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ERRATUM.

Proc. Meeting Dec. 1901, p. 307 form. ($2c$) is to be read:

$$\frac{1}{C_4}\frac{1}{v_k}\frac{dv_{pl}}{dx}=\frac{1}{C_4}(\alpha-\beta)-\frac{\beta-\alpha\left(\frac{\partial\pi}{\partial\tau}\right)}{C_4^3\left(\frac{\partial^3\pi}{\partial\omega^3}\right).\,C_4\left(\frac{\partial^2\pi}{\partial\omega\partial\tau}\right)}\times$$

$$\times\left[C_4\left(\frac{\partial^2\pi}{\partial\omega\partial\tau}\right)\left\{\beta-\alpha\left(\frac{\partial\pi}{\partial\tau}\right)\right\}^2-\right.$$

$$\left.-\,C_4^2\left(\frac{\partial^3\pi}{\partial\omega^2\partial\tau}\right)\left\{\beta-\alpha\left(\frac{\partial\pi}{\partial\tau}\right)\right\}-3\,C_4^2\left(\frac{\partial^2\pi}{\partial\omega\partial\tau}\right)^2\alpha\right].$$

CONTENTS.

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING
of Saturday May 25, 1901.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 25 Mei 1901, Dl. X).

The following papers were read:

Chemistry. — Professor H. W. BAKHUIS ROOZEBOOM presents a communication on "*Cadmium amalgams.*"

The desirability of the investigation of the nature of the cadmium amalgams arises from the use of these amalgams in the WESTON-cell. The discussion [1]) about the irregularities occurring when using this cell could not lead to a satisfactory conclusion until a better insight

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[1]) Compare Proceedings of June 30 and Oct. 27 1900 and Febr. 23 1901.

was obtained into the nature of these amalgams. Mr. BIJL has now brought this investigation to a provisional conclusion and obtained the following results.

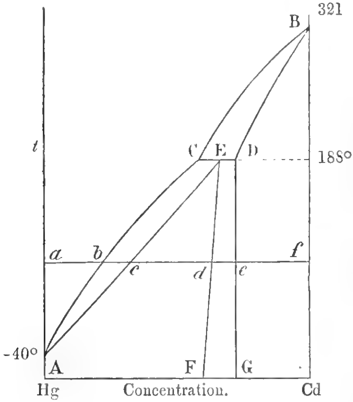

Fig. 1.

Cadmium and mercury are miscible in all proportions when in a liquid state at a sufficiently high temperature. From these fused mixtures crystals are deposited on cooling which are represented in Fig. 1 by the two curves *A C* and *C B*. The first runs from $A = -40°$, the melting point of mercury, to $C = 188°$; the second from this point to $B = 321°$, the melting point of cadmium. The two curves meet at *C* forming an angle. Both the curves which have been determined by the thermometric and by the dilatometric method, indicate the points at which solidifications begins.

None of the liquid amalgams, however, solidify completely at a constant temperature; solidification always takes place through a larger or smaller range of temperatures. The end of this could not be sharply defined thermometrically, but on the other hand the commencement of the fusion of the amalgams which had become solidified by sufficient cooling could be determined dilatometrically.

In this manner the lines *A E*, *E D* and *D B* were obtained as final solidifying, or commencing melting points. From this it appears that the whole solidifying figure belongs to the type of one of the cases which I have formerly distinguished in the systems of two substances which on solidifying yield exclusively mixed crystals.

The lines *A E* and *D B* represent the two possible series of mixed crystals. At 188° they follow each other with a comparatively small hiatus between *E* and *D*, which points correspond approximately with 75 and 77 atoms of cadmium per 100 atoms of the mixture.

The following change must therefore take place when heat is added at 188°:

$$\text{mixed crystals } E \rightarrow \text{liquid } C + \text{mixed crystals } D$$

and the reverse on withdrawing heat.

As the point *C* is situated at 67 percent, all the amalgams below

67 percent Cd. solidify to homogeneous mixed crystals of the mercury type. Similarly all amalgams over 77 percent of cadmium solidify to homogeneous mixed crystals of the cadmium type. The behaviour of the 67—77 percent amalgams is more complicated. These all begin by depositing crystals of the Cadmium type up to 188°. The change:

$$\text{liquid } C + \text{mixed crystals } D \rightarrow \text{mixed crystals } E,$$

then takes place so that 67—75 percent amalgams (at the left of *E*) are converted into mixed crystals *E* + liquid *C* and solidify on further cooling to mixed crystals of the mercury type; on the other hand the 75—77 percent amalgams completely solidify at 188° to a conglomerate of the crystals *E* and *D* of both types.

The very important question in how far the limits of these two series of mixed crystals, which exist together at 188°, change as the temperature falls, has been studied electrically. A change in the composition of such limiting mixed crystals appears as a rule to take place but very gradually, so that it was now, as in former instances, not possible to prove its existence by thermometric or dilatometric means.

With this metallic mixture however, the opportunity presented itself of studying the matter by electrical methods. At the same time all the other percentages were investigated in order to obtain a first instance in which the electrical conduct of a metallic alloy was known for all its various proportions.

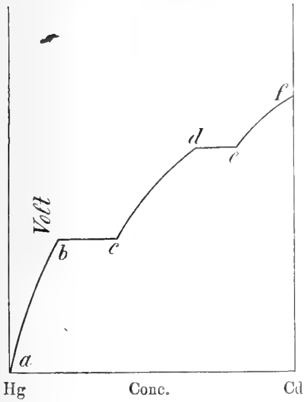

Fig. 2.

The *E M F* of alloys of every degree of concentration was measured in a solution of cadmium sulphate with a reversible mercury electrode as positive pole at temperatures of 25°—75°.

If we draw, in fig. 1, a horizontal line for instance at 25° it will intersect the different areas in such a manner that all the amalgams from *a* to *b* are liquid and therefore, form only one phase. From *b* to *c* they consist of the heterogeneous system consisting of the liquid phase *b* and of the solid phase *c* which consists of mixed crystals. From *c* to *d* every percentage is a homogeneous mixed crystal.

If we assume that the limiting representatives of the mixed crystals which can exist together at 188°, are situated at lower temperatures on the lines EF and DG, we then have at 25° d and e as limits and all the percentages between these are again heterogeneous and consists of both the mixed crystals d and e. On the other hand we have again from e—f a series of homogeneous mixed crystals ending with pure cadmium.

The line indicated in fig. 2 shows the *E M F* of the cells described in which the negative pole consists of the whole series of the amalgams in succession. The letters correspond with those in fig. 1. It therefore appears, that as long as we are dealing with a series of amalgams consisting of one phase (ab, cd, ef) whether liquid (ab) or solid (cd and ef) the *E. M F* varies regularly with the concentration of the amalgam. On the other hand we have in two cases a constant *E M F* accompanying variable concentration, namely from b to c and from d to e. In the first case it corresponds with the fact that the amalgam is a two-phased one (liquid b and solid c): in the second it must, therefore be also two-phased, but both phases are now solid mixed crystals (d and e). In this manner the limits of the horizontal part $d\,e$ could be determined for different temperatures. From this it was found that on cooling below 188° the limits diverge still further until at 25° they have become: 65 and 80 atoms of cadmium percent.

The horizontal line $b\,c$ is of great significance for the WESTON-cells. We can now see plainly why a cell of constant *E M F* may be easily constructed if only the percentage of the amalgam which serves as negative pole is chosen somewhere on this line. We then have an amalgam which is partly liquid and partly solid. Such a mass usually attains equilibrium quickly and sharply when the temperature is changed. Moreover, the composition of the alloy remains on the line for a considerable time even though the total amount of cadmium decreases by the action of the current. This merely causes a slight diminution in the amount of mixed crystals and an increase in the volume of the liquid but the composition of both and consequently the *E M F* remains constant.

The extremities of the line $b\,c$ alter, however, their position with the temperature as shown in fig. 1. This was noticed most plainly when determining the *E M F*: the positions of the points b and c thus found correspond exactly with those in fig. 1.

The many irregularities which have been observed when using WESTON-cells may be easily explained by the fact that up to the present an amalgam has been used containing too large a proportion of cadmium, too near to point c, so that on cooling one soon got outside of the horizontal part, or in other words solid mixed crystals were obtained, the *E M F* of which was different from that of an amalgam represented by the horizontal line and which reached equilibrium much less rapidly than the semi-liquid amalgams.

Microbiology. — "*Further Researches concerning Oligonitrophilous Microbes.*" By Prof. M. W. Beijerinck.

In my first paper on oligonitrophilous microbes [1]) I still left the question unanswered after the forms which develop in the light, in nutrient liquids, which only contain traces of nitrogen compounds, and whose nutrition with carbon can only be effected from the carbonic acid of the air.

The experiments to answer this question were made as follows. Large flasks were plugged with cotton wool or filtering paper, so that the air has free access, or closed in such a way that the air could be renewed, and that, at each renewing, it must pass through strong sulphuric acid in order to be deprived of the nitrogen-compounds. These flasks had been half filled with

100 Tap- or distilled water
0.02 $K^2 H PO^4$

and infected with a not too slight quantity of garden-soil, e. g. 1 to 2 grs. per liter [2]).

They were placed in winter at a window on the south, in spring and in summer on the north-west, and in the beginning they were now and then shaken, in order to sink the floating film of calcium-phosphate, which forms at the surface.

As the rate of nitrogen and carbon compounds is too slight to cause any appreciable development of colourless microbes, no further cloudiness results, but that of the easily precipitating phosphate. But in winter after six to eight, in summer after four to five weeks, *a characteristic flora develops consisting of some species*

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[1]) These Proceedings of March 30, 1901.

[2]) The Delft tap-water contains at present 0.42 mG. nitrogen per L., the garden-soil used 0.56 pCt. nitrogen (analyses of Mr. A. v. Delden); but this nitrogen can only for a minimal portion (as ammonia and nitrate-nitrogen) be assimilated by microbes. The oligonitrophili themselves possess the specific faculty of feeding on the nitrogen from the atmosphere.

of Cyanophyceae, which, once become visible, can promptly give rise to a deep bluish-green colouring of the liquid. In the beginning these Cyanophyceae are seen to develop as free colonies at the sides of the flask, later there also appear floating films, which latter consist chiefly of *Anabaena*, while among the colonies growing on the glass-wall, not only the large flat colonies of *Anabaena*, but likewise the characteristic, but rarer bluish-grey slimy lumps of *Nostoc paludosum* are most striking. A third, very intensely coloured species, which is nearly as common, I determined as *Nostoc sphaericum* [1]).

Motile Cyanophyceae, such as *Oscillaria*, do not result under these conditions, or only in much smaller numbers than those mentioned; probably for them the proportion of organic substances in the said nutrient liquids is still too large and that of nitrogen compounds perhaps too slight. I have also found that *Oscillaria* is microaërophilous [2]) in the dark, so that, at the places fit for its development, at least temporary anaërobiosis should be possible, which is not the case in my experiment.

Chlorophyceae, especially *Chlorococcum* and *Chlorella* are, as might be expected, not wholly absent in these cultures; but their number is so small that they are without any influence on their external character. This fact is the more remarkable because, if to the culture fluid is added

$$0{,}02 \text{ pCt. } NH^4\, NO^3$$

already after a shorter time than the above mentioned, a dense film of Chlorophyceae, in which *Chlorococcum infusionum* is the principal species, grows rapidly on the surface. Only when the nitrogen-compounds added to it have been quite consumed, the green film grows darker, as then again flakes of *Cyanophyceae*, in particular of *Anabaena*, begin to form.

The experiments have essentially the same course when the tap-water-phosphate flasks are not infected with garden-soil, but with a small flake taken from a previous culture of Cyanophyceae. Here I saw however, in some cases appear *Anabaena* only, which under these conditions of culture evidently supplanted the other Cyanophyceae.

If in my experiments I use Delft canal-water, instead of tap-water, and omit the infection with garden-soil, the process is some-

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[1]) Not all the species of Cyanophyceae obtained could be determined. Some of them I think have not been described.

[2]) It is macroaërophilous in the light.

what different. First a rich light-brown culture of *Diatomaceae* takes rise, in which here and there colonies are seen of Chlorophyceae belonging to the genera *Raphidium*, *Scenedesmus*, *Chlorella* and *Chlorococcum*, but without their multiplying sufficiently to alter the brown colour of the culture. After 8 to 9 weeks however, the colour at once grows darker by the then occurring increase of the Cyanophyceae, which increase continues a long time, evidently as long as there is a sufficient quantity of kalium phosphate and other mineral food.

I think the result of this last experiment should be explained as follows. Canal-water contains a greater amount of organic substances than the tap-water cultures; as long as these substances are present the Diatoms are prevailing; they use these substances for their carbon-nutrition, together with the carbonic acid from the air, and at the same time assimilate the nitrogen-compounds. When these are consumed the Cyanophyceae appear.

That the Diatoms can in fact utilise a fairly high rate of organic substances, is well known to the students of that group. The following experiment which, to my knowledge, has not yet been described, proves that the Diatoms are the very coloured microbes, which can, if not assimilate, at least tolerate without injury the full rate of organic matter and of nitrate- and ammonia-nitrogen of fertile garden-soil.

A high glasscylinder is filled for one half with garden-soil, for the other with pure water. After shaking the thus obtained mud is allowed to stand at a sunny window. After some days or weeks, according to season and temperature, one sees at the illumined side of the glass a deep brown film appear, consisting of the Diatoms present in the garden-soil, which slowly creep towards the light. This film increases some months by the multiplication of the Diatoms, but finally there appear in it large green spots of various lower Chlorophyceae, whose propagation becomes only vigorous, when the Diatoms and other microbes (such as bacteria and monads) have for the greater part used the assimilable organic substances and converted them into unassimilable material. Cyanophyceae do not grow under these circumstances, this being prevented by the abundance of nitrogen-compounds in the garden-soil.

Though it is certain, that the flora of Cyanophyceae in my tap- and canal-water experiments only develops with an extremely small proportion of organic matter in the food, I still consider this proportion to be of an essential signification for the experiment. I have already convinced myself that at as complete an absence as possible of organic substances, the development of the flora follows quite a different course, but I am as yet unable thereabout to impart any decisive results.

The experiment now described, is not quite new as to its principle. In another form it was already performed in 1892 by SCHLÖSING fils and LAURENT [1]), not however with a culture liquid, but with a solid sand-soil and under conditions much more complicated than mine. Noteworthy is that also these investigators, cultivating in the light under the exclusion of all compounds of nitrogen, obtained Cyanophyceae belonging to the same or almost the same genera as those resulting from my experiments. They have moreover come to the result that by these Cyanophyceae free nitrogen was assimilated in a slight but distinctly observable quantity, and though they have not completely proved this assertion, as their cultures must have contained other organisms too, e.g. many bacteria, basing also on my own experiences I take their view to be correct.

My experiment throws some light on the two following observations. GRAEBNER [2]) observed that fresh sandy grounds, which are changing into moors, cover in the beginning with a flora of Cyanophyceae; and TREUB [3]), when visiting the isle of Krakatau after its destruction, found that the new flora which first developed on the volcanic ashes, likewise consisted of Cyanophyceae, of which he in particular mentions *Lingbya verbeekiana* and *L. minutissima*. Both, the said heathsand and the ashes of Krakatau, have no doubt been extremely poor in nitrogen-compounds.

If absolutely rejecting the theory of spontaneous generation, it might be assumed that certain Cyanophyceae, carried over from the universe by meteorites, have been the first organisms which peopled the earth, as no other living beings are known which, like the Cyanophyceae, are able to build up their organic constituents from carbonic acid and atmospheric nitrogen.

Once acquainted with the culture conditions of the Cyanophyceae I could easily obtain pure cultures on a solid medium. I therefore used as well silica as agar which by long washing with tap-water had been freed from the soluble organic substances, but saturated with the constituents of the tap-water. Plates of this agar, to

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[1]) Fixation de l'Azote libre par les plantes. Ann. de l'Institut Pasteur T. 6 pag. 832, 1892. The authors make special mention of *Nostoc punctiforme*, *N. minutum* and *Cylindrospermum majus*.

[2]) Studien über die norddeutsche Heide. Bot. Jahrbücher. Bd. 20, 1891.

[3]) Notice sur la nouvelle flore de Krakatau. Ann. d. Jard. Bot. de Buitenzorg. T. 7, 1888.

which nothing else had been added but 0.02 pCt. $K^2 H P O^4$, and on which tap-water cultures of *Anabaena* had been sown out, were placed in the light of a window on the north, and after 10 to 14 days already produced extensive *Anabaena*-colonies free from bacteria. If the plates are not thoroughly washed *Anabaena* does not grow at all on them.

With plates prepared of silica instead of agar I obtained the same results.

The washing of the plates is effected by placing them, after solidification in the glass-box, into a large beaker with water, which is continually renewed during a few days by a current from the tap.

Then kalium phosphate is introduced into the plates by pouring over them a solution of this salt in distilled or tap-water, which solution is renewed a few times. Finally the superfluous water adhering to the plates is removed by heating the glass-box for a short time over a BUNSEN-flame.

Oscillaria and allied species do not grow on the thus prepared media, they even die on it already after some days. Mr. A. VAN DELDEN, however, has succeeded in my laboratory to obtain a pure culture on a solid medium of such a motile form related to *Oscillaria.*

This culture necessitated two other precautions. *First* the organic substance had to be removed from the agar more completely than is wanted for *Anabaena*, and therefore it proved necessary to wash with a current of distilled water. *Second*, the addition of a little of a nitrogen compound, e.g. a trace of ammonium-nitrate proved necessary, or at least favorable. On such agar the growth of the organism remains however very scanty, and, as besides many species of chlorophyceae can develop under these circumstances, we leave herewith the group of oligonitrophili, whose specific faculty consists in their being able to live on the nitrogen from the air, in opposition to the Diatoms and the Chlorophyceae. Hence this faculty seems also peculiar to a part only of the Cyanophyceae.

The question put at the head of this paper should thus be answered as follows.

In culture liquids, containing besides the mineral constituents of the food, a slight quantity of garden-soil, but to which no other nitrogen-compounds have been added, develop, under the influence of the light and the carbonic acid from the air, various species of *Cyanophyceae*, chiefly belonging to the genera *Nostoc* and *Anabaena*. Germs of these are very numerous in garden-soil. The presence of nitrogen-compounds prevents the development of the *Cyanophyceae*, but furthers that of certain *Chlorophyceae* and *Diatomaceae*.

Physiology. — J. W. LANGELAAN on: "*Further investigations on muscle-tone.*" (Abstract). (Communicated by Prof. T. PLACE).

On the assumption of a muscle-reflex arc on which the muscle-tone depends, it was proved in a previous series of experiments [1]), that there is a logarithmic relation between the value of the successive tonicity-quotients and the corresponding increments of the charge. The distensibility was taken as measure of the muscle-tone. From this the analytical expression was derived, which represents the increase in length of the not visibly contracting muscle as function of the increase of the charge.

For this relation was found:

$$l = Ap + Bp\, lgn.\ p,$$

and it was pointed out that this held good only for the interval of the increase of the charge, over which the experiment ranged.

These experiments made on frogs, were afterwards extended to cats, whose spinal cord was cut at a high level. The m. triceps surae was chosen to record distention curves, according to the method shortly described in my previous communication. The operations were done under ether narcosis, but as soon as the medulla was cut, no more narcotic was given. The severing was always followed by spinal shock, but within a few hours the muscles proved, at least partly, to regain their tonicity.

Also in this case the distention curve appeared to be represented by the same analytical expression, but the variation of charge during the experiment was far more considerable. As an illustration we subjoin four tables, representing the measurements of four curves.

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[1]) Proc. Kon. Akad. van Wetensch. Amsterdam, Sept. 29th 1900.

Experiment of 6/XII 1900.

TABLE I. 12th curve.

$A = 74.6 \times 10^{-5}$
$B = -2.6 \times 10^{-5}$

p	l. meas.	l. calc.
0.0 c_1	0.0 c_2	0.0 c_2
6.2	13	10.7
12.6	19	23
19.0	34.5	(34.5)
25.4	45.5	45.1
38.2	67	66.7
45.6	79	(79)

$P = 38$ $p = 127.6$
$c_1 = 2.80$ $c_2 = 10^{-3}$

TABLE II. 14th curve.

$A = 100.5 \times 10^{-5}$
$B = -7.8 \times 10^{-5}$

p	l. meas.	l. calc.
0.0 c_1	0.0 c_2	0.0 c_2
6.2	12.5	13.5
12.6	25	25.5
25.4	48	(48)
38.2	69.5	68
51.0	87.5	87.7
51.9	89	(89)

$P = 38$ $p = 143.8$
$c_1 = 2.77$ $c_2 = 10^{-3}$

Experiment of 4/XII 1900.

TABLE III. 7th curve.

$A = 104.5 \times 10^{-5}$
$B = -6.2 \times 10^{-5}$

p	l. meas.	l. calc.
0.0 c_1	0.0 c_2	0.0 c_2
6.2	14	16.6
12.6	29.5	31.9
19.0	46	46.7
25.4	61	(61)
38.2	92	88.7
46.2	105.5	(105.5)

$P = 38$ $p = 143.1$
$c_1 = 3.10$ $c_2 = 10^{-3}$

TABLE IV. 8th curve.

$A = 103.9 \times 10^{-5}$
$B = -6.3 \times 10^{-5}$

p	l. meas.	l. calc.
0.0 c_1	0.0 c_2	0.0 c_2
6.2	13	15.3
12.6	28	29.4
19.0	43	(43)
25.4	55.5	56.2
38.2	83	81.7
51.4	107	(107)

$P = 38$ $p = 147.3$
$c_1 = 2.87$ $c_2 = 10^{-3}$

The number of tracings I have measured amounts to 22 derived from three cats. It appears from these measurements, that within the given interval of the increase of the charge, the deviations between the measured values and those calculated by means of the given formula are not more considerable than was to be expected in connection with the accuracy of the method. But already in the tables given in my previous communication and also in the tables of these measurements it is clear, that the deviations of the beginning of the curve are most considerable and that they decrease rapidly with increasing charge. All these deviations fall in the same direction. It is therefore probable, that they are due to a definite cause, and in my opinion, it is to be found in the law of FECHNER, which according to its deduction is to be considered only as a "loi limite''. I hope to return to this point in a following series of experiments.

Then I directed my attention to the two constants A and B, which occur in the formula representing the distention curve.

It was proved by the experiments, that when a distention-curve was recorded during the period of shock or with intact medulla in deep narcosis, a straight line was found within a certain interval of time and increase of charge. In a case where the shock was very severe, the rectilinear part of the curve was found to cover an interval of the increase of charge not exceeding 130 grs. This increment of charge took place in 50 sec., the initial charge amounting to 38 grs. Other muscles gave other values which however deviated but little. If the experiment was extended over a greater interval of time and increase of charge, the elastic after-phenomena modified the shape of the curve in a considerable way.

As we know, that under shock and deep narcosis the efferent (motor) part of the muscle-reflex arc (e.g. from the pyramidal tract) remains irritable, while it appeared at the same time that when the shock was disappearing, the linear curve passed into the curve which we know as characteristic for the tonic muscle, we had an indication of the way in which to proceed with the analysis; for the given formula of the distention curve leads to a linear curve for $B = 0$. It is therefore obvious to consider the term $Bp\ lgn.\ p$ as the representation of the influence of the afferent (sensible) part of the muscle reflex arc on the distention curve. It is difficult to decide, whether the linear distention-curve $l = Ap$ is to be considered as typical for a muscle when only under influence of its efferent nerves, or whether this formula is only to be taken as a first approximation, implicitly implied in the law of FECHNER which served us as starting point. For even if in this case the initial linear part was followed by a parabolic distention-curve, as was found for frog-muscles, yet this part of the curve must be modified by the influence of the elastic after-phenomenon. As now this elastic after-phenomenon is not to be eliminated under the conditions of the experiment, the question which we have raised, must be left undecided for the present.

