ and $\mathbf{k}_4\cdot\mathbf{M}^* = -\mathbf{h}$, we have for the lines

$$\tfrac{1}{2}\frac{\left(\sqrt{L^2 + (\mathbf{e}\cdot\mathbf{h})^2} + L\right)\mathbf{e} - (\mathbf{e}\cdot\mathbf{h})\mathbf{h}}{\sqrt{L^2 + (\mathbf{e}\cdot\mathbf{h})^2}}, \qquad \tfrac{1}{2}\frac{\left(\sqrt{L^2 + (\mathbf{e}\cdot\mathbf{h})^2} - L\right)\mathbf{e} + (\mathbf{e}\cdot\mathbf{h})\mathbf{h}}{\sqrt{L^2 + (\mathbf{e}\cdot\mathbf{h})^2}}.$$

These vectors, however, like those mentioned above, are not found to be important in electromagnetic theory.

56 cf. equation (85).

expressing the conservation of electricity. We may write (141) in the form $\Diamond\cdot\mathbf{M}^* = 0$. Expressing $\mathbf{M}^*$ as in (140), this equation becomes

$$\left.\begin{aligned}\overline{\nabla\times\mathbf{e}} + \frac{\partial\mathbf{h}}{\partial t} &= 0,\\ \nabla\cdot\mathbf{h} &= 0.\end{aligned}\right\}$$

Similarly from (142)

$$\left.\begin{aligned}\overline{\nabla\times\mathbf{h}} - \frac{\partial\mathbf{e}}{\partial t} &= 4\pi\rho\mathbf{v},\\ \nabla\cdot\mathbf{e} &= 4\pi\rho.\end{aligned}\right\}$$

These are the well known field equations. Finally (143) gives the continuity equation

$$\nabla\cdot(\rho\mathbf{v}) + \frac{\partial\rho}{\partial t} = 0.$$

It cannot be too strongly emphasized that all these equations follow from the theorems of our four dimensional geometry without any further assumption than that the geometrical vector potential field derived from the locus of an electric charge is the extended electromagnetic vector potential.

57. We have seen that the singular 2-vector field $\mathbf{M}'$ produced by an accelerated electron determines a vector $d\mathbf{g}$ of four dimensional significance involving quantities which may be identified with energy and momentum in the radiation field. A search for similar vectors due to the field $\mathbf{M}$, which in general is not singular, proves, however, to be unsuccessful. In the case of radiation we wrote

$$d\mathbf{g} = (d\mathfrak{S}^*\cdot\mathbf{M}')\cdot\mathbf{M}',$$

or since it is readily shown (see footnote, § 62) that in this case $(d\mathfrak{S}^*\cdot\mathbf{M}')\cdot\mathbf{M}' = (d\mathfrak{S}^*\cdot\mathbf{M}'^*)\cdot\mathbf{M}'^*$ we could have obtained a more symmetrical form

$$d\mathbf{g} = \tfrac{1}{2}[(d\mathfrak{S}^*\cdot\mathbf{M}')\cdot\mathbf{M}' + (d\mathfrak{S}^*\cdot\mathbf{M}'^*)\cdot\mathbf{M}'^*]. \qquad (144)$$

In the case of the vector $\mathbf{M}$ we may write by analogy

$$\tfrac{1}{2}[(d\mathfrak{S}^*\cdot\mathbf{M})\cdot\mathbf{M} + (d\mathfrak{S}^*\cdot\mathbf{M}^*)\cdot\mathbf{M}^*], \qquad (145)$$

where $d\mathfrak{S}$ is the vector volume produced by intersecting a selected portion of the four dimensional field by a planoid. However, this cannot be made to give rise to a real vector in a four dimensional sense, but will only have meaning for the particular planoid chosen.

If we choose a particular $\mathbf{k}_4$ axis and its perpendicular planoid, then $d\mathfrak{S}^* = d\mathfrak{S}\ \mathbf{k}_4$ and the above expression becomes

$$\tfrac{1}{2}[(\mathbf{k}_4\cdot\mathbf{M})\cdot\mathbf{M} + (\mathbf{k}_4\cdot\mathbf{M}^*)\cdot\mathbf{M}^*]\,d\mathfrak{S}. \tag{146}$$

We may perform the operations here indicated upon the expanded form (140) of $\mathbf{M}$ and obtain[57]

$$[\mathbf{e}\times\mathbf{h} + \tfrac{1}{2}(e^2 + h^2)\,\mathbf{k}_4]\,d\mathfrak{S}. \tag{147}$$

Now $\mathbf{e}\times\mathbf{h}$, the complement in three dimensional space of $\mathbf{e}\times\mathbf{h}$, and $\tfrac{1}{2}(e^2 + h^2)$ are the familiar expressions for the Poynting vector and the density of electromagnetic energy, and the above expression therefore represents what is ordinarily regarded as the total electromagnetic momentum and energy in the volume $d\mathfrak{S}$.

Now after the axes have been chosen we may perform similar operations with $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$. Thus

$$\begin{aligned}
\tfrac{1}{2}(\mathbf{k}_1\cdot\mathbf{M})\cdot\mathbf{M} + \tfrac{1}{2}(\mathbf{k}_1\cdot\mathbf{M}^*)\cdot\mathbf{M}^* &= X_x\mathbf{k}_1 + X_y\mathbf{k}_2 + X_z\mathbf{k}_3 - X_t\mathbf{k}_4,\\
\tfrac{1}{2}(\mathbf{k}_2\cdot\mathbf{M})\cdot\mathbf{M} + \tfrac{1}{2}(\mathbf{k}_2\cdot\mathbf{M}^*)\cdot\mathbf{M}^* &= Y_x\mathbf{k}_1 + Y_y\mathbf{k}_2 + Y_z\mathbf{k}_3 - Y_t\mathbf{k}_4,\\
\tfrac{1}{2}(\mathbf{k}_3\cdot\mathbf{M})\cdot\mathbf{M} + \tfrac{1}{2}(\mathbf{k}_3\cdot\mathbf{M}^*)\cdot\mathbf{M}^* &= Z_x\mathbf{k}_1 + Z_y\mathbf{k}_2 + Z_z\mathbf{k}_3 - Z_t\mathbf{k}_4,\\
\tfrac{1}{2}(\mathbf{k}_4\cdot\mathbf{M})\cdot\mathbf{M} + \tfrac{1}{2}(\mathbf{k}_4\cdot\mathbf{M}^*)\cdot\mathbf{M}^* &= T_x\mathbf{k}_1 + T_y\mathbf{k}_2 + T_z\mathbf{k}_3 + T_t\mathbf{k}_4,
\end{aligned}$$

where

$$\begin{aligned}
X_x &= \tfrac{1}{2}(e_1^2 - e_2^2 - e_3^2 + h_1^2 - h_2^2 - h_3^2),\\
Y_y &= \tfrac{1}{2}(e_2^2 - e_3^2 - e_1^2 + h_2^2 - h_3^2 - h_1^2),\\
Z_z &= \tfrac{1}{2}(e_3^2 - e_1^2 - e_2^2 + h_3^2 - h_1^2 - h_2^2),\\
T_t &= \tfrac{1}{2}(e_1^2 + e_2^2 + e_3^2 + h_1^2 + h_2^2 + h_3^2),\\
X_y &= Y_x = e_1e_2 + h_1h_2, \text{ etc.},\\
T_x &= X_t = e_2h_3 - e_3h_2, \text{ etc.}
\end{aligned}$$

In these equations X_x, etc., are the familiar expressions for the components of the Maxwell strains; T_x, T_y, T_z are the components of the Poynting vector; and T_t is that which is ordinarily assumed to be the density of electromagnetic energy. This procedure is essentially that of Minkowski. We may reproduce his procedure exactly with the aid of dyadics. It may readily be shown (see appendix, § 62) that if $\mathbf{M}$ is any 2-vector, and I the unit dyadic or idemfactor, then the dyadics

$$\Phi = (\mathrm{I}\cdot\mathbf{M})\cdot(\mathrm{I}\cdot\mathbf{M}) \qquad \Phi^* = (\mathrm{I}\cdot\mathbf{M}^*)\cdot(\mathrm{I}\cdot\mathbf{M}^*)$$

[57] For abbreviated methods see a footnote in § 53.

are such that

$$\mathbf{a}\cdot\Phi = (\mathbf{a}\cdot\mathbf{M})\cdot\mathbf{M} \qquad \mathbf{a}\cdot\Phi^* = (\mathbf{a}\cdot\mathbf{M}^*)\cdot\mathbf{M}^*,$$

where $\mathbf{a}$ is any 1-vector. The expressions which we obtained from $\mathbf{M}$ and $\mathbf{k}_1, \mathbf{k}_2, \ldots$ in the form

$$\tfrac{1}{2}(\mathbf{k}_1\cdot\mathbf{M})\cdot\mathbf{M} + \tfrac{1}{2}(\mathbf{k}_1\cdot\mathbf{M}^*)\cdot\mathbf{M}^*, \text{ etc.}$$

might therefore equally well have been written

$$\tfrac{1}{2}\mathbf{k}_1\cdot(\Phi + \Phi^*), \text{ etc.}$$

It is these latter expressions which Minkowski obtained. The dyadic $\frac{1}{2}(\Phi + \Phi^*)$ is identical with Minkowski's matrix S, except in as far as he used imaginary space, and distinguished between electric force and displacement and between magnetic force and induction.[58]

While, as we see, the use of the dyadic $\frac{1}{2}(\Phi + \Phi^*)$ yields no results which are not also obtainable by the methods of simple vector analysis, yet to one who is familiar with the dyadic method it frequently affords a considerable gain in simplicity. Thus for example we may obtain an important result by considering the expression $\frac{1}{2}\Diamond\cdot(\Phi + \Phi^*)$, which may be shown to vanish in free space.[59] Now, if Ψ_s be the three dimensional dyadic of the Maxwell strains, if $\mathbf{e}\times\mathbf{h}$ is the Poynting vector, and if T_t is the density of energy, we have

$$0 = \tfrac{1}{2}\Diamond\cdot(\Phi + \Phi^*) = \Diamond\cdot(\Psi_s - \overline{\mathbf{e}\times\mathbf{h}}\,\mathbf{k}_4 - \mathbf{k}_4\overline{\mathbf{e}\times\mathbf{h}} - \mathbf{k}_4\mathbf{k}_4 T_t), \quad (148)$$

or

$$\nabla\cdot\Psi_s - \frac{\partial}{\partial t}(\overline{\mathbf{e}\times\mathbf{h}}) = 0 \qquad \text{and} \qquad \nabla\cdot\overline{\mathbf{e}\times\mathbf{h}} + \frac{\partial}{\partial t}T_t = 0. \quad (149)$$

The first is the important equation of Lorentz connecting the force

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58 The form of the dyadic $\Psi = \frac{1}{2}(\Phi + \Phi^*)$ is

$$\begin{aligned} &X_x\mathbf{k}_1\mathbf{k}_1 + X_y\mathbf{k}_1\mathbf{k}_2 + X_z\mathbf{k}_1\mathbf{k}_3 - X_t\mathbf{k}_1\mathbf{k}_4 \\ &+ Y_x\mathbf{k}_2\mathbf{k}_1 + Y_y\mathbf{k}_2\mathbf{k}_2 + Y_z\mathbf{k}_2\mathbf{k}_3 - Y_t\mathbf{k}_2\mathbf{k}_4 \\ &+ Z_x\mathbf{k}_3\mathbf{k}_1 + Z_y\mathbf{k}_3\mathbf{k}_2 + Z_z\mathbf{k}_3\mathbf{k}_3 - Z_t\mathbf{k}_3\mathbf{k}_4 \\ &- T_x\mathbf{k}_4\mathbf{k}_1 - T_y\mathbf{k}_4\mathbf{k}_2 - T_z\mathbf{k}_4\mathbf{k}_3 - T_t\mathbf{k}_4\mathbf{k}_4. \end{aligned}$$

59 From (158), with $\mathbf{A} = \mathbf{M}$, $\mathbf{A}' = \mathbf{M}$, and from (141) and (142), since in free space $\mathbf{q} = 0$. Where there is electricity the equation would be

$$\tfrac{1}{2}\Diamond\cdot\Psi = 4\pi\mathbf{q}\cdot\mathbf{M}.$$

due to the Maxwell strains and the rate of change of the Poynting vector; the second is Poynting's theorem.[60]

Mechanics of a Material System, and Gravitation.

58. The mechanics of a particle which we have treated in restricted cases in § 21 and § 36 can now be completely generalized. If m_0 is the mass of a particle, and $\mathbf{w}$ the unit tangent to its locus, then

$$m_0\mathbf{w} = m\,(\mathbf{v} + \mathbf{k}_4)$$

is the vector of extended momentum, whose projections on any chosen space and time are $m\mathbf{v}$, the momentum, and m, the mass or energy. If we consider any number of such vectors, we may state the laws of conservation of momentum, mass and energy in a single theorem as follows. *The sum of all the vectors of extended momentum is constant,* that is, the sum of all such vectors cutting any unclosed and continuous three dimensional (γ)-spread is independent of the (γ)-spread chosen. This law is, however, true only when we state that wherever there is energy there is a vector of extended momentum, whether or not this energy is associated with that which is ordinarily known as a material system. Thus in § 51 we have discussed the vector $d\mathbf{g}$ which we have identified with the vector of extended momentum of radiant electromagnetic energy. A *Hohlraum* obeys all the laws of a material system, and must be treated as such. We shall mention presently another form of radiant energy to which also we must assign an extended momentum.

Just as the discrete locus of an electric charge was replaced by a continuously distributed field of density vectors, we might regard a material system as a continuum. Thus if we have a small (δ)-tube parallel to and comprising one or more (δ)-lines of which the resultant vector is $m_0\mathbf{w}$, we may replace this vector by the expression $(d\mathfrak{S}\cdot\mu_0\mathbf{w})^*\mathbf{w}$, where $d\mathfrak{S}$ is the intersection of the tube with any planoid, and $\mu_0\mathbf{w}$ is the vector of the distributed field. If $d\mathfrak{S}$ is taken perpendicular to $\mathbf{w}$, this reduces to $\mu_0\mathbf{w}d\mathfrak{S}$, and therefore μ_0 is the density as it appears to an observer at rest with respect to the system. It must, however,

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[60] In case there is electricity present, these equations become respectively

$$\nabla\cdot\Psi_s - \frac{\partial}{\partial t}\overline{\mathbf{e}\times\mathbf{h}} = 4\,\pi\rho(\mathbf{e} + \overline{\mathbf{v}\times\mathbf{h}}), \qquad \nabla\cdot\overline{\mathbf{e}\times\mathbf{h}} + \frac{\partial}{\partial t}T_t = -\,4\pi\rho\mathbf{v}\cdot\mathbf{e}.$$

Note that if $\mathbf{v}$ is small, the second equation is corrected by the small term $-4\pi\rho\mathbf{v}\cdot\mathbf{e}$, whereas the first has the large correction $4\pi\rho(\mathbf{e} + \mathbf{v}\times\mathbf{h})$, approximately $4\pi\rho\mathbf{e}$.

be borne in mind that when the system in question embraces any energy which is moving with the velocity of light, this method fails completely. And this is an essential difference between a system of electric charges and a system of matter or energy. Indeed a consideration of the properties of a *Hohlraum* shows that it may be unsafe in any case to assume that a material system is not composed wholly or in part of energy moving with the velocity of light.

59. In the study of hydrodynamics cases are considered in which the different portions of the fluid exert forces upon one another, and these forces may be themselves due to a flow of energy with the velocity of light. In fact it is only when we consider a fluid devoid of such mutual forces that we are able to obtain from our continuously distributed field and the law of extended momentum the known equation of hydrodynamics. Let us consider a continuously distributed field divided into infinitesimal tubes in each of which the extended momentum is now written as $(d\mathfrak{S}\times\mu_0\mathbf{w})^*\mathbf{w}$. Then our conservation law leads to the equation

$$\int (d\mathfrak{S}\times\mu_0\mathbf{w})^*\mathbf{w} = \textbf{const.} \tag{150}$$

Or if we consider a portion of the field composed of a number of adjoining tubes and cut off by two different planoids, then since none of the vectors of extended momentum cut the boundary tube the integral of our vector over the whole three dimensional boundary of this four dimensional region is merely the integral over the two planoids namely,

$$-\int (d\mathfrak{S}^*\cdot\mu_0\mathbf{w})\mathbf{w} = 0 = -\int d\mathfrak{S}^*\cdot(\mu_0\mathbf{ww}),$$

by definition of the dyadic $\mu_0\mathbf{ww}$. Now by the application of (65) we may convert this triple integral into a quadruple integral. Thus

$$\int d^*\mathfrak{S}\cdot(\mu_0\mathbf{ww}) = \int d\Sigma^*\Diamond\cdot(\mu_0\mathbf{ww}) = 0.$$

Hence

$$\Diamond\cdot(\mu_0\mathbf{ww}) = 0. \tag{151}$$

If now we set $\mathbf{w} = (\mathbf{v}+\mathbf{k}_4)/\sqrt{1-v^2}$ and $\mu=\mu_0/(1-v^2)$ by (88), this gives by expansion[61]

$$\Diamond\cdot[\mu(\mathbf{v}+\mathbf{k}_4)(\mathbf{v}+\mathbf{k}_4)] = [\Diamond\cdot\mu(\mathbf{v}+\mathbf{k}_4)](\mathbf{v}+\mathbf{k}_4) + [\mu(\mathbf{v}+\mathbf{k}_4)\cdot\Diamond](\mathbf{v}+\mathbf{k}_4) = 0,$$

[61] If $\mathbf{ab}$ is a dyadic, evidently $\Diamond\cdot(\mathbf{ab}) = (\Diamond\cdot\mathbf{a})\mathbf{b} + (\mathbf{a}\cdot\Diamond)\mathbf{b}$.

or

$$\left[\nabla\cdot(\mu\mathbf{v}) + \frac{\partial\mu}{\partial t}\right](\mathbf{v} + \mathbf{k}_4) + \mu\left[(\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\partial\mathbf{v}}{\partial t}\right] = 0,$$

Hence the space and time components both vanish, and

$$\nabla\cdot(\mu\mathbf{v}) + \frac{\partial\mu}{\partial t} = 0, \tag{152}$$

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} + \frac{\partial\mathbf{v}}{\partial t} = 0. \tag{153}$$

The first of these two is the continuity equation, the second is the dynamical equation of hydrodynamics in the present restricted case.[62] The fact that we are thus led not to the general laws of hydrodynamics but merely to the laws for a comparatively trivial case shows the inadequacy of any attempt to distribute the vectors of extended momentum into a continuous field.

Minkowski added to his great memoir on the "Grundgleichungen für die electromagnetischen Vorgänge" an appendix on mechanics which seems to have been more hastily written. In this section he bases his analysis upon two assumptions which must be considered as fundamentally erroneous. The first of these is that $\mu = \mu_0/\sqrt{1 - v^2}$; and the second that Σm_0 is a constant.[63] The results should be that $\mu = \mu_0/(1 - v^2)$ and that Σm is a constant. We have already discussed (§ 23) cases in which m_0 is not a constant.

60. Every locus of a particle to which belongs the vector $m_0\mathbf{w}$ gives rise to the geometric vector fields

$$m_0\mathbf{p} = m_0\mathbf{w}/R \qquad \text{and} \qquad m_0\mathbf{P} = m_0\Diamond\times\mathbf{p}.$$

By replacing the constant ϵ by the constant m_0 we might proceed to reproduce identically all of the formulas which we have obtained for the electromagnetic field. If a suitable unit of mass be chosen, we should then observe that in case axes are so taken that the particle appears at rest, the space vector $m_0\mathbf{P}\cdot\mathbf{k}_4$ becomes identical

62 It may well be that the introduction of additional terms sufficient to give (153) a form as general as that ordinarily used in hydrodynamics would not require serious modifications in (152). For in ordinary units the pressure of light is measured by the density of electromagnetic energy, whereas the mass of the light is its energy divided by the square of the velocity of light. Compare also the fact that the changes in the equations (149) when electricity is present is small in one case and large in the other.

63 The second of these errors has already been pointed out by Abraham, Rend. Circ. Mat. di Palermo, **30**, 45.

in form with gravitational force, and the time component of $m_0\mathbf{p}$ with gravitational potential. When the particle is not at rest it is evident that just as in electromagnetics we must add to the scalar potential a vector potential, and to the (corrected) gravitational force another force which by analogy we may call gravito-magnetic. In every other respect, moreover, the two problems must be completely analogous. Thus an accelerated particle must give rise to a singular vector field which we should expect to be associated with the flow of a new form of radiant energy.[64]

Appendix.

Dyadics.

61. The dyad or formal product of vectors, introduced in 1844 by Grassmann under the name of open product, was given a fundamental position in vector analysis by Gibbs. Gibbs also developed the idea of the dyadic, or sum of dyads, as the most general type of linear vector operator. The dyadic is useful not only in the treatment of the linear vector transformations or strains, but also as a mere formal product (or sum of products) which can later be converted into such determinate products as the outer and inner products of our analysis. We shall outline very briefly the form taken by the theory of dyadics in the vector analysis which we employed.[65]

If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, . . . are 1-vectors, then the product expressed by the mere juxtaposition of $\mathbf{a}$ and $\mathbf{b}$, namely, $\mathbf{ab}$ is called a dyad. The sum of two or more such dyads is called a dyadic, and any such dyadic in an n-dimensional space can be reduced to the sum of n dyads. As the dyad is in part defined by the assumption of the distributive law, every dyadic in four dimensional space may be expressed as a block of sixteen terms analogous to a matrix. Such an expansion is of great

64 It should, however, be noted that there is nothing in electromagnetics corresponding to the vector of extended momentum of energy moving with the velocity of light. It is, furthermore, to be noted that while the radiation fields produced by the acceleration of two electrons, whether of the same or opposite sign, due to their interaction, are cumulative, that produced by the acceleration of two material particles, due to their gravitational attraction, must tend to compensate one another. (Cf. the paper of D. L. Webster, These Proceedings, **47**, 569, 1912.)

65 For further developments we refer to Gibbs's work as set forth in his Scientific Papers, **2**, in the Gibbs-Wilson text on Vector Analysis, and in Wilson's "On the theory of double products and strains in hyperspace," Trans. Conn. Acad., **14**, 1.

convenience when the individual vectors are expressed in terms of coordinate vectors. Thus,

$$\begin{aligned} &a_{11}\mathbf{k}_1\mathbf{k}_1 + a_{12}\mathbf{k}_1\mathbf{k}_2 + a_{13}\mathbf{k}_1\mathbf{k}_3 + a_{14}\mathbf{k}_1\mathbf{k}_4 \\ +\ &a_{21}\mathbf{k}_2\mathbf{k}_1 + a_{22}\mathbf{k}_2\mathbf{k}_2 + a_{23}\mathbf{k}_2\mathbf{k}_3 + a_{24}\mathbf{k}_2\mathbf{k}_4 \\ +\ &a_{31}\mathbf{k}_3\mathbf{k}_1 + a_{32}\mathbf{k}_3\mathbf{k}_2 + a_{33}\mathbf{k}_3\mathbf{k}_3 + a_{34}\mathbf{k}_3\mathbf{k}_4 \\ +\ &a_{41}\mathbf{k}_4\mathbf{k}_1 + a_{42}\mathbf{k}_4\mathbf{k}_2 + a_{43}\mathbf{k}_4\mathbf{k}_2 + a_{44}\mathbf{k}_4\mathbf{k}_4. \end{aligned}$$

The product of a vector **a** and a dyad **bc** is expressed and defined as

$$\mathbf{a}\cdot\mathbf{bc} = \mathbf{a}\cdot(\mathbf{bc}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{c},$$

It is a 1-vector along **c**. Similarly $\mathbf{ab}\cdot\mathbf{c} = (\mathbf{ab})\cdot\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c})$. The product of a vector into any dyadic follows from the distributive law.

The product of two dyads is expressed and defined as follows.

$$\mathbf{ab}\cdot\mathbf{cd} = (\mathbf{ab})\cdot(\mathbf{cd}) = \mathbf{a}(\mathbf{b}\cdot\mathbf{c})\mathbf{d} = (\mathbf{b}\cdot\mathbf{c})\mathbf{ad}.$$

It is another dyad. The product of two dyadics then follows from the distributive law, and is therefore a dyadic.

Since the dyad product is obtained without implying any relation between the sixteen units $\mathbf{k}_i\mathbf{k}_j$, it is the most general product and comprises within itself the more special products which we have designated as the inner and outer products and which we may obtain from it by inserting the special sign of multiplication corresponding to these products, thus giving respectively a scalar or a 2-vector. Hence from any dyadic a scalar or a 2-vector may be obtained by converting each dyad into an inner or outer product. This method was employed in computing $\diamondsuit\cdot\mathbf{p}$ and $\diamondsuit\times\mathbf{p}$ in § 43 and § 44.

A dyadic is said to selfconjugate when for all the coefficients $a_{ij} = a_{ji}$, and anti-selfconjugate when for all the coefficients $a_{ij} = -a_{ji}$. The latter can have no terms in the main diagonal, and therefore has but six degrees of freedom, whereas the selfconjugate dyadic has ten.[66] Except for sign the anti-selfconjugate dyadic not only determines, but conversely is determined by, a 2-vector of the form

$$a_{12}\mathbf{k}_{12} + a_{13}\mathbf{k}_{13} + a_{14}\mathbf{k}_{14} + a_{23}\mathbf{k}_{23} + a_{24}\mathbf{k}_{24} + a_{34}\mathbf{k}_{34},$$

where $a_{12}, \ldots$ are the coefficients of $\mathbf{k}_1\mathbf{k}_2, \ldots$ in the expanded form of the dyadic. This 2-vector is one half the 2-vector obtained by inserting the sign of outer multiplication in the dyads constituting the dyadic.

66 Any dyadic may be written as the sum of two dyadics one of which is selfconjugate, the other anti-selfconjugate.

If Φ is any dyadic, then we have seen that $\mathbf{a} \cdot \Phi$ is another 1-vector. In general $\mathbf{a} \cdot \Phi$ is not equal to $\Phi \cdot \mathbf{a}$. If, however, Φ is selfconjugate, $\mathbf{a} \cdot \Phi = \Phi \cdot \mathbf{a}$; and if Φ is anti-selfconjugate $\mathbf{a} \cdot \Phi = - \Phi \cdot \mathbf{a}$. Hence it may readily be shown that an anti-selfconjugate dyadic turns a vector into a perpendicular vector.

The dyadic which turns a vector into itself is called the idemfactor I. Thus

$$\mathbf{a} \cdot \mathrm{I} = \mathrm{I} \cdot \mathbf{a} = \mathbf{a}; \tag{154}$$

for I is selfconjugate, and when expanded in terms of chosen coordinate vectors is,[67] in the non-Euclidean geometry which we are discussing,

$$\mathrm{I} = \mathbf{k}_1\mathbf{k}_1 + \mathbf{k}_2\mathbf{k}_2 + \mathbf{k}_3\mathbf{k}_3 - \mathbf{k}_4\mathbf{k}_4.$$

62. We could now proceed to develop the theory of dyadics involving vectors of any dimensionalities and their products with each other and with vectors of various dimensionalities. In general if α, β, γ are vectors of any dimensionalities the dyad $\beta\gamma$ may be defined in terms of our inner product by the equation $\alpha \cdot (\beta\gamma) = (\alpha \cdot \beta)\gamma$. This product is itself a dyad unless α, β are of the same dimensionality. Such a discussion, however, would carry us further than is necessary for our present purpose, and we shall therefore consider chiefly one case, which has acquired particular importance through the work of Minkowski.

If $\mathbf{r}$ is any 1-vector, and $\mathbf{A}$ any 2-vector, then the product

$$\mathbf{r}' = \mathbf{r} \cdot \mathbf{A}$$

is a linear vector function of $\mathbf{r}$. It is evident therefore that this multiplication by $\mathbf{A}$ is equivalent to a multiplication by some dyadic Ω. Let us find the relation between this dyadic Ω and $\mathbf{A}$.

If Φ is any dyadic (made up of 1-vectors), we may define the products $\Phi \cdot \mathbf{A}$ and $\mathbf{A} \cdot \Phi$ by first defining the products,

$$(\mathbf{ab}) \cdot \mathbf{A} = \mathbf{a}\,(\mathbf{b} \cdot \mathbf{A}), \qquad \mathbf{A} \cdot (\mathbf{ab}) = (\mathbf{A} \cdot \mathbf{a})\,\mathbf{b},$$

67 As a matrix the idemfactor would be written

$$\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix} \quad \text{instead of} \quad \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix};$$

and the laws of multiplication of matrices would be modified. It is possible, however, to keep the ordinary theory of matrices by the introduction of imaginaries, as Minkowski does.

and then applying the distributive law. The products $\mathbf{A}\cdot\Phi$ and $\Phi\cdot\mathbf{A}$ are therefore themselves dyadics of the same type as Φ. If in place of Φ we use the idemfactor I, then it is easily shown that

$$\mathrm{I}\cdot\mathbf{A}\ (= -\mathbf{A}\cdot\mathrm{I})$$

is the anti-selfconjugate dyadic which is determined by the 2-vector $\mathbf{A}$.

$$\left.\begin{array}{rllll} \Omega = \mathrm{I}\cdot\mathbf{A} = & & -A_{12}\mathbf{k}_1\mathbf{k}_2 & -A_{13}\mathbf{k}_1\mathbf{k}_3 & -A_{14}\mathbf{k}_1\mathbf{k}_4 \\ & +A_{12}\mathbf{k}_2\mathbf{k}_1 & & -A_{23}\mathbf{k}_2\mathbf{k}_3 & -A_{24}\mathbf{k}_2\mathbf{k}_4 \\ & +A_{13}\mathbf{k}_3\mathbf{k}_1 & +A_{23}\mathbf{k}_3\mathbf{k}_2 & & -A_{34}\mathbf{k}_3\mathbf{k}_4 \\ & +A_{14}\mathbf{k}_4\mathbf{k}_1 & +A_{24}\mathbf{k}_4\mathbf{k}_2 & +A_{34}\mathbf{k}_4\mathbf{k}_3 & \end{array}\right\} \quad (155)$$

If we denote by $\Omega_\times$ the 2-vector obtained by inserting the cross in the dyads of Ω, we have $\Omega_\times = (\mathrm{I}\cdot\mathbf{A})_\times = -2\mathbf{A}$.

It is this relation between 2-vectors and linear vector functions or dyadics which enables Minkowski to replace a 2-vector by an anti-selfconjugate (or alternating) matrix and vice versa.

If Ω and Ω' are the two dyadics obtained from the two 2-vectors $\mathbf{A}$ and $\mathbf{A}'$, we may form the product $\Omega\cdot\Omega'$. (This is the product fF of Minkowski). We can then write

$$(\mathbf{r}\cdot\mathbf{A})\cdot\mathbf{A}' = (\mathbf{r}\cdot\Omega)\cdot\Omega' = \mathbf{r}\cdot(\Omega\cdot\Omega'). \quad (156)$$

We employed (§ 57) the selfconjugate dyadic $\Omega\cdot\Omega = (\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A})$, and another dyadic $\frac{1}{2}(\Omega\cdot\Omega + \Omega^*\cdot\Omega^*)$, where Ω^* was defined as $\Omega^* = \mathrm{I}\cdot\mathbf{A}^*$. This dyadic corresponds to the matrix S of Minkowski,[68] and may be regarded as the dyadic representing stress in four dimensional space.

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68 The expression $(\mathbf{r}\cdot\mathbf{A})\cdot\mathbf{A}'$ may be transformed by (38).

$$(\mathbf{r}\cdot\mathbf{A})\cdot\mathbf{A}' = -\mathbf{r}(\mathbf{A}\cdot\mathbf{A}') + \mathbf{A}\cdot(\mathbf{r}\times\mathbf{A}').$$

As $\mathbf{A}\cdot(\mathbf{r}\times\mathbf{A}')$ is a 1-vector, the complement of its complement is itself, by (26). By rules (30) and (24)

$$[\mathbf{A}\cdot(\mathbf{r}\times\mathbf{A}')]^{**} = [\mathbf{A}\times(\mathbf{r}\times\mathbf{A}')^*]^* = [(\mathbf{r}\cdot\mathbf{A}'^*)\times\mathbf{A}]^* = (\mathbf{r}\cdot\mathbf{A}'^*)\cdot\mathbf{A}^*.$$

Hence we obtain the important relation

$$(\mathbf{r}\cdot\mathbf{A})\cdot\mathbf{A}' = -\mathbf{r}(\mathbf{A}\cdot\mathbf{A}') + (\mathbf{r}\cdot\mathbf{A}'^*)\cdot\mathbf{A}^*.$$

By introducing dyadics and canceling the vector $\mathbf{r}$, we have

$$(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}') = -(\mathbf{A}\cdot\mathbf{A})\mathrm{I} + (\mathrm{I}\cdot\mathbf{A}'^*)\cdot(\mathrm{I}\cdot\mathbf{A}^*).$$

If we set

$$\Psi = \tfrac{1}{2}[(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}') + (\mathrm{I}\cdot\mathbf{A}'^*)\cdot(\mathrm{I}\cdot\mathbf{A}^*)],$$

we may write

$$(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}') = \Psi - \tfrac{1}{2}(\mathbf{A}\cdot\mathbf{A}')\mathrm{I}, \qquad (\mathrm{I}\cdot\mathbf{A}'^*)\cdot(\mathrm{I}\cdot\mathbf{A}^*) = \Psi + \tfrac{1}{2}(\mathbf{A}\cdot\mathbf{A}')\mathrm{I}.$$

The dyadic Ψ is precisely the matrix S of Minkowski.

The transformation $\mathbf{r}' = \mathbf{r} \cdot \mathbf{A}$, where $\mathbf{A}$ is a uniplanar 2-vector, can be regarded geometrically as an annihilation of that part of $\mathbf{r}$ which is perpendicular to $\mathbf{A}$, and a replacing of the component of $\mathbf{r}$ in $\mathbf{A}$ by a perpendicular vector magnified in the ratio of A to 1. The transformation $\mathbf{r}' = (\mathbf{r} \cdot \mathbf{A}) \cdot \mathbf{A}$ therefore annihilates components perpendicular to $\mathbf{A}$, and reverses components in $\mathbf{A}$, multiplying them further by $\mathbf{A} \cdot \mathbf{A}$. Hence if $\mathbf{A}$ is a (γ)-plane, the transformation in that plane is rotation through a straight angle combined with a stretch as $A^2 : 1$; whereas if $\mathbf{A}$ is a (δ)-plane, the transformation is one of stretching only, as $\mathbf{A} \cdot \mathbf{A}$ is negative.

In case $\mathbf{A}$ is biplanar we may resolve it into its two completely perpendicular parts, $\mathbf{A} = \mathbf{B} + \mathbf{C}$, where $\mathbf{B}$ is a (γ)-vector and $\mathbf{C}$ a (δ)-vector. Then the equation

$$\mathbf{r}' = (\mathbf{r} \cdot \mathbf{A}) \cdot \mathbf{A} = (\mathbf{r} \cdot \mathbf{B}) \cdot \mathbf{B} + (\mathbf{r} \cdot \mathbf{C}) \cdot \mathbf{C}$$

holds by virtue of the fact that $\mathbf{r} \cdot \mathbf{B}$ is perpendicular to $\mathbf{C}$, and $\mathbf{r} \cdot \mathbf{C}$ perpendicular to $\mathbf{B}$. Hence the transformation $\mathbf{r}' = (\mathbf{r} \cdot \mathbf{A}) \cdot \mathbf{A}$ consists of rotation through a straight angle and stretching in the ratio $B^2 : 1$ for components along $\mathbf{B}$, and of stretching alone in the ratio $C^2 : 1$ for components along $\mathbf{C}$.

The transformation $\mathbf{r}' = (\mathbf{r} \cdot \mathbf{A}) \cdot \mathbf{A} + (\mathbf{r} \cdot \mathbf{A}^*) \cdot \mathbf{A}^*$ is now readily seen to be a stretching of components along $\mathbf{B}$ or $\mathbf{C}$ in the ratio $(B^2 + C^2) : 1$ combined with a reversal of the direction of the components along $\mathbf{B}$. If this transformation were repeated, the result would be to stretch all vectors in space in the ratio $(B^2 + C^2)^2 : 1$. But

$$(B^2 + C^2)^2 = (B^2 - C^2)^2 + 4 B^2 C^2 = (\mathbf{A} \cdot \mathbf{A})^2 + (\mathbf{A} \cdot \mathbf{A}^*)^2.$$

Hence the square of $\frac{1}{2}(\Omega \cdot \Omega + \Omega^* \cdot \Omega^*)$ is $\frac{1}{4}[(\mathbf{A} \cdot \mathbf{A})^2 + (\mathbf{A} \cdot \mathbf{A}^*)^2]$ I, a multiple of the idemfactor. This is the geometric interpretation of a result obtained analytically by Minkowski.

63. From the definition (48) of the differentiating operator $\diamondsuit$,

$$d\mathbf{f} = d\mathbf{r} \cdot \diamondsuit \mathbf{f},$$

it follows that the expression $\diamondsuit \mathbf{f}$, where $\mathbf{f}$ is a 1-vector, is a dyadic. This definition may frequently be applied directly and with ease to determining the dyadic $\diamondsuit \mathbf{f}$, and renders unnecessary the expansion of $\diamondsuit \mathbf{f}$ in terms of its components. For if the value of $d\mathbf{f}$ for four independent displacements $d\mathbf{r}$ can be found, the dyadic is thereby completely determined, and in some cases can immediately be written down by inspection. This was the method pursued in § 44. The dyadic itself, however, was not then desired except for the purpose of deriving the scalar $\diamond\cdot\mathbf{f}$ and the 2-vector $\diamond\times\mathbf{f}$, which are functions of it.

By means of the same defining equation the operator $\diamond$ may be applied to 2-vector functions of position. The result $\diamond\mathbf{F}$ is then a dyadic in which the first vectors of the dyads are 1-vectors and the second vectors 2-vectors. If written out in terms of the coordinate unit vectors, such a dyadic would consist of twenty-four terms, each of the type $\mathbf{k}_i\mathbf{k}_{jk}$, $j \neq k$. By inserting the dot or cross, the 1-vector $\diamond\cdot\mathbf{F}$ and the 3-vector $\diamond\times\mathbf{F}$ are immediately found. In case the 2-vector $\mathbf{F}$ is given as a product $\mathbf{f}\times\mathbf{g}$ of two 1-vectors, the dyadic $\diamond\mathbf{F}$ may be obtained directly by means of the rules of differentiation in terms of the dyadics $\diamond\mathbf{f}$ and $\diamond\mathbf{g}$. For

$$d\mathbf{r}\cdot\diamond\mathbf{F} = d\mathbf{F} = d(\mathbf{f}\times\mathbf{g}) = d\mathbf{f}\times\mathbf{g} + d\mathbf{f}\times d\mathbf{g} = \mathbf{f}\times\mathbf{g} - d\mathbf{g}\times\mathbf{f},$$

$$d\mathbf{r}\cdot\diamond\mathbf{F} = d\mathbf{r}\cdot\diamond\mathbf{f}\times\mathbf{g} - d\mathbf{r}\cdot\diamond\mathbf{g}\times\mathbf{f},$$

$$\diamond\mathbf{F} = \diamond\mathbf{f}\times\mathbf{g} - \diamond\mathbf{g}\times\mathbf{f}.$$

It was such analysis which was used in § 44. It illustrates strikingly the great advantage of the symbol $\diamond$ over such symbols as Div, Rot, Grad, and 𝔇𝔦𝔳.

If Ψ is a dyadic function of position, the equation $d\mathbf{r}\cdot\diamond\Psi = d\Psi$ may be used to define $\diamond\Psi$, which is a triadic, that is, a sum of formal products of which each contains three vectors juxtaposed without any sign of multiplication. By interposing a dot between the first two of the three vectors in the triads, we find the 1-vector $\diamond\cdot\Psi$. The expression $\diamond\cdot\Psi$ corresponds to what Minkowski calls lor Ψ, where Ψ is for him a matrix.

We may compute the expression $\diamond\Psi$ in the case where

$$\Psi = \tfrac{1}{2}[(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}') + (\mathrm{I}\cdot\mathbf{A}'^*)\cdot(\mathrm{I}\cdot\mathbf{A}^*)]. \qquad (157)$$

First we write

$$d\mathbf{r}\cdot\diamond[(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}')] = d[(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}')]$$
$$= [d(\mathrm{I}\cdot\mathbf{A})]\cdot(\mathrm{I}\cdot\mathbf{A}') + (\mathrm{I}\cdot\mathbf{A})\cdot d(\mathrm{I}\cdot\mathbf{A}').$$

The second term may be transformed so that the differential comes to the front. For by the equation found in the previous footnote,

$$(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot d\mathbf{A}') = -\mathbf{A}\cdot d\mathbf{A}'\mathrm{I} + (\mathrm{I}\cdot d\mathbf{A}'^*)\cdot(\mathrm{I}\cdot\mathbf{A}^*).$$

Hence

$$d[(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}')] = -(d\mathbf{A}\cdot\mathrm{I})\cdot(\mathrm{I}\cdot\mathbf{A}') - (d\mathbf{A}'^*\cdot\mathrm{I})\cdot(\mathrm{I}\cdot\mathbf{A}^*) - d\mathbf{A}'\cdot\mathbf{A}\mathrm{I}.$$

Now

$$(d\mathbf{A}\cdot\mathrm{I})\cdot(\mathrm{I}\cdot\mathbf{A}') = d\mathbf{A}\cdot(\mathrm{I}\cdot\mathrm{I}\cdot\mathbf{A}') = d\mathbf{A}\cdot(\mathrm{I}\cdot\mathbf{A}').$$

Hence

$$d\mathbf{r}\cdot\Diamond[(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}')]$$
$$= (d\mathbf{r}\cdot\Diamond\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}') - (d\mathbf{r}\cdot\Diamond\mathbf{A}'^*)\cdot(\mathrm{I}\cdot\mathbf{A}^*) - d\mathbf{r}\cdot\Diamond\mathbf{A}'\cdot\mathbf{A}\mathrm{I}.$$

Hence finally

$$2\Diamond\Psi = -\Diamond\mathbf{A}\cdot(\mathrm{I}\cdot\mathbf{A}') - \Diamond\mathbf{A}'^*\cdot(\mathrm{I}\cdot\mathbf{A}^*) - \Diamond\mathbf{A}'\cdot\mathbf{A}\mathrm{I}$$
$$- \Diamond\mathbf{A}'^*\cdot(\mathrm{I}\cdot\mathbf{A}^*) - \Diamond\mathbf{A}\cdot(\mathrm{I}\cdot\mathbf{A}') + \Diamond\mathbf{A}\cdot\mathbf{A}'\mathrm{I}.$$

If the expression $\Diamond\cdot\Psi$ is desired, care must be exercised to insert the dot between the first two vectors of each triad. Hence [69]

$$2\Diamond\cdot\Psi = 2(\Diamond\cdot\mathbf{A})\cdot\mathbf{A}' + 2(\Diamond\cdot\mathbf{A}'^*)\cdot\mathbf{A}^* - \Diamond\mathbf{A}'\cdot\mathbf{A} + \Diamond\mathbf{A}\cdot\mathbf{A}',$$
$$\Diamond\cdot\Psi = (\Diamond\cdot\mathbf{A})\cdot\mathbf{A}' + (\Diamond\cdot\mathbf{A}'^*)\cdot\mathbf{A}^* + \tfrac{1}{2}(\Diamond\mathbf{A}\cdot\mathbf{A}' - \Diamond\mathbf{A}'\cdot\mathbf{A}). \quad (158)$$

Some Projective Geometry, and Trigonometry.

64. We may discuss very briefly the relations between our non-Euclidean measure of angle and the projective measure as determined by logarithms of cross-ratios. Let us consider Figure 30 first as a Euclidean and second as a non-Euclidean diagram. The two fixed lines α, β are drawn so that they are perpendicular from the Euclidean point of view. The initial line from which angles are measured is taken as the bisector of one of the right angles; this line and its perpendicular through the origin will be taken as axes of x and y. The pseudo-circle appears as a rectangular hyperbola with the equation $x^2 - y^2 = 1$. The angle between the initial line and any radius in the pseudo-circle in Euclidean measure will be called θ, and $\tan\theta = y/x$. Now in non-Euclidean measure, if this angle be called ϕ, we have seen that $\tanh\phi = y/x$. Hence we have the relation

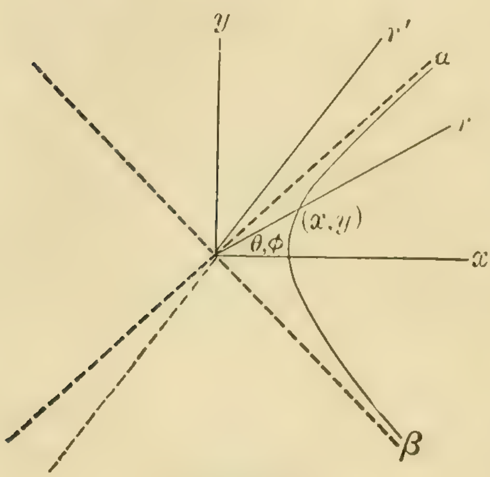

FIGURE 30.

$$\tan\theta = \tanh\phi.$$

[69] The form $\Diamond\mathbf{A}\cdot(\mathrm{I}\cdot\mathbf{A}')$ may be written as a sum of triads of the type $\mathbf{aA}\cdot(\mathbf{ef})$ or $\mathbf{a}(\mathbf{A}\cdot\mathbf{e})\,\mathbf{f}$. Now by (35), $\mathbf{a}\cdot(\mathbf{A}\cdot\mathbf{e}) = -(\mathbf{a}\cdot\mathbf{A})\cdot\mathbf{e}$. Hence the insertion of a dot in $\Diamond\mathbf{A}\cdot(\mathrm{I}\cdot\mathbf{A}')$ gives $-(\Diamond\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}')$ or $-(\Diamond\cdot\mathbf{A})\cdot\mathbf{A}'$. In the form $\Diamond\mathbf{A}\cdot\mathbf{A}'\mathrm{I}$, the dot goes between $\Diamond$ and I, since $\mathbf{A}\cdot\mathbf{A}'$ is a scalar. But as I is the idemfactor, we have simply $\Diamond\mathbf{A}\cdot\mathbf{A}'$ as the result.

The cross-ratio formed by the four lines, x, r, α, β is

$$\lambda = \frac{\sin \angle (\beta, r) \sin \angle (x, \alpha)}{\sin \angle (r, \alpha) \sin \angle (\beta, x)},$$

where the angles are measured in Euclidean fashion. Hence

$$\lambda = \frac{\sin\left(\frac{\pi}{4} + \theta\right)}{\sin\left(\frac{\pi}{4} - \theta\right)} = \frac{1 + \tan\theta}{1 - \tan\theta} = \frac{1 + \tanh\phi}{1 - \tanh\phi} = e^{2\phi}.$$

Or

$$\phi = \tfrac{1}{2}\log\lambda.$$

Hence the non-Euclidean angle is measured by one-half the logarithm of the cross-ratio of four rays. Although the Euclidean point of view has been adopted for simplicity, the final result, depending as it does only on the cross-ratio, is projective; it is therefore independent of the particular assumptions that the rays α and β are perpendicular and that the initial line bisects the angle between them.

Consider next a ray r' such that in the Euclidean sense

$$\angle (\alpha, r') = \angle (r, \alpha).$$

(In the non-Euclidean sense r and r' are perpendicular). In forming the cross-ratio it is evident that $\lambda' = -\lambda$. Hence for the non-Euclidean angle ϕ' between x and r'

$$\phi' = \tfrac{1}{2}\log\lambda' = \tfrac{1}{2}\log(-\lambda) = \phi + \tfrac{1}{2}\log(-1).$$

Hence

$$\phi' = \phi \pm \tfrac{1}{2}\pi i.$$

The angle $\phi' - \phi$, that is, the angle between two lines perpendicular in the non-Euclidean sense is therefore $\pm \frac{1}{2}\pi i$. This result also is projective and independent of our special assumptions. It is only natural that the angle between two lines in different classes should appear as a complex number, owing to the fact that it is impossible to rotate one line into the other.

In setting up a projective measure of angle by means of cross-ratios, it is customary among mathematicians to define the angle as

$$\phi = \frac{1}{2\sqrt{-1}}\log\lambda,$$

where the logarithm of the cross-ratio is divided by $2i$ instead of by 2 as above. The choice of the divisor $2i$ is due to the desire to have the angle real when the fixed lines are conjugate imaginary lines and to have the total angle about a point equal to 2π as in Euclidean geometry; this is not, however, in any way suggested by projective geometry. In our non-Euclidean geometry, where we have taken a different set of postulates for rotation, the real divisor 2 is more natural. We have seen that from the point of view of the postulates of translation or the parallel transformation our geometry and the ordinary Euclidean geometry fall into one class, while such geometries as the Lobatchewskian and the Riemannian belong to another class. With respect to the postulates of rotation, however, the Euclidean and most of the non-Euclidean geometries which have been studied lie in one class, to which our geometry does not belong. The methods of projective geometry are applicable to all these classes.

If the ray r is perpendicular to the rays r' and r'', the latter two being in the same line but oppositely directed, it is evident that we must choose arbitrarily the sign of the angle $\pm \frac{1}{2}\pi i$ between r and r'; but we shall assume that if the sign of the angle rr' has been determined the sign of the angle rr'' will be the same. Thus the angle $r'r''$ is zero. This means that a pair of intersecting lines determine but one angle except for sign; thus any angle is identical, except for sign, with its supplement.

The angle from a line to a second line and the angle from the first line to the perpendicular to the second will be called complementary. The complement of a real angle is a complex angle, and vice versa.

65. Hitherto we have chosen to avoid the use of the term distance, and have used the word interval to represent a positive number expressing the measure of length. If r is a line drawn from the origin, the interval of r has been defined as $\sqrt{x^2 - y^2}$ or $\sqrt{y^2 - x^2}$ according as x is greater than y or y greater than x. This was done to avoid altogether the use of imaginaries. We might, however, have defined distance as

$$\int \pm \sqrt{dx^2 - dy^2},$$

where x is, for example, measured along a (γ)-line, y along a perpendicular (δ)-line. Then every (γ)-line would have a real, and every (δ)-line an imaginary distance. In this case it would be convenient to consider the distance along any vector AB as the negative of the distance along BA. The distance along any singular line is zero.

The preceding ideas can be used to give new definitions of the inner and outer products of two vectors. Namely,

$$\mathbf{a}\cdot\mathbf{b} = \text{distance of } \mathbf{a} \text{ times distance of } \mathbf{b} \text{ times } \cosh \angle\ (\mathbf{a}, \mathbf{b}),$$

$$\mathbf{a}\times\mathbf{b} = \text{distance of } \mathbf{a} \text{ times distance of } \mathbf{b} \text{ times } \sinh \angle\ (\mathbf{a}, \mathbf{b}),$$

it being understood that the latter quantity is not a scalar but a pseudo-scalar. If **a** and **b** are vectors of the same class the angles are real, and the equations are essentially identical with those which have been previously derived. If **a** and **b** are (δ)-vectors the distances are purely imaginary and the product $\mathbf{a}\cdot\mathbf{b}$ is negative if the vectors issue into the same "quadrant." If **a** and **b** are of different classes, and the angle between them complex, we may use in place of these complex angles their complementary real angles by the aid of the familiar formulas

$$\cosh(\phi + \tfrac{1}{2}\pi i) = i\sinh\phi, \qquad \sinh(\phi + \tfrac{1}{2}\pi i) = i\cosh\phi.$$

Massachusetts Institute of Technology,
Boston, Mass., *May*, 1912.

TABLE OF NOTATIONS.

General Symbols.

1-vectors, lower case Clarendons, **a**, **b**, **c**,...;
 their magnitudes, corresponding Italic, a, b, c,...;
 their components (algebraic magnitudes), a_1, a_2, a_3, a_4, etc.;
 their (vector) space components, $\mathbf{a}_s$, $\mathbf{b}_s$, $\mathbf{c}_s$,....
 $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, unit coordinate space vectors;
 $\mathbf{k}_4$, unit coordinate time vector.

2-vectors, Clarendon capitals, **A**, **B**, **C**,...;
 their magnitudes, corresponding Italic, A, B, C,...;
 their components, A_{12}, A_{23},..., A_{34}; etc.;
 $\mathbf{k}_{12}$, $\mathbf{k}_{23}$,..., k_{34}, unit coordinate 2-vectors.

3-vectors, Tudor black capitals, 𝕬, 𝕭, 𝕮,...;
 their magnitudes, corresponding German, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$,...;
 their components, A_{234},..., A_{123};
 $\mathbf{k}_{234}$,..., $\mathbf{k}_{123}$, unit coordinate 3-vectors (the last, "space").

unit pseudo-scalar, $\mathbf{k}_{1234}$.
sign of the outer product, small cross, $\times$.
sign of the inner product, heavy dot, $\cdot$.
sign of the complement, asterisk, $\mathbf{a}^*$, $\mathbf{A}^*$,....
three-dimensional differentiating operator, del, ∇.
four-dimensional differentiating operator, quad, $\Diamond$.
dyadics, Greek capitals, Φ,... (idemfactor, I).

Special Symbols (non-vectorial).

α, β, singular lines (§ 9).

γ, spacial lines; δ, temporal lines (§ 9, 37).

ϵ, electric charge (§ 48).

μ, material density (§ 45); μ_0, density under no relative motion.

ρ, electric density (§ 54); ρ_0, density under no relative motion.

φ, electric scalar potential (§ 48).

m, mass; m_0, mass under no relative motion.

t, time (also x_4).

u, v, velocities.

x, y, z, space coordinates (also x_1, x_2, x_3).

I, idemfactor.

L, Lagrangian function (§ 56).

R, a perpendicular interval (§ 43).

Special Symbols (vectorial).

a, (three-dimensional) ordinary vector potential (§ 48).

b, a special four-dimensional "radiation field" (§ 53).

c, extended curvature (§ 22, 35).

e, (three-dimensional) electric force (§ 49, 50).

f, (three-dimensional) mechanical force (§ 35).

g, as in d**g**, special vector of extended momentum (§ 47).

h, (three-dimensional) magnetic force (§ 49, 50).

l, extended light-vector, singular ray (§ 43).

m, extended (four-dimensional) vector potential (§ 48, 55).

n, unit normal to (δ)-curve (§ 43).

p, geometric potential vector (§ 43).

q, vector of extended electric current density (§ 54).

r, four-dimensional radius vector.

s, as in d**s**, vector element of arc.

v, (three-dimensional) velocity (§ 43).

w, unit tangent to (δ)-curve.

E, electric 2-vector (§ 49).

H, magnetic 2-vector (§ 49).

M, electromagnetic 2-vector (§ 48).

P, geometric 2-vector field (§ 43).

S, as in d**S**, element of (two-dimensional) surface.

$\mathfrak{S}$, as in $d\mathfrak{S}$, element of three-dimensional volume.

Σ, as in $d\Sigma$, element of four-dimensional volume.

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By D. L. Webster.

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In the science of mechanics of ordinary matter we are accustomed to regard velocity as essentially relative but acceleration as absolute; and to say that, if a body is not acted upon in any way by other bodies, its acceleration is zero, but that, if it is acted upon by any other body, the accelerations of the two are opposite, and inversely proportional to their masses. But how can we test this law? and how can we measure the acceleration? If we measure the velocity relative to the earth, or to the sun, or to any star, at any two times separated by a very short interval, how can we be sure that the system of reference has not been accelerated during the time that has elapsed? And if it has, on what system is its acceleration measured?

This difficulty is made still more puzzling if we consider two mechanical systems, such as the solar system, exactly similar in every way, but one of which is removed to a practically infinite distance from all other matter while the other is subject to the attraction of a tremendous mass, so large and far removed that its gravitational field is practically uniform, and at absolute zero temperature so that no radiation would be received from it. These systems would be accelerated relatively to each other, but which of them would be accelerated? No observer on either of them could tell by any mechanical means.

An answer to these questions appears to be given by the electromagnetic equations, which assume the especially simple form with which we are familiar when expressed in terms of the length, mass, and time units of any one of a certain set of systems, any one of which appears to be moving relative to any other with a constant velocity, less than the velocity of light. These systems may all be assumed to be unaccelerated, and assuming the impossibility of any system's moving relative to one of these with a velocity greater than that of light, we say that all other systems are accelerated.

But acceleration means rate of change of velocity, and therefore absolute acceleration means rate of change of absolute velocity, and, if there is no such thing as absolute velocity, how can there be such a thing as rate of change of it? We must, therefore, redefine the absolute acceleration of any system to mean the acceleration relative to another system moving in such a way that the simple electromagnetic equations hold on it, and on which the velocity of the first system is zero at the instant in question.

Even now, however, there is difficulty, because in place of our set of systems with the constant relative velocities, for which the equations hold, we might equally well imagine any other exactly similar set of systems each of which has a certain given acceleration relative to the corresponding one of the first set. And, disregarding the rather arbitrary definition of absolute acceleration given above, it is evident that, if space had no properties other than those of geometry and time, any difference between the laws of nature as observed from two of these relatively accelerated systems would be impossible. But since the observed laws are simpler in one of the first set than in one accelerated relative to it, the space must have other properties than the above mentioned ones; and, because of these properties, it appears highly probable that there must be some sort of a substance, or medium, filling all space, having no acceleration relative to any of the systems for which the simple electromagnetic equations hold, not directly affecting our senses, but having properties which account for all the laws of the phenomena that are directly observable, including the exact mathematical similarity of the expressions for these laws in terms of quantities measured on any system moving with uniform velocity, less than that of light, through it. This is the medium to which we give the name of "ether."

The Ether.—To obtain any knowledge of the properties of this medium, that enable it to show the phenomena of electricity, magnetism, and gravitation, and to account for the laws of motion of matter, the principal of relativity, and the permanent existence of positive and negative electrons in spite of the possibility of collisions between them, it will be necessary to obtain the simplest possible form of the set of laws which govern these phenomena.

Since many of the quantities that we deal with are vectors, we shall find it convenient to use some simple vector analysis, with the following notation, that of Gibbs, in which all vectors will be printed in Clarendon type while scalars are in italic type. The scalar product,

$$(\mathbf{a}_x\mathbf{b}_x + \mathbf{a}_y\mathbf{b}_y + \mathbf{a}_z\mathbf{b}_z),$$

of any two vectors, **a** and **b**, will be denoted by $\mathbf{a}\cdot\mathbf{b}$; while the vector product,

$$\mathbf{i}\,(\mathbf{a}_y\mathbf{b}_z - \mathbf{a}_z\mathbf{b}_y) + \mathbf{j}\,(\mathbf{a}_z\mathbf{b}_x - \mathbf{a}_x\mathbf{b}_z) + \mathbf{k}\,(\mathbf{a}_x\mathbf{b}_y - \mathbf{a}_y\mathbf{b}_x),$$

will be denoted by $\mathbf{a}\times\mathbf{b}$, where **i**, **j** and **k** are the unit vectors in the directions of x, y, and z respectively.

The symbol ∇ will be used for the operator,

$$\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right),$$

so that ∇a = the gradient of scalar a, a vector; $\nabla\cdot\mathbf{a}$ = the divergence of vector **a**, a scalar; and $\nabla\times\mathbf{a}$ = the curl of vector **a**, another vector.

The symbol "Pot" will be used for the operation of taking the Newtonian potential of any function, so that

$$\text{Pot}\,a = \int\!\!\int_{\infty}\!\!\int \frac{a}{r}\,d\tau,$$

where r is the distance from $d\tau$ to the point at which we wish to find Pot a. We may apply the operator Pot to a vector as well as a scalar, and, in either case, Poisson's equation tells us that

$$(\nabla^2\,\text{Pot}) = -4\pi,$$

or the application of the operator ∇^2 to the Pot of any function gives -4π times the original function.

It will be found convenient to indicate differentiation with respect to ct, where c is the velocity of light, by a dot over the letter that stands for the function. Thus

$$\dot{\mathbf{a}} = \frac{\partial\mathbf{a}}{\partial\,(ct)}\cdot$$

For brevity let us assume also, unless otherwise stated, that the functions used in the following work all vanish at infinity and are finite and continuous throughout space.

Laws of the Ether.—To write out the laws of the ether in the form that accounts for all the above mentioned phenomena, we must distinguish between the effects due to positive and negative charges, and, therefore, it will be convenient to call the density of positive electricity at any point $\overset{+}{\rho}$, (a quantity which is always positive), and that of negative electricity $\overset{-}{\rho}$, (always negative). The electric forces due to all the positive and all the negative electricity we may call $\overset{+}{\mathbf{E}}$ and $\overset{-}{\mathbf{E}}$, and the magnetic forces $\overset{+}{\mathbf{H}}$ and $\overset{-}{\mathbf{H}}$, while the velocities of the charges may be represented in terms of $\overset{+}{\boldsymbol{\beta}}$ and $\overset{-}{\boldsymbol{\beta}}$, their ratios to the velocity of light.

These quantities may be supposed to satisfy the following set of equations:

$$(1)\quad \nabla\cdot\overset{+}{\mathbf{E}} = \overset{+}{\rho}, \qquad (2)\quad \nabla\cdot\overset{-}{\mathbf{E}} = \overset{-}{\rho},$$

$$(3)\quad \nabla\times\overset{+}{\mathbf{H}} = \dot{\overset{+}{\mathbf{E}}} + \overset{+}{\rho}\overset{+}{\boldsymbol{\beta}}, \qquad (4)\quad \nabla\times\overset{-}{\mathbf{H}} = \dot{\overset{-}{\mathbf{E}}} + \overset{-}{\rho}\overset{-}{\boldsymbol{\beta}},$$

$$(5)\quad \nabla\cdot\overset{+}{\mathbf{H}} = 0, \qquad (6)\quad \nabla\cdot\overset{-}{\mathbf{H}} = 0,$$

$$(7)\quad \nabla\times\overset{+}{\mathbf{E}} = -\dot{\overset{+}{\mathbf{H}}}, \qquad (8)\quad \nabla\times\overset{-}{\mathbf{E}} = -\dot{\overset{-}{\mathbf{H}}},$$

which determine the positive forces in terms of the positions and velocities of the positive charges, and the negative forces in terms of those of the negative charges. But in addition to these equations we have the following pair,

$$(9)\quad \mathbf{E} + \overset{+}{\boldsymbol{\beta}}\times\mathbf{H} + \overset{+}{\mathbf{K}} - G(\overset{+}{\mathbf{E}} + \overset{+}{\boldsymbol{\beta}}\times\overset{+}{\mathbf{H}}) = 0,$$

$$(10)\quad \mathbf{E} + \overset{-}{\boldsymbol{\beta}}\times\mathbf{H} + \overset{-}{\mathbf{K}} = 0,$$

which must hold at every point of every electron, positive or negative, and in which $\overset{+}{\mathbf{K}}$ and $\overset{-}{\mathbf{K}}$ are forces per unit charge due to the internal stresses of the electron, while G is a very small number whose presence in equation (9) accounts for the phenomena of gravitation.[1]

The laws governing the vectors $\overset{+}{\mathbf{K}}$ and $\overset{-}{\mathbf{K}}$ may be deduced from the fact that the deformation of the electron when its velocity is very great is the same as that of a perfectly flexible and compressible, charged, conducting shell, with no internal stresses, subject to a constant external hydrostatic pressure or internal hydrostatic tension.[2] Therefore, we may assume that $\overset{+}{\mathbf{K}}$ and $\overset{-}{\mathbf{K}}$ are forces such as would result from such a tension, and that they are transmitted by

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1 For further details on this point see D. L. Webster, "On an Electromagnetic Theory of Gravitation," These Proceedings, **47,** 14 (1912). The reasoning and conclusions are changed but little if we introduce a similar term in equation (10), and thereby gain in symmetry in our theory.

2 Poincaré, Comptes Rendus, **140**, 1504–8.

the material of the electron. This tension is, of course, constant throughout its volume only if all the charge of the electron is on its surface, otherwise, it increases as we go nearer the centre of the electron.

Abraham has raised the objection to this theory, that it involves an instability of the shape of the electron,[3] that would soon destroy all such bodies. But this objection is based on the interpretation of the above vectors as mechanical forces per unit charge, tending to accelerate the parts of the electrons involved, and on the idea that a part of the charge may in some way be displaced from the position in which all these forces exactly balance. Such displacements would result in a rapid disruption of the electron, a process in which equations (9) or (10) could not hold indefinitely. But if we take them as expressions of a fundamental law, which would be violated by such a process, we have a reason why this process cannot occur, nor even start to occur, and the problem of stability of shape of the election is solved.

To determine the motion of a whole electron from these equations (9) and (10) we are aided by the fact that the resultant of the internal forces is zero, but we have to remember that the vectors $\overset{+}{\mathbf{E}}$, $\overset{-}{\mathbf{E}}$, $\overset{+}{\mathbf{H}}$, and $\overset{-}{\mathbf{H}}$, that occur in these equations, include not only the contributions from external sources, but internal as well. Therefore, the equations demand motion with constant velocity when the external forces are zero, and the resultant of the actions of different parts of the electron on each other must be zero also. But if the external forces are not zero, each part of the electron must be accelerated in such a manner that the resultant of all the forces, radiated or otherwise, from all other parts, will just balance the resultant of the external forces. Thus we have a reason for the apparent inertia of every electron, and of bodies composed of electrons, so that the laws of motion of matter may be proved to be consequences of the laws of electromagnetic forces.

Simplification of the Laws. Hamilton's Principle.

We may, however, simplify these laws still further, by remembering the fact that there is one dynamical principle that applies to all motions of matter and also to all the phenomena of slow changes of positions of electric charges and of the positions and magnitudes

3 See Lorentz, Theory of Electrons; Chap. V, 1905.

of currents, and expresses the laws of the phenomena perfectly with no other assumptions than equations (1) – (4). Therefore, it seems reasonable to suppose that this same one principle may replace equations (5)–(10), and reduce the number of necessary laws from 10 to 5.

This fundamental principle is Hamilton's Principle, which says that for any dynamical system whose kinetic and potential energies are T and W respectively,

$$\delta \int_{t_1}^{t_2} (T - W)\, dt = 0,$$

where t_1 and t_2 are any two times, and where the variation from the actual motion is any variation, consistent with the constraints of the system, that makes the configurations of the system at the times t_1 and t_2 the same as it is in the actual motion. In the case of the ether, writing $\mathbf{E}$ for $\overset{+}{\mathbf{E}} + \overset{-}{\mathbf{E}}$ and $\mathbf{H}$ for $\overset{+}{\mathbf{H}} + \overset{-}{\mathbf{H}}$, this principle takes the form,

$$(11) \quad \delta \int_{t_1}^{t_2} \iiint_{\infty} \{ (\mathbf{H}^2 - G\overset{+}{\mathbf{H}}{}^2) - (\mathbf{E}^2 - G\overset{+}{\mathbf{E}}{}^2 + 2\,U) \}\, d\tau dt = 0,$$

where U is the sum of the hydrostatic tensions in the positive and negative electrons, if any, in which the element $d\tau$ lies, and which produce the forces $\overset{+}{\mathbf{K}}$ and $\overset{-}{\mathbf{K}}$, and where two configurations are to be considered the same if, and only if, the vectors $\overset{+}{\mathbf{E}}$ and $\overset{-}{\mathbf{E}}$ are the same in one as in the other.[4]

To prove that equations (5)–(6) result from equations (1)–(4) and equation (11) we may write (11) in the form,

$$(12) \quad \int_{t_1}^{t_2} \iiint_{\infty} \{ (\mathbf{H}\cdot\delta\mathbf{H} - G\overset{+}{\mathbf{H}}\cdot\delta\overset{+}{\mathbf{H}}) - (\mathbf{E}\cdot\delta\mathbf{E} - G\overset{+}{\mathbf{E}}\cdot\delta\overset{+}{\mathbf{E}} + \delta U) \}\, d\tau dt = 0,$$

and then suppose that $\delta\mathbf{H}$, $\delta\mathbf{E}$, $\delta\overset{+}{\mathbf{E}}$, and δU are all zero throughout the interval. Now whatever vector $\overset{+}{\mathbf{H}}$ may be we may split it into a sum of two parts, $\overset{+}{\mathbf{H}}_S$ and $\overset{+}{\mathbf{H}}_L$, such that

$$\nabla \cdot \overset{+}{\mathbf{H}}_S = 0 = \nabla \times \overset{+}{\mathbf{H}}_L,$$

[4] For another form of Hamilton's Principle, involving different assumptions see Larmor, "Aether and Matter," Chapter I.

and then write

$$\int\int\int_{\infty} \overset{+}{\mathbf{H}}{}^2 d\tau$$

as

$$\int\int\int_{\infty} (\overset{+}{\mathbf{H}}_S{}^2 + 2\, \overset{+}{\mathbf{H}}_S \cdot \overset{+}{\mathbf{H}}_L + \overset{+}{\mathbf{H}}_L{}^2)\, d\tau.$$

But by Green's Theorem, whatever these parts are, if both vanish at infinity,

$$\int\int\int_{\infty} \overset{+}{\mathbf{H}}_S \cdot \overset{+}{\mathbf{H}}_L d\tau = 0. \tag{13}$$

In this case $\overset{+}{\mathbf{H}}_S$ is completely determined by equation (3), so that, if $\delta\mathbf{E}$, $\delta\overset{+}{\mathbf{E}}$, and δU are zero, $\delta\overset{+}{\mathbf{H}}_S$ is zero, therefore

$$\begin{aligned} \int\int\int_{\infty} \delta\overset{+}{\mathbf{H}}{}^2 d\tau &= 2\int\int\int_{\infty} (\overset{+}{\mathbf{H}}_S \cdot \delta\overset{+}{\mathbf{H}}_L + \overset{+}{\mathbf{H}}_L \cdot \delta\overset{+}{\mathbf{H}}_L)\, d\tau \\ &= 2\int\int\int_{\infty} \overset{+}{\mathbf{H}}_L \cdot \delta\overset{+}{\mathbf{H}}_L d\tau. \end{aligned} \tag{14}$$

But this must be zero whatever $\delta\overset{+}{\mathbf{H}}_L$ is, therefore $\overset{+}{\mathbf{H}}_L$ is zero, as is $\overset{-}{\mathbf{H}}_L$ also. Therefore

$$(5)\quad \nabla \cdot \overset{+}{\mathbf{H}} = 0, \qquad\qquad (6)\quad \nabla \cdot \overset{-}{\mathbf{H}} = 0.$$

To derive equations (7) and (8) we may introduce vectors $\overset{+}{\mathbf{I}}$, $\overset{-}{\mathbf{I}}$, $\overset{+}{\mathbf{P}}$, and $\overset{-}{\mathbf{P}}$, defined by the equations,

$$(15)\qquad \overset{+}{\mathbf{I}} = \int_0^{ct} \overset{+}{\rho} \overset{+}{\boldsymbol{\beta}}\, d(ct), \qquad\qquad \overset{-}{\mathbf{I}} = \int_0^{ct} \overset{-}{\rho} \overset{-}{\boldsymbol{\beta}}\, d(ct),$$

$$(16)\qquad \overset{+}{\mathbf{P}} = \frac{1}{4\pi} \operatorname{Pot} (\overset{+}{\mathbf{E}} + \overset{+}{\mathbf{I}}), \qquad\qquad \overset{-}{\mathbf{P}} = \frac{1}{4\pi} \operatorname{Pot} (\overset{-}{\mathbf{E}} + \overset{-}{\mathbf{I}}).$$

From these equations we may infer

$$\begin{aligned} \nabla^2 \overset{+}{\mathbf{P}} &= -\, (\overset{+}{\mathbf{E}} + \overset{+}{\mathbf{I}}), \\ \nabla^2 \dot{\overset{+}{\mathbf{P}}} &= -\, (\dot{\overset{+}{\mathbf{E}}} + \dot{\overset{+}{\mathbf{I}}}) = -\, (\dot{\overset{+}{\mathbf{E}}} + \overset{+}{\rho} \overset{+}{\boldsymbol{\beta}}). \end{aligned} \tag{17}$$

From equation (3) we know that

$$\nabla \cdot (\dot{\overset{+}{\mathbf{E}}} + \overset{+}{\rho} \overset{+}{\boldsymbol{\beta}}) = 0, \tag{18}$$

so that $$\frac{1}{4\pi}\nabla\cdot\text{Pot}\,(\dot{\overset{+}{\mathbf{E}}} + \overset{+}{\rho}\,\overset{+}{\boldsymbol{\beta}}) = \nabla\cdot\dot{\overset{+}{\mathbf{P}}} = 0;$$

and since, whatever $\dot{\overset{+}{\mathbf{P}}}$ is,

$$\nabla\times\nabla\times\dot{\overset{+}{\mathbf{P}}} = -\nabla^2\dot{\overset{+}{\mathbf{P}}} + \nabla(\nabla\cdot\dot{\overset{+}{\mathbf{P}}}),$$

(19) $$\nabla\times\nabla\times\dot{\overset{+}{\mathbf{P}}} = (\dot{\overset{+}{\mathbf{E}}} + \overset{+}{\rho}\,\overset{+}{\boldsymbol{\beta}}) = \nabla\times\overset{+}{\mathbf{H}},$$

which, combined with (5), gives

(20) $$\nabla\times\dot{\overset{+}{\mathbf{P}}} = \overset{+}{\mathbf{H}}.$$

(21) Similarly $$\nabla\times\dot{\overset{-}{\mathbf{P}}} = \overset{-}{\mathbf{H}}.$$

Equation (12) now takes the form

(22) $$\int_{t_1}^{t_2}\iiint_{\infty}\{(\nabla\times\dot{\mathbf{P}}\cdot\nabla\times\delta\dot{\mathbf{P}} - G\nabla\times\dot{\overset{+}{\mathbf{P}}}\cdot\nabla\times\delta\dot{\overset{+}{\mathbf{P}}}) - \tfrac{1}{2}\delta(\mathbf{E}^2 - G\overset{+}{\mathbf{E}}{}^2 + 2U)\}d\tau dt = 0.$$

(23) But

$$\int_{t_1}^{t_2}\nabla\times\dot{\mathbf{P}}\cdot\nabla\times\delta\dot{\mathbf{P}}dt = \frac{1}{c}\nabla\times\dot{\mathbf{P}}\cdot\nabla\times\delta\mathbf{P}\Big|_{t_2}^{t_2} - \int_{t_1}^{t_2}\nabla\times\ddot{\mathbf{P}}\cdot\nabla\times\delta\mathbf{P}dt,$$

$$= \frac{1}{c}\nabla\times\dot{\mathbf{P}}\cdot\nabla\times\delta\mathbf{P}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}[\nabla\cdot\{\ddot{\mathbf{P}}\times(\nabla\times\delta\mathbf{P})\} + \ddot{\mathbf{P}}\cdot\nabla\times\nabla\times\delta\mathbf{P}]dt.$$

(24) $\therefore$

$$\int_{t_1}^{t_2}\nabla\times\dot{\mathbf{P}}\cdot\nabla\times\delta\dot{\mathbf{P}}dt = \frac{1}{c}\Big[\iiint_{\infty}\nabla\times\dot{\mathbf{P}}\cdot\nabla\times\delta\mathbf{P}\,d\tau\Big]_{t_1}^{t_2}$$
$$-\int_{t_1}^{t_2}\iiint_{\infty}\ddot{\mathbf{P}}\cdot\nabla\times\nabla\times\delta\mathbf{P}\,d\tau dt - \int_{t_1}^{t_2}\lim_{S=\infty}\iint_{S}\ddot{\mathbf{P}}\times(\nabla\times\delta\mathbf{P})\cdot d\mathbf{S}dt,$$

where S is any closed surface that may recede indefinitely in all directions from any interior point, and of which $d\mathbf{S}$ is an element considered as a vector in the direction of the exterior normal. If we now let $\delta\mathbf{I}. = 0$, then $\delta\mathbf{P} = 0$ at t_1 and t_2, and the first term on the right side of equation (24) drops out, and so does the surface integral when there is only a finite amount of charge in the universe. Treating $\overset{+}{\mathbf{P}}$ in the same way, we obtain, if

$$\delta\overset{+}{\mathbf{I}} = \delta\overset{-}{\mathbf{I}} = 0,$$

$$\int_{t_1}^{t_2}\{(\ddot{\mathbf{P}}\cdot\nabla\times\nabla\times\delta\mathbf{P}-G\overset{+}{\ddot{\mathbf{P}}}\cdot\nabla\times\nabla\times\delta\mathbf{P})+(\mathbf{E}\cdot\delta\mathbf{E}-G\overset{+}{\mathbf{E}}\cdot\delta\overset{+}{\mathbf{E}}+\delta U)\}d\tau dt=0, \tag{25}$$

or, since $\delta U = 0$ and

$$\nabla\times\nabla\times\delta\mathbf{P} = -\nabla^2\delta\mathbf{P} = +\delta\mathbf{E}, \tag{26}$$

when no motions of charges have been varied,

$$\int_{t_1}^{t_2}\int\int_{\infty}\int\{(\ddot{\mathbf{P}}+\mathbf{E})\cdot\delta\mathbf{E}-G(\overset{+}{\ddot{\mathbf{P}}}+\overset{+}{\mathbf{E}})\cdot\delta\overset{+}{\mathbf{E}}\}\,d\tau dt=0. \tag{27}$$

Splitting $\mathbf{E}$ into the parts $\mathbf{E}_S$ and $\mathbf{E}_L$, treating $\overset{+}{\mathbf{E}}$ likewise, and applying Green's Theorem as in (14),

$$\int_{t_1}^{t_2}\int\int_{\infty}\int\{(\ddot{\mathbf{P}}+\mathbf{E}_S)\cdot\delta\mathbf{E}_S-G(\overset{+}{\ddot{\mathbf{P}}}-\overset{+}{\mathbf{E}}_S)\cdot\delta\overset{+}{\mathbf{E}}_S\}d\tau kt=0, \tag{28}$$

because $\delta\mathbf{E}_L$ and $\delta\overset{+}{\mathbf{E}}_L$ are zero when no charge motions are varied.

(29) Therefore $$\overset{+}{\ddot{\mathbf{P}}}+\overset{+}{\mathbf{E}}_S=0,$$

(30) and $$\overset{-}{\ddot{\mathbf{P}}}+\overset{-}{\mathbf{E}}_S=0,$$

(31) or $$\overset{+}{\mathbf{E}}_S=-\frac{1}{4\pi}\mathrm{Pot}(\overset{+}{\ddot{\mathbf{E}}}+\dot{\overline{\overset{+}{\rho}\overset{+}{\boldsymbol{\beta}}}}).$$

(32) and $$\overset{-}{\mathbf{E}}_S=-\frac{1}{4\pi}\mathrm{Pot}(\overset{-}{\ddot{\mathbf{E}}}+\dot{\overline{\overset{-}{\rho}\overset{-}{\boldsymbol{\beta}}}}).$$

Applying $\nabla\times$ to (29) and (30) we have

$$\nabla\times\overset{+}{\mathbf{E}}=-\overset{+}{\dot{\mathbf{H}}}, \tag{7}$$ $$\nabla\times\overset{-}{\mathbf{E}}=-\overset{-}{\dot{\mathbf{H}}}. \tag{8}$$

To derive equations (9) and (10) from equation (11) or (12) let us suppose that, for a short time during the interval t_1 and t_2, an *infinitesimal* positive charge de, occupying a small tube of length $d\mathbf{r}'$ and cross section $d\sigma$[5] and lying in the direction of $\overset{+}{\boldsymbol{\beta}}$ is displaced in some other direction through a distance $\delta\mathbf{r}$. To satisfy equation (1) with this variation we may superpose on the actual value of $\overset{+}{\mathbf{E}}$ a straight

[5] Any element of surface may be considered as a vector along its normal, and when its direction is chosen, the positive direction around its boundary is that of a right-handed screw rotation.

tube of the vector $\delta\overset{+}{\mathbf{E}}$ connecting the new position of de to the old, the flux of $\delta\overset{+}{\mathbf{E}}$ being against the direction of $\delta\mathbf{r}$ and of magnitude de. And to satisfy equations (3) and (5) we must also assume a certain variation $\delta\overset{+}{\mathbf{H}}$ which is uniquely defined by these equations and the variations assumed above. We may now assume no variations of the negative forces, and for the positive forces only the necessary variation of $\overset{+}{U}$ and the variations specified above.

With these assumptions (12) becomes

$$(33)\quad \int_{t_1}^{t_2}\iiint_{\infty}\{(\mathbf{H}-G\overset{+}{\mathbf{H}})\cdot\delta\overset{+}{\mathbf{H}}-(\mathbf{E}-G\overset{+}{\mathbf{E}})\cdot\delta\overset{+}{\mathbf{E}}-\delta\overset{+}{U}\}d\tau dt=0.$$

(34) To calculate

$$-\iiint_{\infty}(\mathbf{E}-G\overset{+}{\mathbf{E}})\cdot\delta\overset{+}{\mathbf{E}}d\tau,$$

we need to integrate only over the tube of $\delta\overset{+}{\mathbf{E}}$ defined above, so that, if de is of infinitesimal size, we may take for the result,

$$(35)\qquad (\mathbf{E}-G\overset{+}{\mathbf{E}})\cdot\delta\mathbf{r}de.$$

To calculate

$$(36)\qquad \iiint_{\infty}(\mathbf{H}-G\overset{+}{\mathbf{H}})\cdot\delta\overset{+}{\mathbf{H}}d\tau,$$

we may consider $\delta\overset{+}{\mathbf{H}}$ as the $\overset{+}{\mathbf{H}}$ produced by a current of strength $\overset{+}{\rho}\overset{+}{\boldsymbol{\beta}}\cdot d\sigma$ flowing around the edge of the parallelogram one side of which contains the old position of de and the other side the new position, while the remaining sides are the tubes of the vector $\delta\overset{+}{\mathbf{E}}$ made necessary by the motion of the parallelogram. We may now evaluate the integral, splitting the space up into elements of each of which two sides, $d\mathbf{S}$, are surfaces whose normals are in the direction of $\delta\overset{+}{\mathbf{H}}$ while the remaining dimension, $d\mathbf{s}$, is in the same direction. The integral may now be written

$$(37)\qquad \iiint_{\infty}(\mathbf{H}-G\overset{+}{\mathbf{H}})\cdot\delta\overset{+}{\mathbf{H}}d\mathbf{S}\cdot d\mathbf{s},$$

(38) or $$\int\int_{\infty}\int(\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot d\mathbf{S}\delta\overset{+}{\mathbf{H}}\cdot d\mathbf{s}.$$

But since $$\nabla\cdot\mathbf{H} = \nabla\cdot\overset{+}{\mathbf{H}} = 0,$$

the surface integral

(39) $$\int\int(\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot d\mathbf{S}$$

is the same over any cap of the parallelogram circuit as over any other; and since

(40) $$\nabla\times\delta\overset{+}{\mathbf{H}} = \delta\overset{+}{\mathbf{E}} + \delta(\overset{+}{\rho}\,\overset{+}{\boldsymbol{\beta}}),$$

the line integral

(41) $$\int\delta\overset{+}{\mathbf{H}}\cdot d\mathbf{s}$$

is the same around any line of the vector $\delta\overset{+}{\mathbf{H}}$ as on any other. Therefore the integral (38) is

(42) $$\Big[\underset{\text{Any cap}}{\int\int}(\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot d\mathbf{S}\Big]\Big[\underset{\text{Any line}}{\int}\delta\overset{+}{\mathbf{H}}\cdot d\mathbf{s}\Big].$$

But by Stokes' Theorem, the line integral is

(43) $$\overset{+}{\rho}\,\overset{+}{\boldsymbol{\beta}}\cdot d\sigma,$$

while the surface integral over the plane cap is,

(44) $$d\mathbf{r}'\times(\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot\delta\mathbf{r},$$

so that (36) becomes

(45) $$\overset{+}{\rho}\,\overset{+}{\boldsymbol{\beta}}\cdot d\sigma d\mathbf{r}'\times(\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot\delta\mathbf{r}$$

$$= \overset{+}{\rho}d\mathbf{r}'\cdot d\sigma\overset{+}{\boldsymbol{\beta}}\times(\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot\delta\mathbf{r}$$

(46) $$= \overset{+}{\boldsymbol{\beta}}\times(\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot\delta\mathbf{r}de.$$

Substituting $-\overset{+}{\mathbf{K}}\cdot\delta\mathbf{r}de$ for $\delta\overset{+}{U}$, (33) now becomes

(47) $$\int_{t_1}^{t_2}\{\boldsymbol{\beta}\times(\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot\delta\mathbf{r}de + (\mathbf{E} - G\overset{+}{\mathbf{E}})\cdot\delta\mathbf{r}de + \overset{+}{\mathbf{K}}\cdot\delta\mathbf{r}de\}dt = 0,$$

from which we may infer that

$$(9)\qquad \mathbf{E} + \overset{+}{\beta}\times\mathbf{H} + \overset{+}{\mathbf{K}} - G(\overset{+}{\mathbf{E}} + \overset{+}{\beta}\times\overset{+}{\mathbf{H}}) = 0.$$

Obviously, we may derive equation (10) by an exactly similar process in which the terms involving G do not enter. And if we wish to use an infinitesimal charge of some other shape, we may consider it as divided up into a number of cylinders, not necessarily right cylinders, such as we used above.[6]

Meanings of the Laws.— To find out what we can about the properties of the ether, we may now examine carefully the meanings of these five laws:

$$(1)\quad \nabla\cdot\overset{+}{\mathbf{E}} = \overset{+}{\rho}, \qquad (2)\quad \nabla\cdot\overset{-}{\mathbf{E}} = \overset{-}{\rho},$$

$$(3)\quad \nabla\times\overset{+}{\mathbf{H}} = \dot{\overset{+}{\mathbf{E}}} + \overset{+}{\rho}\overset{+}{\beta}, \qquad (4)\quad \nabla\times\overset{-}{\mathbf{H}} = \dot{\overset{-}{\mathbf{E}}} + \overset{-}{\rho}\overset{-}{\beta},$$

$$(11)\quad \delta\int_{t_1}^{t_2}\iiint_{\infty}\{(\mathbf{H}^2 - G\overset{+}{\mathbf{H}}{}^2) - (\mathbf{E}^2 - G\overset{+}{\mathbf{E}}{}^2 + 2U)\}\,dr\,dt = 0.$$

The first two of these laws contain no reference whatever to time, and deal with quantities whose existence is in no way dependent on motion or change with time. Therefore, we may infer that they probably express relations between the geometrical configurations of different parts of the ether, and show the dependence of these geometrical configurations upon the presence in the ether of the peculiar movable spots called charges, whose indestructibility and ability to be located definitely at different times (specified in equations (3) and (4), as well as the internal forces, suggest that they are due to the presence of some substances not present in the rest of the ether but freely movable through it. Since these substances can be located at any time if the vectors $\overset{+}{\mathbf{E}}$ and $\overset{-}{\mathbf{E}}$ are known at every point, the question arises whether any more information than the value of these vectors needs to be given to determine completely the configuration of the ether. A suggestion of the answer to this question is given by the fact that in applying Hamilton's Principle to problems of ordinary dynamics, the variations must be such as to give the actual configura-

6 To be certain that no equations not derivable from equations (1)–(10) can be derived from (11) and (1)–(4), we need only to consider the facts that any possible variation in equation (11) can be made up of variations of the types treated above, and that the mutual energy of two independent variations of the first order is an infinitesimal of the second order.

tions exactly at the times t_1 and t_2, whereas, in equation (11) they must be such as to give the actual vectors $\overset{+}{\mathbf{E}}$ and $\overset{-}{\mathbf{E}}$. Hence, from analogy, we may say that these vectors are probably sufficient to specify the configuration of the ether completely.

And if this last statement is true, their time derivatives must be sufficient to specify completely, not only the quantities $\overset{+}{\boldsymbol{\beta}}$ and $\overset{-}{\boldsymbol{\beta}}$, but all the motions of the ether; and it seems probable that these motions at any point are specified by the values of $\dot{\overset{+}{\mathbf{E}}}$, $\dot{\overset{-}{\mathbf{E}}}$, $\overset{+}{\rho}\,\overset{+}{\boldsymbol{\beta}}$, and $\overset{-}{\rho}\,\overset{-}{\boldsymbol{\beta}}$ at that point, and not by the values of the vectors $\overset{+}{\mathbf{H}}$ and $\overset{-}{\mathbf{H}}$, which depend on the values of the other vectors at distant points. This hypothesis is further strengthened by the fact that the whole theory of the ether might be developed without any use of these vectors, replacing $\overset{+}{\mathbf{H}}$ wherever it occurs by

$$\frac{1}{4\pi}\,\nabla\times\mathrm{Pot}\,(\dot{\overset{+}{\mathbf{E}}} + \overset{+}{\rho}\,\overset{+}{\boldsymbol{\beta}}),\quad \text{and}\quad \overset{-}{\mathbf{H}}\ \text{by}\ \frac{1}{4\pi}\,\nabla\times\mathrm{Pot}\,(\dot{\overset{-}{\mathbf{E}}} + \overset{-}{\rho}\,\overset{-}{\boldsymbol{\beta}}),$$

and therefore without any use of equations (3) and (4), except as they express the indestructibility of the charges.

Therefore we may consider equations (3) and (4) as merely equations of continuity and partial definitions of two convenient mathematical functions fully defined by equations (3) and (4) and (11) all together, and whose values at any point depend on the motions of the ether at all points, but not in any way on the motions or configurations at the point in question only. And thus, although they contain time derivatives and quantities dependent entirely on motion and existing only when there is motion, they tell us nothing about what is going to happen at some future time from what is happening now, and therefore cannot be considered as laws of motion, but only as mathematical definitions of convenient functions.

Equation (11), however, in form and substance, is essentially an equation of motion, from which no information about the geometrical configurations of the ether can be derived at any time, unless the configuration and motion at some other time, or the configurations at two other times, are specified; but without which no information about the configuration or motion at any time can be derived even if they are given at any number of other times.

Properties of the Ether. — The first question that arises about the properties of the ether is, Is its structure continuous or granular?

To answer this question definitely seems impossible, but at any rate, we can say that if it is granular and if these equations are to hold, the structure must be exceedingly minute compared to the dimensions of the electrons. A further suggestion is given by the fact that in the geometrical equations, (1) and (2), the positive and negative quantities appear very similar, but seem to be more or less independent of each other; while in the equation of motion (11), and in the phenomena of vacuum tube discharges, etc., differences between the actions of the positive and negative quantities appear, that seem to show that not only are the electrons of the different signs made up differently, but that the forces are transmitted by more or less independent, and slightly different, structures in the medium. As this condition of affairs seems to be incompatible with the idea of a continuous medium we are thereby led to the conception of a medium in which there are probably two similar, but slightly different, interlacing, granular structures, whose grains and distances between them are inconceivably small, even compared to the electrons.

The question of solid or fluid character of the ether appears easier to answer; for if it were fluid, that is, if no amount of shear at any point would change the properties at that point in such a way as to affect the subsequent motions around it, a transverse wave would be impossible. And if it were quasi-elastic, with effects analogous to viscosity, that would enable it to transmit wireless telegraph waves as well as the shortest known light waves, electrostatic forces around stationary charges should be due to some effect entirely different from that which produces those of the wireless wave, so that slow continued flow of ether might occur without hindrance. But the changes of electric force near a moving electron may be much more rapid than those of the wireless wave, and yet there appears to be no viscous retardation of its motion. Furthermore, the aberration of light and experiments such as that of H. A. Wilson [7] on the polarization of a dielectric cylinder rotating in a magnetic field seem to show that no flow of ether occurs in moving matter. These considerations and many others compel us to reject the fluid theory, and to say that the structures of the ether are solid. But by "solid" we must not mean possessed of ordinary solid elasticity, but merely that every particle is permanently connected to the particles near it by connections that cannot be deformed indefinitely, or even by a finite amount without affecting the subsequent motion.

[7] H. A. Wilson. "Electric Effect of Rotating a Dielectric in a Magnetic Field," Roy. Soc. Proc., **73**, pp. 490–492. June 22, 1904.

As we must not assume ordinary elasticity, so also we must not assume ordinary inertia of the fundamental particles. For, after all, Newton's laws of motion, that we observe for ordinary matter, appear to be only approximations to the laws that result from equation (11), the more general law of motion. And furthermore, they are by no means the only ones consistent with the relative nature of time and space, nor is there any other *a priori* philosophical reason for assuming that they are true, while there is good philosophical reason for assuming that Hamilton's Principle, the mathematical expression of the perfect efficiency of the fundamental machinery of nature, is at least plausible. Therefore, whatever motions of the parts of the ether it may involve, and whether or not it is easy for us, with our Newtonian mechanical training, to form a mental picture of the dynamics of these motions, the fundamental law of the dynamics of the ether, or of any mental picture of it, must be Hamilton's Principle.

A Model of the Ether. — To get a mental picture of the actions of the ether, we must now make some arbitrary assumptions as to the nature of the two interlacing structures and the strains in them that are represented by the vectors $\overset{+}{\mathbf{E}}$ and $\overset{-}{\mathbf{E}}$. For simplicity we may think of them as nets with cubical meshes with each knot of either net in the centre of a mesh of the other, wherever the electric vectors are zero. The vector $\overset{+}{\mathbf{E}}$ may be a very minute displacement of one of these nets from this position, and the vector $\overset{-}{\mathbf{E}}$ the negative of a similar displacement of the other. If we now suppose the strings of these nets to be hollow and rigid, and the knots to be hollow boxes, so constructed that the displacements of the nets will be those of an incompressible substance, we may suppose an electric charge to be a region in which the pipes and boxes of one of the nets are filled with a liquid of high surface tension, that will expand the boxes into which it flows, and cause a divergence of the displacement of the net. An electron will then be a region of this sort, in the shape of a hollow sphere when at rest, of which every dimension, including the thickness, is very large compared with the meshes of the net. The pipes and boxes of that net that lie inside this region may be filled with a fluid whose only properties are adhesion with everything it touches and a constant hydrostatic tension, independent of its volume. For the connections between the nets we may assume anything we please.

Equations (1) and (2) are satisfied by this model, which also gives an interesting interpretation for (3) and (4). For in free ether the vector $\overset{+}{\mathbf{H}}$ becomes a hydrokinetic flow-function for the motion of the positive net; and where there is any positive charge it is a flow-function for the motion of the net plus that of the charge. Similarly the vector $\overset{-}{\mathbf{H}}$ is the negative of a flow-function of the motion of the negative net and charges. And in each case, equations (5) and (6) tell us that it is the solenoidal flow-function that is required.

The equation of motion is, as we expected, one which we have some difficulty in applying. But if we split it up into equations (5) to (10), and then combine them properly, we may use in electrical problems only the vectors **E** and **H**, representing the *relative* displacement of the positive net from the negative, and the flow-function of the *relative* motion. And in gravitational problems the vectors **E** and **H** disappear entirely.

Collisions of Electrons. — An interesting application of this model is to the problem of collisions of electrons, of the same or opposite signs, as in the case of a cathode particle striking an electron in the metal it hits. If they are of the same kind they will evidently become flattened as they come together. But as soon as they are within about their own length of each other, the side of either of them nearest the other will be effected not only by the displacement due to the presence of the other, but also by the displacements radiated from the other on account of its acceleration. To make the vectors balance, as required by equations (9) and (10), its acceleration must therefore be so much greater than that required by the inverse square law that they can never collide.

In the case of two electrons of different kinds, both are lengthened, and they come together faster than the inverse square law would demand. But since they may go right through each other perfectly freely, there need not be any of the destructive effects that one might expect from other theories.

Retarded Potentials.[8]— In calculating the values of the retarded potentials due to moving electrons it is found necessary to treat each electron as if its charge were not the same as when at rest, but changed in the ratio $(1 - \beta_r)^{-1}$, where β_r is the component of β in the direction towards the point at which we wish to know the potential. This has been interpreted by some writers[9] as indicating that all electromagnetic actions are due to some sort of pulsation of the electrons, and are

8 For information about retarded potentials, see Lorentz, "Theory of Electrons," Chap. 1.

9 L. de la Rive, Phil. Mag., **13**, p. 279.

stronger if the pulsations are more rapid, so that the Doppler effect is introduced if the charge is moving. But with the model it is obvious that any such interpretation is unnecessary; for the important quantity is not the actual charge of the electron, but the volume of the ether in which there was a spreading of the net at such a time as to affect the point in question at the time in question.

Summary.

Because of the apparently absolute nature of acceleration, as well as for other reasons, we find it necessary to assume the existence of the ether, and therefore desirable to learn as much as possible of its properties. To do this, we first reduce the laws of all its phenomena, including gravitation and the relativity-principle, to five equations, and then examine their meanings; and find that two of them are probably laws of the geometrical configurations of the different parts of the ether; two more, equations partially defining two convenient vectors, and stating the indestructibility of electricity; while the fifth, Hamilton's Principle, is a law of motion, expressing the perfectly efficient cooperation of the different parts of the fundamental mechanism of the universe.

