PROCEEDINGS

OF THE

Cambridge Philosophical Society.

VOLUME VIII.

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CONTENTS.

VOL. VIII.

October 30, 1893.

PAGE

November 13, 1893.

November 27, 1893.

January 29, 1894.

February 12, 1894.

February 26, 1894.

March 12, 1894.

April 30, 1894.

May 14, 1894.

May 28, 1894.

October 30, 1893.

October 29, 1894.

November 12, 1894.

November 26, 1894.

February 11, 1895.

February 25, 1895.

April 29, 1895. PAGE

May 13, 1895.

May 27, 1895.

PLATES.

PROCEEDINGS

OF THE

Cambridge Philosophical Society.

October 31, 1892.

ANNUAL GENERAL MEETING.

THE officers and new members of the Council for the ensuing year were elected :—

President.

*Prof. T. McK. Hughes.

Vice-Presidents.

Prof. G. H. Darwin, *Dr Cayley, Dr Hill.

Treasurer.

Mr Glazebrook.

Secretaries.

Dr Hobson, Mr Larmor, Mr Bateson.

Ordinary Members of the Council.

Dr W. H. Gaskell.
Dr A. S. Lea.
Mr A. Harker, St John's College.
Mr L. R. Wilberforce.
Mr H. F. Newall, Trinity College.
Mr C. T. Heycock, King's College.
Mr A. E. H. Love, St John's College.
Mr S. F. Harmer, King's College.
*Dr Shore.
*Prof. J. J. Thomson.
*Mr F. Darwin, Christ's College.
*Mr S. Ruhemann, Caius College.

The names marked with an asterisk denote a new election.

The retiring President, Prof. G. H. Darwin, addressed the Society upon the History of the Society during his tenure of office.

Sir Robert Stawell Ball, Lowndean Professor of Astronomy and Geometry, and Mr R. A. Sampson, M.A., St John's College, were elected Fellows of the Society.

The following communications were made to the Society:

(1) *Note on the Determination of Low Temperatures by Platinum-Thermometers.* By E. H. GRIFFITHS, M.A., Assistant Lecturer at Sidney College, and G. M. CLARK, B.A., Sidney College.

In connexion with Profs. Dewar and Fleming's recent work on the resistance of certain metals and alloys at very low temperatures, and the suggestion that they have made—viz. that the resistance of certain pure metals (amongst which is platinum) vanishes at absolute zero,—the following facts may be of interest.

Having previously published the constants of several platinum-thermometers, whose accuracy has been exposed to severe tests, we have, by assuming the possibility of applying Callendar and Griffiths' method, calculated the temperature at which $R=0$ from the formulæ

$$pt = \frac{R - R_0}{R_1 - R_0} \times 100$$

and

$$t - pt = \delta \left\{ \overline{\frac{t}{100}}\Big|^2 - \overline{\frac{t}{100}}\Big| \right\},$$

where R_1, R_0 are the resistances in steam under normal pressure, and melting ice, respectively, and R is the resistance at temperature t; the value of δ for each thermometer having been determined by observations of the resistance in boiling sulphur

$$(t = 444^\circ{\cdot}53).$$

The following table gives the constants and results:—

The actual numbers used in these calculations are those taken from the papers referred to in column 2.

All the other thermometers mentioned by Griffiths (*Phil. Trans.* A. 1891, pp. 43—72) were made with single electrodes for rapid observations; and thus every observation includes their stem-resistance, which evidently cannot become zero, so that the above investigation with their constants would be of no use.

TABLE I.

Thermometer.	References.	R_0.	R_1/R_0.	δ.	when R = 0. *pt.*	*t.*
Callendar's.	Phil. Trans. A. 1887, sect. 2 & 3.	5·0845	1·3460	1·46	− 289·04	− 274·12
N_{A+B}.	Phil. Trans. A. 1891, pp. 151, 152.	9·8558	1·3484	1·638	− 287·03	− 270·60
N_A.	,, ,,	5·9865	1·3482	1·648	− 287·19	− 269·60
N_B.	,, ,,	3·8749	1·3480	1·639	− 287·34	− 270·86
M_1.	Phil. Trans. A. 1891, pp. 132–136.	4·2267	1·3381	1·57	− 295·81	− 279·15
M_2.	,, ,,	4·1732	1·3383	1·57	− 295·58	− 278·97
H.	Phil. Trans. A. 1891, p. 153.	13·5216	1·3463	1·474	− 288·77	− 273·71
					*Mean.........	− 273·86

The above table seems to corroborate the conclusions arrived at by Profs. Dewar and Fleming, and at the same time is a valuable testimony to the accuracy of the method adopted by Callendar and Griffiths in the papers referred to, for it did not appear probable that their formulæ could bear the strain of extrapolating over a range of nearly 300°. This is more especially the case when it is remembered that the wires used have different resistance coefficients and different values for δ; the origin of such differences is probably due to slight impurities in the platinum.

We have reason to believe, from some recent experiments conducted by us, that the rise in temperature of the wire, caused by the current necessary to determine its resistance, is greater than is usually supposed. If the difference of potential at the ends of the wire is kept constant, the effect of the error thus introduced is to make the absolute zero too low; if, however, the current is kept constant, the absolute zero works out too high. In the determination of the constants of thermometers *N*, a rough attempt was made to keep the current constant; but in the remaining thermometers no alteration was made in the electromotive force as the resistance of the coil, and therefore

* Cf. with the value given by Joule and Thomson, − 273°·7, *Proc. Roy. Soc.*, vol. x., p. 502.

of the bridge, increased. It appears probable that some of the discrepancies in the above table are due to this cause.

A simple method of graduating platinum-thermometers is thus suggested. Assuming, as we are fully entitled to do, that the curve $t - pt$ is a parabola, three points only are necessary for its complete determination: the points hitherto adopted have been 0°, 100°, and 444°·5. The necessity of guarding against loss by radiation, gain by superheating, &c., when determining the resistance in sulphur, renders it a somewhat troublesome operation to those observers who are not provided with the necessary apparatus. In cases where a high order of accuracy is not a *sine quâ non*, and where the platinum is known to be fairly pure, we may assume that when $t = -273°{\cdot}7$ then $R = 0$: therefore, if R_1 and R_0 are determined by direct observations in steam (760 millim.) and melting ice, the instrument may be considered as graduated, since

$$\delta = \frac{-273{\cdot}7 - pt}{\left(\frac{273{\cdot}7}{100}\right)^2 + \frac{273{\cdot}7}{100}},$$

where
$$pt = \frac{-R_0}{R_1 - R_0} \times 100.$$

As an example of the order of accuracy which could be thus obtained, we append the following table, the first column giving the true temperature, the others the error introduced in graduating, by this method, the various thermometers previously mentioned.

TABLE II.

True temp.	Callendar's.	N_{A+B}.	N_A.	N_B.	H.	M_1.	M_2.
0	0	0	0	0	0	0	0
50	− ·01	+ ·08	+ ·08	+ ·08	0	− ·14	− ·14
100	0	0	0	0	0	0	0
150	+ ·03	− ·25	− ·25	− ·23	0	+ ·44	+ ·43

Although the above discrepancies may in some cases appear large, it must be remembered that we have here the whole of the error in each case, and that such causes of error as non-uniform bore, zero changes (both temporary and permanent), unequal graduation, changes of condition, sticking, &c. are totally absent, while the experimental difficulties of the air-thermometer are avoided.

It is thus evident that a platinum-thermometer, especially of the H pattern*, is a convenient instrument for the determination of very low temperatures where an error of $0^\circ{\cdot}5$ is immaterial; and the above investigation strengthens our confidence in the accuracy of the method adopted for the measurement, by platinum-thermometers, of temperatures over the range of -273° to $+700^\circ$.

[Note by E. H. G., Feb. 1893.] The constants of all the thermometers referred to in Table I. (with the exception of Callendar's 1887 thermometer) were determined by means of the same resistance box, which was one of the dial pattern, constructed by Messrs Elliott. Mr Glazebrook has, since the reading of the above communication, been so kind as to make a direct comparison of the coils of this box and the B. A. standards; and although the errors in the individual coils are small their cumulative effect is, for certain values of R, considerable.

I have applied the corrections thus rendered necessary to the values of R_0, R_1, R_5, and redetermined the consequent values of δ.

The results are as follows.

TABLE III.

Thermometer.	R_0.	R_1/R_0.	δ.	when $R=0$. *pt.*	*t.*
N_{A+B}.	9·8636	1·3483	1·641	−287·12	−270·66
N_A.	5·9881	1·3484	1·645	−287·04	−270·55
N_B.	3·8762	1·3481	1·640	−287·33	−270·86
M_1.	4·2280	1·3381	1·57	−295·74	−279·08
M_2.	4·1745	1·3383	1·57	−295·52	−278·91
H.	13·5300	1·3463	1·475	−288·77	−273·73
				Mean	−273·96

* A full description of the construction of H will be found in the British Association Report on Electrical Standards, 1890, reprinted in the *Electrical Review*, No. 670, p. 363; and a short account on p. 153, *Phil. Trans. A.*, vol. clxxxii., 1891.

The accuracy of the corrections is indicated by the close agreement amongst the results deduced from thermometer N, for its coils differ greatly in length and resistance. On reference to my 1890 note-book I find that thermometer H was made of a special sample of wire, which was supposed to be of greater purity than the samples previously supplied, and judging by results, it appears that the assumption as to its purity was justified. It is probable that small traces of impurities could be detected by an application of the above method, and I have little doubt but that the considerable divergence from the mean shown by those values of t deduced from thermometer M, are due to the presence in its coils of other metals than platinum.

(2) *Carnot's principle and animal and vegetable life.* By J. PARKER, M.A., St John's College.

It is commonly stated that an animal has a much greater 'efficiency' than a Carnot's perfectly reversible engine working through the same range of temperature; and it is even compared to an electro-magnetic engine, though there are generally no sensible electric manifestations about the animal body. In consequence it is believed that the changes which take place in the animal world are not subject to the restrictions of Carnot's principle. Hence since Carnot's principle is universally applicable to inanimate objects, it will follow that the laws of nature are not the same for living things as for the rest of the universe. The whole difficulty, however, arises from an oversight, and can be removed with a little care.

In order that Carnot's principle should be applicable to a material system, two conditions are necessary:—

(1) The system must always contain the same matter, so that matter must neither be allowed to enter the system nor leave it.

(2) The system must gain or lose energy only in the forms of heat and mechanical work. The interior of the system may be electrified or magnetized in any way we please, but the system must produce no electric or magnetic effects on external objects, nor have any electric communication with them.

If the first condition is fulfilled, but not the second, the system is an electro-magnetic engine. Carnot's principle is then not generally applicable, nor the principle of energy in its usual form

$$dU = dQ + dW.$$

The difference between an electro-magnetic and a heat engine is well illustrated by taking as our system two pieces, AP, BP, of different metals, joined together at P. Let a *feeble* electric current enter the system at A, and after crossing the junction P, leave at B; and suppose that the current tends to cool the junction, but that by absorbing heat from external objects the temperature of the whole system is kept constantly equal to θ. Then the quantity of heat Q, absorbed by the system in any period, is positive; and therefore Carnot's principle is not satisfied in the form $\frac{Q}{\theta} < 0$. If, however, we extend the system so that it includes all bodies traversed by the current, Carnot's principle becomes applicable.

It is now clear that an animal cannot be compared to an electro-magnetic engine, and that Carnot's principle cannot be expected to apply unless we take account of the flux of matter. In the case of the animal body, our system must therefore be chosen to contain, in addition to the living animal, a sufficient quantity of air, water, food, soil, &c.; so that, in the changes we wish to investigate, the system contains the same matter. When this is done, it will be found that Carnot's principle applies. But even when our system is properly chosen, we cannot speak of the 'efficiency' unless the cycle of operations is complete. If the cycle is incomplete, we must take account of the change of entropy; and it will be presently seen that there is something which may easily be mistaken for 'efficiency.'

If, for example, a system undergoes a reversible operation, at the constant temperature θ, during which the energy and entropy change from U_1 and ϕ_1 to U_2 and ϕ_2, respectively, the heat given out will be equal to $\theta(\phi_1 - \phi_2)$, and therefore the work done by the system will be $U_1 - U_2 - \theta(\phi_1 - \phi_2)$. Here there is nothing which can be called 'efficiency,' if we use the word strictly according to the definition; but the ratio of the work done to the loss of energy is

$$\frac{U_1 - U_2 - \theta(\phi_1 - \phi_2)}{U_1 - U_2}, \text{ or } 1 - \frac{\theta(\phi_1 - \phi_2)}{U_1 - U_2}.$$

This quantity may have any value we please. If no sensible amount of heat is lost or gained, it will be practically equal to unity; and the work obtained from the system will be equal to the energy lost by it.

Now if the same amount of energy had been lost from the system entirely in the form of heat, and the heat so obtained had been used to work a Carnot's perfectly reversible engine between the temperatures θ' and θ'', the work obtained would have been

$$(U_1 - U_2)\,\frac{\theta' - \theta''}{\theta'}, \text{ or } U_1 - U_2 - \frac{\theta''}{\theta'}(U_1 - U_2).$$

This quantity, it is clear, has no connection with the former quantity of work, $U_1 - U_2 - \theta(\phi_1 - \phi_2)$; and it is also clear that the efficiency of the Carnot's engine, $\frac{\theta' - \theta''}{\theta'}$, which is independent of $U_1 - U_2$, is entirely different from the miscalled 'efficiency,'

$$1 - \frac{\theta(\phi_1 - \phi_2)}{U_1 - U_2},$$

in the former case. In one case, it will be observed, the method of obtaining work is entirely reversible; in the other, partly reversible and partly irreversible.

This example illustrates the well-known fact that it is more profitable to give corn to horses than to use it as fuel in a steam-engine.

In applying Carnot's principle to the vegetable world, we must again choose our system so that, while under consideration, it neither gains nor loses matter. It must therefore contain, besides the living vegetation, a sufficient quantity of air, carbonic acid, ammonia, water, soil, &c. Suppose now that the only vegetation in the system at first is a twig; then let it grow into a tree; and then let the whole tree, except a single twig, be destroyed by fire. Suppose also, for simplicity, that the temperature of the system is uniform and constantly equal to θ during the growth of the tree, and that the pressure is uniform and constant during the cycle. Then since water is one of the products of the burning of wood, the air will contain more aqueous vapour after the products of combustion are reduced to the original temperature θ than it did before the conflagration, unless it was saturated before the conflagration. It will therefore be convenient to have the air saturated with aqueous vapour and at the temperature θ before the conflagration. The complete cycle will then consist of the following operations at constant pressure, at the end of each of which the temperature is θ:—

(1) The twig grows into a tree at the temperature θ.

(2) The air is then saturated with aqueous vapour.

(3) The conflagration takes place, and the products of combustion are reduced to the original temperature θ.

(4) The quantity of vapour in the air is made the same as at the beginning of the cycle.

In the third operation, a positive quantity of heat, q say, will be given out, and the corresponding value of $\Sigma \frac{Q}{\theta}$ may be written $-\frac{q}{\theta'}$, where $\theta' > \theta$. If the air neither gains nor loses aqueous vapour during the life of the tree, operation (4) will be the reverse of (2) (both being supposed reversible); and the quantity of heat absorbed and the value of $\Sigma \frac{Q}{\theta}$ for the two operations together, will both be zero. Hence since the total quantity of heat absorbed in the complete cycle is zero, the heat absorbed in the first operation is q, and the value of $\Sigma \frac{Q}{\theta}$ for the whole cycle is $\frac{q}{\theta} - \frac{q}{\theta'}$, which is positive, contrary to Carnot's principle.

The solution of this difficulty appears to be found in what botanists call 'transpiration,' or the exhalation of aqueous vapour. In consequence of this process the air in which the tree grows gradually comes to contain more aqueous vapour. Hence more vapour is deposited in (4) than absorbed in (2); and therefore, in the two operations together, a positive quantity of heat, x say, is given out, and the corresponding value of $\Sigma \frac{Q}{\theta}$ is $-\frac{x}{\theta_0}$, where $\theta_0 < \theta$. If therefore, as before, q be the quantity of heat given out in (3), $q + x$ will be the quantity absorbed in (1), and the value of $\Sigma \frac{Q}{\theta}$ for the complete cycle will be

$$\frac{q+x}{\theta} - \frac{x}{\theta_0} - \frac{q}{\theta}, \quad \text{or} \quad q\left(\frac{1}{\theta} - \frac{1}{\theta'}\right) - x\left(\frac{1}{\theta_0} - \frac{1}{\theta}\right).$$

This will be negative when $x > \frac{\theta_0}{\theta'} \frac{\theta' - \theta}{\theta - \theta_0} q$, and therefore, *à fortiori*, when $x > \frac{\theta_0}{\theta - \theta_0} q$. If, for example, $\theta_0 = \frac{9}{10}\theta$, $\Sigma \frac{Q}{\theta}$ will certainly be negative if $x > 9q$. Now this appears to be actually the case. Thus we may safely put the quantity of aqueous vapour transpired by an oak-tree in 24 hours at 20 litres, or about $4\frac{1}{2}$ gallons, of water [Sir J. D. Hooker's *Primer of Botany*]. Hence since the latent heat of one gramme of water at the freezing-point is a little over 600 calories, the value of x for one day's growth will be roughly 12 million calories. If we suppose 50 growing days in a year, the value of x for 20 years will be about 12 thousand million calories. Again, if we take the heat given out in the combustion of one gramme of dry wood to be 3000 calories, it will require 4 million grammes, or 4000 kilogrammes, or about 4 tons of dry wood to be burnt to evolve 12 thousand million calories of heat. Hence since we can only suppose the increase of the tree to amount to a small fraction of 4 tons of dry wood in 20 years, q will only be a small fraction of x, and thus Carnot's principle is proved to hold for the cycle just described.

It will be noticed that we have only considered the total quantity and not the quality (or wave length) of the radiation absorbed by the plant. We have merely shown that Carnot's principle is satisfied when the plant grows by absorbing heat, without enquiring whether any or what conditions must be satisfied in order that the plant may be able to absorb heat. Now it is observed that a plant cannot grow when all the radiation which falls upon it is *dark*. Thus there is a further question for consideration; but at present we do not appear to possess sufficient data to discuss it. We shall therefore merely point out that the question cannot affect the validity of the preceding reasoning.

It follows from what we have said that 'transpiration' is essential to vegetable life. In fact, a plant would be choked if kept in a place constantly saturated with aqueous vapour. On the earth the atmosphere is prevented from being constantly at the point of saturation by the succession of day and night: in other words, 'cold and heat, day and night*,' are necessary to vegetation.

If a planet be placed so that no heat can reach it from external sources, its temperature will generally become practically constant and its atmosphere saturated with vapour. On such a body vegetable life would be impossible; and we may infer from Carnot's principle that our beds of coal cannot have been produced by the internal heat of the earth alone, but must be due to solar radiation.

If we make a few assumptions, Carnot's principle easily leads us to some further results. Thus if we suppose the condition of the air to be the same in two places, θ and θ' will be the same in both. If therefore we put $x = \frac{\theta_0}{\theta'} \frac{\theta' - \theta}{\theta - \theta_0} q$, we may write $x = \frac{\theta_0}{\theta - \theta_0} \lambda q$, where λ is the same in both cases. Consequently, since $\frac{\theta_0}{\theta}$ is nearly equal to unity, we see that q practically varies as $x(\theta - \theta_0)$. Next, since x depends on the difference between the temperature of the air and the dew-point, and therefore on $\theta - \theta_0$, we will assume that, for a given time, it varies as the intensity of solar radiation and $\theta - \theta_0$ conjointly. Hence, for a given time, q varies as the intensity of solar radiation and $(\theta - \theta_0)^2$ conjointly. For example, Jupiter is about 5·2 times as far away from the sun as the earth, and at this distance the intensity of

* Gen. viii. 22.

solar radiation is about $\frac{1}{27}$th of what it is at the distance of the earth.

Again, Jupiter rotates on his axis in about 10 hours. We will therefore assume that for places similarly on the two planets, $\theta - \theta_0$ for the earth is $27 \times \frac{24}{10}$, or about 65 times as great as for Jupiter. Thus, with the preceding assumptions, it follows that the want of sunshine causes vegetation, and therefore food, to be about $65^2 \times 27$, or about 114000 times as scanty on Jupiter as on the earth. With similar assumptions, it will follow that vegetation and food are far scarcer on the further planets, Saturn, Uranus and Neptune, even than on Jupiter. On such barren wildernesses, animal life, as we know it, could hardly exist.

(3) *Note on the Geometrical interpretation of the Quaternion Analysis.* By J. BRILL, M.A., St John's College.

November 14, 1892.

PROF. T. McK. HUGHES, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

J. Y. BUCHANAN, M.A., Christ's College;
A. HUTCHINSON, B.A., Pembroke College.

The President exhibited (1) a live Tarantula, (2) quartz crystals of unusual form.

Mr J. J. Lister exhibited preparations showing the division of the nuclei in the sporangium of a species of *Trichia*, one of the Myxomycetes. The nuclei divide throughout the sporangium with clearly recognizable karyokinetic figures, immediately before the formation of the spores.

The following communications were made:—

(1) *On the Reproduction of Orbitolites.* By J. J. LISTER, M.A., St John's College.

Mr H. B. Brady has described specimens of Orbitolites, which he obtained in Fiji, showing the margin of the disc crowded with young shells. Mr Brady's material was worked at in the dry state, and it was at his suggestion that the author collected specimens preserved in spirit from the Tonga reefs.

Examination of this material shows that large brood chambers are formed at the margin of the disc during the later stages of growth. These are at first lined with a thin layer of protoplasm. At a later stage the central region of the disc is found to be empty, and the whole of the protoplasm is massed in the brood chambers in the form of spores. The spores have the structure of the 'primitive disc,' which during the early stages of growth of the Orbitolites occupies the centre of the shells. They are liberated by absorption of the walls of the brood chambers, and each becomes the centre of a new disc, which is built up by additions of successive rings of chamberlets at the margin.

The reproduction of Orbitolites therefore takes place by spore formation.

The spore contains a single nucleus, lying in its 'primordial chamber.' After several rings of chamberlets have been added, a stage is reached at which the nucleus appears to be represented by numbers of irregular, darkly staining masses scattered through the protoplasm of the central part of the disc.

In the later stages numbers of oval nuclei are found in the protoplasm, often arranged in pairs, and in favourable preparations they may be seen to be undergoing amitotic division.

(2) *The fragmentation of the oosperm nucleus in certain ova.* By S. J. Hickson, M.A., Downing College.

In a former paper* I have described the early stages in the development of the Hydrocoralline Allopora. The following sentences in that paper indicate the points that I have thought necessary to investigate with especial care in the development of Distichopora.

"I can find no trace whatsoever of any division of the protoplasm in the neighbourhood of the (oosperm) nucleus and no evidence that would lead me to suppose that I have missed any stages of regular segmentation either of the nucleus or the egg protoplasm.

The evidence, so far as it goes, seems to show that the oosperm nucleus after losing its membrana limitans simply breaks up into fragments and that from these fragments a number of embryonic nuclei are formed that wander into the blastodermic area where they rapidly multiply by a process of growth and simple division. The larger irregular lumps of nuclear substance found only in the earliest stages I take to be portions of the oosperm nucleus that have not so fragmented."

In Allopora the ova are scattered about in the coenenchym and are not as a rule visible from the surface. In order to obtain therefore a complete series of stages an enormous number of sections must be made haphazard through the colony and in many cases an immense amount of time and trouble is rewarded with perfectly barren results.

In Distichopora fortunately the element of chance is comparatively speaking very small, for large clusters of ova in all stages of development can be seen from the surface even before the coenosteum is dissolved. One can proceed in this case to cut

* *Q. J. Micr. Soc.* xxix.

sections through a large field of ova with perfect confidence that most of the stages will be represented.

Taking advantage of a large supply of very well preserved Distichoporas that was placed in my hands by Professor Haddon, I have, during the last two years, prepared a very considerable number of series of sections which enable me to review the conclusions I came to in my paper on Allopora.

The ovum of Distichopora like that of Allopora is provided with a large amount of yolk and lies in a cup-shaped trophodisc. In young immature ova, ova that is to say, that can be seen to be immature by the fact that they do not contain their full complement of yolk spheres, the germinal vesicle is spherical, provided with a well-marked limiting membrane, a large germinal spot and a fine network of fibrils with thickened nodes.

In some ova with a full complement of yolk spheres, the germinal vesicle is irregular in shape and the membrana limitans broken in places, so that the intra-nuclear protoplasm is perfectly continuous with the extra-nuclear protoplasm.

These irregular amoeboid germinal vesicles are I believe caught in the act of travelling from the centre of the ovum to the periphery.

The stage which is most frequently found however is one in which the germinal vesicle lies close to or actually on the periphery of the ovum, and many interesting varieties in the form of its outline may be observed.

In the first place, a form in which the inner half is hemispherical in shape with the membrana limitans well defined, while the outer half is irregular and the membrana limitans not well-defined, is very common.

Secondly, there is a very common form in which the membrana of the inner half has disappeared while that of the outer half is indistinct, and the germinal spot absent.

Lastly, numerous forms that may be considered to be intermediate between these two types.

In all of these forms well-stained specimens show a considerable number of very minute irregular chromosomes lying in the central parts of the vesicle.

It seems probable that the discharge of the polar bodies and fertilisation take place while the germinal vesicle is at or near the periphery of the ovum, but I feel that it is impossible to assert with any degree of assurance that any of the structures that I have seen and believed to be polar bodies and male pronuclei are really what I have believed them to be.

I have been unable to find in any of my preparations of these and other stages any trace of regular karyokinetic figures or centrosomata, notwithstanding the fact that my material is excellently well preserved.

The next stage is one in which I have failed to find any trace of a membrana limitans to the egg nucleus: the protoplasm is, from the manner in which it stains, obviously being distributed through the outer pole of the ovum. Extremely minute granules deeply stained in haemotoxylin or carmine can be seen, but no large chromatin rods, nor asters, nor an equatorial plate of any kind or description can be discovered.

During the course of the investigation I failed to find any traces of karyokinetic figures in the first divisions of the oosperm nucleus. Thinking that perhaps this failure was due to imperfect methods of staining my preparations, I refrained from publishing my results until I had experimented with a great many different reagents and combinations of reagents. No new results were however forthcoming, and it seems to me to be possible that karyokinesis—in the ordinary sense of the word—does not occur and that the division of the oosperm nucleus is a true amitotic fragmentation.

It is probable too that this is not the only instance of amitotic division of the oosperm nucleus.

I should point out that neither Balfour nor any of those who preceded him were able to distinguish any karyokinetic figures in the earliest stages of the segmentation of the Elasmobranchs, although at a later stage he found typical spindles in the blastoderm.

More recently Kastschenko has very carefully reinvestigated these early stages with the help of our modern knowledge of stains and preservative reagents and again finds no karyokinesis.

Kastschenko's results seem to me to be particularly interesting and important, because there can be no doubt that his work was very carefully done, that his material was well preserved and that his staining methods were suitable for bringing out the karyokinetic figures.

He shows very clearly that the so-called yolk nuclei do not arise during and in consequence of the segmentation, but before segmentation and probably in consequence of the repeated divisions of the oosperm nucleus which precedes the division of the egg.

He further shows that regular segmentation of the egg does not exist in the Elasmobranchs. Only in rare cases could he observe the first segmentation furrow alone. In most cases numerous segmentation furrows appear simultaneously after fertilisation.

Before segmentation occurs then there is for a very short a true multinucleated plasmodium, the nuclei dividing by karyokinesis.

But how do these nuclei originate? "By repeated division of the first or oosperm nucleus" is the suggested answer to this question; but it is a curious and very suggestive fact that these primary divisions of the oosperm nucleus have never been observed, and further that the chromatin bodies of the undivided oosperm nucleus, although it never returns to a true resting stage, are remarkably small and numerous.

No karyokinetic figures have been described in the earliest stages of the development of birds.

In Peripatus Novæ Zealandiæ Miss Sheldon was unable to find any very definite figures in the first nuclei that were formed, although a large and typical spindle and spheres of attraction have been described in the division of the oosperm nucleus of Peripatus capensis.

In Julus terrestris, according to Heathcote, the oosperm nucleus divides into irregular lumps of considerable size, but no true karyokinetic figures can be seen.

In Insects we find some very conflicting evidence.

It is true that Henking has recently described and figured some rather irregular karyokinetic figures in the first nuclear divisions of the eggs of some insects that do not apparently segment, but in Pieris, Pyrrochoris and Lasius he points out that after the division of the first equatorial plate the chromosomes disappear, and later—and this is the point to which I wish to call particular attention—the chromosomes appear again and fuse together to form larger bodies which exhibit an equatorial plate.

It is noteworthy too that notwithstanding the fact that several observers have carefully studied the development of the blowfly (Musca), we have no very satisfactory account nor figure of the karyokinetic division of the oosperm nucleus, notwithstanding the fact that in later stages the karyokinetic figures are large, readily seen and quite typical.

Blochman, who figures the spindles of the nuclear division to form the polar bodies and also the spindles of the later stages of the embryo, did not apparently observe the first division of the oosperm nucleus.

He says "Als erste Theilung des Eikernes kann man die Bilder wohl nicht auffassen, weil, wie ein Blick auf die späteren Figuren zeigt, bei Theilungen die Tochter kernplatten stets so fort weit aus einander rücken."

Henking, who carefully investigated the early stages of the blowfly, was unable to find the first division of the oosperm nucleus. In his first paper he described the occurrence of free nuclear formation in yolk.

It does not seem at all reasonable to suppose that nuclei are really formed in the substance of the egg cell from substances other than those derived from pre-existing nuclei. It will require at any rate rather much stronger evidence than we have at present to induce naturalists to believe in "free nuclear formation."

The facts that have been observed then—not only in Musca but also in Pieris, Pyrrochoris and Lasius by Henking, in Phryganids by Patten, and by Tichomiroff in the silkworm—can only be interpreted on the lines either that the first divisions of the oosperm nucleus are regular karyokinetic figures and have been missed, or that the nucleus fragments into small particles, too small to be recognised with any decree of certainty, which are distributed through the substance of the ovum, and give rise to the so-called free-nuclei.

It seems to me to be highly improbable that these first spindles could have been missed by so many careful observers if they really occurred; for the spindles in the formation of the polar bodies and in the later stages of nuclear division are to be made out without much difficulty and seem to be of regular occurrence.

Lastly, I may say that I have already described and figured the fragmentation of the nucleus of the fertilised ovum of Millepora, an ovum that does not segment, and that I failed in the case of Allopora, as in Distichopora, to follow the nuclear changes immediately following the fertilisation of the ovum.

Now it will be observed that in all these cases segmentation of the cell substance of the egg does not occur immediately after fertilisation.

In holoblastic eggs, such as Amphioxus, Echinus, Peripatus capensis and many others, well-marked karyokinetic figures can be seen on the first division of the segmentation nucleus, and also in those meroblastic ova in which evidence of cell formation—that is to say, of the drawing together of the protoplasm into blocks—occurs immediately after fertilisation as in Cephalopods (Vialleton), Anguis fragilis (Oppel), Tubularia (Tichomiroff), and others.

In the case of Aphis there is, as Will observed, a distinct accumulation of the protoplasm round the nuclei, although there is no segmentation.

It seems to me to be probable then that the formation of karyokinetic figures may be in some manner connected with the regular cell formation, but that in cases where cells are not immediately formed nuclei *may* divide amitotically. Or in other words, that the force, whatever it may be, that initiates cell formation drags the nuclei into spindle figures, etc.

How are these views supported by other facts of histology? When the cells are commencing to be formed in Elasmobranch embryos we find spindles, and also in the formation of polar bodies.

In many cases, however, where division of the nucleus is not followed by division of the cell substance, we have amitotic division of the nuclei.

In the spermatogenesis of many animals for example, the nuclei of the spermatogonia increase and multiply long before the protoplasm segregates round them to form the cell substance of the spermatogonia.

In recent years we have fragmentation of the nucleus in spermatogenesis described by Verson in Bombyx.

I can find no trace of mitotic nuclear division in the spermatogenesis of Alcyonium, Millepora, Distichopora and Allopora, although by using the same stains I can demonstrate the karyokinetic figures in the divisions of the spermatogonia of Ascaris where segmentation of the cell protoplasm follows division of the nuclei.

Similarly Brandt was unable to find karyokinesis in the divisions of the nucleus to form the nuclei of the spores in certain Radiolaria, and Hertwig too describes a process of nuclear fragmentation in the formation of the nuclei of the spores of Thallassicola.

Many examples could be given of mitotic division of the nuclei of giant cells, such as we find described in the marrow of bone by Denys, inflammatory cells of the cornea and the spleen of the white mouse by Arnold.

All of these facts of histology tend to show that it is possible that in those ova which do not regularly segment the oosperm nucleus may fragment.

I do not wish for one moment to say that it is definitely proved that it does fragment in any one case, for it is never safe to generalise upon purely negative evidence until every precaution has been taken to eliminate all possible sources of error.

In this case of Distichopora for example, we cannot say that a karyokinetic figure does not occur in the first division of the oosperm nucleus until a great variety of preservative reagents have been experimented with; but on the other hand, I think I have shown sufficient reason to justify us in hesitating to accept the view that seems to be only too prevalent at the present time that karyokinesis must always occur in the first nuclear divisions of the developing ovum.

(3) *Gynodiœcism in the Labiatæ.* (*Second paper.*) By J. C. Willis, B.A., Gonville and Caius College (Frank Smart Student).

In a paper read on May 30th, 1892*, a short preliminary account was given of some observations and experiments upon this subject. These observations have been continued during the

* *Proc. Camb. Phil. Soc.* Vol. vii., Part vi., p. 349.

current year, but as it now seems certain that they cannot be completed in a satisfactory way for some years to come, it appears advisable to give a brief statement of some more of the results so far arrived at, leaving the full discussion until the conclusion of the research.

The observations of 1892 have been made chiefly upon female plants of *Origanum vulgare.* From the hermaphrodite plants on the "Labiatae" bed in the Cambridge Botanic Gardens*, seed was gathered in 1890 and a bed of seedlings was planted out. These flowered in the present year, and a number of them turned out female plants. Unfortunately, owing to the creeping of the stems upon the ground, the plants became inextricably mixed with one another, and it was impossible to determine from which original seedling any given head of flowers arose, and in consequence the exact percentage of female plants could not be determined. Out of about 322 stalks of flowers, however, 11 were female.

Though all derived from one original stock, these seedlings varied considerably among themselves, more especially in the colour of the flowers. Every gradation occurred from heads with deep rose coloured flowers and reddish-brown bracts, to heads with snow-white flowers and green bracts. The parent plants had pale pink flowers of the usual type.

Six heads of female flowers, growing, all but three, at some distance apart, and representing four plants, so far as could be found, were marked, and every flower examined, throughout their flowering season, just as was done with the hermaphrodites in 1891. The experiment gave a similar result. Many abnormal (i.e. staminate) flowers appeared, but the season had apparently no influence. It seemed possible that the weather had some effect, but this could not be satisfactorily made out.

On the hermaphrodite plants, more female flowers than intermediate forms occurred, and so also here, more hermaphrodites occurred than intermediate forms. The corolla again seemed to vary with the number of stamens, the hermaphrodite flowers being the largest and the female smallest, the flowers with 1, 2, or 3 stamens intermediate in size.

Several wild female plants from Abington (Cambs.) were transplanted to the same bed, and observed with the rest. They varied in a very similar way.

Considerable difference, in the size of the flower and in other points, was observed between the various female plants. The wild females had fairly large flowers with the aborted stamens represented by small dark-coloured bodies in the throat. The cultivated seedlings varied greatly in these points. One almost

* *l.c.* table on p. 350, batch A.

exactly resembled the wild ones, the rest had flowers of very various sizes, and the rudimentary stamens were in most cases large and light-coloured. They also differed from the wild form in the time of opening of the stigmas. In the wild plant the stigmas separate shortly after the opening of the flower, and so the female flower differs considerably in this respect from the very protandrous hermaphrodite. One of the seedling females resembled the wild form in this respect; the rest differed from it in various degrees. Their stigmas did not open until some time, occasionally as much as two or three days, after the opening of the flowers. They seemed in fact to have recently arisen from hermaphrodites, and to have retained, though now useless, or even disadvantageous, their protandry.

These and other peculiarities, besides their greater yield of abnormal flowers, led to the idea that possibly some, at any rate, of these female plants had sprung, not from the hermaphrodite, but from the female flowers, upon the parent plant. Experiments to test this point are now in progress.

The variation of the numbers of flowers of each kind on one of these seedling females was very remarkable. Seedling B at the end of the first week, bore 7 abnormal flowers, out of a total of 42, and of these 4 were hermaphrodite. At the end of the third week there were 3 abnormals (no hermaphrodites) in 31, and at the end of the fourth week 25 abnormals (22 hermaphrodites) out of only 32 flowers! Eleven of the hermaphrodites occurred on a lateral branch of the inflorescence, which bore no other flowers. [Cf the case of the hermaphrodite plants in the previous paper*.]

Experiments are now being set on foot to determine the relative fertility of the various types of flower, normal and abnormal, and also to discover the type of offspring to which each gives rise, besides other points.

From the observations on the hermaphrodite plants it seemed possible that the presence of abnormal flowers might be due directly to lack of nutriment, as they were most common on the lateral twigs of the inflorescence. An attempt was made to test this as follows. A string was tied tightly round the main axis of the inflorescence, about the middle, ten days before it began flowering. Only one of the plants thus treated survived in a healthy condition, but the results obtained seemed to favour the view expressed above. The number of abnormal flowers above the string was large (17 in 137), that below was small (1 in 98). Further observations will be made upon this point.

F. Möwes† has given instances observed by him of abnormal

* *l. c.* p. 350.

† "Ueb. Bastarde von Mentha arvensis u. M. aquatica, sowie d. sexuellen

flowers in various Labiatae (*Nepeta Glechoma, Thymus serpyllum,* and *Galeopsis Tetrahit*), and points out that in nearly all the cases he observed, the abortion took place in a symmetrical way, i.e. the abnormal flowers were either female, or had lost two stamens, and those either the two posterior or the two anterior. The observations of 1890—91 on hermaphrodite plants have therefore been analysed carefully, and it has been found that on the whole this statement is correct. Out of 265 abnormal flowers, 179 had the abortion symmetrical. Of these, 142 were female, 26 had the two anterior (long) stamens missing, and 11 the two posterior (short). It was noticed that the long anterior stamens were more often aborted than the short posterior ones.

It is hoped to publish a paper after the conclusion of these observations, giving full details, and also a general discussion of the subject of gynodiœcism, with the various theories concerning it that have from time to time been proposed. These will be tested so far as possible, in the course of the experiments now being carried on.

In conclusion, the author's warmest thanks are due to Mr Francis Darwin, for the constant advice and assistance he has given throughout the work, and to Mr W. French, who kindly undertook some of the observations during the author's absence, in Sept. 1891.

November 28, 1892.

PROF. T. M K. HUGHES, PRESIDENT, IN THE CHAIR.

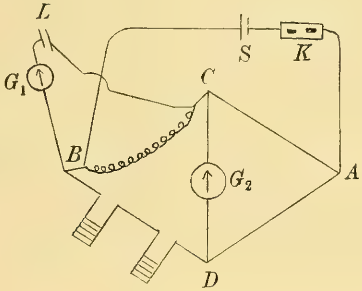
The following communications were made:

(1) *On the rise in resistance of a Conductor when transmitting a Current.* By E. H. GRIFFITHS, M.A., Sidney College, and G. M. CLARK, B.A., Sidney College.

As we mentioned in a paper which we had the honour to read before this Society "On the Measurement of Low Temperature," we have for a long time suspected that the heating effects of small currents necessary to determine a resistance are of more importance than is usually supposed. As these might be sufficient to account for the differences which have occurred in the measurement of the same resistance-coil by different observers, we have thought it advisable to bring before the Society a description of the means we adopted to overcome this difficulty when it presented itself to us, in an exaggerated extent, in our determination of the mechanical equivalent of the heat developed by an electric current. In these experiments the heat being

Eigenschaften hybrider u. gynodiöcischer Pflanzen." Engler's *Botan. Jahrbuch* IV. 1883, p. 189.

developed in a platinum wire, it must of necessity have been at a higher temperature than the water to which it was supplying heat and therefore its resistance at any time was not that at the temperature of the water when no current was passing, but something greater.



A Wheatstone's bridge (Fig.) was constructed of which the arm *BC* contained the calorimeter coil, the corresponding arm *BD* had very nearly the same resistance as *BC*, but was formed of two large German-silver coils, one, belonging to the Cavendish Laboratory, was a triple-strand, containing 1400 ft. of single wire, the other a double strand of stouter wire containing 480 ft.; the mass of metal in this arm was very great, its cooling surface was large, and its temperature coefficient was small, so that the increase in resistance in this arm for currents up to 1 ampère although it could be detected was scarcely measurable: since these arms *BC*, *BD* were equal, the other pair *AC*, *AD* were also equal, these were of the same German-silver wire as the two-strand coil and were each about 5 feet long, and had a resistance of ·3 ohms. They were arranged as near together as possible so as to be equally influenced by air-currents, &c.

The calorimeter coil was furnished with a pair of electrodes at each end, one of each pair was connected with the Clark-cell circuit, the other two being connected directly at *B*, and through the arm *CA* at *C* to the battery circuit in which was placed a rheochord *K* by the adjustment of which we were able to keep the galvanometer G_1 always at its zero, and therefore the difference of potential at the ends of the coil could be made that of 1, 2, or more Clark cells.

[As a full description of the rheochord, and of this "potential balance," as it may be termed, will be described in a paper upon which we are now engaged, we do not think it necessary to inflict any further details on the members of this Society.]

A preliminary rough adjustment of the bridge was made at *A*, so that the bridge was in equilibrium at some temperature within the range of the thermometer (E) used in the calorimeter, the calorimeter was then cooled down and allowed to heat up until the low resistance galvanometer G_2 showed no deflection, the temperature being read at the same moment. This was done in each case whilst the difference of potential at the ends of the coil was increased from 1 to 6 Clark cells: since the resistances of $AD \cdot AC$ bore a constant ratio, and the resistance BD remained unchanged, the resistance of the arm BC, the coil, must also be the same whenever balance is obtained, hence the difference in the observed temperatures gave the number of degrees that the wire was hotter than the water with which it was in contact; or since the temperature coefficient of the wire had been determined [with $E = \cdot 04$ volts] the results could be expressed in terms of increase in resistance.

The following table gives the results of a series of observations taken on Sept. 20, 1892.

No. of Clark Cells	Temperature at which bridge is balanced				Calculated results
	E	C	R at °C.	δR	
1 (·163 Amp.)	376·3, 376·3	21·226	8·7790	x	·0042
2	360·7, 360·2, 360·4	20·818	8·7671	$x + \cdot 0119$	$x + \cdot 0126$
3	332·5, 332·, 332·1	20·100	8·7458	$x + \cdot 0332$	$x + \cdot 0336$
4	292·5, 292·8, 292·3	19·080	8·7157	$x + \cdot 0633$	$x + \cdot 0630$
5	240, 240·5	17·760	8·6771	$x + \cdot 1019$	$x + \cdot 1008$
6	177·5, 176·8	16·202	8·6316	$x + \cdot 1474$	$x + \cdot 1470$

In order to find the value of x these numbers were plotted with E.M.F. as abscissa and δR as ordinate, when they were found to agree with the parabola

$$\delta R = \cdot 0042\, E_c^2 = \cdot 00204\, E^2 = \cdot 1575\, C^2,$$

(where E_c is the E.M.F. of 1 Clark cell) within the limits of experimental error.

From the form of this result, it would appear that the statement

$$\delta R = a C^2$$

would probably hold good in any case, where α is a coefficient dependent upon the nature of the surroundings, and thus it would only be necessary to determine the resistance of any coil with two different E.M.F.'s to find the value of α and hence the limiting value of the resistance when $C = 0$.

This is, of course, a mere suggestion and would require a large number of experiments to verify its truth, especially where, as in standard coils, the wire is almost thermally insulated by being embedded in thick layers of paraffin wax, whilst in our calorimeter coil the thermal conduction was very good, the coil being immersed in well-stirred water. The stirrer revolving at the rate of 2000 revolutions per minute.

We hope to carry out these experiments replacing the calorimeter coil BC, by some paraffin-embedded coil, the Clark cell circuit by a high resistance shunt which would not be appreciably heated by the currents used and would give the means of adjusting the bridge; the large German-silver coils BD could be replaced by a Manganin coil with zero Temperature Coefficient of the same resistance as BC, and the arms AC, AD by two equal and similar coils so chosen as to give maximum sensitiveness to the bridge: a tangent galvanometer in the battery circuit ought to give the current-strength to the required accuracy.

(2) *On the Stability of Maclaurin's Liquid Spheroid.* By A. B. Basset, M.A., F.R.S., Trinity College.

1. One of the most obscure passages in Thomson and Tait's *Natural Philosophy* is the one which relates to the stability of Maclaurin's liquid spheroid. The authors state, without proof, that the spheroid becomes unstable when its excentricity exceeds that of the limiting Jacobian ellipsoid which coalesces with it; that is when its excentricity exceeds ·8127. No allusion is made to the fact that Riemann proved many years previously that for an *ellipsoidal* displacement, the spheroid, if composed of *frictionless* liquid, does not become unstable until its excentricity is equal to ·9529; and as Riemann's result is certainly correct, the natural inference is that the authors were unacquainted with his paper, or were contemplating a disturbance of a more general character. It appears, however, from the papers of Mr Bryan*, that Riemann's result gives the limiting excentricity for which the spheroid is stable, when the disturbance is of the most general possible character; the statement is therefore misleading, and if Mr Bryan's result is correct, it is in its present form inaccurate.

But it would seem from certain recent investigations, that the passage *may* refer to a spheroid composed of *viscous* liquid. There

* *Phil. Trans.* 1889, p. 187; *Proc. Roy. Soc.*, vol. XLVII. p. 367.

is, however, no plain and well-defined statement that the condition in question is intended to apply to a *viscous* and not to a frictionless liquid; and from a careful perusal of the whole passage, I fail to discover any expressions, except one or two of a very vague kind, which indicate that the authors are contemplating the case of a viscous liquid. It is important to recollect that the conditions of stability of a viscous liquid depend upon totally different principles from those of a frictionless liquid, and I propose in the present paper to carefully examine this question.

Stability of Viscous Figures of Equilibrium.

2. When a mass of liquid is rotating as a rigid body about a fixed axis under the influence of its own attraction, it possesses potential energy which may be determined as follows. In the first place let the mass of liquid be supposed to be spherical, and let W_0 be the work which must be done in order to remove every element of the liquid to infinity against the attraction of its parts; in the next place let W be the corresponding work which must be done when the liquid has its actual form; then if U be the potential energy

$$U = W_0 - W \qquad (1).$$

From this result we see that the potential energy in the spherical form is zero, as ought to be the case, for in this form the liquid is incapable of doing any work.

The total energy E is equal to

$$E = \tfrac{1}{2} I\omega^2 + U \qquad (2),$$

where I is the moment of inertia, and ω the angular velocity of the liquid about the fixed axis of rotation; also since

$$I\omega = h,$$

where h is the constant angular momentum, the value of E may be written

$$E = \tfrac{1}{2} h^2/I + U \qquad (3).$$

The first term of E of course represents kinetic energy, but it may also be regarded as potential energy; for the conditions of relative equilibrium will be unchanged, if the rotation is annulled and replaced by a repulsive force $\omega^2 r$ or $h^2 r/I^2$, acting upon every element. If the liquid be supposed to be distributed in the form of an infinitely thin annulus of indefinitely large radius, whose centre coincides with the axis of rotation and whose plane is perpendicular to it, the term $\tfrac{1}{2} h^2/I$ will be indefinitely small; whence the value of this term in the actual state may be regarded as the work which must be done in order to bring the liquid from the annular into the actual form.

But although the presence of the term $\frac{1}{2}h^2/I$, which may be regarded as the potential energy due to centrifugal force, increases the total energy of the system, it diminishes its *capacity to do work upon itself.* For equation (3) may be written

$$E = W_0 - (W - \tfrac{1}{2}h^2/I) \qquad \text{..........(4)},$$

and since W_0 is a constant, the capacity to do work depends upon the last term of (4). Also if (4) be written in the form

$$E = W_0 + (\tfrac{1}{2}h^2/I - W) \qquad \text{..........(5)},$$

it follows that the capacity to do work will be least, when

$$\tfrac{1}{2}h^2/I - W$$

is a minimum. When this is the case, the system will be in stable equilibrium; for if it be disturbed in any manner, its capacity to do work will be increased, and the system will therefore tend to return to its configuration of relative equilibrium, and will perform small oscillations about it.

When the figure is passing through its configuration of relative equilibrium, the portion of the kinetic energy which depends upon the oscillations will be a maximum; but if the liquid is viscous, this portion will gradually be transformed into heat. Hence if $E + \delta E$ be the energy immediately after disturbance, δE must gradually diminish and ultimately vanish; and when δE has been entirely converted into heat, the figure will be restored to its original state before disturbance. We have, therefore, the following rule for determining the steady motion and stability of a mass of *viscous* liquid which is rotating as a rigid body in relative equilibrium. Let the figure be displaced into a slightly different form, and let $W_0 - W$ be the work which must be done against gravitation in order to displace it from the spherical into its disturbed form; also let $\frac{1}{2}h^2/I$ be the kinetic energy of a mass of liquid of the same form as the disturbed figure, which is supposed to be rotating as a rigid body with unchanged angular momentum: then the condition of steady motion is that $\frac{1}{2}h^2/I - W$ should be stationary, and the condition of stability is that this quantity should be a minimum.

3. We are now in a position to give a formal proof of the proposition that Maclaurin's spheroid, when composed of viscous liquid, is unstable for an ellipsoidal displacement if the excentricity exceeds ·8127.

To do this, we must suppose that the liquid is rotating as a rigid body about its least axis, and that the free surface is an ellipsoid which differs very slightly from the spheroidal form, and we have to find the conditions that V should be a minimum, where

$$V = \tfrac{1}{2}h^2/I - W.$$

Now employing the notation of *Hydrodynamics*, §§ 363—367,

$$I = \frac{1}{5} M (a^2 + b^2), \quad W = \frac{2}{5} M \pi \rho abc \int_0^\infty \frac{d\lambda}{P},$$

so that

$$V = \frac{5h^2}{2M(a^2 + b^2)} - \frac{2}{5} M \pi \rho abc \int_0^\infty \frac{d\lambda}{P},$$

whence if the suffixes denote differentiation

$$V_a = -\frac{5h^2 a}{M(a^2+b^2)^2} + \frac{MQ}{5a},$$

$$V_b = -\frac{5h^2 b}{M(a^2+b^2)^2} + \frac{MR}{5b},$$

$$V_{aa} = \frac{5h^2(3a^2 - b^2)}{M(a^2+b^2)^3} + \frac{M}{5a}\left(\frac{dQ}{da} - \frac{c}{a}\frac{dQ}{dc}\right) - \frac{MQ}{5a^2},$$

$$V_{ab} = \frac{20h^2 ab}{M(a^2+b^2)^3} + \frac{M}{5a}\left(\frac{dQ}{db} - \frac{c}{a}\frac{dQ}{dc}\right).$$

Now $\qquad h = \frac{1}{5} M (a^2 + b^2)\,\omega,$

and on substitution in the first two, we obtain the usual conditions of steady motion of a Jacobi's ellipsoid, which includes as a particular case Maclaurin's spheroid. Accordingly when $a = b$, we have

$$h^2 = \tfrac{4}{25} M^2 a^2 Q.$$

Substituting in the last two equations, and omitting the factor $\frac{1}{5}M$, we obtain when $a = b$,

$$V_{aa} = V_{bb} = \frac{1}{a}\left(\frac{dQ}{da} - \frac{c}{a}\frac{dQ}{dc}\right),$$

$$V_{ab} = \frac{1}{a}\left(\frac{dQ}{db} - \frac{c}{a}\frac{dQ}{dc}\right) + \frac{2Q}{a^2}.$$

The value of δV may accordingly be written

$$\delta V = \tfrac{1}{4}(V_{aa} + V_{ab})(\delta a + \delta b)^2 + \tfrac{1}{4}(V_{aa} - V_{ab})(\delta a - \delta b)^2.$$

The value of $V_{aa} + V_{ab}$, as shown in my book, is always positive; accordingly if the displacement were *spheroidal*, so that $\delta a = \delta b$, the steady motion would be stable; but if the displacement is *ellipsoidal*, the steady motion will be unstable unless $V_{aa} - V_{ab}$ is positive. The condition of stability is therefore that

$$a\left(\frac{dQ}{da} - \frac{dQ}{db}\right) - 2Q > 0 \text{..................... (6).}$$

On reduction it will be found that (6) becomes

$$c^2 \int_0^\infty \frac{d\lambda}{(a^2+\lambda)(c^2+\lambda)^{\frac{3}{2}}} - a^4 \int_0^\infty \frac{d\lambda}{(a^2+\lambda)^3 (c^2+\lambda)^{\frac{1}{2}}} > 0,$$

which is the condition that the spheroid should coincide with the limiting Jacobian ellipsoid (*Hydrodynamics*, vol. II. p. 113). The excentricity is determined by the equation

$$e(1-e^2)^{\frac{1}{2}}(3+10e^2) - (3+8e^2-8e^4)\sin^{-1}e = 0,$$

which gives $e = \cdot 8127$.

When the liquid is frictionless, the condition of stability is

$$a\left(\frac{dQ}{da} - \frac{dQ}{db}\right) - Q > 0,$$

giving $e = \cdot 9529$, which shows the difference between the two conditions.

4. We can also show that when the liquid is viscous a Jacobi's ellipsoid, which differs very slightly from the limiting form, is stable for an ellipsoidal displacement. Since V_a, V_b are zero in steady motion, the value of δV is

$$\delta V = \tfrac{1}{2}(V_{aa}\delta a^2 + 2V_{ab}\delta a \delta b + V_{bb}\delta b^2).$$

Let a_0 be the equatorial radius of the spheroid for which

$$e = \cdot 8127,$$

and let the values of a and b in the Jacobi's ellipsoid be

$$a = a_0 + \alpha, \quad b = a_0 + \beta,$$

where α and β are very small quantities; then the value of δV may be written

$$\delta V = \tfrac{1}{4}(V_{aa} + V_{ab})_0 (\delta a + \delta b)^2 + \tfrac{1}{4}(V_{aa} - V_{ab})_0 (\delta a - \delta b)^2$$
$$+ \text{ terms of the third and higher orders,}$$

where the suffix 0 denotes that α and β are to be put equal to zero. But we have already shown that in this case $V_{aa} - V_{ab} = 0$, whence δV is positive unless $\delta a = -\delta b$. It, therefore, follows that the Jacobi's ellipsoid is stable for all ellipsoidal disturbances for which $\delta a + \delta b$ is not zero; but when $\delta a + \delta b$ is zero, the stability depends on terms of the third order, which must be examined. It is, however, probable that the ellipsoid is stable in this case also.

Since the limiting spheroid is a surface of *bifurcation*, we have an example of figure in which there is an exchange of stabilities, the figures belonging to one series becoming unstable at this point, whilst the figures belonging to the other series remain stable.

5. A few years ago, I gave an investigation* which purported to show that a Maclaurin's spheroid, which is composed of frictionless liquid, is stable for an ellipsoidal displacement. This result is known to be erroneous, but at the same time the analysis shows that a Jacobi's ellipsoid composed of frictionless liquid, and which differs very slightly from the spheroidal form, is stable for an ellipsoidal displacement. In the light of Poincaré's investigations this result is interesting; because it shows that a surface of bifurcation does not necessarily involve an exchange of stabilities. On the other hand, we should anticipate that a surface which forms the limit between a stable and an unstable system is a surface of bifurcation, and consequently that the spheroid whose excentricity is ·9529 is a surface of this character.

6. The phrases *ordinary stability* and *secular stability* have been employed by Poincaré in the following sense: let $\epsilon^{\alpha t + \iota\beta t}$ be the time factor of a vibration; then if α is zero, the steady motion (or equilibrium) is *ordinarily stable*, but if α is negative it is *secularly stable*. It appears to me that this distinction is unnecessary, and that the employment of these phrases in this sense is an inaccurate use of language; and that as the physical properties, as well as the conditions of stability, of viscous liquids are different from those of frictionless liquids, the most accurate and intelligible course is to discuss the two species of liquids separately. If α is zero or negative, the system is *absolutely stable*; for in both cases it performs small oscillations about its position of steady motion (or equilibrium), but in the one case the oscillations are permanent, whilst in the other they gradually diminish and ultimately die away. It is, however, possible for the steady motion (or equilibrium) to be such that a small displacement causes the system to perform *finite* oscillations about its undisturbed configuration. In cases of this kind, the time factor in the beginning of the disturbed motion will involve the exponential term $\epsilon^{\alpha t}$, where α is positive, and the system will be unstable in the ordinary sense of the word; but as the *complete* solution of the equations of disturbed motion must necessarily consist of periodic terms, the system may accurately be described as *secularly stable*.

7. Up to the present time we have confined our attention to ellipsoidal displacements; but the stability of a rotating ellipsoid composed of viscous liquid has been investigated in a very able manner by Poincaré†, when the disturbance is of the most general character. He has, however, made a slight slip in the work, in consequence of having omitted to recognize the fact that the angular velocity of the disturbed figure is not the same as that of

* *Proc. Lond. Math. Soc.*, vol. XIX. p. 53.

† *Acta Mathematica*, vol. VII. p. 259.

the original one. The quantity which remains unchanged is the angular momentum. If the angular velocity were supposed to remain unchanged, it would be necessary for the disturbance to consist in part of a couple about the axis of rotation, and a disturbance of this kind could not be applied to a figure of revolution by means of an external pressure, although it might of course be applied to a Jacobi's ellipsoid. The constancy of the angular momentum leads to an additional term in the conditions of stability, and it is the existence of this term which renders Maclaurin's spheroid stable for a *spheroidal* displacement. We shall presently see, that if the disturbance is such that the figure assumes the form of any other surface of revolution which is symmetrical with respect to the equatorial plane, the spheroid will be unstable when its excentricity is sufficiently great.

8. I shall now give Poincaré's investigation.

Let V be the total potential due to gravitation and centrifugal force; then

$$V = \int \frac{dm'}{R} + \frac{1}{2}\,\omega^2 r^2,$$

where dm' is an element of mass, and R its distance from any point x, y, z.

Since the angular momentum h is connected with ω by the equation $$h = I_0 \omega,$$

where I_0 is the moment of inertia of the figure about its axis, the equation for V becomes

$$V = \int \frac{dm'}{R} + \frac{h^2 r^2}{2 I_0^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

The potential energy of the system due to gravitation is

$$W_1 - \frac{1}{2}\iint \frac{dm'dm}{R},$$

where W_1 is the work which must be done in removing a spherical mass of liquid of equal volume to infinity against the attraction of its parts.

In evaluating this integral, Poincaré divides the elements into two parts dm, dm', and $d\mu$, $d\mu'$, where the first two refer to the original figure, and the last two to the stratum of liquid whose superposition may be conceived to constitute the disturbed figure. The integral may, therefore, be written

$$\frac{1}{2}\iint \frac{dm'dm}{R} + \frac{1}{2}\iint \frac{d\mu d\mu'}{R} + \frac{1}{2}\iint \frac{d\mu dm'}{R} + \frac{1}{2}\iint \frac{d\mu' dm}{R} \ldots\ldots\ldots (8).$$

The last two integrals are known to be equal, because they represent the mutual potential energy of the original figure and the stratum. To evaluate this integral, let V' be the gravitation potential of the original figure at an external point; then if λ be a length measured along the normal

$$V' = V_0' - g'\lambda,$$

where g' is the normal component of the attraction at the surface. But if g be the total force in this direction due to gravitation and centrifugal force,

$$g = g' - \omega^2 r_0 l = g' - h^2 r_0 l / I_0^2;$$

whence

$$\iint \frac{dm' d\mu}{R} = \int V_0' d\mu - \int g\lambda d\mu - \frac{h^2}{I_0^2}\int r_0 l\lambda d\mu \quad \text{.............. (9).}$$

Now V is constant at the surface, also since the mass of the stratum is zero,

$$\int d\mu = 0,$$

accordingly from (7) we get

$$\int V_0' d\mu = -\frac{h^2}{2I_0^2}\int r_0^2 d\mu,$$

and (9) becomes

$$\iint \frac{dm' d\mu}{R} = -\frac{h^2}{2I_0^2}\int (r_0 + l\lambda)^2 d\mu - \int g\lambda d\mu + \frac{h^2}{2I_0^2}\int l^2\lambda^2 d\mu \quad \text{...(10).}$$

Now $d\mu = \rho d\lambda dS$, whence the last integral in (10) is of the third order of small quantities and may be neglected; also if σ be the thickness of the stratum, and I the moment of inertia of the disturbed figure about the axis of rotation, the right-hand side of (10) becomes

$$-\frac{h^2(I - I_0)}{2I_0^2} - \frac{1}{2}\rho\iint g\sigma^2 dS.$$

Whence if W be the potential energy of the disturbed figure *due to gravitation* and W_0 be that of the original figure

$$W_0 = W_1 - \frac{1}{2}\iint \frac{dm\, dm'}{R} \quad \text{.................... (11),}$$

and

$$W = W_0 - \frac{1}{2}\iint \frac{d\mu' d\mu}{R} + \frac{1}{2}\rho\iint g\sigma^2 dS + \frac{h^2(I - I_0)}{2I_0^2} \quad \text{......... (12).}$$

The total energy E of the disturbed figure, which represents its capacity to do work upon itself, is obtained by adding the term

$h^2/2I$ to (12), which may be regarded indifferently as kinetic energy or as potential energy due to centrifugal force; whence if E_0 be the value of E for the original figure

$$E = E_0 + \frac{1}{2}\rho\iint g\sigma^2 dS - \frac{1}{2}\iint\frac{d\mu' d\mu}{R} + \frac{h^2(I-I_0)^2}{2I_0^2 I} \quad\ldots\ldots\ldots(13).$$

The second integral in (13) is equal to

$$\rho^2\iint\frac{\sigma\sigma' dS dS'}{R}.$$

Now σ' is a function of the position of a point on the surface of the original figure, and may, therefore, be regarded as a surface distribution of matter; whence if U be its potential

$$\left.\begin{aligned} &\int\frac{\sigma' dS'}{R} = U \\ \text{and}\qquad &\frac{dU_1}{dn} - \frac{dU_2}{dn} = -4\pi\sigma' \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

where U_1, U_2 are the values of U just outside and just inside the surface. The integral may, therefore, be written

$$\rho^2\iint\sigma U dS;$$

also since $(I-I_0)^2$ is a quantity of the second order, we may write $I = I_0$ in the denominator of the term in which this quantity occurs, whence

$$E = E_0 + \frac{1}{2}\rho\iint g\sigma^2 dS - \frac{1}{2}\rho^2\iint U\sigma dS + \frac{h^2(I-I_0)^2}{2I_0^3} \quad\ldots\ldots(15).$$

The right-hand side of this equation represents the capacity of the disturbed figure to do work on itself; and as the condition of stability of the original figure requires that E_0 should be a minimum, the figure will be stable provided the sum of the last three terms is positive, and unstable if it is negative.

Comparing (15) with the result given by Poincaré on p. 315, it will be observed that he has omitted the last term of all. This term, as will be presently shown, leads to the result that Maclaurin's spheroid is stable for a spheroidal displacement.

9. Poincaré has applied (15), with the last term omitted, to investigate the stability of a Jacobi's ellipsoid. The potential U is expressed in a series of Lamé's functions, and the value of σ is then found from (14), and the condition that the sum of the *second and third* terms of (15) should be positive, leads to an

equation by means of which the critical figures which form the borderland between stability and instability are obtained. I do not propose to consider whether any of Poincaré's results for ellipsoids are vitiated by the circumstance that he has failed to take account of the last term of (15), as my object is to discuss the simpler case of a Maclaurin's spheroid.

10. But first of all I wish to make some observations on the question of notation. The various functions which occur in Harmonic Analysis have received great development during recent years; and numerous functions, whose properties were formerly but little known, are now largely employed in physical investigations, and are found to provide the mathematical physicist with a powerful weapon for attacking unsolved problems. It will be admitted on all hands that uniformity of notation is of the utmost importance, although opinions may differ as to the desirability of some particular notation. The notation $P_n(\mu)$ for an ordinary zonal harmonic, where μ is not necessarily less than unity, is well established; and the notation which I always employ, and should recommend others to adopt, is that $P_n^m(\mu)$ should be used to denote the function,

$$(1-\mu^2)^{\frac{1}{2}m}\frac{d^mP_n}{d\mu^m},$$

and should be called an associated function of the first kind of degree n and order m. When $\mu < 1$, it follows with this notation that

$$\sum_{m=0}^{m=n} A_n^m P_n^m(\mu)\cos(m\phi+\alpha_m),$$

where $\mu=\cos\theta$, is the most general form of a spherical surface harmonic of degree n.

The associated function of the second kind may be denoted by Q_n^m; and when $\mu > 1$, as is the case with prolate spheroids, the factor $(\mu^2-1)^{\frac{1}{2}m}$ must be written in the place of $(1-\mu^2)^{\frac{1}{2}m}$. The corresponding spheroidal harmonics which are employed in the case of oblate spheroids may be denoted by $p_n^m(\nu)$ and $q_n^m(\nu)$; and as it is often unnecessary to specify the argument, powers of these quantities may be written $(p_n^m)^2$. The notation $I_m(x)$ and $K_m(x)$, by which I proposed to denote the two kinds of associated Bessel's functions, was adopted a few years ago by a Committee of the British Association which was formed for the purpose of tabulating these functions; and the second solution of Bessel's equation, which frequently occurs in the theory of sound, I should propose to denote by $Y_m(x)$.

11. We shall now apply (15) to investigate the stability of a Maclaurin's spheroid which is composed of viscous liquid.

Let the original surface be confocal with the spheroid

$$\frac{x^2+y^2}{\nu^2+1}+\frac{z^2}{\nu^2}=c^2,$$

and let the system of confocal hyperboloids be

$$\frac{x^2+y^2}{1-\mu^2}-\frac{z^2}{\mu^2}=c^2,$$

and let $\nu=\gamma$ be the free surface of the original figure.

Let ds_1, ds_2 be line elements measured along the normals of the spheroid and hyperboloid, and p the central perpendicular on to the tangent planes to the *former*, then

$$ds_1=h_1^{-1}d\nu, \quad ds_2=h_2^{-1}d\mu,$$

also

$$h_1^{\,2}=\frac{\nu^2+1}{c^2(\nu^2+\mu^2)}, \quad h_2^{\,2}=\frac{1-\mu^2}{c^2(\nu^2+\mu^2)}, \quad \frac{1}{p^2}=\frac{\nu^2+\mu^2}{c^2\nu^2(\nu^2+1)},$$

whence

$$h_1=\frac{p}{c^2\nu}, \quad h_2=\frac{p(1-\mu^2)^{\frac{1}{2}}}{c^2\nu(\nu^2+1)^{\frac{1}{2}}} \quad \text{...............(16)},$$

and

$$dS=c^3\gamma(1+\gamma^2)p^{-1}d\mu d\psi, \quad dn=ds_1=c^2\gamma p^{-1}d\nu \text{.....(17)}.$$

The values of U_1, U_2 may be written

$$U_1=\Sigma A_n^m p_n^m(\gamma)\,q_n^m(\nu)P_n^m(\mu)\cos m\psi,$$

$$U_2=\Sigma A_n^m\,q_n^m(\gamma)\,p_n^m(\nu)\,P_n^m(\mu)\cos m\psi.$$

Accordingly from (14) and (17),

$$\sigma=\frac{p}{4\pi c^2\gamma}\Sigma A_n^m(p'^m_n q_n^m-p_n^m q'^m_n)\,P_n^m(\mu)\cos m\psi.$$

From the formulæ given by Mr Bryan*,

$$t_n^m=p_n^m, \quad u_n^m=(-)^m\frac{n-m!}{n+m!}q_n^m,$$

whence
$$p'^m_n q_n^m-p_n^m q'^m_n=\frac{n+m!}{n-m!}\frac{(-)^m}{1+\gamma^2},$$

and therefore

$$\sigma=\frac{p}{4\pi c^2\gamma(1+\gamma^2)}\Sigma(-)^m A_n^m\frac{n+m!}{n-m!}P_n^m(\mu)\cos m\psi\text{..(18)}.$$

* *Proc. Roy. Soc.*, vol. XLVII. p. 368.

Accordingly

$$\iint \sigma U dS = c\Sigma \frac{A_n^2}{2n+1} p_n q_n$$

$$+\frac{1}{4} c\Sigma (-)^m (A_n^m)^2 \frac{n+m!}{n-m!} p_n^{\bar{m}} q_n^m \int_{-1}^{1} (P_n^m)^2 \, d\mu \quad \ldots (19),$$

where in the last term m has any of the values 1, 2, 3 ... n.

From the ordinary formulæ for relative equilibrium

$$pg = (A - \omega^2) c^2 (1 + \gamma^2), \quad pg = Cc^2\gamma^2,$$

from the last of which we obtain

$$pg = 4\pi\rho c^2 \gamma (1 + \gamma^2) p_1 q_1 \quad \ldots\ldots\ldots\ldots (20);$$

whilst the elimination of pg gives the ordinary equation for ω^2. We thus obtain

$$\iint g\sigma^2 dS = \frac{\rho c}{4\pi} p_1 q_1 \iint \left\{\Sigma (-)^m A_n^m \frac{n+m!}{n-m!} P_n^m(\mu) \cos m\psi\right\}^2 d\mu \, d\psi$$

$$= \rho c p_1 q_1 \Sigma \frac{A_n^2}{2n+1} + \frac{1}{4} \rho c p_1 q_1 \Sigma \left\{\frac{n+m!}{n-m!} A_n^m\right\}^2 \int_{-1}^{1} (P_n^m)^2 d\mu$$

$$\ldots\ldots (21).$$

To calculate the last term of (15) we have

$$I - I_0 = \rho \iint (r_0 + l\lambda)^2 d\lambda \, dS$$

$$= \rho \iint r_0^2 \sigma dS + \rho \iint r_0 \sigma^2 l dS.$$

Since the square of this quantity occurs, we may neglect the last term in the integral, whence

$$I - I_0 = \frac{\rho c^3}{4\pi} (1 + \gamma^2) \iint (1 - \mu^2) \Sigma (-)^m \frac{n+m!}{n-m!} A_n^m P_n^m(\mu) \cos m\psi \, d\mu \, d\psi$$

$$= \frac{1}{2} \rho c^3 (1 + \gamma^2) \iint_{-1}^{1} (1 - \mu^2) \Sigma A_n P_n(\mu) d\mu.$$

Since the mass of the new figure is equal to that of the old, $\int \sigma dS = 0$; whence $A_0 = 0$; also since

$$1 - \mu^2 = \tfrac{2}{3} (P_0 + P_2),$$

we finally obtain

$$I - I_0 = \tfrac{2}{15} \rho c^3 (1 + \gamma^2) A_2 \quad \ldots\ldots\ldots\ldots (22).$$

Substituting from (19), (21) and (22) in (15), the condition of stability is that

$$\frac{1}{2}\rho^2 c\Sigma\,(p_1q_1 - p_nq_n)\,\frac{A_n^{\,2}}{2n+1} + \frac{h^2}{2I_0^{\,3}}\left(\frac{2}{15}\rho c^3\right)^2 (1+\gamma^2)^2 A_2^{\,2}$$

$$+\frac{1}{8}\rho^2 c\Sigma\left\{p_1q_1 - \frac{n-m!}{n+m!}(-)^m p_n^{\,m} q_n^{\,m}\right\}\left(\frac{n+m!}{n-m!}A_n^{\,m}\right)^2\int_{-1}^{1}(P_n^{\,m})^2 d\mu \quad \ldots\ldots(23),$$

should be positive.

Comparing this result with that given by Mr Bryan*, which is Poincaré's so-called condition of secular stability, and having regard to the difference of notation, we see that the condition of stability (except for the simultaneous values of $m=0$, $n=2$) is that

$$p_1q_1 - \frac{n-m\,!}{n+m\,!}(-)^m p_n^{\,m} q_n^{\,m} > 0 \quad \ldots\ldots\ldots\ldots\ldots(24),$$

which agrees.

12. We shall now discuss this result in the special case in which $m=0$, so that the displacement is symmetrical with respect to the axis of revolution, in which case *provided n is not equal to* 2, the condition becomes

$$p_1q_1 - p_nq_n > 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25).$$

If e be the excentricity, $e=(1+\gamma^2)^{-\frac{1}{2}}$, and consequently when $e=0$, $\gamma=\infty$, and when $e=1$, $\gamma=0$. Now Poincaré and Mr Bryan have both shown that (24) is essentially positive provided $n-m$ is odd, and consequently the spheroid is stable when $m=0$, and n is odd, that is for displacements represented by harmonics of odd degree. But, when $m=0$ and n is even, the first term of (25) vanishes when $\gamma=0$, whilst the second term remains finite, and consequently the spheroid becomes unstable for such a displacement, provided its excentricity is sufficiently great. The values of the excentricity for which instability commences are given by the equation

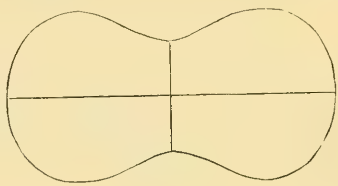
$$p_1q_1 - p_{2n}q_{2n} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26),$$

where $n = 2, 3\ldots.$

From these results we conclude that Maclaurin's spheroid is unstable for a zonal displacement of any even order which is greater than 2, provided the excentricity is sufficiently great; and the inference to be drawn from these results is that the least value of e which makes the left-hand side of (26) vanish and change sign, is a spheroid of bifurcation, which forms the limit between the spheroidal system and another system of surfaces of

* *Proc. Roy. Soc.*, vol. XLVII. p. 369, equation (7).

revolution which are probably stable, at any rate if they do not differ very greatly from the spheroidal form. When the angular velocity is sufficiently great, it seems certain that a ring-shaped figure exists; and consequently if the angular velocity be conceived to diminish, the ring would close up until it assumed the form shown in the figure; and if the angular velocity still further



diminished, the depressions at the poles would disappear, and the liquid would finally assume a spheroidal form. That there must be a spheroidal surface, which forms the limit between the spheroidal form and a series of surfaces which ultimately become annular, appears to me so probable as almost to amount to a certainty.

13. We must now consider the case in which $m = 0$, and $n = 2$. Mr Bryan has noticed that in this case the left-hand side of the inequality (25) vanishes and changes sign for a certain value of the excentricity, and he has endeavoured to account for this by stating that* "it does not indicate that the spheroid in question is secularly unstable for this particular type of displacement. Its meaning is that the spheroid is more oblate than that form for which the angular velocity is a maximum." This explanation is incorrect; for, as we shall now show, the displacement represented by $n = 2$ is a spheroidal displacement for which the figure is stable, and the error into which he has fallen has doubtless arisen from his not having noticed that Poincaré has omitted the last term in (15).

Since $$h = I_0\omega,$$

$$I_0 = \tfrac{8}{15}\pi\rho c^5\gamma(1+\gamma^2)^2 \quad\text{.....................(27)},$$

it follows from (22) that

$$\frac{h^2(I-I_0)^2}{I_0^3} = \frac{\rho c A_2^2\omega^2}{30\pi};$$

also since $\omega^2 = 2\pi\rho\gamma\{(1+3\gamma^2)\cot^{-1}\gamma - 3\gamma\}$,

the condition of stability when $n = 2$ becomes

$$\gamma q_1(\gamma) - p_2(\gamma)q_2(\gamma) + \tfrac{1}{3}\{(1+3\gamma^2)\cot^{-1}\gamma - 3\gamma\} > 0.$$

* *Proc. Roy. Soc.*, vol. XLVII. p. 371.

Substituting the values of q_1, q_2, p_2, the condition becomes

$$3\gamma + 9\gamma^3 - (-\tfrac{1}{3} + 6\gamma^2 + 9\gamma^4)\cot^{-1}\gamma > 0,$$

or $$9e(1-e^2)^{\frac{1}{2}}(3-2e^2) - (27 - 36e^2 + 8e^4)\sin^{-1} e > 0 \quad \ldots(28).$$

This condition is the same as the condition of stability of a Maclaurin's spheroid composed of *frictionless* liquid, when the displacement is spheroidal (see *Hydrodynamics*, vol. II. p. 124), and consequently the spheroid is stable for this kind of displacement.

Stability of Figures composed of Frictionless Liquid.

14. The investigation of the stability of a rotating mass of *frictionless* liquid by means of the energy method depends upon totally different principles. It must be borne in mind that in steady motion it is not essential that the liquid should rotate as a rigid body, and one figure is known (the irrotational ellipsoid) in which the motion does not possess molecular rotation. On the other hand, if the liquid is *viscous*, and is acted upon by no forces except those due to its own attraction, steady motion cannot exist unless the liquid rotates as a rigid body; for if any different motion existed at any particular instant, it would gradually be extinguished by viscosity, and the final state would necessarily be a rigid body motion.

15. Whenever the momenta of a conservative dynamical system, or any quantities in the nature of momenta, are constant throughout the motion, the steady motion and stability may be investigated in the following manner*. A constant momentum always involves an ignored coordinate, and if the velocities corresponding to the ignored coordinates be eliminated from the Lagrangian expression for the kinetic energy, the result will be of the form $\mathfrak{T} + \mathfrak{K}$, where $\mathfrak{T}$ is a homogeneous quadratic function of velocities $\dot{\theta}$ which are the time variations of coordinates which appear in the expression for the total energy of the system, and $\mathfrak{K}$ is a similar function of the constant momenta. Under these circumstances, it follows that if V be the potential energy of the system measured from a configuration of stable equilibrium, the steady motion will be obtained by assigning constant values to the coordinates θ, and making $\mathfrak{K} + V$ stationary; and the steady motion will be stable, provided $\mathfrak{K} + V$ is a minimum. The function $\mathfrak{K}$ is accordingly the kinetic energy of the most general possible motion which the system could assume if the coordinates θ were constrained to remain invariable, and the function $\mathfrak{K} + V$ is the total energy of that motion.

* *Proc. Camb. Phil. Soc.*, vol. VII. p. 351.

16. When a mass of frictionless liquid is rotating in any manner in steady motion, the fact of the motion being steady requires that the form of the free surface should remain invariable, and that it should rotate like a rigid body, although it is not necessary for the particles composing the film of liquid which forms the free surface to move in this manner. The kinetic energy of the steady motion will accordingly be a homogeneous quadratic function of constant quantities which specify the fact that the generalized momenta and vorticity of the liquid are constant. This is the function $\mathfrak{K}$. Now let the liquid be disturbed in any manner, subject to the condition that the generalized momenta remain unchanged; and in contemplating conceivable disturbances, it must be recollected that although it is usually possible to apply a disturbance which will alter one or more of the generalized momenta, it is impossible to alter the vorticity. The new form of the free surface will depend upon certain parameters θ which are coordinates, and which have certain definite values in steady motion; accordingly the kinetic energy of the disturbed motion will be expressible in the form $\mathfrak{T} + \mathfrak{K}$, where $\mathfrak{T}$ is a homogeneous quadratic function of the velocities $\dot{\theta}$. The form of the function $\mathfrak{K}$ in the disturbed motion will not necessarily be the same as in steady motion, since it frequently happens that in steady motion some of the coordinates θ are zero or are equal to one another.

17. The function V, which is the potential energy due to gravitation, can be calculated by Poincaré's method; but the calculation of $\mathfrak{K}$ presents various difficulties. When the liquid is rotating about a fixed axis, it sometimes happens that in the beginning of the disturbed motion the vortex lines remain parallel to that axis, and that the new value of the molecular rotation is independent of the position of particular particles of liquid. Whenever this is the case, the disturbed motion depends upon a velocity potential, and can be determined by means of Prof. Greenhill's method. Let the liquid be enclosed in a case whose form is the same as that of the free surface of the disturbed figure at any particular epoch, and let the liquid be frozen and set in rotation with angular velocity ζ about the fixed axis of rotation; let the liquid now be melted and an additional angular velocity Ω be impressed on the case, then $\mathfrak{K}$ is the kinetic energy of the resulting motion.

The subsequent calculation is as follows. Let the motion be referred to axes x and y fixed in the case, and which are therefore rotating about the fixed axis of z with angular velocity ω, where $\omega = \Omega + \zeta$; then if u, v, w be the velocities of the liquid referred to this set of axes,

$$u = \frac{d\phi}{dx} - \zeta y, \quad v = \frac{d\phi}{dy} + \zeta x, \quad w = \frac{d\phi}{dz} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where ϕ is obviously the velocity potential of the irrotational motion which would take place if the case were filled with liquid at rest, and then set in rotation about the axis of z with angular velocity Ω.

The value of $\mathfrak{K}$ is

$$\mathfrak{K} = \frac{1}{2}\rho \iiint \left\{ \zeta^2 (x^2 + y^2) + 2\zeta \left(x\frac{d\phi}{dy} - y\frac{d\phi}{dx} \right) \right\} dxdydz$$
$$+ \frac{1}{2}\rho \iint \phi \frac{d\phi}{dn} dS \quad \text{..........(2)},$$

also if h be the angular momentum,

$$h = \rho \iiint \left\{ \zeta (x^2 + y^2) + x\frac{d\phi}{dy} - y\frac{d\phi}{dx} \right\} dxdydz \quad \text{............(3)},$$

and since by hypothesis ζ is independent of x, y and z, we obtain

$$\mathfrak{K} = h\zeta - \frac{1}{2} I\zeta^2 + \frac{1}{2}\rho \iint \phi \frac{d\phi}{dn} dS \quad \text{..................(4)},$$

where I is the moment of inertia of the disturbed figure about the axis of z. Since ϕ depends upon the disturbed motion, it must necessarily be a linear function of small quantities of the first order, whence $\phi d\phi/dn$ is of the second order, and consequently the surface integral may be taken over the original surface.

The molecular rotation ζ must be determined from the condition that the vorticity remains constant; and since by hypothesis the vortex lines remain parallel to z in the beginning of the disturbed motion, and ζ is independent of x, y and z, we obtain

$$A\zeta = A_0\zeta_0 = \tau \ \text{(say)} \quad \text{..........................(5)},$$

where A is the area of the curve of intersection of the disturbed surface with the plane (xy).

If, therefore, the velocity potential ϕ can be calculated, Ω and ζ can be eliminated from (4) by means of (3) and (5); and the resulting value of $\mathfrak{K}$ will be a homogeneous quadratic function of h and τ, whose coefficients are functions of the coordinates of the disturbed surface.

18. The reader who is acquainted with the methods by which the steady motion of the ellipsoidal figures is investigated (see *Hydrodynamics*, vol. II. ch. XV.), and the method by which the stability of such figures has been ascertained in the case of an ellipsoidal displacement, will experience no difficulty in following the preceding argument. It will, however, be noticed that the method of § 17 depends upon certain conditions being fulfilled, and when this is not the case a different course of procedure is necessary. Although it is impossible to alter the vorticity of a

given mass of liquid by natural forces or by any operations performed on the boundary, it is usually possible to alter the value of the resultant molecular rotation as well as the configuration of the vortex lines. If the liquid is rotating as a rigid body in steady motion about the axis of z, the vortex lines will be parallel to that axis, and the molecular rotation will be constant throughout the whole mass of liquid; but it is quite possible that certain disturbances may twist the vortex lines into slightly sinuous forms, which differ from straight lines by small quantities depending on the disturbed motion, and also that the new value of the molecular rotation at any point of the liquid may be a function of the position of that point. Whenever this is the case, the velocities during the disturbed motion will not be expressible by means of (1), and the method explained in the last section for calculating $\mathfrak{K}$ will not be applicable. Under these circumstances the method of Poincaré and Mr Bryan*, for calculating the disturbed motion would enable us to find the value of $\mathfrak{K}$ in the form of a series, and the discussion of this series combined with the one for V would lead to the conditions of stability.

* *Phil. Trans.*, 1889, p. 187.

PROCEEDINGS

OF THE

Cambridge Philosophical Society.

Monday, Jan. 30, 1893.

PROFESSOR T. MCK. HUGHES, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *On a new Fern from the Coal Measures.* By A. C. SEWARD, M.A., St John's College.

Abstract.

The specimen described as a new species, *Rachiopteris Williamsoni,* resembles in certain particulars the genus *Myeloxylon,* but possesses distinctive characters not previously recognised in fossil Fern petioles. *Rachiopteris Williamsoni* may be briefly described as a petiole with scattered vascular bundles; those near the periphery appear to be rather collateral than concentric in structure, but the larger bundles have a more decided concentric arrangement of the xylem and phloem. Each group of xylem elements is surrounded by a ring of small secretory canals. The hypoderm is like that of *Myeloxylon,* and gum (?) canals are abundantly distributed in the ground tissue.

(2) *On the Structure and Functions of the Alimentary Canal of* DAPHNIA. By W. B. HARDY, M.A., Caius College, and W. MC DOUGALL, St John's College.

[*Received March* 28, 1893.]

So far as we know none of the workers who have investigated the gut of the Crustacea have described the existence in the case of the lowest members of that group of a differentiation of regions corresponding to the processes of digestion, of absorption, and of elaboration of the fæces. As is well known, the gut of the lower Crustacea differs in a very striking way from that organ in the higher forms. Throughout the entire group the existence of the three main divisions, stomodæum, mesenteron and proctodæum, is very clearly seen, but whereas in the former the mesenteron constitutes the greater part of the gut, in the latter it is limited to a very short region into which the bile-ducts open, and which forms only a very small fraction of the total length of the alimentary canal.

The long mesenteron of the lower forms exists as a simple tube offering no obvious distinction into regions or diversity of structure save the two so-called liver diverticula which spring from its anterior end and extend forwards over the brain.

The œsophagus or stomodæum is a short tube running upwards and forwards from the mouth to open on the ventral surface of the mesenteric tube near its anterior end and a little posterior to the points of origin of the liver diverticula which spring from the lateral aspect.

The proctodæum is rather longer than the stomodæum but differs in this and other respects in the different divisions of the Entomostraca. In the Phyllopoda it occurs as a short simple tube.

Though the mesenteron appears as a simple tube without any obvious anatomical expression of differentiation of function, yet a study of the processes of deglutition and digestion, and of the character and arrangement of the cells lining its wall makes evident the fact that in *Daphnia* this apparently simple tube is divided into three regions, an anterior region devoted to the absorption of the products of digestion, a middle region wherein digestion occurs, and a posterior region in which the fæces are formed.

The fact that digestion occurs in a region of the gut *posterior* to that which is occupied in the absorption of the products is so far as we know without a parallel in the animal kingdom, except perhaps among the simplest Cœlenterates.

For purposes of observation *Daphnia* can be readily fed by pouring beaten yolk of an egg, milk or carmine &c. over the bottom of the dish in which it is living, and the phenomena accompanying the taking in and digestion of food may be easily followed in the living *Daphnia* owing to the transparent nature of the animal. The various events will be described in the order in which they occur.

Deglutition is a rapid act. The food particles, e.g. carmine, or yolk globules, are carried over the mouth in the current of water which is constantly maintained by the movements of the foliaceous appendages and many of them adhere to the sticky surfaces of the mouth appendages. These adherent particles are formed into a bolus by the movements of the appendages.

We have not succeeded however in determining how this is done.

When the bolus is complete it is rapidly carried into the mesenteron by a peristaltic movement of the œsophagus, which in its resting condition is a closed tube flattened dorso-ventrally. The peristaltic movement spoken of above consists of a wave of dilatation which starts at the mouth, and which is followed by a wave of constriction. The bolus is sucked into the œsophagus by the first wave and then thrown with a certain degree of force into the mesenteron.

When the peristaltic wave reaches the mesenteron it is continued backwards over that structure and carries the bolus before it as far as the middle third.

The act of deglutition therefore is brought about by a peristaltic wave which starts at the mouth as a result of the stimulation of the sensory surfaces there and runs backwards carrying the food along the gut to the middle third or digestive region of the mesenteron.

Digestion. The food when swallowed consists of particles, e.g. precipitated proteid, fat globules, carmine grains, or unicellular plants, which are glued together by some sticky substance. As digestion proceeds the insoluble particles are slowly liberated and some are carried into the anterior region of the midgut and even into the liver diverticula. There the nutritive particles such as the fat globules of milk or yolk (but *not* the carmine grains) are ingested by the columnar cells lining that region. If the food contain any soluble colouring matter, such for instance as chlorophyll, that also accumulates in the anterior region. The non-nutritive particles do not accumulate in the anterior region but are from time to time driven backwards. Thus, in the case of an animal which has been fed on green algae we find the anterior third of the mesenteron including the liver diverticula occupied with a bright-green fluid in which are suspended a few solid particles; the middle region occupied by a dark-green mass of still undigested food, while the posterior region contains a mass of brown particles which will form the fæces.

The movements of the gut which bring about this distribution of the contents are twofold. First there is a constantly occurring peristalsis which consists of waves occurring at regular intervals. These start at the junction of mesenteron and proctodæum and run *forwards,* ultimately dying away in the liver diverticula. They are best developed, that is they lead to the most considerable constriction, in the posterior region, and they are apparently conditioned by the presence of food particles or of their soluble products in the lumen of the gut. They, however, are sometimes seen in starving animals.

The second movement consists of a quick contraction of the walls of the liver diverticula which occurs at irregular intervals during the progress of digestion and which is apparently due to the stimulation of the walls of the liver diverticula by innutritious food particles. It is difficult to be certain on this point, but so far as our observations go they agree with this view of the causation of the movement.

It is clear that the first movement is the agent which propels the soluble products of digestion and the freed discrete food particles to the anterior region of the mesenteron. The contractions of the liver diverticula on the other hand serve to drive the innutritious residue of these particles back to the middle and posterior regions.

The effect of these two movements may be made clear by taking the case of an animal fed on a mixture of yolk of egg and carmine. The yolk particles, consisting mainly of proteid precipitate and fat globules, and the carmine grains form at first an agglutinated mass in the middle region of the gut. As solution proceeds the fat globules and carmine grains are liberated and are driven in relatively small numbers into the anterior region, where the former are taken up by the lining epithelium so that the walls become in time thickly studded with fat drops. The carmine particles are not taken up but are at intervals driven back by contractions of the liver diverticula to the middle region.

Absorption. By feeding with food rich in fat globules we can determine the region in which the absorption of solid particles occurs, and we find that fat is most readily taken up by the cells of the anterior region, including the liver diverticula. If the amount of fat given be not great, globules appear practically only in this region. If however a large quantity is ingested, in 24 to 40 hours the cells of the middle region also become considerably loaded but always to a much less extent than those of the anterior region. The globules of fat in the cells differ in size in the two regions, thus in one animal the globules in the cells of the liver diverticula measured 3 to 5 μ in diameter, in the anterior region of the mesenteron they measured 4 to 7 μ; while in the middle region the globules were relatively scanty and too small to measure.

If the fatty diet be pushed to great excess fine globules may occur in the cells three quarters of the way back along the mesenteron.

It is however quite clear that the absorption of fat is practically confined to the anterior third of the mesenteron.