The curves were recorded in sets of three, while the time between the successive curves was noted down. The curves of a same set, which may be indicated as 1st, 2nd and 3rd, according to their succession in time, may be mutually compared; but what took place between the successive sets of three curves was not noted down. The values of A and B, as they are determined from the distention-curves follow in three tables.

TABLE V.

Experiment of 6/XII 1900.

		9.	12.	15.	18.	21.
Ist curves	$A \times 10^5$	87.3	74.6	84.8	69.9	62.5
	$B \times 10^5$	−5.1	−2.6	−5.5	−2.7	−2.7
		11.	14.	17.	20.	23.
IInd curves	$A \times 10^5$	97.5	100.5	84.6	78.1	84.1
	$B \times 10^5$	−6.8	−7.8	−5.8	−5.3	−6.1

TABLE VI.

Experiment of 4/XII 1900.

		7.	13.	19.	22.	
IInd curves	$A \times 10^5$	104.5	107	95.7	127.5	
	$B \times 10^5$	−6.2	−8.1	−6.0	−10.9	
		8.			23.	26.
IIIrd curves	$A \times 10^5$	103.9			120.1	118.8
	$B \times 10^5$	−6.3			−10.9	−8.9

TABLE VII.

Experiment of 11/XII 1900.			
		6.	9.
Ist curves	$A \times 10^5$	163.9	135.0
	$B \times 10^5$	-6.7	-7.9
			10.
IInd curves	$A \times 10^5$		191.0
	$B \times 10^5$		-19.6
		8.	
IIIrd curves	$A \times 10^5$	150.4	
	$B \times 10^5$	-9.9	

It appears from these values in connection with the time-notation, that if the interval of time between two successive distention-curves is shorter than three minutes, the constant B in a succeeding curve is considerably greater. With B the value of A increases. Among the values mentioned there are three cases, viz. Table V curves 15 and 17, Table VI curves 7 and 8, and curves 22 and 23, for which nearly identical constants are found, and in these three cases the interval between two successive curves was always longer than five minutes. In further experiments I hope to return to this last phenomenon, which if generally correct, seems very important to me.

If the conception of a muscle-reflex arc is correct and if it is also true, that B represents the coefficient added to the term, which is the analytical expression of the influence of the afferent part of this muscle-reflex arc on the shape of the distention-curve, then the increment of B caused by a shortly preceding distention, is only a special case of the physiological rule, which is generally less accurately formulated as follows: "shortly after every stimulation the nerve is more irritable." But then the increase of the coefficient A together with the increase of B is not accidental. From the two subjoined tables:

TABLE VIII.

Experiment of 13/XII 1900.

		4.	7.	10.	13.	16.	19.	25.
IInd curves	$A \times 10^5$	76.5	71.6	65.6	76.8	69.4	69.9	56.9
	$B \times 10^5$	0.0	0.0	0.0	0.0	0.0	0.0	0.0

TABLE IX.

Experiment of 18/XII 1900.

		4.	10.	13.	16.	19.	25.
Ist curves	$A \times 10^5$	79.6	76.2	72.5	79.8	79.1	64.1
	$B \times 10^5$	0.0	0.0	0.0	0.0	0.0	0.0

which represent the value of the coefficient A in two sets of experiments, where the cat had not recovered from the shock, it appears that in this case the coefficient A according to its value will join those cases where the coefficient B has the smallest value which can occur in connection with the accuracy of the method. The mutual differences of these coefficients A in the two tables are greater than was to be expected from the method, and the successive deviations are distributed over the series of experiments in such a way, that they can certainly not be brought under a simple rule.

In connection with the discontinuous tonus-tracings I have already pointed out these irregularities which appear after the muscle-reflex arc has been interrupted.

The distention-curves obtained from cats with intact spinal cord and under very slight narcosis are quite different from those which occurred when the medulla was cut at a high level. Ether was used as narcotic. The shape of these curves is in appearance perfectly accordant with the curves obtained by Mosso and Benedicenti for men, with their myotonometer. In this case the measurements gave for most curves increasing tonicity-quotients with increment of charge, within the interval over which the experiment ranged. Under some circumstances these curves show a point of inflection. The place of this inflection-point depends greatly upon the depth of the narcosis, and the extent of the increase of the charge. The tonicity-quotient, which is very low at first, rises quickly, then slowly with increase of charge, reaches its maximum at the inflection-point and begins then to decrease. For this last part of the curve, where decreasing tonicity-quotients correspond with increase of charge, the law of FECHNER proved by approximation, to connect the increase of the charge with its effect, the corresponding value of the tonicity quotient.

Starting from the well-known fact, that a high section of the medulla, after the shock is passed, gives rise to increase of muscle-tone and that therefore a physiological inhibition seems to disappear, the analysis of these curves was attempted in the following way.

The stimulus, originating by extension in the terminations of the afferent nerves of the muscle, ascends along afferent paths and arrived in the spinal cord, it is directly transferred along a short path to the efferent part of the primary muscle-reflex arc. About this we know that the law of FECHNER connects stimulus and its effect. But this same stimulus, ascending along the long path, which as secondary muscle-reflex arc is built upon the first, brings about a change which may be composed by addition with the change in the primary muscle-reflex arc. The supposition of the simple addition of the two influences on the muscle, has been stated very clearly, i. a. by SHERRINGTON. But if the law of FECHNER holds true for that part of the curve which lies beyond the inflection-point, and this part is therefore not different from those tracings, recorded when the stimulus can only pass the primary reflex arc, then the physiological meaning of the inflection-point is this: the inflection-point in the curve appears at the moment, when the influence of the secondary muscle-reflex arc on the primary disappears. But knowing also that in these tonus-curves, the tone is at first very low, that it reaches its maximum at the inflection-point, at which moment the influence of the secondary on the primary muscle-reflex arc ceases, we have also to assume, that the stimulus passing through this secondary muscle-reflex arc, brings about a change in it, the effect of which is seen as the disappearance of a tone-inhibition. We assume now as second supposition that here too the law of FECHNER brings relation between the effect and its cause, the stimulus.

The analysis is now possible in the following way. For a certain increase of charge the corresponding extent of the variation of tone is determined from the segment of the curve which lies beyond the inflection point. This is the basis for the further calculation as from this a series of successive tonicity-quotients is extrapolated according to the law of FECHNER. So this series of tonicity-quotients represents the probable value of the muscle-tone, corresponding to a group of successive increases of charge, if the stimulus had only passed through the primary muscle-reflex arc. Then the inflection point in the curve is determined as accurately as possible. At this point the inhibiting influence which is exerted by the secondary on the primary muscle-reflex arc is zero. Also the real value of the tonicity-quotient for a certain increase of the charge is known from the measurements of the curve. But if by means of the before-mentioned extrapolation the probable value of the tonicity-quotient can be calculated, exclusively depending on the primary muscle-reflex arc and by direct measurements the tonicity-quotient under simultaneous influence of the two muscle-reflex arcs can be determined, and if the premise of the simple addition of these quantities is correct, it is possible to determine by simple subtraction of these quantities the value of the inhibiting influence exerted by the secondary muscle-reflex arc. But if for two given charges the value of this influence is known, it is possible to interpolate a number of terms by means of the second premise. For this, terms were chosen corresponding to those for which the value of the tonicity-quotient was calculated, when the muscle , was only under the influence of its primary muscle-reflex arc. The difference between these tonicity-quotients must then be identical with the tonicity-quotient, determined directly from the curves. We shall elucidate this method by a scheme. Let the curve A B C represent the value of the tonicity-quotients as function of the increase of the charge. In B a maximum is reached and this point coincides with the inflection-point on the distention-curve. Accordingly the influence of the secondary muscle-reflex arc on the primary disappears at point B. Consequently F is the final point of the curve representing the extent of the tone-inhibition of the secondary muscle-reflex arc on the primary. The portion between B and C represents therefore the value of the tonicity-quotient as function of the increase of the charge, when the muscle is exclusively under the influence of its primary muscle-reflec arc. Extrapolating according to the law of FECHNER, BD is obtained, and curve DC becomes then the graphical representation of the value of the tonicity-quotient as function of the increase of charge, if the muscle had been only under the influence of its primary muscle-reflex arc all through the course of the experiment. OA is the value of the tonicity-quotient under influence of the

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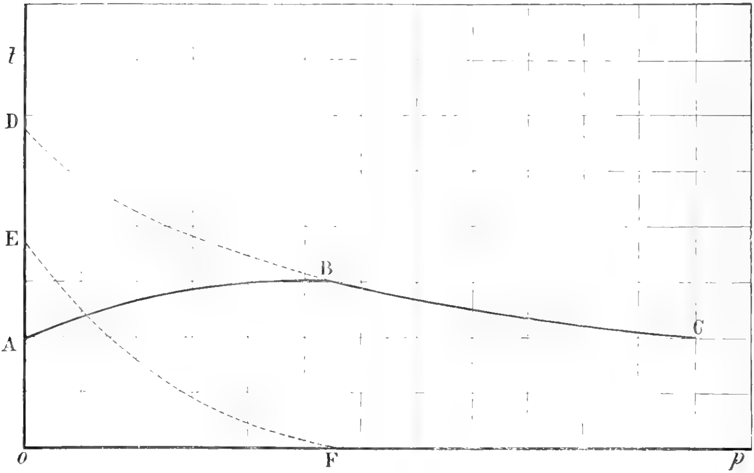
two muscle-reflex arcs, as it is really measured, but then OE = OD — OA is the value of the tone-inhibiting influence exerted by the secondary muscle-reflex arc. This determines point E and construing curve EF between the two points according to the law of FECHNER, this curve becomes the graphical representation of the tone-inhibiting influence exerted by the secondary muscle-reflex arc. Over the interval of the charge OF the difference of the ordinates determined by the two curves DB and EF must now at any moment be equal to the ordinates determined by the curve AB.



It appears clearly from this graphical representation, that there can only be question of a definite problem in these cases, where the curve shows an inflection point and that the possibility of this analysis depends on this.

The result of the measurement and of the calculation of the tonicity-quotients performed on this basis is given for four curves.

Experiment of 30/XI 1900.

TABLE X. 20th curve.

p	Tonicity-quotient meas.	Tonicity-quotient cal.
0.0 c_1		
3.0	3.1 c_3	(3.1 c_3)
6.2	3.7	3.8
12.6	4.5	(4.5)
25.4	3.5	(3.5)
51.0	?	2.5
62.4	2.7	2.3

$P = 38 \quad c_1 = 2.88$

$c_3 = 0.00035$

infl. point at 12.6.

Experiment of 30/XI 1900.

TABLE XI. 2nd curve.

p	Tonicity-quotient meas.	Tonicity-quotient cal.
0.0 c_1		
3.0	2.2 c_3	(2.2 c_3)
6.2	2.5	2.7
12.6	3.2	3.2
25.4	3.6	3.7
38.2	3.9	(3.9)
47.7	3	(3)

$P = 38 \quad c_1 = 2.92$

$c_3 = 0.00034$

infl. point at 38.2.

Experiment of 24/XI 1900.

TABLE XII. 35th curve.

p	Tonicity-quotient meas.	Tonicity-quotient cal.
0.0 c_1		
3.0	2.7 c_3	(2.7 c_3)
6.2	3.2	3.7
12.6	4.5	(4.5)
19.0	4.1	3.7
25.4	3.1	(3.1)
30.0	3	2.8

$P = 45 \quad c_1 = 2.58$

$c_3 = 0.00037$

infl. point at 12.6.

Experiment of 24/XI 1900.

TABLE XIII. 29th curve.

p	Tonicity-quotient meas.	Tonicity-quotient cal.
0.0 c_1		
3.0	4.9 c_3	(4.9 c_3)
6.2	6.7	6.5
12.6	6.6	(6.6)
19.0	5	5.4
25.4	4.6	(4.6)

$P = 45 \quad c_1 = 2.73$

$c_8 = 0.00035$

infl. point at 9.5.

In order to examine whether this analysis might be applicable to curves obtained for men by Mosso, I have measured one of them [1]).

TABLE XIV.

	Tonicity-quotient	
p	meas.	cal.
$0.0 \times 7.2\ c_1$		
6.25	$3\ c_3$	$(3\ c_3)$
12.5	5.5	5
21.9	6.5	(6.5)
25.0	3.5	(3.5)

$$P = 800 \quad p = 2200 \quad c_1 = \frac{2200}{800}$$

$$c_3 = \frac{180}{500 \times 2200}$$

infl point at 21.9.

It seems to me that we may conclude from the values given, that the analysis of the tonus-curves with intact medulla is possible with the aid of the two premises, or inversely we may conclude to the probability of the two premises from the concordance between the measured and the calculated values.

These experiments were made in the physiological laboratory of the medical school of Harvard University at Boston where I enjoyed the hospitality of Professor H. P. Bowditch.

Amsterdam, May 1901.

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[1]) Arch. Ital. de biologie 1896 T. XXV p. 356 fig. 3.

Mathematics. — On *"Special cases of* MONGE'S *Differential Equation"* by Prof. W. KAPTEYN.

When the differential equation of MONGE consists of three terms and has the form

$$s + \lambda t + \mu = 0$$

we formerly found all the cases in which this equation has two intermediate integrals. A further investigation of the equation

$$r - \lambda^2 t + \mu = 0$$

gives the following results.

I. When λ and μ are dependent only on p and q, this equation possesses two intermediate integrals only when

$$\lambda = \frac{a + 2bp + cp^2}{f + 2gq + hq^2} = \frac{P}{Q}$$

and

$$\mu = K\lambda(hQ - cP).$$

Here K represents an arbitrary constant, whilst the six constants a, b, c, f, g, h are bound only by the condition

$$b^2 - ac = g^2 - fh.$$

If we put $b^2 - ac = \alpha^2$, the two intermediate integrals are the following

$$w^{g-\alpha}(w-1)^{2\alpha} e^{2\alpha h K[(g-b)x-cz]} = f\left(w e^{2\alpha K(hx+cy)}\right),$$

where

$$w = \frac{b+cp-\alpha}{b+cp+\alpha}\;\frac{g+hq-\alpha}{g+hq+\alpha}$$

and where f represents an arbitrary function

$$w'^{-g-\alpha}(w'-1)^{2\alpha} e^{-2\alpha h K[(g+b)x+cz]} = f\left(w' e^{2\alpha K(hx-cy)}\right),$$

where

$$w' = \frac{b + c\,p - \alpha}{b + c\,p + \alpha} \quad \frac{g + h\,q + \alpha}{g + h\,q - \alpha}$$

f having the same meaning as above.

II. When λ and μ are only dependent on x, y, z, the equation possesses two intermediate integrals only when

$$\lambda = \frac{X'}{Y'}\,,$$

$$\mu = \frac{X'^2}{4}\left[\frac{2\,X'\,X''' - 3\,X''^2}{X'^4} - \frac{2\,Y'\,Y''' - 3\,Y''^2}{Y'^4}\right] z + \psi\,(x, y)\,,$$

where $\psi\,(x, y)$ represents an arbitrary function of x and y, whilst X denotes a function of x and Y a function of y, of which the derivatives are indicated by dashes.

In this case we find one of the intermediate integrals by eliminating y between

$$X + Y = \text{Const.} = C$$

and

$$\frac{dv}{dx} - \frac{3}{2}\,X'\left(\frac{X''}{X'^2} + \frac{Y''}{Y'^2}\right) v = -\,2\,\frac{X'}{Y'}\,\psi\,(x, y)\,,$$

where

$$v = \left(\frac{\partial\lambda}{\partial x} - \lambda\,\frac{\partial\lambda}{\partial y}\right) z + 2\,\lambda\,(p + \lambda\,q).$$

If we solve the integral of this linear differential equation according to the arbitrary constant C' and if we replace this constant by an arbitrary function of C, the intermediate integral under research will be found if moreover we substitute $X + Y$ for C.

The second intermediate integral is determined in a similar way out of

$$X - Y = C$$

and

$$\frac{dv'}{dx} - \frac{3}{2}\,X'\left(\frac{X''}{X'^2} - \frac{Y''}{Y'^2}\right) v' = -\,2\,\frac{X'}{Y'}\,\psi\,(x, y)\,,$$

where

$$v' = \left(\frac{\partial\lambda}{\partial x} + \lambda\,\frac{\partial\lambda}{\partial y}\right) z + 2\,\lambda\,(p - \lambda\,q).$$

Physics. — J. C. Schalkwijk: "*Precise Isothermals. II. Accuracy of the measurement of pressure by means of the open manometer of* Kamerlingh Onnes." (Communication N⁰. 70 from the Physical Laboratory at Leiden, by Prof. H. Kamerlingh Onnes).

§ 1. As the accuracy, which could be attained in the measurement of pressure, was of the greatest importance for attaining that high degree of accuracy which is required for my determinations of isothermals (Comp. Comm. N⁰. 67), I have made the measurements with the open manometer mentioned the subject of a separate investigation. Although there is little reason to doubt the exactness of the observation, after the necessary corrections described in Comm. N⁰. 44 have been applied yet some reason for doubt might exist in the fact that the correction for the depression of the mercury menisci, and also those resulting from the friction of the mercury in the capillary tubes are not sufficiently known. The former were borrowed from the tables of Mendelejeff and Gutkowsky [1]) and the latter were calculated by means of Poiseuille's law; especially the latter give rise to many difficulties, as I could calculate only the mean bore of the manometer capillaries, whereas in Poiseuille's [2]) equation the mean of the square of the bore is required. It is however a favourable circumstance that the variation of the level of the mercury in each of the manometer tubes hardly ever exceeds the amount of 1 c.m. an hour, and as in the narrowest of the capillaries this would give a difference of pressure of 0.02 c.m. per tube, with our degree of accuracy it may be neglected in comparison with a pressure of 304 c.m. per tube.

§ 2. The lengths of the tubular measuring rods (comp. Communication N⁰. 44) which are of chief importance for the measurement of the pressure, were determined in the following way: the

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[1]) As the reading of the level and the top of the meniscus would double the time of observation the depression is already accounted for in the reading, by estimating and adding for each tube the required correction to the height.

[2]) In order to test the use of the formula for tubes of the bore used (of 0.055 c.m. radius on an average), I have calculated from the velocity of the mercury through a similar tube the value of the co-efficient of friction η and found 0.00001725, differing only a little from the value 0.00001633 found by Warburg (0.01609 in the C.G.S. system).

first rod L_0, graduated over its whole length, and hanging between the limbs of the U-shaped manometer-tube A, was compared with the Standardmeter N^0. 1 of the Physical Laboratory of Leiden, which according to its correction is at 7° C. shorter by 0,005 mm. than the Mètre des Archives. From this we calculate its length at 0° C. 999.91 mm. The length between the marks 0 and 304 (corresponding to 4 atm.) of this measuring rod was measured in two ways and found to be 303.976 c.m. and 303 971 cm., which two values are in sufficient agreement. In order to determine the lengths of the other measuring rods $B_I - B_{VII}$ they were suspended together with the measured rod at short distances from each other, and at 425 c.m. distance from the theodolites to be used for the reading of the mercury levels, the same distance at which they are used. The readings were made with telescopes rotating round perfectly vertical axes; during rotation I could not observe on the very sensitive levels a larger variation than $^1/_4$ mark, corresponding to $^1/_{28}$ mm. on the measuring tube. In four measurements the mean difference of the readings was less than 0.1 mm.

The following corrections have to be applied in the measurements:

A. The correction for the depression, mentioned above.

B. The correction for the friction of the mercury, also discussed above.

C. The corrections for the inclination of the telescope of the theodolite.

But in the most unfavourable case, namely in the outer tube from 56—60 atm., this correction is only $^1/_{160}$ of the vertical distance between the mercury meniscus and the level of the theodolite, and as in the measurements this distance was hardly ever more than 3 c.m. the correction may be neglected, especially for the other tubes where it is much less.

D. The corrections arising from the temperature of the mercury.

The temperature is measured by eight thermometers distributed over different parts of the apparatus. If the mean temperature is t, tables can be calculated for this correction from the formula $\Delta = L\{1 - (a - k)\,t\}$, in which a is the co-efficient of cubic expansion of mercury and k the co-efficient of linear expansion of brass. (The measuring rods are of brass).

E. The corrections for the weight of the air which is compressed between the mercury of the successive manometer tubes. By means of the known values of the pressure belonging to each column, the specific weight of the air and the mean temperature, tables can also be calculated for these corrections.

F. The corrections of the thermometers necessary for the corrections mentioned in D and E. They were determined by suspending all the thermometers in a water-bath at a constant temperature, together with a standard thermometer of which the scale had been tested at the Physikalisch-Technische Reichsanstalt, and of which the zero error had been determined every two months in melting ice.

G. The correction for the compression of the mercury; this is $\Delta = {}^1/_2\, \beta\, P^2$, if the coefficient of compressibility is β. As according to AMAGAT β is only 0.00000392 for mercury if P is expressed in atmospheres, this correction need only be considered above 40 atm.

H. The correction for the lengthening of the measuring rods during the suspension.

The rod L_0 was measured in a horizontal position, whereas the rods are used, suspended at about ${}^1/_3$ of their height from the top by a Cardanus-collar. Hence the upper part will be compressed and the lower part will be lengthened; if l is the distance to the point of suspension we can easily calculate $\Delta = {}^1/_2 \dfrac{S}{E}\, l^2$, for the displacement of the lower end. For brass $S = 8.45$; $E = 1037{,}000{,}000$ while $l' =$ about 215 cm.; so we calculate $\Delta = 0.0002$ cm. which therefore may be entirely neglected.

§ 3. In order to judge of the influence of the uncertainty in the corrections A and B, the open manometer (comp. Communication n°. 50 § 2) was divided into two parts (see Plate Comm. n°. 44). For this tube A was entirely cut off by closing cock K_1 and by loosing the steel capillary of tube B_1 from the **T**-piece T_1, and the steel capillary of tube B_8 from the **T**-piece T_8, while this opening of T_8 was closed tightly by a nut with leather packing. If all the other cocks are open and pressure is very slowly admitted by means of cock X, we shall see the mercury rise in the capillary both in tube B_1 and B_8. If in B_1 the mercury has reached the mark X_1 the cock K_2 is closed, after which the mercury is forced up in B_2, etc. At the same time attention must be paid to the mercury in B_8 and the cock K_9 is closed when the mercury in B_8 has reached the required height, which now however will be higher than the mark X_8 of the figure and which must be determined experimentally beforehand. In this way we can continue until the mercury has reached the lower end of the upper reservoir in tube B_7 and therefore also in B_{14}; then the cock K_8 is closed while the cock X is left open until the mercury in tube B_{14} has reached the upper part of the reservoir; then also the cock X is closed [1]). In this way it was possible to raise the pressure in the second system of tubes more than $^1/_4$ atm. higher than in the first system. At the reopening of the cock K_8 the communication between the two systems was reestablished and thus while the pressure rose for the first system it fell for the second. The influence both of the friction and of the depressions was opposite for the two systems, for in the first system the menisci were rounded at the upper end and flat at the lower end, whereas in the second they were rounded at the lower and flat at the higher end. As during the reading which lasted each time 30 minutes we could expect a variation of the menisci the observations have each time been made symmetrically, and so first in system I at the higher end from the left to the right, then in system II at the higher end from the left to the right, then in system II at the lower end from the right to the left and lastly in system I at the lower end from the right to the left.

§ 4. *Results.* In order to give a survey of the value of the corrections to be applied and the calculation of the pressure this has been carried out for one measurement. For the further determinations which will be communicated in the table of comparisons (p. 28) all the pressures are similarly calculated [2]).