From these laws we may draw certain conclusions about the structure and properties of the ether, which are not, however, enough to enable us to determine exactly what it is. But by a few simple assumptions, we obtain an imaginable model of its actions. And since the model is based directly on the electromagnetic laws, it may be applied, without fear of error, to any electromagnetic problem, to enable us to obtain a qualitative result without mathematical analysis.

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CONTRIBUTIONS FROM THE PHANEROGAMIC LABORATORIES OF HARVARD UNIVERSITY. — Nos. 55-58.

THE HISTORY, COMPARATIVE ANATOMY AND EVOLUTION OF THE ARAUCARIOXYLON TYPE.

By Edward C. Jeffrey.

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THE HISTORY, COMPARATIVE ANATOMY AND EVOLUTION OF THE ARAUCARIOXYLON TYPE.

By Edward C. Jeffrey.

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Part I.

Fossil woods of the Araucarioxylon type are extremely abundant in the Mesozoic deposits. The only living conifers with wood of this type are confined to the Eastern tropical region, to Australasia and to South America and are all included under the two genera Agathis and Araucaria. As a consequence of their habit, which differs from that of all living Conifers, except certain of the Podocarpineae, and of the organization of their woody tissues, the Araucarian Conifers have been most commonly referred to affinities with the Cordaitales, an important gymnospermous group of the Paleozoic. As will be shown in connection with the present investigations, the importance of these features of resemblance has apparently been much exaggerated. The association with the Cordaitales carries with it the implication, that the Araucariineae are either the ancestors of the other existing coniferous tribes, as is quite commonly held, or else that they constitute a separate line of descent, distinct from the ancestral stock of the remaining Conifers, as has been maintained in recent years by Seward and Penhallow. It is obviously a matter of considerable importance to clear up the affinities of the Araucarian stock, not only from the standpoint of its particular origin; but on account of the light thus to be thrown on the vexed subject of evolutionary processes as a whole by reason of the abundant display of the group during so long a period of geological time. The present writer has devoted nearly ten years to the procuring of material of Araucarian Conifers living and extinct and to the developmental, experimental and comparative anatomical investigation of their various organs and tissues.

It cannot be too strongly emphasized in connection with the present work, that general principles in biology are either of universal validity or of little scientific value, and that they cannot in certain cases be admitted and in others denied. There can be little doubt moreover that not a little of the existing reaction against the hypothesis of evolution, is the result of a failure on the part of biologists to apply evolutionary principles clearly, consistently and logically to the elucidation of their investigations, even if only from the standpoint of a working hypothesis. It seems clear that either there are generally valid biological principles, as there are commonly accepted principles in chemistry, physics and the other cognate sciences, or that biology has either not yet reached the scientific stage of development or has ceased to exist on the scientific footing. There appears to be no reason to adopt either of the latter alternatives. Darwin in the prolegomena to his Origin of Species, emphasized the importance of the data supplied by development, history, comparative anatomy and geographical distribution in connection with the study of evolutionary processes in living beings. Since Darwin's time experimental methods have come largely into the foreground and there can be little doubt that evidence derived from this source, especially when controlled by an adequate knowledge of the geological history of beings now living, is of paramount importance. It is proposed in the series of articles of which this is the first, to discuss the origin, affinities and evolution of the Araucarian Conifers, so far as appears profitable, along all the important lines of investigation, indicated above.

It will be convenient here to define the Araucarioxylon type of wood. In the mature secondary wood of the trunk in the living Araucaria and Agathis, we find certain peculiarities, which are taken together unique among living conifers. The tracheids in these two genera are characterized by the presence of pits, which are closely approximated and flattened, or where they occur in two or more rows, alternating in their arrangement and polygonal in their form. The wood of the Araucarian type in respect to its pitting resembles in a marked degree that of the Cordaitales. The remaining tribes of existing Conifers possess a type of tracheary pitting in which the pores are rarely or never closely contiguous and when in several rows are opposite and not alternating. The pits in this type too are often separated by cellulose bars running transversely across the lignified walls of the tracheids and imbedded in their substance. These bars of Sanio are absent in the Araucarian conifers.[1] They should not be confused

[1] Gerry, Eloise, The Distribution of the Bars of Sanio in the Coniferales, Ann. Botany, 24, p. 231.

with the trabeculae of Sanio, lignified processes, crossing the lumen of the tracheids, common to the Gnetales, Coniferales and a few Angiosperms. Another feature of the Araucarioxylon type is the usual absence of wood parenchyma and the smooth walled character of the ray cells. The last two features are less typical than the ones mentioned above since they are shared to a considerable extent by the woods of the remaining tribes of Coniferales. The last character has had recently assigned to it an apparently exaggerated importance.[2] Gothan has recently referred woods, which are strikingly Araucarian in the aggregate of their characteristics, to abietineous affinities on account of their strongly pitted rays, apparently losing sight of the fact that pitted rays occur commonly or sporadically in all the tribes of Conifers. The present article is to be devoted to the historical, comparative anatomical and experimental study of the rays and wood parenchyma in the Araucarian Conifers.

Beginning with the historical aspect, Figure *a*, Plate 1, shows the character of the pitting in the tracheids in an Araucarian wood of the Upper Jurassic, to be described in detail on another occasion. The pits are numerous and in several rows, with the marked alternation, characteristic of the Araucarioxylon type. They are not however as closely approximated as is the case with the pits in the tracheids of the adult wood of the living genera Araucaria and Agathis. Figure *b*, Plate 1, illustrates the ray structure in the same wood. It is clear that the cells of the ray, in contact with one another are very strongly pitted, exactly for example as is commonly the case in the rays of the Abietineae. On account of the pitting of the rays in woods of this type from the Upper Jurassic of King Carl's Land [3] and of the island of Spitzbergen [4] Gothan has recently referred them to abietineous affinities. It is to be pointed out in this connection that Seward has considered woods of a similar type from the Upper Lias of Yorkshire in England [5] to belong to the Araucarian conifers. Moreover Lignier more recently has described woods of a similar or nearly similar horizon, as likewise of Araucarian affinities.[6] About the same time

2 Gothan, Zur Anatomie lebender u. fossiler Gymnospermen-Hoelzer; Abh. d. Koenig. Preuss. geolog. Landesanstalt; Neue Folge, Heft 44, Berlin (1903).

3 Gothan, Fossilen Hoelzer von Koenig Karl's Land, Kung Svensk. Vetenskap. Handlingar, Bd. 42, No. 10.

4 Gothan, Fossilen Holzreste von Spitzbergen, Kung. Svensk. Handlingar, Bd. 45, No. 8.

5 Cat. of Mesozoic Plant, Brit. Museum, Jurassic Flora, Pt. 2, pp. 56, 57, pls. 6, 7, London (1904).

6 O. Lignier, Végétaux Fossiles de Normandie, IV. Bois Divers (Ire. Série), Caen (1907).

the present writer described woods of a similar type with a similar expression of affinities from a horizon, variously estimated from Middle to Lower Cretaceous, displayed at Kreischerville, Staten Island, N. Y.[7] It will be noted that the weight of opinion is against Gothan, in the matter of the reference of woods Araucarian in other respects, which have the strongly pitted rays of the Abietineae, to affinities with that tribe of Conifers, since Professor Seward, Professor Lignier and the writer agree in retaining them with the Araucariineae. Since a correct scientific verdict, however, does not depend on majorities, it will be well to investigate the matter from other standpoints.

A fundamental doctrine of Biology, owing its origin primarily to the deductive methods of the philosopher rather than to the more severe inductive procedure of the sciences, but since strongly confirmed by purely inductive data, is the doctrine of recapitulation. While it is undoubtedly the case that the seedlings and sporelings of the higher plants vouch in the strongest way for the validity of the recapitulation hypothesis, we have on the vegetable side corollaries to that doctrine, not illustrated as a rule by animals. There are organs of the plant for example, even more strongly retentive of ancestral characters than the seedling stem. Perhaps the most conservative organ is the root, which varies so little in its fundamental organization throughout the vascular plants, that one formula will represent the organization of all roots. In the case of the Gymnosperms and other typically coniferous groups, the axis of the cone has likewise been found to be strongly retentive of features which have disappeared entirely in the vegetative stem. Figure *c*, Plate **1**, shows the inner region of the woody cylinder of the cone of *Agathis australis*, in transverse section. It is clear that the cells of the wood rays are in contrast to the typical condition for living Araucarian Conifers, very strongly thickened and even in this unfavorable plane of section, obviously pitted. We have in other words a condition present like that found in certain Jurassic and Lower Cretaceous woods which have been referred by the majority of paleobotamists, who have specially investigated them, to Araucarian affinities. Gothan however as pointed out above, places them on account of their thickened and strongly pitted ray-cells among the Abietineae. Figure *d*, Plate **1**, shows a vertical section of one of the rays of the cone of *Agathis australis*, from which the contents have been removed in order that the sculpture

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[7] Araucariopitys, a new genus of Araucarians, Bot. Gaz., **44**, 1–15, pls. 27–30, (1907).

of the cell walls might stand out more clearly. It is obvious from the pitting of the tracheids seen on the left of the figure, that we have to do with araucarian wood, since the pits are alternating. The ray cells very strongly pitted on all their walls, towards the right of the figure, towards the left thin out and assume the ordinary Araucarian type. Figure *e*, Plate 1, shows part of the foregoing very highly magnified. The nature and abundance of the pits are now very clearly seen.

Not only does the cone of *Agathis australis*, clearly show the strongly pitted rays, which are found in the Jurassic and Lower Cretaceous woods, referred by the majority of recent investigators, to araucarian. affinities, but we find that the Mesozoic type of ray may be recalled by injuries to the root and the *seedling* stem. Figure *f*, Plate 1, illustrates the modification of ray structure which frequently occurs in the old roots of Agathis australis as the result of injury. The cells in this case too are much thickened and strongly pitted. The normal seedling rays of *A. australis* have not been observed to show pitting or thickening on their terminal or horizontal walls in any case. The mature stem rarely shows reversion in ray structure to the earlier Mesozoic type as a result of injury. *Agathis australis* merely furnishes a good illustration of a condition of affairs in normal and traumatic anatomy, which so far as it goes, in accordance with accepted biological principles, vouches for the descent of the existing representatives of the Araucarian stock from ancestors in the Mesozoic, which possessed rays like those of living as well as extinct representatives of the Abïetineae. Similar facts have been observed in other cases not only in the genus Agathis but also in Araucaria. It appears unnecessary to enlarge upon these at the present time.

Attention may now be given advantageously to the question of wood parenchyma in the Araucariineae. As is well known the Cordaitales, from which perhaps the majority of botanists at the present time directly derive the Araucarian Conifers, were characterized by the complete absence of wood parenchyma. The living species of Agathis and Araucaria, manifest this condition likewise in the normal mature wood of the stem and thus present *prima facie* evidence of close affinity with Cordaitales and other ancient Gymnosperms. Here again we may turn with advantage to the historical evidence and then to comparative anatomical and experimental data in the living representatives of the Araucarian stock, Figure *a*, Plate 2, shows a longitudinal section of an Araucarian wood from the Raritan Cretaceous of Kreischerville, Staten Island, N. Y. Certain dark longitudinal stripes are to be noticed particularly to the right of the center of the figure. These represent resiniferous parenchyma. Figure *d*, Plate 2, shows a portion of the last figure more highly magnified, to make clear the transverse partitions separating the resiniferous elements from one another. Figure *b*, Plate 2, shows the same wood in transverse section, the dark spots indicating the presence of the resiniferous cells. Figure *c*, Plate 2, shows a section of the same wood near the pith, making it clear that we have to do with the stem wood of an extinct araucarian conifer. The resiniferous elements can be seen as in the preceding figure scattered throughout the wood. The writer has had the opportunity of examining a number of araucarian woods from the Raritan Cretaceous of the Eastern United States and has found in all true Araucarioxyla an abundance of wood parenchyma. In this respect they present a marked contrast to the normal stem wood of the living Agathis and Araucaria, although resembling them to a striking degree in other respects.

Let us now turn our attention to the conservative organs of the living genera. Figure *e*, Plate 2, illustrates the structure of an old root of *Agathis australis*, near the center. It is to be observed that the wood is thickly sown with parenchyma cells. These, it may be added are most abundant near the center of the root and die out progressively as the outer annual rings of the older root are reached, unless recalled by injury, as is noted below. Figure *a*, Plate 3, shows a longitudinal section of the same root, making it clear that we have really to do with resin cells and not merely with tracheids filled with a resinous or mucilaginous contents such as are not infrequent in coniferous woods of varied affinities. Resin cells are extremely common in the first formed annual rings of the root in the genus Agathis and likewise occur to a less degree in the root organs of Araucaria. In certain species of Agathis, they likewise are found in *the first annual ring of the stem.* This condition may be illustrated by *A. australis* and *A. Bidwillii,* which represent as nearly as possible the extremes of affinity within the genus. Figure *b*, Plate 3, illustrates the mode of occurrence of parenchyma in the first year's growth of *A. Bidwillii.* The dark spots are parenchyma cells. Figure *c*, Plate 3, shows a longitudinal section of the same species. On the left is seen the protoxylem and a little to the right of the center, a row of parenchymatous elements, still retaining their protoplasmic contents. Figure *d*, Plate 3, shows the same conditions in the first annual ring of *A. australis.* Here the parenchyma tends to occupy the face of the summer wood, in the first yearly increment, thus resembling the conditions found by Gothan in certain araucarian woods from King Carl's Land apparently wrongly referred by him to abietineous affinities.[8] Figure *c*, Plate **3**, is a longitudinal section of the same species, illustrating the vertical distribution of the wood parenchyma. Seldom or never does parenchyma make its appearance in the normal wood of outer annual rings. At this point it is convenient to record another feature of interest. Through the kindness of Messrs. Eames and Sinnott, Sheldon fellows of Harvard University, who have recently spent a year in the investigation of the coniferous flora of the Australasian region, the writer has been supplied with seedlings of the genera Agathis and Araucaria. It was found on investigation that in the case of Agathis there was usually no wood parenchyma in the first annual ring in the seedling until it had reached a considerable size. In fact it is only in the vigorous branches that bear cones that the parenchymatous elements appear in any abundance. The recapitulationary phenomena in the case of wood parenchyma are accordingly delayed until the plant has reached a certain vigor, thus presenting an exact homologue with the conditions found for example in certain of the Abietineae, which are normally without resin canals in the wood[9] and in *Sequoia gigantea*.[10] Here the resin canals, so characteristic of the pine-like Abietineae, occur in the first annual ring of vigorous vegetative shoots and in a few cases only in the axis of the cone. The evidence in the case of the genus Sequoia and in the Abieteae, has been accepted by other investigators who have given special attention to the Conifers, as a clear indication that both the Abieteae and Sequoia have come from pine-like ancestors.[11] Mr. Thomson's views in this respect are particularly significant as his attitude in regard to the affinity of the Araucarian conifers is diametrically opposed to that of the present writer. As will be pointed out later, the admission of the validity of certain general principles in the case of certain coniferous tribes, logically implies their application to the whole series. We find then this feature of accord between recapitulationary phenomena in for example Abies and Sequoia on the one hand and Agathis on the

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[8] Gothan, Die Fossilen Hoelzer von Koenig Karl's Land, Kung. Svensk, Vetenskab. Handlingar, Bd. 42, No. 10.

[9] Jeffrey, Comp. Anat. of the Coniferales, No. 2. The Abietineae, Mem. Boston Soc. Nat. Hist., **6**, pp. i–37, pls. 1–7 (1904).

[10] Comp. Anat. Coniferales, No. 1, The Genus Sequoia, Mem. Bost. Soc. Nat. Hist., **5** (1903).

[11] Coulter and Chamberlain, Morphology of Gymnosperms, Chicago (1911), and Thomson, R. B., Megasporophyll of Saxegothea and Microcachrys, Bot. Gazette, **47** (1909).

other, that the resin canals in the case of the former and the resiniferous parenchyma in the case of the latter, do not appear in the first annual ring of the seedlings but only in the first annual increment of older and more vigorous axes. In the case of Agathis it is clear from the fossil data, that we actually have a harking back to ancestral phenomena, presented in the extinct forms as shown above. In Abies and Sequoia we can only infer that their ancestors had resin canals in accordance with accepted principles of biological science. It seems clear also in both types of illustration, that we have in the case of the living representatives, to do with reduction phenomena. The fact that Abies and Sequoia on the one hand and Agathis on the other hand are degenerate descendants of stocks once more vigorous and richly endowed, doubtless furnishes the explanation of why the recapitulationary phenomena in connection with the first annual ring make their appearance not in the seedling; but only after the plant has attained the reproductive age.

Figure *f*, Plate **3**, illustrates the conditions found in connection with the parenchyma of a wounded root of *Agathis australis*. It is to be noticed that most of the parenchymatous cells are thick-walled and in some instances strongly pitted. This figure is to be compared with Figure *c*, Plate **1**, which shows the normal condition of the cone axis. Here both the rays and parenchyma are thick-walled on the side nearer the pith. In the case of *A. australis* the adult stem, when injured, in contrast to the root does not form thick-walled wood parenchyma but only thin walled elements. That this is the case is demonstrated by Figures *a* and *b*, Plate **4**, which show the injured stem wood of *A. australis* in transverse and longitudinal section. Thin-walled parenchyma can be seen in each case.

Figures *c* and *d*, Plate **4**, show the transverse and longitudinal views of the heart wood of *A. australis*, illustrating the presence of resinous exudations in the tracheids of the wood immediately adjacent to the rays. In the longitudinal view the relation of the exudation to the ray cells is particularly well seen. Penhallow has compared these transverse septa, resulting from substances poured out by the ray cells into the tracheids with the trabeculae of Sanio. They have in reality of course nothing to do with these structures.[12] Lignier has described the thickening up of the tracheids adjacent to the rays in certain Araucarian woods from the French Jurassic. It seems entirely probable that he has mistaken resin filled tracheids for thick-walled

12 Penhallow, North American Gymnosperms, Boston (1907), pp. 53–58.

ones, as a result of the bad condition of preservation of his material.[13] Figures *e* and *f*, Plate 4, make this probability practically a certainty. Figure *e* illustrates the transverse view of *Araucarioylon noveboracense* from the Raritan Cretaceous of Staten Island.[14] The tracheids in contact with the rays are apparently distinguished by their very thick walls. Figure *f*, which represents a longitudinal view of the same piece of lignite, makes it clear that the apparently thick walled tracheids are in reality only tracheids more or less occupied by a plugging exudation from the rays.

Although a general statement as to the inferences to be drawn from the series of articles, of which this is the first, will appropriately appear in connection with the last of the series, it is apposite and necessary to point out the particular conclusions to be derived from the observations recorded here. It is clear that there are certain definite structural relations between the Araucarian woods now in existence and those no longer living. In general the structural features of the Mesozoic Araucarioxyla are strongly retained in the cone axis, and the root of living species. They are less strongly retained in the vegetative stem. In the case of the latter, ancestral features may reappear in the first annual ring of axes of unusual vigor or as a result of injury. Injuries to the root result in the recall of more ancient features than those which can traumatically be recalled in the stem. Further it is clear that the comparative developmental and experimental study of living Araucarian conifers is of the greatest value and significance in connection with the accurate diagnosis of fossil forms. A comparison of living with extinct forms, so far as the points considered in this article are involved, shows that certain Mesozoic woods, which have been referred by Seward, Lignier and the present writer to the Araucariineae, in reality have that systematic affinity and are not as has been recently suggested by Gothan, the woods of Abietineous Conifers.

Conclusions.

1. The ancestors of Araucaria and Agathis were characterized by the possession of wood parenchyma.
2. They likewise had strongly pitted rays.
3. The possession of these two features is quite inconsistent with their derivation from Cordaitean ancestry.

13 *Op. cit.*, pl. 17.

14 Hollick and Jeffrey, Cret. Coniferous Remains, Staten Island, Mem. N. Y. Bot. Garden, **3**, pl. 21.

4. Certain woods from the Jurassic and Lower Cretaceous, possessing at once araucarian pitting, of the tracheids, abundantly developed wood parenchyma and strongly pitted medullary rays are in reality aruacarian in their affinities and not abietineous as has recently been asserted by Gothan on the insufficient basis of their ray structure.

5. The characteristic features of Mesozoic araucarian woods are retained to a large degree in the wood of cone axis, root and first annual ring of vigorous branches of living representatives of the Araucariineae.

6. They may be recalled by experimental means particularly in the root and the seedling stem.

PLATE 1.

Fig. *a*. Radial view of an undescribed wood from the Lias of Yorkshire, England, showing the Araucarian type of pitting. × 100.

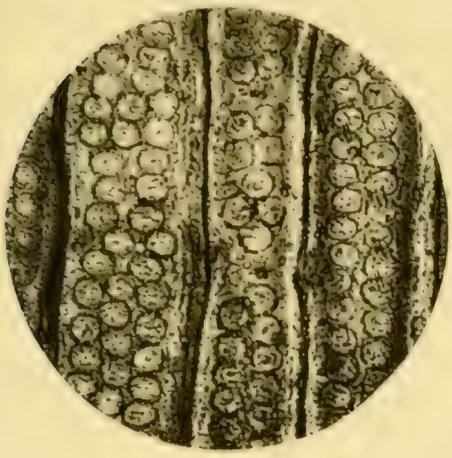
Fig. *b*. Radial view of the wood of the same showing pitted character of the medullary ray × 100.

Fig. *c*. Transverse section of the wood of the cone-axis of *Agathis australis*, showing the strong normal pitting of the ray cells near the pith. × 100.

Fig. *d*. Radial section of the same showing character of the tracheids and the ray cells. × 100.

Fig. *e*. Part of the same. × 300.

Fig. *f*. Injured wood of the root of the same in transverse section, showing the thick-walled, strongly pitted ray cells formed traumatically. × 100.



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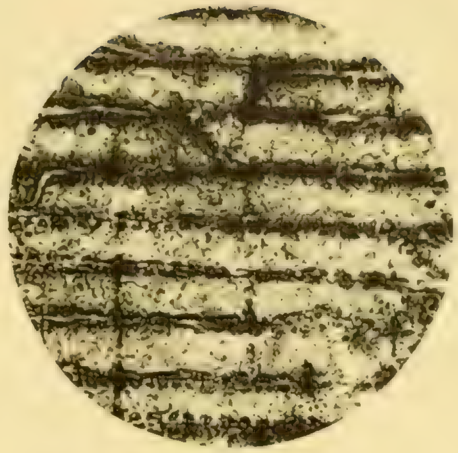

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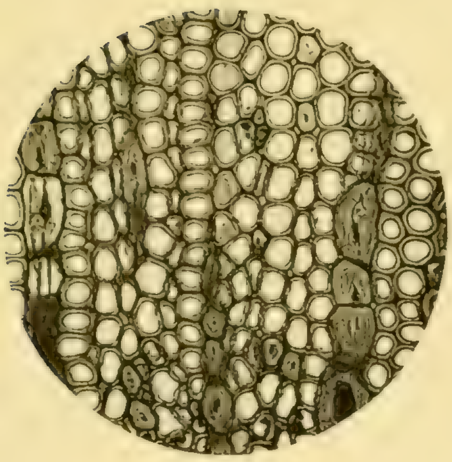

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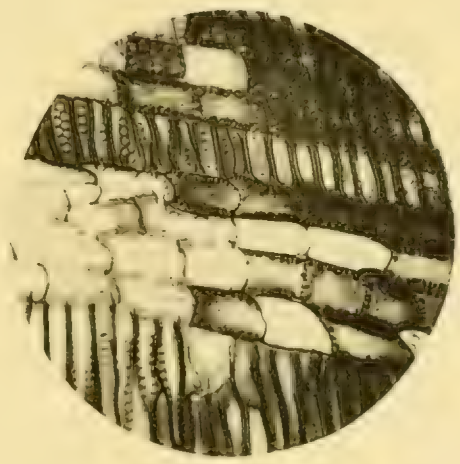

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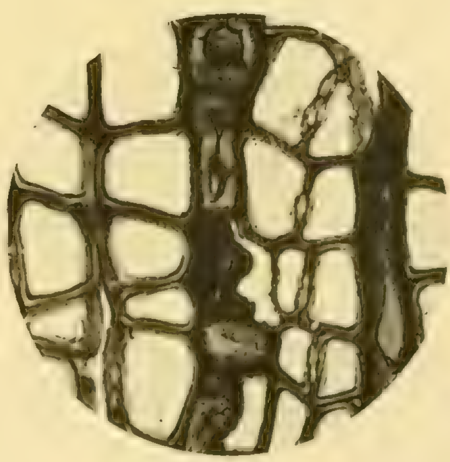


f

PLATE 2.

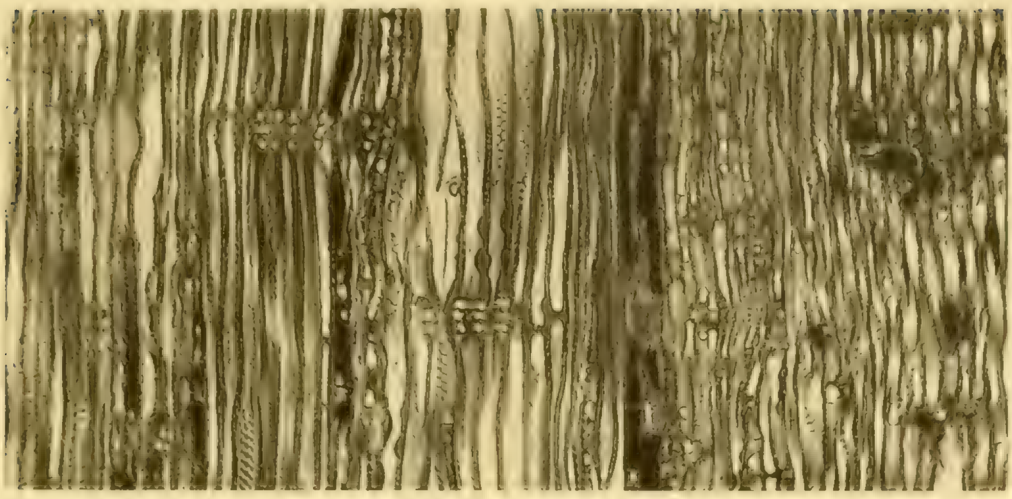
Fig. *a*. Radial view of the wood of *Araucarioxylon noveboracense*. $\times$ 50.

Fig. *b*. Transverse section of the same. $\times$ 40.

Fig. *c*. Transverse section of the same near the pith. $\times$ 40.

Fig. *d*. Long radial section of the same. $\times$ 100.

Fig. *e*. Transverse section of the wood of the root of *Agathis australis*. $\times$ 40.



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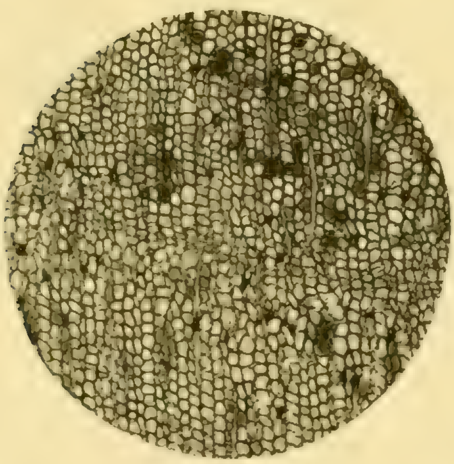

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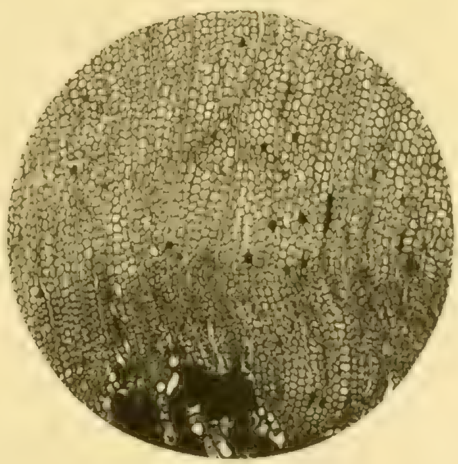

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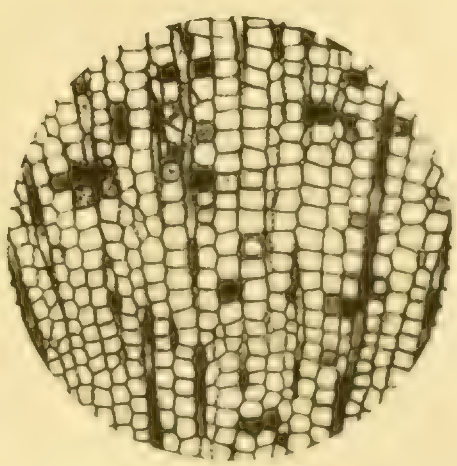

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PLATE 3.

Fig. *a*. Radial section of the same. × 40.

Fig. *b*. Transverse section through first annual ring of stem of *Agathis Bidwillii*, showing resin cells. × 40.

Fig. *c*. Longitudinal section of the same. × 40.

Fig. *d*. Transverse section of old stem of *Agathis australis*, showing first annual ring. × 60.

Fig. *e*. Radial section of the same. × 60.

Fig. *f*. Transverse section of root wood of *Agathis*, showing both pitted and thin walled parenchyma. × 100.

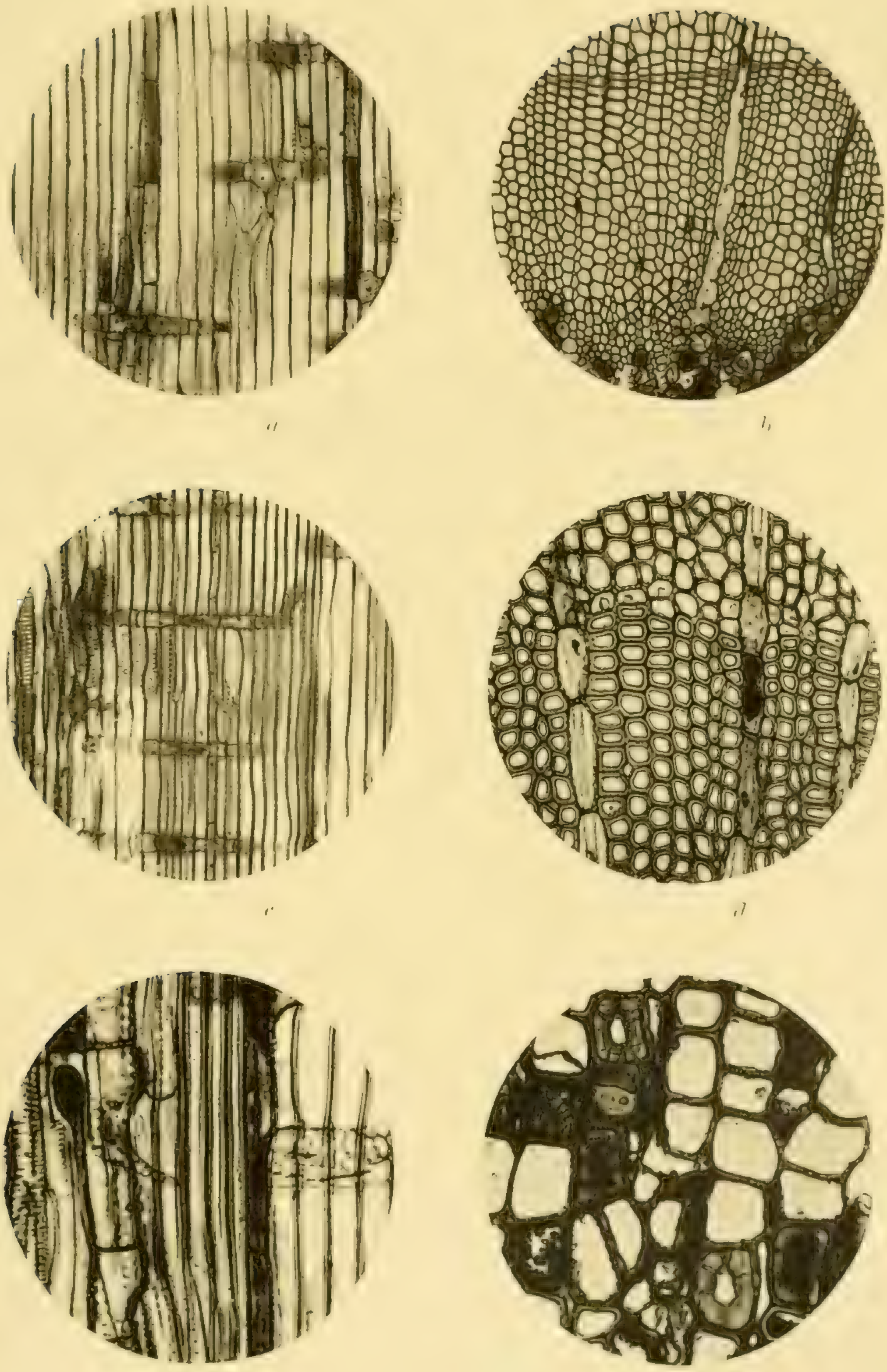


PLATE 4.

Fig. *a*. Wounded wood of the stem of *Agathis australis* in transverse section, showing the return of the ancestral wood parenchyma as the result of injury. × 100.

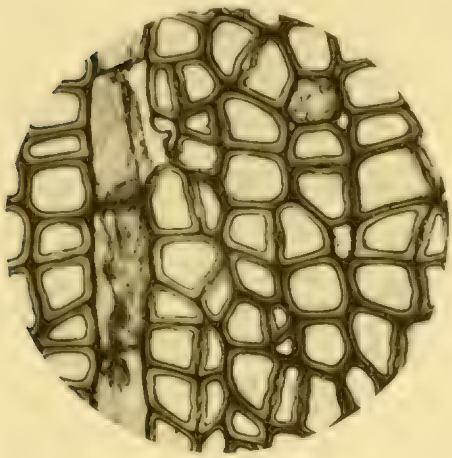
Fig. *b*. The same in radial section. × 100.

Fig. *c*. Wood of *Agathis australis* (normal) in transverse section, showing plugging of tracheids in proximity to the rays. × 40.

Fig. *d*. The same in tangential longitudinal section, showing relation of plugs to the ray cells. × 40.

Fig. *e*. Transverse section of the wood of *Araucarioxylon noveboracense*, for comparison with Figure *c*. × 40.

Fig. *f*. Tangential section of the same for comparison with Figure *d*. × 40.



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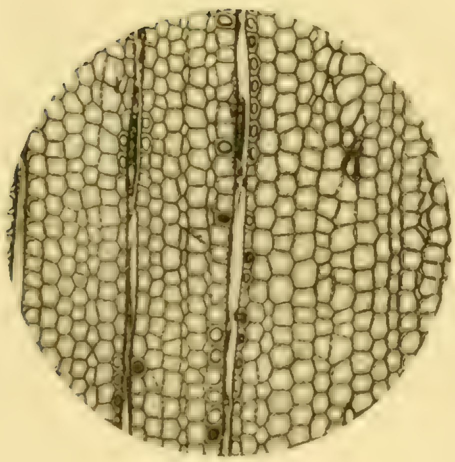

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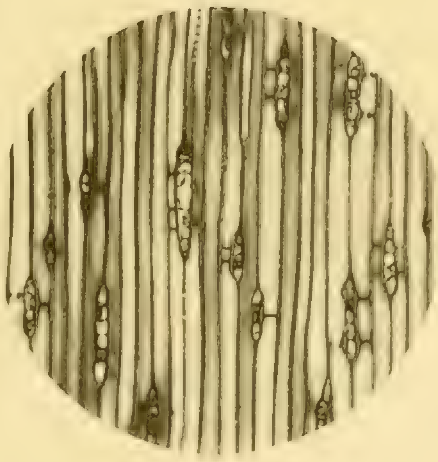

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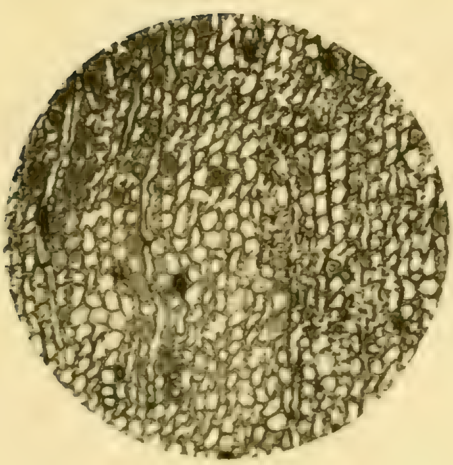

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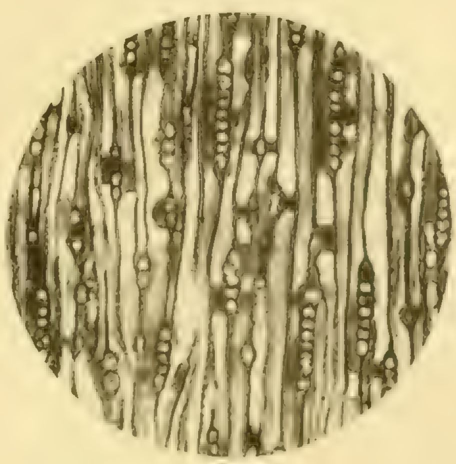

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HELIOTYPE CO., BOSTON

THE HISTORY, COMPARATIVE ANATOMY AND EVOLUTION OF THE ARAUCARIOXYLON TYPE.

By Edward C. Jeffrey.

Part II.

In the first article of the present series the structure of the rays and the parenchyma of woods of Araucarian affinities, was considered. In the present one the characteristic features of the tracheids and the nature of the pitting will be particularly discussed. The pitting of the tracheary elements in the Araucarian Conifers has been considered by practically all writers as an infallible criterion for the diagnosis of their woods as fossils. It is unnecessary to enter upon this matter in detail as the literature on the subject has quite recently been admirably summarized by Gothan.[15] It is universally assumed that crowded radial pits on the tracheid walls either flattened by mutual contact, if the pores are uniseriate, or of somewhat polygonal outline in case the pits are in several rows, indicate Araucarian affinities. Recently however a tendency to question the universal validity of the Araucarian type of tracheary pitting as an indication of Araucarian affinities has made itself felt. On the one hand it has been maintained that woods with typical Araucarian pitting in reality were referable to the Abietineae on the assumed more important character of their ray structure.[16] On the other it has been maintained that woods without typical Araucarian pitting in reality belonged, in consideration of the sum of their characters to the Araucarian Conifers.[17] The

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15 Zur Anatomie lebender u. fossiler Gymnospermen-Hölzer, Berlin (1905).

16 Gothan, Die Fossilen Hölzer von König Karl's Land, Kung. Svensk. Handlingar, Bd. 42, and Gothan, Die Fossilen Hoelzreste von Spitzbergen, Kung. Svensk. Vetenskap Handlingar, Bd. 45.

17 Gerry, Eloise, Distribution of the bars of Sanio in the Conifers. Ann. Bot. **24** (1910); Sinnott, E. W., Paracedroxylon, a new type of Araucarian wood, Rhodora, **11**;Jeffrey, E. C., The Affinities of *Geinitzia gracillima*, Bot. Gazette, **50**.

question of the value of Araucarian pitting as an indication of Araucarian affinities is all the more important because it likewise has been made to involve the relationship of the Araucarian Conifers with the Cordaitales of the Paleozoic, which as regards their pitting strongly resemble the conditions typical of the wood of the living Agathis and Araucaria. Still another important question arises in connection with certain features of organization of the Araucarian tracheid as compared with that found in other Conifers. It has been pointed out by Miss Gerry [18] that the Araucarian Conifers both living and extinct are without the horizontal cellulose bands, between the radial bordered pits, characteristic of all other Conifers. This feature has an added importance from the fact that a similar feature is likewise characteristic of the wood of the Cordaitean gymnosperms. It is the purpose of the present article to deal with these features of the Coniferous tracheids in regard to their value as indications of tribal relationship and evolutional sequence in the Coniferales.

It will be convenient to begin with the subject of Araucarian pitting. Figure *a*, Plate 5, shows the crowded alternating arrangement of the tracheary pores, which is regarded as typically Araucarian. The illustration is taken from the wood of *Araucarioxylon, noveboracense,* from the Raritan Cretaceous of Kreischerville, Staten Island, N. Y.[19] Figure *b*, Plate 5, illustrates the arrangement and propinquity of the radial bordered pits in the *first annual ring* of the same type of lignite. The absence of approximation and consequent flattening of the bordered pits is very apparent. An examination of a considerable number of true Araucarioxyla from the American Cretaceous has led the writer to the general conclusion that no matter how typical the Araucarian arrangement of the pits may be in the mature wood, that in the first annual ring of the stem one always finds a marked tendency to the rounded and well spaced pits which are typical of the wood of the Abietineae and allied Conifers.

Further in connection with recent investigations on woods of the American and European Mesozoic, numerous instances have been described, presenting to a greater or less degree Araucarian characteristics, but with a marked departure from the Araucarian type of pitting This is notably the case, for example, with the recently established genus Brachyoylon.[20]

An even more striking illustration is supplied by the genus Para-

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18 *Op. cit.* 19 *Op. cit.*

20 Hollick and Jeffrey, Cretaceous Coniferous Remains from Kreischerville, Mem. N. Y., Bot. Garden, No. III.

cedroxylon recently described by Sinnott.[21] Perhaps the most conspicuous illustrations of this condition are supplied by Araucariopitys [22] and the so called *Cedroxylon transiens* of Gothan.[23] Here not only does the pitting depart largely from the Araucarian type, but the wood is likewise particularized by strongly pitted rays resembling those of the Abietineae. It is clear from the facts and citations of facts here assembled, that in the Mesozoic there were woods which either had the Araucarian type of pitting very imperfectly developed or if well displayed in the adult wood, not found to be present in the first annual ring. It may be stated in anticipation of conclusions to be drawn later, that it follows that Araucarian pitting was not a characteristic of the primitive stock from which the Araucariineae of to-day and their nearest relatives in the Mesozoic, were derived.

The conditions in the living genera of the Araucariineae as regards pitting, may now advantageously be considered. Figure *c*, Plate **5**, shows the appearance of a radial section of the wood in an old seedling stem of *Agathis australis*, perhaps the most highly specialized species of the genus now in existence. The pits are obviously much crowded and when in a single row strongly flattened, or when multiseriate somewhat polygonal in shape. Exactly similar conditions are found in the case of the wood of species of Araucaria, and as a consequence it is not necessary to illustrate by a figure the wood structure in that genus. Figure *d*, Plate **5**, shows a radial section of the wood at the base of the seedling stem of *Agathis australis*. Here the pits are obviously not crowded or flattened by mutual contact. This condition is found for several inches above the ground in the seedling stem and for a great number of annual rings outwards, as many for example as fifteen. In the main root of the seedling similar conditions are found to a considerable depth but in the secondary roots, the pitting becomes typically Araucarian, with the very many rows of pits, characteristic of root wood in general. In the seedling stem of *Araucaria Bidwillii*, *A. imbricata* and *A. Cookii*, very similar conditions were found, to a less marked degree and lower down in the stem, rather in its hypocotyledonary than its epicotyledonary region. It is an interesting fact that in the seedling stem of the living genera of the Araucarian Conifers, we find perpetuated the type of pitting charcteristic of Brachyoxylon,[24] Araucariopitys,[25] Paracedroxylon,[26] *Cedroxylon transiens* [27] from various levels of the Mesozoic. Here we have illustrated in a

21 *Op. cit.*
22 Jeffrey, Bot. Gazette, **44** (1907).
23 *Op. cit.*
24 *Op. cit.*
25 *Op. cit.*
26 *Op. cit.*
27 *Op. cit.*

remarkable way the validity of the doctrine of recapitulation, in accordance with which the young individuals of living species may pass through in their earlier stages of development the condition found typically in their extinct ancestors of more or less remote geological time.

The first annual ring of the living species of Agathis and Araucaria, unlike the Araucarian woods of the Araucarioxylon type, from the American Cretaceous, shows only slightly and often sporadically the departure from Araucarian pitting characteristic of Brachyoxylon, etc. Not more than two or three tracheids next the protoxylem in the most favorable cases illustrate this feature. In this respect the existing woods of the Araucarian type show themselves, as might be expected, less retentive of ancestral characters than is the case with the similar woods from the Cretaceous. In the case of the normal type of Araucarian wood, not only the approximation but also the alternation of the radial pits of the tracheids are characteristic features. Figure *e*, Plate 5, shows under a comparatively low magnification, the structure of the tracheids adjoining the protoxylem in the cone of *Araucaria Bidwillii.* It is easy to make out that the pits in the tracheary elements of the secondary wood nearest the scalariform elements of the protoxylem, are arranged for the most part in opposite pairs. Moreover even with the low magnification employed it is clear that the pits in question are not flattened by mutual, approximation. In other words we have the conditions present, so far as the radial pitting is concerned, which are typical of the wood of the Abietineae and other tribes of Conifers. Farther away from the protoxylem the pitting passes into the typical araucarian condition. Figure *f*, Plate 5, shows a part of the last more highly magnified. On one side the tracheids still retain some indications of the spiral and reticulate sculpture of the protoxylem. On the other tracheids of the secondary wood have made their appearance. They are characterized, however, by a distinctly non-Araucarian arrangement of the pits and by other remarkable and important features. The pits are separated from each other by appreciable intervals. The most remarkable feature, however, shown by the tracheids in this region is the presence of dark discontinuous stripes crossing the tracheids transversely between the pits. These dark stripes in the photograph often fork at the ends and represent cellulose bands in the substance of the tracheid walls. They are in fact typical bars of Sanio (not to be confused with the 'Balken' or trabeculae of Sanio, which are a very different thing), found normally in all the tribes of Conifers except the Araucariineae.

Before passing to the consideration of the significance of these structural features of the wood of the cone in *Araucaria Bidwillii*, it will be well to examine them more particularly in this species and discover their occurrence and development in other species of Araucaria as well as in species of the allied genus Agathis.

Figure *a*, Plate 6, illustrates the conditions presented in another photograph of a radial section of the wood in *Araucaria Bidwillii*. Here although the magnification is not great the bars of Sanio stand out with great clearness between the pits, which on the whole tend more in their arrangement to the typical Araucarian condition of alternation than in the figures described above. The tracheids are bounded above and below by wood rays, showing that although they lie near the primary wood they are typical elements of the secondary xylem. Figure *b*, Plate 6, shows a very highly magnified view of parts of three tracheids of the secondary wood in proximity to the primary xylem. Here it is possible to distinguish the bars of Sanio with great clearness. They are as a rule, invariably in the figure under discussion, not continuous across the tracheid, but subtend usually the breadth of a single pit. The forking of the cellulose bars at the ends can be clearly made out.

Figure *c*, Plate 6, shows the conditions in the tracheids of the secondary wood, adjacent to the primary xylem in *Araucaria imbricata*, very highly magnified. On one side of the figure can be seen a spirally sculptured element of the primary xylem. On the other, one tracheid in particular shows clear bars of Sanio. In *Araucaria imbricata* which, as will be shown later, as a result of the consideration of a number of lines of evidence is among the least primitive species of the genus, the tracheids showing well spaced pits and clearly discernible bars of Sanio are very few in number. *Araucaria Cookii* and *Araucaria Rulei* were likewise examined, with results intermediate between those found in *A. Bidwillii* and *A. imbricata* which appear in these as in other respects to represent the extreme conditions found in the genus.

For comparison an illustration of the conditions in the mature secondary wood of *Pinus strobus* is shown in Figure *d*, Plate 6. Here the cellulose bars of Sanio are very distinct between the uni- or bi-seriate pits. The pits where they are in two rows are opposite. The occasional forking of the bars at the ends can likewise be made out. The pits are well spaced and rounded.

Figure *e*, Plate 6, shows a tracheid wall of *Araucaria Bidwillii* in tangential section. That the plane of section is in reality tangential and that the element belongs to the secondary wood is vouched for by the presence, on the right and left, of cells of the wood rays in transverse section. The dark transverse sections of the bars of Sanio, embedded in the substance of the lignified tracheid wall, between the radial pits, are easily distinguished. Figure *f*, Plate 6, illustrates the conditions observable in the ordinary secondary wood of the vegetative axis. Obviously the pits as seen in profile are here in close contact and are not separated by bars of Sanio.

The stem of the seedling and the first annual ring of the adult branches of various species of Araucaria, were examined in the region of the primary xylem for the presence of bars of Sanio. Where any evidence of their existence was apparent, however, they were extremely indistinct and ghostly and very evanescent. The same conditions were observed in the root. In accordance with the now widely accepted dictum of comparative anatomy that the leaf trace is very apt to perpetuate ancestral conditions, the foliar traces of several species of Araucaria were investigated, but on account mainly of the small size of the tracheary elements, it was difficult to make out the presence of bars of Sanio, with any distinctness, although their existence in these regions was indicated.

The absence of bars of Sanio in the seedling, where the pits are often widely separated from one another, is of particular significance, in view of the statement of Gothan, that their non-existence in Auracarian woods is to be explained by the close approximation of the pits. Obviously such an explanation will not hold in the case of the undoubtedly Araucarian wood of Araucarian seedlings.[28] It follows that woods from the Mesozoic which are without typical Araucarian pitting, can best be diagnosed as to their affinities not on the basis of their radial pitting or even their ray structure, but by the presence or absence of bars of Sanio, in well preserved material. Where the bars are present in the mature wood, we may certainly assume that the wood is not Araucarian. On the other hand, where the bars of Sanio are distinctly absent in well preserved Mesozoic woods, it may safely be concluded that they are of Araucarian affinites, no matter what may be the nature of their radial pitting or that of the cells of their rays.

In conclusion of the descriptive part of the present article, it is necessary to refer to the pitting and structure of the tracheids in the living genus Agathis. It has been found here, that even in the cone, the tracheids very quickly cease to show opposite pitting and the bars

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28 Gothan, Die fossilen Holzreste von Spitzbergen, Kung. Svensk. Vetenskap, Handlingar, Bd. 45.

of Sanio in all cases are shadowy and difficult to discern, although they can be made out by the eye of expectancy. *Agathis Bidwillii*, material of which was received from the Botanic Garden at Buitenzorg, Java, through the kindness of the late Dr. M. Treub, Director, proved in this respect to be most favorable. Even here, however, in the most favorable instances, the bars of Sanio are scarcely as well developed in the wood of the cone, as they are in the least favorable species of Araucaria, which has been examined and figured in the present connection, viz. *A. imbricata*. It does not seem necessary on that account to present illustrations of Agathis. *A. australis* shows bars of Sanio, less distinctly than any other species examined. No indication whatever of the existence of bars of Sanio has been found in the seedling of *A. australis*, although it has been examined in detail with considerable care.

Finally attempts were made to discover bars of Sanio in the region of the primary wood in species of Araucarioxylon. Here on account of the generally bad state of preservation of the material and also doubtless on account of the delicate nature of the bars in this region, even in living representatives of the Araucariineae, the results were entirely negative. It is well perhaps at this point to indicate the best method of demonstrating bars of Sanio in the wood of cones of living species of the Araucarian tribe. Haidenhain's hematoxylin was found most useful for bringing out the structure in question; but care must be taken to have both the hematoxylin solution and the iron alum solution perfectly fresh. The sections after being subjected to the action of the iron alum for ten or fifteen minutes are washed carefully and rapidly in three changes of distilled water. They are then allowed to remain in fresh distilled water for half an hour or more. Next they are treated for some time with hematoxylin solution of one fourth of one per cent strength. In this they remain for some time, up to half an hour. Unless the solutions are quite fresh they will become fatally overstained. After a washing or two in distilled water, the sections are transferred to a very dilute aqueous solution of safranin and allowed to remain for several hours or over night. If the process has been successfully carried out, the bars of Sanio will appear as intense blue transverse bands on the red background of the lignified cell wall of the tracheid. They are most easily seen nearer the ends of the tracheary elements, just as is the case in those coniferous woods where they are abundantly and normally present in the mature tissue.

It was considered that the appearances described above in connection with the obvious presence of bars of Sanio, might possibly be common to all secondary woods in the region of the primary xylem. Sections of the cone axis, the leaf-strands and roots of Cycas and Zamia were accordingly made and subjected to the same treatment. In no case was there any indication of the presence of horizontal bands of cellulose in the tracheids, between the radial bordered pits. Similar observations were made on the vegetative stem, the leaf strands and the reproductive axes of Ginkgo. Here as in the case of Cycas and Zamia, no bars of Sanio were seen in proximity to the primary wood. In fact in the reproductive axes and in the leaf strands no bars of Sanio were seen at all. In the vegetative stem, however, they appear late in the first annual ring, not in close proximity to the primary xylem. As is well known, Ginkgo resembles the mass of Conifers, in showing bars of Sanio clearly in its mature wood. Pinus, as probably the most primitive living representative of the Coniferales was likewise examined in this connection. Here the conditions closely resemble those found in Gingko, so far as the vegetative shoots are concerned, for the bars of Sanio make their appearance late and not in proximity to the primary wood. In the cone of *Pinus strobus*, bars of Sanio were not found at all. It is to be noted in connection with these results, as contrasted with those found in the case of the Araucarian Conifers that, there is clear evidence, so far as may be judged from the structure of the first annual ring, that Ginkgo and the genus Pinus are directly connected with the Cordaitean stock, in which bars of Sanio are absent and the pitting is alternating, while Agathis and Araucaria have obviously come from ancestors which, in accordance with accepted principles of comparative anatomy, had opposite pitting and bars of Sanio in their tracheids.

It seems to be quite clear so far as the particular features of wood structure, considered in the present article, are concerned, that far from the absence of bars of Sanio and the presence of alternating pitting in the woods of the Araucariineae, being an argument for their direct filiation with the Cordaitales, these features have clearly been secondarily acquired and the Araucarian stock primitively was characterized by the bars of Sanio and opposite pitting, which have been retained in the ligneous structure of all the other living tribes of the Coniferales. It is also quite clear from the fossil evidence that the loss of bars of Sanio, in the case of the Araucariineae, as well as the disappearance of the ancestral opposite pitting, took place at a period relatively remote. That this general inference is justified by a number of other equally important facts will be shown in the later articles.

Summary.

1. The characteristic pitting of the wood in Agathis and Araucaria, the Araucarioxylon type, is not ancestral but more recently acquired.

2. This conclusion is based on the structure of the first annual ring of the stem in Mesozoic Araucarioxyla. It is confirmed strongly by the seedling structure of the living genera and particularly by the anatomical structure of the wood of their cone axes.

3. The cellulose bars of Sanio, characteristic of the mature wood of all living genera of the Coniferales, except Agathis and Araucaria, are clearly present in the secondary tracheids adjacent to the primary wood of the cone axis in these two genera. They are absent in the seedling and cannot be clearly discerned in the leaf traces on account of the small size of the elements.

4. Since bars of Sanio do not occur in similar situations in Cycas and Ginkgo, it cannot be assumed that they are a feature of all gymnospermous woods in proximity to the primary xylem.

5. Since deviations of a significant nature in the pitting and structure of the tracheids occur in primitive regions of the Araucarian axes, which connect them with the remaining tribes of the Coniferales stock, it follows that so far as these features are concerned, the Araucarian Conifers are derived from the common coniferous plexus and are not directly articulated with the Cordaitales.

6. On the basis of comparative studies of the tracheids of the Araucariineae, they cannot be regarded as primitive representatives of the Coniferous order.

7. The real affinities of the Araucariineae can best be defined when all the evidence is considered in the concluding article of this series.

PLATE 5.

Fig. *a*. Radial section of the wood of *Araucarioxylon noveboracense*. × 200.

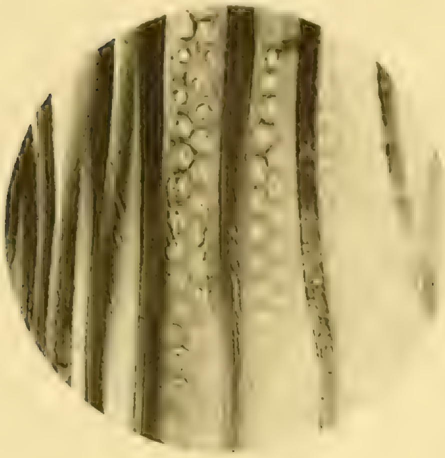
Fig. *b*. Radial section of wood in proximity to the protoxylem in the same. × 200.

Fig. *c*. Radial section of the wood of *Agathis australis*. × 200.

Fig. *d*. Radial section of the wood of the seedling of the same. × 300.

Fig. *e*. Radial section of the wood of the cone of *Araucaria Bidwillii*, in proximity to the protoxylem. × 60.

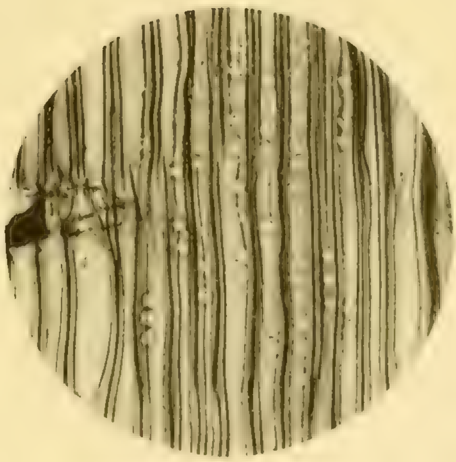
Fig. *f*. The same more highly magnified. × 200.



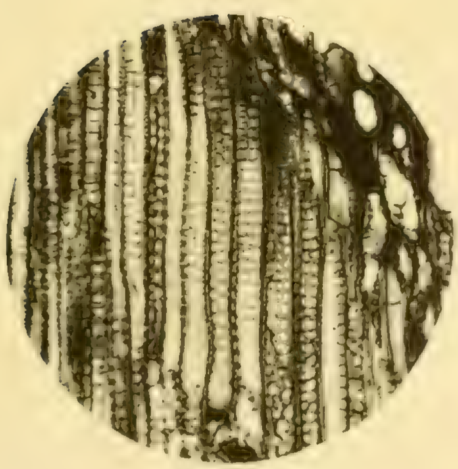
a

b

c



d



e

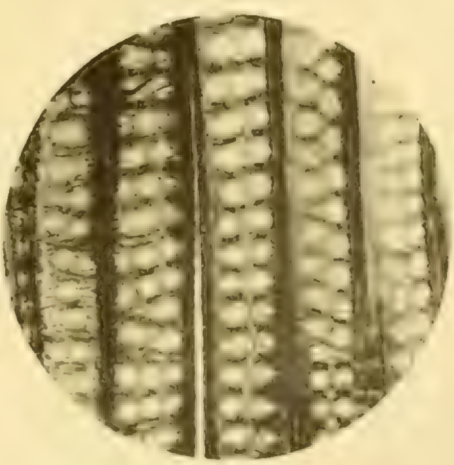


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PLATE 6.

Fig. *a*. Radial section of the same. × 100.

Fig. *b*. Another of the same. × 500.

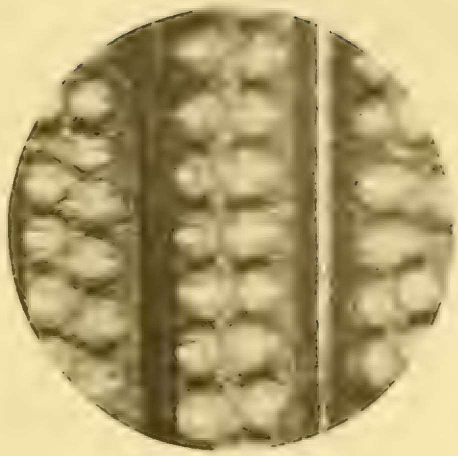
Fig. *c*. Radial section of the xylem of the cone of *Araucaria imbricata*, in proximity to the protoxylem. × 500.

Fig. *d*. Radial section of the wood of *Pinus strobus*. × 150.

Fig. *e*. Tangential section of the wood of the cone of *Araucaria Bidwillii*. × 500.

Fig. *f*. Tangential section of the wood of the vegetative stem of the same. × 300.

a



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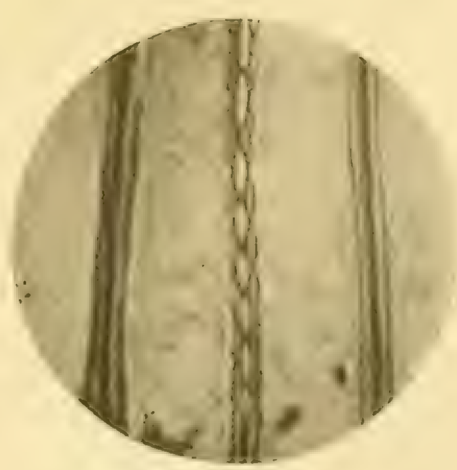

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CONTRIBUTIONS FROM THE PHANEROGAMIC LABORATORIES OF HARVARD UNIVERSITY. NO. 57.

THE HISTORY, COMPARATIVE ANATOMY AND EVOLUTION OF THE ARAUCARIOXYLON TYPE.

By Edward C. Jeffrey.

Part III.

The present article will be devoted to the consideration of resin canals in the wood of the Araucariineae, living and extinct. Some time ago the present writer in collaboration with Dr. Arthur Hollick [29] described the occurrence of resin canals as a result of injury in certain Araucarian woods from the Raritan Cretaceous of Kreischerville, Staten Island. Later a more complete study of this phenomenon was made, which included the consideration of more varied and abundant material from the Kreischerville deposits, as well as from the strata of similar age on the island of Martha's Vineyard and likewise from the much older Cretaceous Potomac deposits of Virginia. In this connection certain leafy twigs of Cretaceous Conifers were described in which the wood clearly showed the formation of resin canals as the result of injury. The twigs in question belonged to the well known Cretaceous genus Brachyphyllum, and for that reason the Araucarian type of wood, producing traumatic resin canals as a result of injury was named Brachyoxylon.[30] Still another type of Araucarian wood, forming traumatic resin canals was described by the present writer in 1907.[31] Here the canals were much more like the traumatic resin-canals of the Abietineae than has proved to be the case with any other araucarian woods, showing wound resin canals from the American

[29] Cretaceous Coniferous Remains from Kreischerville, Mem. N. Y. Botanic Garden, No. III.

[30] Jeffrey, E. C., Wound Reactions of Brachyphyllum, Ann. Botany, **20**, pp. 383–394, pls. 27, 28.

Hollick and Jeffrey, Cretaceous Coniferous Remains from Kreischerville, Mem. N. Y. Bot. Garden, No. III.

[31] Araucariopitys, a new genus of Araucarians, Bot. Gazette, **44**, pp. 435–444.

Cretaceous. Araucariopitys, not only shows resin canals closely resembling those of the Abietineae, but likewise has wood rays, with strongly pitted cells very similar to those found in Abietineous woods. Previously Seward [32] had described woods of similar organization from the Lias (Upper Jurassic) of Yorkshire in England, under the appellation *Araucarioxylon Lindleyi.* These woods, which, so far as it is possible to judge from Professor Seward's description, belong partly to the Brachyoxylon and partly to the Araucariopitys type, were characterized by very well marked Araucarian pitting of the tracheids, accompanied by quite typical traumatic ligneous resin canals. Contemporaneously with the present writer's article on Araucariopitys, Gothan published a description of the fossil woods of the Upper Jurassic of King Carl's Land.[33] These are characterized by the presence of Araucarian pitting of the tracheids of the wood, by strongly pitted rays, resembling those of Abietineous woods, by, for the most part, terminal wood parenchyma, and often by the presence of traumatic resin canals. These woods are in general considered by Gothan to be intermediate between the Araucariineae and Abietineae and to indicate a derivation of the latter tribe from Araucarian ancestry. The writer agrees with Professor Seward in considering that the woods in question are distinctly on the Araucarian side in affinities. He is further of the opinion, which is apparently not shared by Professor Seward, that the Araucariineae on the basis of structure of Mesozoic woods are derived from the Abietineae and not *vice versa,* as is the opinion of the majority of competent investigators at the present time. Recently an overwhelming amount of evidence has been brought to light which appears to strongly support the present writer's contentions. More recently Gothan has published an extensive memoir on the fossil woods of the island of Spitzbergen, in which he described a number of interesting woods from the upper Jurassic (or Lower Cretaceous!!) resembling strongly those of King Carl's Land and in some instances presenting a still more striking combination of Abietineous and Araucarian characters.[34] The author in this memoir restates and emphasizes his opinion that the Abietineae have been derived from Araucarian ancestors.

Figure *a,* Plate 7, shows the transverse section of a wounded speci-

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32 Cat. Mesozoic Plants. Brit. Museum, Jurassic Plants, **2**, pps. 56–59. London (1904).

33 Kung. Svensk. Vetenskap. Handlingar, **42**, No. 10. Berlin (1908).

34 Kung. Svensk. Vetenskap. Handlingar, **45**, No. 8. Uppsala u. Stockholm (1910)

men of Brachyoxylon. On the left appears the wound parenchyma, which universally adjoins a wound in woody tissues. To the right of this appear certain somewhat compressed cavities in the wood, surrounded by cells filled with dark contents. The cavities in question are the traumatic resin canals, which are a feature of the Brachyoxylon type. Figure *b*, Plate 7, from the same specimen shows part of a row of traumatic resin canals farther away from the same wound. It is an interesting fact in woods, in which resin canals are either almost obsolete or may be recalled only by experimental means, that they occur in nearly continuous tangential rows. This for example is the case with the resin canals in the woods of species of the genus Sequoia and with the woods of the genera Abies, Cedrus, Tsuga and Pseudolarix of the Abieteae, which are admitted by those who have recently devoted special attention to the comparative anatomy of the Coniferales, to be descended from ancestors which possessed resin-canals abundantly and normally in their woods. It follows that the presence of traumatic resin-canals in close tangential rows in the genus Brachyoxylon is *prima facie* evidence that this type has come from an ancestry which possessed normal ligneous resin canals. This consideration alone enormously complicates the task of those who endeavor to derive the Abietineae from Araucarian ancestors, for they have to explain the presence of resin-canals in an obsolete and vestigial condition in forms, which they claim to be the direct ancestors of the pine-like Abietineae, in which the resin canals are highly developed. It is scarcely necessary to point out that this is a palpable logical contradiction. It is unnecessary to figure the longitudinal view of the traumatic resin canals in the wood of the Brachyoxylon type of Araucarian woods, since this subject has already been sufficiently dealt with in the articles by the present author cited above. It is well however to point out at this stage that Brachyoxylon has the type of ray which is characteristic of living representatives of the Araucarian stock, namely one in which the cells are without pits and thin walled, except where they are laterally in contact with the tracheids of the wood. The pitting in this type is often strikingly Araucarian and at the same time in many instances the radial pores are widely spaced. In no case is there any indication of the presence of cellulose bars of Sanio, although many of the specimens, which have passed under my notice are in a remarkable condition of preservation, which has been a matter of comment on the part of all who have examined them.

Figure *c*, Plate 7, illustrates part of a transverse section of *Arau-*

cariopitys americana[35] showing a series of traumatic resin canals. The row nearer the center of the stem is larger and better developed, while that farther out is composed of very small canals separated by wider intervals. As has been pointed out in the article cited above, the traumatic resin canals in Araucariopitys are more nearly like those of the Abietineae, than is the case with canals of this type which have been described in any other American wood of Araucarian affinities. In Araucariopitys the rays too are distinctly of the Abietineous type, being composed of thick-walled cells, strongly pitted. The radial bordered pits of the tracheids however are often arranged in the compressed and sometimes in the alternating manner of the Araucarian conifers. Moreover there are no bars of Sanio present, although much of the material on which the genus Araucariopitys has been founded is in a perfect condition of preservation. Gothan, as has been indicated in the earlier paragraphs of this article, has described woods of a similar type from the Jurassic beds of King Carl's Land and Spitzbergen. These have traumatic resin-canals, thick-walled and strongly pitted ray cells. This author does not figure the presence of bars of Sanio in these woods of the arctic regions, so that it may be assumed that they are absent as in Araucariopitys, especially as woods of a similar horizon and identical features of organization, which I have examined, show no indication whatever of these peculiar structures, which constitute transverse bands between the radial pits of the tracheids, in all coniferous except Araucarian woods. Miss Gerry has investigated the distribution of bars of Sanio in the Coniferales in a comprehensive manner and found them to be absent in all Araucarian woods, living or extinct, which she examined.[36] Before discussing the conditions found in the Araucariopitys type, it will be well to consider briefly woods of a similar type from older geological horizons. It is pertinent before doing this, however, to point out that the Araucariopitys type, so far as our present knowledge goes, is rare in the later Mesozoic (*i. e.*, the Cretaceous).

Figure *d*, Plate **7**, shows the presence of two rows of resin canals of the traumatic type in an Araucarian wood from the Upper Jurassic. This wood and others of the same type will be described in detail on another occasion. At the present time only those features, which are of importance in the present connection, will be dealt with. The section from which Figure *d*, Plate **7**, was made, shows a distinct wound cap a few annual rings away from the pith. From this on either side

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35 Jeffrey, Bot. Gazette, **44**, pps. 435–444.

36 Distribution of Bars of Sanio in the Coniferales, Ann. Bot. **24**, pp. 119–124.

extend rows of wound resin canals. Farther out these disappear entirely as is wont to be the case in woods, giving rise to canals of this type, unless the injury is extremely severe. The canals are in some cases obviously very wide in the tangential plane. This feature indicates their lateral fusion with one another, a condition quite typically present in traumatic resin-canals. The canals are surrounded by cells filled with very dark contents, which sometimes makes its way into the lumen of the canal itself. The rays are likewise occupied by a dense dark hued substance. Figure *e*, Plate 7, shows the canals in longitudinal radial section. They are clearly not of equal caliber throughout as is usually the condition in Pinus, but are constricted at intervals even in the short portion shown in the figure. This condition is likewise one, which is characteristic of traumatic resin canals, although it is also more or less observable in the normal canals of the wood in the Abietineous genera, Picea, Pseudotsuga and Larix. To the right and left of the resin canal are medullary rays. The magnification is not sufficient to show their structure, which will be figured in detail subsequently. It is enough to state that the rays in this region of the wood are very strongly pitted, exactly as in Araucariopitys described above. The pitting is Araucarian of the Brachyoxylon type, that is the pits are not only flattened or alternating but also occur in the rounded and well spaced condition characteristic of the Conifers, other than the Araucarian tribe. Bars of Sanio cannot be made out in the wood under consideration or in any similar ones from the same deposits.

It is clear from the foregoing paragraph that there are woods in the Jurassic, which as regards their features of organization combine Abietineous and Araucarian characteristics. They have namely the strongly pitted rays and traumatic resin canals of the Abietineae, combined with the pitting of the tracheids and the absence of bars of Sanio which are recognized, by those who have made a special study of Araucarian woods, as diagnostic of the Araucarian tribe. There are obviously two possible interpretations of the combination of characters referred to above. It is agreed by competent investigators that the features under discussion clearly indicate a close degree of relationship in the Jurassic, between the Abietineous and Araucarian tribes. The disagreement, however, arises as to whether the Abietineae have come from the Araucariineae or the opposite mode of derivation is the correct one. Gothan in the two important memoirs on the Jurassic woods of King Carl's Land and Spitzbergen cited above, takes the view that the transition is from the Araucarian

Conifers to the Abietineae and bases his position on the structure of the rays, which he claims is the most trustworthy diagnostic character of coniferous woods. That is a view to which the present writer cannot subscribe, as from an extended comparative, anatomical, developmental, experimental, and paleobotanical acquaintance with coniferous woods he is in the position to state that wide variations of ray structure occur within all the coniferous tribes, and that as a consequence this feature of the organization of the wood cannot be successfully employed for the diagnosis of the woods of extinct conifers. It has been pointed out in the first article of this series, that even in so highly specialized an Araucarian species as *Agathis australis*, rays of the Abietineous type occur normally in the wood of the cone and may be readily produced traumatically in the wood of the root. A similar illustration may be cited in the case of Sequoia. As is well known the rays in the Sequoiineae are usually composed of thin walled cells without intercommunicating pits. In the cones of both the living species of Sequoia, particularly *S. gigantea*, the cells of the rays of the woody axis are very strongly pitted, especially towards the primary wood. Further in this genus strongly pitted rays appear as the result of injury. It appears quite clear from the conditions in the case of *Agathis australis* and *Sequoia gigantea*, that ray structure may vary greatly within the same genus and even the same species and consequently cannot be as an infallible diagnostic feature. The presence of rays of the Abietineous type consequently cannot be taken as satisfactory proof that the Jurassic woods under discussion are in reality Abietineous. This consideration likewise applies to the presence of traumatic resin canals because if these alone were a sufficient diagnostic character, we would be compelled to put the wounded wood and normal cone axes of *Sequoia gigantea* under the Abietineae, although the sum of characters of that species by common consent justify the placing of it in an entirely distinct tribe. Let us now turn to more constant characters than ray structure or traumatic resin-canals, namely the pitting of the tracheids which has been admitted by all experts, with the sole exception of Dr. Gothan, as an important Araucarian diagnostic feature. In the Jurassic woods under discussion the radial pitting of the tracheids is distinctly of the Araucarian type. Further we have recently had added to the list of utilizable diagnostic characters of Araucarian woods, the absence of the cellulose bars of Sanio, as worked out by Miss Gerry.[37] True

[37] *Op. cit.*

Gothan has stated that the absence of bars of Sanio is to be explained by the close approximation of the radial pits in the tracheids of the Araucarian tribe. It has been shown however that in the seedling, cone-axis and leaf trace of the living Araucarian conifers the pitting is not crowded as is the case in the mature secondary wood of the trunk and root. This is particularly true of the base of the seedling stem, where typical Araucarian pitting appears only after many years. In spite of the free spacing of the pits of the tracheids in the regions just described bars of Sanio are absent, *except in the part of the secondary wood of the cone axis, immediately adjoining the primary xylem,* as has been indicated in the second article of the present series. It follows apparently that Gothan's explanation of the absence of the bars of Sanio in Araucarian woods is not the valid one. On the criteria of the absence of bars of Sanio and presence of Araucarian radial pitting, the Jurassic woods under discussion are clearly of Araucarian affinities. Moreover if we admit for the sake of argument that the Jurassic woods in question are Abietineous, what is to become of the very numerous woods of the Cretaceous of the Brachyoxylon type, which have traumatic resin canals but have not normally at least the strongly pitted rays of the Abietineae? They can scarcely be included on the basis of Gothan's view with the Abietineae on account of their not possessing his sovereign diagnostic, Abietineous ray structure. Professor Seward has agreed that woods of this type are "undoubtedly Araucarian" and it may be assumed that such is the case until serious argument to the contrary can be adduced.[38]

Gothan in his articles cited above, has to assume that practically all the coniferous woods of the high arctics are Abietineous in their affinities, thus leaving no woody structures for the numerous Araucarian conifers, which are known to have flourished in that period. Moreover if we grant his identification of Jurassic woods with strongly pitted rays, traumatic resin canals, Araucarian radial pitting and non existent bars of Sanio, as of Abietineous affinities and indicating a recent derivation of the Abietineae from Araucarian ancestors, what shall we say of the characteristic Cretaceous woods of the Brachyoxylon type, which resemble these in every respect except in the absence of the Abietineous type of ray? If we derive the Araucarian conifers from the Abietineae no such difficulty arises, because we would expect on such an hypothesis, to find the Araucariineae progressively less like the Abietineous stock in later geological time. On the basis of

38 The Araucarieae, Recent and Extinct, Phil. Trans. Roy. Soc. London, Series B. **198**, p. 382.

ray structure alone then we find that the woods which as the result of the consideration of their most reliable diagnostic features are Araucarian, form a logical sequence on the hypothesis of derivation from the Abietineae, from the earlier to the later Mesozoic (Jurassic and Cretaceous). The opposite hypothesis, even taking into consideration the ray structure only, apparently involves us in hopeless confusion.

Having considered the known types of fossil Araucarian woods in regard to the feature of the presence or absence of resin canals, it is now desirable to inquire whether there is any evidence for the normal or traumatic occurrence of resin canals in the wood of existing representatives of the Araucariineae. In this connection the writer has had somewhat exceptional opportunities of securing material. Through the kindness of the late Director of the Botanic Gardens of the Dutch Government at Buitenzorg, Java, an abundant supply of the Malayan representatives of the genus Agathis were secured, including all vegetative parts of the plant, together with the very important cones. Later Dr. Maiden of the Botanic Gardens in Sydney, N. S. W. Australia, and Dr. Baker of the Technological Museum, Sydney, have forwarded abundant material of Australasian and exotic species of Agathis and Araucaria. The writer is likewise indebted to his students, Dr. A. J. Eames and Mr. E. W. Sinnott for collections made in Australia and New Zealand, secured in connection with their tenure of Sheldon Traveling Fellowships of Harvard University. The latest contribution to the writer's stores of valuable material was supplied through the kindness of the Director of the Royal Garden, Kew, England, and through the goodness of Mr. L. A. Boodle of the Jodrell Laboratory, Kew. It should be emphasized here that the abundant material, which has been secured through the kindness of many botanists, covers all the anatomically interesting parts of the two living genera of the Araucariineae and to a remarkable extent their whole geographical range. Not only has normal material been available but also that which has through injury or other causes undergone abnormal development.

It is the writer's purpose to give an account of the organization of the cone of the Araucarian conifers, in its systematic and anatomical aspects in an article distinct from the present series. Only features of special interest in the present connection will be considered here. As a preliminary to a description of these features, a general statement may be made in regard to features of organization of the ovulate strobilus, of importance in the case of this investigation. The writer has found, in those species of Araucaria, which have both the upper and lower systems of fibrovascular bundles present in the cone-scales of their ovulate strobilus, that the axial region of the cone does not show certain remarkable features found in the case of those species in which the upper system of cone-scale bundles has disappeared. In the latter condition there are apparent medullary resin-canals present in the pith of the cone axis. In the lower region of the cone these resin-canals are often surrounded with the tissues of the xylem, which constitute medullary strands joining with the wood of the cylinder of the axis, at the points where the supply of the cone-scales is given off. Without going into the matter here it may be stated that the medullary strands, containing resin canals in certain species of Araucaria and Agathis represent the vanished upper system of cone-scale bundles. The wooden envelope of the resin-canals disappears in the upper region of the cone and is best developed in the peduncular region. In Agathis, only the most primitive species have the medullary vascular strands. In the case of *Agathis Bidwillii*, resin-canals are found not only in the wood of the medullary system of bundles but they likewise not unfrequently make their appearance in the bundles of the lower cone-scale series, which are alone present in this genus. This feature is shown in Figure *f*, Plate **7**, which represents a cone-scale supply in the lower region of the cone, passing out through the wood of the axis. In the upper region of the scale supply and immersed in the elements of the primary xylem, is to be seen a dark mass which represents a resin canal filled with mucilaginous contents, a common accompaniment of the resinous secretion both in Agathis and Araucaria, as well as in the resin passages of extinct representatives of the Araucarian stock. Mucilage is particularly abundant with the resin in the canals of Agathis. The peculiar position of the resin canal in the primary wood, is to be compared with the conditions in living and extinct pines, where the first formed resin canals are often embedded in the elements of the primary wood. The present writer has described similar conditions in the case of the vestigial resin-canals of the cone-axis and cone-scales of the genus Sequoia.[39] Interesting in this connection are likewise the resin canals in the primary wood of the root in the two subtribes of the Abietineae, the Pineae and the Abieteae. The occurrence of resin canals in the outgoing vascular supply of the cone-scales on *Agathis Bidwillii* is an extremely incon-

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39 Jeffrey, Comparative Anatomy of the Coniferales. I. The genus Sequoia, Mem. Boston Soc. Nat. Hist. **6**.

stant feature and shows various stages of degeneracy the canal often for instance being largely or even wholly blocked with tyloses.

The occurrence of resin-canals in the scale bundles in a species of Agathis is a feature of considerable interest phylogenetically. The question at once arises whether it is to be regarded as an inchoative stage in the development of resin canals in the group or a vestigial one. Its place of origin appears to negative the former hypothesis. We may in fact compare the occurrence of vestigial resin-canals in the xylem strands of the peduncle of the cone in Agathis Bidwillii, with the development of vestigial centripetal wood in the peduncular region of the cone of certain Cycads, the very interesting and important discovery of Dr. Scott[40] or the existence of the same ancestral type of xylem development in the strobilar organs of Equisetum, long after it has disappeared in the vegetative axis of the ancient stock from which that genus has been derived.[41] The resin-canals in question are also to be regarded as ancestral on account of the wound reactions of Mesozoic Araucarian woods, which have been discussed above. These interesting vestigial resin-canals appear in the vascular supply of the lowermost abortive cone-scales, attached to the peduncle of the cone, and die out before the cone-scale supply leaves the wood of the peduncular axis. They have as yet been found only in *Agathis Bidwillii.* It does not appear at all likely that they will be discovered in other living species of the genus Agathis. It is probable on other grounds, that this species is the most primitive now in existence.

It naturally has occurred to the writer to investigate the wound reactions of the stem and roots of living species of Agathis and Araucaria. The results of extensive examinations of wounded material from the Australasian and East Indian regions have however been entirely negative. There is reason to suppose however from a series of investigations carried on with another purpose that traumatic reactions in the seedlings, particularly the seedlings of *Agathis Bidwillii,* may yield more favorable results, since it has been found in certain instances that seedlings respond much more readily to experiment than does the adult plant. It seems clear that so far as the mature individuals are concerned, however, that the living representatives of the Araucarian stock have entirely lost their capacity for producing reversionary wound resin canals, and in this respect as in other equally important normal features of structure, differ from a

[40] Scott, D. H., The Anatomical Characters of the Peduncles of Cycadaceae, Ann. Bot. II (1897).

[41] Eames, Centripetal Xylem in Equisetum, Ann. Bot. **23**, (1909).

large number of Araucarian forms, which apparently became extinct with the close of the Mesozoic.

As a consequence of the investigation of the normal and traumatic occurrence of resin canals in the wood of the Araucariineae, living and extinct, the conclusion seems clear, that this tribe of conifers once possessed ligneous resin canals as a normal feature and there is thus added one more argument for deriving them ancestrally from the Abietineae and not directly from the Cordaitales, as is commonly held. This view of the matter is strongly supported by the data described in the previous article, in connection with the pitting of the tracheids and the distribution of bars of Sanio. It is likewise confirmed by the evidence as to the ancestral character of the ray structure in the Araucarian tribe, which strongly resembled that found in the Abietineae, past and present. The ancestral occurrence of wood parenchyma in the Araucarian tribe is likewise a strong argument against their immediate connection with the Cordaitean forms and indicates that they in common with the conifers in general, with diffuse wood parenchyma are of relatively recent origin compared with the Abietineae, which in so many ways show themselves to be a very ancient group.

Summary.

1. Certain Mesozoic woods from the Jurassic and Cretaceous, showing traumatic resin canals are of Araucarian affinities.

2. This is shown to be the case by the structure of their tracheids, both as regards pitting and the absence of cellulose bars of Sanio.

3. Abietineous pitting in the rays of extinct conifers is not in itself a character of sufficient constancy to serve as a reliable diagnostic feature, since pitting of this type can readily be produced as the result of injury and moreover is often normal in the more conservative parts of living representatives of the Araucariineae.

4. Normal resin canals occur embedded in the primary xylem of the traces leading to the abortive cone-scales attached to the peduncular region of the cone of *Agathis Bidwillii*.

5. This fact taken together with the traumatic phenomena presented by certain Mesozoic Araucarian woods, supplies an additional argument for the derivation of the Araucariineae from an Abietineous ancestry.

PLATE 7.

Fig. *a*. Transverse section of the wood of *Brachyoxylon notabile,* showing wound resin canals. × 40.

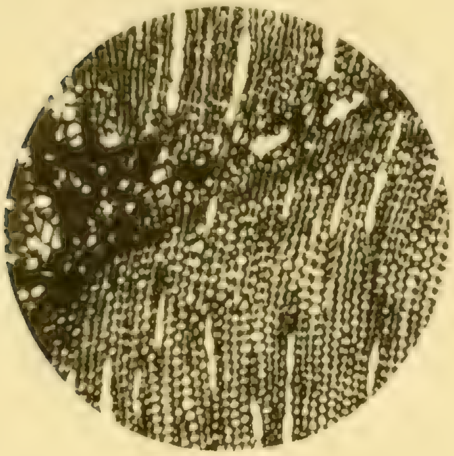
Fig. *b*. Another section of the same showing traumatic resin canals more remote from the wound. × 40.

Fig. *c*. Transverse section of the wounded stem of *Araucariopitys americana,* showing wound resin canals. × 100.

Fig. *d*. Transverse section of a wood from the English Jurassic, showing traumatic resin canals. × 40.

Fig. *e*. Longitudinal section of the same. × 40.

Fig. *f*. Tangential section through the wood of the peduncle of the cone of *Agathis Bidwillii,* showing the presence of a normal resin canal in the vascular supply of the cone-scale. × 100.



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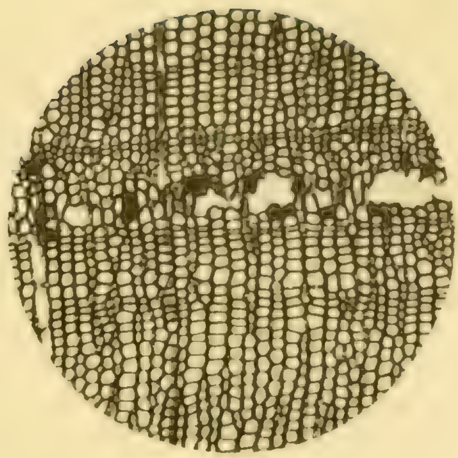

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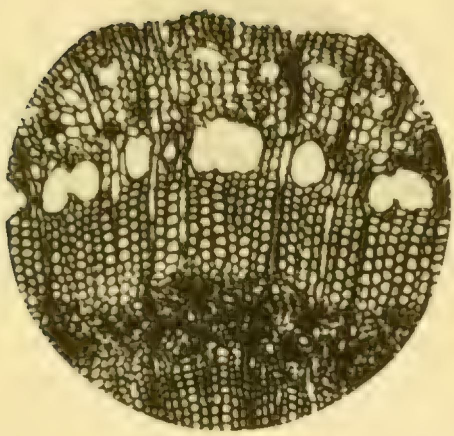

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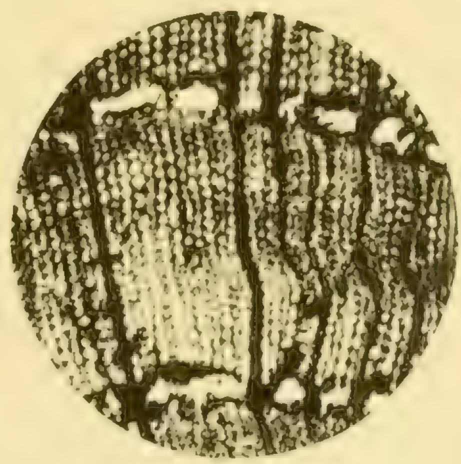

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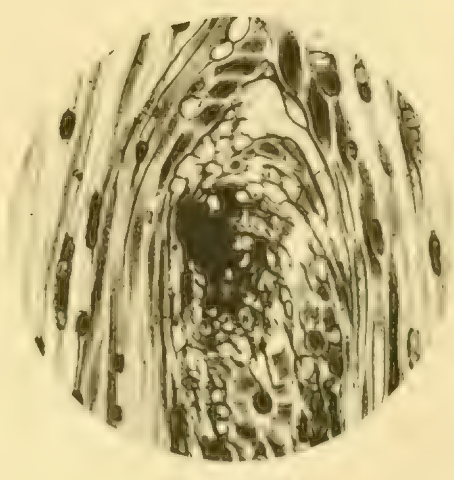

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CONTRIBUTIONS FROM THE PHANEROGAMIC LABORATORIES OF HARVARD UNIVERSITY. NO. 58.

THE HISTORY, COMPARATIVE ANATOMY AND EVOLUTION OF THE ARAUCARIOXYLON TYPE.

By Edward C. Jeffrey.

Part IV.

The present article will be devoted to the consideration of the structure of the pith and the relations of the foliar trace in woods of the Araucarioxylon type and nearly allied Araucarian lignites. Gothan in his second memoir on arctic woods, which deals with the fossil ligneous remains of the island of Spitzbergen, makes the statement that the Cretaceous fossil, which the present writer has described under the name Araucariopitys [42] and considered on the basis of its general organization to belong to the Araucarian alliance, cannot be so referred on account of the abietineous character of its rays and on account of the sclerotic septa in the pith, a character in his opinion likewise exclusively Abietineous.[43] The writer has shown in the first article of the present series, that the presence of Abietineous rays is by no means necessarily an indication of Abietineous affinity, especially in the case of Mesozoic woods. It will be his aim to demonstrate in the present communication that sclerotic diaphragms in the medullary region are equally fallacious criteria of Abietineous affinities. Professor Seward has adduced the persistent leaf trace of the living genera Agathis and Araucaria and of true Araucarioxyla from the Mesozoic deposits, as an argument for the antiquity of the Araucariineae and for their relationship with the extinct Lycopodiales, which he considers likewise as characterized by leaf-traces persistent for a long time in the secondary wood. The writer will attempt to show in the present article that on generally accepted biological principles the leaf trace was not

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42 Jeffrey, Araucariopitys, a new genus of Araucarians, Bot. Gazette **44**, pp. 435–444.

43 Gothan, Die Fossilen Holzreste von Spitzbergen, Kung. Svensk. Vetenskap. Handlingar, Bd. 45, No. 8, Uppsala u. Stockholm (1910).

primitively persistent in the Araucarian stock and consequently cannot be used as an argument for their antiquity or their affinity with any other group in which the persistence of the foliar strands is a feature of structure in the wood.

Figure *a*, Plate 8, illustrates a transverse section of a well preserved Araucarian trunk from the Raritan Cretaceous of Kreischerville, Staten Island, N. Y., which has been described by the author under the name *Araucarioxylon noveboracense*.[44] Through the center of the figure vertically passes the leaf trace. The annual rings are scarcely curved at all, showing that the stem was one of considerable thickness, its age in fact being somewhat over fifty years. This persistence of the leaf trace seems to be a characteristic of all woods of the true Araucarioxylon type and as has been particularly indicated by Dyer [45] and Seward,[46] is likewise a feature of the trunks of the living genera Agathis and Araucaria. Throughout the wood of the figure may be seen numerous dark dots, indicating the position of the true resiniferous parenchyma, which as has been pointed out in the first article of this series, seems to have been a constant feature of structure in the true Araucarioxyla of the Mesozoic, and which interestingly enough persists as a vestige in the wood of the cone, first annual ring of vigorous branches and the root of the living genera Agathis and Araucaria. Woods of the Araucarioxylon type in the stricter sense, have been described recently by Lignier from the Middle and Upper Jurassic of France.[47] They are exceedingly common in the Cretaceous both of Europe and America.

In addition to the true Araucarioxylon type, there exist, particularly in the Cretaceous, woods in which the pitting and general structure of the tracheids, although unmistakably Araucarian, present certain features of divergence from those properly included under the generic appellation Araucarioxylon. These are pitting not invariably flattened or alternating and the presence of wound resin canals in connection with injuries. These woods are further characterized by rays which are frequently of the Abietineous type after injury. Another interesting feature of these woods is the fact that the leaf-traces, instead of being persistent as is the case with the living genera Agathis and Araucaria, endure only for a very short

44 Hollick and Jeffrey, Cretaceous Coniferous Remains from Kreischerville, Mem. N. Y. Bot. Garden, 3.

45 Persistence of Leaf traces in Araucarieae, Ann. Bot. 15, pp. 547 (1901).

46 *Op. cit.*

47 Lignier, Végétaux Fossiles de Normandie, IV. Bois Divers, Ire. Série, Caen (1907).

time, a few years at most. The writer has discussed and figured woods of this type in a memoir on the Coniferous remains found at Kreischerville, Staten Island.[48] It is accordingly unnecessary to do more than call attention to their characteristic features here. It has been pointed out in the second article of this series that peculiarities of pitting and other features, found in Mesozoic woods of this type, to which the writer has given the generic name Brachyoxylon, are likewise found in the seedling axis and the cone axis of the living genera, Agathis and Araucaria. In Agathis the Brachyoxylon type of pitting persists for very many years in the basal region of the seedling. It has occurred to the writer that since the older less typically Araucarian mode of pitting persists in the seedlings and cones of the living genera of the Araucarian conifers, that the evanescent leaf traces, which are likewise a feature of the Brachyoxylon type of wood as contrasted with the persistent ones of the Araucarioxylon type, might be found in the seedling axis of Agathis and Araucaria, in accordance with the general biological law of recapitulation. An examination of the facts resulted in a very gratifying realization of this theoretical expectation. Figure *b*, Plate 8, shows a tangential section through the epicotyledonary region of the seedling stem of *Agathis australis*, material of which was obtained by Messrs. Eames and Sinnott, Sheldon Traveling Fellows of Harvard University, on a journey to the Australasian region. In the center of the figure are seen two leaf traces in transverse section. Of these one is better developed than the other. The smaller one is about to disappear. Figure *c*, Plate 8, shows a section of the same stem a little farther out. The trace which shows smaller in Figure *b*, has now completely disappeared and the persistent one has become much reduced in size. A serial examination of sections showed that the leaf trace came off as a single strand from the region of the protoxylem of the stem and after passing out a very short distance divided into two. The double strand thus produced is of very short duration and finally disappears in both its divisions in the third or fourth annual ring. As one passes up the seedling stem, the leaf traces are seen to become more and more persistent until they reach a condition of permanency like that characteristic of the older stem. It is clear from the facts described that the leaf trace of Agathis, so far as *A. australis* is concerned at any rate, in the seedling is an evanescent structure and only becomes permanent later in life. The conditions are comparable in fact *mutatis mutandis*, with the conditions found

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48 Hollick and Jeffrey, Coniferous Remains of Kreischerville, Mem. N. Y. Bot. Garden, **3**.

in the seedling of our only deciduous occidental conifer the larch, for here in the seedling the leaves are persistent for two or three years and only gradually become annually deciduous. There can be no doubt that in the case of the larch we have to do with a tree, which originally had evergreen leaves, as is the case with the other conifers, and that its seedling perpetuates that condition. *Vice versa* in Agathis we have to do with a coniferous genus, which originally had its leaves moderately persistent as in conifers in general and that only later did the extreme condition of persistence of the leaf trace found only among living conifers in the mature stems and lateral branches of the genera Agathis and Araucaria, become established. It is to be emphasized then as a result of the examination of the seedling anatomy of Agathis that not only the pitting of the older Mesozoic type Brachyoxylon persists in the seedlings of the living genera but also the evanescent character of the foliar trace. Seedlings of *Araucaria Bidwillii* were examined with similar results, the only difference being that the leaf traces here are somewhat more persistent in the lower region of the cotyledonary stem than they are in Agathis. It appears unnecessary to furnish further illustrations, since the facts seem to be so conclusive and so much in accord with the natural theoretical expectation.

Having made it clear that both the anatomical conditions found in the older Mesozoic woods of Araucarian affinities (Brachyoxyla) and the developmental data supplied by the seedlings of the modern forms, vouch for the fact that the persistent leaf trace characteristic of the woody cylinder of the living genera of the Araucariineae and of woods of a similar type from the Mesozoic, (true Araucarioxyla) is not an ancestral feature of the stock, and consequently not phylogenetically important, we may appropriately pass on to the consideration of the pith in the primitive Araucarian type, in connection with the affinities of the Araucarian stock.

Figure *d*, Plate 8, illustrates the structure of the pith in the stems, the wood of which has been described by the author under the appellation *Araucarioxylon noveboracense*.[49] At quite regular intervals the pith is characterized by the presence of lighter bands, which represent regularly recurring transverse diaphragms of sclerotic tissue. Figure *e*, Plate 8, illustrates the same feature in the pith of an undescribed and different species of Araucarioxylon from the Raritan Cretaceous of Cliffwood, New Jersey. Sclerotic diaphragms appear at intervals

49 *Op. cit.*

in the pith and often occupy a somewhat oblique position. Figure *f*, Plate 8, shows a portion of one of these somewhat more highly magnified. The contrast between its organization and that of the ordinary tissues of the pith can clearly be made out. In the memoir on the Conifers of Kreischerville, the writer has called attention to the very frequent occurrence of medullary septa of a sclerotic nature in the pith of branches not only of the Brachyoxylon type, but also of the probably still older type, to which the name Araucariopitys has been applied. It is interesting to consider the organization of the pith in the two Cretaceous Araucarioxyla described above. They have the same tendency to form sclerotic diaphragms. Gothan in a recent memoir on the fossil woods of Spitzbergen [50] has questioned the accuracy of the writer's reference of the genus Araucariopitys to Araucarian rather than to Abietineous affinities, because he thinks it impossible that an Araucarian conifer should have the pith structure and ray structure, which so far as living representatives of the Coniferales are concerned is more characteristic of the Abietineae than of the Araucariineae. It is clear that conclusions as to affinities can only be safely drawn after a full and accurate comparison of Mesozoic and living forms. Most of the results of structural paleobotany, in the case of Mesozoic conifers at any rate, are vitiated by a neglect of this absolutely necessary precaution.