Our reasons for supposing that absorption of the dissolved products of digestion is especially the function of the anterior region cannot be regarded as conclusive. The very small quantity of fluid that is found at any time in the middle and posterior regions of the midgut of a well-fed animal points to this conclusion. In the chlorophyll of the algae which form so large a portion of the diet of Daphnias, we have a substance whose fate we can to a certain extent trace, and we find that as digestion proceeds the food mass in the middle region loses its green tint, while the fluid contents of the anterior region become coloured a vivid green. Further there is evidence that this dissolved chlorophyll is absorbed, for the striated border of the epithelium becomes coloured an intense green and the cells charge themselves with yellow pigment masses.

Defœcation is a sudden act: the fæces, which are formed in the posterior region of the mesenteron, are quickly expelled from the intestine passing rapidly through the proctodæum.

Defæcation is carried out by a wave of contraction which appears to start at some undeterminable point in the mesenteron, and a wave of dilatation and contraction passing backwards over the proctodæum. The movements of the proctodæum are of such a nature that they first aspirate the contents of the posterior region of the mesenteron, and then expel them through the anus.

There is undoubtedly a sphincter muscle at the junction of mid and hind gut in Daphnia.

The proctodæum also exhibits a rhythmical movement which consists of peristaltic waves starting from the anus and travelling forwards to the junction with the midgut. The interval between successive waves varies (e.g. 3 to 9 seconds in one animal). They have no connection with the forward running peristaltic waves of the mesenteron, for the latter occur at quite regular intervals without any reference to the time of occurrence of the proctodeal wave. Thus in the case given above, the proctodeal waves followed one another at irregular intervals varying from 3 to 9 seconds, while the peristalsis of the mesenteron occurred with perfect regularity every $1\frac{4}{5}$ seconds.

The rhythmic movements of the proctodæum appear to be independent of the central nervous system, for they are maintained long after all signs of life have vanished and may be shown by the proctodæum when isolated as completely as possible in normal salt solution. They undoubtedly lead to the entrance and exit of water and, in the absence of any definite knowledge on the subject, we may perhaps regard them as respiratory in character.

It is clear that the stomodeal and proctodeal portions of the intestine of Daphnia, so far as the manipulation of the food stuffs is concerned, take part only in the processes of deglutition and defæcation, and the ingesta and egesta are not lodged in them but merely hurried through. They thus differ profoundly in function from the similar structures in the higher forms of the Crustacea.

Minute structure of the Gut of Daphnia.

The walls of the gut consist of a single-layered epithelium resting on a muscular membrane.

The Mesenteron including the liver diverticula is lined by columnar cells disposed quite regularly and with a well developed hyaline border with vertical striation. When the gut is in a state of average normal distension the cells are short columnar, being only a little deeper than broad. When the gut is much constricted the epithelium is thrown into longitudinal rounded ridges separated by furrows. The cells of the ridges are then very tall and thin, those of the furrows short and broad. The gut may be so distended by food or perhaps artificially during manipulation that the cells are broader than deep.

The depth of the hyaline border varies in different regions of the gut but bears a fairly constant proportion to the depth of the cells in each region. There can generally be seen embedded in the substance of the border more darkly staining rods. They are most commonly wholly embedded in the border, reaching just to its free surface; and are generally separated by spaces about equal to or slightly greater than their own thickness. The distance which separates them from one another varies with the degree of distension of the gut and the consequent breadth of the cells. When the cells are very broad the border with the rods sometimes appears discontinuous, there being patches of striated border on each cell, separated by intervals.

Sometimes no rods or structure of any kind can be made out in the border; this may be due to complete retraction of the rods; but possibly to faulty preparation.

The rods may be sometimes seen to be shorter than the depth of the border and so not quite to reach to its surface.

In other cases the rods may be seen to project freely beyond the border to at least half their length. In such cases the projection is not due to retraction of the border but to projection of the rods, for the border retains its normal depth while the rods are longer proportionally to the height of the cell; they may be more than half the height of the cell when thus projected.

When the rods project through increase in length, they do not seem to be narrower, but rather thicker; if this be so, there must be a passage into them of substance from the cell body.

The rods are not always parallel and not always perpendicular, sometimes they lie at an angle to the surface of the cells.

The rods again may appear to stand freely on the surface of the cells without any embedding hyaline substance.

Sometimes the border contains a number of small clear non-staining patches oval or irregular in shape, generally close to the base of the border; these are commonest opposite the intervals between cells.

The borders of adjacent cells are generally in contact and closely adherent. In cases where the cells are at all displaced the borders may be seen stretched out and still adherent. The border is therefore fairly tough and of course extensible as it changes in shape with the alterations in form of the cell bodies.

The mobility of the rods and of the hyaline substance of the border is thus very obvious in Daphnia, and Heidenhain has attributed a similar mobility to the hyaline border of the cells lining the small intestine of vertebrates*, and parallel changes have been noticed at times in the ciliated cells of the gut of Lumbricus†.

Actual measurements show that the depth of this border forms a greater proportion of the total depth of the cell in the diverticula and anterior portions than in the rest of the gut. Thus in one case the measurements were:

	Depth of cell body.	Depth of border.
Diverticula	5μ	$2{\cdot}5\mu$.
Anterior region of intestine	10μ	$6-7\mu$
Middle to posterior region	$7{\cdot}5\mu$	3μ

Thus we may say generally that the border is best developed in that region in which the absorption of fat was found to take place, and this corresponds to observations made in other animals. Thus in vertebrates the hyaline border is best developed on the cells of the villi of the small intestine, and in Lumbricus‡ it, or rather its homologue, is only found on those cells which ingest fat.

Just posterior to the junction of the stomodæum with the mesenteron, there is a short neck-like region in which the lumen is smaller and the cell border lower in proportion to the depth of the cells than in regions before or behind.

If we turn to the histological characters of the cell substance of the cells lining the mesenteron we obtain further evidence of a differentiation of the epithelium into regions corresponding to those which actual observation of the processes of digestion shows to exist.

The cells of the middle region of the mesenteron are specially characterised by the presence under certain conditions of granules

* Heidenhain, *Pflüger's Archiv.* Bd. XLIII. Suppl.
† M. Greenwood, *Journ. of Physiol.* Vol. XIII. p. 239.
‡ M. Greenwood, *loc. cit.*

which are preserved by osmic vapour and to a less extent by corrosive sublimate. Since these granules accumulate in great numbers in starving animals, and since it can be shown that they make their way into the lumen of the gut there to swell up and dissolve, we are justified in regarding them as secretory granules, and the cells which bear them as gland cells, engaged in the elaboration of a digestive ferment or ferments.

In a series of sections of a gut preserved by most reagents (osmic acid and corrosive sublimate excepted) there are usually seen, apparently between the cells, clear spaces reaching from basement membrane to striated border but usually not into the border; they vary in shape, in a gut taken from an animal during digestion, from a narrow slit, straight or curved, according to the shape of the cell, to broadly oval spaces, which seem to compress the adjacent cells to a dice-box shape.

Occasionally the striated border is discontinuous opposite these spaces; being perforated by a small canal which in oblique sections often resembles a small vacuole.

In cells preserved with osmic acid vapour, no such gaps between the cells occur, or if so, very rarely; but there are seen within the cells granules which are not seen in the specimens where the gaps occur.

These granules appear to stain rather differently with osmic vapour in different cases. Sometimes only lightly, at other times more darkly.

In cells preserved with a saturated solution of corrosive sublimate the granules are generally only partly preserved. In some cases each granule seems to be swollen up and to have become clear and non-staining, but not to have fused with the neighbouring granules, or only to a slight extent; the cell substance then appears as a stained protoplasmic network of very fine meshes, enclosing the clear swollen granules.

The change in places goes further; the granules and the network are no longer seen but in their place appear clear unstained patches chiefly at the bases of the cells.

Absolute alcohol sometimes preserves the granules in part. They then stain deeply with hæmatoxylin. It may then often be seen that in a cell some of the granules are swollen up and coalesced into a clear non-staining mass, while the rest remain and are very clearly seen.

Wherever granules are well preserved they may be seen to be arranged in vertical rows stretching from the basement membrane to the cell border.

In the middle region of a fasting gut preserved by osmic vapour the cells are often very much distended laterally with granules. The rods of the hyaline border then stand much further apart than usual and often do not reach the surface of the border.

In any one section the granules of different cells are rather differently preserved. Some of the cells appear lighter than the rest, and these are generally broader; possibly in them swelling of the granules has begun, while in the others the granules are better preserved.

In sections through fasting animals which have been preserved in absolute alcohol the epithelium shows large clear (i.e. unstained) spaces alternating with narrow threads or columns of staining substance. The spherical nucleus may sometimes be seen attached as it were to one side of the staining column and projecting into the clear space. In many cases the clear space contains scattered granules which have not swollen up.

The discharge of the granules into the lumen of the gut may not unfrequently be seen in osmic vapour preparations especially of fasting animals. They appear in places to be streaming from the cells through the border and to form a homogeneous mass in the lumen which sometimes imbeds still unaltered granules.

These granule-bearing or gland cells occur throughout the length of the gut, but they are most numerous in the middle or digestive region. They are fewest in the short neck-like region, already alluded to, which is just posterior to the junction of œsophagus and mesenteron.

It is interesting to note that quite at the posterior end of the mesenteron gland cells occur which contain remarkably compact groups of granules which stain deeply with osmic vapour. These like the other granules are best seen in starving animals.

We may correlate the existence of these cells in this particular position with the formation of the fæces which takes place in this region. The fæces are composed of the innutritious detritus of the food stuffs glued together by some substance which makes its appearance at a very late period in digestion and in the posterior end of the mesenteron. The effect of the production of this substance is to cause the fæces to change from particles scattered through the lumen to a compact mass which occupies the centre of the gut and is not in contact with the walls.

This concludes the main part of the paper, and in it we have endeavoured to show that the apparently undifferentiated mesenteron of Daphnia is really divided into regions defined by the processes which take place in them and by the character of the cells forming their walls.

In order to complete the account of the histology of the gut we will briefly describe the structure of the stomodæum, proctodæum, and muscular basement membrane of the mesenteron.

Structure of Stomodæum. The stomodæum forms a muscular œsophagus leading upwards and forwards into the gut. It consists of a muscular tube lined by a simple low epithelium covered internally by a cuticle.

The muscle fibres are striated; they are arranged in a single layer of annular fibres and a pair of longitudinal muscles.

The proctodæum is the terminal vertical or recurved portion of the gut. It is narrow, flattened laterally and lined by a low cubical epithelium and cuticle. Its basement membrane is especially well developed.

Structure of basement membrane of the gut. This is the actively contractile organ which brings about peristalsis. Its structure is peculiar and it forms a membrane which is continuous over the whole mesenteron. This membrane is thin, hyaline, structureless, very tough, and very elastic, as is shown in teasing, and highly refractive. In it are embedded protoplasmic strands which are presumably the contractile elements. These are narrow flattened bands arranged in two series, a longitudinal and a circular. The longitudinal bands are continuous for at least half the length of the gut, probably the whole; whether the circular bands are annular or spiral we have not determined.

The circular bands are rather broader and more closely set than the longitudinal, being separated by intervals two to three times as broad as themselves. Each band is enclosed by a splitting of the hyaline membrane which sends a sheath both internal and external to the band. In transverse section the latter therefore appears as a dark oval patch lying embedded in the hyaline substance, and resembling a nucleus of the latter.

The bands show indistinct longitudinal fibrillation in places; and also a faint indication of irregular cross striation; these markings however are probably due to folding.

We have not succeeded in finding any traces of nuclei in the basement membrane. Where the longitudinal and circular bands cross one another, they appear to run straight on without being specially connected.

(3) *On Urobilin.* By A. EICHHOLZ, B.A., Emmanuel College.

In this communication a new method of urobilin extraction was described, by which the pigment is preserved in the state of chromogen. The properties of urobilin in normal and febrile urines were recapitulated in order to compare urobilin with the reduction products from bilirubin and hæmatin. The communication was then devoted to a description of experiments devised to settle the question as to the possibility of artificial production of urobilin from bilirubin and hæmatin. After pointing out how

Maly's hydrobilirubin differs from true urobilin and how consequently the identity of Hoppe Seyler's and Nencki and Sieber's urobilin from hæmatin reduction becomes doubtful, it was shewn, in spite of statements to the contrary by McMunn and Le Nobel, that it is possible by complete reduction of both bilirubin and hæmatin to obtain substances in each case accurately resembling urobilin.

Monday, February 13, 1893.

PROFESSOR T. MCK. HUGHES, PRESIDENT, IN THE CHAIR.

W. G. P. Ellis, M.A., St Catharine's College, was elected a Fellow of the Society.

The following Communications were made to the Society:

(1) *Note on the stability of rotating liquid spheroids.* By G. H. BRYAN, M.A., St Peter's College.

IN his paper "On the stability of Maclaurin's spheroid" Mr Basset has characterized as "incorrect" one of the statements in my paper "On the Stability of a Rotating Spheroid of Perfect Liquid*". The statement in question refers to the negative value of the quantity

$$p_1(\zeta) \cdot q_1(\zeta) - p_2(\zeta) \cdot q_2(\zeta),$$

which enters into the criteria of stability, and is to the effect that "Its meaning is that the spheroid is more oblate than that form for which the angular velocity is a maximum."

While admitting that Poincaré was in error in overlooking the fact that the angular momentum and not the angular velocity should be kept constant, I am unable to see how Mr Basset arrives at this conclusion. On the contrary, the following considerations seem to leave no doubt as to the correctness of my interpretations.

The angular velocity ω is connected with the quantity ζ which defines the eccentricity of the spheroid, by the relation

$$\omega^2 = 2\pi\rho\gamma\,(3\zeta^3 + \zeta)\cot^{-1}\zeta - 3\zeta^2\dagger;$$

$$\therefore \quad \frac{d\omega^2}{d\zeta} = 2\pi\rho\gamma\left[(9\zeta^2+1)\cot^{-1}\zeta - \frac{3\zeta^3+\zeta}{\zeta^2+1} - 6\zeta\right];$$

$$\therefore \quad (\zeta^2+1)\frac{d\omega^2}{d\zeta} = 2\pi\rho\gamma\,[(9\zeta^2+10\zeta+1)\cot^{-1}\zeta - (9\zeta^3+7\zeta)]$$

$$= 2\pi\rho\gamma\,[(3\zeta^2+1)\{(3\zeta^2+1)\cot^{-1}\zeta - 3\zeta\} + 4\zeta\{\zeta^2\cot^{-1}\zeta - 1\}]$$

$$= 8\pi\rho\gamma\,\{p_2(\zeta) \cdot q_2(\zeta) - p_1(\zeta) \cdot q_1(\zeta)\}\ddagger.$$

* *Proceedings Royal Society*, Vol. XLVII. p. 371.
† *Phil. Trans.* Vol. CLXXX. A. p. 197 (50). ‡ *Proc. R. S.* Vol. XLVII. p. 369.

Therefore ω^2 is a maximum when

$$p_2(\zeta) \cdot q_2(\zeta) - p_1(\zeta) \cdot q_1(\zeta) = 0,$$

which proves the correctness of my statement.

The same thing may also be shown by examining Poincaré's alternative method of finding the criteria of stability in § 9 of his paper*. He there considers that the ellipsoid will be in critical equilibrium if, when it has been slightly displaced, the displaced form is also in equilibrium. But he takes the displaced form to have the same angular velocity as the original form whereas it really ought to have the same angular momentum. If the displacement of a spheroid is determined by a spheroidal harmonic other than the zonal harmonic of the second degree, the original and the displaced spheroid have the same moment of inertia (to the first order of small quantities) and therefore their angular momenta are equal if their angular velocities are equal and Poincaré's method gives correct results for the stability. But in the case of a zonal harmonic displacement of the second degree, the moments of inertia in the two forms are no longer equal. And since the displaced form is also spheroidal, the condition obtained by Poincaré is that which must hold when two spheroids with the same angular velocity coalesce, i.e. when the angular velocity is a maximum or minimum. And we know from other considerations that it is a maximum, not a minimum.

Mr Basset's paper suggests another point which it would be well to inquire into more fully. He says that the spheroid is secularly stable if the motion when slightly displaced is determined by terms of the form $e^{(\alpha+\beta\iota)t}$ where α is negative. But is it quite certain that the small motions of viscous rotating mass of liquid can always be expressed by means of terms of this form? In electrodynamics we have instances of dissipative systems in which the general equations and boundary conditions, although linear, are not satisfied by a single solution of this form, and those systems in which the solution can be so expressed have been called "self inductive"†. It would be interesting to inquire whether any such conditions have to be satisfied in the case of a mass of viscous rotating liquid, or whether such a system is always, so to speak, "self inductive" in this sense. It is certain that the equations of the small motion about equilibrium of a Maclaurin's spheroid of viscous liquid do not admit of any such simple solution as those which I have found for the spheroid of perfect liquid, and the cylinder of viscous liquid.

It is, moreover, highly desirable that the general conditions of stability of rotating liquids should be proved by an alternative

* *Acta Mathematica*, Vol. VII. pp. 319—321.

† Watson and Burbury, *Electricity and Magnetism*, Vol. II.

method to that employed by Mr Basset on pp. 24, 25 of his paper. That a mass of gravitating liquid is in stable equilibrium when its form is spherical and that its potential energy is therefore then a minimum, is more obvious than that the latter fact follows as a consequence from the "result" stated in Equation 1. If W_0 had been defined as the work done in separating the elements of liquid in a mass of *any* form (say cubical for example) there is nothing either stated or shown in this argument as it stands in the paper which would preclude its being employed to show that the potential energy of the cubical form would be zero.

Again, the distinction between "energy" and "capacity to do work" is at variance with the custom of defining energy *as* "capacity to do work" in many text-books, and the statement that "the presence of the term $\frac{1}{2}h^2/I$" diminishes the "capacity to do work upon itself" of the liquid is not easily reconcileable with the remark (following equation (5)) that "the capacity to do work will be least when $\frac{1}{2}h^2/I - W$ is a minimum." The simple expedient of placing the minus sign first outside and then inside the bracket is not a very convincing argument on this point.

The object of these remarks is to show that the "obscurities" which exist in the statements in *Natural Philosophy* relating to the stability of rotating ellipsoids have only been partially cleared up, and that there is still room for a fresh investigation in the same direction.

[*Remarks on the above by* Mr A. B. BASSET.

In the first place the surface of a cube is not an equipotential surface for a mass of gravitating liquid, and consequently my argument would not apply if the word *cubical* were substituted for *spherical*.

Secondly, the distinction between *energy* and *capacity to do work* must be obvious to everyone who considers how the state of relative equilibrium of a Maclaurin's spheroid may be generated. The term $\frac{1}{2}h^2/I$ increases the total energy of the system, but diminishes its capacity to do work upon itself. When $\frac{1}{2}h^2/I - W$ is a minimum the system is incapable of doing any work upon itself; when it is not, the figure if free will begin to assume some other form.]

(2) *On the isotropic elastic sphere and spherical shell.* By C. CHREE, M.A., King's College.

In a paper communicated to the Society in 1887 the author gave a mathematically complete solution of the equations of equilibrium for an elastic solid sphere in polar coordinates. The expressions for the elastic displacements contained arbitrary constants to be determined by the surface conditions. These constants were, however, found explicitly only in the case of normal forces over a solid sphere. The first object of the present paper is to determine the arbitrary constants for all forms of surface forces over both surfaces of a shell. The expressions for the typical displacements are given explicitly. The forms taken by the displacements, strains and stresses when the shell becomes very thin are deduced. The order of magnitude of the stress usually assumed to vanish in theories of thin shells relative to the other stresses is determined, and their mode of variation along the thickness of the shell is illustrated by means of curves. The problem when the two surfaces of a shell suffer any given arbitrary displacements is also solved explicitly, the form taken by the results when the shell is very thin being more particularly considered. Several other applications are made of the solution to cases of physical interest.

(3) *On a system of two tetrads of circles; and other systems of two tetrads.* By Prof. CAYLEY.

The investigations of the present paper were suggested to me by Mr Orr's paper, "The Contacts of certain Systems of Circles.*"

1. It is possible to find *in plano* two tetrads of circles, or say four red circles and four blue circles, such that each red circle touches each blue circle: in fact counting the constants, a circle depends upon 3 constants, or say it has a capacity $=3$; the capacity of the eight circles is thus $=24$; and the postulation or number of conditions to be satisfied is $=16$: the resulting capacity of the system is thus primâ facie, $16-24=8$. It will, however, appear that in the system considered the true value is $=9$.

2. The primâ facie value of the capacity being $=8$, we are not at liberty to assume at pleasure three circles of the system. And in fact assuming at pleasure say 3 red circles, then touching each of these we have 8 circles, forming $\frac{1}{24}\,8\,.\,7\,.\,6\,.\,5, =70$, tetrads of circles: taking at random any one of these tetrads for the blue circles, the remaining red circle has to be determined so as to touch each of the four blue circles, that is by four instead of three conditions; and there is not in general any red circle satisfying these four conditions. But the 8 tangent circles do not stand to each other in a relation of symmetry, but form in fact four pairs

* Proc. Camb. Phil. Soc., Vol. VII.

of circles; and it is possible out of the 70 tetrads to select (and that in 6 ways) a tetrad of blue circles, such that there exists a fourth red circle touching each of these four blue circles. We have thus a system depending upon 3 arbitrary circles, and for which, therefore, the capacity is $= 9$. It is (as is known) possible, in quite a different manner, out of the 70 tetrads to select (and that in 8 ways) a tetrad of blue circles such that there exists a fourth red circle touching each of these four blue circles—but the present paper relates exclusively to the first mentioned 6 tetrads and not to these 8 tetrads.

3. I consider in the first instance a particular case in which the three red circles are not all of them arbitrary, but have a capacity $9-1, = 8$; and pass from this to the general case where the capacity is $= 9$. Calling the red circles 1, 2, 3 and 4; I start with the circles 1, and 2 arbitrary, and 3 a circle equal to 2: the radical axis, or common chord, of the circles 2 and 3 is thus a line bisecting at right angles the line joining the centres of the circles 2 and 3, say this is the line Ω. We have then four circles, each having its centre in the line Ω and touching the circles 1 and 2: in fact the locus of the centre of a circle touching the circles 1 and 2 is a pair of conics, each of them having for foci the centres of these circles: the line Ω meets each of these conics in two points, and there are thus on the line Ω four points, each of them the centre of a circle touching the circles 1 and 2. But the equal circles 2 and 3 are symmetrically situate in regard to the line Ω; and it is obvious that the four circles having their centres on the line Ω, will each of them also touch the circle 3; we have thus the four blue circles; each of them with its centre on the line Ω, and touching each of the red circles 1, 2 and 3. And it is moreover clear that taking the red circle 4 equal to 1 and situate symmetrically therewith in regard to the line Ω, then this circle 4 will touch each of the blue circles: so that we have here the four blue circles, each of them touching the four red circles. As already mentioned the blue circles have their centre on the line Ω, that is the line Ω is a common orthotomic of the four blue circles.

4. By inverting in regard to an arbitrary circle we pass to the general case; the line Ω becomes thus a circle Ω, orthotomic to each of the blue circles.

Starting *ab initio*, we have here at pleasure the red circles 1, 2, 3: the circle Ω is a circle having for centre a centre of symmetry of the circles 2 and 3, and passing through the points of intersection (real or imaginary) of these two circles; the circles 2 and 3 are thus the inverses (or say the images) each of the other in regard to the circle Ω. We can then find 4 circles each of them orthotomic to Ω, and touching the circles 1 and 2: but a circle orthotomic to Ω is its own inverse or image in regard to Ω; and it will thus touch the circle 3 which is the image of 2 in regard to Ω. We have thus the four blue circles each of them touching the red circles 1, 2 and 3; and then taking the red circle 4 as the inverse or image of 1 in regard to Ω, this circle 4 will also touch each of the blue circles. Thus starting with the arbitrary red circles 1, 2, 3, we find the four blue circles and the remaining red circle 4, such that each of the blue circles touches each of the red circles. Since in the construction we group together at pleasure the two circles 2, 3 (out of the three circles 1, 2, 3) and use at pleasure either of the two centres of symmetry, it appears that the number of ways in which the figure might have been completed is $=6$.

5. The blue circles have a common orthotomic circle Ω, that is the radical axis or common chord of each two of the blue circles passes through one and the same point, the centre of the circle Ω. The figure is symmetrical in regard to the red and blue circles respectively, and thus the red circles have a common orthotomic circle Ω', that is the radical axis or common chord of each two of the red circles passes through one and the same point, the centre of the circle Ω'.

6. Projecting stereographically on a spherical surface, the four red circles and the four blue circles become circles of the sphere; and then making the general homographic transformation they become plane sections of a quadric surface; we have thus the theorem that on a given quadric surface it is possible to find four red sections and four blue sections such that each blue section touches each red section; and moreover the capacity of the system is $=9$; viz. 3 of the red sections may be assumed at pleasure. But (as is well known) the theory of the tangency of plane sections of a quadric surface is far more simple than that of the tangency of circles: the condition in order that two sections may touch each other is simply the condition that the line of intersection of the two planes shall touch the quadric surface. And we construct as follows the sections touching each of three given sections: say the given sections are 1, 2, 3; through the sections 1 and 2 we have two quadric cones having for vertices say the points d_{12} and i_{12} (direct and inverse centres of the two sections): similarly through the sections 1 and 3 we have two quadric cones vertices d_{13} and i_{13} respectively, and through the sections 2 and 3 we have two quadric cones vertices d_{23} and i_{23} respectively; the points d_{12}, i_{12}, d_{13}, i_{13}, d_{23}, i_{23} lie three and three in four intersecting lines or axes, viz. these are $d_{23}d_{31}d_{12}$, $d_{23}i_{31}i_{12}$, $d_{31}i_{12}i_{13}$, $d_{12}i_{23}i_{31}$ respectively. Through any one of these axes say $d_{23}d_{31}d_{12}$, we may draw to the

quadric surface two tangent planes each touching the three cones which have their vertices in the points d_{23}, d_{31}, d_{12} respectively; and the section by either tangent plane is thus a section touching each of the three given sections 1, 2, 3; we have thus the eight tangent sections of these three sections.

7. Taking as three of the red sections the arbitrary sections 1, 2, 3; and grouping together two at pleasure of these sections, say 2 and 3; we may take for the blue sections the two sections through the axis $d_{23}d_{31}d_{12}$, and those through the axis $d_{23}i_{31}i_{12}$; we have thus the four blue sections touching each of the given red sections 1, 2, 3; and this being so, there exists a remaining red section 4 touching each of the blue sections; we have thus the four blue sections touching each of the red sections 1, 2, 3 and 4. This implies that the vertices or points d_{24} and d_{34} lie on the axis $d_{23}d_{31}d_{12}$, and that the vertices or points i_{24} and i_{34} lie on the axis $d_{23}i_{21}i_{31}$; or what is the same thing that the four sections 1, 2, 3, 4 have in common an axis $d_{23}d_{21}d_{31}d_{24}d_{34}$ and also an axis $d_{23}i_{21}i_{31}i_{24}i_{34}$.

8. If the quadric surface be a flat surface (*surface aplatie*) or conic, then the red sections become chords of the conic; the axes are lines in the plane of the conic, and thus the tangent planes through an axis each coincide with the plane of the conic, and it would at first sight appear that any theorem as to tangency becomes nugatory. But this is not so; comparing with the last preceding paragraph, we still have the theorem: on a given conic, taking at pleasure any three chords 1, 2, 3, it is possible to find a fourth chord 4, such that the four chords have in common an axis $d_{23}d_{21}d_{31}d_{24}d_{34}$ and also an axis $d_{23}i_{21}i_{31}i_{24}i_{34}$. And the analytical theory (although somewhat complex) is extremely interesting. Considering the conic $xz - y^2 = 0$, the coordinates of a point on the conic are given by $x : y : z = 1 : \theta : \theta^2$, or say any point of the conic is determined by its parameter θ; and this being so, considering any three chords 1, 2, 3, I take as for the two extremities of 1 the values ϵ, ζ; for those of 2 the values α, β; and for those of 3 the values γ, δ; the remaining chord 4 is to be determined as above, and I take for its two extremities the values E, Z.

9. Starting with the chords 1, 2, 3, we have each of the points d_{23}, &c. as the intersection of two lines, viz. these are

$$\text{for } d_{23} \begin{cases} x\alpha\delta - y(\alpha+\delta) + z = 0, \\ x\beta\gamma - y(\beta+\gamma) + z = 0, \end{cases} \qquad \text{for } i_{23} \begin{cases} x\alpha\gamma - y(\alpha+\gamma) + z = 0, \\ x\beta\delta - y(\beta+\delta) + z = 0, \end{cases}$$

$$\text{,,}\ d_{12} \begin{cases} x\alpha\zeta - y(\alpha+\zeta) + z = 0, \\ x\beta\epsilon - y(\beta+\epsilon) + z = 0, \end{cases} \qquad \text{,,}\ i_{12} \begin{cases} x\alpha\epsilon - y(\alpha+\epsilon) + z = 0, \\ x\beta\zeta - y(\beta+\zeta) + z = 0, \end{cases}$$

$$\text{,,}\ d_{13} \begin{cases} x\gamma\zeta - y(\gamma+\zeta) + z = 0, \\ x\delta\epsilon - y(\delta+\epsilon) + z = 0, \end{cases} \qquad \text{,,}\ i_{13} \begin{cases} x\gamma\epsilon - y(\gamma+\epsilon) + z = 0, \\ x\delta\zeta - y(\delta+\zeta) + z = 0, \end{cases}$$

and we thence find without difficulty for the axis $d_{23}d_{12}d_{13}$ the equation

$$\begin{aligned} &x\,\{\alpha\beta\,(\delta\epsilon-\gamma\zeta)+\gamma\delta\,(\zeta\alpha-\epsilon\beta)+\epsilon\zeta\,(\beta\gamma-\alpha\delta)\} \\ &+y\,\{(\beta-\alpha)(\gamma\delta-\epsilon\zeta)+(\delta-\gamma)(\epsilon\zeta-\alpha\beta)+(\zeta-\epsilon)(\alpha\beta-\gamma\delta)\} \\ &+z\,\{\quad-\quad(\delta\epsilon-\gamma\zeta)-\quad(\zeta\alpha-\epsilon\beta)-\quad(\beta\gamma-\alpha\delta)\}=0, \end{aligned}$$

and the equation of the axis $d_{23}i_{12}i_{13}$ is obtained herefrom by the interchange of ϵ and ζ.

10. The points d_{24} and d_{34} will lie upon the first mentioned axis if only d_{24} lies upon this axis, viz. if we have

$$\begin{aligned} &\{\quad\alpha\beta\,(\delta\epsilon-\gamma\zeta)+\quad\gamma\delta\,(\zeta\alpha-\epsilon\beta)+\quad\epsilon\zeta\,(\beta\gamma-\alpha\delta)\} \\ &\qquad\qquad\qquad(\beta+E-\alpha-Z) \\ &+\{(\beta-\alpha)(\gamma\delta-\epsilon\zeta)+(\delta-\gamma)(\epsilon\zeta-\alpha\beta)+(\zeta-\epsilon)(\alpha\beta-\gamma\delta)\} \\ &\qquad\qquad\qquad(\beta E-\alpha Z) \\ &+\{\quad-\quad(\delta\epsilon-\gamma\zeta)-\quad(\zeta\alpha-\epsilon\beta)-\quad(\beta\gamma-\alpha\delta)\} \\ &\qquad\qquad\qquad(-\alpha Z(\beta+E)+\beta E(\alpha+Z))=0. \end{aligned}$$

Reducing this equation, the factor $\alpha-\beta$ divides out, and we finally obtain

$$\begin{aligned} &(\gamma-\alpha)(\beta\delta+\zeta Z)(\epsilon-E) \\ &+\ (\beta-\delta)(\alpha\gamma+\epsilon E)(\zeta-Z) \\ &+(\alpha\delta-\beta\gamma)(\epsilon\zeta-EZ) \\ &+(\alpha\beta-\gamma\delta)\ (\epsilon Z-\zeta E)=0; \end{aligned}$$

say this is

$$A+BE+CZ+DEZ=0,$$

where

$$\begin{aligned} A&=\quad.\quad(\gamma-\alpha)\,\beta\delta\epsilon+(\beta-\delta)\,\alpha\gamma\zeta+(\alpha\delta-\beta\gamma)\,\epsilon\zeta, \\ B&=-(\gamma-\alpha)\,\beta\delta\quad.\quad-(\alpha\beta-\gamma\delta)\,\zeta+\quad(\beta-\delta)\,\epsilon\zeta, \\ C&=-(\beta-\delta)\,\alpha\gamma+(\alpha\beta-\gamma\delta)\,\epsilon\quad.\quad+\quad(\gamma-\alpha)\,\epsilon\zeta, \\ D&=-(\alpha\delta-\beta\gamma)-(\beta-\delta)\,\epsilon-(\gamma-\alpha)\,\zeta\quad.\quad, \end{aligned}$$

viz. this is the condition for the existence of the axis $d_{23}d_{13}d_{12}d_{43}d_{42}$.

We interchange herein ϵ, ζ and also E, Z, and we thus obtain

$$A'+C'E+B'Z+D'EZ=0,$$

where

$$\begin{aligned} A'&=\quad.\quad(\beta-\delta)\,\alpha\gamma\epsilon+(\gamma-\alpha)\,\beta\delta\zeta+(\alpha\delta-\beta\gamma)\,\epsilon\zeta, \\ B'&=-(\gamma-\alpha)\,\beta\delta-(\alpha\beta-\gamma\delta)\,\epsilon\quad.\quad+\quad(\beta-\delta)\,\epsilon\zeta, \\ C'&=-(\beta-\delta)\,\alpha\gamma\quad.\quad+(\alpha\beta-\gamma\delta)\,\zeta+\quad(\gamma-\alpha)\,\epsilon\zeta, \\ D'&=-(\alpha\delta-\beta\gamma)-(\gamma-\alpha)\,\epsilon\quad-(\beta-\delta)\,\zeta\quad.\quad, \end{aligned}$$

viz. this is the condition for the existence of the axis $d_{23}i_{13}i_{12}i_{43}i_{42}$.

11. I remark that we have

$$A' = A + \mathrm{a}(\epsilon - \zeta), \quad B' = B + \mathrm{b}(\epsilon - \zeta),$$
$$C' = C + \mathrm{c}(\epsilon - \zeta), \quad D' = D + \mathrm{d}(\epsilon - \zeta),$$

where
$$\mathrm{a} = (\beta - \delta)\,\alpha\gamma - (\gamma - \alpha)\,\beta\delta = \alpha\beta(\gamma + \delta) - \gamma\delta(\alpha + \beta),$$
$$\mathrm{b} = \mathrm{c} = \gamma\delta - \alpha\beta$$
$$\mathrm{d} = \alpha + \beta - \gamma - \delta,$$

and further that

$$B - C = \alpha\beta(\gamma + \delta) - \gamma\delta(\alpha + \beta) + (\gamma\delta - \alpha\beta)(\epsilon + \zeta) + (\alpha + \beta - \gamma - \delta)\,\epsilon\zeta,$$

say this is Π.

12. It thus appears that for the determination of E, Z we have

$$A + BE + CZ + DEZ = 0,$$
$$A' + C'E + B'Z + D'EZ = 0.$$

Eliminating Z we find

$$\frac{A + BE}{A' + C'E} + \frac{C + DE}{B' + D'E},$$

that is

$$(AB' - A'C) + (AD' - A'D + BB' - CC')\,E + (BD' - B'D)\,E^2 = 0\,;$$

upon reducing the coefficients of this equation it appears that they contain each of them the factor Π, and throwing out this factor, the equation is

$$\begin{aligned}\epsilon\,[\alpha\beta(\gamma - \delta) + \gamma\delta(\beta - \alpha) + (\alpha\delta - \beta\gamma)\,\zeta]\\ + [-\alpha\beta(\gamma - \delta) - \gamma\delta(\beta - \alpha) + (\alpha\delta - \beta\gamma)\,\epsilon\\ + (\beta\gamma - \alpha\delta)\,\zeta + (\beta + \gamma - \alpha - \delta)\,\epsilon\zeta]\,E\\ + [\beta\gamma - \alpha\delta - (\beta + \gamma - \alpha - \delta)\,E^2 = 0\,;\end{aligned}$$

this contains obviously the factor $E - \epsilon$, or throwing out this factor, we have for E the simple equation

$$\{\alpha\beta(\gamma - \delta) + \gamma\delta(\beta - \alpha)\} + (\alpha\delta - \beta\gamma)(E + \zeta) + (\beta - \alpha + \gamma - \delta)\,\zeta E = 0,$$

and in a similar manner it may be shown that the two equations give for Z the like simple equation

$$\{\alpha\beta(\gamma - \delta) + \gamma\delta(\beta - \alpha)\} + (\alpha\delta - \beta\gamma)(Z + \epsilon) + (\beta - \alpha + \gamma - \delta)\,\epsilon Z = 0,$$

viz. starting from the chords 1, 2, 3 which depend on the parameters (ϵ, ζ), (α, β), (γ, δ) respectively, these last two equations give the parameters (E, Z) of the chord 4.

Monday, February 27, 1893.

PROF. T. MCK. HUGHES, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:—

(1) *On the histology of the Blood of Rabbits which have been rendered immune to Anthrax*, by LIM BOON KENG, M.B. (Edin.).

Abstract.

The research was conducted in the Pathological Laboratory of the University. The rabbits were rendered immune to anthrax by inoculation with the lymph and blood of frogs which had been subjected to various treatment. Previous observers had succeeded in conferring immunity with the use of similar substances. The object of the investigation, however, was to ascertain the changes in the character and relative number of the white cells of the blood after protective vaccination and after the introduction of virulent anthrax.

From four to several hours after the injection of the vaccine, a great increase in the number of the white cells is noticeable; and the most remarkable feature is the augmentation in number of the coarsely granular (eosinophile) corpuscles. The relative proportion in the numbers of the different varieties of cells is therefore altered, so that instead of forming only from 2 to 4 per cent. of the total number of white cells, the eosinophile corpuscles now constitute about 10 to 25 per cent. This increase persists only a short time, and on the third day the cells may have returned to a normal condition; and at this stage hyaline cells ingesting granular cells may be detected in numbers in certain localities. Although the blood has thus apparently returned to the normal condition, it is found that the state of eosinophile leucocytosis is rapidly reproduced, on the introduction of virulent anthrax. After inoculation with a virulent culture of the microbes, the eosinophile cells appear in great numbers, so that they may form 50 per cent. of the white corpuscles, and in one instance an even higher percentage was found. These cells are not only increased in number but are also larger and have larger granules. Similar changes were observed in Guinea pigs rendered immune by Dr Haffkine to the comma bacillus.

In non-vaccinated rabbits, the introduction of anthrax causes profound leucocytosis, but the cells are all very small and the eosinophile corpuscles are only slightly increased in number. General infection occurs in 36 to 48 hours, rapidly followed by death.

(2) *On numerical Variation in Digits, in illustration of a principle of Symmetry*, by W. BATESON, M.A., St John's College.

Abstract.

An account was given of cases of Variation in number of digits so occurring that the parts are symmetrical about a new axis in the limb. Of these the phenomena seen in the bones of a number of polydactyle Cats were chiefly important. The normal hind foot of the Cat has four toes, each bearing a claw retracted by an elastic ligament to a notch on the *external* side of the second phalanx. This circumstance differentiates digits formed as lefts from those formed as rights. As extra digits are added on the internal side of the limb the symmetry changes. The limb being taken as a *right*, the variations seen are as follows. (1) Hallux present, making five digits: index is then in form *intermediate between right and left*. (2) Six digits present, internal having two phalanges: the three external digits are then normal rights, *the next two are lefts;* the internal, having a non-retractile claw, is indifferent. (3) Six digits present, internal having three fully-formed phalanges and retractile claw: the three externals are then normal rights, and *the three internals are left digits*, thus forming two groups in bilateral symmetry about an axis passing between the digits having the relations of index and medius. Several cases of "double hand" in Man form a similar progressive series, and analogous facts in other animals were instanced.

Monday, March 13, 1893.

PROF. T. M^{c}K. HUGHES, PRESIDENT, IN THE CHAIR.

H. K. Anderson, M.A., Gonville and Caius College, was elected a Fellow of the Society.

The following Communications were made to the Society:—

(1) *On a compound of gold and cadmium.* By C. T. HEYCOCK, M.A., King's College.

(2) *On the Spectra of electric discharges without electrodes through gases.* By H. F. NEWALL, M.A., Trinity College.

(3) *Exhibition of some photographs of the Solar Spectrum by Mr Geo. Higgs.* By R. T. GLAZEBROOK, M.A., F.R.S., Trinity College.

(4) *An experiment on the vibrations of a spiral spring.* By L. R. WILBERFORCE, M.A., Trinity College.

Monday, May 1, 1893.

PROF. McK. HUGHES, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *On the torsional strength of a hollow shaft.* By H. M. MACDONALD, M.A., Clare College.

Saint-Venant in the eleventh chapter of his *Memoir on the Torsion of Prisms*, 1855, deals with the case of hollow prisms, but in the cases there considered the bounding surfaces are concentric.

In the *Phil. Mag.*, Jan. 1892, Mr Larmor investigated the shear at the surface of a hollow in a prism, the dimensions of the hollow being supposed small compared with its distance from the outer surface of the prism.

The object of the following is to consider the case of a prism whose bounding surfaces are any two cylindrical surfaces of circular section.

1. Take as axes of x and y the radical axis and the line of centres of the two circles in which a plane perpendicular to the axis of the prism cuts it, and as axis of z the straight line through their point of intersection parallel to the axis of the prism.

Then the displacements are

$$\left.\begin{aligned} u &= -\tau yz \\ v &= \tau zx \\ w &= \phi \end{aligned}\right\} \qquad (1),$$

where τ is the twist per unit length of the prism and ϕ satisfies the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad (2)$$

throughout the matter of the prism, and

$$l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y} = \tau(ly - mx) \qquad (3)$$

at every point of the boundary of a cross-section, l, m being the direction-cosines of the outward-drawn normal.

Let ψ be chosen so that ϕ and ψ are conjugate functions of x and y; then ψ satisfies the equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \qquad (2')$$

throughout the matter of the prism, and

$$\psi = \frac{\tau}{2}(x^2 + y^2) + C \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3')$$

at the boundary, where C is an arbitrary constant which will not be the same for the two bounding circles.

Transforming to ξ, η where

$$x + \iota y = c \tan \tfrac{1}{2}(\xi + \iota\eta),$$

ψ satisfies

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

from $\eta = \alpha$ to $\eta = \beta$, and

$$\psi = \frac{\tau c^2 \cosh\eta}{\cosh\eta + \cos\xi} + \text{const.} \quad\ldots\ldots\ldots\ldots\ldots(5)$$

when $\eta = \alpha$ and when $\eta = \beta$,

$\eta = \alpha$ being the outer bounding circle, and

$\eta = \beta$ the inner.

Equation (5) may be written

$$\psi = 2\tau c^2 \coth\eta \sum_1^\infty (-)^n e^{-n\eta} \cos n\xi + C \quad\ldots\ldots\ldots(5')$$

when $\eta = \alpha$ or β.

Assume $$\psi = \sum_1^\infty (-)^n (A_n e^{-n\eta} + B_n e^{n\eta}) \cos n\xi \quad\ldots\ldots\ldots(6),$$

then
$$\left.\begin{aligned} A_n &= 2(-)^n \tau c^2 \frac{e^{-2n\beta}\coth\beta - e^{-2n\alpha}\coth\alpha}{e^{-2n\beta} - e^{-2n\alpha}} \\ B_n &= 2(-)^n \tau c^2 \frac{\coth\beta - \coth\alpha}{e^{2n\beta} - e^{2n\alpha}} \end{aligned}\right\}\ldots\ldots\ldots(7).$$

Therefore

$$\psi = 2\tau c^2 \sum_1^\infty (-)^n \frac{e^{-n\beta}\coth\beta \sinh n(\eta-\alpha) - e^{-n\alpha}\coth\alpha \sinh n(\eta-\beta)}{\sinh n(\beta-\alpha)} \cos n\xi \ldots(8),$$

and

$$w = \phi = 2\tau c^2 \sum_1^\infty (-)^n \frac{e^{-n\beta}\coth\beta \cosh n(\eta-\alpha) - e^{-n\alpha}\coth\alpha \cosh n(\eta-\beta)}{\sinh n(\beta-\alpha)} \sin n\xi \ldots(9);$$

hence the displacements are known.

2. To obtain the twisting couple, we have

$$y\frac{\partial\phi}{\partial x}-x\frac{\partial\phi}{\partial y}=\frac{\partial\psi}{\partial\eta}\sinh\eta\cos\xi+\frac{\partial\psi}{\partial\xi}\cosh\eta\sin\xi,$$

$$\iint\left(y\frac{\partial\phi}{\partial x}-x\frac{\partial\phi}{\partial y}\right)dx\,dy$$

$$=2c^2\int_\alpha^\beta\int_0^\pi\left(\frac{\partial\psi}{\partial\eta}\sinh\eta\cos\xi+\frac{\partial\psi}{\partial\xi}\cosh\eta\sin\xi\right)\frac{d\xi\,d\eta}{(\cosh\eta+\cos\xi)^2}$$

$$=4\pi\tau b^2c^2\sum_1^\infty\frac{ne^{-n(\alpha+\beta)}}{\sinh n(\beta-\alpha)}-\pi\tau c^4\left(\frac{\cosh^2\beta}{\sinh^4\beta}-\frac{\cosh^2\alpha}{\sinh^4\alpha}\right).$$

Also

$$\iint(x^2+y^2)\,dx\,dy=\frac{\pi}{2}(a^4-a'^4)+\pi c^4\left(\frac{\cosh^2\alpha}{\sinh^4\alpha}-\frac{\cosh^2\beta}{\sinh^4\beta}\right),$$

where $c\operatorname{cosech}\alpha=a$ the radius of the circle $\eta=\alpha$,

$c\operatorname{cosech}\beta=a'$ the radius of the circle $\eta=\beta$,

and

$c(\coth\alpha-\coth\beta)=b$ the distance between their centres.

Hence if N is the twisting couple, and n the rigidity of the substance of the prism,

$$\frac{N}{\mathrm{n}}=\frac{\pi\tau}{2}(a^4-a'^4)-4\pi\tau b^2c^2\sum_1^\infty\frac{ne^{-n(\alpha+\beta)}}{\sinh n(\beta-\alpha)},$$

$$=\frac{\pi\tau}{2}(a^4-a'^4)-2\pi\tau b^2c^2\{\operatorname{cosech}^2\beta+\operatorname{cosech}^2(2\beta-\alpha)$$
$$+\operatorname{cosech}^2(3\beta-2\alpha)+\dots\}$$

$$=\frac{\pi\tau}{2}(a^4-a'^4)-2\pi\tau b^2\left(a'^2+\frac{a^2a'^4}{(a^2-b^2)^2}\right.$$
$$\left.+\frac{a^4a'^6}{\{(a^2-b^2)^2-a'^2b^2\}^2}+\dots\right).$$

If a' is small, we have

$$\frac{N}{\mathrm{n}}=\tfrac{1}{2}\pi\tau a^4-2\pi\tau b^2a'^2,$$

and if τ_0 is the twist per unit length in a solid cylinder of radius a due to the same twisting couple

$$\tau=\tau_0\Big/\left(1-\frac{4b^2a'^2}{a^4}\right)=\tau_0\left(1+\frac{4b^2a'^2}{a^4}\right).$$

If there should be any number of such cavities placed so that the areas over which their effect in changing the shear is significant do not overlap, the twist is given by

$$\tau = \tau_0 \Big/ \left(1 - \frac{4}{a^4}\Sigma b^2 a'^2\right) = \tau_0 \Big/ \left(1 - 2\frac{I'}{I}\right),$$

where I is the moment of inertia of the area of the cross-section of the cylinder, and I' of the cavities about the axis of the cylinder supposed solid; and if I' is small compared with I

$$\tau = \tau_0 \left(1 + 2\frac{I'}{I}\right).$$

3. To find where the shear is greatest.

It is stationary in value when $\xi = 0$ or π and is then given by

$$s = -\tau c \tanh\frac{\eta}{2} + \frac{\cosh\eta + 1}{c}\frac{\partial\psi}{\partial\eta} \quad \text{when } \xi = 0,$$

and $$s = -\tau c \coth\frac{\eta}{2} - \frac{\cosh\eta - 1}{c}\frac{\partial\psi}{\partial\eta} \quad \text{when } \xi = \pi.$$

Writing

$$\psi = \tau c^2 \coth\beta \left(\frac{\sinh\eta}{\cosh\eta + \cos\xi} - 1\right)$$
$$- 2\tau bc \sum_1^\infty \frac{(-)^n e^{-n\alpha}\sinh n(\eta - \beta)\cos n\xi}{\sinh n(\beta - \alpha)},$$

we have, when $\xi = 0$,

$$s = -\tau c \tanh\frac{\eta}{2} + \tau c \coth\beta$$
$$- 2\tau b(\cosh\eta + 1)\sum_1^\infty \frac{n(-)^n e^{-n\alpha}\cosh n(\eta - \beta)}{\sinh n(\beta - \alpha)},$$

$$\frac{ds}{d\eta} = -\frac{\tau c}{2}\operatorname{sech}^2\frac{\eta}{2} - 2\tau b\sum_1^\infty \frac{n^2(-)^n e^{-n\alpha}\sinh n(\eta - \beta)}{\sinh n(\beta - \alpha)}(\cosh\eta + 1)$$
$$- 2\tau b\sum_1^\infty \frac{n(-)^n e^{-n\alpha}\cosh n(\eta - \beta)}{\sinh n(\beta - \alpha)}\sinh\eta.$$

When $\eta = \beta$,

$$\frac{ds}{d\eta} = -\frac{\tau c}{2}\operatorname{sech}^2\frac{\beta}{2} - 2\tau b\sum_1^\infty \frac{n(-)^n e^{-n\alpha}\sinh\beta}{\sinh n(\beta - \alpha)};$$

putting $p = e^{-\beta+\alpha}, \quad q = e^{-\beta}$,

$$\frac{ds}{d\eta} = -\frac{2\tau c q}{(1+q)^2} + 2\tau b\,(1-q^2)\left(\frac{1}{(1+q)^2} + \frac{p^2}{(1+pq^2)^2} + \frac{p^4}{(1+qp^4)^2} + \dots\right).$$

Now q and p are less than unity; therefore $\frac{ds}{d\eta}$ is positive when $\eta = \beta$ and s diminishes with η.

When $\eta = \alpha$,

$$\frac{ds}{d\eta} = -\frac{\tau c}{2}\operatorname{sech}^2\frac{\alpha}{2} + 2\tau b\,(p^2 - q^2)\left(\frac{1}{(1+pq)^2} + \frac{p^2}{(1+p^3q)^2} + \dots\right),$$

hence when $\eta = \alpha$, $\frac{ds}{d\eta}$ is negative, and s diminishes as η increases; therefore s is greatest when $\eta = \beta$ or $\eta = \alpha$, ξ being zero.

When $\xi = \pi$,

$$s = -\tau c \coth\frac{\eta}{2} + \tau c \coth\beta + 2\tau b\,(\cosh\eta - 1)\sum_1^\infty \frac{n e^{-n\alpha}\cosh n\,(\eta - \beta)}{\sinh n\,(\beta - \alpha)};$$

$\frac{ds}{d\eta}$ and s are positive when $\eta = \beta$, therefore s diminishes with η, and when $\eta = \alpha$, s diminishes as η increases.

When $\xi = 0, \quad \eta = \beta, \quad s = 2\tau b + \tau a' + 2\tau b\,\frac{a^2 a'^2}{(a^2 - b^2)^2} + \dots,$

$\xi = 0, \quad \eta = \alpha, \quad s = \tau a + 2\tau b\,\frac{a'^2}{(a-b)^2} + \dots,$

$\xi = \pi, \quad \eta = \beta, \quad s = 2\tau b - \tau a' + \frac{2\tau b a^2 a'^2}{(a^2 - b^2)^2} + \dots,$

$\xi = \pi, \quad \eta = \alpha, \quad s = -\tau a + \frac{2\tau b a'^2}{(a-b)^2} + \dots.$

The shear at the surface differs from that at the surface of a solid cylinder by a quantity depending on the square and higher powers of the radius of the cavity, and if $a > 2b + a'$, that is, if the cavity is at a distance from the axis of the outer surface less than half the radius of the cylinder, the shear is greatest at the surface and the strength of the cylinder is practically unaltered if the cavity is small. If $a < 2b + a'$, that is, if the distance of the cavity from the axis of the cylinder exceeds half its radius, the shear is greatest at that point of the surface of the cavity which is most distant from the axis and is greater than the greatest shear in the solid cylinder, so that the existence of a small cylindrical cavity at a distance from the axis greater than half the radius reduces the strength of the cylinder in the ratio $a : 2b$ approximately.

4. When $\eta = \beta - m\alpha$, $\xi = 0$, $s = \tau b \left\{1 + \left(\frac{b}{a}\right)^{2m}\right\}$, neglecting powers of a', that is, at a distance from the surface of the cavity $\left\{\left(\frac{a}{b}\right)^m - 1\right\} a'$, the shear is $\tau b \left\{1 + \left(\frac{b}{a}\right)^{2m}\right\}$. This shews that the shear diminishes rapidly in the neighbourhood of the cavity; *e.g.* $a = 2b$, at distance from surface of cavity a', $s = \frac{5\tau b}{4}$, at distance $3a'$, $s = \frac{17}{16}\tau b$ etc., or $a = \frac{3b}{2}$, at distance $\frac{a'}{2}$, $s = \frac{13\tau b}{9}$, at distance $\frac{5a'}{4}$, $s = \frac{97}{81}\tau b$ etc. Hence we may suppose any number of cavities to exist provided that their distance apart is large compared with their radius, and the strength of the cylinder will be determined by the cavity which is farthest from the axis. The geometrical axis of the cylinder will form a helix whose axis will be that of the centroids of the sections.

From the expression for w we observe that the line of centres of each section suffers no displacement parallel to the axis of z, and that every other element of the section does, the displacements of points symmetrical with regard to this line being of opposite sign.

5. If we take as bounding surfaces $\eta = \alpha$ and $\eta = -\beta$, and the axis of z as axis of torsion, we obtain

$$\psi = 2\tau c^2 \coth\alpha \sum_1^\infty (-)^n e^{-n\eta} \cos n\xi \text{ from } \eta = \alpha \text{ to } \eta = \infty,$$

and $\psi = +2\tau c^2 \coth\beta \sum\limits_1^\infty (-)^n e^{+n\eta} \cos n\xi$ from $\eta = -\beta$ to $\eta = -\infty$,

or $\psi = \tau c y \coth\alpha$ from $\eta = \alpha$ to $\eta = \infty$,

and $\psi = -\tau c y \coth\beta$ from $\eta = -\beta$ to $\eta = -\infty$;

hence in this case, which is that of a prism formed of two solid circular cylinders outside one another, $w = \tau c x \coth\alpha$ from $\eta = \alpha$ to $\eta = \infty$, and $w = -\tau c x \coth\beta$ from $\eta = -\beta$ to $\eta = -\infty$.

The twisting couple is given by

$$\frac{N}{n} = \frac{\pi\tau}{2}(a^4 + a'^4).$$

If the cylinders are of equal cross-sections $\alpha = \beta$, and the above are the actual displacements, namely a rotation round the axis of y of amount $\tau c \coth \alpha$ from x to z for the cylinder $\eta = \alpha$, and from z to x for the cylinder $\eta = -\alpha$. If the cylinders are not equal, we have to add to the above displacements a rotation τd round the axis of y, where d is the distance from the origin of the centroid of the section.

(2) *On the Astronomical Theory of glacial periods.* By Dr E. W. HOBSON.

(3) *On Black Globes and some convenient uses of them.* By J. Y. BUCHANAN, M.A., Christ's College.

(4) *A Provisional Theory of Kerr's Experiments on the Reflection of Light from an Electromagnet.* By A. B. BASSET, M.A., F.R.S., Trinity College.

The experiments of Kerr on the reflection of light from an electro-magnet can be explained by transforming the expressions for the amplitudes of the reflected waves, when the medium is transparent (*Phil. Trans.* 1891, p. 371), by assuming that in the case of a metal the refractive index is a complex quantity.

1. Everyone who has devoted any attention to the history of Physical Optics cannot fail to have been struck with the fact that numerous theoretical results have from time to time been obtained, which are in close agreement with experimental facts, either by an imperfect theory or in some cases without anything which in the proper sense of the word can be called a theory at all. Cauchy, for example, towards the middle of the present century arrived at certain formulae giving the amplitudes of the waves reflected at the surface of a metal, which the experiments of Jamin and others have shewn to be fairly in accordance with experimental facts. Cauchy gave no proof of his formulae, and until attempts were made in Germany by von Helmholtz and others to construct theories based upon the mutual reaction between ether and matter, nothing in the shape of a plausible dynamical theory of metallic reflection could be said to exist. At the same time, it will no doubt be conceded, that the attempts made by Cauchy and others to elucidate the difficulties connected with this subject, coupled with the fact that formulae were actually obtained which may be regarded as an empirical representation of the facts, were of distinct scientific value, and have been, and no doubt will be in years to come, of great assistance to men who are desirous of probing this question to the bottom.

Impressed with the importance of employing every possible means of throwing light upon the curious and interesting phenomena observed by Kerr* when polarized light is reflected from a magnetized medium, I have endeavoured to arrive at formulae for the amplitudes of the reflected light which give results in agreement with his experiments, and may therefore be regarded as an empirical representation of the facts. I shall therefore first explain how the formulae may be obtained; I shall then discuss their application with reference to Kerr's experiments; and I shall lastly consider the theoretical difficulties which lie in the way of placing them on a satisfactory dynamical basis.

2. Cauchy's formulae for metallic reflection can be obtained from Fresnel's sine and tangent formulae in the manner explained in §§ 372—5 of my book on *Physical Optics*; and since the principal incidence I and the principal azimuth β can be determined by experiment, the two constants R and α, where $Re^{\iota\alpha}$ is the pseudo-refractive index, can be calculated.

Adopting the notation of my book, it follows from equations (23) and (24) of § 375, that

$$\left.\begin{aligned} Rc\cos I &= \sin^2 I \\ 2\beta &= \alpha + u \end{aligned}\right\} \quad\text{.........................(1).}$$

By (7) of § 372 combined with these equations we readily obtain

$$\tan 2\alpha = \frac{\sin 4\beta}{\cos 4\beta + \cot^2 I} \quad\text{.........................(2),}$$

which determines α.

Eliminating c from (7) of § 372, we get

$$R^2 = \frac{\sin^2 I \sin 4\beta}{\sin(4\beta - 2\alpha)} \quad\text{..........................(3),}$$

which gives R. The values of c and u, *when the angle of incidence is equal to the principal incidence* can now be obtained from (1) and (2); but for other incidences their values must be found from (7) of § 372.

There is another angle which is denoted in § 376 of my book by $\pi - \chi$, which is called by French writers the azimuth at which plane polarization is restored—*l'azimuth de polarisation rétablie.*— This angle has been determined experimentally by Jamin† and others, and an account of the results will be found in Mascart's *Traité d'Optique,* vol. II., p. 524 &c. He denotes this angle by θ_2, and the principal azimuth by C; and it can readily be proved that

$$\tan\theta_2 = \tan^2 C = \tan^2\beta$$

in my notation.

* *Phil. Mag.* May 1877, March 1878.

† *Ann. de Chim. et de Phys.* (3), Vol. XXII., p. 311.

The table given on § 383 of my book was taken from Eisenlohr's paper* referred to below, from which it appears that throughout the visible spectrum, the value of α for steel is less than 45°. I now find that his results are wrong, and that he has published a corrected table in a subsequent paper†, in which he finds that for steel α is about equal to 56°. This result destroys the hopes which I lately entertained of constructing a satisfactory electromagnetic theory of Kerr's experiments by taking into account the conductivity combined with Hall's effect.

As a good deal turns on the values of the constants R, α and u, I have calculated their values independently by means of the formulae (2) and (3).

Since the source of light in Kerr's experiments was the flame of a paraffin lamp, I shall calculate the values of the quantities for line D when the metal is steel. For this line Jamin‡ gives

$$\theta_2 = 16^\circ\, 48', \quad I = 76^\circ\, 40',$$

from which I find, (since $\tan\theta_2 = \tan^2\beta$),

$$\beta = 28^\circ\, 48', \quad \alpha = 55^\circ\, 53', \quad u = 1^\circ\, 43',$$

$$\log R = \cdot 577749, \quad R = 3\cdot 7823.$$

In the paper in *Wied. Ann.* Eisenlohr gives the following values, viz. for line D in the case of steel,

$$\alpha = 56^\circ\, 5', \quad \log R = \cdot 6084.$$

Sir John Conroy§ gives the following values for the principal incidence and azimuth for steel

$$I = 76^\circ\, 48', \quad \beta = 27^\circ\, 53',$$

from which I find

$$\alpha = 54^\circ\, 16', \quad u = 1^\circ\, 30',$$

$$\log R = \cdot 6132602, \quad R = 4\cdot 1045.$$

Sir J. Conroy states (p. 32) that a soda flame was used as the source of light; but it appears from Jamin's experiments that R and α are subject to chromatic variations. There is however a fairly close agreement between the three sets of numbers.

The value of u of course depends on the angle of incidence; the foregoing results give its value at the principal incidence.

* *Pogg. Ann.* Vol. CIV., p. 368.

† *Wied. Ann.* Vol. I., p. 205. His A, H, ϵ and θ are my I, β, α and R. In this paper there appears to be a misprint, viz. *tg* (or tan) ought to be log.

‡ *Ann. de Chim. et de Phys.* (3), Vol. XXII., p. 311; Mascart, *Traité d'Optique* Vol. II., p. 539.

§ *Proc. Roy. Soc.* Vol. XXXV., p. 33.

It is, moreover, worthy of note that u is a very small angle, whose maximum value is less than 2°.

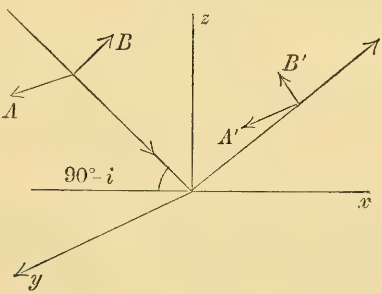
3. In a paper published in the *Phil. Trans.* for 1891 I deduced formulae giving the amplitudes of light reflected at the surface of a *transparent* magnetized medium, by means of the hypothesis that Hall's effect can exist in dielectrics. When the lines of magnetic force are perpendicular to the reflecting surface the equations giving the amplitudes A', B' of the reflected wave in terms of the amplitudes A, B of the incident wave are (30) and (31) of that paper, and are as follows:

$$A' = \frac{A(U\cos i - V\cos r)}{U\cos i + V\cos r} + \frac{2iqBV\cos i}{U(U\cos r + V\cos i)(U\cos i + V\cos r)} \quad \ldots(4),$$

$$B' = -\frac{B(U\cos r - V\cos i)}{U\cos r + V\cos i} + \frac{2iqAV\cos i}{U(U\cos r + V\cos i)(U\cos i + V\cos r)} \quad \ldots(5),$$

where U, V are the velocities of light in air, and in the transparent medium when *unmagnetized*, i and r the angles of incidence and refraction, and q a quantity which is directly proportional to the magnetic force. The magnetic permeability k is supposed to be unity*, and the directions of the various quantities are shewn in the figure.

Fig. 1.



I shall now transform these formulae in the same manner as Eisenlohr has transformed Fresnel's sine and tangent formulae,

* In the case of an iron reflector we have no right strictly speaking to put k equal to unity; but as the formulae which will ultimately be deduced are of a tentative character and do not profess to be founded upon a rigorous dynamical theory, little would be gained by retaining k.

[Prof. J. J. Thomson suggests on p. 422 of his *Electricity and Magnetism* that iron may not retain its magnetic properties when subjected to such rapidly alternating waves as waves of light. Sep. 1893.]

using the notation of §§ 372—376 of my book on *Physical Optics*, and distinguishing the equations referred to by square brackets.

By § 372, the first term of (4) becomes

$$-\mathfrak{A}A\epsilon^{\frac{2\iota\pi e}{\lambda}},$$

where the value of $\mathfrak{A}$ is given by [9], and

$$\tan\frac{2\pi e}{\lambda}=\frac{2Rc\cos i\sin(\alpha+u)}{R^2c^2-\cos^2 i}\quad\text{................(6).}$$

The second term of (4) becomes

$$-Bq(P-\iota Q),$$

where

$$\left.\begin{aligned}P&=\frac{2R^3\cos i}{V^2D}\{R\cos^2 i\sin 2\alpha+Rc^2\sin 2(\alpha-u)\\&\qquad+c\cos i\sin(3\alpha-u)+R^2c\cos i\sin(\alpha-u)\}\\Q&=\frac{2R^3\cos i}{V^2D}\{R\cos^2 i\cos 2\alpha+Rc^2\cos 2(\alpha-u)\\&\qquad+c\cos i\cos(3\alpha-u)+R^2c\cos i\cos(\alpha-u)\}\end{aligned}\right\}\text{...(7),}$$

and

$$D=\{R^2\cos^2 i+c^2+2Rc\cos i\cos(\alpha-u)\}\{R^2c^2+\cos^2 i\\+2Rc\cos i\cos(\alpha+u)\}.$$

By § 374, the first term of (5) is

$$\mathfrak{B}B\epsilon^{\frac{2\iota\pi e'}{\lambda}}.$$

where the value of $\mathfrak{B}$ is given by [15], and

$$\tan\frac{2\pi e'}{\lambda}=\frac{2Rc\cos i\sin(\alpha-u)}{R^2\cos^2 i-c^2}\quad\text{................(8).}$$

Case I. Let the incident light be polarized in the plane of incidence and be of amplitude unity, then $B=0$. Also let

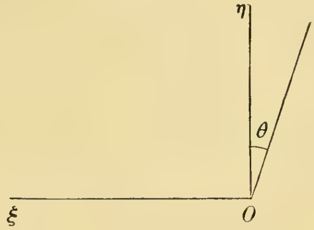
$$\phi=(2\pi/\lambda)(x\cos i+z\sin i-Vt),$$
$$e_1=2\pi e/\lambda,\quad e_1'=2\pi e'/\lambda.$$

Then if ξ, η be the two component vibrations of the reflected wave perpendicularly to, and in the plane of incidence

$$\xi=-\mathfrak{A}\epsilon^{\iota(\phi+e_1)}=-\mathfrak{A}\cos(\phi+e_1),$$
$$\eta=-q(P-\iota Q)\epsilon^{\iota\phi}=-q(P\cos\phi+Q\sin\phi).$$

In fig. 2, the observer is supposed to be looking through an analyser at the point of incidence along the reflected ray; if

Fig. 2.



therefore the analyser be placed in the position of extinction $O\eta$, and be then turned through a very small angle θ towards the *right hand* of the observer, the emergent vibration will be

$$-\xi \sin\theta + \eta\cos\theta = \mathfrak{A}\theta\cos(\phi + e_1) - q(P\cos\phi + Q\sin\phi),$$

since θ is very small. Whence the intensity I^2 of the emergent light is

$$I^2 = \mathfrak{A}^2\theta^2 - 2\mathfrak{A}q(P\cos e_1 - Q\sin e_1)\,\theta + q^2(P^2 + Q^2)\ldots(9).$$

If we put $I = 0$, this is a quadratic equation for determining the values of θ for which the intensity vanishes, and we see that both roots must be of the same sign; also both will be positive provided the coefficient of θ in the second term be positive*. If therefore q be positive, this condition will be satisfied provided

$$P\cos e_1 - Q\sin e_1 > 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

Omitting the extraneous factors in (7), which are positive, the condition becomes

$$R\cos^2 i\sin(2\alpha - e_1) + Rc^2\sin(2\alpha - 2u - e_1) + c\cos i\sin(3\alpha - u - e_1) + R^2c\cos i\sin(\alpha - u - e_1) > 0\ldots(11).$$

Taking the values of R and α furnished by Sir J. Conroy's observations of the principal incidence and azimuth, it can be shewn by actual calculation that this inequality is always satisfied. From [7] it at once follows that $u = 0$ at normal incidence, and I find that at grazing incidence $u = 1°35'$ about; u is therefore an exceedingly small positive angle. At grazing incidence $e_1 = 0$, whilst at normal incidence I find by calculation from (8) that

$$e_1 = 22°\ 48'.$$

* The last term of (9) is so small that it may generally be omitted; at the same time the first experiment described on p. 385 of my book shews that it is capable of producing a sensible effect.

By means of these values it can be seen by inspection that each term of (11) is positive for all incidences. The same result follows from Eisenlohr's corrected table.

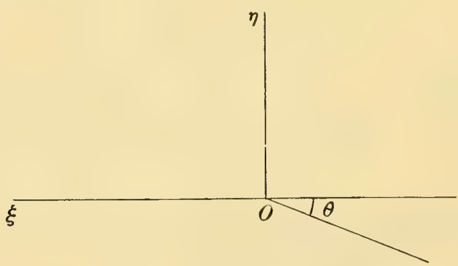
Having regard to the directions in which the quantities are measured in my paper in the *Phil. Trans.*, it follows that when q is positive the amperean current which magnetizes the electro-magnet circulates *from the right hand towards the left hand* of an observer who is looking at the point of incidence through the analyser*; whence θ is positive, and consequently in order to extinguish the light produced by magnetic action the analyser must be rotated *towards the right hand of the observer*, that is in the *opposite* direction to that of the current. On the other hand a rotation in the *same* direction as that of the current, or a reversal of the *direction* of the current strengthens the light restored from extinction. All these results agree with Kerr's experiments.

It ought to be noticed that the reflected light is really elliptically polarized, but as the magnetic term q is exceedingly small the portion of the intensity which depends upon q^2 is comparatively unimportant, and the reflected light is approximately plane polarized.

Case II. We must now consider what happens when the incident light is polarized perpendicularly to the plane of incidence.

In this case $A = 0$, and we may take $B = 1$; whence the re-

Fig. 3.



flected vibrations are

$$\xi = -q(P - \iota Q)\epsilon^{\iota\phi} = -q(P\cos\phi + Q\sin\phi),$$

$$\eta = \mathfrak{B}\epsilon^{\iota(\phi+e_1')} = \mathfrak{B}\cos(\phi + e_1').$$

* [The value of q is $C\mathfrak{H}/4\tau$, where C is Hall's constant, $\mathfrak{H}$ the magnetizing force which acts along the axis of z and τ the period. The value of C for iron is *positive* (see J. J. Thomson, *Electricity and Magnetism*, p. 488). On p. 398 of my book, line 19, read "Hall's constant" for "Hall's effect." Sep. 1893.]

Let the analyser be placed in the position of extinction $O\xi$, and be then turned through a small angle θ towards the right hand of the observer; the emergent vibration will be

$$-\xi\cos\theta-\eta\sin\theta=q\,(P\cos\phi+Q\sin\phi)-\mathfrak{B}\theta\cos(\phi+e_1'),$$

and consequently

$$I^2=\mathfrak{B}^2\theta^2-2\mathfrak{B}q\,(P\cos e_1'-Q\sin e_1')\,\theta+q^2(P^2+Q^2).$$

Whence if q is positive, the rotation of the plane of polarization will be in the opposite or in the same direction as the amperean current, according as the coefficient of θ is positive or negative. We have therefore to examine the sign of

$$P\cos e_1'-Q\sin e_1',$$

or by (7) of

$$R\cos^2 i\sin(2\alpha-e_1')+Rc^2\sin(2\alpha-2u-e_1')+c\cos i\sin(3\alpha-u-e_1')$$
$$+R^2c\cos i\sin(\alpha-u-e_1')\ldots(12).$$

The value of $\tan e_1'$ is given by (8), and is positive at normal incidence, and the value of e_1' as stated in Case I. is about $22°\,48'$. But if i is very nearly equal to $90°$, $\tan e_1'$ becomes a very small negative quantity, and therefore when $i=90°$, $e_1'=\pi$; and under these circumstances the most important term in the above expression is

$$Rc^2\sin(2\alpha-2u-e_1'),$$

which is nearly equal to

$$-Rc^2\sin 2(\alpha-u),$$

that is a negative quantity. We therefore see that if the angle of incidence be supposed to increase from normal incidence to grazing incidence, the rotation of the plane of polarization is at first in the *opposite* direction to that of the current, but it afterwards diminishes and vanishes, and finally takes place in the *same* direction as that of the current.

It would seem that Kerr did not particularly examine the effect produced at nearly grazing incidence; Kundt* however found that in the case of an *iron* electromagnet, the rotation was in the opposite direction to that of the current until the angle of incidence became equal to about 82° when it vanished and changed sign, taking place in the same direction when the angle of incidence lay between 82° and 90°.

To find the exact value of the angle which makes the expression (12) vanish and change sign would necessitate some rather laborious numerical calculations; it is however not difficult to shew that it is positive when the angle of incidence is equal to

* *Berlin. Sitzungsberichte*, July 10th, 1884; translated *Phil. Mag.* Oct. 1884, p. 308.

the principal incidence, that is when $i = 76° 48'$. To do this, we observe that at the principal incidence,

$$e_1' = e_1 + \tfrac{1}{2}\pi,$$

so that the expression to be examined becomes

$$-R\cos^2 i\cos(2\alpha - e_1) - Rc^2\cos(2\alpha - 2u - e_1)$$
$$-c\cos i\cos(3\alpha - u - e_1) - R^2 c\cos i\cos(\alpha - u - e_1) \ldots\ldots(13).$$

From (6) it follows that at the principal incidence

$$\tan e_1 = \frac{\sin^2 2I\sin(\alpha + u)}{2\cos(\pi - 2I)},$$

from which I find that

$$e_1 = 5° 12',$$

whence

$$2\alpha - e_1 = 103° 20', \quad 2\alpha - 2u - e_1 = 100° 20',$$
$$3\alpha - u - e_1 = 156° 6', \quad \alpha - u - e_1 = 47° 34',$$

from which it follows that all four terms of (13) are positive except the last. By calculation I find that

$$-Rc^2\cos(2\alpha - 2u - e_1) = \cdot 75288,$$
$$R^2 c\cos I\cos(\alpha - u - e_1) = \cdot 26251,$$

accordingly the expression (13) is positive, and therefore at the principal incidence the rotation is in the opposite direction of the current, and must vanish and change sign for an incidence greater than 76° 48′.

All the foregoing results, so far as they are qualitative, agree with experiment, and the quantitative result which has just been obtained appears to indicate that more elaborate calculations would shew that the change in the direction of rotation would occur at an angle which does not differ very much from 82°.

4. We must now consider the case when the magnetization is parallel to the reflecting surface and also to the plane of incidence.

The equations which give the amplitudes of the reflected light when the medium is transparent are (47) and (48) of my paper in the *Phil. Trans.* and are as follows:

$$A' = \frac{A(U\cos i - V\cos r)}{U\cos i + V\cos r} - \frac{2\iota qBV\tan r\cos i}{U(U\cos i + V\cos r)(U\cos r + V\cos i)} \ldots\ldots(14),$$

$$B' = -\frac{B(U\cos r - V\cos i)}{U\cos r + V\cos i} + \frac{2\iota qAV\tan r\cos i}{U(U\cos i + V\cos r)(U\cos r + V\cos i)} \ldots\ldots(15),$$

k being put equal to unity as before.

We have now to transform these equations. The first terms on the right-hand sides merely lead to Cauchy's expressions; whilst the coefficient of B in the second term of (14) is

$$-q(P_1 - \iota Q_1),$$

whence

$$\left.\begin{aligned} P_1 &= \frac{R^2 \sin 2i}{V^2 Dc}\{R\cos^2 i \sin(\alpha - u) + Rc^2 \sin(\alpha - 3u) \\ &\qquad + c\cos i \sin 2(\alpha - u) - R^2 c \cos i \sin 2u\} \\ Q_1 &= \frac{R^2 \sin 2i}{V^2 Dc}\{R\cos^2 i \cos(\alpha - u) + Rc^2 \cos(\alpha - 3u) \\ &\qquad + c\cos i \cos 2(\alpha - u) + R^2 c \cos i \cos 2u\} \end{aligned}\right\} \ldots (16).$$

Case III. Let the incident light be polarized in the plane of incidence and be of amplitude unity; then $B = 0$, and we shall find in exactly the same manner as in Case I. that if q be positive, the rotation of the plane of polarization of the reflected light will be towards the right hand of the observer if

$$P_1 \cos e_1 - Q_1 \sin e_1 > 0.$$

Omitting extraneous factors, this condition becomes by (16)

$$R\cos^2 i \sin(\alpha - u - e_1) + Rc^2 \sin(\alpha - 3u - e_1) + c\cos i \sin(2\alpha - 2u - e_1)$$
$$- R^2 c \cos i \sin(2u + e_1) > 0 \ldots (17).$$

Since the magnetic terms contain $\sin 2i$ as a factor, it follows that they must vanish at normal incidence which agrees with experiment. They also vanish at grazing incidence, which also agrees with experiment, for Kerr found that at 85° the magnetic effects were very faint, but grew stronger as the incidence diminished to about 60°, when they began to grow weaker. When the incidence is very nearly grazing, the most important term of (17) is the second which is positive, since $e_1 = 0$, and

$$\alpha - 3u \equiv 49° \, 31'.$$

Now when q is positive the amperean current circulates towards the left hand of the observer, whence the analyser must be rotated towards his right hand; and this agrees with experiment.

When the incidence is nearly normal, the last term of (17), which is negative, has its greatest value. In this case

$$c = 1, \quad u = 0, \quad e_1 = 22° \, 48'$$

and it can be shewn that (17) is satisfied. From this it is easily seen that (17) is satisfied for all angles of incidence, and consequently the rotation of the plane of polarization of the reflected light is in the opposite direction to that of the current. This agrees with experiment.

Case IV. The last case of all that we have to consider arises when the incident light is polarized perpendicularly to the plane of incidence. Here $A=0$, $B=1$,

$$\xi = q(P_1 - \iota Q_1)\,\epsilon^{\iota\phi} = q(P_1 \cos\phi + Q_1 \sin\phi),$$

$$\eta = \mathfrak{B}\epsilon^{\iota(\phi+e_1')} = \mathfrak{B}\cos(\phi + e_1');$$

consequently as in Case II.

$$I^2 = \mathfrak{B}^2\theta^2 + 2\mathfrak{B}q(P_1\cos e_1' - Q_1 \sin e_1')\,\theta + q^2(P_1^2 + Q_1^2).$$

Accordingly when q is positive, so that the current circulates from the right hand towards the left hand of the observer, the value of θ will be *negative*, when

$$P_1 \cos e_1' - Q_1 \sin e_1' > 0 \quad\text{.....................(18).}$$

From this it follows that for angles of incidence which make the left-hand side of (18) positive, the analyser must be rotated in the *same* direction as the current in order to produce extinction; whilst for angles such that the left-hand side is negative, it must be rotated in the *opposite* direction.

When the incidence is very nearly equal to 90°,

$$e_1' = 180°, \quad u = 1°\,35',$$

so that the term $Q_1 \sin e'$ is negligible; also the most important term in P_1 which is $Rc^2 \sin(\alpha - 3u)$ is positive, so that $P_1 \cos e_1'$ is negative. Under these circumstances the analyser must be rotated in the *opposite* direction to that of the current.

The left-hand side of the inequality (18) is equal to

$$R\cos^2 i \sin(\alpha - u - e_1') + Rc^2 \sin(\alpha - 3u - e_1')$$
$$+ c\cos i \sin(2\alpha - 2u - e_1') - R^2 c \cos i \sin(2u + e_1') \text{...(19).}$$

At nearly normal incidence $u=0$, and e_1' is sensibly equal to e_1, so that the above expression becomes sensibly equal to (17), which has been shewn to be positive. In this case the rotation must be in the *same* direction as that of the current.

These results are in fair agreement with Kerr's experiments, for he found that the rotation was in the *opposite* direction to that of the current so long as the angle of incidence lay between 90° and 75° and in the *same* direction when it lay between 75° and 0°. At the same time the agreement as regards numerical results is not quite so close as might be desired, for I find that the value of the expression (19) at the principal incidence is about equal to -7, from which it appears that the direction of rotation ought to change sign at an angle which is somewhat smaller* than 75°. All observers seem to give values of the principal incidence and azimuth which lead to values of the

* When $e_1' = 90°$, $\cos i = c/R$, from which I find $i = 75°\,14'$.

constants R and α which are in fair agreement with one another, but as there is some uncertainty as regards their precise values, perhaps a very close agreement cannot be expected between the theoretical and the experimental value of the angle of incidence at which the direction of rotation vanishes and changes sign.

Although Kerr states that the principal incidence of the piece of soft iron which formed the reflector in his experiments was about 76°, he does not state the value of the principal azimuth. As so much depends upon these quantities, I should suggest that if any further experiments are made, the values of the principal incidence and azimuth should be accurately determined for the particular piece of metal experimented upon. It would also be desirable to obtain quantitative results giving the magnitude of the angle of rotation at different angles of incidence, and for light of certain selected refrangibilities.

5. The formulae which have been discussed in the present paper furnish results which are in such substantial agreement with Kerr's experiments, that I feel strongly persuaded that they are a close approximation to the truth; but as the complete theory has not yet been discovered, I shall conclude this paper by making some observations upon the difficulties which lie in the way of its realization. It is hopeless to attempt to construct a perfectly satisfactory theory of Kerr's experiments until a satisfactory theory of metallic reflection has been discovered, and from the results of the present paper I incline to the opinion that the difficulty does not consist in explaining the magnetic effect, but in explaining metallic reflection. The formulae in question have been obtained by assuming in the first place that Hall's effect is capable of existing in transparent media, and by the aid of this hypothesis the formulae giving the amplitudes of the reflected waves were rigorously deduced two years ago by means of Maxwell's theory of the electromagnetic field. The next step was to transform these formulae by assuming that the pseudo-refractive index is a complex quantity in the case of a metal; and the fact that the resulting formulae agree so closely with Kerr's experiments justifies, I think, the conclusion that such an intimate connection exists between Hall's effect and the discoveries of Faraday, Kerr and Kundt upon the action of a magnetic field upon light, that the two classes of phenomena are in great measure due to the same ultimate cause.

The difficulties, which lie in the way of constructing a satisfactory electromagnetic theory of metallic reflection by taking into account the conductivity of the metal, arise from the fact that the physical conditions imposed by such a theory imperatively require that α should be less than 45°; whereas experiment shews that for many metals $\alpha > 45°$. Exactly the same objection applies to the hypothesis that metallic reflection may be accounted for on the so-called elastic solid theory by the introduction of a viscous term. In this theory the difficulty may to a certain extent be got rid of by an extension of von Helmholtz' theory of anomalous dispersion, as shewn in § 386 of my book; and I entertain very little doubt that if we were better acquainted with the molecular motions which take place when matter is disturbed by the action of electromagnetic waves, this difficulty would be surmounted, and a satisfactory electromagnetic theory of metallic reflection would be obtained. "When," as I stated in § 486 of my book, "electromagnetic waves travel through a medium which is susceptible to magnetic influence, the molecules of matter will be thrown into vibration; and the direction in which we ought to look for a theory, which will take cognizance of these hitherto unexplained phenomena, is one in which account is taken of the mutual reaction of ether and matter, and which will enable us to introduce the free periods of the vibrations of the matter into our equations."

[6. Since this paper was read, Prof. J. J. Thomson's *Recent Researches in Electricity and Magnetism* has been published, which contains a theory of Kerr's experiments, see pp. 482—509. The principles upon which this theory is based appear to be much the same as those of the present paper, but the details of the analysis are sufficiently different to render comparison difficult; moreover he has not entered into numerical calculations to the same extent that I have done. He finds that a result consistent with the experiment discussed in Case IV. cannot be obtained unless the transverse electromotive intensity is proportional to the *polarization* current instead of to the *total* current; whereas in the present paper it has been tacitly assumed that this quantity depends upon the total current.

The reversal of the direction of rotation, which takes place when the incident light is polarized *perpendicularly* to the plane of incidence, appears to arise from the circumstance that e_1' (where $e_1'\lambda/2\pi$ is the change of phase) increases from about 22° at normal incidence to 180° at grazing incidence, and consequently between these incidences $\cos e_1'$ vanishes and changes sign. But when the light is polarized *in* the plane of incidence e_1 (where $e_1\lambda/2\pi$ is the change of phase) diminishes from about 22° at normal incidence to zero at grazing incidence, and therefore $\cos e_1$ can never change sign, but always remains positive. Sep. 1893.]

7. It may not be out of place to call attention to another series of experiments made by Kerr *, in which he shewed that

* *Phil. Mag.* Nov. 1875, p. 339; Dec. 1875, p. 446; March 1880, p. 157.

the effect of electrostatic force upon an isotropic transparent medium is to convert it into one which is optically equivalent to a uniaxal crystal, whose axis is parallel to the direction of the force. Under these circumstances it would appear probable that a strongly charged metallic conductor would behave like a doubly refracting metallic medium, having a single optic axis whose direction is normal to the surface of the reflector. If this conjecture should turn out to be correct, we should anticipate that when polarized light is reflected, the component vibration *in* the plane of incidence would be much more affected than the component perpendicular to that plane; and that the values of the principal incidence and azimuth and also the differences between the changes of phase of the two components would also be affected by electrostatic action. No experiments upon reflection from electrified metallic reflectors appear as yet to have been made; but I am strongly inclined to think that such experiments would repay the trouble involved in making them, and would reveal some entirely novel and interesting phenomena.

Monday, May 15, 1893.

PROF. T. McK. HUGHES, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *Exhibition of abnormal forms of* Spirifera lineata (*Martin*) *from the Carboniferous Limestone.* By F. R. COWPER REED, B.A., Trinity College.

This species, as defined by Davidson, is normally subject to great variation of form and ornamentation, as it includes *Sp. imbricata* and *Sp. elliptica.* Specimens with intermediate characters are however common. The series of abnormal forms exhibited showed the gradual development of a sharp median groove both in the dorsal and ventral valves so as ultimately to produce a bilobed shell. From the nature of these grooves interruption of the shell-secreting action of the mantle seems to have occurred along a definite line: and the cause may have been disease, the presence of a parasite or foreign body, or pressure during life. Similar malformation is seen in some Terebratulas, etc. The normal and regular bilobation of some species of *Orthis, Terebratula*, etc., is comparable.

(2) *Exhibition of Post-Glacial Mammalian bones from Barrington recently acquired by the Museum of Zoology.* By S. F. HARMER, M.A., King's College.

(3) *Exhibition of a specimen shewing Karyokinetic division of the nuclei in a plasmodium of one of the* Mycetozoa. By J. J. LISTER, M.A., St John's College.

(4) *Observations on the Flora of the Pollard Willows near Cambridge.* By J. C. WILLIS, M.A., Frank Smart Student of Gonville and Caius College, and I. H. BURKILL, B.A., Gonville and Caius College, Assistant Curator of the Herbarium.

The observations contained in the following paper were begun in 1890 and completed during the present year, and the results obtained seem worthy of publication, as of some interest in themselves, and as confirming and extending those given by Loew in a recent paper*.

The trees examined are mostly on or near the banks of the Cam and Ouse, from Ely on the north to Dernford on the south, a distance of about 22 miles. They are polled at a height of 8 feet, and stand in rows a few yards apart. Their tops contain large masses of humus, in which occur many plants.

Prof. Babington, in his Flora of Cambridgeshire, gives 950 species of plants as occurring in the County. Of these 350 may be excluded, as water or bog plants or for other reasons, and of the remaining 600 which could possibly be found, 80 species have been observed in the willows. About 4000 trees were examined, and a total number of 3951 records made. Where several plants of the same species occur in one tree, they are counted as one. The tables accompanying this paper show the actual number of records of each plant for each section of the district examined, and also the percentages and totals.

The names given for the plants are those of the "London Catalogue of British Plants," 8th edition.

The districts studied are named A, B, C... and require a more detailed description.

A. This includes the trees found upon Coe Fen, within the town of Cambridge, and in Grantchester, Shelford and Barraway villages. All these regions are well wooded, and are near to numerous gardens. The effect is seen in the figures: the most of the records of *Acer*, *Viburnum*, *Ulmus*, occur in this section; *Rosa* and *Crataegus*, hedge plants, are below the average; *Sambucus* and *Fraxinus* stand high, and also *Ribes*, on account of the nearness of the gardens.

B. Trees along the Cam, for about half a mile up stream from Clayhythe bridge. Along the opposite bank runs a wood of ash trees, &c., with a dense undergrowth of *Anthriscus*, *Urtica*, *Galium*, *Epilobium*, &c. These plants appear in large number in the list. Near the bridge are gardens, accounting for the presence of *Ribes*.

* "Anfänge epiphytischer Lebensweise bei Gefässpflanzen Norddeutschlands," *Verhandl. d. bot. Vereins der Prov. Brandenburg*, XXXIII., 1892, p. 63. Abstr. in *Bot. Centralblatt*, vol. 52, p. 27.

C. Trees along the Cam, left bank, from Chesterton to Horningsea. A district fairly open, with a good many trees in it and many hedges, passing near to gardens in several places. *Ribes* consequently stands high. Elder, Rose and Hawthorn, hedge plants, stand high, but trees scarcely appear in the list at all, none being very near to the willows.

D. Trees on the right bank of the Cam, from Ditton to Baitsbite lock (i.e. opposite to about half of section C) and on both banks from Clayhythe bridge to the lock at Bottisham. A district of similar character to C, but being nearer to many ash trees by the width of the river (30 metres) the records of ash are very much more numerous. Nettles also are more plentiful, there being many of them in fields near Ditton.

E. Trees on the open meadow between Cambridge and Grantchester, and on Dernford Fen. A more open district than the preceding, but with many hedges. Rose, Hawthorn and Elder figure largely in the list, while nettles fall off.

F. Trees along the left bank of the Cam, from Horningsea to the top of the reach above Clayhythe (i.e. beginning of section B). An open district, rather like the last. There being no gardens near, *Ribes* falls very low, as also does Elder, there being very little of it in the hedges.

G. Trees on Lingay Fen, to the south of Cambridge. A fairly open fen, but with hedges and trees within a short distance.

H. Trees on various parts of Bottisham and Soham Fens. These districts are really open fen, with the usual ditches in place of hedges between the fields. Hedges, however, occur round the farms, and these and the gardens account for the presence of Elder, *Ribes* and Hawthorn. *Rosa* is conspicuous by its absence, and the list consists almost entirely of herbaceous plants, the grasses especially occurring in large numbers, forming half the list.

The districts are thus arranged in a rough series, beginning with those most thickly wooded and ending with the most open

The results thus obtained present various points of interest, which will now be discussed.

The plants belong to 28 Natural Orders and include 61 genera. Eleven genera of grasses occur, 6 each of Compositae and Rosaceae, 4 of Umbelliferae.

If we take the actual number of records of each plant, and leave out those forming less than one per cent. of the total, we have left only 21 species (counting each of the four species of *Poa*). *Galium Aparine* heads the list with no less than 16·3 per cent., and is closely followed by *Sambucus* and *Rosa*. *Urtica* and *Crataegus* also form each more than 5 per cent. of the total.

Nineteen species have only one record each, and twenty others have less than five records. The remaining twenty species have from 5 to 40 records.