In order to obtain the uncorrected height at the mean time viz. at 4.36 we need only add to the sum of the lengths of the measuring rods (entered in the table for both system as $\Sigma^I L$ and $\Sigma^{II} L$), the sum of the upper readings (for both systems $\Sigma_1{}^I B$ and $\Sigma_1{}^{II} B$) taking into account that the measuring rod L serves as well for the manometer tube B_1 as for B_2 etc. and to subtract from this sum the sum of the lower readings (for both systems $\Sigma_2{}^I B$ and $\Sigma_2{}^{II} B$). In determining the mean temperature given as $T_1{}^I$ and $T_1{}^{II}$ etc. we must pay attention to the positions of the thermometers; they were arranged in the following way: t_1, t_2 and t_3 were suspended at the upper end, in the middle and at the lower end of tube B_1; t_4 and t_5 were suspended at $^1/_4$ and $^3/_4$ of the tube B_7 and t_6, t_7 and t_8 again

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[1]) In order to exclude unnecessary possibilities of leakage the differential manometer tube C was always loosened from the **T**-piece T_{16} and the crosspiece N, which were both again closed.

[2]) In this equation the height of the barometer can be left out of consideration, because it equally influences both systems.

Readings on June 28 1900.

Thermometer	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	T_1^{I} = 17°.68
Reading.	18°	17°.9	17°.7	17°.7	17° 75	17°.8	17°.7	17°.45	T_1^{II} = 17°.61
Corrected.	17°.87	17°.77	17°.52	17°.6	17°.67	17°.7	17°.62	17°.37	
time	4.26								
Manometer tube B. top left to right.	I.	II.	III.	IV.	V.	VI.	VII.		
	3.62	4.70	0.77	0.68	3.50	4.27	6.14		$\Sigma_1{}^{I}B$ = 23.68
Manometer tube B. top left to right.	VIII	IX.	X.	XI.	XII.	XIII.	XIV		
	4.38	5.66	5.58	6.81	7.27	6.84	5.83		$\Sigma_1{}^{II}B$ = 42.37
time	4.36								
Manometer tube B. bottom right to left.	VIII.	IX.	X.	XI.	XII.	XIII.	XIV.		
	5.87	8.44	9.73	9.75	9.25	8.14	10.17		$\Sigma_2{}^{II}B$ = 61.35
Manometer tube B. bottom right to left	I.	II.	III.	IV.	V.	VI.	VII.		
	7.18	7.49	4.32	4.59	9.97	4.83	3.80		$\Sigma_2{}^{I}B$ = 42.13
time	4.46								
Thermometer	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	T_2^{1} = 17°.63
Reading.	17°.9	17°.8	17°.8	17°.65	17°.7	17°.8	17°.7	17°.5	T_2^{II} = 17°.59
Corrected.	17°.77	17°.67	17°.62	17°.55	17°.62	17°.7	17°.62	17°.42	
Measuring rod L.	I.	II.	III.	IV.	V.	VI.	VII.		$\Sigma^{I}L$ = 2123.36
Length.	302.18	303.45^5	304.19^5	303.70	303.68^5	302.74^5	303.52^5		$\Sigma^{II}L$ = 2123.61

at the higher end, in the middle and at the lower end of tube B_{14}. Hence the mean temperature will be given pretty accurately for the first system by $^1/_8\,(t_1 + 2t_2 + t_3 + 2t_4 + 2t_5)$ and for the second system by $^1/_8\,(2t_4 + 2t_5 + t_6 + 2t_7 + t_8)$. The following table is obtained:

System.	ΣL	$\Sigma_1 B$	$\Sigma_2 B$	Height mercury.	Corr. leakage.	Mean temp.	Corr. for temp.	Corr. for weight air.	Corrected mercury height.
I.	2123.36	23.68	42.13	2104.91	0.00	17°.66	— 6.05	— 2.38	2096.48
II.	2123.61	42.37	61.35	2104.63	0.00	17°.60	— 6.02	— 2.38	2096.23

All the lengths are expressed in cms.

The heights of the mercury found on different days are combined in the following table, which also gives the difference of those calculated heights in the two systems.

Table of comparisons.

Date.	Time.	System I.	System II.	Difference.
28 June	4.36	2096.48 cm.	2096.23 cm.	+ 0.25 cm.
» »	5.14	2095.33 »	2095.45 »	— 0.12 »
29 »	3.50	2107.03 »	2107.15 »	— 0.12 »
» »	4.30	2106.13 »	2106.27 »	— 0.14 »
» »	5.07	2104.25 »	2104.41 »	— 0.16 »
30 »	2.25	2117.55 »	2118.01 »	— 0.46 »
» »	3.02	2117.42 »	2117.73 »	— 0.31 »
» »	3.42	2116.01 »	2116.23 »	— 0.22 〃
» »	4.13	2115.23 »	2115.51 »	— 0.28 »
» »	4.49	2113.81 »	2113.75 »	+ 0.06 »
5 July	4.00	2118.76 »	2118.53 »	+ 0.23 »
» »	4.25	2117.67 »	2117.96 »	— 0.29 »
» »	5.00	2116.59 »	2116.64 »	— 0.05 »

For the *mean difference* at one measurement we find therefore 0.24 c.m., this amounts to $\frac{1}{8800}$ of the pressure measured. But there appears to be *a systematic error* in the observations; for the reading in the second system is on an average 0.13 c.m. higher than in the first system. Perhaps it may be ascribed to the fact that all the tubes of the second system are much narrower than those of the first, so that if the height of the high menisci at the lower end is estimated only a little too low, the depression becomes too small and hence the mercury height read too large.

This systematic error amounts only to $\frac{1}{16000}$ of the pressure measured and moreover will only have any influence at pressures above 32 atm. (for then only the tubes of the second system are used). By reading the real height of the menisci perhaps even this slight deviation might be prevented.

Physics. — J. C. Schalkwijk: "*Precise Isothermals.* III. *A water-jacket of constant ordinary temperature.*" (Communication N⁰. 70, (continued) from the Physical Laboratory at Leiden, by Prof. H. Kamerlingh Onnes).

§ 1. *The necessity of having at disposal a current of water at a constant temperature.* In consideration of the small heat conductivity of gases and the great thickness of the walls of piezometer tubes, used in my experiments (comp. Communication N⁰. 50 of prof. H. Kamerlingh Onnes, June 24th 1899) it is desirable to take care that the temperature of the surrounding water cannot vary more than some hundredths of a degree per hour, and that it can be accurately adjusted at the desired value and be kept almost constant during 5 hours. This offers however many difficulties; for the bath must be more than 80 c. m. high, and we must be able to read accurately the marks over the whole length of the tube, which excludes coating the bath with a badly conducting substance as a protection. Hence there will be a continual large loss of heat from the surface of the bath. The distance of the piezometertube from the glass wall must be very small with a view to the refraction of the rays emerging from the water, as we must be able to read very accurately the difference in the height of the mercury in the piezometer tube C_1 (comp. the plate Proceedings June 24th 1899) and in the measuring glass P [1]); therefore 6 c. m. was taken as

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[1]) A means used by Dr. N. Quint (comp. his thesis for the doctorate 1900, p. 15 and fig. 5) cannot be used in our case. He placed the observation-tube in a rectangular basin containing about 40 liters of water and kept up a constant temperature by means of two liquid-resistances (saturated ammonium chloride) carrying an alternating current, an exterior krüppin resistance to be regulated by hand and rotating bladed wheel. Moreover in accurate determinations of isothermals, the measurements proper occupy each time about three hours and as therefore many readings have to be made, the regulation of the krüppin–resistance would cause an undesirable interruption of the series of observations.

diameter of the jacket. Hence as soon as the temperature is some degrees higher than the temperature of the room, — and such a difference is sure to occur in the course of the year, if we want to adjust at a definite temperature within the limits of the temperature of the room — the cooling will become considerable[1]).

Most of the thermostates used allow an accurate regulation only at a small loss of heat, moreover on account of the given dimensions of the bath and because the bottom cannot be heated (for instance by a gas flame), only electrical heating could be applied. In a narrow high jacket it is very difficult to apply heat by means of an electrically heated spiral wire without damaging the illumination of the observation tube, for that tube must remain visible over a length of 54 c. m. and strong local heating must be avoided as it would not be easy to distribute the heat equally by moderate stirring. Therefore ROTHE's method [2]), followed in the Physikalisch-Technische Reichsanstalt, could not be used. Thus there only remains heating by means of an alternating current, either the bath itself [3]), or separate heating tubes filled with a liquid [4]). But both methods are only fit for large baths, as the regulation of the alternating current[5]) with a sufficient degree of accuracy in the case of small baths with a rapid loss of heat offers difficulties.

So we are obliged to continually replace the slightly cooled water by new water at the exact temperature and therefore to connect the *observation bath* with a *heating bath*. CADY [6]) brings about a circulation by means of a rapidly rotating funnel, in the bath itself, of which means we could not avail ourselves owing to the narrow bore of the vessels. For heating microscope stages convection currents are used [7]) which however gives a much too weak and un-reliable circulation in this case.

For the arrangement used by me the adjustments at every tem-

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[1]) In order to avoid a possible error resulting from the uncertainty in the co-efficient of dilatation, all measurements for the isotherm at an ordinary temperature are made at the same temperature.

[2]) Ein Thermostat mit elektrischer Heizvorrichtung für Temperaturen bis 500°. Zeitschr. f. Instr. 1899.

[3]) DUANE and LORY (Am. Journ. of Sc. (4) 9. p. 179, 1900) use a solution of common salt of 160 liters and send through it an alternating current of 110 Volts.

[4]) Compare the footnote on QUINT's method.

[5]) Done automatically by DUANE and LORY, by QUINT with the hand.

[6]) CADY, Journ. of Phys. Chem. 2, pag. 242, 1898.

[7]) VAN RIJN, Mech. Zeitung 1899. FRESENIUS' Zeitschr. 99, p. 96.

perature [1]) above the highest occurring temperature of the water supply are made by admitting a regular current of water of constant temperature.

§ 2. *The reduction of the variation of temperature in flowing water.* The arrangement for obtaining flowing water at constant temperature by VAN ELDIK in Communication N°. 39 (Proceedings May 29th '97 p. 22 and fig. IV) has served as my starting point. The disadvantage of using large quantities of water does not matter much in my arrangements as the same water is used to move the stirring apparatus. Important modifications had to be introduced into VAN ELDIK's arrangement because it only reduced the variations in the temperature to 0.1 deg. C, and this is insufficient for my purpose (comp. § 1).

The most important of these modifications is the construction of a large well packed *mixing bath*, containing over 84 liters of water. The water streams into this from a smaller heating bath of $\frac{1}{6}$ the capacity and is mixed with the larger quantity and well stirred; if now in the heating bath the temperature is only allowed to vary within given limits, this variation is reduced in the mixing bath to one sixth of its value [2]). Moreover a more reliable thermoregulator was constructed to regulate the temperature of the heating bath.

The apparatus is shown on the annexed plate. The tube K conducts a little more water than is needed to the overflow A, the superfluous water flowing away by the tube L. From the funnel B the water runs to the heating bath D, to the mixing bath F and through the indiarubber tube G_1 to the observation bath H, whence it is conducted by the outlet-tube I. The difference of level amounted to 75 c. m., and 800 c.c. of water ran through per minute. The connecting tubes E and G_1 are protected from cooling by a coating of pure wool; the mixing vessel is surrounded by a second vessel and the space between the two is filled with pure wool. The cover of the inner vessel is provided with 4 openings M, N, O and P; thermometers are passed through corks in M and N, O and P are pro-

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[1]) For the investigation of the Isothermal of hydrogen at ordinary temperature I have chosen 20° C.

[2]) It would have been even better if the heating bath had been smaller and the mixing bath larger; but as apparatus of the description mentioned was at hand no change has been made in the ratio.

vided with loose glass covers; the whole is covered by a double layer of felt.

In order to be able to properly mix the water in the heating bath and in the mixing bath the stirrers Q and R have been made, each provided with six blades [1]). In order to prevent a general rotation of the water which would delay the perfect mixture, vertical baffle plates T have been fixed vertically to the inner wall of each vessel. At their upper ends the axes Q and R are provided with grooved discs U and V, connected by a string consisting of a spirally wound wire. The axis R also carries a disc W, connected by a similar string with the small disc X on the axis of the watermotor Y [2]). In order to prevent strain in the apparatus the axes are connected by strong iron bands Z; while the axis of the motor is supported by the tube a which also serves as oil-reservoir [3]). The water from the water supply streams through the motor to the overflow A.

Stirring is also necessary in order to obtain an equal temperature in the observing bath. For this purpose two brass rings e, are connected by means of three glass rods, and moved by a chord f over a pully (see Plate Comm. N⁰. 50). In order to assist the mixing by convection the water is admitted at g and runs off above at I.

§ 3. *The Xylene-thermo-regulator.* The only important thing left was to construct a suitable regulator for the heating bath. The known regulators of GOUY [4]), DOLEZALEK, GUMLICH [5]) and others, which work electromagnetically, seemed less suitable, since there the regulating flame is repeatedly extinguished and with a view to the large quantity of heat required, this cannot but give rise to too great variations of temperature. I thought it therefore better to return to liquid regulators, as with these small gradual variations could be

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[1]) The plane of these blades is bent at an angle of 45° as shown in the figure for a pair of blades S; they are bent so that in rotating alternate blades move the water upwards and downwards.

[2]) The motor consists of a box Y; at a distance of $\frac{1}{4}$ of the circumference two parallel tubes b and c are fixed. Above these tubes a horizontal plate is fastened to the axis, provided with vertical blades at angles of 45° with the radius, so that they are placed crossly before c and straight before b.

[3]) To facilitate taking the apparatus to pieces the axes Q and R have been cut through; and the parts are fixed to the connecting tube by means of pins d.

[4]) GOUY, Journ. de Phys. (3) 6, p. 479, 1897.

[5]) GUMLICH, Ueber einen Thermoregulator für ein weites Temperaturgebiet. Zeitschr. für Instr. 1898.

produced in the supply of gas. As E. BOSE [1]) has also demonstrated, the chief requirement for this is a large volume with an extensive surface and well conducting walls, while the liquid must have a large coefficient of expansion α, small coefficient of compressibility β and a small specific heat and must be forced up in a rather narrow tube. As an additional requirement might be mentioned the regulation of the exterior pressure of gas by means of a manostate, as used by SMITS [2]) or by TRAUBE and PINCUSSOHN [3]) but as for my apparatus the regulation of the gas supply was sufficient, I could neglect this.

The liquid used by me was xylene; BOSE advises the use of chloroform, for which $\frac{\beta}{\alpha} = 0.064$, specific heat $= 0.235$; xylene however is less dangerous, while $\frac{\beta}{\alpha} = 0.075$, and is in that respect almost as advantageous. The xylene is enclosed in a copper thin-walled spiral h, consisting of 3 layers each of 6 turns; the whole length is 12.5 meter, so that the total surface amounts to about 1950 c. m^2. and the volume to about 240 c. c.

For 1 degree increase of temperature the volume will increase by 0.235 c.c. and this causes in a tube of 2.5 m. m. bore, such as used by me, a displacement of 47 m. m. [4]). By means of the xylene a mercury column must be forced up, which will produce the regulation of the gas. As the mercury may not come into contact with the copper we have soldered on to the end of the spiral i (see the fig. to the left) a flanged tube j, provided with a screw thread and with a groove of 2 m. m. depth into which the glass tube k fits easily. The rest of the groove is filled with some cork rings. Round the glass tube, ground flat at the end, I have sealed a long flanged-tube l with a smooth rim and over this a loose nut m fits into the screw thread of the tube j. If the nut is screwed up the glass tube compresses the cork packing and the closure is perfectly tight, and the xylene does not wet the sealing-wax. Unto the glass tube k is sealed the reservoir n and to this a narrow bent glass tube and reservoir o while above this the narrow tube p of 2.5 m. m. bore projects beyond the heating bath and there joins the wide tube q.

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[1]) BOSE, Leistungsfähigkeit und Konstruktionsprinzipien von Präzisionsthermostaten mit selbstthätiger Regulierung. Mech.-Zeit. 1899.

[2]) SMITS, Manostat. Zeitschr. für phys. Chem. 33, p. 39, 1900.

[3]) TRAUBE und PINCUSSOHN, Ein einfacher Thermostat und Druckregulator. Mech. Zeit. 1897.

[4]) If the temperature remains constant displacement of 1 m. m. is brought about by a variation of 22 c. m. of mercury pressure.

At the lowest temperature that we can expect[1]) the mercury stands at the lower end of the reservoir *o* and at the upper end of the reservoir *n*; at the highest temperature[2]) the mercury must remain in the lower end of the reservoir *n*. It may be easily seen on the plate how the regulator controls the flame. The narrow glass tube *t* is drawn out at its lower end until the bore is less than 2 m. m., then it is ground flat and after this ground at a slant, so that we obtain a lengthened opening. This end of the tube *t* is placed at the narrow opening of the tube *p*; a slight increase of temperature is sufficient to diminish the supply of gas perceptibly but gradually.

In order to fill the regulator the tube *q* was closed at its higher end by means of an india-rubber stopper and carefully exhausted through the side-tube *r* ending in a point which was broken off under xylene. The india-rubber stopper is removed, the spiral is immersed in a bath of over 25° C. and mercury is poured into it. The xylene bubbles through the mercury and is removed; then the bath is slowly cooled to the desired temperature, in my case 20° C., taking care that always sufficient mercury remains; by means of a narrow glass tube as siphon, small quantities of mercury may be removed for adjustment to different temperatures.

This regulator would also be insufficient, if the regulating flame had to be used to heat all the water streaming through as in the case of VAN ELDIK; its only purpose however is to serve as a *regulating flame*, while a constant flame *v* must warm the water to a little below the temperature desired. In order to avoid heating of the mixing bath from the side it is protected by an asbestos plate *z*.

§ 4. *Use and results.* If we want to set the apparatus working, the clip *w* is kept closed, the stirring apparatus are put in motion and we begin to slowly heat the heating bath. Then boiling water must slowly be poured through the open window *O* into the mixing bath, until the desired temperature is reached. When this temperature is almost reached in the heating bath, the dip *w* is opened. First the constant flame is regulated so, that it can heat the water to almost the desired temperature and then it is somewhat diminished: this difference must be supplied by the regulating flame, which even while burning at its highest must be *much smaller* than the *constant flame* and may *never* be extinguished.

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[1]) In the Phys. Lab. at Leiden rooms, when necessary, can be heated night and day; the lowest temperature may therefore be kept higher than 0° C.

[2]) Supposed to be 25° C. the isotherm is determined at 20° C.

It is obvious that changes in the temperature of the room may have a considerable influence on the constancy of the temperature in the *observing bath*: although I could not notice in the heating bath a variation of 0.1 deg., it sometimes happened with a great change in the temperature of the room that the temperature in the observing bath changed some hundredths of a degree. Therefore it is desirable to keep the temperature of the room as constant as possible.

The results obtained with the apparatus described were quite satisfactory; although under some circumstances we could observe variations of almost 0.1 deg. C. in the heating bath, they were not perceptible in the observing bath.

Here follow some readings made during my observations. In the first place I give an instance of the adjustment for a case when sufficient care had not been taken in the regulation of the constant flame, so that the regulating flame was sometimes extinguished. This happened on July 7th.

7 July	time	2.29	2.47	3.09	3.24	3.44	3.59
	temp.	19°.75	19°.77^5	19°.81	19°.80	19°.80	19°.76^5

The accuracy of adjustment is insufficient for my purpose. But if the required precautions are taken, the temperature is accurately maintained to 0.01 deg. C. as appears from the following observations on two days:

25 Aug.	time	2.48	3.09	3.28	3.48	4.07	4.26
	temp.	19°.78^5	19°.78^5	19°.78^5	19°.78^5	19°.78^5	19°.78^5
27 Aug.	time	2.44	2.56	3.07	3.22	3.36	3.48
	temp.	19°.79	19°.78^5	19°.78^5	19°.78^5	19°.80	19°.79^5

Physics. — J. C. SCHALKWIJK: "*Precise Isothermals. IV. The calibration of piëzometertubes.*" (Communication N°. 70 (2nd continuation) from the Physical Laboratory at Leiden, by Prof. H. KAMERLINGH ONNES).

In this paper the method for the calibration of piëzometer tubes, mentioned in Communication N°. 50 is described more in detail [1]).

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[1]) AMAGAT mentions only that he calibrated carefully, but not how and to what degree of accuracy; a source of uncertainty in the calibrations of REGNAULT is mentioned and corrected by LEDUC, who also employed constant temperature and considered the volume of the meniscus in the calibration of his bulbs; BUNSEN calibrated by admitting small measured quantities of mercury.

The piëzometer tube which has been exhausted by means of the mercury pump, hand has been sealed off at the cock-piece fastened to the sealing capillary and filled in a sloping position by breaking off that point under mercury, is carefully centred and placed vertically in a reversed position in the water jacket (comp. Communication N^0. 67. § 1). The positions of the meniscus with regard to the divisions are read by a cathetometer. The marks are made on the side so that the middle of the groove can be read in the same way as in the experiments (comp. Communication N^0. 67, § 2). The distances between the marks are measured outside the bath. For the calibration we chose those marks at which (according to preliminary measurements) the adjustments would be made during the observations.

Mark.	I.	II.	Difference.
—3.2	57.666	57.662	—0.004
—2.8	57.270	57.271	+0.001
0.9	53.599	53.601	+0.002
13.0	41.582	41.585	+0.003
19.2	35.369	35.370	+0.001
19.7	34.868	34.868	0.000
29.1	25.550	25.550	0.000
29.4	25.253	25.255	+0.002
34.6		20.106	
42.1	12.729	12.728	—0.001
42.3	12.534	12.533	—0.001
48.8	6.130	6.131	+0.001
49.3	5.628	5.630	+0.002
52.3	2.665	2.665	0.000

The principal point is the accuracy with which the distances between the marks can be determined. The preceding table gives two independent series of readings for the tube *I A* (corrected for the temperature [1]) and for the reading of the level).

In order to read the mercury menisci very accurately a black

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[1] The lengths and all following measures have been given for 20° C.

paper was fastened behind the back window in the water-jacket a little above the meniscus. Here follows a reading on February 2nd 1900.

	Cathetometer-reading.	Position of level.	Corrected.
Mark 1.8 (U — tube)	94.217	+ 2.25	94.218 (2.35)
Meniscus (top.)	94.230	+ 2.2	94.231 (2.35)
Meniscus (level)	94.117	+ 2.35	94.117 (2.35)
Thermometer.	t_1	t_2	t_3
Reading.	8.11	8.08	8.04
Corrected.	8.04	8.03	8.00

After the reading the mercury is drawn off through the cock at the lower end; to prevent the cock from being heated too much by the hand in doing so, we used paper as a means of insulation; to take care that the mercury should break off each time in the same way, the surface of the mercury was just brought into contact with the point and then taken away again.

The mercury was weighed on a balance which for 1 m.g. moves over about 2 graduations and of which the ratio of the arms had been determined to within $^1/_{1,000,000}$; the corrections of the weights were certain to within $^1/_{100,000}$ by means of repeated determinations. The correction for weighing in the air has been applied.

The knowledge of the quantities of mercury drawn off each time is however by itself not sufficient to give the contents between the two menisci observed; for after each measurement the apparatus had to stand a long time while the water was being stirred, before we could expect that the mercury in the piezometer tube had assumed the temperature of the thermometers; in consequence the temperature sometimes showed important differences, especially in measurements made on different days 1).

And so for each reading we must apply the correction for temperature to the entire quantity of mercury from the cock to the

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1) In one instance even the temperature on February 16th was 7°.24 C. and on February 20th 9°78 C. Owing to the small coefficient of expansion of the mercury it was not necessary to use the apparatus for the regular current of water of constant temperature.

given mark; for this I have taken $\alpha = 0{,}0001814$; to determine the volume I took the specific gravity $s = 13{,}5953$ and to reduce the volume of the glass tube to 20° C. $\alpha = 0{,}0000277$, according to a determination with a similar reservoir after the method of the weight thermometer. The following table gives a determination of the volume from the cock to the different marks for one tube (*I A*); in this table *a*, *b*, and *c* refer to the calibration of the *U*-tube fastened to the large reservoir, *d* to the volume of the large reservoir, *m* to the volume of the small reservoir and *n* to the cock-piece sealed on to the tube before the calibration.