The writer has not observed the presence of true sclerotic diaphragms in either the seedling stem or the cone-axes of any living Araucarian species. Isolated stone cells are typical of Araucaria, and sclerotic nests which never become so extensive as to constitute true diaphragms are found in Agathis.

It appears to be definitely established from the data supplied in the present article that persistent leaf traces cannot in the future be regarded as an infallible diagnostic of Araucarian woods. It seems further clear that foliar traces of this type are not a primitive feature of Araucarian woods, since they are not characteristic of the seedling structure of living representatives of the Araucariineae, and are not found, in what we must regard on the basis of a great many concurrent lines of evidence, as the older Araucarian types, namely Brachyoxyla and woods of the organization of Araucariopitys. It is moreover obvious that medullary diaphragms are equally characteristic of both the older Araucariineae and of the Abietineae living and fossil. Their presence in older Araucarian types, is consequently one more piece

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[50] *Op. cit.*

of evidence in favor of the derivation of the Araucarian tribe from Abietineous ancestors.

Having in the present article and those which have preceded it, considered a number of anatomical features presented by the Araucarioxylon type and by the living genera with the same type of wood, we are in a position to discuss its affinities and evolutionary development. It has been pointed out in the first article that there is the best of evidence, derived from both fossil and living forms, that woods of the Araucarioxylon type were originally characterized by the possession of strongly pitted rays and abundant wood parenchyma. These features are quite inconsistent with a direct connection of this type with the Cordaitean plexus of gymnosperms, since here, we know, that the wood was entirely without wood parenchyma and the rays were composed of cells with unpitted walls. Passing to the next important item of wood structure, we find that there is every reason to believe that the older Araucarian conifers were not characterized by alternating or compressed pitting. On the contrary the radial pits were often opposite and moreover were separated from one another, particularly towards the ends of the tracheids, by cellulose bars imbedded transversely in the lignified wall of the tracheids. Bars of this type do not occur in any Cordaitean woods but are found in the mature wood of all existing Conifers, *except the living Araucariineae.* It follows that on the basis of pitting and the cellulose bars of Sanio, the Araucarian conifers were derived from the same ancestors as the remaining coniferous tribes. It is further clear both from a consideration of comparative anatomy and from the organization of the older woods belonging to the Araucariineae, that the absence of resin canals is not a primitive feature of Araucarian woods, since the progenitors of the stock clearly possessed them. The present article appears moreover to make it clear that persistent leaf traces are not an ancestral feature of organization of the Araucarian stock, both the anatomical conditions found in the older forms and in the seedlings of the living genera, showing beyond any reasonable doubt, on generally accepted biological principles, that the leaf strand in the ancestors of the Araucarian stock, persisted only for a few years, as is characteristically the case in all other living conifers.

It is apposite to consider if other facts justify the conclusion reached in connection with the present investigation, namely that the Araucarian stock is distinctly coniferous and is neither the most ancient tribe of the Coniferales, nor connects them with those ancient Gymnosperms, which the majority of competent morphologists regard as the ancestors of the coniferous stock, namely the Cordaitales. Taking first the very important criterion from the standpoint of the systematic arrangement of the Coniferales, the organization of the female cone, we find little to justify the recent contention of Professor Seward and his students and of Mr. Thomson, that the ovulate cone of the Araucarian conifers is of a different morphological order from that characteristic of the remaining coniferous tribes. It is perfectly clear that not only in the more primitive species of the living genus Araucaria but also in the cones of the Mesozoic representatives of the Araucariineae, described by the present writer either independently or in collaboration with Dr. Arthur Hollick, that the Araucarian female cone, like that of the other tribes of conifers was originally composed of cone-scales with a double system of bundles, independently emanating from the cone axis and of inverse orientation. Consequently whatever explanation is adopted for the double system of bundles in one case must be adopted in all. Attempts to read the Araucariineae out of the conifers must continue so long as the view is adhered to that they represented the primitive elaboration of the coniferous stock. It is a noteworthy fact that Professors Penhallow and Seward as well as Mr. Thomson, who much as they disagree in other matters are in harmony in regarding the Araucariineae as distinct from other coniferous tribes and at the same time as the primitive representatives of the stock. The recent investgations of Mr. A. J. Eames [51] appear to make it perfectly clear that whatever explanation is adopted of the organization of the female strobilus in the Araucariineae, must hold likewise for all the remaining tribes of Conifers.

If we turn our attention now to the gametophytes, we arrive at similar conclusions, if our logical processes are based on the established principles of biological science. Taking first the male gametophyte, we find a method of germination of the microspore unlike that found in any other gymnospermous group, which has been inaptly denominated by Mr. Thomson as 'protosiphonogamic.' Certainly we would not expect to find the primitive type of pollen tube formation in a group in which the pollen no longer reaches the apex of the ovule, as it characteristically does in all other known groups of Gymnosperms, living and extinct. The peculiar germination of the pollen of Agathis and Araucaria, on the cone scale and not on the apex of the young seed is an unmistakable stigma of aberration. The contents of the pollen tube likewise vouch for the highly specialized con-

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51 Ann. Bot. *Ined.*

dition of the Araucariineae. Here the two prothallial cells common to the Abietineae and the equally ancient Ginkgoales become proliferated into a large number, doubtless in correlation with the extreme length and meandering course of the fertilizing tube. Moreover the absence of a stalk cell in connection with the setting off of the body cell, which gives rise to the two sperm cells, is a clear and outstanding feature of aberrancy. Mr. Eames in the memoir, already cited, has shown moreover, that in the organization of the female gametophyte, the structure of the archegonium, the nature and functions of the archegonium neck, as well as in the method of penetration of the pollen tube and the development of the embryo, the Araucarian conifers manifest not a primitive but an extremely aberrant condition. They are in fact comparable to a large degree in their systematic position with the edentate fauna, likewise characteristic of the antarctic region, Developmental investigations on the zoological side have recently shown that the edentulous features which have been until the present time regarded as a primitive feature of this group are in reality marks of aberrancy, since a more abundant dentition, at first makes its appearance in the embryo.

Reviewing all the evidence in the light of many recent investigations both in general morphology and in the morphology of the conifers in particular, it is clear that it is the anatomical features of the reproductive and vegetative organs, which give us the most reliable criteria as to the evolution of the coniferous stock and above all in the present connection, as to the evolution of the Araucarian tribe. The anatomical conditions in the living forms cannot be understood without careful comparison with the organization of those which are now extinct. Basing our conclusions on these criteria, the result is reached that the Araucarioxylon type has been derived from the Pityoxylon type and as a consequence formerly possessed the opposite pitting, the bars of Sanio, the strongly pitted rays and the resin canals of the ancient Abietineous woods. Some of these characters are still to be observed in primitive regions of the existing Araucariineae, while others are to be inferred from a consideration of the organization of Araucarian forms now extinct. It is further clear that the external form of the reproductive structures and the organization of the gametophytes supplies as little light, regarded independently from the anatomical organization of the reproductive and vegetative parts, for the interpretation of the true course of evolution and affinities of the ancient but highly aberrant coniferous tribe, the Araucariineae as is the case with the corresponding structures in the Bennettitean tribe among the Cycadophyta. It is finally clear that morphologists will find it necessary in the future more and more to adopt certain general working principles, as in the case for example in the sister sciences of chemistry and physics. If there prove on trial to be no generally applicable fundamental principles in morphology, that branch of biological science cannot be too soon cast into the outer darkness, which prevails outside the scientific view of nature.

General Conclusions.

1. The Araucariineae cannot have been derived from the Cordaitales since they possessed primitively a number of features which so far as our knowledge goes, never existed in the Cordaitean stock.
2. The Araucarioxylon type is derived from ancestral forms, which possessed opposite pitting, bars of Sanio, strongly pitted rays and horizontal and vertical resin canals.
3. The primitive existence of these features in the ancestral type from which Araucarioxylon has been derived, show clearly that it has taken its origin from the Abietineous Pityoxylon type.
4. This conclusion is entirely confirmed by a consideration of the reproductive structures both sporophytic and gametophytic.
5. Any hypothesis as to the origin of the Coniferales in general must start with the Abietineae as the most primitive tribe.
6. It is absolutely essential to the progress of plant morphology, that investigation be carried on in connection with the elucidation of the general working principles of the biological sciences.
7. The comparative, developmental, paleobotanical and experimental investigation of the Coniferales is likely to throw more light on the stable and sound general principles of biology, than that of any other large group of animals or plants, on account of their great geological age and remarkably continuous and complete display, both as regards external form and internal structure in the strata of the earth.

Botanical Laboratories of Harvard University,
17th, June, 1912.

PLATE 8.

Fig. *a*. Transverse section of the wood of *Araucarioylon noveboracense*, showing the persistent leaf trace. × 15.

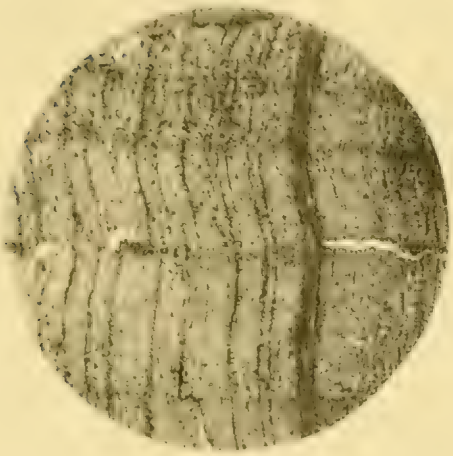
Fig. *b*. Tangential section of the seedling stem of *Agathis australis*. × 60.

Fig. *c*. Another of the same further out in the wood. × 60.

Fig. *d*. Longitudinal section through the pith of a trunk of *Araucarioxylon noveboracense*. × 8.

Fig. *e*. Longitudinal section of an undescribed species of Araucarioxylon from New Jersey, showing the region of the pith. × 15.

Fig. *f*. Part of the same more highly magnified. × 40.



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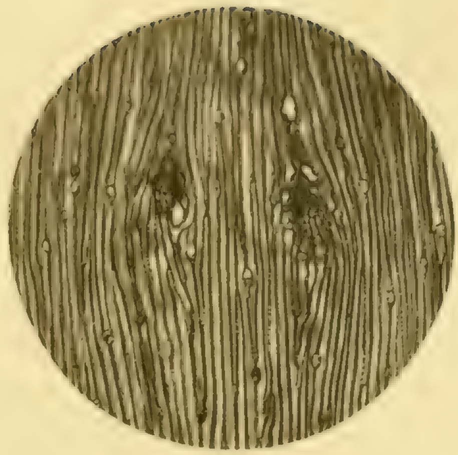

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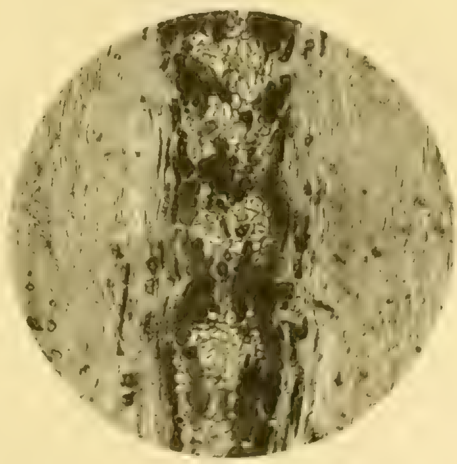

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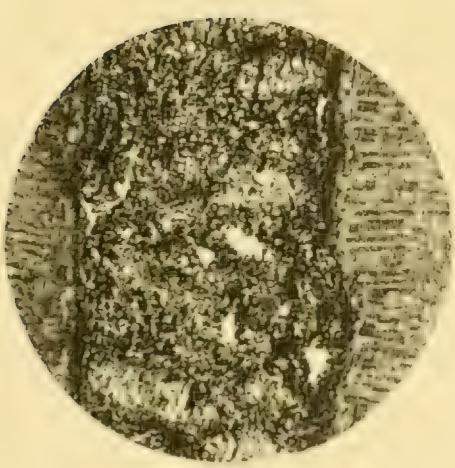

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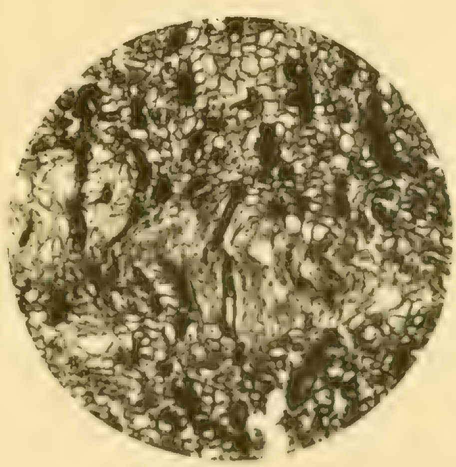

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THE ACTION OF SULPHUR TRIOXIDE ON SILICON TETRACHLORIDE.

By Charles Robert Sanger and Emile Raymond Riegel.

THE ACTION OF SULPHUR TRIOXIDE ON SILICON TETRACHLORIDE.[1]

By Charles Robert Sanger and Emile Raymond Riegel.[1]

Presented by C. L. Jackson, November 13, 1912. Received, October 24, 1912.

The reaction between sulphur trioxide and carbon tetrachloride yields phosgene and pyrosulphuryl chloride.[2]

$$2SO_3 + CCl_4 = COCl_2 + S_2O_5Cl_2$$

Since the two elements carbon and silicon resemble each other so closely, it was reasonable to suppose that a similar reaction might take place if silicon tetrachloride were substituted for the carbon tetrachloride. In order to test this, or to find out what reaction, if any, took place, this research was undertaken.

The only reference to the subject we can find in the literature is the following note of Gustavson, quoted in extenso:

"Silicon tetrachloride gives with sulphur trioxide pyrosulphuryl chloride." [3]

This information Dammer [4] enlarges into the following reaction:

$$4SO_3 + SiCl_4 = 2S_2O_5Cl_2 + SiO_2$$

Our investigation showed that on mixing pure melted sulphur trioxide and silicon tetrachloride, there is at first mere solution, but on standing, a reaction takes place, exceedingly slowly in the cold, but more rapidly at about 50°, resulting in the formation of a liquid which when freed by distillation from the unchanged materials boils between 135° and 150° at atmospheric pressure, whereas sulphur trioxide and silicon tetrachloride boil below 60°. This distillate fumes weakly in

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[1] This research was suggested by the late Professor C. R. Sanger and most of the work was done under his direction, until he was prevented by illness from continuing its supervision, when Professor T. W. Richards took charge of it. The material was prepared for publication with the aid of Professor C. L. Jackson after the lamented death of Professor Sanger, who is therefore in no way responsible for its arrangement or presentation. I am very grateful to Professors Richards and Jackson for their respective aid. E. R. R.

The work described in this paper formed part of a thesis presented to the Faculty of Arts and Sciences of Harvard University for the Degree of Doctor of Philosophy by Emile Raymond Riegel.

[2] If water is present, a certain amount of chlosulphonic acid is formed, very nearly proportional to the quantity of water. See Sanger and Riegel, These Proceedings, 47, 673 (1912); Zeit. anorg. Chem., p. 79 (1912).

[3] Ber., 1872, 5, 332.

[4] Dammer, Inorg. Chem. 1, 667.

moist air and reacts violently with water; it contains sulphur, chlorine, silicon, and oxygen, and the indefiniteness of its boiling point indicates that it is a mixture. We have not succeeded in isolating the substances of which it is composed, although we tried every way we could devise. Nevertheless we are convinced that it is made up of pyrosulphuryl chloride $S_2O_5Cl_2$ and silicon oxychloride Si_2OCl_6, because all our complete analyses of well-established specimens give percentages corresponding to such a mixture; that is, if the amount of sulphur found is assumed to be present as pyrosulphuryl chloride, and the amount of chlorine corresponding to the sulphur in that compound is calculated, the difference between the total chlorine and this amount agrees well with the chlorine necessary to form silicon oxychloride with the silicon found; thus out of 18 analyses, it agrees in 8 cases within 1 percent, in 5 other cases within 2 percent, in one more case within 2.5 percent. The four remaining analyses gave results which did not agree, but this is satisfactorily explained, for these are analyses of fractions boiling at higher temperatures than the usual one, namely above 150°, indicating the presence of other substances. These analytical results are confirmed by other observations. A specimen distilled from a heavy gelatinous residue of silicic acid gave results on analysis differing by only 1.8 percent on the sulphur, and 4 percent on the chlorine from pyrosulphuryl chloride, showing that that substance had been formed; another distilled from a large excess of phosphorus pentoxide, melted at $-40°$ to $-50°$ and crystallized in radiating crystals like pyrosulphuryl chloride which melts at $-37°$, showing its presence again; a third, distilled from an excess of sodium chloride gave analytical results indicating a more impure pyrosulphuryl chloride. It seems therefore that heating the liquid with a large excess of any solid disposes of most of the silicon oxychloride, but reveals the presence of the pyrosulphuryl chloride.

The boiling point of the distillate, 135–150°, is what would be expected of a mixture of pyrosulphuryl chloride and silicon oxychloride, for the former boils at 152.5°–153°, and its boiling-point may be lowered 5–10° by a minute amount of water, and the latter boils at 136–139°. The boiling-point of the mixture is not changed by the addition of one fifth of its weight of pyrosulphuryl chloride. If a great deal of water is added to the original distillate a violent reaction attended with the formation of silicic acid takes place such as would be expected from silicon oxychloride and a heavy liquid separates at the bottom of the vessel, dissolving but slowly. This is the behavior of pyrosulphuryl chloride or of sulphuryl chloride, but the latter is excluded by the boiling point of the mixture, hence it must be the former. Chlor-sulphonic acid, the only other compound of sulphur which might be expected, and then only in those preparations in which hydrous sulphur trioxide was used, was found to be absent by a distillation with salt, when only a very small amount of hydrochloric acid was given off.

It appears from these observations that silicon tetrachloride does not behave like carbon tetrachloride with sulphur trioxide, the principal reaction being represented as follows:

$$2SO_3 + 2SiCl_4 = S_2O_5Cl_2 + Si_2OCl_6 \quad (1)$$

If it did behave like carbon tetrachloride, the reaction would be:

$$2SO_3 + SiCl_4 = S_2O_5Cl_2 + SiOCl_2.$$

We base our contention that $SiOCl_2$, the unknown oxychloride of silicon which would be analogous to phosgene, and might therefore be called silico-phosgene, is not formed on the fact that all the properties of the silicon compound resulting from the reactions point to the oxychloride, Si_2OCl_6; furthermore, on the analyses, and on our failure to find the silico-phosgene in the lower boiling fraction, where it would be expected. The liquid distilled from the reaction mixture below 130° gave on distillation sulphur trioxide vapors, a mixture of sulphur trioxide and silicon-tetrachloride, an almost pure silicon tetrachloride which was on several occasions re-distilled and found to boil at 56-58°, its true boiling point being 57°, and a little of the higher boiling fraction. In the distillation in vacuo, a vessel cooled with liquid air was added to condense any silico-phosgene, but nothing was found there beyond silicon tetrachloride and sulphur trioxide. The weights of the various condensations and residues were always noted; their nature being established as either unchanged substances, or as the mixture of oxychloride and pyrosulphuryl chloride, nearly all the material was accounted for (thus in one case 94%), so that no considerable amount was left which might have formed the silico-phosgene. In nearly everyone of the twenty-six preparations made the proportions taken were those of two molecules of sulphur trioxide to one of silicon tetrachloride, favoring the reaction $2SO_3 + SiCl_4 = SiOCl_2 + S_2O_5Cl_2$; nevertheless every fact points to the formation of silicon oxychloride, and none to that of silico-phosgene. That silico-phosgene is unknown also speaks against the likelihood of its formation in this reaction.

In three cases distillates were obtained which were not very far removed from a mixture of the two products in molecular proportions, as required by reaction (1), but in most cases there was a decided excess of the pyrosulphuryl chloride, especially if the preparation had been allowed to stand some time, in one case three summer months. This excess may be formed by the following reaction:

$$Si_2OCl_6 + 6SO_3 = 2SiO_2 + 3S_2O_5Cl_2 \quad (2)$$

as a considerable amount of silica is deposited during the standing.

It follows from our work that the reaction constructed by Dammer on Gustavson's meagre statement

$$4SO_3 + SiCl_4 = 2S_2O_5Cl_2 + SiO_2$$

is incorrect, inasmuch as there is formed at first the oxychloride:

$$2SO_3 + 2SiCl_4 = S_2O_5Cl_2 + Si_2OCl_6 \quad (1)$$

and only by a secondary reaction, silica,

$$Si_2OCl_6 + 6SO_3 = 2SiO_2 + 3S_2O_5\,Cl_2$$

By combining the two reactions that given by Dammer is indeed obtained, but (2) takes place only to a limited extent and always follows (1). To give Dammer's equation alone would be misleading; the two separate equations (1) and (2) must be given and explained.

There is some evidence that the distillate contains a loose compound of pyrosulphuryl chloride, $S_2O_5Cl_2$, and silicon oxychloride, Si_2OCl_6, formed under the influence of heat. The distillate did not freeze unaided above $-78°$, except in a single case, while a mixture of equal parts of the two substances crystallized easily on cooling and melted at $-40°$ to $-38°$. It is astonishing that this mixture melted at about the same temperature as its constituents, $S_2O_5Cl_2$, melting at $-37°$, Si_2OCl_6 at $-40°$. A mixture of 15.6 grams of $S_2O_5Cl_2$ with 5.2 grams Si_2OCl_6, which therefore had about the same composition as one of our distillates crystallizing at $-78°$, was divided into two parts, one of which was heated for 5 minutes in the Bunsen flame; on cooling the two in the carbon dioxide-alcohol mixture, the portion which had been heated took 20 times as long to begin to solidify as the unheated one. This could hardly be accounted for unless the supposition was made that the two substances had combined under the influence of heat. The assumption of such a compound does not interfere with the other observations made, thus the boiling point might remain that of mixed silicon oxychloride and pyrosulphuryl chloride, because the compound between the two is too weak to exist in the state of vapor, a recombination, however, taking place as they return to the liquid phase; the formation by distillation of nearly pure pyrosulphuryl chloride took place only when the temperature was raised much higher than usual, so that the compound would be decomposed, the silicon oxychloride disappearing completely as such, leaving the pyrosulphuryl chloride; again the reaction of water on such a compound might well be a decomposition into the components accompanied by the destruction of the silicon oxychloride, leaving a part at least of the pyrosulphuryl chloride to react more slowly, for it is less sensitive to water than the oxychloride, as shown quantitatively further on. It must be added, however, that all the distillate which gave distinctly the oily deposit with water were low in silicon, and contained an excess of pyrosulphuryl chloride, so that all of the combined oxychloride and pyrosulphuryl chloride might have been destroyed, leaving only the free pyrosulphuryl chloride to become visible. The assumption of this compound explains to perfection why the distillates nearing in percentage composition an equi-molecular mixture of the two substances did not crystallize even after seeding with pyrosulphuryl chloride, while the distillates low in silicon, containing an excess of pyrosulphuryl chloride, which could exist free, crystallized readily under the same conditions. Reaction (2) is not affected by this assumption, for it takes place on standing at room temperature, or at the most at 50°, so that the loose compound, for which we assume that a temperature of about 130° is needed, is not formed. Our effort, however, was to prove reactions (1) and (2) rather than study this subsequent compound.

EXPERIMENTAL PART.

Materials: The commercial sulphur trioxide marked C. P. contained no impurities. In order to obtain a liquid at room temperature it was melted for some experiments, for others the melted substance was added to fuming sulphuric acid in the proportions necessary to give solutions of various strengths which were ascertained by titration or gravimetrically. The melting was done in a cylindrical copper air bath built for the purpose, and this was extraordinarily easy with a fresh sample;[5] a moderate temperature was required (about 50° in bath), the melting was rapid. no clots formed in the center, and the low temperature caused little boiling, hence little pressure, so that the stopper could be left in place without danger; with an old sample on the other hand the melting was almost impossible; so much heat was

5 Fresh sulphur trioxide melts at 17.7°; old samples do not melt at all, but sublime; Knietsch, Ber., **34**, 4101 (1901). Compare also Schenck, Lieb. Ann. **316**, 1 (1901); Weber, Ber., **19**, 3187 (1886).

needed that the small quantity which had melted boiled away before any more became liquid.

Silicon tetrachloride was prepared by passing dry chlorine over powdered silicon spread out in a hard glass tube, and heated in a combustion furnace.[6] An adapter at the end of the tube (the latter was slightly inclined towards the former) fitting into a receiver set in ice, completed the apparatus. Charges of 50 grams yielded 190 to 225 grams of silicon tetrachloride, that is, 64 to 75 percent of the theoretical yield. Special effort to obtain a large yield was never made; on the contrary economy of time was the only consideration. In this respect this method is ideal; the apparatus could be set up in a few minutes, and a preparation carried through in an afternoon.[7] The silicon contained iron, which caused the presence of ferric chloride in the crude product; the latter also contained free chlorine. The crude product was freed from chlorine by shaking with mercury and was then distilled.[8] It was kept in glass stoppered bottles under a dry bell-jar, or the flasks were sealed off. The bottles were left in the open at first, but the moist air caused a deposit of silica which cemented the stopper in place, and such a bottle could only be opened by breaking it, when it usually exploded. Rubber stoppers were found more satisfactory, but these hardened in time and also became cemented. Bottles nearly empty and imperfectly closed to moisture exploded spontaneously, because of the formation of hydrochloric acid.

Silicon oxychloride was prepared by the method of Troost and Hautefeuille[9], namely, by passing a mixture of chlorine and oxygen over heated metallic silicon in the same apparatus as the one used for silicon tetrachloride. The yield was very small; out of 154 grams a

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6 Hempel and Haasy state that they used this method, but give no details. Zeitschr. anorg. Chem., **23**, 32 (1900).

7 In our earlier work the silicon tetrachloride was made by passing chlorine over silicide of copper as done by G. H. Pratt (M. I. T. thesis, 1897, vol. 68; in the hands of Vigouroux, C. R., **129**, 334 (1899), this method did not give satisfactory results) except that glass retorts were used instead of iron tubes, but this method was much slower, less convenient, and gave a poor yield.

8 The melting point of silicon tetrachloride is −69° (corr.) It was determined several times on different samples, by the beaker method (see note 11) and by complete immersion of the thermometer. W. Becker and J. Meyer (Zeitschr. anorg. Chem. **43**, 251 (1905)) give this point as −89°; they determined it by winding a thermo-element on the outside of the containing vessel, while this was suspended in a Dewar tube containing a little liquid air; the junction was presumably at the bottom of the containing tube. Their material was exceedingly pure, but the method used in obtaining the melting-point is open to objection.

9 Bull. Soc. Chim., **35**, 360 (1881).

yield of 5.5 grams of the oxychloride Si_2OCl_6 was obtained, after several fractionations. The greater part of the crude material was silicon tetrachloride, besides some 35 grams of the higher oxychlorides. The method was nevertheless better than that of Friedel and Ladenburg [10], from which we obtained absolutely no yield. These five grams were found to crystallize readily, melting at —41° to —38°, corr. by the beaker method.[11]

Analyses: Rapid analyses for chlorine and sulphur were made volumetrically. A small bulb containing a known weight was broken in water; the solution made up to a definite volume and aliquot portions used. As a rule part of the silica precipitated. The total acidity was found by titrating against standard potassium hydroxide. The effect of the silicic acid on the indicator (phenol phthalein) was not to be considered, as the method was intended merely for following changes in whole percentages. The chlorine was determined by the Volhard method, with which the silver silicate does not interfere, for it is readily decomposed by all strong acids.[12] The acidity and the chlorine content were expressed in terms of a normal solution; from the difference the percentage of sulphur was calculated.

The accurate determinations were made as follows:

Silicon and Sulphur: A bulb containing a known weight was broken in a freshly prepared solution of sodium hydroxide made from sodium, and which had been shown to contain no silica, chlorine, or sulphur. The solution was filtered from the pieces of glass into a platinum dish, and acidified with sulphate-free hydrochloric acid.[13] The analysis

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10 Lieb. Ann., **147**, 355 (1868); C. R. (1868) **66**, 539; also Troost and Hautefeuille, C. R. (1871) **73**, 563; J. prakt. Chem., (2) **4**, 304 (1871).

11 A rapid method for obtaining melting-points at low temperatures was used. A small melting-point tube, as used in organic work, contains the substance already crystallized by dipping it in liquid air; this is placed in a beaker containing naphtha which has been cooled by immersion in liquid air also. By removing the latter, the bath is permitted to warm up until the substance melts; the temperature is read on a pentane thermometer calibrated in the same way as it is used. A full description will be found in a previous article, Proc. Am. Acad., **47**, p. 699. It will be called the "beaker" method.
In addition to this method the "immersion" method, in which the thermometer is placed in the melting substance, after having been standardized for that use, was employed whenever possible.

12 J. D. Hawkins, Am. Jour. Sc., **139**, 311 (1890).

13 The time during which the alkaline solution was in contact with glass varied between twelve and twenty-five minutes. In order to show that no glass was dissolved, the pieces of glass from one of the bulbs were collected, after the alkaline solution had been removed, and weighed:

Glass recovered 0.7458 gram.
" taken 0.7452 gram.

was continued in the customary way, involving in the earlier analyses two evaporations with intermediate filtrations, and corrections with hydrofluoric acid, but these corrections were found to be so small that the accuracy desired did not justify the work necessitated by them. Later therefore only one filtration was made, and the hydrofluoric acid correction left out. In the filtrate from the silica, the sulphur was determined as barium sulphate.

Chlorine: A bulb containing a known weight was broken in a solution of the sodium hydroxide. The liquid was filtered into a precipitating flask, and, after adding a drop of phenol phthalein, weakly acidified with chlorine-free nitric acid; silver chloride was precipitated from the clear solution, and weighed on a Gooch crucible.

In order to determine how much silica was carried down by the precipitate, a sample of silicon tetrachloride was treated with sodium hydrate, and the alkaline solution evaporated in a platinum dish; the silica was removed, and the chlorine determined. It was found to be 83.3 percent. The same material was then analyzed without removing the silica and there was found 83.6 percent of chlorine. Several other determinations confirmed this result. The amount of inclusion depends mainly upon the dilution at the moment of precipitation, and upon the percentage of silicon in the substance. The dilution was always made considerable, and while silicon tetrachloride contains 16.63 percent of silicon, the material analyzed contained one half to one fourth as much. Therefore it seemed safe to assume that the analyses had an accuracy of 3 parts in 1000, or 0.3 percent, which satisfied the requirement in this work.

The Mixtures.

When sulphur trioxide was added to silicon tetrachloride they mixed at once forming at 32° a clear colorless liquid, which after being sealed and standing in the room deposited long white needles like those of sulphur trioxide from which we inferred little or no reaction had taken place, but after this liquid had been heated in an air bath to 50° for 6 hours a reaction took place as shown by the formation of a product boiling between 135° and 150° at atmospheric pressure, whereas sulphur trioxide boils at 46° and silicon tetrachloride at 57°. A considerable deposit of silica also appeared in many of our experiments.[14]

14 Preliminary experiments tried by Mr. Maurice L. McCarthy led to the formation of distillates similar to those described later.

Solid sulphur trioxide did not dissolve in silicon tetrachloride and liquid mixtures of the trioxide and fuming sulphuric acid containing 8 percent of water or less, did not mix with the tetrachloride, but such hydrous sulphur trioxide solutions were made to react with it by shaking the two for several hours ($5\frac{1}{2}$ to 12) at room temperature, or better still by directing a blast of air warmed to 50°, on the bottle while on the shaker, when the mixing took place in less than an hour. The product after distillation could not be distinguished from that obtained from pure melted sulphur trioxide, except on analysis, when the former was found to contain roughly 14 percent sulphur, the latter 21 percent, and on cooling and seeding with pyrosulphuryl chloride, the latter could be made to crystallize, but not the former. The reason is self-evident, for in the former cases the water was combined with much of the sulphur trioxide, reducing its concentration as such; in the latter, the concentrations remained high, and reaction (2) could take place to a sufficient extent to raise the amount of pyrosulphuryl chloride. The yield (50 percent in the best case) seemed to be better when no water had been used in the sulphur trioxide, and when the proportions were those of two molecules of sulphur trioxide to one of silicon tetrachloride; an excess of either reagent diminished the action.

This product was freed from the unaltered reagents by distillation either at atmospheric or at reduced pressure. Its character and our attempts to prepare a pure substance from it are best made evident by the description of two of our most extended experiments.

101 grams of silicon tetrachloride were added to 100 grams of 93.8 percent sulphur trioxide, being 1 molecule of the former to 2 of the latter, and after shaking $5\frac{1}{2}$ hours a pale brown homogeneous liquid was formed which on standing over night deposited a flocculent precipitate and became colorless. 145 grams of the supernatant liquid were distilled at 16 mm. pressure with two condensers inserted between the receiver and the pump; the first was cooled by solid carbon dioxide mixed with absolute alcohol, and was destined to collect sulphur trioxide and silicon tetrachloride; the second was cooled by liquid air, and could therefore condense hydrochloric acid besides any material escaping the first tube.[15] 76 grams were collected at 42° to 70°. The tube at −78° contained 9 grams, the liquid air tube. 19.5 grams. These were silicon tetrachloride and sulphur trioxide In the subsequent distillations a single condenser cooled by liquid air

15 For description of the kind of tube used in the liquid air, see Sanger and Riegel, These Proceedings, **47**, 697.

was used. A hard residue which weighed 32 grams, was left in the boiling flask.

The 76 grams were distilled again. The material condensed by the liquid air was 1.7 percent of the weight taken, the residue in the flask also 1.7 percent. The 72 grams collected were distilled once more; 70.7 grams were obtained. In this third distillation, the more volatile matter was less than 0.15 percent, the hard residue in flask 0.5 percent. The material distilled at 49° to 70° with a pressure of 17 mm.; the highest temperature of the bath was 120°, the time thirty minutes. The distillations showed that the 76 grams contained no silicon tetrachloride, no sulphur trioxide, and no considerable amount of dissolved or suspended silica; an analysis showed the presence of 17.2 % S and 49.3 % Cl; the silica was not determined. No crystals could be formed, but the material congealed at −120°. This behavior suggested impure chlorsulphonic acid; so in order to determine whether it was present or not, 65 grams of the substance were treated with 40 grams of common salt.[16] (22 grams would have been required if all the sulphur found by analysis had been present as chlorsulphonic acid, hence the enormous excess would be expected to retain mechanically a great deal of the liquid). On adding the salt, no bubbling of hydrochloric acid gas occurred, as is always the case with chlorsulphonic acid; a distillation at low pressure gave 43.5 grams of distillate. In the liquid air condenser there were 3 grams of a liquid which were undoubtedly uncondensed distillate and hydrochloric acid. The large distillate, accompanied by the insignificant amount of hydrochloric acid,[17] established the absence of chlorsulphonic acid as an essential part of the liquid. The composition was

	S	Si	Cl
(1)	16.9%	9.4%	53.3%

not markedly different from that of the liquid before treatment with salt. Nevertheless in subsequent mixtures a distillation from salt was frequently performed, in order to remove even the smallest amount of chlorsulphonic acid that might have been formed.

The 43.5 grams were distilled twice more, at low pressure, in an effort to obtain a constant boiling-point, but in neither case was the temperature steady; the best result was the second fraction in the

16 For this method of removing chlorsulphonic acid, see also Sanger and Riegel, These Proceedings, **47**, 689.

17 A portion of this hydrochloric acid was formed by the action of the vapors on the rubber corks and connections.

second distillation, which weighed 23 grams and boiled from 45.8° to 48.2° at 11–13 mm. This material analyzed gravimetrically contained:

	S	Si	Cl
(2)	14.63%	10.20%	54.87%
(3)	14.50%	10.18%	
$1S_2O_5Cl_2$ $+1Si_2OCl_6$	12.82%	11.31%	56.69%

The composition of an equi-molecular mixture of pyrosulphuryl chloride and silicon oxychloride is given above, and it can be seen that the liquid gives similar figures.

Distilled at atmospheric pressure the temperature rose steadily and evenly from 141° to 150°. No crystals were obtained on cooling. To the distillate one fifth of its weight of pyrosulphuryl chloride was added, and it was then distilled again when the temperature readings were unchanged. On cooling no crystals were formed; the material congealed as before, below −100°.

In our second extended experiment 236 grams of silicon tetrachloride and 221 grams of melted anhydrous sulphur trioxide, that is, one molecule of the former to two of the latter, were heated in the air-bath for six hours at 50° and deposited 8 percent of a white solid. 205 grams of the liquid poured off from this solid were distilled at ordinary pressure and gave

90	grams at	37–44°
62	" "	44–58°
31	" "	59–145°
21	" "	residue.

(Sulphur trioxide boils at 46°, silicon tetrachloride at 57°). The third fraction and the residue gave on a second distillation 20 grams at 135–151° and 16 grams at 151–172°; these 20 grams did not crystallize on cooling, and were therefore distilled again, and separated into four fractions. The second one only, 6 grams collected at 137–145° (4, 5) could be made to crystallize at −83°, melting at −50°; the remaining three fractions were combined and distilled, the distillate being collected in 3 portions, the middle one, 138–143° (8, 9) crystallized spontaneously; the lower one, 136–138° (6, 7), on stirring; the upper one, 143–176° (10, 11) only on seeding from the lower one. As none of these fractions showed any signs of constant boiling-point, the crystallization was studied further. For this purpose each fraction was introduced into a special separator[18], crystallized and the mother liquor drawn off; the crystals were then allowed to melt, drawn off in their turn, and analyzed. Only the fraction 138–143° was not separated into crystals and mother liquor, because its amount was too small.

		S	Si	Cl
(4)	137–145°	21.37%	5.25%	43.68%
(5)		21.33	5.45	43.94
(8)	138–143°	22.88	4.4	41.65
(9)				41.68
(6)	136–138°	21.9	4.9	43.00
(7)		22.0		
(10)	143–174°	22.73	5.28	39.74
(11)		22.89	5.38	
(12)	Mother liquor	22.97	5.19	39.73
(13)	of crystals			39.44

These analyses show that the fractional distillation gives little or no promise of leading to a pure product as the percentages of sulphur and silicon in the different fractions differ by 1.1 percent or less, those of chlorine by 3.56 percent or less Nor is crystallization more promising since there is essentially no difference in composition between the crystals (10, 11) and their mother-liquor (12, 13).

Another preparation similar to the last was allowed to stand for three summer months in a glass stoppered bottle under a bell-jar whose atmosphere was kept dry by phosphorous pentoxide; during this time a solid amounting to 23 percent of the total weight was deposited, and on distillation a fraction of 53 grams or 38 percent boiling at 136–156° was obtained. This fraction when cooled to −78° did not crystallize, but on inoculation with a crystal of pyrosulphuryl chloride it did, so many crystals developing that it became a stiff paste; this had been done in the separator, and after warming a little, the mother liquor was drawn off; the process was repeated twice, the melted crystals serving as starting material each time. After making an analysis (14) of the final crystals, a portion of the original 53 grams was taken and the crystallization was repeated, but this time with an alcohol bath cooled to −65° and −60° instead of −78°, in order to reduce the supercooling. The material was fractionally crystallized 4

18 Sanger and Riegel, These Proceedings, **47**, 710. It consisted essentially of a glass vessel holding a platinum cone, and connections above and below so that suction might be applied below or above, all out of contact with moist air.

times, with a seed of pyrosulphuryl chloride, the separation taking place in order at —65°, —60°, —60°, and —60°; the crystals thus purified weighed 3.5 grams, and their analysis (15, 16) is given below. The mother liquors were then combined, and a crystallization from a silicon oxychloride seed attempted, but it failed, for only at —78° could crystals be obtained, and at that low temperature, crystallization was spontaneous; nevertheless, the crystallization was repeated twice, and the final crystals analyzed (17). The analysis of the original 53 grams is also given (18).

	S	Si	Cl
(14)	23.98%	3.7 %	40.02%
(15)	24.28	3.78	39.57
(16)	24.03	3.85	39.38
(17)	23.62	4.07	40.08
(18)	23.26	4.90	41.61

The composition had not changed markedly; (14) has more sulphur and less silicon than (18) which would point to a concentration of pyrosulphuryl chloride; the fact that it crystallized at —60° rather than at —78° also supports this assumption, but at —60° the supercooling is still considerable; (15) and (16) differ too little from (14) to have a meaning, so that neither the analysis nor the crystallization with pyrosulphuryl chloride show that all the pyrosulphuryl chloride is present as such. The failure of crystallization with silicon oxychloride is reflected in the analysis (17) which hardly differs from (18) the original material, showing that the silicon oxychloride is not present free.

The experiments described above show why we did not succeed in obtaining a pure substance from our product either by fractional distillation or crystallization; an additional experiment might be mentioned as having led to the same result. A preparation similar to the previous one was fractionally distilled 5 times, at atmospheric pressure, using a dephlegmator, and yielded two fractions boiling over several degrees:

		S	Cl
(26)	145–149°	15.3%	50.6%
(27)	149–157°	17.1	50.7

But in spite of our failure to isolate the pure substances, our analyses established the nature of the two compounds of which this mixture is made up. For this discussion we have collected all the complete analyses already given and all the others made by us in a table. Of the analyses not already given the product for (19) was made by heating solid sulphur trioxide and silicon tetrachloride in sealed tubes at 250°; when the tubes were opened there was no pressure, showing that no gas was formed by the reaction. A fraction boiling at 135–157° was purified by distilling it at a pressure of 1.5 mm., and a product obtained boiling at that pressure at 29.5° to 34°. A preparation made almost like that of analysis (2) and (3) yielded a lower fraction analyzed under (20) while a higher fraction which approached pyrosulphuryl chloride in composition and behavior will be discussed later. Finally three analyses were made of fractions obtained from a liquid similar to the preceding ones, but which had been treated with water, in order to destroy if possible one of the constituents; one of these fractions (21) was submitted to two distillations at low pressure; another (22) was this preceding product after treatment with salt, in order to remove any chlorsulphonic acid which might have been formed by the action of the water; the third (23) was collected from the same flask, but after the temperature had been raised from 150° to 210°, with the pressure still 20 mm., a degree of superheating almost certain to cause decomposition, since the average boiling-point at such pressures is below 100°. This action of the study of water will be referred to further on.

The percentages of chlorine marked "calculated" in the table were obtained by computing the percentage of chlorine which would correspond to the percentage of silicon in each analysis if this were present as silicon oxychloride, Si_2OCl_6. The amount of chlorine was then calculated which would be combined with the percentage of sulphur if this was present as $S_2O_5Cl_2$, and subtracted from the percentage of chlorine recorded by the analysis and the remainder entered as "Found" in the table.

From this table it would appear that 13 out of the 18 analyses agree within 2 percent, of the calculated amount of chlorine in a mixture of pyrosulphuryl chloride and silicon oxychloride, and one more (18) within $2\frac{1}{2}$ percent; moreover, among these 14, 8 agree within 1 percent. Of the four which do not agree, all were analyses of fractions boiling at higher temperatures than the average, 135–150°; thus (10) and (12) at 143–174°, (22) at 52–110° with a pressure of 20 mm. and (23) over 110° with the same pressure, the oil-bath around the boiling flask being finally at 210°. These high temperatures indicate that impurities were present, as products of decomposition, or substances due to the action of moisture from the air and of organic matter in the shape of unavoidable rubber stoppers; there would seem to be sufficient ground

TABLE OF COMPLETE ANALYSES.

	S	Si	Cl	Cl Calculated	Cl Found
(1)	16.9%	9.4%	53.3%	35.3%	34.6%
(2)	14.63	10.20	54.87	38.34	38.69
(4)	21.37	5.25	43.68	19.73	20.05
(5)	21.33	5.45	43.94	20.49	20.35
(6)	21.9	4.9	43.00	18.42	18.78
(8)	22.88	4.4	41.65	16.54	16.35
(10)	22.73	5.28	39.74	19.85	14.60
(12)	22.97	5.19	39.73	19.51	14.33
(14)	23.98	3.7	40.02	13.9	14.5
(15)	24.28	3.78	39.57	14.21	12.72
(16)	24.03	3.85	39.38	14.47	12.80
(17)	23.62	4.07	40.08	15.30	13.86
(18)	23.26	4.90	41.61	18.42	15.89
(19)	15.14	10.14	53.15	38.12	36.41
(20)	19.6	6.3	45.2	23.7	24.5
(21)	20.1	6.1	43.4	22.9	21.2
(22)	19.9	6.75	43.4	25.4	21.4
(23)	24.2	3.9	35.2	14.7	8.4

for excluding them, and if this is done, all the complete analyses of the well-established specimens of the mixture show that it is made up of pyrosulphuryl chloride and silicon oxychloride only. This inference is confirmed in a number of different ways.

The boiling-point of this fraction 135–150° is what would be expected for while pyrosulphuryl chloride boils at 152.5–153°, the least trace of moisture causes a considerable portion of it to boil 5° to 10° lower. and the boiling point of silicon oxychloride is 136–139°.[9]

The behavior of the distillate with water also is a valuable indication. When much water was added to it there was a violent reaction accompanied by the formation of silica, such as silicon oxychloride gives, and an oily liquid separated and sank to the bottom of the vessel, where it dissolved very slowly, showing the behavior of pyrosulphuryl chloride; dilute sodic hydrate acted in the same way. As described before, the crystallization could be induced with a comparatively moderate degree of cooling below its freezing-point by a crystal of pyrosulphuryl chloride in those mixtures which had 5 per-cent or less of silicon, and the crystals obtained in this way gave an analysis indicating a concentration of pyrosulphuryl chloride (14) compared with (18), the mother liquor. Finally several distillates were obtained under certain conditions stated further on, which were almost pure pyrosulphuryl chloride; these cannot be used as proof of its presence because they were obtained at temperatures higher than usual; but taken in connection with the other agreeing observations, they help to establish the fact that pyrosulphuryl chloride is one of the two products of the reaction. With pyrosulphuryl chloride established as one of the products, the table of analyses show beyond a doubt that the other is silicon oxychloride.

The more regulated action of water on the distillate was next studied in the effort to destroy one substance and isolate the other. In searching for a neutral diluant it was found that alcohol, acetone, benzol reacted with the distillate, but carbon tetrachloride, chloroform, and carbon disulphide did not; chloroform and carbon tetrachloride were used. 103 grams of chloroform were mixed with 84 grams of the distillate, the diluted material cooled to -5° and water added drop by drop. Each drop caused the formation of a white ball which when pricked, burst with a slight explosion and formation of hydrochloric acid. The action evidently consisted in the coating of the entering drop with a silica shell; while on pricking, the unchanged water was freed and reacted further. 3 grams of water were added, approximately the amount required to change all the pyrosulphuryl chloride present to chlorsulphonic acid. On transferring the material a sediment of 6.5 grams (silica) was found. A distillation at low pressure removed the chloroform and hydrochloric acid; a distillate of 42 grams was collected, while a semi-solid residue was left in the boiling flask. A second distillation at low pressure gave 38 grams, free from hydrochloric acid. The density at 18° was 1.73 (pyrosulphuryl chloride is 1.837 at 20°). An analysis showed the change produced by the water:

	S	Si	Cl
(24) Before treatment with water:	16.7 %	—	49.9 %
(21) After " " "	20.1	6.1	43.4

The figures show that the water attacked mainly the silicon body. A treatment with salt followed and the distillate from the salt mixture was collected, under diminished pressure, in two fractions: the lower one, 9 grams, between 50° and 110°, with oil-bath around the boiling flask raised to 150°; the second, 4.5 grams, with the bath heated to

210°, the pressure being 20 mm. throughout. The analyses resulted as follows:

		S	Si	Cl
(22)	9 grams	19.9 %	6.75 %	43.4 %
(23)	4.5 grams	24.2	3.9	35.2
For comparison, $S_2O_5Cl_2$		29.82	—	32.98

The figures for the 9 grams indicate that no appreciable amount of chlorsulphonic acid had been removed, hence that the water treatment destroyed the silicon body, and yielded the sulphur containing body. That this sulphur body is pyrosulphuryl chloride is indicated by the figures for the 4.5 grams obtained by heating the salt residue to a temperature exceeding 200°, while the pressure was kept at 20 mm.

In another preparation in which hydrous sulphur trioxide was used of such a strength that there was enough water present to form chlorsulphonic acid instead of pyrosulphuryl chloride according to the reaction: $2SO_3 + H_2O + SiCl_4 = 2ClSO_3H + SiOCl_2$, that acid was indeed obtained, but the reaction had taken place in a different way. The water decomposed a portion of the silicon tetrachloride, forming silica to such an extent that the solid residue in the flask after distillation was 50 percent by weight of the materials taken, and hydrochloric acid, which combined with the sulphur trioxide to form chlorsulphonic acid, a portion of the silicon tetrachloride was recovered unchanged in the liquid air condenser; all the silicon not accounted for by it was present in the solid residue.

Three preparations have been observed which approached pyrosulphuryl chloride. The most nearly pure was obtained early in our work by distilling a homogeneous mixture of 115 grams of 94 percent sulphur trioxide and 111 grams of silicon tetrachloride (2 molecules to 1) resulting after 12 hours' shaking at room temperature. A large amount of gelatinous silica was filtered off through glass wool, and the 144 grams of filtrate submitted to a distillation at low pressure; the distillate was collected in four fractions; the analysis of the first one, boiling at 60° to 100° with a pressure of 70 mm., has been given under (20) in the table of analyses; the last, boiled at 105°, with the oil-bath surrounding the flask 240°, and pressure 60 mm., accompanied by violent foaming, and left a solid residue of 50 grams; its analysis follows:

	S	Si	Cl
Found	28%	0.17%	29.4%
$S_2O_5Cl_2$	29.82	—	32.98

The experiment on the action of water on the average distillate diluted by chloroform has been recorded; there too on heating to 210°

with a pressure of 20 mm. and leaving a solid residue mainly sodium chloride in the flask, a last fraction of 4.5 grams rich in pyrosulphuryl chloride was obtained, as shown by its analysis (23) and behavior with water. In the third case 197 grams of a liquid prepared just like the 144 grams above, and not previously distilled, were mixed with the 116 grams of phosphorus pentoxide and distilled at low pressure; only 38 grams could be collected, in three fractions, with the bath surrounding the flask at 220°. On attempting to heat with the free flame, because the distillate came at the rate of a drop per minute, the flask exploded. These fractions were tested as to their melting-point; they all solidified readily in large crystals, radiating from one point, exactly as pyrosulphuryl chloride does; the first and second melted at near —40°, the third near —50°; pyrosulphuryl chloride melts at —37°; one of the fractions was tested for silica and contained none. It would seem as if by distilling from any solid, so that a high temperature is required, the silicon body is destroyed, while the more resisting pyrosulphuryl chloride can be collected.

That no chlorsulphonic acid was present in our original distillate was proved by an experiment performed in our first extended study and already recorded. See page 584.

In an attempt to obtain the ethyl ester of the silicon containing body,[19] the material diluted by carbon tetrachloride was treated with alcohol and after distilling off the solvent there remained in the boiling flask a semi-solid mass which was silica and ethyl sulphuric acid. The experiment was repeated several times, always with the same result. That the ethyl ester of silicon oxychloride was not obtained does not show the absence of silicon oxychloride, for such an ester would hardly be stable in presence of ethyl sulphuric acid and other products of water on pyrosulphuryl chloride.

An examination of the table of complete analyses shows that different specimens of the distillate contain the pyrosulphuryl chloride and silicon oxychloride in different proportions, three of them show percentages approaching those required if the two substances are present in molecular proportions.

	S	Si	Cl
Calculated for $S_2O_5Cl_2 + Si_2OCl_6$	12.82%	11.31%	56.59%
(2)	14.63	10.20	54.87
(19)	15.14	10.14	53.15
(1)	16.9	9.4	53.3

19 Friedel and Ladenburg prepared the ethyl ester of silicon oxychloride $(C_2H_5)_6OSi_2$; reference 5. Compare Friedel and Crafts, Ann. Chim. phys. **9**, 12 (1886); Ladenburg, Lieb. Ann., **173**, 144 (1874).

Besides these, several incomplete analyses show similar proportions:

	S	Si	Cl
(3)	14.50	10.18	—
(24)	16.7	—	49.9
(26)	15.3	—	50.6

The results leave much to be desired but certainly indicate that the two products were formed approximately in the proportions required by the reaction:

$$2SO_3 + 2SiCl_4 = S_2O_5Cl_2 + Si_2OCl_6 \qquad (1)$$

In the majority of analyses percentages were found indicating a decided excess of pyrosulphuryl chloride, and a smaller proportion of silicon oxychloride; the silicon varying from 7.75 to 3.7, instead of being 11.31; the sulphur varied from 19.6 to 24.28 instead of being 12.82. The formation of more pyrosulphuryl chloride, and the destruction of silicon oxychloride is accounted for by the reaction:

$$Si_2OCl_6 + 6SO_3 = 2SiO_2 + 3S_2O_5Cl_2 \qquad (2)$$

An especially large percent of sulphur indicating a great excess of pyrosulphuryl chloride was obtained from a specimen which had stood 3 summer months undistilled, that is, with the low-boiling portions containing the excess of sulphur trioxide still unseparated from the higher boiling portions. It was contained in a glass-stoppered bottle under a bell-jar whose atmosphere was dried by phosphorus pentoxide. Comparing this material with a similar one, which had been heated for 6 hours and had not stood at all, it was found that the higher boiling fraction had increased on standing from 3 percent to 38 percent, the solid deposited rising at the same time from 2.3 percent to 23 percent; in another more favorable case the figures for a similar mixture heated for 6 hours were 15 percent of higher boiling fraction, and 8 percent solid. In all three cases two molecules of sulphur trioxide had been used to each molecule of the silicon tetrachloride. Reaction (1) calls for equal molecules of the two, while (2) requires six molecules of the trioxide to each one of the oxychloride already formed, but no more silicon tetrachloride. This action should be reflected in the lower boiling fraction in an increase in the proportion of silicon tetrachloride, evidenced by a higher percentage of chlorine; an analysis showed that it contained, in the case of the material which stood 3 months, 65.5 percent of chlorine, whereas the original mixture of silicon tetrachloride and sulphur trioxide contained only 42.95 percent; this shows that the action is correctly interpreted by reaction (2). It is important to add that once the higher boiling fraction was removed from the excess of sulphur trioxide, or which is the same, from the low-boiling fraction, no solid was deposited.

Most of the mixtures of pyrosulphuryl chloride and silicon oxychloride obtained by us crystallized at various temperatures, the highest one being $-60°$, the lowest $-78°$; some of them did not crystallize, but merely solidified to a vitreous mass, near $-120°$. Pyrosulphuryl chloride melts at $-37°$, silicon oxychloride near $-40°$. A mixture of equal parts of the prepared pure substances melted at $-40°$ to $-38°$, but on mixing 15.6 grams of the former with 5.2 grams of the latter, and heating half of the mixture on the Bunsen flame, it was found that the heated portion took twenty times longer to crystallize than the unheated one. As stated in the introduction, the only possible explanation is that the two substances form a compound under the influence of heat. All the mixtures were obtained by means of one or more distillations, and submitted to the heat of a flame, and this influence of heat explains the phenomena of crystallization observed. In the mixtures containing approximately one molecule of each substance, the two substances are combined, and not being free, cannot crystallize when a seed of either substance is introduced; in the mixtures containing an excess of pyrosulphuryl chloride, the silicon oxychloride is all combined, but some of the pyrosulphuryl chloride remains free, hence this mixture can crystallize on a seed of pyrosulphuryl chloride, although here again, it will not crystallize on a seed of the oxychloride.

The deposit formed on standing over the summer was placed in a Gooch crucible, washed twice with silicon tetrachloride, pressing down the material with a glass rod, and using a suction pump to remove all the liquid: all as rapidly as possible. The solid was then packed and sealed in tubes, in which it remained without alteration. This material appears perfectly dry; it smokes strongly in the air and attracts moisture rapidly. With water it reacts violently; a few bubbles of gas escape, and a slight yellow color (due to chlorine) develops; the white particles become transparent, but retain their original shape; no visible amount of silica separates out from the liquid. The reaction with dilute sodium hydroxide is the same, but more violent. When moistened with chloroform the solid becomes translucent and filled with bubbles. Heated over the flame it evolves white fumes, and leaves behind a white ash, which no longer reacts with water. The deposit is not homogeneous; the silicon percentage varied between 6 and 10; the portions which had been formed against the wall of the flask contained the higher percentage. One sample contained:

S	Cl	Si
26.8%	18.3%	6.2%

The study of this solid suggests that it is silica enclosing much sulphur trioxide and some silicon oxychloride and pyrosulphuryl chloride. This silica would have been formed by the equation, which has been discussed before,

$$Si_2OCl_6 + 6SO_3 = 2SiO_2 + 3S_2O_5Cl_2$$

Our reasons for contending that silico-phosgene is not formed have been fully stated in the introduction.

SUMMARY.

Melted sulphur trioxide and silicon tetrachloride are miscible; on standing a long time or on heating 6 to 10 hours to 50° a reaction takes place:

$$2SiCl_4 + 2SO_3 = Si_2OCl_6 + S_2O_5Cl_2;$$

an excess of sulphur trioxide causes a further reaction:

$$Si_2OCl_6 + 6SO_3 = SiO_2 + 3S_2O_5Cl_2$$

The most significant result, as regards the relation of carbon and silicon, is the non-formation of silico-phosgene.

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AN ELECTRIC HEATER AND AUTOMATIC THERMOSTAT.

By A. L. Clark.

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AN ELECTRIC HEATER AND AUTOMATIC THERMOSTAT.

By A. L. Clark.

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In a previous paper [1] I have described a form of electric heater and automatic thermostat for control of temperature, capable of a fair degree of accuracy and possessing a wide range. This has been improved recently so that the accuracy with which the heater may be maintained at any given temperature is very much increased. For the work described in the paper mentioned, it was not necessary to regulate more closely than $1/10°$, but subsequent work developed the need for a higher degree of accuracy with certainty of operation, and with no sacrifice of range or capacity. The following is a description of the improved apparatus. It is given because this form of heater and thermostat seems to combine accuracy of control, ease of adjustment, wide range and large size of heating spaces as does no other — at least the writer knows of none.

As mentioned in the previous paper, the device is a modification of the thermostat used by Griffith [2] in his work on the Mechanical Equivalent of Heat. The essential features are as follows: — a cubical cast-iron box 15 cm. on an edge is made with hollow walls and bottom, the solid parts of the walls being 6 mm. thick, while the hollow space is of the same thickness. In this way a chamber is formed in the walls and bottom whose volume is 420 c. c. This is filled with mercury and forms the bulb of a gigantic thermometer, the tube of which is outside the apparatus. This cast-iron box with its enclosed mercury is surrounded by coils of German silver wire, and placed within a larger box for heating. The air in this space is kept in constant and rapid motion by a number of fans, so that the entire space is maintained at uniform temperature. This apparatus is lagged with magnesia and enclosed again in a massive wooden box. It is perhaps unnecessary to state that the body to be heated is placed inside the inner cast-iron box, which is provided with windows of ample size both in front and rear, as are also the enclosing boxes, so that observation is always possible. The outer windows have covers that may be closed to investigate effects of radiation. The mercury space of the inner box is connected by a steel tube with the automatic part of the apparatus which is shown in Fig. 1.

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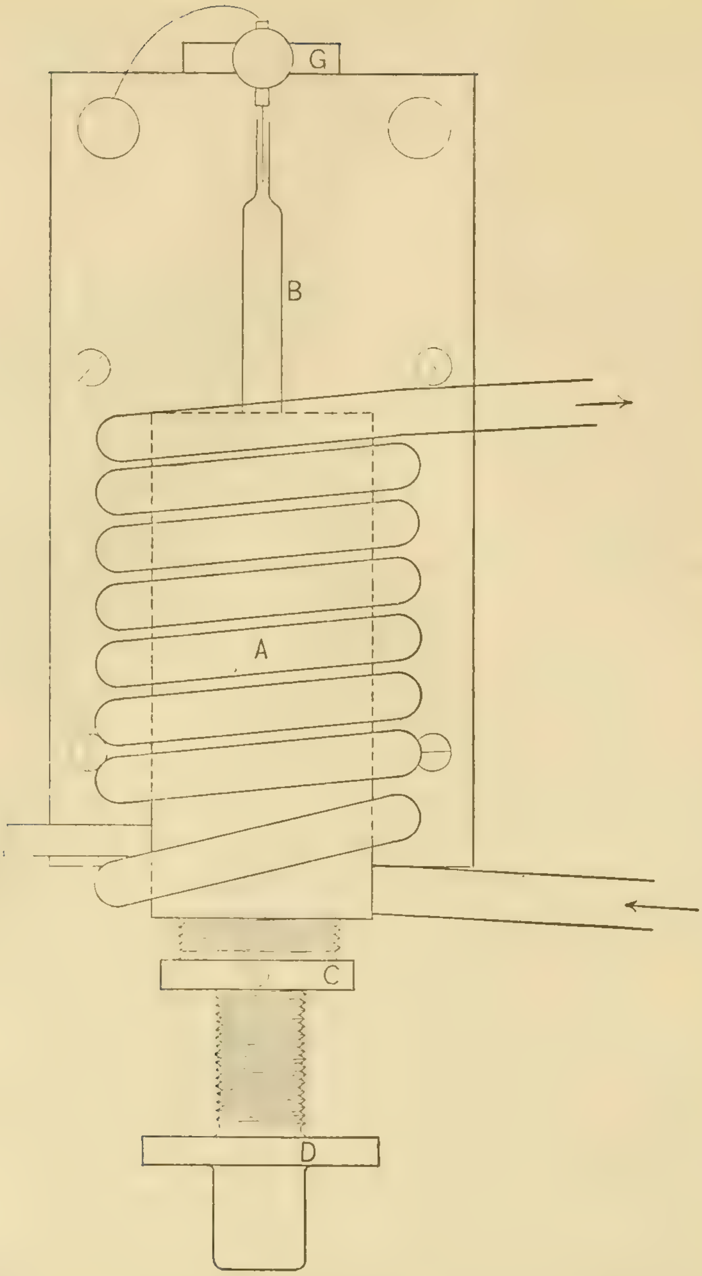
1 These Proceedings, **41,** No. 16, Jan. (1906).
2 Griffiths, Phil. Trans., **184,** 361 (1893).



E. is the steel tube from the mercury space of the cast-iron box. A. is a cylindrical cast-iron chamber or reservoir, opening at the top into the glass tube B, and closed at the bottom by the stuffing box C, into which the screw D may be turned. When the temperature is varied the mercury within the heater expands filling the chamber A and rises eventually into the tube B, until it reaches the end of a platinum wire. This completes the circuit of a relay which cuts off the heating current, either entirely or in part. When the current is cut off, the temperature falls until contact of the relay is broken at the platinum point, when the heating current is thrown on again. If the current is properly adjusted and the change in value caused by the action of the relay be small, the amount through which the temperature rises and falls may be very small indeed. Obviously the temperature at which the relay cuts off the current depends on the actual volume of the reservoir A, or in other words on the position of the screw D. The total capacity of the reservoir is about 18 cm. which equals the expansion of the mercury in heater caused by an elevation in temperature of about 300°. Of course the amount of current used depends on the temperature at which the work is to be done and no more than is actually necessary is used.

When the proper amount of current is used the regulation near 200° is within 1/10° when the entire current is cut off. The adjustment for this accuracy need not be very carefully made. When a portion only of the current is cut off and the adjustment be made with sufficient care the variation in temperature may be made very small. Close regulation at low temperature is much easier than at higher and there is less need of careful adjustment; as the temperature is carried higher regulation becomes more difficult. One source of difficulty in maintaining constancy of temperature is due to the fact that heat is conducted along the mercury in the steel tube connecting the mercury chamber with the reservoir attended by a rise of temperature of the mercury in the reservoir. This rise in temperature has been obviated by surrounding the reservoir with a coil of small lead tubing (shown in Fig. 1) through which a current of cold water is kept circulating.

The table shows the values of the currents necessary to maintain the heater at different temperatures:—

Amps.	*Temp.*
1.43–1.55	67.2
2.35–2.45	100.6
2.88–3.05	198.0
4.30–4.40	221.0

At 198° with the regulator changing the current from 2.88–3.05 a thermometer graduated to 1/5° was watched through a microscope and no motion of the thread was apparent. The regulator worked at about two-minute intervals. One very serious difficulty which gave trouble for a long time was with the lubrication of the bearings of the shafts, driving the fans in the inner box. Ordinary lubricating oils boil out of the bearings at about 180° and condense on the windows of the outer box, obscuring the view of the inside of the box. Below about 180° there is no trouble but above this the distillation of oil occurs. Various oils were tried with no success because there is always this distillation at some temperature. Finally the difficulty was overcome by using paraffin wax which melts at about 50° and does not distil away enough to cause any trouble. Small pieces are placed in the ends of the oil tubes leading to the inner bearings. These are quickly melted by the heat conducted from within and run down to the bearings lubricating them very efficiently. The slight jarring of the whole apparatus causes trembling of the mercury at the relay contact and no sticking of the mercury to the platinum point has been noticed. A little alcohol on top of the mercury helps to keep it clean.

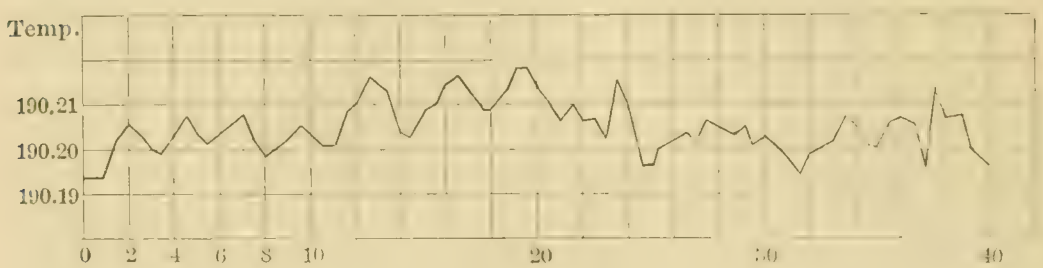
Not only is it important that there shall be no unsteadiness of temperature but there must be no temperature gradients inside the box, more particularly vertical gradients. So an investigation of the distribution of temperature was made. The mercury thermometer was found to be worthless for this work as it does not show very small changes readily. Accordingly a platinum resistance thermometer made by Mr. C. H. Day was used. This is made of about 50 cm. of platinum wire fused on to platinum leads. The resistance wire is wound on a small mica frame in the form of a cross. The cross was first made and cemented together with "cementium." Two slits were cut in the mica, the platinum wire doubled and the loop in the end caught in the slit. Then the wire was wound on double in small cuts in the mica and finally fused to the platinum leads in the oxyhydrogen flame. The two thermometer leads together with the compensating leads were thrust through small mica discs, and the whole placed in a thin walled glass tube. The tube was slightly enlarged at the top so that it might hang in a hole in a piece of vulcanized fibre. Flexible cords were then soldered to the platinum leads and finally a second piece of fibre was fastened to the first by screws holding tube and leads firmly. The compensating leads are connected in series with a good resistance box and the two sets are connected to a slide wire bridge of the circular drum type made by the Leeds and Northrup Co.

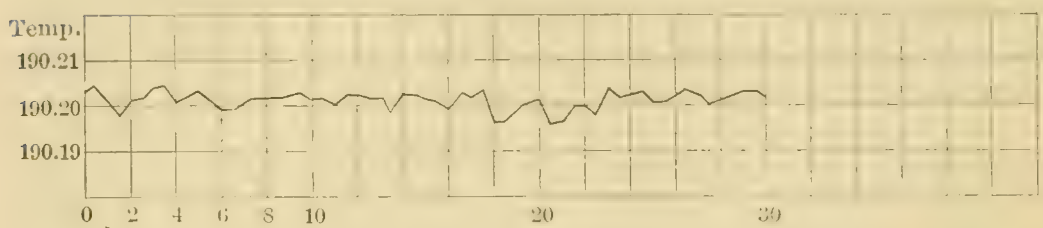
A steady current of .007 amps. is allowed to flow through the thermometer so that it is always slightly higher in temperature than its surroundings, but the amount is very small and is constant. The thermometer was calibrated by immersing in melted ice, in steam, and finally in vapor of boiling aniline which had been redistilled several times. The calibration curve compares very favorably with those given by Callendar. As the thermometer is used, one small division on the galvanometer scale corresponds to a change in temperature of about 1/120° so that the thermometer is easily read to 1/1000°. The platinum thermometer surpasses the mercury thermometer in its ability to follow small changes in temperature, and while the scale of this thermometer in the higher region may be in doubt by as much as 1/10°, its efficiency is in no way impaired. During the warming up stages in any experiment, the current for heating is taken from the 110 volt dynamo circuit, but this is too unsteady for accurate work. So when the temperature rises near the desired point the 110 volt storage battery circuit is thrown in. For work requiring 1/10° accuracy the lighting circuit is ordinarily steady enough.

It is extremely doubtful if the readings of most mercury thermometers can be relied upon to 1/100th of a degree when working at temperatures as high as 200°. The amount of stem exposed, sticking of mercury, etc., bring its indications under suspicion. With the platinum thermometer just described, it is possible to follow fluctuations which ought to be visible in the mercury thermometer, but which are not as a matter of fact. Much interesting and valuable information was gained by use of this instrument. It was discovered that the incandescent lamp behind the box used for illumination caused a rise in temperature of over 1/10°. A lamp in the room which shone into the front window affected the temperature of the thermometer noticeably. For work on liquids near the critical point this fact must not be overlooked. It is essential that the very smallest amount of light possible be used, particularly when the light shines on a portion only of the tube, which contains the liquid under experiment. Most observers have not taken sufficient pains in this matter. Many tests for constancy of temperature have been made. The following (Table I) may be regarded as typical, and show the possibilities of the apparatus. The first set (Temp. I) was obtained by breaking the entire current of 3.96 amperes, the second (Temp. II) when the current varied between 2.62 and 3.90 amperes.

Time in Minutes	Temp. I.	Temp. II.	Time in Minutes	Temp. I.	Temp. II.
0	190.194	190.203	15½	.210	.201
½	.194	.204	16	.215	.199
1	.194	.201	16½	.218	.203
1½	.202	.198	17	.212	.201
2	.206	.200	17½	.208	.203
2½	.203	.202	18	.208	.200
3	.200	.203	18½	.212	.197
3½	190.199	190.204	19	.217	.197
4	.202	.201	19½	.218	.200
4½	.207	.202	20	.213	.201
5	.203	.203	20½	.210	.200
5½	.201	.202	21	.206	.196
6	.203	.200	21½	190.209	190.200
6½	.205	.199	22	.205	.200
7	.207	.201	22½	.207	.198
7½	.200	.202	23	.201	.203
8	.200	.201	23½	.215	.202
8½	.200	.201	24	.210	.202
9	.202	.201	24½	.197	.203
9½	.205	.202	25	.197	.201
10	.202	.201	25½	.200	.201
10½	.200	.201	26	.202	.202
11	.200	.201	26½	.204	.203
11½	190.208	190.202	27	190.201	190.202
12	.210	.202	27½	.206	.200
12½	.216	.202	28	.205	.201
13	.215	.202	28½	.203	.203
13½	.212	.198	29	.205	.203
14	.204	.202	29½	.200	.203
14½	.202	.202	30	.202	.201
15	.209	.201			

If the apparatus could be attached to a storage battery on which there were no other loads, it would be comparatively easy to get closer regulation by using a narrower range of current variation. The battery used in these experiments is liable to have other loads thrown on at any time. The curves (Fig. 2) show the variations plotted from the tables.





Next the existence of a vertical temperature gradient was examined. Readings on thermometer were made at different levels and no sure difference of temperature was observable when the space inside heater is empty. When masses of metal or other obstructions were placed inside, slight differences amounting to one thousandth of a degree were observed. In experiments like this all the windows in the apparatus must be covered as radiation causes noticeable differences in temperature.

The platinum thermometer led to another important discovery. Even after the automatic controlling apparatus becomes steady it was found that the temperature of the air inside the heater continues to rise. This is due to the fact that altogether the wall of the heater box is about 1.8 cm. thick and while the *mean* temperature of the mercury in the wall does not vary, the temperature of the inner part and the wall adjacent to it is rising while that of the outer part is falling. This rise may amount to more than .1° and it is a matter of an hour or so before it disappears. This time has been shortened by placing a small flat heating coil of fine GS wire along the inner wall inside. A small current sent through this helps to establish equilibrium a little more quickly. Considerable judgment must be exercised in its use however.

Finally the effect of the stirring system was investigated and it was found that running at normal speed, the fans gave a rise in temperature of about .1° per hour; so that any slight variation in speed of fans is not important, but great variations may interfere with close regulation.

The ease with which one temperature after another can be obtained is one of the features of the apparatus. Other advantages are the wide range of available temperatures, the precision with which any given temperature may be reached and maintained, the large volume of heating chamber, ease of observation and the certainty of operation. Another advantage in work near the critical point is the small amount of damage caused by explosion. The windows, which are easily replaced, may be blown out but no injury to the essential parts has ever occurred and explosions have not been infrequent.

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CONTRIBUTIONS FROM THE PHANEROGAMIC LABORATORIES OF HARVARD UNIVERSITY.— No. 59.

CRETACEOUS PITYOXYLA FROM CLIFFWOOD, NEW JERSEY.

By Ruth Holden.

With Four Plates.

CRETACEOUS PITYOXYLA FROM CLIFFWOOD, NEW JERSEY.

By Ruth Holden.

Presented by E. C. Jeffrey. Received December 1, 1912.

During the spring of several successive years, Dr. E. C. Jeffrey collected a considerable amount of lignite from the Middle Cretaceous of Cliffwood, New Jersey, which he has since turned over to the writer for investigation. The material was from two localities,— that from the yards of the Cliffwood Brick Company, and that from Cliffwood Beach. The former lot was as a whole badly pyritized and of no value from a structural standpoint; while the latter was often perfectly preserved, revealing all the details of its structure under microscopical examination. The greater part was found to belong to the genera *Cupressinoxylon*, *Araucarioxylon*, and *Brachyoxylon*, and will be described later. There were also specimens representing three types of *Pityoxylon;* the characteristics and affinities of which it is the purpose of this paper to discuss.

Pinus protoscleropitys n. sp.

It will be appropriate to begin with the one which most closely resembles modern forms. Figures *a*, *c*, and *e*, Plate **1**, reveal the general features of the lignite in question. It will be noticed that the tracheides are small and thick walled. The summer elements are few in number, but limit a well marked annual ring, as shown in the lower part of Figure *a*. Resin ducts such as are characteristic of all *Pityoxyla* occur in two planes. Figure *a* includes two vertical canals, and to the right a horizontal one. It is apparent that both are completely filled with tyloses,— a condition more clearly seen in Figures *c* and *e*. Surrounding each, there is a jacket of epithelial parenchyma. The cells composing this jacket are thin walled, heavily pitted, and in general devoid of contents. Figure *d*, on the other hand, illustrates a case where they are filled with a dark, resinous substance. Figure *b* gives the topography next the pith,— at a lower magnification. It will be noticed that, as in the hard pines, there is a double series of resin ducts in the first annual ring, but the inner series in this case lies in the primary wood. This condition is at variance with that of hard pines where both are in the secondary wood. The presence of resin canals in the primary wood is not unparalleled in the coniferous series, — they occur in the primary wood of the root of all the Abietineae (1), of the cone axis of *Sequoia gigantea* (2), and of certain members of the Araucarineae (3). True medullary canals, such as have been described in *Pinus succinifera* (Goepp) Conw. (4) seem to be entirely absent. In the succeeding annual rings, the vertical canals are smaller and less frequent. With the horizontal canals they form a freely anastomosing system (Figure *c*).

Figure *b* also shows the character of the pith. Scattered among the thin walled parenchyma cells there are clusters of very thick-walled sclerified elements. Such a cluster occurs in the upper part of Figure *b*. These show a tendency to be in more or less definite horizontal bands, but do not form true diaphragms.

The medullary rays are of two sorts,— linear and fusiform. The latter are frequent, and always embrace a resin canal (at the left of Figure *e*), a leaf trace (Figure *f*) or both. The linear rays are much more abundant, as may be seen in any of the illustrations. They are usually low, and as in living pines, destitute of resinous content. The walls are thin and heavily pitted. The lateral pits, as shown in Figures *a* and *b*, Plate 2, vary from one to two to each cross field; they are small, the mouths lenticular on the wall of the ray and circular on that of the tracheide. Not infrequently there are indications of fusion where two small pits unite to form one of medium size. At the extreme lower right of Figure *a*, and in the upper part of Figure *b*, such phenomena are represented. The resulting pore is rarely as large as in modern pines such as *Pinus strobus*, though occasionally a single pit occupies almost the entire cross field, as in the lower left of Figure *b*. Both horizontal and end walls are also heavily pitted.

In association with this parenchyma, there are longer and lower cells, always devoid of contents, with bordered pits on lateral, horizontal and end walls. That these are ray tracheides, such sections as are photographed for Figures *c* and *d*, prove beyond question. They may occur only on one margin of a ray, or on both, as in Figure *c*. Rarely they are interspersed, with parenchyma above and below. Projecting in from the horizontal walls, there are well marked teeth. These may be seen in the lower ray tracheide of Figure *c*, the upper one of Figure *d* and better in Figure *e*. These teeth are doubtless analogous to similar appearances in hard pines, though less developed than is usually the case in the latter. Aside from our specimen, there are but two instances where ray tracheides have been described in a fossil,— *Pinus scituatensiformis*, Bailey (5) and *P. succinifera* (Goepp.) Conw. (4). In the former, the walls seem to be smooth like those of living soft pines, while in the latter, Conwentz figures just such a sculptured appearance as is presented by the lignite under consideration.

The pitting of the tracheides is entirely confined to the radial wall. Owing to the small size of the elements, the pores are usually uniseriate. They are normally circular in outline and scattered; rarely toward the end of a tracheide, they become closely approximated and flattened by mutual contact. Figure *d*, Plate **1**, represents a typical condition. In the larger tracheides of the spring wood, the pores are often diseriate. In such instances they are always opposite and separated by well marked "bars of Sanio." In the living condition these bars are formed by the thickening of the cellulose middle lamella, which in the process of fossilization, rots away, leaving an empty space. Consequently the bar appears as a white line. A particularly favorable region is shown in Figure *f*, Plate **2**. Were anything more needed to demonstrate the Abietineous affinities of our lignite, these would suffice, since as shown by Miss Gerry (6) these bars of Sanio are invariably absent in woods of Araucarian affinities.

The short shoots in this fossil are much larger than those in most living pines, though never showing annual rings as in *Ginkgo*, Figure *c*, Plate **4**, represents one of these organs. On careful examination it may be seen that there is a single row of resin ducts in the wood, and that the medulla contains sclerotic nests similar to those of the main axis. This section was cut at some distance from the pith. Figure *d*, Plate **4**, shows, at a lower magnification, a section cut considerably nearer the centre. In the upper part of the photograph the short shoot may be seen; toward the lower limit, there is a dark spot. Figure *f*, Plate **1**, represents this spot at a much higher magnification, and demonstrates its foliar nature. Examination of serial sections has shown that at its departure from the medulla, each brachyblast is subtended by an axillating leaf trace, which dies out after a few years, leaving an apparently unaxillated short shoot. A similar condition has been described by Dr. Jeffrey in the case of *Woodworthia*, an Araucarian from the Triassic (7): in *Woodworthia*, however, these short shoots often branch, which is never the case in this *Pityoxylon*. The short shoot of *Ginkgo* is always axillated, in this case by a double leaf trace; in *Larix* also the short shoots are axillary structures. In the majority of living pines, however, such is not the case. That the primitive condition was for the brachyblast to be subtended by a leaf trace, is further indicated by the occasional presence of an axillating strand in the seedling of certain living pines, — e. g. *Pinus strobus*, and in the mature wood of certain others,— e. g. *Pinus Jeffreyi*. The character of the short shoot thus presents an interesting example of seedling recapitulation. The leaf traces of this *Pityoxylon* are not confined to an axillating position, but are quite numerous near the pith. Their presence would indicate that the leaves of this conifer were of two sorts,— those borne directly on the main axis as in the seedling of living pines, and those on short shoots. Such a condition has been figured by Fontaine (8) in *Leptostrobus*, Heer. The foliar strands are jacketed by parenchyma, the whole forming a fusiform ray (Figure *f*). Not infrequently a resin duct accompanies them in their outward journey,— a condition comparable to that of the vegetative leaves of some of the Abietineae, and of the sporophyll traces of some of the Araucarineae (3).

Having described the salient features in the anatomy of this specimen, it remains to consider its affinities. The presence of resin canals in two planes relegates it at once to the genus *Pityoxylon*, Kraus, and the short shoots narrow its possibilities to *Larix* and *Pinus*. There are a number of reasons for excluding the former,— dentate ray tracheides, thin-walled ray parenchyma, with fusion pits, abundant tylosed resin canals,— none of which occur in the wood of the larch. Further *Larix* has wood parenchyma at the end of the year's growth and tangential pits,— both of which are absent here. It seems clear therefore that our lignite belongs to the genus *Pinus*. Pines may be divided into two great groups,— hard and soft. Aside from certain external criteria,— for the most part unreliable,— the two groups may be differentiated by the following characters. Hard pines have sculptured ray tracheides, two or more rows of resin canals in the first annual ring, sclerified pith (except most of the two-needled varieties) and lack tangential pitting (except in the first few year's growth and the cone axis). Soft pines on the other hand, have smooth walled ray tracheides, a single row of resin canals in the first annual ring, tangential pitting, and lack stone cells in the pith. On all four of these features, our lignite belongs with the hard pines, and since it is the earliest known completely differentiated hard pine, we propose for it the name of *Pinus protoscleropitys*. In using the generic name *Pinus* rather than *Pityoxylon*, we are following the example set by Conwentz and Bailey, since the specimen in question cannot be separated anatomically from living pines.

It is of interest to compare this type with other fossil pines. The only ones described up to now with ray tracheides are *Pinus scituatensiformis*, Bailey (5) and *P. succinifera*, Conw. (4). First let us consider the former, since it is of the same geological age as our specimen. Both have a sclerified pith, large short shoots, and tyloses in the resin canals. *P. situatensiformis* differs from the lignite described in this article in numerous features,— the ray tracheides are smooth walled, the rays and abundant epithelium of the vertical canals are highly resinous, the lateral pits of the rays are small and invariably one per crossfield, the summer tracheides are pitted on their tangential walls, and the short shoot has no axillating leaf trace. While our specimen is a typical hard pine, that described by Mr. Bailey unites the characteristics of both groups,— it has the tangential pitting and smooth ray tracheides of a soft pine, with the sclerified pith of a hard. It seems to be a more generalized type, perhaps representing an ancestral condition before the two groups had become sharply separated.

With *P. succinifera* of the early Tertiary, our lignite has more in common. Both have sculptured ray tracheides in marginal and interspersed positions; thin-walled, non-resinous ray parenchyma; septate tracheides around the resin ducts, which are surrounded by thin-walled heavily pitted epithelium and filled with tyloses. On the other hand, as opposed to our specimen, *P. succinifera* has but a single row of resin ducts in the first annual ring; tangential pits; tyloses in the tracheides; ray cells with sometimes four small piciform pits to each crossfield, sometimes one large fusion pit; resin canals embedded in the pith, and no stone cells. Further, ray tracheides in *P. succinifera* do not occur normally in the first few years' growth, while in our form they are present in the first annual ring.

From this comparison of *P. protoscleropitys* with other similar *Pityoxyla* it is evident that the former represents a higher and more specialized type than either of the others. It has all the features of a living hard pine, while the others present different combinations of the features of both hard and soft. The occurrence of a completely differentiated hard pine as far back as the Middle Cretaceous substantiates the conclusion reached by Jeffrey (9) from a study of the leaves that the two groups had already become separate by the Middle Cretaceous. Zeiller's description of cones of both groups from the Jurassic renders it probable that the separation goes back to that epoch. An interesting corollary to the presence of such a modern type of wood in the Cretaceous is afforded by the modern character of the leaves of Upper Cretaceous pines described by Stopes and Kershaw (10).

These facts indicate that the Abietineae are a much older group geologically than is usually supposed. It is further evident that such forms as these must be the ancestors of living pines, and that such forms as *Pinus scituatensiformis* and *P. succinifera*,— of the same or later geological age, yet less specialized,— are off the main line of development.

Before leaving this specimen, it is convenient to consider the light it throws on the origin of ray tracheides. Jeffrey and Chrysler (11) concluded that ray tracheides were evolved during the early Tertiary, basing their conclusions on the following developmental and palaeobotanical evidence. Ray tracheides are absent from the cone axis of most modern pines, and poorly developed in the seedling; they are absent in *Pinites Ruffordi*, Seward (12) (Wealden), *Pityoxylon statensense* and *P. scituatense* (Middle Cretaceous) and do not appear for several years' growth in *Pinus succinifera* (Early Oligocene or Late Eocene). The discovery of ray tracheides in *P. scituatensiformis* (Middle Cretaceous) led Mr. Bailey to the conclusion that these structures came in during the Middle Cretaceous. In that species they do not appear at all in the first ten to fifteen years' growth and thereafter are but poorly developed. Their occurrence, though rare, in the first annual ring of *P. protoscleropitys* (Middle Cretaceous) and their abundance later, seems to indicate that they are a more ancient feature than has been assumed by any of the above cited investigators. It is probable that they were developed in the Lower Cretaceous if not in the Jurassic.

As regards the origin of ray tracheides, the final word remains to be said. There are two theories which have been advanced to explain the question. Thompson (13) has suggested that tracheary ray cells are derived from vertical tracheides, which by progressive shortening, have taken on a horizontal position. Stages in such a process he found in *Pinus resinosa* and *P. strobus*. As Bailey pointed out there are two objections to this theory,— if these phenomena are recapitualtionary or reversionary, in the first place, why are they more evident in these highly specialized varieties than in such primitive ones as the Nut and Foxtail pines? And, in the second, why are they completely absent in fossil forms? Mr. Bailey was unable to find any trace of such an origin in *P. scituatensiformis*, and I have been unable to find any in *P. protoscleropitys*. Since there is no confirmatory evidence in the case of the primitive living forms, or the two oldest known fossil forms, it seems improbable that Mr. Thompson's interpretation is correct.

On the other hand, Penhallow (14) has suggested that they have been formed from ray parenchyma by a thickening of the cell wall. As Bailey points out, the evolutionary sequence has been from thick- to thin-walled parenchyma, not *vice versa*,— a consideration which immediately invalidates this hypothesis.

Pityoxylon foliosum n. sp.

The next specimen to be considered is much less like modern forms than that just described. Figures *a* and *b*, Plate **3**, represent at different magnifications cross sections of the wood. As may be seen in Figure *b*, the annual rings are very broad and well marked,— the first occurring near the lower limit of the field. Another appears a little below the centre of Figure *a*. There are many concentric arcs extending half way or more around the stem, caused by some external pressure in the process of fossilization. One such is shown in the upper part of Figure *a*. That it is not a true growth ring is proven by the fact that it does not completely encircle the medulla, and that the tracheides composing it are not pitted on the tangential wall,— an invariable characteristic of the summer tracheides of the lignite in question.

Resin ducts are very numerous, extending in two planes,— vertical and horizontal. The vertical canals are surrounded by clusters of highly resinous epithelial parenchyma. Not infrequently a single mass contains three or even four tangentially grouped canals, which as may be seen from longitudinal sections, intercommunicate. The epithelium is moderately thick-walled, and densely perforated by simple pits. The horizontal canals are also numerous, as shown by Figure *e*. Together with the vertical canals they form a freely anastomosing system of resin passages throughout the wood. Both horizontal and vertical canals, especially the latter, are almost invariably filled with thick-walled tyloses. At the extreme right of Figure *b* the proximity of the canals to the pith may with difficulty be ascertained. Figure *d*, a radial longitudinal section, shows the relation more clearly. In fact the canals are often so near to the medulla that in transverse section they appear to be embedded in it. A more careful examination however, reveals the presence of a jacket of metaxylem elements around the duct. This occurrence of canals in the primary wood is unknown in the main axis of living pines, but is similar to that of *Pinus protoscleropitys*.

Toward the right of Figure *b*, a vascular strand may be seen to pass off from the medulla; Figure *c* shows one of these leaf traces in cross section. Such leaf traces are quite abundant in this specimen, but in the limited amount of material available, there was a complete absence of short shoots,— a remarkable condition for an obvious *Pityoxylon.* The trace appears to be always single, like those of the fascicular leaves of all living pines, at the point of departure from the pith. Further, like those traces the xylem is entirely centrifugal. Not infrequently a resin canal accompanies the strand in its outward passage, rarely two.

The rays are of two sorts,— linear and fusiform. The latter are very numerous; they consist of parenchymatous elements embracing a resin duct, a foliar trace, or both. The character of the linear rays may be inferred from the photomicrographs. They are low and highly resinous; the walls are comparatively thin and heavily pitted. The lateral pits are usually one to each cross field, rarely two: they are piciform, with an elliptical opening on the side of the ray and a circular one on that of the tracheide. Unlike living pines, all the cells composing the ray are parenchymatous, although those on the margin are often quite different from the others, being irregular in shape and destitute of resinous content. Figure *c* shows several instances of this condition. At first sight they appear to be ray tracheides, but the unbordered character of the pits negatives that possibility.

The tracheides are uniformly small and thick-walled. The pits on the radial wall are uniseriate and scattered; in places, indications of the so called "bars of Sanio" could be distinguished, but as a rule the indifferent state of preservation obscures this feature. In the majority of cases, the pits are confined to the radial walls, but those tracheides laid down at the end of the year's growth, have pores also on the tangential wall. As is well known, this is characteristic of all the Abietineae except hard pines.

The characteristics of the pith are evident from Figures *b* and *d*. There are two sorts of elements,— thin-walled parenchyma and thick-walled sclerenchyma, the latter standing out as black masses in the photographs. They show a general tendency toward arrangement in horizontal bands, which are not, however, sufficiently localized to form diaphragms.

It remains now to consider the affinities of this lignite. The presence of resin ducts in both horizontal and vertical planes affiliate it with the genus *Pityoxylon,* Kraus. Like all previously described Cretaceous *Pityoxyla* (with the exception of *Pinus scituatensiformis* and possibly *Pinus Nathorsti* Conw. (15), about which however it is impossible to tell, owing to imperfect preservation), the rays are devoid of tracheides, and bear out the conclusion of Jeffrey (11) that the majority of pines of this horizon had not yet acquired them. The affinities of this specimen must, therefore, lie with one of the four Abietineous genera normally possessing resin canals,— *Pinus*, *Picea*, *Larix* and *Pseudotsuga*. Both the last two have well marked wood parenchyma at the end of the year's growth. Since this feature is lacking in our fossil, it cannot be related to them. Between *Pinus* and *Picea*, there is little occasion for hesitation. The abundant, tylosed resin canals, complete absence of spiral thickenings, thin-walled parenchyma forming the epithelium of the canals and the cells of the medullary rays, clearly indicate its connection with *Pinus*. Another criterion for distinguishing the wood of *Pinus* and *Picea* is the wound reaction. As pointed out by Jeffrey (1) dense tangential series of resin canals are an invariable concomitant of injury in the case of *Picea*. One fragment of the lignite under consideration had a large wound cap. This was carefully examined, but no trace of traumatic canals found. That the capacity for such a reversion had been acquired as early as the Cretaceous, is proved by the presence of a traumatic series in *Pinites Ruffordi* (12) from the Wealden of England. If our fossil were related to *Picea*, as severe a wound as it had received would have unquestionably stimulated this characteristic reaction. Against this proposed affiliation with *Pinus*, may be brought forward the absence of short shoots. Fontaine (8) however, has described from the Potomac certain coniferous remains with both fascicular leaves on lateral and terminal short shoots, and also primary leaves, borne directly on the main axis, which are spirally arranged like those of seedling pines. In view of the small amount of available material, it is entirely possible that our specimen really possessed typical short shoot organs. Jeffrey (9) has suggested that *Prepinus* may belong with this *Leptostrobus* of Fontaine's. If such is the case, the lignite under discussion may be referred to *Prepinus*. Its characteristics are, indeed, extremely like those of the wood of the brachyblast of *P. statenensis*. Both have sclerified nests in the pith, resin canals in two planes, highly resinous rays with piciform lateral pitting and tangential pitting of the tracheides. On the other hand, there are certain important differences. Modern pines may be divided into the two classes,— hard and soft. Disregarding the differences in ray tracheides, the characteristics of the two are,— first, hard pines have a double, soft,— a single, leaf trace (in both, however the trace leaves the wood of the brachyblast as one and divides in the cortex); second, hard pines have two or more rows of resin canals in the first annual ring,— soft but one; third, hard pines (except most of the two needled varieties) have stone cells in the pith,— soft have none; fourth, hard pines have not, soft have, tangential pitting of the summer tracheides. Our lignite, then, is nearer the hard than the soft pines, having more than a single row of canals in the first annual ring, and stone cells, but it has also the tangential pitting of a soft pine. On the other hand, *Prepinus statenensis* has the sclerified pith of a hard pine, with the single leaf trace of a soft. Moreover, it has tangential pitting, and but one row of resin canals. It must be borne in mind, however, that we have to do with a brachyblast, and that the second row of resin ducts may be omitted for lack of space. This condition of affairs would be analogous to that of many living hard pines, where, as *Pinus rigida*, there may be but one row of canals in the short shoot, and sometimes none at all in the first annual ring of the seedling. The fact that the leaf trace of *Prepinus statenensis* is mesarch, whereas that of our specimen is endark, need not militate against this suggested relationship. In the first place, we have a record of only the fascicular leaves of *Prepinus*, and of only the primary leaves of this lignite: — there are no grounds for assuming that they must have been alike in this respect. In the second place, it is entirely possible that the strand, though endark in the wood, might acquire mesarch structure in the cortex, or even in the blade of the leaf. An analogous condition is true in the case of the Cycads. Any connection between our specimen and *Prepinus viticetensis* (16) is less likely, because even though the latter has two rows of resin canals, it lacks the highly characteristic medullary stone cells. The identification of this lignite with the wood of *Prepinus*, or with either *Leptostrobus*, Heer or the somewhat similar *Pinites Solmsi* (17) of Seward,— both of which are known only superficially,— must remain very problematical.

Its relation to other *Pityoxyla* should next be considered. As regards other lignites from Cliffwood, it differs from *Pinus protoscleropitys* in the absence of ray tracheides; and from *Pityoxylon hollicki* Knowlton (18) in that the latter has 'punctations contiguous,' 'thick-walled ray cells,' and often two series of pits. Thickness is, of course a relative term, and more material of our specimen might show diseriate pitting. However, Knowlton states that the structure is 'too obscure for accurate description,' so further comparison is impossible. The lignite in question differs from *Pinoxylon dacotense*, Knowlton, in that the latter has only vertical canals, and from *Pityoxylon statenense* in that the latter has no stone cells in the pith. Further, it cannot be identified with *Pinites Ruffordi* which has tyloses in the tracheides and teeth in the ray parenchyma. One character absolutely puts *P. Ruffordi* out of the question,— it contains traumatic resin ducts,— which as pointed out above, were not present here. It cannot be placed with *Pinus Nathorsti* (15), which had thick-walled, unpitted parenchyma around the resin canals and lacked tangential pitting, or with *Protopiceoxylon antiquius* Gothan (19). That species had thick-walled ray parenchyma, thick-walled epithelium around the resin canals, three to four pits to each crossfield on the lateral walls of the rays, and lacked tangential pitting,— features diametrically opposed to those of our fossil. Accordingly, our specimen cannot be identified with any previously described. In view of the fact that the leaves were borne on the main axis exclusively, rather than on short shoots, it may appropriately be called *Pityoxylon foliosum.* The only other forms with such leaves are *Pinus protosclcropitys* and *Prepinus.* With the former it cannot be identified because that form had such abundant short shoots that it would be impossible to miss them, and further it had ray tracheides. As suggested above, it may very probably be the wood of *Prepinus.*

Pityoxylon anomalum n. sp.

The third type of Pityoxylon differs from either of those previously described, though similar to *P. foliosum.* Figure *f*, Plate **3**, shows the general topography of the stem. The annual rings are narrow and indistinct. Resin ducts are present, extending in two planes, but as is evident from a comparison of Figures *b* and *e*, Plate **3**,— equally enlarged — they are much less frequent in this specimen than in the former. A further difference is that there is but one row in the first annual ring; this row occurs in the primary wood. Figure *a*, Plate **4**, shows the character of the ducts. They are surrounded by a large mass of epithelium, which is completely filled with resin. This feature is best brought out in the longitudinal sections (Figures *b*, *f*, and *g*). The cells of this jacket are fairly thin-walled, and very heavily pitted, which doubtless accounts for the abundant tyloses, which are very thick-walled. As a rule there is but a single canal in each cluster of parenchyma,— rarely there are two.