The results are of some interest from the point of view of the means of distribution of seeds. If we roughly classify them according to the various methods employed we obtain the following list:

I. *Fleshy Fruits, distributed by animals.*

Rhamnus, Prunus, Rubus (4 species), *Rosa, Pyrus, Crataegus, Ribes* (3), *Bryonia, Hedera, Sambucus, Viburnum, Lonicera, Solanum, Asparagus.*

Total species 19 (23·75 per cent.), plants 1763 (44·62 per cent.).

II. *Burred Fruits, distributed by animals.*

Geum, Galium Aparine, Avena.

Total species 3 (3·75 per cent.), plants 651 (16·47 per cent.).

III. *Winged and feathered fruit or seed, wind-distributed.*

Acer, Epilobium (2), *Angelica, Heracleum, Senecio* (2), *Cnicus, Leontodon, Taraxacum, Lactuca, Fraxinus, Syringa, Rumex* (3), *Ulmus, Humulus, Alnus,* and the *Gramineae* (14).

Total species 33 (41·25 per cent.), plants 995 (25·18 per cent.).

IV. *Seeds, &c., very small and light, wind-distributed.*

Sisymbrium, Cerastium (2), *Stellaria, Achillea, Veronica* (2), *Urtica, Polypodium.*

Total species 9 (11·25 per cent.), plants 425 (10·75 per cent.).

V. *Explosive mechanisms.*

Geranium.

Total species 1 (1·25 per cent.), plants 2 (·05 per cent.).

This plant must have reached its place in the willow by some other means, as its mechanism is not good enough to throw the seed to such a height.

VI. *Plants whose means of distribution is poor or doubtful.*

Ranunculus (3), *Barbarea, Lathyrus, Chaerophyllum, Anthriscus, Galium Mollugo, Calystegia, Nepeta, Stachys, Lamium* (2), *Plantago, Polygonum.*

Total species 15 (18·75 per cent.), plants 115 (2·9 per cent.).

The percentages of the species distributed in each of the different ways agree fairly closely with those given by Loew.

		W. and B.	Loew.
Animal distributed	(I., II.)	27·5	23·33
Wind „	(III., IV., V.)	53·75	53·33
Doubtful	(VI.)	18·75	23·33

But while in number of species wind-distribution holds the first place, in number of plants animal-distributed species form 61 per cent. of the total.

The figures show very well the effect of possessing a good distribution mechanism. Only those plants which possess such occur in any important number in the willow tops. Even such common plants as *Brassica, Capsella, Trifolium, Bellis* and others, do not appear in the list.

Another important conclusion on distribution that can be drawn from these observations, is that a seed is rarely carried by its distribution mechanism to a distance of more than a few hundred yards.

Plants were never found in willows more than 200 yards from others of the same species upon the ground, except in one instance, a plant of *Ribes Grossularia* in section H, which was more than half a mile from any other, but as it was some years old, probably there may formerly have been a gooseberry plant near at hand.

It is impossible to give the details of every tree examined, within the limits of a short paper, but in making the observations this fact of short-distance carriage was forced upon our notice at every step. A consideration of the Table will to some extent show this. Several plants require also special notes and these may conveniently be here introduced.

Rhamnus. Chiefly found wild in the southern part of the district, where the records occur.

Acer, Fraxinus, Syringa, Ulmus. These only occur in the wooded districts, and always close to trees.

Rosa. Common in all districts except the open fen (H). Even here there were often plenty of roses within half a mile, yet none were found in the willow tops.

Ribes. Common near gardens, but never found far from them, hence its decrease in sections F, G, H.

Epilobium. Though its seeds are so light, and it is so common, yet it only occurs in any number in section B, opposite the wood in which it forms a large part of the undergrowth.

Anthriscus. Occurs in most districts, being very common upon the soil. It forms the chief portion of Class VI. as regards actual numbers.

Hedera. It is probable that many of the records of the occurrence of ivy are due to its having climbed up the tree, taken root in the top and then died away below. Several cases were found where this process was going on, the stems dying away and leaving no trace, but in all the 75 cases recorded the ivy was found simply in the tops.

Sambucus. Extremely common everywhere.

Galium Aparine. This plant heads the list of records. It is extremely common in the whole district. The burr fruit is well adapted for clinging to animals and probably birds carry it largely to the trees, adhering to their feathers; the numbers in Class I. show how largely they visit the trees. *Galium* is also used for nest building, as will be seen below.

Urtica. Common everywhere, but mostly in districts B and D.

Dactylis. Most common in the fen districts.

Comparing our list with that given by Loew for the willows he examined, it appears that of his 31 species, 11 are not present in our list, and Bolle* gives 3 more that do not occur in ours.

A comparison with the flora of the churches of Poitiers† shows that our flora contains far more bird distributed species in proportion to the total, viz. 22 species out of 80, as contrasted with 3 in 76. This is what would be expected in view of the fact that birds so largely visit the willow trees. Of Richard's 76 species, only 14 occur in our list.

Other floras of similar character‡ exist, but need not be specially discussed here.

Birds' nests being common in willow tops, it seemed to us that the plants used in building might become established in large numbers by this means, and we have therefore made an analysis, so far as possible, of a few nests taken at random. Among the materials of the nests were found the following:

(1) Two thrushes' nests from Dernford Fen (section E).
Anthriscus sylvestris, stems.
Galium Aparine, many stems, 13 fruits, and a living seedling, rooted.
Urtica dioica, many stems.
Poa annua and *P. pratensis*, stems, leaves and panicles.
Bromus sterilis, panicles.
Glyceria aquatica, stems and leaves.
Agropyrum repens, „ „

(2) One thrush's nest from Swaffham Lode (H).
Senecio aquaticus, stem.
Lamium purpureum, living plant, rooted in the nest.
Dactylis glomerata, entire plant.

* "Nachtrag zur Florula der Kopfweiden," *Verhandl. d. bot. Vereins d. Prov. Brandenburg*, XXXIII., 1892, p. 72.

† Richard, "Florule des clochers et des toitures des églises de Poitiers." Paris, 1888.

‡ See list in Richard, *loc. cit.*, p. 48.

(3) Three sparrows' nests from section H.

Anthriscus sylvestris, stem.
Carduus sp., large numbers of ripe fruits.
Phragmites communis, Poa annua, pratensis and *trivialis, Lolium perenne, Bromus mollis, Deschampsia caespitosa, Festuca ovina, Dactylis glomerata,* stems and panicles.

(4) Two sparrows' nests from section D.

Much like the last, especially in containing no *Galium*, but including also *Agropyrum repens*, and *Alisma Plantago.*

(5) Three blackbirds' nests from section H.

Galium Aparine, stems, &c.
Phragmites communis, stems, &c.
Urtica dioica, stems.
Anthriscus sylvestris, stems.

(6) One nest (? sparrow) from section H.

Stems, leaves and inflorescences of *Cynosurus cristatus, Triticum vulgare, Alopecurus pratensis, Avena* sp. ?, *Phragmites communis, Glyceria aquatica, Agropyrum repens, Lolium perenne, Poa annua* and *pratensis, Deschampsia caespitosa*, and most interesting, perhaps, of all, pieces of *Elodea canadensis*, which were living when added to the nest.

(7) One wren's nest from section H.

Daucus Carota, stem and fruit.
Galium Aparine and several grasses.

It is thus evident that many grasses are probably distributed by birds, and also other plants, such as *Galium, Lamium, Anthriscus*, &c.

Turning now to the question of the nutrition of these plants, the first point to be noticed is the great size and vigour of many plants. Specimens of *Sambucus* 2—3 metres high and 2—8 cms. thick are common; the largest found was 4 metres high and 16 cms. thick. In this case the roots were observed to have grown right down through the trunk of the willow into the soil, and probably this occurs in other cases. A plant of *Acer* was 5 cms. thick. *Rosa, Ribes, Crataegus*, &c. were often well grown, and bearing many flowers. Herbs such as *Galium* and Grasses also grow well, many trees being simply beds of them.

Of the 80 species, 64 are perennial (23 shrubs or trees), 5 biennial, 11 annual. Probably, however, this fact is not of importance, as the annuals become established in the trees; among them is included *Galium Aparine*, the commonest species of all.

The roots of the perennials go deeply into the humus, and the question arises as to whether they can draw from thence, unaided, all the materials required for their growth. Loew thinks this improbable, and suggests that mycorhiza comes into play; he has observed this on several species. It does not seem to us, however, that this is necessary, considering the age of the humus, much of which must be fully decayed, and considering also the amount of dust which must blow into the bowl-shaped tops of the willows in the course of many years. Höveler* in a recent paper has shown that plants are able to make use of humus without the aid of mycorhiza. We hope to investigate this point further.

With reference to water supply, an important question in regard to epiphytic plants, the large masses of humus present in the willow tops must retain a great deal of water, dry though the climate of Cambridge is. The crown of leafy shoots upon the willow tree itself will also protect the "epiphytes" from transpiration. We have noticed a tendency to a bulbous enlargement of the base of the stem in one or two specimens of *Holcus lanatus* and *Poa annua.*

To sum up, the one character of epiphytes† which is well shown in these plants is a well-marked adaptation of the seeds to distribution. They possess no special methods of clinging to their supports, nor for collecting water or humus, beyond what has been above described. They are, nevertheless, of interest as showing a tendency in the direction of epiphytism. Such an epiphyte as *Æschinanthus* might perhaps be compared with them. This side of the question is discussed more at length in Loew's paper.

In conclusion, one or two miscellaneous points may be mentioned.

The record of *Lactuca muralis* is interesting. It is recorded for the same trees in Babington's Flora, and has thus survived, though an annual, for over 35 years in these trees. There are no plants of this species now living on the soil within a considerable distance.

One or two willows examined had put out roots from the crown into their own humus, and some at least of these had grown right through the willow tree to the soil (cf. the case of the roots of the Elder, above).

* "Ueb. d. Verwerthung d. Humus bei d. Ernährung der chlorophyllführenden Pflanzen," *Prings. Jahrb.* xxiv. p. 283, 1892.

† Goebel, "Pflanzenbiologische Schilderungen," I. pp. 153, 161 &c.

DISTRICT	Well wooded villages A	%	Ditto, few gardens B	%	Hedges, gardens, few trees C	%	Similar, more trees D	%	Open meadows E	%	Open, no gardens F	%	Open fen, few hedges G	%	Open fen, few farms near H	%	Total	%
Total no. of Records	568		412		780		886		423		326		258		298		3951	
1. Ranunculus acris, L.	1	·17	—	—	1	·13	—	—	—	—	2	·61	—	—	—	—	4	·10
2. bulbosus, L.	—	—	—	—	—	—	—	—	—	—	—	—	1	·39	—	—	1	·025
3. Ficaria, L.	2	·35	2	·48	3	·38	—	—	—	—	1	·31	—	—	—	—	8	·20
4. Barbarea vulgaris, R. Br.	1	·17	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	·025
5. Sisymbrium Alliaria, Scop.	2	·35	—	—	2	·26	2	·22	3	·71	1	·31	—	—	—	—	10	·25
6. Cerastium tetrandrum, Curt.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	·33	1	·025
7. triviale, Link.	6	1·06	—	—	2	·26	2	·22	—	—	4	1·22	3	1·16	5	1·68	22	·56
8. Stellaria media, Cyr.	1	·17	10	2·43	18	2·31	18	2·03	—	—	21	6·44	6	2·32	6	2·01	80	2·02
9. Geranium Robertianum, L.	2	·35	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	·05
10. Rhamnus catharticus, L.	5	·88	—	—	—	—	—	—	—	—	—	—	11	4·26	—	—	16	·40
11. Acer Pseudo-platanus, L.	16	2·82	—	—	1	·13	3	·34	—	—	—	—	3	1·16	—	—	23	·58
12. Lathyrus pratensis, L.	1	·17	—	—	—	—	—	—	—	—	1	·31	—	—	—	—	2	·05
13. Prunus Cerasus, L.	1	·17	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	·025
14. Rubus Idaeus, L.	—	—	2	·48	1	·13	—	—	—	—	—	—	—	—	—	—	3	·076
15. rusticanus, Merc.	5	·88	1	·24	4	·51	4	·45	3	·71	6	1·84	—	—	—	—	23	·58
16. corylifolius, Sm. &c.*	8	1·41	12	2·91	12	1·54	11	1·24	11	2·60	25	7·67	3	1·16	1	·33	83	2·10
17. Geum urbanum, L.	1	·17	1	·24	—	—	1	·11	1	·24	—	—	2	·77	—	—	6	·15
18. Rosa canina, L.	33	5·82	16	3·88	77	9·87	81	9·14	91	21·51	46	14·11	66	25·56	—	—	410	10·38
19. Pyrus Aucuparia, Gaert.	1	·17	—	—	1	·13	—	—	2	·47	—	—	—	—	—	—	4	·10
20. Crataegus Oxyacantha, L.	23	4·05	8	1·94	50	6·41	38	4·28	41	9·69	33	10·12	28	10·85	10	3·35	231	5·85

* Under this heading are included other species belonging to this section of *Rubi*, owing to the difficulty of identification.

† *Cynosurus cristatus* was identified among the doubtful grasses for the sections A and C.

†† All four species of *Poa* were about equally common; they are classed together, owing to the difficulty of distinguishing them upon the spot. Among them are also probably included several records of *Lolium*.

‡ Under this head there occur chiefly seedling grasses, most of them belonging to the species above, but possibly including some others.

§ Recorded on the authority of Babington's "Flora of Cambridgeshire." It has since disappeared from this section.

DISTRICT	Well wooded villages A	%	Ditto, few gardens B	%	Hedges, gardens, few trees C	%	Similar, more trees D	%	Open meadows E	%	Open, no gardens F	%	Open fen, few hedges G	%	Open fen, few farms near H	%	Total	%
Total no. of Records...	568		412		780		886		423		326		258		298		3951	
21. Ribes Grossularia, L.	20	·3·52	9	2·18	26	3·33	11	1·24	7	1·65	—	—	—	—	3	1·00	76	1·92
22. rubrum, L.	57	10·03	12	2·91	54	6·92	40	4·50	15	3·54	3	·92	3	1·16	3	1·00	187	4·73
23. nigrum, L.	—	—	—	—	2	·26	2	·22	—	—	1	·31	—	—	—	—	5	·13
24. Epilobium hirsutum, L.	1	·17	—	—	—	—	—	—	—	—	—	—	1	·39	—	—	2	·05
25. parviflorum, Schreb.	2	·35	36	8·74	—	—	9	1·01	—	—	—	—	—	—	—	—	47	1·19
26. Bryonia dioica, L.	—	—	—	—	—	—	1	·11	1	·24	—	—	2	·77	1	·33	5	·13
27. Chaerophyllum temulum, L.	—	—	—	—	1	·13	—	—	—	—	—	—	—	—	—	—	1	·025
28. Anthriscus sylvestris, Hoffm.	8	1·41	11	2·67	13	1·72	15	1·69	16	3·78	—	—	—	—	—	—	63	1·59
29. Angelica sylvestris, L.	3	·53	—	—	—	—	—	—	—	—	—	—	1	·39	—	—	4	·10
30. Heracleum Sphondylium, L.	—	—	—	—	—	—	—	—	1	·24	—	—	—	—	—	—	1	·025
31. Hedera Helix, L.	11	1·94	11	2·67	19	2·43	16	1·80	11	2·60	6	1·84	1	·39	—	—	75	1·89
32. Sambucus nigra, L.	117	20·59	8	1·94	130	16·66	144	16·24	71	16·78	10	3·06	26	10·08	44	14·76	550	13·92
33. Viburnum Opulus, L.	2	·35	—	—	—	—	—	—	1	·24	—	—	—	—	—	—	3	·076
34. Lonicera Periclymenum, L.	1	·17	—	—	1	·13	1	·11	—	—	—	—	—	—	—	—	3	·076
35. Galium Mollugo, L.	—	—	—	—	—	—	—	—	—	—	—	—	4	1·55	1	·33	5	·13
36. Aparine, L.	35	6·17	82	19·90	151	19·36	185	20·87	58	13·71	71	21·77	28	10·85	34	11·49	644	16·30
37. Achillea millefolium, L.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	·33	1	·025
38. Senecio Jacobaea, L.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	·33	1	·025
39. aquaticus, Huds.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	·67	2	·05
40. Cnicus lanceolatus, Hoffm.	2	·35	—	—	1	·13	—	—	—	—	3	·92	—	—	—	—	6	·15
41. Leontodon hispidus, L.	—	—	—	—	—	—	—	—	—	—	—	—	1	·39	—	—	1	·025
42. Taraxacum officinale, Web.	3	·53	18	4·36	16	2·05	12	1·35	7	1·65	13	3·99	8	3·10	5	1·68	82	2·07
43. Lactuca muralis, Fresen.	5	·88	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5	·13
44. Fraxinus excelsior, L.	29	5·09	26	6·31	1	·13	31	3·50	9	2·12	2	·61	1	·39	1	·33	100	2·53
45. Syringa vulgaris, L.	2	·35	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	·05
46. Calystegia Sepium, R. Br.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	·33	1	·025
47. Solanum Dulcamara, L.	4	·70	8	1·94	20	2·56	20	2·25	10	2·36	4	1·22	19	7·36	2	·67	87	2·20
48. Veronica hederaefolia, L.	—	—	—	—	1	·13	—	—	1	·24	—	—	1	·39	—	—	3	·076

49. Chamaedrys, L.	1	·17	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	·025
50. Nepeta Glechoma, Benth.	—	—	1	·24	—	—	—	—	1	·24	2	·61	—	—	2	·67	6	·15
51. Stachys sylvatica, L.	—	—	—	—	1	·13	—	—	—	—	—	—	—	—	—	—	1	·025
52. Lamium purpureum, L.	—	—	—	—	2	·26	—	—	—	—	1	·31	—	—	—	—	3	·076
53. album, L.	5	·88	4	·97	2	·26	3	·34	—	—	1	·31	—	—	1	·33	16	·40
54. Plantago major, L.	—	—	—	—	1	·13	—	—	—	—	—	—	—	—	1	·33	2	·05
55. Polygonum aviculare, L.	—	—	—	—	—	—	—	—	—	—	—	—	1	·39	—	—	1	·025
56. Rumex crispus, L.	1	·17	—	—	1	·13	1	·11	—	—	—	—	—	—	1	·33	4	·10
57. obtusifolius, L.	—	—	1	·24	—	—	—	—	—	—	1	·31	—	—	—	—	2	·05
58. Acetosa, L.	—	—	—	—	3	·33	—	—	1	·24	1	·31	1	·39	—	—	6	·15
59. Ulmus campestris, Sm.	14	2 47	—	—	2	·26	—	—	—	—	—	—	—	—	—	—	16	·40
60. Humulus Lupulus, L.	—	—	—	—	1	·13	—	—	—	—	—	—	—	—	—	—	1	·025
61. Urtica dioica, L.	42	7 42	76	18·45	39	5·00	87	9·81	16	3·78	23	7·05	3	1·16	20	6·71	306	7·74
62. Alnus glutinosa, L.	2	·35	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	·05
63. Asparagus officinalis, L.	—	—	—	—	1	·13	—	—	—	—	—	—	—	—	—	—	1	·025
64. Phleum pratense, L.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	·33	1	·025
65. Deschampsia caespitosa, Beauv..	—	—	—	—	—	—	1	·11	—	—	—	—	—	—	3	1·00	4	·10
66. Holcus lanatus, L.	2	·35	6	1·45	6	·77	4	·45	2	·47	2	·61	5	1·94	8	2·68	35	·89
67. Avena pubescens, Huds.	—	—	—	—	—	—	1	·11	—	—	—	—	—	—	—	—	1	·025
68. Cynosurus cristatus, L.†	?	—	—	—	?	—	—	—	—	—	—	—	—	—	—	—	?	
69. Dactylis glomerata, L.	7	1·23	2	·48	3	·38	17	1·92	1	·24	1	·31	4	1·55	34	11·49	69	1·75
70. Poa annua, L.††																		
71. nemoralis, L.	56	9·85	41	9 95	51	6·54	114	12 86	21	4·96	36	11·02	24	5 42	96	32 20	439	11·11
72. pratensis, L.																		
73. trivialis, L.																		
74. Lolium perenne, L.††	2	·35	8	1·94	19	2·43	1	·11	5	1·18	4	1·22	—	—	2	·67	41	1·04
75. Festuca ovina, L.	4	·70	—	—	—	—	2	·22	—	—	—	—	1	·39	4	1·34	11	·27
76. elatior, L.	—	—	—	—	1	·13	1	·11	—	—	—	—	—	—	1	·33	3	·076
77. Bromus mollis, L.	4	·70	—	—	3	·38	1	·11	3	·71	—	—	—	—	1	·33	12	·30
78. Brachypodium sylvaticum, Roem. et Schult.	—	—	—	—	—	—	1	·11	—	—	—	—	—	—	—	—	1	·025
79. *Doubtful Species of Grasses*‡	17	2·99	—	—	36	4·62	4	·45	13	3·07	1	·31	—	—	1	·33	72	1·82
80. Polypodium vulgare, L.§	—	—	—	—	—	—	1	·11	—	—	—	—	—	—	—	—	1	·025
Total no. of Species in District...	49		29		45		39		28		32		32		35			

(4) *Notes on the Plants distributed by the Cambridge dust-carts.* By I. H. BURKILL, B.A., Gonville and Caius College.

The dust of the Cambridge streets, with the materials which chance to be scattered on them, with the droppings of animals which fall on them and with the seeds which by one agent or another come to lie in them, is now deposited on the low ground of Coe Fen with the object of raising the level. It is this circumstance which has enabled me with comparative ease to collect materials for these notes, and so to see the extent of man's work in the distribution of seeds in and about a town.

On the small area of approximately three-quarters of an acre ninety-nine species whose names follow have been found:—and counting in the well-marked variety *italicum* of *Lolium perenne* we get exactly one hundred forms.

The following signs are used in this table:

- *α*, only one plant;
- *β*, several plants;
- *γ*, fairly abundant;
- *δ*, common;
- *ε*, very common;
- R, a plant habitually growing on our roadsides;
- G, a plant habitually seeding in our gardens;
- T, a plant, whose seeds, etc. man carries about for his own purposes, either for his own food (T. f.), or as a constituent of hay (T. h.), or in cereals or straw (T. s.), or as food for cage-birds (T. b.).

Ranunculus sceleratus L. R.

Ranunculus acris L. R.

Ranunculus repens L. δ. R.

Delphinium Ajacis Reich. G.

Papaver somniferum L. G.

Papaver Rhoeas L. β. T. s.

Cheiranthus Cheiri L. β. G.

Sisymbrium officinale Scop. R.

Erysimum Cheiranthoides L. T. s.

Brassica Napus L. (Rape.) T.

Brassica Sinapis Visiani. γ. R. (or T. s.)

Diplotaxis muralis D.C. γ. R.

Capsella bursa-pastoris Moench. R.

Senebiera Coronopus Poir. R.

Raphanus Raphanistrum L. T. s.

Viola tricolor L. G.

Silene Cucubalus Wibel. a. T. s.

Stellaria media With. R.

Cerastium glomeratum Thuill. R.

Spergula arvensis L. a. T. s.

Malva rotundifolia L. R.

Geranium Robertianum L. R.

Aesculus Hippocastanum L. a. G.

Acer pseudo-platanus L. β. G.

Cytisus Laburnum L.? a. G.

Medicago lupulina L. β. R. (or T. h.)

Melilotus alba Desv. T. h.

Trifolium pratense L. R. (or T. h.)

Trifolium repens L. R.

Trifolium fragiferum L. β. —

Trifolium procumbens L. R.

Trifolium dubium Sibth. R.

Vicia sepium L. a. T. s.

Vicia Faba L. β. T. f.

Pisum sativum L. T. f.

Pyrus malus L.? a. T. f.

Crataegus oxyacantha L. γ. R.

Epilobium sp. R.

Aethusa Cynapium L. T. s.

Daucus Carota L. a. T.

Galium Aparine L. R. (or T. s.)

Bellis perennis L. R.

Helianthus annuus L.? β. G.

Bidens tripartita L. a. —

Chrysanthemum segetum L. T. s.

Chrysanthemum Parthenium Pes. a. G.

Matricaria Chamomilla L. T. s.
Artemisia sp. α. G.
Senecio vulgaris L. R.
Calendula sp. α. G.
Arctium Lappa L. (minus?) α. —
Cnicus arvensis Hoffm. R. (or T. s.)
Centaurea nigra L. R. (or T. h.)
Centaurea Cyanus L. β. G.
Lapsana communis L. R.
Picris hieracoides L. R. (or T. h.)
Taraxacum officinalis Web. R.
Sonchus oleraceus L. R.
Myosotis arvensis Hoffm. R.
Calystegia sepium R. Br. G.
Lycopersicum esculentum Mill. β. T. f.
Solanum nigrum L. R.
Solanum tuberosum L. α. T. f.
arising from a tuber.
Veronica polita Fr. R.
Veronica persica Poir. R.
Veronica Beccabunga L. α. R.
Lamium purpureum L. R.
Plantago major L. R.
Plantago lanceolata L. R. (or T. h.)
Chenopodium album L. ε. R.
Chenopodium murale L. δ. R.
Polygonum Convolvulus L. T. s.
Polygonum aviculare L. ε. *R.*
Polygonum Hydropiper L. β. R.
Polygonum Persicaria L. R.
Fagopyrum esculentum Moench. α. T.
Rumex obtusifolius L. R. (or T. h.)
Rumex crispus L. R. (or T. h.)
Urtica urens L. R.
Cannabis sativa L. T. b.
Populus nigra L. G.
Juncus bufonius L. δ. R.
Zea Mays L. α. T.
Alopecurus geniculatus L. R. (or T. h.)
Phleum canariense L. β. T. b.
Holcus mollis L. R. (or T. h.)
Holcus lanatus L. R. (or T. h.)
Avena sativa L.? α. T. s.
Dactylis glomerata L. δ. R. (or T. h.)
Poa pratensis L. R. (or T. h.)
Poa trivialis L. δ. R. (or T. h.)
Poa annua L. ε. R. (or T. h.)
Bromus mollis L. R. (or T. h.)
Bromus sp. R. (or T. h.)
Bromus sp. R. (or T. h.)
Lolium perenne L. R. (or T. h.)
Lolium perenne v. italicum Braun. α. T. h.
Triticum vulgare Nill.? T. s.
Agropyrum repens Beauv. R.
Hordeum murinum L. R.

We then get three headings (1) Roadside element, consisting of plants which grow naturally so situated that their seeds must fall into the dust of the streets; (2) Garden element, of plants which man maintains in such situations that they are enabled to scatter their seeds on to the roads; (3) Traffic element, consisting of plants whose seeds man himself distributes involuntarily.

How are these represented in the flora of the street sweepings?

Roadside element ...., up to		58 %
Garden element		14 %
Traffic element:		
Man's own food	5 %	at least 25 %
Forage, etc. element by cereal crops	12 %	
Forage, etc. element by hay crops	2 %	
Through cage-birds	2 %	
Miscellaneous	4 %	
Of doubtful origin		3 %
		100

This shews in a most obvious manner to what a great extent (at least 39 %) man acts in promoting the distribution of seeds in and about a town.

Mr Willis and I in studying the flora of the tops of Pollard Willows near Cambridge* have found the importance of bird distribution shewn in a very forcible manner. But from the street sweepings we get a flora of an extremely different type; excluding the fruits of the traffic element, we find only *Solanum nigrum* and *Crataegus Oxyacantha* with fleshy fruits, and these not necessarily distributed by birds, and of burred fruits only *Galium Aparine*, *Bidens tripartita*, *Arctium Lappa* and *Myosotis arvensis*. Of the other 82 plants some have wings or pappus, e.g. *Cnicus*, *Rumex*, and many grasses, but many are without special adaptations for dispersion in their seeds.

Richard, in his *Florule des Clochers et des Toitures des Églises de Poitiers* has shewn how the two churches Saint Pierre and Sainte Rhadegonde, situated nearest to the sandy hills, possessed when he investigated their flora about four times as many plants as any of the five churches remote from these hills. Light seeds, such as those blown across to the Poitiers churches, are by no means absent from this street-sweepings flora. Seeds such as would be carried easily in any one of those clouds of dust which the wind may blow along our streets weigh but little.

Papaver Rhoeas, ·00011 gramme,
Capsella bursa-pastoris, ·00015 gramme †.

Of seeds distributed by man one seed of Rape (*Brassica Napus*, ·0038 gramme) weighs as much as 34 Poppy-seeds, and one seed of Buckwheat (*Fagopyrum esculentum*, ·0267 gramme); as much as 178 Shepherd's-purse seeds.

The Roadside element, which is principally made up of these small-seeded plants, is largely swollen through the rural nature of some of our roads, *e.g.* Queen's Road. It is this fact which causes the Hay element and the roadside element to overlap to such an extent; but the prevalence in the street-sweepings of the capsules of *Rhinanthus Crista-galli*—a plant common enough in the hay-fields on the clay land near the town, but not found in it,—and the nature of the *débris* which accumulates in such places as Tennis-court Road, are facts which in themselves testify strongly to the importance of seed distribution by means of hay, much of which cab-horses scatter in feeding. In fact oats, which are given to such a great extent to these horses, always may be seen lying scattered about cab-stands.

In composition the flora is peculiar; *Poa annua* is the dominant species, and next come *Poa trivialis* and *Dactylis glomerata*. Of

* Observations on the flora of the Pollard Willows near Cambridge. *Proc. Cam. Phil. Soc.* VIII. 1893, p. 82.

† We may compare the seeds of the Epiphytic Phanerogams where the need of effectual distribution is more pressing. Dendrobium attenuatum Lindl. ·00000565 gr. Aeschynanthus ·00002 gr. Goebel, *Pflanzenbiologische Schilderungen*, vol. I., 'Epiphyten.'

other plants but few rise in number of individuals above fifty; while the common Dandelion (*Taraxacum officinale*) is reduced to ten roots. In species, *Gramineae* and *Compositae* are the best represented, each containing more than one-sixth of the total. *Leguminosae, Cruciferae,* and *Polygonaceae* are fairly numerous. These five orders make up three-fifths of the whole; the other two-fifths consisting of representatives of no less than 22 orders, including *Caryophyllaceae, Rosaceae, Umbelliferae, Scrophularineae,* and *Labiatae,* all numerically important in the British Flora. *Cyperaceae* is not represented, nor is *Euphorbiaceae*: the latter we should certainly expect as the soil is rich, so rich that the grasses, essentially a manure-loving order, are very luxuriant, and there is not that keen struggle for existence which is fatal to our common British *Euphorbiae**.

Of published floras the only one which is similar is Vallot's little list of 45 species from the Place du Carroussel in his *Flore du Pavé de Paris.* These plants are distributed through the orders in a manner similar to those with which we are dealing; and we find the same forage element—an element which forms a very considerable percentage of the naturalised plants of any country in the world.

We know how European hay sold at an Australian port has introduced European plants; how *Trifolium repens* has taken possession of the Pampas near Buenos Ayres; how the German army in 1871 established two clovers in the neighbourhood of Paris†, the seeds falling out of the hay which they carried for their horses; and so here we see the same kinds of plants carried through small distances in our own country and our own town.

Monday, May 29, 1893.

Dr E. W. Hobson, Secretary, in the Chair.

The following Communications were made to the Society:

(1) *On the Kinematics of a Plane, and in particular on Three-bar Motion; and on a Curve-tracing Mechanism, with exhibition of apparatus.* By Prof. Cayley.

[To be published in the Transactions.]

(2) *On Contour Integration.* By H. F. Baker, M.A., St John's College.

(3) *Electric stability in crystalline media.* By J. Larmor, M.A., St John's College.

* The seeds of *Euphorbia Peplus* (·00041 gramme) weigh less than White Clover *Trifolium repens* (·00059 gramme), and this plant and others of the same order have been found in various comparable situations. Cf. Vallot, *Flore du Pavé de Paris et des Ruines du Conseil d'Etat.* Deakin, *The Flora of the Colosseum of Rome.* Richard, *Florule des Clochers et des Toîtures des Églises de Poitiers.*

† Vallot, Florule du Panthéon, *Journal de Botanique,* vol. i. 1887.

PROCEEDINGS

OF THE

Cambridge Philosophical Society.

October 30, 1893.

ANNUAL GENERAL MEETING.

THE names of the Benefactors of the Society were recited, viz. Dean Peacock, Rev. F. Martin, Prof. Sedgwick, Prof. Babington, Prof. Adams.

The officers were elected for the ensuing session as follows:

President:

Prof. Hughes.

Vice-Presidents:

Prof. Cayley, Prof. Darwin, Dr A. Hill.

Treasurer:

Mr Glazebrook.

Secretaries:

Mr Larmor, Mr Newall, Mr Bateson.

New Members of the Council:

Prof. Sir G. G. Stokes, Dr Lea, Mr Shipley, Mr Seward.

The President referred to the death of the Rev. Leonard Blomefield (Jenyns) of St John's College, and his services and benefactions to the Society, of which he promised a fuller notice.

THE MASTER OF DOWNING COLLEGE, V.P., IN THE CHAIR.

The following Communication was made to the Society:—

Criticism of the Geological evidence for the Recurrence of Ice Ages. By the PRESIDENT (PROFESSOR HUGHES).

PART I. CONDITION OF THE SURFACE OF THE BOULDERS AND OF THE SOLID ROCK.

i. *Introduction.*

To many of us the most interesting matter which has recently been brought before this Society was the discussion of the cause of the Glacial Epoch. But those who have been watching the development of theory and the controversies which have arisen out of the consideration of the question must feel that although it is too large a subject to be treated as a whole at any one meeting yet the several points in the long chain of evidence are of such a nature as to lend themselves readily to separate discussion, and that no view, founded upon such circumstantial evidence as that on which every glacial theory depends, should be accepted until a searching enquiry has been made into the value of the evidence for each of the facts upon which it is based.

I would state at the outset that I do not wish to divide the advocates of the Geographical explanation and the supporters of the Astronomical theory by a hard and fast line. I will acknowledge that most of those who look upon astronomical combinations as the principal cause allow that the distribution and height of the land must have great influence though they may be prepared to admit the possibility of freezing over the sea and then heaping snow on until the accumulation of ice displaces all the water.

On the other hand, among those who regard geographical conditions as having had most to do with the determination of the time and place of greatest glaciation, there are few who would deny that climatal conditions must be affected by astronomical changes, though they may regard these more or less in the same light as they do lunisolar influences in volcanic eruptions.

Whether we adopt the one view or the other, it is obvious that the recurrence of glaciation may be regarded as probable. It has however been maintained by some advocates of the astronomical theory that, although the results were due to secularly recurring combinations, it was only during the latest periods that the requisite conditions prevailed to produce the full effect in glaciation.

The great difference between the two cases is this. On the astronomical hypothesis the effect should be produced at intervals of different length but of regular recurrence and should always have a definite relation to the poles. Whereas, assuming the geographical view to be the more true we should expect to find less regularity as to the intervals, but still some secularity, and a constant relation to the areas and axes of greatest earth movement.

We have therefore to search the record of the rocks to see whether each great period has its glacial epoch represented in circumpolar regions, and, if not, then before we can accept the astronomical theory we must explain the absence of glacial phenomena by supposing that that particular Geological period was too short to cover the time required for the recurrence of cold-producing combinations or we must grant that glacial action has never extended so far or that all traces of it have been obliterated over the area open to examination.

We may fairly expect that some traces should remain; for, although terminal moraines on land might stand little chance of surviving a period of submergence when each portion of the old land was successively brought within the action of the encroaching waves, still we must remember that a large part of the glacial drift at the present day finds its resting-place far out to sea, whither it is transported, if not by the land ice-foot at any rate by bergs and other forms of sea-borne ice.

If we fail to find distinct evidence of the recurrence of glacial action in the rocks of circumpolar regions then we should examine the known areas of successive upheaval in equatorial and temperate, as well as in arctic or subarctic regions, in order to see whether we can find there proofs of elevation sufficient to give rise to glacier ice and can find evidence in the ancient deposits that glacial conditions did there prevail.

In order to be in a position to discuss this question it will be well first to consider the nature of the evidence upon which we have to rely for the establishment of the fact of glaciation, and I have thought that the best way of bringing this matter before the Society was to exhibit samples of the specimens which have been appealed to in support of special views of glaciation and certain other specimens, of which we know the vicissitudes and the mode of production, in illustration of the characters relied upon.

ii. *Characters of glacial deposits.*

Two principal classes of phenomena have to be studied in connection with ice-borne drift. First the condition of the included fragments, and secondly the arrangement of the material. Many of the inferences with regard to the mode of transport and distribution of the material are founded on the condition of the included boulders, and the most important of the characters impressed upon the rocks by glacial agencies are the polished and grooved surfaces of the solid rock over which the icebound stones have been driven or the smoothed and striated faces of the stones which have been compelled to adjust themselves to the more or less restricted movement of the yielding mass in which they are imbedded: the harder minerals cutting the softer and the "flour of rock" produced by their waste acting as polishing powder on the rock surfaces. But nature has many ways of arriving at analogous results, and before we proceed to enquire whether there is evidence of glaciation on the fragments included in ancient conglomerates or on the solid rock on which such conglomerates rest, it will be useful to guard against cases of mistaken identity by examining as many as we can of the known agents and effects of rock polishing and striation and to endeavour to distinguish clearly between them. With a view to this end I have arranged a series of specimens of rock naturally, accidentally, or artificially, polished and striated. Having regard to the difficulty of indicating on a plate those minute differences upon which we have to rely in endeavouring to discriminate between the various modes of smoothing or producing striations with which we are concerned in this enquiry I have not thought it worth while to give figures of the specimens: but they are all preserved in the Woodwardian Museum with the numbers by which they are referred to in this paper attached to them so that they can be easily consulted.

iii. *Glacially smoothed and polished rock.*

In most cases glaciated rocks are smoothed rather than polished. Inequalities are worn down and the roughness of the original fracture is removed but except when wet there is rarely the glistening surface to which the term polish could be applied. The slates on top of a distant house after a shower flash back the sun's rays but these slates are not polished. Example of such surfaces is shown on Nos. 1, 2, 3.

There are however surfaces over which the ice has forced a heavy mass of stones and earthy material frozen into a solid body and this when supplied with the "flour of rock" as polishing powder has sometimes acted as a burnisher and left the rock with a surface like porcelain, as seen on the fragment of a boulder from the Clwydian Drift No. 4, or on the surface of the Jurassic limestone No. 5. This lay in the path of the great glacier which came down from Mont Blanc and its surrounding snowfields, crossed the Swiss valley, climbed the flanks of the Jura and left a sheen on the rocks that would take it, but simply smoothed the others as we see on Nos. 1, 2, 3. We occasionally find that the original rough surface of the rock is preserved in places while the protuberant parts are polished and striated as in No. 6, from which we may infer that no very large amount of abrasion has taken place on that specimen or it would have been worn down to an even surface. On the principle of "diamond cut diamond" the hardest white quartz is sometimes polished and striated as in No. 7: a similar kind of wear being seen on the face of the siliceous Burrstone which has been used for any length of time for grinding in a mill; see No. 8.

Fragments of a rock-face smoothed and fluted by glacial action seem to have been not uncommonly detached by denudation from the boulder or parent rock and entombed in newer deposits. This takes place now where cliffs of boulder clay are exposed to the action of frost and sun. The included masses of rock break up under their influence and, without suffering much wear and tear, get buried in the clayey mass into which they fall. The grooves on such fragments are cut off with their full breadth at the margin of the stone whereas, if the striations had been produced on the stone as it is, the thin edge would have broken away. An example of such a broken boulder is seen in No. 3 or better still in No. 4.

iv. *Stones naturally smoothed and polished but not by glacial action.*

In earth movements when the parts of a fissured rock are relatively displaced and the sides of a fault are dragged against one another, perhaps even with a to-and-fro motion, if the cut be clean and the rock of a somewhat homogeneous texture, it will be smoothed and polished. Minerals commonly form along the crack or the exterior of the rock-faces in the fault undergoes mineral change. This is a very important character in helping us to distinguish between one of these slickensides as they are called and the smoothed face produced by any surface action. In this way a rock as rough as or rougher than the Molasse (see No. 1) may have a soft sheen given to it, as in the case of some specimens from the New Red Sandstone, see No. 9, on which among other minerals carbonate of copper has been formed, or No. 10, on the face of which a film of mineral matter has produced a surface like a mirror.

Some minerals and rocks which are readily soluble in water, either pure or charged with a small quantity of such acids as are commonly present in surface waters, have a polish produced by the chemical decomposition of their surface. This often happens to masses of carbonate of lime or limestone exposed to the spray from a waterfall or embedded in clay. Nos. 11 to 13 are examples of this process.

Rocks polished in this way are generally fretted into irregular pits as shown on the fragment of limestone No. 14 and more obviously on the piece of gypsum No. 15.

Blown sand also gives a delicate polish but as in the former case, see Nos. 16 and 17, slight differences of texture in the rock are apt to give rise to inequalities which are perpetuated in a fretted surface. This eating away of the rock by the constantly impinging grains of sand is seen on the obelisks of Egypt, on the basalt of Burntisland No. 16, or on the siliceous pebbles sticking in the exposed cliffs of St Davids No. 17. On softer material its action can be more easily studied, as for instance on pieces of broken bottle lying on the shore in the track of the prevalent wind. It has been turned to account in the sand blast used for etching glass.

General McMahon[1] has described how gullies amongst the rocks bring heavy gusts of wind to a focus, as it were, as may be seen at Dosi an isolated hill within the Arvali area, where deep grooves, several feet in depth and in diameter, have been carved out of the sides and faces of huge granitic blocks by sand-laden wind aided by the selective agency of natural decay.

Some remarkable specimens, Nos. 18 and 19, were noticed in 1869 by Mr Hackworth and described by Mr W. T. Travers before the Wellington Society[2] and afterwards by Mr J. D. Enys before the Geological Society of London[3]. These owe their form to the fact that they lay in the route of prevalent winds compelled by the form of the ground to carry the sand in a constant direction back and fore. The exposed part of the stone was thus chamfered off from each side so as to leave a roof-like ridge projecting. These stones have no striæ but generally show some places fretted away below the general level in a manner that proves that they were not planed or ground down by contact with flat surfaces. I have found a siliceous pebble on the Plateau Gravel above Biarritz No. 20 which was beginning to assume a similar form, from having, as it appeared, been bevelled off by the sand blast driven by the Atlantic gales which so often sweep that exposed ground.

v. *Glacially Striated Rocks.*

So far we have been considering chiefly the various agents which produce smooth or polished surfaces and endeavouring to discriminate those due to ice action from those due to other causes. We will now examine the striated surfaces so commonly found on the surface of the rocks over which ice has passed or on the boulders which have been transported by it, and in this case again we shall find that the mere presence of scratches and grooves is not of itself sufficient evidence of glacial action.

[1] *Records Geol. Survey of India*, Vol. xvii. Pt. 3, 1884, p. 101.
[2] *Trans. New Zealand Institute*, Vol. ii. p. 246.
[3] *Q.J.G.S.*, Vol. xxxiv., 1878, p. 86.

The striæ in question are of various character and depth according to the shape and amount of projection of the mineral point that cut them and the form and texture of the rock attacked. They generally run in straight lines across the stone, as, in glacier movement, changes of direction producing curves such as could be observed on a small area must be rare. They seldom run over the edge further than the protruding mineral could reach while the faces of the stone remained in contact. Separate stones however seem to have not infrequently changed their position, so that a fresh set of striæ has been drawn across those first formed.

The grooves on the solid floor of rock are therefore apt to be all parallel as seen on the slab of sandstone from near Edinburgh No. 21 or on the piece of Molasse from Lausanne No. 1; whereas on loose stones the scratches run in different directions on different faces and cross one another at all angles on the same face as well seen on the large rolled mass of chalk No. 22 from the drift of Roslyn Pit near Ely and on the small boulder of black limestone No. 23 from the great moraine of Lausanne. The striated fragment of Silurian No. 24 from the lower boulder clay of Longwathby cutting in the Eden Valley is well scratched on all sides but generally parallel to the longer axis of the stone. No. 25 is from one of the highest patches of boulder clay in the kingdom. I found it at an elevation of 1800 feet on the mountains that bar the southern end of the Eden Valley. It has the grooves and scratches running in the length of the stone and almost all parallel on both sides.

It must not be supposed that all ice-borne stones will be glaciated. Masses that have fallen on to a glacier and travelled down on its surface to the terminal moraine or even to the sea may never have been subjected to any crushing or grinding. Scratched stones are exceedingly rare on the terminal moraine of any existing Alpine glacier, whereas they are abundant in the drift lower down the valleys. Fragments that show ground and striated faces must have had time to get into or under the glacier, except of course those caught in shore ice of which we are not now speaking. The specimen of Shap Granite No. 26 as it lay in the Eden Valley is therefore interesting in view of the controversies as to the mode of transport of the boulders from Wastdale Crag for it is distinctly ice-worn and thus we know that it travelled, at one time, *in* or *under* the ice.

No. 27 from the drift of the coast of Wexford was evidently caught by sea-borne ice, as it was bored by lithodomous molluscs before it was striated. On it the striæ are nearly parallel on the same face but those on one face do not run in the same direction as those on the other.

vi. *Rocks striated by other than glacial agency.*

If the solid rock or any stone firmly fixed in its place have another mass holding minerals hard enough to groove the surface drawn across it, the result must be the same whether it be ice or any other material that clasps the stone or holds the rasp. For instance, if trees be dragged down a hill side, bits of grit get driven into the wood and as it passes over the solid rock they cut striæ exactly like those produced by glacial action as may be well seen on No. 28 and No. 29 which is part of a mass of Jurassic Limestone which stuck out in the road up the Jura from Grenchen to Bürenberg in the canton of Solothurn (Soleure). The grit imbedded in the wood and the carts with breaks and skids have grooved and scratched the stones so that the exposed parts can hardly be distinguished from glaciated surfaces, as may be seen by comparing No. 28 with No. 5 which is a genuine glaciated rock from the same district.

The effect of dragging timber down a hill side is well shown also on Nos. 30 and 31, two specimens of Silurian rock from near Corwen.

Some examples it may be pointed out cannot be offered in explanation of possible modes of producing surfaces simulating those due to glacial action as they are due to the agency of man or other forces which cannot have been in activity at the time. But any case in which the mode of operation is certainly known and the results can be closely studied are of value in such enquiries.

For instance there is a quarry at Rhiwallt near St Asaph in which the Silurian rock has been removed along the planes of bedding so that a long sloping face has been left, down which slide broken rock and masses of earth with stones imbedded, marking their path on the face of the rock as seen in No. 32 and No. 33. Though in this case the face of rock was exposed by man, the same kind of thing must often have occurred at every period of the world's history and stones and gritty material have slid down over long faces of bare rock into deep water and so produced all sorts of difficult complications.

More distinctly due to artificial operations, but most instructive from the deep grooves and fine polish produced, are the two specimens No. 34 and No. 35 from the Mountain limestone of Ribblesdale where the wire rope, dragging in places over remains of drift, which was largely derived from gritty carboniferous beds above, has polished and striated and fluted the rock over which it has run.

No. 36 is a glaciated face of rock but the ice marks are very obscure. The obvious scratching and polishing of the rock is due to the nail boots and caudal corduroys of small Welshmen who amuse themselves between school hours by sliding down the smooth face of the rock.

On the limb bone of a large Saurian, No. 37 and No. 38, from the Kimeridge clay of Littleport, North of Ely, there are scratches and grooves running in various directions. We have several other examples of the same kind of thing. There is in the case of this specimen enough to prevent our referring them to glaciation in the character of the striæ themselves. The short cross cuts for instance which appear here and there and the abrupt termination of the grooves on the face of the bone. But had only a fragment of this specimen been preserved and that perhaps been somewhat rolled, it would not have been so safe to pronounce that the scratches could not have been of glacial origin.

The cuts cannot have been produced at any later date than the deposit in which the specimen was found as they are in places covered by organisms which grew upon the bone as it lay at the bottom of the Jurassic sea, and the fauna of the Kimeridge clay precludes our entertaining the possibility of the occurrence of glacial conditions in our part of the world during its deposition[1]. Occasionally the same kind of thing may be seen on lumps of phosphate of lime from the Cambridge Greensand as in No. 39 on which Plicatula sigillina has grown upon a striated surface.

vii. *Striations and polishing due to crushing of solid rock.*

When the faces of a fissured rock have been rubbed against one another by earth movements under tremendous pressure, either continuously in the same direction or with a to-and-fro motion, as I have already pointed out, there is often a film of some mineral formed over the surface, and I have already considered the cases in which the rock is homogeneous and a smooth polished surface is produced. But when the surfaces in contact are covered by irregularities such as minerals of unequal hardness in the rock or in the fault the harder produce flutings and scratches on the softer along the lines of movement.

The immediate cause of the grooves may often be detected. Sometimes they are due to such apparent trifles as worm tubes, which, being filled with sandy material, are more solid and harder than the rock that is thrust over them along the bedding planes as seen on No. 40. Sometimes the lumps and grains which have caused the fluting can be seen on the fault as in No. 41 which is off the face of a landslip in the Barton Cliff, Hampshire, where the accompaniments of faulting on a small scale may be studied.

Sometimes movements have been repeated after a fault

[1] Vict. Inst., May 6, 1889.

breccia has been formed and minerals of different hardness have crystallized along the fissure. A good example of this is seen in No. 42, which is a piece of fault breccia which has been itself faulted and the included fragments and the matrix have been scored by the passage over them of harder fragments in the opposing face. We may find produced in this way many of the varieties which we observed in the case of true glacial striæ. The rock may be fluted but not polished as in No. 43, which is a fragment from a fault face in Silurian rocks of the Vale of Clwyd, or we may find a coarse grit first deeply grooved and polished and then covered with a glistening mineral film, as in the specimen No. 44 from Nodale, Stansfield Moor, or in the brighter surface of No. 45 from Derbyshire. Sometimes we find evidence of change of direction producing secondary striation oblique to the first, or more rarely, as on the specimen from a quarry north of Aberystwith, No. 46, we may see from the curved lines on the mineral layer and on the rock that the direction of movement has changed.

A fluted surface is produced in some rocks by a combination of thin bands of tougher rock with joints and cleavage so disposed that the unyielding bands, alternating with the portions of the rock which are compressible and susceptible of cleavage, are protruded or displaced at the planes of interruption caused by the joints and thus present parallel ridges, as seen in the specimen of Silurian rock, No. 47, from Tirgwyn in the Vale of Clwyd.

It often happens that when a rock, with fine lines of sediment that weather out more readily, is exposed to denuding agents, the occurrence of those lines of weakness is indicated by finer or coarser parallel grooves, and if such a rock lies in boulder clay, the striæ may very likely be all referred to glacial action. Examples of this may often be found among Silurian and Cambrian rocks, as for instance on the specimen No. 48 from the drift of Maes Mynan near Caerwys. Sometimes such lines and grooves indicate joints which are otherwise quite invisible on the face of the rock. This is well seen on the specimen No. 49 of sandy shale, probably Devonian, from the slopes of Dartmoor, on which the divisional planes which have caused the grooves on the face of the stone are seen as joints on the side.

If a conglomerate be subjected to pressure of the same kind as that which produces crumpling or cleavage in one place, faults in another, or belts of puckered rock in a third, it often happens that the rock, yielding along definite areas, readjusts itself by movement between the portions of different tenacity and a shearing takes place throughout, with distinct thrusts and displacements between the matrix and the included pebbles, see Nos. 50, 51, while smaller fragments of greater hardness imbedded in the paste will scratch and furrow the larger softer pebbles along which they are driven, see Nos. 52, 53, 54, 55, 56. Small tough pebbles will often deeply indent softer pebbles without breaking them or being broken by them. For the permanent deformation of rock without fracture, however, time is an element which has to be taken into account, but for our present purpose we need not further consider this point.

In the case of crushed conglomerates we often find that the pebbles have been broken and the several parts have been separated somewhat or shifted and recemented by mineral matter, see Nos. 57, 58, 59, 60. This I have never known to be the case in glacial deposits. Sometimes the conglomerate has been crushed in such a manner as to cause not only a protrusion of the pebbles through the matrix but also a number of small faults throughout the mass on the face of which it will be seen that the striæ produced by slickensiding run *across the matrix and included pebbles alike,* as seen on Nos. 50, 51. Striated floors have been noticed in glacial drift, but they were all due to more or less horizontal movements, whereas the striated faces in this conglomerate are at all angles to the planes of the bedding.

When however a thrust happens to have taken place between the conglomerate and underlying rock we have of course a polished, striated and grooved floor just where, on the supposition that the conglomerate is of glacial origin, we should look for the glaciated surface on which the ancient boulder drift was deposited. One very striking example of this occurs in the same area as that in which so many striated blocks have been procured from the conglomerate. It might seem that there was cumulative evidence for the glacial origin of the deposit. Scratched stones in a boulder drift resting upon a striated surface of solid rock ! But when we examine the evidence more carefully it all breaks down. This striated surface occurs along a thrust plane which can be clearly made out on the ground. And even were it not so the condition of the surface itself is sufficient to show that it is due to earth movement and not to glaciation. The rock itself can be seen on the specimens Nos. 61 and 62 to be squeezed out in the direction of the movement, and a thin layer of mylonite covers the whole. Now the evidence is turned against the glacialists, and it may be urged that where there were such rock crushings as those which are known to have taken place in Holbeck Gill it would be curious if here and there we did not find that there had been a differential movement between masses of such unequal structure as the conglomerate and the solid grauwacke on which it lay.

The character which is most constant in pebbles which have been subjected to this crushing action in a conglomerate and at the same time most clearly distinguishes them from glacially striated rocks is the manner in which they appear to be pinched to an edge and scored on both sides up to the edge, so that whereas as I have already remarked (see p. 103, and specimens Nos. 22 to 27) the same set of glacial striæ seldom run far over the edge of any glacial boulder, and flat sides are the rule; in crushed conglomerates on the other hand the grooves follow the curvature of the stone, and the edge is sometimes crushed up as if the pebble had been nipped out. This is well seen in Nos. 53 and 54, and less distinctly in Nos. 55 and 56.

Whenever a rock is acted upon by pressure in such a manner as to produce deformation, and there occur here and there in it harder masses which do not yield to the same extent, they are thrust through the surrounding matrix and are apt to be smoothed or striated according to the relative hardness of the surfaces. This must happen whether the included body be a pebble in a conglomerate or a concretion in a clay. The spherical or cylindrical or conical bodies known as *stylolites* are formed in this manner and their slickensided or fluted exterior is produced by the harder less yielding condition of the concretionary mass, as in the examples numbered 63, 64, or even by the obstruction to movement offered by a shell or other foreign body as in No. 65, in both which cases the surrounding matrix is dragged round and along the included unyielding body.

We must remember that many though not all of the abovementioned operations, which produce the form or smoothness or polish or striation, simulating that due to glacial action, are subaerial and that therefore, before such a stone could be transported to any situation where it could be entombed, the chances are that it would have been exposed to so much wear and tear as would have entirely obliterated the superficial characters it had acquired. We admit therefore that it is improbable that many stones polished and striated in any of these accidental manners should be found in a conglomerate. But when we notice that, although a large number of observers have been on the look out all over the world for evidence of this kind of ancient glaciation, the examples found are generally of very rare occurrence and doubtful character, we must not lose sight of those accidental modes of producing polishing and striæ which may be easily mistaken for those which are the result of glacial action.

viii. *Supposed occurrence of glaciated stones in ancient deposits.*

Having thus considered some of the various modes in which stones may have acquired both polish and striation let us examine some of the cases in which it has been maintained that the included fragments in a conglomerate or unconsolidated gravel prove the existence of glacial conditions during the deposition of the beds in which they occur.

Where we might expect to find traces of glacial action is of course where we have evidence of submerged mountain lands, that is at the base of each great system as defined on the table of strata which I laid before the Society on a former occasion[1].

When the sea crept up the pre-Cambrian highlands it must have washed into its depths whatever there was of moraine matter. When the great subsidence commenced that made room for the accumulation of the Carboniferous deposits, the boulders of Silurian, Cambrian and older rocks must have been buried where they were or been swept into the hollows and valleys of the submerged land. When the Carboniferous rocks had been uplifted and acted upon through long ages by all the denuding forces that then as now struggled to reduce the dry land as soon as raised, depression again set in and the land with all its mountain ranges and coast lines again went down; if there were glaciers here we should find their traces. It is likely there were glaciers somewhere and at these horizons we should still search for evidence. But the question now before us is what is the value of the evidence which has been adduced in favour of that discovery having been already made.

ix. *Southern extension of ice in glacial period.*

The extension of the glacial drift in Southern Germany, suggested by the transport of fragments far from the parent rock, has been supposed to be confirmed especially by the occurrence in the widespread gravel and sand of that area of certain faceted and polished stones. See Nos. 66, 67. These are composed of very hard material being generally quartzite, and the manner in which they are supposed to have assumed their peculiar form is by having been held in the ice as in a vice and having one face after another ground down as they shifted their position in the mass. Even if the facets were perfectly flat we need some explanation of the fact that although there has been such tremendous glacial abrasion there are no glacial striæ. But the faces are not flat and the surface of the portions fretted into hollows are in exactly the same *condition* as the parts in relief. In fact these specimens show none of the essential characters of glacially dressed stones. If however we look back through our collection of variously abraded stones (see p. 102) we shall find that there are some, of which we know the origin, in which the very same characteristics appear as those which distinguish the supposed glacial boulders of Copitz by Pirna. In the sand-worn stones of Cook's Straits, Nos. 18, 19, and

[1] *Proc. Camb. Phil. Soc.*, Vol. III. p. 247. *Brit. Assoc.*, Rept. 1875. Trans. Sects., p. 70.

of Biarritz, No. 20, we have the same facets, the same polish, the same fretted surface due to the eating away of the softer parts by the sand and the consequent relief of the veined or harder portion.

Now to turn to the Swiss valleys where it is known that the glaciers once extended far beyond their present limits and where their constantly recurring advance and recession within the memory of man is certain. What limit can we place upon this oscillation? If earth movements are going on, if climatal changes are taking place, the alteration in the volume of ice must be of the same kind but greater in degree than what is observed from half-century to half-century as the result of what we call weather. What wonder then that in the basin around Zurich, which the ancient larger lake once filled, we should find moraines above peat and older morainic débris below. I am satisfied as to the fact. I worked for two hours under ground in the lignite pit till I found for myself undoubted glacially scratched stones, e.g. No. 68, which is identical with those that were found above, e.g. No. 69. But what has this to do with secularly recurring cold periods? It is only on a larger scale and belonging to an older time what the guides will point out as having occurred within their own memory at the foot of the Rhone glacier.

x. *Tertiary Glaciers.*

Of earlier Tertiary age we will take as an example the Miocene deposited on the south of the Alps which Gastaldi argues has evidence of glacial action not only in the character of the deposit but also in the occurrence of glaciated stones. Admitting the accuracy of the observations on which these views are founded, and, though I have been over the ground I have nothing to say for or against them, it is not remarkable to find glaciers in ancient times where there have been glaciers in recent times unless we hold that the last upheaval which raised and left the Alps where they now are was the only great upheaval of which we have any knowledge. That has through many successive periods been an axis of greatest movement, and the higher it was raised the greater, cæteris paribus, must its glacier ice have been. This question of earth movement and its bearing on glacial phænomena I will ask leave to bring before the Society on some other occasion.

xi. *Boulders in Cretaceous Rocks.*

Of still earlier date are the blocks of coal and boulders of granite found in the Chalk for instance or the large masses of gneissic and other rocks found in the Cambridge Greensand or the Potton Beds. Among these we have from the Cambridge

Greensand a boulder, No. 72, of a green sandstone with shells and masses of phosphate attached and covering striations caused by weathering along lines due to rock structure. But these are in no ways glaciated rocks, and the mode of transport of large masses of rock without the agency of ice we will leave also for separate discussion.

I have not opened the question whether a glacial deposit may be recognised by other characters such as the arrangement of the material, the shape and condition of the stones irrespective of striation. These are in the present state of the evidence too vague to be profitably discussed, as is also the suggestion of Agassiz that the paucity of life in some formations might be coincident with cold periods.

xii. *Supposed evidence of glacial action in Poikilitic Rocks.*

Of the older ages, such as the incoming of the Jurassic period or of the Carboniferous as recorded in the conglomerates at the base of the New Red or Permian and in those at the base of the Upper Old Red or Devonian there is more to say. In the case of the Lower New Red or Permian some specimens are preserved in the Jermyn Street Museum on which there are striæ which have been referred to glacial action. These may be divided into two: one represented by a single stone of a greenish colour and flattened oval form and covered with striæ undistinguishable from those seen on stones from the true glacial drift. This stone must have been placed there for comparison and illustration, and having lost its label have got mixed up at last with those actually found in the Haffield conglomerate. I am well acquainted with that rock, and have no hesitation in saying that the stone I refer to was never in it. The others are specimens with obscure striæ such as would be produced by movements in the rock. The general character of this conglomerate and of the fragments included in it may be gathered from an examination of the piece of the Haffield conglomerate No. 70, and the fragment of close textured mudstone No. 71, with a shiny surface produced by a film of oxide of iron, which occurred in the conglomerate.

Professor Sellas[1] has called attention to this source of error in the case of beds of the same age near Portskewet, Monmouthshire. No. 54 is from this locality.

There are some accidents which distribute a few glacially striated stones among others that have never been subjected to ice action: for instance, If a clay is broken up into lumps and these are rolled along the shore, pebbles and shells are driven into the sodden exterior which they can help to protect. In this

[1] *Geol. Mag. N. S.*, Vol. VIII., 1881, p. 79.

manner rounded masses of what looks like conglomerate are formed. When the original mass happens to be a boulder clay, the included striated stones are thus handed on without further wear of the surface, and in this way many curious associations occur of rocks and shells that we should not have expected to find together: 73, 74, 75 are examples of this.

But in such cases as we have now under consideration we must bear in mind that, in a genuine boulder clay, such as any one may examine for himself on the top of Madingley Hill, scratched stones are abundant. Why then should we be expected to take on trust, as evidence of glacial action in ancient times, a few isolated fragments with the "ghosts of scratches" on them, when good observers visiting the same spot afterwards cannot find a trace?

xiii. *Evidence of ice agency in Talchir, Karoo, and Bacchus Marsh beds.*

We will consider the Talchir, Karoo and North Australian beds together because geographically they occur on the margin of one basin, and geologically are referred to approximately the same horizon.

The Talchir beds as described by the officers of the Geological Survey of India[1] consist in part of great masses of clayey silt with boulders somewhat irregularly scattered through it. Griesbach suggests that they were *dropped on the sea bottom and sunk in the soft mud.* Beds of the same character and believed to be of the same age occur in the Salt Range, in Cashmeer, and in Afghanistan. They are said to rest upon a polished and striated rock surface, and polished and striated boulders are recorded from them, but those who have seen the beds in situ say that they rely more on their similarity to boulder clays in general appearance than on the occurrence of a few striated blocks here and there.

The evidence is not altogether satisfactory, as we might expect a larger proportion of scratched stones in such far-transported material if it were of similar origin to the boulder clay of this country, and the sketches of striated blocks given do not recall the commoner forms, but as I have noticed it is not easy to reproduce such characters.

[1] Blanford, W. T., Blanford, H. F. and Theobald, W. Jun., *Mem. Geol. Surv. Ind.* 1859, Vol. I. p. 33. *Record Geol. Surv. Ind.*, Vol. XX. Pt. i. p. 49, 1887. Fedden, *Record G. Surv. Ind.*, Vol. VIII. Pt. i. p. 16, 1875. *Record Geol. Surv. Ind.*, XIII. ii. 83. *Record G. Surv. Ind.*, Vol. IX. Pt. iii. p. 79. Vol. XXI. Pt. i. p. 34, Pl. i. Vol. XXII. Pt. ii. p. 69, Pl. p. 130. Vol. XIX. Pt. ii. p. 133. Cf. also Blanford and Medlicott. *A Manual of the Geology of India*, 1879. See also a good résumé of views in Seward, *Sedgwick Prize Essay*, Cambridge, 1892.

The same kind of reasoning has led others[1] to refer certain beds in New South Wales to ice agency. The earlier observers Sir R. Daintree and Dr Selwyn, though the idea of the glacial origin of the boulders was present to them, said, "grooved or ice-scratched pebbles or rock fragments have however not yet been observed." Their impression was that the character of the conglomerate was very suggestive of the results likely to be produced by *marine glacial transport.* The section given by Mr David is not unlike what may be seen everywhere along our Norfolk coast where the contortions are probably due to stranding sea-borne ice.

After a long time Oldham found "boulders and pebbles, unmistakably striated and polished by ice, in a railway cutting at Branxton, close to the spot where the erratics were first found by Mr Wilkinson." This reminds me strongly of what happened in regard to the conglomerates at the base of the Carboniferous in the north of England. Not a scratched stone could be detected in all the great boulder beds along the Lune nor at the foot of Ullswater, nor anywhere until I found the belt of faulted and disturbed rock near Dent, and from this an abundant supply of well smoothed and striated boulders has been since obtained.

A somewhat similar deposit has been described[2] as occurring in South Africa, and the suggestion that the boulders in it were ice-borne has found increasing favour as the identification of the series with the Talchir beds of India and the Bacchus Marsh beds of Australia proceeded. When that view was first communicated to the Geological Society I ventured to point out sources of error in the observations on which such deposits had been referred to glacial agency. While believing, as must any one who holds the geographical theory of the origin of glacial condition, that there had often in past time been local ice ages, I pointed out that the mere occurrence of subangular boulders did not prove a glacial origin, and that in the case under discussion the Natal deposit was not like our boulder clay, but was a stratified and ripple-marked series, and moreover was shown to have been subjected to such pressure that distinct cleavage was produced. Of course such pressure must have produced a differential movement between the included boulders and the matrix.

Sometimes the form of the stones in a conglomerate rather than their striation or polish is relied on. The form of the boulders in glacial drift, as in water-borne accumulations, is chiefly dependent upon the manner in which the divisional planes of the parent rock

[1] David, *Q. J. G. S.*, Vol. XLIII., 1887, p. 190.

[2] Sutherland, *Q. J. G. S.* XXVI., p. 514, 1870, *ib.* p. 517. Griesbach, *Q. J. G. S.*, XXVII. p. 58. Schenck, A. *Pet. Mittheilung*, 1888, p. 225. *Protokoll der März-Sitzung.* N. Jahrbuch. Stuttgart, 1889, p. 172.

cause it to break up, and a large number of the fragments in the ice have been carried by water on to it or in channels in and under it. It would be difficult to say of any boulder that its flattened sides had been produced by the grinding action of the ice with its included grains. Moreover it would not be easy to ascertain certainly whether any particular stone in our river gravels, or in similar deposits at the foot of the Alps or Himalayas, had been derived directly from the parent rock or second hand from a glacial drift, so that the comparison is attended with many difficulties.

It would appear then that many good observers have arrived at the conclusion that these beds, whether on the south of the great Himalayan range or on the flanks of the mountains which form the watershed in South Eastern Africa or in the unstable region of New South Wales, present many characters in common and that there is evidence, stronger in some cases and weaker in others, that ice in some form or another has played an important part in building up the boulder-bearing beds which they contain. The vertical range and correlation of the beds is still unsettled, the precise manner of intervention of the ice is doubtful. Many a good case is weakened by being supported by weak and easily refuted arguments and we must not dismiss all the evidence advanced by competent observers because we find obvious errors or irreconcilable statements here and there, but still we cannot help bearing in mind the case of the Old Red Conglomerate in England (see pp. 115 to 117).

Supposing we admit the glacial origin of much of these boulder deposits ranging over periods so vast that they are referred to everything from Carboniferous to Trias and perhaps to still newer beds, distributed over regions so far apart as India, Africa and Australia. What then? must we shift the poles? The evidence goes to show that they are marine strata into which the boulders have been dropped. All we have to account for is floating ice, not ice coming down to the sea where we now find the boulders. Stranding ice will striate the solid floor and melting ice will drop blocks whether ice-worn or subaerially weathered that have fallen on it or been frozen up in it. How to account for such ice in those regions is a question which I reserve for a later communication in which I hope to deal with the earth movements, but I would point out that if we admit the evidence that the beds with the Glossopteris fauna belong to one age and prove that glacial conditions then prevailed over the area in which they are found, that is fatal to the astronomical theory which requires *alternate* intensification of cold or heat *at either pole.*

xiv. *Scratched boulders in and Striated floors below the Basement Beds of the Carboniferous System.*

The case of the conglomerate at the base of the Carboniferous System is much more clear and, as near as may be, gives us an opportunity of proving the negative.

The Rev. J. G. Cumming[1] remarked that the Old Red Conglomerate (by which he meant what is now often spoken of as the Basement Bed of the Carboniferous) looked "extremely like a consolidated ancient boulder clay formation" and suggested that the scaly fish "had to endure the buffeting of icy waves," and in an article[2] upon the Geology of Cumberland and Westmorland he extended these remarks to the Old Red Conglomerate of the Lake District.

Professor Ramsay[3] advocated the view that these conglomerates were of glacial origin, founding his opinion upon the following observations: (1) the manner of their occurrence in isolated patches in old valleys; (2) the character of the deposit which is a coarse conglomerate, showing a very irregular accumulation; (3) the shape of the included fragments which is very similar to that of the fragments of the same formations in the glacial drift; and, (4) the occurrence of scratched stones. Upon this I ventured in 1867 to offer the following remarks[4]: "But we must remember that any subaerial or fluviatile deposit, covered by an encroaching sea, must have that patchy character, that in its irregular accumulation, and the shape of the stones, it more resembles the gravel drift of the valleys than the Boulder Clay; and the origin of this gravel drift is, at any rate, doubtfully glacial. I think we must be cautious, too, about referring the scratches on the stones to glacial action—though undoubtedly they are like those found in the true drift, and the shape of the stones is also the same. Yet they have never been found except where we have other evidence that the beds have been much disturbed. We see, where the red conglomerate can be examined close to great faults, that the beds do not get crumpled up, as in the Silurian or any even bedded homogeneous rocks, but that, because the hard and included pebbles resist more than the soft matrix, the whole mass readjusts itself to suit its new position, the included pebbles being often scrunched against one another, scratched, and broken.

"In one place I found, inclined at a small angle to the bedding, a face of jointage, on which were striæ similar to those on the

[1] *Hist. Isle of Man*, 1848, p. 89.

[2] *History and Topography of the Counties of Cumberland and Westmorland.* By W. Whellan. Pontefract, 1860, p. 28.

[3] *Q. J. G. S.*, Vol. XI., 1855, p. 187. *The Reader*, Vol. VI., Aug. 12, 1865, p. 186.

[4] *Notes on the Geology of parts of Yorkshire and Westmorland.* Geol. Polytech. Soc. W. Riding, Yorkshire, July 17, 1867.

scratched stones *running across the soft matrix and included fragments alike.* This of course leaves it open to suppose that the Old Red Conglomerate may be the wreck of a glacial drift; but as we know of no gravel made up of fragments of similar rocks which are not directly or indirectly derived from glacial deposits, the arguments from the shape of the stones &c., cannot, in the present state of our knowledge, go for much."

Considering the evidence as to the origin of these beds derived from their apparent manner of accumulation and their relation to the beds with which they are associated I observed, "Along the cliffs from near Settle to Ingleton the base of the Mountain Limestone may be traced resting with an almost unbroken line of junction on a planed-off surface of Silurian rocks. About Kirkby Lonsdale, Sedbergh and many other places W. and N., thick masses of Old Red, with its coarse drift-like conglomerate, tell of deep valleys filled with the débris of higher land. On the north side of the Howgill Fells, thick beds of red sandstone and conglomerates, alternating with more or less calcareous shales, are evidently the waste of neighbouring land, resorted on the sea bottom, where numerous corals, shells and other forms of life flourished.

"Near Horton-in-Ribblesdale, at Gillet Brae, in Beecroft Hall plantation, and near Dove Cote, we have pockets of Old Red conglomerate in the surface of the Silurian rocks, and the Mountain Limestone, with its own peculiar thin beds and conglomerate, seems to lie on this with an even line, as it does on the Silurian rocks on either side. Fossils occur among the fragments of Silurian, at the very base of the Mountain Limestone. Corals seem to have grown in abundance among the loose stones and on the rocky sea bottom, but no fossil have I ever found in those pockets of Old Red. This will apply also to the larger patches near Kirkby Lonsdale, Sedbergh, Kendal, and the foot of Ullswater, as far as we can observe them in those faulted districts. At any rate we may say that the Mountain Limestone never rests on an irregular surface; all the old valleys and minor inequalities having been filled with coarse conglomerate previous to the deposition of the Carboniferous rocks.

"It would appear from this that we have in the Basement Bed of the Carboniferous (the so-called Old Red Conglomerate) the remains of an earlier formation lying on the irregular surface of an old continent, and that small patches were preserved in the deeper hollows when the Carboniferous sea planed across it.

"But, on the other hand, in Hebblethwaite Gill, near Sedbergh, we find the coarse red conglomerate succeeded by shales, grits and earthy limestones; and in these shales a second bed of red conglomerate occurs, in every respect similar to that below. On the north side of the Howgill Fells near Tebay and Shap, there seems to be a clear passage up from the coarse red conglomerate into finer conglomerates, sandstones, shales, and limestones. The shales and sandstones contain remains of plants and marine shells. On the whole therefore it seems most probable that as the land went down in the early Carboniferous period, the sea kept planing off everything up to the Lake Mountains. Perhaps there was then a more rapid subsidence; at any rate the sea crept up the hills, and found in the recesses great masses of débris, the result of subaerial waste. Some of these got covered up and are still preserved; others were washed out and resorted over previous marine deposits, or on the bare sea bottom, and from these resorted beds there would be a passage up through lower Limestone shale to higher Carboniferous beds.

"This view will quite explain why we seem to have a passage in one place, and in another a sharp line between the Old Red Conglomerate and Carboniferous rocks.

"It would appear probable from the number and character of the corals and the plants imbedded in the earliest deposits of the period, that the climate was temperate or subtropical."

Some pebbles from this conglomerate have the less soluble portions projecting beyond the general level of the surface of the stone, as for instance on the boulder derived from the Bala Limestone, No. 76, on which the coral Halysites catenularius stands out in clear relief, showing all the details of its structure. In this case it is clear that the specimen has been subjected to ordinary weathering not to glaciation.

XV. *Striated Boulders in Cambrian Conglomerate.*

Reusch described and exhibited to the International Geological Congress of Washington some specimens of striated rock from the Cambrian conglomerate of Norway. Admitting for the sake of argument that these are of glacial origin they are of great interest as proving the occurrence of glacial conditions locally in those early ages, and as so far combatting the idea of there having been a perceptibly higher temperature at the period of the deposition of our earliest fossiliferous rocks. But the moderate uniformitarian would expect to find such traces there if anywhere; at the base of a system (see above, p. 109 and footnote); after one of the greatest earth movements of which we have any record; in a latitude where glaciers now exist in Europe and where far more severe conditions still prevail on the other side of the Atlantic. It proves nothing in regard to periods of general lowering of temperature, and lends no support to any theory of circumpolar ice caps.

xvi. *Conclusion.*

I have in this the first part of my communication confined my attention to the evidence derived from the character of the fragments contained in glacially transported deposits and of the rock over which the ice has passed. That is to say I have considered only the polishing and striation of the boulders and of the solid floor. The question I have kept before me has been always, Can such a condition of the surface have been produced in any other way than by the agency of ice? and I have exhibited a series of specimens to illustrate the view that there are many different operations of nature by which polished and smoothed and striated surfaces can be produced, that some of them can in certain cases be distinguished from those due to ice action, but that many of them, especially when weathered, exhibit the "ghosts of scratches" undistinguishable from those left on glacial boulders. I have incorporated a large number of specimens simply in explanation of the manner in which certain conditions of the surface are produced, irrespective of the question whether the particular agent artificial or other that did in that particular case produce them can have existed in past ages or in arctic climates.

I have then criticised the principal cases in which it has been contended that we have evidence of glacial action in ancient boulder deposits, and have shown, by reference to actual specimens of the rocks in question, that, not only is the evidence of palæozoic or mesozoic glaciation in Britain inconclusive, but that the negative can be proved in all the cases hitherto adduced.

Being thus warned against taking on trust evidence for glacial action in ancient times founded upon the form or the condition of the surface of the rock, I venture to throw doubt on the inference that the faceted stones of Copitz by Pirna are of glacial origin. I give the results of some of my own explorations among the ancient boulder clays of Wetzikon, &c. I point out that the Cambrian scratched stones of Norway are in regions still under the influence of glacial conditions in spite of the mild influences of the gulf stream. I then give a sketch of the distribution of boulder-bearing beds in India, Australia and Africa, but have no evidence from personal observation to offer respecting them. I admit that the consensus of many competent observers renders it difficult to believe that these beds do not exhibit evidence of glacial origin.

Another question must however be carefully examined before we can accept the view that glaciers came down to the sea in the region where these boulders have been discovered, that is, the mode of transport of boulders. First, how far from their starting point at the land-ice foot may they have been carried by floating ice, and, secondly, by what means and how far may some of them be handed on by other agents beyond the regions of probable ice transport.

Having thus reduced the *necessary* glaciation of these south regions to more modest dimensions we may enquire what causes may be suggested in explanation of the occurrence of greater cold in those regions in former times. We know approximately what increase of cold to expect as we ascend a mountain range or as a mountain region is upheaved, and we know that some areas are more unstable than others. What evidence have we of great earth movements in the regions from which the ice-borne boulders have been derived?

To these different but connected questions I hope to be allowed to return on future occasions.

xvii. *List of Specimens exhibited in illustration of paper*[1].

1. Smoothed and striated surface of Molasse in situ near the highest part of Lausanne, Lake of Geneva. Collected with Professor Renevier, 1873. The striæ run in a north-westerly direction.
2. Surface of solid rock glaciated. North of Killaloe, Co. Clare, Ireland.
3. Similar surface on fragment from the Clwydian Drift near Ffynon Beuno, St Asaph, North Wales.
4. Boulder of Mountain Limestone which had received polish and striations and was then broken up and the fragments imbedded in the later or Clwydian Drift.
5. Piece of the polished and striated surface of the solid Portlandien Rock which lay in the path of the great Swiss valley glacier above the quarries, north of Solothurn (Soleure), Switzerland.
6. Boulder of limestone with the protuberant parts polished and striated, Drift, Roslyn Pit, Ely.
7. Vein quartz smoothed, polished, and striated by glacial action. Top of Slievwhuallian, Isle of Man (50 yards S.W. of fence corner, S.W. of Cairn $\frac{9}{100}$). Given to me by Mr Lamplugh.
8. Piece of Burr-stone, used in Phosphate mill, Burwell Lode, Cambridge.
9. Slickensides from fault in Lower Keuper Sandstone, Alderley Edge, Cheshire.
10. Do. from the Stockdale Shale of Skelgill, Windermere.
11. A cluster of crystals of Calcite which have been exposed to the spray from a mountain rill. Third Grit near Park House, south of Wray, Lancaster.
12. A mass of banded travertine from the side of a small stream draining the peat near Big Wheel Lode, Penrhyn, Portmadoc.
13. A similar mass which was wedged in among some insoluble stones in such a manner as to be acted upon nearly all over. The points of contact with the other stones remain rough like

[1] These specimens are now arranged, with corresponding numbers attached, in the Woodwardian Museum.

the three points on a Delft plaque, indicating where it rested upon its tripod in the firing.

14. Corner of a rock of Mountain Limestone exposed to the splash of a small waterfall. This specimen shows a fretted surface.
15. Piece of gypsum dug out of damp marl and showing fretted surface. Little Salkeld quarry, near Penrith, Cumberland. (See also two larger specimens in the Museum.)
16. Basalt polished and fretted by blown sand. West of Spa Point, Burntisland, Edinburgh.
17. Quartzite pebble projecting from surface of Cambrian conglomerate and fretted by blown sand. Top of cliff, east of Porth Seli, St Davids. Two specimens.
18. Sand-worn stone from Western side of entrance to harbour of Wellington, Cook's Straits, Southern end of North Island, New Zealand.
19. Another specimen in which veins of harder material project.
20. Pebble worn by blown sand. Top of Plateau gravel, Biarritz.
21. A large slab of red sandstone smoothed and striated: from the glaciated surface of the solid rock near Edinburgh.
22. Large boulder of chalk from the Glacial Drift of Roslyn Pit, Ely.
23. Small boulder of black limestone from the great moraine of Lausanne, Switzerland.
24. Boulder of Silurian rock from the lower Till, Longwathby cutting, Settle and Carlisle Railway.
25. Do. from clay drift 1800 feet above sea level. Top of great Swindale, Weasdale, Ravenstonedale, Westmorland. (29 S.E.)
26. Boulder of Shap granite from the Eden Valley.
27. Fragment of limestone which had been bored by lithodomous molluscs and was afterwards glacially striated. Boulder Clay, N.W. of Greenore Point, Wexford.
28. Piece of large mass of Jurassic limestone which projected in the path along which timber was dragged. Road up Jura from Grenchen to Burenberg, Canton Solothurn.
29. Do. do.
30. Rock scratched by dragging timber over it. Bonwmuchaf, Corwen.
31. Do. do.
32. Surface of bed striated by fragments sliding over it from the top of the quarry. Rhiwallt, St Asaph.
33. Do. do.
34. Mountain Limestone polished and grooved by passage of wire rope over it. Arco Wood Quarry, Horton-in-Ribblesdale.
35. Do. do.
36. Rock scratched and polished by boys sliding down it at right angles to the direction of the glacial striæ, which are obliterated on this specimen. Rhosygwaliau, Bala, N. Wales.
37. Limb bones of Pliosaurus from Littleport, with cuts (teeth marks?). Compare No. 39.
38. Do.
39. Nodule of Phosphate of Lime from the Cambridge Greensand, with Plicatula sigillina growing over small cuts and grooves.

40. Slickensides along bedding plane, interrupted by worm tubes. Bala Beds, Constitution Hill, Aberystwith.
41. Slickenside fluted by lumps and grains seen on the specimen. Landslip in Barton Clay, with some surface material dragged in. Barton Cliff, Hampshire.
42. Fault breccia smoothed and scored by movement against the wall of the fault. Cove below Daddy Hole, Torquay.
43. Part of fault face in Silurian. The Grove, Bodfari, near St Asaph. Showing strong parallel fluting.
44. Fluted and slickensided face of grit with film of mineral. Nodale, Stansfield Moor.
45. Fluted and slickensided face of lode-stuff with film of mineral. Derbyshire.
46. Curved slickenside on slate rock. Quarry north of Aberystwith.
47. Fluted structure due to hard beds in rock, affected by imperfect cleavage and joints. Tirgwyn, three miles E.N.E. of St Asaph.
48. Part of a glaciated Silurian boulder from the drift of Maes Mynan near Caerwys in North Wales. The striæ seen on this fragment however are almost all due to the weathering out of softer lines in the rock.
49. Fragment of Devonian sandy shale with the joints weathered out into grooves. Foot of Rough Tor, about a mile from Okehampton Station by the track leading up to Yes Tor, Dartmoor.
50. Finer bed in the conglomerate at the base of the Carboniferous Rocks showing striation across the matrix and included pebbles. Holbeck Gill, between Dent and Garsdale, Yorkshire.
51. Do. do.
52. Scratched stone from do. do.
53. Do. do.
54. Striated pebble in Triassic conglomerate. Portskewet, Monmouthshire.
55. Hard lump included in the shale at the base of the Cambrian Grit. N.E. of Glyngarth, Menai Straits.
56. Pebble from the conglomerate twisted into the Archæan Gneiss. Obermitweida, Annaberg.
57. Small boulder crushed and broken, and the several parts more or less separated and then cemented together in the conglomerate by mineral matter. Basement bed of the Carboniferous. Holbeck Gill, between Dent and Garsdale, Yorkshire.
58. Do. do. do.
59. Do. do. do.
60. Do. do. do.
61. Surface of Coniston Flags smoothed and striated (slickensided) by thrust at base of Conglomerate (Basement Bed Carboniferous). Holbeck Gill, between Dent and Garsdale.
62. Do. do.
63. Stylolites from the London Clay, Sheppey.
64. Flint which has been forced by earth movement through the enveloping chalk.
65. Fossils thrust through surrounding material.

66. Pebble of quartzite worn by blown sand. Copitz by Pirna.
67. Do. These are good samples of the stones supposed to have been ground on several sides as they occupied different positions under the ice. Similar specimens are exhibited in the Dresden Museum as evidence of glaciation.
68. Glacially striated pebble found by myself *below* the lignite at Wetzikon.
69. Do. on the moraine above the lignite of Wetzikon.
70. A piece of Haffield conglomerate, Haffield.
71. A slightly worn fragment of close textured mudstone with a shining surface produced by a film of iron oxide. Haffield conglomerate, Haffield.
72. Boulder of a green sandstone with shells and masses of the phosphate bed attached and covering striations caused by weathering along line of weakness due to rock structure. Compare Nos. 37, 38, 39.
73. Rolled lump of older boulder clay found in newer boulder clay, Clwydian Drift. St Asaph, North Wales.
74. Lump of newer boulder clay (Clwydian Drift) rolled on shore. Colwyn, North Wales.
75. Lump of London Clay rolled on shore. Sheppey.
76. Pebble of Bala Limestone with Halysites catenularius projecting on the surface. In conglomerate at base of Carboniferous. Holbeck Gill, between Dent and Garsdale, Yorkshire.

November 13, 1893.

PROF. SIR G. G. STOKES IN THE CHAIR.

The following Communications were made to the Society:

(1) *The Application of Newton's Polygon to the Theory of the Singular Points of Algebraic Functions.* By H. F. BAKER, M.A.

(2) *On a Class of Definite Integrals connected with Bessel's Functions.* By A. B. BASSET, M.A., F.R.S.

This paper contains a method for reducing double integrals of the form $\int_0^\infty \lambda^m \epsilon^{-\lambda^2 t} J_n(\lambda r)\, d\lambda$ to single integrals; also various expressions for the different kinds of Bessel's functions are found in the forms of definite integrals.

1. In many investigations connected with the motion of viscous liquids and the conduction of heat, definite integrals occur which depend upon integrals of the form

$$\int_0^\infty \lambda^m \epsilon^{-\lambda^2 t} J_n(\lambda r)\, d\lambda = V_n^m \text{ (say)}.$$

The ordinary expression for J_n in the form of a definite integral is

$$J_n(\lambda r) = \frac{(\lambda r)^n}{\pi \,.\, 1 \,.\, 3 \ldots (2n-1)} \int_0^\pi \cos(\lambda r \cos\theta) \sin^{2n}\theta d\theta \ldots (1),$$

and accordingly when $m+n$ is an even integer, the integral V_n^m depends upon one of the form

$$\int_0^\infty x^{2s}\, \epsilon^{-a^2x^2} \cos 2bx\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The value of this integral when $s=0$ is known to be $\sqrt{\pi}/2a \,.\, \epsilon^{-b^2/a^2}$, from which the value of the integral (2) can be deduced by differentiation with respect to b.

If however $m+n$ is an odd integer, V_n^m depends upon integrals of the form

$$\int_0^\infty x^{2s+1}\, \epsilon^{-a^2x^2} \cos 2bx\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

This last integral cannot be evaluated in finite terms.

There is however another form of J_0, which is given by the equation

$$J_0(\lambda r) = \frac{2}{\pi} \int_0^\infty \sin(\lambda r \cosh\phi)\, d\phi \quad \ldots\ldots\ldots\ldots (4),$$

and consequently V_0^{2m+1} depends upon integrals of the form

$$\int_0^\infty x^{2s+1}\, \epsilon^{-a^2x^2} \sin 2bx\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

which can be deduced from the known value of (2) by differentiating with respect to b.

The integral (2) enables V_1^m to be integrated with respect to λ when m is an odd integer; but when m is even this integral cannot be employed. Now $J_0'(r) = -J_1(r)$, and if it were allowable to differentiate (4) with respect to λr, we should obtain

$$J_1(\lambda r) = -\frac{2}{\pi} \int_0^\infty \cosh\phi \cos(\lambda r \cosh\phi)\, d\phi \ldots\ldots\ldots (6).$$

We shall hereafter prove equation (6) by a different method; accordingly when m is an even integer equation (6) enables V_1^m to be integrated with respect to λ. Since any three Bessel's functions are connected together by the equation

$$J_{n+1}(x) = 2nx^{-1} J_n(x) - J_{n-1}(x) \ldots\ldots\ldots\ldots\ldots (7),$$

it follows that since V_0^m and V_1^m can be integrated with respect to λ for *any* integral value of m, the same process can be performed upon V_n^m.

The advantages of reducing a double integral to a single integral are obvious; and it often happens that when the integration with respect to λ has been performed, the integration with respect to θ or ϕ can also be effected, and the integral completely evaluated.

A good many proofs of (4) have been given, one of which will be found on p. 431 of the fifth volume of these *Proceedings*. Equation (6) may be established in a precisely similar manner by integrating the definite integral

$$\int_0^\infty \int_0^\infty x \sin x \cos u^2 (x^2 - r^2)\, dx\, du,$$

first with respect to x, and then with respect to u and comparing the results.

The function Y_n.

2. In many investigations the second solution of Bessel's equation, which will be denoted by Y_n, is required. In some cases it is better to employ complex quantities throughout and to discard the imaginary part in the final result, whilst in others it is better to employ a function with a *real* argument. It is easily shown that the definite integral

$$\frac{2}{\pi}\int_1^\infty \frac{\epsilon^{-\iota r x}\, dx}{(x^2-1)^{\frac{1}{2}}} \quad\text{.........................(8)},$$

satisfies the same equation as $J_0(r)$, as is otherwise obvious from the theory of linear sources of sound—see Rayleigh, *Theory of Sound*, Vol. II. p. 275. The integral (8) may be written

$$\frac{2}{\pi}\int_0^\infty \cos(r\cosh\phi)\, d\phi - \frac{2\iota}{\pi}\int_0^\infty \sin(r\cosh\phi)\, d\phi = Y_0(r) - \iota J_0(r)$$

by (4), whence

$$Y_0(r) = \frac{2}{\pi}\int_0^\infty \cos(r\cosh\phi) d\phi \quad\text{...............(9)},$$

which may be regarded as the definition of Y_0; also differentiating with respect to r, and recollecting that $Y_1(r) = -Y_0'(r)$ we obtain

$$Y_1(r) = \frac{2}{\pi}\int_0^\infty \cosh\phi \sin(r\cosh\phi)\, d\phi \quad\text{.........(10)}.$$

This result will be obtained by a more satisfactory process later on.

It is proved in Lord Rayleigh's *Sound*, Vol. II. p. 273, that

$$\left(\frac{2}{\pi\iota r}\right)^{\frac{1}{2}} \epsilon^{-\iota r}\left\{1 - \frac{1^2}{1\,.\,8\iota r} + \frac{1^2\,.\,3^2}{2!\,(8\iota r)^2} - \frac{1^2\,.\,3^2\,.\,5^2}{3!\,(8\iota r)^3} + \dots\right\}$$

$$= -\frac{2}{\pi}\left(\gamma + \log \tfrac{1}{2}\iota r\right) J_0(r) - \frac{2}{\pi}\left(\frac{r^2}{2^2}S_1 - \frac{r^4}{2^2\,.\,4^2}S_2 + \frac{r^6}{2^2\,.\,4^2\,.\,6^2}S_3 - \right) \quad \dots(11),$$

where γ is Euler's constant, and $S_n = 1 + 2^{-1} + \dots n^{-1}$. The imaginary part of the right-hand side is obviously equal to $-\iota J_0(r)$, and we must now show that the integral (9) is equal to the real part of either side of the above equation. This we shall do by showing that the integral (8) can be expressed by means of the series on the left-hand side of (11). We shall also establish the legitimacy of equation (10), which was deduced from (9) by differentiation with respect to r.