Measurement.	Drawn off mercury.	Mercury in the tube.	Mean temp.	Density mercury.	Volume mercury.	Volume of corresp. glass at 20° C.
a	$8{,}483^4$	2515,202	8.05	13,5755	185.275	185.333
b	$13{,}017^4$	2506,718	8.03	55	184.650	184.707
c	$17{,}108^2$	2493,701	8.07	54	183.692	183.749
d	1793,582	2476,593	7.65	65	182.418	182.477
	,	,				
e	$24{,}994^7$	683,011	7.78	61	50.310	50.326
f	$81{,}963^5$	658,016	7.99	56	48.470	48.485
g	$42{,}360^4$	576,053	8.59	41	42.437	42.450
h	$67{,}820^4$	533,692	8.62	41	39.317	39.328
i	$36{,}524^2$	465,872	8.68	39	34.321	34.331
j	$51{,}162^6$	429,348	8.71	39	31.630	31.640
k	$45{,}443^7$	378,185	8.79	37	27.862	27.870
l	$24{,}333^1$	332,741	8.17	51	24.511	24.518
m	254,968	308,408	8.70	39	22.721	22.727
n	53,440	53,440	8.57	42	3.937	3.938

In order to obtain the volume of the two menisci, we avail ourselves of the bore of the tube, as it is found approximately by equalising the menisci. This approximate bore is given in the following table. (For the distances between the marks compare the table referring to them on p. 36).

From the table in Communication N⁰. 67, § 9, we can easily derive another which gives the volume of the meniscus expressed by its height and the bore of the tube. And so we can put down

Measurement.	Mark.	Distance of the marks.	Menisc. level below mark.	Distance of the meniscus levels.	Approximate[1]) vol. between the levels.	Approximate mean bore.
a	0.5		0.155			
b	1.8	1.297	0.101	1.243	0.626	0.504
c	3.7	1.885	0.103	1.887	0.958	0.507
d	6.2	2.511	0.109	2.517	1.272	0.506
e	—2.8		0.200			
f	0.9	3.671	0.154	3.625	1.841	0.508
g	13.0	12.020	0.138	12.004	6.035	0.503
h	19.2	6.216	0.141	6.219	3.122	0.502
i	29.1	9.823	0.207	9.889	4.997	0.505
j	34.6	5.446	0.115	5.354	2.691	0.503
k	42.1	7.380	0.151	7.416	3.770	0.508
l	48.8	6.600	0.127	6.576	3.352	0.510
m	52.3	3.467	0.187	3.527	1.791	0.508
n	(1,6)					

the volume of the tube as far as the level of the meniscus and derive from this a more accurate mean bore, which can serve for the volume between the meniscus level and the mark near it.

Now we can determine the exact volume of the tube after it is sealed off, from the point of the capillary. For this we give the volume up to a given place at the beginning of the capillary; these numbers have a constant value; in order to derive from them the total volume, we must each time add the volume of the closing capillary, which becomes smaller by being opened and sealed again when it is necessary to clean and fill with another gas.

This capillary is too short and too narrow to be calibrated in the way described. The cock-piece was placed in a large basin filled with water, so that only the fine capillary projected beyond it and after long stirring the temperature and the height of the mercury were read. Then the temperature was raised a few degrees and the same thing was repeated. These are the results for the tube *I A* mentioned above, at two different measurements, for which different cock-pieces were used.

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[1]) This column gives the differences of the column “volume of corresp.glass at 20° C.”

Measurement.	c.c. Mercury	Increase temp.	Expansion.	Length	Bore.
n	3.937	3°.07	0.001885	1.55	0.0012^2
o	3.958	2°.71	0.001674	1.40	0.0012

As during the calibration discussed, the mercury was standing in the capillary at 1.6 c.m. distance from the reservoir, we must subtract from all volumes found $3.938 + 1.6 \times 0.0012^1 = 3.940$ c.c. Then we find:

Measurement.	Height meniscus.	Volume meniscus.	Exact Volume to level.	Difference.	Bore.	Volume from level to mark.	Volume of tube from capillary to mark.
a	0.101	0.030	185.303			0.079	181.442
b	0.114	0.035	184.672	0.631	0.5075	0.051	180.783
c	0.109	0.033	183.716	0.956	0.5067	0.052	179.828
d	0.100	0.029	182.447	1.269	0.5041	0.055	178.562
e	0.143	0.047	50.279			0.101	46.440
f	0.105	0.031	48.454	1.824	0.5033	0.078	44.592
g	0.120	0.037	42.413	6.041	0.5032	0.069	38.542
h	0.116	0.035	39.293	3.120	0.5017	0.071	35.424
i	0.115	0.035	34.296	4.997	0.5053	0.104	30.460
j	0.130	0.041	31.598	2.698	0.5039	0.058	27.716
k	0.092	0.027	27.843	3.755	0.5064	0.077	23.980
l	0.087	0.025	24.493	3.350	0.5094	0.065	20.618
m	0.108	0.033	22.695	1.798	0.5100	0.095	18.850
n			3.938				

Correction for the closing capillary 0.002.

For each tube a calibration curve can be drawn, belonging to one or more calibrations. In order to keep always the same accuracy of representation, I have drawn as abscissa a length and as ordinate a mean bore, namely those which we should get if the reservoirs were replaced by tubes of the same mean bore as the tube considered; the "reduced" length drawn is then the distance from the division to the end of that imaginary tube and the mean "bore" is the volume divided by the reduced length. The best way to judge of the accuracy attained is by comparing the two calibrations I and II of tube *I A*:

Measurement.	Mark	Distance marks.	Reduced length.	Volume to the capillary.		Mean bore.	Mean bore	Mean bore from curve.	Difference.	
				I.	II.	I.	II.		I.	II.
	0.3		358.422		181.510		0.50650	0.50650	0.00000	0.00000
a	0.5	0.189	358.233	181.442		0.50649		0.50649	0	
b	1.8	1.297	356.936	180.783		0.50649		0.50649	0	
c	3.7	1.884	355.052	179.828		0.50649		0.50649	0	
	4.0	0.299	354.753		179.682		0.50650	0.50649		— 1
	5.9	1.908	352.845		178.719		0.50651	0.50650		— 1
d	6.2	0.303	352.542	178.562		0.50650		0.50650	0	
	—3.2		92.304		46.644		0.50533	0.50530		— 3
e	—2.8	0.395	91.909	46.440		0.50528		0.50531	+ 3	
f	0.9	3.670	88.239	44.592	44.596	0.50535	0.50540	0.50537	+ 2	— 3
g	13.0	12.016	76.223	38.542		0.50564		0.50567	+ 3	
h	19.2	6.214	70.009	35.424		0.50599		0.50605	+ 6	
	19.7	0.501	69.508		35.179		0.50612	0.50607		— 5
i	29.1	9.319	60.189	30.460		0.50607		0.50612	+ 5	
	29.4	0.296	59.893		30.317		0.50618	0.50613		— 5
j	34.6	5.150	54.743	27.716		0.50630		0.50630	0	
k	42.1	7.378	47.365	23.980		0.50627		0.50623	— 4	
	42.3	0.195	47.170		23.877		0.50619	0.50622		+ 3
l	48.8	6.402	40.768	20.618		0.50575		0.50576	+ 1	
	49.3	0.502	40.266		20.363		0.50572	0.50570		— 2
m	52·3	2.964	37.302	18.850		0.50533		0.50533	0	

The mean deviation for the total volume employed in the normal volume amounts to $\frac{1}{350,000}$; for the stem of the manometer tube to $\frac{1}{17000}$; this satisfactory result must be ascribed to the calculation of the volume of the meniscus and the use of the waterbath. (See proceedings June 24 1899, Comm. N⁰. 50, plate II, fig. 4).

Chemistry. — Dr. A. SMITS and L. K. WOLFF: "*On the repelling of the ionisation of solutions of $Na\,OH$, $Na_2\,CO_3$ and $Na\,HCO_3$ by addition of $Na\,Cl$.*" (Communicated by Prof. H. W. BAKHUIS ROOZEBOOM).

In an article entitled: „Globulin als Alkali-eiweis-verbindung"[1]) STARKE has developed a theory based on some chemical phenomena noticed by him, which are so utterly in contradiction with the teachings of physical chemistry, that we found it very desirable to repeat his experiments and trace the cause of these singular results.

1. STARKE found, that, when he added 1 c.c. of a 0.15 percent solution of $Na\,OH$ to 25 c.c. of a 15 per cent solution of $Na\,Cl$ and the same amount to 25 c.c. of distilled water, mixtures were obtained of a different degree of alkalinity, the $Na\,Cl$ solution having always the strongest alkaline reaction.

These experiments were repeated using instead of $NaCl$, solutions of $Na_2\,CO_3$ and $Na\,HCO_3$; the result, however, was exactly the same.

From this, the conclusion was drawn that, contrary to theory, the alkaline reaction of a solution of $Na\,OH$ perceptibly increases instead of decreasing on adding a salt with a homonymous ion.

Another phenomenon observed by STARKE appeared to confirm this.

2. He found namely that the solubility of globulin in a solution of $Na\,OH$ is increased by addition of $Na\,Cl$. Addition of $Na\,Cl$, therefore, appeared to have the same effect as addition of more $Na\,OH$.

3. Another experiment which strengthened him in the conviction that theory utterly deserted us here was the following: Whilst convincing himself that $Na\,Cl$ may be precipitated from its concentrated solutions by adding a strong solution of $H\,Cl$, he did not find it possible to cause a precipitation by adding a strong solution of $Na\,OH$.

4. On repeating the experiments (1) we found that on using water, which is not free from carbon dioxide, we noticed the same thing as observed by STARKE. Before proceeding further we may state that STARKE in his experiments always used red litmus paper to judge which solution was the most alkaline, whereas we have

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[1]) Zeitschr. f Biologie B. XXII, S. 419 (1900).

used the colorimetric method with litmus and phenolphthalein as indicators.

As already stated, we got by a more accurate method the same surprising result as STARKE. When, however, we used instead of water containing carbon dioxide water, which was absolutely free from the same (obtained by passing air free from carbon dioxide for some hours through distilled water) we obtained just the *opposite* result and the Na OH solution always had a stronger alkaline reaction than the Na OH—Na Cl solution, which agrees with the theory which requires, that the ionisation of the Na OH solution be repelled by addition of Na Cl.

5. The explanation of the result obtained by STARKE is now as follows:

On dissolving Na Cl in distilled water a portion of the dissolved carbon dioxide is expelled, because the solubility of carbon dioxide in water is greater than its solubility in a solution of Na Cl. Distilled water has, therefore, a stronger acid reaction than a solution of *neutral* Na Cl [1]) in this same water and this stronger acid reaction of the water is so predominant that in STARKE's experiment it exceeded the decrease of the ionisation caused by addition of Na Cl and thus caused that perplexing result.

6. After obtaining the above result, we have tried to estimate the CO_2 in water and solution of salt colorimetrically by adding baryta water. Here we observed the peculiar phenomenon that, while the water became distinctly opalescent, this did not happen with the 15 percent solution of Na Cl. Another experiment where we started from 100 cc. of water containing CO_2 to which was added 100 cc. of a 15 percent solution of Na Cl and then baryta water gave the same result; no opalescence was observed and it was not until the solution had been diluted many times with water containing CO_2, that barium hydroxide caused a turbidity. A colorimetric estimation was, therefore, not possible in this manner.

The explanation of this phenomenon must be found in the reversibility of the reaction

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[1]) It is plain that, if we want to find out the reaction of a salt, this should be dissolved in water absolutely free from carbon dioxide. We experienced the fact that Na Cl, which was thought to be neutral on account of its neutral reaction in distilled water showed a decided alkaline reaction when dissolved in water free from carbon dioxide.

$$BaCO_3 + 2\,NaCl \rightleftarrows BaCl_2 + Na_2CO_3$$

which in our case, where a small quantity of $BaCO_3$ could react with a very large excess of NaCl, practically took place only from the left to the right.

7. Being thus obliged to abandon this method, we have taken another course.

A current of air free from carbon dioxide was first passed through a washbottle containing 300 cc. of distilled water and then through a little washbottle filled with baryta water; the current of air being thus again freed from carbon dioxide was passed through a washbottle containing 300 cc. of a 15 percent solution of NaCl and then finally through a small washbottle containing baryta water. If now a piece of black paper was put underneath the bottles containing the baryta it was plainly noticed, that, after 5 minutes the first bottle contained already more $BaCO_3$ than the second one, which difference was maintained even after passing the gas for several hours, thus freeing the water and the solution completely from carbon dioxide.

In this manner we had succeeded after all in demonstrating that the 15 percent solution of NaCl contained less dissolved carbon dioxide than the distilled water from which it had been prepared. On dissolving NaCl in water some of the CO_2 must, therefore, have been expelled.

8. As regards experiment 3 we may state that STARKE has made another mistake. NaCl may be precipitated from a concentrated solution by addition of either HCl or NaOH. The solution of the latter if used for this purpose should, however, be much more concentrated than the HCl solution. There exists therefore, only a quantitative difference which may be explained by the difference in the degree of dissociation of equally concentrated solutions of HCl and NaOH.

From the foregoing we see how, in this subtile question, STARKE has wandered from the right path through not sufficiently criticising his own results, and has arrived at totally wrong conclusions. His theory being based on erroneous principles must, therefore, be entirely rejected.

Amsterdam, Chem. Lab. University, May 1901.

Physiology of Plants. — Professor BEIJERINCK presents a paper on: *„Photobacteria as a Reactive in the Investigation of the Chlorophyll-function."*

If in a mortar leaves of some neutrally reacting plant, e. g. of white clover are crushed, diluted with destilled water, and filtered, a green filtrate is obtained, in which are found that portion of the living protoplasm which is soluble in water, and many chlorophyll-granules which give the filtrate a green colour.

If this green liquid is mixed with a culture of phosphorescent bacteria in fish-broth with 3 pCt common salt, or with sea-water[1]) rendered phosphorescent by a „luminous bouillon", and if this mixture is filled into a test-tube or stoppered bottle, the liquid becomes dark as soon as the oxygen has been used by the physiological processes of the phosphorescent bacteria and of the living protoplasm of the clover-leaves in the filtrate.

If the dark liquid is exposed to light, it is evident that the chlorophyll and the living protoplasm have not become inactive by the said treatment, for, by production of oxygen, they again cause the luminosity of the bacteria. If the plant-juice is fresh and the bottle is placed for a minute or longer in the full sun, then so much oxygen is formed, that the bacteria, transferred to the dark can continue phosphorescing for some minutes.

This experiment is of an extreme sensibility, for even the lighting of a match is sufficient, after part of a second already, to produce a distinct phosphorescence which, of course, can only be observed when by remaining long enough in the dark, the eye has become sensible to feeble light.

If 'the liquid is left to stand for some hours, either as such or after mixing with the phosphorescent culture, the power of decomposing carbonic acid gets quite lost. Evidently the presence ot living protoplasm is necessary for it. Concequently, FRIEDEL's[2]) experiment, wherein clear, filtered juice of squeezed spinage-leaves, mixed with powdered leaves of the plant, dried at 100° C., causes decomposition of carbonic acid, does not prove, as FRIEDEL thinks, that the function of chlorophyll reposes on the action of enzymes, but on the fact, that the portion of the protoplasm concerned in the

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[1]) By "sea-water" is meant tap-water with 3 pCt. Cl Na.

[2]) J. FRIEDEL, l'Assimilation chlorophyllienne réalisée en dehors de l'organisme vivant. Comptes rendus T. 132, pag. 1138, 6 Mai 1901.

process, occurs in the liquid state and is not solid, — hence, a new argument for the more and more prevailing opinion, that the living protoplasm is, if not quite, at least partly liquid. That the juice can be precipitated with alcohol, without the precipitate becoming inactive, proves nothing for the enzyme-hypothesis, as in many other cases the living protoplasm is proof against the action of alcohol.

If it be thought desirable to use the name of „protoplasm" only for the mixture of the living matter such as it occurs in the cell, and to connect with that term the idea of a special structure, I can quite well share this view, and will allow that, in this case, the decomposition of the carbonic acid is brought about by something else but "the protoplasm", namely by a portion of it. To this portion, or rather, to this particular constituent of the protoplasm, the name of "oxy-biophores" or "oxy-pangens" might be given, in accordance with the theory of biophores or pangenesis. With what has always been understood by enzymes, the properties of the biophores do not coincide but, of course, they do with those of the protoplasm itself [1]).

With crushed algae I could also perform the above experiment, but the secretion of oxygen was much slighter than with the sap of the examined land-plants.

On the other hand, algae which have not been crushed, whether enclosed in a mixture of culture-gelatin and luminous bacteria, or simply in sea-water rendered luminous by phosphorescent bouillon, are very well fit to study the secretion of oxygen in the light and its relation to the colour of the light.

Some years hence, Prof. KAMERLINGH ONNES, at Leiden, had the kindness to enable me to make an investigation thereabout in his laboratory. Our experiment was conducted as follows.

Between two glass-plates was enclosed fish-bouillon-gelatin diluted with sea-water, and thus containing 3 pCt. Cl Na, which by a great number of phosphorescent bacteria (*Photobacter phosphorescens*), mixed with it, was highly luminous at sufficient access of oxygen. In the middle of the gelatin I had placed, before the solidification, a broad stripe of a sea-*Ulva*.

In the dark the gelatin quickly loses its luminosity, the glass-plates rendering access of air impossible. When exposed to the light, the *Ulva* produces oxygen through the decomposition of

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[1]) This observation holds also good with regard to BUCHNER's „alcohol-enzyme", of which the active agent consists in „alcohol-biophores"

carbonic acid, and a local spot of light appears, which may be caused to come and to vanish at will as often as desired.

This apparatus was set up in a simple camera and could be locally illumined by withdrawing a slide. When the slide was closed the camera was quite dark, by which the eye of the observer became sensible to the light. Prof. ONNES himself supplied spectral colours of known refrangebility, taken from the spectrum of an electric arc-light, and projected them on the *Ulva* in the gelatin. By me was then observed what coloured lights were well, and what were not able to cause the decomposition of carbonic acid. The result was the following: Only red light decomposes carbonic acid, for only in it the phosphorescent bacteria emit a strong light; the maximum of decomposition was found near the chief absorption-band of the chlorophyll-pigment, situated between B and C, and this maximum coincides about with C itself, certainly it was somewhat out of the middle of B—C. Decomposition of carbonic acid in other coloured lights could not be detected.

If the Chlorophycee was replaced by a Rhodophycee, which I determined as *Porphyra vulgaris*, and which, like the *Ulva*, is common on the stone piers at Scheveningen, the process was nearly the same, but with this difference that the maximum of decomposition does not coincide with C but lies quite in the orange.

As the chromatophores of *Porphyra*, besides the chlorophyll-pigment, contain a red pigment soluble in water, and of which two chief absorption-bands are situated in the yellow, it is obvious that the maximum of carbonic-acid decomposition is in this case determined by the co operation of the coloured rays which both pigments by preference absorb.

Our results, accordingly, correspond in the main point with those obtained by Prof. ENGELMANN [1]), by his method based on the motion of bacteria, with this difference that the production of oxygen in two other absorption-bands, situated in the blue, as described by him, could not be observed by us.

In opposition to the sea-algae and likewise to the crushed leaves of landplants, whole leaves of the latter, immersed in luminous fish-bouillon, or in gelatin mixed with phosphorescent bacteria, do not distinctly, or only for a very short time, produce oxygen, when they are illumined after being freed from the air enclosed in their tissues.

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[1]) Botanische Zeitung. 1883 pag. 1, 1884 pag. 81.

In the following way, however, the experiments with them went very satisfactorily.

Instead of enclosing the leaf *in* the strongly phosphorescent gelatin it is simply laid *on* the surface, and firmly pressed to it by means of a solid glass-plate.

Kept in the dark, after some time all the oxygen originally enclosed in the tissues of the leaf is utilised by the phosphorescent bacteria and everything under the glass-plate grows dark. If now the leaf is illumined, oxygen is formed, and when transferred to the dark, the bacteria will be seen to continue emitting light for some time [1]).

These experiments confirm the results obtained by STAHL [2]), which demonstrate that the stomata are the ways by which the gases enter and leave the leaf. For when suitable leaves are selected with about an equal number of stomata on both surfaces, and examined after our method, it appears to be all the same, whether the leaf is pressed with its under or upper side against the gelatin, in both cases a luminous spot of the shape of the leaf appears, after illumination. If, on the other hand, the stomata are only, or for the greater part, at the under surface, and the leaf is pressed with its *upper* surface on the gelatin, thus with its underside against the glass-plate, then the oxygen accumulates between the latter and the leaf, and does not, or only partly pass through the lamina but, reaching the gelatin along the margin of the leaf, a luminous *line* following this margin is produced.

If such a leaf is illumined after being pressed with its *under* surface on the gelatin, the oxygen issuing from the stomata directly comes into contact with the gelatin, and a luminous *spot* appears shaped like the leaf.

In performing this experiment it is advisable to cut one and the same leave into two halves and press at once both parts on the gelatin, one with the upper- the other with the underside.

The process can, however, become very complicated by the closing of the stomata, which are extremely sensitive to the contact of the salt-containing culture-gelatin, and evidently also to the absence of oxygen in their surrounding, when kept in the dark.

The fact that nyctitropic leaves evaporate the most vigorously

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[1]) For the right performance of this experiment some practice is required as the layers of air, adhering to the leaves, and which are greatly different at the upper and under surface, influence largely on the course of the process.

[2]) Botan. Zeitung. 1894 pag. 117, 1897 pag. 71.

at that side, which is covered during the night, has been confirmed by the photobacteria-method respecting the secretion of oxygen. So, the clover-leaf closes at night by putting the upper surfaces of the leaflets against one another: hence these surfaces must exhibit a more energetic secretion of water-vapour, and in the light, of oxygen, then the under surfaces, which has been confirmed by the experiment.

For *Robinia pseud-acacia*, where at night the under surfaces cover each other, the most vigorous secretion of oxygen is to be expected in the light from the underside, which is likewise confirmed by the experiment. But with *Robinia* the difference is less considerable than with clover.

Physics. — Mr. Fred. Schuh on: "*Plane waves of light in an homogeneous, electrically and magnetically anisotropic dielectric.*" (First part).

1. If P, Q and R are the components of the electric force $\mathfrak{E}$, and f, g and h the components of the electric induction $\mathfrak{D}$ (4π times the dielectric displacement) both expressed in electric units, then we have, in the case of an electrically anisotropic medium, the relations:

$$P = k_{xx} f + k_{xy} g + k_{xz} h \ , \quad \ldots \ldots \quad (1)$$

$$Q = k_{yx} f + k_{yy} g + k_{yz} h \ , \quad \ldots \ldots \quad (2)$$

$$R = k_{zx} f + k_{zy} g + k_{zz} h \ , \quad \ldots \ldots \quad (3)$$

in which $k_{xy} = k_{yx}$, $k_{xz} = k_{zx}$, $k_{yz} = k_{zy}$. The electric (potential) energy per unit volume is:

$$U_e = \frac{1}{8\pi} (P f + Q g + R h) \ . \quad \ldots \ldots \quad (4)$$

The surface:

$$k_{xx} x^2 + k_{yy} y^2 + k_{zz} z^2 + 2\, k_{yz}\, yz + 2\, k_{xz}\, xz + 2\, k_{xy}\, xy = 1$$

is an ellipsoid (with O as centre), because $U_e > 0$, if f, g and h are not all zero. I shall call this the electric ellipsoid. If $\mathfrak{D}$ is a radius vector of this ellipsoid, then $U_e = \frac{1}{8\pi}$; $\mathfrak{E}$ is then normal to the diametral plane conjugate to $\mathfrak{D}$, which we shall call the electrically conjugate diametral plane. The radius vector r_e of this ellipsoid has the following meaning:

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$$\frac{1}{r^2_{e_\mathfrak{D}}} = \frac{\mathfrak{E} \cos (\mathfrak{D}\,\mathfrak{E})}{\mathfrak{D}}, \quad \ldots\ldots\ldots \quad (5)$$

if $\mathfrak{D}$ is in the direction of this radius vector, as is indicated by the index $\mathfrak{D}$ of r_e. If an electric induction in the direction I does not require a component of electric force in the direction II, then I and II are conjugate diameters of the electric ellipsoid, or briefly electrically conjugate diameters; we may also say that an electric induction in the direction II does not require a component of electric force in the direction I.