The tracheides are badly collapsed, and the lumen usually completely obliterated. At times, however, in the better preserved regions, the characteristics of the pitting may be made out. In the lower part of Figure *g*, for example, a single row of pores may be observed. This is universally the case,— the tracheides being too narrow to accommodate a double series; in no case was the preservation sufficiently good to make out the bars of Sanio. Tangential pitting also is present, rather infrequently, on the face of the summer wood.

The rays are of two sorts,— linear and fusiform. Their highly resinous condition obscures the pitting, which in favorable localities is seen to be piciform. The pores are one to each crossfield, circular on the wall of the tracheide, and elliptical on that of the ray. In no case was there evidence of pit fusion.

The section photographed for Figure *f*, Plate **3**, was cut at the region of the exit of a brachyblast. Figure *e*, Plate **4**, shows its structure in cross section. The enlargement is the same as that of Figure *c*, which represents *Pinus protoscleropitys*. In the case of both, the short shoots are much larger than those of living pines, and in the medulla of each, there are aggregations of sclerified tissue similar to that of the main axis.

The affinities of this specimen are rather difficult to determine. The presence of short shoots and the absence of wood parenchyma relegate it definitely to *Pinus*. Further it is impossible to go, for it has the characteristics of neither a hard nor a soft pine exclusively,— the presence of tangential pitting and single row of resin canals excludes the former, and the presence of stone cells excludes the latter. As regards other fossil forms, its affinities are equally indefinite. It lacks the ray tracheides of *Pinus scituatensiformis*, *P. succinifera* or *P. protoscleropitys*, and the tracheary tyloses and toothed ray parenchyma of *Pinites Ruffordi;* unlike *Protopiceoxylon antiquius* and *Pinus Nathorsti*, there is tangential pitting. On the other hand, *Pityoxylon statenense* has no sclerenchyma in the pith, and *P. foliosum* has abundant leaf traces. Granted that *Prepinus* really belongs with *Leptostrobus* this cannot be the wood of *Prepinus*, because it has no primary leaves. In other characteristics, its general resemblance to *Prepinus* is quite striking. The woods look alike,— both have stone cells in the pith, resinous rays, piciform ray pitting,— further both have numerous small crystals,— a feature of neither of the other specimens.

In view of these apparent points of difference from other forms, it is suggested that this fossil be called *Pityoxylon anomalum*.

SUMMARY.

1. The Pityoxyla of Cliffwood, New Jersey, include the following previously undescribed varieties:

(1) *Pinus protosclcropitys,*— probably the earliest form with all the characters of a modern hard pine, yet retaining certain ancestral features, as the association of primary and fascicular leaves, the latter borne on brachyblasts subtended by a foliar trace.

(2) *Pityoxylon foliosum,*— possibly the wood of *Prepinus,* with all its leaves borne directly on the main axis, and presenting mingled characteristics now confined exclusively to either hard or soft pines.

(3) *Pityoxylon anomalum,*— with ligneous features extremely like those of *Prepinus,* yet with all its leaves borne on short shoots.

2. The absence of tangential pitting in the first described *Pityoxylon*, and its presence in the other two, confirm the conclusions of Jeffrey and Chrysler that tangential pitting is a primitive feature now lost in the more highly specialized hard pines.

3. The absence of evidence confirming thé origin of ray tracheides from vertical tracheides of the wood, renders it unlikely that this hypothesis is correct.

4. The occurrence of a completely differentiated hard pine as far back as the Middle Cretaceous is an argument for the great geological antiquity of the pines as such.

In conclusion, I wish to thank Professor E. C. Jeffrey for all the material used in this investigation, for an opportunity to examine sections of *Prepinus,* and for his helpful advice throughout the course of the work. To Professor I. W. Bailey, I am indebted for opportunity to study sections of *Pinus scituatensiformis,* and to Mr. E. W. Sinnott for sections of various living pines.

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DESCRIPTION OF PLATES.

PLATE 1.

a. *Pinus protoscleropitys*, transverse section of wood. × 40.
b. Same, transverse section near pith. × 15.
c. Same, radial section. × 40.
d. Same. × 80.
e. Same, tangential section of wood. × 40.
f. Same, showing leaf trace. × 60.

PLATE 2.

a. Same, radial section, showing ray pitting. × 150.
b. Same. × 500.
c. Same, showing ray tracheides. × 150.
d. Same. × 500.
e. Same, showing teeth in tracheide. × 500.
f. Same, showing radial pitting of tracheide. × 600.

PLATE 3.

a. *Pityoxylon foliosum*, transverse section of wood. × 40.
b. Same, transverse section at pith. × 12.
c. Same, radial section of wood. × 40.
d. Same, radial section at pith. × 12.
e. Same, tangential section of wood. × 40.
f. *Pityoxylon anomalum*, transverse section at pith. × 12.

PLATE 4.

a. Same, transverse section of wood. × 40.
b. Same, tangential section of wood. × 40.
c. *Pinus protoscleropitys*, tangential section including short shoot. × 15.
d. Same, including leaf trace and short shoot, cut nearer pith. × 12.
e. *Pityoxylon anomalum*, tangential section, including short shoot. × 15.
f. Same, radial section of wood. × 40.
g. Same. × 80.


a
b
c
d
e
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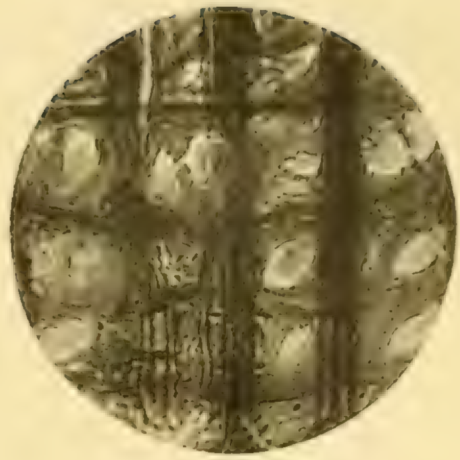
a



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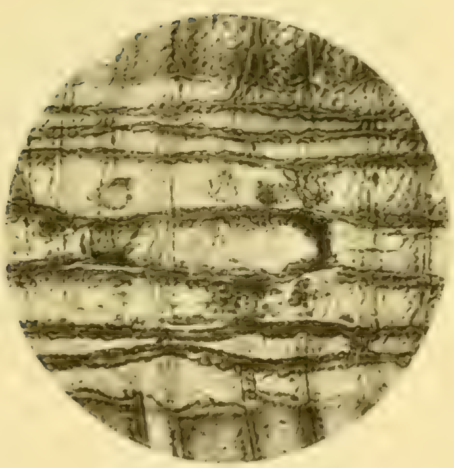

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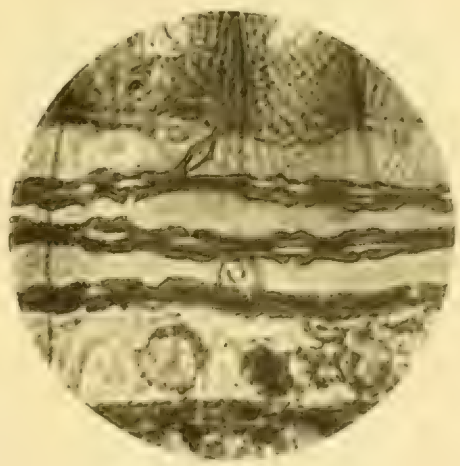

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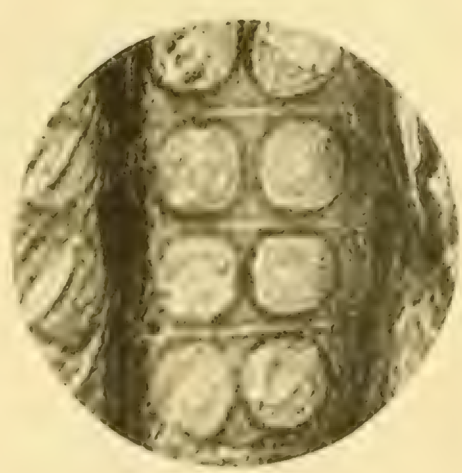

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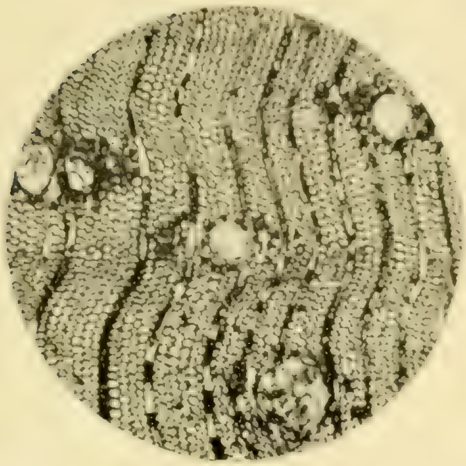

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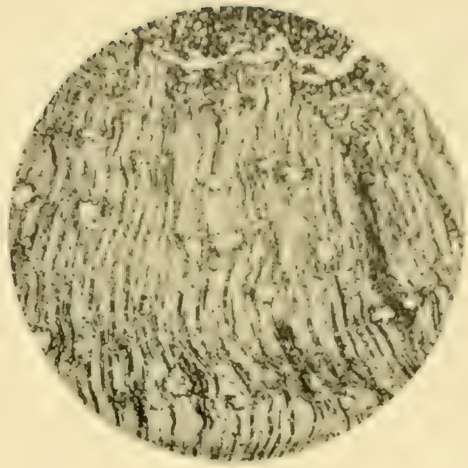

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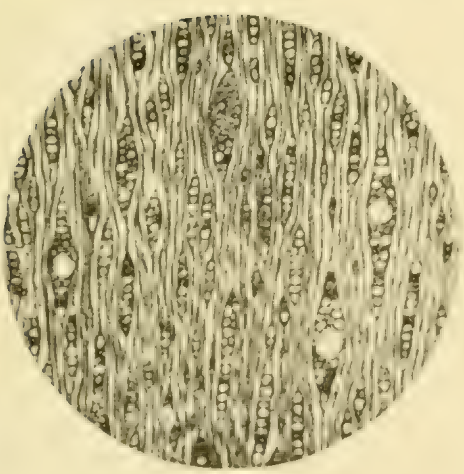

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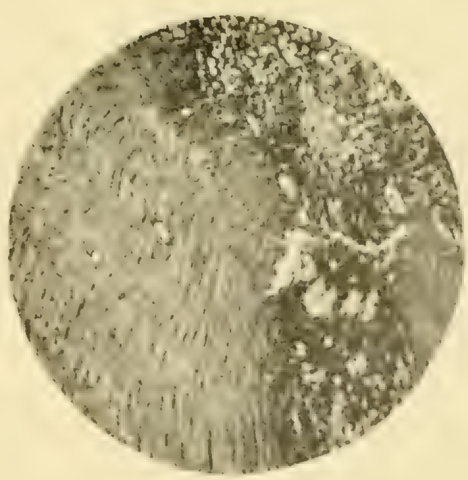

f


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ON THE SCALAR FUNCTIONS OF HYPER COMPLEX NUMBERS.

SECOND PAPER.

By Henry Taber.

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§ 1.

In this paper I shall denote by γ_{ijk}, for $i, j, k = 1, 2, \ldots m$, the constants of multiplication of a given non-nilpotent hyper complex number system $(e_1, e_2, \ldots e_m)$.[1] We then have

$$(1) \qquad e_i e_j = \sum_{k=1}^{m} \gamma_{ijk} e_k \qquad (i, j = 1, 2, \ldots m).$$

In These Proceedings, vol. 41 (1905), p. 59, I have shown that there are two functions of the coefficients of any number

$$(2) \qquad A = a_1 e_1 + a_2 e_2 + \ldots + a_m e_m$$

of the system $(e_1, e_2, \ldots e_m)$ constituting generalizations of the scalar function of quaternions, to which they reduce, becoming identical when $m = 4$, and, at the same time, the system (e_1, e_2, e_3, e_4) is equivalent to the system constituted by the four units of quaternions. These functions, in designation the *first* and *second scalar* of A, are defined as follows:

$$(3) \qquad S_1 A = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \gamma_{ijj},$$

$$(4) \qquad S_2 A = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \gamma_{jij},$$

and conform to *theorem I* given below. In this paper I shall employ these functions to establish a simple criterion for the existence of an

[1] A number $A = \sum_{i=1}^{m} a_i e_i$ of any hyper complex system $(e_1, e_2, \ldots e_m)$ is *idempotent* if $A^2 = A \neq 0$; A is *nilpotent*, if $A \neq 0$ but $A^p = 0$ for some positive integer p > 1. A system is *nilpotent*, if it contains no idempotent number; otherwise, *non-nilpotent*. Every number of a nilpotent system is nilpotent. See B. Peirce, Am. Journ. Maths., **4**, 113, (1881); cf. H. E. Hawkes, Trans. Am. Math. Soc., **3**, 321 (1902).

invariant nilpotent sub system of $(e_1, e_2, \ldots e_m)$, and a method of determining the maximum invariant nilpotent sub system, if any exist.[2] These results are embodied in *theorem II.*

Theorem I. *Let* γ_{ijk}, for $i, j, k = 1, 2, \ldots m$, *be the constants of multiplication of any given hyper complex number system* $(e_1, e_2, \ldots e_m)$. *Let*

$$A = a_1 e_1 + a_2 e_2 + \ldots + a_m e_m$$

be any number of the system; and let

$$S_1 A = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \gamma_{ijj},$$

$$S_2 A = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \gamma_{jij}.$$

Then both $S_1 A$ *and* $S_2 A$ *are invariant to any linear transformation of the system: that is, if*

$$e'_i = \tau_{i1} e_1 + \tau_{i2} e_2 + \ldots + \tau_{im} e_m \qquad (i = 1, 2, \ldots m),$$

the determinant of transformation not being zero, and if

$$e'_i e'_j = \sum_{k=1}^{m} \gamma'_{ijk} e'_k \qquad (i, j = 1, 2, \ldots m),$$

and

$$A = a_1 e_1 + a_2 e_2 + \ldots + a_m e_m = a'_1 e'_1 + a'_2 e'_2 + \ldots + a'_m e'_m,$$

then

$$S_1 A = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a'_i \gamma'_{ijj},$$

$$S_2 A = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} a'_i \gamma'_{jij}.$$

---

2 A sub system $B_1, B_2, \ldots B_p$ of any hyper complex number system $(e_1, e_2, \ldots e_m)$ is said to be *invariant* if the product in either order of each number of $(e_1, e_2, \ldots e_m)$ and each number of $(B_1, B_2, \ldots B_p)$ belongs to the sub system, for which the necessary and sufficient conditions are

$$e_i B_j = g'_{1ij} B_1 + g'_{2ij} B_2 + \ldots + g'_{pij} B_p,$$
$$B_j e_i = g''_{1ij} B_1 + g''_{2ij} B_2 + \ldots + g''_{pij} B_p$$
$$(i = 1, 2, \ldots m;\ j = 1, 2, \ldots p).$$

An invariant sub system $(B_1, B_2, \ldots B_p)$ is an *invariant nilpotent* sub system if its units by themselves constitute a nilpotent system; and in that case is a *maximum invariant nilpotent* sub system if it contains every invariant nilpotent sub system of $(e_1, e_2, \ldots e_m)$.

If ρ is any scalar, and

$$B = b_1 e_1 + b_2 e_2 + \ldots + b_m e_m$$

any second number of the system, we have

$$S_1 \rho A = \rho S_1 A, \qquad S_2 \rho A = \rho S_2 A,$$
$$S_1 (A \pm B) = S_1 A \pm S_1 B, \qquad S_2 (A \pm B) = S_2 A \pm S_2 B,$$
$$S_1 AB = S_1 BA, \qquad S_2 AB = S_2 BA.$$

If ϵ is a modulus of the system,

$$S_1 \epsilon = 1 = S_2 \epsilon.$$

If A is nilpotent,

$$S_1 A^p = 0, \qquad S_2 A^p = 0,$$

for every positive integer p; and conversely, if either

$$S_1 A^p = 0 \qquad (p = 1, 2, \ldots m)$$

or

$$S_2 A^p = 0 \qquad (p = 1, 2, \ldots m),$$

A is nilpotent. Moreover, A is nilpotent if

$$S_1 A e_1 = S_1 A e_2 = \ldots = S_1 A e_m = 0,$$

or

$$S_2 A e_1 = S_2 A e_2 = \ldots = S_2 A e_m = 0.$$

If A is idempotent, there are $m\, S_1 A > 0$ linearly independent numbers of the system satisfying the equation

$$AX = X,$$

in terms of which every number of the system satisfying this equation can be expressed linearly, also $m\, S_2 A > 0$ linearly independent numbers satisfying the equation

$$XA = X,$$

in terms of which every solution of this equation can be expressed linearly.[3]

Let

$$X = x_1 e_1 + x_2 e_2 + \ldots + x_m e_m, \tag{5}$$

3 See paper by the author cited above, pp. 61, 69, and 70, also Trans. Am. Math. Soc., **5**, 522, (1904).

and let the number system $(e_1, e_2 \ldots e_m)$ contain at least one number satisfying the system of equations

$$(6) \qquad S_1 X e_i = x_1 S_1 e_1 e_i + x_2 S_1 e_2 e_i + \ldots + x_m S_1 e_m e_i = 0 \qquad (i = 1, 2, \ldots m).$$

The resultant of this system being the determinant

$$(7) \qquad \Delta_1 \equiv \begin{vmatrix} S_1 e_1 e_1, & S_1 e_2 e_1, & \ldots & S_1 e_m e_1 \\ S_1 e_1 e_2, & S_1 e_2 e_2, & \ldots & S_1 e_m e_2 \\ \cdots & \cdots & \cdots & \cdots \\ S_1 e_1 e_m, & S_1 e_2 e_m, & \ldots & S_1 e_m e_m \end{vmatrix},$$

we then have $\Delta_1 = o$. Let $X = B$ be any solution of equations (6). Then, by *theorem I*, B is nilpotent. Moreover, for any number A of $(e_1, e_2, \ldots e_m)$, both BA and AB are also solutions of equations (6). For, for any number

$$Y = y_1 e_1 + y_2 e_2 + \ldots + y_m e_m$$

of $(e_1, e_2, \ldots e_m)$, we now have

$$S_1 B Y = y_1 S_1 B e_1 + y_2 S_1 B e_2 + \ldots + y_m S_1 B e_m = 0;$$

in particular,

$$S_1 (BA \cdot e_i) = S_1 (B \cdot A e_i) = 0,$$

$$S_1 (AB \cdot e_i) = S_1 (A \cdot B e_i) = S_1 (B e_i \cdot A) = S_1 (B \cdot e_i A) = 0 \qquad (i = 1, 2, \ldots m).$$

Since both BA and AB are solutions of equations (6), they are both nilpotent.

Further, since, for $1 \leqq i \leqq m$, $B e_i$ is nilpotent, it follows from *theorem I* that $S_2 B e_i = 0$, and thus any solution B of the system of equations (6) is also a solution of the system of equations

$$(8) \qquad S_2 X e_i = x_1 S_2 e_1 e_i + x_2 S_2 e_2 e_i + \ldots + x_m S_2 e_m e_i = 0 \qquad (i = 1, 2, \ldots m),$$

of which the resultant is

$$(9) \qquad \Delta_2 \equiv \begin{vmatrix} S_2 e_1 e_1, & S_2 e_2 e_1, & \ldots & S_2 e_m e_1 \\ S_2 e_1 e_2, & S_2 e_2 e_2, & \ldots & S_2 e_m e_2 \\ \cdots & \cdots & \cdots & \cdots \\ S_2 e_1 e_m, & S_2 e_2 e_m, & \ldots & S_2 e_m e_m \end{vmatrix}.$$

By *theorem I* every solution of equations (8) is nilpotent. Let B' be any solution of this system of equations. Precisely as above, we may show that B' is nilpotent, and that both $B'A$ and AB' are also solutions of these equations for any number A of the system $(e_1, e_2, \ldots e_m)$; and, therefore, both $B'A$ and AB' are nilpotent. Since, in particular, for $1 \leq i \leq m$, $B'e_i$ is nilpotent, it follows from *theorem I* that B' is a solution of the system of equations (6).

Let now the nullity [4] of the determinant Δ_1 be m', where $0 < m' < m$. There is then a set of just m' linearly independent numbers, $B_1, B_2 \ldots B_{m'}$ of the system $(e_1, e_2 \ldots e_m)$ satisfying equations (6); therefore, just m' linearly independent numbers satisfying equations (8), whence it follows that the nullity of Δ_2 is m'. For $1 \leqq j \leqq m'$, the product of B_j in either order with any number A of the system is a solution of equations (6) and, therefore, both B_jA and AB_j are expressible linearly in terms of $B_1, B_2 \ldots B_{m'}$; otherwise, there is a set of more than m' linearly independent solutions of equations (6) which is contrary to supposition. Moreover, since

$$\begin{aligned} S_1(\rho_1 B_1 + \rho_2 B_2 + \ldots + \rho_{m'} B_{m'})\, e_i \\ = \rho_1 S_1 B_1 e_i + \rho_2 S_1 B_2 e_i + \ldots + \rho_{m'} S_1 B_{m'} e_i = 0 \\ (i = 1, 2, \ldots m), \end{aligned}$$

every number linear in the B's is a solution of equations (6), and is, therefore, nilpotent. Whence it follows that $B_1, B_2 \ldots B_{m'}$ constitute an invariant nilpotent sub system of $(e_1, e_2 \ldots e_m)$.

Further, the sub system $(B_1, B_2 \ldots B_{m'})$ contains every invariant nilpotent sub system of $(e_1, e_2 \ldots e_m)$, and is therefore the maximum invariant nilpotent sub system of the latter. For, let $(C_1, C_2 \ldots C_p)$ be any invariant nilpotent sub system of $(e_1, e_2 \ldots e_m)$. Since every number of this sub system is nilpotent, in particular,

$$S_1 C_j = 0 \qquad (j = 1, 2, \ldots p).$$

Moreover, since

$$\begin{aligned} C_j e_i = g_{ji1} C_1 + g_{ji2} C_2 + \ldots + g_{jip} C_p \\ (i = 1, 2, \ldots m;\ j = 1, 2, \ldots p), \end{aligned}$$

we have

$$\begin{aligned} S_1 C_j e_i = g_{ji1} S_1 C_1 + g_{ji2} S_1 C_2 + \ldots + g_{jip} S_1 C_p = 0 \\ (i = 1, 2, \ldots m;\ j = 1, 2, \ldots p); \end{aligned}$$

4 The nullity of a matrix or determinant of order m is m' if every $(m' - 1)$th minor (minor of order $m - m' + 1$) is zero but not every m'th minor (minor of order $m - m^1$). Nullity of order m' is equivalent to rank (*Rang*) $m - m'$.

and thus each of the C's is a solution of equations (6). Therefore, each of the C's is inexpressible linearly in terms of $B_1, B_2 \ldots B_{m'}$.

Let

$$(10) \quad B_j = b_{j1} e_1 + b_{j2} e_2 + \ldots + b_{jm} e_m \qquad (j = 1, 2, \ldots m').$$

We may take the b's to be rational functions with respect to the domain $R(1)$ of the constituents of Δ_1 (or of Δ_2) which are integral quadratic functions, rational with respect to R (1), of the constants of multiplication of the number system $(e_1, e_2 \ldots e_m)$. If this number system *belongs to* the domain R', that is, if its constants of multiplication lie in the domain R', the b's may be so chosen as to lie in this domain. We may take the B's as m' new units of the number system. Thus let

$$(11) \qquad e'_{m-m'+j} = B_j \qquad (j = 1, 2, \ldots m'),$$

and let $e'_1, e'_2 \ldots e'_{m-m'}$ be any $m-m'$ numbers of $(e_1, e_2 \ldots e_m)$ which constitute with the B's a set of m linearly independent numbers. By what has just been said the coefficients of the transformation

$$(12) \qquad e'_i = \tau_{i1} e_1 + \tau_{i2} e_2 + \ldots + \tau_{im} e_m \quad (i = 1, 2, \ldots m)$$

of the number system can be taken rational in any domain to which the number system belongs.

If the number system is transformed by the preceding substitution (12), and if we put

$$(13) \qquad \Delta'_1 = \begin{vmatrix} S_1 e'_i e'_j \\ (i, j = 1, 2, \ldots m) \end{vmatrix},$$

then, since

$$S_1 e'_i e'_j = \sum_{h=1}^{m} \sum_{k=1}^{m} \tau_{ih} \tau_{jk} S_1 e_h e_k \qquad (i, j = 1, 2, \ldots m)$$

we have

$$(14) \qquad \Delta'_1 = T^2 \Delta_1,$$

where T is the determinant of the substitution. Similarly, if

$$(15) \qquad \Delta'_2 = \begin{vmatrix} S_2 e'_i e'_j \\ (i, j = 1, 2, \ldots m) \end{vmatrix},$$

we have

$$(16) \qquad \Delta'_2 = T^2 \Delta_2.$$

Therefore, the equations $\Delta_1 = 0$, $\Delta_2 = 0$ are invariant to any transformation of the units of the system.

Let now $\Delta_1 \neq 0$, in which case $\Delta_2 \neq 0$, and there is no number of the system satisfying equations (6), or equations (8); and, therefore, the system contains no invariant nilpotent sub system. In this case, therefore, if

$$S_1 A e_i = S_1 B e_i \qquad (i = 1, 2, \ldots m),$$

we have $A = B$; otherwise, $A - B \neq 0$ is a solution of equations (6). Similarly, if

$$S_2 A e_i = S_2 B e_i \qquad (i = 1, 2, \ldots m),$$

then $A = B$.

We have now the following theorem.

Theorem II. *Let $(e_1, e_2 \ldots e_m)$ be any non-nilpotent hyper complex number system; let*

$$X = x_1 e_1 + x_2 e_2 + \ldots + x_m e_m,$$

and let

$$\Delta_1 \equiv \left| \begin{matrix} S_1 e_i e_j \\ (i, j = 1, 2, \ldots m) \end{matrix} \right|, \quad \Delta_2 \equiv \left| \begin{matrix} S_2 e_i e_j \\ (i, j = 1, 2, \ldots m) \end{matrix} \right|$$

be the resultants, respectively, of the two systems of equations

$$(\alpha) \qquad S_1 X e_i = x_1 S_1 e_1 e_i + x_2 S_1 e_2 e_i + \ldots + x_m S_1 e_m e_i = 0 \qquad (i = 1, 2, \ldots m),$$

and

$$(\beta) \qquad S_2 X e_i = x_1 S_2 e_1 e_i + x_2 S_2 e_2 e_i + \ldots + x_m S_2 e_m e_i = 0 \qquad (i = 1, 2, \ldots m).$$

Then, if the number system is transformed by the substitution

$$e'_i = \tau_{i1} e_1 + \tau_{i2} e_2 + \ldots + \tau_{im} e_m \qquad (i = 1, 2, \ldots m),$$

and if

$$\Delta'_1 = \left| \begin{matrix} S_1 e'_i e'_j \\ (i, j = 1, 2, \ldots m) \end{matrix} \right|, \quad \Delta'_2 = \left| \begin{matrix} S_2 e'_i e'_j \\ (i, j = 1, 2, \ldots m) \end{matrix} \right|,$$

we have

$$\Delta'_2 = T^2 \Delta_1, \qquad \Delta'_2 = T^2 \Delta_2,$$

where T is the determinant of the substitution. Further, the condition, necessary and sufficient, that the number system shall contain no invariant nilpotent sub system is that $\Delta_1 \neq 0$, or $\Delta_2 \neq 0$. In this case, if either

$$S_1 A e_i = S_1 B e_i \qquad (i = 1, 2, \ldots m)$$

or

$$S_2 A e_i = S_2 B e_i \qquad (i = 1, 2, \ldots m),$$

we have $A = B$. If $\Delta_1 = 0$, then $\Delta_2 = 0$, and conversely; moreover, the nullity of Δ_1 is equal to the nullity of Δ_2. Every number of the system satisfying equations (α) is a solution of equations (β), and conversely. If B is any solution of equations (α) (or of equations (β)), then, for any number A of the system $(e_1, e_2 \ldots e_m)$, both $B A$ and $A B$ are solutions of these equations. If the nullity of Δ_1 is m', there is a set of just m' linearly independent solutions of equations (α) (or equations (β)); and any such set of m' numbers of $(e_1, e_2, \ldots e_m)$ constitute an invariant nilpotent sub system containing every invariant nilpotent sub system of $(e_1, e_2, \ldots e_m)$.

Let the system $(e_1, e_2, \ldots e_m)$ contain a nilpotent sub system $(C_1, C_2, \ldots C_p)$ such that

$$C_j e_i = \sum_{h=1}^{p} g_{ijh} C_h \qquad (i = 1, 2, \ldots m;\ j = 1, 2 \ldots p).$$

For $1 \leqq j \leqq p$, we then have, by *theorem I*,

$$S_1 C_j e_i = \sum_{h=1}^{p} g_{ijh} S_1 C_h = 0 \qquad (i = 1, 2, \ldots m);$$

therefore, $\Delta_1 = 0$, and thus $(e_1, e_2, \ldots e_m)$ contains an invariant nilpotent sub system to which the sub system $(C_1, C_2, \ldots C_p)$ belongs. Similarly, we may show that, if the system $(e_1, e_2, \ldots e_m)$ contains a nilpotent sub system $(C_1, C_2, \ldots C_p)$ such that

$$e_i C_j = \sum_{h=1}^{p} g_{ijh} C_h \qquad (i = 1, 2 \ldots m;\ j = 1, 2 \ldots p),$$

it then contains an invariant nilpotent sub system which includes the sub system $(C_1, C_2, \ldots C_p)$.

If $(e_1, e_2, \ldots e_m)$ contains a sub system $(C_1, C_2, \ldots C_p)$ such that

$$S_1 C_1 = S_1 C_2 = \ldots = S_1 C_p = 0$$

or

$$S_2 C_1 = S_2 C_2 = \ldots = S_2 C_p = 0,$$

this sub system is nilpotent, since then, by *theorem I*, every number of the sub system is nilpotent. Thus, if

$$C = g_1 C_1 + g_2 C_2 + \ldots + g_p C_p$$

is any number of the sub system, we have

$$C^q = g_1^{(q)} C_1 + g_2^{(q)} C_2 + \dots + g_p^{(q)} C_p;$$

therefore,

$$S_1 C^q = g_1^{(q)} S_1 C_1 + g_2^{(q)} S_1 C_2 + \dots + g_p^{(q)} S_1 C_p = 0,$$

for any positive integer q.

§ 2.

For any given number

$$A = \sum_{i=1}^{m} a_i e_i$$

of the non-nilpotent system $(e_1, e_2, \dots e_m)$ there is a linear relation between $A, A^2, \dots A^{m+1}$; therefore, a smallest positive integer $\mu \leqq m + 1$ for which $A, A^2, \dots A^\mu$ are linearly related, and thus for which we have

$$(17) \qquad \Omega(A) \equiv A^\mu + p_1 A^{\mu-1} + \dots + p_{\mu-1} A = 0,$$

where the p's are functions of the a's. Let $\rho_1, \rho_2, \dots \rho_r$, respectively of multiplicity $\mu_1, \mu_2, \dots \mu_r$, be the distinct non-zero roots, if any, of $\Omega(\rho) = 0$; when we have

$$(18) \qquad \Omega(\rho) \equiv \rho^{k_0} (\rho - \rho_1)^{k_1} (\rho - \rho_2)^{k_2} \dots (\rho - \rho_r)^{k_\rho},$$

where $k_0 \geqq 1$. Further, let

$$(19) \qquad W(\rho) \equiv \rho(\rho - \rho_1)(\rho - \rho_2) \dots (\rho - \rho_r).$$

Let now

$$f(A) = \sum_{h=1}^{p} c_h A^h$$

be any polynomial in A. If $f(A) = 0$, then $p \geqq \mu$ and $f(\rho)$ contains $\Omega(\rho)$; otherwise, there is a linear relation between $A, A^2, \dots A^{\mu-1}$, which is contrary to supposition. Wherefore, if $f(A)$ is nilpotent, $f(\rho)$ contains $W(\rho)$. Conversely, if $f(\rho)$ contains $W(\rho)$, $f(A)$ is nilpotent; and, if $f(\rho)$ contain $\Omega(\rho)$, then $f(A) = 0$.

Let A be non-nilpotent. Corresponding respectively to the $r \geqq 1$ distinct non-zero roots of $\Omega(\rho) = 0$, are r linearly independent numbers $I_1, I_2, \dots I_r$, linear in powers of A, which are severally idempotent and mutually nilfactorial: thus we have

$$(20) \quad I_u I_u = I_u \neq 0, \quad I_u I_v = 0 \qquad (u, v = 1, 2, \dots r;\ v \neq u).$$

If, for $1 \leqq u \leqq r$,

(21) $$\phi_o^{(u)}(\rho) \equiv \left(\frac{(\rho - \rho_u)^{k_u} - (o - \rho_u)^{k_u}}{-(o - \rho_u)^{k_u}}\right)^{k_o},$$

$$\phi_v^{(u)}(\rho) \equiv \left(\frac{(\rho - \rho_u)^{k_u} - (\rho_v - \rho_u)^{k_u}}{-(\rho_v - \rho_u)^{k_u}}\right)^{k_v}$$
$$(v = 1, 2, \ldots u - 1, u + 1, \ldots r),$$

and

(22) $$f_u(\rho) \equiv \phi_o^{(u)}(\rho)\,\phi_1^{(u)}(\rho) \ldots \phi_{u-1}^{(u)}(\rho)\,\phi_{u+1}^{(u)}(\rho) \ldots \phi_r^{(u)}(\rho),$$

we may write

(23) $$I_u = f_u(A) \qquad (u = 1, 2, \ldots r).$$[5]

I shall denote by $\bar{r}$ the greatest value of r for any number A of the system. Then $\bar{r}$ is the greatest number of idempotent numbers, mutually nilfactorial, contained in the system $(e_1, e_2, \ldots e_m)$. For, if possible, let the system contain $p > \bar{r}$ numbers $K_1, K_2, \ldots K_p$ satisfying the conditions

$$K^2_u = K_u \neq 0, \qquad K_u K_v = 0$$
$$(u, v = 1, 2, \ldots p;\ v \neq u).$$

The K's are then linearly independent. If now

$$A = \lambda_1 K_1 + \lambda_2 K_2 + \ldots + \lambda_p K_p,$$

where the λ's are any p distinct scalars other than zero, the equation $\Omega(\rho) = 0$ has $p > \bar{r}$ distinct non-zero roots, which is contrary to supposition.

Let A be non-nilpotent and, for any positive integer p, let

(24) $$N^{(p)} \equiv \varpi_p(A) \equiv A^p - \sum_{u=1}^{r} \rho^p{}_u I_u;$$

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5 For then, in the first place, $f_u(\rho)$ contains ρ as a factor; therefore, $f_u(A)$ is linear in powers of A. Moreover, for $1 \leq u \leq r$, $f_u(\rho)$ does not contain $\Omega(\rho)$, whereas $(f_u(\rho))^2 - f_u(\rho)$ does contain $\Omega(\rho)$; and, therefore, $I_u = f_u(A) \neq 0$, $I_u^2 - I_u = 0$. Further, for any two distinct integers u and v from 1 to r, $f_u(\rho) f_v(\rho)$ contains $\Omega(\rho)$; and, therefore, $I_u I_v = 0$. By the aid of the above two equations, we may show that $I_1, I_2, \ldots I_r$ are linearly independent. Thus, if

$$J \equiv c_1 I_1 + c_2 I_2 + \ldots + c_r I_r = 0,$$

then, for $\quad 1 \leq u \leq r, \qquad c_u I_u = I_u J I_u = 0;$

and, therefore, $c_u = 0$.

in which case $N^{(p)}$ is nilpotent, since

$$\varpi_p(\rho) \equiv \rho^p - \sum_{u=1}^{r} \rho_u{}^p f_u(\rho)$$

contains $W(\rho)$: therefore, by *theorem I*,

$$(25) \qquad \begin{aligned} S_1 A^p &= \sum_{u=1}^{r} \rho_u{}^p S_1 I_u + S_1 N^{(p)} = \sum_{u=1}^{r} \rho_u{}^p S_1 I_u, \\ S_2 A^p &= \sum_{u=1}^{r} \rho_u{}^p S_2 I_u + S_2 N^{(p)} = \sum_{u=1}^{r} \rho_u{}^p S_2 I_u. \end{aligned}$$

If possible, let

$$S_1 A^p = S_1 A^{p+1} = \ldots = S_1 A^{p+r-1} = 0$$

for some positive integer p. By (25), we then have

$$\rho_1{}^{p+h} S_1 I_1 + \rho_2{}^{p+h} S_1 I_2 + \ldots + \rho_r{}^{p+h} S_1 I_r = 0 \qquad (h = 0, 1, 2, \ldots r - 1);$$

and since, by *theorem I*, neither $S_1 I_1$, $S_1 I_2$, ... nor $S_1 I_r$ is zero, it follows that

$$\begin{vmatrix} \rho_1{}^p, & \ldots & \rho^{p+r-1} \\ \ldots & \ldots & \ldots \\ \rho_r{}^p, & \ldots & \rho_r{}^{p+r-1} \end{vmatrix} = 0,$$

which is impossible, since by supposition the ρ's are distinct and other than zero. *A fortiore*, we cannot have

$$S_1 A^p = S_1 A^{p+1} = \ldots = S_1 A^{p+\bar{r}-1} = 0$$

for any positive integer p. Similarly, we may show that we cannot have

$$S_2 A^p = S_2 A^{p+1} = \ldots = S_2 A^{p+\bar{r}-1} = 0$$

for any positive integer p if A is non-nilpotent.

We have now the following theorem.

Theorem III. *Let $(e_1, e_2, \ldots e_m)$ be any given non-nilpotent number system; and let r be the maximum number of idempotent numbers, mutually nilfactorial contained in the system. Then, if for any number*

$$A = a_1 e_1 + a_2 e_2 + \ldots + a_m e_m$$

of the system, we have, for some positive integer p,

$$S_1 A^{p+h} = 0 \qquad (h = 0, 1, 2, \ldots r - 1),$$

or

$$S_2 A^{p+h} = 0 \qquad (h = 0, 1, 2, \ldots r - 1),$$

A is nilpotent. Conversely, if A is nilpotent, these equations are all satisfied for any positive integer p.[6]

With respect to the idempotent numbers $I_1, I_2, \ldots I_r$, linear in powers of any non-nilpotent number A, the number system may be *regularized* as follows. Let Γ_{oo} denote the aggregate of numbers

$$e_i - \sum_{u=1}^{r} I_u e_i - \sum_{u=1}^{r} e_i I_u + \sum_{u=1}^{r} \sum_{v=1}^{r} I_u e_i I_v$$

for $i = 1, 2, \ldots m$. For any assigned integer u from 1 to r, let Γ_{uo} and Γ_{ou} denote, respectively, the aggregates

$$I_u e_i - \sum_{v=1}^{r} I_u e_i I_v \quad \text{and} \quad e_i I_u - \sum_{v=1}^{r} I_v e_i I_u$$

for $i = 1, 2, \ldots m$; and, for any assigned pair of integers u, v from 1 to r, let Γ_{uv} denote the aggregate of numbers $I_u e_i I_v$ for $i = 1, 2, \ldots m$. Further, for u and v any two integers from o to r, let m_{uv} denote the greatest number of linearly independent numbers of the aggregate Γ_{uv}; and, if $m_{uv} \neq 0$, let J_{uhv}, for $h = 1, 2, \ldots m_{uv}$, denote any system of m_{uv} linearly independent numbers of Γ_{uv}. We then have, by (20),

(26) $\quad I_u J_{uhv} = J_{uhv} = J_{uhv} I_v, \qquad I_u J_{uh'o} = J_{uh'o} \qquad J_{oh''v} I_v = J_{oh''v}$
$(u, v = 1, 2, \ldots r;\ h = 1, 2, \ldots m_{uv};\ h' = 1, 2, \ldots m_{uo};\ h'' = 1, 2, \ldots m_{ov})$,

(27) $\quad I_{u'} J_{uhv} = 0 = J_{uhv} I_{v'}$
$(u, v = 0, 1, 2, \ldots r;\ h = 1, 2, \ldots m_{uv};\ u'v' = 1, 2, \ldots r;\ u' \neq u, v' \neq v)$.

We may now show that the J's are linearly independent. For, if

$$J \equiv \sum_{p=1}^{r} \sum_{q=1}^{r} \sum_{h=1}^{m_{pq}} g_{phq} J_{phq} + \sum_{p=1}^{r} \sum_{h=1}^{m_{po}} g_{pho} J_{pho}$$
$$+ \sum_{p=1}^{r} \sum_{h=1}^{m_{op}} g_{ohp} J_{ohp} + \sum_{h=1}^{m_{oo}} g_{oho} J_{oho} = 0,$$

6 Cf. paper by the author in the Trans. Am. Math. Soc., **5**, 545, note.

then, for any pair of integers u, v from 1 to r,

$$\sum_{h=1}^{m_{uv}} g_{uhv} J_{uhv} = I_u J I_v = 0;$$

and, since by supposition J_{u1v}, J_{u2v}, etc., are linearly independent, we have

$$g_{uhv} = 0 \qquad (u, v = 1, 2, \dots r;\ h = 1, 2, \dots m_{uv}).$$

Whence it follows that

$$J \equiv \sum_{p=1}^{r} \sum_{h=1}^{m_{po}} g_{pho} J_{pho} + \sum_{p=1}^{r} \sum_{h=1}^{m_{op}} g_{ohp} J_{ohp} + \sum_{h=1}^{m_{oo}} g_{oho} J_{oho} = 0;$$

and, therefore, for $1 \leqq u \leqq r$,

$$\sum_{h=1}^{m_{uo}} g_{uho} J_{uho} = I_u J = 0, \qquad \sum_{h=1}^{m_{ou}} g_{ohu} J_{ohu} = J I_u = 0.$$

From these equations we derive

$$g_{uho} = 0 \qquad (u = 1, 2, \dots r;\ h = 1, 2, \dots m_{uo}),$$

$$g_{ohu} = 0 \qquad (u = 1, 2, \dots r;\ h = 1, 2, \dots m_{ou}).$$

Thus, ultimately, we have

$$J \equiv \sum_{h=1}^{m_{oo}} g_{oho} J_{oho} = 0;$$

whence follows

$$g_{oho} = 0 \qquad (h = 1, 2, \dots m_{oo}).$$

Since

$$(28) \quad e_i = \sum_{u=1}^{r} \sum_{v=1}^{r} I_u e_i I_v + \sum_{u=1}^{r} \Big(I_u e_i - \sum_{v=1}^{r} I_u e_i I_v\Big) + \sum_{u=1}^{r} \Big(e_i I_u - \sum_{v=1}^{r} I_v e_i I_u\Big) + \Big(e_i - \sum_{u=1}^{r} I_u e_i - \sum_{u=1}^{r} e_i I_u + \sum_{u=1}^{r} \sum_{v=1}^{r} I_u e_i I_v\Big) \qquad (i = 1, 2, \dots m),$$

it follows that each unit of $(e_1, e_2, \dots e_m)$, and thus that any number of this system, can be expressed linearly in terms of numbers in the $(r + 1)^2$ aggregates $\Gamma_{uv}\,(u, v = 0, 1, 2, \dots r)$, and, therefore, linearly in terms of the J's. Whence it follows that we may take the J's as new units, and the number system thus transformed is *regularized* with respect to the idempotent numbers $I_1, I_2, \ldots I_r$.[7]

Since, for $1 \leqq u \leqq r$, I_u belongs to Γ_{uu}, we may put

$$(29) \qquad I_u = J_{u m_{uu} u} \qquad (u = 1, 2, \ldots r).$$

If now B' is any number of the system $(e_1, e_2, \ldots e_m)$ satisfying the equation $I_u B' = B'$, then, by (26) and (27),

$$B' = \sum_{v=0}^{r} \sum_{h=1}^{m_{uv}} b'_{vh} J_{uhv};$$

similarly, if $B'' I_u = B''$, we have

$$B'' = \sum_{v=0}^{r} \sum_{h=1}^{m_{vu}} b''_{vh} J_{vhu}.$$

Therefore, by *theorem I*,

$$(30) \qquad \begin{aligned} m S_1 I_u &= \sum_{v=0}^{r} m_{uv}, \\ m S_2 I_u &= \sum_{v=0}^{r} m_{vu}. \qquad (u = 1, 2, \ldots r). \end{aligned}$$

Let (u, v), for u, v any two integers from 0 to r denote a number of the aggregate Γ_{uv}. From (26) and (27), it then follows that the non-vanishing products of numbers in the several aggregates are given by the following equations:

$$(31) \qquad (u, v)\ (v, w) = (u, w) \qquad (u, v, w = 0, 1, 2, \ldots r);$$

and we further have

$$(32) \qquad (u, v)\ (v', w) = 0 \qquad (u, v, v', w = 0, 1, 2, \ldots r;\ v' \neq v).$$[8]

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7 When the number system is thus transformed each of the new units is in one or other of Peirce's four "groups" or aggregates with respect to each of the r idempotent numbers $I_1, I_2, \ldots I_r$. Thus, if u is any integer from 1 to r and v, w any two integers from 0 to r other than u, then the units J_{uh_1u} $(1 \leqq h_1 \leqq m_{uu})$, J_{uh_2v} $(1 \leqq h_2 \leqq m_{uv})$, J_{vh_3u} $(1 \leqq h_3 \leqq m_{vu})$, and $J_{vh\,w}$ $(1 \leqq h_4 \leqq m)_{vw}$ are respectively in the first, second, third, and fourth groups with respect to I_u. See B. Peirce, loc. cit., p. 109.

We have now

$$m = \sum_{u=0}^{r} \sum_{v=0}^{r} m_{uv}.$$

8 Cf. B. Peirce, loc. cit., p. 111.

Therefore, if in the square array,

$$\begin{array}{ccccc} \Gamma_{11}, & \Gamma_{12}, & \ldots & \Gamma_{1r}, & \Gamma_{10} \\ \Gamma_{21}, & \Gamma_{22}, & \ldots & \Gamma_{2r}, & \Gamma_{20} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Gamma_{r1}, & \Gamma_{r2}, & \ldots & \Gamma_{rr}, & \Gamma_{r0} \\ \Gamma_{01}, & \Gamma_{02}, & \ldots & \Gamma_{0r}, & \Gamma_{00} \end{array}$$

we strike out any p rows or any p columns, the units of the aggregates in the resulting array constitute a sub system of $(e_1, e_2, \ldots e_m)$. In particular, for $0 \leqq u \leqq r$, the units of Γ_{uu} constitute a sub system. Since, by (32), (u, v) is nilpotent if $u \neq v$, we have

$$(33) \quad S_1(u, v) = 0, \quad S_2(u, v) = 0 \qquad (u, v = 0, 1, 2, \ldots r;\ v \neq u).$$

Let now A be so chosen that $r = \bar{r}$, where, as above, $\bar{r}$ is the greatest value of r for any number A of the system. The units of Γ_{00} then constitute a nilpotent sub system; and, since every number of a nilpotent system or sub system is nilpotent, we now have

$$(34) \qquad S_1(0, 0) = 0, \qquad S_2(0\ 0) = 0.$$

For, otherwise, if Γ_{00} contains an idempotent number I_0, we have

$$I_0 I_u = 0 = I_u I_0 \qquad (u = 1, 2, \ldots \bar{r})$$

by (27); and thus the number system $(e_1, e_2 \ldots e_m)$ contains $\bar{r} + 1$ idempotent numbers mutually nilfactorial, which is impossible, as shown above p. 19. Moreover, for $1 \leqq u \leqq \bar{r}$, there is now but one idempotent number in the aggregate Γ_{uu}. For, if possible, let Γ_{uu} contain a second idempotent number $I'_u = \sum_{h=1}^{m_{uu}} c_h J_{uhu}$ other than I_u, in which case we have $I'^2_u = I'_u$; let

$$I''_u = I_u - I'_u,$$

when we have, by (20) and (26),

$$I''^2_u = I^2_u - I_u I'_u - I'_u I_u + I'^2_u = I_u - 2I'_u + I'_u = I''_u,$$
$$I'_u I''_u = I'_u(I_u - I'_u) = 0 = (I_u - I'_u)I'_u = I''_u I'_u,[9]$$

and, by (32),

$$I_v I''_u = I_v(I_u - I'_u) = 0 = (I_u - I'_u)I_v = I''_u I_v \quad (v \neq u).$$

9 Cf. B. Peirce, loc. cit., p. 112.

Wherefore, there are then at least $r+1$ idempotent numbers mutually nilfactorial, namely, I'_u, I''_u and I_v for $v = 1, 2, \ldots u-1, u+1, \ldots \bar{r}$, which is impossible.

The number system when regularized with respect to $\bar{r}$ idempotent numbers, so that Γ_{00} contains no idempotent number, and each of the aggregates Γ_{11}, Γ_{22}, $\ldots$ $\Gamma_{\bar{r}\bar{r}}$ but a single idempotent number, is said to be *completely regularized.*

For $1 \leqq u \leqq \bar{r}$, we may now take the $m_{uu} - 1$ units other than I_u of the aggregate or system Γ_{uu} so that they shall all be nilpotent; in which case they constitute by themselves a nilpotent sub system, every number of which is, therefore, nilpotent.[10] I shall assume that in each of the aggregates Γ_{uu} $(u = 1, 2, \ldots \bar{r})$ the units have been so chosen.

Let

$$A = \sum_{u=0}^{\bar{r}} \sum_{v=0}^{\bar{r}} \sum_{h=1}^{m_{uv}} a_{uhv} J_{uhv}. \tag{35}$$

By equations (29), (30), (33), and (34), and by what has just been stated, we now have

$$\begin{aligned} S_1 A &= \sum_{u=1}^{\bar{r}} a_{um_{uu}u} S_1 I_u \\ &= \frac{1}{m} \sum_{u=1}^{\bar{r}} \sum_{v=0}^{\bar{r}} a_{um_{uu}u} m_{uv}, \end{aligned} \tag{36}$$

$$\begin{aligned} S_2 A &= \sum_{u=1}^{r} a_{um_{uu}u} S_2 I_u \\ &= \frac{1}{m} \sum_{u=1}^{\bar{r}} \sum_{v=0}^{\bar{r}} a_{um_{uu}u} m_{vu}. \end{aligned} \tag{37}$$

I shall say that the two idempotent units I_u and I_v $(1 \leqq u \leqq \bar{r}$, $1 \leqq v \leqq \bar{r}$, $v \neq u)$ are *connected* if there are two numbers $(u, v)'$ and $(v, u)'$ such that

$$S_1 (u, v)' (v, u)' \neq 0;$$

otherwise, *not connected.* If I_u and I_v are not connected, then

$$S_1 (u, v) (v, u) = 0$$

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10 This theorem is due to B. Peirce, loc. cit., p. 118. His proof is defective. The first proof, I believe, of the theorem without the aid of the theory of groups was given by me in the Transactions American Mathematical Society, **5**, p. 547, by employing the generalized scalar function.

for any two numbers (u, v) of Γ_{uv} and (v, u) of Γ_{vu}. Let $(u, v)'$, $(v, u)'$ be any two numbers of Γ_{uv}, Γ_{vu} respectively. Then

$$(u, v)' \, (v, u)' = \rho I_u - N_u$$

by (31), where N_u is linear in the nilpotent units of Γ_{uu} and is, therefore, either zero or nilpotent, and thus $N_u{}^{p+1} = 0$ for some positive integer p. Furthermore,

$$S_1(u, v)' \, (v, u)' = \rho S_1 I_u.^{11}$$

If now I_u and I_v are connected, then, for a proper choice of $(u, v)'$, $(v, u)'$, we have $S_1(u, v)' \, (v, u)' \neq 0$, in which case $\rho \neq 0$: therefore, we may put

$$(v, u)'' = \frac{1}{\rho^{p+1}} (v, u)' \, (\rho^p I_u + \rho^{p-1} N_u + \ldots + \rho N_u{}^{p-1} + N_u{}^p),$$

when we have

$$\begin{aligned}(u, v)'(v, u)'' &= \frac{1}{\rho^{p+1}} (\rho I_u - N_u) \, (\rho^p I_u + \rho^{p-1} N_u + \ldots + \rho N_u{}^{p-1} + N_u{}^p) \\ &= \frac{1}{\rho^{p+1}} (\rho^{p+1} I_u - N_u{}^{p+1}) = I_u;\end{aligned}$$

and since

$$\begin{aligned}[(v, u)'' \, (u, v)']^2 &= (v, u)''. \, (u, v)' \, (v, u)''. \, (u, v)' \\ &= (v, u)'' \, I_u (u, v)' = (v, u)'' \, (u, v)',\end{aligned}$$

it follows that

$$(v, u)'' \, (u, v)' = I_v,$$

otherwise, there is more than one idempotent unit in Γ_{vv}, which is contrary to supposition. Wherefore, *if I_u and I_v are connected, there are two numbers $(u, v)'$ and $(v, u)'$*, of Γ_{uv} and Γ_{vu} respectively, such that

$$(u, v)'(v, u)' = I_u, \qquad (v, u)'(u, v)' = I_v;$$

and conversely, since in this case

$$S_1(u, v)'(v, u)' = S_1 I_u \neq 0.$$

If I_u and I_v are connected, and I_v and I_w are also connected, then I_u and I_w are connected, where u, v, w are any three distinct integers from

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11 Further, $S_2 \, (u, v)' \, (v, u)' = \rho S_2 I_u$;
therefore, if $S_1 \, (u, v)' \, (v, u)' \neq 0$, then $S_2 \, (u, v)' \, (v, u)' \neq 0$, and conversely.

1 to r. For, in this case there are two pairs of numbers, namely, $(u, v)'$, $(v, u)'$ and $(v, w)'$, $(w, v)'$ such that

$$(u, v)'\,(v, u)' = I_u, \qquad (v, u)'\,(u, v)' = I_v,$$
$$(v, w)'\,(w, v)' = I_v, \qquad (w, v)'\,(v, w)' = I_w.$$

Therefore, if

$$(u, w)' = (u, v)'\,(v, w)', \qquad (w, u)' = (w, v)'\,(v, u)',$$

we have, by (26),

$$\begin{aligned}(u, w)'\,(w, u)' &= (u, v)'.\,(v, w)'\,(w, v)'.\,(v, u)' \\ &= (u, v)'\,I_v\,(v, u)' = (u, v)'\,(v, u)' = I_u,\end{aligned}$$

$$\begin{aligned}(w, u)'\,(u, w)' &= (w, v)'.\,(v, u)'\,(u, v)'.\,(v, w)' \\ &= (w, v)'\,I_v\,(v, w)' = (w, v)'\,(v, w)' = I_w.\end{aligned}$$

For u, v any two distinct integers from 1 to r, let I_u and I_v be connected. Thus let

$$(u, v)'\,(v, u)' = I_u, \qquad (v, u)'\,(u, v)' = I_v.$$

Let $k = m_{uu} - 1$; and let the nilpotent units of Γ_{uu} be denoted by $N_u^{(1)}, N_u^{(2)}, \ldots N_u^{(k)}$. Then $(u, v)'$ and the products $N_u^{(h)} \cdot (u, v)'$, for $h = 1, 2, \ldots k$, are numbers of the aggregate Γ_{uv} linearly independent. For, if

$$g_0 \cdot (u, v)' + \sum_{h=1}^{k} g_h N_u^{(h)} \cdot (u, v)' = 0,$$

then

$$g_0 I_u + \sum_{h=1}^{k} g_h N_u^{(h)} = [g_0 \cdot (u, v)' + \sum_{h=1}^{k} g_h N_u^{(h)} \cdot (u, v)']\,(v, u)' = 0$$

which is impossible, unless the g's are all zero. Therefore,

$$m_{uv} \geqq k + 1 = m_{uu}.$$

Moreover, there is no number in the aggregate Γ_{uv} linearly independent of these $k + 1$ numbers of this aggregate. For, if (u, v) is any number of this aggregate, since $(u\ v)\ (v, u)'$ belongs to the aggregate Γ_{uu}, we have

$$(u, v)\,(v, u)' = c_0 I_u + \sum_{h=1}^{k} c_h N_u^{(h)};$$

and, therefore,

$$\begin{aligned}(u, v) = (u, v)\, I_v &= (u, v) \cdot (v, u)' \,(u, v)' \\ &= (c_0 I_u + \sum_{h=1}^{k} c_h N_u^{(h)})\,(u, v)' \\ &= c_0\,(u,v)' + \sum_{h=1}^{k} c_h\, N_u^{(h)}\,(u, v)'.\end{aligned}$$

Whence it follows that m_{uv} cannot exceed $m_{uu} = k + 1$; and, therefore, $m_{uv} = m_{uu}$. Similarly, we may show next that $(v, u)'$ and the product $(v, u)' N_u^{(h)}$, for $h = 1, 2, \ldots k$, are linearly independent, and that in terms of these numbers every number of the aggregate Γ_{vu} can be expressed linearly. Finally, that I_v and the k products $(v, u)' N_u^{(h)} (u, v)'$, for $h = 1, 2, \ldots k$, are linearly independent, and that in terms of these numbers every number of the aggregate Γ_{vv} can be expressed linearly. Therefore, in particular, *if I_u and I_v are connected,*

$$m_{uu} = m_{uv} = m_{vu} = m_{vv}.$$

For $1 \leqq i \leqq m$ and u, v any two integers from 0 to $\bar{r}$, let $(u, v)_i$ denote the component of e_i in Γ_{uv}. We then have

$$(38) \qquad e_i = \sum_{p=0}^{r} \sum_{q=0}^{r} (p, q)_i \qquad (i = 1, 2, \ldots m).$$

Whence, from (32), we derive

$$\begin{aligned}(39) \quad S_1 (u, v)\, e_i &= \sum_{p=0}^{r} \sum_{q=0}^{r} S_1 (u, v)\,(p, q)_i \\ &= \sum_{q=0}^{r} S_1 (u, v)\,(v, q)_i \\ &= \sum_{q=0}^{r} S_1 (v, q)_i\,(u, v) = S_1 (v, u)_i\,(u,v) = S_1 (u, v)\,(v, u)_i \\ &\qquad (u, v = 0, 1, 2 \ldots \bar{r};\ i = 1, 2, \ldots m).\end{aligned}$$

We may now show first that if, for $0 \leqq u \leqq \bar{r}$, the aggregate Γ_{0u} contains any unit, that is, if $m_{u0} > 0$, the number system $(e_1, e_2, \ldots e_m)$ contains an invariant nilpotent sub system. For, let $(u, o) \neq 0$, and let

$$(o, u)_i (u, o) = (o, o)_i' \qquad (i = 1, 2, \ldots m);$$

when, by (34) and (39), we have

$$\begin{aligned} S_1(u, o)\, e_i &= S_1(u, o)\,(o, u)_i \\ &= S_1(o, u)_i\,(u, o) = S_1(o, o)_i' = 0 \\ & \qquad\qquad (i = 1, 2, \ldots m), \end{aligned}$$

and thus (u, o) satisfies equations (6). Similarly, if $m_{ou} > 0\,(1 \leqq u \leqq r)$, we may show that $(e_1, e_2, \ldots e_m)$ contains an invariant nilpotent sub system.

Again, if $\Gamma_{uu}\,(1 \leqq u \leqq \bar{r})$ contains more than one unit, that is, if $m_{uu} > 1$, the system $(e_1, e_2, \ldots e_m)$ contains an invariant nilpotent sub system. For, in this case, there is a nilpotent number (u, u) of Γ_{uu} whose product with any number of this aggregate is, therefore, nilpotent;[12] and thus $(u\, u)\ (u, u)_i$, for $i = 1, 2, \ldots m$, is nilpotent: therefore,

$$S_1(u, u)\, e_i = S_1(u, u)\,(u, u)_i = 0 \qquad (i = 1, 2, \ldots m),$$

and thus (u, u) is a solution of equations (6). If, for u, v any two distinct integers from 1 to $\bar{r}$, I_u and I_v are connected, and either Γ_{uv} or Γ_{vu} contains more than one unit; that is, if either $m_{uv} > 1$ or $m_{vu} > 1$, the system $(e_1, e_2, \ldots e_m)$ contains a nilpotent sub system. For then, by the theorem p. 645, we have $m_{uu} > 1$. Further, if I_u and I_v are not connected, and either Γ_{uv} or Γ_{vu} contains one or more units, that is, if $m_{uv} > 0$ or $m_{vu} > 0$, the number system contains an invariant nilpotent sub system. For let $(u, v) \neq 0$: in this case, by the theorem given, p. 642, we have

$$S_1(u, v)\,(v, u)_i = 0 \qquad (i = 1, 2, \ldots m);$$

therefore,

$$S_1(u, v)\, e_i = S_1(u, v)\,(v, u)_i = 0 \qquad (i = 1, 2, \ldots m),$$

and thus (u, v) satisfies equations (6). Finally, if I_u and I_v are not connected and $m_{vu} > 0$, $(e_1, e_2, \ldots e_m)$ contains an invariant nilpotent sub system.

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12 Namely, when $m_{uu} > 1$, any number (u, u) linear in the nilpotent units of Γ_{uu} is such a number. For since I_u is a modulus of the system Γ_{uu}, these nilpotent units constitute an invariant nilpotent sub system of Γ_{uu}. Wherefore, the products of (u, u) and any number of Γ_{uu} belongs to this nilpotent sub system, and is, therefore, nilpotent.

I shall now assume that the number system $(e_1, e_1, \ldots e_m)$ contains no invariant nilpotent sub system, in which case, by what has just been proved, we have

$$(40) \qquad m_{uo} = m_{ou} = m_{oo} = 0 \qquad (u = 1, 2, \ldots \bar{r}),$$

that is, no number of the system is contained in Γ_{oo} nor in either of the aggregates Γ_{uo}, Γ_{ou} for $u = 1, 2, \ldots \bar{r}$. Further,

$$(41) \qquad m_{uu} = 1 \qquad (u = 1, 2, \ldots \bar{r}),$$

that is, I_u is the only unit in Γ_{uu} for $1 \leqq u \leqq \bar{r}$. Finally, for u and v any two distinct integers from 1 to $\bar{r}$, if I_u and I_v are connected,

$$m_{uv} = m_{vu} = 1;$$

whereas, if I_u and I_v are not connected,

$$m_{uv} = m_{vu} = 0.$$

In the present case, the number system contains a modulus, viz.,

$$(42) \qquad \epsilon = I_1 + I_2 + \ldots + I_{\bar{r}},$$

since, for u, v any two integers from 1 to $\bar{r}$, if Γ_{uv} contains a unit J_{u1v}, we have

$$\epsilon J_{u1v} = J_{u1v} = J_{u1v}\epsilon$$

by (26) and (27).

It is, with the present assumption, convenient to modify our notation to indicate the connection which may exist between certain of the idempotent numbers, I_1, I_2, $\ldots$ $I_{\bar{r}}$. I shall, therefore, suppose these numbers arranged in ν aggregates, $1 \leqq \nu \leqq \bar{r}$, containing respectively μ_1, μ_2, $\ldots$ μ_ν of the I's, where $\sum_{p=1}^{\nu} \mu_p = \bar{r}$, any two idempotent numbers in the same aggregate being connected, but no pair of idempotent numbers in different aggregates being connected; and, for $1 \leqq p \leqq \nu$, I shall denote by $I_u^{(p)}$ $(u = 1, 2, \ldots \mu_p)$ the idempotent numbers in the p^{th} aggregate. The $\bar{r}^2$ aggregates of numbers, formerly denoted by $\Gamma_{u,v}$ for $u, v = 1, 2, \ldots \bar{r}$, into one or other of which the units fall when the system is regularized as above and contains no invariant nilpotent sub system, will now be denoted by $\Gamma_{u,v}^{(p,q)}$ for $p, q = 1, 2, \ldots \nu$, and for $u = 1, 2, \ldots \mu_p$ and $v = 1, 2, \ldots \mu_q$;

and the number of linearly independent numbers in $\Gamma_{u,v}{}^{(p,q)}$ will be denoted by $m_{u,v}{}^{(p,q)}$[13]. By what is shown above we now have

$$(44) \qquad m_{uv}{}^{(p,p)} = 1 \qquad (p = 1, 2, \dots \nu;\ u, v = 1, 2, \dots \mu_p),$$

$$(45) \qquad m_{uv}{}^{(p,q)} = 0$$
$$(p, q = 1, 2, \dots \nu;\ q \neq p;\ u = 1, 2, \dots \mu_p;\ v = 1, 2, \dots \mu_q)^{14}.$$

For $1 \leqq p \leqq \nu$ and u and v any two distinct integers from 1 to μ_p, we may now, in harmony with the preceding notation, denote the single unit of $\Gamma_{uv}{}^{(p,p)}$ by $J_{uv}{}^{(p)}$; and if, further, we denote by $J_{uu}{}^{(p)}$ the idempotent unit $I_u{}^{(p)}$ of $\Gamma_{uu}{}^{(p,p)}$, we shall have as the multiplication table of the system

$$(46) \qquad J_{uv}{}^{(p)} J_{vw}{}^{(p)} = \rho_{uvw} J_{uw}{}^{(p)}, \qquad J_{uv}{}^{(p)} J_{v'w}{}^{(p)} = 0$$
$$(p = 1, 2, \dots \nu;\ u, v, v', w = 1, 2, \dots \mu_p;\ v' \neq v),$$

$$(47) \qquad J_{uv}{}^{(p)} J_{u'v'}{}^{(q)} = 0$$
$$(p, q = 1, 2, \dots \nu;\ q \neq p;\ u, v = 1, 2, \dots \mu_p;\ u', v', = 1, 2, \dots \mu_q)$$

by (31), (32), and (44), where $\rho_{uuv} = \rho_{uvv} = 1$. For $1 \leqq p \leqq \nu$, and for u, v any two integers from 1 to μ_p, it follows from (44) that

$$\rho_{uvu}{}^{(p)} S_1 J_{uu}{}^{(p)} = S_1 J_{uv}{}^{(p)} J_{vu}{}^{(p)} \neq 0,$$

and thus $\rho_{uvu}{}^{(p)} \neq 0$, otherwise $J_{uu}{}^{(p)} = I_u{}^{(p)}$ and $J_{vv}{}^{(p)} = I_v{}^{(p)}$ are not connected; and, since

$$\begin{aligned}(\rho_{uvu}{}^{(p)})^2 J_{uu}{}^{(p)} &= (\rho_{uvu}{}^{(p)} J_{uu}{}^{(p)})^2 = (J_{uv}{}^{(p)} J_{vu}{}^{(p)})^2 \\ &= J_{uv}{}^{(p)} \cdot J_{vu}{}^{(p)} J_{uv}{}^{(p)} \cdot J_{vu}{}^{(p)} = \rho_{vuv}{}^{(p)} J_{uv}{}^{(p)} J_{vv}{}^{(p)} J_{vu}{}^{(p)} \\ &= \rho_{vuv}{}^{(p)} J_{uv}{}^{(p)} J_{vu}{}^{(p)} = \rho_{vuv}{}^{(p)} \rho_{uvu}{}^{(p)} J_{uu}{}^{(p)} \neq 0,\end{aligned}$$

we have $\rho_{uvu}{}^{(p)} = \rho_{vuv}{}^{(p)}$. Further, for $1 \leqq w \leqq \mu_p$,

$$\begin{aligned}\rho_{uvw}{}^{(p)} \rho_{vuw}{}^{(p)} J_{vw} &= \rho_{uvw}{}^{(p)} J_{vu}{}^{(p)} J_{uw}{}^{(p)} = J_{vu}{}^{(p)} \cdot J_{uv}{}^{(p)} J_{vw}{}^{(p)} \\ &= J_{vu}{}^{(p)} J_{uv}{}^{(p)} \cdot J_{vw}{}^{(p)} = \rho_{vuv}{}^{(p)} J_{vv}{}^{(p)} J_{vw}{}^{(p)} \\ &= \rho_{vuv}{}^{(p)} J_{vw}{}^{(p)} \neq 0;\end{aligned}$$

and, therefore, $\rho_{uvw}{}^{(p)} \neq 0$.

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13 Thus, whereas, formerly Γ_{uv} denoted the aggregate of numbers $I_u e_i I_v$ for $i = 1, 2, \dots m$, of which m_{uv} were linearly independent, $\Gamma_{uv}{}^{(p,q)}$ is now the aggregate of numbers $I_u{}^{(p)} e_i I_v{}^{(q)}$ for $i = 1, 2, \dots m$, of which $m_{uv}{}^{(p,q)}$ are linearly independent.

14 Therefore,

$$m = \sum_{p=1}^{\nu} \sum_{q=1}^{\nu} \sum_{u=1}^{\mu_p} \sum_{v=1}^{\mu_q} m_{uv}{}^{(p,q)} = \sum_{p=1}^{\nu} \sum_{u=1}^{\mu_p} \sum_{v=1}^{\mu_p} m_{uv}{}^{(p,p)} = \sum_{p=1}^{\nu} \mu_p{}^2.$$

Let

$$(48)\qquad \overline{J}_{uv}{}^{(p)} = \frac{1}{\sqrt{\rho_{1u1}\rho_{1v1}}} J_{u1}{}^{(p)} J_{1v}{}^{(p)}$$
$$(p = 1, 2, \ldots \nu;\ u, v = 1, 2, \ldots \mu_p).$$

Then

$$(49)\qquad J_{uv}{}^{(p)} = \frac{1}{\rho_{u1v}{}^{(p)}} J_{u1}{}^{(p)} J_{1v}{}^{(p)} = \frac{\sqrt{\rho_{1u1}{}^{(p)}\rho_{1v1}{}^{(p)}}}{\rho_{u1v}{}^{(p)}} \overline{J}_{uv}{}^{(p)}$$
$$(p = 1, 2, \ldots \nu;\ u, v = 1, 2, \ldots \mu_p)$$

by (46); and, therefore, we may take the $\overline{J}$'s as new units. We now have

$$(50)\qquad \overline{J}_{uv}{}^{(p)}\overline{J}_{vw}{}^{(p)} = \frac{1}{\sqrt{\rho_{1u1}{}^{(p)}\rho_{1v1}{}^{(p)}\cdot\rho_{1v1}{}^{(p)}\rho_{1w1}{}^{(p)}}} J_{u1}{}^{(p)} J_{1v}{}^{(p)}\cdot J_{v1}{}^{(p)} J_{1w}{}^{(p)}$$
$$= \frac{1}{\sqrt{\rho_{1u1}{}^{(p)}\rho_{1w1}{}^{(p)}}} J_{u1}{}^{(p)} J_{1w}{}^{(p)}\,[15] = \overline{J}_{uw}{}^{(p)},$$

$$\overline{J}_{uv}{}^{(p)}\overline{J}_{v'w}{}^{(p)} = \frac{1}{\sqrt{\rho_{1u1}{}^{(p)}\rho_{1v1}{}^{(p)}\cdot\rho_{1v'1}{}^{(p)}\rho_{1w1}{}^{(p)}}} J_{u1}{}^{(p)} J_{1v}{}^{(p)}\cdot J_{v'1}{}^{(p)} J_{1w}{}^{(p)} = 0$$
$$(p = 1, 2, \ldots \nu;\ u, v, v', w = 1, 2, \ldots \mu_p;\ v' \neq v),$$

$$(51)\qquad \overline{J}_{uv}{}^{(p)}\overline{J}_{u'v'}{}^{(q)} = \frac{1}{\sqrt{\rho_{1u1}{}^{(p)}\rho_{1v1}{}^{(p)}\cdot\rho_{1u'1}{}^{(q)}\rho_{1v'1}{}^{(q)}}} J_{u1}{}^{(p)} J_{1v}{}^{(p)}\cdot J_{u'1}{}^{(q)} J_{1v'}{}^{(q)} = 0$$
$$(p, q = 1, 2, \ldots \nu;\ q \neq p;\ u, v = 1, 2, \ldots \mu_p;\ u', v' = 1, 2, \ldots \mu_q).$$

For $1 \leqq p \leqq \nu$, the units $\overline{J}_{uv}{}^{(p)}$ for $u, v = 1, 2, \ldots \mu_p$ constitute a quadrate of order μ_p; and, therefore, in the present case, the number system is constituted by ν mutually nilfactorial quadrates.[16] For the modulus ϵ of the system we now have

$$(52)\qquad \epsilon = \sum_{p=1}^{\nu}\sum_{u=1}^{\mu_p} I_u{}^{(p)} = \sum_{p=1}^{\nu}\sum_{u=1}^{\mu_p} \overline{J}_{uu}{}^{(p)}$$

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15 For

$$J_{u1}{}^{(p)} J_{1v}{}^{(p)}\cdot J_{v1}{}^{(p)} J_{1w}{}^{(p)} = J_{u1}{}^{(p)}\cdot J_{1v}{}^{(p)} J_{v1}{}^{(p)}\cdot J_{1w}{}^{(p)} = \rho_{1v1}{}^{(p)} J_{u1}{}^{(p)} J_{11}{}^{(p)} J_{1w}{}^{(p)}$$
$$= \rho_{1v1}{}^{(p)} J_{u1}{}^{(p)} J_{1w}{}^{(p)}.$$

16 A *quadrate* is a hyper complex number system with $m = \overline{m}^2$ units ϵ_{uv} $(u, v = 1, 2, \ldots \overline{m})$ which can be so chosen that

$$\epsilon_{uv}\epsilon_{vw} = \epsilon_{uw}, \qquad \epsilon_{uv}\epsilon_{v'w} = 0 \qquad (u, v, v', w = 1, 2, \ldots \overline{m};\ v' \neq v).$$

B. Peirce, Am. Journ. Maths., 4, 217.

By (30), (33), (40), (44), and (45), we now have

$$(53)\qquad m S_1 \bar{J}_{uu}^{(p)} = \sum_{q=1}^{\nu} \sum_{v=1}^{\mu_q} m_{uv}^{(p,q)} = \sum_{v=1}^{\mu_p} m_{uv}^{(p,p)} = \mu_p$$

$$= \sum_{v=1}^{\mu_p} m_{vu}^{(p,p)} = \sum_{q=1}^{\nu} \sum_{v=1}^{\mu_q} m_{vu}^{(q,p)} = m S_2 \bar{J}_{uu}^{(p)}$$

$$(p = 1, 2, \dots \nu;\ u = 1, 2, \dots \mu_p),$$

$$(54)\qquad S_1 \bar{J}_{uv}^{(p)} = 0, \qquad S_2 \bar{J}_{uv}^{(p)} = 0$$

$$(p = 1, 2, \dots \nu;\ u, v = 1, 2, \dots \mu_p;\ v \neq u).$$

And since, for any number $A = \sum_{i=1}^{m} a_i e_i$, we may now put

$$(55)\qquad A = \sum_{p=1}^{\nu} \sum_{u=1}^{\mu_p} \sum_{v=1}^{\mu_p} c_{uv}^{(p)} \bar{J}_{uv}^{(p)},$$

we have

$$(56)\qquad S_1 A = \sum_{p=1}^{\nu} \sum_{u=1}^{\mu_p} \sum_{v=1}^{\mu_p} c_{uv}^{(p)} S_1 \bar{J}_{uv}^{(p)}$$

$$= \sum_{p=1}^{\nu} \sum_{u=1}^{\mu_p} \sum_{v=1}^{\mu_p} c_{uv}^{(p)} S_2 \bar{J}_{uv}^{(p)} = S_2 A.$$

Therefore, in particular,

$$(57)\qquad S_1 e_i e_j = S_2 e_i e_j \qquad (i, j = 1, 2, \dots m),$$

and thus we have $\Delta_1 = \Delta_2$ also in the case now considered, when the system $(e_1, e_2, \dots e_m)$ contains no invariant nilpotent sub system and neither Δ_1 nor Δ_2 is zero.

From the conditions, necessary and sufficient, that the m^3 constants γ_{ijk} $(i, j, k = 1, 2, \dots m)$ shall constitute the constants of multiplication of a hyper complex number system in m units, viz.,

$$(58)\qquad \sum_{k=1}^{m} \gamma_{ijk} \gamma_{khl} = \sum_{k=1}^{m} \gamma_{ikl} \gamma_{jhk}$$

$$(i, j, h, l = 1, 2, \dots m),$$

we derive

$$(59)\quad m^m\Delta_1 = \begin{vmatrix} mS_1(e_ie_j) \\ (i,j=1,2,\ldots m) \end{vmatrix} = \begin{vmatrix} \Sigma_h\Sigma_k\gamma_{ijk}\gamma_{khh} \\ (i,j=1,2,\ldots m) \end{vmatrix} = \begin{vmatrix} \Sigma_h\Sigma_k\gamma_{ikh}\gamma_{jhk} \\ (i,j=1,2,\ldots m) \end{vmatrix}$$

$$= \begin{vmatrix} \gamma_{i11}, \ldots \gamma_{i1m}, \ldots \gamma_{im1}, \ldots \gamma_{imm} \\ (i=1,2,\ldots m) \end{vmatrix} \begin{vmatrix} \gamma_{j11}, \ldots \gamma_{jm1}, \ldots \gamma_{j1m}, \ldots \gamma_{jmm} \\ (j=1,2,\ldots m) \end{vmatrix},$$

$$(60)\quad m^m\Delta_2 = \begin{vmatrix} mS_2e_je_i \\ (i,j=1,2,\ldots m) \end{vmatrix} = \begin{vmatrix} \Sigma_h\Sigma_k\gamma_{jik}\gamma_{hkh} \\ (i,j=1,2,\ldots m) \end{vmatrix} = \begin{vmatrix} \Sigma_h\Sigma_k\gamma_{hjk}\gamma_{kih} \\ (i,j=1,2,\ldots m) \end{vmatrix}$$

$$= \begin{vmatrix} \gamma_{1i1}, \ldots \gamma_{1im}, \ldots \gamma_{mi1}, \ldots \gamma_{mim} \\ (i=1,2,\ldots m) \end{vmatrix} \begin{vmatrix} \gamma_{1j1}, \ldots \gamma_{mj1}, \ldots \gamma_{1jm}, \ldots \gamma_{mjm} \\ (j=1,2,\ldots m) \end{vmatrix}.$$

A number system containing no invariant sub system is termed by Cartan a *simple system* (*systeme simple*), and he shows that such a system is what is here termed a quadrate. A non-simple system containing no invariant nilpotent sub system Carten terms *semi-simple*.[17] Such a system is constituted by nilfactorial quadrates of which the invariant sub systems are any $p(1 \leqq p < \nu)$ of these quadrates. By what is shown above it appears that $\Delta_1 \neq 0$ or $\Delta_2 \neq 0$ is the condition necessary and sufficient that a number system shall be either simple or semi-simple. We have, therefore, the following theorem:

Theorem IV. *Let $e_1, e_2, \ldots e_m$ be the units of any hyper complex number system, and let*

$$\Delta_1 \equiv \begin{vmatrix} S_1e_ie_j \\ (i,j=1,2,\ldots r) \end{vmatrix}, \qquad \Delta_2 \equiv \begin{vmatrix} S_2e_ie_j \\ (i,j=1,2,\ldots r) \end{vmatrix}$$

Then $\Delta_1 = \Delta_2$. If $\Delta_1 \neq 0$, the number system contains a modulus and is either simple or semi-simple, that is, is constituted by $\nu \geqq 1$ mutually nilfactorial quadrates; and, conversely, in this case, $\Delta_1 = \Delta_2 \neq 0$.

17 Comptes Rendus, **124**, 1218 (1897).

§ 3.

It has been shown by C. S. Peirce that any given hyper complex number system $(e_1, e_2, \ldots e_m)$ is a sub system of a quadrate of order n, where the greatest value n need assume is $m+1$. This is, of course equivalent to the theorem that any given number system can be represented by a matrix whose order need not exceed $m+1$.[18]

Let now $(e_1, e_2, \ldots e_m)$ be any given number system; let $e_{uv}\,(u, v = 1, 2, \ldots n)$ be the units of the quadrate of which $(e_1, e_2, \ldots e_m)$ is a sub system, when we have

$$\epsilon_{uv}\epsilon_{vw} = \epsilon_{uw}, \qquad \epsilon_{uv}\epsilon_{v'w} = 0 \qquad (61)$$
$$(u, v, v', w = 1, 2, \ldots n;\ v' \neq v);$$

and let

$$e_i = \sum_{u=1}^{n}\sum_{v=1}^{n} \theta_{uv}^{(i)}\epsilon_{uv} \qquad (i = 1, 2, \ldots m). \qquad (62)$$

The units $e_1, e_2, \ldots e_m$ may then be regarded as represented, respectively, by the m linearly independent matrices $E_1, E_2, \ldots E_m$, where E_i, for $i = i, 2, \ldots m$, is defined by the system of equations,

$$(\xi_1', \xi_2', \ldots \xi_n') = \begin{pmatrix} \theta_{11}^{(i)}, & \theta_{12}^{(i)}, & \ldots & \theta_{1n}^{(i)} \\ \theta_{21}^{(i)}, & \theta_{22}^{(i)}, & \ldots & \theta_{2n}^{(i)} \\ \ldots & \ldots & \ldots & \ldots \\ \theta_{n1}^{(i)}, & \theta_{n2}^{(i)}, & \ldots & \theta_{nn}^{(i)} \end{pmatrix} (\xi_1, \xi_2, \ldots \xi_n), \qquad (63)$$

and any number $x = \sum_{i=1}^{m} x_i e_i$ of $(e_1, e_2, \ldots e_m)$ by the matrix of the linear substitution

(64)

$$(\xi_1', \xi_2', \ldots \xi_n') = \begin{pmatrix} \sum_{i=1}^{m} x_i\theta_{11}^{(i)}, & \sum_{i=1}^{m} x_i\theta_{12}^{(i)}, & \ldots & \sum_{i=1}^{m} x_i\theta_{1n}^{(i)} \\ \sum_{i=1}^{m} x_i\theta_{21}^{(i)}, & \sum_{i=1}^{m} x_i\theta_{22}^{(i)}, & \ldots & \sum_{i=1}^{m} x_i\theta_{2n}^{(i)} \\ \ldots & \ldots & \ldots & \ldots \\ \sum_{i=1}^{m} x_i\theta_{n1}^{(i)}, & \sum_{i=1}^{m} x_i\theta_{n2}^{(i)}, & \ldots & \sum_{i=1}^{m} x_i\theta_{nn}^{(i)} \end{pmatrix} (\xi_1, \xi_2, \ldots \xi_n).$$

18 Loc. cit., p. 221; also These Proceedings, **10,** 392 (1875). In certain cases, as shown by Peirce, we may take $n \leqq m$; in other cases, n must be greater than m. See § 4.

For any number of the quadrate $\epsilon_{uv}(u, v = 1, 2, \ldots n)$ the two scalar functions with respect to this number system defined in *theorem I* are equal as shown in § 2; and, therefore, but a single symbol is required for these functions. I shall denote by $\bar{S}\bar{A}$ the two equal scalar functions of any number

(65) $$\bar{A} = \sum_{u=1}^{n} \sum_{v=1}^{n} a_{uv}\epsilon_{uv} = \begin{vmatrix} a_{11}, & a_{12}, & \ldots & a_{1n} \\ a_{21}, & a_{22}, & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}, & a_{n2}, & \ldots & a_{nn} \end{vmatrix}$$

of the quadrate; and, by *theorem I*, we then have

(66) $$\bar{S}\epsilon_{uu} = \frac{1}{n}, \quad \bar{S}\epsilon_{uv} = 0 \qquad (u, v = 1, 2, \ldots n;\ v \neq u),$$ [19]

and, therefore,

(67) $$\bar{S}\bar{A} = \sum_{u=1}^{n} \sum_{v=1}^{n} a_{uv}\bar{S}\epsilon_{uv} = \frac{1}{n}\sum_{u=1}^{n} a_{uu}.$$

I shall denote simply by 1 the modulus of the quadrate, and $\rho 1$, for any scalar ρ, simply by ρ. We have

(68) $$1 = \sum_{u=1}^{n} \epsilon_{uu}.$$

Any number $\bar{A} = \sum_{u=1}^{n} \sum_{v=1}^{n} a_{uv}\epsilon_{uv}$ of the quadrate satisfies an equation

(69) $$\phi(\bar{A}) \equiv (\bar{A} - \rho_1)(\bar{A} - \rho_2) \ldots (\bar{A} - \rho_n) = 0,$$

where the ρ's are scalars; and we have

(70) $$\phi(\rho) \equiv \begin{vmatrix} \rho - a_{11}, & -a_{12}, & \ldots & -a_{1n} \\ -a_{21}, & \rho - a_{22}, & \ldots & -a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ -a_{n1}, & -a_{n2}, & \ldots & \rho - a_{nn} \end{vmatrix} \equiv (\rho - \rho_1)(\rho - \rho_2) \ldots (\rho - \rho_n).$$ [20]

19 For the number of linearly independent numbers X of the quadrate satisfying the equation $e_{uu}X = X$ is n, since every such number is linearly expressible in $\epsilon_{u1}, \epsilon_{u2}, \ldots \epsilon_{un}$, and each of these numbers satisfies this equation. Therefore, by *theorem I*, $n^2\bar{S}_1\epsilon_{un} = n$. Similarly, the number of linearly independent numbers X of the quadrate satisfying the equation $X\epsilon_{uu} = X$ is also n; and, therefore, $n^2\bar{S}_2\epsilon_{uu} = n$. Since, for $v \neq u$, ϵ_{uv} is nilpotent, $\bar{S}_1\epsilon_{uv} = \bar{S}_2\epsilon_{uv} = 0 \quad (v \neq u)$.

20 Cayley: Philosophical Transactions, p. 800 (1858).

The polynomial $\phi(\rho)$ is termed the "characteristic function" of $\bar{A}$, and $\phi(\rho) = 0$ the "characteristic equation" of $\bar{A}$. Since, by (67), $n\,\bar{S}\,\bar{A}$ is the sum of the constituents in the principal diagonal of the matrix representing $\bar{A}$, it follows that *$n\,\bar{S}\,\bar{A}$ is equal to the sum of the roots of the characteristic equation of $\bar{A}$.*

If $\bar{A}$ is idempotent, the roots of its characteristic equation are 0 and 1. Wherefore, *if $\bar{A}$ is idempotent, $n\,\bar{S}\,\bar{A}$ is equal to the multiplicity of the root 1 of the characteristic equation of $\bar{A}$.*

In conformity with the notation employed in § 2, let

$$\bar{\Omega}(\bar{A}) \equiv \bar{A}^{\mu} + \bar{p}_1\bar{A}^{\mu-1} + \ldots + \bar{p}_{\mu-1}\bar{A} = 0 \tag{71}$$

be the syzygy of lowest order in powers of $\bar{A}$. Then $\rho\phi(\rho)$ contains $\bar{\Omega}(\rho)$. Whence it follows that n is the maximum number of distinct non-zero roots of the equation $\bar{\Omega}(\rho) = 0$. Therefore, by *theorem III*, and what was proved p. 636, *$\bar{A}$ is nilpotent if, for some positive integer p,*

$$\bar{S}\,\bar{A}^{p+h} = 0 \qquad (h = 0, 1, 2, \ldots n-1).$$

Conversely, by *theorem I, if $\bar{A}$ is nilpotent, these equations are satisfied for any positive integer p.*

For the scalar functions defined in § 1 of any number $A = \sum_{i=1}^{n} a_i e_i$ of the system $(e_1, e_2, \ldots e_m)$ I shall write $S_1 A$ and $S_2 A$ as in § 1 and § 2. The symbol $\bar{S}$ also is significant when prefixed to any letter denoting a number of the system $(e_1, e_2, \ldots e_m)$, since any such number belongs to the quadrate ϵ_{uv} $(u, v, = 1, 2, \ldots n)$. We have, by (62) and (67),

$$\bar{S}e_i = \frac{1}{n}\sum_{u=1}^{n} \theta_{uu}^{(i)} \qquad (i = 1, 2, \ldots n); \tag{72}$$

and, therefore

$$\bar{S}A = \sum_{i=1}^{m} a_i\bar{S}e_i = \frac{1}{n}\sum_{i=1}^{m}\sum_{u=1}^{n} a_i\theta_{uu}^{(i)}. \tag{73}$$

Let now

$$X = \sum_{i=1}^{m} x_i e_i = \sum_{i=1}^{m}\sum_{u=1}^{n}\sum_{v=1}^{n} x_i\theta_{uv}^{(i)}\epsilon_{uv}; \tag{74}$$

and let the number system $(e_1, e_2, \dots e_m)$ contain at least one number satisfying the system of equations

(75) $$\bar{S} X e_i = x_1 \bar{S} e_1 e_i + x_2 \bar{S} e_2 e_i + \dots + x_m \bar{S} e_m e_i = 0 \qquad (i = 1, 2, \dots m).$$

(76) $$\nabla \equiv \begin{vmatrix} \bar{S} e_1 e_1, & \bar{S} e_2 e_1, & \dots & \bar{S} e_m e_1 \\ \bar{S} e_1 e_2, & \bar{S} e_2 e_2, & \dots & \bar{S} e_m e_2 \\ \dots & \dots & \dots & \dots \\ \bar{S} e_1 e_m, & \bar{S} e_2 e_m, & \dots & \bar{S} e_m e_m \end{vmatrix},$$

we, therefore, now have $\nabla = 0$, Let $X = B$ be any number of $(e_1, e_2, \dots e_m)$ satisfying equations (75). Then B is nilpotent; moreover, the product, in either order, of B and any number $A = \sum_{k=1}^{m} a_k e_k$ of the system $(e_1, e_2, \dots e_m)$ is also a solution of equations (75). For, for any number

$$Y = y_1 e_1 + y_2 e_2 + \dots + y_m e_m$$

of the system $(e_1, e_2, \dots e_m)$, we now have

$$\bar{S} B Y = y_1 \bar{S} B e_1 + y_2 \bar{S} B e_2 + \dots + y_m \bar{S} B e_m = 0:$$

wherefore, in particular,

$$\bar{S} B^{2+h} = \bar{S} B B^{h+1} = 0 \qquad (h = 1, 2, \dots n - 1),$$

and thus, by the theorem given on p. 654, B is nilpotent; further,

$$\bar{S}.(BA \cdot e_i) = \bar{S}(B \cdot A e_i) = 0,$$

$$\bar{S}(AB \cdot e_i) = \bar{S}(e_i \cdot AB) = \bar{S}(e_i A \cdot B) = \bar{S}(B \cdot e_i A) = 0 \qquad (i = 1, 2, \dots m).$$

Since both BA and AB are solutions of equations (75), it follows by what has just been proved that both BA and AB are nilpotent. In particular, for $1 \leqq i \leqq m$, Be_i is nilpotent; and, therefore, by *theorem I*, $S_1 B e_i = 0$. Whence it follows that B is a solution of the system of equations

(77) $$S_1 X e_i = x_1 S_1 e_1 e_i + x_2 S_1 e_2 e_i + \dots + x_m S_1 e_m e_i = 0 \qquad (i = 1, 2, \dots m).$$

Wherefore, we now have

$$\Delta_1 = \Delta_2 = 0.$$[21]

Conversely, if $B = \sum_{k=1}^{m} b_k e_k$ is any solution of equations (77), Be_j $(1 \leqq j \leqq m)$ is by *theorem II* then also a solution of these equations, and thus Be_j, by *theorem I*, is nilpotent: therefore, by the theorem of p. 654, $\overline{S}Be_j = 0$ for $j = 1, 2, \ldots m$; that is, B is a solution of equations (75). Let the nullity of ∇ be m', where $1 \leqq m' \leqq m$. There is then a set of just m' linearly independent numbers $B_1, B_2, \ldots B_{m'}$ of the system $(e_1, e_2, \ldots e_m)$ satisfying equations (75); therefore, just m' linearly independent numbers of this system satisfying equations (77): whence it follows that the nullity of Δ_1 is m'. And since each of the B's satisfies equations (77) it follows, from *theorem II*, that $B_1, B_2, \ldots B_{m'}$ constitute an invariant nilpotent sub system of $(e_1, e_2, \ldots e_m)$ containing every invariant nilpotent sub system of $(e_1, e_2, \ldots e_m)$.

Let now $\nabla \neq 0$. In this case, if, for any two numbers

$$A = \sum_{i=1}^{m} a_i e_i, \qquad B = \sum_{i=1}^{m} b_i e_i$$

of $(e_1, e_2, \ldots e_m)$, we have

$$\overline{S}Ae_i = \overline{S}Be_i \qquad (i = 1, 2, \ldots m),$$

then $A = B$; otherwise, there is a number $A - B \neq 0$ of the system satisfying equations (77). In this case, $\Delta_1 \neq 0$ and the number system $(e_1, e_2, \ldots e_m)$ contains a nodulus but no invariant nilpotent sub system.

Let now the number system $(e_1, e_2, \ldots e_m)$ be transformed by the substitution

$$(78) \qquad e'_i = \tau_{i1}e_1 + \tau_{i2}e_2 + \ldots + \tau_{im}e_m \qquad (i = 1, 2, \ldots m);$$

and let

$$(79) \qquad \nabla' \equiv \begin{vmatrix} \overline{S}e'_i e'_j \\ (i, j = 1, 2, \ldots m) \end{vmatrix}.$$

Then, since

$$\overline{S}e'_i e'_j = \sum_{h=1}^{m} \sum_{k=1}^{m} \tau_{ih}\tau_{jk}\overline{S}e_h e_k \qquad (i, j = 1, 2, \ldots m),$$

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21 See p. 630.

we have

$$\nabla' = T^2 \nabla, \tag{80}$$

where T is the determinant of the transformation. Therefore, the equation $\nabla = 0$ is invariant to any transformation of the units of the system $(e_1\, e_2, \ldots e_m)$.

We have now the following theorem:

Theorem V. *Let $(e_1, e_2, \ldots e_m)$ be any given number system constituting a sub system of the quadrate $\epsilon_{uv}\,(u, v = 1, 2, \ldots u)$: thus let*

$$e_i = \sum_{u=1}^{n} \sum_{v=1}^{n} \theta_{uv}\epsilon_{uv} \qquad (i = 1, 2, \ldots m).$$

For any given number

$$\bar{A} = \sum_{u=1}^{n} \sum_{v=1}^{n} a_{uv}\epsilon_{uv}$$

of the quadrate, let

$$\bar{S}\bar{A} = \frac{1}{n} \sum_{u=1}^{n} a_{uu},$$

when, for any given number

$$X = \sum_{i=1}^{m} x_i e_i = \sum_{i=1}^{m} \sum_{u=1}^{n} \sum_{v=1}^{n} x_i \theta_{uv}{}^{(i)} \epsilon_{uv}$$

of the system $(e_1, e_2, \ldots e_m)$, we have

$$\bar{S}X = \frac{1}{n} \sum_{i=1}^{m} \sum_{u=1}^{n} x_i \theta_{uu}{}^{(i)}.$$

Let

$$\nabla \equiv \begin{vmatrix} \bar{S} e_i e_j \\ (i, j = 1, 2, \ldots m) \end{vmatrix}$$

denote the resultant of the system of equations

$$\bar{S} X e_i = \sum_{j=1}^{m} x_j \bar{S} e_j e_i = 0. \qquad (i = 1, 2, \ldots m).$$

Then, if the number system be transformed by the substitution

$$e_i' = \tau_{i1} e_1 + \tau_{i2} e_2 + \ldots + \tau_{im} e_m \qquad (i = 1, 2, \ldots m),$$

and if

$$\nabla' \equiv \begin{vmatrix} \bar{S} e'_i e'_j \\ (i, j = 1, 2, \ldots m) \end{vmatrix},$$

we have

$$\nabla' = T^2 \nabla,$$

where T is the determinant of the substitution. If $\nabla \neq 0$, the system $(e_1, e_2, \ldots e_m)$ contains a modulus but no invariant nilpotent sub system; and, in this case, if for any two numbers

$$A = \sum_{i=1}^{m} a_i e_i, \qquad B = \sum_{i=1}^{m} b_i e_i$$

of the system we have

$$\bar{S} A e_i = \bar{S} B e_i \qquad (i = 1, 2, \ldots m),$$

then $A = B$. If $\nabla = 0$ and m' $(0 < m' \leqq m)$ is the nullity of ∇, the system $(e_1, e_2, \ldots e_m)$ contains a maximum invariant nilpotent sub system with m' units constituted by any m' linearly independent solutions of the equations $\bar{S} X e_i = 0 \quad (i = 1, 2, \ldots m)$.