In (8) put $x = 1 + y$ and it becomes

$$\frac{2}{\pi}\int_0^\infty \frac{\epsilon^{-\iota r - \iota r y}\,dy}{y^{\frac{1}{2}}(2+y)^{\frac{1}{2}}} = \frac{\sqrt{2}\,\epsilon^{-\iota r}}{\pi}\int_0^\infty y^{-\frac{1}{2}}\,\epsilon^{-\iota r y}\left\{1 - \tfrac{1}{2}\left(\frac{y}{2}\right) + \dots (-)^n H_n \left(\frac{y}{2}\right)^n \dots\right\} dy,$$

where $$H_n = \frac{1\,.\,3 \dots 2n-1}{2\,.\,4 \dots 2n}.$$

Now $$\int_0^\infty \epsilon^{-\iota r y}\,y^{n-\frac{1}{2}}\,dy = \frac{1\,.\,3 \dots (2n-1)}{(2\iota r)^n}\,\frac{\sqrt{\pi}}{(\iota r)^{\frac{1}{2}}},$$

so that the integral

$$= \left(\frac{2}{\pi\iota r}\right)^{\frac{1}{2}} \epsilon^{-\iota r}\left\{1 - \frac{1^2}{8\iota r} + \frac{1^2\,.\,3^2}{2!\,(8\iota r)^2} - \dots\right\} \quad \dots\dots(12).$$

The latter series is therefore equal to $Y_0(r) - \iota J_0(r)$; and by realizing and equating the real and imaginary parts we shall obtain the two series for Y_0 and J_0 which are given in Rayleigh's *Sound*, Vol. II. p. 275 and Vol. I. p. 264.

Equations (10) and also (6) may be deduced by differentiating the series (12) with respect to r and then summing the two resulting series to which the differentiation leads.

3. The circumstance that two distinct forms of the J functions exist in one of which the limits are 1 and 0, whilst in the other they are ∞ and 1, for equation (4) is equivalent to

$$\int_0^1 \frac{\cos rx\,dx}{(1-x^2)^{\frac{1}{2}}} = \int_1^\infty \frac{\sin rx\,dx}{(x^2-1)^{\frac{1}{2}}},$$

suggests whether a similar form does not exist in the case of the Y functions. We shall now proceed to show that this is the case, and the method which we shall adopt is that of Lipschitz[1], and depends upon the proposition in the theory of complex functions that $\int f(z)\,dz$ taken round any closed curve which does not surround a pole of the function $f(z)$ is zero.

Consider the integral

$$\int \frac{\epsilon^{-rz}\,dz}{(1+z^2)^{\frac{1}{2}}}$$

taken round a rectangle bounded by the axes and the lines $x = h$, $y = 1$. No pole lies within the rectangle, and the corner $z = \iota$ of the rectangle is the only pole of the function which lies on the boundary; we must therefore exclude this pole by drawing a quadrant of a circle round it, whose radius ultimately vanishes. The portion of the integral taken along this quadrant will ultimately be found to vanish, so that no difficulty arises in consequence of integrating through this point. The integral accordingly leads to four integrals taken *along* the four sides of the rectangle, whose sum is zero; and if we make $h = \infty$, we shall obtain in the limit the following equation

$$\iota \int_0^1 \frac{\epsilon^{-\iota ry}\,dy}{(1-y^2)^{\frac{1}{2}}} = \int_0^\infty \frac{\epsilon^{-rx}\,dx}{(1+x^2)^{\frac{1}{2}}} - \int_0^\infty \frac{\epsilon^{-r(x+\iota)}\,dx}{\{1+(x+\iota)^2\}^{\frac{1}{2}}} \;\ldots\ldots(13).$$

The first integral on the right-hand side is real, whilst the second is complex; and by means of the equation

$$\int_0^\infty \epsilon^{-au^2}\,du = \tfrac{1}{2}(\pi/a)^{\frac{1}{2}}$$

it may be written

$$\int_0^\infty \frac{\epsilon^{-r(x+\iota)}\,dx}{x^{\frac{1}{2}}(x+2\iota)^{\frac{1}{2}}} = \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^\infty x^{-\frac{1}{2}} \epsilon^{-r(x+\iota)-(x+2\iota)u^2}\,dx\,du$$

$$= 2\int_0^\infty (r+u^2)^{-\frac{1}{2}} \{\cos(r+2u^2) - \iota\sin(r+2u^2)\}\,du$$

$$= \int_0^\infty \frac{\cos rv - \iota \sin rv}{(v^2-1)^{\frac{1}{2}}}\,dv,$$

if $r + 2u^2 = rv$. Equating the real and imaginary parts in (13) we get

$$\int_0^1 \frac{\cos ry\,dy}{(1-y^2)^{\frac{1}{2}}} = \int_1^\infty \frac{\sin rv\,dv}{(v^2-1)^{\frac{1}{2}}}$$

$$\int_0^1 \frac{\sin ry\,dy}{(1-y^2)^{\frac{1}{2}}} = \int_0^\infty \frac{\epsilon^{-rx}\,dx}{(1+x^2)^{\frac{1}{2}}} - \int_1^\infty \frac{\cos rv\,dv}{(v^2-1)^{\frac{1}{2}}}.$$

[1] *Crelle*, Vol. LVI.

The first equation is a reproduction of (4); whilst the second equation leads to

$$Y_0(r) = \frac{2}{\pi}\int_0^\infty \cos(r\cosh\chi)\,d\chi$$

$$= \frac{2}{\pi}\int_0^\infty \epsilon^{-r\sinh\phi}\,d\phi - \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\sin(r\cos\theta)\,d\theta \ldots\ldots(14).$$

This equation gives the second form of $Y_0(r)$. It will be observed that the second integral on the right-hand side is analogous to the integrals by which the first form of the J functions are expressed, whilst the second is analogous to one of the forms of the K functions; and it is worthy of note that the last class of functions are also capable of being expressed in two different forms which closely resemble those of the J functions. The connection is expressed by the equation

$$K_0(r) = \int_0^\infty \epsilon^{-r\cosh\theta}\,d\theta = \int_0^\infty \cos(r\sinh\phi)\,d\phi;$$

see *Hydrodynamics*, Vol. II. p. 18.

4. We shall now consider the functions J_1 and Y_1.

By integration by parts we obtain

$$\int_0^1 (1-x^2)^{\frac{1}{2}}\cos rx\,dx = \frac{1}{r}\int_0^1 \frac{x\sin rx\,dx}{(1-x^2)^{\frac{1}{2}}},$$

whence
$$J_1(r) = \frac{2}{\pi}\int_0^1 \frac{x\sin rx\,dx}{(1-x^2)^{\frac{1}{2}}}.$$

Now consider

$$\int \frac{z\epsilon^{-rz}\,dz}{(1+z^2)^{\frac{1}{2}}},$$

taken round the rectangle as before. This will be found to lead to the following equations:

$$\int_0^\infty \sinh\phi\,\epsilon^{-r\sinh\phi}\,d\phi - \frac{1}{r}\int_0^\infty \frac{\cos(r\cosh\phi)\,d\phi}{1+\cosh\phi}$$

$$-\int_0^\infty \sin(r\cosh\phi)\,d\phi + \int_0^1 \frac{y\cos ry\,dy}{(1-y^2)^{\frac{1}{2}}} = 0 \ldots\ldots(15).$$

$$J_1(r) = \frac{2}{\pi r}\int_0^\infty \frac{\sin(r\cosh\phi)\,d\phi}{1+\cosh\phi} - \frac{2}{\pi}\int_0^\infty \cos(r\cosh\phi)\,d\phi \ldots(16).$$

Integrating the first integral on the right-hand side of (16) by parts, and assuming that $\sin\infty = 0$, which is justified by our previous results, we get

$$J_1(r) = -\frac{2}{\pi}\int_0^\infty \cosh\phi\cos(r\cosh\phi)\,d\phi,$$

which we have already obtained, whilst (15) leads to

$$Y_1 = \frac{2}{\pi}\int_0^{\infty} \sinh\theta \epsilon^{-r\sinh\theta}\, d\theta + \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \cos\theta \cos(r\cos\theta)\, d\theta \dots (17).$$

The last result might be immediately obtained from (14) by differentiation.

(3) *On Isotropic Elastic Solids of nearly Spherical Form. Part I. Equilibrium.* By C. CHREE, M.A. [*Published in the Transactions.*]

November 27, 1893.

PROF. T. M^C^K. HUGHES, PRESIDENT, IN THE CHAIR.

Dr F. H. H. Guillemard, Mr R. Assheton and Mr J. W. Capstick were elected Fellows of the Society.

The following Communications were made to the Society:

(1) *The action of Light on Bacteria.* By Dr H. MARSHALL WARD.

By throwing the spectrum on various bacteria suspended in films of agar, it is possible to obtain photographic records of the action of the various rays; because, after incubation, those spores or bacilli &c. which are killed by certain rays remain invisible, whereas the colonies which are formed by the germination of those still left capable of development render the agar opaque. The experiments show that those germs which are struck by the infra-red, red, orange and yellow develop as rapidly as those not exposed to light at all. The action begins as we leave the green, and rises to a maximum in the blue-violet and violet, falling off as we pass into the ultra-violet.

For the solar spectrum a heliostat and glass lenses and prism were used; for the electric spectrum a quartz train, and quartz covering to the film. Even a thin plate of clear glass blocks out much of the effective region of the spectrum, and especially ultra-violet rays.

The author records his thanks to Prof. O. Lodge, F.R.S., for kindly exposing the plates to the electric spectrum for him, with most successful results.

The author also found that the water of the Thames, examined in August and in October respectively, shows the following interesting results.

1. The number of bacteria per c.c. was distinctly smaller in the bright August weather than in the duller days of October, and differences were observable in the aspect of the colonies on plate-cultures.

2. Suspecting that this was concerned with light action, experiments showed that insolation not only kills off large numbers of the bacteria in the water, but in some cases shows

its effects in diminishing the liquefying (i.e. enzyme action) power of certain forms, and even in altering their mode of growth, so that the aspect of the colonies on gelatine plates is affected.

These changes in aspect are not a mere matter of preventing or increasing the production of pigment, &c., but are due to effects on the *growth* of the colonies. As regards pigments, the author has examined a pigment of one of these river-species, which pigment, though so resistent as to bear solution in alcohol, evaporation and drying at 100° C., and re-solution, without apparent alteration, is destroyed in an hour or two on exposure to light.

(2) *On Gynodiœcism* (third paper), *with a preliminary note upon the origin of this and similar phenomena.* By J. C. WILLIS, M.A., Gonville and Caius College.

The experiments begun in 1890, upon which two preliminary notes have been published (*Proc. Camb. Phil. Soc.*, VII. 1892, p. 348, and VIII. 1892, p. 17) have been continued during 1893. The chief point of immediate interest in the present year's observations was the change of type of some plants. It has hitherto been taken for granted, so far as can be found from a careful study of the literature, that a "female" plant remains female throughout its life history, and a hermaphrodite likewise. This year's experiments have shown that this is not the case. A wild female plant of *Origanum vulgare* which was transplanted from Abington (see 2nd paper, p. 18) and which remained female during last year, came out hermaphrodite in the present season. On July 15, 1893, it was in full flower, and was covered with normal hermaphrodite flowers, with here and there a stray female flower, just like a normal hermaphrodite plant. The few female flowers were almost entirely borne on the lower lateral tufts. On July 22, it was in a similar condition, and the female flowers were comparatively large. Presently, however, a change occurred, and by Aug. 10, the plant looked like a normal female, only bearing a few hermaphrodite flowers. (The flower of Origanum usually lasts about a week.) In this condition it remained for the rest of the season, at times, however, bearing a considerable number of hermaphrodite flowers.

The bed of seedlings of 1890 (see 2nd paper, p. 18) flowered vigorously again this year; but although in 1892 there were several female plants among them, during this year no stalks appeared which could be termed female, though some of the hermaphrodites bore several female flowers. As the female plants of 1892 were not specially marked, it was impossible to determine whether they had turned into hermaphrodites, but in view of the observations above, and of the improbability of them all having died, it seems quite likely that they have done so.

Similar facts are given in the preceding papers (see especially the case mentioned in 2nd paper, p. 19).

Various observations have been made on other gynodiœcious plants, chiefly for purposes of comparison with the work of other writers upon this subject. A few of these are worth recording here.

Capsella Bursa pastoris, Mœnch.—On March 12, 1893, specimens of this plant were observed on a wall near Grantchester Church, bearing female flowers only, and on April 16th, at Clayhithe, similar plants were found by Mr Burkill and the author.

The Grantchester plants had just begun to flower, and showed an almost total abortion of the stamens. The Clayhythe plants, however, possessed stamens which were often fertile, but too short to reach and pollinate the stigma. Every stage from fully female flowers with aborted stamens, to normal hermaphrodites, was observed upon these plants. The lowest 15—20 flowers in some racemes were unfertile; above these were a few that had set small shrivelled-looking capsules, and then came fully-developed fruit, with normal flowers above.

In November of the present year, racemes of Capsella were again found bearing female flowers. Their presence would thus seem to be dependent upon the light or temperature.

Capsella may thus, under certain circumstances, appear gynomonœcious or possibly gynodiœcious, as has indeed been observed already by Breitenbach[1].

Hippuris vulgaris L.—Plants have been observed in the Ouse, near Ely, bearing female flowers only. It has been recorded[2] as diœcious, but the author has not seen any but hermaphrodite and female plants.

Asperula cynanchica L.—This plant, growing upon the steep dry bank of the Cherryhinton chalk-pit, exhibits the two forms described by Müller[3]. On July 30, 1892, and again on August 5, 1893, it was observed that the bulk of the flowers were female, with small aborted stamens.

Scabiosa arvensis L.—The gynodiœcism of this plant is well known. Müller describes it in such a way as to give the idea that the female form is frequent, if not abundant, in Low Germany. On the other hand, it appears from the recent observations of Macleod[4] that the female form does not occur at all

[1] "Einige neue Fälle von Blütenpolymorphismus." *Kosmos* 1884, p. 206.

[2] Kirchner, "Flora von Stuttgart und Umgebung." Stuttgart (Ulmer), 1888.

[3] "Fertilisation of Flowers," English edition, p. 302.

[4] "Over de bevruchting der bloemen in het Kempisch gedeelte van Vlaanderen." *Botanisch Jaarboek*, Gent, v. 1893, p. 394.

in West Flanders. Darwin[1] found it frequent in Kent, while in Cambridgeshire during the last few years, the author has found it very much more common than the hermaphrodite form.

Prunella vulgaris L.—The author has never observed a female plant in England, though great numbers have been examined. On the Continent it seems to be frequently female.

If we sum up the results of these and other observations upon gynodiœcism and gynomonœcism, we may come to the following conclusions. They are very wide spread phenomena, but in a large number of cases where they occur, seem only to be sporadic. In a few cases, e.g. many of the Labiatæ, and possibly other orders, the phenomena are pretty regularly shown, and appear to be fixed by heredity. In all cases, however, they are very variable; this is well seen in the observations given above, and in those of Schulz[2], Ludwig, Loew and other writers. Causes of variation appear to be soil, temperature, light, climate and causes internal to the plant itself, all acting upon and determining the immediate cause, which may be differences in nutrition of the flower primordia.

The female flowers in the Labiatæ are as much visited as the hermaphrodite, and set many seeds; according to Darwin more than the hermaphrodite, according to Schulz not so. In these cases it is therefore probable that it is a distinct advantage to the plant to possess the two forms. In other cases the advantage seems rather doubtful. It is noteworthy that a large proportion of gynodiœcious plants are perennials with good vegetative reproduction.

With regard to the origin of these phenomena, there has been much discussion. Müller's view has now been entirely abandoned and need not be considered. Hildebrand and Ludwig regard it as an outcome of dichogamy: most of the plants exhibiting it are protandrous and the stamens of the first flowers are therefore useless and may become aborted. This view Ludwig supports by his observation that in Thymus the female plants bear a larger proportion to the hermaphrodites at the beginning of the season than later on. The author made some observations upon Nepeta Glechoma (see 1st paper) which tended to confirm this view. These have been continued and now afford the following Table [the numbers represent percentage (of the total number of plants in flower) of female plants in flower].

Week	1	2	3	4	5	6	7	8
1891	50·	16·	35·8	28·5	23·4	19·2	28·3	—
1892	57·	30·9	25·8	29·2	26·6	incomplete		
1893	37·5	25·	16·8	14·3	6·8	31·3	31·	20

[1] "Forms of Flowers," p. 305.

[2] "Beiträge zur Kenntniss d. Bestäubungseinrichtungen u. d. Geschlechtervertheilungen bei den Pflanzen." *Bibliotheca Botanica*, Heft 11 u. 17.

[In the first week, 1891, only 8 plants were out, in 1892 only 7, in 1893, 24.]

It is thus evident that on the whole the female plants flower a little sooner than the hermaphrodite, but the proportion, after the first week, does not fall much, and has a tendency to rise again later in the season. Schulz[1], on the other hand, states that the proportion of female to hermaphrodite plants is not dependent on the season. Hence the support given to the hypothesis of Hildebrand and Ludwig by these observations is, if any, very slight. The author's (not yet published) observations on dichogamy in Thymus serpyllum show that, though it is fully dichogamous late in its flowering season (i.e. the male period is over when the female begins), it is not so at first, there being a considerable overlap, during which the flower is hermaphrodite, and *may* fertilise itself. This phenomenon, if general, would also be an argument against the hypothesis under consideration.

Darwin[2] looks on the greater fertility of the female form as the great factor in producing gynodiœcism, but leaves it as undecided whether this was the first cause or whether the stamens began to abort and so raised the fertility of those flowers. Schulz denies the greater fertility of the female, and it should be observed that in any case, a Labiate flower can only give four seeds, which it does when it is properly pollinated, whether hermaphrodite or female.

Taking together all the facts, it would seem that the proximate cause of one flower being female, another hermaphrodite, is some difference in nutrition; in the case of gynodiœcism, between two plants, in the case of gynomonœcism between flowers on the same plant. The phenomena vary largely with the nature of the soil and climate, with the season of the year, and other conditions. Females are apparently more common in very wet situations, though occasionally the reverse is the case.

Gynodiœcism and gynomonœcism are almost invariably associated with pronounced dichogamy. It seems probable therefore that in dichogamous plants, the various causes mentioned above may produce a considerable degree of gynomonœcism, or even gynodiœcism. This may prove advantageous and become perpetuated by natural selection, as seems to have occurred in the Labiatæ and other orders.

Many other phenomena are apparently due to similar causes acting upon the plant. These may be briefly considered.

Androdiœcism, which is very rare, is almost certainly due[3] to lack of nourishment of the male plant, and so too, andromonœcism to some flowers receiving a small supply.

[1] *Loc. cit.* [2] *Loc. cit.* p. 304.
[3] Müller, "Alpenblumen," p. 41. (*Veratrum.*)

Diœcism, in angiosperms, is descended from hermaphroditism, and although it is hereditary, it has been shown that the sex of a seedling can to some extent be determined in advance by its conditions of nutrition. Thickly sown[1] seedlings of *Lychnis diurna* gave 200 males to 100 females. Thinly sown, on the other hand, 77 males were obtained per 100 females. Various similar observations have been made. The phenomena of diœcism show considerable variation in nature, like those of gynodiœcism (see Darwin, *loc. cit.*).

Fragaria, hermaphrodite in England, becomes diœcious or polygamous in the United States. Diœcism in some cases may have come from heterostylism (Mitchella, &c.). Monœcism presents similar phenomena.

Cleistogamy is also a very variable phenomenon, appearing sporadically in many plants, constantly in others. It appears to vary with the time of year, soil, climate, light and temperature (according to some unpublished observations of the author's upon *Salvia Verbenaca* L.). Vöchting[2] has recently shown in a very striking way the effect of light upon the production of cleistogamy. It is noteworthy that cleistogamous plants are not usually very dichogamous. It does not usually occur in gynodiœcious genera (except Salvia).

Lastly dichogamy is a phenomenon closely bound up with some of the above, and largely dependent on external conditions.

Meehan[3] has tried to show that it is largely dependent on temperature, and the author inclines to the same view, though in a modified form. It varies with season, soil and other conditions, both in different plants of the same species, and (as observed by the author) in the same plant at different times.

To sum up, it appears probable that all these phenomena are closely allied to one another, depending very largely upon external causes, and capable of being called out, or modified, by these, in a very marked degree, but at the same time, fixed to a great extent by natural selection and heredity. The author is now engaged in working up the literature relating to these subjects, and hopes to publish a review of it, discussing the subject of the origin of these phenomena in some detail.

In conclusion, the author desires to thank Mr F. Darwin, Mr R. I. Lynch, and Mr I. H. Burkill for valuable advice and assistance rendered.

[1] Hoffmann, "Ueber Sexualität." *Bot. Zeit.* vol. 43, 1885, p. 145.
[2] Prings. Jahrb. xxv., 1893, Heft 2.
[3] Various papers, chiefly in *Proc. Acad. Nat. Sci. Philadelphia*, 1885—1893.

January 29, 1894.

PROF. T. MCKENNY HUGHES, PRESIDENT, IN THE CHAIR.

Mr E. W. MacBride was elected a Fellow of the Society.

The following Communications were made to the Society:

(1) *Electricity of Drops.* By Professor J. J. THOMSON.

(2) Mr GRIFFITHS described an easy method of making absolutely air-tight joints between glass and metal tubes, by means of an alloy which has a low melting-point. The use of this alloy was suggested by Mr F. Thomas. An illustration was given of the ease and certainty of the method.

(3) A compensating open-scale barometer was then exhibited and described by Mr GRIFFITHS.

The principle of this instrument is the same as that of Professor Callendar's long-distance air thermometer. An air bulb is placed within a second bulb and the annular space between them is filled with sulphuric acid. The air and the H_2SO_4 have a common surface in a tube connecting the two bulbs, the H_2SO_4 also communicates with the air by means of a vertical tube partially filled with acid. The masses of air and sulphuric acid are so adjusted that when the temperature of the instrument is raised, the increase in pressure due to the increased length of the sulphuric acid column in the vertical tube exactly counterbalances the increase in pressure of the contained air and thus the position of the common surface is unchanged by alterations in temperature, although at once affected by alterations in the external pressure. The resulting scale is about six times as open as the scale of a mercury barometer, and the readings give the pressure expressed in terms of the length of a column of mercury at 0° C. in latitude 45°, without any preliminary calculations.

(4) *On the condition of the interior of the earth; a correction and addition to a former paper.* By Mr O. FISHER, Hon. Fellow of Jesus College.

I wish to correct an error in a paper "On the hypothesis of a liquid condition of the earth's interior in connection with Professor Darwin's theory of the genesis of the moon," read in May last year. It is unnecessary to go into all the geological arguments which favour that view. Many of them will be found well set forth in an article in the *Fortnightly Review*[1] by Dr Alfred Russel Wallace, F.R.S. The strongest argument on the other side is admitted to be the existence of ocean tides; and it was in

[1] Nov. 1892, "Our Molten Globe."

that connection that my mistake occurred. Desiring to ascertain the amount of diminution of the tide in an equatorial canal, I chose for the tidal deformation of a liquid globe an inapplicable value, viz. $1\frac{3}{4}$ feet from highest to lowest. I took this estimate out of Thomson and Tait's *Nat. Phil.*[1] but, as Mr Becker of the U.S.A. geological survey has pointed out, I unfortunately overlooked the condition on which it had been obtained, which was that there should be no mutual attraction among the particles of the liquid[2] interior. The use of this number led to the incorrect result, that the diminution of the ocean tide would be but small, and I concluded that more importance had been attached to this point than it possessed.

I have lately calculated what would be the tidal deformation of a liquid earth owing to the attraction of the moon, supposing Laplace's law of density to obtain. The method followed was simply to substitute the moon's potential in place of that of the centrifugal force in the usual calculation of the earth's figure by means of Laplace's functions. The result that I obtained was a deformation of 3·45 feet, or 6·90 feet from highest to lowest. Whether this is to be found in text books I do not know.

The first three pages of the paper referred to consequently lose their force. But if the tidal objection can momentarily be waived, I think the remaining portion of the paper contains among other things a forcible argument in favour of liquidity, owing to the accumulation of the heat generated in the central parts by tidal friction during the lengthening of the lunar day in the lapse of ages[3].

This being so I am encouraged to make another attempt to find a way of escape from the tidal argument for rigidity.

The potential at a distance r from the centre of a liquid sphere deformed by the moon's attraction may be expressed by the formula

$$\frac{ga^2}{r} + ga\epsilon\left(\tfrac{1}{3} - \mu^2\right) + \tau\left(\tfrac{1}{3} - \mu^2\right),$$

where a is the earth's mean radius (*i.e.* of the undeformed sphere), ϵ the ellipticity caused by the moon's attraction (a negative quantity), and μ is the cosine of the angle between r and the line joining the centres of the earth and moon, and

$$\tau = \tfrac{3}{2} \times \frac{\text{moon's mass}}{(\text{distance})^3} \times a^2,$$

$$= \tfrac{3}{2}\frac{M}{D^3}a^2.$$

[1] 2nd Ed. § 804. [2] *Ibid.* § 799.

[3] There is an obvious erratum at p. 340 regarding the lunar day. For "The present" read "Suppose the".

Now, if any tide exists, supposing the water to be of mean depth h, and y to be depth of the wave below its mean level, we shall have for the distance from the earth's centre to the surface of the water,

$$r = a\{1 + (\tfrac{1}{3} - \mu^2)\,\epsilon\} + h - y.$$

Substituting this value, we have for the potential of the deformed earth at the surface of the water,

$$g[a\{1 - (\tfrac{1}{3} - \mu^9)\,\epsilon\} - h + y] + ga\epsilon\,(\tfrac{1}{3} - \mu^2) + \tau\,(\tfrac{1}{3} - \mu^2),$$

or

$$g\,(a - h + y) + \tau\,(\tfrac{1}{3} - \mu^2).$$

If we differentiate this with respect to $a\theta$, it will give the horizontal attraction of the deformed earth in the direction away from the moon, which, since $\mu^2 = \cos^2\theta$, will be

$$\frac{\tau}{a}\sin 2\theta.$$

Again, the moon's potential at the same point is

$$\frac{M}{D} - \tau\,(\tfrac{1}{3} - \mu^2).$$

This, when differentiated, gives the moon's attraction on the same particle of water in the same direction as before

$$-\frac{\tau}{a}\sin 2\theta.$$

These two forces being equal and of opposite signs will balance one another, and the water will not be disturbed, and there will be no tide.

It follows from the above—in fact it is almost self-evident—that in order that the argument for rigidity derived from the ocean tides should be complete, the surface of the earth should, if the interior is liquid, be deformed to the exact shape that the moon's attraction would produce in a liquid globe; or, in other words, the mean form of the surface of the solid crust should be, under the attractive force, one of the equipotential surfaces, just as the ocean surface would be one of them, if not interfered with by land. But it seems certain that in nature the solid surface would not so conform, because on the hypothesis of liquidity there must needs be deep depressions, which I have called roots of mountains, answering to the elevations, and extending to many times their height down into the heavier liquid in order to support them. The presence of these roots explains the observed deficiency of gravity beneath elevated tracts. Now seeing that a tide is caused by the local accumulation of liquid by differential horizontal flow, these roots would so break up and deflect the tide wave in the substratum, that the exact form of the tidal spheroid at the mean under-surface of the floating crust would not generally be maintained, just as it is not maintained at the surface of the ocean owing to the deflection caused by coast lines; whence arise the irregularities known as the establishment of ports. The result would be, that the upper surface of the crust would not conform to the rule of the equipotential surfaces, and ocean tides would be possible, for it is highly improbable that the establishments of the tide in the substratum should exactly agree with those of the ocean.

If there is truth in this suggestion, it would seem likely that, where there is an exceptionally large unbroken area of ocean, there would also be a comparatively smooth extent of under-surface to the crust, and the form of the tidal equipotential surfaces might be more nearly preserved both for the subjacent liquid and for the water. In such an area the ocean tides would consequently be small. Now this is the case in the central parts of the Pacific, scarcely any tide being observable at the Sandwich Islands[1].

If however tidal waves derived from the main equatorial tide in the substratum by reflection at the mountain roots were to be propagated into higher latitudes, they would cause undulations every six lunar hours in the crust; and it is a question whether these would not be noticed at observatories.

And first of the effect upon the clock. Suppose the point of suspension of a pendulum to be carried downwards, or upwards, with a mean velocity of b feet in six hours. The average acceleration of the bob with reference to the point of suspension would be $g \mp b/6 \times 60^2$ feet per second. In 12 hours the gain would compensate the loss; each of which would be $b/64$ seconds. For example, if the crust tide from highest to lowest was six feet, the gain or loss in six hours would be about $\frac{1}{10}$th of a second. This would not be noticed.

As regards the levels, the passing wave would scarcely affect the direction of gravity at all, because the direction of gravity would still be towards the centre of the earth. But the supports of instruments would be slightly tilted. To estimate roughly how much, suppose a tidal wave, which when n days old arrives in latitude λ, to be represented by the formula

$$y = \frac{b}{2} \cos \frac{2n\pi}{a\lambda} x;$$

that is, suppose it to be modified in its passage as if it had a height $b/2$ at the equator, and ran up as a simple harmonic wave of n loops to the place in question.

[1] Herschel's *Physical Geography*, § 78.

Then $$\frac{dy}{dx} = -\frac{2n\pi}{a\lambda}\frac{b}{2}\sin\frac{2n\pi}{a\lambda}x.$$

Hence $\frac{2n\pi}{a\lambda}\frac{b}{2}$ will represent the maximum tilting, a being the earth's radius, and b the range of the wave from highest to lowest. For illustration suppose the latitude to be 60°, and the age of the tide two days, and the range of the wave six feet. This would indicate a tilt of 0·35 of a second, which would be noticeable with Mr H. Darwin's pendulum.

Attention has lately been given to small disturbances of the surface, and it was only last week that a short article on "earth movements" by Prof. Milne appeared in *Nature*, in which he tells us of a "tide-like movement of the surface of the earth." He adds that the effect is "too large for a terrain tide produced by lunar attraction." This can only mean that the deformation which has been observed is larger than the deformation which lunar attraction would produce in the earth: because any deformation directly produced by lunar attraction would not be indicated by any effect on Mr Darwin's pendulum, unless the earth was rigid. But disturbances of the surface by deflected tides in the substratum in the manner now suggested *would* be noticeable by means of that pendulum, and they would have a greater rise and fall in comparison with a "terrain tide" caused by lunar attraction, just as the tides on our shores rise and fall much more than would happen in the case of a rigid earth wholly covered with water. In the case of deflected tides as suggested, the tilt would in general be in some direction differing from east and west, always the same at the same locality, but altering from one locality to another. All these points are for observation. The disturbances would need to be disentangled from those arising from barometric changes, and their periodicity made out. If caused in the manner suggested, they ought to alternate every six lunar hours, and to be most marked at spring tides.

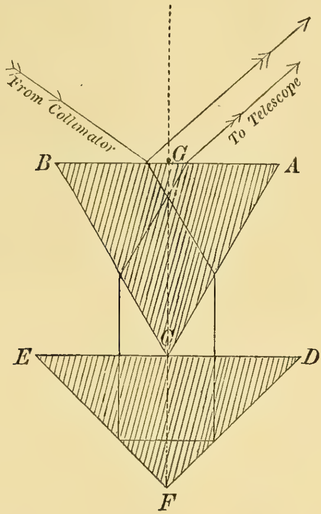
(5) *On a combination of prisms for a stellar spectroscope.* By H. F. Newall, M.A., Trinity College.

The arrangement of prisms, which is the subject of this note, has not so far as I am aware been described before, and its conveniences, for astronomical purposes in particular, are so numerous that I propose to give some details concerning it.

ABC is a strongly dispersing prism; all three faces are accurately worked, and the angles at *A* and *B* are equal.

DEF is an ordinary double total reflexion prism.

These two prisms are relatively fixed as shown in the accompanying section with the faces AB and DE parallel (or



Prisms relatively fixed. Combination capable of being turned round G.

approximately so), the edges of the prisms are also parallel and further the edges C and F lie in a plane GCF which passes through the middle, G, of the face AB and is perpendicular to AB.

The combination thus formed is symmetrical about the plane GCF and is made capable of being turned about an axis which is the intersection of the plane GCF with the face AB. Thus in the figure, the section turns in the plane of the paper round G.

Light from a collimator falls on the face AB as indicated in the figure (the *central* ray however falls on G), and the incidence is suitably adjusted by turning the combination about G; the usual telescope is adjusted to view the spectrum. The light from the collimator suffers deviation and dispersion in passing through the prism of refracting angle A; it is then twice reflected within the reflecting prism and again suffers deviation and dispersion in passing through the prism of refracting angle B.

The spectrum seen with the telescope is therefore similar to what would be seen in a two-prism spectroscope whose prisms were of the same material as that used for the prism ABC and had angles equal to A and B respectively.

The telescope receives, when it and the prism-combination are properly adjusted, not only the light which has passed *through* the combination but also the light reflected from the surface AB at primitive incidence. The reflected light gives rise to a simple image of the slit, which appears to be superposed upon the spectrum; for ease in description and for an obvious reason this will be called the "pointer." If the prism-combination is turned through a small angle $\delta\theta$, the pointer moves through the angle $2\delta\theta$ but the spectrum moves through a much smaller angle, whose magnitude varies with the part of the spectrum considered. Hence the pointer can be made to coincide with any line in the spectrum and its change of position is known in terms of the corresponding change of position of the prism-combination. If therefore a suitable micrometer movement is used to move the prisms, the position of the pointer may be read off the micrometer.

The line in the spectrum which coincides with the pointer is always that which is due to rays which have passed symmetrically through the prism combination. The movement of the prism gives symmetrical passage to different rays in turn and the pointer indicates which ray has passed symmetrically.

The course of rays which pass symmetrically through the prism-combination is shown in the figure; such rays emerge from the face AC in a direction parallel to the plane of symmetry, and consequently the deviation in passing through the prism A is equal to the angle of primitive incidence:

$$D = \phi = \phi + \psi - A,$$

whence $A = \psi$, or the angle of emergence from the face AC is equal to the angle A. The angle of emergence can be made, under readily calculable conditions, equal to the angle of primitive incidence, but generally only with rigid accuracy for rays of one refrangibility; hence the rays which pass symmetrically will not in general pass with *absolute* minimum deviation.

The 'pointer' may be considered as being connected with the prism, and independent of the observing telescope. It is thus attached, so to speak, to the strongest part of the instrument instead of the weakest, where the micrometer is usually placed, namely the eye end of the telescope. The telescope is used merely as a magnifier. The need for carefully worked surfaces for the prisms forms perhaps the strongest objection to the use of this combination; curvature in any one surface of either prism must throw the points and the spectrum into different focal planes in the observing telescope and so introduce parallax difficulties which can only be eliminated by reworking the faulty surface. Two prisms which I have in my possession and which were worked by Hilger, have been used to test the capabilities of the combination, and give excellent results.

The observing telescope is pointed towards G and is mounted so as to turn about the same axis through G as that about which the prisms turn; this single motion is enough to bring every part of the spectrum in turn into view. Thus the advantages of a two-prism spectroscope are obtained without the disadvantages arising from the usual double adjustment necessary in directing the observing telescope. The combination which I describe may therefore replace a grating, in a diffraction spectroscope.

If a bright star is observed, both spectrum and pointer are bright; for a faint star, the brightness of the pointer is appropriately subdued. The fact that the brightness of the pointer maintains a suitable proportion to the brightness of the spectrum to be investigated is a great convenience.

In astronomical work, the object-glass of an equatorial is used to throw an image of the star, whose spectrum is to be studied, on the slit of the spectroscope. If the slit is widened, the image of the star itself is seen in place of the pointer. This is a great convenience in as much as in most cases the star may be thus identified. When the star is recognized amongst its neighbours, the slit is closed to a suitable width and the 'pointer' then appears as a short narrow line in the spectrum. In practice it is preferable to have the pointer not actually superposed on the spectrum, but displaced so as to be a little above or below the spectrum. This end may be attained by slightly tilting the reflecting prism.

February 12, 1894.

PROF. T. MCKENNY HUGHES, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society.

(1) *On a suggested case of Mimicry in the Mollusca.* By A. H. COOKE, M.A., King's College.

The species concerned were *Strombus mauritianus* L., and *S. luhuanus* L., the shells of which differed from those of all other *Strombus* in their close resemblance to the shell of *Conus*, a genus with which they are known to live: *Strombus* being a frugivorous animal with small and weak teeth, and *Conus* on the other hand being carnivorous, with very large and barbed teeth, provided with a poison bag and duct. It was suggested that this resemblance must tend greatly to the advantage of the *Strombus*, since the dangerous properties of *Conus* would tend to prevent its being touched by predatory fishes.

(2) *On the evidence as to the extent of Earth movements and its bearing upon the question of the cause of glacial conditions.* By the PRESIDENT.

[Publication deferred till next number.]

(3) *On the Fertilisation of some Species of* Medicago *L. in England. By* I. H. BURKILL, B.A., Gonville and Caius College.

Historical. The explosive mechanism found in the flower of *Medicago* has long been known. A. P. de Candolle[1] considered it an outcome of the maturity of the flower whereby the stigma was spontaneously pollinated. Hildebrand[2] was the first to show how it had for an object the fertilisation of the flower by insects. In the succeeding year Delpino[3] and G. Henslow[4] both published papers on the subject. Delpino advanced our knowledge by locating the explosive force, and Henslow noted that the Hive bee (*Apis mellifica* L.) does not explode the flower; at the same time he communicated an observation made by Charles Darwin on the fertility of *M. lupulina* in the absence of insect-visitors. Urban and Hermann Müller have also studied this genus. Urban[5] attributes the work of fertilisation to bees, and at variance with Hildebrand states that the flower of *M. sativa* is not fertile if allowed to remain unexploded. And in the year in which the last was published appeared Müller's description of the flower; to him we owe lists of visitors to three species.[6]

The species here concerned all belong to Seringe's[7] section Lupularia, or to Urban's[8] two sections Lupularia and Falcago. Urban following Doll[9] has united, not without some justification, under the name of *M. sativa* three species—*M. sativa* L., *M. fal-*

[1] Physiologie, II. p. 548. Paris, 1832. 'Certaines corolles contribuent même d'une manière indirecte à la fécondation :...lorsque leur développement s'achève, ces crochets (referring to the processes of the alae in Indigofera and Medicago) se détachent, la carène, n'étant plus fixée, se déjette avec élasticité, et imprime aux faisceaux des étamines une secousse qui détermine la chute du pollen.'

[2] Ueber d. Vorrichtungen an einigen Blüthen zur Befruchtung durch Insektenhülfe. Bot. Zeitung, XXIV. p. 71, 1866.

[3] Sugli Apparecchi della Fecondazione nelle Piante antocarpee (Fanerogame). Firenze 1867. Reviewed by Hildebrand. Bot. Zeitung, XXV. p. 283, 1867.

[4] Notes on the structure of *Medicago sativa* as apparently affording facilities for the intercrossing of distinct species. Journ. Linn. Soc. (Bot.) IX. p. 327, 1867, and Notes on the structure of Indigofera...with additional notice of Dr Hildebrand's paper. Journ. Linn. Soc. (Bot.), IX. p. 355, 1867.

[5] Prodromus einer Monographie d. Gattung Medicago L. Verhandl. d. Bot. Vereins d. Provinz Brandenburg, XV. p. 1, 1873.

[6] (1) Die Befruchtung d. Blumen durch Insekten, Leipzig 1873, p. 225. (2) Weitere Beobachtungen ü. Befruchtung d. Blumen durch Insekten. Verhandl. d. Naturhist. Vereins d. preuss. Rheinlands u. Westfalens, XXXVI. p. 252, 1879. (3) Alpenblumen, Leipzig, 1881, p. 248. (4) The Fertilisation of Flowers translated by D'Arcy Thompson, London, 1883, p. 175 (no additions here to the German of 1873).

[7] De Candolle's Prodromus, II. p. 172, 1825.

[8] *Loc. cit.*

[9] Rheinische Flora. Frankfurt, 1843, p. 802.

cata L. and *M. silvestris* Fries; but here I use *M. sativa* to signify the segregate and not the aggregate species.

M. sativa L.

Mechanism. The flower is typically Papilionaceous, always (and in this respect very different from *M. maculata* Sibth.) with the vexillum standing above and the carina below. The carina and alae are firmly united together by the usual combining processes at 5 mm. from the base of the flower. The total length of the alae averages about 9 mm., and of this 4 mm. belongs to the claw. Further each ala possesses a basal-process 2 mm. long; and these basal processes of the two alae project towards the base of the flower, lying almost parallel above the stamens.

The stamens are diadelphous and on either side of the upper free stamen at the base of the flower lies a passage—1 mm. long—giving access to the honey secreted inside the staminal tube. Beyond these passages the edges of the free stamen do not merely touch those of the other fused nine, but fit into grooves along their margin. This is plainly seen in a transverse section of the sexual organs. The explosive action of the flower depends on the uppermost stamens of the fused nine, as Delpino first and Müller afterwards shewed; these by having the cells of their filaments intensely turgid tend to make the whole staminal tube assume a curved form, whereby stigma and anthers are forced against the vexillum. This explosive force is resisted by the paired combining processes of the alae and carina, and according to my observation *not* by the basal processes of the alae: for working carefully I have at various times and under various atmospheric conditions found it quite possible not only to remove both these basal processes, but further to cut the claws of both alae; and in consequence of this no other resistance remained to prevent the flying up of the staminal tube, except that of the combining processes. Further these combining processes are fixed exactly where required:—the sexual organs extend to a length of 8 mm.; after explosion there is no curvature, observed in the basal 2 mm.[1] or in the terminal 3 mm., but the bending is greatest at a point about 4 mm. from the base—or just short of where the combining processes press on the stamens. I see in the basal processes nothing but two triggers by which the flower is, as it were, fired-off.

The surface of the alae on both sides is covered with papillae affording a good foot-hold to any insect visitor. The carina is perfectly smooth, but on the inner surface of the vexillum along

[1] Therefore the nectar-passages cannot be of any use in allowing, as Henslow suggests, a free curvature of the staminal tube. Note on the structure of Medicago sativa. Journ. Linn. Soc. (Bot.) IX. p. 329, 1867.

the margin is a band of papillate epidermis, broadest at the apex and narrowing downwards; this too may afford a foot-hold to any long-legged insect. The centre and base of the vexillum are perfectly smooth.

Pollen is shed in the bud, and lies round the stamens and stigma in a little lens-shaped space made by the carina.

The stigma is at this time, as Henslow says of *M. denticulata*, mature[1]; but I have covered a considerable number of flowers with nets to prevent insect-visits and get results agreeing with those of Urban[2], i.e. no seeds are set in the unexploded flower in spite of the pollen in contact with the stigma. This is explained by the fact that the stigma does not become receptive until rubbed or until its cells are injured in some manner. My proof is I think conclusive. Firstly, the stigma appears not to be moist, but when rubbed on glass leaves a sticky mark. Secondly, I have caused flowers to set seed though unexploded, (1) by pinching the stigma through the keel, (2) by perforating the keel with a needle and scratching the stigma, and (3) by cutting off the tip of the keel and rubbing the stigma with a stiff paint-brush. An insect visitor exploding the flower will injure the stigmatic papillae and bring about fertilisation[3]. There appears to be a prepotency in foreign pollen, to judge from Urban's observations[4] on the hybridisation of *M. sativa* and *falcata*. Patches of these two plants were grown by him close together; and amongst the seedlings derived from them only two from *M. sativa* and none from *M. falcata* proved true; the rest being the hybrid form *M. media*.

Insect visitors, observed in and near Cambridge.

Hymenoptera aculeata:—

1. *Apis mellifica* L. ☿ very abundant.
2. *Bombus pratorum* L.
3. *B. lapidarius* L.
4. *B. hortorum* L. not uncommon.
5. *B. muscorum* L.
6. *Megachile centuncularis* L. ♀
7. *Vespa vulgaris* L. ☿.
8. *Andraena convexiuscula* Kirby ♀.
9. *A extricata* Smith ♂.

[1] Henslow. Self-fertilisation of plants. Popular Science Review, New Series, III. p. 13, 1879.

[2] *Loc. cit.*

[3] Seringe's variety *tumida* of *M. falcata*, which appears from the description as if it might be cleistogamic, is a gall—the work of a *Cecidomyia*, whose larvæ mature inside the hypertrophied bud. The gall is common on *M. sativa* at Cambridge. De Candolle's Prodromus, II. p. 173.

[4] Verhandlungen d. Bot. Vereins d. Provinz Brandenburg, XIX. p. 125, 1877. I endeavoured by timing the rate of withering of cross and self-fertilised flowers, to obtain some evidence on the question of prepotency, but failed to find anything satisfactory. Only the younger flowers withered more rapidly than the older ones.

Lepidoptera:—Rhopalocera:—

10. *Pieris brassicae* L. abundant. 11. *P. rapae* L.
12. *P.* napi L. 13. *Vanessa urticae* L.
14. *Polyommatus phloeas* L. 15. *Lycæna icarus* L.

Heterocera:—

16. *Triphaena pronuba* L. 17. *Plusia gamma* L.
18. *Strenia clathrata* L.

Coleoptera:—19. *Meligethes viridescens* F.
Diptera:—Syrphidae:—

20. *Platychirus manicatus* Mg. 21. *P. albimanus* F.
22. *P. scutatus* Mg. 23. *Syrphus balteatus* Deg.
24. *S. corollae* F. 25. *S. ribesii* L.
26. *Sphaerophorea scripta* L. 27. *Erystalis pertinax* Scop.
28. *Myiatropa florea* L. 29. *Syritta pipiens* L.

Muscinae:—30. *Lucelia sericata* Mg.
Anthomyidae:—31. *Caricea tigrina* F.

Act of Fertilisation. Of these insects all seek honey, but the flies seem rarely to obtain it and cannot obtain pollen while the flower is unexploded. As Müller and Henslow have already observed, the Hive bee does not explode the flower, but inserts its proboscis obliquely over the basal processes and not between them. It is necessary to produce explosion for an insect to insert its proboscis between these basal processes;—any outward movement imparted to them is imparted to the combining processes, and the flower explodes. Müller attributed the work of fertilisation to butterflies, but never saw the act. Urban disqualifies these insects on account of the flexibility of their probosces; and my observations lead to the same view. I have watched several hundred separate butterfly visits, often at very close distances, and have never seen these insects cause any explosion. *Bombus* generally sucks like *Apis*, sometimes visiting the flower first on one side, and then on the other, like a *Dicentra* flower. But on one very hot afternoon I observed B. hortorum in the act of exploding the flowers in great numbers, and on two occasions I have seen Apis deliberately exploding the flowers. I have not seen the calyx bitten through by Bombus, but such has been recorded [1].

When the flower is exploded the stamens free themselves with a jerk which scatters the pollen [2], some of which will adhere to the insect and some may fall into neighbouring flowers. If the stigma strikes the insect's body but obtains no pollen, there still

[1] Schulz, A. Beiträge zur Kenntniss d. Bestäubungseinrichtungen u. Geschlechtsvertheilung bei den Pflanzen. Cassel, Heft II. p. 208, 1890.
[2] Cf. De Candolle, Physiologie, *loc. cit.*

remains sufficient in the surrounding anthers to ensure self-fertilisation.

Further, if the stigma fails to strike against the insect-visitor, the impact on the vexillum is sufficient in many cases to render the flower fertile. For if the flower be exploded by simply pulling the alae apart it may set seed. The stigma strikes against the smooth and *not* the rough part of the vexillum. In order to make certain that this was due to the vexillum and not to the edges of the keel, I removed in above fifty flowers the lamina of the vexillum and then exploded them; out of these none set seed, while out of 34 in which the vexillum remained, exploded at the same time as a control experiment, 12 set seed. Therefore should the insect visitor fail to bring about fertilisation, impact on the vexillum in some extent (here 35°/₀) ensures it.

That the separation of the basal processes is the legitimate and almost only natural method of exploding the flower is obvious from the following consideration. By means of a fine wire hung on to the alae, weights to a known extent were suspended from them. In September 1892 flowers obtained near Poulton (Gloucestershire) were found to explode with an average weight of 1·68 grammes (maximum and minimum 2·37 and ·93). Now an insect visiting the flower rests its weight on the points whence these weights were hung. The worker of Apis I find to weigh about ·096 and Bombus hortorum (large specimens) ·199 grammes. The mere weight of these two insects is therefore quite insufficient to explode the flower. Moreover the pedicel of the flower bends under a weight insufficient to explode the flower, so that in these experiments I found it necessary always to fix the flower by a wire hooked into the standard; and again the Hive bee so settles as to hold the parts of the flower together with its feet.

By the same method of experiment I discovered that the flower is not always in the same degree of explosiveness; the hotter the weather the more explosive is the flower. In cold weather the flower frequently remains unexploded for eight or nine days, after which it withers, but in hot sunny weather I found three days to be the maximum duration; for explosion is brought about,—often within 24 hours from the opening of the bud. We must remember in this connection that *M. sativa* is of Persian origin and has only traversed Europe northwards by slow degrees[1].

Shaking by the wind cannot explode the flowers. Pieces of paper with a surface of 18½ and 22 square inches were tied to stalks of this plant, in order to give more power to the wind, but no effect was observable from the shaking it produced.

[1] Alph. De Candolle. Origin of Cultivated Plants. London, 1884, p. 103.

The flowers of *M. sativa* exhibit no sleep movements; and *Plusia gamma,* which I have observed visiting the flowers at 9 p.m., may be taken as evidence of night visitors. My hours of observation were between 6 a.m. and 9.15 p.m.

M. falcata L.

I find in this flower nothing but a yellow image of *M. sativa,* with a distinct scent and in which, as Müller points out, the flowers become more explosive. This greater explosiveness is determined by warmth as in *M. sativa,* but is carried to such an extent that the settling of an insect must explode it, for the combining processes allow the stamens half to escape. The basal-processes then no longer touch each other, and some *Syrphidae* or even *Anthomyidae* may reach the honey at the risk of an explosion. Heating a flower brings on this very explosive stage. In the very explosive state a shower of rain (as may easily be demonstrated with a watering-can) causes explosion.

Before the flower passes into this state it is impossible for any common British insect (except perhaps some moths) to explode the flower by merely resting on the alae.

From plants grown in the Cambridge Botanic Gardens I obtained the following figures, using wire weights as given above. Average weight required to explode the flower 1·48 grammes, maximum and minimum 2·46 and ·60 grammes.

The duration of the flower is slightly less than that of *M. sativa.*

Insect-visitors. My list of visitors is small, the plants not flowering well.

Hymenoptera aculeata:—1. *Apis mellifica* L ☿. 2. *Bombus hortorum* L. 3. *Formica rufa* L. Terebrantia:—4. *Cryptus analis* Gr.?

Diptera:—Syrphidae:—5. *Syrphus balteatus* Deg. 6. *S. luniger* Mg. 7. *Syritta pipiens* L.

M. prostrata Jacq.

This flower is like the last in mechanism, but is smaller and explodes when weights are hung on the alae more readily. Average ·53, extremes ·81 and ·33 grammes.

M. silvestris Fries.

The flowers are green, and their colour acts on a photographic plate only in the same degree as the leaves of the plant. Urban[1] has placed this plant under *M. falcata,* but, if the method of ferti-

[1] Prodromus, *loc. cit.* p. 56.

lisation is any criterion, it is nearer to *M. sativa*; for it never enters the 'very-explosive state' of *M. falcata*. The Hive-bee was an extremely abundant visitor, almost enjoying a monopoly. The flowers are scented and nearly the whole of an inflorescence is in blossom at the same time. But, in spite of the abundant visitors, 99°/₀ of the flowers, after remaining open for 8 or 9 days, withered unfertilized. The bees sucked the flowers from one side as in *M. sativa*. Anxious to make out what particular attraction this plant offered to the bees (these being ten times as abundant on it as on a similar neighbouring plant of *M. sativa*), I visited the Botanic Gardens on several occasions at 6 a.m. and observed the first bees return to work after the night. Many plants compete for the bees' attention on the 'Leguminosae' bed, but, except on a single occasion when one bee repeatedly tried to force open some closed *Melilotus* flowers, *M. silvestris* not only received the first, but also the greatest number of visits. Experiments were then made to see if the buzzing of one bee will attract others to the plant on which it is buzzing. Bees were imprisoned in flowerless plants in the neighbourhood of the patch of *M. silvestris*, where they were allowed to buzz freely. The results were partly affirmative. Evidently the bees had learnt by experience that this medick offers abundant honey.

The list of insect visitors bears out the fact that it is the flowers which the inconstant butterflies fertilise, that need the brilliant colours: *M. silvestris* certainly was not freely visited by Lepidoptera.

Insect-visitors in the Cambridge Botanic Gardens, none of which were seen exploding the flower.

Hymenoptera aculeata:— 1. *Apis mellifica* L ☿ very abundant; 2. *Bombus hortorum* L. 3. *B. lucorum* L. 4. *Odynerus parietum* L ♀.

Lepidoptera:—Rhopalocera:—5. *Pieris brassicae* L.

Diptera:— Syrphidae:— 6. *Platychirus manicatus* Mg. 7. *Syrphus balteatus* Deg. 8. *S. luniger* Mg. 9. *S. corollae* F. 10. *S. ribesii* L. 11. *Sphaerophorea scripta* L. 12. *Erystalis pertinax* Scop. 13. *Syritta pipiens* L. Sarcophagidae:— 14. *Sarcophaga carnaria* L. Muscinae:—15. *Lucelia sericata* Mg. Anthomyidae:—16. *Caricea tigrina* F.

M. lupulina L.

Mechanism. The explosive arrangements in this little flower are just as in *M. sativa*, but the tension is much less, the stigma being only just brought up to the vexillum in explosion. The flowers are small and massed together. The rough areas on the petals are similar in position and shape to those on *M. sativa*. As a rule the flower closes about 5 p.m.; but, while investigating the fertilisation of this plant, I noticed that, at Scarborough, there were certain plants growing with the others, of more erect habit, which did not close their flowers at night, or only partially so. This variation I have under study.

The stigma when young requires rubbing to render it fertile, but as it gets older it becomes receptive. Hence plants covered with a net set seed, as Darwin[1] pointed out. Repeating the experiment I get the same result: while in the open 95% of the flowers set seed: under a net about 75% did so, the seed of which germinated well.

Insect visitors at Scarborough (Sc.) between June 22nd and July 3rd, 1893, and at Cambridge (Camb.) through July and August.

Hymenoptera aculeata:—1. *Apis mellifica* L ☿ (Sc. and Camb.) seldom. 2. *Bombus hortorum* L. (Sc.). 3. *Andraena parvula* Kirby. ♀ (Sc.). 4. *Halictus morio* F. ♂ ♀ (Camb.) not unfrequent. 5. *H. minutissimus* Kirby ♂ (Camb.). Terebrantia. 6—11. Six species undetermined (Sc.)

Lepidoptera:—Heterocera:— 12. *Miana fasciuncula* Haw. (Sc.). Tortrices:—13. *Tortrix* sp.? (Camb.). 14. *Tortrix* sp.? (Sc.). Crambi:—15. *Crambus pratellus* L. (Sc.). Pyralides:— 16. *Porrectaria* sp.? (Camb).

Coleoptera:— 17. *Anthobium torquatum* Marsh (Sc.). 18. *Meligethes aeneus* F. (Sc.). 19. *Ceuthorhynchidius floralis* Payk. (Sc.).

Diptera:—Bibionidae:—20 *Scatopse brevicornis* Mg. (Sc. and Camb). Cheironomidae:—21. *Cheironomus* sp. (Sc.). Tabanidae:—22. *Symphoromyia crassicornis* Pz. (Sc.). Empidae:— 23. *Empis punctata* Mg. (Sc.). Syrphidae:—24. *Paragus tibialis* Fln. (Camb.). 25. *Pipizella virens* F. (Camb.). 26. *Platychirus manicatus* Mg. (Sc.). 27. *P. albimanus* F. (Sc.). 28. *P. scutatus* Mg. (Camb.). 29. *Syrphus balteatus* Deg. (Camb.). 30. *S. corollae* F. (Sc. and Camb.). 31. *Syritta pipiens* L. (Camb.). Tachinidae:—32. *Myobia inanis* Fln. (Camb.). 33. *Siphona geniculata* Deg. (Camb.). 34. *S. cristata* F. (Camb.). Sarcophagidae:— 35. *Sarcophaga* sp. (Camb.) very abundant. Anthomyidae:—36. *Hydrotea irritans* Fln. (Sc.) 37. *Pogonomyia alpicola* Rud.? (Sc.). 38. *Hylemyia pullula* Ztt. (Sc. and Camb.). 39. *Anthomyia* sp. (Camb.). 40. *Chortophila cinerella* Fln. (Camb.). 41. *C. sepitorum* Meade. (Camb.). 42. *Chortophila* sp. (Sc.). 43. *Homolomyia armata* Mg.? (Sc.). 44. *Caricea tigrina* F. (Sc. and Camb.). Scatophagidae:—45. *Scatophaga stercoraria* L. (Sc.). Sepsidae:—46. *Sepsis cynipsea* L. (Camb.). 47. *Hydrellia griseola* Fln. (Camb.). Chloropidae:—48, 49, 50. *Chlorops* 3 sp. (Camb.). 51. *Ossinis* sp.?

[1] Henslow. On the Structure of Medicago. J. Linn. Soc. (Bot.) IX. p. 328, 1867.

(Sc. and Camb.) very abundant. Muscinae:—52 and 53, 2 species not identified (Sc.).

Hemiptera:—54. *Siphonophora artemisiae* Koch (Camb.). 56. *Aphis* sp. (Sc.).

Neuroptera:—56. *Thrips* sp. (Sc.).

Besides these insects *Heterocordylus* sp. juv. (Hemiptera) and *Homalonotus aeneicollis* Sharp (Coleoptera) were frequently observed running about the spikes of flowers, and may occasionally explode the flowers with their legs.

Fertilisation. This is my list of visitors to a plant which Henslow quotes as being a widely distributed self-fertilising flower[1].

By finding the average rate of withering of a flower after explosion and consequent fertilisation, I was able in June 1893 at Scarborough to make the following calculation:—At times when the flowers required 2 to 5 five hours to wither after explosion (fine warm weather), the following percentages of open flowers were found to be exploded: 29%, 24%, 4%, 8%, 18%—the average being 18%. Taking the average rate of withering to be 4 hours, then 4·5% of the flowers in blossom were exploded every hour, and in such weather the average duration of 22 hours (nearly 2 days of 12 hours) should be sufficient to permit of the explosion of every flower. Such about is their duration, but they ensure themselves against accidents by becoming self-fertilised.

Now comes the question—what insects do the work of fertilisation? Müller[2] in Germany and Knuth[3] on the Friesian islands attribute this work to the Hive-bee: it is not so in England as far as my observations extend.

At Scarborough *Platychirus manicatus*, and at Cambridge *Halictus morio* and *Scatophaga*, were the most efficacious visitors. The flowers were watched at all hours of the day from 4 a.m. to 10 p.m.

Of these insects which I have seen on the flower nearly all are capable of causing explosion. I have seen *Scatopse brevicornis* both at Scarborough and Cambridge, and a *Cryptus* at Scarborough, entrapped by the explosion of stamens of the flower, shewing that these—almost the least insects in this list—were just strong enough to effect it.

In order to see if those flowers which remained open all night received visitors, spikes were marked and flowers on them found exploded in the early morning; and further, in the twilight (9.30 and 9.35 p.m.) I have twice seen a beetle settle on the heads and apparently seek for honey.

[1] On the Self-fertilisation of Plants. Trans. Linn. Soc. Series II. Vol. I. (Bot.) p. 392.

[2] Fertilisation of Flowers, p. 180.

[3] Blumen u. Insekten auf den Nordfriesischen Inseln, Kiel, 1894.

Winds, except by knocking spike against spike, are not likely to explode the flower, as only very violent shaking effects it.

Conclusion. Comparing the number of species in these lists with those given by Müller, we get the following table:—

	Hymenoptera				Lepidoptera	Diptera		Coleoptera		
	Apis	Bombus	Other Hymenoptera Aculeata	Terebrantia		Syrphidae Bombylidae Conopidae	Other Diptera[1]		Other Insects	Total no. of Species
M. sativa										
Germany	1	1	12	—	13	—	—	—	—	27
England	1	4	4	—	9	10	2	1	—	31
M. falcata										
Germany	1	2	4	—	9	2	—	—	—	23
England	1	1	1	1	—	3	—	—	—	7
M. lupulina										
Germany	1	2	3	—	4	2	—	—	—	12
England	1	1	3	6	5	8	26	3	3	56

From this it appears *not unlikely,* as might be expected, that flies take in England the place which other more specialised insects occupy in Germany[2].

The second table indicates the number of individuals observed to visit the flowers. The Hive-bees were too numerous and too assiduous to permit of accurate counting for *M. sativa* and *M. sylvestris*; they were therefore estimated in these two cases. It is impossible to state the number of individual flowers visited.

The last two lines show that, except for *Halictus,* there is but little difference between Cambridge and Scarborough in the relative proportions of the insect visitors to *M. lupulina.*

Finally, the mechanism of the flower of *Medicago* seems nearer to that of *Trifolium* than of *Melilotus* ; for the basal processes do not press on the staminal tube. The combining processes possess a new function—a necessary consequence of the new explosive property of stamens, which we do not find in other genera of Trifolieae. With regard to the stigma, it is such an easy transition from the normal glandular form to one in which the secretion is hindered and receptivity only brought about by injury, that we

[1] This contains all the 'allotropous' or short-tongued flies. Cf. E. Loew, Beiträge zur blüthenbiologischen Statistik. Verhandl. d. Bot. Vereins d. Provinz Brandenburg, xxxi. 1889, p. 1.

[2] Further observations made with Mr J. C. Willis near Plinlimmon and by myself at Scarborough support this; as also do Scott Elliot's observations in South Scotland, Flora of Dumfrieshire, Parts I. and II., Trans. and Journal of Proceedings of Dumfriesshire and Galloway Nat. Hist. and Antiquarian Soc. 1891—92.

might expect such to arise in any family of the Leguminosae. The

	Hymenoptera				Lepidoptera			Diptera						
	Apis	Bombus	Other Aculeate Hymenoptera	Terebrantia	Rhopalocera	Heterocera	Crambi Pyralides etc.	Syrphidae[1]	Other Diptera	Coleoptera	Hemiptera	Orthoptera	Total no. of Individuals	
M. sativa	357	29	8	—	75	3	1	32	4	1	—	—	510	during $31\frac{1}{4}$ hours
M. falcata	14	6	3	1	—	—	—	8	1	—	—	1	34	during $21\frac{1}{2}$ hours
M. silvestris	612	6	—	—	2	—	—	41	19	—	—	—	680	during $22\frac{3}{4}$ hours
M. lupulina	9	1	19	11	—	1	7	64	193	5	5	3	318	during 41 hours
do. at Cambridge	4	—	18	2	—	—	2	15	69	—	1	1	112	during 11 hours
at Scarborough	5	1	1	9	—	1	5	49	124	5	4	2	206	during 30 hours

stigmas of Cyclopia[2] in the Podalyrieae, Vicia[3] and Lathyrus[4] in

[1] Conopidae and Bombylidae do not occur in my lists.
[2] Scott Elliot, On the Fertilisation of South African and Madagascan Flowering Plants. Annals of Bot. v. p. 341, 1891.
[3] Müller, Fertilisation of Flowers, p. 203. Weitere Beobachtungen, p. 259.
[4] Müller, Fertilisation of Flowers, p. 209. Knuth, Vergleichende Beobachtungen ü. d. Insekten-besuch an Pflanzen d. Sylter Haide u. d. Schleswigschen Festlandshaide. Bot. Jaarboek, IV. p. 26, 1892.

the Vicieae—Lotus[1] and Anthyllis[2] in the Loteae, Tephrosia[3], Oxytropis[4] and Astragalus[5] in the Galegeae—for all of which such a mechanism has been suggested—and of Medicago in the Trifolieae may equally be outcomes of a parallel modification; but on the other hand it is very probable that many similar stigmas exist among the plants of this extensive order.

To Dr D. Sharp, Mr G. H. Verrall and Mr J. C. Willis I owe very many thanks for invaluable help in naming insects or in making kind suggestions, and to Mr R. I. Lynch for many services.

(4) *Contributions to the Geology of the Gosau Beds of the Austrian Salzkammergut.* By H. KYNASTON, B.A., King's College, Cambridge.

(*Abstract.*)

This paper, after treating of the previous literature of the subject, gives an account of the situation and physical aspects of the Gosau Valley. The valley is situated in the central portion of the Salzkammergut, in Upper Austria, and forms a more or less trough-like depression to the West of Hallstatt Lake. The Gosau beds, while they may be said to be typically developed in the Gosau Valley, have on the whole a fairly extensive distribution in the Eastern Alps, and chiefly on the Northern flanks of the chain. They nearly always occur in basin-shaped or trough-shaped areas in the Alpine limestone, of which the Gosau Thal may be taken as a fair type, or in small, narrow, loftily situated valleys, like that of Zlam, near Aussee in Steiermark. Except at Obersiegersdorf, S.E. of the Chiem See in Bavaria, they always occur flanking the Northern Zone of Alpine limestone, and they never encroach on the central axial portion of the chain. Everywhere they occur in the form of isolated outliers, resting unconformably on the older Alpine Trias, and they are never associated with either younger or older cretaceous beds, except at Ruhpolting, W. of Salzburg, where they are exposed in a section resting on beds of the age of the Gault. The beds of Gosau Thal are not confined to that valley; but, constituting as they do the whole of the hills on its Western side from the Zwiesel Alp on the South to the Southern slopes of the Russberg on the North, they are continued into the adjoining valley of Russbachthal as far as a mile or so S.W. of Russbachsag, and

[1] Delpino, *loc. cit.* p. 25. Müller, Fertilisation of Flowers, p. 170.

[2] Müller, Fertilisation of Flowers, p. 173.

[3] Robertson, "Flowers and Insects," No. IV. Bot. Gazette, XV. p. 84, 1890.

[4] Müller, Alpenblumen, p. 233. E. Loew, Ueber d. Bestäubungseinrichtungen u. d. anatomischen Bau d. Blüthe von Oxytropis pilosa D. C. *Flora* LXXIV. p. 86, 1891.

[5] Heinsius. Eenige waarnemingen en eschouwingen over de bestuiving van bloemen der Nederlandsche flora door insecten. Bot. Jaarboek, IV. p. 54, 1892.

they also extend up the tributary valley of the Randaubach as far as Neu Alp, between the Gamsfeld and the Hohe Platten. The beds themselves consist of conglomerates, thin limestones, bluish and greyish marls, bluish sandstones and flaggy beds, and red and grey sandy marls. The average dip of the series is South, and varies from almost horizontal up to an angle of about 50°. The beds of Russbachthal are more disturbed than those of Gosau Thal. The total thickness of the group probably does not fall far short of 3000 feet. The following classification was found to be the most convenient:—

Upper Gosau Beds.

1. *Grey, red, and variegated sandy marls*, here and there, especially towards the upper part, alternating with sandstones, grits, and conglomerates. Well seen on the sides of the Horn, Hennarkogl, and Zwiesel Alp.
2. *Grey sandstones and flags, with some sandy shales*, with obscure plant remains, worm-tracks, and ripple-marks. Well seen on sides of Ressenberg, and below Bibereck Alp.

Lower Gosau Beds.

3. *Bluish-grey marls*, with some limestones; very fossiliferous. Hippurites organisans, Reef-building Corals, Gasteropods, Lamellibrauchs, &c.
4. *Estuarine beds* of Neu Alp.
5.
   - *a.* *Limestone* with Nerinæa—Traunwand, Neu Alp.
   - *b.* ,, ,, Actæonella—Traunwand.
   - *c.* ,, ,, Hippurites cornuvaccinum—Traunwand, &c.
   - *d.* *Coarse conglomerate*, forming basement bed, sometimes alternating with grits, sandstones, and marls.

The Gosau beds of this district bear a marked correspondence, both in lithological and palæontological characteristics, with beds of the same age in other localities in the Eastern Alps, such as Neue Welt near Wiener Neustadt, Hieflau, Gamsthal, Zlam, &c.

The Lower Gosau beds contain a remarkably rich and varied fauna; the upper beds, however, in the Gosau Valley are monotonous and barren, the only organic remains present being represented by obscure worm-tracks and vegetable remains. The fossils of the lower beds have been described by Reuss, Zittel, Zekeli, Stoliczka, and Von Hauer.

Most of the fossils occur in the fossiliferous marl series, while a good many are found in the Estuarine group and the limestone beds below it. Hippurites are extremely abundant, and build up great banks or reefs of hard limestone. Hipp. cornuvaccinum is the commonest and most characteristic species. Reef-building Corals are extremely plentiful, especially at Nefgraben. It is rather curious to notice the almost entire absence of Echinoderms, which are such a characteristic feature of the Upper Cretaceous rocks of North-western Europe; also the great scarcity of Cephalopods and Brachiopods, while Lamellibranchs and Gasteropods are extremely abundant.

As regards the exact geological horizon of these beds, no very precise conclusions can be drawn on purely stratigraphical, much less on lithological, grounds. But for information on this point we must turn to the study of the included organic remains.

We evidently have in the Gosau district two distinct Hippurite limestones, a lower one characterised essentially by Hippurites cornuvaccinum, and an upper characterised by Hipp. organisans.

Now Toucas (see "Synchronisme des Étages Turonien, Senonien, et Danien, dans le nord et dans le midi de l'Europe," *Bull. Soc. Géol. Franç.*, Ser. 3, vol. x., pp. 200—202) clearly recognises two distinct Hippurite zones in the South of France, the lowermost of which, viz. the zone of Hipp. cornuvaccinum, he places at the top of the Turonian system, while the uppermost, which contains Hipp. organisans, he places at the top of the Senonian. Comparing the Upper Cretaceous fossils of the South of France with those of the Gosau beds, by far the larger proportion of the Gosau forms are found to be Senonian, while some Turonian forms also occur. Toucas concludes that the Gosau beds represent the whole of the Senonian of the South of France: but, since it is evident that we have in the Gosau district the zone of Hipp. cornuvaccinum, which, according to Toucas, is Turonian, the author of the present paper feels quite justified in saying that these Gosau beds are represented in the South of France by Upper Cretaceous rocks from the zone of Hipp. cornuvaccinum to the zone of Belemnitella inclusive. Comparing these zones with those of the Paris basin, and these again with the English Upper Cretaceous zones, the author concludes that the Gosau beds represent, on the whole, the zones of Holaster planus, Micrasters, Marsupites, and Belemnitella mucronata, i.e. from the horizon of the Chalk Rock to the top of the Chalk with flints inclusive; and this conclusion is further confirmed on palæontological grounds.

Possibly the upper unfossiliferous portion of the Gosau series may represent part of the so-called Danian system of North-western Europe.

The Gosau beds are on the whole of fairly shallow water origin, with beds indicating Estuarine conditions near their base, and were deposited in narrow bays in the Upper Cretaceous sea of Southern and Central Europe on the Northern flanks of the Eastern Alps.

Probably towards the close of Upper Cretaceous times the Southern area of the Gosau district was cut off from free communication with the sea, so as to constitute a lake-basin, in which the upper unfossiliferous marls were deposited.

From the stratigraphical position of the beds, and the occurrence of a calcareous conglomerate at their base, it is evident that the older secondary rocks of the Eastern Alps on which they rest had undergone elevation and denudation with a considerable amount of earth-movement, accompanied by contortion and plication, previous to the period of depression when the Gosau beds were deposited. During the deposition of the latter, the Eastern Alps probably existed as fairly high land along the central portion of the chain, with a very irregular coast-line along its Northern flanks. At the close of Cretaceous times there was considerable elevation, which was followed by depression in the Eocene period at the time of the deposition of the Nummulitic Rocks and the Flysch. Then followed the period of the great Alpine uplift, a movement intense in the Western portion and gradually dying out towards the East in the direction of the Vienna Basin; and it is to this period that the present magnitude of the mountains is chiefly due, and the present isolated and elevated position of the several small areas occupied by the Gosau Beds.

February 26, 1894.

PROFESSOR HUGHES, PRESIDENT, IN THE CHAIR.

Philip Lake, M.A., St John's College, was elected a Fellow of the Society, and W. H. Rivers, M.D. (Lond.), St John's College, was elected an Associate.

The following Communications were made to the Society:

(1) *On Current Sheets, especially on Ellipsoids and Anchor-Rings.* By R. H. D. MAYALL, B.A., Sidney Sussex College, Cambridge.

In the following paper I propose to consider the electric currents induced in thin sheets of a conducting substance by the variation of a magnetic field in which they are placed. For those particular cases when the conducting sheet takes the form of an infinite plane, a sphere, an ellipsoid, or a cylinder whose cross-section is of the second degree, the results have been worked out by several different investigators with slightly different methods and are to be found in various scientific journals. My object is to shew that the currents in the sheet and the state of the magnetic field surrounding it may be completely determined in all the above-mentioned cases by the solution of a general equation connecting the current function with the potential due to the current sheet and the external magnetic field, which equation holds for the most general case and is theoretically capable of solution as soon as the attendant circumstances are known.

In order to solve this equation we assume the external field of magnetic force to be given by the product of e^{ipt} and a function of the co-ordinates used. This is sufficiently general since any function of the time t can be expanded in powers of $e^{\pm it}$. We then assume the magnetic potential due to the system of induced currents to be of a similar form, and determine the arbitrary constants involved in this assumption by means of the general equation above referred to and the necessary conditions of the problem, viz. that the magnetic potential is discontinuous at the conducting surface by an amount $4\pi\Phi$ where Φ is the current function, while the rate of variation of the potential in a direction normal to the surface, i.e. the normal differential coefficient, is the same on both sides of the surface.

The co-ordinates used throughout are those known as orthogonal, and are defined by the equation

$$ds^2 = A^2da^2 + B^2db^2 + C^2dc^2,$$

where $a = \text{const.}$, $b = \text{const.}$, $c = \text{const.}$ are the equations of three families of surfaces cutting each other orthogonally, as is easily seen from the equation itself; A, B, C are here functions of a, b, c and ds is the distance between two points

$$(a\,.\,b\,.\,c) \text{ and } (a+da,\ b+db,\ c+dc).$$

In Section (1) the general equation connecting the current function with the magnetic potential is found.

In Sections (2) and (3) the well-known results for the infinite plane and the sphere are deduced and found to be in agreement with those of Maxwell and of Larmor (*Phil. Mag.*, 1884).

Section (4) deals with the infinite right circular cylinder.

Section (5) contains the solution for the ellipsoid. I am not aware that any result has been obtained previously for the case when the ellipsoid has three unequal axes except that by Prof. Lamb for free currents of the type $\Phi = Cz$.

In Section (6) an attempt is made to deduce results for the anchor-ring, but here we are met by difficulties which did not present themselves in the solution of the previous cases, and it is only for a simple form of magnetic disturbance that a complete solution is arrived at.

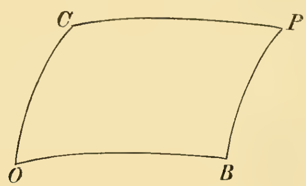
(1) It is first of all necessary to find the equations connecting the vector potential due to a magnetic field of force with the scalar potential. With fixed rectangular axes, these are of course

$$\left.\begin{aligned}\frac{dH}{dy}-\frac{dG}{dz}&=-\frac{d\Omega}{dx}\\ \frac{dF}{dz}-\frac{dH}{dx}&=-\frac{d\Omega}{dy}\\ \frac{dG}{dx}-\frac{dF}{dy}&=-\frac{d\Omega}{dz}\end{aligned}\right\},$$

where F, G, H are the components of the vector potential along the co-ordinate axes, and Ω is the scalar potential; but their form is altered if F, G, H are the components along the normals to the mutually orthogonal systems of surfaces $a=$ const., $b=$ const., $c=$ const. instead of three fixed directions.

Let $OBPC$ be an element of the surface $a=$ const. bounded by two pairs of surfaces

$$b=\text{const.},\ c=\text{const.},\ b+db=\text{const.},\ c+dc=\text{const.};$$



OB being normal to $b=$ const., OC to $c=$ const.; and let F, G, H be the components of the vector potential at O along the normals to the surfaces a, b, c respectively.

Then the line integral of the vector potential round $OBPC$ is equal to the magnetic induction through the same area, which gives

$$\begin{aligned}GBdb+\left(HCdc+\frac{d}{db}\,HCdc\,db\right)&\\ -\left(GBdb+\frac{d}{dc}\,GBdb\;dc\right)&\\ -HCdc=-\frac{d\Omega}{da}\,BdbC\,dc,&\end{aligned}$$

the orthogonal surfaces being given as before mentioned by the equation

$$ds^2=A^2da^2+B^2db^2+C^2dc^2.$$

Hence and similarly
$$\left.\begin{aligned}\frac{d}{db}(HC)-\frac{d}{dc}(GB)&=-\frac{BC}{A}\cdot\frac{d\Omega}{da},\\ \frac{d}{dc}(FA)-\frac{d}{da}(HC)&=-\frac{CA}{B}\cdot\frac{d\Omega}{db},\\ \frac{d}{da}(GB)-\frac{d}{db}(FA)&=-\frac{AB}{C}\cdot\frac{d\Omega}{dc}\end{aligned}\right\}\quad\ldots\ldots\ldots(1).$$

Suppose now that one of the orthogonal surfaces, e.g. $a=a_0$, is formed of a conducting substance whose specific superficial resistance at any point is σ.

Let Φ be the current function at that point, F, G, H the components of the vector potential due to the current sheet, $F_0G_0H_0$ those due to the external disturbing system, Ω and Ω_0 the corresponding scalar potentials, and ψ the function known as the potential of free electricity when the motion is steady. Then the equations of electromotive force are

$$\left.\begin{aligned}\sigma\frac{d\Phi}{Cdc}&=-\frac{d}{dt}(G+G_0)-\frac{d\psi}{Bdb}\\ -\sigma\frac{d\Phi}{Bdb}&=-\frac{d}{dt}(H+H_0)-\frac{d\psi}{Cdc}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(2).$$

Again, equation (1) gives

$$\frac{d}{db}(H+H_0C)-\frac{d}{dc}(G+G_0B)=-\frac{BC}{A}\frac{d}{da}(\Omega+\Omega_0),$$

hence on eliminating $G+G_0$ and $H+H_0$ by means of (2), we have

$$\frac{d}{db}\left(\frac{\sigma C}{B}\frac{d\Phi}{db}\right)+\frac{d}{dc}\left(\frac{\sigma B}{C}\frac{d\Phi}{dc}\right)=-\frac{BC}{A}\frac{d}{dt}\frac{d}{da}(\Omega+\Omega_0)\ldots(3),$$

in which a must be put equal to a_0 after the differentiation with respect to a has been performed.

Since Ω is known in terms of Φ this equation is sufficient to determine either of them, but it is more convenient in applying the equation to particular cases to use the conditions that Ω is discontinuous at the surface $a=a_0$, while $\frac{d\Omega}{da}$ is continuous; thus

and
$$\left.\begin{aligned}4\pi\Phi&=\Omega_1-\Omega_2,\\ \frac{d\Omega_1}{da}&=\frac{d\Omega_2}{da}\end{aligned}\right\}(a=a_0),$$

where Ω_1 is the potential due to the sheet on its positive side, and Ω_2 that on its negative side.

These conditions with equation (3) are necessary for the solution of the problem, and they are also sufficient; for the forms assumed for Ω_1, Ω_2 each contains an arbitrary constant, so that we have three equations to determine these and Φ.

We proceed now to the examination of some simple cases.

(2) Suppose the surface $a = a_0$ is an infinite plane, e.g. the plane of xy, using Cartesian co-ordinates, and let us further suppose that this plane is made of a conducting substance whose specific resistance is the same at every point. Then we have

$$ds^2 = dx^2 + dy^2 + dz^2,$$

and equation (3) becomes

$$\sigma\left(\frac{d^2\Phi}{dx^2} + \frac{d^2\Phi}{dy^2}\right) = -\frac{d}{dt}\frac{d}{dz}(\Omega + \Omega_0) \qquad\text{.................(4).}$$

Let P be the potential due to a distribution of density Φ over the sheet, then

$$\left.\begin{aligned}\Phi &= -\frac{1}{2\pi}\frac{dP}{dz}\\ \Omega &= -\frac{dP}{dz}\end{aligned}\right\} \quad (z=0),$$

P being here the value of the potential on the positive side of the sheet, hence if $\Omega_0 = -\dfrac{dP_0}{dz}$ (4) becomes

$$-\frac{\sigma}{2\pi}\frac{d}{dz}\left(\frac{d^2P}{dx^2} + \frac{d^2P}{dy^2}\right) = \frac{d}{dt}\frac{d^2}{dz^2}(P + P_0),$$

or since $\nabla^2 P = 0$

$$\frac{\sigma}{2\pi}\frac{d^3P}{dz^3} = \frac{d}{dt}\frac{d^2}{dz^2}(P + P_0);$$

and this is satisfied if

$$\frac{\sigma}{2\pi}\frac{dP}{dz} = \frac{d}{dt}(P + P_0),$$

which is the equation given by Maxwell, having for a solution

$$P = F\left\{x,\ y,\ \left(z + \frac{\sigma t}{2\pi}\right)\right\}.$$

We can, however, find a solution in a different way, for suppose

$$\left.\begin{aligned}\Omega_0 &= A_0 e^{ipt} J_m(qr)\, e^{-qz} \cos m\phi\\ \Omega_1 &= A_1 e^{ipt} J_m(qr)\, e^{-qz} \cos m\phi\\ \Omega_2 &= A_2 e^{ipt} J_m(qr)\, e^{+qz} \cos m\phi\end{aligned}\right\} \qquad\text{.................(5),}$$

where Ω_1 is the potential due to the sheet on its positive side and Ω_2 that on its negative side, J_m denoting a Bessel's function of order m. The co-ordinates used here are cylindrical and

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2.$$

Now
$$\begin{aligned} 4\pi\Phi &= \Omega_1 - \Omega_2 \, (z=0) \\ &= (A_1 - A_2)\, e^{ipt} J_m(qr) \cos m\phi, \end{aligned}$$

and (3) becomes

$$\frac{\sigma}{4\pi}(A_1 - A_2)\left\{\frac{d}{dr}\left(r\frac{d}{dr}\right) + \frac{1}{r} \cdot \frac{d^2}{d\phi^2}\right\} J_m \cos m\phi$$
$$= ipqr\,(A_0 + A_1)\, J_m \cos m\phi,$$

or
$$\frac{\sigma}{4\pi}(A_1 - A_2)\left(\frac{d^2 J}{dr^2} + \frac{1}{r} \cdot \frac{dJ}{dr} - \frac{m^2}{r^2}\right) = ipq\,(A_0 + A_1)\, J,$$

i.e. since
$$\frac{d^2 J}{dr^2} + \frac{1}{r} \cdot \frac{dJ}{dr} + \left(q^2 - \frac{m^2}{r^2}\right) J = 0$$

$$-\frac{\sigma q}{4\pi}(A_1 - A_2) = ip\,(A_0 + A_1) \;\ldots\ldots\ldots\ldots (6).$$

Again, because $\dfrac{d\Omega}{dz}$ is continuous at the sheet

$$A_1 = -A_2,$$

whence substituting in (6) we find

$$A_1 = -A_2 = -\frac{2\pi ip}{q\sigma + 2\pi ip} \cdot A_0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

and $\Omega_1\,\Omega_2$ and thence Φ are completely determined.

If $A_0 = 0$ there is no external magnetic disturbance, and we must then have by (7)

$$q\sigma + 2\pi ip = 0,$$

or
$$ip = -\frac{q\sigma}{2\pi};$$

therefore
$$\begin{aligned} \Omega_1 &= \quad A_1 J_m(qr)\, e^{-q\left(z + \frac{\sigma}{2\pi} t\right)} \cos m\phi, \\ \Omega_2 &= -A_1 J_m(qr)\, e^{+q\left(z - \frac{\sigma}{2\pi} t\right)} \cos m\phi. \end{aligned}$$

Since any function of r, θ, ϕ can be expanded in a series of terms of this type, we see that if at the time $t = 0$, $\Omega_1 = F(r, \theta, z)$, then at the time t

$$\Omega_1 = F\left\{r, \theta, \left(z + \frac{\sigma}{2\pi} t\right)\right\},$$

and if when $t=0$, $\Omega_2 = f(r\theta z)$, then at the time t

$$\Omega_2 = f\left\{r,\, \theta,\, \left(z - \frac{\sigma}{2\pi}\, t\right)\right\},$$

so that the effect on either side of the sheet is the same as if the currents in it remained constant, while the sheet itself receded with a uniform velocity $\frac{\sigma}{2\pi}$, as Maxwell shewed.

(3) Next suppose the current sheet to be spherical, σ being again constant. Here

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2,$$

and equation (3) is now

$$\sigma\left(\frac{d}{d\theta}\sin\theta\,\frac{d\Phi}{d\theta} + \frac{1}{\sin\theta}\cdot\frac{d^2\Phi}{d\phi^2}\right)$$
$$= -\,a^2\sin\theta\,\frac{d}{dt}\cdot\frac{d}{dr}(\Omega+\Omega_0)\ \ldots\ldots\ldots\ldots(8),$$

in which a the radius of the sphere is substituted for r after differentiation.

Let $$\Omega_0 = A_0 e^{ipt}\left(\frac{r}{a}\right)^n Y_n,$$

and assume $$\Omega_1 = A_1 e^{ipt}\left(\frac{a}{r}\right)^{n+1} Y_n,$$

$$\Omega_2 = A_2 e^{ipt}\left(\frac{r}{a}\right)^n Y_n,$$

Y_n being a spherical surface harmonic of order n.

Then $$4\pi\Phi = \Omega_1 - \Omega_2\,(r=a)$$
$$= (A_1 - A_2)\,e^{ipt}\,Y_n\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

or on substituting in (8)

$$\frac{\sigma}{4\pi}(A_1 - A_2)\left(\frac{d}{d\theta}\sin\theta\,\frac{dY_n}{d\theta} + \frac{1}{\sin\theta}\cdot\frac{d^2Y_n}{d\phi^2}\right)$$
$$= -\,ipna\sin\theta\,(A_2 + A_0)\,Y_n,$$

or $$-\frac{\sigma}{4\pi}(A_1 - A_2)\,n\,(n+1)\,Y_n = -\,ipna\,(A_2 + A_0)\,Y_n,$$

i.e. $$(n+1)\frac{\sigma}{4\pi}(A_1 - A_2) = ipa\,(A_2 + A_0)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

Again, since $\frac{d\Omega}{dr}$ is continuous at the surface

$$(n+1)\,A_1 + nA_2 = 0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(11);$$

therefore
$$\frac{A_1}{n} = \frac{A_2}{-(n+1)} = \frac{A_1 - A_2}{2n+1}$$
$$= \frac{ipa\,(A_2 + A_0)}{(n+1)(2n+1)\,\dfrac{\sigma}{4\pi}} \text{ by (10)},$$

and therefore
$$= \frac{ipaA_0}{(n+1)\left\{(2n+1)\,\dfrac{\sigma}{4\pi} + ipa\right\}} \quad\text{.................(12).}$$

Thus both the unknown constants are determined and Φ and Ω are completely known; for example,

$$\Phi = \frac{A_1 - A_2}{4\pi} e^{ipt} Y_n \text{ by (9)}$$
$$= \frac{(2n+1)\,ipaA_0}{(n+1)\,\{(2n+1)\,\sigma + 4\pi ipa\}} e^{ipt} Y_n.$$

Again, the whole potential inside the sphere is

$$(A_2 + A_0)\, e^{ipt} \left(\frac{r}{a}\right)^n Y_n = \frac{(2n+1)\,\sigma A_0}{(2n+1)\,\sigma + 4\pi ipa} e^{ipt} \left(\frac{r}{a}\right)^n Y_n$$
$$= A_0 \cos \chi e^{i(pt-\chi)} \left(\frac{r}{a}\right)^n Y_n,$$

where
$$\chi = \tan^{-1} \frac{4\pi pa}{(2n+1)\,\sigma},$$

shewing a field diminished in the ratio $\cos\chi : 1$, and lagging behind by a time equal to $2\pi/\chi$ of a complete period. This agrees with the result given by Mr Larmor (*Phil. Mag.* 1884).

For free currents of the type $e^{ipt}Y_n$ we must put $A_0 = 0$ in (12), and we then have

$$(2n+1)\,\sigma + 4\pi ipa = 0,$$

or
$$ip = -\frac{(2n+1)\,\sigma}{4\pi a},$$

giving a value for the modulus of decay which is well known.

The results given by Prof. C. Niven (*Phil. Trans.* 1881) for the law of decay of such currents can also be found from our equations.

(4) Let the current sheet be an infinite right circular cylinder of radius a, the specific resistance being constant at every point of its surface.

Here
$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

with cylindrical co-ordinates and equation (3) is

$$\sigma \left(\frac{1}{r^2} \cdot \frac{d^2\Phi}{d\theta^2} + \frac{d^2\Phi}{dz^2}\right) = -\frac{d}{dt} \cdot \frac{d}{dr} (\Omega + \Omega_0)\,(r = a) \quad\text{.......(13).}$$

Suppose $$\Omega_0 = A_0 e^{ipt} J_m(qr)\, e^{\pm qz} \begin{matrix}\sin\\ \cos\end{matrix}\, m\theta,$$
where J_m denotes a Bessel's function of the first kind and of order m. We may then assume

$$\left.\begin{aligned}\Omega_1 &= A_1 e^{ipt} K_m(qr)\, e^{\pm qz} \begin{matrix}\sin\\ \cos\end{matrix}\, m\theta,\\ \Omega_2 &= A_2 e^{ipt} J_m(qr)\, e^{\pm qz} \begin{matrix}\sin\\ \cos\end{matrix}\, m\theta\end{aligned}\right\}\ldots\ldots\ldots\ldots(14);$$

K_m being the corresponding Bessel's function of the second kind, and equal to

$$J_m(qr)\frac{\displaystyle\int_r^\infty \frac{dr}{rJ_m^2}}{\displaystyle\int_a^\infty \frac{dr}{rJ_m^2}},$$

so that $K_m = J_m$ at the surface of the cylinder and vanishes when r is infinitely large.