2. If we solve f, g and h from the equations (1), (2) and (3), we find:

$$f = k'_{xx} P + k'_{xy} Q + k'_{xz} R, \quad \ldots\ldots\ldots \quad (6)$$

$$g = k'_{yx} P + k'_{yy} Q + k'_{yz} R, \quad \ldots\ldots\ldots \quad (7)$$

$$h = k'_{zx} P + k'_{zy} Q + k'_{zz} R, \quad \ldots\ldots\ldots \quad (8)$$

where $k'_{xy} = k'_{yx}$, $k'_{zx} = k'_{xz}$, $k'_{yz} = k'_{zy}$. I shall call the surface:

$$k'_{xx} x^2 + k'_{yy} y^2 + k'_{zz} z^2 + 2\, k'_{yz}\, yz + 2\, k'_{zx}\, zx + 2\, k'_{xy}\, xy = 1$$

the reciprocal electric ellipsoid. Its axes have the same direction as those of the electric ellipsoid, but the axes of the one are the reciprocal values of the axes of the other. If $\mathfrak{E}$ is a radius vector of the reciprocal electric ellipsoid, then $U_e = \frac{1}{8\pi}$, and $\mathfrak{D}$ is normal to the diametral plane that is conjugate to the direction of $\mathfrak{E}$.

The radius vector r'_e of the reciprocal electric ellipsoid is:

$$\frac{1}{r'^2_{e_\mathfrak{E}}} = \frac{\mathfrak{D} \cos (\mathfrak{D}\,\mathfrak{E})}{\mathfrak{E}}, \quad \ldots\ldots\ldots \quad (9)$$

if $\mathfrak{E}$ falls in the direction of that radius vector. If the directions I′ and II′ are conjugate diameters of the reciprocal electric ellipsoid, then an electric force in one direction does not produce an electric induction in the other direction.

3. We take the medium as not permanently magnetizable. If α, β, γ are the components of the magnetic force $\mathfrak{H}$, and a, b, c those

of the magnetic induction $\mathfrak{B}$, both expressed in magnetic units, then for a medium that is magnetically anisotropic:

$$\alpha = \mu_{xx}\, a + \mu_{xy}\, b + \mu_{xz}\, c\,, \quad \ldots \ldots \quad (10)$$

$$\beta = \mu_{yx}\, a + \mu_{yy}\, b + \mu_{yz}\, c\,, \quad \ldots \ldots \quad (11)$$

$$\gamma = \mu_{zx}\, a + \mu_{zy}\, b + \mu_{zz}\, c\,, \quad \ldots \ldots \quad (12)$$

where $\mu_{xy} = \mu_{yx}$, $\mu_{xz} = \mu_{zx}$, $\mu_{yz} = \mu_{zy}$, . Hence, by solving a, b and c:

$$a = \mu'_{xx}\, \alpha + \mu'_{xy}\, \beta + \mu'_{xz}\, \gamma\,, \quad \ldots \ldots \quad (13)$$

$$b = \mu'_{yx}\, \alpha + \mu'_{yy}\, \beta + \mu'_{yz}\, \gamma\,, \quad \ldots \ldots \quad (14)$$

$$c = \mu'_{zx}\, \alpha + \mu'_{zy}\, \beta + \mu'_{zz}\, \gamma\,, \quad \ldots \ldots \quad (15)$$

where $\mu'_{xy} = \mu'_{yx}$, $\mu'_{xz} = \mu'_{zx}$, $\mu'_{yz} = \mu'_{zy}$. The magnetic (kinetic) energy U_m is per unit volume:

$$U_m = \frac{1}{8\pi}\,(a\,\alpha + b\,\beta + c\,\gamma). \quad \ldots \ldots \quad (16)$$

I call further the ellipsoid:

$$\mu_{xx}\, x^2 + \mu_{yy}\, y^2 + \mu_{zz}\, z^2 + 2\,\mu_{yz}\, yz + 2\,\mu_{zx}\, zx + 2\,\mu_{xy}\, xy = 1$$

the magnetic ellipsoid, and

$$\mu'_{xx}\, x^2 + \mu'_{yy}\, y^2 + \mu'_{zz}\, z^2 + 2\,\mu'_{yz}\, yz + 2\,\mu'_{zx}\, zx + 2\,\mu'_{xy}\, xy = 1$$

the reciprocal magnetic ellipsoid. Everything that has been said above of the electric ellipsoid, holds also good, if we substitute „magnetic" for „electric". If r_m and r'_m are again the radii vectores of the two magnetic ellipsoids, than

$$\frac{1}{r^2_{m\,\mathfrak{B}}} = \frac{\mathfrak{H}\cos(\mathfrak{B}\,\mathfrak{H})}{\mathfrak{B}}\,, \quad \ldots \ldots \ldots \quad (17)$$

$$\frac{1}{r'^2_{m\,\mathfrak{H}}} = \frac{\mathfrak{B}\cos(\mathfrak{B}\,\mathfrak{H})}{\mathfrak{H}} \quad \ldots \ldots \ldots \quad (18)$$

if in the first case $\mathfrak{B}$ falls in the direction of r_m, and in the second $\mathfrak{H}$ in the direction of r'_m.

The principal axes of the two electric ellipsoids will be called electric principal axes, and those of the two magnetic ellipsoids, magnetic principal axes; we shall suppose the direction of the two sets of principal axes to be perfectly arbitrary with respect to each other.

4. In the case of a non-conductive medium, the equations of the electromagnetic field become, if a direct system of axes is used:

$$\frac{df}{dt} = v\left(\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\frac{dg}{dt} = v\left(\frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$\frac{dh}{dt} = v\left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$-\frac{da}{dt} = v\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$-\frac{db}{dt} = v\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$-\frac{dc}{dt} = v\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where v represents the ratio of the magnetic and electric units of electricity. From equations (19), (20) and (21) one deduces

$$\frac{d}{dt}\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) = 0.$$

Let us suppose the medium to be uncharged, then:

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

If the medium is charged, this makes only this difference that we have a superposition of a variable field and a static field.

5. Let us now investigate whether a plane polarized wave of light can satisfy equations (19) to (26). For this we substitute:

$$\frac{f}{f_0}=\frac{g}{g_0}=\frac{h}{h_0}=\frac{P}{P_0}=\frac{Q}{Q_0}=\frac{R}{R_0}=\frac{a}{a_0}=\frac{b}{b_0}=\frac{c}{c_0}=$$

$$=\frac{\alpha}{\alpha_0}=\frac{\beta}{\beta_0}=\frac{\gamma}{\gamma_0}=F\left(t-\frac{lx+my+nz}{s}\right),$$

where l, m and n represent the direction-cosines of the normal to the wave-front, while t is the time and s the normal velocity of propagation. f_0, g_0 etc. are all constants.

If the substitution is carried out, we find:

$$f=\frac{v}{s}(n\,\beta-m\,\gamma), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$g=\frac{v}{s}(l\,\gamma-n\,\alpha), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$h=\frac{v}{s}(m\,\alpha-l\,\beta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$$-a=\frac{v}{s}(n\,Q-m\,R), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$-b=\frac{v}{s}(l\,R-n\,P), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$-c=\frac{v}{s}(m\,P-l\,Q), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

$$l\,f+m\,g+n\,h=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$l\,a+m\,b+n\,c=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The last two equations are a direct consequence of the preceding formulae. Equations (27) to (32) may also be written as follows:

$$\mathfrak{D}=\frac{v}{s}\sin_{\pi}(\mathbf{1}_{-n}\,\mathfrak{H}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

$$\mathfrak{B}=\frac{v}{s}\sin_{\pi}(\mathbf{1}_{-n}\,\mathfrak{E}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

where $\underline{sin}_{\pi}$ is the symbol for the sine-product. This is the "vector product" of two vectors; it is itself a vector normal to the plane through the two vectors and equal to the area of the parallelogram described on the two vectors; the product is drawn in such a direction, that if the two vectors are viewed from the extremity of that product, the rotation of the first-mentioned vector to the second-mentioned one (by an angle smaller than 180°) is seen as the motion of the hands of a clock. As to $\underline{1}_n$, it represents the unit vector in the direction of the positive normal to the wave-front.

6. We see from (35) and (36) that $\mathfrak{B}$ and $\mathfrak{D}$ are both in the wave-front; that $\mathfrak{B}$ is normal to $\mathfrak{E}$, so that $\mathfrak{B}$ and $\mathfrak{D}$ are electrically conjugate; further $\mathfrak{D}$ is normal to $\mathfrak{H}$, so that $\mathfrak{B}$ and $\mathfrak{D}$ are also magnetically conjugate. I call $\mathfrak{B}$ and $\mathfrak{D}$ doubly conjugate. In the same way $\mathfrak{E}$ and $\mathfrak{H}$ are reciprocally electric and reciprocally magnetic conjugates or doubly reciprocal conjugates. If the wave-front is given, we find the directions of $\mathfrak{B}$ and $\mathfrak{D}$ as the common conjugate diameters of its sections with the electric and the magnetic ellipsoid, the wave-front being supposed to pass through O. In general only one such a set is possible, of which either diameter can represent the electric induction. If the direction of the electric induction $\mathfrak{B}$ is given, but not the wave front, we find the direction of $\mathfrak{D}$ as the intersection of the plane which is electrically conjugate and that which is magnetically conjugate to $\mathfrak{B}$. The direction of $\mathfrak{D}$ and the wave-front will be determined in this way if the two planes do not coincide. These planes coincide only if $\mathfrak{B}$ falls in the direction of one of the common set of three conjugate diameters of the electric and the magnetic ellipsoid. We shall call these three directions the principal directions, which we indicate by 1, 2 and 3. The same holds if $\mathfrak{D}$ is given and we want to determine $\mathfrak{B}$. A principal direction may be defined as follows: if $\mathfrak{D}$ and $\mathfrak{B}$ fall in the same direction and at the same time $\mathfrak{E}$ and $\mathfrak{H}$ have the same direction, then the direction of $\mathfrak{D}$ and $\mathfrak{B}$ is called a principal direction; the direction of $\mathfrak{E}$ and $\mathfrak{H}$ will be called the corresponding reciprocal principal direction, and we shall indicate it by the same number, but distinguish it by an accent.

If now $\mathfrak{B}$ falls along a principal direction, $\mathfrak{D}$ is indefinite in the plane of the two others. We shall only discuss the general case that the three principal directions are perfectly determinate; the more special cases are then easily treated.

The same considerations apply to the vectors $\mathfrak{E}$ and $\mathfrak{H}$ with respect to the two reciprocal ellipsoids; instead of the wave-front we have then to do with the plane through $\mathfrak{E}$ and $\mathfrak{H}$, which we shall call the ray-plane. It is easily seen that the reciprocal principal directions are a set of three conjugate diameters of the two reciprocal ellipsoids.

Let the distances in the principal directions from O to the points of intersection with the electric ellipsoid be ε_e, ζ_e and η_e and to the points of intersection with the magnetic ellipsoid ε_m, ζ_m and η_m. We use the same symbols but distinguished by accents, for the distances along the reciprocal principal directions, cut off by the reciprocal ellipsoids.

Let us now put $\frac{\varepsilon_e}{\varepsilon_m} = b_1$, $\frac{\zeta_e}{\zeta_m} = b_2$ and $\frac{\eta_e}{\eta_m} = b_3$ and let us take $b_1 > b_2 > b_3$. If b_1, b_2 and b_3 all differ, which we shall assume, then the electric and the magnetic ellipsoid have only one set of three conjugate diameters.

Now $U_e = \frac{1}{8\pi}$, if $\mathfrak{D}$ is a radius vector of the electric ellipsoid, so that $\varepsilon_e \varepsilon'_e \cos(1, 1') = 1$ and in the same way $\varepsilon_m \varepsilon'_m \cos(1, 1') = 1$. Therefore $\varepsilon_e \varepsilon'_e = \varepsilon_m \varepsilon'_m$ and $\frac{\varepsilon'_e}{\varepsilon'_m} = \frac{1}{b_1}$ and likewise $\frac{\zeta'_e}{\zeta'_m} = \frac{1}{b_2}$, $\frac{\eta'_e}{\eta'_m} = \frac{1}{b_3}$. So, if the b's are all different, it is also seen that the reciprocal ellipsoids have only one set of three conjugate diameters.

We have seen that, for a given wave-front, $\mathfrak{B}$ and $\mathfrak{D}$ are found as common conjugate diameters of the sections of the wave-front with the electric and the magnetic ellipsoid. These directions are always determinate, except when the two sections are similar and have O as centre of similitude. If this be the case, we shall call the section a section of similitude; there are two of these sections, both passing through the principal direction 2, which we shall call the middle principal direction, and their position is such that the planes through 2 and 1 and through 2 and 3 are diametral planes for both, resp with 3 and 1 as direction of chords. If the wave-front is such a section of similitude, we can choose $\mathfrak{D}$ arbitrarily, but then $\mathfrak{D}$ is determined, being doubly conjugate to $\mathfrak{B}$.

The same thing holds good for $\mathfrak{E}$ and $\mathfrak{H}$ in the ray-plane; if this is a section of similitude of the two reciprocal ellipsoids, passing through the middle reciprocal principal direction 2′, then $\mathfrak{E}$ en $\mathfrak{H}$ are indeterminate in the plane; however, if one is given, the other may be found.

It is easy to find:

A principal direction is normal to the plane of the two not corresponding reciprocal principal directions and vice versa.

If $\mathfrak{D}$ or $\mathfrak{B}$ lie in a plane through two principal directions, then $\mathfrak{E}$ or $\mathfrak{H}$ lie in a plane through the two corresponding reciprocal principal directions.

In order to facilitate somewhat what precedes a transformation of affinity may be applied, in which O remains fixed and the magnetic ellipsoid is transformed to a sphere with radius $= 1$, a transformation which we shall lateron apply frequently. If we take the magnetic principal axes as coordinate-axes, the transformation is represented by the formulae:

$$x' = x\sqrt{\mu_x} \qquad y' = y\sqrt{\mu_y} \qquad z' = z\sqrt{\mu_z} \quad . \quad . \quad (37)$$

if $\mu_x x^2 + \mu_y y^2 + \mu_z z^2 = 1$ is the equation of the magnetic ellipsoid on its axes. By this transformation the electric ellipsoid becomes an ellipsoid with b_1, b_2 and b_3 as axes. Now we transform also the wave-front and the electric and magnetic induction (but not the electric and magnetic force). Now, in the transformation the relation of conjugate diameters is preserved, so that the magnetically conjugate diameters pass into mutually perpendicular lines. So the three principal directions pass into the axes of the transformed electric ellipsoid, and the section of similitude into a circular section of the transformed electric ellipsoid.

7. The normal to the wave-front being perpendicular to the electric and the magnetic induction, we have:

$$\frac{l}{gc-hb} = \frac{m}{ha-fc} = \frac{n}{fb-ga} = \frac{1}{A}.$$

If this is substituted in equation (27), we find:

$$A = \frac{v}{8}(a\alpha + b\beta + c\gamma) = \frac{v}{8} \,.\, 8\pi U_m.$$

We find the same, if we substitute in (28) and (29), but if we substitute in (30), (31) or (32), we find:

$$A = \frac{v}{8}(fP + gQ + hR) = \frac{v}{8} \,.\, 8\pi U_e.$$

If U is the total energy per unit volume, we find:

$$a\alpha + b\beta + c\gamma = fP + gQ + hR = 4\pi U = 8\pi U_e = 8\pi U_m \quad . \quad (38)$$

Expressed physically, the relation between the amplitudes of the electric and the magnetic vibrations is such that at any moment $U_e = U_m$. Expressed geometrically this is: If the extremity of $\mathfrak{D}$ lies on the electric ellipsoid, the extremity of $\mathfrak{B}$ lies on the magnetic ellipsoid. We may add that the extremities of $\mathfrak{E}$ and $\mathfrak{H}$ will lie on the two reciprocal ellipsoids.

So we have found:

$$l = \frac{s}{v} \frac{gc - hb}{4\pi U}, \quad \ldots\ldots\ldots \quad (39)$$

$$m = \frac{s}{v} \frac{ha - fc}{4\pi U}, \quad \ldots\ldots\ldots \quad (40)$$

$$n = \frac{s}{v} \frac{fb - ga}{4\pi U}, \quad \ldots\ldots\ldots \quad (41)$$

which we may also write:

$$4\pi U \frac{v}{s} 1_n = \sin_\pi (\mathfrak{B}\,\mathfrak{D}). \quad \ldots\ldots \quad (42)$$

From this we derive:

$$s = v \frac{4\pi U}{\mathfrak{B}\,\mathfrak{D} \sin (\mathfrak{B}\,\mathfrak{D})}$$

and so according to (38):

$$s^2 = v^2 \frac{\mathfrak{D}\,\mathfrak{E} \cos (\mathfrak{D}\,\mathfrak{E}) \,.\, \mathfrak{B}\,\mathfrak{H} \cos (\mathfrak{B}\,\mathfrak{H})}{\mathfrak{B}^2\,\mathfrak{D}^2 \sin^2 (\mathfrak{B}\,\mathfrak{D})} = \frac{v^2}{\sin^2 (\mathfrak{B}\,\mathfrak{D})} \frac{\mathfrak{E} \cos (\mathfrak{D}\,\mathfrak{E})}{\mathfrak{D}} \frac{\mathfrak{H} \cos (\mathfrak{B}\,\mathfrak{H})}{\mathfrak{B}}$$

and by (5) and (17):

$$s = \frac{v}{r_{e_\mathfrak{D}}\, r_{m_\mathfrak{B}} \sin (\mathfrak{B}\,\mathfrak{D})} \quad \ldots\ldots\ldots \quad (43)$$

According to this equation the normal velocity s is equal to v, divided by the area of the parallelogram on the radii vectores of the electric and magnetic ellipsoids, resp. in the direction of the electric and magnetic induction.

If the directions in the wave-front, along which the electric and magnetic induction can fall are I and II, then we get:

$$s_1 = \frac{v}{r_{e\text{I}}\, r_{m\text{II}} \sin(\text{I}, \text{II})},$$

if the electric induction falls along I, and:

$$s_2 = \frac{v}{r_{e\text{II}}\, r_{m\text{I}} \sin(\text{I}, \text{II})},$$

if the magnetic induction falls along I, so that $\frac{s_1}{s_2} = \frac{r_{e\text{II}}\ r_{m\text{I}}}{r_{e\text{I}}\ r_{m\text{II}}}$. The two values s_1 and s_2 are the same if $r_{e\text{I}} : r_{m\text{I}} = r_{e\text{II}} : r_{m\text{II}}$, i.e. if the wave-front is a section of similitude, and $\mathfrak{B}$ and $\mathfrak{D}$ are therefore indeterminate in it.

8. We shall now derive an equation which we shall use in order to determine the wave-surface. We apply the before-mentioned transformation, which if we take the axes of the magnetic ellipsoid as axes of the coordinates, is represented by (37). We distinguish by accents the transformed quantities, so that we get:

$$f' = f\sqrt{\mu_x}, \qquad g' = g\sqrt{\mu_y}, \qquad h' = h\sqrt{\mu_z},$$
$$a' = a\sqrt{\mu_x}, \qquad b' = b\sqrt{\mu_y}, \qquad c' = c\sqrt{\mu_z}.$$

The equation:

$$lx + my + nz = s$$

represents a plane parallel to the wave-front at a distance s from O. This plane is changed by the transformation into:

$$\frac{x'l}{\sqrt{\mu_x}} + \frac{y'm}{\sqrt{\mu_y}} + \frac{z'n}{\sqrt{\mu_z}} = s,$$

or

$$x'\,l' + y'\,m' + z'\,n' = s',$$

where l' m' and n' are the direction-cosines of the normal, and s' the distance from O to the transformed plane. In this:

$$l' = \frac{l}{p\sqrt{\mu_x}}, \qquad m' = \frac{m}{p\sqrt{\mu_y}}, \qquad n' = \frac{n}{p\sqrt{\mu_z}}, \qquad s' = \frac{s}{p},$$

where:

$$p^2 = \frac{l^2}{\mu_x} + \frac{m^2}{\mu_y} + \frac{n^2}{\mu_z}.$$

If in equations (27)—(32) only the electric and the magnetic induction is introduced, they become:

$$f = \frac{v}{s}\,(n\,b\,\mu_y - m\,c\,\mu_z),$$

$$g = \frac{v}{s}\,(l\,c\,\mu_z - n\,a\,\mu_x),$$

$$h = \frac{v}{s}\,(m\,a\,\mu_x - l\,b\,\mu_y),$$

$$a = \frac{v}{s}\left\{m\,(k_{zx}\,f + k_{zy}\,g + k_{zz}\,h) - n\,(k_{yx}\,f + k_{yy}\,g + k_{yz}\,h)\right\},$$

$$b = \frac{v}{s}\left\{n\,(k_{xx}\,f + k_{xy}\,g + k_{xz}\,h) - l\,(k_{zx}\,f + k_{zy}\,g + k_{zz}\,h)\right\},$$

$$c = \frac{v}{s}\left\{l\,(k_{yx}\,f + k_{yy}\,g + k_{yz}\,h) - m\,(k_{xx}\,f + k_{xy}\,g + k_{xz}\,h)\right\}.$$

The electric ellipsoid changes by the transformation into:

$$\frac{k_{xx}}{\mu_x}\,x'^2 + \frac{k_{yy}}{\mu_y}\,y'^2 + \frac{k_{zz}}{\mu_z}\,z'^2 + 2\,\frac{k_{yz}}{\sqrt{\mu_y\,\mu_z}}\,y'\,z' + 2\,\frac{k_{xz}}{\sqrt{\mu_x\,\mu_z}}\,x'\,z' + 2\,\frac{k_{xy}}{\sqrt{\mu_x\,\mu_y}}\,x'\,y' = 1,$$

or

$$a_{xx}\,x'^2 + a_{yy}\,y'^2 + a_{zz}\,z'^2 + 2\,a_{yz}\,y'\,z' + 2\,a_{xz}\,x'\,z' + 2\,a_{xy}\,x'\,y' = 1,$$

where $a_{xx} = \frac{k_{xx}}{\mu_x}$ etc. If in the 6 above equations we introduce quantities that are distinguished by accents, we find:

$$f' = \frac{v}{s''}\,(n'\,b' - m'\,c'), \quad \ldots \ldots \quad (44)$$

$$g' = \frac{v}{s''}\,(l'\,c' - n'\,a'), \quad \ldots \ldots \quad (45)$$

$$h' = \frac{v}{s''}\,(m'\,a' - l'\,b'), \quad \ldots \ldots \quad (46)$$

$$a' = \frac{v}{s''} \left\{ m' (a_{zx} f' + a_{zy} g' + a_{zz} h') - n' (a_{yx} f' + a_{yy} g' + a_{yz} h') \right\},$$

$$b' = \frac{v}{s''} \left\{ n' (a_{xx} f' + a_{xy} g' + a_{xz} h') - l' (a_{zx} f' + a_{zy} g' + a_{zz} h') \right\},$$

$$c' = \frac{v}{s''} \left\{ l' (a_{yx} f' + a_{yy} g' + a_{yz} h') - m' (a_{xx} f' + a_{xy} g' + a_{xz} h') \right\},$$

where $s'' = \frac{s'}{\sqrt{\mu_x \mu_y \mu_z}}$. From these equations everything has disappeared that depended on the special choice of axes of the coordinates; they are therefore of general application, and they will still hold if we choose the axes of the transformed electric ellipsoid as axes of the coordinates by which the equation of the ellipsoid becomes: $a_x x'^2 + a_y y'^2 + a_z z'^2 = 1$. The last three equations become then:

$$a' = \frac{v}{s''} (m' a_z h' - n' a_y g'), \quad . \quad . \quad . \quad . \quad . \quad (47)$$

$$b' = \frac{v}{s''} (n' a_x f' - l' a_z h'), \quad . \quad . \quad . \quad . \quad . \quad (48)$$

$$c' = \frac{v}{s''} (l' a_y g' - m' a_x f'). \quad . \quad . \quad . \quad . \quad . \quad (49)$$

If we eliminate a', b' and c', we find:

$$f' (a_x v^2 - s''^2) = v^2 \; l' (a_x l' f' + a_y m' g' + a_z n' h'), \quad . \quad . \quad (50)$$

$$g' (a_y v^2 - s''^2) = v^2 \; m' (a_x l' f' + a_y m' g' + a_z n' h'), \quad . \quad . \quad (51)$$

$$h' (a_z v^2 - s''^2) = v^2 \; n' (a_x l' f' + a_y m' g' + a_z n' h'). \quad . \quad . \quad (52)$$

Multiplying (50) by $\frac{l'}{a_x v^2 - s''^2}$, (51) by $\frac{m'}{a_y v^2 - s''^2}$ and (52) by $\frac{n'}{a_z v^2 - s''^2}$ and adding them, we find:

$$\frac{l'^2}{a_x v^2 - s''^2} + \frac{m'^2}{a_y v^2 - s''^2} + \frac{n'^2}{a_z v^2 - s''^2} = 0. \quad . \quad . \quad . \quad (53)$$

This is a quadratic equation in s'', which, with given l', m' and n', furnishes 2 values for s'' and so also for s'.