In precisely the same way we may now prove the following theorem of which the preceding theorem is a special case:

Theorem VI. *Let $(e_1, e_2, \ldots e_m)$ be any given hyper complex number system constituting a sub system of the number system $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ whose constants of multiplication are $\bar{\gamma}_{uvw}$ for $u, v, w = 1, 2, \ldots n$, so that*

$$\epsilon_u \epsilon_v = \sum_{w=1}^{n} \bar{\gamma}_{uvw} \epsilon_w;$$

and let

$$e_i = \sum_{u=1}^{n} \theta_{iu} \epsilon_u \qquad (i = 1, 2, \ldots m).$$

For any number $\bar{A} = \sum_{u=1}^{n} \bar{a}_u \epsilon_u$ of the system $(\epsilon_1, \epsilon_2, \ldots \epsilon_n)$, let

$$\bar{S}_1 \bar{A} = \frac{1}{n} \sum_{u=1}^{n} \sum_{v=1}^{n} \bar{a}_u \bar{\gamma}_{uvv}, \qquad \bar{S}_2 \bar{A} = \frac{1}{n} \sum_{u=1}^{n} \sum_{v=1}^{n} \bar{a}_u \bar{\gamma}_{vuv},$$

in which case, for any number

$$A = \sum_{i=1}^{m} a_i e_i = \sum_{i=1}^{m} \sum_{u=1}^{n} a_i \theta_{iu} \epsilon_u$$

of the system $(e_1, e_2, \ldots e_m)$, *we have*

$$\bar{S}_1 A = \frac{1}{n} \sum_{i=1}^{m} \sum_{u=1}^{n} \sum_{v=1}^{n} a_i \theta_{iu} \bar{\gamma}_{uvv}$$

$$\bar{S}_2 A = \frac{1}{n} \sum_{i=1}^{m} \sum_{u=1}^{n} \sum_{v=1}^{n} a_i \theta_{iu} \bar{\gamma}_{vuv}.$$

Finally, let $X = \sum_{i=1}^{m} x_i e_i$, *and let*

$$\nabla_1 \equiv \begin{vmatrix} \bar{S}_1 e_i e_j \\ (i\, j, = 1, 2, \ldots m) \end{vmatrix}, \qquad \nabla_2 = \begin{vmatrix} \bar{S}_2 e_i e_j \\ (i, j = 1, 2, \ldots m) \end{vmatrix}$$

be, respectively, the resultants of the systems of equations

$$(\alpha) \quad \bar{S}_1 X e_i = x_1 \bar{S}_1 e_1 e_i + x_2 \bar{S}_1 e_2 e_i + \ldots + x_m \bar{S}_1 e_m e_i = 0 \qquad (i = 1, 2, \ldots m),$$

$$(\beta) \quad \bar{S}_2 X e_i = x_1 \bar{S}_2 e_2 e_i + x_2 \bar{S}_2 e_2 e_i + \ldots + x_m \bar{S}_2 e_m e_i = 0 \qquad (i = 1, 2, \ldots m).$$

Then, if the number system $(e_1, e_2, \ldots e_m)$ *is transformed by the substitution*

$$e'_i = \tau_{i1} e_1 + \tau_{i2} e_2 + \ldots + \tau_{im} e_m \qquad (i = 1, 2, \ldots m),$$

and if

$$\nabla'_1 \equiv \begin{vmatrix} \bar{S}_1 e'_i e'_j \\ (i, j = 1, 2, \ldots m) \end{vmatrix}, \qquad \nabla'_2 \equiv \begin{vmatrix} \bar{S}_2 e'_i e'j \\ (i, j = 1, 2, \ldots m) \end{vmatrix}$$

we have

$$\nabla_1' = T^2 \nabla_1, \qquad \nabla'_2 = T^2 \nabla_2,$$

where T is the determinant of the substitution. If $\nabla_1 \neq 0$, *in which case* $\nabla_2 \neq 0$, *and conversely, the system* $(e_1, e_2, \ldots e_m)$ *contains a modulus, but no invariant nilpotent sub system; and in this case, if for any two numbers*

$$A = \sum_{i=1}^{m} a_i e_i, \qquad B = \sum_{i=1}^{m} b_i e_i$$

of this system, we have

$$\bar{S}_1 A e_i = \bar{S}_1 B e_i \qquad (i = 1, 2, \ldots m),$$

or

$$\bar{S}_2 A e_i = \bar{S}_2 B e_i \qquad (i = 1, 2, \ldots m),$$

then $A = B$. *If the nullity of* ∇_1 *is* m' $(0 < m' \leqq m)$, *in which case* $\nabla_1 = 0$, *the nullity of* ∇_2 *is* m', *and conversely; and the system* $(e_1, e_2, \ldots e_m)$ *then contains a maximum invariant nilpotent sub system constituted by any* m' *linearly independent numbers of* $(e_1, e_2, \ldots e_m)$ *satisfying equations* (α), *or equations* (β), *every solution of equations* (α) *being a solution of equations* (β), *and conversely.*

§ 4.

Let $(e_1, e_2, \ldots e_m)$ be any given number system; let ϵ_{uv}, for $u, v = 1, 2, \ldots n$, constitute a quadrate of which $(e_1, e_2, \ldots e_m)$ is a sub system; and let

$$(81) \qquad e_i = \sum_{u=1}^{n} \sum_{v=1}^{n} \theta_{uv}{}^{(i)} \epsilon_{uv} \quad (i = 1, 2, \ldots m).$$

The units of the system $(e_1, e_2, \ldots e_m)$ are then represented, respectively, or may be identified, respectively, with the m linearly independent matrices defined by equations (63).

The number system $(e'_1, e'_2, \ldots e'_m)$ reciprocal to $(e_1, e_2, \ldots e_m)$ is then also a sub system of the quadrate: that is,

$$(82) \qquad e'_i = \sum_{u=1}^{n} \sum_{v=1}^{n} \eta_{uv}{}^{(i)} \epsilon_{uv} \qquad (i = 1, 2, \ldots m)$$

for a proper choice of the η's. For the m numbers $e'_1, e'_2, \ldots e'_m$ of the quadrate defined by equations (82) may be identified, respectively, with the m matrices $E'_1, E'_2, \ldots E'_m$, where E'_i, for $1 \leqq i \leqq m$, is defined by the equations

$$(83) \quad E'_i(\xi_1, \xi_2, \ldots \xi_n) = \begin{pmatrix} \eta_{11}{}^{(i)}, & \eta_{12}{}^{(i)}, & \ldots & \eta_{1n}{}^{(i)} \\ \eta_{21}{}^{(i)}, & \eta_{22}{}^{(i)}, & \ldots & \eta_{2n}{}^{(i)} \\ \ldots & \ldots & \ldots & \ldots \\ \eta_{n1}{}^{(i)}, & \eta_{n2}{}^{(i)}, & \ldots & \eta_{nn}{}^{(i)} \end{pmatrix} (\xi_1, \xi_2, \ldots \xi_n)\,;$$
$$(i = 1, 2, \ldots m);$$

and, therefore, if we put

$$(84) \qquad \eta_{uv}^{(i)} = \theta_{vu}^{(i)} \qquad (i = 1, 2, \ldots m;\ u, v = 1, 2, \ldots n),$$

we then have

$$(85) \qquad E'_i = tr.\, E_i \qquad (i = 1, 2, \ldots m):$$[22]

whence it follows that $E'_1, E'_2, \ldots E'_m$ are linearly independent, and

$$(86) \quad E'_i E'_j = tr.\, E_i \cdot tr.\, E_j = tr.\,(E_j E_i)$$
$$= tr.\,(\sum_{k=1}^{m} \gamma_{jik} E_k) = \sum_{k=1}^{m} \gamma_{jik}\, tr.\, E_k = \sum_{k=1}^{m} \gamma_{jik} E'_k$$
$$(i, j = 1, 2, \ldots m);$$

that is to say, the numbers $e'_1, e'_2, \ldots e'_m$ of the quadrate are then linearly independent, and

$$(87) \qquad e'_i e'_j = \sum_{k=1}^{m} \gamma_{jik} e'_k \qquad (i, j = 1, 2, \ldots m).$$

We may take $n = m$, and, at the same time, put

$$(88) \qquad \theta_{uv}^{(i)} = \gamma_{ivu},\ \eta_{uv}^{(i)} = \gamma_{iuv} \qquad (i, u, v = 1, 2, \ldots m),$$

unless, for $a_1, a_2, \ldots a_m$ not all zero, we have, simultaneously,

$$(89) \qquad \sum_{i=1}^{m} a_i \gamma_{ivu} = \sum_{i=1}^{m} a_i \theta_{uv}^{(i)} = 0,$$
$$(u, v = 1, 2, \ldots m),$$

in which case, neither the m matrices $E_1, E_2, \ldots E_m$ of order m representing, respectively, $e_1, e_2, \ldots e_m$ nor the m matrices $E'_1, E'_2, \ldots E'_m$ representing, respectively, $e'_1, e'_2, \ldots e'_m$, are linearly independent.[23]

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22 I here follow Cayley in denoting by $tr.\, M$ the transverse (or conjugate) of any given matrix M. Loc. cit., p. 31.

23 If $n = m$ and $\theta_{uv}^{(i)} = \gamma_{ivu}$ for $i, u, v = 1, 2, \ldots m$, the constituent of $E_i E_j$ in the uth row and vth column is

$$\sum_{w=1}^{m} \theta_{uw}^{(i)} \theta_{wv}^{(j)} = \sum_{w=1}^{m} \gamma_{iwu} \gamma_{jvw} = \sum_{w=1}^{m} \gamma_{ijw} \gamma_{wvu} = \sum_{w=1}^{m} \gamma_{ijw} \theta_{uv}^{(w)}$$

by (54); and, therefore,

$$E_i E_j = \sum_{w=1}^{m} \gamma_{ijw} E_w.$$

In this case, there is some number $A = \sum_{i=1}^{m} a_i e_i \neq 0$ of the system $(e_1, e_2, \ldots e_m)$ such that $AX = 0$ for any number $X = \sum_{i=1}^{m} x_i e_i$ of this system; since we should then have

$$AX = \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} a_i x_j \gamma_{ijk} e_k$$
$$= \sum_{j=1}^{m} \sum_{k=1}^{m} (\sum_{i=1}^{m} a_i \gamma_{ijk}) x_j e_k = 0.$$

Conversely, if $A = \sum_{i=1}^{m} a_i e_i \neq 0$ and $AX = 0$ for every number X of $(e_1, e_2, \ldots e_m)$, equations (89) are satisfied for at least one system of values $a_1, a_2, \ldots a_m$ not all zero, and we cannot assign to the θ's, nor to the η's, the values given by equations (88). In this case, we have

$$S_1 A e_i = 0 = S_2 A e_i \qquad (i = 1, 2, \ldots m);$$

and, therefore,

$$\Delta_1 = \Delta_2 = 0.$$

It is to be noted that equations (89) are the conditions, necessary and sufficient, that the reciprocal system shall contain a number $A' = \sum_{i=1}^{m} a_i e'_i \neq 0$ such that $X'A' = 0$ for any number $X' = \sum_{i=1}^{m} x_i e'_i$ of this system.

Further, we may take $n = m$ and put

$$(90) \qquad \theta_{uv}^{(i)} = \gamma_{uiv}, \ \eta_{uv}^{(i)} = \gamma_{viu} \qquad (i, u, v = 1, 2, \ldots m),$$

unless for $b_1, b_2, \ldots b_m$ not all zero, we have

$$(91) \qquad \sum_{i=1}^{m} b_i \gamma_{uiv} = \sum_{i=1}^{m} b_i \theta_{uv}^{(i)} = 0 \qquad (u, v = 1, 2, \ldots m),$$

in which case $E_1, E_2, \ldots E_m$ are not linearly independent, nor are

$E'_1, E'_2, \ldots E'_m$ linearly independent.[24] In this case, there is some number $B = \sum_{i=1}^{m} b_i e_i \neq 0$ of $(e_1, e_2, \ldots e_m)$ such that $XB = 0$ for every number $X = \sum_{i=1}^{m} x_i e_i$ of this system; and there is also a number $B' = \sum_{i=1}^{m} b_i e'_i \neq 0$ of the reciprocal system such that $B'X' = 0$ for every number X' of the reciprocal system. Conversely, if there is some number $B = \sum_{i=1}^{m} b_i e_i \neq 0$ of $(e_1, e_2, \ldots e_m)$ such that $XB = 0$ for every number X of this system (or if, for $B' = \sum_{i=1}^{m} b_i e'_i \neq 0$ and for any number X' of the reciprocal system, we have $B' X' = 0$) equations (91) are satisfied for some system of values $b_1, b_2, \ldots b_m$ not all zero, and we cannot assign to the θ's, nor to the η's, the values given by equations (90). When equations (91) are satisfied,

$$S_1 B e_i = 0 = S_2 B e_i \qquad (i = 1, 2, \ldots m);$$

and, therefore,

$$\Delta_1 = \Delta_2 = 0.$$

When the system $(e_1, e_2, \ldots e_m)$ contains a modulus it is not possible to satisfy equations (89) nor equations (91).

We may distinguish three cases. *First*, the given number system $(e_1, e_2, \ldots e_m)$ may contain both a number $A = \sum_{i=1}^{m} a_i e_i \neq 0$ and a number $B = \sum_{i=1}^{m} b_i e_i \neq 0$ such that $AX = 0$, $XB = 0$ for every number $X = \sum_{i=1}^{m} x_i e_i$ of the system, in which case the system does not contain a modulus and $\Delta_1 = \Delta_2 = 0$. In this case it is not possible to assign to the θ's the values given by either equations (88) or (90), nor to assign to the η's the values given by either of these equations. Nevertheless, it may be possible in this case to put $n = m$, provided $m > 2$, but not otherwise. Thus let $m = 3$, and let

$$e_1^2 = e_1, \qquad e_1 e_2 = 0, \qquad e_1 e_3 = e_3,$$

24 If $n = m$ and $\theta_{uv}^{(i)} = \gamma_{uiv}$ for $i, u, v = 1, 2, \ldots m$, it follows from (54) that $E_i E_j = \sum_{w=1}^{m} \gamma_{ijw} E_w$. Cf. note 23.

$$e_2e_1 = e_2e_2 = e_2e_3 = 0, \qquad e_3e_1 = e_3e_2 = e_3e_3 = 0:$$

if

$$A = a_2e_2 + a_3e_3 \neq 0, \qquad B = e_2,$$

we have

$$Ae_i = 0, \qquad e_iB = 0 \qquad (i = 1, 2, 3);$$

and we may now put $n = m = 3$, and

$$e_1 = \epsilon_{11}, \qquad e_2 = \epsilon_{23}, \qquad e_3 = \epsilon_{13}.$$

On the other hand, let $m = 2$ and let (e_1, e_2) contain a number $A \neq 0$ such that

$$A\,e_1 = A\,e_2 = 0.$$

In this case, we may, without loss of generality, put $A = e_1$, when we have

$$e_1^2 = 0, \qquad e_1e_2 = 0.$$

If now

$$e_i = \begin{pmatrix} \theta_{11}^{(i)}, & \theta_{12}^{(i)} \\ \theta_{21}^{(i)}, & \theta_{22}^{(i)} \end{pmatrix} \qquad (i = 1, 2),$$

we then have, since $e_1^2 = 0$,

$$\begin{pmatrix} \theta_{11}^{(1)}, & \theta_{12}^{(1)} \\ \theta_{21}^{(1)}, & \theta_{22}^{(1)} \end{pmatrix} = \varpi \begin{pmatrix} 0, & k \\ 0, & 0 \end{pmatrix} \varpi^{-1},$$

where $k \neq 0$ and the determinant of the matrix ϖ is not zero; and, therefore, since $e_1e_2 = 0$,

$$\begin{pmatrix} \theta_{11}^{(2)}, & \theta_{12}^{(2)} \\ \theta_{21}^{(2)}, & \theta_{22}^{(2)} \end{pmatrix} = \varpi \begin{pmatrix} \alpha, & \beta \\ 0, & 0 \end{pmatrix} \varpi^{-1},$$

where, without loss of generality, we may put $\alpha = 1$, $\beta = 0$, giving

$$e_2e_1 = e_1, \qquad e_2^2 = e_2.$$

This system, however, contains no number $B \neq 0$ for which

$$e_1\,B = e_2\,B = 0.$$

Second, the number system $(e_1, e_2, \ldots e_m)$ may contain either a number $A \neq 0$ such that $Ae_i = 0$ for $i = 1, 2, \ldots m$, or a number $B \neq 0$ such that $e_iB = 0$ for $i = 1, 2, \ldots m$, but not both. In this case, we may put $n = m$ and assign to the θ's and η's either the values given by equations (90) or equations (88) respectively.

Third, the system $(e_1, e_2, \ldots e_m)$ may contain neither a number $A \neq 0$ such that $Ae_i = 0$ for $i = 1, 2, \ldots m$ nor a number $B \neq 0$ such that $e_iB = 0$ for $i = 1, 2, \ldots m$, for which a sufficient, but not necessary condition, is the existence of a modulus, and, *a fortiori*, that $\Delta_1 \neq 0$. In this case, we may put $n = m$ and assign to the θ's the values given by equations (88), and to the η's the values given by equations (90). We then have

$$(92)\qquad \begin{aligned} A &= \sum_{i=1}^{m} a_i e_i = \sum_{i=1}^{m} \sum_{u=1}^{m} \sum_{v=1}^{m} a_i \gamma_{ivu} \epsilon_{uv}, \\ A' &= \sum_{i=1}^{m} a_i e'_i = \sum_{i=1}^{m} \sum_{u=1}^{m} \sum_{v=1}^{m} a_i \gamma_{viu} \epsilon_{uv}; \end{aligned}$$

and, therefore,

$$(93)\qquad \begin{aligned} S_1 A &= \frac{1}{m} \sum_{i=1}^{m} \sum_{u=1}^{m} a_i \gamma_{iuu} = \bar{S} A, \\ S_2 A &= \frac{1}{m} \sum_{i=1}^{m} \sum_{u=1}^{m} a_i \gamma_{uiu} = \bar{S} A'. \end{aligned}$$

On the other hand, if we assign to the θ's the values given by (90) and to the η's the values given by (88), which is now possible, we shall have

$$(94)\qquad \begin{aligned} A &= \sum_{i=1}^{m} a_i e_i = \sum_{i=1}^{m} \sum_{u=1}^{m} \sum_{v=1}^{m} a_i \gamma_{uiv} \epsilon_{uv}, \\ A' &= \sum_{i=1}^{m} a_i e'_i = \sum_{i=1}^{m} \sum_{u=1}^{m} \sum_{v=1}^{m} a_i \gamma_{iuv} \epsilon_{uv}; \end{aligned}$$

whence follows

$$(95)\qquad \begin{aligned} S_1 A &= \frac{1}{m} \sum_{i=1}^{m} \sum_{u=1}^{m} a_i \gamma_{iuu} = \bar{S} A', \\ S_2 A &= \frac{1}{m} \sum_{i=1}^{m} \sum_{u=1}^{m} a_i \gamma_{uiu} = \bar{S} A. \end{aligned}$$

When either the representation of the number system $(e_1\, e_2, \ldots e_m)$ and its reciprocal system given by equations (88) or by equations (90) fails, and indeed in any case, we may proceed as follows. Let $n = m + 1$, and let

$$(96\,a)\qquad \theta_{uv}^{(i)} = \gamma_{ivu}, \qquad \theta_{m+1,v}^{(i)} = \theta_{m+1,m+1}^{(i)} = 0 \qquad (i,\, u,\, v = 1, 2, \ldots m),$$

$$(96\,b)\qquad \theta_{u,m+1}^{(i)} = 0, \qquad \theta_{i,m+1}^{(i)} = 1 \qquad (i,\, u = 1, 2, \ldots m;\; u \neq i);$$

moreover, let

$$(97\,a)\qquad \eta_{uv}{}^{(i)} = \gamma_{viu}, \qquad \eta_{m+1,v}{}^{(i)} = \eta_{m+1,m+1}{}^{(i)} = 0 \qquad (i, u, v = 1, 2, \ldots m),$$

$$(97\,b)\qquad \eta_{u,m+1}{}^{(i)} = 0 \qquad \eta_{i,m+1}{}^{(i)} = 1 \qquad (i, u = 1, 2, \ldots m;\ u \neq i).$$

The m matrices $E_1, E_2, \ldots E_m$ which we thus obtain have the same multiplication table as the units of the system $(e_1, e_2, \ldots e_m)$ and are, moreover, linearly independent. For, if

$$c_1 E_1 + c_2 E_2 + \ldots + c_m E_m = 0,$$

then

$$\sum_{i=1}^{m} c_i \theta_{uv}{}^{(i)} = 0 \qquad (u, v = 1, 2, \ldots m+1);$$

and, therefore, in particular

$$c_u = \sum_{i=1}^{m} c_i \theta_{u,m+1}{}^{(i)} = 0 \qquad (u = 1, 2, \ldots m).$$

Further, the m matrices determined by the above values of the η's are also linearly independent and have the same multiplication table as the system $(e'_1, e'_2, \ldots e'_m)$ reciprocal to $(e_1, e_2, \ldots e_m)$. We now have

$$(98)\qquad \begin{aligned} A &= \sum_{i=1}^{m} a_i e_i = \sum_{i=1}^{m} a_i \Big(\sum_{u=1}^{m}\sum_{v=1}^{m} \gamma_{ivu} e_{uv} + e_{i,m+1}\Big), \\ A' &= \sum_{i=1}^{m} a_i e_i' = \sum_{i=1}^{m} a_i \Big(\sum_{u=1}^{m}\sum_{v=1}^{m} \gamma_{viu} \epsilon_{uv} + \epsilon_{i,m+1}\Big); \end{aligned}$$

and, therefore,

$$(99)\qquad \begin{aligned} S_1 A &= \frac{1}{m}\sum_{i=1}^{m}\sum_{u=1}^{m} a_i \gamma_{iuu} = \frac{m+1}{m}\bar{S} A, \\ S_2 A &= \frac{1}{m}\sum_{i=1}^{m}\sum_{u=1}^{m} a_i \gamma_{uiu} = \frac{m+1}{m}\bar{S} A'. \end{aligned}$$

We may also proceed as follows. Let $n = m + 1$, and let

$$(100\,a)\qquad \theta_{uv}{}^{(i)} = \gamma_{uiv}, \qquad \theta_{u,m+1}{}^{(i)} = \theta_{m+1,m+1}{}^{(i)} = 0 \qquad (i, u, v = 1, 2, \ldots m),$$

(100 *b*) $$\theta_{m+1,v}{}^{(i)} = 0, \qquad \theta_{m+1,i}{}^{(i)} = 1 \qquad (i, v = 1, 2, \ldots m;\ v \neq i);$$

moreover, let

(101 *a*) $$\eta_{uv}{}^{(i)} = \gamma_{iuv}, \qquad \eta_{u,m+1}{}^{(i)} = \eta_{m+1,m+1}{}^{(i)} = 0 \qquad (i, u, v = 1, 2, \ldots m),$$

(101 *b*) $$\eta_{m+1,v}{}^{(i)} = 0, \qquad \eta_{m+1,i}{}^{(i)} = 1 \qquad (i, v = 1, 2, \ldots m;\ v \neq i).$$

The m matrices $E_1, E_2, \ldots E_m$ thus obtained are linearly independent, as are also the m matrices $E'_1, E'_2, \ldots E'_m$; and the former have the same multiplication table as the units of the system $(e_1, e_2, \ldots e_m)$, while the latter have the same multiplication table as the units of the reciprocal system. We now have

(102) $$A = \sum_{i=1}^{m} a_i e_i = \sum_{i=1}^{m} a_i \left(\sum_{u=1}^{m} \sum_{v=1}^{m} \gamma_{uiv} \epsilon_{uv} + \epsilon_{m+1,i} \right)$$
$$A' = \sum_{i=1}^{m} a_i e_i' = \sum_{i=1}^{m} a_i \left(\sum_{u=1}^{m} \sum_{v=1}^{m} \gamma_{iuv} \epsilon_{uv} + \epsilon_{m+1,i} \right);$$

and, therefore,

(103) $$S_1 A = \frac{1}{m} \sum_{i=1}^{m} \sum_{u=1}^{m} \gamma_{iuu} = \frac{m+1}{m} \bar{S} A',$$
$$S_2 A = \frac{1}{m} \sum_{i=1}^{m} \sum_{u=1}^{m} \gamma_{uiu} = \frac{m+1}{m} \bar{S} A.$$

The fundamental properties of the scalar functions given in *theorem I* are more readily proved for the special case in which the number system is a quadrate than in the general case. What precedes in this section indicates how the properties of these functions may be made to depend upon the properties of the single scalar function of a quadrate.

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PRELIMINARY STUDY OF THE SALINITY OF SEA-WATER IN THE BERMUDAS.

By Kenneth L. Mark.

PRELIMINARY STUDY OF THE SALINITY OF SEA-WATER IN THE BERMUDAS.[1]

By Kenneth L. Mark.

Presented by E. L. Mark, January 8, 1913. Received February 3, 1913.

The objects of this investigation of the salinity and of the temperature of the waters in and about the Bermudas were the collection of data which would supplement those recorded for other parts of the Atlantic Ocean, especially by the "Conseil Permanent International pour L'Exploration de la Mer," and the study of the relation of the salinity to the depth below the surface, to the depth of the sea, and to the locality. A knowledge of these relations was desired as a part of the basis for studies on the distribution of oceanic organisms at the Bermudas.

For these purposes, therefore, samples of water were collected at various places and depths and the temperature of the water was noted in each case. The salinity of these samples was determined by the method used by the "Conseil International." This consists of the complete precipitation of the halides of the sea-water by the requisite amount of a standard solution of silver nitrate. The salinity and density of the samples are then calculated from the analytical results by the aid of the Hydrographical Tables of Knudsen.

Procedure.

The water was collected in a Buchanan[2]-Nansen[3] stop-cock water-bottle, as modified by Dr. H. B. Bigelow,[4] which allows the free passage of water through it during its descent, but can be made to enclose a sample of water at any desired depth. The water was immediately transferred through a brass cock to glass bottles. Care was taken to allow as little evaporation as possible during this transfer. The glass bottles were provided with porcelain stoppers with rubber rings, held on by wire, like the old-fashioned beer-bottle stop-

[1] Contributions from the Bermuda Biological Station for Research. No. 25.

[2] Challenger Report, Narrative, Vol. I, Part 1, p. 112–117.

[3] The Norwegian Sea, its Physical Oceanography based upon the Norwegian Researches 1900–1904, by B. Helland-Hansen and F. Nansen. Christiania 1909, *in* Report on Norwegian Fishery and Marine-Investigations, Vol. II, 1909, No. 2, p. 55.

[4] Dr. Bigelow's modification consists chiefly in the substitution of a messenger for the propeller used by Nansen, and will be described in a forthcoming report to be published in the Bull. Mus. Comp. Zoöl., Cambridge Mass.

pers. They were the so called "citrate of magnesia" bottles made by the Whitall, Tatum Co. The water was often stored in these bottles for several days before analyzing it.

The temperature of the water was determined by a Negretti and Zambra deep-sea thermometer, which was attached to the cable carrying the water bottle and directly beneath it. This thermometer had previously been compared with a thermometer standardized by the Deutsche Physikalische Technische Reichsanstalt.

The volumes of sea-water taken for analysis and the volumes of silver nitrate solution required to react with them were measured in a Knudsen pipette and a Knudsen burette respectively; both were made by R. Goetze, Leipzig. The former is an ordinary pipette of about 25 c. c. capacity, provided with a three-way cock at the top. This arrangement allows the liquid to pass beyond the cock when the pipette is being filled; but upon turning the cock so that the body of the pipette is in connection with the air through its third opening, the pipette empties itself and the excess of liquid remains behind. Thus an exact filling is always attained. The Knudsen burette also has a three-way cock at the top, which is used in the same way. It is filled through a side tube entering at the bottom. The lower part is graduated in terms of the standard used in Knudsen's Tables.[5] The volume between the smallest graduation marks is about .05 c. c. and the total capacity of the burette is about 42 c. c. The burette used in this investigation was carefully standardized and the graduations were found to be equal within the limit of accuracy of the readings.

A silver nitrate solution, containing about 42 grams of the salt per liter, was prepared and stored in a large bottle of brown glass. This bottle, which was placed on a shelf several feet above the table, was provided with a two-hole stopper, through one hole of which a glass tubule extended from the bottom of the bottle to the inlet tube at the bottom of the Knudsen burette. The other hole of the stopper was kept closed except during the filling of the bottle.

The solution was standardized as follows. A tube of standard sea-water, obtained from the "Conseil International" at Copenhagen, was opened and the Knudsen pipette was immediately filled from it. The water was run from the pipette into a beaker, allowing one minute for drainage, and three drops of a one percent sodium chromate solution were added as indicator. Silver nitrate solution was then

[5] Knudsen, Martin: Hydrographical Tables etc. Copenhagen, G. E. C. Gad, and London, Williams & Norgate. 1901. v + 63 pp.

run in from the burette, at first rapidly, but at the end drop by drop, until a faint reddish tinge in the precipitate was permanent for thirty seconds. This was taken as the end point. The difference between the volume of silver nitrate used, as expressed in burette divisions, and the figure accompanying the standard sample was the value "*a*" of the Knudsen tables. Obviously this method of standardization shows only the strength of the solution as compared with the standard upon which the Knudsen tables are based; but since the analyses also are expressed in terms of this standard, no further knowledge of the concentration is required.

A secondary standard sea-water was prepared by diluting ordinary sea-water till approximately the same volume of silver nitrate was required to react with it as was required to react with the original standard. The exact ratio of these standards was determined with great care, since the secondary standard was the one constantly used during the investigation. At the end of the work, however, the silver nitrate was again compared with the original standard and was found to be unchanged.

In making a series of analyses, the silver nitrate solution was each day titrated against the standard water, as just described. The various samples of water were then titrated in an exactly similar way, and finally the solution was again compared with the standard. The temperature of the room was noted during the progress of the work, but in no case did it vary enough to require any correction of the results. All determinations were made in duplicate.

Accuracy.

Since the method of analysis consists of comparing the amount of silver nitrate solution necessary to react with a definite amount of sea-water with that necessary to react with the same amount of sea-water of known composition, no standardization of the pipette used for measuring the water was necessary. The amount delivered by the pipette was constant, as the time allowed for drainage was always the same.

The determination of the capacity of the burette in absolute units was not required. Only the relation of the divisions to each other had to be known, and these were found to be equal within the limit of accuracy to which the volume could be read. These readings could be relied upon to one one-hundredth of a unit. As the total volume of solution used in a determination was about twenty units, the

Map of S. W. third of the Bermuda Islands.

N. B. The latitude and longtitude of this and following maps is that of the British Ordinance Survey published in 1902.

The latitude of all places is about 18 seconds of arc less on the chart of the Hydrographic Office.

proportional error was thus one in two thousand. This measurement limited the reliability of the whole analysis, which was thus trustworthy to five hundredths of one percent.

Corrections for change of temperature are unnecessary when the standardization and analyses are carried out under conditions sufficiently similar. As the limit of accuracy of reading the burette was one in two thousand, this allowed a variation in temperature of 8° C., which was a greater change than ever took place.

That no other sources of incidental error existed was shown by the facts that duplicate analyses always agreed to one part in two thousand or better, and that comparisons between the silver nitrate solution and the standard water always showed the same ratio to exist.

Table of Results.

Water Sample No.	Date	Locality: Name	Locality: Latitude N.	Locality: Longitude W.	Depth: below surface	Depth: of bottom	Temp. F.	Salinity
	July		32°	64°	fm.	fm.		
2	12	Brackish Pond Flats	21′15″	47′20″	1	7	82	36.47
3	12	Between No. 2 & Cobbler's Cut	19′45″	48′00″	7	8	81	36.58
4	13	Great Sound Sta. 1	16′10″	49′40″	6	7	82	36.83
5	16	" " " 2	17′10″	51′00″	8½	9	82	36.83
6	16	" " " 2			1		83	36.83
7	16	" " " 3	17′10″	50′20″	3½	4½	83.5	36.83
8	16	" " " 3			Surface		84	36.83
9	17	Little " " 2	15′40″	51′15″	5	6	84	36.87
10	17	" " " 1	15′05″	50′05″	Surface	6		34.92
11	19	Off Nonsuch Id. Sta. 0	19′46″	38′30″	10	ca. 12	81	36.51
12	19	" " " " 0			Surface		84	36.42
13	19	" " " " 1	19′44″	38′10″	18	ca. 20	80.5	36.45
14	20	Castle Harbor Sta. 0	19′50″	40′20″	6		83	36.49
15	20	" " " 1	20′05″	41′13″	6		83	36.51
16	20	" " " 1			Surface		84	36.51
17	20	Off Nonsuch Id. Sta. 2	19′36″	37′30″	50	ca. 275*	67	36.42
18	21	" " " " 1	19′44″	38′10″	20	ca. 30	81.5	36.45
19	22	" " " " 3	18′00″	36′10″	100	ca. 650*	64.5	36.42
20	28	Harrington Sd. Sta. 1	19′14″	42′56″	Surface	13	82	35.07
21	28	" " " 1			10		82	36.51
22	28	" " " 2	19′27″	42′30″	Surface	10½	82	35.52
23	28	" " " 2			9½		81	36.45
	Aug.							
24	2	" " " 1	19′14″	42″56″	Surface	12½	?	36.27
25	2	" " " 1			11½		83	36.45

The positions were usually determined by sighting conspicuous objects on shore.

The "depth below surface" and the "depth of bottom" were measured directly on the iron cable which carried the bottle. For the positions marked with an asterisk the depth of the bottom was not determined, but is that marked on the chart of the "Bermuda Islands" — issued by the Hydrographic Office, Washington, D. C., and corrected

Map of N. E. third of Bermuda Islands.

to 1900 — for the position indicated. The temperature and salinity were determined as described in the preceding pages.

Samples numbers 10, 20 and 22 were taken after heavy rains and therefore do not indicate the normal condition of the water. Samples numbers 24 and 25 were collected by E. L. Mark and were brought to Cambridge, where they were analyzed.

The pipette and burette used in Cambridge were not the ones used in Bermuda. The silver nitrate solution also was different and it was standardized against a different sample of Danish water. The agreement in the analytical results of samples 21 and 25, which were thus determined absolutely independently, serves to increase confidence in the reliability of all the analyses.

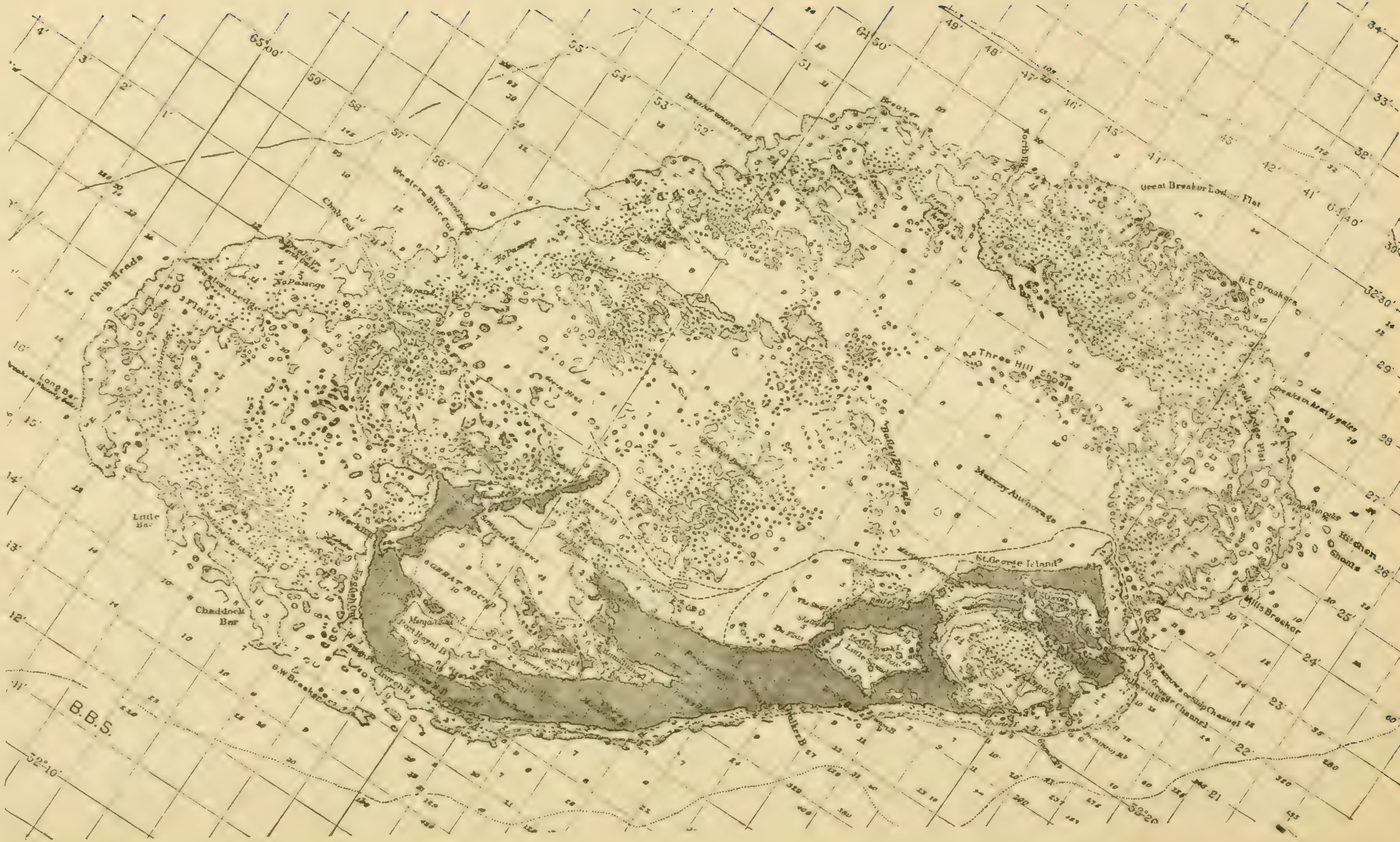
Discussion of Results.

The salinity of the water of the open ocean in the vicinity of Bermuda is undoubtedly that of the samples obtained off Nonsuch Island, namely 36.43 grams of salt per 1000 grams of sea-water. These samples were all collected outside the reefs, in positions exposed to the unbroken swell of the ocean from the south. In taking an average of the results, however, No. 19 has been omitted, as that sample was collected under unfavorable conditions. The depth below the surface, even down to 100 fathoms, appears to make no difference in salinity, except after recent rainfall.

The water of the shallow enclosed bays was found to increase in salinity with remoteness from the open ocean. This becomes particularly noticeable by comparing samples 2, 3, 4 and 9, where the successive samples were collected farther and farther within the shelter of the reefs and islands. The samples taken in Castle Harbor. also, were in good agreement with predictions based upon the connection of that bay with the ocean. The salinity of the water from the bottom of Harrington Sound, on the contrary, was surprisingly small, as compared with that of other enclosed bodies of water. It was found to be nearly the same as that of the open ocean, although the inlets to this sound are so narrow that the tide rises only about one fourth as much as it does outside.

Summary.

Data concerning the salinity and temperature of sea-water in the Bermudas are presented. These indicate that the salinity is independent of depth even down to 100 fathoms, but increases considerably as the water becomes more and more enclosed.



Map of the Bermuda Islands, reduced from the map of the Ordinance Survey.

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ON CERTAIN FRAGMENTS OF THE PRE-SOCRATICS:

CRITICAL NOTES AND ELUCIDATIONS.

By William Arthur Heidel,

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ON CERTAIN FRAGMENTS OF THE PRE-SOCRATICS:

CRITICAL NOTES AND ELUCIDATIONS.

By William Arthur Heidel.

Presented April 9. Received February 28, 1913.

The collection of notes here presented owes its origin to a request for suggestions from Professor Hermann Diels when he was engaged in revising *Die Fragmente der Vorsokratiker* for the third edition, since published (1912). In response to his courteous invitation I sent, together with a list of errors noted in the second edition, a number of proposals for the emendation of texts and the interpretation of doubtful passages. Had I then had the requisite leisure it would have been my duty to explain and defend my suggestions; since that was impossible, the notes then submitted were in effect mere marginalia, to notice which as fully as Professor Diels has done required uncommon courtesy. To be permitted to contribute even in a small measure to so excellent an instrument of scholarship is an honor not lightly to be esteemed. The renewal of certain suggestions previously made but not accepted by Professor Diels is due solely to the desire to enable him and other scholars to judge of their merits when the case for them is properly presented; others, in the correctness of which I still have confidence, are here left unnoticed because, as referred to in the third edition, they are already recorded and bear on their face such credentials as are necessary for a proper estimate of their claims. But I here present for the first time a considerable number of proposed readings and interpretations, the importance of which, if approved by the judgment of competent scholars, must be at once apparent to the historian of Greek thought. If it were customary to dedicate such studies, I should dedicate these notes to my honored teacher and friend, Professor Diels, to whom I owe more for instruction and inspiration during a quarter of a century than I can hope to repay. In the following pages reference is made to chapter, page, and line of his second edition (V^2), because the pages of this edition are noted also in the margin of the third (V^3).

c. 2. Anaximander.

V^2 12, 28. Plin. N. H. 2. 31. Obliquitatem eius [sc. zodiaci] intellexisse, hoc est rerum foris aperuisse, Anaximander Milesius traditur primus.

Perhaps the full significance of the clause 'hoc. . .aperuisse,' whatever the source of the sentiment, is hardly appreciated. The Delphin edition refers to Plin. N. H. 35. 36 'artis foris apertas ab Apollodoro Zeuxis intravit'; but that is not a real parallel. For such we turn rather to Lucret. 1, 66 sq.

> Graius homo [sc. Epicurus] . . .
> . . . . eo magis acrem
> irritat animi virtutem, effringere ut arta
> naturae primus portarum claustra cupiret.
> ergo vivida vis animi pervicit, et extra
> processit longe flammantia moenia mundi
> atque omne immensum peragravit mente animoque,
> unde refert nobis victor quid possit oriri
> quid nequeat, finita potestas denique cuique
> quanam sit ratione atque alte terminus haerens.

The same conception recurs Lucret. 3, 14 sq.

> nam simul ac ratio tua coepit vociferari
> naturam rerum, divina mente coorta,
> diffugiunt animi terrores, moenia mundi
> discedunt, totum video per inane geri res.

For these passages I would refer the reader to my essay, *Die Bekehrung im klassischen Altertum, mit besonderer Berücksichtigung des Lucretius,* Zeitschrift für Religionspsychologie, Bd. III, Heft 11, p. 13 sq. Heinze's parallels to Lucret. 3, 14 sq. ought to have made clear to him that there is here an allusion to the ecstatic ἐποπτεία of the mysteries evoked, as I pointed out, by the pronouncement of the ἱερὸς λόγος (ratio. . .divina mente coorta), coming as the climax of the rites of initiation, when the mystae catch a vision and seize the significance of the world (ἐποπτεύειν δὲ καὶ περινοεῖν τήν τε φύσιν καὶ τὰ πράγματα), according to Clem. Alex. Strom. 5. 11. Müller on Lucil. 30, 1 compared Lucret. 1, 66 sq., and the editors of Lucretius have copied the reference, although the resemblance is altogether superficial and without significance. Recently Professor Reid, *Lucretiana*, Harvard Studies in Class. Philology, Vol. 22, p. 2, has once more drawn attention to Sen. Dial. 8. 5. 6, Cogitatio nostra caeli munimenta perrumpit nec contenta est id, quod ostenditur, scire: illud, inquit, scrutor, quod ultra mundum iacet, utrumne profunda vastitas sit an et hoc ipsum terminis suis cludatur, etc. I doubt, however, the correctness of his statement that Seneca was here imitating Lucretius. It seems to me more probable that both authors are reproducing with some freedom the thought of an earlier, perhaps Stoic, writer, who may have been Posidonius. Be that as it may, the thought common to Lucretius, Seneca, and Pliny (and I may add, Bishop Dionysius, ap. Euseb. P. E. 14. 27. 8) is that a great revelation has come, rending as it were the curtain or outer confines of the world and permitting a glimpse into the utmost secrets of nature. Such a revelation, according to Pliny, ensued upon the discovery of the obliquity of the ecliptic; and a study of early Greek cosmology clearly demonstrates the capital importance attached to it. To some aspects of this question I drew attention in my article, *The Δίνη in Anaximenes and Anaximander*, Class. Philol., Vol. 1, p. 279 sq. Very much more remains to be said, but I shall have to reserve the matter for a future occasion.

V² 13, 2. *Ἀναξίμανδρος . . . ἀρχήν τε καὶ στοιχεῖον εἴρηκε τῶν ὄντων τὸ ἄπειρον.*

For the meaning of ἀρχή Diels refers in V³ to the preliminary statement in my Περὶ Φύσεως, Proceed. of Amer. Acad. of Arts and Sc., Vol. 45, p. 79, n. 3. The subject has now received a fuller treatment in my essay *On Anaximander*, Class. Philol., Vol. 8 (1912), p. 212 sq. To the statement there given, though much might be said by way of enlargement and confirmation, I think it unnecessary to add anything, except to say that the results of my investigations dovetail admirably into certain other observations recently made by different scholars. I refer among others to the views of Otto Gilbert as to the original meaning of the 'elements' set forth in his *Griech. Religionsphilosophie*, 1911, which reached me at the same time with the off-prints of my essay; and to Mr. Cornford's conception of Μοῖρα as developed in *From Religion to Philosophy*, 1912. Unfortunately both these authors accept the Peripatetic tradition regarding the meaning of Anaximander's ἀρχή; consequently their observations remain fruitless when they proceed to interpret the early history of Greek philosophy.

V[2] 13, 7. *διδόναι γὰρ αὐτὰ δίκην καὶ τίσιν ἀλλήλοις τῆς ἀδικίας κατὰ τὴν τοῦ χρόνου τάξιν.*

In his note on this passage (V[3] 15, 28) Diels repeats his former explanation, "ἀλλήλοις: *dativus commodi*: das Untergehende dem Überlebenden und dieses wieder untergehend dem künftig Entstehenden. Vgl. Eur. Chrysipp. fr. 839, 13." This interpretation, which is that now currently accepted, rests obviously on the assumption that the preceding sentence in Simplicius, *ἐξ ὧν δὲ ἡ γένεσίς ἐστι τοῖς οὖσι, καὶ τὴν φθορὰν εἰς ταῦτα γίνεσθαι κατὰ τὸ χρεών*, preserves the authentic words of Anaximander and that, in consequence, it is individual things or objects (*τὰ ὄντα*) that mutually exact and pay the penalty for injustice done to one another. On that view Diels's elaboration of the implications of ἀλλήλοις is both obvious and necessary. I believe, however, that in my essay *On Anaximander*, p. 233 sq., I showed conclusively (1) that it is not individual objects but the contraries, hot and cold, that encroach on one another and suffer periodic punishment inflicted by each on the other (wherefore ἀλλήλοις is here to be interpreted as a strict reciprocal and not as Diels proposes), and (2) that when this mutual *κόλασις* is said to recur *κατὰ τὴν τοῦ χρόνου τάξιν*, reference is had to the seasonal excess of the hot in summer and of the cold in winter. The strict limitations of space imposed upon my essay led to the exclusion of many things which I reluctantly omitted, and did not admit of a full statement of my views. I propose, therefore, here to add a few points which may serve to explain and confirm them. Zeller insists that for Anaximander one pair of contraries only, the hot and the cold, existed, at least as primarily proceeding from the *ἄπειρον*; this would rule out the moist and the dry, which are mentioned with the first pair by Simplicius, as due to Aristotle. This may be true, but it is not necessarily so; for the Empedoclean and Hippocratic group of four contraries is too well attested, and if, as seems certain, Anaximander had in mind the seasonal changes it is hard to conceive of him as overlooking the differences in drought and moisture which Simplicius mentions with those of heat and cold. A passage strikingly illustrating and interpreting that of Simplicius is found in Philo, De Anim. Sacrif. Idon. II. 242 Mang. *ἡ δὲ εἰς μέλη τοῦ ζῴου διανομὴ δηλοῖ, ἤτοι ὡς ἓν τὰ πάντα ἢ ὅτι ἐξ ἑνός τε καὶ εἰς ἕν· ὅπερ οἱ μὲν κόρον καὶ χρησμοσύνην ἐκάλεσαν, οἱ δ' ἐκπύρωσιν καὶ διακόσμησιν· ἐκπύρωσιν μὲν κατὰ τὴν τοῦ θεοῦ δυναστείαν τῶν ἄλλων ἐπικρατήσαντος, διακόσμησιν δὲ κατὰ τὴν τῶν τεττάρων στοιχείων ἰσονομίαν, ἣν ἀντιδιδόασιν ἀλλήλοις.* Philo is of course far from thinking of Anaximander and has in mind Heraclitus and the Stoics only; but we know that the conception of Heraclitus was older than the fifth century, being traceable to Alcmaeon, a contemporary of Anaximander. The *ἰσονομία τῶν δυνάμεων* (Alcmaeon, fr. 4), as the condition of health, and the *ἐπικράτεια* and *πλεονεξία* of the several constituents of the human body as the cause of disease, are fixed factors of practically the whole medical tradition of Greece. We may therefore confidently affirm that the *ἰσονομία* <*τῶν στοιχείων* or rather *τῶν ἐναντιοτήτων*> *ἣν ἀντιδιδόασιν ἀλλήλοις*, which Philo attributes to Heraclitus and the Stoics, applies with equal propriety to Anaximander, and explains his meaning. These different factors, correlated also with the seasonal changes, are mentioned by Plato, Legg. 906 C, *φαμὲν δ' εἶναί που τὸ νῦν ὀνομαζόμενον ἁμάρτημα, τὴν πλεονεξίαν, ἐν μὲν σαρκίνοις σώμασιν νόσημα καλούμενον, ἐν δὲ ὥραις ἐτῶν καὶ ἐνιαυτοῖς λοιμόν, ἐν δὲ πόλεσιν καὶ πολιτείαις τοῦτο αὐτό, ῥήματι μετεσχηματισμένον, ἀδικίαν.* The connection, here hardly more than suggested, is clearly noted by Plato, Symp. 188 A, *ἐπεὶ καὶ ἡ τῶν ὡρῶν τοῦ ἐνιαυτοῦ σύστασις μεστή ἐστιν ἀμφοτέρων τούτων, καὶ ἐπειδὰν μὲν πρὸς ἄλληλα τοῦ κοσμίου τύχῃ ἔρωτος ἃ νυνδὴ ἐγὼ ἔλεγον, τά τε θερμὰ καὶ τὰ ψυχρὰ καὶ ξηρὰ καὶ ὑγρά, καὶ ἁρμονίαν καὶ κρᾶσιν λάβῃ σώφρονα, ἥκει φέροντα εὐετηρίαν τε καὶ ὑγίειαν ἀνθρώποις καὶ τοῖς ἄλλοις ζῴοις τε καὶ φυτοῖς, καὶ οὐδὲν ἠδίκησεν· ὅταν δὲ ὁ μετὰ τῆς ὕβρεως Ἔρως ἐγκρατέστερος περὶ τὰς τοῦ ἐνιαυτοῦ ὥρας γένηται, διέφθειρέν τε πολλὰ καὶ ἠδίκησεν.* On this passage cp. Hirzel, *Themis, Dike und Verwandtes*, p. 220 sq. The medical doctrine expounded by Eryximachus in the Symposium, although perhaps slightly colored with Heraclitean thought, is that of the Hippocratic treatises, notably of *Περὶ φύσιος ἀνθρώπου*, from which we may quote one passage, c 7 (6.48 L.), *κατὰ φύσιν γὰρ αὐτέῳ ταὐτά ἐστι μάλιστα τοῦ ἐνιαυτοῦ . . . ἔχει μὲν οὖν ταῦτα πάντα αἰεὶ τὸ σῶμα τοῦ ἀνθρώπου, ὑπὸ δὲ τῆς περιισταμένης ὥρης ποτὲ μὲν πλείω γίνεται αὐτὰ ἑωυτῶν, ποτὲ δὲ ἐλάσσω, ἕκαστα κατὰ μέρος* [= *ἐν μέρει*] *καὶ κατὰ φύσιν* [sc. *τοῦ ἐνιαυτοῦ*] . . . *ἰσχύει δὲ ἐν τῷ ἐνιαυτῷ ποτὲ μὲν ὁ χειμὼν μάλιστα, ποτὲ δὲ τὸ ἦρ, ποτὲ δὲ τὸ θέρος, ποτὲ δὲ τὸ φθινόπωρον· οὕτω δὲ καὶ ἐν τῷ ἀνθρώπῳ ποτὲ μὲν τὸ φλέγμα ἰσχύει, ποτὲ δὲ τὸ αἷμα, ποτὲ δὲ ἡ χολή, πρῶτον μὲν ἡ ξανθή, ἔπειτα δ' ἡ μέλαινα καλεομένη.* Not to repeat what I have elsewhere said in regard to the doctrines of Heraclitus and Empedocles, I refer the reader to my essay *Qualitative Change in Pre-Socratic Philosophy*, Archiv für Gesch. der Philos., Vol. 19. pp. 360 sq. and 365. Since the *ἀδικία* and the *δίκη καὶ τίσις* of Anaximander refer not to the origin and destruction of individual objects but to the successive encroachment of the elemental opposites one on another in the seasonal changes of the year, it follows that the words of Anaximander cannot be used to support the interpretation of his ἄπειρον-ἀρχή as a metaphysical world-ground in which the sin of individual existence is punished by the reabsorption of the concrete objects of experience. For this see *On Anaximander*, p. 225, n. 3, and my review of James Adam, *The Vitality of Platonism and Other Essays*, Amer. Journ. of Philol., Vol. 33 (1912), p. 93 sq.

> **V**[2] **13, 34.** [Plut.] Strom. 2, *φησὶ δὲ τὸ ἐκ τοῦ ἀιδίου γόνιμον θερμοῦ τε καὶ ψυχροῦ κατὰ τὴν γένεσιν τοῦδε τοῦ κόσμου ἀποκριθῆναι καί τινα ἐκ τούτου φλογὸς σφαῖραν περιφυῆναι τῷ περὶ τὴν γῆν ἀέρι ὡς τῷ δένδρῳ φλοιόν. ἧστινος ἀπορραγείσης καὶ εἴς τινας ἀποκλεισθείσης κύκλους ὑποστῆναι τὸν ἥλιον καὶ τὴν σελήνην καὶ τοὺς ἀστέρας.*

The words *τὸ . . . ψυχροῦ* have been much discussed and variously interpreted. Zeller, I[a] 220, n. 1, pronounces the text corrupt and suggests *φησὶ δ' ἐκ τοῦ ἀιδίου τὸ γόνιμον θερμόν τε καὶ ψυχρόν*, rejecting Neuhäuser's obviously correct proposal to take the genitives *θερμοῦ* and *ψυχροῦ* as depending on *γόνιμον*. Burnet, *Early Greek Philosophy*[2], p. 66, retaining the traditional text, renders, "Something capable of begetting hot and cold was separated off from the eternal." If we were dealing with a poet we might take such liberties, but we may safely dismiss the interpretation as impossible for prose. Diels gives no definite indication of his understanding of the words, but claims *γόνιμον* as possibly belonging to Anaximander, certainly to Theophrastus, referring in support of his contention to Porphyr. De Abstin. 2. 5. The text of Porphyry, however, throws no light on ours, and there is good reason to doubt whether we may attribute the word to Theophrastus. In all probability we are dealing with a Stoic source, however related to Theophrastus; for *γόνιμον* seems to be a congener to the *λόγος σπερματικός* of the Stoics. Cp. Marc. Aurel. 9. 1. 4, *λέγω δὲ τὸ χρῆσθαι τούτοις ἐπίσης τὴν κοινὴν φύσιν ἀντὶ τοῦ συμβαίνειν ἐπίσης κατὰ τὸ ἑξῆς τοῖς γινομένοις καὶ ἐπιγινομένοις ὁρμῇ τινι ἀρχαίᾳ τῆς προνοίας, καθ' ἣν ἀπό τινος ἀρχῆς ὥρμησεν ἐπὶ τήνδε τὴν διακόσμησιν, συλλαβοῦσά τινας λόγους τῶν ἐσομένων καὶ δυνάμεις γονίμους ἀφορίσασα ὑποστάσεών τε καὶ μεταβολῶν καὶ διαδοχῶν τοιούτων.* It seems fairly certain that *τὸ . . . γόνιμον θερμοῦ τε καὶ ψυχροῦ* is the Stoic *ἄποιος ὕλη* which contains *δυνάμει* the hot and the cold of the cosmos. We thus find masked in Stoic phraseology the *φύσις ἀόριστος* of Theophrastus. This *γόνιμον θερμοῦ τε καὶ ψυχροῦ* is, at least in extent, not identical with the *ἄπειρον* itself, but was "separated off" from it at the origin of our cosmos. It must, therefore, be that portion of the *ἄπειρον-ἀρχή* which gave rise to the present world. Tannery, Zeller, Burnet, and others regard *ἐκ τοῦ ἀιδίου* as referring to the *ἄπειρον*, thinking perhaps of certain passages referring to Xenophanes, Melissus, and Anaxagoras; but Zeller at least perceived that this was not to be accepted without considerable violence to the text. I maintain the correctness of my suggestion, *On Anaximander*, p. 229, n. 2, that we are to supply *ἀπὸ τοῦ ἀπείρου* with *ἀποκριθῆναι*, whether it ever stood in the text or not, and that the phrase *ἐκ τοῦ ἀιδίου*, which stands just where it belongs, means "from eternity." We are familiar with *ἐς ἀΐδιον*, "forever," and Marc. Aurel. 2. 14; 4. 21; 10. 5 thrice uses *ἐξ ἀιδίου* in that sense, and numerous other instances might be cited. It happens that I cannot point to another instance of *ἐκ τοῦ ἀιδίου*, but the analogy of parallel expressions occurring with and without the article would render it not at all surprising if such should be found in late authors. The expression under consideration may be taken with confidence to mean "*The eternal substratum capable by dynamic evolution of producing hot and cold.*"

The remainder of this interesting passage also deserves renewed consideration. It speaks of a 'sphere of flame,' and this appears to be generally accepted as establishing the sphericity of Anaximander's cosmos. Diels has not, to my knowledge, expressed himself in unmistakable terms; but his description of the *φλογὸς σφαῖρα* as a "Waberlohe" would be best taken as applicable to a circle. A conclusion so opposed to the apparent meaning of the word *σφαῖρα* will surprise no one who is familiar with the general ambiguity of words in Greek meaning 'round' and the uncritical habit among later authors of attributing Eudoxian notions to earlier cosmologists and astronomers, provided that the remainder of the statement points to a circle rather than a sphere. I have no intention of discussing here the whole subject, which would require a connected examination of all the data of early Greek cosmology, but propose to confine my attention to this one passage. It is pertinent, however, to remark that on other grounds I have elsewhere found reasons for doubting the correctness of the Aristotelian account, which places the earth in Anaximander's scheme at the center of a sphere; for if Aristotle's authority is accepted as final, the interpretation here offered will be ruled out of court without a hearing. See my essay, *The Δίνη in Anaximenes and Anaximander*, Class. Philol., Vol. 1, p. 279 sq., especially p. 281.

Let us then address ourselves to the text: *καί τινα ἐκ τούτου φλογὸς σφαῖραν περιφυῆναι τῷ περὶ τὴν γῆν ἀέρι ὡς τῷ δένδρῳ φλοιόν· ἧστινος ἀπορραγείσης καὶ εἴς τινας ἀποκλεισθείσης κύκλους ὑποστῆναι τὸν ἥλιον καὶ* τὴν σελήνην καὶ τοὺς ἀστέρας. The orthodox view appears to be that a sphere of flame is somehow exploded and (rather curiously!) reduced to a succession of circles of flame confined within an envelope of mist; these circles being those which constitute sun, moon, and stars. We have come to expect definite analogies and clear 'Anschauung' among the early Greek philosophers; and the severe strain which the current view puts on the imagination would of itself cast suspicion on it. We might nevertheless feel compelled, however reluctantly, to accept it, if the details of the account itself pointed to it or were even consistent with it. It will probably be conceded that — the term σφαῖρα apart — it is vastly simpler to conceive of a wide annular mass breaking up into annular parts than to imagine the same result ensuing from the destruction of a sphere. But as a matter of fact our text says nothing that may fairly be interpreted as implying the breaking or exploding of the sphere. The crucial words are περιφυῆναι and ἀπορραγείσης. Perhaps the real force of neither word has been appreciated. Here περιφυῆναι means that the "sphere" at first "snugly fitted" or was "closely attached to" the "air" which encircles the earth; whereas ἀπορραγείσης states merely that subsequently it became detached, as even a superficial attention to the normal meaning of the terms will convince the reader. The contrast may be illustrated by Arist. Hist. Animal. 5. 19. 552^a 3, ταῦτα δὲ χρόνον μέν τινα κινεῖται προσπεφυκότα, ἔπειτ' ἀπορραγέντα φέρεται κατὰ τὸ ὕδωρ, αἱ καλούμεναι ἀσκαρίδες. Besides, ἀπορρηγνύναι is not the proper word to use of the tearing of such an envelope as a sphere of flame; Greek writers so use ῥηγνύναι, διαρρηγνύναι, and περιρρηγνύναι, especially the last-mentioned, as might be shown by a long list of examples derived from Aristotle and other authors. The same general conception is implied in the simile ὡς τῷ δένδρῳ φλοιόν. We may not press similes beyond the immediate point of comparison, which in this instance is the snugness of the fit; but if one is to press it, it is obvious that the bark of a tree is annular rather than spherical. It will hardly serve the interest of the objector to refer to Anaximander's notion of the prickly integument of the first animals, V^2 17, 18, ἐν ὑγρῷ γενηθῆναι τὰ πρῶτα ζῷα φλοιοῖς περιεχόμενα ἀκανθώδεσι . . . περιρρηγνυμένου τοῦ φλοιοῦ; for there, as περιρρηγνυμένου sufficiently shows, the conception is altogether different. It is quite possible, as later Greek thinkers prove, to conceive of the cosmos and the human embryo as equally inclosed in a ὑμήν without pressing the comparison beyond reason. I have noted with some interest another passage in which the meaning of ἀπορρηγνύναι has been similarly misconceived.

Arist. Hist. Animal. 5.18. 549ᵇ 31 sq. the spawning of the octopus and the development of its young are described. There we read 550ᵃ 3, *τὰ μὲν οὖν τῶν πολυπόδων μεθ' ἡμέρας μάλιστα πεντήκοντα γίνεται ἐκ τῶν ἀπορραγέντων πολυπόδια, καὶ ἐξέρπει, ὥσπερ τὰ φαλάγγια, πολλὰ τὸ πλῆθος.* Professor Thompson in his recent translation renders it thus: "Some fifty days later, *the eggs burst and the little polupuses creep out*" [italics mine]. In fact there is no reference to the bursting of the eggs. Aristotle's meaning is that that which develops into the individual polyp becomes detached from the vine-like mass which he has previously described, and that the young crawl forth (not from the eggs, but) from the hole or vessel in which the spawn was deposited.

To return to the cosmology of Anaximander: the words *καὶ εἴς τινας ἀποκλεισθείσης κύκλους* refer not specifically to *σφαῖρα* but to *φλόξ*. The Waberlohe by some means, doubtless identical with that which detached the envelope of flame from the envelope of "air" was segregated into a number of annular masses, each like the earth inclosed in an envelope of "air." This segregation is not specifically mentioned but must be inferred; and we can guess only at the immediate cause of it. Now it is fairly certain that Anaximander knew the obliquity of the ecliptic or, as the early Greeks seem regularly to have called it, the inclination or dip of the zodiac or ecliptic. Pliny, as we have seen, attached great significance to its discovery, and so far as we know all the early Greek philosophers regarded it as an actual dipping resulting from some cause subsequently to the origin of the cosmos. Such an event would amply explain the initial break between the respective envelopes of "air" and flame; what caused the subsequent disintegration of the circle of flame into separate rings we do not know and perhaps it were idle further to speculate.

V² 17, 18. Aet. 5. 19. 4, *'Αναξίμανδρος ἐν ὑγρῷ γενηθῆναι τὰ πρῶτα ζῷα φλοιοῖς περιεχόμενα ἀκανθώδεσι, προβαινούσης δὲ τῆς ἡλικίας ἀποβαίνειν ἐπὶ τὸ ξηρότερον καὶ περιρρηγνυμένου τοῦ φλοιοῦ ἐπ' ὀλίγον μεταβιῶναι.*

In V¹ ᵃⁿᵈ ² the word *χρόνον* was omitted by mistake after *ἐπ' ὀλίγον*; his attention having been called to the omission by me, Diels has restored it in V³. Ordinarily a fact of this sort would hardly deserve to be noted; but since the false reading has found its way into Kranz's *Wortindex*, s. v. *μεταβιοῦν*, and has been quoted without question by various writers, as e. g. by Otto Gilbert, *Die meteorol. Theorien des gr. Altertums*, p. 332, n. 1, and Kinkel, *Gesch. der Philos.*, I. p. 7*, it calls for more than a tacit correction. This is the more necessary because the text has been very generally misunderstood and false conclusions have been drawn from it. It is perhaps unnecessary to recount in detail this chapter of curious errors. I have no means of knowing what interpretation Diels now puts on the text; but in the absence of any indication in his notes it seems reasonable to assume that he still adheres to the view briefly set forth in the index to his *Doxographi Graeci*, s. v. μεταβιοῦν: "mutare vitam [cf. μεταδιαιτᾶν]." This may be said to have been the common view of recent interpreters, until Burnet, *Early Greek Philosophy²*, p. 72 sq., correcting the version of his first edition, returned to the correct rendering of Brucker, "ruptoque cortice non multum temporis supervixisse," which Teichmüller with characteristic ignorance of Greek sharply condemned, *Studien zur Gesch. der Begriffe*, p. 64, n. Tannery, *Pour l'histoire de la science hellène*, pp. 87 and 117, gives in effect two renderings, each incorrect. The important point to note is that ἡλικία can refer to nothing but the age of the individual; and that ἐπ' ὀλίγον χρόνον can have but one meaning, to wit, "for a short time only." The force of μεταβιῶναι must, therefore, be determined with reference to these known quantities of the problem. This once granted, the decision between the rival claims of *vitam mutasse* and *supervixisse* is easy and certain. To be sure, μετά in composition far more frequently implies change than it denotes 'after'; but μεταδειπνεῖν is as well attested as μεταδιαιτᾶν. However if, as seemed plausible from Diels's earlier editions, it were possible to conceive that the correct text was ἐπ' ὀλίγον μεταβιῶναι, one might have inclined to take ἐπ' ὀλίγον in the sense of "to a small extent," as in Arist. Meteor. 350^b 28 and Marcellinus, Vita Thucyd. 36, and to interpret μεταβιῶναι as referring to a change in the mode of life. Another possibility, which I have considered, would be to take ἐπ' ὀλίγον and μεταβιῶναι in the sense just indicated and to read χρόνῳ for χρόνον, thus obtaining the sense "they changed their mode of life to a small extent in course of time." This suggestion was very tempting to one who was prepared to find an anticipation of Darwinism in Anaximander; but against all these proposals ἡλικία stands with its inexorable veto. The sort of change contemplated would require more than one life-time, and ἡλικία limits the action of μεταβιῶναι to the life-period of the individual. We must therefore content ourselves with the rendering "As they advanced toward maturity the first animals proceeded from the wet on to the drier ground and as their integument burst (and was sloughed off) they survived but a little while." Perhaps this interpretation may be further supported by a comparison of the view thus obtained with that of the origin of animal life attributed to Archelaus, V^2 324, 18, *περὶ δὲ ζῴων φησίν, ὅτι θερμαινομένης τῆς γῆς τὸ πρῶτον ἐν τῷ κάτω μέρει, ὅπου τὸ θερμὸν καὶ τὸ ψυχρὸν ἐμίσγετο, ἀνεφαίνετο τά τε ἄλλα ζῷα πολλὰ καὶ οἱ ἄνθρωποι, ἅπαντα τὴν αὐτὴν δίαιταν ἔχοντα ἐκ τῆς ἰλύος τρεφόμενα (ἦν δὲ ὀλιγοχρόνια)· ὕστερον δὲ αὐτοῖς ἡ ἐξ ἀλλήλων γένεσις συνέστη.*

c. 3. Anaximenes.

V^2 17, 37. *οὗτος ἀρχὴν ἀέρα εἶπεν καὶ τὸ ἄπειρον.*

In his note in V^3 Diels says: "Missverständnis oder Verderbnis statt *καὶ τοῦτον ἄπειρον.*" This suggestion is plausible, but far from certain. As I showed in my study of *ἀρχή, On Anaximander,* various vestiges of an earlier cosmological, non-metaphysical, sense of that word survive in Aristotle; it can hardly be thought impossible that the same should be true of Theophrastus, from whom this statement of Diogenes ultimately derives. Indeed, as we shall see when we discuss Diogenes's account of the cosmology of Leucippus (cp. p. 732, on V^2 343, 1), there is at least one such vestige, though almost obliterated by the unintelligence of excerptors or copyists. But, leaving that for the present aside, we are credibly informed that Anaximenes regarded the outer "air" as boundless, upon which fact Diels relies for his proposed correction; and we know that Anaximenes held the doctrine of the cosmic respiration, in accordance with which the cosmos subsists, as it arises, by receiving its substance from the encircling *ἄπειρον* in the form of *πνεῦμα* or breath. This *πνεῦμα* comes from and returns to the *ἄπειρον,* which is therefore nothing else but an *ἀρχὴ καὶ πηγή,* or reservoir, of *πνεῦμα.* We thus have a complete parallel, so far as concerns the *πνεῦμα-ἀήρ,* to the doctrine of the early Pythagoreans reported by Aristotle. Cp. my *Antecedents of Greek Corpuscular Theories,* p. 139 sq. In V^3 I. 354, 16 sq. Diels has corrected the text of Aristotle along the lines I suggested. I cannot, however, approve of the bracketing of *χρόνου,* ib. 22, as proposed by Diels.

V^2 18, 30 sq. Hippolytus, Ref. 1.7.

The corrupt state of the text of Hippolytus's Philosophumena, especially in the first book, is well known. With the aid of Cedrenus Diels has been able to set many passages right; yet much remains to be done. In 1. 7, the chapter devoted to Anaximenes, several additions or interpolations which ought to be removed or bracketed still encumber the text, though we cannot determine to whom they are due. Diels formerly bracketed *πυκνότατον* (V² 18, 39), but now contents himself with characterizing it as an inaccuracy of the late compiler. There are, however, two larger additions which are false and misleading. V² 18, 31, *ἀέρα ἄπειρον ἔφη τὴν ἀρχὴν εἶναι, ἐξ οὗ τὰ γινόμενα καὶ τὰ γεγονότα καὶ τὰ ἐσόμενα καὶ θεοὺς καὶ θεῖα γίνεσθαι, τὰ δὲ λοιπὰ ἐκ τῶν τούτου* [so Diels, following C: *τούτων* T] *ἀπογόνων*. It is obvious that in the statement of Theophrastus the *ἀπόγονοι* were those of the first generation, and not the absurd list we here have presented to us. The primary forms of existence are afterwards mentioned, V² 18, 35–40: the report of Theophrastus is even better preserved by Cic. Acad. 2. 37. 118 (V² 19, 16), "Anaximenes infinitum aëra, sed ea, quae ex eo orerentur, definita: *gigni autem terram, aquam, ignem*, tum ex iis omnia. The variant readings above noted are probably due to the intrusion of the impertinent clause, which clearly does not derive from Theophrastus. Whether Hippolytus or some other made the addition I find it difficult to decide. A second instance of the same kind occurs V² 18, 35, *κινεῖσθαι δὲ ἀεί· οὐ γὰρ μεταβάλλειν ὅσα μεταβάλλει, εἰ μὴ κινοῖτο.* This sentence is awkward and intervenes between two parts of the exposition of the changes to which "air" is subject. What we expect from Theophrastus is something about the *κίνησις ἀίδιος*, and doubtless he did refer to it here. The clause *κινεῖσθαι δὲ ἀεί* in all probability is sound and derives from him; but the sentence *οὐ γὰρ . . . κινοῖτο* introduces a foreign element. Perhaps Hippolytus found it in his immediate source.

I add here a note on V² 19, 2, where the MSS read *ἀνέμους δὲ γεννᾶσθαι, ὅταν ἐκπεπυκνωμένος ὁ ἀὴρ ἀραιωθεὶς φέρηται*, and Diels prints *ὅταν ᾖ πεπυκνωμένος ὁ ἀὴρ καὶ ὠσθεὶς φέρηται.* This reading seems to me to depart farther than necessary from the MS. text. I would propose *ὅταν ἢ π. ὁ ἀὴρ ἢ ἀραιωθεὶς φέρηται.* Though a greater degree of rarefaction or condensation would, according to Anaximenes, result in fire or cloud respectively, it does not appear why he might not have held that a more moderate change in either direction gave rise to wind.

c. 11. Xenophanes.

V² 34, 16. Diog. L. 9 .19, (*φησὶ*) *τὰ νέφη συνίστασθαι τῆς ἀφ' ἡλίου ἀτμίδος ἀναφερομένης καὶ αἰρούσης αὐτὰ εἰς τὸ περιέχον.*

Diels still regards this doxography preserved by Diogenes as derived from Theophrastus through the biographical line of tradition.

The whole account is, as Diels, *Doxographi Graeci*, p. 168, pointed out, remarkable for its curious statements. I confess that, if it be really derived from Theophrastus, it seems to me to have suffered changes similar in character to those of the doxography of Hippolytus (V^2 41, 25 sq.), which owes much of its data to the Pseudo-Aristotelian treatise De Melisso, Xenophane, Gorgia. But first let us speak of the passage transcribed above. What Xenophanes taught concerning the origin of clouds is clearly stated by Aet. 3. 4. 4 (V^2 43, 20), *ἀνελκομένου γὰρ ἐκ τῆς θαλάττης τοῦ ὑγροῦ τὸ γλυκὺ διὰ τὴν λεπτομέρειαν διακρινόμενον νέφη τε συνιστάνειν ὁμιχλούμενον καὶ καταστάζειν ὄμβρους ὑπὸ πιλήσεως καὶ διατμίζειν τὰ πνεύματα.* Cp. also fr. 30. It is clear that Theophrastus simply stated the theory of the meteoric process, according to which clouds originate from vapors rising under the action of solar heat and lifting skyward. In the text of Diogenes we readily note two inaccuracies. We should doubtless read *ὑφ'* for *ἀφ'*, since vapors rising *from* the sun are sheer nonsense. The other difficulty is at first more puzzling; for a vapor lifting clouds skyward is nonsense likewise. The vapor condensed to mist or fog (*ὁμιχλούμενον*) *is* cloud. I therefore suggested to Professor Diels that we bracket *αὐτά* and take *αἰρούσης* in its intransitive sense: he records, but does not accept, the proposal in his third edition. It is at once clear that this would remove all difficulties from the passage. Probably Professor Diels was doubtful about the intransitive use of *αἴρω*, which the lexica almost entirely ignore. Of that usage I gave examples in a *Note on Menander, Epitrepontes 103 sq.*, published in Berl. Philol. Wochenschr., 1909, No. 16, col. 509 sq. I there cited Plato, Phaedr. 248 A, Arist. Respir. 475^a 8 and 479^a 26, Sophocl. Philoct. 1330. To these instances I would now add Sophocl. O. R. 914 and the Schol. to Sophocl. ad loc. and p. 239, 4; Proclus in Tim. I. 78, 2 Diehl. Other examples, concerning which there may be some doubt, I now omit, but may recur to the subject another time. There can be no question, therefore, that *αἴρειν* was used intransitively, and in our passage the change appears to be demanded by the sense. Probably some one not familiar with the usage added *αὐτά* in order to supply an object, but in so doing he gave us nonsense.

In this same paragraph occur the words (V^2 34, 18) *ὅλον δὲ ὁρᾶν καὶ ὅλον ἀκούειν, μὴ μέντοι ἀναπνεῖν.* I discussed this passage briefly in *Antecedents of Greek Corpuscular Theories*, p. 137 sq., pointing out its agreement with Plato, Tim. 32 C–33 C. I ought in justice to say that the parallel had been previously noted by Tannery, *Pour l'histoire de la science hellène*, p. 121, though the fact had slipped from my memory.

Since my previous discussion I have come to doubt whether the words of the Timaeus may be used to support the statement of Diogenes. About the agreement itself there can be no question. Plato does not, however, mention Xenophanes, and there is no indication in his text that what he says is to be taken as a correct statement of his doctrine. If we were quite sure that the report of Diogenes came materially unchanged from Theophrastus, the parallel would unquestionably prove that Xenophanes expressly denied the doctrine of the cosmic respiration. Tannery would then be justified in holding, as he did, that the brief notice of Diogenes was a precious document showing beyond question that Xenophanes was engaged in a sharp polemic against the Pythagoreans, whose doctrine, amply attested by Aristotle, he emphatically denied. Tannery's position would be untenable except on the assumption that Pythagoras himself proposed the theory of cosmic respiration: the testimony of Aristotle, however, who refers (as always) not to Pythagoras but to the Pythagoreans, is scarcely adequate to establish it. On the other hand, as has already been said, the accuracy and integrity of the account of Diogenes is subject to grave suspicion. The statement with which it opens, that Xenophanes held the doctrines of the four physical elements (*στοιχεῖα*) and of innumerable worlds, cannot be reconciled with other data unquestionably derived from Theophrastus. Again, the sentence V^2 34, 19, *πρῶτός τε ἀπεφήνατο ὅτι πᾶν τὸ γινόμενον φθαρτόν ἐστι,* in which Otto Gilbert, *Die meteorol. Theorien des gr. Altertums,* p. 98, n. 1, sees "nur ein ungenauer Ausdruck für die Rückbildung der Elemente in den Urstoff" (!), appears to be nothing but an echo of the anecdote related by Arist. Rhet. 2.23 1399^b 6 (V^2 35, 21), *οἷον Ξενοφάνης ἔλεγεν ὅτι "ὁμοίως ἀσεβοῦσιν οἱ γενέσθαι φάσκοντες τοὺς θεοὺς τοῖς ἀποθανεῖν λέγουσιν,"* and of De Melisso, Xenophane, Gorgia, 977^a 14 sq., which latter passage in turn incorporates arguments derived from Plato. This fact should give us pause, and suggests that Diogenes's account of the philosophy of Xenophanes is derived from a source which, like that of Hippolytus (V^2 41, 25 sq.) and Simplicius (V^2 40, 21 sq.), sought to eke out the scanty Theophrastean summary with information coming from the spurious De Melisso, Xenophane, Gorgia, and ultimately from the Timaeus and Parmenides of Plato. I am therefore inclined to believe that the statement of Diogenes, *μὴ μέντοι ἀναπνεῖν,* rests solely on the Timaeus, which the compiler regarded as a trustworthy source for the philosophy of Xenophanes.

I may add a brief note on the word *πρῶτος* in the sentence just quoted (V^2 34, 19). Diels long ago observed that the claim of

Xenophanes to be the originator of this doctrine is absurd and opposed to statements of Aristotle and Theophrastus. How came the claim to be made? During the sixth and fifth centuries B. C., as we well know, much interest attached to the inventors of contrivances and the first propounders of ideas, as was entirely natural in the fine burst of individualism characteristic of the epoch. We commonly think of the passionate quest for εὑρήματα during the Alexandrian Age, but Herodotus (1.25; 1.171; 2.4; 2.24; 2.109; 3.131; 4.42; 4.44) and the earlier logographers display the same interest. The exaggerations to which claims of this nature led have been well illustrated by Professor J. S. Reid, *Lucretiana,* Harvard Studies in Class. Philol., Vol. 22 (1911), p. 1 sq. in his note on Lucret. 1, 66 sq. Certain peculiarities of phrase used in such connections deserve attention. Thus Herod. 1.25 says, Γλαύκου τοῦ Χίου, ὃς μοῦνος δὴ πάντων ἀνθρώπων σιδήρου κόλλησιν ἐξεῦρε, using μοῦνος, where we might have expected πρῶτος, to denote the sole original authorship of Glaucus. When data were collected for the later compilations such turns may have given rise to errors. In some such way we may perhaps account for the embarrassment of Simplicius (V^2 18, 19) in regard to Anaximenes: ἐπὶ γὰρ τούτου μόνου Θεόφραστος . . . τὴν μάνωσιν εἴρηκε καὶ πύκνωσιν, δῆλον δὲ ὡς καὶ οἱ ἄλλοι τῇ μανότητι καὶ πυκνότητι ἐχρῶντο. Here Diels formerly accepted Usener's suggestion of πρώτου for μόνου, but has latterly with good reason returned to the MS. reading, which the context requires.

V^2 36. De Melisso, Xenophane, Gorgia 977^a 18, ταὐτὰ γὰρ ἅπαντα τοῖς γε ἴσοις καὶ ὁμοίως ὑπάρχειν πρὸς ἄλληλα.

Here Diels follows the reading of L, except that he rightly changes ταῦτα to ταὐτά: R, which is second only to L, gives ἴσοις ἢ ὁμοίοις. Probably neither reading is correct. Arist. De Gen. et Corr. 1. 7. 323^b 5 has πάντα γὰρ ὁμοίως ὑπάρχειν ταὐτὰ τοῖς ὁμοίοις. Both passages, however, rest upon Plato, Parm. 139 E–140 D, where the implications of the ὅμοιον and ἀνόμοιον are first considered, then those of the ἴσον and ἄνισον. In view of this fact I think we should read τοῖς γε ἴσοις καὶ <ὁμοίοις> ὁμοίως.

c. 12. Heraclitus.

V^2 61, 35. Fr. 1, ὁκοίων ἐγὼ διηγεῦμαι διαιρέων ἕκαστον κατὰ φύσιν καὶ φράζων ὅκως ἔχει.

These words have been variously interpreted. So far as I am aware everybody has regarded *φύσις* as meaning "nature" in some one of its numerous acceptations and *ἕκαστον* as being the immediate object of *διαιρέων*. With respect to neither word, I believe, is the current opinion correct. The phrase *ἕκαστον κατὰ φύσιν*, which has been misinterpreted in various connections, means "each after its kind." We shall have to discuss a similar phrase in Empedocles, fr. 110, 5. The object of *διαιρέων*, as of *διηγεῦμαι*, is contained in *ὁκοίων*, which *ἕκαστον* distributes: "Making trial of such arguments and facts as I recount, distinguishing them each after its own kind and declaring the nature of each." I have rendered *ὅκως ἔχει* ambiguously with "nature," for the phrase occurs frequently in Hippocrates where the *φύσις* of things is to be explained, when nothing but the context, and often not even that, makes it possible to decide whether *φύσις* has regard primarily to the process of growth or to the constitution of the thing in which the process eventuates. In this fragment the precise implication of *ὅκως ἔχει* cannot be determined; below (V^2 91, 23) in Epicharmus, fr. 4, 6, we shall find an instance of *ὡς ἔχει* in which the process is obviously intended. I referred briefly to this question in my *Περὶ Φύσεως*, p. 126, n. 180 and p. 127, n. 185, and illustrated the scientific ideal of dividing and simplifying complex problems by distinguishing between classes and individuals, ibid. pp. 123–125. Perhaps the most noteworthy text is the following, Hippocr. *Περὶ διαίτης ὀξέων*, 1 (2. 226 L.), *ἀτὰρ οὐδὲ περὶ διαίτης οἱ ἀρχαῖοι ξυνέγραψαν οὐδὲν ἄξιον λόγου, καίτοι μέγα τοῦτο παρῆκαν. τὰς μέντοι πολυτροπίας τὰς ἐν ἑκάστῃ τῶν νούσων καὶ τὴν πολυσχιδίην αὐτέων οὐκ ἠγνόεον ἔνιοι· τοὺς δὲ ἀριθμοὺς ἑκάστου τῶν νουσημάτων σάφα φράζειν ἐθέλοντες, οὐκ ὀρθῶς ἔγραψαν· μὴ γὰρ οὐκ εὐαρίθμητον εἴη, εἰ τουτέῳ τις σημανεῖται τὴν τῶν καμνόντων νοῦσον, τῷ ἕτερον ἑτέρου διαφέρειν τι, καί, ἢν μὴ τωὐτὸ νούσημα δοκέῃ εἶναι, μὴ τωὐτὸ οὔνομα ἔχειν.*

V^2 65, 10. Fr. 18, *ἐὰν μὴ ἔλπηται, ἀνέλπιστον οὐκ ἐξευρήσει, ἀνεξερεύνητον ἐὸν καὶ ἄπορον.*

Here, as in fr. 27, Diels and Nestle translate *ἔλπομαι* with "hope." Burnet here renders the word with "expect," there with "look for," in either case correctly. I am not sure, however, that he understands our fragment as I do. It is well known that *ἐλπίς* may signify any degree of expectation ranging from vague surmise to lively hope or fear. In reading this fragment I am constantly reminded of a story which Tyndall tells of Faraday, who required to be told precisely what to look for before observing an experiment which was in preparation. All scientific observation, whether assisted or not assisted by carefully controlled experimentation, presupposes an ἐλπίς — surmise or clearly formulated anticipation — of that which observation will show. To form such a conception is to exercise the scientific imagination, and the findings anticipated assume the shape of a theory or an hypothesis. Early Greek philosophy was so prolific of nothing else as of hypotheses, and the philosophy of Heraclitus in particular is nothing but a bold hypothesis, whatever concrete observations may have led him to propound it. Now, that is precisely what I conceive our fragment to mean: "*Except a man venture a surmise, he will not discover that which he has not surmised; for it is undiscoverable and baffling.*" Fr. 123, *φύσις κρύπτεσθαι φιλεῖ*, 'the processes of nature are not to be read by him who runs, for the true inwardness of things does not appear on the surface', is probably to be understood in the same sense; for *ἁρμονίη ἀφανὴς φανερῆς κρείττων* (fr. 54). So, too, fr. 86, *ἀπιστίῃ διαφυγγάνει μὴ γιγνώσκεσθαι*, probably refers not to faith in a dogma or a revelation but to the scientific faith which is the evidence of things not seen.

V² 64, 1. Fr. 10, *συνάψιες ὅλα καὶ οὐχ ὅλα, συμφερόμενον διαφερόμενον, συνᾷδον διᾷδον, καὶ ἐκ πάντων ἓν καὶ ἐξ ἑνὸς πάντα.*

I do not recall seeing anywhere a reference to the evident reminiscence of this fragment in Seneca, De Otio, 5. 6, utrum contraria inter se elementa sint, an non pugnent, sed per diversa conspirent.

V² 66, 13. Fr. 28, *δοκεόντων γὰρ ὁ δοκιμώτατος γινώσκει φυλάσσειν· καὶ μέντοι καὶ δίκη καταλήψεται ψευδῶν τέκτονας καὶ μάρτυρας, ὁ Ἐφέσιός φησιν.*

The text of this fragment is regarded by all critics as desperate, and desperate measures have been taken to restore it. I have no desire to canvass them, but shall offer an interpretation which, with a minimal alteration, appears to render it intelligible and quite as defensible as the texts obtained by introducing more radical changes. First of all, it seems clear that *γάρ* is due to Clement, who quotes the sentence, and must be set aside as not belonging to Heraclitus. This is the view of Bywater, who omits the word. If that be true, what is there to hinder our taking *δοκεόντων* as an imperative? It wants a subject, but that was doubtless supplied by the context from which the sentence was obviously wrested. A plausible conjecture is made possible by the reference in the last clause to the inventors and supporters of lies, who are clearly contrasted with those who receive the philosopher's scornful permission to hold an opinion. If δοκεόντων has that meaning, it is transitive as in Herod. 9. 65, δοκέω δέ, εἴ τι περὶ τῶν θείων πρηγμάτων δοκέειν δεῖ. Whether we shall read ὃ for ὁ or assume that ὃ was omitted by haplography before ὁ δοκιμώτατος is difficult to decide; for, as Diels has remarked, Heraclitus is sparing in the use of the article. I incline to insert <ὃ>, or possibly <ἃ>, the only change I consider necessary in the text. Critics appear to consider γινώσκει φυλάσσειν impossible or unintelligible. It is well known, however, that οἶδα and ἐπίσταμαι are used with the infinitive in the sense of "knowing how" to do anything, and in some cases the nuance given by these verbs is so slight as to be best disregarded in translating the thought into English. It is difficult to see why γινώσκω should not be used in the same construction as οἶδα and ἐπίσταμαι. In fact we have two passages which are calculated to support the assumption that it was so used. Sophocl. Ant. 1087,

ἵνα
τὸν θυμὸν οὗτος ἐς νεωτέρους ἀφῇ
καὶ γνῷ τρέφειν τὴν γλῶσσαν ἡσυχωτέραν.

Eurip. Bacch. 1341,

εἰ δὲ σωφρονεῖν
ἔγνωθ', ὅτ' οὐκ ἠθέλετε, τὸν Διὸς γόνον
εὐδαιμονεῖτ' ἂν σύμμαχον κεκτημένοι.

Goodwin, *Greek Moods and Tenses*, 915, 3 (c), mentions the first passage only and takes γιγνώσκω (ἔγνων) in the sense of "learning." The ingressive aorist naturally bears this sense; but it does not exclude the same construction with the present, as may be seen by comparison with ἐπίσταμαι, which shows the same meaning in the ingressive aorist, Herod. 3. 15, εἰ δὲ καὶ ἠπιστήθη μὴ πολυπραγμονέειν. This line of argument would perhaps not suffice to justify a conjectural introduction of γινώσκει into the text, but it is an adequate defense of a MS. reading. We have then to consider the meaning of φυλάσσειν. Here we are thrown upon the fragment itself as our only resource, since the verb has a great variety of meanings. There seems to be a slight clue in the last clause. Diels appears to be right in assuming that Homer, Hesiod, and the like, are the ψευδῶν τέκτονες καὶ μάρτυρες. If this conjecture be true, it is not difficult to see that ψευδῶν τέκτονας characterizes them as inventors of lies, and that ψευδῶν μάρτυρας can hardly mean those who commit perjury, but must rather refer to the witness they bear to falsehoods by recording them in their verse. In other words, the woe pronounced upon the poets is for originating and perpetuating false views, whether they relate to the gods, to the desirability of banishing discord, or what not. But φυλάσσειν does bear this precise sense of "perpetuating," and we may be justified in accepting it as referring to the παράδοσις of poetical tradition. I think it probable that ὁ δοκιμώτατος refers to Homer as the coryphaeus of the group of false teachers of the multitude whom Heraclitus is denouncing, and that the epithet signifies nothing more than that he is held in the highest esteem, although fr. 57 would perhaps rather suggest Hesiod. The subject of δοκεόντων, then, is the uncritical multitude, who live according to the tradition of the fathers (fr. 74) and may be pardoned for what they do in ignorance, though woe shall be unto those through whom offence cometh. Accordingly I should translate the fragment rather freely somewhat after this manner: "*Ay, let them think as he who is most highly esteemed among them contrives to report; but verily, judgment shall overtake those who invent and attest falsehoods.*" It is hardly necessary to add that Heraclitus was not threatening Homer with hell-fire, as Clement would have us suppose.

V² 68, 11. Fr. 41, *ἓν τὸ σοφόν, ἐπίστασθαι γνώμην, ὁτέη ἐκυβέρνησε πάντα διὰ πάντων.*

Here I accept the text, but not the interpretation of Diels, who renders the fragment thus: "In Einem besteht die Weisheit, die Vernunft zu erkennen, als welche alles und jedes zu lenken weiss." Nestle translates γνώμην with "Geist"; and Burnet, with "thought." In order to arrive at the thought of Heraclitus, it is needful first of all to note how in a number of his fragments, which are concerned with his conception of true wisdom, he surcharges with meaning the terms for knowledge in contradistinction to sense-perception or opinion. Fr. 17, *οὐ γὰρ φρονέουσι τοιαῦτα πολλοί, ὁκόσοι* [so Diels, V³] *ἐγκυρεῦσιν, οὐδὲ μαθόντες γινώσκουσιν, ἑωυτοῖσι δὲ δοκέουσι,* "The majority of mankind [this, I think must be the meaning of πολλοί, whether or not with Bergk we add οἱ], so far as they meet such problems, do not comprehend them even when instructed, though they think they do." Fr. 34, "They that lack understanding (ἀξύνετοι) hear, but are like unto them that are deaf." Fr. 35, "Men who are lovers of wisdom must have acquired true knowledge of full many matters" (εὖ μάλα πολλῶν ἵστορας εἶναι). But Heraclitus is well aware that much instruction (cp. μαθόντες, fr. 17) does not impart understanding (fr. 40, πολυμαθίη νόον ἔχειν οὐ διδάσκει· Ἡσίοδον γὰρ ἂν ἐδίδαξε καὶ Πυθαγόρην αὖτίς τε

Ξενοφάνεά τε καὶ Ἑκαταῖον), else would the champions of the new, self-styled ἱστορίη and Hesiod, their coryphaeus, have got understanding. The same pregnancy of meaning as in fr. 17 attaches to γινώσκειν in fr. 108, to be discussed more at length below, and in fr. 57, where Heraclitus says that Hesiod, whom men regard as most knowing, did not really comprehend (οὐκ ἐγίνωσκεν) day and night; for, contrary to his opinion, they are one. It is thus clearly shown that by understanding Heraclitus means a cognitive faculty or act which penetrates beyond superficial differences and distinctions, present to sense and uncritical fancy, to an inner core of truth, and is characterized by the apprehension of a fundamental unity. Again, the same point of view finds expression in fr. 56, where he likens mankind, readily duped when it comes to a true understanding of the surface show of things (ἐξηπάτηνται οἱ ἄνθρωποι πρὸς τὴν γνῶσιν τῶν φανερῶν), to Homer, who could not read a foolish riddle propounded to him by *gamins*. Above, in discussing fr. 18, I have already touched on fr. 86, ἀπιστίῃ διαφυγγάνει μὴ γιγνώσκεσθαι, maintaining that Heraclitus meant to imply that the true meaning of things is missed for want of a confident act of imaginative anticipation, whereby that which does not obtrude itself on our senses is brought home to the understanding. It is perhaps not too fanciful to detect the same distinction between sense and understanding, where understanding involves the synthesis of apperception, in fr. 97, κύνες γὰρ καταβαΰζουσιν ὧν ἂν μὴ γινώσκωσι. Heraclitus would thus be merely repeating the distinction of Alcmaeon, fr. 1[a] (V^2 103, 25), ἄνθρωπον γάρ φησι τῶν ἄλλων (sc. ζῴων) διαφέρειν ὅτι μόνον ξυνίησι, τὰ δ' ἄλλα αἰσθάνεται μέν, οὐ ξυνίησι δέ.

Returning now to fr. 41 after a considerable *détour*, we naturally pause again before the phrase ἐπίστασθαι γνώμην, which is the real crux. Scholars appear to be fairly unanimous in holding that, whether it means "Vernunft," "Geist," or "thought," γνώμην is an accusative of the external object, being, in fact, the divine entity which rules the world. Heraclitus ὁ κυκητής does not much encourage fine distinctions, but to me this interpretation seems to yield a Stoic rather than a Heraclitean thought. In obvious reminiscence of our fragment and of fr. 32, ἓν τὸ σοφὸν μοῦνον λέγεσθαι οὐκ ἐθέλει καὶ ἐθέλει Ζηνὸς ὄνομα, Cleanthes, H. in Iov. 30 could say,

δὸς δὲ κυρῆσαι
γνώμης, ᾗ πίσυνος σὺ δίκης μέτα πάντα κυβερνᾷς.

But Cleanthes was clearly writing from a different, and a later, point of view, for which the οὐκ ἐθέλει of Heraclitus had no real significance. Following him and having regard to Antipho Soph. fr. 1 (V^2 591, 18, γνώμη γινώσκει, and V^2 592, 4, γνώμη νῶσαι) one might incline to propose to emend γνώμην and read γνώμῃ ἐπίστασθαι in Heraclitus. I should regard that, however, as an error; for I hold that γνώμην is an accusative of the inner object. In other words, ἐπίστασθαι γνώμην is a periphrasis for γινώσκειν. In the time of Heraclitus ἐπίστασθαι had not yet acquired the technical sense which it later bore in philosophical prose: in fr. 57, τοῦτον ἐπίστανται πλεῖστα εἰδέναι, it means to "fancy"; in fr. 19, ἀκοῦσαι οὐκ ἐπιστάμενοι οὐδ' εἰπεῖν, to "be skillful." The latter sense is common from Homer onward, the former in Herodotus. It is not surprising, therefore, that Heraclitus should wish to reinforce it with a cognate substantive. A similar turn recurs in Ion of Chios, fr. 4 (V^2 222, 28 sq.),

> ὣς ὁ μὲν ἠνορέῃ τε κεκασμένος ἠδὲ καὶ αἰδοῖ
> καὶ φθίμενος ψυχῇ τερπνὸν ἔχει βίοτον,
> εἴπερ Πυθαγόρης ἐτύμως ὁ σοφὸς περὶ πάντων
> ἀνθρώπων γ ν ώ μ α ς ᾔδεε κἀξέμαθεν.

Here Diels, whose emendation, ᾔδεε for εἶδε I heartily approve, renders γνώμας ᾔδεε κἀξέμαθεν with "Einsichten erworben und erforscht hat." I believe we have a sort of hysteron proteron, and that Ion (for, herein differing from Diels, I believe the verses are his) meant "if Pythagoras was well informed and really knew whereof he spoke." This interpretation of Ion's phrase is proved correct beyond a doubt by Theognis, 59,

> ἀλλήλους δ' ἀπατῶσιν ἐπ' ἀλλήλοισι γελῶντες,
> οὔτε κακῶν γνώμας εἰδότες οὔτ' ἀγαθῶν.