Then $$\begin{aligned}4\pi\Phi &= \Omega_1 - \Omega_2\\ &= (A_1 - A_2)\, J_m(qa)\, e^{ipt} e^{\pm qz} \begin{matrix}\sin\\ \cos\end{matrix}\, m\theta\ldots\ldots(15);\end{aligned}$$

hence substituting in (13) and denoting differentiation by dashes, we have

$$\frac{\sigma}{4\pi}(A_1 - A_2)\left(-\frac{m^2}{a^2} + q^2\right) J_m = -ip\,(A_2 + A_0)\, J_m' \;\ldots(16),$$

also since $\dfrac{d\Omega}{dr}$ is continuous at the surface

$$A_1 K_m' = A_2 J_m' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17),$$

therefore

$$\left.\begin{aligned}\frac{A_1}{J_m'} = \frac{A_2}{K_m'} &= \frac{A_1 - A_2}{J_m' - K_m'}\\ &= (A_1 - A_2)\, aJ_m \int_a^\infty \frac{dr}{rJ_m^2}\\ &= (A_2 + A_0)\frac{\displaystyle -ipaJ_m' \int_a^\infty \frac{dr}{rJ_m^2}}{\displaystyle \frac{\sigma}{4\pi}\left(-\frac{m^2}{a^2} + q^2\right)} \text{ by (16),}\\ \text{and therefore}\qquad &= A_0 \frac{\displaystyle -ipaJ_m' \int_a^\infty \frac{dr}{rJ_m^2}}{\displaystyle \frac{\sigma}{4\pi}\left(-\frac{m^2}{a^2} + q^2\right) + ipaJ_m'K_m' \int_a^\infty \frac{dr}{rJ_m^2}}\end{aligned}\right\}\ldots(18),$$

and the constants are determined.

Thus $$\Phi = \frac{1}{4\pi}(A_1 - A_2) J_m e^{ipt} e^{\pm qz} \frac{\sin}{\cos} m\theta$$

$$= A_0 \frac{- ipJ_m' e^{ipt} e^{\pm qz} \frac{\sin}{\cos} m\theta}{\sigma\left(-\frac{m^2}{a^2} + q^2\right) + 4\pi ipaJ_m'K_m' \int_a^\infty \frac{dr}{rJ_m^2}}.$$

Again, the whole effect inside the cylinder is given by

$$\Omega + \Omega_0 = (A_2 + A_0) J_m(qr) e^{ipt} e^{\pm qz} \frac{\sin}{\cos} m\theta$$

$$= A_0 \frac{\sigma\left(-\frac{m^2}{a^2} + q^2\right) J_m(qr) \ldots}{\sigma\left(-\frac{m^2}{a^2} + q^2\right) + 4\pi ipaJ_m'K_m' \int_a^\infty \frac{dr}{rJ_m^2}}$$

$$= A_0 \cos\chi J_m(qr) e^{i(pt-\chi)} e^{\pm qz} \frac{\sin}{\cos} m\theta,$$

where $$\tan\chi = \frac{4\pi paJ_m'K_m' \int_a^\infty \frac{dr}{rJ_m^2}}{\sigma\left(-\frac{m^2}{a^2} + q^2\right)},$$

shewing as in the case of the sphere a field diminished in the ratio $\cos\chi : 1$ with a lag in phase equal to $2\pi/\chi$ of a complete period.

For free currents of the above type we may put $A_0 = 0$ in (18), and we then have

$$\sigma\left(-\frac{m^2}{a^2} + q^2\right) + 4\pi ipaJ_m'K_m' \int_a^\infty \frac{dr}{rJ_m^2},$$

or $$ip = \frac{-\sigma\left(\frac{m^2}{a^2} + q^2\right)}{4\pi aJ_m'K_m' \int_a^\infty \frac{dr}{rJ_m^2}},$$

which gives a real value for the modulus of decay.

(5) Let the current sheet be now an ellipsoid, taking as orthogonal co-ordinates of any point the semi-major-axes $(\rho\mu\nu)$ of the quadrics confocal with the given ellipsoid which pass through that point, so that

$$ds^2 = L^2 d\rho^2 + M^2 d\mu^2 + N^2 d\nu^2,$$

where
$$\left.\begin{aligned} L^2 &= \frac{(\rho^2-\mu^2)(\rho^2-\nu^2)}{(\rho^2-h^2)(\rho^2-k^2)},\\ M^2 &= \frac{(\rho^2-\mu^2)(\mu^2-\nu^2)}{(\mu^2-h^2)(k^2-\mu^2)},\\ N^2 &= \frac{(\rho^2-\nu^2)(\mu^2-\nu^2)}{(h^2-\nu^2)(k^2-\nu^2)},\\ h^2 &= a^2-b^2,\ k^2=a^2-c^2, \end{aligned}\right\}$$

a, b, c being the semi-axes of the ellipsoidal sheet.

Equation (3) is in this case
$$\frac{d}{d\mu}\frac{\sigma N}{M}\frac{d\Phi}{d\mu}+\frac{d}{d\nu}\frac{\sigma M}{N}\frac{d\Phi}{d\nu}=-\frac{MN}{L}\frac{d}{dt}\frac{d}{d\rho}(\Omega+\Omega_0)\text{.........}(19).$$

This equation may be easily solved if $\sigma=\kappa L$ where κ is constant. This is equivalent to supposing the current sheet to be a homœoidal shell, for the thickness of such a shell at any point is proportional to the perpendicular from the centre on the tangent plane at that point, i.e. inversely proportional to L, so that if the sheet be made of a homogeneous conducting substance its specific resistance at any point will vary as L.

Let $\qquad \Omega_0=A_0E_n(\rho)\,E_n(\mu)\,E_n(\nu)\,e^{ipt},$

and assume $\qquad \Omega_1=A_1F_n(\rho)\,E_n(\mu)\,E_n(\nu)\,e^{ipt},$

$$\Omega_2=A_2E_n(\rho)\,E_n(\mu)\,E_n(\nu)\,e^{ipt},$$

where E_n and F_n denote Lamé's functions of the first and second kind respectively, and
$$F_n(\rho)=(2n+1)\,E_n(\rho)\int_\rho^\infty\frac{d\rho}{E_n{}^2\sqrt{(\rho^2-h^2)(\rho^2-k)^2}}\text{......}(20).$$

Then $\qquad 4\pi\Phi=(A_1F_n-A_2E_n)\,E_n(\mu)\,E_n(\nu)\,e^{ipt}$

[E_n and F_n are here written as abbreviations for $E_n(a)$, $F_n(a)$].

Substitute this value of Φ in (19) and there results
$$\frac{\kappa}{4\pi}(A_1F_n-A_2E_n)\left\{\frac{d}{d\mu}\frac{NL}{M}\frac{d}{d\mu}+\frac{d}{d\nu}\frac{LM}{N}\frac{d}{d\nu}\right\}E_n(\mu)\,E_n(\nu)$$
$$=-\frac{MN}{L}\,ip\,(A_2+A_0)\,E_n{}'E_n(\mu)\,E_n(\nu),$$

which if we introduce two quantities η, ζ defined by
$$\eta=\int_b\frac{d\mu}{\sqrt{(\mu^2-h^2)(k^2-\mu^2)}},\quad \zeta=\int_0^\nu\frac{d\nu}{\sqrt{(h^2-\nu^2)(k^2-\nu^2)}}$$

may be written

$$\frac{\kappa}{4\pi}(A_1F_n - A_2E_n)$$

$$\frac{(\rho^2-\nu^2)\dfrac{d^2}{d\eta^2}+(\rho^2-\mu)^2\dfrac{d^2}{d\zeta^2}}{\sqrt{(\rho^2-h^2)(\rho^2-k^2)(\mu^2-h^2)(k^2-\mu^2)(h^2-\nu^2)(k^2-\nu^2)}}E_n(\mu)E_n(\nu)$$

$$= -i\dot{p}\frac{(\mu^2-\nu^2)\sqrt{(\rho^2-h^2)(\rho^2-k^2)}}{\sqrt{(\mu^2-h^2)(k^2-\mu^2)(h^2-\nu^2)(k^2-\nu^2)}}$$

$$(A_2+A_0)E_n'E_n(\mu)E_n(\nu),$$

or since $\rho = a$ at the sheet

$$\left.\begin{aligned}\frac{\kappa}{4\pi}(A_1F_n - A_2E_n)\Big\{(\rho^2-\nu^2)&\frac{d^2}{d\eta^2}\\ +(\rho^2-\mu^2)&\frac{d^2}{d\zeta^2}\Big\}E_n(\mu)E_n(\nu)\\ = -ipb^2c^2(\mu^2-\nu^2)(A_2+A_0)&E_n'E_n(\mu)E_n(\nu)\end{aligned}\right\}\text{.........(21).}$$

Now (see Heine, *Kugelfunctionen*)

$$\frac{d^2E_n(\mu)}{d\eta^2} = -\{n(n+1)\mu^2 - q(h^2+k^2)\}E_n(\mu),$$

and
$$\frac{d^2E_n(\mu)}{d\zeta^2} = \{n(n+1)\nu^2 - q(h^2+k^2)\}E_n(\nu),$$

where q is a constant which is known when the form of E_n is known. Hence

$$\left\{(\rho^2-\nu^2)\frac{d^2}{d\eta^2}+(\rho^2-\mu^2)\frac{d^2}{d\zeta^2}\right\}E_n(\mu)E_n(\nu)$$
$$= -(\mu^2-\nu^2)\{n(n+1)\rho^2 - q(h^2+k^2)\}.$$

Substituting this result in (21) we have

$$\left.\begin{aligned}\frac{\kappa}{4\pi}(A_1F_n - A_2E_n)\{n(n+1)a^2 - q(h^2+k^2)\}\\ = ipb^2c^2(\mu^2-\nu^2)(A_2+A_0)E_n'\end{aligned}\right\}\text{............(22).}$$

Since $\dfrac{d\Omega}{d\rho}$ is continuous in crossing the surface of the ellipsoid

$$A_1F_n' = A_2E_n';$$

therefore
$$\frac{A_1}{E_n'} = \frac{A_2}{F_n'} = \frac{A_1F_n - A_2E_n}{E_n'F_n - F_n'E_n}\text{ ....(23),}$$

but since $$F_n = (2n+1) E_n \int_\rho^\infty \frac{d\rho}{E_n^2 \sqrt{(\rho^2 - h^2)(\rho^2 - k^2)}},$$

$$F_n' = (2n+1) \left\{ E_n' \int_a^\infty \frac{d\rho}{E_n^2 \sqrt{(\rho^2 - h^2)(\rho^2 - k^2)}} - \frac{1}{E_n bc} \right\} \text{ when } \rho = a$$

therefore $$E_n' F_n - F_n' E_n = \frac{2n+1}{bc},$$

and (23) becomes

$$\left.\begin{aligned}
&\frac{A_1}{E_n'} = \frac{A_2}{F_n'} = \frac{bc}{2n+1} (A_1 F_n - A_2 E_n) \\
&= \frac{ipb^3c^3 (A_2 + A_0) E_n'}{(2n+1) \frac{\kappa}{4\pi} \{n(n+1) a^2 - q(h^2 + k^2)\}} \text{ by (22)}; \\
&\text{and therefore} \\
&= \frac{ipb^3c^3 E_n' A_0}{(2n+1) \frac{\kappa}{4\pi} \{n(n+1) a^2 - q(h^2 + k^2)\} - ipb^3c^3 E_n' F_n'}
\end{aligned}\right\} \text{......(24)},$$

which gives A_1 and A_2 completely. The expression for the current function is

$$\Phi = \frac{1}{4\pi} (A_1 F_n - A_2 E_n) E_n(\mu) E_n(\nu) e^{ipt}$$

$$= \frac{(2n+1) ipb^2c^2 E_n' A_0 E_n(\mu) E_n(\nu) e^{ipt}}{(2n+1) \kappa \{n(n+1) a^2 - q(h^2 + k^2)\} - 4\pi ipb^3c^3 E_n' F_n'},$$

and the effect inside the shell is given by

$$\Omega + \Omega_0 = (A_2 + A_0) E_n(\rho) E_n(\mu) E_n(\nu) e^{ipt}$$

$$= \frac{(2n+1) \kappa \{n(n+1) a^2 - q(h^2 + k^2)\} A_0}{(2n+1) \kappa \{n(n+1) a^2 - q(h^2 + k^2)\} - 4\pi ipb^3c^3 E_n' F_n'} E_n(\rho) \ldots$$

$$= A_0 \cos \chi E_n(\rho) E_n(\mu) E_n(\nu) e^{i(pt - \chi)},$$

where $$\tan \chi = \frac{4\pi pb^3c^3 E_n' F_n'}{(2n+1) \kappa \{q(h^2 + k^2) - n(n+1) a^2\}},$$

again as in the previous cases of the sphere and cylinder shewing a diminished intensity and change of phase.

The modulus of decay, viz. $-1/ip$ for free currents of the type considered, is found as before by putting $A_0 = 0$ in equation (24). Thus

$$(2n+1)\frac{\kappa}{4\pi}\{n(n+1)a^2 - q(h^2+k^2)\} - ipb^3c^3E_n'F_n' = 0,$$

or
$$ip = \frac{(2n+1)\kappa\{n(n+1)a^2 - q(h^2+k^2)\}}{4\pi b^3c^3E_n'F_n'}.$$

When $\Phi = Cz$ we have $E_n(\rho) = \sqrt{\rho^2 - k^2}$, and we easily find by substituting this value in the equation satisfied by $E_n(\rho)$ that

$$q(h^2+k^2) = h^2.$$

Thus for free currents

$$ip = \frac{\kappa(2a^2 - h^2)}{4\pi b^3c^3E_n'\left\{E_n'\int_a^\infty \frac{d\rho}{E_n^2\sqrt{(\rho^2-h^2)(\rho^2-k^2)}} - \frac{1}{E_nbc}\right\}},$$

or since
$$E_n' = \frac{a}{\sqrt{a^2-k^2}} = \frac{a}{c},$$

$$ip = \frac{\kappa(a^2+b^2)}{4\pi ab^3c^2\left\{\frac{a}{c}\int_a^\infty \frac{d\rho}{E_n^2\sqrt{(\rho^2-h^2)(\rho^2-k^2)}} - \frac{1}{bc^2}\right\}}$$
$$= \frac{\kappa(a^2+b^2)}{ab^2\{N-4\pi\}},$$

where
$$N = 4\pi abc\int_a^\infty \frac{d\rho}{(\rho^2-h^2)^{\frac{1}{2}}(\rho^2-k^2)^{\frac{3}{2}}},$$

and the modulus of decay is

$$-\frac{1}{ip} = -\frac{1}{\kappa}\cdot\frac{ab^2}{a^2+b^2}\cdot(N-4\pi),$$

and this will be found to agree with Prof. Lamb's result (*Phil. Trans.* 1887), when we take into account the difference between his ρ and our κ.

(6) We proceed now to the case of the anchor-ring. The notation used is that of Dr W. M. Hicks in his paper on Toroidal Functions, *Phil. Trans.* 1881, except that we use σ and θ instead of u and v respectively, and we have

$$ds^2 = \frac{k^2}{(\sigma\theta)^4}(d\sigma^2 + d\theta^2 + \sinh^2\sigma d\psi^2),$$

(see Heine, *Kugelfunctionen*), where

$$(\sigma\theta)^2 = \cosh\sigma - \cos\theta.$$

Let $\sigma = \sigma_0$ be the equation of the current-sheet and suppose the external magnetic field to be symmetrical about the axis of the ring, and therefore $\Omega_0\Omega$ and Φ independent of ψ. Then denoting the specific resistance in this case by ρ, equation (3) becomes

$$\frac{d}{d\theta}\rho\sinh\sigma\frac{d\Phi}{d\theta} = -\frac{k\sinh\sigma}{(\sigma\theta)^2}\cdot\frac{d}{dt}\cdot\frac{d}{d\sigma}(\Omega+\Omega_0),$$

or

$$\frac{d}{d\theta}\rho\frac{d\Phi}{d\theta} = -\frac{kip}{(\sigma\theta)^2}\cdot\frac{d}{d\sigma}(\Omega+\Omega_0)\ldots\ldots\ldots\ldots\ldots(25),$$

if Φ, Ω and Ω_0 vary as e^{ipt}.

The type of function which satisfies Laplace's equation for the co-ordinates here used is $(\sigma\theta)\,P_n(\cosh\sigma)\sin n\theta$ where $P_n(\cosh\sigma)$ is a zonal spherical harmonic of degree $\frac{2n+1}{2}$ with $i\sigma$ as argument; its properties as well as those of the allied function $Q_n(\cosh\sigma)$ are fully investigated by Dr Hicks in the paper to which allusion was made above.

If however we attempt to solve (25) in the same way as the previous cases, viz. by assuming Ω_0 to be of the form

$$A_0(\sigma\theta)\,Q_n(\cosh\sigma)\sin n\theta,$$

Ω_1 of the form $\quad A_1(\sigma\theta)\,P_n(\cosh\sigma)\sin n\theta,$

and Ω_2 of the form

$$A_2(\sigma\theta)\,Q_n(\cosh\sigma)\sin n\theta,$$

we find that these values of Ω_1 and Ω_2 cannot represent the potential due to the current sheet, for their differential coefficients with respect to σ are

$$\left\{\frac{1}{2}\sinh\sigma P_n/(\sigma\theta) + (\sigma\theta)\,P_n'\right\}A_1\sin n\theta,$$

and

$$\left\{\frac{1}{2}\sinh\sigma Q_n/(\sigma\theta) + (\sigma\theta)\,Q_n'\right\}A_2\sin n\theta,$$

and these cannot be equal when $\sigma = \sigma_0$ for all values of θ unless A_1 and A_2 are both zero.

The point in which this case differs from those already discussed lies chiefly in the fact that the expression for the typical harmonic instead of being the product of three terms each of which is a function of one co-ordinate only, contains in addition to these a term which is a function of two co-ordinates σ, θ.

Let us find an expression which can represent the potential due to the sheet. Assume

$$\left.\begin{aligned}\Omega_1 &= \Sigma a_n (\sigma\theta) P_n \sin n\theta,\\ \Omega_2 &= \Sigma b_n (\sigma\theta) Q_n \sin n\theta\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(26),$$

where the Σ denotes summation with respect to n, and a_n, b_n are constants; Ω_1 and Ω_2 will then satisfy Laplace's equation and since P_n and Q_n are finite and continuous and P_n does not become infinite at any point outside the surface nor Q_n within it, Ω_1 and Ω_2 may represent the potential due to a current sheet provided a_n and b_n are so determined as to make $\frac{d\Omega}{d\sigma}$ continuous at the surface. This requires that

$$\Sigma \frac{d}{d\sigma} \{(\sigma\theta)(a_n P_n - b_n Q_n) \sin n\theta\} = 0$$

for all values of θ when $\sigma = \sigma_0$,

i.e. $$\Sigma \left\{\frac{1}{2} \frac{S}{(\sigma\theta)} (a_n P_n - b_n Q_n) + (\sigma\theta)(a_n P_n' - b_n Q_n')\right\} \sin n\theta = 0,$$

where $S = \sinh\sigma$ and $P_n' Q_n'$ denote $\frac{dP_n}{d\sigma}$ and $\frac{dQ_n}{d\sigma}$ respectively, or

$$\Sigma \{S (a_n P_n - b_n Q_n) + 2(\sigma\theta)^2 (a_n P_n' - b_n Q_n')\} \sin n\theta = 0 \ldots(27).$$

In place of $a_n b_n$ introduce now two new constants λ_n and μ_n defined by

$$\left.\begin{aligned}a_n &= \lambda_n Q_n + \mu_n Q_n'/S\\ b_n &= \lambda_n P_n + \mu_n P_n'/S\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28),$$

in which S, P_n, Q_n etc. have their values at the surface, then

$$a_n P_n - b_n Q_n = \frac{\mu_n}{S} (P_n Q_n' - P_n' Q_n) = -\frac{\pi\mu_n}{S^2},$$

$$a_n P_n' - b_n Q_n' = \lambda_n (P_n' Q_n - P_n Q_n') = \frac{\pi}{S} \lambda_n,$$

and (27) becomes

$$\Sigma \{\mu_n - 2\lambda_n (\sigma\theta)^2\} \sin n\theta = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(29),$$

or $$\Sigma \{\mu_n - 2\lambda_n (C - \cos\theta)\} \sin n\theta = 0,$$

i.e. $$\Sigma \{\mu_n \sin n\theta - 2C\lambda_n \sin n\theta + \lambda_n \sin \overline{n+1}\,\theta$$
$$+ \lambda_n \sin \overline{n-1}\,\theta\} = 0,$$

and this can only be satisfied for all values of θ when

$$\left.\begin{aligned}
\mu_1 &= 2C\lambda_1 - \lambda_2,\\
\mu_2 &= 2C\lambda_2 - \lambda_3 - \lambda_1,\\
\mu_3 &= 2C\lambda_3 - \lambda_4 \quad - \lambda_2,\\
&\dots\dots\dots\dots\dots\dots\\
\mu_n &= 2C\lambda_n - \lambda_{n+1} - \lambda_{n-1}\\
&\dots\dots\dots\dots\dots\dots
\end{aligned}\right\}\dots\dots\dots\dots(30).$$

Any values of λ and μ which satisfy this set of equations give on substituting in (28) values for a, b which determine proper forms for Ω_1, Ω_2.

The value of Φ is given by

$$\begin{aligned}
4\pi\Phi &= \Omega_1 - \Omega_2\,(\sigma = \sigma_0)\\
&= \Sigma\,(a_n P_n - b_n Q_n)\,(\sigma\theta)\sin n\theta\\
&= \Sigma\,\frac{\mu_n}{S}\,(P_n Q_n' - P_n' Q_n)\,(\sigma\theta)\sin n\theta\\
&= -\frac{\pi}{S^2}\,\Sigma\mu_n\,(\sigma\theta)\sin n\theta \dots\dots\dots\dots(31)\\
&= -\frac{2\pi}{S^2}\,\Sigma\lambda_n\,(\sigma\theta)^3\sin n\theta \text{ by (29)},
\end{aligned}$$

or

$$\Phi = -\frac{1}{2S^2}\,\Sigma\lambda_n\,(\sigma\theta)^3\sin n\theta \dots\dots\dots\dots(32).$$

Using this value of Φ, equation (25) becomes

$$\frac{1}{2S^2kip}\,\frac{d}{d\theta}\,\rho\,\frac{d}{d\theta}\,\Sigma\lambda_n\,(\sigma\theta)^3\sin n\theta$$
$$= \frac{1}{(\sigma\theta)^2}\,\frac{d}{d\sigma}\,[\Sigma b_n\,(\sigma\theta)\,Q_n\sin n\theta + \Omega_0].$$

Let $\rho = kR/(\sigma\theta)^2$ so that the specific resistance varies as the distance between the two consecutive surfaces

$$\sigma = \sigma_0 \text{ and } \sigma = \sigma_0 + d\sigma_0$$

(this is analogous to the assumption made above as to the resistance in an ellipsoidal shell), and the equation is now

$$\frac{R}{2S^2ip}\,\frac{d}{d\theta}\,\frac{1}{(\sigma\theta)^2}\,\frac{d}{d\theta}\,\Sigma\lambda_n\,(\sigma\theta)^3\sin n\theta$$
$$= \frac{1}{(\sigma\theta)^2}\,\frac{d}{d\sigma}\,[\Sigma b_n\,(\sigma\theta)\,Q_n\sin n\theta + \Omega_0]\dots(33).$$

The left-hand side of this without the factor $\dfrac{R}{2S^2ip}$ is

$$\frac{d}{d\theta}\,\Sigma\lambda_n\left\{\frac{3}{2}\,\frac{\sin\theta}{(\sigma\theta)}\sin n\theta+n\,(\sigma\theta)\cos n\theta\right\}$$

$$=\Sigma\lambda_n\left\{\frac{3}{2}\,\frac{\cos\theta}{(\sigma\theta)}\sin n\theta-\frac{3}{4}\,\frac{\sin^2\theta}{(\sigma\theta)^3}\sin n\theta\right.$$
$$\left.+\,2n\frac{\sin\theta}{(\sigma\theta)}\cos n\theta-n^2\,(\sigma\theta)\sin n\theta\right\}$$

$$=\Sigma\lambda_n\left\{\frac{3}{4}\,\frac{2\cos\theta\,(\sigma\theta)^2-\sin^2\theta}{(\sigma\theta)^3}\sin n\theta\right.$$
$$\left.+\,2n\frac{\sin\theta}{(\sigma\theta)}\cos n\theta-n^2\,(\sigma\theta)\sin n\theta\right\}$$

$$=\Sigma\lambda_n\left\{\frac{3}{4}\,\frac{S^2-(\sigma\theta)^4}{(\sigma\theta)^3}\sin n\theta+2n\,\frac{\sin\theta}{(\sigma\theta)}\cos n\theta-n^2\,(\sigma\theta)\sin n\theta\right\}$$

$$=\frac{1}{(\sigma\theta)^3}\,\Sigma\lambda_n\left\{-\left(\frac{3}{4}+n^2\right)(\sigma\theta)^4\sin n\theta+2n\,(\sigma\theta)^2\sin\theta\cos n\theta+\frac{3}{4}\,S^2\sin n\theta\right\}$$

$$=\frac{1}{(\sigma\theta)^3}\,\Sigma\lambda_n\left[(\sigma\theta)^2\left\{-\left(\frac{3}{4}+n^2\right)(C-\cos\theta)\sin n\theta\right.\right.$$
$$\left.\left.+\,n\,(\sin\overline{n+1}\,\theta-\sin\overline{n-1}\,\theta)\right\}+\frac{3}{4}\,S^2\sin n\theta\right]$$

$$=\frac{1}{(\sigma\theta)^3}\,\Sigma\lambda_n\left[(\sigma\theta)^2\left\{-\left(\frac{3}{4}+n^2\right)C\sin n\theta\right.\right.$$
$$+\,\frac{1}{2}\left(\frac{3}{4}+n^2+2n\right)\sin\overline{n+1}\,\theta$$
$$\left.\left.+\,\frac{1}{2}\left(\frac{3}{4}+n^2-2n\right)\sin\overline{n-1}\,\theta\right\}+\frac{3}{4}\,S^2\sin n\theta\right]$$

$$=\frac{1}{(\sigma\theta)^3}\,\Sigma\left[(\sigma\theta)^2\left\{-\left(\frac{3}{4}+n^2\right)C\lambda_n+\frac{1}{2}\left(n^2-\frac{1}{4}\right)(\lambda_{n-1}+\lambda_{n+1})\right\}\right.$$
$$\left.+\,\frac{3}{4}\,S^2\lambda_n\right]\sin n\theta$$

$$=\frac{1}{(\sigma\theta)^3}\,\Sigma\left[(\sigma\theta)^2\left\{-C\lambda_n-\frac{1}{2}\left(n^2-\frac{1}{4}\right)\mu_n\right\}+\frac{3}{4}\,S^2\lambda_n\right]\sin n\theta\ \text{by (30)}$$

$$=-\frac{1}{(\sigma\theta)^3}\,\Sigma\left[\frac{1}{2}\,C\mu_n+\frac{1}{2}\left\{\left(n^2-\frac{1}{4}\right)C\mu_n-\frac{1}{2}\left(n^2+2n+\frac{3}{4}\right)\mu_{n+1}\right.\right.$$
$$\left.\left.-\,\frac{1}{2}\left(n^2-2n+\frac{3}{4}\right)\mu_{n-1}\right\}-\frac{3}{4}\,S^2\lambda_n\right]\sin n\theta.$$

The last line is got by finding the coefficient of $\sin n\theta$ in the preceding line and using (30), and the expression holds for all values of n down to unity provided we consider $\lambda_0 = \mu_0 = 0$. The left-hand side of (33) is then

$$-\frac{R}{4S^2 ip(\sigma\theta)^3}\sum_{n=1}\left[C\mu_n\left(n^2+\frac{3}{4}\right)-\frac{1}{2}\left(n^2+2n+\frac{3}{4}\right)\mu_{n+1}\right.$$
$$\left.-\frac{1}{2}\left(n^2-2n+\frac{3}{4}\right)\mu_{n-1}-\frac{3}{2}S^2\lambda_n\right]\sin n\theta$$
$$=-\frac{R}{4S^2 ip(\sigma\theta)^3}\sum_{n=1}\left[\frac{1}{2}\left(n^2+\frac{3}{4}\right)(2C\mu_n-\mu_{n-1}-\mu_{n+1})\right.$$
$$\left.+n(\mu_{n-1}-\mu_{n+1})-\frac{3}{2}S^2\lambda_n\right]\sin n\theta \quad \ldots(34).$$

For the right-hand side of (33) suppose Ω_0 is of the form

$$A_0(\sigma\theta)\,Q_m(\cosh\sigma)\sin m\theta,$$

then we have

$$\frac{1}{(\sigma\theta)^2}\left[\frac{d}{d\sigma}\Sigma b_n(\sigma\theta)\,Q_n\sin n\theta+\frac{d}{d\sigma}A_0(\sigma\theta)\,Q_m\sin m\theta\right]$$
$$=\frac{1}{(\sigma\theta)^2}\left[\Sigma b_n\left\{\frac{1}{2}\,\frac{S}{(\sigma\theta)}\,Q_n+(\sigma\theta)\,Q_n'\right\}\sin n\theta\right.$$
$$\left.+A_0\left\{\frac{1}{2}\,\frac{S}{(\sigma\theta)}\,Q_m+(\sigma\theta)\,Q_m'\right\}\sin m\theta\right]$$
$$=\frac{1}{2(\sigma\theta)^3}\left[\Sigma\{b_n(SQ_n+2CQ_n')-b_{n-1}Q'_{n-1}-b_{n+1}Q'_{n+1}\}\sin n\theta\right.$$
$$\left.+A_0\{(SQ_m+2CQ_m')\sin m\theta-(\sin\overline{m+1}\,\theta+\sin\overline{m-1}\,\theta)\,Q_m'\}\right],$$

(33) written in full is therefore

$$-\frac{R}{2S^2 ip}\Sigma\left[\frac{1}{2}\left(n^2+\frac{3}{4}\right)(2C\mu_n-\mu_{n-1}-\mu_{n+1})\right.$$
$$\left.+n(\mu_{n-1}-\mu_{n+1})-\frac{3}{2}S^2\lambda_n\right]\sin n\theta$$
$$=\Sigma\{b_n(SQ_n+2CQ_n')-b_{n-1}Q'_{n-1}-b_{n+1}Q'_{n+1}\}\sin n\theta$$
$$+A_0\{(SQ_m+2CQ_m')\sin m\theta$$
$$-(\sin\overline{m+1}\,\theta+\sin\overline{m-1}\,\theta)\,Q_m'\}\ldots(35).$$

By equating to zero the coefficients of $\sin n\theta$ in this equation we arrive at a system of equations which in conjunction with (30) furnish the complete solution of the problem.

These are

$$\frac{R}{2S^2ip}\left[\frac{1}{2}\left(n^2+\frac{3}{4}\right)(2C\mu_n-\mu_{n-1}-\mu_{n+1})+n(\mu_{n-1}-\mu_{n+1})-\frac{3}{2}S^2\lambda_n\right]$$
$$+\{b_n(SQ_n+2CQ_n')-b_{n-1}Q'_{n-1}-b_{n+1}Q'_{n+1}\}=0\ \ldots(35\text{ a}),$$

where n may have any positive integral value except $m-1$, m, or $m+1$; in the first and third cases we have instead of zero on the right A_0Q_m', and in the second case

$$-A_0(SQ_m+2CQ_m').$$

If $m=1$ there is a solution given by

$$\left.\begin{aligned}\lambda_n&=\quad AP_n'+BQ_n'\\ \mu_n&=-S(AP_n+BQ_n)\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots(36),$$

for all values of $n>1$, A and B being constants, or more simply

$$\left.\begin{aligned}a_n&=\quad A\frac{\pi}{S}\\ b_n&=-B\frac{\pi}{S}\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(36\text{ a}),$$

and when $n=1$

$$\left.\begin{aligned}\lambda_1&=\quad AP_1'+BQ_1'\\ \mu_1&=-S(AP_1+BQ_1)+AP_0'+BQ_0'\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(37),$$

or

$$\left.\begin{aligned}a_1&=\quad A\frac{\pi}{S}+\frac{Q_1'}{S}(AP_0'+BQ_0'),\\ b_1&=-B\frac{\pi}{S}+\frac{P_1'}{S}(AP_0'+BQ_0')\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(37\text{ a}).$$

For these values of λ, μ, satisfy equations (30) by reason of the relations

$$\left.\begin{aligned}P'_{n-1}+P'_{n+1}&=2CP_n'+SP_n\\ Q'_{n-1}+Q'_{n+1}&=2CQ_n'+SQ_n\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots(38)$$

(*Phil. Trans.* 1881, p. 646) and they will also satisfy (35) if A and B have suitable values. To prove this it will be sufficient to substitute P_n' for λ_n and $-SP_n$ for μ_n as the relations between the Q terms are always similar to those between the P^s terms. The left-hand side of (35) then becomes

$$-\frac{R}{2S^2ip}\Sigma\left[-\frac{1}{2}\left(n^2+\frac{3}{4}\right)S(2CP_n-P_{n-1}-P_{n+1})\right.$$
$$\left.-nS(P_{n-1}-P_{n+1})-\frac{3}{2}S^2\lambda_n\right]\sin n\theta$$
$$=\frac{R}{4Sip}\Sigma\left[\left(n^2+\frac{1}{4}\right)(2CP_n-P_{n-1}-P_{n+1})+2CP_n\right.$$
$$\left.+(2n-1)P_{n-1}-(2n+1)P_{n+1}+3SP_n'\right]\sin n\theta.$$

Now the following relations hold between the P's

$$\left.\begin{aligned}\frac{2S}{2n+1}P_n' &= P_{n+1} - CP_n,\\ \frac{2S}{2n-1}P_n' &= CP_n - P_{n-1}\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(39);$$

hence

$$-\frac{4S}{4n^2-1}P_n' = P_{n-1} + P_{n+1} - 2CP_n,$$

or
$$\left(n^2 - \frac{1}{4}\right)(2CP_n - P_{n-1} - P_{n+1}) = SP_n' \quad\ldots\ldots(40).$$

Again we have also from (39)

$$4SP_n' = (2n+1)P_{n+1} - (2n-1)P_{n-1} - 2CP_n \quad\ldots\ldots(41),$$

and if we substitute in the expression found above from (40) and (41) it is seen to vanish.

The right-hand side of (35) as far as it depends on the potential due to the sheet is

$$\begin{aligned}&\Sigma\{b_n(SQ_n + 2CQ_n') - b_{n-1}Q'_{n-1} - b_{n+1}Q'_{n+1}\}\sin n\theta\\ &= -\frac{\pi B}{S}\Sigma\{SQ_n + 2CQ_n' - Q'_{n-1} - Q'_{n+1}\}\sin n\theta\\ &= 0.\end{aligned}$$

These results do not hold for the cases $n=1$ or 2, but by equating the coefficients of $\sin\theta$ and $\sin 2\theta$ in (35) to zero we shall have two equations which will determine the values of the constants A, B, in order that (35) may be satisfied by the forms assumed for $\lambda\mu$ in (36) and (36 a), and we shall thus have found the current function when the external magnetic disturbance is of the form

$$A_0(\sigma\theta)\,Q_1(\sigma)\sin\theta.$$

The coefficient of $\sin\theta$ on the left of (35) is

$$-\frac{R}{2S^2ip}\left\{\frac{7}{8}(2C\mu_1 - \mu_2) - \mu_2 - \frac{3}{2}S^2\lambda_1\right\} \quad\ldots\ldots\ldots\ldots(42),$$

now putting for the moment $\mu_0 = -S(AP_0 + BQ_0)$ we may shew as above that

$$\frac{7}{8}(2C\mu_1 - \mu_2 - \mu_0) + \mu_0 - \mu_2 - \frac{3}{2}S^2\lambda_1 = \frac{7C}{4}(AP_0' + BQ_0'),$$

hence the expression (42) is equal to

$$-\frac{R}{2S^2ip}\left\{\frac{7C}{4}(AP_0'+BQ_0')+\frac{S}{8}(AP_0+BQ_0)\right\}.$$

The coefficient of $\sin\theta$ on the right of (35) is

$$b_1(SQ_1+2CQ_1')-b_2Q_2'+A_0(SQ_1+2CQ_1')$$
$$=b_1(Q_0'+Q_2')-b_2Q_2'+A_0(Q_0'+Q_2')$$
$$=-\frac{\pi B}{S}Q_0'+\frac{P_1'}{S}(AP_0'+BQ_0')(Q_0'+Q_2')+A_0(Q_0'+Q_2')$$

by (36 a) and (37 a).

Equating then the two coefficients of $\sin\theta$, we have

$$\left.\begin{aligned}&-\frac{R}{2S^2ip}\left\{\frac{7C}{4}(AP_0'+BQ_0')+\frac{S}{8}(AP_0+BQ_0)\right\}\\&=-\frac{\pi B}{S}Q_0'+\frac{P_1'}{S}(AP_0'+BQ_0')(Q_0'+Q_2')+A_0(Q_0'+Q_2')\end{aligned}\right\}\text{ ...(43),}$$

which is the first relation between A and B.

Again, the coefficient of $\sin 2\theta$ on the left of (35) is

$$-\frac{R}{2S^2ip}\left\{\frac{19}{8}(2C\mu_2-\mu_3-\mu_1)+2(\mu_1-\mu_3)-\frac{3}{2}S^2\lambda_2\right\}$$
$$=-\frac{R}{2S^2ip}\left(-\frac{19}{8}+2\right)(AP_0'+BQ_0')\text{ by (36) and (37)}$$
$$=\frac{3R}{16S^2ip}(AP_0'+BQ_0'),$$

and on the right we have

$$b_2(SQ_2+2CQ_2')-b_1Q_1'-b_3Q_3'-A_0Q_1'$$
$$=-\frac{P_1'Q_1'}{S}(AP_0'+BQ_0')-A_0Q_1',$$

hence

$$\frac{3R}{16S^2ip}(AP_0'+BQ_0')=-\frac{P_1'Q_1'}{S}(AP_0'+BQ_0')-A_0Q_1',$$

or $$AP_0'+BQ_0'=\frac{-A_0SQ_1'}{\dfrac{3R}{16Sip}+P_1'Q_1'}\text{ ....(44);}$$

(43) and (44) determine AB in terms of A_0 and thence Φ, $\Omega_1\Omega_2$.

For free currents A_0 is zero, and (43), (44) must reduce to the same equation. This will be so if

$$Q_0' \left[-\frac{R}{2S^2ip}\left\{\frac{7C}{4}P_0' + \frac{S}{8}P_0\right\} - \frac{P_1'P_0'}{S}(Q_0'+Q_2')\right]$$

$$= P_0'\left[-\frac{R}{2S^2ip}\left\{\frac{7C}{4}Q_0' + \frac{S}{8}Q_0\right\} + \frac{\pi Q_0'}{S} - \frac{P_1'Q_0'}{S}(Q_0'+Q_2')\right],$$

i.e. if
$$-(P_0Q_0' - P_0'Q_0)\frac{R}{16Sip} = \frac{\pi}{S}P_0'Q_0',$$

or
$$\frac{R}{16Sip} = P_0'Q_0';$$

whence the modulus of decay is

$$-\frac{16SP_0'Q_0'}{R},$$

the corresponding current function being

$$\Phi = -\frac{1}{2S^2}\Sigma\lambda_n(\sigma\theta)^3 \sin n\theta\, e^{ipt}$$

$$= -\frac{1}{2S^2}\Sigma(AP_n' + BQ_n')(\sigma\theta)^3 \sin n\theta\, e^{ipt}$$

$$= A'\,\Sigma(Q_0'P_n' - P_0'Q_n')(\sigma\theta)^3 \sin n\theta\, e^{Rt/16SP_0'Q_0'},$$

since $AP_0' + BQ_0' = 0$, A' being an arbitrary constant and the summation extending from $n = 1$ to $n = \infty$.

(2) *The Complete System of Quaternariants for any Degree.* By D. B. Mair, B.A., Christ's College.

[Published in the *Transactions.*]

(3) *The Configuration of a pair of equal and opposite hollow straight vortices, of finite cross-section, moving steadily through fluid.* By H. C. Pocklington, B.A., St John's College.

The motion of a fluid that circulates about an annular hollow, which itself has a motion of translation, has been worked out by Dr Hicks in a paper in *Phil. Trans.* 1884. He there finds the shape of the cross-section and the velocity of the hollow, together with other circumstances of the motion. He, however, assumes the ratio of the diameter of the cross-section of the hollow to that of the ring to be a small quantity of which the higher powers may be neglected. As his results are thus only approximate it may be interesting to give an investigation of the two-dimensional case to which his three-dimensional one is analogous. This investigation the greater power of the methods of two dimensions enables us to make accurate.

Here the fluid, which is at rest at infinity, contains two hollows, and its circulations about these are equal and of opposite signs. The hollows themselves move, without change of form or relative position, in a direction parallel to a line which is an axis of symmetry and with a constant velocity V. We shall prove that this motion is possible, and find equations to give the shape of either hollow and the value of V. Impressing on the whole system a velocity equal and opposite to V, and regarding the motion on one side only of the above-mentioned axis of symmetry, we reduce the problem to that of the steady motion of fluid circulating about and flowing past a fixed hollow in a half-plane. Before proceeding with the solution, which is obtained by the method expounded by Mr Love in *Proc. Camb. Phil. Soc.* Vol. vii., p. 175 *et seq.*, the cyclic region occupied by the fluid must be converted into an acyclic one. This is done if we draw the shortest distance between the hollow and the straight boundary, which latter we will take as the axis of x, and consider it part of the boundary of the region dealt with. The line just drawn is easily seen to be the other axis of symmetry of the system, which, since the pressure at the surface of the hollow is unaltered by reversing all the velocities, must possess such an axis. We accordingly take it as the axis of y. The boundary in the plane of the complex variable $z = x + \iota y$ is now as shown in fig. 1, where the arrows show the direction of motion of the fluid.

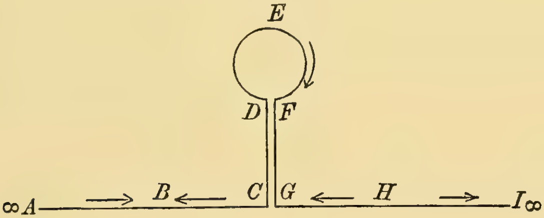


Fig. 1. Plane of z.

They are inserted on the strength of information given by the case where the hollows are small compared with their distance apart.

If we define w as $\phi + \iota\psi$, where ϕ and ψ are respectively the velocity-potential and the flow-function of the fluid, it is a function of z, as the equations satisfied by ϕ and ψ show. Hence if we put $\Omega = \log \frac{dz}{dw} = \log \frac{1}{q} + \iota\theta$, where $q \cos \theta$, $q \sin \theta$ are the components of velocity of the fluid, Ω also is a function of z. It follows that the region in the plane of z is represented conformably on the planes of w and Ω. The boundaries in these planes can be found. For, proceeding along AC, ψ is constant and ϕ increases from $-\infty$ at A to a certain value at B, and then diminishes to its value at C. Along CD, ϕ is constant and ψ decreases to a value which the arbitrary constant contained in it enables us to take as zero. Along DEF, ψ is constant and ϕ increases. Along FG, ϕ is constant and ψ increases till at G it has its original value, and finally along GI, ψ is constant while ϕ first decreases and then increases till at I it is infinite. The boundary in the w-plane indicated by the above is the symmetrical figure shown in fig. 2. Again, from A to B, $\theta = 0$ and $\log \frac{1}{q}$ increases from

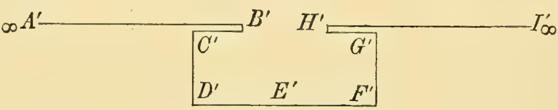


Fig. 2. Plane of w.

$\log \frac{1}{V}$ to ∞. From B to D, $\theta = \pi$ and $\log \frac{1}{q}$ decreases from ∞ to $\log \frac{1}{U}$, where U is the velocity of the fluid at the surface of the hollow. This velocity is constant, since the pressure there has the constant value zero. From D to F, $\log \frac{1}{q}$ is thus constant, while θ decreases from π to $-\pi$. From F to H, $\theta = -\pi$ while $\log \frac{1}{q}$ increases from $\log \frac{1}{U}$ to ∞. From H to I, $\theta = 0$ and $\log \frac{1}{q}$ de-

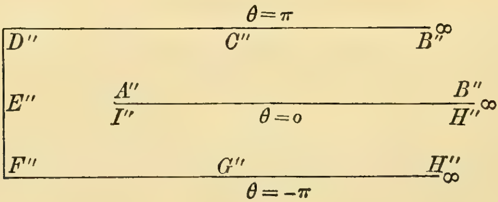


Fig. 3. Plane of Ω.

creases from ∞ to $\log \frac{1}{V}$. The boundary in the Ω-plane is therefore as shown in fig. 3.

The relation between w and t that transforms the boundary in the plane of w into the real axis in that of t is given by the theory of Schwarz and Christoffel in the form

$$w = k\lambda \int dt \frac{a^2 - t^2}{\sqrt{(1-t^2)\left(\frac{1}{k^2} - t^2\right)}},$$

where we have assumed that the points ± 1, $\pm k$, $\pm a$ are those corresponding to F' and D', G' and C', H' and B' respectively, and where $a > \frac{1}{k} > 1$.

To integrate this, put $t = \operatorname{sn} u$ (mod. k), and then

$$w = \lambda \int du\, (k^2a^2 - k^2 \operatorname{sn}^2 u)$$

$$= \lambda \left\{ Z(u) + \left(\frac{E}{K} + k^2a^2 - 1\right) u \right\} \quad \text{...............(1)},$$

if we chose the coefficient of integration so that $w = 0$ at E'. A consequence, utilized below, of this equation is that the points A', I' correspond to the points at infinity on the real axis in the plane of t. Between the three constants, λ, k, a, that enter into this equation, we can find two relations. For if $\phi_0 + \iota\psi_0$ is the value of w corresponding to G, so that $2\phi_0$ is the circulation about the hollow, ψ_0 the amount of fluid that in unit time flows between it and the axis of x, we have

$$\phi_0 + \iota\psi_0 = \lambda \left\{ Z(K + \iota K') + \left(\frac{E}{K} + k^2a^2 - 1\right)(K + \iota K') \right\}.$$

Putting for $Z(K + \iota K')$ its value

$$-\frac{\pi}{2K}\iota = -\iota K' \left(\frac{E}{K} + \frac{E'}{K'} - 1\right),$$

this is

$$\phi_0 + \iota\psi_0 = \lambda \left[\{E + (k^2a^2 - 1)K\} + \iota (k^2a^2K' - E') \right],$$

giving

$$\phi_0 = \lambda \{E + (k^2a^2 - 1)K\} \quad \text{..................(2)}$$

$$\psi_0 = \lambda (k^2a^2K' - E') \quad \text{.........................(3)}.$$

The relation between Ω and t by which the boundary in the Ω-plane is transformed into the real axis in the t-plane, so that the points D'' and F'', B'' and H'', A'' or I'' become the points $t = \pm 1$, $t = \pm a$, $t = \infty$, is given by

$$\Omega = \mu \int \frac{dt}{\sqrt{t^2 - 1}} \frac{1}{t^2 - a^2}$$

$$= \frac{\mu}{2a} \int \frac{dt}{\sqrt{t^2 - 1}} \left(\frac{1}{t - a} - \frac{1}{t + a} \right)$$

$$= \frac{\mu}{2a\sqrt{a^2 - 1}} \log \left\{ \left(\frac{at - 1 + \sqrt{a^2 - 1}\sqrt{t^2 - 1}}{t - a} \right) \left(\frac{at + 1 - \sqrt{a^2 - 1}\sqrt{t^2 - 1}}{t + a} \right)^{-1} \right\} + \nu.$$

Noticing that when t increases through the value a, the imaginary part of Ω diminishes by $\pi\iota$, we put

$$\frac{\mu}{2a\sqrt{a^2 - 1}} = 1.$$

Giving now to t the values 0 and ∞, and to Ω the corresponding values $\log \frac{1}{U}$ and $\log \frac{1}{V}$, we have the equations

$$\log \frac{1}{U} = \pi\iota + \nu,$$

$$\log \frac{1}{V} = \pi\iota + \nu + \log \frac{a + \sqrt{a^2 - 1}}{a - \sqrt{a^2 - 1}},$$

giving finally

$$\frac{V}{U} = \frac{a - \sqrt{a^2 - 1}}{a + \sqrt{a^2 - 1}} = 2a^2 - 1 - 2a\sqrt{a^2 - 1} \quad \text{.........(4)},$$

and

$$\Omega = \log \frac{1}{U} \left(\frac{at - 1 + \sqrt{a^2 - 1}\sqrt{t^2 - 1}}{t - a} \right) \left(\frac{at + 1 - \sqrt{a^2 - 1}\sqrt{t^2 - 1}}{t + a} \right)^{-1}$$

or, making the same substitution, $t = \operatorname{sn} u$, as before, and putting for Ω its value $\log \frac{dx}{dw}$,

$$\frac{dx}{dw} = \frac{1}{U} \frac{a \operatorname{sn} u - 1 + \iota\sqrt{a^2 - 1} \operatorname{cn} u}{\operatorname{sn} u - a} \Bigg/ \frac{a \operatorname{sn} u + 1 - \sqrt{a^2 - 1} \operatorname{cn} u}{\operatorname{sn} u + a}$$

$$= \frac{1}{U} \frac{(2a^2 - 1) \operatorname{sn}^2 u - a^2 + 2\iota a\sqrt{a^2 - 1} \operatorname{sn} u \operatorname{cn} u}{\operatorname{sn}^2 u - a^2}.$$

From (1) $\frac{dw}{du} = k^2\lambda(a^2 - \text{sn}^2 u)$, and so

$$\frac{dz}{du} = \frac{\lambda}{U}\left[(2a^2-1)\,\text{dn}^2 u - (2a^2 - a^2k^2 - 1) - 2\iota a\sqrt{a^2-1}\,k^2\,\text{sn}\,u\,\text{cn}\,u\right]$$

$$z = \frac{\lambda}{U}\left[(2a^2-1)\,Z(u) + \left\{(2a^2-1)\left(\frac{E}{K}-1\right) + a^2k^2\right\}2 \right.$$
$$\left. + 2\iota a\sqrt{a^2-1}\,\text{dn}\,u\right] + \text{const.}$$

Now z has the same value for D and F, that is, for $u = K$ and $u = -K$. This requires that the coefficient of u should vanish, that is

$$(2a^2-1)\left(\frac{E}{K}-1\right) + a^2k^2 = 0,$$

or

$$a^2 = \frac{K-E}{2(K-E) - k^2K} \quad \text{......................(5).}$$

Further, since $z = 0$ if $u = K + \iota K'$, the value corresponding to G, we find that the constant of integration is

$$= -\frac{\lambda}{U}(2a^2-1)\,Z(K+\iota K')$$

$$= \quad \frac{\lambda}{U}(2a^2-1)\,\frac{\pi\iota}{2K}.$$

Putting this value in the formula for z, and equating separately the real and imaginary quantities involved in the resulting equation,

$$x = \frac{\lambda}{U}(2a^2-1)\,Z(u) \quad \text{...............................(6),}$$

$$y = \frac{\lambda}{U}\left\{(2a^2-1)\frac{\pi}{2K} + 2a\sqrt{a^2-1}\,\text{dn}\,u\right\} \quad \text{.........(7).}$$

These equations with those numbered (1), (2), (3), (4) and (5), give the form of the free surface and the other details of a complete solution of the problem we started with.

We can in two special cases find the form of this curve without the aid of tables of elliptic functions. The first is that of k small. Using the approximate formulae

$$K = \frac{\pi}{2}\left(1 + \frac{k^2}{4} + \frac{9}{64}k^4\right),$$

$$E = \frac{\pi}{2}\left(1 - \frac{k^2}{4} - \frac{3}{64}k^4\right),$$

we find

$$a^2 = \frac{4}{K^2}$$

$$\lambda = \frac{\phi_0}{2\pi},$$

from (5) and (2) respectively. Also

$$\mathrm{dn}\, u = 1 - \tfrac{1}{2} k^2 \sin^2 u = 1 - \frac{k^2}{4} + \frac{k^2}{4} \cos 2u,$$

$$Z(u) = \int du \left\{ 1 - k^2 \sin^2 u - \left(1 - \frac{k^2}{2}\right)\right\}$$

$$= \frac{k^2}{4} \sin 2u,$$

giving

$$x = \frac{\phi_0}{\pi U} \sin 2u,$$

$$y = \frac{\phi_0}{\pi U}\left(\frac{8}{k^2} - 2 + \cos 2u\right),$$

and showing that the hollows are circles whose distance apart is great compared with their radii. The velocity of translation of the vortex is $V = (2a^2 - 1 - 2a\sqrt{a^2 - 1})\, U = \frac{k^2}{32} U$, a value agreeing with that which we can obtain by elementary methods.

The second tractable case is that in which k is nearly equal to unity. Here $K = \log \frac{2}{k'}$, where k' is the modulus complementary to k, and $E = 1$, giving

$$a^2 = 1 + \frac{1}{\log \frac{2}{k'}}, \quad \lambda = \frac{\phi_0}{2},$$

and thus

$$y = \frac{\phi_0}{2U}\left\{\frac{\pi}{2 \log \frac{2}{k'}} + 2 \sqrt{\left(\frac{1}{\log \frac{2}{k'}}\right)} . \,\mathrm{dn}\, u\right\}.$$

Since the limits of $\mathrm{dn}\, u$ are 1 and 0, and $\log \frac{2}{k'}$ is large, those of y are two small quantities, the larger of which, however, has a ratio to the smaller of the order $\sqrt{\log \frac{2}{k'}}$.

The two hollows of our original problem are therefore in this case of small breadth, while the length, since the perimeter is $\frac{2\phi_0}{U}$, is the finite quantity $\frac{\phi_0}{U}$. They are very close, their distance apart being, in fact, of the second order of small quantities if their breadths are of the first. The velocity of translation is

$$V = (2a^2 - 1 - 2a\sqrt{a^2-1})\,U = U,$$

which is the greatest value it can have.

It has been suggested to me that it would be of interest to discuss the shape of the sharp extremity of the vortex in this case. If we confine our attention to what takes place near the extremity, we may consider the case as that of a jet, proceeding from infinity with velocity U, which meets an infinitely broad

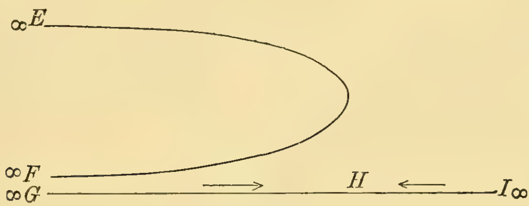
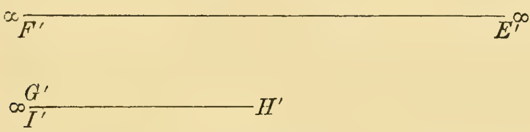


Fig. 4. Plane of z.

stream of fluid moving in the opposite direction with the same velocity, both the jet and the stream being bounded by the axis of x. The diagram in the plane of $z = x + \iota y$ is that given in fig. 4, where the arrows indicate the direction of motion of the fluid. Since FE and GHI are flow lines, and ϕ increases from $-\infty$

Fig. 5. Plane of w.



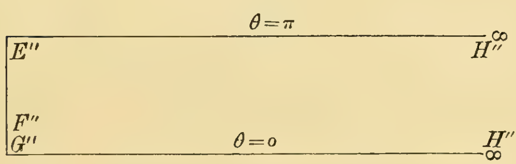


Fig. 6. Plane of Ω.

at G to a certain value at H, and then diminishes to $-\infty$ at I, the boundary in the plane of $w=\phi+\iota\psi$ is that shown in fig. 5. Again, along FE $\log\frac{1}{q}$ is equal to the constant $\log\frac{1}{U}$, while θ increases from 0 to π; along GH $\log\frac{1}{q}$ increases from $\log\frac{1}{U}$ to ∞, while θ is equal to zero; and along HI $\log\frac{1}{q}$ diminishes from ∞ to $\log\frac{1}{U}$ while θ is equal to π. The boundary in the plane of Ω is therefore that shown in fig. 6.

The formula transforming the boundary in the plane of w into the real axis in that of t is

$$w=\lambda\int dt\,\frac{t-1}{t},$$

where we have made F' or G', H' and E' or I' correspond to the points $t=0$, $t=1$, $t=\infty$.

The formula transforming the boundary in the plane of Ω into the real axis in that of t is

$$\Omega=\mu\int dt\,\frac{1}{(t-1)\sqrt{t}},$$

if we make F'' or G'', H'' and E'' or I'' correspond to the points $t=0$, $t=1$ and $t=\infty$. On integration this is

$$\Omega=2\mu\log\frac{\sqrt{t}-1}{\sqrt{t}+1}+\nu.$$

Noting that $\Omega=\log\frac{1}{U}$ when $t=0$, and increases by $\pi\iota$ when t increases through 1, we find $2\mu=-1$, and $\nu=\log-\frac{1}{U}$, and thus

$$\Omega=\log\frac{1}{U}\frac{1+\sqrt{t}}{1-\sqrt{t}}.$$

Since $\Omega=\log\frac{dz}{dw}$, this gives

$$\begin{aligned}\frac{dz}{dt}&=\frac{dz}{dw}\frac{dw}{dt}\\&=-\frac{\lambda}{U}\frac{t+1+2\sqrt{t}}{t},\end{aligned}$$

and so

$$z=-\frac{\lambda}{U}(t+4\sqrt{t}+\log t-5),$$

choosing the constant of integration so that the origin is at the point H which corresponds to $t = 1$.

The free surface is given by negative values of t. Putting therefore $t = -u$, and dividing the equation into two by considering separately the real and imaginary parts,

$$x = -\frac{\lambda}{U}(\log u - u - 5),$$

$$y = -\frac{\lambda}{U}(\pi + 4\sqrt{u}),$$

where the value of the positive quantity $-\frac{\lambda}{U}$ is determined by the breadth of the jet. Since the value of y corresponding to E is infinite, we conclude that the hollows of our original problem are, in the case when they are very close, much flatter on their near than on their remote sides.

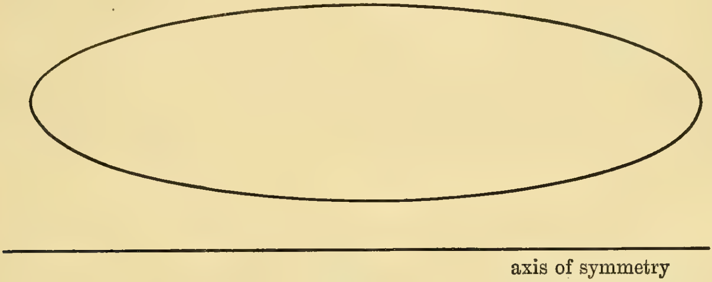


Fig. 7.

In fig. 7, is drawn an intermediate case of a pair of vortices in which k has the value $\sin 89° = \cdot 9998$. The Elliptic Functions are taken from Legendre's Tables. Only one hollow is drawn, the other being the image of this in the axis of symmetry. It is remarkable that, even for a value of k so near the limit unity as this is, the limiting case of linear hollows is not closely approximated to, although the velocity of translation, which is here $= \cdot 97\, U$, differs little from its limit U.

The artifice of making the region occupied by the fluid acyclic may be applied to any other case of the circulation of fluid about a hollow, provided that the system has an axis of symmetry. The only other case, besides that here treated, in which the integrations can be effected, seems to be that of a stationary hollow between two parallel planes, a case discussed by Prof. Michell in *Phil. Trans.* 1890.

Monday, March 12, 1894.

Prof. T. McK. Hughes, President, in the chair.

Malcolm Laurie, B.A., King's College, was elected a Fellow of the Society.

The following Communications were made to the Society:

(1) Dr W. H. R. Rivers showed apparatus devised by Prof. Hering to illustrate (1) colour-blindness of peripheral retina; (2) mirror-contrast; (3) influence of strength of illumination and of contrast on quality of colour; (4) diagnosis of colour-blindness.

(2) Mr J. C. Willis exhibited a plant of *Deherainea smaragdina* in flower. The flowers are interesting on account of their green colour, their large size and disagreeable smell. They are extremely protandrous. In the early stage the extrorse anthers completely surround and hide the stigma; later on the stamens bend away and come to rest on the corolla, and the flower is now female. From its colour, scent, &c., it is probably adapted to large flies.

(3) *Notes on the Bunbury Collection of Fossil Plants, with a list of type specimens in the Cambridge Botanical Museum.* By A. C. Seward, M.A., St John's College.

Between the years 1846 and 1861 several important communications on fossil plants were read before the Geological Society of London by Sir Charles Bunbury. In the case of some of these the value of the contributions seems to have been frequently overlooked by more recent writers. Bunbury's work bears obvious signs of careful investigation, and a cautious handling of the difficult problems involved in palæobotanical studies. His illustrations were faithfully drawn, and, so far as they are in any way open to criticism, it is that in some cases the artist has hardly done justice to the original specimens.

Through the generosity of Lady Bunbury, by whom the botanical department has previously been considerably enriched, the whole of the Bunbury collection of fossil plants has been presented to the University Botanical Museum.

The chief interest of the collection lies in the type specimens which it includes, and in the fact that it contains representative specimens from floras which are widely separated both geologically and geographically.

It is not proposed in these notes to attempt any critical *résumé* of Bunbury's work as a whole, but rather to call attention to his type specimens, and especially to one specimen of a Jurassic fern [*Klukia* (*Pecopteris*) *exilis*] described by him in 1851.

In illustration of the sound principles which influenced Bunbury's work it may not be out of place to quote a few sentences from one or two of his papers. Discussing the question of geological climates he thus expresses himself :—"If in all departments of geology it is necessary to advance with caution, it is more especially necessary in the department of fossil botany, where so much of the evidence we possess is fragmentary and imperfect[1]." This is, indeed, but the statement of a truism, but nevertheless it is a truism which bears repetition, and one which has been too often overlooked by students of geological floras. Again, on the subject of the distribution of Carboniferous ferns, he writes in the following words, which are still worthy of attention after an interval of forty-seven years :—"We may, I think, conclude that the wide diffusion of the same forms of vegetation through the ancient Carboniferous deposits of Europe and America, is less extraordinary than it would appear if we neglected to observe the large proportion of ferns in this ancient vegetation, and their distribution at the present day. Still it must be admitted that the uniformity of this ancient vegetation over so large an area, extending from Scotland to Alabama in latitude, and in longitude from Bohemia to Ohio, is greater than can be found at the present day; and I quite agree with Mr Lyell in believing that this indicates a greater uniformity and equability of climate, depending probably on a different distribution of land and sea. I believe we are fully justified by analogy in saying, that if such continents as Europe and North America had not existed at that period, but in their stead groups of islands, large and small, and if the ocean which now intervenes between the two continents had been thickly studded with such groups, like the Southern Pacific at present, there would have been nothing unnatural or surprising in such a uniformity of vegetation throughout those regions, as we now meet with in the Coal-formation. But I suggest this merely as a hypothesis, which, if admitted, would serve to explain a remarkable fact, and I do not wish to build much upon it; being aware, as I observed on a former occasion, of the danger of resting a large theory on such uncertain foundations as are supplied by fossil botany, at least in the present state of our knowledge[2]."

[1] Quart. Journ. Geol. Soc. Vol. II. 1846, p. 90.
[2] Q. J. G. S. Vol. III. 1847, p. 436.

List, with notes, of type specimens.

Pecopteris elliptica Bunb.

Quart. Journ. Geol. Soc., Vol. II. 1846, p. 84, Pl. vii.

Frostburg, Maryland. Coal-Measures.

Schimper[1] includes *Pecopteris* (*Cyath.*) *distans* Lesq. as a synonym of this species, but Lesquereux [2] himself, on the grounds of a few differences in detail, prefers to retain both specific names. The fructification on Bunbury's specimen is very indistinct, and affords no real clue as to the nature of the sori or sporangia.

Pecopteris bullata Bunb.

Quart. Journ. Geol. Soc.. Vol. III. 1847. p. 283, Pl. ii. fig. 1.

Richmond, Virginia. Trias.

In describing this fern Bunbury draws attention to the round pits in which the sori are placed, as the peculiar and distinctive character of the species.

He goes on to say that "in one or two instances I think I have observed an appearance resembling the reniform indusium of the Genus *Nephrodium*." He suggests that, in all probability, the plant belongs to the *Aspideœ* and may possibly be a genuine species of *Nephrodium*. The type specimen does not permit any very definite conclusion as to the nature of the fertile pinnæ. Most of the pinnules exhibit two rows of more or less oval projections contiguous with one another, and occupying nearly the whole breadth of the segments between the midrib and the margin of the pinnule. In one pinna, owing to a difference in the manner of preservation, the projections are replaced by pits on either side of the midrib. A careful examination of the specimen has convinced me that the indusium-like appearance spoken of by Bunbury is merely the outline of a small circular sorus with a slight central depression. There is nothing to justify a reference to the recent genus *Nephrodium*.

Prof. Fontaine[3] refers to Bunbury's species in his monograph on the older Mesozoic flora of Virginia; he changes the generic name to *Mertensides* and figures several examples of frond fragments. The sori are described as globular, and as consisting of 5—6 sporangia radially arranged. This description appears to agree with the appearance presented by Bunbury's specimen, but

[1] Trait. pal. vég. Vol. I. p. 506.

[2] Geol. Pennsylvania, H. D. Rogers, Vol. II. Pt. ii. 1858, p. 866. See also, Second Geol. Surv. Pennsylvania, Rep. Progress P. Vol. I. 1880, p. 245.

[3] U. S. Geol. Surv. Monograph, VI. 1883, p. 35, Pls. xv.—xix.

the pinnules are too imperfect to show the outlines of any individual sporangia. Fontaine considers that the plants grouped by him under his genus *Mertensides* have a great resemblance to the *Mertensia* group of the *Gleicheniaceæ,* and may be regarded as the precursors of that family. He adds that "the only point of difference between our plants and *Mertensia* is in the absence of dichotomous branching in *Mertensides.*"

One distinctive feature of *Pecopteris* (*Mertensides*) *bullata* lies in the existence of a relatively large and spatulate pinnule at the basal end on the lower side of each ultimate pinna. This character is clearly seen in several of Fontaine's figures, but cannot be traced in the incomplete fragment on which Bunbury founded the species.

There can be no doubt that Fontaine's figures represent the same species as Bunbury's specimen, but I cannot recognise any satisfactory evidence in the numerous figures in Fontaine's monograph for the adoption of such a generic term as *Mertensides.* The absence of dichotomous branching is in itself an important deviation from the characteristic habit of the recent *Mertensia.* As regards the sporangia in the "globular sori" there are no indications in Fontaine's figures of any detailed structure whatever.

I have elsewhere[1] protested against the use of recent generic names by this author in the case of Potomac (? Wealden) plants on much too slender grounds; and in the present instance we have yet to seek sufficiently trustworthy data, on which to found such sweeping statements as those of Fontaine on the relationship of *Pecopteris bullata* to the genus *Mertensia.* It should be noted that Stur[2] regards *P. bullata* as synonymous with *Oligocarpia robustior* Stur, from the Lunzer beds.

Filicites fimbriatus Bunb.

Quart. Journ. Geol. Soc. Vol. III. 1847, p. 283, Pl. ii. fig. 2.

Richmond, Virginia. Trias.

Bunbury regards this very imperfect fossil as a fertile portion of a fern frond, and designates it by the "vague and comprehensive name of *Filicites.*"

Fontaine[3] refers to this specimen as a badly preserved portion of a compound fertile pinna of *Asterocarpus virginiensis* Font. Possibly such a view may be tenable; there is certainly a distinct similarity between some of Fontaine's figures, e. g. Pl. xxiii. fig. 2, and the obscure specimen described by Bunbury, but the preservation is too imperfect to permit of any definite determination.

[1] The Wealden Flora, Pt. I. 1894, pp. 56 and 57 (British Museum Catalogue).

[2] Verhand. k. k. Geol. Reichs. Wien, No. 10, 1888, pp. 6 and 8.

[3] *loc. cit.* p. 44.

Neuropteris rarinervis Bunb.

Quart. Journ. Geol. Soc. Vol. III. 1847, p. 425, Pl. xxii.

Cape Breton, Nova Scotia. Coal-Measures.

Bunbury's figure represents a portion of a frond with a length of 32 cm. The pinnules and venation are clearly preserved. Solms-Laubach[1] refers to this species as an example of a fern possessing *Aphlebiæ*. Lesquereux[2] has figured several examples of *N. rarinervis* from Illinois, some of which show cyclopteroid aphlebia—leaflets; and more recently Zeiller[3] has figured an aphlebia-bearing specimen from the Valenciennes Coal-Measures. In Bunbury's example we have no actual proof of the occurrence of such cyclopteroid leaflets. This species has been recorded from several English localities, and Zeiller[4] has lately recognised it among the fragments of Coal-Measure plants from the Dover boring.

Odontopteris subcuneata Bunb.

Quart. Journ. Geol. Soc. Vol. III. 1847, p. 427, Pl. xxiii. fig. 1.

Cape Breton, Nova Scotia. Coal-Measures.

In the first notice of this fragment it is described as probably the extremity of a frond, and compared to *Odontopteris obtusa* Brong. Schimper[5] recognises the species in his *Traité de paléontologie végétale*, but Bunbury[6], in a later paper, prefers to consider it as a terminal portion of some large *Neuropteris*; "at any rate," he adds with his usual caution, "a species ought not to be founded on so imperfect a fragment as the only one I have seen of this supposed *Odontopteris*." In his *Studien über Odontopteriden*, Weiss[7] includes *Odontopteris Subcuneata* in his subgenus *Xenopteris*. Lesquereux[8] retains Bunbury's species, and speaks of it as possessing well-marked distinctive characters. It would seem then that we are justified in accepting this terminal fragment as the type of a distinct species.

Pecopteris tæniopteroides Bunb.

Quart. Journ. Geol. Soc. Vol. III. 1847, p. 428, Pl. xxiii. fig. 2.

Cape Breton, Nova Scotia. Coal-Measures.

The small specimen accurately represented in Bunbury's figure possesses a very well-defined venation. It is much too small to

[1] Fossil Botany (Engl. trans. 1891) p. 134.
[2] Geol. Surv. Illinois, Vol. IV. 1870, Pl. viii. figs. 1—6.
[3] Flor. foss. bass. houill., Valenciennes, 1886, Pl. xlv. fig. 2.
[4] Compt. rend. Vol. cxv. p. 628.
[5] Trait. pal. vég. Vol. I. p. 461.
[6] Q. J. G. S. Vol. XIV. 1858, p. 247.
[7] Zeitschr. deutsch. geol. Ges. Vol. XXII. 1870, p. 863.
[8] Second Geol. Surv. Pennsylvania, Vol. I. 1880, p. 134.

afford any adequate idea as to the nature of the complete frond; the form of the pinnules suggests rather an alethopteroid than pecopteroid type of fern.

Lepidodendron ? binerve Bunb.

Quart. Journ. Geol. Soc. Vol. III. 1847, p. 431, Pl. xxiv. fig. 2.

Cape Breton, Nova Scotia. Coal-Measures.

This species is founded on the occurrence of two longitudinal ribs in the small leaves of the terminal branch fragments figured by Bunbury. In one of the specimens the ribs are apparently not shown; in the other they are faintly indicated. It is doubtful how far this character affords sufficient grounds for the institution of a new species; possibly, as Kidston[1] suggests, in the case of *Lepidophyllum binerve,* figured by Lebour[2], the supposed existence of two veins rests on an accident of fossilization. It is conceivable, however, that the "double" leaf-trace of *Lepidodendron Harcourtii* With. may be recognised in well preserved leaves as two distinct vein-like ribs.

Lepidodendron tumidum Bunb.

Quart. Journ. Geol. Soc. Vol. III. 1847, p. 432, Pl. xxiv. fig. 1.

Cape Breton, Nova Scotia. Coal-Measures.

Bunbury speaks of this species as "one of those ambiguous forms which would be referred by some to *Lepidodendron,* and by others to *Sigillaria.*" He compares it with *L. ottonis* Goepp. and *Sigillaria Brardii* Brong. Schimper[3] refers Bunbury's plant to the genus *Lepidophloios,* and Macfarlane[4] apparently accepts this opinion. Kidston[5] considers, on the other hand, that *Sigillaria* should be substituted for *Lepidodendron* as a more fitting designation for this "ambiguous form": in this I thoroughly agree with him. The leaf-scars are rather farther apart than in many specimens of *Sigillaria Brardii,* but the great variation in the appearance of this species has been well illustrated by Zeiller[6] and Weiss[7], and should serve as a warning not to attach any great importance to the distance between individual leaf-scars on small fragmentary specimens.

Bunbury's figure gives one the impression of prominent and more or less well-defined leaf-cushions, each terminated by a leaf-

[1] Trans. R. Soc. Edinburgh, Vol. XXXV. Pt. I. 1888, p. 323.
[2] Illust. fossil plants, 1877, Pl. lii.
[3] *Loc. cit.* Vol. II. p. 52.
[4] Trans. Bot. Soc. Edinburgh, Vol. XIV. p. 182.
[5] Annals Mag. Nat. Hist. Series 5, Vol. XV. 1885, p. 359.
[6] Bull. Soc. Géol. France, Sér. 3, Vol. XVII. 1889, p. 603.
[7] Zeitsch. deutsch. geol. Ges. 1888, p. 565. (For an abstract of the two last papers, see Seward, Geol. Mag. Dec. iii. Vol. VII. 1890, p. 213.)

scar. As a matter of fact the leaf-scars occupy depressions, and there are no typical lepidodendroid cushions. The real nature of the specimen is best seen by taking a cast of the type specimen; we have then a surface view of the sigillarian stem, the leaf-scars project slightly, and short downward curved ridges extend for a short distance from the two lateral angles. The general appearance is very like that figured by Zeiller in *S. Brardii*[1].

Kidston[2] compares his species, *S. McMurtriei*, with Bunbury's specimen, and points out that in the description of the latter there is nothing said with regard to the form of the "lateral cicatricules" on the leaf-scar; these are not very well marked in the specimen, but sufficiently so to show that they are of the typical elongated oval form, and are situated relatively to the leaf-trace scar similarly to those in Kidston's species. The leaf-trace scar distinctly bears out Bunbury's description that each is made up of "two vascular points, placed close together, and often confluent."

My own impression is that Bunbury's specimen should be referred to Brongniart's species, *Sigillaria Brardii;* in any case the specific name *tumida* would be unsuitable as founded on an erroneous interpretation of the fossil.

Baiera gracilis Bunb.

Quart. Journ. Geol. Soc. Vol. VII. 1851, p. 182, Pl. xii. fig. 3.

Scarborough, Yorks. Oolite.

Bunbury describes this species as not uncommon in the Lower Sandstones of the Scarborough series. He considers it to be closely allied to *Cyclopteris digitata* L. and H.; it had previously been designated *Schizopteris gracilis* by Bean (MS. name). Bunbury includes the plant among fossil ferns, and compares it with *Acrostichum peltatum* Sw. Zigno[3] prefers to make use of the generic designation *Cyclopteris.* In the third edition of Phillips' "Illustrations of the Geology of Yorkshire" an inferior woodcut is given of Bunbury's species[4]. Saporta[5] figures several English examples of *Baiera gracilis,* and holds the view that the genus *Baiera* is closely allied to *Ginkgo.* The association of seeds and flowers with certain forms of *Baiera,* shows that at all events some species must be assigned to the Gymnosperms; it may, perhaps, be found that some of the dichotomously forked Jurassic leaves present a close resemblance to the recent Cycadean genus *Macrozamia.* In the Royal Gardens, Kew, there are some plants

[1] Zeiller *loc. cit.* Pl. XIV.
[2] *Loc. cit.* p. 360.
[3] Flor. foss. Oolith. Vol. I. 1856, p. 104.
[4] p. 199.
[5] Pal. Franç. Plantes Jurassiques, Vol. III. 1884, p. 277, Pl. clvii. fig. 4 and Pl. clviii. figs. 1—3.

of *Macrozamia heteromera* with divided leaves which suggest a comparison with such fossil forms as occur in the genus *Baiera*. I do not wish to imply that we have at present any grounds for emphasizing such resemblances, but would merely draw attention to the existence of somewhat similar leaves among recent Cycads.

Dictyopteris obliqua Bunb. (= *D. Brongniarti* Gutb.).
Quart. Journ. Geol. Soc., Vol. III. 1847, p. 427, Pl. xxi., fig. 2.

Cape Breton, Nova Scotia. Coal-Measures.

Bunbury describes the detached pinnules of this fern as "oblong, very obtuse, slightly convex, usually more or less curved, and sometimes in a remarkable degree; at the base they are oblique and slightly cordate, and evidently were attached to the stalk at one point only, as in *Neuropteris*. The midrib is very faint, often obsolete, and always vanishing far below the extremity of the leaflet; the lateral veins prominent and strongly marked, forming a regular and beautiful network with small meshes, which are longest and narrowest near the middle of the leaflet, becoming shorter and rounder towards the margins." The author of the species suggests that possibly these pinnules may be identical with *Dictyopteris Brongniarti* Gutb. On the whole I agree with Kidston[1] that it would be better to include Bunbury's species as a synonym of *D. Brongniarti*. Zeiller[2] has recently figured some detached *Dictyopteris* pinnules which he does not refer to any species; they appear to be practically identical with those described by Bunbury.

Figured specimens other than types of species.

Neuropteris in circinate vernation.
Quart. Journ. Geol. Soc., Vol. XIV. 1858, p. 243.

The figure of what Bunbury considered, and probably correctly, to be a young frond of *Neuropteris gigantea* Sternb. hardly does justice to the beautifully preserved specimen from the Oldham Coal-Measures. The hairs on the rachis mentioned in the description are not obvious, but it is reasonable to suppose that such were actually present. The figure shows an apparently shaded structure at right angles to the frond tip; this is merely a fractured surface in the stone.

After describing this example of circinate vernation, Bunbury proceeds to discuss the affinities of *Neuropteris;* the following remarks are worth quoting as being in close agreement with the views of the late Dr Stur. "It is certainly not very easy to

[1] Cat. Pal. plants, B. M. 1886, p. 103.
[2] Bass. houill. et Perm. Brive. 1892, p. 47, Pl. ix. fig. 2.

prove positively that the Neuropterids may not belong to that family" (i.e. *Cycadeæ*). And further on:—"I shall not be surprised to find that the Neuropterids differ considerably in their real affinity from those recent ferns to which they have most likeness in outline and veining[1]."

Neuropteris cordata, Brong. (Bunbury's specimens referred to Brongniart's species must be removed to *N. Scheuchzeri* Hoff.)

Quart. Journ. Geol. Soc., Vol. III. 1847, p. 423, Pl. xxi. fig. 1.

Cape Breton, Nova Scotia. Coal-Measures.

Bunbury's specimens of this species no doubt represent the same plant as that figured by Lesquereux[2] as *Neuropteris hirsuta*. Kidston[3] includes *N. hirsuta* Lesq. as a synonym of *N. Scheuchzeri* Brong. The hairs on the surface of the pinnules are shown very clearly in Bunbury's specimens.

Bunbury figures a pinnule which he describes as presenting "appearances somewhat resembling fructification"; but, he adds, they may be due to some disease of the parenchyma or to a fungus. The figures represent fairly accurately the appearances in question; they have the form of elliptical depressions *between* the veins of the pinnule; each pit is surrounded by a slightly raised rim.

Exactly similar marks have been figured by Fontaine and White[4]; these authors compare the oval depressions to the sori and indusia of *Scolopendrium*. The position of these pits between the veins as shown by the figures of Bunbury and Fontaine and White, is scarcely consistent with the idea of sori.

In any case, in the absence of all structure and detail, we can only say that if the marks are really traces of fructification they throw little or no light on the taxonomic position of the species.

There is no evidence whatever to warrant the assumption that we have to do with fern sori, and we are still in the dark as to the nature of the sporophylls of this form of *Neuropteris*.

Kidston[5] has shown that the fern pinnæ figured by Bunbury as *N. cordata* must be referred to Hoffmann's species *N. Scheuchzeri*. The same writer expresses his belief that the so-called sori of Bunbury, and Fontaine and White are simply the result of fungal parasitism.

[1] Q. J. G. S. Vol. XIV. 1858, p. 245.

[2] Geol. Pennsylvania 1858, Pl. iii. Also Second Geol. Surv. Penn. 1880, Vol. i. p. 88, Pl. viii. fig. 1.

[3] Trans. R. Soc. Edinburgh, Vol. XXXIII. p. 356. (It should be noted that Kidston in this paper, p. 359, corrects an opinion expressed by him in his Cat. Palæozoic plants.)

[4] Second Geol. Surv. Penn. Rep. Progress P. P. 1880, p. 47, Pl. viii. figs. 7 and 8.

[5] Trans. R. Soc. Edinburgh, Vol. XXXIII. p. 151.

Pecopteris exilis Phill. [= *Klukia exilis* (Phill).].

Quart. Journ. Geol. Soc., Vol. VII. 1851, p. 188, Pl. xiii. fig. 5.

In 1829 Phillips[1] described this species of fern from the Upper Sandstones and Shales of the Yorkshire coast; his figure shows two fertile pinnæ which afford but a poor idea of the actual plant. The pinnules are described as "marked with round swellings over the seed vessels."

Bunbury gives a figure of a single sterile pinnule, and another showing sporangia on the under surface of a fertile segment; he describes the sporangia as not in sori, but singly in regular rows on each side of the midrib. He speaks of the sporangia as large in proportion to the size of the leaflet, of an ovate and spherical form, with regular striæ radiating from a central point; they are compared with those of *Schizaceæ*, and especially with the sporangia of *Anemia* and *Mohria*. Among fossil forms *Senftenbergia* presents the closest resemblance, but its sporangia differ in having apical rings of two layers instead of one.

Solms-Laubach[2], in referring to *Senftenbergia* and Bunbury's description of the fertile specimen of *Pecopteris exilis*, suggests the desirability of a re-examination of the latter. More recently Raciborski[3] has made an exceedingly important contribution to our botanical knowledge of Mesozoic ferns. He describes several specimens of fertile fern fronds from Jurassic strata in the neighbourhood of Cracow, and *Pecopteris exilis* is discussed at some length as a Jurassic representative of the *Schizaceæ*.

Raciborski points out that Bunbury was the first to give any accurate account of the sporangia of Phillips' species; but apparently Schimper and some other palæobotanists have overlooked Bunbury's paper.

Several fertile segments of this species have been found in the Cracow beds[4], and an examination of the sporangia enables Raciborski to confirm the description of the English specimen, as well as its inclusion in the family *Schizaceæ*. As our knowledge of the fructification of fossil ferns gradually increases, we are able to dispense with such provisional generic terms as *Pecopteris*, *Sphenopteris*, &c., and are in a position to institute new names expressive of some definite botanical affinity.

In the case of *Pecopteris exilis* Raciborski has proposed a new genus *Klukia*, to be included with *Anemia* and others in the *Schizaceæ*. The introduction of a new term is usually to be avoided, especially in a subject like fossil botany, which is hopelessly overloaded with synonym lists; but in cases where

[1] Illust. Geol. Yorks. 1829, Pl. viii. fig. 16.
[2] Fossil Botany, p. 152.
[3] Bot. Jahrb. (Engler) Vol. XIII. 1890, p. 1.
[4] Ibid. Pl. i. figs. 17—19.

our knowledge has been definitely advanced, and we are able to transfer a fossil fern from a purely provisional genus to a definite taxonomic position, it is undoubtedly wise to mark such advance by the institution of a new term.

It is fortunate that Raciborski has chosen a new generic title and has not attempted to include the Jurassic species in any existing Schizaceous genus. Nothing is more misleading than the utterly unscientific and unjustifiable use of recent generic names for fossil specimens which present some superficial resemblance to living forms, but which afford no trustworthy evidence as to their connection with any existing genus or even family.

Bunbury's specimen shows portions of four pinnæ attached to a rachis, and one more clearly preserved detached pinna; the latter is represented, natural size, in fig. 5, the small dots on the pinnules show the position of partially developed sporangia which are visible on the upper surface.

In fig. 4 we have a slightly magnified (× 3) drawing of the under surface of a fertile pinnule with fully developed sporangia;

KLUKIA EXILIS (Phill).

Fig. 1—3 × 40. Fig. 4 × 3. Fig. 5, nat. size.

there appear to be five in each row, and the sporangia show more or less distinctly the characteristic apical annulus. In some of the pinnules the sporangia have a clearly marked line along which longitudinal dehiscence has taken place.

The detailed structure of the sporangia is shown in figs. 1—3 (× 40). In fig. 1 we have an apical view of a sporangium showing about fourteen cells in the annulus; at the apex there is a slight central depression, which was no doubt originally occupied by thin walled cells as in the recent *Anemia*[1]. Fig. 2 presents a

[1] See Luerssen, Grundzüge der Botanik 1885, p. 319, fig. 170.

still clearer view of the annulus, and what appears to be an indication of longitudinal splitting.

In some cases, e.g. fig. 1, there are faint indications of the walls of the sporangial cells below the annulus.

Fig. 3 shows very clearly a gaping longitudinal slit which stands out conspicuously owing to the filling up of the sporangial cavity by light-coloured sandy material. On the whole the sporangia agree very closely with those of *Anemia*; they are slightly broader in proportion to their length than in the recent genus, but not so nearly spherical as in *Mohria*[1]; and in the latter the annulus cells are less clearly differentiated from the thinner walled cells of the sporangium wall[2].

The characters of the fossil sporangia correspond to those described by Raciborski in the Cracow specimens.

At present I will content myself with this confirmation of Bunbury's surmise as to the systematic position of Phillips' *Pecopteris exilis;* on another occasion I hope to discuss the difficult question of the relation between such forms as *Klukia exilis* (Phill.), *Pecopteris obtusifolia* L. and H., *Sphenopteris serrata* L. and H., *P. exiliforme* as figured by Geyler from Japan, *P. exilis* as represented by Yokoyama from the same country, and the Wealden species *Cladophebis Dunkeri* (Schimp.).

I desire to express my thanks to Miss Dorothea F. M. Pertz for contributing the sketches which were utilised by Mr Wilson in his preparation of the woodblock.

(4) *Note on the Liver Ferment.* By Miss M. C. Tebb.

By extraction with glycerin Claude Bernard[3] obtained from liver a ferment which converted glycogen to sugar, but the properties of this sugar were not described. Miss Eves[4] extracted from liver a ferment which was active on starch and glycogen, but she states that the product of action was certainly not dextrose, but a sugar of less reducing power, and she considers it very probable that she was dealing with the ordinary amylolytic ferment, converting starch and glycogen to *maltose,* and obtainable from most tissues of the body.

In the present research pig's liver was rapidly dried at 35°—40° C. and finely shredded, and the sugar present initially was removed by dialysis. It was found that this dried liver produced dextrose when allowed to act on starch or glycogen,

[1] See Hooker, Gen. Filicum 1842, Pl. civ. B.

[2] Cf. *Trochopteris elegans* as regards the arrangement of the sporangia, Hooker, *loc. cit.* Pl. civ. A.

[3] Comptes Rendus 1877, T. LXXXV. p. 519.

[4] Journal of Physiology, 1884, Vol. v, p. 342.

and this whether the blood was previously washed out of the liver or not. Several experiments were performed, of which the following is an example: 10 grammes of dried liver, free from blood and sugar, was allowed to act on a solution of glycogen for 22 hours at 25° C. The sugar produced was removed by dialysis, and on adding phenyl hydrazin to a portion of the dialysate, typical crystals of phenyl glucosazone were formed. The reducing power of another portion was estimated, and after boiling the solution with 2% hydrochloric acid for 30 minutes, the reducing power was found to have increased but slightly—in the ratio 65 to 68—which change is probably due to the presence of dextrin, with or without maltose.

Extracts were made by soaking the dried tissue in 5% sodium sulphate, the sugar initially present being subsequently removed by dialysis, and these were found to be active in the same way.

Also an extract was made from fresh liver, and this was found to produce dextrose from starch, but no experiment was made with glycogen. In most of the experiments digestion was allowed to go on at 37° C. for about 20 hours, though sometimes for not more than 4 hours; and the resulting sugar was in some cases removed by extraction with alcohol instead of by dialysis.

Digestion was always carried on in neutral or faintly alkaline media, and antiseptics, usually chloroform, were used throughout.

In all cases, whether an extract or the dried tissue itself was used, the product of the action on starch or glycogen always gave crystals of phenyl glucosazone with phenyl hydrazin, and the reducing power increased *only slightly* on boiling with acid; hence the conclusion is drawn that one product of the action is *dextrose*; and, as far as they have gone, the experiments with fresh liver have yielded the same result.

April 30, 1894.

PROF. T. McK. HUGHES, PRESIDENT, IN THE CHAIR.

D. B. Mair, B.A., Fellow of Christ's College, R. H. D. Mayall, B.A., Sidney Sussex College, and H. C. Pocklington, B.A., St John's College, were elected Fellows of the Society.

The following Communications were made to the Society:

(1) *The modifications of Fresnel's optical laws, that would apply to circumstances of magnetic as well as electric aeolotropy.* By Mr J. Larmor.

(2) *On the Application of the Theory of Matrices to the Discussion of Linear Differential Equations with Constant Coefficients.* By J. BRILL, M.A., St John's College.

1. Let p and q stand for the two matrices

$$\begin{matrix} \alpha_1, & \beta_1 \\ \gamma_1, & \delta_1 \end{matrix} \quad \text{and} \quad \begin{matrix} \alpha_2, & \beta_2 \\ \gamma_2, & \delta_2. \end{matrix}$$

Then we have the three identities:

$$p^2 - (\alpha_1 + \delta_1) p + \alpha_1\delta_1 - \beta_1\gamma_1 = 0,$$

$$q^2 - (\alpha_2 + \delta_2) q + \alpha_2\delta_2 - \beta_2\gamma_2 = 0,$$

$$pq + qp - (\alpha_2 + \delta_2) p - (\alpha_1 + \delta_1) q + \alpha_1\delta_2 + \alpha_2\delta_1 - \beta_1\gamma_2 - \beta_2\gamma_1 = 0.$$

If we now form the identical equation satisfied by the matrix $x + py + qz$, we shall obtain

$$(x + py + qz)^2 - \{2x + y(\alpha_1 + \delta_1) + z(\alpha_2 + \delta_2)\}(x + py + qz) + (x + \alpha_1 y + \alpha_2 z)(x + \delta_1 y + \delta_2 z) - (\beta_1 y + \beta_2 z)(\gamma_1 y + \gamma_2 z) = 0,$$

which may be written in the form

$$\{-x + (p - \alpha_1 - \delta_1) y + (q - \alpha_2 - \delta_2) z\}(x + py + qz) + x^2 + (\alpha_1\delta_1 - \beta_1\gamma_1) y^2 + (\alpha_2\delta_2 - \beta_2\gamma_2) z^2 + (\alpha_1\delta_2 + \alpha_2\delta_1 - \beta_1\gamma_2 - \beta_2\gamma_1) yz + (\alpha_2 + \delta_2) zx + (\alpha_1 + \delta_1) xy = 0.$$

Thus we see that the theory of matrices will enable us to resolve the result of the multiplication of the matrix unity by a scalar ternary quadric into linear factors. In fact, we have

$$ax^2 + by^2 + cz^2 + fyz + gzx + hxy = \{ax + (h - ap) y + (g - aq) z\}(x + py + qz),$$

where p and q identically satisfy the three equations

$$ap^2 - hp + b = 0,$$

$$aq^2 - gq + c = 0,$$

$$a(pq + qp) - gp - hq + f = 0.$$

It is further to be remarked that these two factors are commutative.