It is easy to deduce the equation of the wave-surface in tangential coordinates from this equation. We first seek the transformed wave-surface. Let $0 = A'x' + B'y' + C'z' + 1$ be a tangent plane to the transformed wave-surface; we consider A', B' and C' as the coordinates of that plane. The direction-cosines l', m' and n' and the distance s' to O are for that plane:

$$l' = \frac{A'}{q'}, \qquad m' = \frac{B'}{q'}, \qquad n' = \frac{C'}{q'}, \qquad s' = \frac{1}{q'},$$

where $q'^2 = A'^2 + B'^2 + C'^2$. If this is substituted in equation (53), taking into account that $s' = s'' \sqrt{\mu_x \mu_y \mu_z}$, we find:

$$\frac{A'^2}{a_x v^2 \mu_x \mu_y \mu_z q'^2 - 1} + \frac{B'^2}{a_y v^2 \mu_x \mu_y \mu_z q'^2 - 1} +$$

$$+ \frac{C'^2}{a_z v^2 \mu_x \mu_y \mu_z q'^2 - 1} = 0.$$

If we now direct our attention to the wave-surface itself, we find easily, the coordinates of a plane being the reciprocal value of the portions cut off from the axes with inverse sign, that if we choose the principal directions as axes of the coordinates:

$$A' = \varepsilon_m A \quad , \quad B' = \zeta_m B \quad , \quad C' = \eta_m C.$$

Further $a_x = \frac{1}{b_1^2} = \frac{\varepsilon^2_m}{\varepsilon^2_e}$ and also $a_y = \frac{\zeta^2_m}{\zeta^2_e}$, $a_z = \frac{\eta^2_m}{\eta^2_e}$. Since $\mu_x x^2 + \mu_y y^2 + \mu_z z^2 = 1$ is the equation of the magnetic ellipsoid on the axes, $\sqrt{\mu_x \mu_y \mu_z}$ is the inverse value of the product of the axes or of the volume of a parallelopiped on three conjugate diameters; if we take for these diameters the principal directions, we find $\sqrt{\mu_x \mu_y \mu_z} = \frac{1}{w \varepsilon_m \zeta_m \eta_m}$, where w is the volume of a parallelopiped on the principal directions, the sides of which are all 1. The tangential equation of the wave-surface changes therefore, after removing the fractions, into:

$$\frac{1}{\varepsilon^2_e\,\zeta^2_e\,\eta^2_e\,e^2_m\,\zeta^2_m\,\eta^2_m}\,(A^2\,\varepsilon^2_m + B^2\,\zeta^2_m + C^2\,\eta^2_m)\,(A^2\varepsilon^{\,2}_e + B^2\,\zeta^2_e + C^2\,\eta^2_e) -$$

$$-\left\{A^2\left(\frac{1}{\zeta^2_e\,\eta^2_m} + \frac{1}{\eta^2_e\,\zeta^2_m}\right) + B^2\left(\frac{1}{\eta^2_e\,\varepsilon^2_m} + \frac{1}{\varepsilon^2_e\,\eta^2_m}\right) + \right.$$

$$\left. + C^2\left(\frac{1}{\varepsilon^2\,\zeta^2_m} + \frac{1}{\zeta^2_e\,\varepsilon^2_m}\right)\right\}\frac{w^2}{v^2} + \frac{w^4}{v^4} = 0. \quad . \quad . \quad (54)$$

As was to be expected this equation is symmetrical in the electric and the magnetic quantities, though the deduction was asymmetrical.

9. It is now possible to deduce the equation in the coordinates of a point from this equation in tangential coordinates, but in order to find at the same time some properties of the radius vector of the wave-surface, — which we shall call ray of light, I take the following course. We first determine the transformed wave-surface. Let λ, μ and ν be the direction cosines of the ray of light and ϱ the radius vector of the wave-surface, which we call the velocity of the rays; we take the same symbols for the transformed surface, but distinguished by accents.

If further $\varrho'' = \dfrac{\varrho'}{\sqrt{\mu_x\,\mu_y\,\mu_z}}$, then:

$$l'\,\lambda' + m'\,\mu' + n'\,\nu' = \frac{s'}{\varrho'} = \frac{s''}{\varrho''} \quad . \quad . \quad . \quad , \quad . \quad . \quad (55)$$

If now l', m', n' and s'' are made to change a little, so that equations (53) and $l'^2 + m'^2 + n'^2 = 1$ continue to be satisfied, then λ', μ', ν' and ϱ'', do not change, being the coordinates of a point of the envelope, so that:

$$\lambda'\,dl' + \mu'\,dm' + \nu'\,dn' - \frac{ds''}{\varrho''} = 0,$$

$$l'\,dl' + m'\,dm' + n'\,dn' = 0,$$

$$\frac{l'\,dl'}{a_x\,v^2 - s''^2} + \frac{m'\,dm'}{a_y\,v^2 - s''^2} + \frac{n'\,dn'}{a_z\,v^2 - s''^2} +$$

$$+ s''\,ds''\left\{\frac{l'^2}{(a_x\,v^2 - s''^2)^2} + \frac{m'^2}{(a_y\,v^2 - s''^2)^2} + \frac{n'^2}{(a_z\,v^2 - s''^2)^2}\right\} = 0.$$

After multiplying the 2nd equation by A and the 3rd by B, we can add them to the first equation, and then we conclude to:

$$\lambda' - Al' + B\frac{l}{a_x v^2 - s''^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

$$\mu' - Am' + B\frac{m'}{a_y v^2 - s''^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

$$\nu' - An' + B\frac{n'}{a_z v^2 - s''^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

$$\frac{1}{s'' \varrho''} = B\left\{\frac{l'^2}{(a_x v^2 - s''^2)^2} + \frac{m'^2}{(a_y v^2 - s''^2)^2} + \frac{n'^2}{(a_z v^2 - s''^2)^2}\right\}, \quad . \quad (\delta)$$

$$l'^2 + m'^2 + n'^2 = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\varepsilon)$$

$$\lambda'^2 + \mu'^2 + \nu'^2 = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\zeta)$$

From these equations and from (53) and (55) we have to eliminate A, B, l', m', n' and s''. We can eliminate A and B by substituting λ', μ' and ν' from (α), (β) and (γ) in (ζ) and (55).

We find:

$$A = \frac{s''}{\varrho''},$$

$$B = \frac{s''}{\varrho''}(\varrho''^2 - s''^2).$$

If we substitute this in (α), (β) and (γ), we find:

$$\frac{\lambda' \varrho''}{a_x v^2 - \varrho''^2} = \frac{l' s''}{a_x v^2 - s''^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

$$\frac{\mu' \varrho''}{a_y v^2 - \varrho''^2} = \frac{m' s''}{a_y v^2 - s''^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

$$\frac{\nu' \varrho''}{a_z v^2 - \varrho''^2} = \frac{n' s''}{a_z v^2 - s''^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

If we multiply (56) by $\dfrac{\lambda'(a_x v^2 - s''^2)}{\varrho''}$, (57) by $\dfrac{\mu'(a_y v^2 - s''^2)}{\varrho''}$ and (58) by $\dfrac{\nu'(a_z v^2 - s''^2)}{\varrho''}$, and add, we find after some reduction:

$$\frac{a_x \lambda'^2}{a_x v^2 - \varrho''^2} + \frac{a_y \mu'^2}{a_y v^2 - \varrho''^2} + \frac{a_z \nu'^2}{a_z v^2 - \varrho''^2} = 0. \quad . \quad (59)$$

This is again a quadratic equation in ϱ'', so that for every direction of the ray radius there are two velocities of ray. The wave-surface consists therefore of 2 sheets, which both enclose O.

From these last equations we can easily find the equation of the wave-surface in coordinates of points. In order to find those of the transformed wave-surface, we have only to substitute $\lambda' = \frac{x'}{\varrho'}$, $\mu' = \frac{y'}{\varrho'}$, $\nu' = \frac{z'}{\varrho'}$ and $\varrho'^2 = x'^2 + y'^2 + z'^2$.

If x, y and z are the coordinates of a point of the wave-surface with the principal directions as axes of the coordinates, we find easily:

$$x = \varepsilon_m x' \quad , \quad y = \zeta_m y' \quad , \quad z = \eta_m z',$$

whereas $\sqrt{\mu_x \mu_y \mu_z} = \frac{1}{w \varepsilon_m \zeta_m \eta_m}$, $a_x = \frac{\varepsilon^2_m}{\varepsilon^2_e}$, $a_y = \frac{\zeta^2_m}{\zeta^2_e}$, $a_z = \frac{\eta^2_m}{\eta^2_e}$.

The equation of the wave-surface becomes then:

$$\left(\frac{x^2}{\varepsilon^2_e} + \frac{y^2}{\zeta^2_e} + \frac{z^2}{\eta^2_e}\right)\left(\frac{x^2}{\varepsilon^2_m} + \frac{y^2}{\zeta^2_m} + \frac{z^2}{\eta^2_m}\right) \varepsilon^2_e \zeta^2_e \eta^2_e \varepsilon^2_m \zeta^2_m \eta^2_m -$$

$$- \left\{ x^2 (\zeta^2_e \eta^2_m + \eta^2_e \zeta^2_m) + y^2 (\eta^2_e \varepsilon^2_m + \varepsilon^2_e \eta^2_m) + \right.$$

$$\left. + z^2 (\varepsilon^2_e \zeta^2_m + \zeta^2_e \varepsilon^2_m) \right\} \frac{v^2}{w^2} + \frac{v^4}{w^4} = 0. \quad . \quad . \quad . \quad . \quad (60)$$

This is the same equation as the equation in tangential coordinates except that the quantities ε_e, ζ_e, η_e, ε_m, ζ_m, η_m, v and w are replaced by their inverse values. The surface must therefore be dualistic with itself. From every property which is invariant with respect to transformations of affinity, a dualistic one may be deduced, by replacing a point by a plane with the same coordinates and vice versa, replacing at the same time ε_e, ζ_e, η_e, ε_m, ζ_m, η_m, v and w by their reciprocal values.

The wave-surface is of the 4th degree and of the 4th class with the coordinate planes as diametral planes, with the opposite axis of coordinates as direction of chords.

Every line through O intersects the wave-surface on either side of O in two real points, while parallel to a definite plane on either side of O two real tangent planes may be drawn.

Bacteriology. — Prof. PLACE presents a paper from Mr. ALEX. KLEIN on: "*Bacteriologic Researches of Human Faeces.*" (1st Part).

In the estimation of quantitative bacteriologic relations in faeces, attention should be paid to the very irregular and unequal distribution of the lower organisms in this substance; in a great number of places they are accumulated in enormous masses, in other portions of the same faeces they are relatively scarce. In these researches two expedients were used in order to neutralize this irregularity, namely:

1. by examining at each time a relatively great quantity of faeces; this offering more probability that the various unequalities are proportionately represented than smaller volumes. In these researches 10 grs. (on the average thus $^1/_{15}$ part of the total quantity of faeces deposed in 24 hours by a full-grown person) was fixed as minimum; and

2. by making an extremely fine and equal emulsion in sterilized water of the faeces destined for the estimation. This emulsion was thus prepared: the weighed faeces were for a fairly long time rubbed fine in a sterilized mortar and in sterilized water (mostly 100 cM^3), by means of a sterilized pestle; when thus the substance was properly mixed, a certain quantity (10 cM^3) was withdrawn from the mortar and thoroughly shaken for some time in a flask with a great number of small porcelain balls, under continual addition of known quantities of sterilized water.

Of such an emulsion was then determined:

a. by the culture-method the number of germs fitted for development, and

b. by the microscopic counting-method the total number of bacteria present; for both, plates and counting-preparations, was always used the same platinum loop.

Since the first publication[1]) of the microscopic counting-method I have made some modifications in the method, which are not devoid of importance for its practical application.

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[1]) ALEX. KLEIN. Eine neue mikroskopische Zählungsmethode der Bacteriën. Centralblatt für Bact. und Par., 1 Abth., 1900, Bd XXVII, S. 834.

Already before[1] I called attention to the fact, that preparations, colored after the moist way, are liable to lose their pigment through decoloring agents. In the counting-preparations also, this difficulty is met with, and, as Dr. F. H. HEHEWERTH points out[2], a slight acid reaction of the preparations or of the Canada balsam can already call forth a rather prompt decoloration. By using a somewhat concentrated, neutrally reacting solution of xylol-Canada balsam, the decoloring can be sufficiently prevented during the time of the counting.

I have now found that this decoloration should be attributed to the detaching from the cover-glass of the whole preparation, which the non-removed pigment has rendered consistent; the xylol-Canada balsam, beginning at the periphery of the cover-glass, enters between it and the preparation, and, in so doing detaches the latter from the glass. Hence, viscous Canada balsam has a less prejudicial effect than a thinly liquid solution; acids too, further this softening-process, so that hereby the influence on the decoloring of the preparations is sufficiently explained. The decoloration can be wholly prevented by making the substance adhere more firmly to the cover-glass by means of some viscous matter; I commonly use therefore a clarified 4—5 % solution of gelatin in water. Before using, the solution is melted and a very small loopful of the liquid gelatin is deposed on the cover-glass; the colored bacteria-emulsion is mixed on the cover-glass with the gelatin-solution and spread over the glass. After drying the preparation is not flamed, but directly enclosed in xylol-Canada balsam; the consistency or the reaction of the balsam has no more any influence: the preparations remain colored for weeks.

A second modification refers to the choice of the optic fields which are to be counted. To this end are beforehand marked on a scheme of millimeter paper the places of the preparation to be counted; the centre of the cover-glass serves as fixed point. The optic fields (they are numbered from 1 to 50) are chosen in such a succession that they may easily be reached one after another by means of a movable object-stage, whilst the respective distances of the optic fields are taken in such a way that for example, 1 c.m. on the scheme corresponds with 1 m.m. displacing of the stage.

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[1] ALEX. KLEIN, Eine einfache Methode zur Sporenfärbung. Centralbltt. für Bact. und Par., 1e Abth., 1899, Bd XXV, S. 376.

[2] F. H. HEHEWERTH, De microscopische telmethode der bacteriën van ALEX. KLEIN en eenige van hare toepassingen. Diss. Amsterdam 1900. Die mikroskopische Zählungsmethode der Bacteriën von ALEX. KLEIN und einige Anwendungen derselben. Archiv für Hygiene. Bd XXXIX, 1901, S. 321—389.

The scheme is pasted on card-board and for each counting covered with tracing-paper, whereon of each field the number of the counted organisms is recorded; in this way the same scheme-table can be used every time again (provided the cover-glasses be of the same size).

The preparation is set in on the centre of the cover-glass; the position of the two verniers of the object-stage is marked on the tracing-paper, so that this point can always be found back; around it the several fields are grouped.

When human faeces are treated in this way, there mostly appears to exist a great disproportion between the number of bacteria which can be counted microscopically, and that which can be grown on some fitted medium; this disproportion is so great that it is obvious even in the ordinary staining-method after KOCH, although numerous organisms are then washed away. Accordingly, some investigators as BUCHNER, KUISL, EBERLE, and others, drew attention to this fact. As however, no exact method was known to microscopically determine the number of bacteria, and thus, partly no quantitative estimations were made, partly imperfect methods could only be worked with the importance of the fact was not acknowledged, still less tried to explain it.

When applying the microscopic counting-method and the culture-method to alkalin gelatin at 22° C., the differences often prove exceedingly great; to give a few instances: in 1 mgr. of faeces, were counted microscopically 74.959000 bacteria, found by the culture-method 356; 2nd instance: microscopic counting 165.614000, cultivated 9900; etc.

At first view it might be presumed that the greater part of the enormous number of organisms, which are seen microscopically, are in the same condition as the few cultivable bacteria, i. e. also living and fitted for development, but that the difference might be quite, or at least greatly explained by the distribution in clumps of the faeces-bacteria each clump producing but one single colony, whilst in the counting-method each organism is reckoned separately. It is in fact very troublesome to make a good emulsion of faeces, wherein the bacteria are equally distributed, but if following the above method, it nearly always succeeds, which may directly be controlled from the microscopic preparation. Also the excessively great differences which are found, likewise a priori contradict this view. In order, however, completely to exclude this possibility, a series of determinations was made, where in the microscopic preparation not only each organism was counted, but moreover in each counted field the number (reduced number) was determined in such a way, that each clump of bacteria, and all lower organisms, too, found within a field of 15 microns square, were recorded as one single individual.

TABLE I [1])

N° of the research.	Quantity of examined faeces in gr.	Percentage of solid substance of the faeces.	*Total* number of bact. found by *micr. counting* and estim. on 1 mgr. of faeces.	*Reduced* numb. found by *microsc. counting* and estimated on 1 mgr. of faeces.	Percent. difference between total and reduced number	Number of bacteria found by *culture* (alk. gel. at 22° C.) estim. on 1 mgr. of faeces.	Proportional number with resp. to the total numbers.	Proportional number with respect. to the reduc. number.
1	10.890	25.65	52.400000	33.570000	$-36^{0}/_{0}$	1.083000	47	30
2	11.915	30.41	21.285000	15.737000	$-29^{0}/_{0}$	800	26605	19670
3	10.555	22.61	55.468000	35 147000	$-36^{0}/_{0}$	20000	2772	1756
4	11.200	24.21	165.614000	71.304000	$-57^{0}/_{0}$	9900	16728	7201
5	11 200	13.92	43.842000	28.666000	$-35^{0}/_{0}$	58000	755	493

From this Table is seen that, by the second way of counting, the total number of microscopically found bacteria, as might well be expected, decreases, in n°. 2 with $29^{0}/_{0}$ as minimum, in n°. 4 with $57^{0}/_{0}$ as maximum. In spite of this diminution the microscopic numbers remain far superior to those of the colonies (grown at 22° C. on alk. gelatin), which is clearly demonstrated by the two last columns, where for both cases of microscopic counting are stated the so-called "proportional numbers" (denoting the number of times that the microscopic counting method indicates more organisms than Koch's plate-method). If thus it cannot be denied that the conglomeration and respective distance of the lower organisms must diminish the number of colonies on the plates, — on the average with $38{,}6\,^{0}/_{0}$, a number fairly corresponding with the value $(40{,}2^{0}/_{0})$ found by Dr. Hehewerth [2]) in the use of pure cultures, — still

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[1]) The faeces of this Table, as also those of all the following investigations, are derived from different healthy, full-grown individuals, nourished with mixed food, where no further attention was paid to the nearer nature of the nutrition; the faeces were examined as fresh as possible, mostly within 2–3 hours after the deposing.

[2]) L. c. S. 340.

the grouping of the lower organisms seems unable to level the differences between counting- and culture-method.

To explain this there are but two possibilities, 1st the greater number of lower organisms occurring in human faeces cannot develop under the given conditions (alkalin gelatin at 22° C.), or 2nd the excessive majority of the microscopically visible bacteria have died, or at least, are so much attenuated in their vital functions as no more to be fitted for multiplication.

In order to test the former explanation on the experiment, the bacteria of the faeces were allowed to multiply under various external circumstances, whose influence on the development of other bacteria-species is commonly known; among these are: *a.* the nature of the nutrients, *b.* the reaction of the medium; *c.* the absence or presence of free oxygen, and *d.* the temperature of growth.

TABLE II.

N° of the research	Quantity of examined faeces in grs.	Percentage of solid substance of the faeces.	Number of bact. found by *microsc. counting* and estimated on 1 mgr. of faeces,	Number of bact. found by culture and estimated for 1 mgr. of faeces.				Proportional number to alkalin gelatin.
				Alkalin nutrient gelatin at 22° C.	Acid malt gel. at 22° C.	1/4 % glycose gelatin, anaërobic 1) at room temperature.	Agar-agar at 37° C.	
1	10.830	15.70	77.602000	6.396000	5.299000	—	2.735000	12
2	10.480	15.49	45.299000	692300	552900	754800	605800	65
3	11.327	23.88	31.709000	101000	89500	115900	24100	314
4	12 270	16 59	86.972000	7410	5450	7410	9780	11737
5	11.495	20.13	60.363000	432000	394000	595000	307000	139
6	11 185	24.83	47.662000	12700	11700	13200	12700	3753
7	12.110	24.72	20.162000	379300	300700	35800	45600	53
8	10 780	30.60	74.959000	356	0	2848	1068	210559
9	10.735	18.66	39.961000	6400	5000	3600	5400	6244

From Table II we see that colonies, developing on alkalin gelatin and acid malt gelatin at 22° C, on 1/4 pCt. glycose-gelatin under anaërobic relations, and on agar-agar at 37° C., are different in

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1) Alex. Klein, Ein Apparat zur bequemen Herstellung von anaëroben Plattenculturen. Centralbltt. für Bact. u. Paras. 1e Abth. Bd. XXIV, 1898, s. 967.

number, and that in the several samples of faeces, under some or other circumstances, a greater or smaller number of bacteria may be produced; alkalin gelatin at 22° C. yields in general the greatest number of colonies, or at least very near to it, so that the thereby found estimations are also for the future used as criterion for the proportional numbers. The differences in the numbers of cultivable organisms prove, however, relatively slight under the given relations, and very certainly are reduced to nothing when opposed to the great number stated by the microscopic counting-method. We should accordingly conclude, that the applied external conditions are not able even partly to level the great difference between counting- and culture-method. If nevertheless we would maintain this view in order to explain the influence of nutriment and other external conditions, we should be obliged to admit that the greater part of bacteria occurring in human faeces, can only develop on very special media and in relations quite unknown as yet. For a few microscopically perceptible organisms — some finer spirillum- and comma-shaped species — this possibility must be allowed; these forms are however rarely met with in the preparations, so that they can well be neglected, and most certainly they can by no means compensate the lack of cultivable organisms.

The percentage of solid matter in the faeces is in no observable relation to the proportional numbers, in other words, the number of cultivable bacteria in human faeces does not depend on the amount of water (or state of dessication) of the faeces; the proportional numbers themselves have in the examined faeces most diverging values.

Table III gives a survey of a mean, a maximum and a minimum number, estimated from the 14 determinations of Table I and Table II.

TABLE III.