The couplet was reproduced with slight modifications by an unintelligent imitator, Theognis 1113,

> ἀλλήλους δ' ἀπατῶντες ἐπ' ἀλλήλοισι γελῶσιν,
> οὔτ' ἀγαθῶν μνήμην εἰδότες οὔτε κακῶν.

Here we must without doubt adopt Hecker's emendation γνώμην for μνήμην. The imitator did not perceive the true significance of the original, which sought to hold up to scorn the blissful Edenic ignorance of good and evil characteristic of the new-made lords of Megara, who but recently, clad in goat-skins, lived like pasturing deer in the wilds without the city walls, but now in the city light-heartedly hoodwink one another. Clearly γνώμας εἰδέναι is a mere periphrasis for εἰδέναι. A similar reinforcement of εἰδέναι occurs in the LXX. account of Eden, Gen. 2. 9, *τὸ ξύλον τοῦ εἰδέναι γνωστὸν καλοῦ καὶ πονηροῦ*, where, but for the confirmation of the MS. text by Philo Jud. 1. 55, 27, one might be inclined to suspect that *γνωστόν* was a corruption of *γνῶσιν* or *γνώμην*. If Ion's phrase reminds us of such Homeric locutions as *νοήματα ἤδη* (*β* 121) and *μήδεα οἶδε* (Σ 363), we find something closely analogous to that of Heraclitus in Plato, Apol. 20 E, *οὐ γὰρ δὴ ἔγωγε αὐτὴν* (sc. *τὴν σοφίαν*) *ἐπίσταμαι*. In this last phrase, however, the comparison with 20 D, *κινδυνεύω ταύτην εἶναι σοφός*, may suggest that Plato had in mind the old force of *ἐπίστασθαι*, "be skillful." However, Theognis 564, *σοφίην πᾶσαν ἐπιστάμενον*, has the same construction. Cp. ibid. 1157. If, then, we so interpret *ἐπίστασθαι γνώμην*, we cannot take the relative *ὁτέη* so closely with *γνώμην* as the ordinary view requires. I should rather say that *ὁτέη* was roughly equivalent to *ᾗ γε*, *quippe quae*, as *ὅστις* in fr. 57 means *ut pote qui*, and render the fragment somewhat as follows: "*One thing only is wisdom: to get Understanding: she it is that pervades all things and governs all.*"

V² 69, 2. Fr. 48, *τῷ οὖν τόξῳ ὄνομα βίος, ἔργον δὲ θάνατος*.

Diels, *Die Anfänge der Philologie bei den Griechen*, Neue Jahrbücher, xxv (1910), I. Abteilung, p. 3, says, "Der Gleichklang der Worte *βιός* (Pfeil) und *βίος* (Leben) war ihm ein äusseres Zeichen für seine Lehre, dass die Gegensätze Leben and Tod im Grunde eins seien." Zeller I, 640, n. 2, expresses himself in much the same way. I have no desire to controvert this interpretation, so far as it goes; but it seems to me that the words of Heraclitus imply much more. In V³ Diels properly refers to Hippocrates, *Περὶ τροφῆς*, 2 (V² 86, 1 sq.), *τροφὴ οὐ τροφή, ἢν μὴ δύνηται, οὐ τροφὴ τροφή, ἢν οἷόν τε ᾖ τρέφειν· οὔνομα τροφή, ἔργον δὲ οὐχί· ἔργον τροφή, οὔνομα δὲ οὐχί*. With this passage of undoubtedly Heraclitean origin we should take fr. 37, sues caeno, cohortales aves pulvere vel cinere lavari; for the thought apparently is that mud and dust are not *ὀνόματι* water, but are *ἔργῳ* identical with it. Fr. 13, *δεῖ γὰρ τὸν χαρίεντα μήτε ῥυπᾶν μήτε αὐχμεῖν μήτε βορβόρῳ χαίρειν καθ' Ἡράκλειτον*, where *βορβόρῳ χαίρειν* alone seems to belong to Heraclitus, may conceivably have reference to the same problem, the philosopher meaning to imply that we should call things and men by names conformable to their *ἔργον*: by their fruits ye shall know them! Plotinus Enn. 1. 6. 6, *ἔστι γὰρ δή, ὡς ὁ παλαιὸς λόγος, καὶ ἡ σωφροσύνη καὶ ἡ ἀνδρεία καὶ πᾶσα ἀρετὴ κάθαρσις καὶ ἡ φρόνησις αὐτή· διὸ καὶ αἱ τελεταὶ ὀρθῶς αἰνίττονται τὸν μὴ κεκαθαρμένον καὶ εἰς* [an *ἐν*?] *ᾅδου κείσεσθαι ἐν βορβόρῳ, ὅτι τὸ μὴ καθαρὸν βορβόρῳ διὰ κάκην φίλον· οἷα δὴ* *καὶ ὗες, οὐ καθαραὶ τὸ σῶμα, χαίρουσι τῷ τοιούτῳ*, obviously glancing at fr. 13, suggests the possibility that Heraclitus used the words in connection with a discussion of the mysteries, with the intent of which he seems to have been satisfied, while he denounced their forms. Thus, fr. 5, *καθαίρονται δ' ἄλλως αἵματι μιαινόμενοι οἷον εἴ τις πηλὸν ἐμβὰς πηλῷ ἀπονίζοιτο*, we find a context in which he may have distinguished between the form and the substance, the *ὄνομα* and the *ἔργον*. Be that as it may, there is abundant evidence that Heraclitus had grasped the fruitful principle that the true nature of a thing is to be understood in relation to its function or *ἔργον*. We are familiar enough with his interest in etymologies, which reveals the desire to detect the true meaning of objects in the derivation of their names; but the study of homonyms, which our fragment reveals, almost necessarily involved a corresponding attention to synonyms, in which words of very different origin and etymology are shown to have a common meaning. The test of identity or difference of meaning Heraclitus found in the *ἔργον* of the thing. Plato, in a passage clearly under the influence of Heraclitus, Crat. 394 A sq., develops this twofold principle, which underlies the study of homonyms and synonyms, referring to the law of uniformity in nature, in accordance with which like begets like, and concludes therefrom that, as the physician recognizes drugs by their physiological action (*δύναμις* = *ἔργον*), not allowing himself to be deceived by their several disguises, so the philosopher must apply the same name to parent and offspring, or at any rate he must learn to detect the identity of concepts by whatever names they may go. Plato is obviously developing ideas derived from Heraclitus, partly such as are expressed in the fragments above cited, partly those of fr. 67, which we shall presently discuss more at length. In Tim. 50 A–51 B Plato combines in a highly suggestive way Heraclitean and Eleatic concepts, very much as he develops the law of uniformity, mentioned in the Cratylus, into the principle of interaction (*ποιεῖν καὶ πάσχειν*) in Gorg. 476 B sq. In the living tissue of so vital a tradition as Greek philosophy presents we expect to find continuous developments of this kind. What is more difficult is the task of discriminating the stages marked by the individuals who contributed to the total result. In regard to the particular question with which we are now concerned, it is clear that Heraclitus and the Heracliteans laid the foundations for the Socratic procedure of definition by noting the essential importance of the *ἔργον* in determining the meaning of a concept. It was Socrates, however, who elaborated the method of definition on the basis of dialectic, thus in turn laying the foundations of the science of logic.

V² 69, 10. Fr. 50, *Ἡράκλειτος μὲν οὖν <ἓν> φησιν εἶναι τὸ πᾶν διαίρετον ἀδιαίρετον, γενητὸν ἀγένητον, θνητὸν ἀθάνατον, λόγον αἰῶνα, πατέρα υἱόν, θεὸν δίκαιον· οὐκ ἐμοῦ, ἀλλὰ τοῦ λόγου ἀκούσαντας ὁμολογεῖν σοφόν ἐστιν ἓν πάντα εἶναι ὁ Ἡράκλειτός φησι.*

It is agreed that the authentic words of Heraclitus begin with *οὐκ ἐμοῦ*: what precedes we owe to Hippolytus, who obviously modeled his introductory statement on fr. 67. The comparison of the two passages shows that Bergk's <*ἓν*>, which Diels adopts, is unnecessary. The predicates of *τὸ πᾶν* are, as one sees at a glance, arranged in contrasted pairs. In the fourth pair, *λόγος* is of course the intelligible principle, virtually the *κόσμος νοητός*, opposed to *αἰών* which is the *κόσμος αἰσθητός*. The next pair, *πατέρα υἱόν*, is of course of Christian origin. Apparently the last, *θεὸν δίκαιον*, has puzzled Professor Diels; for he now (V³) proposes to insert [*ἄδικον*] after *δίκαιον*. I long ago saw that this pair was suggested to Hippolytus or his source by Plato, Crat. 412 C–413 D, but had taken for granted that this was a matter of common knowledge and not worthy of special notice, until Diels's note undeceived me. I observe that Otto Gilbert, *Griech. Religionsphilosophie*, p. 62, n. 1, also noticed the connection. He there proposes a different interpretation of *αἰών*, but his suggestion I take to be too clearly mistaken to require refutation. In reference to *θεὸν δίκαιον*, it ought to be said that Hippolytus possibly wrote *διαϊόν* (= *ἥλιον*), and that *δίκαιον* may be due to the copyist; but there is no sufficient justification for making a change in the text. Diels is probably right in adopting Miller's *εἶναι* for the *εἰδέναι* of Par.; but *εἰδέναι* may possibly have been originally a gloss on *ὁμολογεῖν*; for if *ὁμολογεῖν* is sound it must be interpreted here, as in fr. 51, with reference to Heraclitean etymology, as "sharing in the (a) common *λόγος*."

V² 71, 15. Fr. 67, *ὁ θεὸς ἡμέρη εὐφρόνη, χειμὼν θέρος, πόλεμος εἰρήνη, κόρος λιμός (τἀναντία ἅπαντα· οὗτος ὁ νοῦς), ἀλλοιοῦται δὲ ὅκωσπερ <πῦρ>, ὁπόταν συμμιγῇ θυώμασιν, ὀνομάζεται καθ' ἡδονὴν ἑκάστου.*

This is the text of Diels. I hope to make it clear that it is not correct, and to show also what Heraclitus wrote and what he meant. In order to understand and reconstruct this fragment we must compare two passages from Plato, in which he obviously alludes to it. Crat. 394 A, *οὐκοῦν καὶ περὶ βασιλέως ὁ αὐτὸς λόγος; ἔσται γάρ ποτε ἐκ βασιλέως βασιλεύς, καὶ ἐξ ἀγαθοῦ ἀγαθός, καὶ ἐκ καλοῦ καλός, καὶ τἆλλα* *πάντα οὕτως, ἐξ ἑκάστου γένους ἕτερον τοιοῦτον ἔκγονον, ἐὰν μὴ τέρας γένηται· κλητέον δὴ ταὐτὰ ὀνόματα. ποικίλλειν δὲ ἔξεστι ταῖς συλλαβαῖς, ὥστε δόξαι ἂν τῷ ἰδιωτικῶς ἔχοντι ἕτερα εἶναι ἀλλήλων τὰ αὐτὰ ὄντα· ὥσπερ ἡμῖν τὰ τῶν ἰατρῶν φάρμακα χρώμασιν καὶ ὀσμαῖς πεποικιλμένα ἄλλα φαίνεται τὰ αὐτὰ ὄντα, τῷ δέ γε ἰατρῷ, ἅτε τὴν δύναμιν τῶν φαρμάκων σκοπουμένῳ, τὰ αὐτὰ φαίνεται, καὶ οὐκ ἐκπλήττεται ὑπὸ τῶν προσόντων. οὕτω δὲ ἴσως καὶ ὁ ἐπιστάμενος περὶ ὀνομάτων τὴν δύναμιν αὐτῶν σκοπεῖ, καὶ οὐκ ἐκπλήττεται εἴ τι πρόσκειται γράμμα ἢ μετάκειται ἢ ἀφῄρηται, ἢ καὶ ἐν ἄλλοις παντάπασιν γράμμασίν ἐστιν ἡ τοῦ ὀνόματος δύναμις. ὥσπερ ὃ νυνδὴ ἐλέγομεν, "Ἀστυάναξ" τε καὶ "Ἕκτωρ" οὐδὲν τῶν αὐτῶν γραμμάτων ἔχει πλὴν τοῦ ταῦ, ἀλλ' ὅμως ταὐτὸν σημαίνει. καὶ "Ἀρχέπολίς" γε τῶν μὲν γραμμάτων τί ἐπικοινωνεῖ; δηλοῖ δὲ ὅμως τὸ αὐτό· καὶ ἄλλα πολλά ἐστιν ἃ οὐδὲν ἀλλ' ἢ βασιλέα σημαίνει· καὶ ἄλλα γε αὖ στρατηγόν, οἷον "Ἆγις" καὶ "Πολέμαρχος" καὶ "Εὐπόλεμος". καὶ ἰατρικά γε ἕτερα, "Ἰατροκλῆς" καὶ "Ἀκεσίμβροτος"· καὶ ἕτερα ἂν ἴσως συχνὰ εὕροιμεν ταῖς μὲν συλλαβαῖς καὶ τοῖς γράμμασι διαφωνοῦντα, τῇ δὲ δυνάμει ταὐτὸν φθεγγόμενα.* The general connection of this passage with the Heraclitean doctrine of the *ἔργον* was noted above in the discussion of fr. 48. The *δύναμις* or specific physiological action of the drug is compared to the *δύναμις* of a word, its "force" or meaning. The identity of meaning in words that are different (*διαφωνοῦντα, τἀναντία ἅπαντα*), and the methods employed to produce variation (*ποικίλλειν, ἀλλοιοῦται*), — these are the themes common to Heraclitus and Plato. We naturally think of Heraclitus, fr. 15, *ὡυτὸς δὲ Ἀίδης καὶ Διόνυσος*, and fr. 57, *ὅστις ἡμέρην καὶ εὐφρόνην οὐκ ἐγίνωσκεν· ἔστι γὰρ ἕν.* The second passage from Plato, to which I referred above, is Tim. 49 sq., where the relation of the elements to the *δεξαμενή* or the *ἐκμαγεῖον* is under discussion. It will suffice for our purpose to quote a sentence from 50 E, *διὸ καὶ πάντων ἐκτὸς εἰδῶν εἶναι χρεὼν τὸ τὰ πάντα ἐκδεξόμενον ἐν αὑτῷ γένη, καθάπερ περὶ τὰ ἀλείμματα ὁπόσα εὐώδη τέχνῃ μηχανῶνται πρῶτον τοῦτ' αὐτὸ ὑπάρχον, ποιοῦσιν ὅτι μάλιστα ἀώδη τὰ δεξόμενα ὑγρὰ τὰς ὀσμάς· ὅσοι τε ἔν τισιν τῶν μαλακῶν σχήματα ἀπομάττειν ἐπιχειροῦσι, τὸ παράπαν σχῆμα οὐδὲν ἔνδηλον ὑπάρχειν ἐῶσι, προομαλύναντες δὲ ὅτι λειότατον ἀπεργάζονται.* Plato here employs two comparisons to illustrate the relation of the substratum to the elemental forms, borrowing one from the manufacture of unguents, the other from the art of moulding figures in a matrix. The first of these is obviously similar to that above quoted from the Cratylus, and was repeated by Lucret. 2, 847 sq.

> sicut amaracini blandum stactaeque liquorem
> et nardi florem, nectar qui naribus halat,
> cum facere instituas, cum primis quaerere par est,
> quoad licet ac possis reperire, inolentis olivi
> naturam, nullam quae mittat naribus auram,
> quam minime ut possit *mixtos in corpore odores*
> concoctosque suo contractans perdere viro,
> propter eandem rem debent primordia rerum
> non adhibere suum gignundis rebus odorem, etc.

Heeding the suggestions afforded by these passages from Plato and Lucretius, which seem to me clearly to reproduce, however freely, the thought of Heraclitus in our fragment, it should be possible with considerable certainty to restore the text and to determine its meaning. It is obvious that in the Cratylus Plato slightly changed the figure, substituting drugs for unguents, because of the advantage of thus being able to appeal to the expert knowledge of the physician. He may have been influenced also by certain Heraclitean elements in the medical literature, such as we find in Hippocrates *Περὶ διαίτης* and *Περὶ τροφῆς*. At all events, it is clear that <*πῦρ*>, which Diels has adopted from the conjecture of Dr. Thomas Davidson, and <*οἶνος*>, which Bergk proposed, are alike inadmissible. The latter part of the fragment and the use of *θύωμα*, which Hesychius defines with *μύρον* and *ἄρωμα*, point clearly to the conclusion that Heraclitus, as we should infer from Plato and Lucretius, referred to an unguent. The instances of *θύωμα* (Herod. 2. 86; Lucian, De Dea Syra, 8 and 46) refer to unguents. If one or the other of the passages in Lucian should be doubtful, there can be no question in regard to Hippocr. *Γυναικείων β*, 209 (8, 404 L.), *ἑψεῖν τὰ θυώματα ἃ ἐς τὸ μύρον ἐμβάλλεται*, with which compare ibid. 202 (8, 386 L.) and 206 (8, 398 L.) In the making of unguents (see Blümner, *Technologie und Terminologie der Gewerbe und Künste*[2], I., 359 sq.), the neutral base, as well as the product resulting from the union of aromatic substances with it, was called *μύρον* or *ἔλαιον*. The finished product bore a variety of names determined by the volatile ingredients. Theophrastus, *Περὶ ὀσμῶν*, gives ample information, from which we may quote a few sentences. V. 25, *πρὸς ἕκαστον δὲ τῶν μύρων ἐμβάλλουσι τὰ πρόσφορα τῶν ἀρωμάτων, οἷον εἰς μὲν τὴν κύπρον καρδάμωμον, ἀσπάλαθον ἀναφυράσαντες τῷ εὐώδει.* VI. 27, *ἅπαντα δὲ συντίθενται τὰ μύρα τὰ μὲν ἀπ' ἀνθῶν τὰ δὲ ἀπὸ φύλλων τὰ δὲ ἀπὸ κλωνὸς τὰ δ' ἀπὸ ῥίζης τὰ δ' ἀπὸ ξύλων τὰ δ' ἀπὸ καρποῦ τὰ δ' ἀπὸ δακρύων. μικτὰ δὲ πάνθ' ὡς εἰπεῖν.* In intention, therefore, the conjecture of Bernays, *συμμιγῇ <θύωμα> θυώμασι*, was better than either of those which we noticed above; but Diels is right in assuming that the desiderated word is to be supplied after *ὅκωσπερ*. The only point in favor of <*πῦρ*> is that its omission can so easily be explained; but with almost equal ease we can account for the loss of <*μύρον*>, which is obviously required by the sense and by the Platonic and Lucretian parallels.

But we must now return to the earlier part of the fragment. The words *τἀναντία ἅπαντα· οὗτος ὁ νοῦς* have been a stumbling-block. Bywater and Diels bracket them, since they can make nothing of them. Mullach accomplished the same result by making two fragments instead of one, and omitting the troublesome words. But a reference to the passage from the Cratylus should prove beyond question that they belong just where they stand; only one slight change is required, viz, *ὡυτὸς* for *οὗτος*, as Bergk perceived. He says, *Kleine Philol. Schriften*, II. 86, n. 4, "Ceterum etiam verba illa *τἀναντία ἅπαντα, οὗτος ὁ νοῦς* non interpretis, sed ipsius Heracliti esse existimo, quae ita videntur corrigenda: *ὁ θεὸς . . . κόρος, τἀναντία ἅπαντα· ὡυτὸς νόος· ἀλλοιοῦται δέ, ὅκωσπερ οἶνος κτλ.*" Unfortunately Bergk did not interpret his proposed text; but judging by his punctuation and the absence of any remark about the force of *νόος*, I venture to suggest that what he had in mind was something like this: "Gott ist... Überfluss und Hunger, mit einem Worte, alle Gegensätze. Es ist derselbe Geist," usw. If this suggestion does him justice, it will be seen that he did not really anticipate my proposal except in regard to the change of *οὗτος* into *ὡυτός*; and working with the text of Diels, who did not even record the proposal, I did not come upon his emendation until I had reached the same conclusion independently and by a different route. As a matter of fact, it was the passage from the Cratylus which disclosed the connection of ideas and led me to the obviously correct text and interpretation; for I saw at once that *νοῦς* had no reference whatever to *θεός* and did not mean "Geist," but, as in Herod. 7. 162, *οὗτος δὲ ὁ νόος τοῦ ῥήματος*, signified "sense" or "meaning." But, this point once cleared up, it followed at once that we must read *ὡυτός* for *οὗτος*, and that *τἀναντία ἅπαντα* did not merely add a generalization to sum up the bill of particulars which precedes. In short, *τἀναντία ἅπαντα* is the plural form of *τοὐναντίον ἅπαν*, which occurs, Plato, Polit. 310 D, as a variant for the more usual phrase *πᾶν τοὐναντίον*; cp. Xen. Mem. 3. 12. 4 and (for the adverbial force of *πᾶς* or *ἅπας*) Plato. Protag. 317 B.

Restoring to Heraclitus what rightfully belongs to him, we should therefore write the fragment thus: *ὁ θεὸς ἡμέρη εὐφρόνη, χειμὼν θέρος, πόλεμος εἰρήνη, κόρος λιμός· τἀναντία ἅπαντα, ὡυτὸς ὁ νοῦς· ἀλλοιοῦται δὲ ὅκωσπερ <μύρον>, ὁπόταν συμμιγῇ θυώμασιν, ὀνομάζεται καθ' ἡδονὴν ἑκάστου.* "*God is day and night, winter and summer, war and peace, satiety and hunger,— opposites quite, but the sense is the same; he changes, however, just as the neutral base employed in making unguents, when it is mixed with volatile essences, receives a name in accordance with the odor of each.*"

In regard to the philosophical interpretation of the fragment, which thus assumes a rank of capital importance for the thought of Heraclitus, it is hardly necessary to say more at present, than that we must henceforth build upon the foundations laid by Plato, Tim. 48 E–52 C. Plato and Lucretius prove that the same thought lay at the core of the atomic theory, and it is evident that Heraclitus here touched one of the basic conceptions of metaphysics in so far as it is concerned with the relation of the One and the Many. We are therefore called upon to consider the questions which crowd upon us with sobriety and careful discrimination, unless we are to efface the mile-stones that mark the progress of speculation. Such an inquiry is, however, too far-reaching to admit of discussion in this connection.

V² 72, 18. Fr. 71, *μεμνῆσθαι δὲ καὶ τοῦ ἐπιλανθανομένου ᾗ ἡ ὁδὸς ἄγει.*

The meaning, apparently missed by some scholars, is made clear by fr. 117, *οὐκ ἐπαΐων ὅκῃ βαίνει.* He forgets *whither he is going.*

V² 73, 14. Fr. 77, *ψυχῇσι . . . τέρψιν ἢ θάνατον ὑγρῇσι γενέσθαι.*

It seems very probable that we are here dealing, if one may so express it, with a conflate text; that is to say, two utterances of Heraclitus, otherwise essentially identical, but differing in this, that one related to *τέρψις*, the other to *θάνατος*, appear to have been merged in one. Either statement, taken by itself, is entirely intelligible; but it is improbable that Heraclitus combined them in the manner of this 'fragment.'

V² 73, 19. Fr. 78, *ἦθος γὰρ ἀνθρώπειον μὲν οὐκ ἔχει γνώμας, θεῖον δὲ ἔχει.*

The word *ἦθος* is difficult and improbable. I suspect that we should write *ἔθνος*; cp. Eurip. Orest. 976,

ἰὼ ἰώ, πανδάκρυτ' ἐφαμέρων
ἔθνη πολύπονα.

The iambic movement of the fragment is obvious, and the position of μέν appears somewhat forced. One is tempted to write the sentence as verse,

ἔθνος μὲν ἀνθρώπειον οὐ γνώμας ἔχει,
θεῖον δ' ἔχει.

This may, of course, be nothing more than the work of chance; but the entire cast of the sentence suggests that we are dealing with verse converted into prose. Now we know that there were those who versified the philosophy of Heraclitus. One of their number, Scythinus, a writer of the fourth century, is known by name; and one of the fragments of Scythinus (fr. 2, V^2 86, 22 sq.) has come down to us reconverted into prose, which Wilamowitz has again rendered in verse. I do not suggest, though it is possible, that we have before us another reconverted version of Heraclitus by Scythinus; for the cases of Cleanthes, whose Stoic verses are in part little more than paraphrases of Heraclitus, and of 'Epicharmus,' among whose fragments there are some which reproduce the thought of Heraclitus as others do that of Plato, caution us to avoid hasty conclusions. Nevertheless, I incline to think that fr. 78 is in fact a thinly disguised prose rendering of a verse original; for there are at least two other 'fragments' of Heraclitus (80 and 100) whose form suggests a versified original. As it is best to discuss them separately, I will add only that one of them, like fr. 78, is quoted by Origen *Against Celsus*. If my suggestion be approved by scholars, an interesting question arises, to wit, how accurately the versifier, if he was actually trying to reproduce the thought of Heraclitus, as Celsus or his source supposed, succeeded in rendering it. In the case of fr. 78, it is a nice question whether Heraclitus would have said what is here imputed to him. Origen seems to be clearly right in interpreting γνώμας with σοφία; but Heraclitus, whose doctrine of τὸ σοφόν we considered above in the note on fr. 41, although unsparing in his denunciation of the stupidity of the crowd, clearly believed that he had attained to wisdom. We naturally think of him as declaring with the Hebrew prophet that he alone was left.

We may note that fr. 78 seems to have served as a model for the spurious fragment of Epicharmus, 57, 7, which Diels (V^2 99, 4) writes thus:

οὐ γὰρ ἄνθρωπος τέχναν τιν' εὗρεν, ὁ δὲ θεὸς τοπάν.

In the same way Epicharmus, fr. 64 (V^2 100, 5 sq.), likewise spurious,

εἰμὶ νεκρός· νεκρὸς δὲ κόπρος, γῆ δ' ἡ κόπρος ἐστίν·
εἰ δ' ἡ γῆ θεός ἐστ', οὐ νεκρός, ἀλλὰ θεός,

glances at Heraclitus, fr. 96, νέκυες γὰρ κοπρίων ἐκβλητότεροι, and also at the anecdotes relative to the manner of his death, V^2 54, 29 sq., and to the anecdote about the oven, where also there were gods (V^2 58, 36 sq.). It seems altogether likely that the case of Heraclitus is in this a close parallel to that of Pythagoras, that myth soon began to weave legends about his name, and that forgeries sprang up which were supported by other forgeries. For the relation of the late Pythagoreans to Heraclitus, see Norden, *Agnostos Theos*, p. 345, n. 1. The examples given above and to be discussed presently make it extremely probable that some of these were written in verse and current as adespota, becoming in time attached to various names, such as Epicharmus. Others went under the name of Heraclitus, and it is probably to them that the Vita in Suidas refers (V^2 56, 46), ἔγραψε πολλὰ ποιητικῶς.

V^2 73, 23. Fr. 80, εἰδέναι δὲ χρὴ τὸν πόλεμον ἐόντα ξυνόν, καὶ δίκην ἔριν, καὶ γινόμενα πάντα κατ' ἔριν καὶ χρεώμενα.

This fragment has been discussed times innumerable, more particularly with reference to the last word, which is conceded to be impossible. If the sentence be regarded as an authentic prose fragment of Heraclitus, we probably cannot do better than accept Schuster's conjecture, καταχρεώμενα for χρεώμενα, and take it as complementary to γινόμενα. Diels, however, has rightly refused to admit into his text any of the numerous substitutes proposed for χρεώμενα. First of all it should be noted that καὶ γινόμενα πάντα κατ' ἔριν does not look so much like an utterance of Heraclitus as like an attempt to summarize details; this impression is confirmed by fr. 8, Arist. Eth. Nic. 1155^b 4, Ἡράκλειτος τὸ ἀντίξουν συμφέρον καὶ ἐκ τῶν διαφερόντων καλλίστην ἁρμονίαν καὶ πάντα κατ' ἔριν γίνεσθαι, which is itself quite obviously not a verbatim quotation but a summary. Long ago I was struck by the similarity in thought between καὶ δίκην ἔριν, καὶ γινόμενα πάντα κατ' ἔριν and Cleanthes, H. in Iov. 36,

δὸς δὲ κυρῆσαι γνώμης, ᾗ πίσυνος σὺ δίκης μέτα πάντα κυβερνᾷς,

and in a letter to Professor Diels I proposed instead of χρεώμενα to read χρεὼν μέτα, after Eurip. Herc. F. 20,

εἴθ' Ἥρας ὕπο
κέντροις δαμασθεὶς εἴτε τοῦ χρεὼν μέτα.

He replied that the anastrophe of μέτα was impossible in prose. This is of course true, as I well knew, assuming that we are dealing with real prose. At that time, having nothing more definite than the vague impression that the diction and movement of certain fragments of Heraclitus were distinctly poetic, and the statement in the Vita of Suidas, which I then interpreted as referring in a general way to poetic diction, I dropped the matter, though I still felt that χρεὼν μέτα was probably the true reading. Recently Dr. Bruno Jordan, Archiv für Gesch. der Philos., 24 (1911), p. 480, has independently made the same suggestion. In view of the probability that in this 'fragment,' as in fr. 78, we have a versified version of Heraclitus reconverted into prose, I regard my emendation as all but certain. I do not think it feasible to recover the verse original throughout, because, as I indicated above, καὶ γινόμενα πάντα κατ' ἔριν appears to be a summarizing formula; but it is easy to pick out parts of the sentence which fall almost without change into iambic verse:

εἰδέναι δὲ χρὴ
τὸν πόλεμον ὄντα ξυνόν . . . . . .
. . . . καὶ δίκην ἔριν
. . . . <τοῦ> χρεὼν μέτα.

It must be said that the text of the fragment is not absolutely certain, as the Mss. of Origen *Against Celsus* read εἰ δὲ χρὴ and δίκην ἐρεῖν; but the emendations adopted by Diels and reproduced above are so obvious that we may with confidence make his text the basis of our study. Regarded in the light of the poetic tags which have just been noted, we have again a close parallel to the prose paraphrase of Scythinus, fr. 2; but I hazard no guess as to the author of the versified version.

V² 76, 12. Fr. 100, ὥρας αἳ πάντα φέρουσι.

This fragment is preserved by Plutarch, who again alludes to it. The movement is clearly dactylic, and one may suspect that it formed part of an hexameter, though its brevity forbids dogmatic conclusions. In view of the experiments of Cleanthes it is not improbable that there were versions of certain Heraclitean sayings in heroic verse. It is, of course, possible that this fragment owes its rhythmical or metrical form to chance or to unconscious poetical influences not unnatural in the early stages of prose when verse was still the prevailing medium of artistic expression. This is perhaps the most probable explanation of the hexameter ending of fr. 5, θεοὺς οὐδ᾽ ἥρωας οἵτινές εἰσι, which I noted long ago and find referred to Homeric influence by Norden, *Agnostos Theos,* p. 88, n. 1. Dactylic movement, due to epic models, is much more easily thus accounted for than iambic or trochaic, such as have been noted above in fragments 78 and 80. Of the latter sort there is perhaps another example in fr. 120, quoted by Strabo, ἠοῦς καὶ ἑσπέρας τέρματα ἡ ἄρκτος καὶ ἀντίον τῆς ἄρκτου οὖρος αἰθρίου Διός. The general trochaic or iambic rhythm is at once apparent, and the close at least is faultless and strikingly suggestive of a trochaic verse. See infra, p. 714 sq. One may recast it into trochaics quite as easily as Wilamowitz did the second fragment of Scythinus, —

ἠοῦς [possibly ἕω δὲ] χἀσπέρας
τέρματ᾽ ἄρκτος κἀντί᾽ ἄρκτου οὖρος αἰθρίου Διός.

V² 77, 11. Fr. 108, ὁκόσων λόγους ἤκουσα, οὐδεὶς ἀφικνεῖται ἐς τοῦτο, ὥστε γινώσκειν ὅτι σοφόν ἐστι πάντων κεχωρισμένον.

This fragment has been much discussed; cp. Schuster, pp. 42, 44; Zeller, I. 629, n. 1. Gomperz proposed to bracket ὅτι σοφόν κτλ. as an interpolation. All those who retain the words regard them as an object clause, whatever interpretation they may put upon it. Diels identifies (τὸ) σοφόν with God, and understands the fragment as declaring the divine transcendence. This view has naturally provoked vigorous protests; for it is incompatible with all that we otherwise know of the thought of Heraclitus. I think λόγους is here used as Heraclitus uses λόγος of his own philosophic message or gospel: it refers to the Weltanschauungen of the great teachers and philosophers; for ἤκουσα does not necessarily refer to actual hearing of the person who sets forth his views, but includes the reading (by himself or by a slave) of written records. The pregnant force of γινώσκειν was sufficiently explained above in the discussion of fr. 41. Heraclitus, then, says: "*Of all those whose message regarding the nature of things it has been my fortune to learn about, not one has attained to the point of true knowledge.*" So much seems to be clear from a survey of the conception of knowledge which he is continually proclaiming. But, once we seize the import of his use of γινώσκειν, it is equally clear that ὅτι is not "that"; it is causal, and the obvious conclusion to his sentence follows: "*for wisdom is far removed from all*" ("men" or "of them"). One may illustrate this use of κεχωρισμένον by a passage from Cleanthes quoted by Sext. Empir. 9. 90, *ὥστε οὐ τέλειον ζῷον ὁ ἄνθρωπος, ἀτελὲς δὲ καὶ πολὺ κεχωρισμένον τοῦ τελείου.* The questionable fragment of Philolaus, quoted by Diels, and the quotation from Philostratus ap. Euseb. P. E. 4. 13, *ἑνί τε ὄντι καὶ κεχωρισμένῳ πάντων,* made by Norden, *Agnostos Theos*, 39, n. 3, afford but weak support for so unlikely a theory as that of Diels. In printing the fragment, I should place a colon between *γινώσκειν* and *ὅτι.* The sentence thus furnishes a new illustration of the difficulty, noted by Aristotle, of phrasing Heraclitus. Diels mentions, but does not adopt, my interpretation in V³.

V² 77, 19. Fr. 112, *σωφρονεῖν ἀρετὴ μεγίστη, καὶ σοφίη ἀληθέα λέγειν καὶ ποιεῖν κατὰ φύσιν ἐπαΐοντας.*

The Mss. here, as in fr. 116, show *σωφρονεῖν.* Diels here substitutes *τὸ φρονεῖν,* there *φρονεῖν,* in order to adapt the diction to that of Heraclitus. He renders: "Das Denken ist der grösste Vorzug, und die Weisheit besteht darin, die Wahrheit zu sagen und nach der Natur zu handeln, auf sie hinhörend." Besides changing *σωφρονεῖν* to *τὸ φρονεῖν,* he gives a forced rendering of *ἀρετή* and *ἐπαΐοντας* which serves to conceal the obvious Stoic character of the saying. Again, there is no other instance of *σοφίη* in the supposedly genuine fragments of Heraclitus, who seems to have used (*τὸ*) *σοφόν* instead: it does recur in fr. 129, which Diels reckons doubtful or spurious but others accept as genuine. Yet, granting that it is genuine, *σοφίη* there means something very different: it is, like *πολυμαθείη* and *κακοτεχνίη,* a term of reproach. One who reads the sentence without bias will readily admit that *ἀρετή* means an ethical virtue. As for *ἀληθέα λέγειν,* one may perhaps defend it by citing the denunciation of the *ψευδῶν τέκτονας καὶ μάρτυρας* in fr. 28; but it is doubtful whether so obviously an ethical virtue would have counted as a mark of *σοφίη* in the days of Heraclitus. In opposition to this it may be said that Ἀλήθεια was the ideal of the Greek philosophers from the beginning. True; but it was objective Truth which they sought, and not the virtue of truthfulness. The juxtaposition of *ἀληθέα λέγειν* and *ποιεῖν κατὰ φύσιν* does not suggest a reference to abstract or objective truth. Finally, *ποιεῖν κατὰ φύσιν ἐπαΐοντας* bears all the marks of Stoic doctrine; for it is hardly defensible to render *ἐπαΐοντας* with "auf sie hinhörend." The word has here, as in fr. 117, *οὐκ ἐπαΐων ὅκῃ βαίνει,* the sense which it regularly bears in Plato, to wit, "knowing"; cp. Xen. Mem. 1. 1. 9, *δαιμονᾶν δὲ καὶ τοὺς μαντευομένους ἃ τοῖς ἀνθρώποις οἱ θεοὶ μαθοῦσι διακρίνειν.* The words then clearly mean "*to act in accordance with* *nature consciously and with full knowledge."* This thought is, however, in substance and in form entirely Stoic, corresponding in the ethical sphere to the injunction to submit willingly to Fate, in the religious sphere, as expressed in Cleanthes's lines to Fate. One may, of course, discover the germs of this view in genuine fragments of Heraclitus; but Diels's alterations in the text and his interpretation do not meet the reasonable objections long since urged by others to the genuineness of this fragment.

V² 78, 8. Fr. 116, *ἀνθρώποισι πᾶσι μέτεστι γινώσκειν ἑωυτοὺς καὶ σωφρονεῖν.*

This fragment, like the preceding, is derived from Stobaeus, and like it, too, has been by many regarded as spurious. As I have already stated, Diels writes *φρονεῖν* for *σωφρονεῖν,* in order to meet an obvious criticism. This procedure would be justifiable, however, only if the passage as a whole created a presumption in favor of Heraclitean authorship, which is supported solely by the lemma of Stobaeus. In fact all indications point to the period after Socrates. Whoever attributed the saying to Heraclitus doubtless did so in view of fr. 101, *ἐδιζησάμην ἐμεωυτόν,* but the interpretation of the Delphic *γνῶθι σαυτόν* as an injunction to recognize one's limitations and to occupy oneself with that which lies within one's proper scope and power,— this is, so far as we know, Socratic: he who would claim it for Heraclitus must assume the burden of proof. But no unbiased reader of our fragment will doubt that *γινώσκειν ἑωυτοὺς καὶ σωφρονεῖν* was intended to express that precise thought. I cannot justify the changing of *σωφρονεῖν* to *φρονεῖν,* and cannot accept the fragment as genuine. Bywater was clearly right in marking both 112 and 116 as doubtful. Since they come to us from Stobaeus, who quotes them under widely different heads, it is plain that their assignment to Socrates is not due to a mere mistake in the lemmata of his text, but the error must be charged to his sources.

V² 78, 16. Fr. 120, *ἠοῦς καὶ ἑσπέρας τέρματα ἡ ἄρκτος καὶ ἀντίον τῆς ἄρκτου οὖρος αἰθρίου Διός.*

In V³ Diels briefly notes my interpretation of *οὖρος αἰθρίου Διός* as "wind of heaven," which was proposed in my review of his *Herakleitos von Ephesos*², in Class. Philol., 5. p. 247; but he appears still to prefer his own suggestion that Heraclitus referred to Mt. Olympus. As I regard my proposal as almost certainly right, I offer here a few additional observations to supplement my former statement, which exigencies of space then compelled me to omit. For the meaning of οὖρος, "wind," I would refer to Schmidt's *Synonymik*. See also Bonitz, *Index Aristotelicus*, s. v. ἄρκτος. It was common to say καὶ πρὸς ἄρκτον καὶ πρὸς νότον. The phrases employed by Herodotus in speaking of the cardinal points are especially interesting; I have made a complete list of them, and they seem to me to be decisive. I will refer, however, to but a few by way of illustration: 1. 148, πρὸς ἄρκτον τετραμμένος . . . πρὸς ζέφυρον ἄνεμον; 2. 8, φέρον ἀπ' ἄρκτου πρὸς μεσεμβρίης τε καὶ νότου; 3. 102, πρὸς ἄρκτου τε καὶ βορέου ἀνέμου. Cp. Hesiod, Theog. 378–82.

Though I do not accept the suggestion of Diels that the οὖρος Διός is Mt. Olympus, I will refer to a passage which might possibly be used to support it, to wit, Hippocr. Περὶ ἑβδομάδων, 48 (9. 462 L.), *Definitio autem superiorum partium et inferiorum corporis umbilicus*. It would be interesting to know the Greek text: perhaps Helmreich or some other ransacker of medical manuscripts may yet recover it! It occurs in a part of the treatise much discussed of late; see Roscher, *Über Alter, Ursprung und Bedeutung der hippokr. Schrift von der Siebenzahl*, p. 37, n. 67, who of course, in relating this to his "Weltkarte," refers to the ὄμφαλος γῆς or θαλάττης, and believes that the writer had in mind (not Delphi, but) Delos or Teos. Mt. Olympus might well serve as a landmark to divide the "upper" or northern parts of the earth from the "lower" or southern; but it does not seem so suitable for a zero meridian. I doubt, moreover, whether Heraclitus had any "Greenwich" in mind: what he seems to have meant is merely this, that "east" and "west" are relative terms and are delimited by a north and south line drawn through any point that may be in question. Various special meridians, useful to the geographer and mariner, were recognized at a comparatively early date, as may be seen from Herodotus; but a zero meridian, so far as I know, was not thought of before the time of the Alexandrian geographers. For the suggestion of a possible verse original for the fragment, see above on fr. 100. This would readily account for the use of οὖρος in the sense of wind.

V² 80, 10. Fr. 128, *δαιμόνων ἀγάλμασιν εὔχονται οὐκ ἀκούουσιν, ὥσπερ ἀκούοιεν, οὐκ ἀποδιδοῦσιν, ὥσπερ οὐκ ἀπαιτοῖεν.*

In regard to the text of this spurious fragment I agree with Diels, except that I would set a colon after ἀκούοιεν; from his interpretation I dissent, because it seems to me obviously at fault. In some unaccountable way he appears to have overlooked my note in Class. Philol. 5. p. 247, for he renders the text thus: "Sie beten zu den Götterbildern, die nicht hören, als ob sie Gehör hatten, die nichts zurückgeben, wie sie ja auch nichts fordern könnten," The saying is a close parallel to fr. 127, likewise spurious, in that it charges men with inconsistency in their dealings with the gods. Hence *οὐκ ἀποδιδοῦσιν* (= *ἀποδιδόασιν*; not the partic.!) answers to *εὔχονται* as *ὥσπερ οὐκ ἀπαιτοῖεν* answers to *ὥσπερ ἀκούοιεν*, and the meaning, as I said in my former note, is: "*They make vows to the images of the gods, that hear not, as if they heard; they pay not their vows, as if they* (*the gods*) *required it not.*" Everyone can supply the necessary classical examples for *εὔχονται, ἀποδιδοῦσιν*, and *ἀπαιτοῖεν*. I will quote one from the LXX., Deuter. 23. 21, *ἐὰν δὲ εὔχῃ εὐχὴν κυρίῳ τῷ θεῷ σου, οὐ χρονιεῖς ἀποδοῦναι αὐτήν, ὅτι ἐκζητῶν ἐκζητήσει κύριος ὁ θεός σου, καὶ ἔσται ἐν σοὶ ἁμαρτία.*

[Hippocrates.]

V² 81, 36—82, 16. For this passage, see my *Antecedents of Greek Corpuscular Theories*, Harvard Studies in Class. Philol., 22 (1911), p. 148 sq. It is to this article, and not to "Class. Philol. 22. 158," that Diels should have referred V³ 106, 16, note.

c. 13. Epicharmus.

V² 91, 23. Fr. 4. 6,

τὸ δὲ σοφὸν ἁ φύσις τόδ' οἶδεν ὡς ἔχει
μόνα· πεπαίδευται γὰρ αὐταύτας ὕπο.

Diels renders, "Doch wie sich's mit dieser Weisheit verhält, das weiss die Natur allein. Denn sie hat's ganz von selbst gelernt." It is, perhaps, a matter of no great consequence, but I believe his translation rests on a misconception of *τὸ σοφὸν τόδε* and *ὡς ἔχει*. As to the former, it has little in common with (*τὸ*) *σοφόν* of Heraclitus, but, like the familiar phrase *οὐδὲν ποικίλον οὐδὲ σοφόν*, denotes something recondite or cunningly devised. In regard to *ὡς ἔχει*, I remarked above, in my note on Heraclitus, fr. 1, that it here refers to the process of becoming, "how it comes about." The words of the fragment mean, "Nature alone knows the secret of this cunning device, or the way in which this mysterious result is brought about." This use of *ὡς ἔχει* and related phrases appears to have escaped many scholars. Possibly it baffled the copyists also in certain instances. Thus Xen. Mem. 1. 1. 11, *οὐδὲ γὰρ περὶ τῆς τῶν πάντων φύσεως, ἧπερ τῶν ἄλλων* οἱ πλεῖστοι, διελέγετο σκοπῶν, ὅπως ὁ καλούμενος ὑπὸ τῶν σοφιστῶν κόσμος ἔχει, καὶ τίσιν ἀνάγκαις ἕκαστα γίνεται τῶν οὐρανίων κτλ. Here the Mss. are divided between ἔχει and ἔφυ, and the editors find it difficult to decide. I believe that ἔχει, which has the better credentials, is the true reading, though one may question whether the unfamiliar force of ἔχει or the similarity of sound led to the substitution of ἔφυ. As I pointed out in my study Περὶ Φύσεως, the same duplicity as appears in the force of ὡς ἔχει occurs also in the use of φύσις, which predominantly signifies that which a thing is, but, pursuant to a constant habit of the human mind, is most frequently and naturally defined by recounting the story of its birth.

c. 18. Parmenides.

V² 105, 34. Diog. L. 9. 22, *γένεσιν ἀνθρώπων ἐξ ἡλίου πρῶτον γενέσθαι· αὐτὸν δὲ ὑπάρχειν τὸ θερμὸν καὶ τὸ ψυχρόν, ἐξ ὧν τὰ πάντα συνεστάναι.*

Various proposals have been made for the emendation of ἡλίου, of which ἰλύος is the most probable. It is obvious, however, that ἐξ ἡλίου, or whatever we may substitute for it, was not intended to denote the elemental constituents of man, since they are expressly mentioned later in the sentence. If the writer had in mind merely the source of the force which led to the origin of man, ἐξ ἡλίου, however singular, may be allowed to stand. But Diels is quite right in regarding αὐτὸν as corrupt. The language of Aristotle and his commentators suggests the obvious correction, αὐτοῖς δ' ἐνυπάρχειν, referring to the στοιχεῖα ἐνυπάρχοντα.

V² 115, 10. Fr. 1, 28,

> χρεὼ δέ σε πάντα πυθέσθαι
> ἠμὲν Ἀληθείης εὐκυκλέος ἀτρεμὲς ἦτορ
> ἠδὲ βροτῶν δόξας, ταῖς οὐκ ἔνι πίστις ἀληθής.

Something depends upon the precise meaning of πίστις ἀληθής; for it must to a considerable extent determine our conception of the attitude of Parmenides toward the βροτῶν δόξαι, which seem to have occupied his thought in much the larger part of his philosophical poem. The phrase recurs, fr. 8, 26 sq.,

> αὐτὰρ ἀκίνητον μεγάλων ἐν πείρασι δεσμῶν
> ἔστιν ἄναρχον ἄπαυστον, ἐπεὶ γένεσις καὶ ὄλεθρος
> τῆλε μάλ' ἐπλάχθησαν, ἀπῶσε δὲ πίστις ἀληθής.

Diels renders it with "verlässliche Wahrheit" and "wahre Überzeugung"; Burnet and Nestle do not vary the phrase but give "true belief" and "des Wahren Gewissheit" in both cases. Two other passages of the poem ought to be compared, to wit, fr. 8, 12,

οὐδέ ποτ' ἐκ μὴ ἐόντος ἐφήσει πίστιος ἰσχὺς
γίγνεσθαί τι παρ' αὐτό,

and fr. 8, 17,

οὐ γὰρ ἀληθὴς
ἔστιν ὁδός.

In the passage last mentioned ἀληθὴς ὁδός is clearly equivalent to Ἀληθείης ὁδός, as in fr. 4, 4 we have Πειθοῦς ἐστι κέλευθος. So in Sophocl. O. R. 500,

ἀνδρῶν δ' ὅτι μάντις πλέον ἢ 'γὼ φέρεται,
κρίσις οὐκ ἔστιν ἀληθής,

where the meaning obviously is that "there is no proving the truth of the contention that a seer outstrips me." This use of κρίσις calls to mind the fact that Parmenides employs the same word, fr. 8, 15,

ἡ δὲ κρίσις περὶ τούτων ἐν τῷδ' ἔστιν·
ἔστιν ἢ οὐκ ἔστιν· κέκριται δ' οὖν, ὥσπερ ἀνάγκη,
τὴν μὲν ἐᾶν ἀνόητον ἀνώνυμον (οὐ γὰρ ἀληθὴς
ἔστιν ὁδός), τὴν δ' ὥστε πέλειν καὶ ἐτήτυμον εἶναι.

Here the context appears to me to furnish the clue to the meaning of πίστις; for Parmenides clearly has in mind an action at law in which the issue is sharply drawn and judgment is rendered. So fr. 8, 27 sq. the πίστις ἀληθής sends γένεσις and ὄλεθρος into banishment. The juxtaposition of κρίσις and πίστις shows that πίστις means such evidence or proof as may be adduced in court, a meaning which the word quite regularly bore in legal argumentation. Aristotle, the logician, feeling that forensic oratory employed the enthymeme rather than the syllogism, and that in consequence its deductions were less cogent, continued to use πίστις for rhetorical proof in contradistinction to ἀπόδειξις, the stricter proof of logic or science. Thus πίστις is for him πειθοῦς κέλευθος, the method proper to a procedure which, like the plea of the rhetor, has for its object the establishment of the εἰκός. In much the same way the σήματα of Parmenides, fr. 8, 2, are the σημεῖα of forensic argumentation, which Aristotle in like manner and for the same reason distinguished from the more certain *τεκμήρια*. Thus we see that the dialectic of Parmenides, which eventuated in the Aristotelian logic, employed the forms and terminology of forensic rhetoric, though with an evident effort to reduce argumentation to the exactitude of demonstration; and *πίστις ἀληθής* is just this demonstration of truth. When, therefore, Parmenides objects to the *βροτῶν δόξαι*, it is because they do not carry the force of logical or dialectic evidence, or that such evidence is against them.

V² 115, 19. Fr. 1, 37,

μόνος δ' ἔτι θυμὸς ὁδοῖο
λείπεται.

V² 118, 38. Fr. 8, 1,

μοῦνος δ' ἔτι μῦθος ὁδοῖο
λείπεται, ὡς ἔστιν.

It appears to be generally conceded that *θυμός* and *μῦθος* are corruptions of one and the same word; *θυμός*, at any rate, is unintelligible. Of the numerous emendations proposed Platt's *οἶμος* is doubtless the best, though Diels seems to prefer *ῥυμός*; but *ῥυμός* does not so well explain the corruption as *οἶμος*. I am about to propose a correction, which seems to me all but certain. The stress on *μόνος* and *λείπεται* suggests that we are reduced to a way that barely remains. Similarly Plato, Symp. 184 B, *μία δὲ λείπεται τῷ ἡμετέρῳ νόμῳ ὁδός*, reinforced by 184 E, *μοναχοῦ ἐνταῦθα . . . ἄλλοθι δὲ οὐδαμοῦ*, like the Aristotelian dictum, *τὸ ἁμαρτάνειν πολλαχῶς ἔστι, τὸ κατορθοῦν μοναχῶς*, calls to mind the Gospel saying, *στενὴ ἡ πύλη καὶ τεθλιμμένη ἡ ὁδὸς ἡ ἀπάγουσα εἰς τὴν ζωήν*. I take it for granted that Parmenides regarded and characterized the way of Truth as a strait and narrow path, just as, fr. 6, 2 sq., he obviously thinks of the way of Error as broad, since "mortals, knowing nought, stagger (*πλάττονται*) along it with unsteady minds." I can think of nothing so suitable for his purpose, or so likely to give rise to the corruptions *θυμός* and *μῦθος*, as the word *ἰσθμός*. Plato, Tim. 69 E, uses it of the human neck, Emped. fr. 100, 19, of the narrow orifice of the clepsydra, and Hom., σ 300, uses *ἴσθμιον* of a necklace. The Homeric scholiast says that the throat is called *ἰσθμός, ἀπὸ τοῦ εἰσιέναι τὴν τροφὴν δι' αὐτοῦ*. The corresponding use of *αὐχήν* (Herod. 7. 223) and of *fauces* in Latin in speaking of a narrow defile or 'isthmus' is sufficiently well known. Now it happens that in Emped. fr. 100, 19, *ἰσθμός* has become corrupted in a part of the MS. tradition, and in Sophocl., fr. 145,

ἁ δὲ μνᾶστις
θνατοῖς εὐποτμόταтα μελέων
ἀνέχουσα βίου βραχὺν ἰσθμόν,

where ἰσθμός refers to "the narrow span of life," modern scholars have ignorantly sought to substitute something else. Nauck here proposed οἶμον, as Platt does for Parmenides. But the MS. reading is confirmed by Aelian, V. H. 2. 41, ὅτε αὐτῷ τὸ ἐκ Βουτοῦς μαντεῖον ἀφίκετο προλέγον τὴν τοῦ βίου στενοχωρίαν, and by Cicero's use of *angustiae temporis.*

I should therefore read ἰσθμὸς ὁδοῖο in both fragments. Lest anyone be disturbed by the hiatus between ἔτι and ἰσθμός, I remark that we find another instance of it in fr. 4, 6,

τὴν δή τοι φράζω παναπευθέα ἔμμεν ἀταρπόν,

in each case in the bucolic diaeresis. Diels, *Parmenides Lehrgedicht,* p. 67, in his note on the latter passage, well says: "Der Hiat in der bukolischen Diärese nicht anzutasten!" Indeed, the collision of words ending and beginning with the same vowel was even regarded by ancient grammarians as peculiarly justifiable. See Christ, *Metrik der Griechen and Romer*[2], p. 41, § 55, and the remarks of ancient grammarians on Hom. Od. λ 595, Verg. Georg. 1, 281, and Hor. C. 1. 28, 24. Herwerden, *Lexicon Gr. Suppletorium,* p. 400, suggests that ἰσθμός may have had the digamma, referring to Pindar, Isth. 1. 10, 32 and Bacchyl. 2, 7 Blass., but continues, "Sed fortasse hiatus nominum propriorum licentiae tribuendus. Cf. O. Schroeder, *Prol. Pind.* II. p. 14 et p. 17. Nec sane digamma habere potuit, si desscendit a verbo ἰέναι." I do not believe it had the digamma.

V[2] **117, 7.** Fr. 5, τὸ γὰρ αὐτὸ νοεῖν ἐστίν τε καὶ εἶναι.

The construction of this sentence has occasioned difficulties. It is obvious, however, that it is identical in meaning with fr. 8, 34, to be discussed below. I think we have here a case of brachylogy, and that we must supply νοεῖν before εἶναι from the preceding νοεῖν. "*For it is one and the same thing to think and to think that it is.*" See the examples cited by Kühner-Gerth, II. p. 565, § 597, h. Burnet, *Early Greek Philosophy*[2], p. 198, notes 1 and 3, propounds syntactical doctrines and puzzles which one ought in kindness to ignore. Any good grammar will supply abundant examples of the substantive use of the infinitive, with or without the article, earlier than the date of Parmenides. For Greek lyric poets, see Smyth, *Greek Melic Poets,* note on Alcman, fr. XII. For the articular infinitive in general, consult the articles of Professor Gildersleeve in Amer. Journ. of Philol.

V² 117, 14. Fr. 6, 1,

> *χρὴ τὸ λέγειν τε νοεῖν τ' ἐὸν ἔμμεναι· ἔστι γὰρ εἶναι,*
> *μηδὲν δ' οὐκ ἔστιν.*

The view of Diels and Burnet, which takes *ἔστι* and *ἔστιν* as equivalent to *ἔξεστι*, appears to me to be unsatisfactory; for the sentence thus becomes weak and out of character. Parmenides says: "*For existence exists, and nought is not.*" The absence of the article with *εἶναι* and *μηδέν* makes no difference. In regard to the first sentence, we must, perhaps, acquiesce in the view of Diels, who regards *τό* as the epic pronoun, and renders: "*Dies ist nötig zu sagen und zu denken, das nur das Seiende existiert*"; but this use of *τό* would be unique in Parmenides, in whom we expect the articular infinitive. It is possible that he meant "*Speech and thought must be real*"; for, though we do not otherwise find the recognition of the corporeal existence of thought and speech clearly expressed before the Stoics and Epicureans, it is by no means certain that Parmenides would not be called upon to defend his 'materialistic' doctrines by asserting the corporeality of thought and speech, since he expressly concerned himself with predication, fr. 8, 35 sq.

V² 117, 21. Fr. 6, 8,

> *οἷς τὸ πέλειν τε καὶ οὐκ εἶναι ταὐτὸν νενόμισται*
> *κοὐ ταὐτόν.*

Burnet, *Early Greek Philosophy*², p. 198, n. 3, tortures this passage in order to eliminate the articular infinitives and the solecism *τὸ . . . οὐκ εἶναι*; but his interpretation is impossible, and, as we have seen, his reluctance to admit the articular infinitive is indefensible. As to *τὸ . . . οὐκ εἶναι*, others before him have found in it a rock of offence; but the responsibility rests with Parmenides. If he could say, *οὕτως ἢ πάμπαν πελέναι χρεών ἐστι ἢ οὐχί* (fr. 8, 11) alongside *ἡ δ' ὡς οὐκ ἔστιν τε καὶ ὡς χρεών ἐστι μὴ εἶναι* (fr. 4, 5) it is difficult to see why he should not have said *τὸ οὐκ εἶναι* instead of *τὸ μὴ εἶναι*.

V² 119, 6. Fr. 8, 9,

> *τί δ' ἄν μιν καὶ χρέος ὦρσεν*
> *ὕστερον ἢ πρόσθεν, τοῦ μηδενὸς ἀρξάμενον, φῦν.*

Diels renders ὕστερον ἢ πρόσθεν with "früher oder später"; Burnet, correctly I believe, with "later rather than sooner"; for I regard the phrase as a sort of *comparatio compendiaria.* The question was repeated and amplified by later philosophers; cp. Lucret. 5, 165–180; Cic. N. D. 1. 9. 21; V² 305, 16 sq.; Diels, *Dox. Gr.*, p. 301, 2, *καὶ οὔτε κατὰ τὸ πρῶτον μακάριός ἐστιν ὁ θεός, τὸ γὰρ ἐλλεῖπον εἰς εὐδαιμονίαν οὐ μακάριον, οὔτε κατὰ τὸ δεύτερον· μηδὲν γὰρ ἐλλείπων κ ε ν α ῖ ς ἔμελλεν ἐπιχειρεῖν πράξεσιν.* In the last passage I think we should clearly read *καιναῖς* for *κεναῖς*; cp. Lucret. 5, 168 sq.,

> Quidve novi potuit tanto post ante quietos
> inlicere ut cuperent vitam mutare priorem?
> nam gaudere novis rebus debere videtur
> cui veteres obsunt; sed cui nil accidit aegri
> tempore in anteacto, cum pulchre degeret aevum,
> quid potuit novitatis amorem accendere tali?

I may add that Parmenides, fr. 8, 7, *πῇ πόθεν αὐξηθέν,* and 8, 32 sq.,

> *οὕνεκεν οὐκ ἀτελεύτητον τὸ ἐὸν θέμις εἶναι·*
> *ἔστι γὰρ οὐκ ἐπιδευές, ἐὸν δ' ἂν παντὸς ἐδεῖτο,*

is expanded by Plato, Tim. 32 C–34 A, with an obvious addition 33 A, which is apparently drawn from the Atomists. Cp. V² 343, 4 sq., and my *Antecedents of Greek Corpuscular Theories,* Harvard Studies in Class. Philol., 22 (1910), p. 139. See also the discussion above (p. 693 sq.) of V² 34, 18.

V² 120, 13. Fr. 8, 34, *ταὐτὸν δ' ἐστὶ νοεῖν τε καὶ οὕνεκέν ἐ σ τ ι νόημα.*

So far as I am aware, all interpreters of Parmenides have taken *οὕνεκεν* in the sense of "that for the sake of which." This is, of course, quite possible; but we thus obtain no satisfactory sense unless we are to adopt the Neo-Platonic conceptions which obviously suggested the accepted rendering. Probably no student of ancient philosophy who has learned the rudiments of historical interpretation would go so far afield. Only the natural obsession that we must take our cue from the ancients, whose incapacity in this regard should no longer be a secret, can account for the failure of some one to make the obvious suggestion that we take *οὕνεκεν* as *ὅτι,* and read *ἔστι*; for it seems clear that Parmenides meant, "*Thinking and the thought that the object of thought exists, are one and the same.*" Kühner-Gerth, II. p. 356, and the lexicons give the examples for this use of *οὕνεκα*; for the dependence of a substantive clause on a verbal substantive, Stahl, *Krit.-histor. Syntax des gr. Verbums der klass. Zeit*, p. 546, § 2, gives abundant examples, to which a careful reader will be able to add largely in a week. The parallelism of infinitive and substantive is no closer than Mimnermus, 2, 10,

αὐτίκα τεθνάμεναι βέλτιον ἢ βίοτος.

If the inverted order of words should cause any one to hesitate, let him recall Xenophanes, fr. 34, 2,

καὶ ἅσσα λέγω περὶ πάντων,

and Sophocl. O. R. 500 sq., quoted above, p. 718, on fr. 1, 28 sq. I regard this construction as of especial importance, because the frank equivalence of the infinitive with the substantive would seem to render for all time impossible the strange acrobatic feats performed by Burnet in his endeavor to eliminate the substantival infinitive, with or without the article, from the text of Parmenides.

c. 19. Zeno.

V² 133, 8. Fr. 1, *καὶ περὶ τοῦ προὔχοντος ὁ αὐτὸς λόγος. καὶ γὰρ ἐκεῖνο ἕξει μέγεθος καὶ προέξει αὐτοῦ τι. ὅμοιον δὴ τοῦτο ἅπαξ τε εἰπεῖν καὶ ἀεὶ λέγειν. οὐδὲν γὰρ αὐτοῦ τοιοῦτον ἔσχατον ἔσται οὔτε ἕτερον πρὸς ἕτερον οὐκ ἔσται. οὕτως εἰ πολλά ἐστιν, ἀνάγκη αὐτὰ μικρά τε εἶναι καὶ μεγάλα· μικρὰ μὲν ὥστε μὴ ἔχειν μέγεθος, μεγάλα δὲ ὥστε ἄπειρα εἶναι.*

The question discussed in the portion of the fragment here reproduced concerns the second alternative, μεγάλα δὲ ὥστε ἄπειρα εἶναι. There is some difference of opinion among scholars regarding the precise conception of τὸ προὖχον. For some years I have been accustomed to think of the προὖχον ἔσχατον of Zeno as the *extremum quodque cacumen* of Lucretius 1, 599; or, more exactly, I have held and still hold that the Epicurean doctrine of the *partes minimae*, of which the definition of the *extremum cacumen* is a part, owed its origin in part to this argument of Zeno's. The discussion of the *partes minimae* by Giussani had never satisfied me; the view of Pascal, *Studii Critici sul Poema di Lucrezio* (1903), p. 49 sq., seemed to me essentially sound (see Amer. Journ. of Philol., 24, p. 332). He drew attention to Aristotle's arguments (De Anim. 409ª 13 sq., De Gen. et Corr. 326ᵇ 1 sq., Phys. 240ᵇ 8 sq.) to prove that the ἀμερές cannot have motion, or at most can have motion *κατὰ συμβεβηκός* only, which would be fatal to the older Atomism. Pascal himself did not see that Aristotle (and MXG. 977^b 11 sq.) derived his arguments from Plato, Parm. 138 BC. With these we must clearly associate the questions touching the rotation of a circle or a sphere, Arist. Phys. 240^a 29 sq., 265^b 7; Simpl. Phys. 1022; [Arist.] Qu. Mech. c. 1; Plotin. Ennead. 2. 2. 1. But Plato clearly had in mind positions taken by the younger Eleatics, which he was developing. What these were in detail I am unable to say; but the argument of Zeno which we are considering seems to me to present the same problem from another angle; if the criticisms of Plato and Aristotle, applied to the atom, as an *ἀμερές*, rendered motion, which the Atomists regarded as inherent in it, apparently impossible, the criticism of Zeno made it necessary that there should be a limit to the number and the divisibility of the parts of which a revised atomism might concede that it was composed. In fr. 1, therefore, I regard *αὐτοῦ* in *προέξει αὐτοῦ τι* as a partitive genitive, and accept the emendation of Gomperz, *ὥστε ἕτερον πρὸ ἑτέρου* for *οὔτε ἕτερον πρὸς ἕτερον*. As I conceive the matter, Zeno does not think of a *cacumen* as being added; but, since every extended part is susceptible of division, that which we regard as the *προὔχον* must always have an outer and an inner half, and so by the division *ad infinitum* of the *προὔχον* itself there is crowded between it and the next inward 'unit' an infinitude of parts which, from Zeno's point of view, must in effect advance the *προὔχον* or *cacumen* outward *ad infinitum*. Consequently things become *μεγάλα ὥστε ἄπειρα εἶναι*.

c. 20. Melissus.

V^2 145, 10. Fr. 7. 3, *ἀλλ' οὐδὲ μετακοσμηθῆναι ἀνυστόν· ὁ γὰρ κόσμος ὁ πρόσθεν ἐὼν οὐκ ἀπόλλυται οὔτε ὁ μὴ ἐὼν γίνεται. ὅτε δὲ μήτε προσγίνεται μηδὲν μήτε ἀπόλλυται μήτε ἑτεροιοῦται, πῶς ἂν μετακοσμηθὲν τῶν ἐόντων εἴη; εἰ μὲν γάρ τι ἐγίνετο ἑτεροῖον, ἤδη ἂν καὶ μετατκοσμηθείη.*

A careful reading of this passage will convince any scholar that there is something wrong with it. The difficulty, however, lies entirely in the clause *πῶς . . . εἴη*, where the MSS. read *μετακοσμηθέντων ἐόντων τι ᾖ*. Mullach and Ritter-Preller present the same text as Diels, except that they read *τι εἴη*. Diels renders the clause thus: "wie sollte es nach der Umgestaltung noch zu dem Seienden zählen?" Burnet, apparently accepting the text of Mullach and Ritter-Preller, translates "how can any real thing have had its order changed?" I do not believe this rendering, which agrees with that of Mullach, is possible, for I know of no such periphrastic form as *μετακοσμηθὲν εἴη* (*ἀπαρνηθείς*, Plato, Soph. 217 C, is aor. pass. in form only); that of Diels, on the other hand, though clearly necessary if one adopts his text, does not yield the thought required in the context. I incline to think that *τι* and *ᾖ* are marginal corrections which have been misread and misplaced, and that we should read *πῶς ἂν μετακοσμηθείη τι τῶν ἐόντων*; "*How should anything real suffer change of order?*"

V[2] 149, 1. Fr. 9, *εἰ μὲν οὖν εἴη, δεῖ αὐτὸ ἓν εἶναι· ἓν δὲ ὂν αὐτὸ σῶμα μὴ ἔχειν. εἰ δὲ ἔχοι πάχος, ἔχοι ἂν μόρια, καὶ οὐκέτι ἓν εἴη.*

Although Simplicius twice so quotes Melissus, and we cannot therefore doubt that his text so read, I cannot believe that Melissus wrote *σῶμα μὴ ἔχειν*. That the Neo-Platonists understood him as holding that the existent is incorporeal is of course well known, but is insufficient warrant for attributing the doctrine to him. Zeller and Burnet seek to obviate the difficulty by referring the fragment, not to the Eleatic One, but to the Pythagorean Unit. Against this view there are two objections which appear to me to be fatal to it: first, we should have to suppose that Simplicius, who read this passage in its context, did not grasp its import, which must have been fairly clear; second, even if Simplicius should have erred in this respect, the argument of Melissus must have been applicable to the Eleatic One, and so Simplicius would be substantially right in quoting the words in order to prove that the Eleatic One was incorporeal. This very conception of Eleatic doctrine, however, would sufficiently account for a corruption of the text, such as reading *ἔχειν* for *εἶναι*. That is what I conceive to have occurred. Melissus, understanding *σῶμα* as an *ἄθροισμα* of parts which, because divisible *ad infinitum*, must be tridimensional or "have thickness," says that a true Unit (whether Eleatic or Pythagorean) cannot be conceived as a *σῶμα* or *ἄθροισμα*. See Amer. Journ. of Philol., Vol. 28, p. 79. At the beginning of the same clause the MS. tradition clearly points to the reading *ἓν δ' ἐὸν* rather than *ἓν δὲ ὄν*. This correction, which I had noted several years ago, has now been made by Diels in V[3].

c. 21. Empedocles.

V[2] 203, 13 sq. Arist. De Anima 1. 2. 404^b 8 sq., asserts that Empedocles regarded the soul (*ψυχή*) as compounded of all the elements, and quotes fr. 109 to prove it. So far as I can recall, all scholars have been content to accept this deduction of Aristotle, although the words quoted offer not the slightest confirmation of it and the doctrine thus ascribed to Empedocles is diametrically opposed to his conception of *ψυχή* in matters of religion. This conflict has been often noted, but no one seems to have seen that the solution of the difficulty lies in the simple fact that Empedocles did not connect these functions with the *ψυχή*, which he, like many other early Greeks, thought of as the entity only which escapes from man at the moment of death and survives the body. Fr. 110, 10,

πάντα γὰρ ἴσθι φρόνησιν ἔχειν καὶ νώματος αἶσαν,

shows what language Empedocles used: everything has *φρόνησις* and *νόημα*, but not *ψυχή*. See my remarks in Amer. Journ. of Philol., 33, p. 94 sq., and Journ. of Philos., Psychol. and Scient. Methods, 10, p. 107.

V² 203, 34. Fr. 110,

εἰ γάρ κέν σφ' ἀδινῇσιν ὑπὸ πραπίδεσσιν ἐρείσας
εὐμενέως καθαρῇσιν ἐποπτεύσῃς μελέτῃσιν,
ταῦτά τέ σοι μάλα πάντα δι' αἰῶνος παρέσονται,
ἄλλα τε πόλλ' ἀπὸ τῶνδ' ἐκτήσεαι· αὐτὰ γὰρ αὔξει
ταῦτ' εἰς ἦθος ἕκαστον, ὅπῃ φύσις ἐστὶν ἑκάστῳ.
εἰ δὲ σύ γ' ἀλλοίων ἐπορέξεαι, οἷα κατ' ἄνδρας
μυρία δειλὰ πέλονται ἅ τ' ἀμβλύνουσι μερίμνας,
ἦ σ' ἄφαρ ἐκλείψουσι περιπλομένοιο χρόνοιο
σφῶν αὐτῶν ποθέοντα φίλην ἐπὶ γένναν ἱκέσθαι·
πάντα γὰρ ἴσθι φρόνησιν ἔχειν καὶ νώματος αἶσαν.

The text of this fragment as given by Hippolytus is extremely corrupt; but I accept the text given by Diels everywhere except in verses 4 and 5. Here the MSS. read *αὔξει* and *ἔθος*: Diels retains the former and adopts Miller's suggestion of *ἦθος* for the latter. This text I think is clearly wrong, as the difficulties experienced by Diels in rendering the passage ought to convince any reader. But v. 8 sq. seem to me to show what we require; for they obviously contain the converse of the statement which the poet made in the sentence we are considering. I am convinced that Empedocles wrote *ἄξει*, not *αὔξει*; with regard to *ἔθος*, one may hesitate before deciding between the claims of *ἔθνος* and *ἦθος*. In favor of *ἔθνος* one may quote Hippocr. *Περὶ τόπων τῶν κατὰ ἄνθρωπον*, 1 (6, 278 L.), *τοῦτο δ' ὁποῖον ἄν τι πάθῃ,* *τὸ σμικρότατον ἐπαναφέρει πρὸς τὴν ὁμοεθνίην ἕκαστον πρὸς τὴν ἑωυτοῦ, ἤν τε κακὸν ἤν τε ἀγαθὸν ᾖ· καὶ διὰ ταῦτα καὶ ἀλγέει καὶ ἥδεται ὑπὸ ἔθνεος τοῦ σμικροτάτου τὸ σῶμα, ὅτι ἐν τῷ σμικροτάτῳ πάντ' ἔνι τὰ μέρεα, καὶ ταῦτα ἐπαναφέρουσιν ἐς τὰ σφῶν αὐτῶν ἕκαστα, καὶ ἐξαγγέλλουσι πάντα.* Other passages which may be compared are the following. Hippocr. *Περὶ φύσιος ἀνθρώπου*, 3 (6, 38 L.), *καὶ πάλιν γε ἀνάγκη ἀποχωρέειν ἐς τὴν ἑωυτοῦ φύσιν ἕκαστον, τελευτῶντος τοῦ σώματος τοῦ ἀνθρώπου, τό τε ὑγρὸν πρὸς τὸ ὑγρὸν καὶ τὸ ξηρὸν πρὸς τὸ ξηρὸν καὶ τὸ θερμὸν πρὸς τὸ θερμὸν καὶ τὸ ψυχρὸν πρὸς τὸ ψυχρόν. τοιαύτη δὲ καὶ τῶν ζῴων ἐστὶν ἡ φύσις καὶ τῶν ἄλλων πάντων· γίνεταί τε ὁμοίως πάντα καὶ τελευτᾷ ὁμοίως πάντα· ξυνίσταταί τε γὰρ αὐτέων ἡ φύσις ἀπὸ τουτέων τῶν προειρημένων πάντων, καὶ τελευτᾷ κατὰ τὰ εἰρημένα ἐς τωὐτὸ ὅθεν περ ξυνέστη ἕκαστον, ἐνταῦθα οὖν καὶ ἀπεχώρησεν.* *Περὶ φύσιος παιδίου* 17 (7, 496 L.), *ἡ δὲ σὰρξ αὐξομένη ὑπὸ τοῦ πνεύματος ἀρθροῦται, καὶ ἔρχεται ἐν αὐτέῃ ἕκαστον τὸ ὅμοιον ὡς τὸ ὅμοιον, τὸ πυκνὸν ὡς τὸ πυκνόν, τὸ ἀραιὸν ὡς τὸ ἀραιόν, τὸ ὑγρὸν ὡς τὸ ὑγρόν· καὶ ἕκαστον ἔρχεται ἐς χώρην ἰδίην κατὰ τὸ ξυγγενές, ἀφ' οὗ καὶ ἐγένετο.* Plato, Tim. 63 E, *ἡ πρὸς τὸ συγγενὲς ὁδός.* Ibid. 90 A, *πρὸς τὴν ἐν οὐρανῷ συγγένειαν.* Herod. 4. 147, *ἀποπλεύσεσθαι ἐς τοὺς συγγενέας.* Plotin. Ennead. 4. 3. 24, *εἰς τὸν προσήκοντα αὐτῷ τόπον.* Hermias, Irris. 7 (V^2 19, 14), *εἰς δὲ τὴν αὑτοῦ φύσιν ἐπανιὼν ἀήρ.* Menand. Epitrep. 105,

> *εἰς δὲ τὴν αὑτοῦ φύσιν*
> *ἄρας ἐλεύθερόν τι τολμήσει ποεῖν.*

Lucret. 2, 1112,

> nam sua cuique locis ex omnibus omnia plagis
> corpora distribuuntur et ad sua saecla recedunt.

These examples sufficiently prove that one can draw no inference from *εἰς* which would serve to decide the respective claims of *ἦθος* and *ἔθνος*; besides, the epic use of *εἰς* with reference to persons as well as places (Il. 7, 312; 15, 402; Od. 14, 126 sq.), which would obtain in Empedocles, leaves the question open. The poet means to say that Pausanias, to whom he addresses his poem as Lucretius addressed his to Memmius, if he gives heed to the instruction of his master, will find that it will lead him into all truth, since each truth will seek its fellows, each after its own kind; but if he deserts the living truth, it will in turn desert him, each truth, as before, longing to join its kindred. There are two passages in which Lucretius has plainly derived inspiration and suggestion from these words of Empedocles.

1, 400 Multaque praeterea tibi possum commemorando
argumenta fidem dictis corradere nostris.
verum animo satis haec vestigia parva sagaci
sunt per quae possis cognoscere cetera tute.
namque canes ut montivagae persaepe ferarum
naribus inveniunt intectas fronde quietes,
cum semel institerunt vestigia certa viai,
sic alid ex alio per te tute ipse videre
talibus in rebus poteris caecasque latebras
insinuare omnis et verum protrahere inde.

1, 1114 Haec sei pernosces parva perductus opella
. . . .
namque alid ex alio clarescet nec tibi caeca
nox iter eripiet quin ultima naturai
pervideas: ita res accendent lumina rebus.

After 1, 1114, with Munro, I assume a lacuna; for it appears obvious that the sentence is incomplete. But in the absence of more certain indications I refrain from speculating as to what and how much may have perished in the breach. Yet *perductus*, which is clearly right and ought not to be changed to *perdoctus*, and *iter*, like the words of Empedocles, suggest guidance on the way of truth: it is possible that Lucretius may have taken a hint, as 2, 75 sq., from ancient relay torch races, in which one runner handed over his torch or ignited that of his team-mate, to illustrate the way in which a truth once known flashes light far along paths hitherto shrouded in night. In 1, 400 sq. Lucretius cleverly adapts a conception to his own uses. As he did not accept the doctrine of the ubiquity of intelligence in nature, which underlies the thought of Empedocles, he was obliged to introduce a simile in lieu of the bold personification of facts and truths which renders memorable the passage of his predecessor. We naturally ask whether there was anything in his model to suggest the particular simile which he chose. Now, it must be confessed that there is a possible point of contact, if Empedocles wrote ἦθος rather than ἔθνος; for in that case ἦθος would certainly not mean "character" or "heart," as has been supposed, but "haunts" or "lair," according to a usage familiar in Greek. In that event we should have to think of facts or truths as having, like mountain-ranging beasts, their lairs where they hide their young and to which they themselves return and guide the man who follows them. If Empedocles used the word ἦθος, one might see in v. 4, ἄλλα τε πόλλ' ἀπὸ τῶνδ' *ἐκτήσεαι*, a reference to *τόκος*, usury; for, as one may perceive by Aeschin. S. 35, *δανείσματα οὐκ ὀλίγα, ἀφ' ὧν ἐκεῖνος τόκους ἐλάμβανε*, the phraseology suggests it. Ancient writers, however, were fully aware of the metaphor, which was still alive, and played on the word, as Ar. Thesmoph. 842 sq., Plato, Repub. 555 E, Arist. Pol. 1. 10. 1258ᵇ 5 sq.. This metaphor would well lead up to that of *ἦθος*, as the lair of wild beasts. From this too, it would be easy to explain the figure of Lucretius, who substitutes mountain-ranging hounds tracking the beasts to their lairs (*quietes*, 1, 405, and *caecas latebras*, 408). Indeed, it is possible that Empedocles may have used the simile of the hound in this very connection, fr. 101,

> *κέρματα θηρείων μελέων μυκτῆρσιν ἐρευνῶν*
> *<ὀσμᾶθ'> ὅσσ' ἀπέλειπε ποδῶν ἁπαλῇ περὶ ποίῃ.*

But the context in which the fragment is quoted by our ancient authorities, as well as Lucret. 4, 680 sq., suggest rather that Empedocles was there illustrating his doctrine of universal *ἀπορροιαί*. I find it difficult, therefore, to decide between the claims of *ἔθνος* and *ἦθος*; but incline on the whole to favor the former because of v. 9,

> *ποθέοντα φίλην ἐπὶ γένναν ἱκέσθαι.*

I may add that Mr. Cornford, *From Religion to Philosophy*, p. 64, makes an interesting suggestion in regard to Emped. fr. 17, 28,

> *τιμῆς δ' ἄλλης ἄλλο μέδει, παρὰ δ' ἦθος ἑκάστῳ,*

where he renders *παρὰ . . . ἑκάστῳ*, 'each has its wonted range.' See ibid., p. 34.

Now that the general sense of Emped. fr. 110 is clear, there can be no doubt about the meaning of v. 5, *ὅπῃ φύσις ἐστὶν ἑκάστῳ*. It is *prout cuique natura est*, "*each after its kind*."

c. 32. Philolaus.

V² 239, 31. Fr. 1, *ἁ φύσις δ' ἐν τῷ κόσμῳ.*

In V³ Diels adopts certain suggestions made in my *Notes on Philolaus*, Amer. Journ. of Philol., 28, p. 79, to which he refers, but rightly retains *δ' ἐν τῷ κόσμῳ* instead of *δὲ τῷ κόσμῳ*, which I formerly proposed; but in sense *τῷ κόσμῳ* was more nearly right than his rendering "bei der Weltordnung." In the notes he now cites parallels, which I furnished, for *φύσις ἐν τῷ κόσμῳ*. They sufficiently explain the phrase and fix its meaning. I will now add another, Plotin. Ennead. 3. 8. 1, παίζοντες δὴ τὴν πρώτην πρὶν ἐπιχειρεῖν σπουδάζειν εἰ λέγοιμεν πάντα θεωρίας ἐφίεσθαι καὶ εἰς τέλος τοῦτο βλέπειν, οὐ μόνον ἔλλογα ἀλλὰ καὶ ἄλλογα ζῷα καὶ τὴν ἐν τοῖς φυτοῖς φύσιν καὶ τὴν ταῦτα γεννῶσαν γῆν κτλ. Thus ἡ ἐν τῷ κόσμῳ φύσις = ἡ τοῦ κόσμου φύσις. In Plotinus there is probably a suggestion of the common, universal φύσις as manifesting itself in plant-life; but all these passages alike prove that the phrase does not mean "bei der Weltordnung."

V² 240, 5. Fr. 2, δηλοῖ δὲ καὶ τὰ ἐν τοῖς ἔργοις.

Since Diels has now (V³) adopted my interpretation of these words, I might allow the matter to rest there; but the observation that this and similar phrases have been unduly pressed in other contexts leads me to illustrate it further. Nestle, in Philol., 67, 544, writing as it seems in ignorance both of Newbold's article and of mine, arrived at substantially the same conclusion with myself. It would carry us too far afield to consider in detail the passages which I have studied; hence I will give a list of those only which serve to illustrate Greek usage. It will be seen that ἐν τοῖς ἔργοις and ἐπὶ τῶν ἔργων are generally used when appeal is made to facts of common observation or knowledge, as opposed to theory, argument, or unsupported statement. As a matter of fact, these references are usually so general that they amount to nothing but the bald assertion that observation or knowledge confirms or contradicts the proposition in question. In very few cases which I have noted does the context suffice to enable one to specify the particular facts to which the writer affects to appeal: many passages are open to different interpretations and competent scholars find it difficult to agree about them. They are therefore especially valuable for our purposes. See Plato, Protag. 352 A, Soph. 234 E, Gorg. 461 D, Repub. 396 A, 599 B, Phaedo 110 A, Tim. 19 E, Legg. 679 D, Axiochus 369 A; Xenoph. Hiero 9. 3; Bonitz, *Index Arist.* 286ᵃ 27 sq., 40 sq.; Bywater, on Arist. Poet. 1453ᵃ 17. Cp. Arist. De Gen. Animal. 3. 11. 762ᵃ 15, οὐθὲν γὰρ ἐκ παντὸς γίνεται, καθάπερ οὐδ' ἐν τοῖς ὑπὸ τῆς τέχνης δημιουργουμένοις. Meteor. 4. 3. 381ᵃ 10, καὶ οὐδὲν διαφέρει ἐν ὀργάνοις τεχνικοῖς ἢ φυσικοῖς, ἐὰν γίγνηται· διὰ τὴν αὐτὴν γὰρ αἰτίαν πάντα ἔσται. Such general references to the similarity of products of art and of nature abound in certain works of the Corpus Hippocrateum. See also Hippocr. Περὶ φυσέων, 5 (where, after stating his theory, the writer says), περὶ μὲν οὖν ὅλου τοῦ πρήγματος ἀρκεῖ μοι ταῦτα· μετὰ δὲ ταῦτα πρὸς αὐτὰ τὰ ἔργα τῷ αὐτῷ λόγῳ πορευθεὶς ἐπιδείξω *τὰ νοσήματα τούτου ἔκγονα πάντα ἐόντα.* In this instance the particular "facts" to which he appeals are mentioned. It is interesting to hear his conclusion, c. 15, *ὑπεσχόμην δὲ τῶν νούσων τὸ αἴτιον φράσειν· ἐπέδειξα δὲ τὸ πνεῦμα καὶ ἐν τοῖς ὅλοις πρήγμασι δυναστεῦον καὶ ἐν τοῖς σώμασι τῶν ζῴων· ἤγαγον δὲ τὸν λόγον ἐπὶ τὰ γνώριμα τῶν ἀρρωστημάτων, ἐν οἷς ἀληθὴς ἡ ὑπόσχεσις* (v. l. *ὑπόθεσις*) *ἐφάνη· εἰ γὰρ περὶ πάντων τῶν ἀρρωστημάτων λέγοιμι, μακρότερος μὲν ὁ λόγος ἂν γένοιτο, ἀτρεκέστερος δὲ οὐδαμῶς οὐδὲ πιστότερος.*

V² 241, 12. Fr. 6, *ἰσοταγῆ.*

Diels has now adopted my emendation *ἰσοταγῆ* for MS. *ἰσοταχῆ*. When I proposed it, I ventured the suggestion relying on the analogy of *ὁμοταγής*, not knowing that *ἰσοταγής* itself was attested. I now observe, however, that Sophocles, *Greek Lexicon*, s. v. cites it from Nicom. 51.

c. 46. Anaxagoras.

V² 319, 19. Fr. 13, *καὶ ἐπεὶ ἤρξατο ὁ νοῦς κινεῖν, ἀπὸ τοῦ κινουμένου παντὸς ἀπεκρίνετο, καὶ ὅσον ἐκίνησεν ὁ νοῦς, πᾶν τοῦτο διεκρίθη· κινουμένων δὲ καὶ διακρινομένων ἡ περιχώρησις πολλῷ μᾶλλον ἐποίει διακρίνεσθαι.*

It seems to me clear that *ὁ νοῦς* is the subject of *ἀπεκρίνετο* in the second clause. "After the *νοῦς* gave the initial impulse to the motion of the world, it began to withdraw from all that was set in motion; and all that to which the movement initiated by the *νοῦς* extended, was segregated. As this motion and segregation continued, the revolution greatly increased the segregation." The *νοῦς* gives the first impulse only, then withdraws to its condition of isolation; the revolution, once started, of itself accelerates and its effects in the segregation of like to like in the *πάντα ὁμοῦ* increase. Cp. *ἡ περιχώρησις αὐτή*, fr. 12, V² 319, 4 sq.

c. 51. Diogenes of Apollonia.

V² 334, 2. Fr. 1, *λόγου παντὸς ἀρχόμενον δοκεῖ μοι χρεὼν εἶναι τὴν ἀρχὴν ἀναμφισβήτητον παρέχεσθαι.*

With this statement compare Hippocr. *Περὶ σαρκῶν*, 1 (S. 584 L.), *Ἐγὼ τὰ μέχρι τοῦ λόγου τούτου κοινῇσι γνώμῃσι χρέομαι ἑτέρων τε τῶν ἔμπροσθεν, ἀτὰρ καὶ ἐμεωυτοῦ· ἀναγκαίως γὰρ ἔχει κοινὴν ἀρχὴν ὑποθέσθαι* *τῇσι γνώμῃσι βουλόμενον ξυνθεῖναι τὸν λόγον τόνδε περὶ τῆς τέχνης τῆς ἰητρικῆς.* Περὶ τέχνης, 4 (6. 6 L.), *ἐστὶ μὲν οὖν μοι ἀρχὴ τοῦ λόγου, ἣ καὶ ὁμολογηθήσεται παρὰ πᾶσιν.* Περὶ τόπων τῶν κατὰ ἄνθρωπον, 2 (6. 278 L.), *φύσις τοῦ σώματος, ἀρχὴ τοῦ ἐν ἰητρικῇ λόγου.* Ion of Chios, fr. 1 (V^2 222, 1 sq.), *ἀρχὴ δέ μοι τοῦ λόγου· πάντα τρία καὶ οὐδὲν πλέον ἢ ἔλασσον τούτων τῶν τριῶν· ἑνὸς ἑκάστου ἀρετὴ τρίας· σύνεσις καὶ κράτος καὶ τύχη.*

c. 54. Leucippus.

V^2 343, 1. *τὸ μὲν πᾶν ἄπειρόν φησιν, ὡς προείρηται· τούτου δὲ τὸ μὲν πλῆρες εἶναι, τὸ δὲ κενόν, <ἃ> καὶ στοιχεῖά φησι, κόσμους τε ἐκ τούτων ἀπείρους εἶναι καὶ διαλύεσθαι εἰς ταῦτα.*

For some time I have felt that there was some confusion and corruption in the text, and that the last sentence must refer to the rise of the worlds out of the *ἄπειρον* and their return into it at dissolution. The well-known difficulties of the text of Diogenes alone deterred me from proposing a change. Now Diels, apparently from the MSS., restores *ἐκ τούτου* for *ἐκ τούτων*. That is obviously the correct reading, whatever its source; but with it should of course go the complementary reading *εἰς τοῦτο* for *εἰς ταῦτα*. The preceding sentence, however, has likewise suffered. The *ἄπειρον* is clearly conceived as the Aristotelian *ἀρχὴ καὶ στοιχεῖον* by the interpolator or epitomator who supplied the clause <*ἃ*> *καὶ στοιχεῖά φησι*; for to his mind the words *τούτου τὸ μὲν πλῆρες, τὸ δὲ κενόν* do not suggest spatial regions of the extended *ἄπειρον*, but ontological *γένη* of the metaphysical *ἀρχή*. His addition was absurdly misplaced, as were many in the text of Diogenes; but once there, it corrupted the following sentence. See above, p. 691, on V^2 17, 37.

V^2 344, 14. Arist. De Gen. et Corr. 1. 8. 324^b 35, *ὁδῷ δὲ μάλιστα καὶ περὶ πάντων ἑνὶ λόγῳ διωρίκασι Λεύκιππος καὶ Δημόκριτος.*

The meaning of the phrase *ἑνὶ λόγῳ* has here been strangely misconceived. Prantl renders it "in einer Begründung"; Zeller, 1^b 847, n. 1, "aus den gleichen Principien"; Döring, *Gesch. der gr. Philos.*, I. 238, "die von einem Princip ausgehende Lösung"; Burnet, *Early Greek Philosophy*^{2}, 385, "on the same theory." I have failed to find this passage noted in Kranz's *Wortindex*, but in a similar one (V^2 83, 8, *ἑνὶ δὲ λόγῳ πάντα κτλ.*), omitting to quote *πάντα*, he gives the meaning of *λόγος* as "Vernunft" (V^2 II. 2, 357, 30)! Similarly

Burnet, in his note on Plato, Phaedo 65 D, gives a false emphasis and in effect a false interpretation, because he overlooks, what is obvious, that in the phrase *καὶ τῶν ἄλλων ἑνὶ λόγῳ ἁπάντων*, the phrase *ἑνὶ λόγῳ* is to be taken as emphasizing *ἁπάντων*; and Capps, on Menander, Epitrep. 197 sq.

> *καταμενῶ,*
> *αὔριον ὅτῳ βούλεσθ' ἐπιτρέπειν ἑνὶ λόγῳ*
> *ἕτοιμος,*

wrongly takes *ἑνὶ λόγῳ* with *ἕτοιμος* instead of *ὅτῳ βούλεσθ'*. Curiosity, awakened by the false points made by scholars in connection with the Aristotelian passage we are considering, led me to make a collection of cases of *ἑνὶ λόγῳ*, which grew to considerable proportions. I will not print a list here, since such collections possess no value in my sight except as an examination of the context serves to determine the sense of the locution in question. Suffice it to say that in almost every instance the immediate context contained a comprehensive or universal expression, such as *πᾶν, οὐδέν, μυρία*, etc. But *ἑνὶ λόγῳ* does not stand alone, for there is a considerable number of phrases similarly used; of these I give a few which should serve to illustrate the construction. Aeschyl. P. V. 46, *ὡς ἁπλῷ λόγῳ . . . οὐδέν*; ibid. 505, *βραχεῖ δὲ μύθῳ πάντα συλλήβδην μαθέ*; ibid. 975, *ἁπλῷ λόγῳ πάντας ἐχθαίρω θεούς*; Herod. 2. 24, *ὡς μέν νυν ἐν ἐλαχίστῳ δηλῶσαι, πᾶν εἴρηται*; ibid. 225, *ὡς δὲ ἐν πλέονι λόγῳ δηλῶσαι, ὧδε ἔχει*; ibid. 2. 37, *μυρίας ὡς εἰπεῖν λόγῳ*; ibid. 3. 6, *ἓν κεράμιον οἰνηρὸν ἀριθμῷ κεινὸν οὐκ ἔστι ὡς λόγῳ εἰπεῖν ἰδέσθαι*; ibid. 3. 82, *ἑνὶ δὲ ἔπεϊ πάντα συλλαβόντα εἰπεῖν*; Plato Apol. 22 B, *ὡς ἔπος εἰπεῖν ὀλίγου αὐτῶν ἅπαντας*; Xenoph. Mem. 4. 3. 7, *ὡς γὰρ συνελόντι εἰπεῖν, οὐδέν κτλ.*; Amphis, fr. 30, 7 Kock, *ἅπαντες ἀνδροφόνοι γάρ εἰσιν ἑνὶ λόγῳ*. Adverbs like *ἔμβαχυ* are similarly employed. After reciting this list of passages I think we may be sure that in the passage we are considering Aristotle merely meant to say that the procedure of Leucippus and Democritus was not only exceedingly methodical (*ὁδῷ μάλιστα*), but also comprehensive (*περὶ πάντων ἑνὶ λόγῳ*). Possibly those who have been reading something more into Aristotle's words might receive some comfort from Hippocr. *Περὶ ἑπταμήνου*, 3 (7. 438 L.), *χρῶνται δὲ πᾶσαι ἑνὶ λόγῳ περὶ τουτέου· φασὶ γάρ κτλ.* But the context shows that *ἑνὶ λόγῳ* means "one formula of expression." Even if one should insist on taking Aristotle's words as a parallel to this, it would greatly affect the traditional interpretations of the passage.

V² 344, 21. Arist. De Gen. et Corr. 1. 8. 325ᵃ 25, ὁμολογήσας δὲ ταῦτα μὲν τοῖς φαινομένοις, τοῖς δὲ τὸ ἓν κατασκευάζουσιν ὡς οὐκ ἂν κίνησιν οὖσαν ἄνευ κενοῦ, τό τε κενὸν μὴ ὂν καὶ τοῦ ὄντος οὐθὲν μὴ ὄν φησιν εἶναι. τὸ γὰρ κυρίως ὂν παμπλῆρες ὄν.

I cannot understand how scholars have been so long content to retain this text, which yields no sense and so clearly suggests the true reading. With it we must compare other passages in which the same matter is under consideration. Arist. Met. 1. 4. 985ᵇ 4 (V² 343, 44), Λεύκιππος δὲ καὶ ὁ ἑταῖρος αὐτοῦ Δημόκριτος στοιχεῖα μὲν τὸ πλῆρες καὶ τὸ κενὸν εἶναί φασι, λέγοντες τὸ μὲν ὂν τὸ δὲ μὴ ὄν, τούτων δὲ τὸ μὲν πλῆρες καὶ στερεὸν τὸ ὄν, τὸ δὲ κενὸν καὶ μανὸν τὸ μὴ ὂν (διὰ καὶ οὐθὲν μᾶλλον τὸ ὂν τοῦ μὴ ὄντος εἶναί φασιν, ὅτε οὐδὲ τὸ κενὸν <ἔλαττον Diels> τοῦ σώματος), αἴτια δὲ τῶν ὄντων ταῦτα ὡς ὕλην. Whether Diels was right in proposing to insert ἔλαττον we shall have presently to inquire. Simpl. Phys. 28, 11 (V² 345, 5), ἔτι δὲ οὐδὲν μᾶλλον τὸ ὂν ἢ τὸ μὴ ὂν ὑπάρχειν, καὶ αἴτια ὁμοίως εἶναι τοῖς γινομένοις ἄμφω. τὴν μὲν γὰρ τῶν ἀτόμων οὐσίαν ναστὴν καὶ πλήρη ὑποθέμενος ὂν ἔλεγεν εἶναι καὶ ἐν τῷ κενῷ φέρεσθαι, ὅπερ μὴ ὂν ἐκάλει καὶ οὐκ ἔλαττον τοῦ ὄντος εἶναί φησι. We are familiar with the pun which Democritus employed to enforce this point of doctrine, fr. 156 (V² 413, 11), μὴ μᾶλλον τὸ δὲν ἢ τὸ μηδὲν εἶναι. It seems to me obvious that in the passage under consideration μὴ ὄν is a corruption by itacism for μεῖόν. Indeed, I am inclined to think that the pun τό τε κενὸν μὴ ὂν καὶ τοῦ ὄντος οὐθὲν μεῖον derives from the same fertile brain as μὴ μᾶλλον τὸ δὲν ἢ τὸ μηδέν, and that we have thus found another fragment of Democritus partially converted into the Attic dialect. If this be conceded, it seems more probable that we should supply μεῖον than ἔλαττον (with Diels) in Met. 985ᵇ 9. Aristotle used the word, Eth. Nic. 5. 1. 1129ᵇ 8, δοκεῖ καὶ τὸ μεῖον κακὸν ἀγαθόν πως εἶναι, where the true reading, corrupted in the MSS., had to be recovered from the commentaries and versions. Cp. Aeschyl. P. V. 508, ὡς ἐγὼ | εὔελπίς εἰμι τῶνδέ σ' ἐκ δεσμῶν ἔτι | λυθέντα μηδὲν μεῖον ἰσχύσειν Διός; Xenoph. Ages. 6. 3, τρόπαια μὴν Ἀγεσιλάου οὐχ ὅσα ἐστήσατο ἀλλ' ὅσα ἐστρατεύσατο δίκαιον νομίζειν. μεῖον μὲν γὰρ οὐδὲν ἐκράτει κτλ.; Herondas 3, 59, ἕξει γὰρ οὐδὲν μεῖον; ibid. 15, 2, ὃς δ' ἔχει μεῖον | τούτου τι.