The above result enables us to replace the operator

$$a\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial y^2} + c\frac{\partial^2}{\partial z^2} + f\frac{\partial^2}{\partial y\partial z} + g\frac{\partial^2}{\partial z\partial x} + h\frac{\partial^2}{\partial x\partial y}$$

by its equivalent

$$\left\{a\frac{\partial}{\partial x}+(h-ap)\frac{\partial}{\partial y}+(g-aq)\frac{\partial}{\partial z}\right\}\left\{\frac{\partial}{\partial x}+p\frac{\partial}{\partial y}+q\frac{\partial}{\partial z}\right\}.$$

Any matrical function of x, y, z, which is made to vanish by either of the two linear operators, may be considered as a matrical solution of the equation

$$a\frac{\partial^2\theta}{\partial x^2}+b\frac{\partial^2\theta}{\partial y^2}+c\frac{\partial^2\theta}{\partial z^2}+f\frac{\partial^2\theta}{\partial y\partial z}+g\frac{\partial^2\theta}{\partial z\partial x}+h\frac{\partial^2\theta}{\partial x\partial y}=0.$$

2. Let λ_1, λ_2 be the latent roots of the matrix m, then the identical equation satisfied by the matrix may be expressed in the form

$$m^2-m(\lambda_1+\lambda_2)+\lambda_1\lambda_2=0.$$

Differentiating, we have

$$m\,.\,dm+dm\,.\,m-dm(\lambda_1+\lambda_2)-m(d\lambda_1+d\lambda_2)+\lambda_1 d\lambda_2+\lambda_2 d\lambda_1=0.$$

We will now assume

$$dm=\frac{m-\lambda_2}{\lambda_1-\lambda_2}d\lambda_1+\frac{m-\lambda_1}{\lambda_2-\lambda_1}d\lambda_2+d\omega.$$

Substituting this value for dm in the above equation, we obtain

$$(2m-\lambda_1-\lambda_2)\left\{\frac{m-\lambda_2}{\lambda_1-\lambda_2}d\lambda_1+\frac{m-\lambda_1}{\lambda_2-\lambda_1}d\lambda_2\right\}+(m-\lambda_1-\lambda_2)\,d\omega$$
$$+d\omega\,.\,m-(m-\lambda_2)\,d\lambda_1-(m-\lambda_1)\,d\lambda_2=0,$$

which reduces to

$$2(m-\lambda_1)(m-\lambda_2)\frac{d\lambda_1-d\lambda_2}{\lambda_1-\lambda_2}+(m-\lambda_1-\lambda_2)\,d\omega+d\omega\,.\,m=0.$$

The first term of this equation vanishes by virtue of the identical equation satisfied by m, and consequently the equation reduces to

$$(m-\lambda_1-\lambda_2)\,d\omega+d\omega\,.\,m=0,$$

which may also be written in the form

$$m\,.\,d\omega\,.\,m=\lambda_1\lambda_2 d\omega.$$

It is not necessary for our present purposes to find the general value of $d\omega$ which satisfies this equation, it will be sufficient to note that a particular solution is $d\omega=0$.

Making this assumption, which is equivalent to assuming that dm shall be commutative with m, we have

$$dm=\frac{m-\lambda_2}{\lambda_1-\lambda_2}d\lambda_1+\frac{m-\lambda_1}{\lambda_2-\lambda_1}d\lambda_2 \quad\ldots\ldots\ldots\ldots\ldots(1),$$

which is equivalent to

$$d\left(\frac{2m-\lambda_1-\lambda_2}{\lambda_1-\lambda_2}\right)=0,$$

or

$$\frac{2m-\lambda_1-\lambda_2}{\lambda_1-\lambda_2}=\text{const.}$$

Now if $f(x)$ be any scalar function which may be expanded in powers of x, we may, by substituting m for x, obtain a matrical function which may be spoken of as framed on the model of the scalar function $f(x)$. Writing $f(m)$ for this matrical function, we have by Sylvester's Interpolation Theorem

$$f(m)=\frac{m-\lambda_2}{\lambda_1-\lambda_2}f(\lambda_1)+\frac{m-\lambda_1}{\lambda_2-\lambda_1}f(\lambda_2).$$

Differentiating this, we obtain

$$\begin{aligned} d\,.f(m)=&\frac{dm-d\lambda_2}{\lambda_1-\lambda_2}f(\lambda_1)+\frac{dm-d\lambda_1}{\lambda_2-\lambda_1}f(\lambda_2)\\ &-\frac{d\lambda_1-d\lambda_2}{(\lambda_1-\lambda_2)^2}\{(m-\lambda_2)f(\lambda_1)-(m-\lambda_1)f(\lambda_2)\}\\ &+\frac{m-\lambda_2}{\lambda_1-\lambda_2}f'(\lambda_1)\,d\lambda_1+\frac{m-\lambda_1}{\lambda_2-\lambda_1}f'(\lambda_2)\,d\lambda_2. \end{aligned}$$

If we assume for dm the value given by equation (1), then it is easily proved that the earlier part of the expression for $d\,.f(m)$ vanishes, and the above equation reduces to

$$d\,.f(m)=\frac{m-\lambda_2}{\lambda_1-\lambda_2}f'(\lambda_1)\,d\lambda_1+\frac{m-\lambda_1}{\lambda_2-\lambda_1}f'(\lambda_2)\,d\lambda_2.$$

But, multiplying equation (1) by $m-\lambda_2$, we have, in virtue of the identical equation satisfied by m,

$$\begin{aligned} dm\,.(m-\lambda_2)&=\frac{(m-\lambda_2)^2}{\lambda_1-\lambda_2}\,d\lambda_1\\ &=\frac{m^2-2\lambda_2 m+\lambda_2^2}{\lambda_1-\lambda_2}\,d\lambda_1\\ &=\frac{(\lambda_1-\lambda_2)(m-\lambda_2)}{\lambda_1-\lambda_2}\,d\lambda_1\\ &=(m-\lambda_2)\,d\lambda_1, \end{aligned}$$

or

$$(dm-d\lambda_1)(m-\lambda_2)=0.$$

Similarly we shall obtain

$$(dm-d\lambda_2)(m-\lambda_1)=0.$$

Therefore, we have

$$d\,.f(m) = dm\,.\left\{\frac{m-\lambda_2}{\lambda_1-\lambda_2}f'(\lambda_1) + \frac{m-\lambda_1}{\lambda_2-\lambda_1}f'(\lambda_2)\right\}$$
$$= dm\,.f'(m),$$

where $f'(m)$ is the matrical function framed on the model of the scalar function $f'(x)$.

Supposing m to vary only as depending on the three scalar variables x, y, z, we may write the above equation in the form

$$\left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right).f(m) = \left(\frac{\partial m}{\partial x}dx + \frac{\partial m}{\partial y}dy + \frac{\partial m}{\partial z}dz\right).f'(m).$$

This being true for all possible values of the ratios $dx : dy : dz$, we see that the equation resolves itself into the three

$$\frac{\partial}{\partial x}f(m) = \frac{\partial m}{\partial x}f'(m),\quad \frac{\partial}{\partial y}f(m) = \frac{\partial m}{\partial y}f'(m),\quad \frac{\partial}{\partial z}f(m) = \frac{\partial m}{\partial z}f'(m).$$

Thus if we write

$$\Delta \equiv \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial z},$$

we have
$$\Delta f(m) = \Delta m\,.f'(m).$$

Hence if m be such that

$$\frac{2m-(\lambda_1+\lambda_2)}{\lambda_1-\lambda_2} = \text{const., and } \Delta m = 0,$$

then we have also $\Delta f(m) = 0$.

3. If we write

$$u = y - px,\quad v = z - qx,$$

then we have $\Delta u = 0$, and $\Delta v = 0$. Further, writing $m = \xi u + \eta v$, where ξ and η are scalar constants, we have $\Delta m = 0$.

If λ_1 and λ_2 be the latent roots of m, we have

$$\lambda_1 + \lambda_2 = 2(\xi y + \eta z) - \frac{x}{a}(h\xi + g\eta),$$

$$\lambda_1\lambda_2 = (\xi y + \eta z)^2 - \frac{x}{a}(\xi y + \eta z)(h\xi + g\eta)$$
$$+ \frac{x^2}{a}(b\xi^2 + c\eta^2 + f\xi\eta).$$

Therefore

$$2m - (\lambda_1 + \lambda_2) = \frac{x}{a}\{\xi(h - 2ap) + \eta(g - 2aq)\},$$

and $$(\lambda_1 - \lambda_2)^2 = \frac{x^2}{a^2}\{(h\xi + g\eta)^2 - 4a(b\xi^2 + c\eta^2 + f\xi\eta)\}.$$

Consequently

$$\frac{2m - (\lambda_1 + \lambda_2)}{\lambda_1 - \lambda_2} = \pm\frac{\xi(h - 2ap) + \eta(g - 2aq)}{\{(h\xi + g\eta)^2 - 4a(b\xi^2 + c\eta^2 + f\xi\eta)\}^{\frac{1}{2}}}$$
$$= \text{const.}$$

Thus we have $\Delta f(m) = 0$.

We see, therefore, that for the solution of the equation $\Delta w = 0$ we may write

$$w = e^{u\frac{\partial}{\partial\xi} + v\frac{\partial}{\partial\eta}} \, . \, F(\xi, \eta),$$

where $$F(\xi, \eta) = A + B\xi + C\eta + \frac{1}{2!}(D\xi^2 + 2E\xi\eta + F\eta^2)$$
$$+ \frac{1}{3!}(G\xi^3 + 3H\xi^2\eta + 3K\xi\eta^2 + L\eta^3) + \&c.,$$

the coefficients A, B, C, &c. being matrices.

We will now expand the operative symbol, and make ξ and η zero after the operations have been performed. The only term which gives any value when operated upon by $\frac{\partial^{r+s}}{\partial\xi^r\,\partial\eta^s}$ is, leaving out the matrical coefficient,

$$\frac{1}{(r+s)!} \cdot \frac{(r+s)!}{r!\,s!}\xi^r\eta^s = \frac{1}{r!\,s!}\xi^r\eta^s.$$

This is reduced to unity by the said operation. Also, the coefficient of $\frac{\partial^{r+s}}{\partial\xi^r\partial\eta^s}$ in the expansion of the operating symbol contains the sum of all the products that can be formed involving u and v respectively r and s times. Thus we shall finally obtain as our solution

$$w = A + uB + vC + \frac{1}{2!}\{u^2D + (uv + vu)E + v^2F\}$$
$$+ \frac{1}{3!}\{u^3G + (u^2v + uvu + vu^2)H$$
$$+ (uv^2 + vuv + v^2u)K + v^3L\} + \&c.$$

Further, we have

$$dw = \left(du\frac{\partial}{\partial \xi} + dv\frac{\partial}{\partial \eta}\right) e^{u\frac{\partial}{\partial \xi} + v\frac{\partial}{\partial \eta}} . F(\xi, \eta)$$

$$= du \,.\, e^{u\frac{\partial}{\partial \xi} + v\frac{\partial}{\partial \eta}} . \frac{\partial F}{\partial \xi} + dv \,.\, e^{u\frac{\partial}{\partial \xi} + v\frac{\partial}{\partial \eta}} . \frac{\partial F}{\partial \eta},$$

where ξ and η have to be made to vanish after the operations have been performed. Thus we shall have an equation of the form

$$dw = du \,.\, U + dv \,.\, V,$$

where U and V are formed from $\partial F/\partial \xi$ and $\partial F/\partial \eta$ in the same manner as w is formed from $F(\xi, \eta)$.

4. It can be proved that the sum of all the products involving u and v respectively r and s times can be expressed in the form

$$k_1 + k_2 u + k_3 v,$$

where k_1, k_2, k_3 are scalars; or, substituting for u and v their values, in the form

$$k_4 + k_5 p + k_6 q.$$

Further, the matrical coefficients, A, B, C, &c., can each be expressed in the form

$$l_1 + l_2 p + l_3 q + l_4 pq,$$

where l_1, l_2, l_3, l_4 are scalars. Thus our function w may be expressed in the form

$$T + Xp + Yq + Zpq,$$

where X, Y, Z, T are scalar functions of x, y, z. Hence, if we write

$$\nabla \equiv a\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial y^2} + c\frac{\partial^2}{\partial z^2} + f\frac{\partial^2}{\partial y \partial z} + g\frac{\partial^2}{\partial z \partial x} + h\frac{\partial^2}{\partial x \partial y},$$

we have

$$\nabla T + \nabla X \,.\, p + \nabla Y \,.\, q + \nabla Z \,.\, pq = 0,$$

from which we may conclude that

$$\nabla T = \nabla X = \nabla Y = \nabla Z = 0,$$

provided that the determinant

$$\begin{vmatrix} 1, & \alpha_1, & \alpha_2, & \alpha_1\alpha_2 + \beta_1\gamma_2 \\ 0, & \beta_1, & \beta_2, & \alpha_1\beta_2 + \beta_1\delta_2 \\ 0, & \gamma_1, & \gamma_2, & \gamma_1\alpha_2 + \delta_1\gamma_2 \\ 1, & \delta_1, & \delta_2, & \gamma_1\beta_2 + \delta_1\delta_2 \end{vmatrix}$$

does not vanish. Rejecting this special case, we see that X, Y, Z, T are solutions of the differential equation mentioned in Art. 1.

Writing out the equation $\Delta w = 0$ at full length, and making use of the equations satisfied by p and q, we obtain

$$a^2\frac{\partial T}{\partial x} - ab\frac{\partial X}{\partial y} - af\frac{\partial X}{\partial z} - ac\frac{\partial Y}{\partial z} - hc\frac{\partial Z}{\partial z}$$
$$+ pa\left\{a\left(\frac{\partial T}{\partial y} + \frac{\partial X}{\partial x}\right) + h\frac{\partial X}{\partial y} + g\frac{\partial X}{\partial z} + c\frac{\partial Z}{\partial z}\right\}$$
$$+ q\left\{a^2\left(\frac{\partial T}{\partial z} + \frac{\partial Y}{\partial x}\right) + ah\frac{\partial X}{\partial z} + ag\frac{\partial Y}{\partial z} - ab\frac{\partial Z}{\partial y} + (gh - af)\frac{\partial Z}{\partial z}\right\}$$
$$+ pqa\left\{a\left(\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial z} + \frac{\partial Z}{\partial x}\right) + h\frac{\partial Z}{\partial y}\right\} = 0,$$

which furnishes us with four linear relations connecting the first differential coefficients of X, Y, Z, T.

5. Proceeding in the same manner as in Art. 1, we obtain the result

$$\{a_1x_1 + (f_{1,2} - a_1p_1)x_2 + \ldots$$
$$+ (f_{1,n} - a_1p_{n-1})x_n\}(x_1 + p_1x_2 + \ldots + p_{n-1}x_n)$$
$$= \Sigma a_rx_r^2 + \Sigma f_{r,s}x_rx_s \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where the p's satisfy $n-1$ equations of the form

$$a_1p_r^2 - f_{1,r+1}p_r + a_{r+1} = 0,$$

and $\frac{1}{2}(n-1)(n-2)$ equations of the form

$$a_1(p_rp_s + p_sp_r) - f_{1,s+1}p_r - f_{1,r+1}p_s + f_{r+1,s+1} = 0.$$

Further, if we write

$$u_1 = x_2 - p_1x_1,\quad u_2 = x_3 - p_2x_1,\quad \ldots,\quad u_{n-1} = x_n - p_{n-1}x_1,$$
$$m = \xi_1u_1 + \xi_2u_2 + \ldots + \xi_{n-1}u_{n-1},$$

and
$$\Delta \equiv \frac{\partial}{\partial x_1} + p_1\frac{\partial}{\partial x_2} + \ldots + p_{n-1}\frac{\partial}{\partial x_n},$$

we have

$$\frac{2m - (\lambda_1 + \lambda_2)}{\lambda_1 - \lambda_2} = \text{const.},\quad \Delta m = 0.$$

It is therefore possible to develope a theory exactly similar to that we have constructed for the case of three variables.

It has now to be remarked that in the case of more than four variables there will be relations among the coefficients of the expression on the right-hand side of equation (2), the case of five variables introducing a single relation. In these cases the equations given above furnish us with the laws obeyed by a set of non-commutative symbols which will enable us to factorize the given expression, as may be verified by the direct multiplication of the factors given above; but it is only in special cases, where relations exist among the coefficients, that these symbols can be identified with matrices of the second order. This, however, will not affect the validity of our theory. We have, in fact, hit upon a generalization of the theory of matrices of the second order. It is easily verified that any expression of the form

$$X_1 + p_1 X_2 + p_2 X_3 + \ldots\ldots + p_{n-1} X_n,$$

where the X's are ordinary algebraical quantities, satisfies an equation of the second degree with scalar coefficients, so that the theory of articles 2 and 3 will apply. The only difference occurs in the derivation of a set of scalar solutions from the solution involving non-commutative symbols. In the case of more than four variables, we have more than four scalar solutions. However, as the full discussion of this point would make the present communication too lengthy, I have reserved it for another paper.

6. It now only remains to point out the application of the foregoing theory to several of the equations that occur in the application of Mathematics to Physics.

As particular examples of the case we have worked out in full, we have the equations

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} = 0,$$

$$\frac{\partial^2\theta}{\partial t^2} = a^2\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right).$$

I have given the theory for Laplace's Equation in a former communication to the Society*. To adapt the foregoing theory to the case of the second equation we must write

$$u = x - pt, \quad v = y - qt,$$

where the matrices p and q satisfy the equations

$$p^2 = a^2, \quad q^2 = a^2, \quad pq + qp = 0.$$

Also for the linear relations connecting the differential coefficients

* *Proc. C. P. S.*, vii., 151—156. See also vii., 120—126.

of the four scalar solutions obtained from the matrical solution, we have

$$\frac{\partial T}{\partial t} + a^2\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = 0,$$

$$\frac{\partial T}{\partial x} + \frac{\partial X}{\partial t} - a^2\frac{\partial Z}{\partial y} = 0,$$

$$\frac{\partial T}{\partial y} + \frac{\partial Y}{\partial t} + a^2\frac{\partial Z}{\partial x} = 0,$$

$$\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} + \frac{\partial Z}{\partial t} = 0.$$

Another equation that may be discussed with the aid of the theory as applied to three variables, is the equation

$$\frac{\partial \theta}{\partial t} = a^2\frac{\partial^2\theta}{\partial x^2}.$$

We have

$$a^2x^2 - ty = a^2(x - pt - qy)(x + pt + qy),$$

where p and q satisfy the equations

$$p^2 = 0, \quad q^2 = 0, \quad pq + qp = \frac{1}{a^2}.$$

Putting $y = 1$, we have

$$a^2x^2 - t = a^2(x - pt - q)(x + pt + q),$$

and consequently

$$a^2\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \equiv a^2\left(\frac{\partial}{\partial x} - p\frac{\partial}{\partial t} - q\right)\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial t} + q\right).$$

To adapt our theory to this case we have, therefore, to write

$$u = t - px, \quad v = 1 - qx.$$

Also, in this case, the four solutions will be connected by the linear relations

$$a^2\frac{\partial T}{\partial x} + X = 0, \quad \frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0,$$

$$a^2\left(T + \frac{\partial Y}{\partial x}\right) + Z = 0,$$

$$\frac{\partial Y}{\partial t} + \frac{\partial Z}{\partial x} - X = 0.$$

Proceeding to the case of four variables, we have the equation

$$\frac{\partial^2\theta}{\partial t^2} = a^2\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right).$$

In this case we shall have for our matrical solution a function of the form

$$\begin{aligned} &A + uB + vC + wD \\ &\quad + \frac{1}{2!}\{u^2E + v^2F + w^2G + (vw + wv)H + (wu + uw)K + (uv + vu)L\} \\ &\quad + \frac{1}{3!}\{u^3M + v^3N + w^3P + (v^2w + vwv + wv^2)Q + \&c.\} + \&c., \end{aligned}$$

where $\qquad u = x - pt, \quad v = y - qt, \quad w = z - rt,$

p, q, r being matrices satisfying the equations

$$p^2 = q^2 = r^2 = a^2,$$

$$qr + rq = rp + pr = pq + qp = 0.$$

By means of the theory as applied to four variables we may also discuss the equation

$$\frac{\partial\theta}{\partial t} = a^2\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right).$$

In this case we must write

$$u = y - px, \quad v = t - qx, \quad w = 1 - rx,$$

where p, q, r satisfy the equations

$$p^2 = -1, \quad q^2 = r^2 = 0,$$

$$qr + rq = \frac{1}{a^2}, \quad rp + pr = pq + qp = 0.$$

Finally we have the equation

$$\frac{\partial\theta}{\partial t} = a^2\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right).$$

In this case we shall have for our solution a function of the above type constructed with the four compound variables

$$y - px, \quad z - qx, \quad t - rx, \quad 1 - sx,$$

where the symbols p, q, r, s obey the laws expressed by the equations

$$p^2 = q^2 = -1, \quad r^2 = s^2 = 0,$$

$$pq + qp = pr + rp = ps + sp = qr + rq = qs + sq = 0,$$

$$rs + sr = \frac{1}{a^2}.$$

In this case our non-commutative symbols cannot be identified with matrices.

May 14, 1894.

THE MASTER OF DOWNING COLLEGE, VICE-PRESIDENT, IN THE CHAIR.

F. F. Blackman, B.A., St John's College, was elected a Fellow of the Society.

The following Communications were made to the Society:

(1) Mr S. J. HICKSON exhibited a specimen of *Chelifer* from Celebes shewing a remarkable sense-organ on the coxæ of the last legs.

(2) *Notes on a Dog's Heart infested with Filaria immitis.* By ARTHUR E. SHIPLEY, M.A.

THE specimen which I have the honour of showing the Society this evening is the heart of a large dog of unknown breed which was sent to the Pathological Museum from Fiji, and I am indebted to the kindness of Sir George Humphry for the opportunity of investigating it.

The right ventricle is literally stuffed with a tangled mass of Nematode worms, which extend into the pulmonary arteries and project from the cut ends of the smaller pulmonary vessels. To such an extent are the lumina of the ventricle and the vessels occluded that it seems impossible to realize that the dog lived any time with so great an obstruction to its circulation.

The Nematodes which I believe belong to the species *Filaria immitis*, often termed the "cruel worms," are so matted and twisted together that it was a matter of considerable difficulty to disentangle a specimen; more especially as I was anxious not to derange the mass, as the heart is destined to become a museum preparation. With some difficulty however I managed to withdraw a single female from the cluster, and this proved to be 26 cms. in length. The males which live along with the females are about half the length, varying from 12 to 15 cms. According to Méguin (9) there is about one male to three females; but this proportion is not very accurately determined, as he refrained, for the same reason that deterred me, from pulling the cluster to pieces.

The parasite was named by Leidy (6) and is the same as the Filaria papillosa haematica of Gruby and Delafond (4) or the Filaria haematica of Galeb and Pourquier (2). It attacks various breeds of dogs and sometimes causes a wide-spread mortality: for instance, a certain Mrs Towne of Beaufort, South Carolina, states (6) that she "lost several dogs of different breed, age and birthplace," whilst "a gentleman living in a neighbouring island (the

Sea islands of South Carolina) lost over thirty hunting dogs in two or three years" from the same cause.

The disease seems to be most common in China and Japan, but it is also recorded in Brazil by Dr Silva Aranjo (1), and two cases have been described in France, one at Montpellier by Galeb and Pourquier (2), and the other by Gruby and Delafond (4): the present example comes from Fiji. The parasite has recently been described by Janson (5) in the heart of a Japanese wolf, the only instance on record of its occurrence in any other animal than a dog.

Most of the investigators who have recently worked at this parasite have sought for some connection between the adult worms found living in the right half of the heart and the large pulmonary vessels, and the minute nematode larvae found so commonly in the blood of the dog. These Haematozoa are of such minute size as to be able to freely circulate in the capillary blood-vessels; they were first described by Dr Lewis, who found them in about 25 per cent. of the dogs examined by him in Calcutta (8).

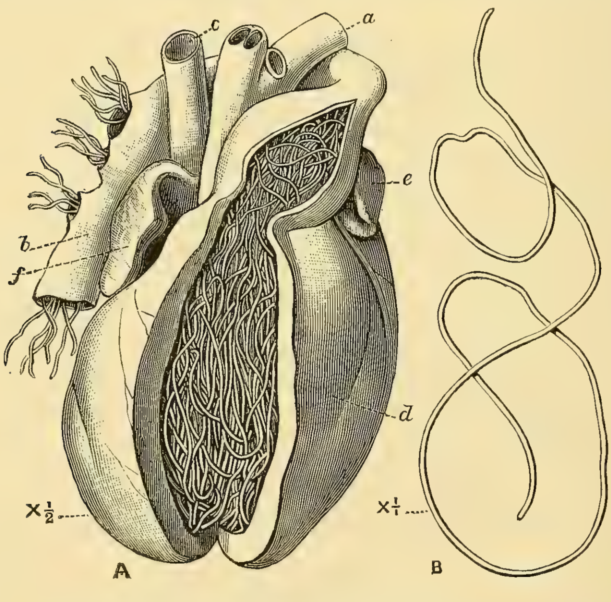


Fig. A. View of the heart of a Dog infested with *Filaria immitis*, the right ventricle and base of the pulmonary artery have been laid open. *a*, aorta. *b*. pulmonary arteries. *c*, vena cava. *d*, right ventricle. *e*, appendix of left ventricle. *f*, appendix of right auricle. $\times \frac{1}{2}$

Fig. B. One of the Filariae, a female, removed from the heart, to show its length. Nat. size.

There seems to be however no doubt according to the researches of Grassi (3) that the Haematozoa of Lewis have their second host in the dog-flea, *Pulex serraticeps,* and have nothing to do with the larvae of the *F. immitis,* although they closely resemble them. The former have a curious habit of hanging themselves by means of their oral aperture on to the cover slips or glass slides, and at the same time swelling out their anterior end: by this habit they can be easily distinguished from the embryos of *F. immitis.* These are also invariably much more numerous in the blood of the dog than the Haematozoa of Lewis. Grassi found the embryos of *F. immitis* in three dogs which he dissected in Milan, and in two of which he found the heart full of adult Filariae; the third owing to an accident he was unable to examine. Two others which he was able to dissect later had embryos in the blood, adults in the heart, and more fully developed embryos under the skin: at the same time he investigated more than 300 fleas gathered from the dogs he had killed and dissected, but found no evidence that they formed the second host of this parasite as was at one time thought.

Galeb and Pourquier (2) found the same embryos in the blood of a gravid bitch whose heart was full of the adults; they also found similar embryos in the blood of the foetal puppies, and satisfied themselves that the latter were infected from the mother, and that the embryos pass from the blood-vessels of the uterine walls into those of the fœtus.

Grassi's researches render it highly improbable that the flea is the second host of *F. immitis,* and he points out that the parasite occurs only in such districts as are very well supplied with streams, marshes etc.; and further that the parasite is most common in such dogs as are used for sport, and which habitually drink water from marshes, ditches and streams, and he concludes from this that we must look for the second host of *F. immitis* amongst the small freshwater Crustaceans or Molluscs.

It is obvious that the presence of such a mass of worms in the heart and large vessels, as our specimen shows, must be accompanied by very serious functional disturbance of the circulatory system, which must ultimately end in the death of the host. The symptoms which the presence of the parasite call forth are thus described by Mrs Towne (6): "I watched my two remaining dogs closely. They were a large Newfoundland (mixed) and a small terrier. Both had the peculiar cough, which was excited by any movement, especially after sleeping. It always ended, after a few coughs, in a violent effort to bring something up from the throat. ......The two dogs had another symptom. When they began to run violently, as at hogs, or a strange dog, they fell down, became stiff and insensible, but in a short time would get up and resume the chase." "The little dog died of haemorrhage from the bladder or kidneys; but no post mortem examination was made."

"The large dog soon began to cough up bloody phlegm, with considerable fresh blood at times. I found in the phlegm one morning two Filariae alive, and at least six inches long.......My large dog grew so ill that I had him shot. His symptoms were drowsiness, sleeping with the upper eyelids raised, and the inner lining showing very red, holding his head to one side, one ear drooped; dragging of one hind leg, turning round and round whenever he attempted to go anywhere; and finally spasms, in which he rolled over and over and drew his head backward. He was fat and had a good appetite to the last."

The heart which Méguin described (9) was that of a Newfoundland dog sent to him from China, preserved in spirit. He states that the animal succumbed with every symptom of serious cardiac disorder, and that before death it suffered from palpitations and fits resembling epilepsy.

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4. Gruby and O. Delafond. *Comptes Rendus*, 1856, p. 11.

5. Janson. *Berl. Tierärztl. Wochenschr.*, 1892, No. 29, v. *Centralblatt für Bakteriologie u. Parasitenkunde*, Bd. XIV. 1893, p. 499.

6. Leidy, Prof. *Proceedings of the Acad. of Nat. Sci. Philadelphia*, 1856, p. 55.

7. Leidy, Prof. Notice of the Cruel Thread-worm, Filaria immitis of the Dog. *Proc. of the Acad. of Nat. Sci. Philadelphia*, 1880, p. 10.

8. Lewis, Dr. On Filaria sanguinolenta, Echinorhyncus etc., from the Dog. Calcutta, 1874.

9. Méguin, P. Mémoire sur les Hématozoaires du Chien. *Journal de l'Anatomie et de Physiologie*, 19me Année, 1883, p. 172.

(3) *Variations in the Larva of* Asterina gibbosa. By E. W. MacBride, B.A., Fellow of St John's College.

The larva of Asterina gibbosa, when fully grown, but before any external traces of the metamorphosis are visible, presents the following features, in longitudinal, horizontal sections, viz.: there is a great præoral lobe; the gut is a simple elongated sac devoid of an anus, but with a mouth opening into it through a vertically placed œsophagus. At the level of the œsophagus a transverse septum divides the body cavity into an anterior portion, filling the præoral lobe, and a right and left posterior portion; these latter being separated by the gut and the dorsal and ventral mesenteries, in which the alimentary canal is suspended. The anterior cœlom is prolonged right and left into two diverticula which overlap the right and left posterior cœloms respectively. Of these the left has already become five-lobed, and is the rudiment of the water-vascular ring and radial canals in the adult. It opens by a slightly constricted neck into the anterior cœlom, and a ciliated groove which has been constricted off from this neck, is the future stone-canal. The right diverticulum is, as we shall see, a rudimentary fellow to the water-vascular system. Its walls are continuous with those of the anterior cœlom, but its cavity, though occasionally opening into the anterior cœlom by a very fine slit, is usually completely closed. It persists throughout life as a completely closed sac, embedded in the madreporic plate. The stone-canal does not open to the exterior; it only leads from the water-vascular rudiment, or "hydrocœle," into the anterior cœlom; the latter, however, opens to the exterior through a short ciliated "pore-canal," the outer aperture of which, situated at first to the left of the mid-dorsal line, is as the larva grows displaced to the right. This pore-canal is the first of the numerous canals which traverse the madreporite of the adult, and the anterior cœlom persists in part as the axial sinus of the adult. The variations which I wish to describe concern chiefly the right "hydrocœle." Its normal appearance is that of a completely closed sac lined by a flattened epithelium. In two specimens I have found it two-lobed, these lobes having the shape and cylindrical epithelium characteristic of the lobes of the left hydrocœle. In both cases it opens by a narrow slit into the anterior enterocœle, and the stone-canal and madreporic pore are normal.

In another specimen it is large and five-lobed, better developed in every way than its left fellow, differing only in its slightly more dorsal position, and its narrower communication with the anterior enterocœle. In this specimen, owing no doubt to the development of the organs of the right side being equal that on the left, the pore is median.

In another and very remarkable specimen the right hydrocœle was represented by two rudiments, one more dorsally situated and one more ventral. The more dorsal rudiment has the ordinary appearance, and it opens by a slit into the anterior cœlom. This latter possesses close to this opening a second pore-canal, much smaller than the first or normal pore-canal, which is in its normal position. The more ventral rudiment consists of two lobes having the distinct hydrocœle structure and opening into the anterior cœlom also.

The metamorphosis of the larva is brought about by three processes. (1) The differentiation of the body into a disc and a stalk, and the absorption of the latter. (2) The growth of the left hydrocœle till its two ends meet and it forms a ring. (3) The similar growth of the left posterior cœlom to form a ring, this giving off at the same time five diverticula, which form the future arms and become applied to the primary lobes of the hydrocœle which form the radial canals of the water-vascular system. Now it will be easily seen that process (2), and with it the whole metamorphoses, can be held in check by the right hydrocœle developing more than usual. In the very remarkable specimen which I now describe the process (1) is almost complete, so that I took the animal for a starfish, the metamorphosis of which was concluded. Sections however revealed the fact that the ends of the left hydrocœle had not met, but were in fact prevented from doing so by the right hydrocœle, which was large, with three distinct lobes. There was a single pore-canal, but two stone-canals, one opening into each hydrocœle, and meeting above close to the inner end of the pore. Another specimen showed just the reverse of this, viz.: two pore canals on opposite sides of the body, joining and opening by a common pore close to the inner end of the single stone-canal opening into the left hydrocœle. In another case the metamorphosis was complete, and the right hydrocœle was a thin walled sac. A second short stone-canal connected it with the anterior cœlom, and a second pore-canal also opened into the latter, but united with the normal pore-canal to open by a single pore.

In a short communication to the Royal Society I compared the cœlomic cavities of the larva of Asterina to those of the Balanoglossus larva. On this view, the madreporic pore would correspond to the proboscis pore, and the collar pores would be unrepresented, as of course neither hydrocœle has a direct communication with the exterior. In one larva, however, to my great surprise I found such a pore co-existing with the ordinary madreporic pore. It opened between the third and fourth lobes of the hydrocœle.

Finally, a larva should be mentioned in which the only hydrocœle present was small and two-lobed, communicated by only a slit with the cœlom, and was placed far in front of the stomach.

In conclusion, I think one may fairly say that these variations prove at any rate beyond all doubt that the sac I have called the right hydrocœle is a structure of the same nature as the left hydrocœle. Since the larva is in its early stages quite bilaterally symmetrical, we may assume that the free-swimming pelagic ancestor of the Echinoderms had two equally developed hydrocœles; and hence the variations I have described come in the majority of cases under the head of atavism, using the word without prejudice to any theory as to the actual cause of the phenomena.

(4) *On a new method of preparing culture media.* By Dr LORRAIN SMITH.

The author described a method for preparing media suitable for the cultivation of Bacteria. The principle of the method consists in the addition of a small percentage of alkali to fluids which contain proteid such as egg-white and serum of blood. The fluid is then heated to the boiling point or over it in the autoclave. By this means it is converted into a clear transparent jelly. It is then a medium suitable for the growth of a large variety of germs.

Monday, *May* 28, 1894.

The following Communications were made to the Society:

(1) *Exhibition of nest of* Trochosa picta *and of certain well-marked varieties of this spider.* By C. WARBURTON, M.A., Christ's College.

(2) *Exhibition of Magnetic Rocks.* By Mr S. SKINNER.

(3) *Geometrical proof of a Theorem of Convergency.* By Mr A. C. DIXON, Trinity College.

Let $u_1 + u_2 + u_3 + \ldots$ be a convergent or oscillating series of quantities which may be complex, and $a_1, a_2, a_3 \ldots$ a series of real positive quantities diminishing continually and without limit. Then the series $a_1u_1 + a_2u_2 + a_3u_3 + \ldots$ will converge. (*Chrystal, Algebra,* Chap. XXVI, § 9, Theorem IV.)

Let us write

Σu_n for $u_1 + u_2 + \ldots + u_n$ and $\Sigma_r u_n$ for $u_r + u_{r+1} + \ldots + u_n$.

Since the series $u_1 + u_2 + \ldots$ does not diverge, a circle of finite radius (R) can be drawn on Argand's Diagram which will include the point Σu_n for *every* value of n.

The point $a_1\Sigma u_n$ will therefore always lie within a certain circle of radius a_1R, which may be derived from the former by drawing lines from the origin to its circumference and multiplying each by a_1. If now the origin is moved to the point a_1u_1, the quantity a_1u_1 is subtracted from every quantity represented, and the point $a_1\Sigma_2u_n$ therefore always lies within the circle as considered from the new origin. Also the new origin is inside this circle, for it was the point $a_1\Sigma u_1$.

The point $a_2\Sigma_2u_n$ will therefore lie within a circle of radius a_2R formed by diminishing every line drawn from the new origin to the circumference of the last circle in the ratio $a_2 : a_1$. This circle is inside the last.

Thus, if we go back to the old origin, we find that the point $a_1u_1 + a_2\Sigma_2 u_n$, for every value of n, lies within a circle of radius a_2R, which is contained in the former circle of radius a_1R; the centre of similitude of the two circles is the point a_1u_1.

Let us now move the origin to the point $a_1u_1 + a_2u_2$ and diminish the lines from this point in the ratio $a_3 : a_2$. It will follow in the same way that for all values of n the point

$$a_1u_1 + a_2u_2 + a_3\Sigma_3 u_n$$

lies within a circle of radius a_3R, contained in the former one of radius a_2R, and the centre of similitude of the two is the point $a_1u_1 + a_2u_2$.

Carrying on this process we find that the point

$$a_1u_1 + a_2u_2 + \ldots + a_{r-1}u_{r-1} + a_r\Sigma u_n$$

lies within a circle of radius a_rR which is contained in every former circle.

Now by hypothesis a_r decreases without limit, so that this circle can be made as small as we please by making r great enough.

It follows that the series $\Sigma a_n u_n$ converges to a definite limit.

It is clear that the argument will hold just as well if any or all of the points Σu_n lie *on* the circumference of the circle of radius R.

In the important special case when

$$u_n = \cos(n\theta + \phi) + \iota \sin(n\theta + \phi),$$

the successive points Σu_n are arranged at equal intervals round a circle of radius $\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}$, which passes through the origin.

We may take $\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}$ as the value of R; it will be finite unless θ is a multiple of 2π and therefore, except in that case, the series $\Sigma a_n \cos(n\theta + \phi)$ and $\Sigma a_n \sin(n\theta + \phi)$ are convergent if a_n is positive and diminishes without limit as n increases.

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Criticism of the Geological evidence for the Recurrence of Ice Ages[1]. By T. McKenny Hughes, M.A., F.R.S.

Part II. (For Part I, see p. 98.)

The Mode of Transport of Boulders and other Drift.

Having considered the nature of the evidence which is relied upon in proof of the glacial origin of the fragments included in certain clays and conglomerates and having exhibited a large collection of rocks polished and striated in different ways, both natural and artificial, but known to be unconnected with glacial action, and having endeavoured to indicate the points of resemblance and of difference between these and the undoubtedly glaciated surfaces, I now go on to consider other sources of error in the observations upon which we have to rely before we attempt to construct a natural explanation of the phenomena of recurrent glaciation[2].

That the boulders were *left by the ice in the place where they now lie* is not always to be received with unquestioning faith. In hurried explorations and especially in those of remote regions the observers can rarely be expected to be familiar with all the operations of nature which move and bury blocks in each particular district and we may admit that landslips or mudflows or the many other processes such as we commonly see on a coast like that of Norfolk[3], with its cliffs of various drifts, may frequently have resulted in the burial of transported material in unexpected places and in positions difficult of explanation.

We have to enquire not only how nature moulds and sculptures the stones on which we rely for evidence of glacial action but also how far the occurrence of a boulder clay containing such stones, even when associated with marine beds, necessarily implies glaciers coming down to the sea in the area in which the boulder clay is now found.

If any region, in which glaciers were generated on the high ground and descended to within say 5000 or 10,000 feet only of sea-level, were afterwards depressed to that number of feet, the glaciers would be gone, but the moraines of those glaciers might often be covered by marine deposits, and cliffs of boulder clay

[1] See page 142.

[2] Since writing the above I have had an opportunity of examining a striated rock from the Indian boulder-beds which is preserved in the museum at Zurich. I was satisfied from an examination of the specimen itself that the markings on it were not due to glacier action, an opinion in which Heim fully concurred.

[3] See *Nature*, Vol. 50, May 3, 1894, p. 5. "On Some Sources of Error in the Study of Drift."

might be undermined and the débris sink into the deeper hollows without much washing and sorting of the material or any destruction of the glacial markings, and thus give rise to the supposition that the glaciers once descended to the sea. It needs much more evidence and an answer to several obvious questions before the inference as to the former descent of the glaciers *to sea-level* can be accepted as satisfactory. Has it in any case been shown that the fauna of the deposits associated with the Carboniferous or Permian boulder beds has an arctic or even a more northern facies than that which occurs in beds of approximately the same age elsewhere? Certainly not in the case of the conglomerates at the base of the Carboniferous rocks of this country, where large corals grew among the boulders, while in other beds of the same series the same corals are associated with the remains of a tropical vegetation.

Further, assuming that the boulders referred to are really of glacial origin and certainly in place, before we can accept them as proof of extreme glacial conditions *on the site and at the level* where they are now found, we must examine the various ways in which nature transports glaciated boulders and other drift, far beyond the region of original glaciation without calling in the aid of earth-movements to depress glaciated hill-tops to sea-level.

The start of the boulders is interesting but need not take long. They fall on the upper ice and, where the ice goes, there they are carried as long as they remain on it or in it and if the journey be long on a glacier[1] most of them will get into the crevasses and eventually find their way to the bottom, where some will be washed down in glacier streams and some help to grind the other included stones and the solid rock.

When they get to the end of the glacier all polish and striæ are soon lost in the streams that flow from it unless the ice breaks off in a lake or in the sea when it will bear its burden of drift about till it is melted down into too small a raft for the load and then the whole will go to the bottom.

The question we are now considering is not how high must the land have been for the snow to accumulate and turn to glacier ice in any part of the tropical or temperate regions, nor how near the equator may glaciers have come down to the sea, nor how near to the sea may they have occurred provided there were a great uplift of mountain ranges and abundant precipitation, but we will now confine our enquiry to the mode of transport *after* the boulder or drift has been launched.

Within the last few weeks the story of icebergs in the South Atlantic in latitude 54° has been going the round of the papers.

[1] "Notes on the Geology of Parts of Yorkshire and Westmorland," *Geol. Polytech. Soc. W. Riding, Yorks.*, July 17, 1867, p. 11.

A ship laden with wheat from San Francisco for Liverpool was caught amongst them. One day 415 were in view from the vessel's maintop. Some of them were 5 or 6 miles in length and between 500 and 700 feet high. This was in the latitude of Holderness and the icebergs were so large that they could not have got near that coast unless there had been a subsidence sufficiently great to submerge the highest mountains in the British Isles if the depression had been uniform over the whole area. Darwin long ago recorded that a fragment of granite was seen on an iceberg between 49° and 50° S. latitude, that is on the latitude of the English Channel.

Though the sun might shine and the southern breezes might rapidly melt away all that projected above the waves, it would be long before such ice islands finally disappeared. Icebergs that have grounded have been observed to remain for years in the same place[1]. Generally they would be carried by the currents with the wind or against the wind, and where there were no currents would be driven by the wind. It is hard to put a limit on the possibilities of transport of drift and boulders in favourable combinations of ice, coasts, and currents.

While on the subject of icebergs I would point out in passing how unsafe it is to infer that all grooves and striae even on the solid rock must be due to glacier ice. When we remember the enormous mass of some of those ice islands and the velocity with which they travel, and compare this with what we know of the weight and movement of glacier ice, we may well credit the ice-berg grounding on a boulder-strewn rock with the power of producing grooves and striae.

Lyell[2] describes an icefield seen by Capt. Belcher on the coast of Newfoundland carrying mud and gravel, and explains how, from observations made on the rate of drifting of such masses in those regions, boulders might be transported 1800 miles or so and be dropped as far south as latitude 48°, i.e. the latitude of the northern part of the Bay of Biscay. Hays[3] gives examples of icebergs drifted towards equatorial regions as far as 40° N. latitude, that is south of the latitude of Naples, and as far as 36° S. latitude, that is the latitude of North Africa. He offers abundant testimony as to many of these being laden with drift and boulders.

If the main mass of this drift is derived from one region we should infer that it was transported to the sea by glacier ice, but if an argument is founded upon a few boulders only of a well-marked rock occurring sporadically here and there in a mass of material derived from other and various sources then we may

[1] Hays, *Assoc. Amer. Geologists and Naturalists*, Ap. 1843. Howorth, *The Glacial Nightmare and the Flood*, I. 155.

[2] Presidential Address, *Proc. Geol. Soc.* 1836, p. 384.

[3] *Op. cit.*

invoke the aid of the many accidents of river ice and of ground ice to help them if necessary on their way from the glacier foot to the pack ice of the coast and further distribute and mix them by the agency of sea-borne icebergs.

It is not only in Arctic regions that coast ice of considerable transporting power is formed. This year I saw in the Wash the frozen snow lifted by the tide and, with whatever was tangled in it, carried up and down with the ebb and flow. The sea for several hundred yards from the shore was covered with pack ice on which sometimes sat large sea birds.

Some years ago I saw in the estuary of the Dee and along the coast of North Wales, i.e. between latitude 53° and 54°, masses of pack ice floating seaward which were more than twelve feet thick and large enough to have carried several cottages. Large two-masted vessels were frozen up in it. Boulders and earth were transported by some of these ice-floes and were shifted up and down with the tide along the shores of the estuary below Connah's Quay.

But many of the stones which lie on the shores of North Wales are glacial boulders derived from the various drifts which occur inland and in the cliffs along the coast—so that ice now carries glaciated boulders along the coast of North Wales—the glaciation having taken place long ago, the transport going on still.

Many other agents besides ice help to carry them. Down the Clwyd, in flood, stumps of trees, with stones from the glacial drifts firmly grasped among the roots, are continually carried and are met with on the shore far from the mouth of any river. Fragments thus dropped on the sand or mud, after lying there long enough to allow the great bladder-like base and the long flat leaves of the laminaria to grow, are further dragged by them in storm, and, with their striæ protected by each agent of transport, find a resting place at last among the shells and other remains of a temperate climate. All along that coast, far from the mouths of the Clwyd or Dee, there are cliffs of boulder clay of various age and origin and these are being perpetually undermined and the boulders, large and small, and the lumps of clay are transported along the shore by the tide and wind waves. Where masses of clay containing rock fragments are rolled along the beach, the softened outside of the mass is covered with recent shells and shingle which have stuck into it, whereas, on breaking it open, we find the more solid clay with fragments, sometimes striated, which have never seen the light since they were buried in the boulder drift.

It is not uncommon to find even single scratched stones detached from their parent cliff and carried a considerable distance without losing their striæ, but where the boulder is surrounded by clay, packed up in the gnarled roots of a tree, or covered by marine growths, it may travel far without being exposed to any wear and tear that would obliterate the ice-marks.

In the observations just recorded I have endeavoured to explain the preservation of the grooves and scratches, but if we are seeking only an explanation of the presence of a block far from its source of origin and in material of very different coarseness the explanation is easier. Stones travel steadily along a coast with every tide and wind-directed shore current and it is a matter of everyday occurrence to find a large boulder or two appear and disappear in a few tides along a sandy or gravelly beach. Although there is a definite proportion between the velocity of a current and the coarseness of the material which it will move, this does not apply to a single large stone resting on sand or small gravel. A boulder 2 feet in diameter would not be moved from its place among boulders all of somewhat the same size by a current which would scour out the sand around it if it lay isolated on a sandy beach and would trundle it along the shore—leaving it at last perhaps buried in clay in some depth out of reach of the wind-waves and currents.

These are only a few examples of the many modes of transport of boulders. We must also remember that they have often been derived from older boulder-bearing rocks and that they may even have come originally from rocks now covered by newer deposits as many of the boulders in the Cretaceous beds of East Anglia have probably been derived from the subterranean plateau now becoming so well known by the numerous borings for water for the supply of London.

With all these doubts as to the glacial origin of the striations on rocks and stones, with so many modes of transport of boulders and drift, with the abundant proofs of depressions of sufficient extent to bring down to sea level the moraines and other traces of glaciers which once occupied the mountain tops, we may be pardoned for not accepting as proved all the reports of extreme glaciation having occurred in equatorial regions with geographical conditions the same as now. A theory involving such difficulties as to have driven some of its advocates in desperation even to shift the axis of the earth back and fore as required so as to bring a portion of the equatorial region within reach of circumpolar glaciation.

Part III.

The Evidence as to the extent of Earth movements, and their relation to Glacial Phenomena.

To recall the line of argument followed. It is admitted that we have abundant evidence of the local occurrence of land ice and of the transport of material by sea-borne ice over areas where, with the present geographical distribution and the existing climate, glacial conditions do not and could not prevail. But the extent of ice action has been pushed too far, and the recurrence of glacial conditions in past time has been urged on insufficient and often false evidence. The principal sources of error have been pointed out, and some particular cases of wrong attribution to ice action have been discussed.

Having thus got rid of some of the difficulties which extreme views of glaciation entail, we proceed to consider what geographical conditions prevailed over certain areas where it is admitted that glacial conditions did exist, or where the indirect influence of neighbouring glaciation is shown to be probable.

We find as the result of this enquiry abundant independent evidence of repeated earth movements over the very areas indicated by the distribution of glacial phenomena.

There have been great upheavals where we see evidence of increase of cold, great depressions where glaciated areas have been brought down to sea-level in warm regions, and great complex movements causing ocean currents to run where land had been, or mountain ranges to rise where the tide had flowed.

To this point I would now direct attention. Many collateral questions of great interest naturally arise out of its discussion, such as the *continuity* rather than the *permanence* of continental and oceanic areas, &c. But I propose to confine myself to the consideration of the evidence of the nature, amount, and effect of the changes of level which can be proved to have taken place over the areas where glacial conditions formerly prevailed.

In examining the nature and extent of the geographical causes of the more intense vicissitudes of temperature it may be well at first to confine our attention to two principal basins, (1) the North Atlantic basin, round which we have evidence of the severe Pleistocene climate which is generally spoken of as the Glacial Period, and on the existing shore of which the Conglomerates of Cambrian, Carboniferous, and Poikilitic Age occur in which some have seen proofs of ice action, and (2) the Indian Ocean, round which it is maintained by many observers that evidence exists of similar conditions having prevailed in times not long preceding our own, and also, according to some, during the deposition of the Lower Carboniferous rocks.

Measures of Earth movement.

The study of the sedimentary rocks brings out most clearly that there have been oscillations of level over large areas—that after a long period of depression and continuous deposition there has always come a reversal of the direction of movement and a continuous uplift for another long period.

The first question is—what is the known extent of such movements in the districts under consideration? A minimum estimate can be obtained by measuring the edges of the deposits that have been continuously depressed to admit of the accumulation of more sediment on top of them, and as these formations can be observed only when afterwards folded up into plateaux and mountains in the second period of their history we have in them a measure of the amount of uplift. It does not follow that there ever existed an ocean as deep as would be implied by the depth to which the first-laid sediment descended, because the filling up of the abyss would go on during the period of subsidence. Nor should we be justified in assuming that at any time there was land as high as the folds of the earth's crust would seem to have carried the strata, because the waste of the high lands would be going on during the whole of the period of upheaval. But the quantities we are dealing with are so vast that there is plenty of margin and a small excess of elevation over denudation would give us mountain regions high enough to account for all the phenomena we seek to explain.

Eastern side of Atlantic basin.

First we will examine the evidence as to the amount of earth movements certainly known to have taken place in our own country. How much material was removed from the Archaean Rocks, before the great Cambrian depression submerged them all, cannot be accurately ascertained owing to the absence of sufficiently well-marked horizons in that ancient crystalline series. But the amount of the depression which went on during the deposition of the Cambrian System cannot be less than the total thickness of the Cambrian Strata, and, as in some areas there is no great discordancy between the Cambrian and Silurian, such as would imply any considerable reversal of the movement, we may estimate the depression as represented by the thickness of the Cambrian and Silurian Systems together[1].

Now taking the thicknesses as deduced from the Geological

[1] *Camb. Phil. Soc.*, Vol. III. p. 247.

Survey Sections, we find that where the sediment is small we have about 32,500 feet, but different members swell out in areas where sediment was more abundant at the time they represent, or depression more rapid, and there we get 63,000 feet for the Cambrian and Silurian. In these calculations we are less likely to obtain wrong results if we take all our measurements from as nearly as possible the same district, so as to avoid errors arising out of mistaken determination of boundaries or compensating accession of thickness.

Here then taking the lowest estimate we have evidence of the descent of a land surface to depths greater than the deepest abyss of ocean. The fact that these beds are now above sea-level and exposed to observation is sufficient proof that a great elevation in due course followed the subsidence we have just described. But the upheaval did not stop there. The Cambrian and Silurian Rocks were thrown into great folds, the tops of which were cut off in time by denudation. That 30, 40, 50, or 60 thousand feet of sediment should be uplifted into arches above sea-level implies earth movement on a scale amply sufficient to account for any climatal changes possible within the periods recorded in the rocks. But when denudation kept pace with upheaval, there would not be any very high lands to produce intense cold, and when no traces of glacial conditions are found it may be that the land was reduced by denudation as fast as uplifted within its influence. The result of all the combined operations is that when the area again subsides, the new sediment is laid upon the planed-down surface of the older rocks, and a measure of the time and extent of the movements may be obtained by restoring the curves of the truncated strata.

I have applied this method to the examination of the break between the Silurian and the Carboniferous Periods in the north of England, confining my observations within an area so small that it is not necessary to make allowance for the thinning out of the strata in any direction, and I found that strata the aggregate thickness of which amounted to 27,000 feet were thrown into folds above sea-level and remained there long enough to be all planed off at sea-level. The height to which the crest of these folds must have risen if they had not been destroyed *pari passu* with the upheaval could not be less than 27,000 feet and may have been much more.

The time for the reversal of the direction of movement came, and throughout long ages shallow water conditions were maintained over the area by the accumulation of sediment keeping pace with depression until the old land surface had reached a depth of from 10,000 to 15,000 feet.

Again the engines are reversed and all this sediment mostly converted into solid rock comes up and is thrown into folds as were the rocks of the old land from which it was derived. The minimum height is the total thickness without allowing anything for the folds, or for the vast masses of older strata pushed up at the same time.

Again after a somewhat commensurate interval during which the folds are planed off and the edges of the Carboniferous Rocks with their Devonian basement are exposed, the direction of movement is again changed, and all goes down to receive the great Jurassic system with its Poikilitic basement series, and so the story is repeated again and again.

We have taken one district on the margin of the Atlantic basin for an approximate estimate of thicknesses and of therefrom-inferred amount of upward and downward movement. But the same reasoning holds everywhere along the axes of principal disturbance, and here and there some members of the series are enormously thicker than their representatives in the sections I have referred to. For instance, in the central basin of Scotland the lower division of the Old Red Sandstone is said to have reached a depth of more than 20,000 feet, while in North Germany the Coal Measures attain a similar thickness. The Alps and Pyrenees offer proofs of upward and downward movement along axes of yielding at many successive periods down to late Tertiary times. As Le Conte[1] has shown, sediment is apt to be thicker and evidence of earth movements greater along the axes of mountain ranges; still there have been wider submergences and elevations, as proved by the vast thickness of sediment such as those of which examples have been given above, and these are more important as showing the instability of the continental areas.

Heim has shown that the crumpling up of the Alps has reduced the breadth of the sediments of which it is composed at least 120,000 metres or 74 miles and, though his figures have been questioned by later observers, all admit an enormous reduction of their original horizontal extent[2]. More recently it has been suggested that the upper part of an immense fold of rock has been carried from the districts south of Mont Blanc and Monte Rosa to the northern slopes of the Alps, and that the movement extended all along the northern edge of the Swiss and Bavarian Highlands[3].

Every author who has described the Alps and given us sections across them bears witness to the vast extent of the earth movements that have recurred there down to quite late times. For this we have only to look at the sections of Alphonse Favre, Renevier, and many others, besides the more recent observations

[1] *Journal of Geology*, I. 1893, p. 544.

[2] Heim, *Mechanismus der Gebirgsbildung*, Vol. II. 1878, p. 213.

[3] Schardt, "Sur l'origine des Préalpes Romandes," *Arch. des Sciences Phys. et Natur.*, Dec. 1893, Geneva.

just quoted. The thickness of the strata thus folded is very great. The Mesozoic and Tertiary strata alone have been estimated at more than 50,000 feet[1].

One controverted point in connection with this question of the periodic oscillation of all known land areas is the inference that these movements have been continued down to quite recent times. Heim has recently shown that the up-valley slope of some terraces along the Lake of Zurich indicates that the hollow in which the lake lies holds the water in consequence of a depression of the region in post-glacial times. But the strongest argument in favour of this is derived from the occurrence of marine deposits of post-glacial age at various levels up to at least 1350 feet, on the flanks of the hills in Britain and Scandinavia. The shell beds of Uddevalla and the fossiliferous sands and clays of Macclesfield and of North Wales are the most important examples. I have already elsewhere described those of North Wales[2]. It has been generally recognised that the occurrence of marine terraces at various levels along the marginal mountain regions would be a strong argument in favour of elevation of the land having had much to do with the occurrence of glaciers in that region. At any rate the proof thus offered of earth movements to that extent in post-glacial times would be a complete answer to the objection that the advocates of the geographical theory were calling in hypothetical changes on the earth's surface of which they had no proof. The extreme glacialists therefore received with great favour the view that these marine deposits did not represent old beaches, but were the sweepings of the sea bottom pushed by the advancing ice up the hill sides, and left there when it receded. The objection that the beds were distinctly stratified and the material sorted by water was met by the suggestion that it was rearranged by the fresh water which flowed from the end of the melting ice.

But an examination of the beds renders this explanation extremely improbable. The drift is not thrown out in cones of dejection and fan-shaped masses, nor arranged in terraces along the possible course of glacier-born streams, but occurs in the bays where it was thrown up by the sea and protected afterwards from destruction, or lies in deep valleys which during the submergence were invaded by the tides. The boulders and shells are thrown ashore and buried in the slips of boulder clay cliffs and the waste of older marine deposits. Sometimes the scratched stones from these boulder clay cliffs are not washed far enough to lose their striæ but, carefully packed in clay, have been preserved unworn.

[1] Judd, *Volcanoes*, p. 295.

[2] "On the Drifts of the Vale of Clwyd and their Relation to the Caves and Cave Deposits," *Q. J. G. S.*, Nov. 17, 1886, Vol. XLIII. p. 73.

Sometimes they are broken and the pieces separated. Rolled balls of boulder clay with pebbles and shells stuck on them get buried with the rest.

Exactly the same process is now going on along the coast of North Wales, and an exactly similar deposit is the result[1]. Dead shells of species that live along the shore, of those that haunt the rocks, of those that burrow in the mudbanks, and of those that frequent the deeper waters are all thrown up together.

The shells of the Moel Tryfaen, Colwyn, Clwyd, or Minera terraces are not arctic, though there are some of more northern, say Scandinavian, aspect than now inhabit our coast, and, what seems to be a conclusive refutation of the Ice-push theory, the facies is different in the different terraces. On the extreme glacial theory, the ice must have crept across the temperate sea and pushed in front of it, once for all, the remains of all the forms of life that could not escape and then left some at one level and some at another on adjoining parts of the coast. It could not go back to fetch any more, and, if it did recede in parts, surely the forms of life would then be more arctic in character than we find in these marine terraces.

The West Side of the Atlantic Basin.

Now if we turn to the other side of the basin we shall find the same kind of evidence of enormous alternate upheaval and depression. We are not now concerned with the causes of the movements but only with their recurrence and their extent.

The Uinta Mountains[2] rise as a single great fold and though 30,000 feet of the uplifted strata have been swept away, they still rise some 10,000 feet. The upper Palæozoic and Mesozoic are, according to Powell, 30,000 feet thick. The vast scale of the movements is further shown by the fold being cut off on its northern and steeper side by a fault of 20,000 feet downthrow. As in the case of the Sierra, where after a gradual ascent to a height of 15,000 feet, there is a sudden drop on the East with a fault-cliff nearly 11,000 feet high. The primary fold of the Sierra was formed towards the close of the Jurassic period, but the great uptilts of its eastern side are referred to an age as late as the end of the Tertiary period.

The thickness of the strata involved in these great folds is immense. According to Clarence King the Palæozoic and Mesozoic strata of the Wasatch are about 50,000 feet thick. The Trias of Western America is sometimes as much as 15,000 feet thick.

[1] See above, p. 222.

[2] Le Conte, *Journ. Geol.*, i. 1893, pp. 547, 553: Clarence King, *Report Geol.* 40th *Parallel*, Vol. I.

The Cretaceous alone in the Coast Range of California near the Bay of San Francisco are according to Whitney 20,000 feet thick and 30,000 according to Diller in Shasta County. These are a few examples of the extent of the earth movements; and nearer the margin of the Atlantic basin we have the same evidence.

In the Appalachian chain, according to Claypole[1], 96 miles of sediment has been crowded into a breadth of 16 miles. When we measure the enormous thickness of this sediment and remember that there must have been vertical extension to make room for that amount of horizontal compression we realise the vast magnitude of the earth movements implied by those figures. To give an idea of the amount of depression we have only to consider that in the Palæozoic rocks of the Appalachians having according to Hall a thickness of 40,000 feet, nearly every bed gives evidence by its fossils of shallow water, and often by shore marks of very shallow water[2].

When we have evidence of movements on such a scale within each geological period and remember that the lowest depth of ocean known is only 27,930 feet, we may well hesitate before we accept any theories involving the permanence of oceanic areas and may feel justified in calling in elevation and depression in explanation of any phenomena which they would account for.

The Indian Ocean Basin.

The regions round the Indian Ocean basin show here and there traces of glacial action in quite recent times. New Zealand may have come within the influence of antarctic ice; India is close to some of the highest and coldest mountains in the world; but African glaciation calls for further explanation.

Capt. Aylward[3] says, "It will be interesting to geologists and others to learn that the entire country, from the summits of the Quathlamba to the junction of the Vaal and Orange Rivers, shows marks of having been swept over, and that at no very distant period, by vast masses of ice from east to west. The striations are plainly visible, scarring the older rocks, and marking the hill sides—getting lower and lower and less visible as, descending from the mountains, the kopkies (small hills) stand wider apart; but wherever the hills narrow towards each other, again showing how the vast ice fields were checked, thrown up and raised against their eastern extremities." On this account Wallace[4] remarks, "The country described consists of the most extensive and lofty plateau in South Africa, rising to a mountain knot with peaks

[1] *American Naturalist*, Vol. XIX. p. 257.
[2] Le Conte, p. 552.
[3] *Transvaal of to-day*, p. 171.
[4] *Island Life*, p. 157.

more than 10,000 feet high, thus offering an appropriate area for the condensation of vapour and the accumulation of snow. At present however the mountains do not reach the snow-line, and there is no proof that they have been much higher in recent times, since the coast of Natal is now said to be rising. It is evident that no slight elevation would now lead to the accumulation of snow and ice in these mountains, situated as they are between 27° and 30° S. Lat.; since the Andes, which in 32° S. Lat. reach 23,300 feet high and in 28° S. Lat. 20,000, with far more extensive plateaux, produce no ice fields. We cannot therefore believe that a few thousand feet of additional elevation, even if it occurred so recently as indicated by the presence of striations, would have produced the remarkable amount of glaciation above described." But Wallace's argument from the present direction of movement on the coast goes for nothing, seeing that it, as I have endeavoured to emphasise above, is a recurring phenomenon and that levels have been rapidly changing in quite recent times in many places. Moreover we must bear in mind that we require not only great cold but also abundant precipitation to produce ice fields, and a plateau may be *too* high or out of the way of the moisture-bearing currents of air.

Mr G. W. Stow[1] describes similar phenomena in the same mountains, and also mounds and ridges of unstratified clay packed with angular boulders; while further south the Sturmberg mountains are said to be similarly glaciated with immense accumulations of morainic matter in all the valleys.

The fact of glaciation in tropical regions is itself evidence of former great elevation, for even the astronomical theory requires high land for the collection of snow and the formation of glaciers.

But this is not the only proof of changes of level in that part of the world.

The uplift of earth above water or the sinking of the land below the sea brings about the greatest change in the continuity of life and of deposit and therefore furnishes the most important data for stratigraphical classification, and perhaps of the two it is the icy mountain range that forms the most effective barrier. On such isolation of species Wallace has founded many of his happiest generalizations.

That there did exist along the East African side of the basin a continuous range of very high ground is proved by the large number of north temperate genera of plants in South Africa. It is clear, says Wallace[2], "that South Africa has received its European plants by the direct route through the Abyssinian highlands and the lofty equatorial mountains, and mostly at a distant

[1] *Q. J. G. S.*, Vol. XXVII. p. 539. [2] *Island Life*, p. 492.

period when the conditions for migration were somewhat more favourable than they are now."

Another argument for the elevation, more or less recent, of the eastern part of Africa and of the great chain of islands which connect Australia and the south-eastern promontories of Asia is that the dividing hollows between these islands themselves and between them and the mainland seem to be chiefly valleys of subaerial denudation, and to bring the bottom of the deepest valley up to the surface would imply an upheaval of at least 15,000 feet.

The same kind of evidence as that founded on the thickness and known movements up and down of the sedimentary rocks in the Atlantic basin is forthcoming also in the basin of the Indian Ocean. The Jurassic rocks of many parts of India and Australia have been estimated at from 6000 to 10,000 feet, and the Basic rocks of Cretaceous age in the Deccan amount in the aggregate to 6000 feet. Series referred to the Tertiary period attain a thickness of 10,000 feet, but, more important for our present purpose, Tertiary beds have been lifted up in the Himalayas to more than 16,000 feet above sea-level.

Effect of Elevation on Climate.

We have now to consider what would be the effect on temperature and precipitation of such earth movements as those which we certainly know did repeatedly occur round the areas just described.

First we must bear in mind that it is not necessary to have very great cold in order to obtain glacial conditions. All that is required is, that the temperature shall be below freezing-point. Great masses of ice encroaching on the lowlands and icebergs drifting into temperate regions maintain a low temperature around themselves at the expense of their mass. All that is wanted is a large collecting ground and abundant precipitation to keep up the supply. If the cold is excessive it arrests the moisture before it reaches the névé.

Now it is a well-known fact that as we climb the higher mountains in any part of the world the cold increases, and the forms of life proper to more northern regions appear successively along the belts where they find the temperature that suits them. The rate of increment is not quite constant, but a fall of 1° F. for every 250 to 300 feet additional height is about the average. If then we start from the mountain foot in tropical regions it is long before we reach the freezing isotherm, in northern climates we reach it sooner; but sooner or later it must be found if the mountains rise to a sufficient height. In high northern latitudes the freezing-point is found at sea-level. In the British Isles it just touches the tops of our highest mountains. In the Alps it lies between 7000 and 9000 feet above sea-level. Exceptional geographical conditions bring it down to sea-level in South-West Greenland on the latitude of Southern Norway.

It was quite possible that denudation might have proceeded *pari passu* with upheaval in every case and no high ground be the result of the long continued upheavals of which we have such abundant evidence. It might have been that this was one of the automatic compensations provided by nature and that no great elevations were ever thus produced. But there they are—mountains and table-lands—existing in the present and ever recurring in the past.

The upheaval is a fact; what we have to ask is only whether in any particular case the balance of uplifted land left after ages of denudation is enough to explain the occurrence of glacial conditions. If we find traces of glaciation we say—you have in elevation an ordinary operation of nature sufficient to account for it. If we find no evidence of excessive cold then we say—in denudation is another well-known operation of nature which has proved in this case sufficient to counteract the effect of the vast upheaval.

If the Scotch mountains, the tops of which are now about freezing-point, were uplifted 3000 feet we should not only get a fall of 10° to 12° in temperature due to the greater height, but the enormous accumulation of snow and ice would still further lower it and glaciers would protrude their cold mass far over the now warm lowlands. An elevation of 30,000 feet with the greater precipitation from adjacent warmer seas would go far to furnish all that has yet been *proved* of tropical glaciation.

Summary.

I have endeavoured to get rid of some sources of error arising out of a wrong reference of certain phenomena to glacial action and want of due regard to the various modes of transport of glaciated material, showing that there are so many ways in which stones are accidentally striated that the greatest caution is necessary with regard to the character and origin of the scratches observed upon them, and that there are so many modes of transport and imbedding of boulders that we require the clearest evidence as to all the circumstances in which they are found.

I explained that I had had exceptional opportunities of forming an opinion as to the value of the evidence in the principal cases relied upon in proof of the occurrence of glacial conditions in the Palæozoic and Mesozoic rocks of this country, and had come to the conclusion that not only were the examples of striated boulders in those rocks of doubtful origin, but that the negative could be proved, and I laid upon the table of the Society a series of examples in proof of the contention that these at any rate must all be referred to other causes than ice action.

I showed that the ancient deposits in which evidence of glacial action had been observed in India, Africa and Australia, viz. the Talchir, Karoo, and Marsh Bush beds, belonged to one geological period, and occurred round one geographical basin. But I submitted that the evidence of their glacial origin, though cumulative, was not sufficiently precise and mostly amounted to no more than an impression derived from their general appearance, rather than a proof in particular cases, and that there was not sufficient evidence as to their mode of formation. Admitting the glacial origin of some of them, I urged that the occurrence of boulder clays did not imply that there had been extreme cold over the areas where they were now found but that many such deposits were transported by icebergs and other agents from the area of glaciation. I pointed out that the beds in question occurred in areas of known earth movement on an enormous scale, so that regions which must have been glaciated when at a great elevation must often have been afterwards depressed below sea-level. I further maintained that, even if the glacial origin of all these beds were to be proved, that fact would lend no support to the theory of circumpolar glaciation seeing that this basin lay entirely in equatorial and temperate regions, some being north, some south of the equator and, far from being circumpolar, did not touch arctic regions at all.

In like manner our well-established Pleistocene glaciation occurred round a basin, in a general way coinciding with the Atlantic basin of to-day, but in no sense circumpolar.

The evidence offered by Reusch of Palæozoic glaciers in Norway, although, if established, it would prove the repetition of the phenomena does not help to settle the question of the relative importance of the geographical and astronomical causes, as the region in which they occurred is within the area of glaciation at the present time.

The question thus becomes reduced to a simpler form. It is stated by those who refer the recurrence of glacial conditions to extra-terrestrial causes that certain astronomical combinations *must* have produced an effect on climate, now intensifying the cold over one hemisphere now over the other, and if traces of these alternations cannot be detected in the strata of the earth's crust, it must be attributed to the imperfection of the record or the incompleteness of the observations.

Let us leave it so. But, granting all that is here asked for, if it can be shown that throughout the ages earth movements have been repeated of such a kind and degree as would fully account for the most severe cold of which we have any record, which with such precipitation as can be easily accounted for by the geographical distribution of the regions affected, would furnish the largest ice-sheets of which we have any suggestion, it follows that we have another cause in operation which also must produce recurrent glacial conditions and we have to ask whether the astronomical or the geographical cause is the most potent.

The amount of the effect which could be produced by the astronomical combinations has been calculated and various intensifying circumstances have been taken account of.

So also, in the case of the geographical causes, on the hypothesis of recurring elevation and depression over any area, the rate of increase of cold as we ascend into higher regions of the atmosphere is a matter of observation, and, given sufficient precipitation, the effect is cumulative as the ice creeps down to lower regions. We can thus approximately estimate the height at which in any latitude glaciers must be formed.

In the case of the astronomical theory the most favourable conditions for the production of glacial conditions would be entirely neutralized by a distribution of land and water having an opposite tendency.

Whereas the geographical theory is not dependent on astronomical conditions, but the most unfavourable combination of them may be overcome by a slight increase in the amount of upheaval, while keeping well within the limit of observed earth movement.

The astronomical effects are small and contingent, the geographical large and independent.

The astronomical theory that the cause of ice ages must be the coincidence of winter Aphelion with great excentricity of the orbit, with concomitant aggravations, requires the secular recurrence of circumpolar glaciation which is not confirmed by observation.

The geographical theory allows the probability of the recurrence, and perhaps the secular recurrence, of glacial conditions but in connection with earth movements, the existence of which is a matter of observation. But the glacial conditions need not be and are not circumpolar except so far as the temperature generally falls as we approach the poles.

PROCEEDINGS

OF THE

Cambridge Philosophical Society.

October 29, 1894.

ANNUAL GENERAL MEETING.

PROFESSOR T. M^{c}KENNY HUGHES, PRESIDENT, IN THE CHAIR.

The officers for the ensuing session were elected as follows:—

President:

Prof. J. J. Thomson.

Vice-Presidents:

Prof. Sir G. G. Stokes, Bart., Prof. Hughes, Mr F. Darwin.

Treasurer:

Mr Glazebrook.

Secretaries:

Mr Larmor, Mr Newall, Mr Bateson.

New Members of Council:

Dr Glaisher, Prof. Ewing, Mr F. H. Neville, Mr E. H. Griffiths, Mr W. B. Hardy, Mr H. F. Baker.

Before vacating the Chair the retiring President, Professor Hughes, read an address to the Society, referring to the history of the Society during his term of office; commenting upon the bearing of recent geological observations upon some of the vexed questions of the day, such as the age of the earth, and ancient vicissitudes of climate; and containing a biographical notice of the Rev. Leonard Blomefield, of which the following is an abridgment.

"The Rev. Leonard Blomefield (Jenyns) was one of our earliest contributors, our first chronicler and one of the founders of the Society's museum. We find in our Publications many papers written by him, some dating back as far as 1825. They referred chiefly to the anatomy of birds, their plumage and the structure of their features; their migrations, and their systematic classification. Besides these he communicated the results of his observations on other zoological groups from mammals to molluscs and many interesting descriptions of natural phenomena. Elsewhere also he published numerous results of scientific investigations in Meteorology and Zoology, retaining his activity and continuing his work to within 18 months of his death in his 94th year.

Born in 1800, educated at Eton and St John's College in this University, he was associated throughout his early life with people of culture and scientific tastes, for his maternal grandfather was a distinguished physician, his grandmother a Wollaston, and Chepstow, the naturalist, his uncle. Besides which his father was an agriculturist of note and a keen sportsman. Leonard Jenyns, as he was called before he assumed the surname of Blomefield, when only just 23 was ordained to the curacy of Swaffham Bulbeck, not far from his father's place, Bottisham Hall. He was soon after appointed to the living and held it for 30 years. He had joined the Philosophical Society in 1822 and being so near Cambridge often met his old friend Henslow, as well as Sedgwick, Whewell, and many others.

After his marriage in 1844 he frequently visited Oxford and, at the house of his wife's relatives Dr Charles Daubeny, the well-known Oxford Professor, and Dr Bulley, the President of Magdalen, he used to meet Phillips and Rolleston, and Westwood, and many another leader of Science, and thus he was stimulated to devote what time he could spare from the duties of his calling to the pursuit of scientific research.

Of the 58 or more books and papers which he published a large number contain original observations of permanent value.

He was soon widely known as a man of extensive knowledge and sound judgment, and was invited to draw up a Report on Zoology for the British Association in 1834, and to write the article on Yarrell's British Birds in the *London and Westminster Review* in 1840. At Darwin's request he described the Fishes obtained during the voyage of the Beagle.

His Manual of British Vertebrate Animals was published by the University Press in 1836. This was followed, ten years later, by his Observations on Natural History, which was not, however, sufficiently popular for the general public: nor were his Observations on Meteorology, which followed in 1858.

When his friend Henslow died in 1861 Blomefield wrote his Memoir.

He was an original member of the Zoological, of the Entomological, and of the Ray Societies, and was elected a member of the Linnæan Society in 1822 and of the Geological Society in 1835.

He enriched many a local institution. To the Ipswich Museum he gave his valuable collections of birds' eggs and of mammalian skulls. His Herbarium in 40 large folio volumes, besides some smaller ones, as well as his scientific library of more than 2000 volumes, he presented to the Royal Literary and Scientific Institution at Bath, in or near which he resided during the last 43 years of his life, and where he also left his mark in the Field Club which he founded, the papers he read, and the stimulus which he personally gave to scientific investigations.

He presented to the museum of our Society the collection of fish obtained by him on the south coast of England and of Diptera mostly caught in the neighbourhood of Cambridge, and he must always be regarded as one of our benefactors, not only for his gifts to the museum, out of which our existing admirable Natural History collections have been developed, but also for the part which he took in promoting the success of the Society in its early days."

The President elect, Prof. Thomson, on taking the Chair, referred to the loss sustained by science in the death of Professor von Helmholtz.

The names of the Benefactors of the Society were recited, viz. Dean Peacock, Rev. F. Martin, Prof. Sedgwick, Prof. Babington, Prof. Adams.

The following proposals received the assent of the Society:

(i) That to the Bye-Laws Chap. XVI. there be added the following paragraph:—

"Associates, being either Teachers recognized by the University or engaged in investigations of a scientific character, may upon application made to the Council through one of the Secretaries receive permission to borrow books from the Society's Library on the same conditions as Fellows of the Society. Associates availing themselves of such permission shall pay a subscription of half a guinea per annum."

(ii) That in the Rules for the Cambridge Philosophical Library there be inserted in Rule 4 after the words "*Fellows of the Philosophical Society*" the following:—

"and Associates who have received special permission from the Council of the Society."

The following Communications were made to the Society:

(1) *Note on Geometrical Mechanics.* By Prof. Sir ROBERT S. BALL.

The kinetic energy of a material system must always submit to a certain identical equation. Take as a particular case a body rotating around a fixed point. Let x_1, x_2, x_3 be the coordinates (supposed small) expressing the position of the body as defined by rotations about three intersecting axes and $\dot{x}_1, \dot{x}_2, \dot{x}_3$ the angular velocities. If the body before being set to spin had been displaced by a small rotation about the instantaneous axis and then set to spin with the original angular velocity, the effect on the kinetic energy will be nil. For the displacement merely altering the azimuth of the body about the axis cannot affect the energy. This alteration however does affect the coordinates x_1, x_2, x_3 by changing them into $x_1 + \epsilon\dot{x}_1$, $x_2 + \epsilon\dot{x}_2$, $x_3 + \epsilon\dot{x}_3$ when ϵ is small. Introducing these values into T we have

$$\dot{x}_1 \frac{dT}{dx_1} + \dot{x}_2 \frac{dT}{dx_2} + \dot{x}_3 \frac{dT}{dx_3} = 0;$$

more generally it can be shown that in any material system whatever

$$\dot{x}_1 \frac{dT}{dx_1} + \ldots + \dot{x}_n \frac{dT}{dx_n} = 0,$$

when screw coordinates are employed. In ordinary coordinates an equivalent equation holds, but its form is complicated. Whenever the equation admits of the above simple form the coordinates must be screw coordinates. The function is an evectant retaining its form for any linear transformation inasmuch as $\dot{x}_1$, $\dot{x}_2$, &c. $\dot{x}_n$ are cogredient with x_1, x_2, &c. x_n. See *T.R.I.A.* Vol. XXIX. p. 613.

Here we give a dynamical illustration. Let the case taken be a rigid body with three degrees of freedom, and let the screws of reference be coreciprocal. Then we have

$$\dot{x}_1 \frac{dT}{dx_1} + \dot{x}_2 \frac{dT}{dx_2} + \dot{x}_3 \frac{dT}{dx_3} = 0.$$

The wrench with coordinates

$$\frac{1}{p_1} \cdot \frac{dT}{dx_1}, \quad \frac{1}{p_2} \cdot \frac{dT}{dx_2}, \quad \frac{1}{p_3} \cdot \frac{dT}{dx_3},$$

is such that if applied to the body it will prevent the body from forsaking the instantaneous screw about which it is twisting. It acts so to speak as a restrainer.

It appears to be of interest to find the geometrical connection between each screw about which the body can twist and the corresponding restraining screw. We proceed thus.

The screws of the three systems can be represented by the several points in a plane. All the points on one conic (I.) correspond to screws of zero pitch; all the points on another conic (II.) correspond to screws about which the body would twist with zero kinetic energy (of course this is imaginary). Let P be any screw, draw the polar of P with regard to II., then Q, the pole of this ray with regard to I., indicates the screw, an impulsive wrench on which would make the body commence to twist about P.

There is a certain homography in the plane such that if P' be the correspondent of P, then the pole of the ray PP' with regard to I. points out the restraining screw, corresponding to P, while the pole of PP' with regard to II. indicates the screw, a twist on which is the acceleration of the body twisting around P.

The three double points of this homography are the points corresponding to the three permanent screws, i.e. those on which the body twists without any immediate tendency to depart.

(2) *On the Construction of a model of* 27 *Straight Lines upon a Cubic Surface.* By W. H. Blythe, M.A., Jesus College, Cambridge.

Every cubic surface contains twenty-seven straight lines real or imaginary, these lie in forty-five planes, in sets of three, there are consequently one hundred and thirty-five points of intersection of these lines.

Examining the position of these points two things are evident, first, that ten of them lie in each of the twenty-seven straight lines, and secondly, that since any two of the planes intersect in a straight line, the straight lines forming any two independent triangles intersect in pairs in three points lying in a straight line.

It is clear therefore that by taking sufficient points to determine the constants in the general equation, we can at the same time by means of these points, find the remainder of the hundred and thirty-five by simple geometrical construction.

It is shewn (Art. 3) that seven points of intersection can be so taken as to determine all constants in the general equation given by Taylor, and that these also are sufficient to find all remaining points of intersection.

The necessary drawing is however very complicated and difficult, but if we take an equation given by Cayley, and suppose the tetrahedron of reference to have three edges equal and at right angles to each other, and further take another plane (one of the forty-five) at right angles to the base of the tetrahedron, the construction is much simpler.

TABLE OF REFERENCE.

[*Numbers indicate straight lines. Three numbers in a bracket indicate three straight lines in a plane.*]

(4, 6, 5), (13, 10, 3), (9, 8, 7), (4, 13, 9), (6, 10, 8), (5, 3, 7), (12, 25, 18), (24, 14, 17), (19, 16, 1), (12, 24, 19), (25, 14, 16), (18, 17, 1), (2, 21, 22), (20, 15, 27), (23, 26, 11), (2, 20, 23), (21, 15, 26), (22, 27, 11), (4, 12, 2), (13, 14, 15), (9, 1, 11), (5, 14, 11), (3, 1, 2), (7, 12, 15), (6, 15, 1), (10, 11, 12), (8, 2, 14), (4, 27, 16), (13, 23, 18), (9, 21, 24), (4, 26, 17), (13, 22, 19), (9, 20, 25), (5, 18, 21), (3, 24, 27), (7, 16, 23), (6, 22, 25), (10, 20, 17), (8, 26, 19), (6, 23, 24), (8, 27, 18), (10, 21, 16), (5, 19, 20), (3, 25, 26), (7, 17, 22).

N.B. *To find whether two straight lines intersect, observe whether they occur in the same plane.*

One side of every triangle intersects with one side of any other triangle.

Art. 1. Geometrical construction to find the twenty-seven straight lines.

Art. 2. Notes on the form of the surface.

Art. 3. Geometrical construction to find the twenty-seven straight lines from the general equation given by Taylor.

[Fig. I. *represents the plane of the base of the tetrahedron of reference, the straight lines* 4, 9, 13 *being the edges of the base. On the sides of this diagram are marked the points at which the projections of the straight lines cut them.*

The rings (7, 9), (8, 9) *indicate where the straight lines* 7 *and* 8 *cut* 9. (18, 1); (11, 12); (9, 25) *are the projections of these points upon the plane of the figure.*

Fig. II. *represents a plane containing the straight lines* 9, 20, 25 *indicated by numerals placed at the end of them, and is at right angles to* I.

Figs. III. *and* IV. *are two planes representing faces of the tetrahedron of reference, also at right angles to* I. *containing respectively the straight lines* 4, 12, 2 *and* 13, 14, 15.

In Figs. II., III., IV. *the numbered points indicate where the straight lines meet these sections.*]

Art. 1. The following rules derived from the equation of the surface are sufficient to determine the straight lines 4, 9, 13, 20, 25, 12, 2, 14, 15, 1 and 11.

1. The heights of the intersections, (14, 25), (2, 20), (12, 15) above the plane of (4, 9, 13) are in the proportion $l : m : 1$. In this case 3 : 2 : 1.

2. The straight lines 20 and 25 each divide the edge 9 in the proportion $lmn : 1$. In this case $n = 2\frac{1}{12}$.

The edges 4 and 13 are divided in a similar manner by 2, and 12, and 14 and 15 respectively in the ratios $mn : 1$ and $ln : 1$.

3. The projections of the straight lines 1, and 11 in Fig. I. divide the edge 9 externally in proportion $1 : lm$ and the line 4 internally in the ratio $1 : m$.

4. Now to find any other straight line we proceed as follows. Take any plane containing the required straight line (e.g. 18) and two other known straight lines (e.g. 12, 25).

Next take any other two triangles already found 9, 11, 1 and 14, 15, 13.

We find that 25 intersects with 9, 12 with 11, and 18 with 1. Since these are the intersections of the sides of two independent triangles they are in one straight line. Therefore joining (9, 25) to (11, 12) in Fig. I. and producing the straight line we find the point at which 18 meets 1.

Similarly by joining (14, 25) and (12, 15) in Fig. IV. we find the point at which 18 meets 13. Join 18, 1 to 18, 13, the projection of 18 in I. is determined.

It is known that 18 meets 25 and 12 from the table of reference, therefore we mark on 25 and 12 in II. and III. points vertically above those in which its projection meets 9 and 4 in I.

Measuring horizontal distances from I. and verticals from II. and III. we can construct the position of 18 in a vertical plane drawn through it, as in V.

The remaining straight lines can be found in the same way, using those already determined to fix the position of others. On the plate opposite the straight lines are all given.

The simplest method of constructing a model of the straight lines is as follows.

First make drawings of I., II., III. and IV. to a scale of three times that given in the plate, being very careful that the lines cut each other exactly in the ratios stated.

Next draw each of the remaining lines to the same scale in the same manner as in V. Then take a drawing-board of any convenient size, probably it may be better to make the model exactly double the scale of the drawings for convenience of measurement.

On this drawing-board make a figure similar to I., it will not be necessary to draw the lines 1, 11, 18, but 4, 9, 13 should be marked, and the points along the edges at which all lines cut them.

Consider the plane of the drawing-board as being a certain distance, about five or six inches at least, below the plane of I.

It will be easy by the help of the drawings to determine the heights of any points on the straight lines, either on the edges of the figure, or at any other convenient points within it.

When choosing these points the position of the projection of the line on the drawing-board may be obtained by fixing two drawing pins where it meets the edges, and joining them by a silk thread.

A small model may be made on a much smaller board, the lines being passed through the eyes of needles fixed into corks at the requisite heights, the ends being tied down to the sides of the boards. In a larger model however it will be found more convenient, when the points have been marked down on the drawing-board, and the height of each indicated, to get blocks of wood in which uprights of the required height have been fixed. An upright may be of metal, or wood with a screw-eye at the top of it. If the measured height be not quite accurate these blocks can be moved backwards or forwards till the string is exactly in position. The wire representing the line, passing through the screw-eyes, is firmly fixed at its ends to the edges of the board, and the blocks can be fixed by means of hinges, or pieces of wood glued at the side.

The lines in the figures II., III., IV., should be first attached to uprights carefully measured and fixed in position, for three being in each vertical plane adjustment is more difficult.

It will be found that some lines require to be attached to the board at one end, the other only requiring an upright support.

The positions of the straight lines may be found by analysis by taking the equations given by Cayley of the forty-five planes: express the condition that the plane (9, 20, 25) that is the plane $lmn(p-\beta)z+[lm(p-\alpha)+2n]w=0$ is at right angles to $w=0$, and then take $l=-3$, $m=-2$, $n=2\frac{1}{12}$. Since α and β are known in terms of l, m, n we can find the value of p, and k, and all constants are known. Use areal coordinates.

[*See Collected Math. Papers by Cayley*, Vol. I. Number 76.]

Art. 2. The form of the surface, except near and within the tetrahedron of reference, approximates very closely to a cone of two sheets.

I. and XII. shew the shape of two horizontal sections of this cone at equal distances above and below the plane of the lines 4, 9, 13.

XIII. and XIV. are vertical sections through the origin making angles 45° and 150° with the plane of 4, 2, 12.

One sheet of the surface formed by joining all points on the oval to the origin is in the shape of an ordinary oblique cone.

The second sheet is traced out by the thick line drawn in XIII. and XIV. If we consider a vertical section through the origin first as passing through 4, and then revolving round axis z, we find this line coincides at first with 4, then traces out a portion of the surface above the plane 4, 9, 13 till the section has revolved through an angle 90°, the line now coincides with 13, it then passes below the plane of 4, 9, 13. At 135° the line coincides with 9.

Between 135° and 180° it is found above the plane of 4, 9, 13. At 180° it coincides with 4. At the same time the line produced backwards through the origin is tracing out similar portions respectively below, above, and below the plane of 4, 9, 13 from 180° to 360°.

The surface only coincides with this cone when the tetrahedron of reference is so small that it may be regarded as a point, but as before stated, the sections of the surface and the cone have very much the same shape except near the tetrahedron of reference.

In general then, horizontal sections of the surface consist of an oval and three infinite hyperbolic branches touching asymptotes parallel to 4, 9, 13. Vertical sections parallel to the planes of (4, 2, 12), (9, 20, 25), (13, 14, 15), except in the vicinity of the tetrahedron consist of one infinite branch touching one horizontal asymptote (e.g. parallel to 4) corresponding to the second sheet of the surface, and two hyperbolic branches touching asymptotes parallel to the remaining sides (e.g. 2, and 12) caused by intersection with the first sheet of the surface.

Next to consider the shape of the surface in the vicinity of the tetrahedron of reference.

Figs. numbered II. to XI. represent sections parallel to the plane 4, 9, 13. The right-angled triangle in each figure represents the projection of these lines upon the plane of the section.

Dotted lines represent asymptotes.

In II. the section is above the plane 4, 9, 13, and when moved a little nearer to this plane is as in III.; it is found that a node has taken place at the point marked by the asterisk and the oval exists no longer but has become part of one of the hyperbolic branches.

The changes are easily traced as the section moves nearer to 4, 9, 13. An asterisk is used to indicate that a node is about to take place.

In X. the section coincides with 4, 9, 13.

The oval again appears for any section below this plane, as in XI.

The equation of the surface in Cartesian coordinates can be found. For if we denote the equation to the plane 4, 9, 13 by (4, 9, 13)...the equation to the surface becomes

$$M(4, 12, 2)(13, 14, 15)(9, 7, 8) = (4, 9, 13)(12, 7, 15)(2, 14, 8).$$

Take 4 and 13 as axes x and y, using rectangular coordinates We can either by measuring the intercepts of these planes on the axes or the coordinates of three points in each plane determine the equations to the planes. M can then be found by substituting in the equation the coordinates of any point on any other of the twenty-seven straight lines.

If we take an edge of the tetrahedron in 4 or 13 as being $12\frac{1}{2}$ inches the equation to the surface in the case under consideration becomes—approximately—

$$60\,xy\,(x+y+(4\tfrac{1}{6})z-12\tfrac{1}{2}) = z\,(11x+10y+(27\tfrac{1}{2})z-110)(10x+14y-(12\tfrac{1}{2})z-24).$$

The equation to the cone to which the surface approximates is

$$60\,xy\,(x+y+(4\tfrac{1}{6})z) = z\,(11x+10y+(27\tfrac{1}{2})z)\,(10x+14y-(12\tfrac{1}{2})z).$$

It will be observed that in either case the equation to any section found by putting z constant reduces to the form

$$(x-A)(y-B)(x+y-C) + \text{terms of the first degree} = 0,$$

which shews that the asymptotes are parallel to 4, 9, 13. A and B vary with z, and the sum of A, B, C, in the case of the surface is very nearly $12\frac{1}{2}$, unless z is very large, and in the case of the cone is nearly zero.