	Percentage of solid matter.	Numb. found by *micr. counting* and estimated on 1 mgr. of faeces.	Numb. found by *culture* and estimated on 1 mgr. of faeces.	Numbers excreted in 24 hours after *microsc. counting*.	Numbers excreted in 24 hours after *culture*-mothod
Mean.	21.96	58.800000	658500	8800 milliards.	99 milliards.
Maximum.	30.60	165.614000	6.396000	24800 »	960 »
Minimum.	13.92	20.162000	356	3000 »	54 millions.

The mean number of organisms cultivable on our ordinary nutrient media thus amounts to only 1.1 pCt. of the total number of bacteria present; the mean proportional number is 89.

The whole number of bacteria excreted with the faeces in 24 hours by a healthy, full-grown individual, — on the average 8800 milliards, — is much larger than was hitherto known. GILBERT and DOMINICI [1]) for instance, note an excretion of 12—15 milliards in 24 hours; SUCKSDORFF [2]) gives as mean result above 55 milliards, as maximum nearly 408 milliards, and as minimum somewhat less than 2 milliards; these numbers thus, remain far behind those obtained by the microscopic counting-method. This enormous yield of bacteria is still better grasped when making a rough calculation of the weight of the bacteria excreted in 24 hours, of the percentage of solid substance occupied by the bacteria, and of the quantity of nitrogen which they contain; we here assume that 30 milliards of bacteria weigh 1 mgr. (NAEGELI), that $15^0/_0$ of the bacterium consists of solid matter, and that $10^0/_0$ of this matter is occupied by *N*.

TABLE IV.

	Percentage of solid matter.	Numbers of bacteria after *micr. counting* excreted in 24 hours.	Weight of bacteria excreted in 24 hours.	Percentage of solid substance of the faeces occupied by bacteria.	Quantity of nitrogen excreted in 24 h. with the faeces in the bacteria
Mean	21.96	8800 milliards.	293 mgrs.	0.13 $^0/_0$	4.39 mgrs.
Maximum	—	24800 »	826 »	0.34 $^0/_0$	12.39 »
Minimum	—	3000 »	100 »	0.039 $^0/_0$	1.5 »

Whereas, on the average $98{,}9^0/_0$ of the bacteria present in human faeces, can under no known circumstances be grown on our artificial media, it is obvious that we should now consider the second alternative, alleged before, viz. that these $98{,}9^0/_0$ of the faeces-bacteria have died, or are at least so much attenuated, that they can no more multiply on our artificial media. Generally it is much more troublesome to state the death than the life of an organism; the existence of vital functions is in most cases easily observable, whilst on the

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[1]) GILBERT et DOMINICI, Semaine médicale 1894 p. 76.

[2]) SUCKSDORFF. Das quantitative Vorkommen von Spaltpilzen im menschlichen Darmkanale. Archiv. f. Hygiene Bd. IV.

other hand, their absence is difficult to prove. If we could, for instance, with absolute certainty point out that the 98,9% of the faeces-bacteria are no more able to propagate, only then we should have a right to speak of the death, or at least of an attenuation of these organisms. But in order to demonstrate this, we should transfer these bacteria to a medium of which we know with the same absolute certainty, that these lower organisms, if alive, would easily propagate thereon. Now, in the faeces themselves these bacteria are found in thousands of milliards, accordingly these faeces must at a special stadium in the human intestinal canal, be an extremely favorable nutriment for our organisms; it is thus obvious that we should observe the behaviour of the bacteria in the same faeces out of the human body.

A larger quantity of faeces was for a longer time placed in the thermostat at 37° C.; by keeping the environment sufficiently damp the faeces were prevented from drying, which was further controlled by repeated determinations of solid substance. After different periods a certain quantity of the faeces (10 grs. at the least) was examined in the ordinary way.

TABLE V. Faeces N°. 1, placed at 37° C.

Periods at 37° C.	Quantity of examined faeces in grs.	Percentage of solid subst.	Number of bacteria found by *micr. counting* and estim. on 1 mgr. of faeces.	Percent. diminution of the micr. counted bacteria.	Number of bact. *cult.* on alk. gel. and estimated on 1 mgr.	Proportional numbers.
Direct examinat.	**11.885**	**18.09**	**64.697000**	**100**	**182400**	**355**
After **1** day	**10.275**	**16.31**	**43.719000**	— **33%**	**208400**	**210**
» 3 days	**10.215**	**15.58**	**26.544000**	— **59%**	**36000**	**737**
» 5 »	**10.690**	**13.51**	**25.490000**	— **61%**	**19700**	**1294**
» 7 »	**10.112**	**16.11**	**28.949000**	— **56%**	**1.284000**	**23**

We see that the number of bacteria found by microscopic counting continually decreases, so that after 7 days already 56 pCt. has vanished; but also the cultivable bacteria have diminished in the first 5 days, whilst only the seventh day a relatively slight increase may be observed. The proportional numbers for the same faeces accordingly show important mutations (from 23 to 1294), which throw some light on the considerable variations of these numbers for the faeces of Tables I and II.

In Table VI the results of a second sample of faeces are recorded.

TABLE VI. Faeces N⁰ 2, placed at 37° C.

Periods at 37° C.	Quantity of examined faeces in grs.	Percentage of solid subst.	Numbers of bact. found by *micr. counting* and estimated on 1 gr. of faeces.	Percent diminution of microsc. counted bacteria	Number of bact. *cultiv.* on alk. gel. and estimated on 1 mgr.	Proportional numbers.
Direct examination	10.440	15.15	44.579000	100	57000	782
After 1 day	11 138	19.87	27.857000	— 39 %	61300	454
» 3 days	11.070	19.18	24.494000	— 45 %	298500	82
» 5 »	10.510	19.09	35.297000	— 20 %	75600	467
» 8 »	10.830	16.01	12.580000	— 73 %	50300	250

Here again a strong decrease (after 8 days with 73 pCt.) of the total number of present bacteria, and also no increase of cultivable organisms (after 8 days there are as many as at first).

It appears thus that in the faeces circumstances occurred, which counter-acted the propagation also of the cultivable, consequently of the certainly living organisms; about the being, or not being fitted for increase of the non-cultivable bacteria, these experiments could accordingly not teach us anything. We should hence try to improve the conditions of increase for the bacteria in the faeces.

Starting from the supposition, that perhaps a special substance (or substances) in the faeces caused the death of the cultivable and decidedly living bacteria, or at least prevented their propagation, it might be expected that dilution of the faeces would also attenuate this anti-bacterial influence; this expectation has been completely realized. In Tabel VII a third sample of faeces is referred to, treated in quite the same way; even after 112 days the number of bacteria cultivable on alkalin gelatin is not increased; this number continues mutating from the beginning to the end, between 322 and 743 on 1 mgr. of faeces, which mutations may be reckoned to fall quite within the limits of the errors of the method. The total number of bacteria has again regularly diminished and, after 112 days amounts to only 5 pCt. of the primitive number.

Quite different, however, is the process in the dilute faeces (Table VIII); the faeces-emulsion, diluted with 100 cm^3. water, obtained in the „direct examination" (Table VII) was likewise placed at

TABLE VII. Faeces N⁰. 3, placed at 37° C.

Periods at 37° C.	Quantity of examin. faeces in grs.	Percentage of solid substance.	Number of bacteria found by *micr. counting* and estimated on 1 mgr.	Percent diminution of microsc. counted bacteria.	Number of bact. (on 1 mgr. of faeces) *cultivated* on: alk. gel. 22° C.	acid malt gel. 22° C.	anaërobiontic.	agar-agar at 37° C.	Proportional numbers to alk. gel.
Direct examinat.	11.930	18.79	39.466000	100	322	—	—	—	Very great (122.566)
After 1 day	10.650	15.89	28.753000	—28%	0	—	—	—	
″ 3 days	10.160	16.19	32.662000	—18%	189	—	—	—	″
» 5 ″	10.690	15.72	23.977000	—41%	179	—	—	—	″
» 7 ″	10.800	15.00	21.734000	—46%	178	—	—	—	″
» 9 ″	10.940	14.86	18.373000	—54%	0	—	—	—	″
» 18 ″	10 680	15.34	14.272000	—64%	359	39520	0	0	″
″ 38 »	10.870	15.84	17.002000	—56%	353	5650	0	—	″
″ 112 ″	10.320	16.11	1.830000	—95%	743	0	0	87000	»

body-temperature and examined after different periods; the results are again calculated on 1 mgr. of the primitive faeces.

TABLE VIII. Dilute Faeces N⁰. 3, placed at 37° C.

Periods at 37° C.	Quantity of examined faeces in grs.	Percentage of solid substance.	Number of bacteria found by *micr. count.* and estimated on 1 mgr. of faeces	Increase with	Number of bacteria cult. on alk. gelatin and estimated on 1 mgr. of faeces.	Increase with	Proportional numbers.
Direct examination	11.930	18.79	39.466000	above	322	above	122.566
After 2 days			31.435000	30	2.700000	30	12
″ 4 ″			70.107000	millions	30.700000	millions	$2^1/_4$

Here we directly see an increase of the cultivable organisms on alk. gelatin), and after 4 days a proportional number is reached already nearly corresponding with that of pure cultures ($1^2/_3$): the anti-bacterial effect of the faeces is thus sufficiently removed by the dilution. Besides, something else appears from this Table; after 4 days the number of cultivable organisms has increased with beyond 30 millions, but the number of bacteria, found by microscopic counting, too, is in that time augmented with more than 30 millions. We may therefrom deduce that only the cultivable organisms have multiplied and that the greater majority of microscopically countable bacteria was no more able of propagation.

Not all faeces, however, display a distinct anti-bacterial action, and in Table IX an investigation is referred to, where, directly in the faeces themselves, an increase of the cultivable bacteria could be observed.

TABEL IX. Faeces N° 4, placed at 37° C.

Periods at 37° C.	Quantity of examined faeces in grams.	Percentage of solid substance in the faeces.	Number of bacteria found by *microsc. counting* and estimated on 1 mgr. of faeces.	Increase with	Number of bacteria *cultivat.* on alk. gelatin and estimated on 1 mgr. of faeces.	Increase with	Proportional numbers.
Direct examinat.	11.770	33.07	17.765000	above 9 millions	79200	above 9 millions	224
After 1 day	10.372	36.84	27.053000		9.839000		3
» 3 days	10.130	39.24	28.533000		8.535000		3

In these faeces, too, we find the preceding experiment confirmed. After one day the number of cultivable organisms has increased with more than 9 millions, the total number shows the same rising. If all the 17 millions, thus *also* the large number of non-cultivable ones, had come to multiplication, we should have found, after 1 day, a much greater total number than is now the case; where the increase of the total number is again in accordance with that of the cultivable number, only the cultivable ones can have propagated.

The same faeces diluted (Table X) demonstrate the corresponding fact; after 3 days the number of cultivable organisms has increased with 21 millions, the total number with 20 millions. A slight anti-bacterial action appears neither to lack in these faeces: in the dilute faeces the number of cultivable bacteria has increased after 3 days more than in the non-dilute

TABLE X. Dilute Faeces N° 4, placed at 37° C.

Periods at 37° C.	Quantity of examined faeces in grs.	Percentage of solid substance.	Number of bact. found by *microsc. counting* and estimated on 1 mgr. of faeces.	Increase with	Number of bact. *cultivated* on alk. gel. and estimated on 1 mgr. of faeces.	Increase with	Proportional numbers.
Direct examinat.	11.770	33.07	17.765000	above 20 millions	79200	above 21 millions	224
After 3 days			47.250000		21.310000		1.7

In accordance with all these researches we may conclude, that by far the majority (on the average nearly $99^0/_0$) of the thousands of milliards of microscopically countable bacteria, excreted in 24 hours with the faeces, by a healthy, full-grown individual, are no more able to multiply; hence, these bacteria must be either dead, or at least so much attenuated that they can no more perform this important vital function.

Between these two alternatives it is not difficult to make a choice. We see namely (Tables V, VI and VII), that the total number of bacteria in the faeces, kept at body-temperature, decreases regularly with great rapidity, which can only be explained by the decomposing and vanishing of dead organisms; in fact, in these faeces, too, the number of bacteria showing post-mortal alterations is seen steadily to increase. [1])

There is no risk in venturing to assume, that the same circumstances, which in most faeces out of the human body cause the number of living bacteria to decrease, respectively prevent their multiplication, that these same circumstances also have their part in the death of that enormous number in the faeces, whilst these are still enclosed in the intestinal canal; future researches will have to throw more light on this point.

The results of the foregoing researches are in brief as follows:

1st By a healthy, full-grown individual are excreted in 24 hours with the faeces a *much greater* number of bacteria than was hitherto known (on the average 8800 milliards, occupying 0,31 pCt. of the solid substance of the faeces)

2nd By far the greater majority of these bacteria have *died* (on the average nearly 99 pCt.), only a small percentage are living (on the average above 1 pCt.).

3rd In most faeces *anti-bacterial* actions may be observed, which, out of the human body, at 37° C., often diminish the number of living germs, or at least prevent their (vigorous) increase.

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[1]) This, and numerous other microscopic observations cannot be entered upon here.

Chemistry. — Dr. C. PREY: "*Synthesis of trioxybutyric acid (erythric acid)*". (Communicated by Prof. C. A. LOBRY DE BRUYN).

The synthesis of erythric- or trioxybutyric acid

$$CH_2OH\ CHOH\ CHOH\ COOH$$

from a substance with three C-atoms
(or more correctly, that of one or more of its four possible isomers) obtained by oxidation of erythrite, had not as yet been accomplished. Its realisation would not only be interesting in itself but also extend our knowledge of the sugars with four atoms of carbon. Then if one or more trioxybutyric acids should have become readily accessible, the transition to the corresponding tetroses could present but few difficulties.

The starting point in Dr. PREY's research was acroldehyde (acrolein). In different ways it could be attempted to realize the synthesis of trioxybutyric acid by means of this substance of which about 3 5 Kilos were used in the investigation.

First of all the acroldehyde was converted into the dibromide, the latter treated with nascent HCN and the nitrile thus obtained treated with strong hydrochloric acid. It appeared that the dibromonitrile is very unstable and very readily eliminates HCN; the numerous experiments made with this nitrile did certainly lead to the formation of the dibromo-amide: $CH_2Br\ .\ CHBr\ .\ CHOH\ .\ CONH_2$, but the yield was so small that the object could not be satisfactorily gained in this way.

Therefore, Dr. PREY now used the method which had been applied by VAN DER SLEEN [1]. From the acroldehyde was first prepared the nitrile of α-hydroxybutenoic acid: $CH_2 : CH\ .\ CHOH\ CN$ and this was then converted by concentrated hydrochloric acid into the amide [2]) and from the latter the dihalogen derivatives $CH_2X\ CHX\ CHOH.CONH_2$ were obtained. Besides the well defined crystals of the dibromo-amide already known by VAN DER SLEEN, Dr. PREY prepared the crystalline dichloro- and diiodo derivatives. The first is formed by acting on the amide with one mol. of chlorine dissolved in carbontetrachloride; a simultaneously occurring substitution diminishes the yield. The diiodo derivative is formed quantitatively and very rapidly

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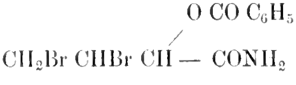
1) See following article.

2) From 2.8 Kg. acroldehyde ± 250 grs. of amide.

under the influence of sunlight; in the dark the addition takes place very slowly and almost imperceptibly.

Dr. PREY has tried in many ways, with sacrifice of much time but in vain, to prepare trioxybutyric acid from the dihalogen compounds starting with the dibromo-derivative. The replacement of the two Br (or I) atoms by hydroxyl or by the residues of acetic or benzoic acid did not succeed; this is caused by the great unstability of the halogen derivatives which very readily lose HBr (or HI) and often break up with evolution of carbondioxide.

Some experiments with the benzoylester of the dibromo-amide



$$CH_2Br\ CHBr\ \overset{O\ CO\ C_6H_5}{CH} - CONH_2$$

which was prepared by direct action of benzoyl-chloride did not lead to any result. These efforts will however be repeated in different directions.

The synthesis of trioxybutyric acid finally succeeded by treating the amide with an alkaline-solution of potassium permanganate, which converted the

$$CH_2 = CH—CHOH—CONH_2$$

directly into

$$CH_2OH — CHOH—CHOH\ COOH,$$

which was isolated in the form of the brucinesalt; from the latter the crystallised lactone [1]) was obtained.

The yield, however, is but very small.

It appears from the study of Drs. VAN DER SLEEN [2]) and PREY that an unsaturated oxyacid, such as α-hydroxybutenoic acid,

$$CH_2\ CH\ .\ CHOH\ .\ COOH,$$

is a very unstable substance capable of decomposing and breaking up in various directions.

The investigation of this substance will be continued.

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[1]) RUFF. Ber. 32 3678.

[2]) See following article.

Chemistry. — Mr. G. van der Sleen: "*On α-hydroxybutenoic acid (vinylglycollic acid) and its decompositions*". (Communicated by Prof. Lobry de Bruyn).

The investigation of vinylglycollic acid $CH_2:CH.CH(OH)COOH$, commenced in 1885 by Lobry de Bruyn [1]) has been again taken up [2]) in 1898; the various preparations were then obtained in large quantities. 600 grams of acroldehyde (acrolein) could be easily prepared in one day and converted by addition of HCN into the nitrile of α-hydroxybutenoic acid $CH_2:CH.CH(OH)CN$.

This substance, when pure, forms a colourless liquid b. p. (93—94° at 16—17 mm.), and when freshly distilled it is free from HCN. It does not solidify at a temperature of — 70° to — 80° and is soluble in water in any proportion. By heating with acetic anhydride it was converted into the acetate.

Both the nitrile and its acetate are converted by very careful treatment with fuming hydrochloric acid into the amide $CH_2:CH.CH(OH)CONH_2$, which melts at 80.8°. This pure amide has served as the starting point for all further experiments. It may be converted into α-hydroxybutenoic acid by different means, such as by boiling with normal sulphuric acid or strong aqueous sodiumhydroxide. This acid was also obtained in fairly large quantities as a bye-product in the preparation of the said amide and purified by distillation in vacuum (125°—130° and 12—13 mm.) although not without partial decomposition into lactone-like substances.

The acid crystallises in the form of needles which, however, cannot be obtained perfectly anhydrous by distillation; it is exceedingly hygroscopic and but sparingly soluble in the ordinary organic solvents. The ester of α-hydroxybutenoic acid was obtained by treating the alcoholic solution of the nitrile with HCl: b.p. 173° (at 756.5 mm.). This ester is isomeric with that of aceto-acetic acid but gives no reaction with $FeCl_3$.

Although the nitrile and the acetate absorb bromine with difficulty (they however strongly reduce an alkaline solution of potassium permanganate), the dibromide may be readily prepared from the amide and the acid. The dibromo-amide does not melt but chars at a higher temperature, the dibromo-acid melts at 121—121.5° and yields α-hydroxybutyric acid when reduced with sodium amalgam. α-Hydroxybutenoic acid is one of the strongest organic acids composed of C, H and O; its dissociation-constant was found to be 0.046.

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[1]) Rec. **4**. 221.

[2]) Preliminary communication Rec. **18**. 302.

If we decompose the amide by heating it with strong acids or by boiling it with dilute alkalis a small quantity of an isomeric ketonic acid is produced which proved to be α-ketobutenoic (propionylformic) acid $CH_3 . CH_3 . CO\ COOH$. The statements found in the literature on the properties of this acid are mostly somewhat incorrect; the different authors have, apparently not obtained it in a pure condition, or else in too small a quantity and as a liquid. The acid which I prepared by decomposing the silver salt with dilute hydrochloric acid was purified by distillation in vacuum (b.p. 73°—75° at 15 mm., or 85° at 21 mm.). The acid is crystallisable, and melts at 31°.5—32°.

Even when highly diluted, the solution of the acid or of its salts yields with phenylhydrazine a compound melting at 143°—144°.

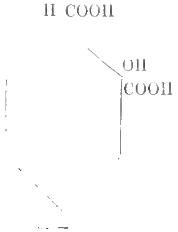
The ethylester of propionylformic acid was obtained by the action of ethyliodide on the silver compound: 66--67° (16 m.m.), m. p. of the hydrazone 191°. This ester is also isomeric with that of acetoacetic acid but gives no reaction with $FeCl_3$. The oxime of propionylformic acid $C_2H_5 . C : N (OH) COOH$ melts with decomposition at 154°.

That the ketonic acid really possesses the said constitution was proved by the fact that the CO group cannot occupy the β- or γ-position and also by its conversion, by reduction, into oxybutyric acid.

The conversion of α-hydroxybutenoic acid into propionylformic (α-ketobutenoic) acid under the influence of acids is not complete; very soon we meet with a state of equilibrium in which the α-hydroxybutenoic acid remains for the greater part unchanged (with small quantities of lactone-like substances) [1]).

The chief product of the action of alkalis is a new acid which was also obtained crystalline and has the composition of two mols. of α-hydroxybutenoic acid minus one mol. of water.

A further investigation showed that to this acid must, probably, be assigned the constitution:



Δ-3-tetrahydro-2-oxyphthalic acid (1,2)

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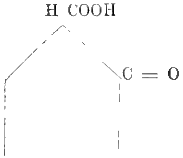
[1]) Perhaps a lactone has formed analogous to that recently obtained by Dr. DE JONG from pyruvic acid. Rec. **20**. 81.

From the increase of the boiling point of the solution in acetone its average molecular weight was found to be 185, 186 being the calculated figure. The equivalent weight was found to be about 97 by titration with phenolphthalein, theory requiring 93, but the end reaction is not very distinct. A lactone-like compound does not exist in the molecule.

The crystals are monoclinic and may be obtained by the simultaneous use of three solvents: acetone, ether and ligroin. They are but little soluble in most of the organic solvents excepting water, acetone and alcohol. A methylester could not be obtained, at least not for certain; no OH group could be detected by means of aceticanhydride or phenylisocyanate. This is caused by the little stability of the acid; at the temperature of its melting point (142—142.5°) it loses one mol. of CO_2; even the aqueous solutions, particularly those of its salts, very readily lose CO_2. The barium salt is very soluble and anhydrous, the zinc salt is also very soluble, silver solutions are rapidly reduced.

By the elimination of 1 mol. of CO_2 a ketonic acid is formed which has, very probably, the constitution:

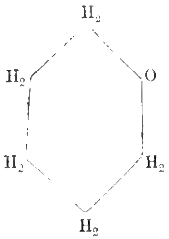


orthoketahexahydrobenzoic acid and was obtained by distillation in vacuum as a colorless glycerol-like liquid.

The phenylhydrazine compound melts at 159°. An oxime was prepared by warming the dibasic oxyacid with hydroxylamine in the presence of Na_2CO_3; it decomposes at about 177°.

The solution of the acid gives a violet coloration with $FeCl_3$, a reaction which agrees with the β-position of the CO group.

By further elimination of CO_2 from this ketonic acid an unsaponifiable oil is formed, sparingly soluble in water, which probably contains cyclohexanone.



The investigation of these latter products could not be brought to a finish for want of material; moreover, they are only produced in small quantities.

From the foregoing it appears that the action of acids and alkalis on unsaturated α-oxyacids does not always take the course indicated by FITTIG. No γ-ketonic acid was formed by the action of acid but only a small quantity of an α-ketonic acid, although FITTIG had observed the formation of γ-ketonic acid from α-oxypentenoic acid. Apparently a shifting of the double bond has taken place in the $\Delta^{\beta\gamma}$ α-hydroxybutenoic acid, under the influence of acids or alkalis causing the formation of intermediate $\Delta^{\alpha\beta}$ α-hydroxybutenoic acid:

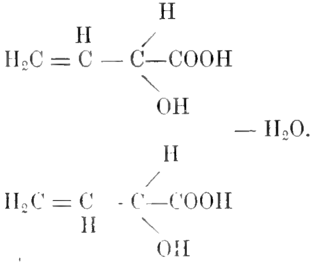
$$CH_2 : CH.\ CH\ (OH)\ COOH \rightarrow CH_3\ CH : C(OH)\ COOH$$

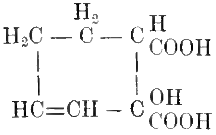
a change already repeatedly noticed in the case of other acids.