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THE STRUCTURE OF THE GORGONIAN CORAL PSEUDOPLEXAURA CRASSA WRIGHT AND STUDER.

By Wayland M. Chester.

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THE STRUCTURE OF THE GORGONIAN CORAL PSEUDOPLEXAURA CRASSA WRIGHT AND STUDER.[1]

By Wayland M. Chester.

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CONTENTS.

Introduction.

Pseudoplexaura crassa is found on the reefs of Florida, of the West Indies, and of the Bermuda Islands. It is very abundant in the shallow water of the inner reefs of Bermuda, and is there one of the two or three very common sea whips; but it is found in the deeper waters of the outer reefs as well. The range in depth, to include the greater number of colonies, is from a position near the surface at low water to seven or eight meters.

Ellis and Solander (1786) described this colony under the name of Gorgonia crassa. Kölliker (1872) placed under the name of Plexaura branched, sea-rod forms in which the polyps completely retract into a comparatively thick coenenchyma, in which club-shaped and spiny spindle-shaped spicules appear. The different species were divided into two groups: Plexaura durae and Plexaura molles. Hargitt and Rogers (:01, p. 285) follow Verrill ('65, p. 34) in describing this form as Plexaura crassa. Wright and Studer ('89, p. 141–143), from observations of Bermuda specimens, created for this species a new

[1] Contributions from the Bermuda Biological Station for Research. No. 27.

genus, Pseudoplexaura. The new genus is characterized by them as follows: "axis horny, with a central calcareous portion, the outer layer of coenenchyme is soft and when dry friable; the inner layer contains a number of light purple or violet coloured irregularly stellate spicules or spindles with few rays." It is to be distinguished from Plexaura, in addition, by the following features, among others: colony feebly branched, older portions of horny axis solid, younger portions with calcareous particles in the center; polyps placed close together in an irregular spiral, completely retractile tentacles without spicules or having a circlet of them at their base; spicules mostly spiny spindles, with numerous pink stellate forms and a few club-shaped with attenuated foliaceous expansions.

The important characters of the colony are: the relative smallness of the spicules; spicules in the outer cortex, and irregularly stellate forms in the inner cortex; the massing of the latter to such an extent as to make the inner cortex firmer when dried, while the outer is friable; the absence of spicules in the tentacles and polyps; the sluggish but complete retraction of the polyps within the cortex; and the smooth cortex surface without projecting calyces in the contracted or dried colony. The polyps are numerous. When they are completely expanded the tentacles of adjacent polyps overlap, and the coenenchyme is hidden. Each tentacle has ten to twelve pairs of pinnae.

Of the three groups of alcyonarian corals,— Alcyonacea, Pennatulacea and Gorgonacea,— only representatives of the first and second have had their minute structure studied recently; the Gorgonacea, to which Pseudoplexaura belongs, have received little attention except from von Koch ('87) in his very important but early comparative study. Studies on the Alcyonacea have been relatively numerous. Von Koch ('82[a]) described briefly the structure of Clavularia and other alcyonacean forms. Bourne ('95) described Heliopora coerulea and later made a very complete study of the origin and structure of its skeleton ('99). Ashworth ('99) studied the minute structure of Xenia Hicksonii Ash. and Heteroxenia elizabethae Köll. He found gland cells in the stomodaeum and correlated their presence there with the absence of the ventral and lateral mesenterial filaments. Hickson ('95) has given a detailed account of the cell structure of Alcyonium digitatum, and Pratt (:05) has described the digesting and mesogloea cells in several members of the Alcyonidae. She found a relatively large number of granular gland cells in the stomodaeum of feeding colonies and very few or none in starved ones. She held the mesogloeal network of cells to be neuro-phagocytic in function. By feeding with colored material, she proved the ingestion and the carriage of such material by the amoeboid movements of the mesogloeal cells. Kassianow (:08) reviewed the literature for the muscle and nerve systems in Alcyonaria digitatum, studied these systems and described in detail the cells of the ectoderm, endoderm and mesogloea with reference to them. He denied a nervous function for the neuro-mesogloeal cells of Pratt.

Among the Pennatulacea, studies have been made by Korotneff ('87) and by Bujor (:01) on Veretillum. They described the cells of the ectoderm and endoderm carefully.

The only complete study of the cell structure of representatives of the group Gorgonacea is by von Koch ('87), who made a comparative study of the structure and minute anatomy of the forms found in the Bay of Naples, giving most attention to Unicella (Gorgonia) cavolinii.

Wilson ('84) studied the mesenterial filaments of a number of species representing the three groups. He described the difference in structure of the ventral and dorsal mesenterial filaments and the origin of each from different germ layers.

Bourne ('99), in a paper giving the result of his study of the origin of the skeleton in Anthozoa, describes the origin and minute structure of the alcyonarian spicule and the structure of the massive skeleton of the alcyonarian Heliopora. He further studied the structure and origin of "holdfasts," or desmocytes, in Heliopora, as well as in madreporarian forms.

Woodland (:05) reviewed the literature on the origin of the alcyonarian spicule and made a very complete study of it for Alcyonium. The names of von Koch ('78, '82[b]), Studer ('87, :06), and Alfred Schneider (:05) are important in the history of researches on the origin of the horny skeleton. Kinoshita (:10) has seen the origin of axis epithelium in young forms of Anthoplexaura and has confirmed von Koch's account of its ectodermal origin in the young form.

The study of this gorgonian coral (Pseudoplexaura crassa) was pursued during the summers of 1909 and 1910 at the Bermuda Biological Station for Research, and during the winter of 1909–1910 at the Zoölogical Laboratory of Harvard University. I wish to express my great indebtedness to Dr. E. L. Mark, the Director of these Laboratories for guidance and generous assistance.

Methods.

Small colonies were kept alive in large aquaria of running water for a short time. Small tips, 5 to 10 cm. long, were easily kept in smaller dishes of running water, if care was taken to keep them upright. Both Bouin's fluid and a five per cent formalin were successfully used as fixing agents. The fixing fluids were taken in some instances to the reefs and large tips or other pieces cut off from the colony and quickly transferred to the fluid; upon returning to the laboratory these were cut into small pieces. Both neutral formalin and Vom Rath's picric-osmic-acetic-platinic chloride fluid were used for nerve fixation. Corrosive sublimate and Wilson's fluid were both used, but did not give any better results than the formalin or Bouin's fluid. Decalcification was effected by 1 % acetic acid in absolute alcohol. Maceration by Hertwig's method gave good results. Delafield's haematoxylin and iron-alum haematoxylin could be used with decalcified material. Many sections of the soft tips were made without decalcification; but in these the haematoxylin overstained the spicules and the axial skeleton, obscuring the neighboring cells. By far the best general stain for these was Mallory's phospho-tungstic haematoxylin, which stained well after formalin and better after Bouin's fluid. This has the advantage that, while it does not obscure the spicules, it differentiates other structures.

General Structure.

A colony of Pseudoplexaura crassa presents a loosely divided group of long branches on a short stem; or the short stem may have short branches which divide and divide again, terminating in long whips. A drawing of a very young colony (Fig. A) illustrates the character of the branching. The short basal shaft spreads out on the coral rock or on old coral masses, secreting a skeleton that becomes very firmly attached to its substratum. The whips, which rise in many planes, are cylindrical, long and flexible, and taper somewhat gradually to the tips, which, however, end bluntly, or even with a slight enlargement. The brownish polyps stand at right angles to the branches (Plate 1, Fig. 1), closely crowded against one another, except at the tips of the branches, where the coenenchyma can often be seen even when all the polyps are expanded. Because the polyps when expanded stand at right angles to the branches, and are crowded, the colony in this condition looks like a miniature leafless shrub with unusually thick branches; because the tip is often bare of polyps or these are there contracted, the colony when seen at a distance below the surface may resemble very superficially a huge compound

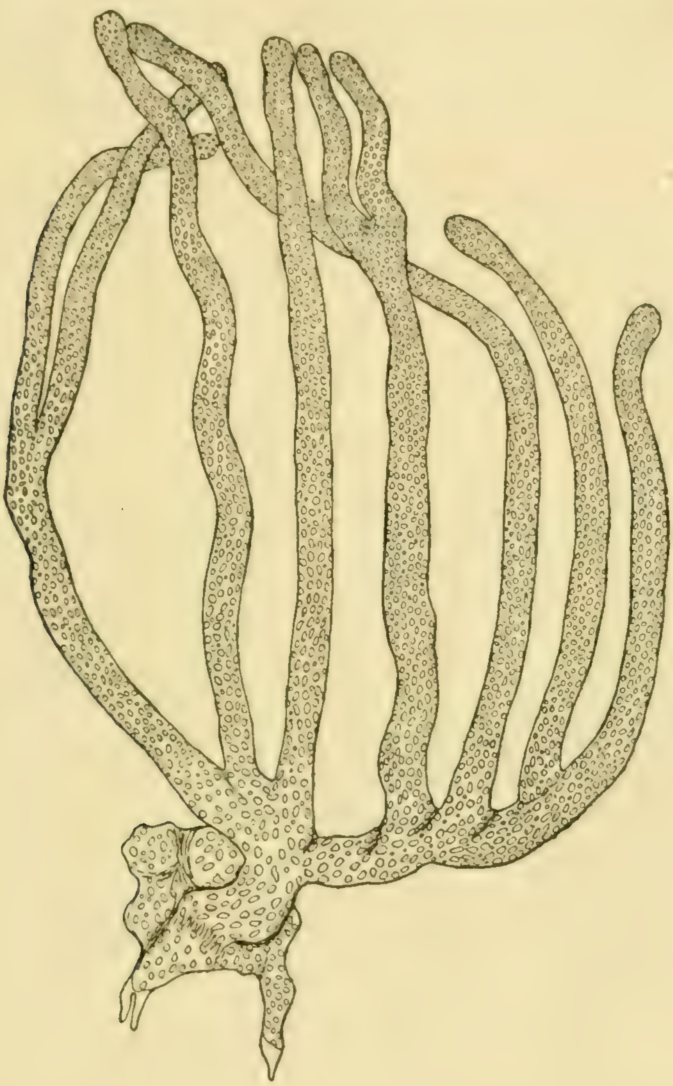

Fig. *A*. Young colony of Pseudoplexaura crassa, showing method of branching and the calycle openings in the coenenchyma.

tube sponge. But Figure *A*, although it shows the character of the branching, does not show the great number of branches, or whips, that occur in larger forms. As found on the Bermuda reefs, the species varies greatly in size. A small colony stood 90 cm. high, its terminal 60 branches spreading over a circle 60 cm. in diameter. Another colony was 110 cm. high with 250 terminal branches. A third measured 200 cm. in height and had 300 terminal branches. These three, taken from one place, are representative of the forms inside the outer reefs. On the outer reefs, forms 225 cm. high have been found. In the first colony (90 cm. high) the diameters of the stems were measured. Half way between the ground and the tips, with the polyps fully expanded, the branches were 4 cm. in diameter; at the base, 5 cm., and near the tip, a little more than 3 cm. After the polyps were fully contracted, the measurements were 1.5 cm. for the base, and .5 cm. for the tip.

A transverse section of a branch, or whip (Plate 1, Fig. 1), shows the structure. In common with that of the branches of Gorgonacea generally, there are recognizable three zones: a central axis of skeletal material (*ax.*), a fleshy enveloping layer,— the coenenchyma,— and a zone of polyps. The polyps can completely retract into spaces in the coenenchyma, whereupon two zones only are evident. The horny axis is very hard at the base but quite soft at the tip of the branches and, except at the basal end, is very flexible. To the naked eye, it is composed of two parts, a central, soft marrow, light in color (white in the figure), and an outer, harder, brown or black tubular shaft or cortex. The marrow has nearly the same thickness in all parts of the colony.[2] While its diameter is sometimes slightly smaller at the tip, it is not always so. Its variations are not wholly dependent on age, for it is slightly larger or smaller in parts of the stem and these parts occur irregularly. It is composed of a number of chambers filled with loosely branching threads (compare Plate 4, Fig. 58), and having walls of horny material, the chambers being generally arranged one above the other (Plate 4, Fig. 57, *med. ax.*). The loosely branching threads in the chambers are not shown in this figure, but are seen in the small chamber of the axis-cortex shown in Figure 58. The walls of the medullary chambers are very thin. Those of the axis-cortex chambers are very thin at the tip of a branch, while at the base they are very thick and hard. This is due to the fact that while the marrow chambers are laid down axially (i. e., at the end of the axis) in the branch, the cortex grows radially. The latter is composed of smaller chambers, which in longitudinal sections appear crescent shaped (Fig. 57, *ctx. ax.*). Not only is the cavity smaller, but the walls are thicker than those of the marrow. The first crescents laid down are adjacent to the marrow and very short, but as the axis-cortex increases in

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2 This cortex of the axis will be called axis-cortex to distinguish it from the more superficial coenenchymal cortex.

thickness, the outer ones are very long and have much thicker walls and thinner cavities. To the naked eye this gives the appearance of a solid cortex.

Where a branch is formed there is a break between the marrow of the branch and that of the stem (Plate **4**, Fig. 61), and the axis-cortex forms a thick knee-like union at the stem for strength. At the base of the colony the axis-cortex is spread upon the substratum and the marrow is lacking. Many of the Bermuda colonies that were located on the shallow reefs, where the tips were exposed at the lowest tides, had lost the fleshy envelope of the tips of the branches for the distance of a few centimeters, the horny axis being here covered by diatoms. The fleshy coenenchyma was growing loosely around the old axis and taking the shape of the original branch. In such regenerated tips no new marrow was formed. In a few instances the old axis had been completely covered, but the coenosarc had not grown enough beyond the axis to give evidence of the presence or absence of a marrow distal to the old dead axis.

The coenenchyma is composed of two regions, an inner, exhibiting longitudinal canals, and an outer, containing the calycles or polyp-chambers. The longitudinal-canal region is characterized by a number of large canals running parallel to the axis (Plate **1**, Fig. 1, *can. lg.*), and by the presence of purple, irregularly stellate spicules (Fig. *B*, 5–7) loosely massed in groups (*spc.'*). The spicules are represented diagrammatically in the left half of Figure 1, to show their positions. The longitudinal canals are less numerous at the tips of the branches — where there may be eight to ten — than they are at the bases, where twenty or more may be found. The diminution in number from the base to the tips is due to the running together of two adjacent canals or to the abrupt ending of one or more. Some of them run continuously from the basal half of the stem into the lower (abaxial) face of a branch. On the axial face of the branch they may be continuous with those of the stem, or the canal may begin abruptly in the branch without such connection. At the tips and on the branches the longitudinal canals connect with one another and with the polyp cavities by smaller canals (*can.*). At the base of the colony they run out radially and end blindly, or are connected with one another or with polyps by the smaller canals. At the tips the purple spicules are loosely arranged in two concentric cylinders separated from each other by the zone of canals. Where the branches are older, however, the inner cylinder breaks up into small groups, which lie between the axis and two adjacent canals (Fig. 1, *spc.'*); here the spicules interlock in a close mass. The diagrammatic nature of Figure 1 does not permit one to show the closeness of the interlocking of these spicules.

The outer region of the coenenchyma is a thick zone with polyp chambers (*cam. pyp.*), into which the polyps can completely retract. These chambers communicate with each other by a greatly branched system of small canals (*can.*). These canals are represented in only one half of Figure 1 (Plate 1). The actual canals are much more numerous and are more complexly branched than the diagram shows. Between the chambers white spindle-shaped and spiny spicules (Fig. *B*, 1–4), longer than the purple ones of the inner region, are found (*spc.*). Some of the deeper of these may be purple. Indeed there is considerable variation in both color and form of the spicules of different colonies and even of the same colony. These spicules have been described and figured by Wright and Studer ('89), by Hargitt and Rogers (:01) and by Verrill (:07). Figure *B* gives the relative proportions in size, as well as the differences in shape for the spicules of both the inner and outer regions of the coenenchyma. Figure *B*, 5–7, presents spicules of different shapes found near the long canals, where they are often locked together. In Figure *B*, 1, 2, are seen long white forms from the outer part, and in Figure *B*, 3, 4, spicules from the deeper part of that region.

The polyp-zone consists of the exposed part of the polyps or anthocodia (Fig. 1, α–γ). These when expanded are cylindrical. The mouth is oval and the eight tentacles of the crown stand at right angles to the column. The tentacles are conical, relatively long when fully expanded, and carry ten to twelve pairs of conical pinnae, arranged in two longitudinal lines, one on either side. No spicules are found in the polyps.

The anthocodia are brown and, when expanded, give the colony its prevailing color, which is caused by the presence of the Zooxanthellae (Plate 2, Figs. 20, 21, *zox.*) that crowd the endoderm cells and give a lighter or darker brown color in accordance with the degree of the animal's contraction. A little magnification shows that each disk is white and also that along the eight longitudinal lines corresponding to the union of the mesenteries with the body wall the color is white. The white appearance in both cases is due to the absence of these algae. The color of the coenenchyma between the anthocodia is distinctly white because of the absence of Zooxanthellae near the surface and the presence of white spicules.

The wall of the column of the anthocodium (Plate 1, Fig. 1, β) merges into that of the somewhat larger chamber in the coenenchyma and its endodermal epithelium is continuous with that of the chamber. The stomodaeum (Fig. 1, *η*, *stmd.*) is long, reaching when the polyp is expanded, nearly if not quite to the level of the surface of the coenenchyma; it is broadest in the dorso-ventral plane and has at its ventral angle a prominent siphonoglyph (Fig. 2, *sipy.*), which is not

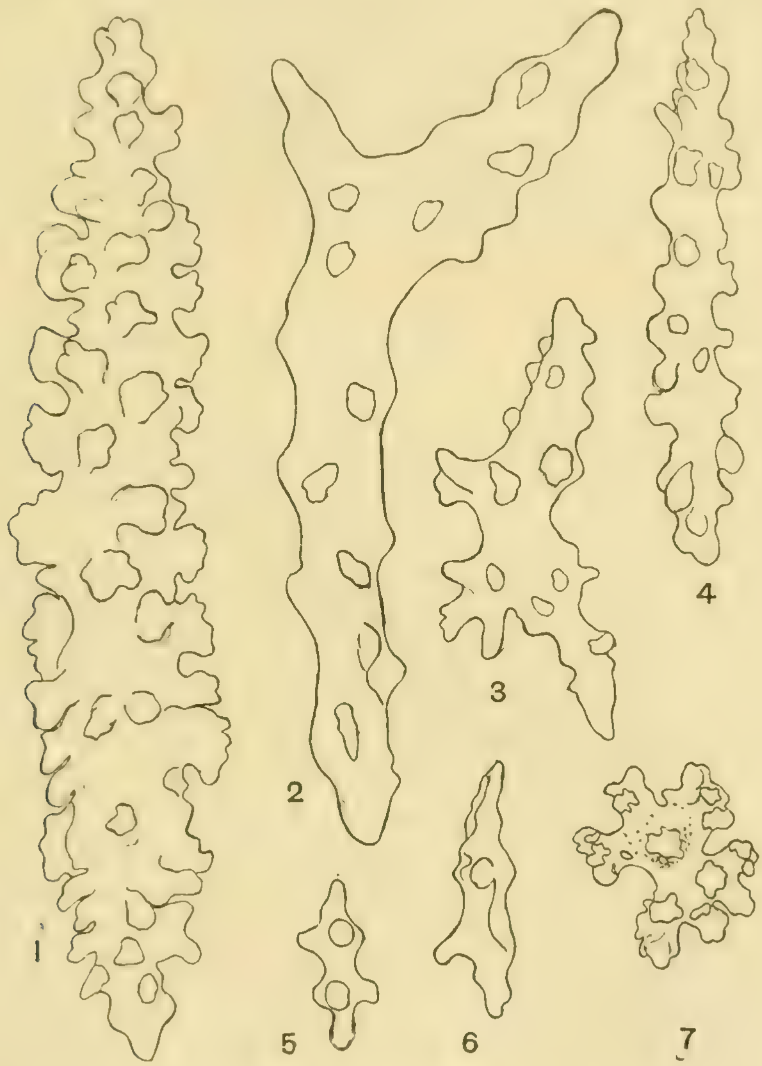

Fig. *B*. Calcareous spicules: × 90. 1–4, white spicules of outer cortex region; 5–7, purple spicules from the region of the longitudinal canals.

visible on the disk, but reaches nearly to the deep end of the stomodaeum. The structure is that usually found in the siphonoglyph of Alcyonaria. A transverse section through the stomodaeum is shown in Figure 2. The siphonoglyph is always ventral, and the longitudinal muscles of the eight mesenteries, as is usual in Alcyonaria, are located on the ventral sides of the mesenteries. Figure 3 shows a section made below the level of the first cut and near the level of the lower end of the stomodaeum — but still through the anthocodium — very close to the coenenchyma. The mesenterial filaments are cut in this section. The filaments belonging to the six ventral and lateral mesenteries are very short (never more than two millimeters long, and usually less than one), and appear as very slight thickenings of the edge of each mesentery. The filaments of the two dorsal mesenteries are seen to be deeply grooved; they are very long and pass with the mesentery from the stomodaeum to the very base of the chamber, being much convoluted at their lower ends. The section shown in Figure 4 is cut through the lowest part of the polyp cavity. The ventral and lateral mesenteries here have no filaments, but the dorsal filaments, cut across several times, show their grooved condition. In the lower region of the polyp chamber are also many ova, or else (the colonies being dioecious) masses of sperm mother cells covered by the endoderm. The ova are attached singly to the sides of the six ventral and lateral mesenteries. The eggs in July were large, but there was no evidence of fertilized eggs or of matured sperm.

Phases in the retraction of the polyp have been described by Wright and Studer ('89) and appear to be similar to those of Alcyonium (Hickson, '95). When the colonies were transferred from the conditions of the sea to those of the laboratory, it was seldom that they remained as fully expanded as at first. The polyps were appreciably shorter, but they had the shape and character described (Plate 1, Fig. 1, α). Sometimes the mouth was open and observation showed a slight current of water passing into the stomodaeum; at other times, or in other parts of the colony, the mouth was tightly closed. Sometimes the tentacles and pinnae were contracted to form eight short cones at the top of a column comparatively well expanded, the mouth being either open or closed. A circular constriction of a narrow region then appeared just below the disk, while the remaining part of the body bulged like a flask (Fig. 1, γ). A slow withdrawal of the polyp then occurred, evidently by means of the longitudinal muscles of the mesenteries, together with a slow inturning of the tentacles, until the disk and tentacles were at the level of the calycle. These now drew within and the oval or ovate calycle opening became visible (Fig. 1, δ). Sometimes in this method of contraction, the tentacles were first rolled toward the mouth, giving the disk and tentacles the appearance of a circle with eight indented radii (Fig. 1, β). The column was then contracted into the polyp cavity of the coenenchyma, as already outlined. But this method of contraction is not the invariable rule.

Sometimes, although the tentacles are contracted, the mouth remains open and the column is brought down nearly to the level of the coenenchyma, where it remains for a long time, the current of water to and from the stomodaeum evidently not being checked (Fig. 1, ϵ). Probably the coenenchyma is also capable of slight contraction and expansion, due principally to the action of the muscles of the mesenteries. When the tip of a branch is cut off, the coenosarc, with the polyps near the line of the cut, becomes appreciably of smaller diameter, thus partially covering the cut surface; and the same result is seen in the regenerating branches on the reefs, where the polyps have been turned from their radial position and are held in such a way as to lessen the area of the cut surface.

Ectoderm.

The ectoderm of the polyp wall in the expanded condition is more than one cell thick, showing an epithelial and a subepithelial part (Plate **2**, Figs. 8, 18). In the epithelial part, the cover cells (*cl. teg.*) are conical, with a round or polygonal external surface as a base and with the opposite end, as apex, extending into the mesogloea as a process. In the expanded state of the polyp many of the cells are peltate or mushroom shaped, since they have the appearance of an external plate supported by a rapidly narrowing stem. This stem soon takes a thread-like character and may be branched (Figs. 9, 11). The outer surface is often convex, like a mushroom, but it may be indented, so that the section of it is cusp like (Fig. 18, *cl. teg.*). In the expanded condition of the polyp, the plate-like surface covers a relatively large area and the process becomes shorter; in the contracted condition, the cells approach a columnar shape, though the process is still thread-like at the mesogloeal end (Fig. 20, *cl. teg.*). The protoplasm is finely granular and sometimes large granules also occur (Figs. 10, 11). Although sometimes without vacuoles, these cells usually contain many small ones. The nuclei are large and granular and are situated in the outer half of the cells or near the point of the cusp. Among the covers cells are found nettle-cells or thread-cells with their nematocysts (Fig. 18, *nm'cys.*) and sense cells (*cl. sns.*).

The nematocysts are of two kinds. The larger and more numerous ones (Plate **2**, Fig. 18 and Plate **4**, Figs. 43–46) occur in batteries of six to ten, but they may be found in much larger numbers. The cyst is sharply outlined and is an elongated oval with a length three times its greatest width. Its length varies from 10 to 14 micra when unexploded. Around the cyst is a thin layer of granular protoplasm containing the flattened nucleus of the cell at the region of the greatest width of the cyst. The nucleus stains deeply with basic stains, the chromatin being abundant and granular. There is no trace of a cnidocil or of a cytoplasmic process projecting from the free surface of the cell. After staining in Mallory's phospho-tungstic acid haematoxylin or methylen blue a thick thread may be distinguished within the cyst, though not easily. It passes in a zigzag line from its attachment for a third of the length of the cyst and fills the remaining two thirds of the cyst in a tight coil (Fig. 44). When exploded the thread is quite thick and short and without barbs. Nematocysts of the second type (Fig. 47) are not found so abundantly. They are ovate and much smaller than those of the first type. The length is 5 to 8 micra. The nucleus is at the broad end of the enclosing cell. The thread with four or five spirals is sharply outlined even without stains. Exploded, it is long and slender, without barbs.

Sense cells (Plate 2, Figs. 8, 13, *cl. sns.*) occur among the cover cells, but are readily found only in the nematocyst batteries, where their bristle tips can be seen in favorable sections and where their nuclei reveal them by the size, which is smaller than that of the cover-cell nucleus. The sense cell is narrow and spindle-shaped with a long process extending into the mesogloea. Its protoplasm is slightly more evenly granular than that of the cover cells, though not greatly different from it; but its nucleus is small and round and can be readily distinguished from the larger nucleus of the cover cells. The bristle point is exposed between the cover cells, or between these and the nematocyst cells.

The subepithelial part of the ectoderm is made up of interstitial cells, globular cells with granules, nematocyst cells, and ganglion cells. It is not a sharply defined layer, since the cover cells and the sense-cell processes pass through it. The interstitial cells (Plate 2, Figs. 18, 20, *cl. in.*) are globular or have a central body with branching processes. In fact, the cells, or some of them, may change their shape in an amoeboid manner. The cytoplasm is similar to that of the cover cells; the nucleus may be smaller, though it varies greatly in size. The granular cells are filled with comparatively large granules and stain sharply with eosin. They have the appearance of the granular cells found in the mesogloea of the coenenchyma. Besides cells containing fully formed nematocysts, there are some which show stages in the formation of the nematocysts (Fig. 18, *cl. nm'cys.*).

Other cells, lying close to the mesogloea with processes parallel to it, are interpreted as ganglion cells (Fig. 20, *cl. gn.*). The difficulty with this interpretation is due to the presence in the adjoining mesogloea of the mesogloeal cells, which sometimes resemble the supposed ganglion cells in shape. In preserved material, the fibers of these ganglion cells sometimes possess many minute varicosities. The cytoplasm is more evenly and closely granular than in surrounding cells.

The ectoderm of the tentacles (Plate 2, Fig. 5) and pinnules (Fig. 6) is relatively thicker than that of the polyp wall, but shows the same superficial and deep layers. It also differs from that of the body wall in the smaller number of the large kind of nematocysts and in the presence of muscle cells in the ectoderm of the oral face of tentacle and pinnule. The deep end of the muscle cell (Figs. 16, 17) is elongated into a process which is perpendicular to the axis of the body of the cell, and runs lengthwise of the tentacle or pinnule. Each process contains in its axis a single highly refractive contractile fiber, or myoneme. The nucleus is small and finely granular. Usually the cell body is flattened in the plane of the muscle fiber; but the flattening may be at right angles to that plane. It is only occasionally that these muscle cells reach to the surface of the ectoderm and thus present the typical condition of an epithelio-muscle cell. Since all the muscle fibers run lengthwise of the tentacle, a transverse section of the tentacle (Fig. 7) shows, adjacent to the mesogloea of the oral surface and the sides, a very definite row of dots — the cut ends of muscle fibers. In sections dyed in Mallory's stain these fibers, being deeply colored, appear as dark dots, or, if cut somewhat obliquely, as short lines. In the partially contracted condition of the tentacle the dots no longer occupy a plane surface, for the originally plane surface is so folded that the line of dots is very sinuous, and even forms in some regions a series of pinnate figures.

The ectoderm of the polyp wall passes gradually into that of the coenenchyma, where, between the polyps, the cover cells (Plate 3, Figs. 23, 24) are very long and conical, but never show the indented peltate shape seen in the ectoderm of the polyps. The ectoderm of the coenenchyma shows a thicker subepithelial region, which gradually merges with the mesogloea. As a rule, the interstitial cells are so numerous in the thick mesogloea and the mesogloeal elements are so near the long cover cells that it is hard to find a boundary between the two layers. The nematocysts of the larger kind are arranged in very large and numerous batteries; those of the smaller kind occur individually and are very few. Both kinds of nematocysts have the same structure as those of the polyp wall. Sense cells, like those in the body wall and tentacles, are found in the batteries and increase in number with the increase in the number of cysts in the batteries.

Some of the ectoderm cells of the coenosarc between the polyps contain a prominent, highly refractive, homogenous fiber (Plate 3, Figs. 24, 26, *fbr. sst.*), beginning near the nucleus and extending to the base of the cell, which is implanted in the mesogloea. The nature of these fibers is not perfectly certain, but it is probable that they are the same as the "Stützfasern" described by K. C. Schneider (:02, p. 622) for Anemonia; however, I have never seen evidence of their fibrilation, such as Schneider has shown to exist in the case of typical "Stützzellen." Figure 26 shows the distribution of these Stützfasern in the coenenchyma and their relation to the grooves which occur in these regions.

The ectoderm of the disk is, in its content and cell structure, like that of the oral face of the tentacle. When the disk is fully expanded, the stomodaeal epithelium reaches over on to the disk. The siphonoglyph and the dorso-lateral regions of the stomodaeum are both characterized by extremely long columnar ciliated cells. The siphonoglyph (Plate 3, Fig. 32) has neither gland cells nor nematocysts, both of which are abundant in the other regions of the stomodaeum (Figs. 30, 31). The columnar cells of the siphonoglyph are long and of small caliber; each has a single strong cilium, or flagellum, which is longer than the cell. The cytoplasm is finely and rather densely granular, and each oval or elongated nucleus has one or two prominent nucleoli and a small number of chromatin granules. The nuclei occupy different levels in adjacent cells and are so situated that collectively they form a definite layer. Four layers, then, are recognizable in the ectoderm of the siphonoglyph (Fig. 32): (1) subnuclear, composed of the bases of the columnar cells, between which may be found occasional nutrition cells (*cl. nut.*) and ganglion cells, (2) the nuclear layer, (3) the layer between the nuclei and the basal granules of the cilia, (4) the cilia layer, a wide border characterized by the long and prominent basal granules. No ectodermal muscle cells are found in this region.

In the dorso-lateral portion of the stomodaeum (Plate 3, Figs. 30, 31) four types of cells occur abundantly, supporting cells (*cl. sst.*), mucus cells (*cl. muc.*), granular gland cells (*cl. grn.*), and the small nematocyst cells (*cl. nm'cys.'*). The same four layers as in the siphonoglyph are found, but the nuclear zone is much wider, and the cell bases do not always terminate so sharply against the mesogloea. The cilia are short. The supporting cells (*cl. sst.* and Fig. 39, Plate 4) are columnar, finely granular, with very small vacuoles, and with a short cilium that has a prominent, inverted cone shaped basal granule. The nuclei vary from oval to globular, but are more often oval, stain lightly, and have one or two prominent nucleoli. The mucus cells (Figs. 30, 37, *cl. muc.*) are columnar or flask like, not always reaching the full depth of the layer, sometimes staining evenly with eosin, at other times showing a loose network of protoplasm. This network stains deeply with muci-carmine. The nucleus is in the basal end, is globular or nearly so, and stains deeply. The granular digesting cells (*cl. grn.*) are very numerous in some colonies, few in others, and particularly few in starved colonies. They also are columnar or flask shaped with a small round nucleus, that stains lightly, near the middle of the cell. Large granules, staining deeply with haematoxylin, sometimes partly, sometimes completely fill the cytoplasm. Nematocyst cells of the smaller type (Fig. 31, *cl. nm'cys.'*) are abundant between the columnar cells in the cilia layer, but the cysts are smaller than those at the surface of the body. They are here nearly globular with a deeply staining nucleus at the lower end. The border, or cilia, region of the supporting cells (Fig. 39) sometimes shows the presence of variously shaped, but very small, digestive or nutritive granules. Below the nuclear layer and between the supporting cells are found many globular cells (Fig. 30, *cl. gl.'*), containing large granules that do not stain with haematoxylin. In a few cases these cells have shown karyokinetic figures. I interpret them to be young stages of gland cells and think they are the same as those described by Kassianow (:08). Ganglion cells are found not far from the mesogloea, but they are very few.

The dorsal filaments and the axis epithelium are described later in this paper.

Mesogloea.

The mesogloea is very thin in the body wall of the polyp and in the tentacles, as well as in the stomodaeum and mesenteries. The boundary between it and the ectoderm or endoderm is sharply marked wherever muscle fibers are found; where there are no muscle fibers the division is not clear. In the pinnules (Plate 2, Fig. 6) evidence of mesogloea is seldom found, and even in the tentacle (Fig. 5) cells are not often seen imbedded in it. In the stomodaeum the layer is made out with difficulty, but it is thicker than in the tentacles. In the mesenteries (Plate 3, Fig. 33) it is very evident, though thin, being sharply outlined in regions where there are muscles; but only occasionally, and then in the thicker parts of the layer, are there any included cells. A thin layer of it is also found between the endoderm of the mesenteries and the egg or the sphere of sperm mother cells. In the body wall, particularly at the base of the polyp (Fig. 19), it is thicker and here a few included cells appear. These cells (*cl. ms'gl.*) are smaller than the ectodermal cells and have a correspondingly small nucleus; they have several elongated, more or less branched processes. Cells in the coenosarc (Fig. 28, *cl. ms'gl.*) which I interpret to be the same as these are very numerous and show plate-like expansions of the terminal branches, which have been described by Kassianow (:08, p. 525) for Alcyonium. The mesogloea layer in the coenosarc is very thick (Fig. 22); here the newer mesogloea, that which is near the ectoderm, is less dense than the portion which occupies the deeper layers. Sections stained in either eosin or Mallory's phospho-tungstic haematoxylin show well the differences between these regions. The ectoderm cells of the coenosarc are very long and the interstitial cells at their bases very numerous. The latter have the appearance of being pushed away from the ectoderm as growth proceeds, and they arrange themselves, or are arranged, in masses or cords (Figs. 22, 28, *cl.cd.*), in which the individual cells are often only loosely associated. The interstitial cells (Fig. 24, *cl. in.*) which are still near the ectoderm, are globular or irregularly branched, but otherwise they are not different in appearance from the ectoderm cells of the coenosarc or of the polyp wall and tentacle; but they, together with the deeper and more specialized cells constituting the cords (Fig. 22), reach down even to the axis epithelium. A transverse section of such a cord is given in Figure 29; other sections are shown in Figures 27 and 28. In these cords some of the cells (*cl. in.*) are like the interstitial cells near the ectoderm. Others (*cl. grn.'*) contain few or many granules, which vary in size and staining capacity, but always stain, either slightly or heavily, in haematoxylin or in eosin. The granules vary in size in different cells. In other respects these cells resemble interstitial cells. Some of the cells at the edges of the cord are partially surrounded by the jelly of the coenosarc, and these are appreciably smaller and more elongated than the others. The loosely arranged cells of the cord show no extra-cellular matter except at the edges of the cord. Where such matter is evident, the characteristic finely granular mesogloea cells (*cl. ms'gl.*) are to be found with their greatly elongated processes and terminal branchings.

I believe the cells enumerated below constitute a genetic series:

(1) the interstitial cells (*cl. in.*), or some of them; (2) the loosely arranged cells (*cl. grn'.*) with few granules, and those with larger granules; (3) the cells on the outside of the cords, which are either partially or wholly surrounded by secreted matter; and (4) the smaller mesogloea cells (*cl. ms'gl.*); because the interstitial cells and the series of loosely arranged cord cells are continuous with each other. The cells of each series may change their shape by amoeboid movements, as sectioned living material has shown me. Any of the cells, except the mesogloeal cells, may have granules. The cytoplasm and the nuclei of all these cells are alike, except in regard to the size and shape of the cell and nucleus and the presence of granules. But it would not be correct to argue from this series of cells found in the coenosarc that the mesogloea is of exclusively ectodermal origin. The bases of the ectoderm cells seem active in the formation of mesogloea in all parts of the colony and there is evidence of the secretion of the same substance by cells associated with the axis epithelium in the axis region. The evidence drawn from Pseudoplexaura does not exclude the probability that the endoderm is also active in the formation of mesogloea in the tentacle, polyp wall, and particularly in the mesenteries. The small cells at the end of my series of four given above occur in most abundance where the jelly layer of the coenenchyma is most dense; they simply represent, it seems to me, the ultimate condition of cells whose usefulness may not be limited to the formation of mesogloea, but which in all of the earlier stages have been more or less active in the formation of such substance.

But not all of the mesogloeal cells belong in this series, nor are all those in the cords secreting cells. Some of the interstitial cells develop nematocysts and some are spicule-forming cells. Those forming nematocysts are very abundant in the coenosarc, and not only are stages in the development of the cyst found, but cysts (*nm'cys.*) as large and as fully formed as any near the surface are very abundant, not only among the interstitial cells near the ectoderm, but also in the cell cords and in the deep parts of the coenenchyma. These, I believe, have been carried in from the ectoderm by the rapid growth of the outer layer and by the amoeboid action of the cells around them.

In growing regions at the tip of the branches some of the interstitial cells, pushed deeper by growth, secrete the spicules, probably during special periods. The formation of the spicules is not different from that described by von Koch ('87), Bourne ('99), or Woodland (:05). The more or less rounded spicule-cell first shows a small calcareous mass (Plate 3, Fig. 23, *spc.*), which increases in size with the division of the cell and takes on a characteristic shape (Figs. 25, 28). Several nuclei are to be found in the cytoplasm enveloping most of the spicules, thus showing that the spicule-cell usually divides more than once. Figure 25 shows the arrangement of the organic matter of the spicule after decalcification; this is similar to the condition described by Bourne.

Certain of the cells in the mesogloea (Plate 4, Figs. 48–51) show ovoid or globular bodies similar to those described by Bourne ('99) and Woodland (:05) as possibly stages in nematocyst formation. Rounded nutritive cells also occur, sometimes few, but in the coenosarc of some colonies very abundantly. They are also found occasionally scattered among the ectoderm cells of the outer edge and also among endoderm cells, where they probably originate. They stain more deeply with eosin than the surrounding cells.

Endoderm.

The endoderm lines the coelenteric side of the stomodaeum, the disk, the polyp wall, the mesenteries and all the canals. It is composed of three types of cells; supporting, mucus, and granular gland cells. Muscle cells are found in some parts. The cell characters are similar in the endodermal lining of the anthocodia (Plate 2, Fig. 20), the polyp chamber (Plate 4, Fig. 56) and the connecting canals (not including the long nutritive canals). The supporting cells are narrow and columnar, in contact with each other proximally and distally. In partially contracted individuals (Fig. 20) the supporting cells and the less numerous gland and mucus cells appear crowded into close contact; but in slightly expanded individuals (Figs. 55, 56) after fixation frequent spaces occur separating individual cells except at their two ends. The cytoplasm is coarsely vacuolated. A large nucleus is found somewhere in the basal two thirds of the cell. The cells are sometimes crowded with Zooxanthellae (*zox*), which are usually very numerous in and near the polyps, but are not so abundant in the deeper canals. Three, four, or more of these algae are common in the sections of each cell in the tentacles, polyp wall or outer coenosarc. Each endoderm cell has a single weak cilium inplanted in its free end, and at its attached end a myoneme, which runs circularly in the wall of the anthocodia and polyp chamber, and generally so in the canals. Mucus glands are abundant (*cl. muc.*); they appear as columnar cells with the cytoplasm in the form of a large-meshed reticulum that stains deeply with muci-carmine, and each has a darkly staining nucleus. There are occasionally found also gland cells (Fig. 55, *cl. grn.*) similar in character to the granular gland cells of the stomodaeum. The feeble staining of the nucleus and the presence of large granules in the cytoplasm show the likeness. The cells may be shorter or even spherical They are not limited to particular regions, but are scattered throughout the endoderm of the polyp wall, the çanals and the mesenteries.

The endoderm of the polyp wall (Figs. 20, 55) has longer cells and stronger myonemes than that of the polyp chamber or canals. However, in the pinnules and in the tentacles, except at their very bases, the myonemes and the granular gland cells are lacking (Figs. 5, 6); otherwise the layer is here like that of the body wall.

The epithelial cells and the muscle elements of the endoderm of the mesenteries (Plate 3, Fig. 33) are specialized, at least in certain regions. On the so-called ventral surface of the mesentery the longitudinal myonemes belong to cells that are entirely below the free surface of the epithelium. These muscle cells are long and spindle-shaped (Plate 4, Fig. 38), with a small amount of cytoplasm enveloping the fiber (myoneme) and most abundant around the elongated nucleus. The cells are very numerous and the fibers are so arranged that in the cross section of the mesentery they form wavy rows of black dots adjacent to the ventral side of the mesogloea (Fig. 33, *my'nm.*). Other muscle fibers, that run radially on the mesentery, are found in both layers of the endoderm and their cells are epitheliomuscular, though in some cases the cell body may be slightly sunk below the surface. Where the endodermal epithelium covers the genital cells, the epithelial cells are shorter and bear no muscle fibers.

The endoderm of the longitudinal canals (Plate 4, Fig. 59) has very long cells as compared with that of the other canals; they are, however, of the same type, viz. supporting cells; they are slender, columnar, vacuolated, and slightly separated from one another except at their proximal and distal ends. No muscle cells, or myonemes, are found in this endoderm.

As compared with corresponding structures in other alcyonarians, it may be said, in brief, that the ectoderm of Pseudoplexaura is like that of the other members of this group described by previous authors, in having an epithelial and a subepithelial layer; in the shape and character of its cover cells it is like Alcyonium (Hickson, '95, Kassianow,

:08), and is not greatly different in its cell characters from Xenia (Ashworth, '99) or Veretillum (Buvor, :01). The nematocysts of the larger kind are about the size of those of Clavularia (von Koch, '82[a]), and are generally slightly larger than those of Xenia or Alcyonium. Those of the smaller kind seem significantly numerous in the stomodaeum. Sense cells like those of Alcyonium (Kassianow, :08) are associated with the nettle batteries; but the number of the batteries in the coenosarc and at the base of the polyps appears to be greater than in Xenia or Alcyonium, and the fewness of the nematocysts on the tentacles and pinnules seems unusual. Evidently the surface of the coenosarc between the polyps is an important region for the work that the large nematocysts do. The polyp often contracts to the level of the coenosarc with the mouth still open and the tentacles still spread, and I have seen food particles passed along from the coenosarc to the mouth by the tentacles. The nematocysts of the smaller sort are more evenly distributed on the outside of the colony, but they are very few. I failed to find gland cells on the outer surface, except perhaps in the coenosarc. Von Koch ('87) does not include them in his list of ectoderm cells for Gorgonacea, but they are present in representatives of the two other alcyonarian groups. It seems hardly probable that the slime which is given off by Pseudoplexaura when it is handled has come from the mucus cells of the endoderm. The fibers of certain of the ectoderm cells of the coenosarc (Plate 3, Fig. 24) are, as has already (p. 750) been suggested, possibly supporting fibers, such as K. C. Schneider (:02, p. 622, Fig. 510) has described for Anthozoa and other invertebrates, and figured for a sea anemone.

The mesogloea of the colony is very thin except in the coenosarc region; but here is thicker than that of the forms heretofore described, except the Gorgonacea.

The endoderm is similar to that of Alcyonium, and shows no significant features, except the absence of muscle fibers in the longitudinal canals. Menneking (:05), from the study of Stachodes and other forms, reached the conclusion that the longitudinal canals have originated as inter-mesenterial chambers of a terminal polyp. The absence of muscles in the walls of the longitudinal canals of Pseudoplexaura, in contrast with their presence in the mesenteries, together with the fact that the canals are sometimes traced to solenia without polyps, suggests that in this form the longitudinal canals have not originated in this way. Kinoshita (:10) did not succeed in finding muscle fibers in the endoderm of the longitudinal canals of the developing colonies of Anthoplexaura, but in the growing part he found solenia, which were sometimes continuations of the longitudinal canals. The structure of adult Pseudoplexaura supports the conclusion of Kinoshita, that the longitudinal canals have not always developed from inter-mesenterial chambers.

Structures Concerned in Nutrition.

Though I often experimented with small portions of a colony in the laboratory, I saw very little feeding. Plankton was given, but I saw none of it stunned, and only the smaller less active organisms, such as sea-urchin eggs, were swallowed. Sea-urchin eggs and small pieces of the flesh of fish were placed near the polyps and were often taken into the stomodaeum. Sometimes a polyp kept large pieces of sea-urchin ovary against its mouth for a long time. Usually the whole colony was quite fully expanded, except when it had been vigorously treated. On the reefs colonies with all the polyps contracted were very seldom seen. In the laboratory I could not find any difference in the condition of a colony at night and in the daytime in this respect. Individual expanded polyps may have the peristome closed, or polyps that are contracted so that the tentacles are spread out on the coenosarc may show it open; but there seems to be no special time for feeding. I think the food is undoubtedly from the plankton, and particularly the smaller and more sluggish forms.

The nettle cells of the smaller kind (Plate 2, Fig. 31, *cl. nm'cys.'*; Plate 4, Fig. 47) are very abundant in the ectoderm of the stomodaeum, while less numerous on the tentacles. Those of the larger kind (Plate 2, Fig. 26, *cl. nm'cys.*; Plate 4, Figs. 43–46) are most abundant in the coenosarc between the polyps and are seldom found in the tentacles. When sea-urchin eggs are scattered with a pipette over the tip of a branch whose polyps are expanded, they fall slowly, and do not seem to be stopped by tentacle or polyp, but collect in the grooves of the coenosarc. Associated with the falling of sluggish material on the coenosarc, adjacent individual polyps often contract down to the level of the coenosarc with the tentacles still partly spread and the mouth widely open. In such cases the eggs are often drawn into the current of the siphonoglyph. The supporting cells of the stomodaeum have at times small irregular granules at the distal end. These may be zymogen granules or, more probably, products of metabolism destined for other than enzyme use. Food material may be taken into these cells, and even algae have been found in them in a partially digested condition. Some of the food, then, is probably digested here, both in an intra-cellular and extra-cellular fashion. After the remaining food passes the stomodaeum it is in contact with the six ventral mesenterial filaments. These (Plate 4, Fig. 64) are very short thickenings of the margins of the six mesenteries, and occupy a position immediately below the stomodaeum. They are less than two millimeters long and in preserved material may be less than one. They begin at the deep end of the stomodaeum, but their gland cells may be found on the mesenteries a little above this. The cross section shows that this thickened margin is nearly cylindrical. The cells are mostly gland cells that are not different from the granular cells of the stomodaeum. A few supporting cells occur among the others and these may contain food matter. There are no nettle cells.

Until 1899, the stomodaeum was considered as merely a passage for the food, the mesenterial filaments being regarded as the only digestive organs. Wilson ('84) described the filaments of eleven genera from the three groups of Alcyonaria and concluded that the six lateral and ventral filaments are derived from endoderm and that the two dorsal ones are from ectoderm. The former contain gland cells and sometimes nettle cells, and are digestive in function; while the latter have two kinds of cells, are ciliated and are used for the production of currents. In 1899 Ashworth found mucous gland cells in the stomodaeum of Xenia, and correlated their presence with the absence of the ventral filaments. Miss Pratt (:05), by a very thorough and complete study of the feeding, in which she employed colored food, found that food was ingested, not only by the cells of the stomodaeum and filaments, but also by the mesogloeal cells. But no ingulfing of food was observed in the cells containing the granules. Gland cells were abundantly present in the stomodaeum of many members of the Alcyonaria, but the granular cells were met with in starved individuals only. Pseudoplexaura agrees with the forms studied by Pratt in the presence of gland cells in the stomodaeum and the abundance of the granular cells in the tips of individuals starved in filtered sea water. In Miss Pratt's experiments particles of fish artificially colored were also engulfed by stomodaeum cells, by the network of interstitial cells in the polyp wall, and by the mesogloeal cells near the outer surface of the coenosarc. Both the stomodaeum and the ventral filaments, then, are digestive structures; while the granular gland cells, which are quite abundantly scattered in the endoderm of the coelentera, including the canals and the mesenteries, may considerably aid in extra-cellular digestion.

Dorsal Mesenterial Filaments.

These filaments differ in origin, structure, and use from the ventral and lateral filaments. As a whole, the filament is a long, deeply grooved ribbon or cord, attached to the margin of the corresponding mesentery, and reaches from the stomodaeum to the depths of the polyp cavity; if there is a large basal canal connecting polyps with one another, it may even be continued into such canals. In cross section (Plate 4, Figs. 62, 63) the filament is much thicker than the mesentery and is deeply notched at its free margin. Consequently, in cross sections the mesogloea has the form of the letter Y. The epithelial cells occupying the space between the arms of the Y are of two kinds. The outer ones (*cl. fil.*), those nearer the ends of the arms of the Y, are the more numerous and are similar to the supporting cells of the stomodaeum. They are columnar, of small diameter and so closely packed that their nuclei are arranged in several rows. Each cell has a very strong cilium, and these cilia are so long that those of one side of the groove touch or cross those of the opposite side. The remaining cells, those occupying the base of the filament groove (*cl. fil.c.*), are few but larger, having broad bases and tapering slender necks. Their cell boundaries usually cannot be demonstrated. Near the base of each cell is a large, lightly staining nucleus. They possess no cilia. The cytoplasm is sometimes evenly granular, but often shows large vacuoles that stain with muci-carmine. The mucus, which they evidently have secreted, may sometimes be found between the cilia of the other cells.

Wilson ('84) has described in detail these dorsal filaments for other alcyonarians. But neither in his eleven genera, nor in the figures of Alcyonium by Hickson ('95), nor of Xenia by Ashworth ('99), are the cells represented to be as large and prominent as they are in Pseudoplexaura. I did not observe the effect of the presence of mucus in this groove, save that sometimes very minute particles, presumably of food, may be found in it; the mucus is probably for the purpose of catching material entering the polyp from another polyp or from the long canals. Portions of colonies kept for some days in the dark or in weak light lost their Zooxanthellae. The polyps of these portions of colonies are then translucent and the direction of the currents produced by the cilia can be detected. The current formed by the cilia of the filaments flows from the base of the polyp to the stomodaeum, while that of the siphonoglyph is in the opposite direction.

Growth.

Growth being both terminal and radial, the polyps may increase in either direction. The tips formed in summer are of two types. One type shows no polyps on the terminal two or three millimeters of the branch, which is crowded with purple spicules. In only a very few instances were polyps formed at the tip of the stem in this type in any other position than the radial one. They were usually large and of the same size. This is not an area of reproduction of polyps at this time. The other type of stem shows a tip denuded of polyps for a relatively long region, one half to one or more centimeters. The coenosarc wall of this tip is smooth and many of the polyps nearest to the denuded region are small. Under the surface of this tip is found an extensive network of canals. Very small polyps are also often found in the coenosarc at other regions than that of the tip. Young polyps, then, may be found in the growing stem in all parts of the colony.

Muscles and Nerves.

The arrangement of muscles into systems is not markedly different from that described for Alcyonium by Kassianow (:08). The systems are: (1) The tentacle and disk system. This is ectodermal. The muscle fibers (Plate 2, Fig. 5, *my'nm.*) run longitudinally on the pinnules (Fig. 6) and on the tentacles (Figs. 5, 7) and are continued on the disk toward the mouth, but the lateral strands of each of the eight bands bend outward to be inserted in the mesogloea of the mesentery. The median strand on the oral side of the tentacle is continued to the mouth, but these muscles are fewer than in Alcyonium. The aboral surface of the tentacle, as is shown by a transverse section (Fig. 7; compare Fig. 21), bears no muscles, and muscles are lacking on a very small portion of the aboral surface of the pinnules (Fig. 6). (2) The polyp-wall system embraces muscles that are endodermal and are arranged circularly (Fig. 20). They are strongest where the polyp wall and tentacles meet, and they may pass a slight distance on to the base of the tentacles. In the small canals (as in Fig. 56, *my'nm.*) they are generally circular. (3) The stomodaeum system. Endodermal muscles are feebly developed in the stomodaeum, where they run circularly (Plate 3, Fig. 30). At the oral end, and to a less extent at the coelenteric end, they are larger and more numerous, but hardly enough so to be termed sphincters. (4) The mesentery system. These are, of course, endodermal. The longitudinal muscles, on the ventral side of the mesogloea (Fig. 33. Compare Plate 1, Figs. 2–4), are independent of the epithelium. The folding of the mesogloea, which in cross sections appears branched, is such as to accommodate a large number of fibers without a corresponding increase in the width of the mesenteries. Transverse muscles are found on both sides of the mesentery (Fig. 33); they are comparatively few and are arranged in a single sheet, i. e., without foldings.

Physiologically the muscles may be divided into, first, the longitudinal muscles of the ectoderm of the tentacle and disk and the strong longitudinal endodermal muscles of the mesenteries; secondly, the circular endodermal muscles of the polyp wall and canals together with the transverse muscles of the mesenteries.

The nerves can hardly be said to be arranged in a system, as they surely are in colonies of more active alcyonarians. Sense cells are found, particularly in connection with the nettle batteries, and ganglion cells are scattered in the deeper layer of the ectoderm of both column and stomodaeum. But there is no conspicuous nerve layer, such as that found by Kassianow (:08) in Alcyonium.

The weakness of the nerve layer accords with the slowness of the polyps in contracting. These do not respond to touch as quickly as many other related forms living near them, such, for example, as Euniceopsis, Plexaura and Gorgonia. The tentacles show no response to a single light touch, but a sharp touch, or one repeated, gives a reaction, which is always toward the mouth, as is to be expected from the fact that the muscles are limited to the oral side. The response of one tentacle, however, is accompanied by a response of the other seven. The disk and column respond to touch, and the column responds more quickly and vigorously near its base than elsewhere. But the coenosarc between the polyps is the most sensitive part of the colony to touch. When this region is stimulated, the adjacent polyps respond by a slow contraction toward the level of the coenosarc; the response, however, is more certain than when the column is touched. There seems to be no nerve system connecting polyps with one another, since touching one does not result in a response from another. One can draw a pencil across a branch and get a contraction of the polyps only in that line, if he does not shake the branch. When a branch is shaken, all polyps begin to contract, although very slowly. I saw no reactions that would indicate taste as contrasted with touch. Food particles on the coenosarc cause the contraction of the polyps near it, the mouth and tentacles remaining expanded; but clean filter paper does the same. Neither in the field nor in the laboratory did I find muscular response to light. The polyps were expanded night and day alike. In the laboratory, away from the sunlight they lost the Zooxanthellae and became white after a week's time.

Skeleton and Axis Epithelium.

The structure of the axis skeleton has already been described under General Structure (p. 742). I find the axis epithelium (Plate **4**, Figs. 54, 58, (*e'th. ax.*) always present and made up of two types of cells, the secreting cells and the holding cells, or desmocytes. The secreting cells are long and cylindrical or prismatic. Of the two ends, the one directed toward the skeletal axis may be designated as axis-end and the other as mesogloea-end; the former is flat, the latter tapers and is more or less rounded (Fig. 54). The large feebly staining nucleus is nearly in the middle, but typically somewhat closer to the axis-end of the cell. The cytoplasm is vacuolated at the mesogloea-end, but near the axis it is finely granular. This type of axis cell is always found at the tip of a branch, where the horny rim of the axis chambers is very thin; and I interpret this as a place of most active secretion. In any region of the colony, except at the very tip, some of the epithelial cells — sometimes only one, sometimes a comparatively large area of them — are modified into desmocytes (Plate **4**, Fig. 41, *dsm'cy.*). These cells are broader than the secreting cells at the axis-end, and relatively shorter. At the axis-end they show a prominent border of striations perpendicular to the surface. These striations are due to slender rod-like differentiations of the cell, which seem to be the means by which the cells hold firmly to the axis, even when, in sections cut either free hand or after imbedding, the other cells are detached. Where the outline of the cell is complete (*dsm'cy.*), a nucleus like that of the secreting cells is present. Often, however, the cell has united with the mesogloea so that the boundary between the two is gone, and then the nucleus may have disappeared (*dsm'cy'.*). The axis-face of the cell (Figs. 41, 60) is usually flat, but may be concave (Fig. 58) or convex (Fig. 42). The desmocytes arise, or at least attain their differentiation, in the secreting epithelial layer. Cells in contact with the axis (Figs. 41, 58, *dsm'cy.*), that apparently are at first not different from the secreting cells, broaden their axis-end, pushing other cells away from the axis. To such a cell a mesogloeal process, probably secreted largely by adjacent epithelial cells, becomes applied, so that the cell then appears to be simply a prolongation of the mesogloea. The nucleus of the cell persists for a long time, but often it degenerates. Meantime the differentiation of the broad end of the cell shows it to be a desmocyte. Secretion on the part of the surrounding cells may continue around these desmocytes. Figure 42 shows that in this case much of the secreted layer of the axis was formed after the differentiation of the desmocyte and while it was still functioning as a hold fast. Figure 52 (*dsm'cy'.*), compared with Figure 42, shows evidence that the axial portion of the desmocyte may lose its connection with the mesogloea owing to the constriction of its neck by the formation of the horny secretion. It is in this way that some of the smaller chambers of the axis-cortex are formed. When this has taken place, other desmocytes appear in the same region peripheral to it.

In places where a great many desmocytes have been formed (Plate **4**, Figs. 58, 60), the secreting cells are pushed back from the secreting surface in disarray. The displacement is perhaps a necessary result of the broadening of the ends of the desmocytes. At a later time, perhaps in response to the same stimulus that causes the beginning of a new skeletal chamber in places where desmocytes do not occur, such displaced secreting cells rearrange themselves preparatory to the secretion of a new lamella, leaving a lenticular space between themselves and the previously secreted portions of the axis. Later still, some sort of stimulus may then cause other desmocytes to appear among these secreting cells, probably as the result of the differentiation of a part of their own number. I consider these holding cells to be homologous to those seen by Fowler in a madreporarian coral and to those whose origin was described by Bourne ('**99**) for the madreporarians and for Heliopora, an alcyonarian with a calcareous skeleton; but I find no reference to similar cells for any other alcyonarian, except that possibly A. Schneider (:**05**, p. 128) found them; but if so, he evidently thought them artifacts. I have found them in all the colonies of Pseudoplexaura studied, and I have also seen them in the species of Euniceopsis and Gorgonia which are associated with them on the Bermuda reefs. In Pseudoplexaura the stimulus for the change from the secreting cell to the desmocyte must be irregular; it is not associated with any particular position of the polyps or with any structure that would give a regular pull or strain, since the cells occur sometimes in broad patches and sometimes singly; the latter are completely united with the mesogloea and are therefore fully formed. They remind strongly of the desmocytes described for the madreporarians by Bourne ('99), but there is no trace of the membrane which Bourne found between cell and axis. In their origin they also differ from those described by Bourne for Heliopora, where the stimulus for the striations occurred before the cells were in contact with the axis, to which they became adjacent secondarily; for in Pseudoplexaura the first trace of the striations is in cells already touching the axis. It should be noticed that in the present paper the desmocytes have been shown clinging to a horny skeleton, whereas previous researches have shown them only in connection with calcareous skeletons. Probably further study will show desmocytes present in a large number of alcyonarian forms.

The origin of the horny skeleton of the Gorgonacea has been the subject of much controversy, with which the names of von Koch, Studer, and A. Schneider have been prominently associated. A. Schneider (:05) has reviewed the literature carefully, and has shown that Ehrenberg, Dana, Milne-Edwards et Haime, and von Koch have maintained an ectodermal origin; while Lacaze-Duthiers, Kölliker, Studer, and Heider have not found the ectoderm involved. The arguments against the ectodermal origin, as summed up by Schneider and strengthened by his researches, have to do with (1) the presence of calcareous spicules within the horny skeleton, (2) the character of the union between the axis and its branches, (3) the existence of extra-axial horny masses in the cortex independent of epithelium, (4) the increase in size of the adult axis, (5) the embryonic origin of the skeleton.

Kölliker ('65, pp. 163–167) argued in part as follows: since the axis skeleton in certain forms (Mopsea) is composed exclusively of fused calcareous spicules, and since these spicules are not produced by epithelium, the skeleton is not an epithelial product. Studer ('87) and A. Schneider (:05) found numerous calcareous spicules in the axis, and thought the axis made up principally of them. I have found no evidence of such spicules in the axis of Pseudoplexaura, though I have found one or two instances of cellular matter that I conceive to have been included in the axis owing to the rearrangement of the secreting cells of the axis epithelium over a mass of desmocytes, and I can account for the possible enclosure of spicules in an axis in the same abnormal manner.

A. Schneider found that in Eunicella the axis of the branch (Nebenaxis) is at first separated from that of the stem, with which, however, it is later united. It is difficult to see how such an axis can be explained as the result of the secreting activity of an ectodermal epithelium, except in cases in which the branch is secondarily united (by anastomosis) to a stem, as in fan corals; but Eunicella does not usually have such a secondary union of branch and stem. Perhaps, however, the conditions in Eunicella are not essentially different from those which are met with in Pseudoplexaura, where I find that a sharp demarcation line between the axis of the branch and that of the stem also occurs (Plate 4, Fig. 61); here it is due to the fact that all branches of the axis are adventitious in respect to the marrow. At the region of branching, the marrow of the stem-axis is separated from that of the branch by the secreted cortex of the stem-axis. The walls of the marrow chambers in the branch were therefore formed after the axis-cortex of the main stem possessed an appreciable thickness (Fig. 61, *ctx. ax.*). But the existence of a stem-cortex between the marrow chambers of the stem and those of the branch is not inconsistent with an ectodermal origin of the epithelium secreting the axis of the branch, because axis-cortex is formed in the same manner as axis-marrow. In both cases the horny matter is laid down in the form of walls of chambers; and these differ only in the size and shape of the cavity and in the thickness of the wall. The chambers of the axis-cortex are smaller than those of the marrow, nevertheless they vary greatly among themselves in size (Fig. 57, *ctx. ax.*). It is assumable that, after some of these axis-cortex chambers of the stem had been formed (Fig. 61, *ctx. ax.*), other chambers with the characteristically thinner walls and larger cavities of the marrow, may have arisen at the place where a branch was about to be produced. The walls of these chambers would, then, be secreted by the same epithelium that recently had been building smaller chambers as an axis-cortex of the stem. The epithelial patch at the distal end of the axis of the branch would be composed of cells which had changed somewhat the character of their secretions, so that henceforth they would produce the larger thin-walled chambers characteristic of the marrow, whereas the remaining cells (at first situated in the periphery of this terminal patch) would continue to produce the smaller chambers, with thicker walls, such as they had been producing as axis-cortex of the stem; but now as the axis-cortex of the branch.

In no part of the axis of a stem or branch has the marrow grown in opposite directions, part distally, part proximally, for the walls of the chambers are always convex toward one end of the stem or branch — the distal end (Plate 4, Figs. 57, 61). The marrow must therefore have grown from the base of the branch toward the tip, just as it does in the stem from base toward tip, and not, as maintained by some writers, in the form of a separately established axis which grows in two directions: partly toward the tip and partly toward the stem to which, in their view, it is destined to be attached secondarily.

This type of axis — with the axis-cortex interposed between the marrow chambers of stem and branch — is the natural one for all axes except such as may have been formed by the dichotomous branching of a main stem. Such branches may possibly occur, but my dissections have not shown any. Occasionally, in colonies that had attained a height of ninety or more centimeters, the beginning of a branch was found on some of the whips. These, as short as five millimeters, had a soft axis that was continuous with the main axis and was formed of the characteristic marrow (Figs. 57, 61, *med. ax.*) and a very thin cortex. The marrow chambers were separated from those of the main stem by the cortex region of the stem-axis and, as has already been stated, were convex toward the free end of the branch, as in the main branch they were convex toward its free end. Although the earliest stage in the formation of the axis of a branch has not been seen in Pseudoplexaura, I am convinced that the axis skeleton of the whole colony in this species is not produced by a coalescence of separately established axes.

Pseudoplexaura gives no evidence on the third of Schneider's points, for no horny substance has been found in the coenenchyma. But the real issue between the two theories of ectodermal or non-ectodermal origin hinges on the results of observation as to the origin of the axis and as to the method of its subsequent growth; whether it is an epithelial secretion, as argued by von Koch ('78, '87), or results from a massing of mesogloeal material which is to be resorbed and replaced either by horny substance or horn and lime. Von Koch has described ('87) a larval stage of Eunicella a week old and has shown sections having the ectoderm continuous with the axis epithelium. His results have recently been confirmed by Kinoshita (:10) in embryos of Anthoplexaura. Kinoshita not only found the ectoderm of the pedal disk continuous with the axis epithelium, but he also has described and figured (Fig. 3–5) the beginning of the axis as a secretion product of the thickened ectoderm of the pedal disk; however, this primitive axis (a single case) did not by its upward growth push before it the floor of the digestive cavity of the primary polyp, but rather grew upward in the wall of the column at one side, so that the primary polyp had the appearance of being a lateral outgrowth from the axis. From this he concludes that the stem of the colony in Anthoplexaura apparently does not belong to the primary polyp, but to the coenosarc (at its base), just as in Pseudaxonia.

The existence of a secreting epithelium in the adult of Pseudoplexaura cannot be doubted. The axis-secreting cells are large, and this cell layer, which is evident, can be traced in free-hand sections from the tip of the axis to the spreading base near the substratum. I always find an unbroken axis epithelium around the tip. This seems to me to be irreconcilable with the method of growth outlined by Studer, and represented by the sections shown in A. Schneider's paper. For Studer's theory demands a mass of spicules at the tip, as well as in other places, perhaps,— spicules which are later to be resorbed. These spicules must of course develop in the mesogloea, and for their incorporation into the axis would require a break in the epithelium around the tip; but such a break I have not seen in Pseudoplexaura. There is often a massing of spicules at the tip *outside* of this epithelium, but there is no trace of their inclusion in the axis, nor of their conversion into it. The spicules are here pushed aside by the growth of the axis and remain as spicules in the mesogloea. A. Schneider holds that the axis epithelium as figured by von Koch, with which the epithelium of my figures is undoubtedly homologous, is the endodermal lining of the digestive cavity of the axial polyp, into which the axis has been pushed, and that the longitudinal canals are mesenterial chambers. But so far as regards Pseudoplexaura, the cells of the axis epithelium are not like endoderm cells. Moreover, the longitudinal canals vary considerably in numbers in Pseudoplexaura tips, being eight or *more*; besides, as Kinoshita found for Anthoplexaura, they have no muscles and sometimes end in solenia.

Pseudoplexaura, then, affords no evidence of spicules included in the axial skeleton; a secreting axis epithelium is present, the cells of which are unlike those of the endoderm in their arrangement and structure. Even when pushed aside by the spreading of the desmocytes, they are not easily to be mistaken for endoderm cells. Kinoshita's evidence in the embryos of Anthoplexaura is a strong support for the ectoderm theory. The results from the study of the adult of Pseudoplexaura are not in themselves complete evidence, but so far as they go speak strongly for the ectodermal origin of the horny axis, as indicated by von Koch.

Summary.

Pseudoplexaura crassa, an alcyonarian of the group Gorgonacea, shows the character of Gorgonacea so far as regards the regions recognizable in cross sections of the branches; the branches have a central horny axis, a thick coenenchyma and an outer zone of polyps. The horny axis shows a marrow composed of large chambers arranged end to end, and a peripheral layer of smaller less regularly shaped ones arranged side by side and irregularly overlapping one another. The coenenchyma has, not far from the axis, a region of large longitudinal canals. These are sometimes prolonged at their tips into solenia. The polyps are long, and have ten to twelve pairs of pinnae on each of their tentacles. They are crowded, so that when expanded they hide the coenenchyma. Groups of small, crowded, irregularly stellate, purple spicules occupy the deeper parts of the coenenchyma, and larger, spiny and spindle-shaped, usually white spicules are in its outer part. No spicules are found in the polyps.

The ectoderm has the usual cover cells, nematocysts, sense cells, and interstitial, ganglion, and muscle cells. Small nematocysts are found in the ectoderm of the polyp's column, tentacles and stomodaeum. Large ones in considerable numbers are grouped into batteries in the coenosarc. Ganglion cells are very few, and muscle cells are found on the oral side of the tentacles and disk only. In the ectoderm of the coenosarc between the polyps some of the ectoderm cells have each a prominent supporting fiber, which runs from near the nucleus perpendicularly to the mesogloea.

The mesogloea is thin, except in the coenosarc regions, where it is very thick. Cords of cells like the interstitial cells of the ectoderm can be traced from the ectoderm to the deeper layers of the mesogloea. In these cords there are partly formed and fully formed nematocysts, spicule cells, and cells having an irregular shape and either containing granules or destitute of them. These irregularly shaped cells form a transition to the jelly-secreting cells, which are small and have many long branches. Large spicules are produced by characteristic secreting cells with large granules and one to many nuclei. Spheroidal nutrition cells occur in many colonies, but these are found in both ectoderm and endoderm; they probably originate in the endoderm of the canals which form a network through the mesogloea.

The endoderm cells are of characteristic form, being united with each other at the proximal and distal ends, but, in fixed material,

separate elsewhere. In the tentacles, in the polyp wall and in many canals they contain large numbers of the alga Zooxanthella. Except in the tentacles and the longitudinal canals, they have myonemes running circularly. Unicellular mucus glands and granular cells, that are probably digestive in function, are numerous. The cells of the longitudinal canals differ from other endodermal cells in being much longer and in having no trace of myonemes.

Digestion is accomplished by cells of the stomodaeum, by the six ventral and lateral mesenterial filaments, and by scattered gland cells in the walls of the polyp cavity and the canals. The stomodaeum has, beside its supporting cells, mucus and granular gland cells. The mesenterial filaments, except the dorsal pair, are very short and their epithelium is composed of granular gland cells only, which give some evidence of intracellular digestion. I found no special feeding time and no regular alternation of contraction and expansion of polyps. Slow-moving organisms, which serve as food, are often transferred from the surface of the coenosarc between the polyps, where large nettle cells abound, to the mouth of a polyp that independently contracted to the level of the coenosarc with its mouth open. The two dorsal mesenterial filaments are very long and sinuous and their cell structure is peculiarly significant. The sides of the groove are lined by cells with strong cilia. The central cells, however, show the character of mucus cells and produce a mucous secretion.

The muscle system is similar to that of Alcyonium. The colony is characterized by the weakness of its responses and by the fewness of its nerve elements. The response to touch is not quick, and the coenosarc between the polyps is more sensitive than the polyps themselves.

The axis skeleton is surrounded by an epithelium consisting of elongated secreting cells, and in places, of desmocytes, or holding cells, these being shorter and wider, and exhibiting striations at the axial end. These cells become connected with the mesogloea secondarily. They may become isolated as the result of being completely enveloped in the secretion of horny material by the secreting cells. The desmocytes have already been described for Heliopora and for the madreporarians. The evidence in Pseudoplexaura favors an ectodermal origin of the axis skeleton.

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EXPLANATION OF PLATES.

Abbreviations.

ax.	axis.
cam. pyp.	polyp chamber.
can.	canal.
can. lg.	longitudinal or nutritive canal.
cl. cd.	cords of cells in mesogloea.
cl. fil.	ciliated cells of the mesenterial filament.
cl. fil. c.	central cells of mesenterial filament.
cl. gl.	gland cell.
cl. gl.'	incipient gland cell of stomodaeum.
cl. gn.	ganglion cell.
cl. grn.	granular digesting cell.
cl. grn.'	granular cell of mesogloea.
cl. in.	interstitial cell.
cl. ms'gl.	mesogloeal cell.
cl. muc.	mucus cell.
cl. nm'cys.	nematocyst cell, large kind.
cl. nm'cys.'	nematocyst cell, small kind.
cl. nut.	nutrition cell.
cl. sns.	sense cell.
cl. spc.	spicule-producing cell.
cl. sst.	supporting cell.
cl. teg.	cover cell.
ctx. ax.	cortex of axis skeleton.
dsm'cy.	desmocyte.
dsm'cy'.	desmocyte, showing union with mesogloea.
ec'drm.	ectoderm.
en'drm.	endoderm.
e'th. ax.	axis epithelium.
fbr. sst.	supporting fiber.
fil. d.	dorsal mesenterial filament.
fil. v.	ventral (or lateral) mesenterial filament.
med. ax.	marrow of axis skeleton.
ms'enr. d.	dorsal mesentery.
ms'enr. v.	ventral mesentery.
ms'gl.	mesogloea.
my'nm.	myoneme.
nm'cys.	large kind of nematocyst.
nm'cys'.	small kind of nematocyst.
or.	oral.
ov.	ovum.
sipy.	siphonoglyph.
spc.	spicule.
spc'.	spicule of outer coenenchyma.
stmd.	stomodaeum.
zox.	zooxanthella.

Figures.

All drawings, except Figures 1 and 61, were carefully outlined with a camera lucida, and the details filled in afterwards. The magnification (except in Figures 1 and 61) is 675 diameters, unless otherwise indicated in the description of the Figure.

PLATE 1.

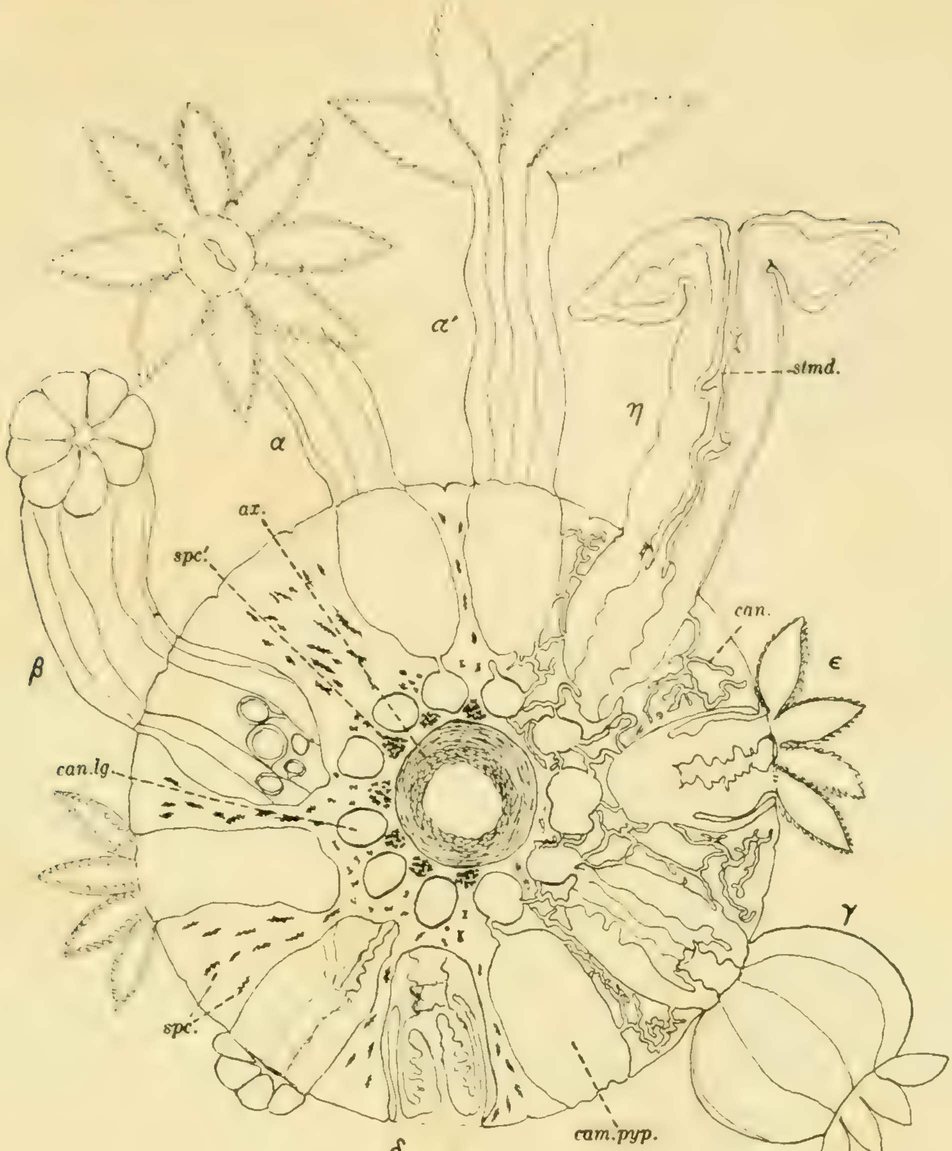
Fig. 1. Diagram of a transverse section through a branch of a colony, showing polyps in various stages of retraction.

Fig. 2. Transverse section of a polyp through the stomodaeum. $\times$ 15.

α, expanded polyp, oral aspect
α', expanded polyp, seen from the side
β, expanded polyp with inrolled tentacles
γ, polyp partially retracted
δ, polyp completely retracted
ϵ, polyp with column retracted, but with expanded tentacles and with open mouth
η, polyp in longitudinal section

Fig. 3. Transverse section of a polyp just below the stomodaeum. $\times$ 15.

Fig. 4. Transverse section of a polyp through the lower polyp cavity. $\times$ 15.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ax.
spc'.
can.
ε
β
can.lg.
γ
spc'.
cam.pyp.
δ

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sipy.

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ms'enr.v.

3


ms'enr.d.
ms'enr.v.
ov.

4

W.M.C. del.

PLATE 2.

Fig. 5. Longitudinal section from the oral wall of a tentacle.
Fig. 6. Transverse section of pinnule. × 600.
Fig. 7. Transverse section of a tentacle, showing the arrangement of the muscles. × 85.
Fig. 8. Longitudinal section of ectoderm of polyp wall.
Figs. 9–12. Isolated ectoderm cells of polyp wall, drawn from maceration preparations.
Figs. 13–17. Ectoderm cells of tentacle isolated by maceration.
Fig. 13. Two cover cells and a sense cell.
Figs. 14, 15. Interstitial cells.
Figs. 16, 17. Muscle cells of the ectoderm.
Fig. 18. Longitudinal section of the ectoderm of polyp wall, in state of expansion.
Fig. 19. Section of ectoderm of the body wall of a polyp near the coenosarc.
Fig. 20. Longitudinal section of polyp wall, partly contracted. × 600.
Fig. 21. Longitudinal section of tentacle wall, aboral side. × 600.

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my'nm.
ms'gl.
cl.ms'gl.
zox.
or.

W.M.C. del.

PLATE 3.

Fig. 22. Section through the coenosarc to show cords of interstitial cells. × 67.

Fig. 23. Section of epidermis of coenosarc.

Fig. 24. Ectoderm of coenosarc in the region shown in Figure 26.

Fig. 25. Section of decalcified spicule, showing spicule-producing cells.

Fig. 26. Outline of portion of coenosarc between polyps, to show the distribution of supporting fibers and of nettle batteries. × 67.

Fig. 27. Section to show mesogloea.

Fig. 28. Section showing mesogloea cells. × 400.

Fig. 29. Section of "a cord" in mesogloea of coenosarc.

Fig. 30. Longitudinal section of epithelium (ectodermal) lining the stomodaeum and of mesogloea.

Fig. 31. Longitudinal section of the wall of stomodaeum, showing both ectoderm and mesogloea.

Fig. 32. Transverse section of a part of siphonoglyph.

Fig. 33. Transverse section of a portion of a mesentery.


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nm'cys.
fbr.sst.
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ms'gl.
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ec'drm.
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cl.nm'cys'.
cl.nut.
cl.gn.
32
my'nm.
en'drm.
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zox.

W.M.C. del.

PLATE 4.

Figs. 34–37. Cells isolated by maceration.
Fig. 34. Gland cell of endoderm.
Figs. 35, 36. Supporting cells of endoderm.
Fig. 37. Mucus cell of stomodaeum.
Fig. 38. Muscle cells of mesentery.
Fig. 39. Supporting cell of stomodaeum.
Fig. 40. Granular gland cell of stomodaeum.
Figs. 41, 42. Axis epithelium, with desmocytes.
Fig. 43. Large kind of nematocyst cell, partly exploded.
Fig. 44. Large kind of nematocyst cell, unexploded.
Figs. 45, 46. Large kind of nematocyst cell exploded.
Fig. 47. Small kind of nematocyst cell.
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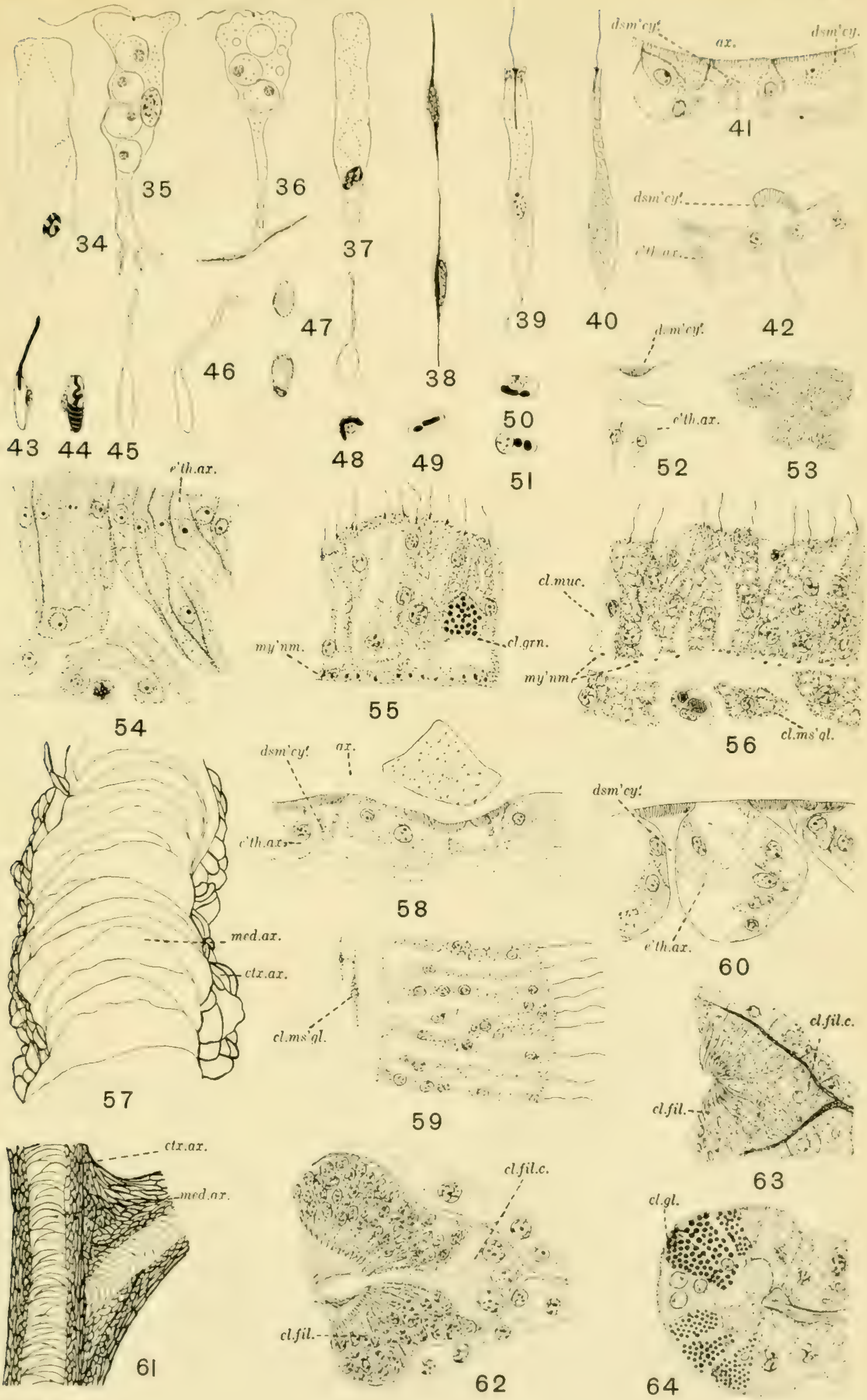
Fig. 52. Desmocyte enveloped in horny matter.
Fig. 53. Face view of the striated ends of six desmocytes.
Fig. 54. Secreting cells of axis epithelium.
Fig. 55. Longitudinal section of endoderm of polyp wall in the region of the column.
Fig. 56. Longitudinal section of endoderm of a polyp chamber.
Fig. 57. Longitudinal section of the axial skeleton, near the tip of a branch.
Fig. 58. Section of a portion of axis with its axis epithelium and desmocytes.
Fig. 59. Longitudinal section through the endoderm of a longitudinal canal.
Fig. 60. Epithelium of axial skeleton, with desmocytes.
Fig. 61. Diagram of a longitudinal section of the axial skeleton to show the nature of the skeleton in the region of a branch.
Fig. 62, 63. Cross sections of two dorsal mesenterial filaments.
Fig. 64. Cross section of a ventral mesenterial filament.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dsm'cy!
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d.m'cy!
e'th.ax.
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e'th.ax.
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my'nm.
cl.grn.
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cl.muc.
my'nm
cl.ms'gl.
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med.ax.
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dsm'cy!
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ctx.ax.
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cl.fil.c.
cl.fil.
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cl.fil.c.
cl.fil.
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cl.gl.

W.M.C. del.

Proceedings of the American Academy of Arts and Sciences.

Vol. XLVIII. No. 21.—September, 1913.

RECORDS OF MEETINGS, 1912–13.

OFFICERS AND COMMITTEES FOR 1913–14.

LIST OF THE FELLOWS AND FOREIGN HONORARY MEMBERS.

BIOGRAPHICAL NOTICES.

Robert Amory. By R. H. Fitz.

Abbott Lawrence Rotch. By R. DeC. Ward.

Charles Robert Sanger. By C. L. Jackson.

STATUTES AND STANDING VOTES.

RUMFORD PREMIUM.

INDEX.

(Title Page and Table of Contents.)

RECORDS OF MEETINGS.

One thousand and sixteenth Meeting.

OCTOBER 9, 1912.— STATED MEETING.

The Academy met at its House.

The PRESIDENT in the Chair.

There were twenty-five Fellows and two guests present.

The following letters were received: — from G. R. Agassiz, S. E. Baldwin, L. A. Bauer, W. H. Bixby, P. W. Bridgman, E. W. Brown, H. L. Chapman, G. H. Chase, R. H. Chittendon, D. F. Comstock, W. H. Dall, A. L. Day, Frederic Dodge, Wilberforce Eames, A. W. Evans, Irving Fisher, Desmond FitzGerald, Simon Flexner, G. W. Goethals, L. J. Henderson, H. L. Higginson, M. A. DeW. Howe, E. P. Joslin, A. L. Kroeber, Waldemar Lindgren, L. S. Marks, S. P. Mulliken, Hanns Oertel, G. H. Palmer, R. S. Peabody, C. P. Putnam, A. P. Rugg, W. B. Scott, M. deKay Thompson, J. E. Thayer, W. J. Tucker, Williston Walker, S. B. Wolbach, F. S. Woods, J. H. Wright, accepting Fellowship; from Svante Arrhenius, J. A. A. J. Jusserand, Augusto Rhigi, H. A. Lorentz, accepting Foreign Honorary Membership; from Louis Cabot, John Fritz, R. B. Richardson resigning Fellowship; from the President and Trustees of the Rice Institute, inviting delegates to the opening of the Institute on October 10, 11 and 12; from the American Antiquarian Society, giving the order of exercises at its centennial celebration to be held October 15 and 16, 1912; from the Académie des Sciences, Lettres et Arts de Bordeaux, inviting delegates to its centenary celebration, November 11 and 12, 1912; from the Secretary of the Société de Pathologie Comparée, inviting delegates to the first International Congress of Comparative Pathology, to be held October 17–23, 1912 at Paris; from the

Director of Congresses of the Panama-Pacific International Exposition, suggesting attendance at the Exposition; a notice of the death of Eduard Strasburger, from his family.

The following deaths were announced by the chair: — William Watson Goodwin, Fellow in Class III., Section 2, and President of the Academy from May, 1903 to May, 1908; Jean Leon Gérôme, Foreign Honorary Member in Class III., Section 4 (died in 1904); Lewis Boss, Fellow in Class I., Section 1; Eduard Strasburger, Foreign Honorary Member in Class II., Section 2, Jules Henri Poincaré, Foreign Honorary Member in Class I., Section 1.

The President appointed Mr. Henry H. Edes as delegate to the celebration of the American Antiquarian Society. He also appointed Professor G. L. Goodale to represent the Academy at Amherst.

The following communication was given:—

Dr. Edwin H. Hall. A Brief Account of the Recent Royal Society Celebration.

One thousand and seventeenth Meeting.

NOVEMBER 13, 1912.

The Academy met at its House.

The PRESIDENT in the chair.

There were thirty Fellows present.

The following letters were read: — from Franz Boas, accepting Fellowship; from the Secretary of the British Academy, inviting the Academy to send a delegate to the third International Congress of Historical Studies to be held in London, April 3–9, 1913; from the President of the Accademia Reale delle Scienze di Torino, giving the conditions of the Avogadro prize; from the Secretary of the Iron and Steel Institute, giving the conditions of the Andrew Carnegie Research Scholarship.

The following deaths were announced by the chair: — Arthur Tracy Cabot, Class II., Section 4; Oliver Clinton Wendell, Class I., Section 1; Horace Howard Furness, Class III., Section 4.

The Corresponding Secretary announced that the Council had granted the use of the Academy Building to the Thursday Evening Club for December 5, 1912; to The Colonial Society of Massachusetts for its regular meetings until further notice; to the M. P. Club (Mathematical-Physical Club) for its regular meetings, the third Monday of the month, until further notice.

The following letter was read:

ACADEMY OF ARTS AND SCIENCES,
Boston, Mass.

CHARLES R. CROSS, Chairman Rumford Committee.

Dear Sir:— Among the bequests in the Will of the late Mrs. Griffith, the second clause and seventh article is as follows:— "To the Academy of Arts and Sciences of Boston all the Rumford mementos, correspondence and papers of the Count Rumford and of his daughter the Countess of Rumford, to be examined and culled by my cousin Baldwin Coolidge, viz: The Count's Study Clock, Coat of Arms, Silver Knife, Fork and Spoon, Seal, Cameo Brooch, Diamond and Topaz Ring, given him by the King of Bavaria, Portrait of the Count painted by the Countess, the Countess' Seal, Portrait of Lady Palmerston, daughter of the first Lord Melbourne, and widow of Earl Cowper, mounted as a Brooch, a small Mother-of-Pearl and Sapphire miniature Opera Glass, a green woolen hearth rug with "C. B." in yellow woven on it, and a small pair of Silver Sugar Tongs, both of which belonged to Sir Charles Blagden."

In accordance with the above I write to say that the Executors are now ready to carry out the above provisions in the Will on receiving a notice of their acceptance.

Awaiting your reply I remain,

Yours truly,

LOAMMI F. BALDWIN,
for the Executors and Trustees.

BALDWIN COOLIDGE, EXECUTOR IN CHARGE OF BEQUEST.
410a Boylston Street, Boston, Mass.

On the recommendation of the Rumford Committee, it was

Voted, That the Academy accept the Rumford mementos mentioned in the letter, and that the Executors be notified.

On the recommendation of the Council, a committee consisting of Henry H. Edes and Robert DeC. Ward, was appointed to consider the amendment of the Statutes in such a way as to add to the Council, *ex officio*, the Chairman of the House Committee and such other Chairmen of Standing Committees as it may seem desirable to have as members of the Council.

The following communications were given: —

Biographical notice of Professor Abbott Lawrence Rotch. By R. DeC. Ward.

The Geographic Origin of Life in Newfoundland and the Magdalen Islands. By M. L. Fernald.

The following papers were presented by title: —

"On the Scalar Functions of Hyper Complex Numbers." Second Paper. By Henry Taber.

"Thermodynamic Properties of Twelve Liquids between 20° and 80° and up to 12000 kgm. per Cm." By P. W. Bridgman.

"The Action of Sulphur Trioxide on Silicon Tetrachloride." By C. R. Sanger and E. R. Riegel. Presented by C. L. Jackson.

One thousand and eighteenth Meeting.

December 11, 1912.

The meeting was held at the House of the Academy.

The President in the chair.

There were Twenty-eight Fellows and guests present.

The following letters were read: — from Elihu Root, accepting Fellowship; from Richard Olney, declining Fellowship; from the Secretary of the ninth International Congress of Zoölogy, to be held at Monaco, March 25–30, 1913, inviting delegates; from the Secretary of The Colonial Society, thanking the Academy for the offer of its building for the meetings of the Society.

The President called attention to Count Rumford's study clock just received as a bequest from Mrs. Griffith.

The following death was announced by the chair: — Sir George Howard Darwin, Foreign Honorary Member in Class I., Section 1.

The following communication was given: —

"Dana's Contribution to Darwin's Theory of Coral Reefs," by Professor W. M. Davis. This was followed by discussion.

Dr. W. S. Bigelow showed for Professor Percival Lowell, a miniature earth or globe, suspended between the two poles of a horse-shoe magnet, which revolved when a lighted candle was placed near it, illustrating the theory of the German scientist, Albert Lotz, that magnetic forces, in conjunction with the sun's heat cause the earth to revolve.

One thousand and nineteenth Meeting.

JANUARY 8, 1913.— STATED MEETING.

The meeting was held at the House of the Academy.

VICE-PRESIDENT Walcott in the chair.

There were thirty-three Fellows present.

The following letters were read: — from Lady Darwin announcing the death of her husband, Sir George Darwin; from the family of Jules Henri Poincaré, announcing his death; from C. S. Hastings accepting Fellowship.

On the recommendation of the Council, it was

Voted, To appropriate from the income of the General Fund: — for House expenses, seven hundred ($700) dollars; for further protection of the Library from fire risk, seven hundred and seventy ($770) dollars.

It was also

Voted, To appropriate from the income of the General Fund, one hundred and fifty ($150) dollars for the use of the Treasurer's office.

The following report of the Committee on the amendment of the Statutes was read and accepted: —

The Committee to whom was referred the Amendment of the Statutes proposed by Dr. Tyler at the November meeting recommend its adoption in the following form: —

The third paragraph of Article I, of Chapter IV is hereby amended by inserting after the word "named" the words "and the Chairman of the House Committee, *ex officio*," so as to read: —

The Councillors, with the other officers previously named and the

Chairman of the House Committee, *ex officio*, shall constitute the Council.

Respectfully submitted,

HENRY H. EDES,
ROBERT DEC. WARD,
Committee.

Boston, 8 January, 1913.

It was

Voted, To amend the Statutes in accordance with the above report.

The following gentlemen were elected Fellows of the Academy:—

In Class I., Section 1 (Mathematics and Astronomy):—

George Cary Comstock, of Madison; Edwin Brant Frost, of Williams Bay.