It is interesting to trace the points in which the twenty-seven straight lines intersect the different sections, and how nodes take place to allow of the intersection of lines on different sheets and branches with each other. Space does not permit of these being given.

At a considerable distance from the tetrahedron:

The straight lines in the first sheet in order are 2, 5, 14, 25, 6, 12, 21, 24, 15, 10, 20, 3.

In the second sheet we find, 4, 22, 27, 18, 17, 13, 23, 1, 19, 9, 7, 8, 26, 11, 16.

Art. 3. The general equation given by Taylor may be put into the form

$$(L+l)(M+m)(N+n)(P+p)\,xyzu = (-lx+My+Nz+Pu)\times$$
$$(Lx-my+Nz+Pu)(Lx+My-nz+Pu)(Lx+My+Nz-pu).$$

It is evident that this surface can be constructed if we determine the ratios the constants L, M, N, P, l, m, n, p bear to one another.

If the tetrahedron of reference be a regular one, having all the edges equal, certain points of intersection seven in number taken upon the edges of the tetrahedron divide them in the ratios of these constants. If the tetrahedron be not regular these points still determine the ratios of the constants, but the lengths of the edges have to be considered.

Take the tetrahedron as regular, the base being ADC, B the centre of the triangle marking the projection of the vertex of the tetrahedron upon the plane of ADC.

In BD mark the point (12, 10). This determines the ratio $M:p$.
In BD produced .... (2, 8). .... $M:P$.
In AD produced .... (4, 9). .... $L:P$.
In DC .... (8, 9). .... $n:P$.
In DC produced .... (6, 1). .... $N:P$.
In AC .... (1, 3). .... $l:N$.
In BA .... (4, 5). .... $L:m$.

(4, 9), (4, 5) produced gives (4, 6) in BD,
(6, 1), (1, 3) .... (1, 2) in AD,
(8, 2), (2, 1) .... (2, 3) in AB,
(4, 9), (8, 9) .... (7, 9) in AC,
(8, 2), (8, 9) .... (7, 8) in BC,
(7, 9), (7, 8) .... (7, 12) in AB,
(7, 12), (10, 12) .... (11, 12) in AD,
(6, 1), (4, 6) .... (5, 6) in BC,
(5, 6), (4, 5) .... (5, 11) in AC,
(5, 11), (11, 12) .... (10, 11) in DC,
(1, 3), (2, 3) .... (3, 10) in BC.

These points determine the positions of the lines numbered 1 to 12. The lines 13, 14, 15 can be found by the method described in Art. 1.

Since however the lines numbered 16 to 27 do not occur in any triangle with two other known lines, that is, those numbered from 1 to 12, the method now fails.

Therefore, since we know that 16 meets 1, 4, 7, 10 and 14 we can draw these lines on paper, or arrange them on a model, and find the position of 16 by a method of adjustment.

When 16 is found, the remaining lines can be determined in the former way.

[*On a special form of the General Equation of a Cubic Surface by H. M. Taylor, M.A. Philosophical Transactions of the Royal Society of London*, Vol. 185. (1894).]

(3) *Exhibition of some photographs shewing the marks made by stars on photographic plates exposed near the focus of a visual telescope.* By H. F. NEWALL, M.A., Trinity College.

[*See Monthly Notices, R. Astron. Soc.*, Vol. LIV.]

November 12, 1894.

PROFESSOR J. J. THOMSON, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *On the inadequacy of the Cell Theory and on the development of Nerves.* By A. SEDGWICK, M.A., Trinity College.

The author pointed out that the cell-theory in so far as it implied that the organism was composed of cell-units derived by division from a single primitive cell-unit, the ovicell, would not bear the scrutiny of modern embryology, and that in fixing men's attention too much upon the cell as a unit of structure, it had had a retarding influence on the progress of the knowledge of structure. He illustrated this latter point by reference to the current ideas on two important subjects : the structure of the embryonic tissue called mesenchyme, and the development of nerves. The mesenchyme is not composed of separate branched cells, but has rather a spongy or reticulate structure, and is continuous both with ectoderm and endoderm. Nerves do not develop as outgrowths of the central organ, but arise in situ from the mesenchyme.

(2) *Note on the evolution of gas by water-plants.* By F. DARWIN, M.A., Christ's College.

November 26, 1894.

PROFESSOR J. J. THOMSON, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *On Benham's Artificial Spectrum.* By Professor G. D. LIVEING.

Professor Liveing exhibited one of Benham's "Artificial Spectrum tops," and a variety of discs with figures in black disposed on a white ground, and with white figures on a black ground, which when revolved in a bright light shewed remarkable bands of colour of various shades of red, green and blue. The general result of his observations of these discs was that if a succession of black and white objects were presented to the eye with moderate, but not too great, rapidity, then, when black was followed by white, an impression of a more or less red colour was perceived, while when white was succeeded by black a more or less blue colour was perceived. If the succession of black and white was very rapid the appearance presented to the eye was of a more or less neutral green or drab.

He found that for different people the rate of rotation necessary to produce the impression of a particular tint varied somewhat. When excentric or spiral bands of black were used, the impression on the eye was that of fringed bands, reddish on one side and greenish on the other. Sometimes the coloured bands seemed mottled or the colours appeared in flashes radiating from the centre of revolution.

He did not feel at all satisfied with the explanation published by Mr Benham which, if he understood it, depends on a sort of Doppler principle, and is to the effect that the *average* number per second of stimuli affecting the eye is increased, or diminished, according as a black portion of a band followed, or preceded, a white ground, in the revolution of the disc.

The only explanation Professor Liveing could himself offer, was based on the known facts that the impression produced on the retina by a bright object remained for an appreciable time after the light from the object had been cut off, and that the duration of that impression was different for different colours; and on a supposition, which he did not know to have been as yet verified experimentally, that the rapidity with which the eye perceives colours was greater for one end of the spectrum than for the other. From this point of view the explanation of the blue colour seen when white is followed by black would be that the impression of blue on the retina lasts a little longer than that of the other colours; while the red colour seen when white succeeds black is due to the greater rapidity with which the eye perceives red light than that with which it perceives blue. If however the alternations of white and black succeed each other with sufficient rapidity, the new impression of a white patch will be produced before that of its predecessor has vanished, and there will be an overlapping of impressions, and the sensation will be that of a mixture of colours, or of a more or less neutral tint. This is in accordance with the observation that if several circular bands, each partly black and partly white in equal proportions, are on the same disc, but in some of the bands the whole of the black parts is in one patch, while in others it is divided up into several patches, then, when the disc is rotated so that the band which presents only two alternations at each revolution is seen coloured, the bands which present a greater number of alternations are seen of a neutral tint. Also if the rotation is rapid enough the bands all appear of a neutral tint.

So far as he could test the theory by his own eyes it appeared to him that the residual impression, left when the light from a white object was suddenly cut off, was at first green and faded out through a more or less blue or slate colour. The object must not however be so bright as to dazzle the eye, the duration and colour of the residual impression might in that case be very different. He could see the colours of the discs best in bright diffuse daylight, and could see hardly any colour when the discs were in direct sunshine.

(2) *A simple test case of Maxwell's Law of Partition of Energy.* By G. H. Bryan, M.A., Peterhouse.

1. By Maxwell's Law of Partition of Kinetic Energy is meant the statement that, if the kinetic energy of a given system be expressed as a sum of squares, the mean values of these several squares taken over a large number of systems distributed according to a certain permanent or stationary law are equal.

The following illustration will, I think, show very clearly, or at any rate will throw some light on, how far this law is (i) a possible, (ii) a necessary law, when the systems considered do or do not collide with one another.

2. Consider a rigid body moveable about a fixed point or about its centre of gravity and acted on by no forces. The equations of motion are

$$A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3 = 0, \text{ \&c. } \ldots\ldots\ldots\ldots\ldots(1),$$

and the kinetic energy takes the form of a sum of squares, viz.

$$T = \tfrac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2).$$

The angular velocities ω_1, ω_2, ω_3 and the corresponding angular momenta $A\omega_1$, $B\omega_2$, $C\omega_3$ are proportional to what Boltzmann in his important paper "On the Equilibrium of Vis Viva"* calls "momentoids" of the body, and they cannot be regarded as true generalised velocities or momenta to which Lagrange's equations are directly applicable.

Now let Ω_1, Ω_2, Ω_3 denote the initial angular velocities of the body about its principal axes, ω_1, ω_2, ω_3 being its angular velocities at any time t. Then ω_1, ω_2, ω_3 are functions of Ω_1, Ω_2, Ω_3 and t, and we may readily show that the Jacobian

$$\frac{\partial(\omega_1, \omega_2, \omega_3)}{\partial(\Omega_1, \Omega_2, \Omega_3)} = 1,$$

the differentiations being made on the supposition $t = \text{constant}$.

3. To prove this relation we have

$$\frac{d}{dt}\left\{\frac{\partial(\omega_1, \omega_2, \omega_3)}{\partial(\Omega_1, \Omega_2, \Omega_3)}\right\} = \frac{\partial(\dot\omega_1, \omega_2, \omega_3)}{\partial(\Omega_1, \Omega_2, \Omega_3)} + \frac{\partial(\omega_1, \dot\omega_2, \omega_3)}{\partial(\Omega_1, \Omega_2, \Omega_3)} + \frac{\partial(\omega_1, \omega_2, \dot\omega_3)}{\partial(\Omega_1, \Omega_2, \Omega_3)}.$$

But by the equations of motion (1),

$$A\frac{\partial(\dot\omega_1, \omega_2, \omega_3)}{\partial(\Omega_1, \Omega_2, \Omega_3)} \text{ or } A\sum \pm\left(\frac{\partial\dot\omega_1}{\partial\Omega_1}, \frac{\partial\omega_2}{\partial\Omega_2}, \frac{\partial\omega_3}{\partial\Omega_3}\right)$$

$$= (B - C)\sum \pm\left(\omega_2\frac{\partial\omega_3}{\partial\Omega_1} + \omega_3\frac{\partial\omega_2}{\partial\Omega_1}, \frac{\partial\omega_2}{\partial\Omega_2}, \frac{\partial\omega_3}{\partial\Omega_3}\right).$$

If this determinant be written in full, with the operators

$$\frac{\partial}{\partial\Omega_1}, \frac{\partial}{\partial\Omega_2}, \frac{\partial}{\partial\Omega_3}$$

arranged in columns, the first row is equal to ω_3 times the second row plus ω_2 times the third, therefore the determinant vanishes, or

$$\frac{\partial(\dot\omega_1, \omega_2, \omega_3)}{\partial(\Omega_1, \Omega_2, \Omega_3)} = 0.$$

Therefore $$\frac{d}{dt}\left\{\frac{\partial(\omega_1, \omega_2, \omega_3)}{\partial(\Omega_1, \Omega_2, \Omega_3)}\right\} = 0,$$

or $$\frac{\partial(\omega_1, \omega_2, \omega_3)}{\partial(\Omega_1, \Omega_2, \Omega_3)} = \text{constant}$$

$$= 1,$$

since its initial value is unity.

* Translated in the *Phil. Mag.* for March, 1893.

4. Now let there be a very large number N of such rigid bodies all perfectly independent of one another. Let them be initially set in motion, in such a way that the number of bodies whose angular velocities about their principal axes initially lie between Ω_1 and $\Omega_1 + d\Omega_1$, Ω_2 and $\Omega_2 + d\Omega_2$, Ω_3 and $\Omega_3 + d\Omega_3$ is

$$Nf(T)\, d\Omega_1 \,.\, d\Omega_2 \,.\, d\Omega_3,$$

where the "frequency factor" $f(T)$ is any function whatever of the kinetic energy.

Then at any subsequent time t, it follows from the determinantal relation proved above and the constancy of the kinetic energy for each body that the distribution is given by

$$Nf(T)\, d\omega_1 \,.\, d\omega_2 \,.\, d\omega_3,$$

and is therefore the same as before.

Hence the distribution is a permanent one, being independent of the time.

5. Now $$T = \tfrac{1}{2}\{A\omega_1^2 + B\omega_2^2 + C\omega_3^2\},$$

and we shall now show that with the above distribution the average kinetic energy is distributed equally among the momentoids corresponding to ω_1, ω_2, ω_3, or that the average values of

$$\tfrac{1}{2}A\omega_1^2, \quad \tfrac{1}{2}B\omega_2^2, \quad \tfrac{1}{2}C\omega_3^2$$

are equal.

For the average value of $\frac{1}{2}A\omega_1^2$ for all the bodies at any instant

$$= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\frac{1}{2}A\omega_1^2 + \frac{1}{2}B\omega_2^2 + \frac{1}{2}C\omega_3^2) \,.\, \frac{1}{2}A\omega_1^2 d\omega_1 d\omega_2 d\omega_3}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\frac{1}{2}A\omega_1^2 + \frac{1}{2}B\omega_2^2 + \frac{1}{2}C\omega_3^2)\, d\omega_1 d\omega_2 d\omega_3}.$$

Writing $$\omega_1\sqrt{\tfrac{1}{2}A} = \xi, \quad \omega_2\sqrt{\tfrac{1}{2}B} = \eta, \quad \omega_3\sqrt{\tfrac{1}{2}C} = \zeta,$$

this becomes $$= \frac{\iiint f(\xi^2 + \eta^2 + \zeta^2)\, \xi^2 d\xi d\eta d\zeta}{\iiint f(\xi^2 + \eta^2 + \zeta^2)\, d\xi d\eta d\zeta},$$

and the average values of $\frac{1}{2}B\omega_2^2$, $\frac{1}{2}C\omega_3^2$ are evidently equal to the same expression. This expression may be put in the form

$$\frac{1}{3}\int_0^{\infty} f(T)\, T^{\frac{3}{2}} dT \div \int_0^{\infty} f(T)\, T^{\frac{1}{2}} dT.$$

Hence Maxwell's law of partition of energy holds good in this case.

6. We now proceed to show that other distributions exist which satisfy the condition of permanency and in which the energy is not distributed according to Maxwell's law.

For the equations of motion have *two* integrals, one of these expressing the constancy of energy being

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = 2T = \text{const.},$$

the other expressing the constancy of angular momentum being

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = G^2 = \text{const.}$$

If now, we assume the frequency factor to be a function of G instead of T, say $f(G)$, so that the number of bodies whose angular velocities lie between ω_1 and $\omega_1 + d\omega_1$, ω_2 and $\omega_2 + d\omega_2$, ω_3 and $\omega_3 + d\omega_3$ is

$$f(G)\, d\omega_1 \,.\, d\omega_2 \,.\, d\omega_3,$$

then precisely the same argument as before shows that this distribution is also a permanent one, being independent of the time.

Also precisely the same argument as before shows that in this case the mean values of the quantities

$$A^2\omega_1^2, \quad B^2\omega_2^2, \quad C^2\omega_3^2$$

are equal to one another, each being equal to

$$\frac{1}{3}\int_0^\infty f(G^2)\, G^4 dG \div \int_0^\infty f(G^2)\, G^2 dG.$$

Therefore *the mean values of the portions of the energy*

$$A\omega_1^2, \quad B\omega_2^2, \quad C\omega_3^2$$

are inversely proportional to A, B, C *and are unequal,* so that Maxwell's law does *not* hold good.

7. If we were to take the frequency factor to be a function both of T and G^2, we should have the most general distributions consistent with permanency, in none of which would the kinetic energy be distributed according to Maxwell's law.

The present investigation shows that in order that Maxwell's law may hold, *the frequency factor f must be a function of the energy only,* at least so far as concerns the generalized velocities, momenta, or momentoids which enter into it.

Where, as in the present instance, other forms of the frequency factor are consistent with permanency Maxwell's law of partition although a possible law is not unique.

8. When the different systems move about freely and collide with each other, as the molecules of a gas, it has been shown by Watson*, by Burbury† and by Boltzmann‡ that if the velocities and angular velocities about the principal axes be arranged according to the Boltzmann-Maxwell distribution

$$Ne^{-hT}\,du\,dv\,dw\,d\omega_1\,d\omega_2\,d\omega_3,$$

that is

$$N\exp\left[-\tfrac{1}{2}h\left\{M(u^2+v^2+w^2)+A\omega_1^2+B\omega_2^2+C\omega_3^2\right\}\right]du\ldots d\omega_3,$$

this distribution will be unaffected by collisions. From the present paper we see that it will also be unaffected by the free motion of the bodies between collisions, and therefore it satisfies all the necessary conditions of permanence. The mean values of Mu^2, Mv^2, Mw^2, $A\omega_1^2$, $B\omega_2^2$, $C\omega_3^2$, are equal. The other distribution of § 6 above, in which the mean kinetic energies due to the three principal rotations are unequal is, in general, no longer permanent when collisions can take place. In fact Boltzmann starts by assuming a distribution of the form

$$N\frac{\sqrt{h^3\,.\,k_1k_2k_3}}{\pi^3}\exp\left[-h\left\{u^2+v^2+w^2\right\}-k_1\omega_1^2-k_2\omega_2^2-k_3\omega_3^2\right]du\ldots d\omega_3,$$

and deduces that

$$\frac{h}{M}=\frac{k_1}{A}=\frac{k_2}{B}=\frac{k_3}{C}.$$

9. An exceptional case is that in which each molecule (regarded as a perfectly smooth hard body) is symmetrical about an axis through its centre of inertia. The angular velocity about this axis will be unaffected both by collisions and by the free motion of the molecules, this angular velocity might, therefore, follow any law of distribution whatever.

Any distribution of the form

$$f(\omega_3)\exp\left[-\tfrac{1}{2}h\left\{M(u^2+v^2+w^2)+A\omega_1^2+A\omega_2^2\right\}\right]$$

will therefore be permanent and $f(\omega_3)$ will not necessarily be proportional to $\exp\left[-\tfrac{1}{2}hC\omega_3^2\right]$.

The chief interest of this case lies in the fact, pointed out by Boltzmann, that since partition of energy only takes place among

* *Nature*, Vol. XLV. March 31, 1892, p. 512.

† *Ibid.* Vol. XLV. April 7, 1892, p. 533.

‡ 'Ueber das Gleichgewicht der lebendigen Kraft zwischen progressiver und Rotations-Bewegung bei Gasmolekülen,' *Sitz. d. k. Akad. zu Berlin*, LII. Dec. 1888.

five of the six degrees of freedom, the ratio of the two specific heats

$$1+\frac{2}{m}=1+\frac{2}{5}=1{\cdot}4$$

agreeing closely with the value found for air and most gases*.

10. Another exceptional case is that in which the molecules are regarded as smooth spheres or as centrobaric bodies repelling one another according to any law of force which is a function of the distance between their centres of inertia. Here, even if the bodies are not dynamically symmetrical about their centres of inertia, (as may be exemplified by smooth spheres containing hollow concentric ellipsoidal cavities,) the forces between them in a collision or encounter always pass through their centres of inertia and cannot affect their angular velocities about their principal axes. It is therefore not necessary in such cases that the mean kinetic energies of rotation about the principal axes should be equal to one another or to the kinetic energies of the components of translation.

* *Sitz. der k. Wiener Akad.*, LXXIV. (II.), 1876. *Phil. Mag.*, March, 1893.

PROCEEDINGS

OF THE

Cambridge Philosophical Society.

MONDAY, 28 *January*, 1895.

PROFESSOR J. J. THOMSON, PRESIDENT, IN THE CHAIR.

The following resolutions were proposed by Professor Sir G. G. STOKES, seconded by Mr GLAZEBROOK, and passed unanimously:

I. "That the Cambridge Philosophical Society desires to express its sense of the great loss sustained by the University and the Society by the death of Professor CAYLEY; whose eminence conferred honour on the Society, which reckoned him amongst its Presidents, and whose simple and earnest character was an example to all, and endeared him to those who knew him."

II. "That the Society do now adjourn without transacting the business of the meeting, as a mark of respect to Professor Cayley."

III. "That the President be requested to convey the foregoing resolutions to Mrs Cayley."

MONDAY, 11 *February*, 1895.

PROFESSOR J. J. THOMSON, PRESIDENT, IN THE CHAIR.

The Treasurer's accounts were presented, duly audited.

P. H. Cowell, B.A., Fellow of Trinity College, was elected Fellow of the Society.

The following Communications were made to the Society:

(1)* *A Method of Comparing the Conductivities of Badly Conducting Substances for rapidly Alternating Currents. By* Professor J. J. THOMSON.

THE method employed in the following experiments consists in making a secondary circuit of standard form made of the substance whose conductivity is to be investigated, and observing the effect produced by its presence on the alternating currents in a primary circuit.

Before proceeding to consider the results of the experiments it will be convenient to discuss some points connected with the theory of the reaction of the secondary circuit on the primary. The first case we shall consider is that of the ordinary transformer when the electromotive force acting on the primary circuit is given. If this electromotive force is $E\cos pt$, and if L, M, N are respectively the coefficient of self-induction of the primary, the coefficient of mutual induction between the primary and secondary and the coefficient of self-induction of the secondary; R the resistance of the primary, S that of the secondary, x the current through the primary, y that through the secondary: then we have

$$L\frac{dx}{dt} + M\frac{dy}{dt} + Rx = E\cos pt,$$

$$M\frac{dx}{dt} + N\frac{dy}{dt} + Sy = 0.$$

Solving these equations we find

$$x = \frac{E\cos(pt-\alpha)}{\sqrt{L'^2p^2 + R'^2}} \quad \text{..........................(1)},$$

where

$$L' = L - \frac{M^2Np^2}{S^2+N^2p^2},$$

$$R' = R + \frac{M^2Sp^2}{S^2+N^2p^2},$$

$$\tan\alpha = \frac{L'p}{R'}.$$

Substituting these values of L' and R' in (1) we get

$$x = \frac{E\cos(pt-\alpha)}{\left\{L^2p^2 + R^2 + \frac{M^2p^2}{S^2+N^2p^2}\{2RS - (2LN - M^2)p^2\}\right\}^{\frac{1}{2}}}.$$

* The chair was taken by Professor Sir G. G. Stokes, Vice President, whilst Professor Thomson read his paper.

Thus until S is less than

$$\frac{(2LN-M^2)p^2}{2R},$$

the maximum current through the primary is less than

$$\frac{E}{\{L^2p^2+R^2\}^{\frac{1}{2}}},$$

which is the maximum value of the current when S is infinite or when the secondary circuit is broken. Thus we get the somewhat curious result that the current through the primary of a transformer is less when the transformer has a slight load than when it has *no load at all.*

The work absorbed by the transformer in unit time is

$$\frac{1}{2}\frac{E^2\cos\alpha}{\sqrt{L'^2p^2+R'^2}}$$

$$=\frac{1}{2}\frac{E^2R'}{(L'^2p^2+R'^2)}$$

$$=\frac{1}{2}\frac{E^2\{S(L^2p^2-R^2)+p^2R(2LN-M^2)\}M^2p^2}{(S^2+N^2p^2)(L'^2p^2+R'^2)(L^2p^2+R^2)}+\frac{1}{2}\frac{E^2R}{L^2p^2+R^2}.$$

The second term on the right-hand side of this equation is the work absorbed when the secondary circuit is open; if Lp is greater than R, the first term is positive for all values of S; hence the work absorbed in the primary is in this case always greater when the secondary is closed than when it is open.

If R is small compared with Lp, we find from the preceding expression that the absorption of energy is greatest when

$$S=\left(N-\frac{M^2}{L}\right)p.$$

Thus if there is any magnetic leakage between the primary and secondary, *i.e.* if LN does not equal M^2, there is a definite resistance for which the work expended in the primary is greatest.

We shall now consider the case when the currents which circulate through the primary are those produced by discharging a Leyden jar; we shall suppose that the primary circuit connects the inside and outside coatings of a jar whose capacity is C. Let x now denote the charge in this jar at any time, the rest of the notation being the same as before; then the equations giving the currents through the coats are,

$$L\frac{d^2x}{dt^2}+M\frac{dy}{dt}+R\frac{dx}{dt}+\frac{x}{C}=0 \quad \dots\dots\dots\dots\dots(1),$$

$$M\frac{d^2x}{dt^2} + N\frac{dy}{dt} + Sy = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Eliminating y we get

$$A\frac{d^3x}{dt^3} + B\frac{d^2x}{dt^2} + D\frac{dx}{dt} + Ex = 0 \quad\ldots\ldots\ldots\ldots\ldots(3),$$

where

$$A = LN - M^2,$$

$$B = LS + NR,$$

$$D = SR + \frac{N}{C},$$

$$E = \frac{S}{C}.$$

If we had eliminated x we should have obtained a precisely similar equation for y.

Now suppose we have a fixed coil in series with the primary circuit and we insert in this an exhausted bulb, which is the same in all experiments, a bright discharge will pass through the bulb; the brightness of this discharge changes with the resistance of the secondary circuit.

Since dx/dt is the current flowing through the primary circuit, the number of lines of magnetic force passing through the coil in series with the primary is proportional to dx/dt; thus the rate of change of the number of lines of magnetic force passing through the bulb, and therefore the electromotive force acting on the bulb, is proportional to d^2x/dt^2, we may therefore take

$$\int\left(\frac{d^2x}{dt^2}\right)^2 dt,$$

as a measure of the brightness of the discharge in the bulb, the integration extending over the whole time of discharge of the Leyden jar.

To find the value of this quantity multiply (3) by $\frac{d^2x}{dt^2}$ and integrate with respect to t, remembering that initially $dx/dt = 0$, and that finally x and all its differential coefficients vanish: performing the integration we find

$$-\tfrac{1}{2}A\left(\frac{d^2x}{dt^2}\right)_0^2 + B\int\left(\frac{d^2x}{dt^2}\right)^2 dt - E\int\left(\frac{dx}{dt}\right)^2 dt = 0\ldots\ldots(4),$$

where $\left(\frac{d^2x}{dt^2}\right)_0$ denotes the initial value of d^2x/dt^2. Now multiply

(3) by $\frac{dx}{dt}$ and integrate; we get

$$-A\int\left(\frac{d^2x}{dt^2}\right)^2 dt + D\int\left(\frac{dx}{dt}\right)^2 dt - \tfrac{1}{2}Ex_0^{\,2} = 0 \ldots\ldots(5)$$

when x_0 denotes the initial value of x.

We get from (4) and (5)

$$\int\left(\frac{d^2x}{dt^2}\right)^2 dt = \tfrac{1}{2}\,\frac{AD\left(\frac{d^2x}{dt^2}\right)_0^2 + E^2x_0^{\,2}}{BD - AE}.$$

Since initially $\frac{dx}{dt}$ and y both vanish, we get from (1) and (2)

$$\left(\frac{d^2x}{dt^2}\right)_0 = -\frac{N}{C}\frac{x_0}{A}.$$

Hence substituting the values of A, D, B, E and $\left(\frac{d^2x}{dt^2}\right)_0^2$ we find

$$\int\left(\frac{d^2x}{dt^2}\right)^2 dt = \tfrac{1}{2}\,\frac{x_0^{\,2}}{C^2}\,\frac{\left\{\dfrac{N^2\left(RS + \frac{N}{C}\right)}{LN - M^2} + S^2\right\}}{(NR + LS)\left(\frac{N}{C} + SR\right) - (LN - M^2)\frac{S}{C}} \ldots(6).$$

The expression on the right-hand side of the equation is least when

$$S\left\{1 - \frac{RC}{\sqrt{C\left(L - \frac{M^2}{N}\right)}}\right\} = \frac{N}{\sqrt{C\left(L - \frac{M^2}{N}\right)}}.$$

If, as in the following experiments, RC is small compared with

$$\sqrt{C\left(L - \frac{M^2}{N}\right)},$$

we have

$$\frac{S}{N} = \frac{1}{\sqrt{C\left(L - \frac{M^2}{N}\right)}}.$$

Now N/S is the time constant of the secondary circuit when alone by itself in the field, while

$$\sqrt{C\left(L - \frac{M^2}{N}\right)} = T/2\pi,$$

where T is the periodic time of the electrical oscillations produced when the jar is discharged, the secondary circuit being in position. Thus $\int\left(\frac{d^2x}{dt^2}\right)^2 dt$, and therefore the brightness of the discharge in the bulb will be a minimum when the time constant of the secondary circuit is $1/2\pi$ times the time of the electrical oscillations. In the actual experiments the secondary circuit was a bulb which was filled with the substance to be examined; if the radius of the bulb is a and if σ is the specific resistance of the substance with which it is filled, then the time constant for the most persistent distribution of current is $4a^2/\pi\sigma$. Thus the specific resistance of the substance which has the greatest effect in diminishing the brightness of the discharge through the bulb is equal to $8a^2/T$. Hence if we are using oscillations whose period is one-millionth of a second and if $a^2 = 10$, then the specific resistance when the brightness of the bulb is a minimum is 8×10^7. Though the method will not be sensitive for comparing resistances close to this value, yet it is important to know the resistance when the dimness of the bulb is a maximum, as for resistances of quite a different order the effect of a change in the resistance will be too small to be appreciable. As the frequency of the electrical oscillations or the size of the bulb is increased the specific resistance of the substance which gives the maximum effect also increases.

The heat produced in the secondary circuit is equal to

$$S\int y^2 dt;$$

the value of the integral can be calculated by the method used to calculate $\int\left(\frac{d^2x}{dt^2}\right)^2 dt$, we find

$$S\int y^2 dt = \tfrac{1}{2}\frac{x_0^{\,2}}{C^2}\frac{M^2 S}{\left\{S^2 LR + S\left(NR^2 + \frac{M^2}{C}\right) + N^2\frac{R}{C}\right\}}.$$

This is a maximum when

$$\frac{S}{N} = \frac{1}{\sqrt{LC}}.$$

Method of making the Experiments. The method consists in observing the effect produced on the currents in the primary circuit by introducing a secondary circuit of standard shape made of the substance to be examined. The first arrangement I tried for effecting this object is shown in Fig. (1).

The primary circuit $CDEF$ connects the outside coatings of two Leyden jars, the inside coatings of which are connected with the terminals of a Wimshurst machine; these terminals are furnished with brightly polished knobs, between which sparks pass discharging the jars and starting alternating currents in the primary circuit; *D* and *E* are two coils wound on ebonite cylinders and made as nearly equal as possible in every respect.

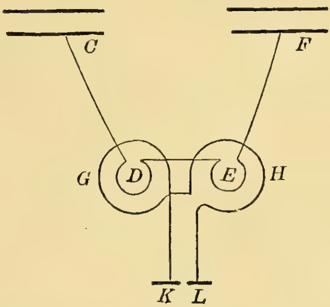


Fig. 1.

G and *H* are two equal coils placed in series, and so arranged that the electromotive force due to the induction between *G* and *D* and between *H* and *E* acted in the opposite direction round the coils, the ends of the coils were connected to the spark gap *KL*, and the equality of the electromotive forces round the two coils was tested by the absence of sparks in the spark gap. One of the coils, *G*, was provided with an arrangement by means of which portions of it could be short-circuited so as to enable the induction between it and the primary coil to be slightly altered, this served as a fine adjustment, the object of which was to make the induction between *G* and *D* equal to that between *H* and *E*. I found it however impossible to make the adjustment so accurately that there was a complete absence of sparks across the spark gap, the most I could do was to adjust the coils so that the length of the spark was a minimum. When this had been done a vessel containing the substance whose resistance was to be investigated was placed inside one of the primary coils, say *D*, the effect of this was to increase the length of the spark gap; a similar vessel containing a solution of an electrolyte was then placed inside the other primary coil, *E*, and the strength of the solution changed until the spark length was reduced to its former value; when this was the case the specific resistance of the substance in *D* was the same as that of the electrolyte in *E*: so that the resistance of the substance in *D* could be compared with that of an electrolyte of known strength. After many trials with this method I abandoned it in favour of the following, which I found much more sensitive.

In this there were two coils, A and B, in the primary circuit; in one of these the substance to be examined was placed, and in the other a bulb containing gas at a low pressure; when the jars were discharged the alternating currents produced in the primary induced a luminous discharge through the rarefied gas in the bulb, and the brightness of the discharge served as an indication of the

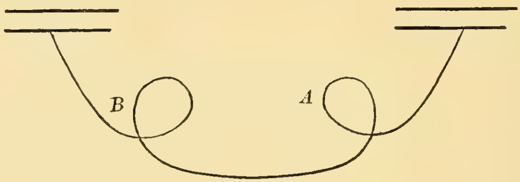


Fig. 2.

strength of the currents flowing through the primary circuit. The substances to be compared were introduced in spherical vessels of constant size inside the coil A, and the diminution their introduction produced in the discharge B observed; when two substances produce the same diminution they have the same specific resistance. The more easily the discharge passes through the bulb B the more sensitive is this method; for this reason it is advisable to take some trouble in the preparation of this bulb; the most sensitive bulbs I obtained were made by adding a little bromine vapour to air at a very low pressure. With these bulbs when a sphere about 3 inches in diameter filled with a mixture of 1 c.c. of H_2SO_4 to a litre of water was introduced into A the diminution in the brilliancy of the discharge was quite marked; if the sphere were filled with distilled water no effect was produced. This arrangement serves very well to illustrate the laws expressed by equation (6). Suppose that we introduce into A a secondary circuit made of a good conductor, say a portion of a copper cylinder which nearly fits the coil. When this cylinder is introduced the discharge through B is brighter than when it is absent. This follows at once from equation (6). The brilliancy of the discharge, I, is measured by the value of $\int\left(\frac{d^2x}{dt^2}\right)^2 dt$. Now we can get from equation (6) the value of I when the secondary circuit is absent by putting either $M = 0$, or $S = \infty$, in the right hand of the equation; doing this we find

$$I = \frac{1}{2}\frac{x_0^{\,2}}{C^2}\frac{1}{LR} \qquad \text{(7).}$$

When the secondary circuit is present, then if S is very small, the value of I by equation (6) is given approximately by the equation

$$I = \tfrac{1}{2}\frac{x_0^2}{C^2}\frac{1}{\left(L - \frac{M^2}{N}\right)R} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8);$$

it is thus greater than the value of I when the secondary circuit is absent given by (7). Thus in this case the diminution in the effective self-induction of the primary circuit due to the presence of the secondary increases the brightness of the discharge.

If we begin with the cylinder some way off and gradually lower it into the coil A we observe that at first the approach of the cylinder diminishes the brilliancy of the discharge, but after the cylinder gets within a certain distance of the coil the brilliancy of the discharge increases as the cylinder is moved towards the coil. Thus at first the diminution in the brilliancy due to the resistance of the cylinder overpowers the increase due to the diminution of the effective self-induction of the primary circuit, but when the cylinder gets close to the coil the latter effect gets the upper hand. This result follows from (6). In that equation $1/C$ is such a very large quantity that it is only when the resistances of the secondary and primary circuits are exceedingly great that the terms which do not contain $1/C$ are comparable with those which do. When the secondary and primary circuits have as in the present case only small resistances, we may omit from the right-hand side of (6) all the terms which do not contain $1/C$; doing this we find

$$I = \frac{x_0^2}{C}\frac{\frac{N^3}{(LN - M^2)}}{(N^2R + M^2S)}.$$

The differential coefficient with respect to M^2 of the denominator of the right-hand side of the equation is proportional to

$$-2M^2S + N(LS - NR),$$

as this is positive until

$$M^2 > \tfrac{1}{2}N\left(L - N\frac{R}{S}\right).$$

We see that until M attains this value an increase in M such as that caused by the approach of the cylinder to the coil will diminish I, while after M has attained this value the value of I will increase as the cylinder approaches the coil.

Resistance of Electrolytes for very rapidly alternating Electromotive forces. I determined by this method the relative resistances of various electrolytes; to do this bulbs of the same shape and size filled with the different electrolytes were placed in the coil, and the effect they produced on the exhausted bulb in coil *B* Fig. (2) compared. In these experiments the electrolytes are exposed to electromotive forces of exceedingly high frequency, the frequency being that of the electrical oscillations produced by the discharge of the jar; this was more than a million per second. The electromotive forces while they last are exceedingly intense, so that the circumstances under which the electrolyte is placed in these experiments are very different from those which obtain in the ordinary determinations of the resistance of electrolytes. I found however, as in my previous experiments on this point made by a different method (*Proc. Roy. Soc.* Vol. XLV. p. 269 (1889)), that the proportion between the resistances determined in this way was the same as for steady currents. A point I specially investigated was to see whether solutions of sulphuric acid showed the same peculiarities in the relation between the conductivity and the strength of solution as are found when the conductivities are measured under steady currents. I found, starting with pure water and gradually adding sulphuric acid, that at first each fresh addition of acid produced an increase in the conductivity, this went on until the solution contained about 15°/₀ by volume of H_2SO_4, at this point the addition of acid produced too small an effect to be appreciable by this method; the conductivity did not seem to change much until the solution contained about 60°/₀ by volume of H_2SO_4; at this point the addition of fresh acid produced an appreciable diminution in the conductivity of the solution, and this diminution went on as the strength of the solution was increased. Thus except that the method is not sufficiently delicate to detect the small differences of conductivity with the strength of the solution which occur in the neighbourhood of the state of maximum conductivity, the results obtained by this method agree exactly with those obtained with steady currents.

The agreement of resistances obtained by using small electromotive forces with those obtained by this method when large electromotive forces are used is interesting inasmuch as it shows that these large electromotive forces do not appreciably increase the amount of dissociation of the electrolyte.

Conductivity of Flames. This is very easily shown by means of this apparatus. All that is necessary is to place in *A* an ordinary lamp-glass. When the flame from a Bunsen burner is placed under this a very marked diminution in the luminosity of the bulb in *B* is produced.

Variation of the Conductivity of a Gas with the Pressure. This was determined by comparing the effect produced by a bulb filled with air at different pressures with the effect produced by bulbs of the same size filled with solutions of sulphuric acid of different strengths; the conductivity of the gas was estimated by finding which of the solutions produced the same effect on the luminosity of the discharge of the bulb in *B* as the bulb containing the gas. The bulb was connected to the air-pump by a flexible glass spiral which allowed it to be lifted out of, or inserted, in the coil without disturbing the connection with the pump. When the air in *A* was at atmospheric pressure it produced no effect on the bulb in *B*, indeed no effect was observed until the pressure in the bulb in *A* was reduced to that due to 3 or 4 millimetres of mercury; at this pressure, though no discharge of any kind was visible in the bulb, the pressure of the bulb produced a distinct effect upon the luminosity of the bulb in *B*, and this effect rapidly increased until at a pressure of between 1 and 2 millimetres it attained its maximum value. During these stages there was no discharge visible in the bulb in the coil *A*; on further diminishing the pressure the effect on the bulb in *B* diminished until the pressure got so low that the ring discharged began to appear in the bulb. When this stage was reached a diminution in the pressure increased the effect on the bulb in *B*; the ring discharge in *A* got brighter and brighter, and the conductivity seemed to go on increasing as the pressure was reduced. It was only after very protracted pumping that a stage was reached when a further diminution in pressure produced a diminution in the conductivity.

Thus if we represent the effect of the bulb in *A* upon the luminosity of the one in *B* by the ordinate of a curve of which the abscissa represents the rarefaction, we get a curve similar to that shown in Fig. 3. The existence of the first maximum

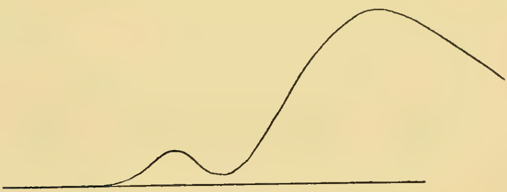


Fig. 3.

is very remarkable especially as it is not accompanied by any luminosity of the gas. I have come to the conclusion that although at this pressure the electromotive force is not sufficient to send a ring discharge through the gas, yet there is a dark discharge from the inside of the glass in the neighbourhood of the coil of wire, due to the high potential which this coil attains at some phases of the discharges of the jars. That there is an effect of this kind is shown by the appearance at a slightly lower pressure, which however is still too high to allow the ring discharge to pass, of a velvety glow over this part of the glass. This electric discharge from the glass sends charged atoms into the body of the bulb, and these moving under the electromotive forces endow the gas with the conductivity which is evidenced by the dimming of the discharge through the bulb placed in the coil *B*.

Conductivity arising from Chemical Action between Two Gases. When there are free charged atoms in a gas, the gas will be a conductor, and its conductivity can easily be tested by the method described in this paper. I thought it would be of interest to test whether chemical action between gases was accompanied by a temporary disengagement of free charged atoms, if this were the case the gases would be conductors while the chemical action was going on. The first experiments I tried consisted in letting hydrochloric acid and ammonia gases mix in a bulb placed inside the coil *A*, but though there was evidently vigorous chemical combinations going on between the gases I was unable to detect any sign of conductivity. A similar want of success was met with when a mixture of hydrogen and chlorine at atmospheric pressure was exposed to diffuse light. Here again no conductivity could be detected. In other cases I was more successful, for when H_2S and Cl_2 were mixed at atmospheric pressure and chemical combination in consequence took place, a most decided effect was produced on the bulb in *B*, showing that the gas in *A* was now a conductor; the effect on *B* ceased after a time, but if there was an excess of one of the gases to begin with it would always be renewed by bringing in a fresh supply of the other gas. When NO was being liberated in the bulb in *A* by the action of nitric acid on copper and oxidised in the air, considerable effect was again produced on the bulb in *B*, showing that this oxidation was accompanied by conductivity. In another case of oxidation I also observed the same effect, as I found that the steam from boiling water containing phosphorus possessed considerable conductivity. These examples are sufficient to show that in many cases of chemical combination the charged atoms are liberated to an extent sufficient to allow the positive atoms to move one way, the negative atoms the other, under the influence of the strong electromotive forces acting upon them in these experiments.

Another experiment I tried was to see whether the formation of a rain-cloud was attended by any separation of positive and negative electricity. A bulb containing air saturated with water-vapour was placed inside *A* and connected with a Fleuss pump, a few strokes of the pump produced a cloud in the bulb, but

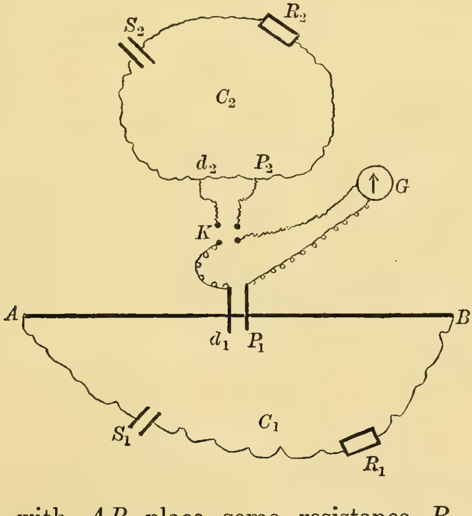
I could not detect any trace of conductivity during the formation of the cloud. In this experiment the apparatus was sensitive enough to detect the conductivity of a mixture of 1 c.c. of H_2SO_4 in a litre of water.

(2) *The Calibration of a Bridge Wire.* By E. H. GRIFFITHS, M.A., Sidney Sussex College.

THE ordinary methods of calibration involve changes in the arrangement of the connections, during the operations, and thus extreme care has to be observed to secure equality in the resistance-contacts. It is chiefly on this account that the process of calibration is invariably a lengthy and tedious business. In the method I am about to describe the only contacts altered throughout the observations are potential-contacts, and thus no attention need be devoted to securing their equality.

Let AB be the wire that is to be calibrated. (Its total resistance should first be approximately determined in the ordinary manner.)

Let S_1 and S_2 be two cells of nearly equal electromotive force, preferably two large storage cells. (Their absolute equality is not essential.)



In series with AB place some resistance R_1 sufficient to prevent any *rapid* fall in the E. M. F. of the cell S_1.

Make the total resistance of the circuit C_2 approximately equal to the *total* resistance of the circuit C_1.

We may assume that under these conditions the *rate of fall* in the E. M. F. of the two cells S_1 and S_2 will be the same. (In practice I have found that a rough approximation in the total resistance of the two circuits is sufficient to secure the necessary equality in the rate of fall.)

Having decided as to the minimum lengths of AB which are to be compared, place two contact makers on the bridge wire (as at d_1 and P_1) separated by (approximately) this minimum length. Connect one of the wires from the contact maker P_1 with a high resistance galvanometer (G) and then with the circuit C_2, to some portion of which it should be soldered (at P_2). The wire from d_1 must pass directly to the second circuit and be shifted along it until G shows no deflection; it should then be soldered in place in order to prevent any movement during the subsequent operations.

The contacts d_1 and P_1 are then moved along AB, their distances being re-adjusted whenever G shows a deflection.

If a reversing key is placed at K the final calculations are simplified by proceeding as follows. Place d_1 at A and adjust P_1. Next lift d_1 over P_1 towards B, reverse K, and then adjust d_1 leaving P_1 unaltered and so on. Thus lengths of equal resistance can be determined without any *overlapping* of the parts thus found, a matter of great practical convenience.

The only contacts moved throughout the operations are those at d_1 and P_1, and also those (if the reversing method has been adopted) of the key at K. All these contacts are potential ones and therefore any variation in them is of no consequence. It is advisable to place the resistances R_1 and R_2 in the same tank of water or paraffin, and thus the results will be practically unaffected by changes in their temperature.

The contact d_1 should occasionally be placed in its original position, and if the reading of P_1 is then found to be unaltered we have proof that the rate of fall in the E. M. F. of the two cells is identical. If any alteration is shown a time chart can be constructed in the manner usually adopted when comparing cells by the Poggendorf method, and any corrections introduced which may be found necessary.

The accuracy of the calibration thus performed depends on

(i) The potential gradient down AB.

(ii) The sensitiveness of the galvanometer.

(iii) The accuracy with which it is possible to measure the length d_1P_1.

It is not advisable to make the potential gradient steeper than is necessary, as the inequalities of the wire would be exaggerated by any excessive rise in temperature, that rise being greatest when the wire is thinnest. Now (i) can be altered at will by adjustment of R_1 whereas (ii) and (iii) are not under control to the same extent. The greatest value of R_1 should be used that is compatible with the limits given by (ii) and (iii).

Let E be the D. P. of the storage cells.

Let e be the smallest D. P. which will visibly influence the galvanometer.

Let r be the resistance of the bridge wire.

Let l be the length of the bridge wire (in mm.).

Let λ be the smallest difference in the distance between the points d_1 and P_1 which can be directly determined by the linear measurements.

Let R_1 be the total resistance of the circuit C_1, exterior to the bridge wire.

Now the value of R_1 should be so selected that the D. P. at the ends of a portion of the bridge wire of length λ shall be e.

This will be the case if

$$\frac{e}{E} = \frac{\frac{r\lambda}{l}}{R_1 + r},$$

hence $$R_1 = r\left(\frac{E}{e} \cdot \frac{\lambda}{l} - 1\right) \quad\text{.........................}(a).$$

A good high resistance galvanometer will (according to my experience) when properly adjusted be visibly affected when the D. P. of its terminals is 10^{-6} volts, if the conditions as to absence of vibration etc. are favourable; while a difference of 10^{-5} volts will give a decided swing. It is safe to assume that a D. P. of 5×10^{-6} volts can always be detected. If the contact makers are supplied with verniers giving 0·1 mm. then 0·05 mm. is about the limit of accuracy of the linear measurements. I assume therefore that under ordinary conditions $e = 5 \times 10^{-6}$ and $\lambda = 0{\cdot}05$ mm.

Suppose that a wire of length 1 metre and approximate resistance 1 ohm is to be calibrated, and that storage cells are used, we have $$E = 2, \quad l = 1000, \quad r = 1.$$

Hence eq. (a), we get $R = 19$ ohms.

And, as we may neglect the internal resistance of the storage cells, the resistance R_1 should be 19 ohms, while the total resistance of the circuit C_2 should be 20 ohms.

Under ordinary circumstances a wire is not calibrated until it has been finally fixed into the instrument of which it is to form a part. A slight modification of the method I have here described enables a rough determination of the evenness of the wire to be made before placing it *in situ*, and thus much trouble may be saved, if the wire is found to be so unequal as to lead to its rejection.

In this case the contact makers d_1 and P_1 can be fixed by shellac to a slip of ebonite at any convenient distance apart. They are then brought in contact with the bridge wire and d_2 adjusted on the other circuit, as before described. If the ebonite is then slipped along AB so that d_1 and P_1 are continually in contact with the bridge wire the swings of the galvanometer will serve as a sufficient indication of the equality of the different portions of the wire *.

Note, February 25th, 1895.

Since the communication of the preceding paper to the Philosophical Society, my attention has been directed to the fact that the account of a somewhat similar mode of calibration, due to Von Helmholtz, was published by Giese in *Wied. Ann.*, Vol. XI. p. 440, 1880. The method there described is dependent on the fall of potential down the wires of two separate circuits. When I made my communication I was, of course, unaware that this principle, obvious as it is, had been previously applied. The mode of application is, however, entirely distinct from the method I have adopted, which is free from many causes of error inseparable from that described by Giese. The chief points of difference are as follows. Suppose a third point of contact m between d_1 and P_1, and let this point m be connected with a key in such a manner that the wires leading from the galvanometer and from d_2 can be placed first in connection with d_1 and m and secondly in connection with m and P_1. If the deflection of the galvanometer is the same in both cases, then the resistance of d_1m is equal to the resistance of mP_1, and thus by slightly shifting m the resistance d_1P_1 can be bisected.

Hence it is obvious that changes in the resistance of the contacts at d_1m and P_1 will affect the galvanometer-deflection as also will changes in the resistance of the key-contacts by which the connections are shifted. Also changes in temperature of the galvanometer would affect the resistance of its coils, and thus influence the swing. Again, as pointed out by Giese himself, any falling off in the electromotive force of the cells S_1 and S_2 will

* In the discussion on this communication it was pointed out by Prof. Ewing that this method would be a convenient one to use in those cases where the wire was to be made approximately uniform by scraping.

affect the galvanometer readings even if the rate of fall of both batteries is the same.

I do not see, therefore, that the Von Helmholtz method, as described by Giese, has any advantage over the ordinary Wheatstone's-Bridge method of Calibration, except in so far as it affords an independent mode of investigation.

My method was specially designed to eliminate the causes of error above enumerated, and, although I now find that I cannot lay claim to priority as far as the use of a double circuit is concerned, the methods are in all other respects so essentially different that I leave my communication unaltered in the hope that it may be of use to others.

MONDAY, 25 *February*, 1895.

MR R. T. GLAZEBROOK, TREASURER, IN THE CHAIR.

Mr C. J. Lay, Fellow of St Catharine's College, was elected a Fellow of the Society.

The following Communications were made to the Society:

(1) *On Binocular Colour-mixture.* By W. H. R. RIVERS, M.D. Lond., St John's College.

There has been much difference of opinion among workers in physiological optics on the question of the occurrence of binocular colour-mixture. Völkers, Dove, Regnault, Foucault, Brücke, Fechner, Ludwig, Panum, Hering, Aubert, Förster and Chauveau are among those who have seen the appropriate mixture-colour when a different colour-stimulus is presented to each eye; but several observers of the greatest eminence, viz. Wheatstone, Meyer, Volkmann, Meissner, Funke and Helmholtz have been unable to satisfy themselves that binocular mixture occurs.

The object of the present paper is to bring forward the fact that binocular colour-mixture may be observed in the after-image, and to consider some possible causes of the different results which different observers have obtained.

I may mention first that there are two methods by which I can most readily see, and show to others, binocular mixture, viz. that described by Hering[1], of which I show a modification devised by Mr E. T. Dixon, and by means of Wheatstone's stereoscope. I see the mixture better with this than with Brewster's

[1] Hermann's *Handbuch d. Physiologie*, Bd. III. S. 593.

form of the instrument, a mixture-colour appearing which is not to be distinguished in colour-tone from that obtained by ordinary methods of mixture.

I also see good binocular mixture when the combination of two coloured patches is brought about by appropriate convergence or divergence of the visual axes; and it is chiefly by the help of this method that I have observed mixture in the after-image. If two patches, a red and a blue for instance, are combined by directing the visual axes so that they would meet beyond the patches, three patches are seen. In the central patch, the rivalry of the visual fields shows itself as a change from one colour to the other, with a transient mixture in the intervals of change. In my own case, after 10 to 20 seconds, the rivalry ceases or becomes almost imperceptible, and I see a purple patch corresponding to the proper mixture-colour, often with slight variations of colour-tone in the direction of one or other of its components. If, after further fixation, I close my eyes, or look at a grey surface, I see under favourable circumstances three after-images; blue-green on one side from the red; yellow on the other side from the blue; and a green or yellowish-green in the centre from the combined purple patch. In many cases the three after-images cannot be seen simultaneously, but a green or yellow-green image is seen which can be recognized to differ in colour from the after-image of either the red or the blue. In other cases, especially when the original mixture has not been good, and the rivalry has not ceased, change of colour may be seen to take place in the central image; *i.e.* the phenomenon of rivalry of the retinal fields may be observed in the after-image. The experiment succeeds with any colours, including those complementary or approximately complementary to each other; thus, with a green and red patch, I obtained as binocular mixture a yellowish-grey; the central after-image was a bluish-white which differed very greatly both in tone and saturation from the blue-green and pink lateral images. The experiments require some practice to enable one to keep up steady convergence or divergence of the eyes for a sufficient time to obtain good after-images, but the same results may be obtained when the binocular combination is brought about by other means, as by Wheatstone's stereoscope. The nature of the experiment does not admit of satisfactory comparison of the colour of the after-images with objective colours; but, so far as I can judge, the central after-image has the colour which would arise from mixture of the colours of the lateral images, and also, so far as I can judge, is complementary to the mixture-colour of the two patches used, and it seems to me impossible to explain this in any other way than by binocular colour-mixture.

In the literature of the subject I have only been able to find mention of experiments in this direction in papers by Fechner[1] and Chauveau[2]. Fechner looked at a white patch on a black ground, holding a coloured glass before each eye, blue before one, and yellow before the other. The patch appeared white, and this was ascribed to binocular neutralization of the two complementary colours. He then removed the glasses, and the patch remained white, which was held to prove that the after-images also neutralized each other by binocular mixture; on doubling the patch by convergence or divergence of the visual axes, each patch appeared coloured; on uniting again, the patch appeared white. I have repeated the experiment and it is a very striking one, but I do not think it affords satisfactory evidence of binocular mixture in the after-image. Under the conditions of the experiment with glasses complementary in colour to each other, a colourless after-image is necessary to prove the existence of mixture, and in order to obtain it a white patch is looked at with its whiteness heightened by contrast with a black background. The influence of a white surface in masking the perception of colour is well known, and in using such a white patch Fechner was taking the best means of obscuring any colour which might otherwise appear. In other words; in order to prove the existence of a subjective white, Fechner looked at an objective white.

This is not merely a theoretical objection. I repeated Fechner's experiment using glasses not complementary to each other, green and yellow, red and blue. The combined after-images, which should now have occurred, were violet or green, but using Fechner's conditions the patch remained white, and I could see no trace of colour in it. I then repeated the experiments using grey patches of different degrees of darkness instead of white, and by their means I obtained appearances which I think showed the existence of binocular mixture in the after-image, but I obtained no results as definite as with the method I have described above. Chauveau's experiments were almost identical with those of Fechner and are open to the same objections. The experiments of Fechner and Chauveau are mentioned by Titchener[3], who has raised another objection to them, viz. that the appearances described may partly at any rate depend on contrast.

Ebbinghaus[4] has described "binocular after-image," but this expression is used by him to describe quite a different phenomenon, viz. the after-image which appears in one visual field when the stimulus has been presented to the other eye.

[1] *Abhand. d. königl. sächs. Gesellschaft*, Bd. LXX. S. 469. 1861.
[2] *Comptes rendus de l'Acad. des Sciences*, T. CXIII. p. 394. 1891.
[3] *Philos. Studien*, Bd. VIII. S. 243. 1893.
[4] *Pflüger's Archiv*, Bd. XLVI. S. 498. 1890.

One of Helmholtz's[1] objections to the reality of binocular colour-mixture was that the appearance obtained might depend on after-images. He describes an experiment in which he combined a pink and a green patch; after fixation of some duration, he was able to call the combined image grey, but on closing each eye, he found that his grey was monocular and not binocular: that the retina exposed to pink had become fatigued to pink and consequently saw the pink as grey, and so with the other eye to green. If he had gone one step further and had closed both eyes, he would probably have seen a colourless image, which could only have been explained by binocular colour-mixture.

I bring forward this after-image phenomenon chiefly as an additional fact in favour of the existence of binocular mixture, but it might also be held to throw light on the vexed question of the seat of after-images, and might be held to favour the view that they are central phenomena. This does not however seem to me to be necessary. If an after-image depends upon a definite physiological process in the retina, the occurrence of mixture in the after-image would be explained by the same process of central fusion as will explain binocular mixture in general, a process however which we do not yet understand.

As regards the second point of my paper, various suggestions have been offered to explain the failure of some observers to see binocular mixture. Bezold[2] and Dobrowolsky[3] believed that the difficulty in seeing the mixture depended on the difference of focus for different colours. They found that a binocular mixture of red and blue occurred more readily when a weak concave glass was held before the eye exposed to blue. It is possible that equal refraction may be the cause of failure in some cases, and possibly in that of Helmholtz, for in the account of his experiments, he describes one colour as seen through the other. In my own case, however, and in the case of several others who have made observations for me, the equality or inequality of refraction has no influence, and mixture is equally well seen whether the concave glass is placed before the eye exposed to red or to blue.

Dobrowolsky also suggested that difference in the power of unequal accommodation of the two eyes in different individuals might be of importance. It is a disputed point whether unequal accommodation ever occurs. Experiments of Donders and Hering are generally held to have disproved its occurrence. A. E. Fick[4]

[1] *Handbuch d. phys. Optik*, 1te Aufl. S. 779.
[2] *Annal. d. Physik. u. Chemie*, 1874. Jubelband, S. 585.
[3] *Pflüger's Archiv*, Bd. x. S. 56.
[4] *Archiv f. Augenheilkunde*, Bd. xix, S. 123.

has more recently advocated its existence, but his results were criticised by Hess[1], and the latter has described experiments which go far towards disproving the occurrence of unequal accommodation sufficient to overcome a difference of as little as ·25 D between the two eyes.

From observations I have made, it seems possible however that the refractive condition of the eyes may influence results in another way. I see binocular mixture better with my abnormal refraction uncorrected, and better still when my myopia is increased by convex glasses. Others who have made observations for me have seen the mixture better when their normal vision has been made artificially myopic or hypermetropic by convex or concave glasses. The improvement seems to depend on diminution of the rivalry of the visual fields owing to blurring of outline, and it is possible that neglect to take account of the refractive condition of the eyes may be one cause of the difference of opinion, and in the accounts of the experiments of most of the observers I mentioned at the beginning of this paper. I have been unable to find any reference to the refraction of their eyes. The influence already considered of difference of focus for different colours would be most marked in those whose refraction was normal.

Some observers, as Völkers[2] and Chauveau[3], on the other hand have insisted on equality of the two eyes as a necessary condition of success. Chauveau equalised the two eyes by placing before each five plates of white glass 1 mm. in thickness. He found this sufficient in most cases to correct any slight difference between the two eyes; and, when not sufficient, he removed one or more of the plates from one side. Chauveau was obviously doing more than equalising the two eyes, and may have been favouring mixture by blurring outline, and thus diminishing rivalry in the way I have suggested above.

(2) *On a new Parasite probably allied to Echinorhynchus.* By A. E. SHIPLEY, M.A., Christ's College.

The specimens described came from the skin of a bird *Hemignathus procerus*, taken by Mr Perkins in the Island of Kauai, one of the Sandwich Islands.

The body of the parasite consisted of three well-marked regions, (i) the head which was pitted and reticulated in a characteristic manner, (ii) the collar, and (iii) the trunk. No spines or hooks were found in the head, but it is possible a hooked proboscis may have been left behind in the skin of the bird.

[1] *Archiv f. Ophth.*, Bd. xxxv. Abh. I. S. 157.
[2] *Archiv f. Anat. u. Phys.* 1838. S. 60. [3] *loc. cit.*

The most striking feature in which these parasites resemble Echinorhynchus is in the structure of their skin and in the presence of the extensions of the skin known as Lemnisci. The skin consists of an outer fibrous layer, with no cell outlines. This is pierced by numerous channels or spaces and is continuous with two Lemnisci or processes hanging down in the body cavity. The fluid which is found in the spaces of the skin can be withdrawn into the Lemnisci. Within this layer is a muscular sheath.

The body cavity contained numerous ova and aggregations of small cells whose exact nature was not clear. There were traces of ducts leading to the exterior at the posterior end. Although owing to the state of preservation of the parasites, many details could not be made out, the structure of the skin and the presence of Lemnisci strongly support the view that we have in these parasites animals which if they do not belong to the family of the Echinorhynchidæ are at any rate nearly allied to them.

(3) *Notes on Pachytheca* (with exhibition of specimens). By A. C. Seward, M.A. St John's College.

The genus *Pachytheca* from Silurian and Devonian rocks of Britain and Canada has been a subject of discussion among palæontologists ever since its discovery in 1853. Several writers have placed the fossil among algæ, and this position has been assigned to it on the grounds of a supposed resemblance of its histological structures to that of certain recent genera. An examination of a series of microscopic sections prepared by Mr Storrie of Cardiff has led me to doubt the sufficiency of the evidence on which the comparison with any existing alga has been based, and to regard *Pachytheca* as an organism of uncertain position which might well receive attention at the hands of zoologists.

MONDAY, 29 *April*, 1895.

MR F. DARWIN, VICE-PRESIDENT, IN THE CHAIR.

Mr E. T. Dixon, Trinity College, was elected a Fellow of the Society.

Mr E. J. Bles was elected an Associate.

The following Communications were made to the Society:

(1) *Exhibition of Palophus tiaratus* (a Stick-insect from Mashonaland). By Dr D. SHARP.

(2) *A new method for the estimation of the Specific Gravity of Tissues.* By WALTER S. LAZARUS-BARLOW, M.D., Demonstrator of Pathology in the University. (From the Pathological Laboratory, Cambridge.)

In a communication by the author to the Royal Society upon the Pathology of the Oedema which accompanies Passive Congestion published in the *Philosophical Transactions* of last year, a method of obtaining the specific gravity of muscle is given which consists of a modification of Roy's method for estimating the specific gravity of blood. It was pointed out that amongst the difficulties in obtaining a correct result, one of the chief depends upon the fact that the glycerine with which Roy's solutions are made up so rapidly abstracts water from the muscle that after a few seconds the tissue invariably sinks in a fluid in which perhaps it floated at first. A second and also important difficulty was that so many pieces of muscle were necessary in order to arrive at a satisfactory result that the process not only took some little time but also involved considerable destruction of muscle. It was in order to obviate as far as possible these two difficulties that the method to be described was adopted.

The ideal method is one which should require only one piece of muscle from which no fluid should be abstracted or added during the process of estimation of the specific gravity. This could of course be obtained by using superposed layers of fluids of different specific gravity which have no affinity for water, but since the pathologist or physiologist wishes above all to deal with the tissues in a state as little altered from that in which they are in the body it is clear that the fluids used should be such as would not damage the muscle or other tissue.

Many varieties of miscible fluids of different specific gravities were thought of and tried, for it is only necessary to have two fluids on either side of the extremes of the specific gravity of the tissues in order to be able by mixing them in different proportions to obtain a series of fluids of known specific gravities, but since these different fluids acted injuriously on the tissues themselves, it is unnecessary to mention them. The material to be used was indicated to me in reading a paper by Heffter[1] in which he speaks of gum-arabic as being a suitable nutrient fluid for experimental work on the heart-muscle. He says that the amount of work the frog's heart can perform when fed with a solution of gum-arabic is greater than that which it can perform with many apparently more nearly physiological nutrient media, such as solutions of egg-albumen, &c. It follows therefore that we have here a material which dissolved in water can be made up into solutions of different specific gravity and which moreover will do as little damage as possible to the fresh tissue immersed in them. Further, though water certainly diffuses into gum solutions it diffuses with great slowness so that it was probable that the specific gravity of the tissue might be obtained before a sufficient amount of water had been abstracted to lead to any considerable modification of the initial specific gravity.

Solutions of gum-arabic in water were therefore made of specific gravity (as estimated by standard hydrometers) varying from 1050 to 1080, viz. those most usually required. Alternate fluids were coloured with a minute quantity of solid methylene blue. The fluids were placed in bottles similar to the ordinary "wash" bottle and the cork was covered with paraffin. A crystal of thymol was placed in each bottle to prevent growth of micro-organisms. Fluids have been kept in this way without modification in specific gravity for over a year. When an estimation was required the fluids were carefully placed in order in a large test-tube, the fluid having the highest specific gravity being placed first and the others above it in their order. There was thus obtained a column of fluid consisting of sharply-defined blue and yellowish bands each of which corresponded to a different specific gravity. The piece of tissue being now removed from the body and rapidly cleaned from blood, &c. on filter paper a portion is placed gently in the uppermost fluid, care being taken to avoid the entanglement of bubbles of air. It sinks rapidly through the upper layers but more slowly as it reaches those layers with which its own specific gravity more nearly corresponds until in some one layer or more commonly at the line of junction between two layers it obviously finds some obstacle to its further descent

[1] Heffter, Ueber die Ernährung des arbeitenden Froschherzens. *Arch. f. exp. Path. u. Pharm.*, 1892, Vol. XXIX. p. 50.

and almost completely comes to rest. This layer corresponds to the specific gravity of the tissue.

The advantages of the method so far as concerns the tissue have already been given, it is necessary to notice how the actual fluids behave and especially how far they diffuse into one another. If such a column of fluid be made and left undisturbed the bands will be absolutely distinct after 24 hours though the edges of course are not so clearly defined, while after 48 hours it obviously consists of deeper and paler bands of blue. Since even a long experiment of the kind in which the method would be employed rarely lasts above 8 hours at the most, diffusion of the layers of fluid into one another may be neglected[1].

It is plain that when several pieces of tissue have been placed in the same column of fluid there is a tendency for portions of the upper layers to be dragged downwards by the falling mass of tissue and therefore for the layers to become mixed, but it is remarkable how little of the superincumbent layers is thus carried down. This can be seen by watching a piece of tissue in its course through the yellow bands. A small amount of the superjacent blue band is carried down, but this rapidly returns to its own proper region by reason of its inferior specific gravity so that it is quite possible to take ten or twelve estimations in the same column of fluid. The principal disadvantage in the method is that diffusion of water though slow nevertheless takes place from the muscle into the gum solutions and therefore that the piece of muscle does not absolutely come to rest until it reaches the bottom of the test-tube. After a little practice however it is quite possible to distinguish a difference of one degree in specific gravity. Since the differences actually noticed in experiment vary from about two degrees to eight or ten degrees when variations do occur, the disadvantage is not nearly so serious as might appear, while the possible margin of error is not so great with gum solutions, since a piece of muscle takes as many minutes to sink through the same number of degrees after it has reached that layer which coincides with its own specific gravity as in the case of glycerine solutions it takes seconds.

[1] The possibility that diffusion of the *colour* might not be an indication of the diffusion of the solutions was borne in mind, and therefore by small pipettes fluid was taken from the various layers after the column had stood for some time and the specific gravity of each estimated by Roy's method. It was found that this possible source of error does not exist, and that after several hours the specific gravity of any individual layer is the same as it was when it was first placed in the column, provided the fluid taken up in the pipette did not come from either the upper or lower margin of the layer.

(3) *On a Collection of Crania from the North-West Provinces of India.* By Prof. A. MACALISTER, F.R.S.

The University Museum has received from Professor Havelock Charles, of Calcutta, a most interesting and valuable series of Crania, including twenty-three from the North-West Provinces. I must in the first instance return my best thanks to Professor Charles for his kindness in supplying a much-felt want in our University Collection.

There are many features of interest presented by these crania, and they are well worthy of a careful description. As, however, the pressure of official work has prevented me from giving the time and labour necessary for the doing of this work I have asked Mr Corner, B.A., of Sidney Sussex College, to undertake the task. This he has willingly and skilfully done, and I therefore desire to present to the Society his monograph in place of my own. I have gone over some of the critical part of his work, but his measurements have been done so carefully that they have needed no revision.

Mr Raymond Horton-Smith, B.A. of St John's College has kindly examined the crania from Bengal, and has with great judgment and skill made all the measurements necessary and has drawn up a most careful and useful account of them. I have added his paper also, which supplements the description of the other series.

On Crania from the North-West Provinces of India. By E. M. CORNER, B.A., B.Sc. Lond., Scholar of Sidney Sussex College.

At Professor Macalister's suggestion I made an examination of the skulls in the Anatomical Museum from the North-West Provinces of India. There are twenty-three of these. The age, caste and sex are recorded upon each as well as the place from which it came. Fourteen of these skulls come from Panjab; five from Patna; one each from Peshawur, Kabul and near the Khyber Pass. And again one, indefinitely, from the North-West Provinces of India. Of this last skull the caste is not recorded. It is taken into this paper in order to complete the set of skulls. The Patna series do not strictly speaking belong to the North-West Provinces.

The castes, of which there are representatives among these skulls, are as follows:

1 skull of each of the following; Kabuli, Rajput, Dusadh, Brahmin, Dhanuk, Musahar, Sansi (gipsy).

2 skulls of each of the following; Mussulman Jat, Chuhra and Pathan.

4 skulls of low caste Hindoos.

5 skulls of Panjabi Mussulmans.

The sexes are as follows: twenty male; two female.

The following is a short summary of the peculiar points in each skull:

1213. Skull of male Hindoo of low caste from Panjab. The age is 60 years.

Synostosis of the sutures has occurred at the bregma, the sagittal suture, the lambda and the upper portions of the lambdoid suture. The portions of the coronal sutures, from just below the stephanion, are synostosed. The nasal suture is also synostosed. The skull is symmetrical and cryptozygous. The estimation of the latter point was conducted as described by Turner (*Challenger Reports*, Vol. x., Pt. xxix.). Supraorbital foramina are present. Supraciliary ridges slightly marked.

1218. Skull of male Hindoo of low caste from Panjab. Age 28.

Sutures are complicated. There is a large epactal bone present and four lambdoid wormian bones. The last molar tooth on the left side of the upper jaw projects directly backward, having pierced the maxillary tuberosity; that on the right side is undeveloped. The supraciliary ridges are hardly marked. There is a small depression just above the lambda. The skull is symmetrical, rounded and cryptozygous.

1222. Skull of male of Chuhra caste from Panjab. Age 30.

Sutures very complicated. There is a wormian bone at the lower end of the coronal suture on the right side. There is a small pteriac bone at the upper end of the squamo-sphenoidal suture. Three lambdoid wormian bones are present. The naso-frontal suture is situated at the bottom of a very deep groove. The sagittal suture lies in a depression near the bregma but is on a ridge near the lambda. The supraciliary ridges are slightly marked. The angle of the lower jaw is everted. The skull is symmetrical, rough, heavy and phænozygous.

1223. Skull of male of Sansi caste from Panjab. Age 30.

Sutures are complicated. One epactal bone, four lambdoid and three coronal wormian bones are present. The nasal suture is synostosed. The lambda is raised. A median frontal ridge extends forward from the bregma towards the ophryon. The coronal suture is on a ridge near bregma. The supraciliary ridges are slightly marked. The skull is heavy, very irregular, rough and phænozygous. The right occipital condyle is divided into two separate portions.

1227. Skull of Panjabi Mussulman. Age 40.

Synostosis has occurred at the sagittal suture, lambda, lambdoid suture (incompletely); also the lower part of the coronal suture, from just below stephanion, and the fronto-sphenoidal sutures are synostosed. The styloid process is constituted of two parts. A supraorbital foramen is present on the left side. The supraciliary ridges are well marked. The occiput is prominent. The skull is symmetrical and phænozygous.

1228. Skull of Panjabi Mussulman. Age 45.

The sutures are simple. One small lambdoid wormian bone is present. There is a depression behind the coronal suture. The styloid process is long. The supraciliary ridges are slightly marked. The skull is symmetrical and phænozygous.

1229. Skull of Panjabi Mussulman. Age 25.

The sutures are complicated. Four lambdoid wormian bones are present. Skull is flattened above the lambda. Supraciliary ridges slightly marked. The occiput is prominent. Skull symmetrical and cryptozygous.

1230. Skull of Panjabi Mussulman. Age 50.

Sutures of cranium almost completely synostosed. Skull is flattened above the lambda. A supraorbital foramen is present on left side. Supraciliary ridges slightly marked. Skull symmetrical and phænozygous.

1249. Skull of Panjabi Mussulman Jat. Age 38.

Sutures are complicated. One lambdoid wormian bone present. Sagittal suture is on a crest. Coronal and lambdoid sutures slightly raised. A depression is present behind the coronal suture; the occiput is prominent, the supraciliary ridges are slightly marked, and there are supraorbital foramina on both sides. The angle of the lower jaw is everted. The skull is symmetrical and phænozygous.

1707. Skull of Mussulman Jat from Panjab. Age 45.

Sutures are complicated. Six lambdoid wormian bones are present, and the portions of coronal suture below stephanion are synostosed. There is a depression behind the coronal suture.

A supraorbital foramen is present on the right side. The supraciliary ridges are small. The occiput is prominent. The skull is symmetrical and phænozygous.

1701. Skull of Panjabi Mussulman. Age 40.

There is partial synostosis of coronal, sagittal, lambdoid and nasal sutures. The synostosis of the coronal suture is complete below the stephanion. The parietal eminences are large. A depression exists above the lambda. The occiput is very irregular and prominent. Supraciliary ridges slightly marked. Skull symmetrical and phænozygous.

1700. Skull of male of Chuhra caste from Panjab. Age 45.

Synostosis as in skull 1701, but less complete. A depression exists behind the coronal suture. There is a supraorbital foramen on the right side. The occiput is prominent. Supraciliary ridges small. Angles of lower jaw are everted. Skull symmetrical and phænozygous.

1212. Skull of low caste Hindoo from Panjab. Age 40.

Sutures are complicated. Six lambdoid and one large coronal wormian bones present. Nasal suture is very asymmetrical. Supraorbital foramina on both sides. A flattening is present just above lambda. The occiput is prominent; supraciliary ridges slightly marked. Skull asymmetrical (left side large) and cryptozygous.

1211. Skull of low caste Hindoo from Panjab. Age 30.

Sutures are complicated. A slight flattening exists just above the lambda. A pteriac wormian bone is present on the right side and three on left side. Two lambdoid wormian bones present. Nasal suture is irregular. Parietal tubera prominent, and supraciliary ridges slightly marked. Skull symmetrical and sphænozygous.

1220. Skull of male of Musahar caste from Patna. Age 60.

Sutures are complicated. The interparietal bone, two lambdoid wormian bones and one pteriac wormian bone are present. The occiput is prominent. A small knob of bone is present just below and in front of the stephanion on the right side; nasal suture is irregular and the supraciliary ridges very slightly marked. Skull symmetrical and cryptozygous.

1221. Skull of male of Dhanuk caste from Patna. Age 55.

Sutures are complicated. One epactal bone is present. Supraorbital foramina present on both sides. Nasal suture is asymmetrical. Occiput is prominent. Supraciliary ridges very slightly marked. Skull is symmetrical and phænozygous.

1224. Skull of female from the N.-W. Provinces. Age 26 ?

Sutures are fairly complicated. One interparietal bone and three lambdoid wormian bones present. The portions of coronal suture just below the stephanion, and the fronto-sphenoidal sutures are synostosed. The nasal suture is irregular. Depressions are present behind the coronal suture and just above the lambda; supraciliary ridges are small and smooth. Skull symmetrical and phænozygous.

1225. Skull of male Rajput from Patna. Age 40.

Sutures are simple, and partly synostosed. The coronal suture below the stephanion is completely synostosed, and the nasal suture is irregular. Slight depressions above the lambda and behind the coronal suture on the right side. Parietal eminences are large. Supraciliary ridges slightly marked. Skull is symmetrical and phænozygous.

1226. Skull of Brahman from Patna. Age 55.

Sutures simple. Three interparietal bones, one epactal and two lambdoid are present, also two pteriac bones. The bregma is raised, and from it a median frontal ridge extends towards the ophryon. Supraciliary ridges very slightly marked. The skull is symmetrical and phænozygous.

1250. Skull of male Pathan from near the Khyber Pass. Age 40.

Sutures simple. Two epactal bones are present. The vomer is much displaced to the left. A pteriac bone is present on the left side. Supraciliary ridges are well marked. Skull is large, symmetrical and cryptozygous.

1703. Skull of female of Dusadh caste from Patna. Age 30.

Sutures are very simple, and a right coronal wormian bone is present. The occiput is very prominent. A supraorbital foramen is present on the left side. Supraciliary ridges very slightly raised. Skull is symmetrical and phænozygous.

1752. Skull of male Pathan from Peshawur. Age 30.

Sutures fairly complicated. Portions of coronal suture below stephanion are partially synostosed. Parietal eminences are prominent. The centres of the squamous portion of the temporal bone are exceedingly thin. The nasal suture is irregular, and the supraciliary ridges are well marked. The skull is symmetrical and cryptozygous.

1702. Skull of male Kabuli from Kabul. Age 30.

Sutures are simple, the lambdoid suture is complicated by very numerous wormian bones. Supraorbital foramina are present on both sides. The occiput is irregular and prominent. Supraciliary ridges are very slightly raised. Skull asymmetrical (left side large) and cryptozygous.

In most of these crania from persons over 30 years of age, those portions of the coronal suture that lie just below the stephanion are synostosed. Eight of the skulls are cryptozygous and fifteen phænozygous. There is no relation between the cryptozygous condition and the size (i.e. cranial capacity) of the skull; No. 1220, that of a Musahar from Patna, with a capacity of 1005 c.c., and No. 1250, a Pathan from near the Khyber Pass, of 1747 c.c. capacity, were both cryptozygous.

Two of the skulls present a median frontal crest, extending from the bregma forwards and fading away towards the ophryon. The other cranial sutures of the skull are also sometimes ridged. These ridges may be seen in skulls 1222, and 1249. Behind the coronal suture in six skulls, on both sides, and in one, on one side only, there is a depression.

In sixteen skulls supplementary bones are present. These consist of interparietal or epactal bones, or wormian bones, in the lambdoid, sagittal, coronal, squamo-sphenoidal and squamo-parietal sutures.

The supraciliary ridges were prominent in only three skulls. In the remaining twenty skulls they were slightly marked and the brow usually smooth.

In one skull (1233) of male of the Sansi (or gipsy) caste a third occipital condyle was present upon the right side.

Only two of the skulls presented marked asymmetry. In both of these the bulging was on the left side.

The greatest glabello-occipital length was 195 mm. This was in the skull of a low caste Hindoo male, 1213. The next to this was 192 mm. in two skulls, both of males, of Chuhra (1222) and Pathan (1250) respectively. The smallest glabello-occipital diameter was 168 mm., in the skull of a male low caste Hindoo (1218). The next were 169 mm. in the skull of a male Dhanuk (1221) and 170 mm. in the skull of a male Musahar (1220).

The greatest ophryo-occipital diameter is 194 mm. in a low caste Hindoo male skull (1213). The least, 168 mm. in the skulls of a low caste Hindoo male (1218) and in the Dhanuk male skull (1221).

The greatest breadth was 141 mm., in two skulls, one of a Panjabi Mussulman (1701) and one of a male Pathan (1250). The least was 121 mm., in two skulls, one a male Musahar (1220) and the other in a female Dusadh (1703).

TABLE I. MEASUREMENTS.

Skull	Caste	age	glabello-occipital length	ophryo-occipital length	breadth	basi-nasal	basi-alveolar	nasio-alveolar	nasal length	nasal breadth	interorbital width	fronto-malar chord	minimum frontal	bimaxillary	orbital length	orbital breadth	palatal length	palatal breadth	basio-bregmatic height	bizygomatic width	nasio-mental	circumference	glabella-bregma arc	glabella-lambda arc	glabella-inion arc	glabella-opisthiac arc	glabella-basion arc	glabella-alveolar arc	sagittal circumference	fronto-malar arc	cranial capacity
1213	Low caste Hindu male, Panjab	60	195	194	133	108	97	72	53	25	25	95	92	92	40	35	—	—	141	131	—	530	127		320	380	420	510	590	102	1485
1218	ditto	28	168	168	133	93	86	56	40	27	23	94	94	91	35	29	45	55	129	116	—	485	113	225	300	345	378	464	532	100	1245
1212	ditto	40	188	185	133	102	99	68	52	22	18	92	91	89	38	36	51	63	130	121	104	505	122	248	338	375	413	508	578	100	1430
1211	ditto	30	175	173	125	98	92	73	50	25	17	93	97	93	37	32	53	64	137	124	112	480	125	255	313	363	398	478	557	100	1245
1222	Chuhra male, Panjab	30	192	191	134	100	86	66	47	25	21	96	97	93	40	31	44	57	141	124	100	528	142	294	345	401	436	522	594	98	1540
1700	ditto	45	186	184	127	100	91	68	50	22	17	90	91	89	38	33	54	58	135	121	—	502	128	265	338	375	413	508	578	102	1410
1223	Sansi male, Panjab	30	187	182	138	105	98	75	54	25	22	101	97	105	40	32	52	65	131	135	113	529	120	228	310	357	398	485	583	110	1347
1249	Mussulman Jat male, Panjab	38	187	187	131	103	101	73	53	22	19	96	99	93	40	33	57	58	141	128	113	513	125	260	330	381	421	518	595	105	1420
1707	ditto	45	181	181	134	101	91	70	50	27	20	96	94	99	38	32	52	61	137	132	116	513	128	255	310	360	408	497	573	102	1420
1227	Mussulman male, Panjab	40	176	174	126	96	92	67	49	23	20	97	92	85	39	33	52	60	119	124	106	491	120	240	290	346	384	475	548	111	1235
1228	ditto	45	180	179	130	97	93	71	49	24	21	98	94	97	39	34	54	50	127	130	104	502	116	248	328	360	402	492	568	108	1410

1229	ditto	25	177	175	129	99	92	71	54	29	20	94	90	87	38	31	51	60	128	123	98	490	110	245	308	348	388	475	555	104	1245
1230	ditto	50	183	181	134	100	94	74	55	26	19	90	93	99	40	34	56	62	132	132	—	500	115	235	317	358	392	480	565	100	1415
1701	ditto	40	183	183	141	98	97	68	51	28	22	101	93	98	41	36	55	65	133	126	104	502	125	260	335	375	410	502	575	111	1500
1220	Musahar male, Patna	60	170	170	121	95	89	67	45	20	23	95	87	97	35	31	51	63	120	114	110	468	112	237	290	342	376	452	520	103	1005
1221	Dhanuk male, Patna	55	169	168	134	93	94	67	56	25	19	91	89	94	38	31	—	63	127	125	—	488	115	240	305	346	378	467	542	97	1275
1224	female, N.-W. Provinces	26?	174	174	123	93	90	69	46	21	18	87	89	91	35	34	50	57	128	116	—	478	110	235	305	345	380	462	535	97	1100
1225	Rajput male, Patna	40	181	181	139	100	100	77	49	24	19	96	94	94	40	39	56	54	131	111	112	503	125	250	320	362	400	500	575	105	1340
1226	Brahmin male, Patna	55	170	170	129	93	91	61	48	25	18	91	87	90	37	33	49	57	130	117	99	477	115	—	293	352	387	473	545	100	1140
1250	Pathan male, Khyber Pass	40	192	191	141	105	95	64	54	29	24	101	104	102	40	38	50	65?	144	136	112	533	130	260	345	397	432	520	597	110	1747
1252	Pathan male, Peshawur	30	182	179	135	102	97	66	43	22	21	90	89	88	38	32	57	58	130	120	—	530	125	250	315	361	400	493	564	96	1447
1703	Dusadh female, Patna	40	174	174	121	94	89	63	47	24	18	91	87	90	38	33	47	61?	123	117	—	475	115	236	297	346	362	468	538	99	1107
1702	Kabuli male, Kabul	30	171	171	146	95	94	67	48	23	20	97	94	95	39	30	55	57?	137	133	102	512	125	260	330	371	408	499	565	102	1563
251 *	Pathan male, Peshawur	50	178	177	150	100	97	75	54	24	23	103	98	96	42	35	53	64	141	133	118		118	242	330	362	405	495	577	112	1540

TABLE II. INDICES, NUMERICAL.

SKULL	CASTE	age	cephalic	vertical	gnathic	nasal	orbital	palatine	nasio-mental and bizygomatic	nasio-mental and bimaxillary	nasio-alveolar and bizygomatic	nasio-alveolar and bimaxillary
1213	Low caste Hindu male, Panjab	60	68.20	72.30	89.81	47.17	87.50	101.85	—	—	54.97	78.25
1218	ditto	28	74.40	76.78	92.46	67.50	82.85	114.58	—	—	48.27	61.53
1212	ditto	40	70.70	69.15	97.06	42.30	94.74	113.20	85.95	116.85	56.19	76.40
1211	ditto	30	71.43	78.78	93.97	50.00	86.49	125.49	90.32	120.43	50.80	78.49
1222	Chuhra male, Panjab	30	69.79	74.43	86.00	44.68	77.50	121.27	80.64	107.52	53.22	70.96
1700	ditto	45	68.28	72.58	91.00	44.00	87.50	103.68	—	—	56.19	78.40
1223	Sansi male, Panjab	30	73.79	75.05	93.33	46.29	80.00	130.00	84.44	107.61	55.55	71.42
1249	Mussulman Jat male, Panjab	38	70.05	75.40	98.00	41.51	82.50	105.55	88.28	121.50	57.03	78.49
1707	ditto	45	74.03	75.69	91.09	54.00	82.05	120.00	87.87	117.17	53.03	60.55
1227	Mussulman male, Panjab	40	71.59	67.61	95.73	46.92	84.61	111.76	84.48	124.70	54.03	78.82
1228	ditto	45	72.62	70.39	95.87	53.70	87.18	105.66	80.00	107.21	54.61	73.19
1229	ditto	25	72.88	72.72	91.90	48.98	81.58	118.00	79.67	112.64	57.72	81.60
1230	ditto	50	73.22	72.13	94.00	47.17	87.50	112.96	—	—	56.06	74.74
1701	ditto	40	77.04	72.73	98.98	54.90	87.80	118.57	82.53	106.32	53.96	67.59
1220	Musahar male, Patna	60	71.17	70.58	93.69	44.44	88.57	121.56	96.49	113.40	58.77	79.49
1221	Dhanuk male, Patna	55	79.70	75.15	101.06	41.07	81.58	—	—	—	53.60	71.27
1224	female, N.-W. Provinces	26?	70.69	73.15	96.77	45.65	97.14	110.00	—	—	59.46	75 82
1225	Rajput male, Patna	40	76.79	72.37	100	49.00	97.50	93.10	100.90	119.17	69.36	81.91
1226	Brahmin male, Patna	55	75.88	76.47	97.85	52.08	89.17	120.82	84.61	110.00	52.13	61.55
1250	Pathan male, nr. Khyber Pass	40	73.44	75.00	94.76	53.70	95.00	119.60	82.35	109.80	47.05	62.74
1251 *	Pathan male, Peshawur	50	84.3	79.2	97.00	44.4	83.3	120.7	88.7	122.9	56.4	78.1
1252	Pathan male, Peshawur	30	74.17	71.43	95.09	51.16	83.21	100.00	—	—	55.00	75.00
1703	Dusadh female, Patna	40	69.54	70.68	94.68	51.06	86.84	111.76	—	—	53	70.00
1702	Kabuli male, Kabul	30	85.32	80.12	98.95	47.91	76.92	111.32	76.69	107.36	50.37	70.52

The greatest height was 144 mm. in the skull of a male Pathan (1250). Three skulls measured 141 mm., viz. Hindoo male (1213), Chuhra male (1222), and Mussulman Jat (1249). The least height was 119 mm. in the skull of a male Panjabi Mussulman. That of a male Musahar was 120 mm.

The greatest bizygomatic breadth was 136 mm. in the skull of a male Pathan (1250). The skull of a male Sansi (1223) came next with 135 mm. The least were in the skull of a Rajput male (1225) 111 mm., and a male Musahar (1220) 114 mm.

The greatest bimaxillary diameter is 105 mm. in the skull of a Sansi male (1223). The next is 102 in that of a male Pathan (1250). The least is 85 mm. in that of a Panjabi Mussulman (1227).

The largest minimum frontal diameter is 104 mm. in a male Pathan skull (1250): the least is 87 mm. in four skulls.

The greatest fronto-malar chord was 101 mm. in three skulls; the longest fronto-malar arc was 111 mm. The least chord was 87 mm. in the skull of a female of indefinite caste from the N.-W. Provinces (1224); and the smallest arc 96 mm. in the skull of a male Pathan (1252).

The interorbital length, as measured by distance from dacryon to dacryon, ranged from 29 mm. in the male Pathan skull (1250), to 17 mm. in two skulls; in seventeen it varied between 20 mm. and 25 mm.

The longest nasal length was 56 mm., the shortest 40 mm. The greatest width was 29 mm., and the least 20 mm.

The greatest height of the orbit was 41 mm., and the least 35 mm. The greatest width was 30 mm., and the least 29 mm.

The greatest length of the palate was 58 mm. and occurred in one skull, the least was 47 mm. The greatest width was 64 mm., and the least 54 mm. The greatest nasio-mental length was 116 mm., the least nasio-mental length was 98 mm.

The greatest nasio-alveolar length was 77 mm., the least was 56 mm.

The greatest basio-alveolar length was 101 mm., the least was 86 mm., in two skulls.

The greatest nasio-basial length was 108 mm., the least was 93 mm. and occurred in four skulls.

The greatest horizontal circumference was 533 mm. Next to this come two skulls with 530 mm. The least was 468 mm., and next to it come 475 mm., 477 mm., and 478 mm.

The greatest longitudinal circumference was 597 mm. Next to this come 594 mm. and 590 mm. The least was 520 mm., and next to it come 532 mm., 535 mm. and 542 mm.

Cephalic index.

This index ranges from 68·20 to 85·32. Eighteen of the skulls are dolichocephalic, four are mesaticephalic, and one is brachycephalic. This last skull is that of a male Kabuli, from Kabul. The indices of the majority lie between 68 and 74.

Vertical index.

This ranges from 67·61 to 80·2. Seven of the skulls are tapeinocephalic; fourteen are metriocephalic; two are akrocephalic. The most akrocephalic is the skull of the male Kabuli. The indices of the majority lie between 70 and 75.

Gnathic index.

This ranges from 86·00 to 101·06. Eighteen of the skulls are orthognathous, five are mesognathous, and none are prognathous. There are two skulls with indices of 101·06 and 100 respectively. These belong to a Dhanuk male and a Rajput male respectively.

Nasal index.

This ranges from 41·07 to 67·50. Twelve of the skulls are leptorhine, six are mesorhine, and five are platyrhine. In one low caste Hindoo male the index is 67·52. It is separated widely from the next, which is 54·90.

Risley, in his book on the *Tribes and Castes of Bengal*, (Vol. I., Ethnographic Glossary, p. xxxiv), says, "Thus, it is scarcely a paradox to lay down as a law of the caste organisation in Eastern India that a man's social status varies in inverse ratio to the width of his nose." He was led to this conclusion from a vast number of observations taken on the living in several districts, including the Panjab, the North-West Provinces and Oudh. While this may be true for the nose in the flesh, the present observations on the dry skull do not show that there is a corresponding length of the nasal skeleton in the higher castes, but as they are only taken from a series of twenty-three skulls their value is small. The following is a list of the skulls, with their castes, arranged in order of the value of their nasal indices.