In Class I., Section 2 (Physics):—

Ernest Fox Nichols, of Hanover; Robert Williams Wood, of Baltimore.

In Class I., Section 3 (Chemistry): —

Wilder Dwight Bancroft, of Ithaca; Bertram Borden Boltwood, of New Haven.

In Class I., Section 4 (Technology and Engineering): —

John Ripley Freeman, of Providence; Alfred Noble, of New York.

In Class II., Section 3 (Zoölogy and Physiology): —

Leland Ossian Howard, of Washington; Charles Atwood Kofoid, of Berkeley; William Emerson Ritter, of Berkeley.

In Class II., Section 4 (Medicine and Surgery): —

David Linn Edsall, of Boston.

In Class III., Section 1 (Theology, Philosophy and Jurisprudence): —

Ezra Ripley Thayer, of Boston.

In Class III., Section 3 (Political Economy and History): —

William Milligan Sloane, of New York; Thomas Franklin Waters, of Ipswich.

In Class III., Section 4 (Literature and the Fine Arts): —

Okakura Kakuzo, of Boston.

The following gentlemen were elected Foreign Honorary Members: —

In Class II., Section 4 (Medicine and Surgery): —

Adam Politzer, of Vienna.

In Class III., Section 2 (Philology and Archaeology): —

Eduard Seler, of Berlin.

The following communications were given: —

"The Study of Infantile Paralysis in Massachusetts by the State Board of Health." By Dr. R. W. Lovett.

"Entomological Studies in connection with Epidemics of Poliomyelitis." By Mr. C. T. Brues.

"Experimental Evidence of the Transmission of Infantile Paralysis." By Dr. M. J. Rosenau.

The following papers were presented by title: —

"Preliminary Study of the Salinity of Sea-water in the Bermudas." By K. L. Mark. Presented by E. L. Mark.

"Cretaceous Pityoxyla from Cliffwood, New Jersey." By Ruth Holden. Presented by E. C. Jeffrey.

One thousand and twentieth Meeting.

FEBRUARY 12, 1913.

The meeting was held at the House of the Academy.

The PRESIDENT in the chair.

There were ninety-eight gentlemen present: — sixty-eight Fellows and thirty guests.

The following death was announced by the chair: —

Francis Blake, Fellow in Class I., Section 2, and Treasurer of the Academy from 1899 to 1905.

Professor C. R. Cross, Chairman of the Rumford Committee, stated the grounds on which the Rumford Medal was to be awarded to Mr. Frederic Eugene Ives.

The President then presented the medals to Mr. Ives.

Mr. Ives on receiving the medals, spoke of the encouragement he felt in the recognition of the value of his work by the Academy and gave an account of his long work in Color Photography, of his struggles and of his successes.

The following papers were presented by title: —

"The Maximum Value of the Magnetization Vector in Iron." By B. O. Peirce.

"Buddhaghosa's Treatise entitled The Way of Salvation, an Analysis of the second Part, on Concentration." By C. R. Lanman.

After the meeting the following exhibits were shown in the reading room: —

F. E. Ives: Specimens of work in color photography, and apparatus for color measurement.

S. I. Bailey: Stellar photographs, showing examples of variable stars having a more rapid rate of variation than any hitherto known.

Outram Bangs (invited by H. B. Bigelow): Birds from the Altai Mountains, collected in the summer of 1912 by Prof. Theodore Lyman, and presented by him to the Museum of Comparative Zoology.

P. W. Bridgman: Specimens of metals illustrating ruptures under pressures up to 30,000 atmospheres.

Henry H. Edes: Mementos of Count Rumford, recently bequeathed to the Academy by Mrs. C. B. Griffith.

L. J. Johnson: Photographs of bent beams, showing novel results of recent experiments.

Alfred C. Lane: Thin sections of igneous rocks, showing variations of grain.

W. C. Lane: Two unique fragments of a book in an otherwise unknown South American language, lately found in the Harvard College Library.

D. C. Lyon: One of the books of Nebuchadnezzar, King of Babylon, recording his building operations in that city about 600 B. C.

G. W. Pierce: The talking arc, reproducing speech transmitted by telephone.

W. T. Sedgwick: Frozen Kansas eggs now two and one-half years old, Chinese and other eggs, and some egg products.

J. E. Wolff: Specimens of a stony meteorite which fell in Arizona last summer.

One thousand and twenty-first Meeting.

FEBRUARY 24, 1913.— SPECIAL MEETING.

A special meeting of the Academy was held at its House, at half past eight o'clock, P. M. in honor of Professor Henri Bergson, of the Collège de France.

Professor Barrett Wendell spoke of the Collège de France as an exponent of the catholicity of the intellectual life; and presented the greetings of the Academy to the distinguished visitor.

Professor Bergson in his address of acknowledgment spoke of the pleasure in meeting a body of scholars and outlined his views of the true function of philosophy.

After the conclusion of Professor Bergson's address a reception was held in the Reading-room. There were present about two hundred Fellows and guests, including ladies.

One thousand and Twenty-second Meeting.

MARCH 12, 1913.— STATED MEETING.

The meeting was held at the House of the Academy.

The PRESIDENT in the chair.

There were twenty-four Fellows and four guests present.

The following letters were read: — from E. B. Frost, W. D. Bancroft, E. R. Thayer, L. O. Howard, D. L. Edsall, E. F. Nichols, R. W. Wood, J. R. Freeman, Okakura-Kakuzo, G. C. Comstock, B. B. Boltwood, Alfred Noble, C. A. Kofoid, W. E. Ritter, and T. F. Waters, accepting Fellowship; from Eduard Seler, accepting Foreign Honorary Membership; from John A. Aiken, declining Fellowship; from the Committee of the International Geological Congress, inviting delegates to its 12th session.

The following deaths were announced by the chair: —

John William Mallet, Fellow in Class I., Section 3; Henry Leland Chapman, Fellow in Class III., Section 4.

On the recommendation of the Council, the following appropriations were made for the ensuing year: —

from the General Fund, $5475. to be used as follows: —

for House expenses	\$1700.
for Library expenses	1800.
for Books, periodicals and binding	1200.
for Expenses of Meetings	200.
for Treasurer's Office	175.
for General Expenses	400.

from the Publication Fund, \$2500. to be used for publication.

from the Rumford Fund, \$1800, to be used as follows:

for research	\$1000.
for periodicals, books and binding	200.
for publication	600.

and to be used at the discretion of the Committee, the balance of available income for the year.

from the Warren Fund, \$500. for the Committee.

An appropriation of \$800. was made from the Publication Fund for publication during the present year.

A proposed amendment to Chapter XI., Article 4, of the Statutes was referred to a Committee consisting of H. H. Edes and J. H. Beale.

The President appointed the Committee on Nominations, consisting of the following Fellows: —

Dr. R. H. Fitz,
Prof. G. F. Swain,
Mr. H. H. Edes.

It was

Voted, To suspend, for the next election, the rule adopted February 8, 1911, restricting the rate of increase of Massachusetts membership of the Academy.

The following letter was presented to the Academy by the Council.

American Academy of Arts and Sciences,
Boston, Massachusetts.

February 4, 1913.

To the Honorable the Senate and House of Representatives of the United States.

The American Academy of Arts and Sciences having learned that a society calling itself the American Academy of Arts and Letters is seeking an incorporation in the House of Representatives and the Senate, desires to enter a protest against the use of the words, American Academy of Arts. The American Academy of Arts and Sciences has been known for more than one hundred and twenty-five years as the American Academy. It has always had a Section of Letters. Benjamin Franklin, George Washington, the Adamses, Winthrop and many other distinguished men have been members: today it includes literary men as well as men in Arts and Science. It fulfils the same purposes as the contemplated Academy, and the taking of the essential part of its name will lead to great confusion in correspondence and in all matters relating to the conduct of a learned Academy.

JOHN TROWBRIDGE, *President*,
CHARLES P. BOWDITCH, *Treasurer*,
HENRY P. WALCOTT, *Vice-President*.

It was remarked that, as the Congress to which this letter was addressed had expired without completing the incorporation of the Academy in question, formal action by the Academy on this letter was unnecessary. It was, however,

Voted, That, if similar occasion shall arise, the officers be instructed to address a similar protest to the proper quarter.

The following paper was presented by title: —

"The Structure of the Gorgonian Coral Pseudo-plexaura crassa Wright and Studer." By W. M. Chester. Presented by E. L. Mark.

The following communication was given: —

"Doctrine of Protection to young Industries, as illustrated by the growth of the American Silk Manufacture." By Professor F. W. Taussig.

Remarks on the subject were made by Howell Cheney, Esq., of South Manchester, Conn.

One thousand and twenty-third Meeting.

APRIL 9, 1913.

The Academy met at the Harvard Medical School.

The PRESIDENT in the chair.

On motion of Dr. Bradford the reading of the records of the last meeting was dispensed with.

A card from the Carnegie Institution of Washington, announcing the death of Dr. John Shaw Billings, Fellow in Class II., Section 4, was presented by the Corresponding Secretary.

Professor R. P. Strong gave an illustrated lecture on the recent Manchurian Epidemic of Pneumonic Plague.

At the conclusion of this paper, remarks were made by Mr. H. L. Higginson as follows: —

Ladies and Gentlemen:

Dr. Strong has told us a deeply interesting tale, and now I will tell you one thing which he cannot tell. He has described his work done under the most difficult circumstances, but has not mentioned the dangers accompanying this work.

Dr. Strong and his colleague went alone to Manchuria, lived in a very dirty town, and fought the terrible disease which threatened their own lives, through infection or through a possible scratch, and also ran constant risk of death at the hands of the Chinese, who hate all work with dead bodies. Dr. Strong and Dr. Teague worked without the usual conveniences of hospitals or the ordinary comforts of life, saved many patients from death, and discovered the means of combating with success this terrible epidemic. It was the work of a hero, and nothing less. One can understand the courage of the fireman as he runs up a ladder to save a woman and her children, or of the soldier in the desperate attack on the enemy. In each case these men have the habit, and perform their work cheered on by the brilliancy of the deed; they do not stop to consider such risks. But in cool blood, through many weeks and under such conditions, to study this fell disease and treat the multitude of patients was a noble act, and we thank Dr. Strong and his colleagues with all our hearts. It was heroism of the highest kind.

Professor F. B. Mallory gave an account of the Pathological Lesion in Whooping Cough and the Relation of the Whooping Cough Bacillus to the Lesion. (Illustrated by lantern slides.)

The following paper was presented by title: —

"On Certain Fragments of the Pre-Socratics: Critical Notes and Elucidations." By W. A. Heidel.

On motion of Professor Webster, it was

Voted, That the thanks of the Academy be given to the members of the Faculty of the Medical School who arranged the exhibitions and have made possible this most interesting and instructive meeting.

One thousand and twenty-fourth Meeting.

APRIL 23, 1913.

The meeting was held at the House of the Academy.

The PRESIDENT in the Chair.

There were fifteen Fellows, with guests present.

Dr. Percival Lowell gave the following paper: —

"The Origin of the Planets."

This was followed by extended discussion on the part of Fellows of the Academy.

One thousand and twenty-fifth Meeting.

MAY 14, 1913.— ANNUAL MEETING.

The Academy met at its House.

The PRESIDENT in the chair.

There were fifty-one Fellows present.

The following letters were read: — from the Reale Accademia delle Scienze, Bologna, giving the conditions of Elia De Cyon prize; from the Institut International de Physique Solvay, Bruxelles, enclosing the Statutes of the Institute.

The annual report of the Council was read: —

REPORT OF THE COUNCIL.

Since the last report of the Council, there have been reported the deaths of nine Fellows: — William Watson Goodwin, Lewis Boss, Arthur Tracy Cabot, Oliver Clinton Wendell, Horace Howard Furness, Francis Blake, John William Mallet, Henry Leland Chapman and John Shaw Billings; and of four Foreign Honorary members: — Jean Léon Gérôme, Eduard Strasburger, Sir George Howard Darwin, and Jules Henri Poincaré.

Three Fellows have resigned: — Louis Cabot, John Fritz and R. B. Richardson.

Sixty-one Fellows have been elected, of which number two have declined Fellowship and one has not replied to his notice of election and six Foreign Honorary Members, of which number one has not yet accepted.

The roll now includes 336 Fellows and 54 Foreign Honorary Members.

The annual report of the Treasurer was read, of which the following is an abstract: —

GENERAL FUND.

Receipts.

Balance, April 1, 1912	$2,035.38	
Investments	2,319.82	
Assessments	2,360.00	
Admissions	560.00	
Sundries	165.00	$7,440.20

Expenditures.

Expenses of Library	$2,800.92	
Expense of House	2,139.36	
Expense of Meetings	184.07	
Treasurer	178.00	
General Expenses of Society	357.51	
Moving	127.75	
Insurance	628.43	
Sundries	226.96	
Interest on Bonds, bought	43.20	
Income transferred to Principal	191.84	
Charged to cancel premium on Bond	45.00	$6,923.04
Balance, April 1, 1913		517.16
		$7,440.20

RUMFORD FUND.

Receipts.

Balance, April 1, 1912	$1,387.92	
Investments	3,070.35	
Sale of Publications	33.75	$4,492.02

Expenditures.

Research	$1,450.00	
Books, periodicals and binding	212.61	
Publication	555.05	
Medals	400.00	
Sundries	1.00	
Income transferred to principal	137.71	$2,756.37
Balance, April 1, 1913		1,735.65
		$4,492.02

C. M. Warren Fund.

Receipts.

Balance, April 1, 1912	$377.34	
Investments	745.84	$1,123.18

Expenditures.

Research	$290.00	
Vault rent, part	4.00	
Interest on Bonds, bought	61.11	
Income transferred to principal	31.03	$386.14
Balance, April 1, 1913		737.04
		$1,123.18

Publication Fund.

Receipts.

Balance, April 1, 1912	$715.35	
Appleton Fund investments	842.06	
Centennial Fund investments	2,432.84	
Sale of Publications	560.35	$4,550.60

Expenditures.

Publications	$3,267 .53	
Sundries: Moving	15 .20	
Vault Rent	12 .50	
Interest on Bonds, bought	49 .55	
Income transferred to Principal	138 .78	$3,483 .56
Balance April 1, 1913		1,067 .04
		$4,550 .06

May 14, 1913.

The following reports were also presented:

Report of the Library Committee.

During the past year the books on Arts and Sciences, the Periodicals and Society Publications, the books on Mathematics and those on Astronomy — these forming the first four of our 32 classes — have been transferred from the stack to the fourth floor of the main building. The space released in the stack has been utilized by rearranging the books of the remaining 28 classes. It is estimated that the available shelf-room will suffice for fifteen years' growth at the present rate.

The question of protection against fire has given the committee serious concern, in view of the close proximity of our stack to the backs of the Boylston Street buildings.

The best remedy was believed to be the substitution of wired glass in the east wall of the stack, and this change has been made at an expense of $757.

Pressure of other work has prevented any progress in the important task of filling gaps in our serial publications. The arrangement of the unbound pamphlets is nearly completed. The folios have been transferred temporarily to the broader shelves of the entrance hall.

A complete set of the Academy publications has been placed in the reading-room, together with the International Catalogue of Scientific Literature.

87 books have been borrowed from the library during the year by 19 persons, including 16 Fellows and 4 libraries. All but one book has been returned for examination, or satisfactorily accounted for.

The number of bound volumes on the shelves at the time of the last report was 32,068. 647 volumes have been added during the past year, making the number now on the shelves, 32,715. This includes 527 received by gift and exchange, 84 purchased by the General Fund, and 36 by the Rumford Fund.

603 volumes have been bound, and 150 have been stamped and plated during the financial year, May 1, 1912 to April 1, 1913, at a cost of $835.45.

The expenses charged to the library for the eleven months ending April 1 are:

Miscellaneous (including $108.75 for cataloguing) . . .	$775.18
Binding	
General Fund	737.45
Rumford Fund	98.10
Purchase of periodicals and books	
General Fund	288.38
Rumford Fund	114.51

The committee begs to remind members of the desirability that copies of their own published works be donated to the library. The value of the library would be greatly increased by a general response to this invitation.

It is the desire of the committee to increase the use of the library by making its resources better known. Suggestions and coöperation in this direction from members of the Academy will be most welcome.

H. W. Tyler, *Librarian.*

May 14, 1913.

Report of the Rumford Committee.

During the present year grants have been made in aid of researches as follows:—

June 5, 1912, to Professor Norton A. Kent of Boston University, for the purchase of a lens to be used in his investigation on the "Effect of the Magnetic Field on the Spectra of Gases, (additional). $375

To Professor Frederick A. Saunders of Syracuse University, for his research, "Spectroscopic Studies in the Ultra-violet Region" 100

October 9, 1912, to Mr. William O. Sawtelle of Harvard University, in aid of his research on the "Spectra of the Light from the Spark in an Oscillatory Discharge" 250

The Committee voted to transfer to Professor Edward L. Nichols of Cornell University the unexpended balance of the appropriation ($100) made to Professor Willard J. Fisher in 1908 for his research on the "Viscosity of Gases," together with the apparatus used by him, as Professor Fisher is not likely to be able to continue the research.

November 13, 1912, to G. W. Ritchey of Pasadena, for the construction of a reflecting telescope employing mirrors with new forms of curves $500

November 13, 1912, as modified May 14, 1913, to Professor Edward L. Nichols of Cornell University, in aid of the research of Mr. W. P. Roop on the "Effect of Temperature on the Magnetic Susceptibility of Gases 250

May 14, 1913, to Frederick G. Keyes of the Massachusetts Institute of Technology, to be used for the payment of assistants in the computation of thermodynamic tables for ammonia 300

It was also voted at this meeting, in accordance with the desire of the Council of the Academy, that an appropriation of $100 be made to Professor Theodore W. Richards to be used in aid of the publication of the Annual International Table of Constants 100

The following papers have been published in Volume 48 of the Proceedings of the Academy with aid from the Rumford Fund since the last annual meeting.

No. 1. "On the Ultra Violet Component in Artificial Light." By Louis Bell.

No. 5. "A Study with the Echelon Spectroscope of Certain Lines in the Spectra of the Zinc Arc and Spark at Atmospheric Pressure." By Norton A. Kent.

No. 9. "Thermodynamic Properties of Liquid Water to 80° and 12000 kgm." By Percy W. Bridgman.

No. 15. "An Electric Heater and Automatic Thermostat." By Arthur L. Clark.

The Committee has also prepared and caused to be printed a pamphlet Supplement to the publication entitled "The Rumford Fund" published in 1905, which contains the record of the awards of the Premium and of researches and papers aided from the Fund to the close of the year of the Academy ending May 8, 1912, together with some other matters of permanent interest.

The necessary photographs or other fac-simile copies of the inscriptions upon all the earlier Rumford Medals having been secured, replicas of the medals will be made shortly.

Reports of progress in their several researches have been received from the following persons: P. W. Bridgman, W. W. Campbell, A. L. Clark, D. F. Comstock, H. C. Hayes, L. R. Ingersoll, N. A. Kent, F. E. Kester, G. N. Lewis, C. E. Mendenhall, E. F. Nichols, E. L. Nichols, J. A. Parkhurst, T. W. Richards, G. W. Ritchey, M. A. Rosanoff, F. A. Saunders, W. O. Sawtelle, M. deK. Thompson, F. W. Very, R. W. Wood.

At the meeting of November 13, 1912, the Committee voted to recommend to the Academy the acceptance of the bequest of the late Mrs. Griffith described in a letter of Loammi F. Baldwin, Esq., representing her executors and trustees, dated October 8, 1912.

At the meeting of February 12, 1913, it was unanimously voted for the first time and at the meeting of March 12, 1913 for the second time to recommend to the Academy that the Rumford Premium be awarded to Professor Joel Stebbins of the University of Illinois for his development of the selenium photometer and its application to astronomical problems.

Chas. R. Cross, *Chairman.*

May 14, 1913.

Report of the C. M. Warren Committee.

The C. M. Warren Committee begs to report that one grant has been made during the year of $140 to Professor Arthur B. Lamb of Harvard University, for work on the rhodiumamines. It now has at its disposal for the current year an unexpended balance of $860. During the year Professor H. G. Byers has published two papers on the passivity of iron, the work on this subject having been carried on in part through the grants from the Warren Fund. Reports of progress have been received from Dr. Gilpin and Professor Lamb and Dr. Washburn.

The Committee has in preparation a circular regarding the purposes of the Warren Fund which it is hoped will occasion renewed interest in the opportunities which it affords for the support of research.

H. P. Talbot, *Chairman.*

May 14, 1913.

Report of the Publication Committee.

Between April 1, 1912, and April 1, 1913, there were published one number of Volume XLVII (No. 22) and seventeen numbers of Volume XLVIII of the Proceedings. There were also published two obituary notices. The total publication for this period amounted to 771 pages. The expense of publishing three of these numbers and a part of a fourth number has been assumed by the Rumford Committee.

There was available for the use of the Publication Committee an unexpended balance from last year of $428.70, an appropriation of $2500, and an additional appropriation of $800, and an amount of $560.35 from the sales of publications — in all, $4289.05 from the Publication fund and sales. Bills against this appropriation to the amount of $3267.53 have been approved by the Chairman. This leaves an unexpended balance of $1021.52.

Bills aggregating $555.05, incurred in publishing papers on light and heat, have been referred to the Rumford Committee for payment in accordance with their authorization.

G. W. Pierce, *Chairman.*

May 14, 1913.

Report of the House Committee.

The House Committee submits the following Report for the year 1912–1913: The Committee had at its disposal a balance of $108.54 from last year. The appropriations by the Academy for the past year have been $2240, making a total of $2348.54 for the use of the Committee. Of this sum, $2348.32 has been expended. These expenditures include approximately $500 which may properly be regarded as unusual expenditures incidental to the establishment of the Academy in its new house. The larger of the latter items are those for window screens, the electric lamp bulbs for the entire building, the installation of a telephone and electrically operated lock on the front doors, alterations in the electric lighting of the stack and stack rooms, additional shelving and cupboards, a residual payment of rental at 711 Boylston St., and the cost of moving. While certain additions to equipment, and some repairs, will necessarily be made every year, the amount of expenditures for equipment should be materially less than during the past year.

The Academy has held seven regular and two special meetings in the building since May, 1912. The small rooms have also been used for eight Council and ten committee meetings.

The Council has authorized the use of the building by the Thursday Evening Club, and for a meeting of teachers of geology on one occasion, and by The Colonial Society and the Mathematical and Physical Club for their regular meetings. The Colonial Society has held four meetings in the late afternoon and the "M. P. Club" three meetings in the evening. Both of these organizations have made payments, determined by the Treasurer, sufficient to reimburse the Academy for the cost of light, heat and attendance.

The present janitor, who with his wife occupies the janitor's apartment in the building, is the third employed during the year. He is, at present, rendering excellent service.

The experience of the year has shown that the Academy building is, in most respects, well adapted to meet the needs of the Academy. The provisions for the use of the lantern in the meeting-room are not as satisfactory as could be desired, especially with respect to the use of the screen, which is rather unsightly in appearance, suggestive of an emergency rather than a permanent arrangement. The Committee expects to provide a better equipment as soon as the necessary expenditures seems to be warranted and the best device can be selected.

With a desire to avoid unnecessary duplication of effort, the House Committee has taken over the charge of the simple collations served after the evening meetings of the Academy, which are provided from funds under the charge of the Committee on Meetings. The House Committee has not undertaken, and would prefer not to undertake, to provide for the more elaborate collations necessary on special occasions.

The building has been open during the year from 8 A. M. to 5 P. M. except on Saturdays, when it has been closed at 1 P. M. No suggestions have been received from Fellows of the Academy regarding more acceptable hours, but such suggestions would be welcomed.

The Committee desires to express its sense of obligation to the Assistant Librarian, Mrs. Holden, for her constant coöperation with the work of the Committee and her care of details for which it would otherwise have been very difficult to provide. Mr. Charles Wilder has also coöperated most helpfully with the work of the Committee.

H. P. Talbot, *Chairman.*

May 14, 1913.

On the recommendation of the Rumford Committee, it was

Voted, To award the Rumford Premium to Professor Joel

Stebbins, of the University of Illinois, for his development of the selenium photometer and its application to astronomical problems.

The following report of the Committee on the Amendment of the Statutes was read and accepted: —

Boston, Mass., 14 May, 1913.

The undersigned, a Committee to which was referred an amendment to the Statutes offered at the Stated Meeting in March, has attended to the duty assigned to it, and begs leave to report as follows:

Your Committee recommends that there be added to Article 4 of Chapter XI., at the end, the words "The Council, in its discretion, by a duly recorded vote, may delegate its authority in this regard to one or more of its members."

If the amendment is adopted by the Academy, the Article will then read as follows:

"Article 4. No report of any paper presented at a meeting of the Academy shall be published by any Fellow without the consent of the author; and no report shall in any case be published by any Fellow in a newspaper as an account of the proceedings of the Academy without the previous consent and approval of the Council. The Council, in its discretion, by a duly recorded vote, may delegate its authority in this regard to one or more of its members."

Respectfully submitted,

Henry H. Edes,
Joseph H. Beale,
Committee.

On the recommendation of the Committee, it was

Voted, To amend the Statutes in accordance with the recommendation contained in the foregoing report.

On motion of the Treasurer, it was

Voted, To appropriate from the income of the General Fund, the sum of $112., to pay for accident insurance for 1912–13, and 1913–14.

On motion of the Treasurer, it was

Voted, That the Annual Assessment be ten (10) dollars.

The Council reported that in accordance with the provisions of Article 1 of Chapter IX of the Statutes, the Reverend Dr. Timothy Dwight, a Fellow in Class III., Section 2, and the Reverend Drs. William Wallace Fenn, Edward Caldwell Moore, George Herbert Palmer, James Hardy Ropes, William Jewett Tucker and Williston

Walker, Fellows in Class III, Section 4, had been transferred to Class III., Section 1.

A marble bust of Dr. Jacob Bigelow and an inkstand used by him were presented to the Academy by his grandson, Dr. William Sturgis Bigelow.

The President in receiving the gifts for the Academy made the following remarks: —

"Dr. Jacob Bigelow was President of this Academy from 1846 to 1863, and was the eighth in a distinguished line of Presidents — James Bowdoin, John Adams, Edward A. Holyoke, John Quincy Adams, Nathaniel Bowditch, James Jackson, and John Pickering. Dr. Bigelow was an eminent writer on botanical and medical subjects; and his great services to science and to the community are set forth in volume 14 of the Proceedings of the Academy. He was greatly interested in technological education and was the first to advocate the foundation of an Institute of Technology in Boston. Dr. Bigelow was also Rumford Professor in Harvard University; and it seems very fitting that the Academy should receive these remembrances of him at this meeting, when the Rumford medals are to be conferred."

In moving the thanks of the Academy, Professor A. G. Webster hoped that similar gifts in honor of distinguished members would be received.

It was

Voted, That the thanks of the Academy be given to Dr. W. S. Bigelow for his valuable gifts.

The Rumford Medal which had been awarded to Professor James M. Crafts, was presented to him in his absence through Professor Charles R. Cross.

The following draft of certain sections in the tariff act, was sent to the Academy by Francis E. Hamilton of 32 Broadway, New York. It was presented to the Council and was referred to a Committee of one — Professor F. W. Taussig.

SUBSTITUTE FOR SECTIONS 517–519–650–714–715.

Books, maps, music, engravings, photographs, etchings, bound or unbound, and charts, which shall have been printed more than twenty years at the date of importation, and all hydropgraphic charts, and publications issued for their subscribers or exchanges, by scientific and literary associations or academies, or publications of individuals for gratuitous private circulation, and public documents issued by foreign governments; ALSO, books, maps, music, photographs, etchings, lithographic prints, and charts specially imported not more than two copies in any one invoice, in good faith for the use and by order of any society or institution incorporated or established solely for religious, philosophical, educational, scientific, or literary purposes, or for the encouragement of the fine arts, or for the use and by order of any college, academy, school, or seminary of learning in the United States, or any State or Public Library; ALSO, philosophical and scientific apparatus, utensils, instruments, and preparations including bottles and boxes containing the same, specially imported in good faith for the use and by order of any society or institution incorporated or established solely for religious, philosophical, educational, scientific, or literary purposes, or for the encouragement of the fine arts, or for the use and by order of any college, academy, school, or seminary of learning in the United States, or any State or Public Library; ALSO, works of art, drawings, engravings, photographic pictures, and philosophical and scientific apparatus, for use temporarily for exhibition and in illustration, promotion and encouragement of art, science, or industry in the United States; ALSO, works of art, collections in illustration of the progress of the arts, sciences, or manufactures, photographs, works in terra cotta, parian, pottery, or porcelain, antiquities and artistic copies thereof in metal or other material, imported in good faith for exhibition at a fixed place by any State or by any Society or institution established for the encouragement of the arts, science, or education, or for a municipal corporation, and all like articles imported in good faith by any society or association, or for a municipal corporation, for the purpose of erecting a public monument. Any and all of the above imported in good faith only for the purposes mentioned and not for sale, shall be admitted free of duty upon oath from an authorized officer of the society, institution, college, academy, school, seminary of learning, corporation, association, and without bond, under regulations to be prescribed by the Secretary of the Treasury: PROVIDED, that the privileges of this and the preceding section shall not be allowed to associations or corporations engaged in or connected with business of a private or commercial character.

The following report was given by Professor Taussig.

The draft submitted to the Academy by Francis E. Hamilton of New York of certain sections in the tariff act relating to the free importation of books, scientific apparatus and works of art, is, in the main, a consolidation of scattered sections as they now stand in the tariff act of 1909. The only changes of substance are in the direction of making more liberal certain provisions concerning the importation of works of art, and the like, for temporary exhibition. These are to be brought in without requirement of a bond, and without requirement that they shall be in charge of professional artists or lecturers. I see no reason why the Academy should not allow its name to be used in favor of the proposed rearrangement, and recommend that it allow the use of its name.

F. W. Taussig.

May 14, 1913.

It was then

Voted, to recommend the proposition made by Mr. Hamilton.

The annual election resulted in the choice of the following officers and commitees: —

John Trowbridge, *President.*

Elihu Thomson, *Vice-President for Class I.*

Henry P. Walcott, *Vice-President for Class II.*

A. Lawrence Lowell, *Vice-President for Class III.*

Edwin H. Hall, *Corresponding Secretary.*

William Watson, *Recording Secretary.*

Charles P. Bowditch, *Treasurer.*

Harry W. Tyler, *Librarian.*

Councillors for Four Years.

Desmond FitzGerald, of Class I.

John Collins Warren, of Class II.

George L. Kittredge, of Class III.

Finance Committee.

John Trowbridge, Gardiner M. Lane,

John Collins Warren.

Rumford Committee.

Charles R. Cross,
Edward C. Pickering,
Arthur G. Webster,
Erasmus D. Leavitt,
Elihu Thomson,
Louis Bell,
Arthur A. Noyes.

C. M. Warren Committee.

Henry P. Talbot,
Charles L. Jackson,
Arthur A. Noyes,
Walter L. Jennings,
Gregory P. Baxter,
James F. Norris,
William H. Walker.

Publication Committee.

George W. Pierce, of Class I.
Walter B. Cannon, of Class II.
Albert A. Howard, of Class III.

Library Committee.

Harry W. Tyler,
Harry M. Goodwin, of Class I.
Samuel Henshaw, of Class II.
William C. Lane, of Class III.

House Committee.

Henry P. Talbot,
Louis Derr,
Hammond V. Hayes.

Committee on Meetings.

The President,
William M. Davis,
The Recording Secretary,
Wallace C. Sabine,
Arthur Fairbanks.

Auditing Committee.

Eliot C. Clarke,
Worthington C. Ford.

The following gentlemen were elected Fellows of the Academy,— a printed list of nominees having been sent to all Voting Fellows with the notice of the April meeting, in accordance with Chapter III., Article 3 of the Statutes: —

In Class I., Section 1 (Mathematics and Astronomy): —

George David Birkhoff, of Cambridge; Julian Lowell Coolidge, of Cambridge; Edward Vermilye Huntington, of Cambridge.

In Class I., Section 2 (Physics): —

Henry Crew, of Evanston, Ill.; Norton Adams Kent, of Cambridge.

In Class I., Section 3 (Chemistry): —

Arthur Dehon Little, of Brookline; William Albert Noyes, of Urbana, Ill.

In Class I., Section 4 (Technology and Engineering): —

Harold Pender, of Boston.

In Class II., Section 4 (Medicine and Surgery): —

Henry Asbury Christian, of Boston; Frank Burr Mallory, of Brookline; Edward Hall Nichols, of Boston.

In Class III., Section 1 (Theology, Philosophy and Jurisprudence): —

Frederick Perry Fish, of Brookline; William Lawrence, of Boston; Henry Newton Sheldon, of Boston; Moorfield Storey, of Boston.

In Class III., Section 2 (Philology and Archaeology): —

Charles Hall Grandgent, of Cambridge; Charles Burton Gulick, of Cambridge; Hans Carl Gunther von Jagemann, of Cambridge; James Richard Jewett, of Cambridge; Edward Kennard Rand, of Cambridge.

In Class III., Section 3 (Political Economy and History): —

Charles Jesse Bullock, of Cambridge; Davis Rich Dewey, of Cambridge; Edwin Francis Gay, of Cambridge; Albert Bushnell Hart, of Cambridge; Charles Homer Haskins, of Cambridge; William Bennett Munro of Cambridge.

In Class III., Section 4 (Literature and the Fine Arts): —

George Whitefield Chadwick, of Boston; Samuel McChord Crothers, of Cambridge; Franklin Bowditch Dexter, of New Haven, Conn.; Arthur Foote, of Brookline; Daniel Chester French, of

Cambridge; Robert Grant, of Boston; John Torrey Morse, Jr., of Boston; Bela Lyon Pratt, of Boston; George Edward Woodberry, of Beverly.

The following communication was given: —

Dr. Theodore Lyman. "A Journey in the Highlands of Siberia."

The following papers were presented by title: —

"Passivity of Iron under Boiler Conditions." By H. G. Byers and F. T. Vores. Presented by H. P. Talbot.

"Relation between the Magnetic Field and the Passive State of Iron." By H. G. Byers and S. C. Langdon. Presented by H. P. Talbot.

Contributions from the Gray Herbarium. New Series XLI. I. A Redisposition of the Species heretofore referred to Leptosyne. II. A Revision of Encelia and some Related Genera. By S. F. Blake.

Contributions from the Gray Herbarium. New Series XLII. I: A Key to the Genera of the Compositae Eupatoricae. By B. L. Robinson. II: Revisions of Alomia, Ageratum, Ctenopappus and Oxylobus. By B. L. Robinson. III: Some new Combinations required by the International Rules. By C. A. Weatherby. IV: On the Graminae collected by Professor Morton C. Peck, in British Honduras, 1905–1907. By F. F. Hubbard. V: Diagnoses and Transfers among the Spermatophytes. By B. L. Robinson.

BIOGRAPHICAL NOTICES.

ROBERT AMORY.

Robert Amory A. M., M. D., was born in Boston, May 3, 1842, and died in Nahant, Aug. 27, 1910. He was graduated from Harvard College in 1863 and from the Harvard Medical School in 1866. After the medical degree was conferred he continued his studies for a year in Europe and while in Paris became especially interested in the experimental study of the action of drugs.

He began the practice of medicine in Brookline and soon opened a small laboratory for experimental research in the stable adjoining his residence in Longwood. He then interested a number of medical students in physiological investigations, especially with reference to the action of medicines. Dr. Edward H. Clarke, professor of materia medica in the Harvard Medical School encouraged his undertaking and recommended his appointment to a lectureship on the physiological action of drugs. Dr. Amory later opened a larger and more convenient laboratory in La Grange St., Boston, for the use of his students and for the benefit of those physicians who were interested in experimental methods of biological study. A centre thus was established for advanced students of medical problems and the laboratory became the meeting place of the Boston Society of Medical Sciences of which Dr. Amory was one of the founders. During this early period of his career were published his researches on hydrocyanic acid, caffein and thein, absinth, the bromide of potassium and ammonium and on nitrous oxide. In connection with Dr. S. G. Webber he published a paper on veratrum viride and veratria, and, with Dr. E. H. Clarke, a monograph on the physiological and therapeutical action of the bromide of potassium and the bromide of ammonium.

His reputation as a scientific investigator along physiological lines thus being established he was appointed in 1872 lecturer on physiology at the Medical School of Maine and in the following year was made professor of physiology in that institution. At this time he translated the lectures in physiology given by Professor Küss of the university of Strasbourg. He also accepted the editorship of the section on poisons in the third edition of the Medical Jurisprudence of Wharton and Stillé. In connection with Professor E. S. Wood, and later with Dr. R. L. Emerson he edited the chapters on poisons in the subsequent editions of this treatise.

He was elected a Fellow of the American Academy of Arts and Sciences in 1871 and in 1875 presented a communication on photographs of the solar spectrum which he had made with the assistance of Mr. J. G. Hubbard who then was working in his laboratory. Communications also were presented by him on the action of dry, silver bromide collodion to light rays of different frangibility and on the theory of absorption bands in relation to photography and chemistry.

In 1874 he resigned his professorship and devoted his time largely to medical practice and to such laboratory studies as his various obligations would permit. He was appointed the medical examiner of his district, held various positions in the medical staff of the Massachusetts Volunteer Militia and in 1880 was President of the National Decennial Convention for the Revision of the United States Pharmacopoeia. During this period he contributed a paper on the haematinic properties of dialyzed iron, with Dr. G. K. Sabine made a study of an epidemic of typhoid fever in Brookline and, in 1886, published a treatise on Electrolysis and its therapeutical and surgical use.

For a number of years he had been in the habit of spending his summers in Bar Harbor, Me., where he also practised medicine. Then having become interested in the telephone he was persuaded to withdraw from medical practice and to devote himself to commercial affairs. He identified himself with telephone, electricity and gas, and became President and Manager of the Brookline Gas Company, from which he retired in 1898.

Dr. Amory, while engaged in scientific pursuits, was an earnest, diligent worker, with high ideals. He gave liberally of his time, the freedom of his laboratory and apparatus for the encouragement of others. He was a pioneer in the introduction into this country of the study of the physiological action of drugs by experiments on animals and apart from his individual researches thus contributed to the advancement of exact knowledge.

R. H. Fitz.

ABBOT LAWRENCE ROTCH.

Abbott Lawrence Rotch was born in Boston, January 6, 1861, the son of Benjamin Smith and Anna Bigelow (Lawrence) Rotch. He was graduated from the Massachusetts Institute of Technology (S.B.) in 1884. In 1891 Harvard recognized the importance of the work which he had already accomplished by bestowing upon him the honorary degree of A.M. From 1888 to 1891, and again from 1902 to 1906, he held the appointment of assistant in meteorology at Harvard, a position which involved no teaching and in which no salary was paid. In 1906 he was appointed professor of meteorology, an honor which he prized very highly, and which gave him the position on the teaching staff of the university to which he was in every way fully entitled. He was the first professor of meteorology who has occupied that position at Harvard, and he served in this professorship without pay. In the year 1908–09, at the request of the department of geology and geography, he generously put the splendid instrumental equipment and library of Blue Hill Observatory at the service of the university, by offering a research course ("Geology 20f") to students who were competent to carry on investigations in advanced meteorology. This action on the part of Professor Rotch gave Harvard a position wholly unique among the universities of the United States. It brought about a close affiliation, for purposes of instruction and of research, between the university and one of the best-equipped meteorological observatories in the world. To his work as instructor Professor Rotch gladly gave of his time and of his means. He fully realized the unusual advantages which he was thus enabled to offer those students who were devoting themselves to the science of meteorology, and the experience of the men who had the privilege of his advice and help in the work at Blue Hill shows clearly how much they profited by this opportunity. Only a short time before his death he had expressed the wish to bring about a still closer connection, for purposes of instruction, between the university and Blue Hill Observatory. He thus showed his appreciation of the importance of the new field of work which he had undertaken.

While thus planning still further usefulness for his observatory; in the midst of a life singularly active; with an ever-widening sphere of scientific influence and a constantly increasing importance of his contributions to meteorology, Professor Rotch died suddenly in Boston on April 7, 1912, in the fifty-second year of his age. His wife, who was

Miss Margaret Randolph Anderson, of Savannah, Ga., and three children survive him.

Professor Rotch early developed that absorbing interest in meteorology which caused him to devote his life to the advancement of that science. Possessed of large means, he preferred to work persistently, and not infrequently to undergo discomfort and hardship in his chosen field of research, rather than to live a life of ease. Realizing the need of an institution which could be devoted to the collection of meteorological observations, and to meteorological research, free from any entanglements, he established, in 1885, Blue Hill Observatory. This was first occupied by Mr. Rotch and his observer, Mr. W. P. Gerrish, on February 1, 1885. This observatory he not only equipped and maintained until his death, but he made provision in his will for having the work there carried on without a break. Blue Hill Observatory is to-day one of the few private meteorological observatories in the world, and there is not one which is better equipped. In fact, it is probably safe to say that there is no private scientific establishment which is better known for the high standard of its work. The Blue Hill Observatory was, with the exception of the municipal meteorological station in New York, the first in this country to be equipped with self-recording instruments, and it is to-day one of the comparatively few in the world where nearly every meteorological element is continuously recorded. Beginning with 1886, hourly values have been printed. Professor Rotch took a splendid pride in his observatory, and in its equipment, and his library, to which he devoted constant care, was one of the most complete and valuable in the world.

Professor Rotch early realized that the advance of meteorology must come through a study of the free air, and with keen and prophetic judgment he planned and carried out the remarkable series of investigations which have made Blue Hill so famous. He secured assistants who were well fitted to carry out the researches which he planned and supervised. He thus showed his ability to judge the value of men, as well as his capacity to organize the work for them to do. Mr. H. H. Clayton became a member of the Observatory staff in 1886, and served as observer and meteorologist, with some interruptions, for twenty-three years. His work brought distinction to himself and to the observatory. Mr. S. P. Fergusson joined the staff in 1887, and remained there until 1910. Many new instruments were devised by him, and perfected with care and success. Mr. A. E. Sweetland died after eight years of service and was succeeded, in 1903, by Mr. L. A. Wells, who is now observer-in-charge. Year after year the Blue Hill publications have contained results of far-reaching importance. It is not an exaggeration to say that much of the recent rapid advance of meteorological science is due to the pioneer work which was done at Blue Hill.

Under an arrangement entered into between Blue Hill Observatory and the Astronomical Observatory of Harvard College, Professor Rotch was, for nearly twenty-five years, closely associated with the latter institution. All of the observations made at Blue Hill were published in the *Annals* of the Harvard Observatory, and fill eight quarto volumes. The international form of publication, and metric units, were first used in the United States in the publications of the Blue Hill Observatory.

It was one of Professor Rotch's most striking characteristics that he never neglected any opportunity which might help him to keep his observatory not only abreast of the times but ahead of the times. He thought nothing of the time and the expense of taking a trip to Europe in order to attend some scientific meeting, meteorological or aeronautical, if he believed, as he most firmly did, that he might by so doing gain inspiration and new ideas. Few scientific men are so regular in their attendance at congresses and meetings; few contribute so much that is new, or gain as much inspiration as he did at such gatherings. It was not the blind following of the dictates of his New England conscience that prompted him to be so regular in his meetings with his scientific colleagues. His motive was a higher one than that. It was his absorbing desire to advance his science by every means within his power. An English colleague (Dr. H. R. Mill) has written of him that he was "the most widely travelled and best-known of meteorologists. It would be hard to name a meteorological observatory or institution in any country which he had not visited, or a meteorologist with whom he was not on terms of personal friendship.... He was not only a name but a friend to all his colleagues in the meteorological world." The list of scientific bodies of which he was a member was a long one, but every one of them gained much from his membership and from his presence at its meetings. He was regular in his attendance; always ready to contribute papers; always modest in his estimate of the importance of his own work; always generous in his appreciation of the work of others; always ready with a word of sympathy, or encouragement, or fellowship.

The productivity of Blue Hill Observatory has been remarkable, especially when it is remembered that this activity was the result of the support and inspiration of one man. The study of cloud heights, velocities, movements, and methods of formation, at Blue Hill, was one of the most complete investigations of the kind ever undertaken. The first series of measurements in America of the height and velocity of clouds, by trigonometrical and other methods, was made at Blue Hill in 1890–91. These measurements were repeated in 1896–97, as a part of an international system.

It was at Blue Hill that the modern methods of sounding the air by means of self-recording instruments lifted by kites were first developed and effectively put into practise (1894), methods which have now been adopted by meteorological services and scientific expeditions in all parts of the world. The use of cellular kites flown with steel wire and controlled by a power windlass originated at Blue Hill. Grants for carrying on this kite work were obtained from the Hodgkins Fund. The success of this exploration of the free air at Blue Hill led, more than anything else, to the establishment of the *Observatoire de la Météorologie dynamique* at Trappes, under the direction of M. Léon Teisserenc de Bort, and of the *Aeronautisches Observatorium* of the Royal Meteorological Institute, near Berlin, under Professor Richard Assmann.

It was Rotch who, in 1901, during a voyage across the Atlantic, first obtained meteorological observations by means of kites flown from the deck of a moving steamer, thus indicating the feasibility of a new way of securing information concerning the conditions of the free air over oceans and lakes. It was Rotch who, in 1904, secured the first meteorological observations by means of sounding balloons from heights of 5 to 10 miles over the American continent, and who, in 1909, made the first trigonometrical measurements of the flight of pilot balloons in the United States. In 1905–06 he joined his colleague, Teisserenc de Bort, in fitting out and taking part in an expedition to explore the tropical atmosphere over the Atlantic Ocean by means of kites and pilot balloons, an undertaking which resulted in the collection of important data regarding the temperatures and movements of the upper air, and especially concerning the existence of the anti-trades. But Rotch was not content with merely sending up kites and balloons. His enthusiasm in the study of the free air, and his desire to visit the mountain observatories of the world, led him to become a mountain climber of no mean ability. He ascended to the summit of Mont Blanc at least five times, and in South America and elsewhere he himself made meteorological observations at considerable altitudes on mountains, and carefully observed the physiological effects of the diminished pressure. He also took part in several balloon ascents, taking important observations during these trips, notably on that of October 24, 1891, starting from Berlin, when he carried out a series of comparisons between the sling thermometer and Assmann's aspiration thermometer. He was a member of more than one solar eclipse expedition. His studies of eclipse meteorology are among the most complete which have been made. Among his many contributions to the advancement of meteorology must also be mentioned his invention of an instrument for determining the true direction and velocity of the wind at sea.

Professor Rotch was naturally intensely interested in the recent rapid development of aeronautics. His earlier training at the Massachusetts Institute of Technology, and his untiring zeal in the exploration of the upper air, combined to give him this interest. He turned his attention largely in that direction of late years. It was characteristic of him that, not content with the mere collection of data, and with investigations of theoretical interest, he always strove to make these results of practical use. Thus, soon after the establishment of his observatory, the issue of local weather forecasts was begun, and one of the last things which he published (in association with Mr. A. H. Palmer) was a set of "Charts of the Atmosphere for Aeronauts and Aviators" (1911), a pioneer work, embodying many of the results of observations made at Blue Hill in a practical form for the use of airmen.

Professor Rotch originally suggested the issue of a cyclostyle weather map, and himself paid the expenses of the first publication of such maps, which was on May 1, 1886, at the Boston office of the United States Signal Service, Sergt. O. B. Cole, who was then in charge of the station, cooperating in the undertaking. This was the first printing of a synoptic chart outside of the Central Office at Washington, and the Signal Service soon extended this method of issuing maps to several of its other stations. The local weather predictions were first made at Blue Hill on July 1, 1886. Their superiority over the Washington predictions made by the Signal Service was soon apparent, and in February, 1887 (*American Meteorological Journal*), Professor Rotch suggested that the United States Signal Service "discontinue its Washington predictions by having the district indications made at the chief station of each district by a competent person and from the data of the synoptic charts." This plan was soon thereafter adopted by the Signal Service at Boston, and was later generally extended over the country.

Forecasts made at Blue Hill were first published in the Boston Evening Transcript from January 4, 1887, until March 7, 1887. From May 2, 1887, until April 30, 1888, and from January 1, 1889, until

October 16, 1891, the Blue Hill forecasts were given to the Associated Press and published in the papers of Boston and neighboring cities. Since October 16, 1891, forecasts have been signaled by flags from Blue Hill, and since July 9, 1911, local forecasts have been displayed at the Observatory gate daily.

Professor Rotch's list of published papers and books comprises 183 titles. These cover a wide range of subjects, by no means strictly confined to meteorology, and show most emphatically how varied were their author's interests; how extended was his reading; how alert and progressive he was in all he undertook. These 183 titles in themselves furnish a satisfactory outline of the development of meteorological science during the past 25 years. In addition to the "Charts of the Atmosphere" just referred to, he published two other books, "Sounding the Ocean of Air," (1900) and "The Conquest of the Air" (1909).

Professor Rotch gave his support freely to a large number of scientific societies and undertakings. He was one of the pioneer and most enthusiastic members of the New England Meteorological Society. He was, for more than ten years (1886–96), one of the associate editors and one of the mainstays of the *American Meteorological Journal*, which did a unique work for American meteorology.

He was elected a Fellow of the American Academy of Arts and Sciences March 14, 1888, and served as Librarian from May 10, 1899, until his death. He was a member of the Astronomical and Astrophysical Society of America; a member and trustee of the Boston Society of Natural History; a member of the American Philosophical Society, of the Physical Society of London, of the International Solar Commission, of the International Commission for Scientific Aeronautics, of the International Meteorological Committee; fellow and later Honorary Member of the Royal Meteorological Society (London); member of the Société Météorologique de France, of the Deutsche Meteorologische Gesellschaft, of the Oesterreichische Gesellschaft für Meteorologie, corresponding member of the Deutscher Verein für Förderung der Luftschiffahrt, and member of many other societies.

He was lecturer at the Lowell Institute, in Boston, in 1891, and again in 1898. He was a member of the International Jury of Awards at the Paris Exposition (1889), and was then made a Chevalier of the Legion of Honor. He received the Prussian Orders of the Crown (1902) and Red Eagle (1905) of the Third Class in recognition of his services in advancing the knowledge of the atmosphere. Further evidence of the high regard in which his scientific work was held abroad was his selection, by the French ministry of public instruction, as exchange professor at the Sorbonne for the year 1912–13. The official letter announcing this selection arrived in this country within a very few days after Professor Rotch's death.

He was a pioneer in a new science; an investigator, whose name is known wherever meteorological work is done; a loyal teacher who served without salary; a generous benefactor, who left to the university an enduring monument of his enthusiasm and untiring devotion to the science which he himself did so much to advance. His life and labor have been an inspiration to his scientific colleagues everywhere, but especially to those who were most closely associated with him in the work of his observatory, and in the department of the university of whose staff he was a valued member.

Robert De C. Ward.

CHARLES ROBERT SANGER

The most important achievement of Charles Robert Sanger grew out of an incident, which occurs in the life of almost every young chemist. While he was Assistant in Chemistry at Harvard College, Professor H. B. Hill was consulted by a literary colleague in regard to a number of cases of obscure poisoning in his family. At first he suggested that they might be due to carbonic oxide from the furnace and referred the question for investigation to Sanger, who found however that the air of the house was free from carbonic oxide, and therefore turned his attention to the other surroundings of the family, when it appeared the wall papers were heavily charged with arsenic, and, after these had been removed, the unpleasant symptoms gradually disappeared. In this way Sanger's attention was called to the relation of arsenic to common life, but instead of contenting himself with the study of this particular case, as most men would have done, he took up the general subject, made this field of research especially his own, and produced in it his most important additions to the science.

In attacking the subject he determined, with characteristic love of truth, to place it on a secure experimental foundation by looking for arsenic in the excreta of people suffering from the disorders commonly attributed to poison from wall papers. Before doing this however it was necessary to improve the methods of testing for arsenic, so that the quantity of poison could be detected with accuracy, even when it was present in very minute amounts. Owing to its frequent use in criminal cases very delicate tests for arsenic had been already worked out, but these showed only its presence or absence, not how much existed in the object tested; for further development therefore Sanger adopted the best of these — the Berzelius-Marsh test — in which the arsenic was detected by a stain (mirror) on a capillary tube; and his improvement consisted in producing all mirrors under identical conditions, when by comparing that from the object under examination with a set made from known weights of arsenic the quantity could be determined with surprising accuracy. Armed with this delicate quantitative method he studied the amount of arsenic in the excreta of persons living in arsenical surroundings, and found that this depended on the amount of exposure to the wall papers, curtains, carpets, or other sources of the poison. In one case even the quantity of arsenic obtained from one patient was half as great as that obtained from another exposed to the same conditions twice as long each day. Further, when the sources of the poison were removed, the arsenic gradually disappeared from the excreta at the same rate as the morbid symptoms vanished.

He was now ready to take part in the battle raging between the two camps, into which chemists at that time were divided, one maintaining that the connection between the morbid disturbances and an arsenical environment was proved, the other with equal vigor asserting that it was not. The frequent discussions of the question up to this time had consisted of a lively fusillade of assumptions and theories from both sides, which like a sham fight with blank cartridges had little result except noise. Sanger's thoroughly established facts therefore, thrown into this wordy warfare like a volley of shot, swept opposition from the field and converted to his views all, not too prejudiced to be open to conviction.

This establishment of the connection between these obscure diseases and arsenic was a service of great importance to the world as well as to chemistry, since it gave the physician a means of secure diagnosis and a certain cure for them; and further his results were used in an important study of the general relation between nervous disorders and chronic poisoning with small quantities of various agents.

It will be of interest next to consider how he had been fitted for this triumph by inheritance and training. His taste for study came directly from a line of scholarly ancestors, graduates of Harvard

College — his great grandfather Zedekiah Sanger, minister at Duxbury and South Bridgewater, Ralph Sanger his grandfather, the last town minister of Dover, so eminent that he was remembered last year by a celebration of the one hundredth anniversary of his ordination, and in the generation immediately preceding him from his father George Partridge Sanger who was judge of the court of common pleas and later United States District Attorney for Massachusetts, and from an aunt, who kept a successful girls' school in Boston, so that on this side he inherited with these scholarly instincts a love of truth and the judicial faculty for weighing evidence. On the other hand he undoubtedly owed his accuracy, his executive ability, his power of discipline, and the neat orderliness so characteristic of him to the family of Portsmouth sea captains from which he was descended through his mother, Elizabeth Sherburne (Thompson) Sanger; while from both sides he drew that faithfulness, which was his most prominent characteristic.

It was to be expected from this family history that he should choose the life of a student, but it is strange that he turned to chemistry rather than to some branch of literary work. Perhaps the practical ability inherited from his mother's ancestors gave this direction to his energies. However this may be, the call of science to him was irresistible, and even when he entered Harvard College, his taste for chemistry was strongly developed. I remember well the marked impression he produced on me in his first chemical recitation, and throughout his course he was an eminent student in that subject, which occupied a large part of his time.

On graduating in 1881 he began the higher study of chemistry, and for the first time came into intimate relations with Professor H. B. Hill, who was to have such a determining influence on his life; for, although he passed the second year after his graduation (1882 1883) in Europe studying at Munich, and at Bonn, where Professor Anschütz, struck by his ability, devoted special attention to him, and thus became an important factor in his higher education, Hill was his chemical father. During four of the five years, when he was growing into a chemist, he shared Professor Hill's private laboratory, working the entire day in his company, and part of the time in the even closer intercourse of a common research. Upon Hill therefore he modelled his methods of research, and views of chemistry, and this was the easier, since the two men naturally resembled each other as closely as father and son in aims, mental habits, and ideals. This warm and beautiful friendship was broken only by the death of the older man.

His work for the Ph. D. consisted of an investigation of substituted pyromucic acids, but the research on arsenic, already described, soon removed him from this field of pure organic chemistry cultivated so successfully by his master. Continuing study in his chosen line, after he had proved the reality of arsenical poisoning from wall papers, he attacked a puzzling mystery, which had baffled all attempts to penetrate it, but with this he proved less fortunate. The symptoms of wall paper poison are divided into two classes, one consisting of irritations of the mucous membrane obviously produced by arsenical dust, the other appearing in far reaching disturbances of the nervous system. Disorders of this latter class have been observed, when poisonous dust was nearly excluded, since the arsenic was contained in a glazed paper, or even, when its formation was impossible, because the arsenical paper was covered by one or more free from arsenic, so that in these cases the poisoning could have been due only to a gas; but here was the mystery—all attempts to detect an arsenical gas had failed (with two exceptions) whether in rooms with poisonous wall papers, or in mixtures of arsenic with organic matter, which should be even more efficient. During the earlier theoretical stage of the discussion those contending against the arsenical source of the nervous disorders were fond of arguing, that if arsenical they could be due to a gas only, as this gas could not be detected, it did not exist, and therefore the symptoms were not caused by arsenic. I think this is a fair statement of this argument, which in spite of its want of logic carried much weight, until Sanger destroyed it, by his discovery of arsenic in the excreta. But, although he proved in this way the existence of an arsenical gas, the puzzle still remained, as to what the gas was, how it was formed, and why it escaped detection. To the study of this problem he devoted a great deal of time, but, as he followed the methods of his predecessors, he was no more successful than they, and in spite of the most careful work did not succeed in detecting a trace of an arsenical gas. The truth was a new line of attack was needed, and this came from cryptogamic botany instead of chemistry, when Gosio announced his discovery that an evil-smelling gas containing arsenic was given off by three sorts of moulds growing in contact with arsenic and organic matter. Sanger at once repeated Gosio's experiments with the only one of these moulds accessible to him (*mucor mucedo*), but without success. Later however with a specimen of the most efficient sort (*penicillium brevicaule*) sent him by Gosio he succeeded in confirming the Italian's results. This important confirmation of the efficiency of moulds in the production of an arsenical gas was his last contribution to the study of poisoning from wall papers, because he felt obliged to retire from the field in order not to interfere with Gosio. This was certainly unfortunate, since his earlier work justifies the conviction that he would have solved this problem also, if he had not been compelled to relinquish the study of it. As it is, the mystery remains; Biginelli has found, it is true, that the gas formed by the moulds is an arsine, a substance related to the alkaloids and therefore probably more poisonous than most other compounds of arsenic, but it has not been shown how this, or any other gas can be formed from wall papers, which only in exceptional cases are in situations moist enough to favor the growth of moulds.

When in this way Sanger was shut out from the practical side of this investigation, he turned his attention to the purely chemical side of the work, extending his analytical method to the quantitative determination of antimony; and later applying this system of determining the amounts of arsenic, or antimony to the method of Gutheit, which in his hands became the most accurate and delicate method known for such work, and even displaced his own earlier Berzelius-Marsh process, admirable as that was. I think he considered this the best piece of work that he did, but I must give the preference to his work on arsenical poisoning from wall papers on account of its great practical importance, and because in this connection he worked out the general principle at the bottom of all these methods.

Two important papers, which occupied the last years of his life, belong to a different line of work. His object here was to prepare the silicon compound corresponding to phosgene, a well known derivative of carbon; but a reaction, which should have led to it in view of the strong resemblance between these elements, gave different products, the identification of which, simple as it seems at first sight, required an unusual amount of ingenuity, chemical insight, and skill in manipulation. This research brought out the entirely unexpected fact, that our knowledge of pyrosulphurylchloride and chlorsulphonic acid — two compounds supposed to be thoroughly established — rested on an inadequate experimental foundation, and in the first of these papers accordingly he placed them on a secure basis with his usual faithful accuracy. The skill in devising apparatus, and overcoming obstacles shown in these papers adds to our regret, that he was not spared to carry on other researches in this somewhat neglected field of inorganic preparations.

Among the work he left unfinished was a much needed method for the quantitative determination of small amounts of fluorine, a beautiful application of the general principle, that had proved so useful in his work on arsenic. It is hoped that this (and some other papers) can be brought into a state fit for publication, and, although shortly before his death he told me it was far from ready, I feel sure that even then it had been tested as carefully, as most chemists think necessary for their work; and this leads me to speak of Sanger's most marked characteristic, admirable in itself, but developed to such an extent, that it reduced the amount of his work very materially. This was an accuracy and care truly phenomenal. Most chemists are satisfied, when they have followed the work of their students closely, and tested it at certain commanding points. A few think it necessary to repeat all the work of their students, of these Hill was one, and in this single respect I must feel his influence was unfortunate, as his precept and example developed this side of Sanger's character to such an excess, that he was never willing to publish, until he had repeated the work of his students not once but many times. This is the principal reason why the list of his papers is short, and does no justice to the amount of work he did, or to his chemical ability; but on the other hand the wonderful accuracy of every published statement of his gives his work unusual authority. Other reasons for the comparatively small number of his papers are, that much of his time was taken up by work in industrial chemistry, which could not be published, and still more the almost over-faithful performance of his duties as teacher and Director of the Laboratory. In this last capacity he was always ready to sacrifice at the expense of his own investigations unlimited time for the purpose of advancing the researches of his colleagues by providing special apparatus, or material for them.

Apart from his chemical work Sanger's life, like those of most scientific men, was barren in striking events. He was born in Boston, August 31, 1860, but early in his boyhood his father moved to Cambridge, where he was fitted for college at the High School. He soon became an important member of the Class of 1881 at Harvard, partly because of his prominence in the societies, and as a member of his class nine, still more because his warm affectionate nature endeared him to his classmates, and enriched him with many lasting friendships. In his senior year he was elected Class Secretary — the important permanent officer of the class — and he met the duties of this office with the same enthusiasm he showed in his chemical work, while his characteristic methodical thoroughness and devotion made his work a model for all class secretaries.

His first year after graduation was passed in study for the degree of Master of Arts with Professor Hill, to whom he returned after his year (1882–1883) in Europe. He took his degree of Doctor of Philosophy in 1884, after which he served as Assistant in Chemistry in Harvard College, until in 1886 he was appointed Professor of Chemistry at the United States Naval Academy at Annapolis, a post for which he was especially fitted by nature, or perhaps rather by inheritance. In 1892 he accepted the better position of Eliot Professor of Chemistry at the Washington University of St. Louis.

In 1899 Professor Hill found the duties of Director of the Chemical Laboratory of Harvard College so exacting, that he was forced to give up the large elective in qualitative analysis (Chemistry 3) which he had taught for many years. We considered this course, as developed by him, our most precious treasure, since it trained men in observation and inductive reasoning better than any other known to us, but on the other hand, if improperly taught, it would sink to a mechanical routine worthless for educational purposes. It became therefore a matter of grave anxiety with us to find a successor for Professor Hill in this course, who should be able to carry it on worthily; and after a careful search of the whole field we decided that Sanger was by far the best man, and accordingly he was called to Harvard University as Assistant Professor of Chemistry in 1899; and in keeping the work in qualitative analysis on its previous high level he more than justified our faith in him.

As a teacher he was somewhat austere; all his students were expected to live fully up to his own standard, and he always retained some touch of the naval discipline. In particular research with him was no easy matter — the same accuracy, the same thoroughness, the same limitless patience, that he showed in his own work, he demanded of his students, but, as they saw he required nothing from them, which he did not exact from himself in even greater measure, they worked with enthusiasm, and felt for him an affection perhaps even deeper and stronger, than would have been inspired by an easier teacher.

An additional reason for his appointment at Cambridge had been that he was excellently fitted to act as director of the laboratory, should this become necessary. The death of Professor Hill in 1903 brought this necessity only too soon, and led to his appointment as Director, and promotion to a full professorship. I have already dwelt on the self-sacrificing devotion shown by him in this position. In all other respects too he proved an ideal choice, wise, and prudent in planning the work, methodical, thorough, and efficient in doing it.

At first it was hoped that he would take charge of the teaching of industrial chemistry in Harvard University; and in 1902 he went abroad for the summer semestre to fit himself better for this work. There he studied at Dresden with Professor Hempel, but with little result beyond a very pleasant and long continued friendship, for it was found that the great labor involved in the directorship rendered it impossible for anyone to give more than a single course in addition, and in his case this could be no other than qualitative analysis. He was not convinced of this impossibility however, until for several years he had made a gallant effort to carry the industrial chemistry on his already overburdened shoulders.

His uncommon administrative ability made him very useful on committees, especially in the Administrative Board of the Lawrence Scientific School, of which he was one of the pillars, but this also robbed him of much time, which would otherwise have been devoted to research.

The care and thoroughness shown in his work appeared also in his amusements, and made him an unusually skilful photographer and successful gardener.

On December 21, 1886 he married Almira Starkweather Horswell, who died January 6, 1905, leaving three children, Mary (married to H. A. Bellows), Eleanor Sherburne and Richard. On May 2, 1910 he married Eleanor Whitney Davis, the daughter of Andrew Mc Farland Davis, who survives him.

He was a member of the German Chemical Society, the Society for Chemical Industry, and the American Chemical Society (Vice-president of the New England Section 1902–1903). He was elected a fellow of our Academy, January 14, 1891; served on the C. M. Warren Committee from 1904, until his death; and was Chairman of the Publication Committee, that is editor of the Proceedings, from 1909 to 1910. His service in this last capacity showed his usual efficiency. Its short duration was due to the fact that he was already stricken with the disease, which led to his death, in fact the most prominent symptom of this was his nervous eagerness to add new undertakings to the load which already weighed him down, for in addition to our Proceedings he took sole charge of raising money for a new laboratory at Cambridge, and, when the American Chemical Society met in Boston and Cambridge in 1909, he was most active in arranging for its reception, and organized an interesting exhibit of the chemical activities of Harvard College. This was the finishing touch however, and at the end of that academic year he was so completely broken down that he was obliged to give up his regular work. Then followed a weary chase after health. A journey to Europe that summer did no good, nor was he more fortunate in the next winter spent on leave of absence, or in the following summer. In the autumn of 1911, although no better, he took up his teaching again, for his physicians decided that, if work were forbidden, the longing for it would do him more harm than the work itself. Accordingly he began to lecture in spite of agonizing attacks of pain, giving us the spectacle of duty triumphing over suffering, as before it had led him to disregard his own ease and advantage; but this heroism was in vain, the attacks grew more frequent, until in the middle of the year lecturing became impossible; but even then, as before, he filled up every cranny of his life with work on his papers feeling that rest was impossible, while anything remained undone, until death found him working at his post on February 25, 1912. The faithfulness, which had moulded every action of his life, reached a fitting climax in the heroic devotion to duty to its close.

C. L. Jackson.

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Woodrow Wilson . . . . Princeton, N. J.

Section II.— *Philology and Archæology.*— 32.

Franz Boas . . . . New York
Charles Pickering Bowditch . . . . Jamaica Plain
Franklin Carter . . . . Williamstown
George Henry Chase . . . . Cambridge
Roland Burrage Dixon . . . . Cambridge

William Curtis Farabee . . . . Cambridge
Jesse Walter Fewkes . . . . Washington, D. C.
Basil Lanneau Gildersleeve . . . . Baltimore, Md.
Charles Hall Grandgent . . . . Cambridge
Charles Burton Gulick . . . . Cambridge
William Arthur Heidel . . . . Middletown, Ct.
Albert Andrew Howard . . . . Cambridge
James Richard Jewett . . . . Cambridge
Alfred Louis Kroeber . . . . Berkeley, Cal.
Charles Rockwell Lanman . . . . Cambridge
Thomas Raynesford Lounsbury . . . . New Haven, Ct.
David Gordon Lyon . . . . Cambridge
Clifford Herschel Moore . . . . Cambridge
George Foot Moore . . . . Cambridge
Hanns Oertel . . . . New Haven, Ct.
Charles Pomeroy Parker . . . . Cambridge
Frederick Ward Putnam . . . . Cambridge
Edward Kennard Rand . . . . Cambridge
Edward Robinson . . . . New York
Fred Norris Robinson . . . . Cambridge
Edward Stevens Sheldon . . . . Cambridge
Herbert Weir Smyth . . . . Cambridge
Franklin Bache Stephenson . . . . Pittsfield
Charles Cutler Torrey . . . . New Haven, Ct.
Alfred Marston Tozzer . . . . Cambridge
Andrew Dickson White . . . . Ithaca, N. Y.
John Williams White . . . . Cambridge

SECTION III.— *Political Economy and History.*— 25.

Charles Francis Adams . . . . Lincoln
Henry Adams . . . . Washington, D. C.
Charles Jesse Bullock . . . . Cambridge
Thomas Nixon Carver . . . . Cambridge
Edward Channing . . . . Cambridge
Archibald Cary Coolidge . . . . Boston
Andrew McFarland Davis . . . . Cambridge
Davis Rich Dewey . . . . Cambridge
Ephraim Emerton . . . . Cambridge
Irving Fisher . . . . New Haven, Ct.
Worthington Chauncey Ford . . . . Boston

Edwin Francis Gay	Cambridge
Abner Cheney Goodell	Salem
Arthur Twining Hadley	New Haven, Ct.
Henry Cabot Lodge	Nahant
Abbott Lawrence Lowell	Cambridge
Alfred Thayer Mahan	New York
William Bennett Munro	Cambridge
James Ford Rhodes	Boston
William Mulligan Sloane	New York
Charles Card Smith	Boston
Henry Morse Stephens	Berkeley, Cal.
Frank William Taussig	Cambridge
Frederick Jackson Turner	Cambridge
Thomas Franklin Waters	Ipswich

SECTION IV.— *Literature and the Fine Arts.*— 30.

James Burrell Angell	Ann Arbor, Mich.
Francis Bartlett	Boston
Arlo Bates	Boston
William Sturgis Bigelow	Boston
Le Baron Russell Briggs	Cambridge
George Whitefield Chadwick	Boston
Samuel McChord Crothers	Cambridge
Wilberforce Eames	New York
Henry Herbert Edes	Cambridge
Arthur Fairbanks	Cambridge
Arthur Foote	Brookline
Kuno Francke	Cambridge
Daniel Chester French	Stockbridge
Robert Grant	Boston
Henry Lee Higginson	Boston
Mark Antony DeWolfe Howe	Boston
George Lyman Kittredge	Cambridge
Gardiner Martin Lane	Boston
William Coolidge Lane	Cambridge
Albert Matthews	Boston
Okakura-Kakuzo	Boston
Robert Swain Peabody	Boston
Bela Lyon Pratt	Boston
Herbert Putnam	Washington, D. C.

Denman Waldo Ross . . . . Cambridge
John Singer Sargent . . . . London, Eng.
William Robert Ware . . . . Milton
Herbert Langford Warren . . . . Cambridge
Barrett Wendell . . . . Boston
George Edward Woodberry . . . . Beverly

FOREIGN HONORARY MEMBERS.— 54.

(Number limited to seventy-five).

Class I.— *Mathematical and Physical Sciences.*— 17.

Section I.— *Mathematics and Astronomy.*— 5.

Svante August Arrhenius	Stockholm
Arthur Auwers	Berlin
Sir David Gill	London
Felix Klein	Göttingen
Émile Picard	Paris

Section II.— *Physics.*— 6.

Oliver Heaviside	Torquay
Sir Joseph Larmor	Cambridge
Hendrik Antoon Lorentz	Leyden
Augusto Righi	Bologna
John William Strutt, Baron Rayleigh	Witham
Sir Joseph John Thomson	Cambridge

Section III.— *Chemistry.*— 4.

Adolf, Ritter von Baeyer	Munich
Emil Fischer	Berlin
Wilhelm Ostwald	Leipsic
Sir Henry Enfield Roscoe	London

Section IV.— *Technology and Engineering.*— 2.

Heinrich Müller-Breslau	Berlin
William Cawthorne Unwin	London

Class II.— *Natural and Physiological Sciences.*— 17.

Section I.— *Geology, Mineralogy, and Physics of the Globe.*— 4.

Sir Archibald Geikie . . . . Haslemere, Surrey
Julius Hann . . . . Vienna
Albert Heim . . . . Zurich
Sir John Murray . . . . Edinburgh

Section II.— *Botany.*— 3.

Adolf Engler . . . . Berlin
Wilhelm, Pfeffer . . . . Leipsic
Hermann, Graf zu Solms-Laubach . . . . Strassburg

Section III.— *Zoölogy and Physiology.*— 5.

Ludimar Hermann . . . . Königsberg
Hugo Kronecker . . . . Bern
Sir Edwin Ray Lankester . . . . London
Elie Metchnikoff . . . . Paris
Magnus Gustav Retzius . . . . Stockholm

Section IV.— *Medicine and Surgery.*— 5.

Emil von Behring . . . . Marburg
Sir Thomas Lauder Brunton, Bart . . . . London
Angelo Celli . . . . Rome
Sir Victor Alexander Haden Horsley . . . . London
Adam Politzer . . . . Vienna

Class III.— *Moral and Political Sciences.*— 20.

Section I.— *Theology, Philosophy and Jurisprudence.*— 4.

Arthur James Balfour . . . . Prestonkirk
Heinrich Brunner . . . . Berlin
Albert Venn Dicey . . . . Oxford
Sir Frederick Pollock, Bart . . . . London

Section II.— *Philology and Archæology.*— 8.

Ingram Bywater	London
Friedrich Delitzsch	Berlin
Hermann Diels	Berlin
Wilhelm Dörpfeld	Athens
Henry Jackson	Cambridge
Hermann Georg Jacobi	Bonn
Sir Gaston Camille Charles Maspero	Paris
Eduard Seler	Berlin

Section III.— *Political Economy and History.*— 5.

James Bryce	London
Adolf Harnack	Berlin
John Morley, Viscount Morley of Blackburn	London
Sir George Otto Trevelyan, Bart	London
Pasquale Villari	Florence

Section IV.— *Literature and the Fine Arts.*— 3.

Georg Brandes	Copenhagen
Jean Adrien Aubin Jules Jusserand	Paris
Rudyard Kipling	Burwash

STATUTES AND STANDING VOTES

STATUTES

Adopted November 8, 1911: *amended May* 8, 1912, *January* 8, *and May* 14, 1913

CHAPTER I

The Corporate Seal

Article 1. The Corporate Seal of the Academy shall be as here depicted:

Article 2. The Recording Secretary shall have the custody of the Corporate Seal.

See Chap. v. art. 3; chap. vi. art. 2.

CHAPTER II

Fellows and Foreign Honorary Members and Dues

Article 1. The Academy consists of Fellows, who are either citizens or residents of the United States of America, and Foreign Honorary Members. They are arranged in three Classes, according to the Arts and Sciences in which they are severally proficient, and each Class is divided into four Sections, namely:

Class I. *The Mathematical and Physical Sciences*
- Section 1. Mathematics and Astronomy
- Section 2. Physics
- Section 3. Chemistry
- Section 4. Technology and Engineering

Class II. *The Natural and Physiological Sciences*
- Section 1. Geology, Mineralogy, and Physics of the Globe
- Section 2. Botany
- Section 3. Zoölogy and Physiology
- Section 4. Medicine and Surgery

Class III. *The Moral and Political Sciences*
- Section 1. Theology, Philosophy, and Jurisprudence
- Section 2. Philology and Archaeology
- Section 3. Political Economy and History
- Section 4. Literature and the Fine Arts

Article 2. The number of Fellows shall not exceed Six hundred, of whom not more than Four hundred shall be residents of Massachusetts, nor shall there be more than Two hundred in any one Class.

Article 3. The number of Foreign Honorary Members shall not exceed Seventy-five. They shall be chosen from among citizens of foreign countries most eminent for their discoveries and attainments in any of the Classes above enumerated. There shall not be more than Twenty-five in any one Class.

Article 4. If any person, after being notified of his election as Fellow, shall neglect for two months to accept in writing and to pay his Admission Fee (unless he be at that time absent from the Commonwealth) his election shall be void; and if any Fellow resident within fifty miles of Boston shall neglect to pay his Annual Dues for twelve months after they are due, provided his attention shall have been called to this Article of the Statutes in the meantime, he shall cease to be a Fellow; but the Council may suspend the provisions of this Article for a reasonable time.

With the previous consent of the Council, the Treasurer may dispense (*sub silentio*) with the payment of the Admission Fee or of the Annual Dues or both whenever he shall deem it advisable. In the case of officers of the Army or Navy who are out of the Commonwealth on duty, payment of the Annual Dues may be waived during such absence if continued during the whole financial year and if notification of such expected absence be sent to the Treasurer. Upon similar notification to the Treasurer, similar exemption may be accorded to Fellows subject to Annual Dues, who may temporarily remove their residence for at least two years to a place more than fifty miles from Boston.

If any person elected a Foreign Honorary Member shall neglect for six months after being notified of his election to accept in writing, his election shall be void.

See Chap. vii art. 2.

ARTICLE 5. Every Fellow hereafter elected shall pay an Admission Fee of Ten dollars.

Every Fellow resident within fifty miles of Boston shall, and others may, pay such Annual Dues, not exceeding Fifteen dollars, as shall be voted by the Academy at each Annual Meeting, when they shall become due; but any Fellow shall be exempt from the annual payment if, at any time after his admission, he shall pay into the treasury Two hundred dollars in addition to his previous payments.

All Commutations of the Annual Dues shall be and remain permanently funded, the interest only to be used for current expenses.

Any Fellow not previously subject to Annual Dues who takes up his residence within fifty miles of Boston, shall pay to the Treasurer within three months thereafter Annual Dues for the current year, failing which his Fellowship shall cease; but the Council may suspend the provisions of this Article for a reasonable time.

Only Fellows who pay Annual Dues or have commuted them may hold office in the Academy or serve on the Standing Committees or vote at meetings.

ARTICLE 6. Fellows who pay or have commuted the Annual Dues and Foreign Honorary Members shall be entitled to receive gratis one copy of all Publications of the Academy issued after their election.

See Chap. x. art. 2.

ARTICLE 7. Diplomas signed by the President and the Vice-President of the Class to which the member belongs, and countersigned by the Secretaries, shall be given to all the Fellows and Foreign Honorary Members.

ARTICLE 8. If, in the opinion of a majority of the entire Council, any Fellow or Foreign Honorary Member shall have rendered himself unworthy of a place in the Academy, the Council shall recommend to the Academy the termination of his membership; and if three fourths of the Fellows present, out of a total attendance of not less than fifty, at a Stated Meeting, or at a Special Meeting called for the purpose, shall adopt this recommendation, his name shall be stricken from the Roll.

See Chap. iii.; chap. vi. art. 1; chap. ix. art. 1, 7; chap. x. art. 2.

CHAPTER III

ELECTION OF FELLOWS AND FOREIGN HONORARY MEMBERS

ARTICLE 1. Elections of Fellows and Foreign Honorary Members shall be by ballot, and only at the Stated Meetings in January and May. Three fourths of the ballots cast, and not less than twenty, must be affirmative to effect an election.

ARTICLE 2. Candidates must be proposed in writing by two Fellows of the Section for which the proposal is made. These signed nominations shall be sent to the Corresponding Secretary and shall be retained by him until the fifteenth of the following October or February, as the case may be, when all nominations then in his hands shall be immediately sent in printed form to every Fellow having the right to vote, with the names of the proposers in each case, and with a request to send to the Corresponding Secretary written comments on these names not later than the fifth of November or the fifth of March respectively.

All the signed nominations, with the comments thereon, received up to the fifth of November or the fifth of March shall be sent at once to the appropriate Class Committees, which shall report their decisions to the Council at a special meeting to be called to consider nominations, not later than two days before the meeting of the Academy in December and April respectively.

ARTICLE 3. All nominations approved by the Council shall be read to the Academy at a meeting in December or in April, or be sent to the

Fellows in print with the official notice of the meeting, and shall then be posted in the Hall of the Academy until the balloting.

Not later than two weeks after any nomination is reported to the Academy, the Corresponding Secretary shall send to every Fellow having the right to vote a brief printed account of the nominee.

See Chap. ii.; chap. vi. art. 1; chap. ix. art. 1.

CHAPTER IV

Officers

Article 1. The Officers of the Academy shall be a President (who shall be Chairman of the Council), three Vice-Presidents (one from each Class), a Corresponding Secretary (who shall be Secretary of the Council), a Recording Secretary, a Treasurer, and a Librarian, all of whom shall be elected by ballot at the Annual Meeting, and shall hold their respective offices for one year, and until others are duly chosen and installed.

There shall be also twelve Councillors, one from each Section of each Class. At the Annual Meeting in 1912 three Councillors, one from each Class, shall be elected by ballot to serve for one year, three for two years, three for three years, and three for four years. At each subsequent Annual Meeting three Councillors, one from each Class, shall be elected by ballot to serve for the full term of four years and until others are duly chosen and installed. The same Fellow shall not be eligible for two successive terms.

The Councillors, with the other officers previously named, and the Chairman of the House Committee, *ex officio,* shall constitute the Council.

See Chap. x. art. 1.

Article 2. If any office shall become vacant during the year, the vacancy may be filled by the Council in its discretion for the unexpired term.

Article 3. At the Stated Meeting in March, the President shall appoint a Nominating Committee of three Fellows having the right to vote, one from each Class. This Committee shall prepare a list of nominees for the several offices to be filled, and for the Standing Committees, and cause it to be sent to the Recording Secretary not later than four weeks before the Annual Meeting.

ARTICLE 4. Independent nominations for any office, if signed by at least twenty Fellows having the right to vote, and received by the Recording Secretary not less than ten days before the Annual Meeting, shall be inserted, together with the list of nominees prepared by the Nominating Committee, in the call therefor, and shall be mailed to all the Fellows.

See Chap. vi. art. 2.

ARTICLE 5. The Recording Secretary shall prepare for use in voting at the Annual Meeting a ballot containing the names of all persons duly nominated for office.

CHAPTER V

THE PRESIDENT

ARTICLE 1. The President, or in his absence the senior Vice-President present (seniority to be determined by length of continuous fellowship in the Academy), shall preside at all meetings of the Academy. In the absence of all these officers, a Chairman of the meeting shall be chosen by ballot.

ARTICLE 2. Unless otherwise ordered, all Committees which are not elected by ballot shall be appointed by the presiding officer.

ARTICLE 3. Any deed or writing to which the Corporate Seal is to be affixed, except leases of real estate, shall be executed in the name of the Academy by the President or, in the event of his death, absence, or inability, by one of the Vice-Presidents, when thereto duly authorized.

See Chap. ii. art. 7; chap. iv. art. 1, 3; chap. vi. art. 2; chap. vii. art. 1; chap. ix. art. 6; chap. x. art. 1; 2; chap. xi. art. 1.

CHAPTER VI

THE SECRETARIES

ARTICLE 1. The Corresponding Secretary shall conduct the correspondence of the Academy and of the Council, recording or making an entry of all letters written in its name, and preserving for the files all official papers which may be received. At each meeting of the Council he shall present the communications addressed to the Academy which have been received since the previous meeting, and at the next meeting of the Academy he shall present such as the Council may determine.

He shall notify all persons who may be elected Fellows or Foreign Honorary Members, send to each a copy of the Statutes, and on their acceptance issue the proper Diploma. He shall also notify all meetings of the Council; and in case of the death, absence, or inability of the Recording Secretary he shall notify all meetings of the Academy.

Under the direction of the Council, he shall keep a List of the Fellows and Foreign Honorary Members, arranged in their several Classes and Sections. It shall be printed annually and issued as of the first day of July.

See Chap. ii. art. 7; chap. iii. art. 2, 3; chap. iv. art. 1; chap. ix. art. 6; chap. x. art. 1; chap. xi. art. 1.

Article 2. The Recording Secretary shall have the custody of the Charter, Corporate Seal, Archives, Statute-Book, Journals, and all literary papers belonging to the Academy.

Fellows borrowing such papers or documents shall receipt for them to their custodian.

The Recording Secretary shall attend the meetings of the Academy and keep a faithful record of the proceedings with the names of the Fellows present; and after each meeting is duly opened, he shall read the record of the preceding meeting.

He shall notify the meetings of the Academy to each Fellow by mail at least seven days beforehand, and in his discretion may also cause the meetings to be advertised; he shall apprise Officers and Committees of their election or appointment, and inform the Treasurer of appropriations of money voted by the Academy.

He shall post in the Hall a list of the persons nominated for election into the Academy; and after all elections, he shall insert in the Records the names of the Fellows by whom the successful candidates were nominated.

In the absence of the President and of the Vice-Presidents he shall, if present, call the meeting to order, and preside until a Chairman is chosen.

See Chap. i.; chap. ii. art. 7; chap. iv. art. 3, 4, 5; chap. ix. art. 6; chap. x. art. 1, 2; chap. xi. art. 1, 3.

Article 3. The Secretaries, with the Chairman of the Committee of Publication, shall have authority to publish such of the records of the meetings of the Academy as may seem to them likely to promote its interests.

CHAPTER VII

The Treasurer and the Treasury

Article 1. The Treasurer shall collect all money due or payable to the Academy, and all gifts and bequests made to it. He shall pay all bills due by the Academy, when approved by the proper officers, except those of the Treasurer's office, which may be paid without such approval; in the name of the Academy he shall sign all leases of real estate; and, with the written consent of a member of the Committee on Finance, he shall make all transfers of stocks, bonds, and other securities belonging to the Academy, all of which shall be in his official custody.

He shall keep a faithful account of all receipts and expenditures, submit his accounts annually to the Auditing Committee, and render them at the expiration of his term of office, or whenever required to do so by the Academy or the Council.

He shall keep separate accounts of the income of the Rumford Fund, and of all other special Funds, and of the appropriation thereof, and render them annually.

His accounts shall always be open to the inspection of the Council.

Article 2. He shall report annually to the Council at its March meeting on the expected income of the various Funds and from all other sources during the ensuing financial year. He shall also report the names of all Fellows who may be then delinquent in the payment of their Annual Dues.

Article 3. He shall give such security for the trust reposed in him as the Academy may require.

Article 4. With the approval of a majority of the Committee on Finance, he may appoint an Assistant Treasurer to perform his duties, for whose acts, as such assistant, he shall be responsible; or, with like approval and responsibility, he may employ any Trust Company doing business in Boston as his agent for the same purpose, the compensation of such Assistant Treasurer or agent to be fixed by the Committee on Finance and paid from the funds of the Academy.

Article 5. At the Annual Meeting he shall report in print all his official doings for the preceding year, stating the amount and condition of all the property of the Academy entrusted to him, and the character of the investments.

Article 6. The Financial Year of the Academy shall begin with the first day of April.

Article 7. No person or committee shall incur any debt or liability in the name of the Academy, unless in accordance with a previous vote and appropriation therefor by the Academy or the Council, or sell or otherwise dispose of any property of the Academy, except cash or invested funds, without the previous consent and approval of the Council.

See Chap. ii. art. 4, 5; chap. vi. art. 2; chap. ix. art. 6; chap. x. art. 1, 2, 3; chap. xi. art. 1.

CHAPTER VIII

The Librarian and the Library

Article 1. The Librarian shall have charge of the printed books, keep a correct catalogue thereof, and provide for their delivery from the Library.

At the Annual Meeting, as Chairman of the Committee on the Library, he shall make a Report on its condition.

Article 2. In conjunction with the Committee on the Library he shall have authority to expend such sums as may be appropriated by the Academy for the purchase of books, periodicals, etc., and for defraying other necessary expenses connected with the Library.

Article 3. All books procured from the income of the Rumford Fund or of other special Funds shall contain a book-plate expressing the fact.

Article 4. Books taken from the Library shall be receipted for to the Librarian or his assistant.

Article 5. Books shall be returned in good order, regard being had to necessary wear with good usage. If any book shall be lost or injured, the Fellow to whom it stands charged shall replace it by a new volume or by a new set, if it belongs to a set, or pay the current price thereof to the Librarian, whereupon the remainder of the set, if any, shall be delivered to the Fellow so paying, unless such remainder be valuable by reason of association.

Article 6. All books shall be returned to the Library for examination at least one week before the Annual Meeting.

Article 7. The Librarian shall have the custody of the Publications of the Academy. With the advice and consent of the President, he may effect exchanges with other associations.

See Chap. ii. art. 6; chap. x. art. 1, 2.

CHAPTER IX

The Council

Article 1. The Council shall exercise a discreet supervision over all nominations and elections to membership, and in general supervise all the affairs of the Academy not explicitly reserved to the Academy as a whole or entrusted by it or by the Statutes to standing or special committees.

It shall consider all nominations duly sent to it by any Class Committee, and present to the Academy for action such of these nominations as it may approve by a majority vote of the members present at a meeting, of whom not less than seven shall have voted in the affirmative.

With the consent of the Fellow interested, it shall have power to make transfers between the several Sections of the same Class, reporting its action to the Academy.

See Chap. iii. art. 2, 3; chap. x. art. 1.

Article 2. Seven members shall constitute a quorum.

Article 3. It shall establish rules and regulations for the transaction of its business, and provide all printed and engraved blanks and books of record.

Article 4. It shall act upon all resignations of officers, and all resignations and forfeitures of fellowship; and cause the Statutes to be faithfully executed.

It shall appoint all agents and subordinates not otherwise provided for by the Statutes, prescribe their duties, and fix their compensation.

They shall hold their respective positions during the pleasure of the Council.

ARTICLE 5. It may appoint, for terms not exceeding one year, and prescribe the functions of, such committees of its number, or of the Fellows of the Academy, as it may deem expedient, to facilitate the administration of the affairs of the Academy or to promote its interests.

ARTICLE 6. At its March meeting it shall receive reports from the President, the Secretaries, the Treasurer, and the Standing Committees, on the appropriations severally needed for the ensuing financial year. At the same meeting the Treasurer shall report on the expected income of the various Funds and from all other sources during the same year.

A report from the Council shall be submitted to the Academy, for action, at the March meeting, recommending the appropriation which in the opinion of the Council should be made.

On the recommendation of the Council, special appropriations may be made at any Stated Meeting of the Academy, or at a Special Meeting called for the purpose.

See Chap. x. art. 3.

ARTICLE 7. After the death of a Fellow or Foreign Honorary Member, it shall appoint a member of the Academy to prepare a Memoir for publication in the Proceedings.

ARTICLE 8. It shall report at every meeting of the Academy such business as it may deem advisable to present.

See Chap. ii. art. 4, 5, 8; chap. iv. art. 1, 2; chap. vi. art. 1; chap. vii. art. 1; chap. xi. art. 1, 4.

CHAPTER X

STANDING COMMITTEES

ARTICLE 1. The Class Committee of each Class shall consist of the Vice-President, who shall be chairman, and the four Councillors of the Class, together with such other officer or officers annually elected as may belong to the Class. It shall consider nominations to Fellowship in its own Class, and report in writing to the Council such as may receive at a Class Committee Meeting a majority of the votes cast, provided at least three shall have been in the affirmative.

See Chap. iii. art. 2.

Article 2. At the Annual Meeting the following Standing Committees shall be elected by ballot to serve for the ensuing year:

(i) *The Committee on Finance*, to consist of three Fellows, who, through the Treasurer, shall have full control and management of the funds and trusts of the Academy, with the power of investing the funds and of changing the investments thereof in their discretion.

See Chap. iv. art. 3; chap. vii. art. 1, 4; chap. ix. art. 6.

(ii) *The Rumford Committee*, to consist of seven Fellows, who shall report to the Academy on all applications and claims for the Rumford Premium. It alone shall authorize the purchase of books publications and apparatus at the charge of the income from the Rumford Fund, and generally shall see to the proper execution of the trust.

See Chap. iv. art. 3; chap. ix. art. 6.

(iii) *The Cyrus Moors Warren Committee*, to consist of seven Fellows, who shall consider all applications for appropriations from the income of the Cyrus Moors Warren Fund, and generally shall see to the proper execution of the trust.

See Chap. iv. art. 3; chap. ix. art. 6.

(iv) *The Committee of Publications*, to consist of three Fellows, one from each Class, to whom all communications submitted to the Academy for publication shall be referred, and to whom the printing of the Proceedings and the Memoirs shall be entrusted.

It shall fix the price at which the Publications shall be sold; but Fellows may be supplied at half price with volumes which may be needed to complete their sets, but which they are not entitled to receive gratis.

Two hundred extra copies of each paper accepted for publication in the Proceedings or the Memoirs shall be placed at the disposal of the author without charge.

See Chap. iv. art. 3; chap. vi. art. 1, 3; chap. ix. art. 6.

(v) *The Committee on the Library*, to consist of the Librarian, *ex officio*, as Chairman, and three other Fellows, one from each Class, who shall examine the Library and make an annual report on its condition and management.

See Chap. iv. art. 3; chap. viii. art. 1, 2; chap. ix. art. 6.

(vi) *The House Committee*, to consist of three Fellows, who shall have charge of all expenses connected with the House, including the general expenses of the Academy not specifically assigned to the care of other Committees or Officers.

See Chap. iv. art. 1, 3; chap. ix. art. 6.

(vii) *The Committee on Meetings*, to consist of the President, the Recording Secretary, and three other Fellows, who shall have charge of plans for meetings of the Academy.

See Chap. iv. art. 3; chap. ix. art. 6.

(viii) *The Auditing Committee*, to consist of two Fellows, who shall audit the accounts of the Treasurer, with power to employ an expert and to approve his bill.

See Chap. iv. art. 3; chap. vii. art. 1; chap. ix. art. 6.

Article 3. The Standing Committees shall report annually to the Council in March on the appropriations severally needed for the ensuing financial year; and all bills incurred on account of these Committees, within the limits of the several appropriations made by the Academy, shall be approved by their respective Chairmen.

In the absence of the Chairman of any Committee, bills may be approved by any member of the Committee whom he shall designate for the purpose.

See Chap. vii. art. 1, 7; chap. ix. art. 6.

CHAPTER XI

Meetings, Communications, and Amendments

Article 1. There shall be annually four Stated Meetings of the Academy, namely, on the second Wednesday of January, March, May, and October. Only at these meetings, or at adjournments thereof regularly notified, or at Special Meetings called for the purpose, shall appropriations of money be made, or amendments of the Statutes or Standing Votes be effected.

The Stated Meeting in May shall be the Annual Meeting of the Corporation.

Special Meetings shall be called by either of the Secretaries at the request of the President, of a Vice-President, of the Council, or of ten

Fellows having the right to vote; and notifications thereof shall state the purpose for which the meeting is called.

A meeting for receiving and discussing literary or scientific communications may be held on the second or the fourth Wednesday, or both, of each month not appointed for Stated Meetings, excepting July, August, and September; but no business shall be transacted at any meeting which may be held on the fourth Wednesday.

ARTICLE 2. Twenty Fellows having the right to vote shall constitute a quorum for the transaction of business at Stated or Special Meetings. Fifteen Fellows shall be sufficient to constitute a meeting for literary or scientific communications and discussions.

ARTICLE 3. Upon the request of the presiding officer or the Recording Secretary, any motion or resolution offered at any meeting shall be submitted in writing.

ARTICLE 4. No report of any paper presented at a meeting of the Academy shall be published by any Fellow without the consent of the author; and no report shall in any case be published by any Fellow in a newspaper as an account of the proceedings of the Academy without the previous consent and approval of the Council. The Council, in its discretion, by a duly recorded vote, may delegate its authority in this regard to one or more of its members.

ARTICLE 5. No Fellow shall introduce a guest at any meeting of the Academy until after the business has been transacted, and especially until after nominations to Fellowship have been read and the result of the balloting for candidates has been declared.

ARTICLE 6. The Academy shall not express its judgment on literary or scientific memoirs or performances submitted to it, or included in its Publications.

ARTICLE 7. All proposed Amendments of the Statutes shall be referred to a committee, and on its report, at a subsequent Stated Meeting or at a Special Meeting called for the purpose, two thirds of the ballot cast, and not less than twenty, must be affirmative to effect enactment.

ARTICLE 8. Standing Votes may be passed, amended, or rescinded at a Stated Meeting, or at a Special Meeting called for the purpose, by a vote of two thirds of the members present. They may be suspended by a unanimous vote.

See Chap. ii. art. 5, 8; chap. iii.; chap. iv. art. 3, 4, 5; chap. v. art. 1; chap. vi. art. 1, 2; chap. ix. art. 8.

STANDING VOTES

1. Communications of which notice has been given to either of the Secretaries shall take precedence of those not so notified.

2. Fellows may take from the Library six volumes at any one time, and may retain them for three months, and no longer. Upon special application, and for adequate reasons assigned, the Librarian may permit a larger number of volumes, not exceeding twelve, to be drawn from the Library for a limited period.

3. Works published in numbers, when unbound, shall not be taken from the Hall of the Academy without the leave of the Librarian.

RUMFORD PREMIUM

In conformity with the terms of the gift of Sir Benjamin Thompson, Count Rumford, of a certain Fund to the American Academy of Arts and Sciences, and with a decree of the Supreme Judicial Court of Massachusetts for carrying into effect the general charitable intent and purpose of Count Rumford, as expressed in his letter of gift, the Academy is empowered to make from the income of the Rumford Fund, as it now exists, at any Annual Meeting, an award of a gold and a silver medal, being together of the intrinsic value of three hundred dollars, as a Premium to the author of any important discovery or useful improvement in light or heat, which shall have been made and published by printing, or in any way made known to the public, in any part of the continent of America, or any of the American Islands; preference always being given to such discoveries as, in the opinion of the Academy, shall tend most to promote the good of mankind; and, if the Academy sees fit, to add to such medals, as a further Premium for such discovery and improvement, a sum of money not exceeding three hundred dollars.

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