Nasal index.	Caste etc.	Sex.	Skull.
41·07	Dhanuk. Patna	male	1221
41·57	Mussulman Jat. Panjab	male	1249
42·30	low caste Hindoo. Panjab	male	1212
44·00	Chuhra. Panjab	male	1700
44·44	Musahar. Patna	male	1220
44·68	Chuhra. Panjab	male	1222
45·65	N.-W. Provinces	female	1224
46·29	Sansi. Panjab	male	1223
46·92	Mussulman. Panjab	male	1227
47·17	low caste Hindoo. Panjab	male	1213
47·17	Mussulman. Panjab	male	1230
47·91	Kabuli. Kabul	male	1702

48·98	Mussulman. Panjab	male	1229
49·00	Rajput. Patna	male	1225
50·86	low caste. Hindoo Panjab	male	1211
51·06	Dusadh. Patna	female	1703
51·16	Pathan. Peshawur	male	1252
52·08	Brahmin. Patna	male	1226
53·70	Mussulman. Panjab	male	1228
53·70	Pathan. Near Khyber Pass	male	1260
54·00	Mussulman Jat. Panjab	male	1707
54·90	Mussulman. Panjab	male	1701
67·50	low caste Hindoo. Panjab	male	1218

Observation on the position occupied by low caste Hindoos in this table shows that the width of nose test of Risley seems here to be inapplicable to skulls. The first low caste Hindoo is the third, the next the tenth most leptorhine noses. Both these should then rank above, for instance, the Rajput, and with another low caste Hindoo above the Brahmin. The Brahmin barely escapes the epithet of plathyrhine. The differences between him and the three low caste Hindoos higher than him in their indices are 10·41, 4·92 and 1·12 respectively.

These numbers indicate that many castes are heterogeneous. Risley, in the volume mentioned above, shows how that some men, having become independent landed proprietors, etc., enrol themselves in one of the leading castes, the favourite one being the Rajput. They finally intermarry and become recognized Rajputs, and thus the difficulty is increased manifold. On this point see Prof. Charles's paper, *Journal Asiatic Soc. of Bengal,* Vol. LXIII., Part iii. No. 1.

Orbital index.

This ranges from 76·92 to 97·50. Nine of the skulls possess microseme orbits, nine mesoseme, and five megaseme.

Palatine index.

This ranges from 77·49 to 103·70. Hence all skulls, in which this index was at all reliable, *i.e.* twenty in number, are markedly dolichuranic. The clusters are at 80, 85 and 96.

Facial indices.

i. Nasio-mental and bizygomatic index.

This ranges from 93·10 to 130. The indices cluster at 80 to 85.

ii. Nasio-mental and bimaxillary index.

This ranges from 106·12 to 124·70.

iii. Nasio-alveolar and bizygomatic index.

This ranges from 47·05 to 69·36.

iv. Nasio-alveolar and bimaxillary index.

This ranges from 60·55 to 81·91.

Cranial capacities.

These are very various and range between 1005 c.c. to 1747 c.c. The former is the skull of a male Musahar (1220), and the latter of a male Pathan (1250). Eleven skulls are microcephalic; seven are mesocephalic; five are macrocephalic. Four of the microcephalic skulls are very small. The capacity of these are 1005 c.c., 1100 c.c., 1107 c.c., 1140 c.c. The second and third of these belong to females, one of the N.-W. Provinces (caste not mentioned) and one of Dusadh. The second most macrocephalic skull is of a Kabuli (1702). Eight skulls range from 1410 c.c. to 1485 c.c. Five skulls have capacities between 1235 and 1275.

Palate indices.

Ten skulls are brachyuranic, six are mesuranic, and six are dolichuranic.

Prof. Macalister has directed my attention to two papers by Professor Charles in the *Journal of Anatomy* (Vol. XXVI. p. 1, and XXVII. p. 5) on Indian Crania. In these he points out that many of the caste divisions are religious and not tribal, as members of some castes can raise themselves in the social scale, and he quotes a native proverb, "Last year I was a Sweeper, this year I am a Shekh, next year if prices rise I shall be a Saiyad."

Professor Charles believes the microcephalic crania to be those of aboriginal races and the macrocephalic to be descendants of more recent invaders coming from the north-west. This seems rather borne out by the greater capacity of some of my N.-W. skulls contrasted with the Bengal skulls; those from the extreme N.-W. Peshawur and Kabul being the largest. The measurements given by Professor Charles on the whole agree with mine; and he has also noticed the occurrence of supplementary bones in 64 per cent. of his crania.

I find that I had overlooked the skull of a male Pathan, No. 1231, as it was out of its proper place in the Museum. I have however inserted its measurements in the table, it is large, brachycephalic, and has the highest facial index of any in the collection.

TABLE III. CLASSIFICATION ACCORDING TO INDEX.

	Cephalic Index	Vertical Index	Gnathic Index	Orbital Index	Cranial Capacity	Nasal Index	Palatal Index
Low caste Hindoos							
1213	dolichocephalic	metriocephalic	orthognathous	mesoseme	macrocephalic	leptorhine	dolichuranic
1218	dolichocephalic	metriocephalic	orthognathous	microseme	microcephalic	platyrhine	mesuranic
1212	dolichocephalic	tapeinocephalic	orthognathous	megaseme	mesocephalic	leptorhine	mesuranic
1211	dolichocephalic	akrocephalic	orthognathous	mesoseme	microcephalic	mesorhine	brachyuranic
Chuhra							
1222	dolichocephalic	metriocephalic	orthognathous	microseme	macrocephalic	leptorhine	brachyuranic
1700	dolichocephalic	metriocephalic	orthognathous	mesoseme	mesocephalic	leptorhine	dolichuranic
Sansi							
1223	dolichocephalic	tapeinocephalic	orthognathous	microseme	microcephalic	leptorhine	brachyuranic
Mussulman Jat							
1249	dolichocephalic	metriocephalic	mesognathous	microseme	mesocephalic	leptorhine	dolichuranic
1707	dolichocephalic	metriocephalic	orthognathous	microseme	mesocephalic	platyrhine	brachyuranic
Mussulman Panjabi							
1227	dolichocephalic	tapeinocephalic	orthognathous	mesoseme	microcephalic	leptorhine	mesuranic
1228	dolichocephalic	tapeinocephalic	orthognathous	mesoseme	mesocephalic	platyrhine	dolichuranic
1229	dolichocephalic	metriocephalic	orthognathous	microseme	microcephalic	mesorhine	brachyuranic
1230	dolichocephalic	metriocephalic	orthognathous	mesoseme	mesocephalic	leptorhine	mesuranic
1701	mesaticephalic	metriocephalic	mesognathous	mesoseme	macrocephalic	platyrhine	brachyuranic
Musahar							
1220	dolichocephalic	tapeinocephalic	orthognathous	mesoseme	microcephalic	leptorhine	brachyuranic
Dhanuk							
1221	mesaticephalic	metriocephalic	mesognathous	microseme	microcephalic	leptorhine	—
N.-W. Provinces							
1224 ♀	dolichocephalic	metriocephalic	orthognathous	megaseme	microcephalic	leptorhine	mesuranic
Rajput							
1225	mesaticephalic	metriocephalic	mesognathous	megaseme	microcephalic	mesorhine	dolichuranic
Brahmin							
1226	mesaticephalic	metriocephalic	orthognathous	megaseme	microcephalic	mesorhine	brachyuranic
Pathan							
1250	dolichocephalic	metriocephalic	orthognathous	megaseme	macrocephalic	platyrhine	brachyuranic
1252	dolichocephalic	tapeinocephalic	orthognathous	microseme	mesocephalic	mesorhine	dolichuranic
Dusadh							
1703 ♀	dolichocephalic	tapeinocephalic	orthognathous	mesoseme	microcephalic	mesorhine	mesuranic
Kabuli							
1702	brachycephalic	akrocephalic	mesognathous	microseme	macrocephalic	leptorhine	mesuranic
* Pathan	brachycephalic	akrocephalic	orthognathous	microseme	macrocephalic	leptorhine	brachyuranic

(2) *A Description of Bengal Crania.* By R. J. HORTON-SMITH, B.A., Scholar of St John's College, Cambridge.

The following eight crania from Bengal, of which I propose to give a brief description, were presented to the University Museum by Prof. Havelock Charles. They belonged to individuals who died in hospital and whose history is consequently known. Prof. Macalister exhibited some of them at a meeting of the Philosophical Society a short while ago, and at his request I have undertaken to describe them.

Of the eight skulls six are male and two female. The measurements and indices are given below, but a few words about each here may not be out of place.

Skull 1214. Belonged to a low caste Hindoo, 40 years old and of the male sex. On the right side are two epipteric bones and one wormian bone in the lambdoid suture; there is a very small epipteric bone on the left side, as well as another wormian bone in the left lambdoid suture. The sutures, on the whole, are rather simple.

Skull 1215. This cranium belonged to a male Hindoo of the high Kayasth caste, age 32. Two wormian bones exist in the left lambdoid suture, near the asterion; there are also two in the right lambdoid suture. The greater part of the right occipito-mastoid suture is obliterated.

Skull 1216. This was the cranium of a male Hindoo of the Kayasth caste, age 40. The sutures are simple. On both sides wormian bones are to be found in the lambdoid and squamous sutures; epipteric bones exist also on either sides and wormian bones in the right occipito-mastoid suture.

Skull 1217. Male Hindoo: coolie by occupation, aged 18. The sutures are rather complicated, especially in the case of the coronal suture. On the right side there is a wormian bone in the squamous suture, near the pterion. Three large and many minute wormian bones exist in the right lambdoid suture, while there are five in the left suture.

Skull 1219. Low caste Hindoo female, aged 30. The coronal suture below the stephanion is nearly obliterated. The sutures are very simple. There is only one wormian bone present; this is to be found in the right lambdoid suture.

Skull 1253. Skull of a Mussulman, aged 30; by occupation a coachman. There is a very large interparietal bone, 93·5 mm. broad and 55 long, as well as two wormian bones at the right asterion and a few minute bones of a similar kind in the right half of the coronal suture, and in the suture joining the interparietal and supraoccipital bones. The sutures are fairly simple, with the exception of those between the parietal and interparietal bones; these are complicated, though somewhat obliterated.

Skull 1704. Belonged to a low caste Hindoo female, 25 years old. The sutures are moderately simple, but there are three epactal bones, one wormian bone in the right lambdoid and four in the right parieto-mastoid sutures. There is also a similar bone at the right asterion. On the left side are five wormian bones in the parieto-mastoid suture and two in the lambdoid suture.

Skull 1705. This is a Mussulman's skull. The man died at the age of 23 or 24. The sutures are fairly simple. There are two wormian bones in the right lambdoid suture and one in the left, as well as a wormian bone in the right parieto-mastoid suture.

A striking point about these skulls is the enormous quantity of supplementary bones present. There is not one of them but has at least one such bone, while the great majority have always a large number. It would almost seem to be exceptional for them to be absent.

On turning to the various indices the most noticeable feature is the smallness of the cranial capacity. With one exception (No. 1214) they are all microcephalic; No. 1219, a female skull, has the exceedingly small capacity of 970 c.c. This is the smallest capacity of any skull in the Museum, and represents a brain-weight of 843 grammes.

Cephalic index. Six out of the eight skulls are dolichocephalic, with indices varying from 74·4 to 69. They belong indiscriminately to the low and high castes. Of the other two skulls, one has a high index of 83·6, and the second a slightly lower index of 79·9. The former skull was that of a Mussulman, the latter belonged to a low caste Hindoo female.

Height index. There is a good deal of variation among these skulls with regard to this index. The highest index is 83·3, and the lowest 71·5. Broadly speaking, they may be described as metriocephalic. There does not seem to be any difference in accordance with caste.

Gnathic index. They are all orthognathous with the exception of 1219, which has an index of 101·7. Here again there does not seem to be any caste difference.

The nasal index is of interest owing to Risley's suggestion that the high castes have a leptorhine and the low caste a platyrhine nose. These skulls do not accord with this view; but at the same time it must be remembered that while I am dealing with a very few skulls, Risley made his deduction from an enormous number of observations on the living. It may be as well, however, to put down the nasal indices in order from lowest to highest and see how far they agree or disagree with Risley's results:

41·6	...index of	Low caste Hindoo
43·4	....	Mussulman
48·4	....	High caste Hindoo (Kayasth caste)
50	....	„ „
50	....	Low caste Hindoo
53·1	....	„
53·5	....	„
56·8	....	Mussulman.

As these indices show, the high castes have, here at any rate, not leptorhine, but mesorhine noses, while of the low castes, two are platyrhine, one mesorhine and one very leptorhine. In fact the skulls do not seem to show Risley's caste differentiation of nose at all.

The faces of all are long and narrow, some more so than others. In no case does Kollmann's Upper Facial index fall below 49·4, while in one skull it rises to 58·8. The longest faces are to be found in the Kayasth caste, that is to say, high caste Hindoos. Risley found the same thing among the living.

As regards the Palatal index, all these skulls are brachyuranic with the exception of No. 1219, which is mesuranic, with an index of 111·2: the latter index is that of a low caste Hindoo female.

The Naso-Malar indices are at variance with the results of Risley. According to Risley, the high castes should be prosopic, the low mesopic or nearly platyopic; neither of the two high caste crania however, that I have examined, are prosopic; in fact the only prosopic face in the whole series is that of a low caste Hindoo (No. 1214). Of the three other low caste crania, two are mesopic and one platyopic; but, as a whole it cannot be said that, in this index, these crania are in accordance with Risley's results.

The following Tables may be of use in enabling one to appreciate at a glance the differences of the various skulls. They are arranged in accordance with their caste:

TABLE I. CLASSIFICATION ACCORDING TO INDEX.

Number of Skull	Caste, etc.	Age and Sex	Cephalic Index	Height Index	Gnathic Index	Orbital[1] Index	Nasal Index	Capacity	Palatal Index	Facial Index	Naso-Malar Index
1215	Kayasth	♂ 32	dolichocephalic	tapeinocephalic	orthognathous	mesoseme	mesorhine	microcephalic	brachyuranic	leptoprosopic	mesopic
1216	Kayasth	♂ 40	dolichocephalic	metriocephalic	orthognathous	mesoseme	mesorhine	microcephalic	brachyuranic	leptoprosopic	platyopic
1214	Low caste Hindoo	♂ 40	dolichocephalic	metriocephalic	orthognathous	microseme	platyrhine	megacephalic	brachyuranic	chamæprosopic	prosopic
1217	Low caste Hindoo	♂ 18	dolichocephalic	acrocephalic	orthognathous	mesoseme	leptorhine	microcephalic	brachyuranic	leptoprosopic	mesopic
1219	Low caste Hindoo	♀ 30	mesaticephalic	metriocephalic	mesognathous	mesoseme	mesorhine	microcephalic	mesuranic	leptoprosopic	mesopic
1704	Low caste Hindoo	♀ 25	dolichocephalic	tapeinocephalic	orthognathous	mesoseme	platyrhine	microcephalic	brachyuranic	chamæprosopic	platyopic
1253	Mussulman	♂ 30	dolichocephalic	tapeinocephalic	orthognathous	microseme	platyrhine	microcephalic	brachyuranic	chamæprosopic	mesopic
1705	Mussulman	♂ 23 or 24	brachycephalic	acrocephalic	orthognathous	mesoseme	leptorhine	microcephalic	brachyuranic	leptoprosopic	mesopic

[1] Mean of both orbital indices.

TABLE II. MEASUREMENTS.

Number of Skull	Name of Person, etc.	Sex, Age and Occupation	Glabello-Occipital Length	Ophryo-Occipital Length	Least Frontal Breadth	Greatest Breadth	Basibregmatic Height	Horizontal Circumference	Sagittal Circumference	Nasion to Bregma	Bregma to Lambda	Lambda to Opisthion	Opisthion to Basion	Basion to Nasion
1214	Nilmoni, Hindoo Low caste, Bengal	♂ 40 Cultivator	182·5	182	101·5	134·5	140·5	508	518	130	133	113	41	101
1215	Rama Nath, Hindoo, Bengal Kayasth caste	♂ 32	176	174·5	87	121·5	126	480	484	124	121	101	35·5	102·5
1216	RamaNath, Hindoo, Bengal, Kayasth caste.	♂ 40	179·5	178·5	90	133·5	136·5	504	499·5	123·5	135	103·5	35·5	102
1217	Dehki, Hindoo, Bengal	♂ 18 Coolie	179	176	93·5	130	138·5	493	502·5	131	125	107	36	103·5
1253	Shekh Peer Beksh, Bengal Mussulman	♂ 30 Coachman	189·5	186·5	93	132	135·5	513	521	125	119	138	32·5	106·5
1705	Jamrathi, Bengal Mussulman	♂ 23 or 24	158·5	158·5	96·5	132·5	132	466·5	465·5	112	122	100	32·5	99
1219	Mohiny, Bengal Low caste	♀ 30	159·5	156·5	93	127·5	117	460	443·5	120·5	101·5	104	31	86·5
1704	Herimoti, Bengal Low caste	♀ 25 Prostitute	179·5	179·5	94	129·5	129	500	501·5	124	112	139	34	92·5

Capacity	Orbital Height	Orbital Width	Interorbital Distance	Nasal Height	Nasal Width	Naso-Malar Measurements	Basialveolar Length	Basinasal Length	Bimaxillary Width	Bizygomatic Width	Nasio-Alveolar Length	Nasio-Mental Length	Palatal Length	Greatest Palatal Breadth
1510	Right 35·5 / Left 35·5	Right 43 / Left 43	19·5	50·5	27	109 / 99	90·5	101	96·5	131·5	65·5	111	51	64·5
1175	Right 32 / Left 32·5	Right 37 / Left 38	17	51	25·5	98·5 / 90·5	95·5	102·5	92·5	119·5	68·5	112·5	50	63·5
1340	Right 35 / Left 34·5	Right 40·5 / Left 39	17·5	53·5	26	102 / 96	98	102	98	122·5	72	—	52	62·5
1285	Right 33 / Left 33	Right 38 / Left 37·5	20	50·5	21	102·5 / 95	97	103·5	100·5	125	69·5	120·5	54	62·5
1235	Right 30 / Left 30	Right 41·5 / Left 40·5	25	47·5	27	114 / 104	99	106·5	104·5	130·5	65	116·5	53	69·5
1160	Right 34 / Left 34	Right 40 / Left 40	18·5	49·5	21·5	105 / 96	95	99	90	122·5	66	110·5	51·5	62·5
970	Right 30 / Left 31	Right 37 / Left 35·5	23·5	43	21·5	101 / 93	88	86·5	91·5	117·5	61·5	109	53·5	59·5
1280	Right 32 / Left 33	Right 38·5 / Left 38	21	49	26	102 / 96	89	92·5	95	125·5	62	—	49·5	62

TABLE III. INDICES.

Number of Skull	Name of Person	Sex	Age	Cephalic Index	Height Index	Gnathic Index	Nasal Index	Orbital Index Right	Orbital Index Left	Capacity	Palatal Index	Kollmann's Upper Facial Index	Kollmann's Total Facial Index	Virchow's Upper Facial Index	Virchow's Total Facial Index	Naso-Malar Index
1214	Nilmoni, Low caste Hindoo	♂	40	73·2	77	89·6	53·5	82·6	82·6	1510	126·5	49·8	84·4	67·9	115	110·1
1215	Rama Nath, Hindoo, Kayasth caste	♂	32	69	71·6	93·2	50	86·5	85·5	1175	127	57·3	94·1	74·1	121·6	108·8
1216	Rama Nath, Hindoo, Kayasth caste	♂	40	74·4	76	96·1	48·6	86·4	88·5	1340	120·2	58·8	—	73·5	—	106·3
1217	Dehki, Hindoo, Low caste	♂	18	72·6	77·4	93·7	41·6	86·8	88	1285	115·7	55·6	96·4	69·2	119·9	107·9
1253	Shekh Peer Baksh, Mussulman	♂	30	69·7	71·5	93	56·8	72·3	74·7	1235	131·1	49·8	89·3	62·2	111·5	109·6
1705	Jamrathi, Mussulman	♂	23 or 24	83·6	83·3	96	43·4	85	85	1160	121·4	53·9	90·2	73·3	122·8	109·4
1219	Mohiny, Hindoo, Low caste	♀	30	79·9	73·4	101·7	50	81·1	87·3	970	111·2	52·3	92·8	67·2	119·1	108·6
1704	Herimoti, Hindoo, Low caste	♀	25	72·1	71·9	96·2	53·1	83·1	86·8	1280	125·3	49·4	—	65·3	—	106·3

Monday, 13 *May*, 1895.

PROFESSOR J. J. THOMSON, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *Exhibition of some recent Photographs of the Moon.* By H. F. NEWALL, M.A., *Trinity College.*

The following photographs were exhibited, (i) a print by Dr Weinek from an original negative taken by MM. Loewy and Puiseux with the *coudé* equatorial of the Paris Observatory; (ii) an enlargement made by Dr Weinek from the same negative, showing the crater Linné and the surrounding region; (iii) a transparent enlargement from the same negative, sent from the Paris Observatory, of the region near Archimedes, on such scale that the diameter of the Moon would be 3·9 metres; (iv) some direct enlargements made by Mr Newall with a camera attached to the 25-inch visual refractor of the Cambridge Observatory.

(2) *On the 'Volume Heat' of Aniline.* By E. H. GRIFFITHS, M.A., *Sidney Sussex College.*

We know little as to the influence of temperature upon the specific heat of bodies. It is true that books of reference occasionally give (to four significant figures) the values of the specific heat of mercury over the temperature range 0° to 100° C., as well as a large number of imposing formulæ by means of which the specific heat of various metals at any temperature may be calculated by the student to as many "significant" figures as he pleases. The accuracy is, however, more apparent than real. Observers have been so dependent on the method of mixtures or some other mode of comparison with water that our present values are almost entirely relative and depend upon the experiments of Regnault.

It is generally assumed that our knowledge of the changes in the capacity for heat of water is sufficiently exact, but I think that this assumption is untenable. The evidence of recent observers (for example Rowland, Bartoli and Stracciati, Joly, and Griffiths), is in direct conflict with Regnault over ranges of temperature below 34°, and I have now in the press an account of some experiments which lead to the conclusion that Regnault's values at higher temperatures are also inexact.

True, that determinations made by methods similar to those employed by Bunsen are less affected by changes in the capacity for heat of water, but, on the other hand, they involve some assumption as to the ratio of the "mean thermal unit" to the thermal unit at other temperatures.

While these doubts exist regarding the standard, the data obtained by the comparison of other bodies with that standard are of little absolute value.

In the *Philosophical Magazine* for January, 1895, I have given an account of some observations on the specific heat of aniline over the temperature range 15° to 52° C. To whatever objections the method there described may be open, it is certain that the values there given are independent of any assumption as to the capacity for heat of water. The following facts may, therefore, be of some interest.

During last autumn Mr C. Green, of Sidney Sussex College, was so kind as to make for me a series of determinations on the *density* of aniline over the above temperature range. The instrument used was Sprengel's Pyknometer. I can from personal observation answer for the care with which the observations were made. The expansion of the glass was determined by a series of preliminary experiments, and three independent sets of determinations gave the following values for the density of Aniline.

TABLE I.

Temperature	Density	Mean
°		
15	1·0259, 1·0251, 1·0260	1·0257
20	1·0218	1·0218
30	1·0130, 1·0129, 1·0130	1·0130
40	1·0042, 1·0042, 1·0043	1 0042
50	·9955, ·9955, ·9956	·9955

One determination only was made at 20°.

If we multiply the capacity for heat at different temperatures by the density at those temperatures, we get the capacity for heat of equal volumes, a quantity which I propose to distinguish by the term "volume heat."

In the following table, col. II (extracted from page 77 of the paper above referred to) gives the values of the specific heat (S_1) of Aniline at temperatures θ_1.

Column III gives the density (d_1) at the same temperature as determined by Mr Green.

Column IV gives the "volume heat."

TABLE II.

I θ_1	II S_1	III d_1	IV $S_1 \times d_1$
°			
15	·5137	1·0257	·5269
20	·5156	1·0218	·5269
30	·5198	1·0130	·5267
40	·5244	1·0042	·5267
50	·5294	·9955	·5270

It would thus appear that the "volume heat" of aniline over the above range is practically constant.

Of course it would be useless to found any hypothesis on an isolated fact of this kind, and (for the reasons above given) I know of no other body to which the enquiry can, at present, be extended.

Where our knowledge is so small, however, even a solitary example may be of use as indicating the direction in which investigation may be profitable.

(3) *Exhibition of Goldstein's Experiments on Kathode Rays.* By J. W. CAPSTICK, M.A., Fellow of Trinity College.

Mr Capstick shewed Goldstein's experiments on the effect of a stream of kathode rays on salts of the alkalies. When the rays are directed on potassium chloride, for instance, the salt becomes of a heliotrope colour and retains the colour for several days if kept out of contact with moisture. The effect appears to be due to a chemical change in the substance—probably the formation of a subchloride—but the layer of altered salt is so exceedingly thin that it is difficult to get unequivocal chemical evidence as to its nature.

(4) *On a curious Dynamical Property of Celts.* By G. T. WALKER, M.A., Fellow of Trinity College.

Mr G. T. Walker exhibited celts which possessed the property of spinning in only one direction upon a horizontal surface. He pointed out that the peculiarity was due to the fact that the direction of the line of least curvature at the point of contact did not exactly coincide with that of either of the axes of dynamical symmetry, and exhibited a dynamical top in which this deviation from parallelism could have any value assigned to it.

Such a top displayed the characteristic properties of a celt, and a deviation of 15′ displayed itself in the motion: with a deviation of 6° each direction of rotation was unstable for either longitudinal or transverse oscillations and an angular velocity imparted was reversed three times before motion ceased. Several other peculiarities of the top were likewise pointed out.

(5) *On the Formation of Cloud in the Absence of Dust.* By C. T. R. Wilson, B.A., Sidney Sussex College.

The cloud-formation is brought about as in the experiments of Aitken and others by the sudden expansion of saturated air. A form of apparatus is used in which a very sudden and perfectly definite increase in volume is produced, and in which all danger of the entrance of dust from the outside is avoided. If we start with ordinary air, after a small number of expansions to remove dust particles by causing condensation to take place upon them, it is found that the expansion has now to be pushed to a certain definite limit in order that condensation may take place. With expansion greater than this critical amount (working with a constant initial temperature) there is invariably a cloud produced, and none with less expansion.

Some preliminary experiments have given the following results.

$$\frac{v_2}{v_1} = 1{\cdot}258, \text{ when initial temperature} = 16^\circ{\cdot}7\text{C.}$$

Here $\frac{v_2}{v_1}$ is the ratio of the final to the initial volume, when condensation just takes place.

This corresponds to a fall of temperature of about 26°C., and to a vapour pressure about 4·5 times the saturation pressure.

In order that water drops should be in equilibrium with this degree of supersaturation their radii must be equal to about $8{\cdot}3 \times 10^{-8}$ cm., assuming the surface tension for such small drops to have its ordinary value.

MONDAY, 27 *May*, 1895.

PROFESSOR J. J. THOMSON PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

(1) *On Graphical Methods in Geometrical Optics.* By J. LARMOR, M.A., St John's College.

1. The fundamental problem of geometrical optics relates to the modification produced in a filament of light on passing across a series of different media. From the geometrical standpoint the filament is made up of a narrow pencil of rays, which are straight when the medium is homogeneous: and it may be considered as defined by the focal lines of the pencil. The direct analytical method therefore hinges on the determination of these focal lines, after each refraction, from their already known positions before refraction: but the formulæ, even for a single refraction, are complicated, and, when there is a question of combining a number of successive refractions, almost prohibitive. In such circumstances however, as in other physical questions where we have to do with linear relations, the use of graphic methods will be found to lead to results of intrinsic simplicity, and thus give direct insight into the general relations of the subject.

2. *Uniplanar System.* In the sketch here to be given of a geometrical method of treatment, it will be convenient to begin with the simple case of a narrow pencil of rays in one plane, and having therefore only one focus: that is, the optical system will at first be a columnar or cylindrical one.

Since a single focus now always corresponds to a single focus as image to object, and, the pencil being narrow, all its rays may be taken to a sufficient approximation as passing exactly through the focus, the elementary geometrical theory of Möbius and Maxwell may be applied to the filament. Thus we may determine on its axes at incidence and emergence a pair of principal points, and a pair of principal foci, and we may construct by aid of them the focus conjugate to any other assigned one[1].

In this way, or by the more usual analytical processes, we see that conjugate foci are connected by a linear relation, as might have been expected *à priori*. Now let us draw the axis of the filament as incident on the optical system, and its axis as emergent from it. Conjugate foci on these two axes are homographically related; therefore the line connecting any pair of them envelopes a conic section which also has contact with the axes themselves. Suppose, by experiment or otherwise, that three pairs of conjugate foci have been determined: it will be possible to

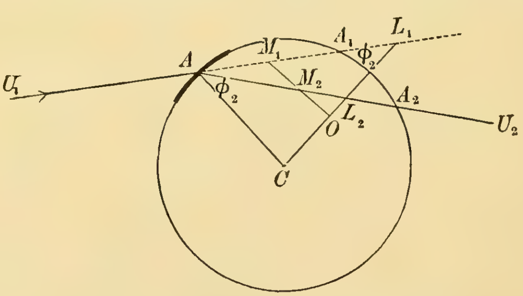
[1] Cf. *Proc. Lond. Math. Soc.*, xx., 1889, p. 182.

deduce all other pairs by linear construction without any knowledge of the internal constitution of the optical system. It has in fact been shown by Pascal how, given five points on a conic section, the other point in which any line through one of them meets it again may be determined by linear construction; and the conjugate or polar theorem, known as Brianchon's, solves the present problem. The lines connecting the three given pairs of conjugate foci, and the two axes of the pencil, form five tangents; and, simply by drawing three lines, the other tangent through any point on one of the axes is constructed, and this meets the other axis in the conjugate focus.

3. There is an important case in which this general construction reduces to a very simple form. Suppose the problem relates to refraction of the filament of light at a single interface; or to passage across an optical system whose total thickness can be neglected in comparison with the focal distances of the pencil, for example, eccentric refraction across a thin lens, or across a thin plate of a medium with any kind of curvatures in its two faces. The incident and emergent axes meet on this plate, and if the incident focus is at their point of intersection, the conjugate one is obviously the same point. Thus the two axes of the pencil touch the conic at the same point; therefore the conic must be a finite line, and all tangents to it pass through one of its extremities. The line connecting conjugate foci in all such cases therefore passes through a fixed point.

4. *Single Refraction in Primary Plane.* The position of this point is determined if the positions of two pairs of conjugate foci are known.

For the case of a single refraction at a curved surface (including of course reflexion) it is desirable to specify precisely its position, as this case is fundamental in the ordinary analytical theory. Now for a spherical interface there are always two exactly



aplanatic points on each radius: the equation of the spherical surface can in fact be thrown into the form $r_1 - \mu r_2 = 0$, the origins from which r_1 and r_2 are measured being inverse points with respect to it.

Let then U_1AU_2 be the path of the central ray of an optical filament refracted at A, at a spherical interface whose centre of curvature is C.

We can find the aplanatic points L_1, L_2 on the path of the filament by inflecting the line CL_1 to make the angle at L_1 equal to the angle of refraction ϕ_2. These points will be exact conjugate foci, no matter how wide the optical beam may be. The easiest construction for these points is to bisect by CL_1 the angle between the radii CA_1 and CA_2 drawn to the points in which the axes of the filament again meet the spherical surface.

As above, the point A is its own exact conjugate focus; for any pencil diverging from the point A continues to do so after refraction.

Further, we can find another pair of conjugate foci, not in this case exact, by constructing a beam such that the angles of incidence and refraction are the same for consecutive rays. For if the ray incident at B is to have the same angle of refraction as the ray incident at A, the circle drawn through C, A, B must meet the refracted ray AU_2 in the point where that ray intersects the ray refracted from B. Hence, passing to the limit when B ultimately coincides with A, the points in which the circle on AC as diameter meets an incident and refracted ray are conjugate foci; and the beam from one of these points diverges from the other, after refraction,—at the same angle, since the deviation of each ray is the same. This pair of conjugate foci are M_1, M_2, the middle points of the intercepts made by the spherical surface on the paths of the central ray; and it is noticeable that they are conjugate foci whatever value the index of refraction may have.

As then A is its own conjugate, and the relation between conjugate foci is homographic, the line joining any pair of conjugates must pass through a fixed point. This is the point O in which the line connecting M_1 and M_2 meets CL_1, which is the perpendicular drawn to it from C. The march of conjugate foci in the primary plane is now open to inspection.

The primary focal lengths AF_1, AF_2 of the refracting surface for a pencil incident along the given direction U_1A are the intercepts made on each of the lines AU_1, AU_2 by vectors from O parallel to the other one.

The construction thus obtained for the primary focus after oblique refraction has, I find, been already stated without proof by Thomas Young[1]. "It approaches the nearest to Maclaurin's construction[2], but is far more convenient."

[1] "On the Mechanism of the Eye," *Phil. Trans.*, 1801.

[2] *Treatise on Fluxions*, § 413.

The following construction, ascribed to Newton, is given by Barrow, in his *Lectiones Opticae*[1], and is no doubt the earliest solution of the problem:

Draw AR at right angles to the incident ray meeting CP_1 in R, P_1 being the incident focus; and draw AQ perpendicular to the refracted ray on the side towards C, such that

$$AQ : AR = AA_1 : AA_2;$$

then the primary focus conjugate to P_1 lies on CQ.

5. As a corollary we have a construction for the centre of curvature at any point P of a Cartesian oval of which S and H are foci. Draw any circle, centre O, touching the oval at P and meeting PS, PH in L, M; find the point R in which the line joining the middle points of these chords meets the bisector of the angle LOM; let SH meet PR in X; draw XF parallel to LM meeting PS in F, and draw FC perpendicular to PS meeting the normal in C, which is the centre of curvature required.

When the oval degenerates into a conic section, this becomes a well-known construction: from G the foot of the normal draw GF perpendicular to it, meeting PS in F, and draw FC perpendicular to PS meeting the normal in C, which is the centre of curvature required.

6. *Surface of Double Curvature: principal incidence.* The relation between the positions of conjugate foci in the secondary plane is of the same kind as above; they are in a line through C.

These results may be at once extended to the case of any beam incident in a plane of principal curvature of a non-spherical surface, provided the focal lines of the beam are in and at right angles to this plane. The relation between the conjugate primary foci is given by the above construction by aid of the circle of curvature in the plane of incidence; while the line joining conjugate secondary foci always passes through the centre of the circle of curvature in the perpendicular plane.

7. *General case of single interface.* The solution of the problem of the determination of the form of a general pencil after refraction at an interface of double curvature had been achieved long ago by Malus; but until Maxwell's adaptation[2] of the Hamiltonian method of the characteristic function of the pencil, the theory remained too lengthy and complicated to be of service in optical applications. It is now well known, after Maxwell, that if the rays of the incident pencil are normals to the surface

$$z_1 - \frac{x_1^2}{2A_1} - \frac{y_1^2}{2B_1} - \frac{x_1 y_1}{C_1} + \ldots = 0,$$

[1] See also Newton's *Optical Lectures*, read in 1669, published posthumously in 1728.

[2] Maxwell, *Proc. Lond. Math. Soc.*, IV. 1872; VI. 1874.

and those of the refracted pencil normals to the surface

$$z_2 - \frac{x_2^2}{2A_2} - \frac{y_2^2}{2B_2} - \frac{x_2 y_2}{C_2} + \ldots = 0,$$

the origin being on the interface, the axis of z along the axis of the corresponding pencil and the axis of x in the plane of refraction, then A_1 is connected with A_2 and B_1 with B_2 by equations of the same form as in the special case of § 6, while C_1 is connected with C_2 by a similar equation. If lengths equal to A_1 and A_2 are laid off along the axes of the pencil, the line connecting their extremities passes through a fixed point; a similar statement applies to the other two pairs of points, and the coordinates of the three fixed points thus obtained have been given by Maxwell. The graphical solution may now be completed by assigning geometrical constructions for them. The fixed point connected with A_1 and A_2 is determined by Young's construction (§ 4) in which the circle is the circle of curvature of the section of the interface by the plane of refraction. The fixed point connected with B_1 and B_2 is the centre of curvature of the normal section perpendicular to the plane of refraction.

The connexion between C_1 and C_2 is[1] through the equation

$$\frac{\cot\phi_1}{C_1} - \frac{\cot\phi_2}{C_2} = \frac{\cot\phi_1 - \cot\phi_2}{S},$$

where ϕ_1 and ϕ_2 are the angles of incidence and refraction, and S^{-1} is the coefficient of the product term in the equation of the interface. Thus, when C_1 is infinite, C_2 is equal to

$$S \sin\phi_1 \cos\phi_2 / \sin(\phi_1 - \phi_2);$$

and when C_2 is infinite, C_1 is equal to

$$-S \cos\phi_1 \sin\phi_2 / \sin(\phi_1 - \phi_2).$$

Hence, draw round the point of incidence a circle of radius S, in the plane of incidence: from the point where each axis meets it draw a perpendicular to the normal at the interface, and from its foot a parallel to the other axis of the pencil: these parallels meet in the fixed point required, which is in fact also on the line connecting the points in which the two axes meet the circle.

8. *Any system of negligible thickness.* It follows (§ 3) from the elements of the theory of perspective, that when a general pencil crosses any optical system whose thickness is negligible, the refracted pencil may still be deduced from the incident one by the construction of § 7; except that the fixed points must now be determined either once for all experimentally, or else by a succession of linear constructions, one for each refraction.

[1] Maxwell, *Proc. Lond. Math. Soc.*, IV. 1872; VI. 1874.

A construction of this kind enables us with great facility to trace the course of the refraction as the incident focal lines gradually alter their positions. For example, we see that there is always one and only one position of the focus for an incident stigmatic pencil, along a given axis, such that the refracted pencil is also stigmatic; a proposition which is obviously of importance in a general theory of photographic combinations of lenses.

9. *General Problem.* When the thickness of the optical system across which the filament of light passes is not negligible, the graphical treatment is not so simple. It has however been shown[1] that the optical effect of such a system may be precisely imitated by the effect on a straight filament of two definite thin astigmatic lenses mounted on the same axis at a definite distance apart; and the relations of this simple system can be represented in a graphical manner by aid of the constructions above. It will be more convenient however to combine deviation with the convergence produced by the lenses, so that the axes of incident and emergent pencils may not be identical. The constants of the filament after refraction through the first thin lens may now be graphically constructed by aid of three fixed points; then they must be transformed to a new origin at the second lens; and finally by another linear construction the constants of the emergent filament may be obtained.

The complexity inherent in the treatment of optical systems of sensible thickness arises solely from the trouble involved in transferring the analytical constants of the filament of light from one point to another of its axis as origin. This operation can be done graphically just as well as analytically, or better: but the process is necessarily clumsy. Probably the best available construction for this purpose is the one indicated by Maxwell[2].

[10. The development of the construction by Pascal's hexagram in § 2 places the theory in a very simple light. The problem under consideration is that of the conjugate focus in two-dimensional or cylindrical optical systems; or of the conjugate primary focal line in systems which have a plane of symmetry, or which possess the (for this purpose) equivalent property that the focal lines of all pencils which are stigmatic at incidence have common directions. We are supposed to know from observation the positions of three pairs of conjugate points, on the straight axes of the pencil in the uniform media at the two ends of the optical system. Let the three incident foci be marked off on any line at their proper distances apart, say A_1, B_1, C_1; and let A_2, B_2, C_2 represent their conjugates marked off at their proper mutual distances on any

[1] *Proc. Lond. Math. Soc.*, XXIII. 1892, p. 172.
[2] *loc. cit.*

other line. The problem is to find pairs of points on the two lines which are in homography with these pairs. The graphical solution is to regard the three lines A_1A_2, B_1B_2, C_1C_2, the two given lines $A_1B_1C_1$ and $A_2B_2C_2$, and the unknown line through D_1, arranged in any order, as the sides of a hexagram, when the lines connecting opposite corners will meet in a point. We might then find the elongation corresponding to the conjugate points D_1 and D_2, by determining the conjugate E_2 of a point E_1 very near to D_1, and taking the limit of the ratio D_2E_2 to D_1E_1. And then the transverse magnification will be derived from the theorem, which might be conveniently named after Maxwell, that the elongation of any finite segment on the axis is equal to the ratio of the extreme indices multiplied by the product of the transverse magnifications corresponding to its two ends.

But there is a much simpler course open. Let the given foci be laid off so that a pair of them coincide at the point of intersection of the lines: then the lines connecting all other pairs will pass through a fixed point, which is at once determined. The principal foci will be obtained by drawing parallels to each of these lines through the fixed point; and the product of the principal focal lengths will be equal to the product of the distances of any pair of conjugate foci from the respective principal foci. As the ratio of the focal lengths is that of the extreme indices, their values, involving the positions of the Gaussian principal points, are known except as to sign. Whether the positive or negative sign is to be taken, requires the further knowledge of whether the image of some one object is erect or inverted; there being always two optical systems, with equal and opposite focal lengths, that give the same relation of conjugate foci along the axes. The cardinal points being thus determined, everything else follows as usual.]

(2) *Note on the Steady Motion of a Viscous Incompressible Fluid.* By J. Brill, M.A., St John's College.

1. My object in the present communication is to obtain the analogues for the viscous fluid of certain well-known theorems relating to the perfect fluid. I suppose the fluid to be homogeneous and incompressible, and the motion to be steady; and proceed first to consider the case of two-dimensional motion. If we write

$$\phi = \frac{p}{\rho} + \overline{V} + \tfrac{1}{2}q^2,$$

the equations of motion may be put into the form

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x}-2v\zeta&=-2\nu\frac{\partial\zeta}{\partial y},\\ \frac{\partial\phi}{\partial y}+2u\zeta&=\quad 2\nu\frac{\partial\zeta}{\partial x}\end{aligned}\right\}\text{......................}(1);$$

in addition to which we have

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=2\zeta.$$

Equations (1) may be written in the form

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x}-2\zeta\left(v-\nu\frac{\partial\log\zeta}{\partial y}\right)&=0,\\ \frac{\partial\phi}{\partial y}+2\zeta\left(u-\nu\frac{\partial\log\zeta}{\partial x}\right)&=0\end{aligned}\right\}\text{......................}(2).$$

If we write

$$\left(u-\nu\frac{\partial\log\zeta}{\partial x}\right)dy-\left(v-\nu\frac{\partial\log\zeta}{\partial y}\right)dx=md\chi,$$

and therefore

$$\left.\begin{aligned}u-\nu\frac{\partial\log\zeta}{\partial x}&=\quad m\frac{\partial\chi}{\partial y},\\ v-\nu\frac{\partial\log\zeta}{\partial y}&=-m\frac{\partial\chi}{\partial x}\end{aligned}\right\}\text{......................}(3),$$

then equations (2) become

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x}+2m\zeta\frac{\partial\chi}{\partial x}&=0,\\ \frac{\partial\phi}{\partial y}+2m\zeta\frac{\partial\chi}{\partial y}&=0\end{aligned}\right\}\text{........................}(4).$$

Eliminating ϕ from these we obtain

$$\frac{\partial(m\zeta,\ \chi)}{\partial(x,\ y)}=0,$$

or

$$2m\zeta=f(\chi)\text{..........................}(5).$$

Multiplying the first of equations (4) by dx, and the second by dy, and adding, we have

$$d\phi+f(\chi)\,d\chi=0,$$

from which by integration we obtain

$$\frac{p}{\rho} + V + \tfrac{1}{2}q^2 + F(\chi) = \text{const.} \quad \text{.................(6)},$$

where

$$F(\chi) = \int f(\chi)\,d\chi.$$

We also have, from equations (3),

$$\frac{\partial}{\partial x}\left(m\frac{\partial \chi}{\partial x}\right) + \frac{\partial}{\partial y}\left(m\frac{\partial \chi}{\partial y}\right) + 2\zeta = 0,$$

or

$$m\frac{\partial}{\partial x}\left(m\frac{\partial \chi}{\partial x}\right) + m\frac{\partial}{\partial y}\left(m\frac{\partial \chi}{\partial y}\right) + f(\chi) = 0 \quad \text{..........(7)}.$$

Also, from equations (3), combined with the equation of continuity, we obtain

$$\frac{\partial(m,\ \chi)}{\partial(x,\ y)} + \nu\nabla^2\log\zeta = 0,$$

which may be written in the form

$$\frac{\partial(m,\ \chi)}{\partial(x,\ y)} + \nu\left\{\nabla^2\log f(\chi) - \nabla^2\log m\right\} = 0 \quad \text{.........(8)}.$$

The motion is therefore given by the three equations (6), (7), (8). These equations however fail to give a minimum theorem, except of a very restricted kind.

2. The particular case in which the vorticity is so distributed that we have

$$\nabla^2\log\zeta = 0,$$

is worthy of note. In this case we have m a function of χ, and the equations reduce to the same form as they assume in the case of the perfect fluid, viz.

$$\frac{p}{\rho} + V + \tfrac{1}{2}q^2 + f_1(\chi) = 0,$$

$$\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} + f_1'(\chi) = 0.$$

We also have

$$2\zeta = f_1'(\chi).$$

In the general case of the preceding article, if we write down the variation of the expression

$$\iint dx\,dy\left\{\tfrac{1}{2}(u^2+v^2) - F(\chi)\right\},$$

and transform it according to the usual method, then equations (7) and (8) will cause the bulk of the terms under the integral sign to vanish. If we seek to determine the condition that the remainder may vanish also, we find either that m is a function of χ, which leads to the case we have just considered, or that the form of the stream-lines is not to be varied.

In connection with the above special case, it is interesting to note the case in which, the motion not being steady, the vorticity is distributed so as to obey the law

$$\frac{\partial \log \zeta}{\partial t} = \nu \nabla^2 \log \zeta.$$

This equation may be written in the form

$$\frac{\partial \zeta}{dt} = \nu \nabla^2 \zeta - \frac{\nu}{\zeta} \left\{ \left(\frac{\partial \zeta}{\partial x} \right)^2 + \left(\frac{\partial \zeta}{\partial y} \right)^2 \right\};$$

and, combining it with the equation

$$\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} = \nu \nabla^2 \zeta,$$

we have
$$u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} = \frac{\nu}{\zeta} \left\{ \left(\frac{\partial \zeta}{\partial x} \right)^2 + \left(\frac{\partial \zeta}{\partial y} \right)^2 \right\},$$

or
$$\left(u - \nu \frac{\partial \log \zeta}{\partial x} \right) \frac{\partial \zeta}{\partial x} + \left(v - \nu \frac{\partial \log \zeta}{\partial y} \right) \frac{\partial \zeta}{\partial y} = 0.$$

Adopting the same notation as before, we have

$$\frac{\partial (\zeta, \chi)}{\partial (x, y)} = 0,$$

or ζ a function of χ and t, so that at any instant ζ is constant along any of the curves $\chi = \text{const.}$, as in the above special case of steady motion.

3. We now pass on to discuss the case of steady motion symmetrical about an axis. In this case the equations of motion assume the form

$$\left. \begin{aligned} &\frac{\partial \phi}{\partial r} - 2V\omega + 2\nu \frac{\partial \omega}{\partial z} = 0, \\ &\frac{\partial \phi}{\partial z} + 2U\omega - 2\nu \left(\frac{\partial \omega}{\partial r} + \frac{\omega}{r} \right) = 0 \end{aligned} \right\} \quad \text{..........(9).}$$

In addition to these we have

$$\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{\partial V}{\partial z} = 0,$$

and
$$\frac{\partial V}{\partial r} - \frac{\partial U}{\partial z} = 2\omega.$$

Equations (9) may be written in the form

$$\left.\begin{aligned} \frac{\partial\phi}{\partial r} - 2\omega\left(V - \nu\frac{\partial}{\partial z}\log r\omega\right) = 0, \\ \frac{\partial\phi}{\partial z} + 2\omega\left(U - \nu\frac{\partial}{\partial r}\log r\omega\right) = 0 \end{aligned}\right\} \qquad \text{.............(10).}$$

If we write

$$\left(U - \nu\frac{\partial}{\partial r}\log r\omega\right)dz - \left(V - \nu\frac{\partial}{\partial z}\log r\omega\right)dr = m\,d\chi,$$

and therefore

$$\left.\begin{aligned} U - \nu\frac{\partial}{\partial r}\log r\omega &= m\frac{\partial\chi}{\partial z}, \\ V - \nu\frac{\partial}{\partial z}\log r\omega &= -m\frac{\partial\chi}{\partial r} \end{aligned}\right\} \qquad \text{..................(11),}$$

then equations (10) may be written in the form

$$\left.\begin{aligned} \frac{\partial\phi}{\partial r} + 2m\omega\frac{\partial\chi}{\partial r} = 0, \\ \frac{\partial\phi}{\partial z} + 2m\omega\frac{\partial\chi}{\partial z} = 0 \end{aligned}\right\} \qquad \text{........................(12).}$$

Eliminating ϕ from equations (12) we obtain

$$\frac{\partial\,(m\omega,\ \chi)}{\partial\,(r,\ z)} = 0,$$

or

$$2m\omega = f(\chi) \qquad \text{........................(13).}$$

Multiplying the first of equations (12) by dr, and the second by dz, and adding, we have

$$d\phi + f(\chi)\,d\chi = 0,$$

from which by integration we readily deduce

$$\frac{p}{\rho} + \overline{V} + \tfrac{1}{2}q^2 + F(\chi) = \text{const.} \qquad \text{...............(14),}$$

where

$$F(\chi) = \int f(\chi)\,d\chi.$$

Also from equations (11) we readily obtain

$$\frac{\partial}{\partial r}\left(m\frac{\partial\chi}{\partial r}\right) + \frac{\partial}{\partial z}\left(m\frac{\partial\chi}{\partial z}\right) + 2\omega = 0,$$

or

$$m\frac{\partial}{\partial r}\left(m\frac{\partial\chi}{\partial r}\right) + m\frac{\partial}{\partial z}\left(m\frac{\partial\chi}{\partial z}\right) + f(\chi) = 0 \qquad \text{.........(15).}$$

We also have, from the same equations combined with the equation of continuity,

$$\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\left(m\frac{\partial \chi}{\partial z}+\nu\frac{\partial}{\partial r}\log r\omega\right)-\frac{\partial}{\partial z}\left\{m\frac{\partial \chi}{\partial r}-\nu\frac{\partial}{\partial z}\log r\omega\right\}=0,$$

or

$$\frac{1}{r}\frac{\partial(mr,\ \chi)}{\partial(r,\ z)}+\nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\log r\omega=0.$$

This last equation may be written

$$\frac{\partial(mr,\ \chi)}{\partial(r,\ z)}+\nu r\nabla^2\log r\omega=0,$$

but since

$$\nabla^2\log r=0,$$

it reduces to the form

$$\frac{\partial(mr,\ \chi)}{\partial(r,\ z)}+\nu r\nabla^2\log\omega=0,$$

i.e.

$$\frac{\partial(mr,\ \chi)}{\partial(r,\ z)}+\nu r\left\{\nabla^2\log f(\chi)-\nabla^2\log m\right\}=0 \quad\ldots\ldots(16).$$

The motion is therefore given by equations (14), (15), (16).

4. For the special case of motion symmetrical about an axis in which the vorticity is distributed according to the law

$$\nabla^2\log\omega=0,$$

we have mr a function of χ, and (14) and (15) reduce to the forms

$$\frac{p}{\rho}+\overline{V}+\tfrac{1}{2}q^2+f_1(\chi)=\text{const.},$$

$$\frac{\partial^2\chi}{\partial r^2}-\frac{1}{r}\frac{\partial\chi}{\partial r}+\frac{\partial^2\chi}{\partial z^2}+r^2 f_1'(\chi)=0,$$

or the theorem takes the same form as in the case of the perfect fluid. We have also $2\omega=rf_1'(\chi)$.

We will now consider the case in which, the motion not being steady, the vorticity is distributed according to the law expressed by the equation

$$\frac{\partial\log\omega}{\partial t}=\nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right).\log\omega.$$

This equation may be written in the form

$$\frac{\partial\omega}{\partial t}=\nu\left(\frac{\partial^2\omega}{\partial r^2}+\frac{1}{r}\frac{\partial\omega}{\partial r}+\frac{\partial^2\omega}{\partial z^2}\right)-\frac{\nu}{\omega}\left\{\left(\frac{\partial\omega}{\partial r}\right)^2+\left(\frac{\partial\omega}{\partial z}\right)^2\right\};$$

and, combining it with the equation

$$\frac{\partial \omega}{\partial t}+U\frac{\partial \omega}{\partial r}+V\frac{\partial \omega}{\partial z}=U\frac{\omega}{r}+\nu\left\{\frac{\partial^2 \omega}{\partial r^2}+\frac{1}{r}\frac{\partial \omega}{\partial r}+\frac{\partial^2 \omega}{\partial z^2}-\frac{\omega}{r^2}\right\},$$

we obtain

$$U\frac{\partial \omega}{\partial r}+V\frac{\partial \omega}{\partial z}=U\frac{\omega}{r}+\frac{\nu}{\omega}\left\{\left(\frac{\partial \omega}{\partial r}\right)^2+\left(\frac{\partial \omega}{\partial z}\right)^2\right\}-\nu\frac{\omega}{r^2},$$

or

$$\frac{\partial \omega}{\partial z}\left\{V-\frac{\nu}{\omega}\frac{\partial \omega}{\partial z}\right\}+\left(\frac{\partial \omega}{\partial r}-\frac{\omega}{r}\right)\left\{U-\frac{\nu}{\omega}\left(\frac{\partial \omega}{\partial r}+\frac{\omega}{r}\right)\right\}=0,$$

i.e.

$$\frac{\partial}{\partial z}\left(\frac{\omega}{r}\right)\left\{V-\nu\frac{\partial}{\partial z}\log r\omega\right\}+\frac{\partial}{\partial r}\left(\frac{\omega}{r}\right)\left\{U-\nu\frac{\partial}{\partial r}\log r\omega\right\}=0.$$

Adopting the same notation as before, the above equation becomes

$$\frac{\partial\left(\frac{\omega}{r},\ \chi\right)}{\partial(r,\ z)}=0.$$

Thus we have ω/r a function of χ and t, being therefore at any instant constant along any of the curves $\chi=\text{const.}$, as in the above special case of steady motion.

5. In the three-dimensional case, the equations of motion may be written in the form

$$\left.\begin{aligned}\frac{\partial \phi}{\partial x}+2(w\eta-v\zeta)+2\nu\left(\frac{\partial \zeta}{\partial y}-\frac{\partial \eta}{\partial z}\right)=0,\\ \frac{\partial \phi}{\partial y}+2(u\zeta-w\xi)+2\nu\left(\frac{\partial \xi}{\partial z}-\frac{\partial \zeta}{\partial x}\right)=0,\\ \frac{\partial \phi}{\partial z}+2(v\xi-u\eta)+2\nu\left(\frac{\partial \eta}{\partial x}-\frac{\partial \xi}{\partial y}\right)=0\end{aligned}\right\}\text{.........(17).}$$

In addition to these we have

$$2\xi=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z},\quad 2\eta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x},$$

$$2\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y},$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

Suppose that we seek to determine three quantities l, m, n such that

$$l\frac{\partial \phi}{\partial x}+m\frac{\partial \phi}{\partial y}+n\frac{\partial \phi}{\partial z}=0\text{.....................(18).}$$

To do this we shall have to satisfy the equation

$$l\left\{w\eta - v\zeta + \nu\left(\frac{\partial\zeta}{\partial y} - \frac{\partial\eta}{\partial z}\right)\right\}$$
$$+ m\left\{u\zeta - w\xi + \nu\left(\frac{\partial\xi}{\partial z} - \frac{\partial\zeta}{\partial x}\right)\right\}$$
$$+ n\left\{v\xi - u\eta + \nu\left(\frac{\partial\eta}{\partial x} - \frac{\partial\xi}{\partial y}\right)\right\} = 0 \quad\ldots\ldots\ldots\ldots\ldots(19).$$

Now consider the congruence of curves given by the differential equations

$$\frac{dx}{l} = \frac{dy}{m} = \frac{dz}{n}.$$

Let λ and μ be the parameters of two families of surfaces belonging to this congruence. Then we have

$$l = k\frac{\partial(\lambda,\ \mu)}{\partial(y,\ z)};\ m = k\frac{\partial(\lambda,\ \mu)}{\partial(z,\ x)};\ n = k\frac{\partial(\lambda,\ \mu)}{\partial(x,\ y)}.$$

Substituting these values for l, m, n in equation (18) we have

$$\frac{\partial(\phi,\ \lambda,\ \mu)}{\partial(x,\ y,\ z)} = 0,$$

or ϕ a function of λ and μ. We may write this result in the form

$$\phi + f(\lambda,\ \mu) = \text{const.},$$

or
$$\frac{p}{\rho} + V + \tfrac{1}{2}q^2 + f(\lambda,\ \mu) = \text{const.}$$

Now equation (19) gives only one relation between the quantities l, m, n, and we are therefore at liberty to assume another. We see that the equation will be satisfied if we assume l, m, n to be connected by the two equations

$$l(w\eta - v\zeta) + m(u\zeta - w\xi) + n(v\xi - u\eta) = 0,$$
$$l\left(\frac{\partial\zeta}{\partial y} - \frac{\partial\eta}{\partial z}\right) + m\left(\frac{\partial\xi}{\partial z} - \frac{\partial\zeta}{\partial x}\right) + n\left(\frac{\partial\eta}{\partial x} - \frac{\partial\xi}{\partial y}\right) = 0.$$

Thus the congruence of curves given by the differential equations

$$\frac{dx}{(u\zeta - w\xi)\left(\frac{\partial\eta}{\partial x} - \frac{\partial\xi}{\partial y}\right) - (v\xi - u\eta)\left(\frac{\partial\xi}{\partial z} - \frac{\partial\zeta}{\partial x}\right)}$$
$$= \frac{dy}{(v\xi - u\eta)\left(\frac{\partial\zeta}{\partial y} - \frac{\partial\eta}{\partial z}\right) - (w\eta - v\zeta)\left(\frac{\partial\eta}{\partial x} - \frac{\partial\xi}{\partial y}\right)}$$
$$= \frac{dz}{(w\eta - v\zeta)\left(\frac{\partial\xi}{\partial z} - \frac{\partial\zeta}{\partial x}\right) - (u\zeta - w\xi)\left(\frac{\partial\zeta}{\partial y} - \frac{\partial\eta}{\partial z}\right)},$$

is such that any one of the surfaces belonging to the family $\phi = \text{const.}$, may be considered as the locus of a singly infinite series of the curves constituting the congruence.

It is clear that there are an indefinite number of congruences satisfying this condition. Another would be that given by the differential equations

$$\frac{dx}{\left(u\eta - \nu\frac{\partial\eta}{\partial x}\right)\left(u\zeta - \nu\frac{\partial\zeta}{\partial x}\right) - \left(v\xi - \nu\frac{\partial\xi}{\partial y}\right)\left(w\xi - \nu\frac{\partial\xi}{\partial z}\right)}$$

$$= \frac{dy}{\left(v\zeta - \nu\frac{\partial\zeta}{\partial y}\right)\left(v\xi - \nu\frac{\partial\xi}{\partial y}\right) - \left(w\eta - \nu\frac{\partial\eta}{\partial z}\right)\left(u\eta - \nu\frac{\partial\eta}{\partial x}\right)}$$

$$= \frac{dz}{\left(w\xi - \nu\frac{\partial\xi}{\partial z}\right)\left(w\eta - \nu\frac{\partial\eta}{\partial z}\right) - \left(u\zeta - \nu\frac{\partial\zeta}{\partial x}\right)\left(v\zeta - \nu\frac{\partial\zeta}{\partial y}\right)}.$$

Suppose that l, m, n and l', m', n' refer to two distinct congruences of this type, then

$$l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y} + n\frac{\partial\phi}{\partial z} = 0,$$

and
$$l'\frac{\partial\phi}{\partial x} + m'\frac{\partial\phi}{\partial y} + n'\frac{\partial\phi}{\partial z} = 0.$$

Therefore

$$\frac{\frac{\partial\phi}{\partial x}}{mn' - m'n} = \frac{\frac{\partial\phi}{\partial y}}{nl' - n'l} = \frac{\frac{\partial\phi}{\partial z}}{lm' - l'm} = h \text{ say};$$

from which it readily follows that

$$d\phi + h\,\{l'(mdz - ndy) + m'(ndx - ldz) + n'(ldy - mdx)\} = 0 \quad\ldots\ldots\ldots\ldots\ldots(20).$$

Now
$$mdz - ndy = k\left\{\frac{\partial(\lambda,\mu)}{\partial(z,x)}dz - \frac{\partial(\lambda,\mu)}{\partial(x,y)}dy\right\}$$

$$= k\left\{\frac{\partial\mu}{\partial x}\left(\frac{\partial\lambda}{\partial x}dx + \frac{\partial\lambda}{\partial y}dy + \frac{\partial\lambda}{\partial z}dz\right) - \frac{\partial\lambda}{\partial x}\left(\frac{\partial\mu}{\partial x}dx + \frac{\partial\mu}{\partial y}dy + \frac{\partial\mu}{\partial z}dz\right)\right\}$$

$$= k\left(\frac{\partial\mu}{\partial x}d\lambda - \frac{\partial\lambda}{\partial x}d\mu\right);$$

and similarly $ndx - ldz = k\left(\frac{\partial \mu}{\partial y} d\lambda - \frac{\partial \lambda}{\partial y} d\mu\right)$,

and $$ldy - mdx = k\left(\frac{\partial \mu}{\partial z} d\lambda - \frac{\partial \lambda}{\partial z} d\mu\right).$$

Thus equation (20) becomes

$$d\phi + hk\left\{\left(l'\frac{\partial \mu}{\partial x} + m'\frac{\partial \mu}{\partial y} + n'\frac{\partial \mu}{\partial z}\right) d\lambda - \left(l'\frac{\partial \lambda}{\partial x} + m'\frac{\partial \lambda}{\partial y} + n'\frac{\partial \lambda}{\partial z}\right) d\mu\right\} = 0.$$

Comparing this with the equation

$$d\phi + \frac{\partial f}{\partial \lambda} d\lambda + \frac{\partial f}{\partial \mu} d\mu = 0,$$

we have $$hk\left(l'\frac{\partial \mu}{\partial x} + m'\frac{\partial \mu}{\partial y} + n'\frac{\partial \mu}{\partial z}\right) = \frac{\partial f}{\partial \lambda},$$

and $$hk\left(l'\frac{\partial \lambda}{\partial x} + m'\frac{\partial \lambda}{\partial y} + n'\frac{\partial \lambda}{\partial z}\right) = -\frac{\partial f}{\partial \mu}.$$

An expression for h can be readily calculated in the case of any particular pair of congruences. The quantity k is to a certain extent analogous to the quantity m of the preceding investigations. In addition to the above equations we have

$$\frac{\partial (k, \lambda, \mu)}{\partial (x, y, z)} = \frac{\partial l}{\partial x} + \frac{\partial m}{\partial y} + \frac{\partial n}{\partial z}.$$

(3) *On a certain automorphic function.* By Mr H. F. Baker, Fellow of St John's College.

On an infinite plane, and lying entirely in the finite part of the plane, are drawn $2p$ circles, each outside all the others. They are divided into p pairs of associated circles, C_i', C_i denoting a pair. The limiting points of the pair C_i', C_i being represented, by complex variables in the ordinary way, by the symbols A_i, B_i, the transformation

$$\frac{\zeta' - B_i}{\zeta' - A_i} = \mu_i \frac{\zeta - B_i}{\zeta - A_i}$$

connects the complex variable of a point ζ, on C_i', with the complex variable ζ' of a point on C_i. We suppose A_i within C_i' and B_i

within C_i; then μ_i is a real quantity less than unity. This transformation may also be written in the form

$$\zeta' = \frac{\alpha_i \zeta + \beta_i}{\gamma_i \zeta + \delta_i}, \; = \vartheta_i(\zeta) \text{ say,}$$

where
$$\begin{pmatrix} \alpha_i, & \beta_i \\ \gamma_i, & \delta_i \end{pmatrix}$$

$$= \begin{pmatrix} (B_i \mu_i^{-\frac{1}{2}} - A_i \mu_i^{\frac{1}{2}})/(B_i - A_i), & -A_i B_i (\mu_i^{-\frac{1}{2}} - \mu_i^{\frac{1}{2}})/(B_i - A_i) \\ (\mu_i^{-\frac{1}{2}} - \mu_i^{\frac{1}{2}})/(B_i - A_i), & -(A_i \mu_i^{-\frac{1}{2}} - B_i \mu_i^{\frac{1}{2}})/(B_i - A_i) \end{pmatrix},$$

so that $\alpha_i \delta_i - \beta_i \gamma_i = 1$. The sign of $\mu_i^{\frac{1}{2}}$ is immaterial so far as the transformation ϑ_i is concerned; but it is of importance when we define functions in terms of the quantities $\alpha_i, \beta_i, \gamma_i, \delta_i$. Hence we define $\mu_i^{\frac{1}{2}} = (-)^{h_i - 1} \mid \mu_i^{\frac{1}{2}} \mid$, h_i being 0 or 1, and at our disposal.

By successive application of the p fundamental transformations, and their inverses, we obtain a group of substitutions. We may use ϑ_i to denote the general substitution of the group, and write $\zeta_i = \vartheta_i(\zeta)$. The p fundamental substitutions may be distinguished by the suffix n. Then ϑ_i is a product of positive and negative powers of the p fundamental substitutions ϑ_n. An automorphic function of ζ, in the narrowest sense, is a function which is unaltered when ζ is replaced by any of the transformations ζ_i. In a more general sense, we consider also, under that name, functions which are multiplied by a definite factor when ζ is replaced by ζ_i. Of such, a function of great importance is that given by the definition

$$E(\zeta, \gamma) = (\zeta - \gamma) \prod_i{}' \frac{(\zeta_i - \gamma)(\gamma_i - \zeta)}{(\zeta_i - \zeta)(\gamma_i - \gamma)},$$

wherein the product extends to every substitution of the group except the identical substitution and the substitutions which are inverse to others occurring.

This function is infinite only at the place $\zeta = \infty$. It vanishes at $\zeta = \gamma$ and at all the places $\vartheta_i(\gamma)$. The function has an essential singularity at the singular points of the group. Except for the places named the function is everywhere finite and different from zero.

Further the function has the property expressed by the equation

$$\frac{E(\zeta_n, \gamma)}{E(\zeta, \gamma)} = \frac{e^{-2\pi i (v_n^{\zeta, \gamma} + \frac{1}{2}\tau_{nn})}}{\gamma_n \zeta + \delta_n} (-)^{g_n + h_n},$$

wherein ϑ_n is one of the p fundamental substitutions, $v_n^{\zeta, \gamma}$ is one of p every-where-finite functions of definite character, τ_{nn} is one of

p constants, whose value is defined precisely by a barrier curve drawn to connect the circles C_n', C_n, and g_n is 0 or 1 according to the way in which this barrier is drawn. The product

$$e^{-\pi i \tau_{nn}} (-)^{h_n}$$

is independent of the form of this barrier; the quantities

$$\gamma_n (-)^{h_n}, \quad \delta_n (-)^{h_n}$$

are independent of h_n.

The product which defines the function $E(\zeta, \gamma)$ is known to be convergent when the circles are suitably situated. More precisely the circles of any pair, C_n', C_n, must be sufficiently small, or their limiting points must be sufficiently distant. Under the same conditions the series

$$\lambda(\zeta, m) = \sum_i \frac{[\gamma_i \zeta + \delta_i]^{-1}}{\zeta_i - m},$$

wherein the summation refers to every substitution of the group, is convergent. The object of this note is to prove that its value is given by

$$\lambda(\zeta, m) = \frac{\Theta\left[v_i^{\zeta m} - \tfrac{1}{2}(g_i + h_i)\right]}{E(\zeta, m)\, \Theta\left[\tfrac{1}{2}(g_i + h_i)\right]},$$

wherein $\Theta(u_i)$ is Riemann's Θ series defined by

$$\Theta(u_i) = \sum_{n=-\infty}^{n=\infty}\sum \dots \sum e^{2\pi i (u_1 n_1 + \dots + u_p n_p) + i\pi (\tau_{11} n_1^2 + \dots + 2\tau_{12} n_1 n_2 + \dots)}.$$

There are clearly 2^p functions $\lambda(\zeta, m)$, according to the signs chosen for

$$(-)^{h_1}, \ (-)^{h_2}, \ \dots, \ (-)^{h_p}.$$

The function $\lambda(\zeta, m)$ is immediately seen, in the ordinary way, to satisfy the equations

$$\lambda(\zeta_n, m) = (\gamma_n \zeta + \delta_n)\, \lambda(\zeta, m),$$

$$\lambda(\zeta, m_n) = (\gamma_n m + \delta_n)\, \lambda(\zeta, m).$$

The function $\lambda(\zeta, m)$ vanishes at $\zeta = \infty$, and at p other places outside the $2p$ circles C_n', C_n. We shall denote these by

$$m_1, \ m_2, \ \dots, \ m_p.$$

The function is infinite at m and at all the places $\vartheta_i(m)$. If the space outside the circles be denoted by S, the function is infinite, within the space $\vartheta_i(S)$, at the place which is the analogue of m within that space, and is zero at the p places

$$\vartheta_i(m_1), \ \dots, \ \vartheta_i(m_p).$$

These results are obvious by inspection, and the ordinary Riemann process of considering the contour integral

$$\frac{1}{2\pi i}\int \lambda(\zeta, m)\, d\zeta.$$

Denoting now by v one of the p every-where-finite functions, the quotient

$$\lambda^2(\zeta, m) \Big/ \frac{dv}{d\zeta}$$

(i) is automorphic, since the factor $[\lambda(\zeta_n, m)/\lambda(\zeta, m)]^2$ is the same as $d\zeta/d\zeta_n$;

(ii) vanishes, within the space S, to the second order at the p zeros, $m_1, \ldots, m_p$, of $\lambda(\zeta, m)$;

(iii) is infinite, within the space S, at $\zeta = m$ (which we suppose within S), to the second order, and at the zeros of $\dfrac{dv}{d\zeta}$, which are known to be $2p-2$ in number. Denote these zeros by

$$\zeta_1, \ldots, \zeta_{2p-2}.$$

Hence, by Abel's theorem, denoting by m_1^2 the repetition of the place m_1, the places $m_1^2, m_2^2, \ldots, m_p^2$ are coresidual with the places $m^2, \zeta_1, \ldots, \zeta_{2p-2}$.

Consider next, within the space S, the function

$$\lambda(\zeta, m)\, E(\zeta, m).$$

By the properties already enunciated this function has no poles: and has zeros at $m_1, \ldots, m_p$. It satisfies the equation

$$\frac{\lambda(\zeta_n, m)\, E(\zeta_n, m)}{\lambda(\zeta, m)\, E(\zeta, m)} = (-)^{g_n + h_n} e^{-2\pi i (v_n^{\zeta, m} + \frac{1}{2}\tau_{nn})}.$$

It is known that, corresponding to any place m, p places

$$n_1, \ldots, n_p$$

can be chosen, such that the function

$$\Theta(v_i^{\zeta, m} - v_i^{z_1, n_1} - \ldots - v_i^{z_p, n_p})$$

has $\zeta = z_1, \ldots, z_p$ for its zeros. The places $n_1, \ldots, n_p$ are definite; they are such that $n_1^2, \ldots, n_p^2$ are coresidual with

$$m^2, \zeta_1, \ldots, \zeta_{2p-2}.$$

Hence there is an equation of the form

$$v_i^{m_1, n_1} + \ldots + v_i^{m_p, n_p} = \tfrac{1}{2}(k_i + \lambda_1 \tau_{i,1} + \ldots + \lambda_p \tau_{i,p}),$$

where $\tau_{i,1}\ldots, \tau_{i,p}$ are the, definite, periods of v_i, and

$$k_1, \ldots, k_p, \lambda_1, \ldots, \lambda_p$$

are rational integers; the values of these integers are quite definite when the barriers, spoken of, are drawn.

Hence the function

$$F(\zeta)=\frac{\lambda(\zeta, m)E(\zeta, m)}{\Theta(v_i^{\zeta, m}-v_i^{m_1, n_1}-\ldots-v_i^{m_p, n_p})}e^{\pi i(\lambda_1 v_1^{\zeta, m}+\ldots+\lambda_p v_p^{\zeta, m})}$$

has no poles, and no zeros, and, by the properties of the Θ function, is such that

$$F(\zeta_n)/F(\zeta)=(-)^{g_n+h_n-k_n}.$$

Thus the square of $F(\zeta)$ is a constant; thus $F(\zeta)$ is a constant, and

$$k_n=g_n+h_n-2K_n,$$

where $K_1, \ldots, K_p$ are unknown rational integers; and

$$\lambda(\zeta, m)E(\zeta, m)$$

$$=Ae^{-\pi i(\lambda_1 v_1^{\zeta, m}+\ldots+\lambda_p v_p^{\zeta, m})}\Theta\left[v_i^{\zeta, m}-\frac{g_i+h_i}{2}-\tfrac{1}{2}(\lambda_1\tau_{i1}+\ldots+\lambda_p\tau_{ip})\right],$$

where A is independent of ζ.

By taking ζ round the circle C_n, and considering the factors of the two sides, we can infer that λ_n is even, $=2L_n$ say; and can then write

$$\lambda(\zeta, m)E(\zeta, m)$$

$$=Ae^{-\pi i\Sigma L_n(g_n+h_n)-\pi i(\tau_{11}L_1^2+\ldots+2\tau_{12}L_1L_2+\ldots)}\Theta\left[v_i^{\zeta, m}-\tfrac{1}{2}(g_i+h_i)\right],$$

where $L_1, \ldots, L_p$ are integers depending on m and $h_1, \ldots, h_p$, and A does not depend on ζ.

But, using the equation

$$\lambda(\zeta, m)=-\lambda(m, \zeta),$$

we can infer that this equation is equivalent to

$$\lambda(\zeta, m)E(\zeta, m)=C\Theta\left[v_i^{\zeta, m}-\tfrac{1}{2}(g_i+h_i)\right],$$

where C does not depend on ζ *or* m.

Hence, putting $\zeta=m$, using the equations

$$[E(\zeta, m)/(\zeta-m)]_{\zeta=m}=1,$$

$$[\lambda(\zeta, m).(\zeta-m)]_{\zeta=m}=1,$$

which are easy to prove, we infer

$$C = 1/\Theta\left(\tfrac{1}{2}(g_i + h_i)\right),$$

which proves the theorem as originally stated.

For the case $p = 1$, putting

$$v = \frac{1}{2\pi i}\log\frac{\zeta - B}{\zeta - A}, \quad v_0 = \frac{1}{2\pi i}\log\frac{m - B}{m - A}$$

we find from the definitions,

$$E(\zeta, m) = \frac{B - A}{2i}\,\frac{\sin\pi(v - v_0)}{\sin\pi v \sin\pi v_0}\prod_{i=1}^{\infty}\frac{1 - 2q^{2i}\cos 2\pi(v - v_0) + q^{4i}}{(1 - q^{2i})^2},$$

$$= -\tfrac{1}{4}(B - A)\,\frac{e^{\pi i(v - v_0)}}{\sin\pi v \sin\pi v_0}\,\frac{\Theta\left(v - v_0 + \frac{1}{2} + \frac{1}{2}\tau\right)}{\Pi(1 - q^{2i})^3},$$

where $q = |\sqrt{\mu}|$ and $e^{\pi i\tau} = q$, so that $g_1 = 1$; and

$$\lambda(\zeta, m) = \frac{1}{\zeta - m}\sum_{r=1}^{\infty}\left\{1 + 4(-)^{r(h_1 - 1)}\frac{(1 + q^r)q^r\sin^2\pi(v - v_0)}{1 - 2q^{2r}\cos 2\pi(v - v_0) + q^{4r}}\right\},$$

where $h_1 = 1$ or 2.

For example when $h_1 = 1$

$\lambda(\zeta, m)$

$$= \frac{1}{\zeta - m}\,\frac{\Theta'\left(\frac{1}{2} + \frac{1}{2}\tau\right)}{\pi\Theta\left(\frac{1}{2}\right)}\sin\pi(v - v_0)\,e^{-\pi i(v - v_0)}\,\frac{\Theta\left(v - v_0 + \frac{1}{2}\right)}{\Theta\left(v - v_0 + \frac{1}{2} + \frac{1}{2}\tau\right)},$$

and in virtue of the fact that

$$\Theta'\left(\tfrac{1}{2} + \tfrac{1}{2}\tau\right) = 2\pi i\prod_{1}^{\infty}(1 - q^{2i})^3,$$

these agree with the formula found in the general case.

Reply to a Paper by Mr Bryan. By A. B. Basset, M.A., F.R.S.

My attention has recently been drawn to a note by Mr Bryan on page 51 of the present volume of the *Proceedings*; and I wish in the first place to state that until October 5, 1895, I had never seen the note in question, and was ignorant of its having been written. It was therefore impossible for me to have made any *remarks* upon this note; and the paragraph at the end of page 53 has consequently been inserted without my knowledge, authority or consent.

In replying to my criticisms, Mr Bryan has neglected one of the cardinal rules of controversy; that is, to quote your opponent's statements correctly. The passage from his paper which I criticized was the following, "it does not indicate that the spheroid in question is secularly unstable for this particular type of displacement. Its meaning is that the spheroid is more oblate than that form for which the angular velocity is a maximum." And in his reply Mr Bryan has omitted the first sentence, which contains the pith of the whole matter. From the passage at the commencement of § 3 of his paper in the *Proc. Roy. Soc.*[1], I understood him to mean that when a Maclaurin's spheroid is composed of *viscous* liquid, the steady motion will be stable provided

$$p_1(\zeta)\, q_1(\zeta) - t_n{}^s(\zeta)\, u_n{}^s(\zeta) > 0 \qquad\qquad (1),$$

and my object was to show that this result was wrong for the simultaneous values of

$$s = 0, \quad n = 2.$$

For a zonal displacement of degree 2, the condition (1) becomes

$$p_1 q_1 - p_2 q_2 > 0 \qquad\qquad (2),$$

and the spheroid is well-known to be stable; whereas for a certain value of the excentricity the above expression, as Mr Bryan has shown, becomes negative. Consequently if the inequality (2) gave the correct condition of stability, the spheroid would become unstable at the critical value of the excentricity which causes the left-hand side of (2) to vanish and become negative. Mr Bryan, being aware that the spheroid is stable for a spheroidal displacement, tries to get out of the difficulty to which a wrong result has led him by arguing, that the fact of the left-hand side of (2) becoming negative does not indicate instability, but means something totally different. Under these circumstances I am justified in characterizing his explanation as incorrect. The true explanation is that the condition in question indicates instability, but it is an erroneous one. The correct condition, as I have shown in my paper, leads to stability.

Another misquotation occurs on p. 52. Mr Bryan states that I have said "that the spheroid is secularly stable if the motion when slightly displaced is determined by terms of the form $\epsilon^{(\alpha+\iota\beta)\,t}$ where α is negative." I have said nothing of the kind. The above definition of secular stability is Poincaré's and not mine. I have objected to it, criticized it, and have said that I consider it an inaccurate use of language. What I said was that when the disturbed motion is of this character, the system is *absolutely* stable. Mr Bryan then goes on to suggest that possibly the

[1] Vol. XLVII., p. 369.

small motions of a viscous rotating mass of liquid cannot always be expressed by terms of this character. Perhaps not; and whenever this is the case the system will be *ordinarily* unstable, but nothing can be asserted with regard to its *secular* stability without further investigation. It is however possible that for certain types of displacement the liquid may perform *finite oscillations* about the spheroidal form; whilst for other types the liquid may continue to deviate further and further from this form until it breaks up into two or more detached masses. When the disturbed motion is of the former kind, I consider that the steady motion may properly be termed *secularly* stable for these particular types of displacement; and it is in this sense that I employ this phrase. When motion is *secularly* stable according to the above definition, the *approximate* solution which gives the state of things in the beginning of the disturbed motion will certainly not be of the form $\epsilon^{\alpha t + \iota\beta t}$, α negative; but since the particular kind of disturbed motion we are considering is periodic, the *complete* solution of the equations of motion will necessarily be expressed by periodic terms.

October 6, 1895.

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INDEX TO VOLUME VIII.

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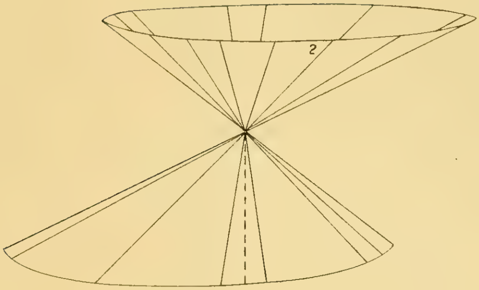

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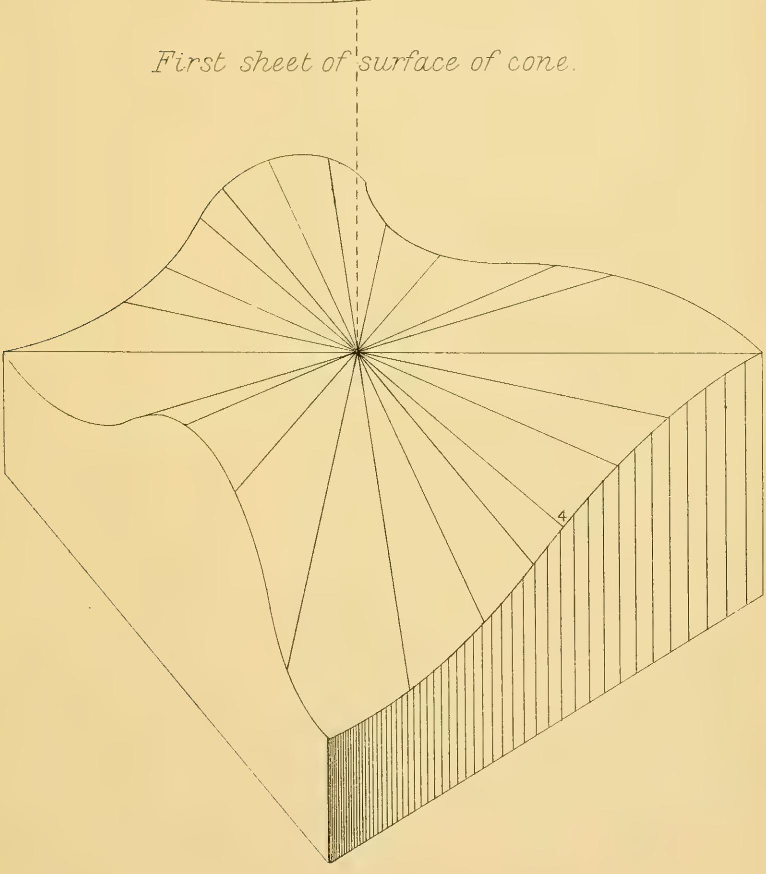

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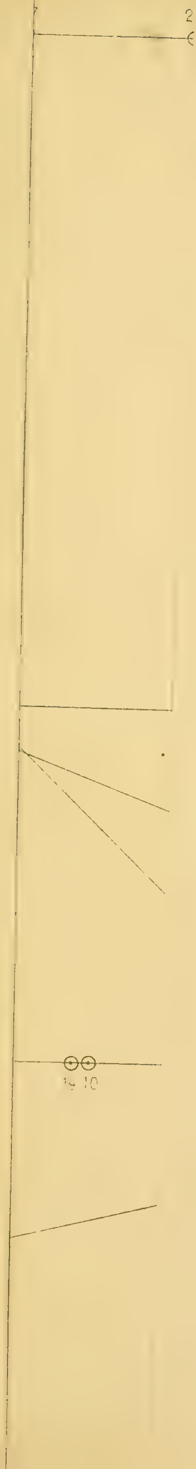
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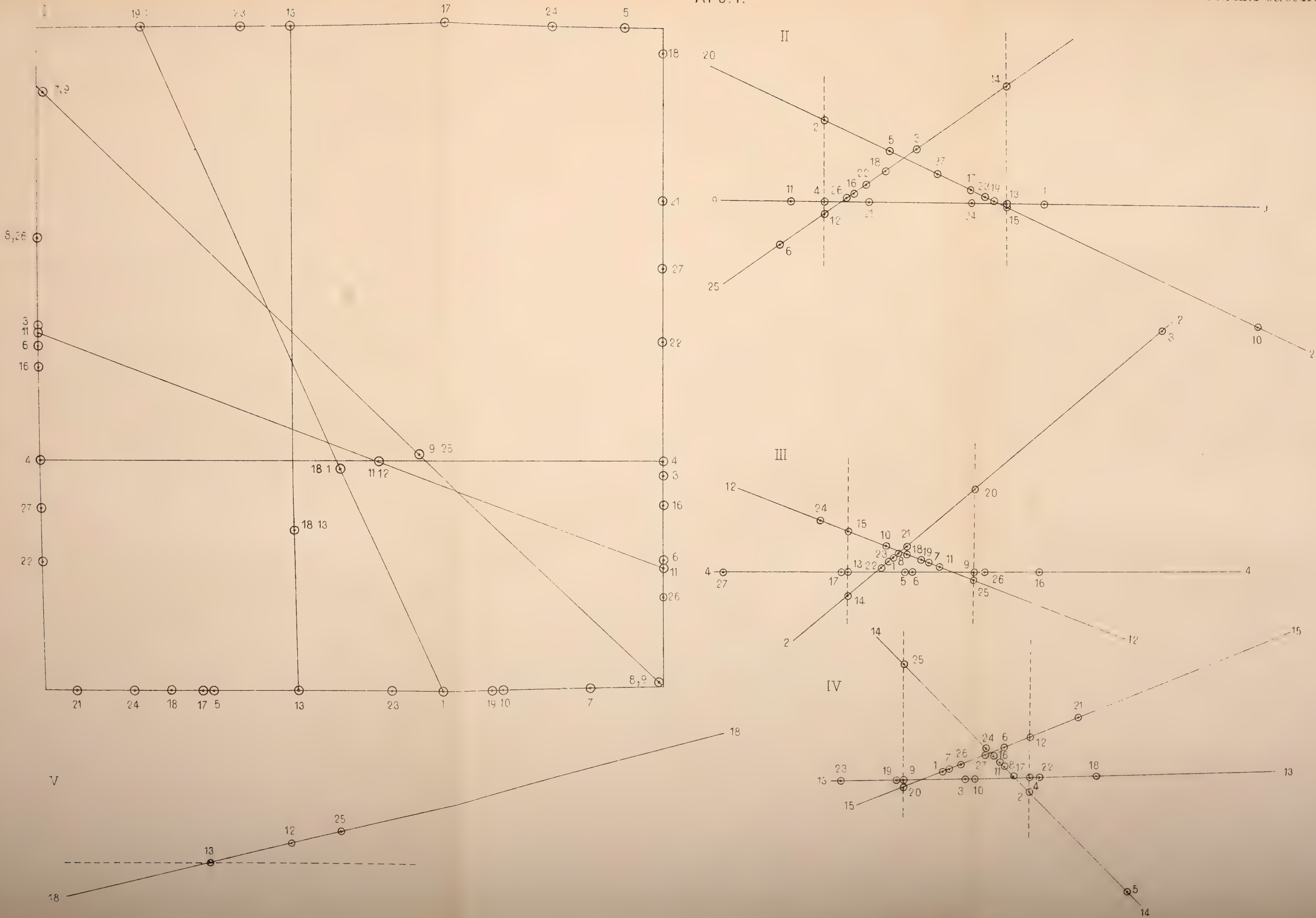

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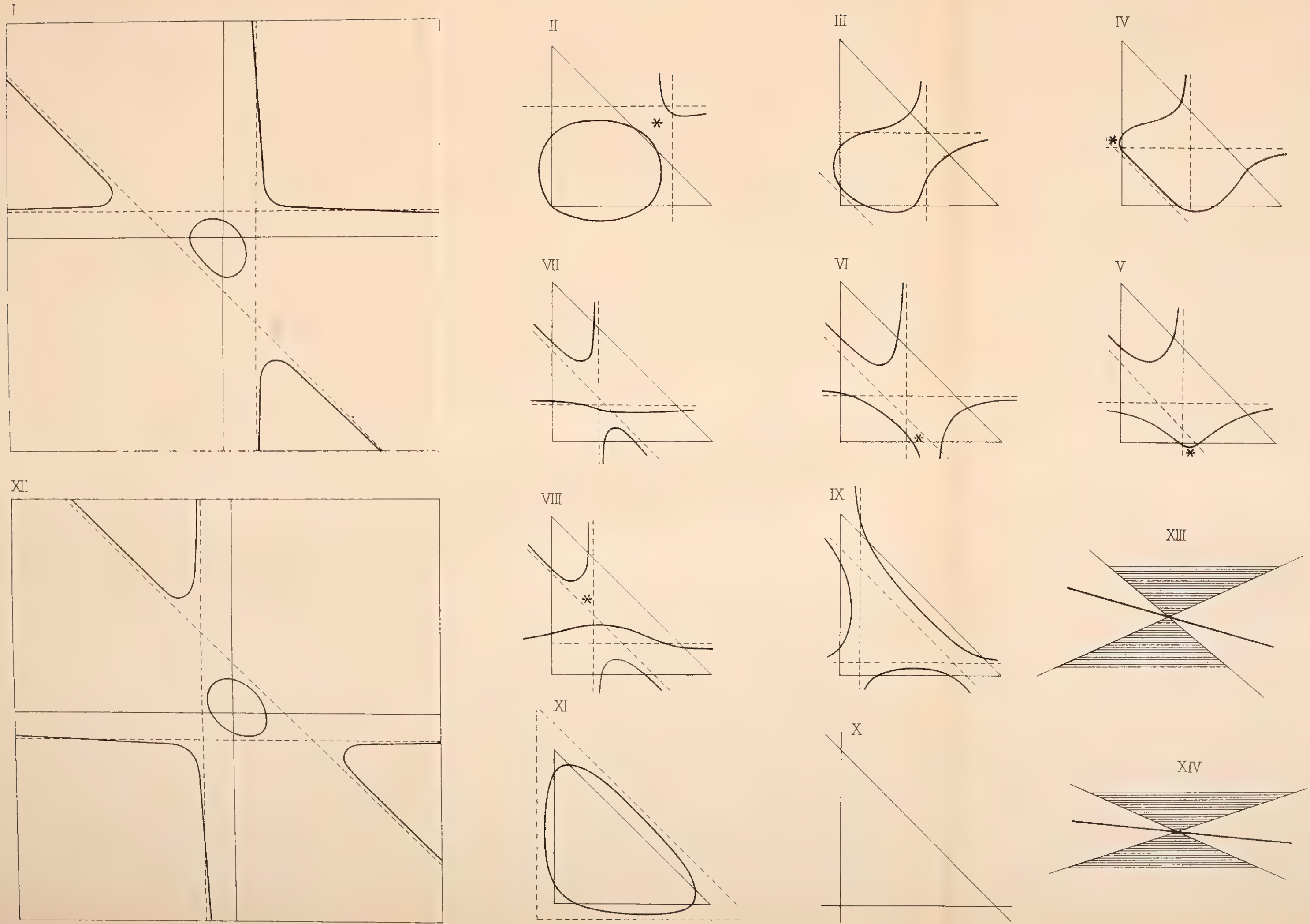

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