Such a combination of OH with the double-bonded C-atom is, however, not stable and must pass into the isomeric keto-compound.

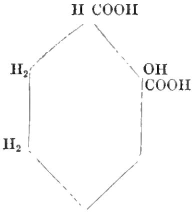
$$CH_3\ .\ CH = COH - COOH \rightarrow CH_3 - CH_2 - CO - COOH$$

By the action of alkalis on phenyl-α-oxycrotonic acid, FITTIG obtained this same change almost quantitatively, whilst in the case of α-hydroxybutenoic acid a dibasic acid is formed (probably a hydride of oxyphthalic acid), for instance:

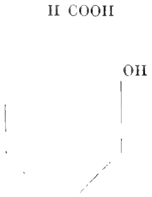




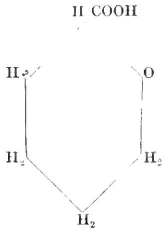
or



The formation of the ketonic acid from this latter acid may be explained in quite a similar way by assuming that the double bond is shifted towards the carbonyl group, CO_2 being eliminated at the same time:



A compound is then formed with OH attached to a double-bonded C-atom which must change directly into a keto-compound:



A more extensive report of this research will appear later on in the „Recueil".

(June 26, 1901).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday June 29, 1901.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige

Afdeeling van Zaterdag 29 Juni 1901, Dl. X).

The following papers were read:

Astronomy. — Mr. H. G. VAN DE SANDE BAKHUYZEN president of the committee for the organisation of the observations of the solar eclipse being absent, Mr. E. F. VAN DE SANDE BAKHUYZEN reads a communication "*on the results of the observations of the Dutch eclipse-party*".

Some days ago letters have been received from India on the results of the observations of the Dutch eclipse party at Karang Sago, and to our regret we have learned from them that the circumstances during the eclipse have been even more unfavourable than we had inferred from the telegrams received.

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The sky has continually been for the greater part overcast by alto-cumuli, and only now and then the sun was uncovered during some moments. During totality the sun was uninterruptedly behind the clouds which only towards the end became thinner.

This was the cause that, after the considerable dispersion it had to undergo in the great spectrograph, the light of the corona was too faint to make a perceptible impression on the photographic plate, and hence, contrary to what we first inferred from the telegrams, the very important observations with this instrument, from which we hoped to derive data on the mode of rotation of the corona, have entirely failed.

Some results are obtained with the small spectrograph and with the prismatic camera, even though they have suffered much from the unfavourable circumstances. The object of these two instruments was to obtain images of the whole spectrum of the corona, and of that of the „flash'' at the beginning and the end of totality. This purpose has been attained reasonably well with regard to the second flash and perhaps also to some extent with the corona-spectrum. Moreover the flash has been observed in the visual slit-spectroscope.

The images of the corona have better succeeded than the spectral-plates. The 6 plates exposed in the focus of the 40-feet-lens are all satisfactory and 3 of them are even good.

Six photographs have been made with the telescope of 10 inches aperture without using the BURCKHALTER-apparatus, of which 3 have succeeded. During the three exposures with that apparatus the images have been slightly displaced, yet they can partly be used. Of the 6 photographs made with each of the smaller telescopes and camera's, among which DALLMEYER's telescope, half can probably be used. - The latter result may relatively speaking be called satisfactory; the very unfavourable condition of the weather appears among others from the fact that on one of the plates exposed in the focus of the portrait-lens nothing but clouds are seen and no trace of the corona is visible.

Observations with the polarimeter were made before, during and after totality. The observations during the partial phases at a distance from the sun equal to the sun's diameter gave no polarisation. During totality the light near the sun was distinctly partly polarised, and this was strongest at some distance from the sun; it may be however that the clouds have had a perceptible disturbing influence on this result.

With Prof. JULIUS' apparatus for the measurement of the radiation of heat observations were made from the moment of the first contact until 3 quarters of an hour after totality. The apparatus worked satisfactorily, but the clouds rendered its results very variable. Nothing has been obtained by means of the photometer, as the intensity of light of the corona weakened by the clouds remained below the minimum that could be measured with the apparatus. Finally the meteorological and the magnetical observations have partly succeeded. Shadow-bands were not observed.

The totality began a few seconds earlier than was expected; but, as had been agreed upon, the observers were warned by a signal from the ship of one of the English observing parties, which anchored west of the Dutch eclipse station. Also the duration of the totality differed from the value predicted, it lasted 6 m. 22 sec. i.e. 10 seconds less than according to the Nautical Almanac. During the totality it was not very dark, which possibly was partly caused by the reflection of light from the clouds. Mercury, Venus, Aldebaran and some stars of Perseus were visible.

The Dutch observers would not have obtained better results if they had selected another place on "Sumatra's West coast" near the central line for their observing station. At different points near the coast the conditions were almost the same as those at Karang Sago, and at Solok, situated at a considerable height on the railway to the interior, they were still more unfavourable; there the phenomenon was rendered entirely invisible by thick clouds.

On the other hand at Fort de Kock also in the interior, but near the northern limit of totality, the weather was very bright. There the diffraction grating of ROWLAND of the Dutch expedition, with which a spectrograph had been constructed, was mounted and given in charge of Mr. KERKHOF Captain Indian army, who had first practised with this instrument at Karang Sago. As has been said before, the intention was to study in this way more closely the reversing of the FRAUNHOFER-lines at the beginning and the end of totality, the so-called flash. But unhappily the inaccuracy in the prediction of the phenomenon proved fatal in this case. While everything had been arranged to photograph one long lasting flash the phenomenon appeared to be different: there was a double flash interrupted by some moments of real totality. However at Fort de Kock a good series of corona-drawings has been made by pupils of the trainingschool for native teachers.

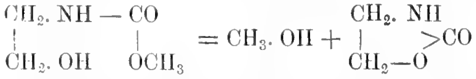
It is much to be regretted that everywhere the weather has been so unfavourable to the observations of this very interesting eclipse. Otherwise we might have expected that very important results would have been obtained. For, to mention only the Dutch expedition, everything had been very carefully organised, through the management of the East-Indian committee and by the observers themselves who, notwithstanding the short time at their disposal, had succeeded in properly mounting and adjusting all the instruments, while, thanks to the collaboration of some officers of the army and the assistance of the officers and the men of H. M. S. Sumatra sent for the purpose to the road of Painan, there was a sufficient number of observers and assistants for each instrument.

Chemistry. — Professor A. P. N. Franchimont presents a communication on "*A new class of nitramines*".

On January 26 1895 I had the honour of communicating to the Academy that, acting under my directions, Mr. van Breukeleveen had prepared a representative of a then unknown class of urea derivatives, *ureo alcohols* or *ureols*. The object was to subject the compound (*ureo-ethanol*) to nitration and in this way to prepare a nitramino-alcohol; this, however, did not succeed as ureo-ethanol in contact with $NO_3\,H$ at the ordinary temperature at once evolves N_2O and CO_2.

At that time only nitraminohydrocarbons, nitramides and derivatives of nitramino acids were known. A free nitramino acid, nitramino-acetic acid has been since prepared by Hantzsch and Metcalf. As the study of nitramino-alcohols and nitramino-aldehydes still interests me very much I have now tried, jointly with Dr. Lublin, to prepare a *nitramino-alcohol* in a different way and this has given us the desired result.

As on the previous occasion, we again started from amino-ethanol which was first converted by means of methyl chloro-formate into methyl oxethylamino-formate. This substance is, at the ordinary temperature, a colourless liquid which cannot be submitted to distillation but is decomposed by the action of heat even in vacuo into methyl alcohol and the internal ester of oxethylaminoformic acid, which crystallises beautifully from benzene, although but little soluble therein.



$$\begin{array}{l}CH_2.\,NH - CO\\ |\qquad\qquad\quad |\\ CH_2.\,OH\quad\; OCH_3\end{array} = CH_3.\,OH + \begin{array}{l}CH_2.\,NH\\ |\qquad\quad >CO\\ CH_2 - O\end{array}$$

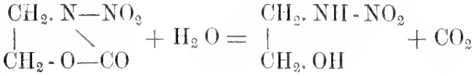
It is a remarkable fact that the ethyl ester of oxethylaminoformic acid, which we have also prepared and obtained as a liquid, does not when heated, yield ethyl alcohol or ethylene and water but distils unaltered, at least in vacuo.

The above mentioned internal ester of oxethylaminoformic acid had been previously obtained by Gabriel in a different and less expensive manner which we have afterwards imitated.

Viewed in connection with oxazol, we may call this heterocyclic compound: μ ketotetrahydro-oxazol, μ keto-oxazolidin or tetrahydro-oxazolon.

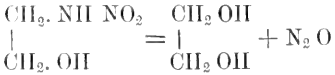
It proved to be a better starting point for the preparation of nitramino-alcohol than the methyl ester. Poured into real nitric acid it at once yields quantitatively a mononitroderivative which is not further acted upon by nitric acid, so that it may be evaporated with four times its weight of this acid to dryness on the waterbath without suffering any decomposition; the same has been previously stated for hydantoïne and its derivatives. This mononitroderivative is more soluble in hot benzene than the internal ester and crystallises on cooling in beautiful, long, lustrous needles which melt at 111°.

This nitrocompound in which the nitro-group is undoubtedly linked to the nitrogen yields on boiling with water quantitatively CO_2 and *nitramino-ethanol* (oxethylnitramine) the desired nitramino-alcohol, according to the equation:



The product which remains dissolved in the water is obtained on evaporation as a syrupy liquid. On boiling the aqueous solution with mercuric oxide a mercury salt was prepared which is but very sparingly soluble in cold water and crystallises in fine needles. A silver salt was obtained in beautiful, nacreous plates by heating the aqueous solution with silver carbonate. Both salts have been analysed. On heating they explode, the mercury salt more violently than the silver compound. The mercury salt is not decomposed by boiling with water but the silver salt is reduced. We have prepared both these salts because there is then no danger that the alcoholic OH-group will yield a metallic derivative. Nitramino-ethanol does not produce at once a precipitate in a solution of silver nitrate as is the case with methylnitramine and also ethylnitramine, thence the preparation of the silver salt with silver carbonate.

Boiled with dilute sulphuric acid of, say, 5 percent strength, nitramino-ethanol yields quantitatively N_2O according to the equation:



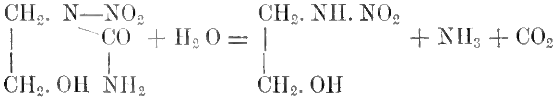
$$\begin{array}{l}CH_2.NH\,NO_2\\ |\\ CH_2.OH\end{array} = \begin{array}{l}CH_2\,OH\\ |\\ CH_2\,OH\end{array} + N_2O$$

Presumably there is also formed some glycol, but at all events no ethylene, no ethylene oxide or aldehyde; but after removal of the sulphuric acid the solution gives with copper sulphate and potassium hydroxide a dark blue liquid like the polyhydric alcohols.

We are now engaged in preparing the nearest homologue and will then try to state the other properties of the nitramino-alcohols.

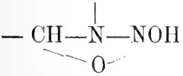
Nitramino-ethanol is perhaps formed from the nitroderivative of the internal ester of oxethylaminoformic acid by another process which is not so simple but not less important on account of the intermediate products. If dry NH_3 is passed through a cold 1 pCt. solution in benzene a white precipitate is formed which to judge from its behaviour, is a NH_3 compound of nitro-ureo-ethanol (oxethylnitro-urea). This compound when boiled with absolute alcohol, in which it is but little soluble, parts with its NH_3, which as we know from experience happens in the case of all compounds of nitramines with NH_3, and on evaporating the solution nitro-ureo-ethanol is obtained which is very soluble in water and alcohol and very slightly so in benzene and chloroform. From these solutions it crystallises in beautiful thin glossy plates which melt at 86°. The aqueous solution gives like urea itself immediately a white precipitate with mercuric nitrate but not with mercuric chloride or silver nitrate unless an alkali is added.

Nitro-ureo-ethanol which, as has been stated above, I had tried in vain to obtain from ureo-ethanol, yields on boiling with barium hydroxide NH_3 and CO_2. Perhaps nitramino-ethanol is formed simultaneously according to the equation:



$$\begin{array}{lll}CH_2. & N\!-\!NO_2 & \\ | & \quad\diagdown & CO\\ | & & |\\ CH_2.OH & & NH_2\end{array} + H_2O = \begin{array}{l}CH_2.NH.NO_2\\ |\\ |\\ CH_2.OH\end{array} + NH_3 + CO_2$$

unless nitrous acid is formed first from the nitro-urea derivative as is the case with nitrohydantoïne, but no certainty exists on this point.

Attention must also be called to the acid character of nitro-ureo-ethanol which not only yields a compound with NH_3 but may be readily titrated in aqueous solution with potassium hydroxide, using phenolphtalein as indicator. It then saturates one mol. of potassium hydroxide and consequently behaves, apparently, like a monobasic acid, such as nitrohydantoïne for instance. If we do not wish to assume that the hydrogen of the group CH_2 is replaceable by metals under the influence of the negative group $-N<\begin{matrix}NO_2\\CO\end{matrix}$, or in other words, that it causes the acid character of the substance, then similar substances and also nitrohydantoïne, ethylenedinitro-ureïne etc. would have a structure totally different from that of the nitramines, which from the manner in which they are prepared have one hydrogen atom linked to the nitrogen. When in these substances the presence of the group  $\underset{\diagdown O \diagup}{-N-NOH}$ is assumed, instead of $-NHNO_2$, nitrohydantoïne etc. ought to contain the group  $\overset{\quad|}{-CH-N-NOH}$ (with O bridging CH and N) unless, as has been rendered highly probable in the case of nitrohydantoïne, the alkali simply forms and eliminates one mol. of nitrous acid. Should oxethylnitro-urea prove to be better suited than nitrohydantoïne to settle this point we will not neglect to have this matter further investigated.

Chemistry. — Professor FRANCHIMONT reads a paper from Miss E. VAN AKEN on: "*The oxidation of organic nitrogen-compounds and the estimation of the carbon and nitrogen therein by the moist process.*"

The oxidation of organic nitrogen compounds is a subject about which a good deal has already been and is still being written. The knowledge thereof is also of practical importance as it is applied not only to the quantitative determination of the nitrogen but also to that of the carbon and nitrogen. The result is that the nitrogen is either liberated or converted into ammonia. This does not only depend on the nature of the oxidizer and the circumstances under which the oxidation takes place, but also, if not in the first place, on the chemical structure of the nitrogen compound. Thence it follows that none of the methods for the quantitative determination of the nitrogen can be applied in every case and even the method of DUMAS, which is the best of all, has given in the hands of some workers bad results with some substances. The same applies to KJELDAHL's method which has now superseded the process of WILL and VARRENTRAPP, but which is still constantly being modified and of which it has been proved that it cannot be employed in the case of substances in which the nitrogen is combined with oxygen or a second nitrogen atom.

The great advantages occurring when an organic nitrogen compounds may be oxidised in such a manner that the carbon and nitrogen are estimated in the same portion, have already led to some investigations but the methods resulting therefrom are not always satisfactory. Those according to which the nitrogen is converted into ammonia by the moist process, such as KJELDAHL's sulphuric acid process, do not yield the total carbon as carbon dioxide, so that other means have to be resorted to in order to obtain the desired result. But neither the electrolytic method of BUDDE and SCHOU, nor the addition of strong oxidising agents, such as chromic acid, do away with the necessity of combining this process with the dry method, namely the passing of the gaseous products over red hot oxidising substances and the separation of the carbon dioxide from sulphur dioxide or sulphuric acid vapour. Even when by this combination the estimation of the carbon is satisfactory, the nitrogen estimation is not always so and particularly not in such cases where KJELDAHL's method does not give good results. BUDDE and SCHOU themselves admit that for cyclic compounds, which have more than one nitrogen atom in the ring, their electrolytic method is quite as unsuitable as KJELDAHL's process. This is, however, not quite correct, for from the results of former investigators it appears that KJELDAHL's method may give very reliable figures indeed in this case. It remains possible, however, that neither the electrolytic method nor the use of oxidising agents give good results in cases where accurate results are obtained by KJELDAHL's method, as has been proved by Miss VAN AKEN in the second instance.

PAUL FRITSCH (LIEBIG's Annalen 1897, 194, 79) describes a method for the determination of carbon and nitrogen in organic compounds by the moist process. It is, he says, a combination of MESSINGER's carbon and KRÜGER's nitrogen process with a slight modification and its essential feature is that the substance is placed in strong sulphuric acid and potassium dichromate added in small portions at the time. The gas evolved is passed by means of a current of air through a red hot tube filled with lead chromate and copper oxide and the CO_2 after having been dried is absorbed in a potash apparatus and weighed. From the residue, which is supposed to contain the nitrogen in the form of ammonia, the latter is expelled by heating with an alkali and estimated by titration.

His conclusion is that this method gives trustworthy figures for the carbon but not always so for the nitrogen; or, as he states a little further on, it is only adapted for such substances which, according to DAFERT, may be safely analysed by KJELDAHL's nitrogen process, such as amides etc.

But already the first amide which Miss VAN AKEN analysed in this manner, namely *urea*, gave wrong figures for the nitrogen; only half the theoretical quantity was obtained. This not only caused the investigation of other amides but also and particularly the application of the process to many other kinds of urea derivatives. In the case of those substances which behaved abnormally a nitrogen determination was also made by the KJELDAHL method as modified by GUNNING, which consists in the simple addition of a certain quantity of potassium sulphate, just to see in what cases the oxidising agent is the cause of the deficiency in nitrogen. A mutual comparison of those cases where a deficiency of nitrogen is found might then perhaps show why one amide gives good and another one bad results.

The selection of the substances was guided by a number of questions which at once presented themselves on considering the matter.

1st. Is the result influenced by the hydrogen atoms of the urea? In order to solve this question substances were analysed in which they were replaced by methyl-groups namely *symetric* and *asymetric dimethylurea*, also *tetramethylurea*.

All three only yielded half of their nitrogen in the form of ammonia, consequently behaved exactly like urea itself. *Acetylurea* in which a negative atom-group replaces an H-atom behaved similarly; the H-atoms, therefore, exercise no influence.

2nd. Is the result influenced by the oxygen of the urea? To answer this, *thiourea*, in which sulphur is substituted for oxygen, was analysed but this also yielded but half of the nitrogen as ammonia. The oxygen therefore, exercises no influence.

3rd. What about cyclic urea derivatives? Of these rings with five and six polyvalent atoms were tested: viz. *hydantoïne*, *parabanic acid*, *dimethylparabanic acid*, *alloxan*, *alloxantin*, *methyluracil* and *malonureïd* which all yielded half their nitrogen as ammonia. Further, dicyclic urea derivatives, combinations of a 5 and a 6 ring namely *uric acid* and *caffeine* which, strange to say, yielded invariably more, although not much more than half the theoretical amount. Finally, *isocyanuric acid*, which may be considered as a cyclic urea derivative with a 6 ring only yielded one-third. Thus while the simple cyclic urea derivatives behave like urea the more complex ones behave in a different manner.

4th. Is it caused by the relative position of the two nitrogen atoms in the urea 1.1 or in other words by their close proximity? To throw some light on this problem analyses were made first of all of *oxamide* and *symmetric dimethyloxamide*, in which two nitrogen atoms are linked to CO but in the position 1.2, therefore not to the same C-atom. The result was that, although by KJELDAHL's method all the nitrogen was converted into ammonia, the oxidation process, although yielding more than one-half, still showed a considerable deficit. *Malonamide* and *succinamide* were also investigated. The first one with the position 1.3 gave less NH_3 than obtained by KJELDAHL's method, but the deficit was not nearly so large as in the case of oxamide. The second one with the position 1.4 showed a somewhat larger deficit.

It therefore, appears that the relative distance of the N-atoms from each other has an influence on the deficit.

This agrees with the result yielded by an amide of an amino-acid, *asparagine*, which also showed a deficit; also with that of *guanidine oxalate* which yielded only one-third of the theoretical quantity.

5th. Is the phenomenon also noticed in the case of amides of other acids than carboxylic acids such as sulphonic acids? Both *benzenesulphonamide* and *benzenesulphondimethylamide* were analysed and each showed a deficit.

From the results obtained it is evident that the method of FRITSCH cannot be used for the estimation of nitrogen without due control and that he is wrong in his conclusion that it may be employed in all cases where good results are obtainable by KJELDAHL's process; this is not the case with all amides and not at all with urea derivatives. It also appears that, when using KJELDAHL's method, an oxidising agent such as chromic acid may cause a danger of finding too little nitrogen.

Although this phenomenon depends perhaps within certain limits on circumstances which it is not always possible to control and which will cause the figures obtained to constantly vary, still it appears that the structure of the nitrogen compound, namely the relative proximity of the N atoms exercises a great influence on the same, a proximity which may be greater or smaller in open as well as in closed chains and depends on their relative position to each other in the space. This result appears sufficiently important to warrant its early publication.

Undoubtedly there are still other influences which are caused by the structure of the nitrogen compounds, but in order to study these a great many more substances will have to be tested. The research will, therefore, be continued.

Microscopy. — Prof. J. W. Moll describes: "*An apparatus for focussing the projecting-microscope from a distance.*"

In the new Botanical Laboratory of the University at Groningen the lecture-room is entirely arranged for the purpose of demonstration by means of projecting-apparatus. I intend to describe elsewhere the principal features of the general arrangements. Here I will only mention one special point. An important part of the projecting-apparatus is formed by the projecting-microscope, of which the magnifying power is 5000 diameters and more. With such highly magnified images it is of special importance that any part of the object can be brought to a sharp focus, which cannot be done by a person who is at some distance from the screen, so as to be unable to distinguish the finest details. Moreover the demonstrator who stands next to the screen must be able during his explanation continually to alter the adjustment of the fine motion-screw, for the same reason which necessitates this adjustement when working with the ordinary microscope.

If therefore high magnifications are often used in the projection, the sharp focussing cannot be left to the assistant at the apparatus, as is generally done in projecting photographic images. It is absolutely necessary that the lecturer himself has a complete control over the adjusting apparatus. In the laboratory at Groningen the distance between the screen and the projecting-apparatus is 6 M., while moreover the latter is placed in a separate small room adjoining the lecture-room.

The idea, which naturally first presents itself, is that of an arrangement by which the adjusting-screw of the microscope can, from the position of the lecturer, be moved at will in either direction. This could be done either by electrical transfer of power [1]), or

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[1]) At the Dutch scientific and medical congress held at Rotterdam in April 1901 an apparatus constructed on this principle was used by Dr. W. Einthoven, of which however I have not seen the details.

mechanically by a connecting rod or by means of so called "flexible tubing", such as is used in boring apparatus of different kinds, e. g. by dental surgeons.

In the present case however there is an objection to such a solution of the problem, *viz.* the bad construction of the adjusting-screw of the projecting-microscope. The projecting apparatus used at Groningen was made by the well known firm NEWTON & CY., London. It is of excellent design [1]) and is, according to my experience, preferable to others. A great drawback however is that the workmanship of the metallic parts is not always of the highest standard. Thus especially the screw for the fine adjustment is far inferior in quality to what is generally found in modern microscopes.

In the case of high magnifications and with a distance of the screen of 6 M. extremely minute alterations of this distance correspond to large differences in focussing. No satisfactory result could thus be expected in this way. Also it proved to be very difficult to replace the bad screw by a better one.

Thus I was led to another solution of the problem which appears to me to be very simple and effective. It is well known that by changing the distance between the ocular- and objective-glasses in an ordinary microscope, different levels of the object can be brought to a sharp focus. In this way the same end is obtained as by turning the adjusting-screw, and even somewhat better, as a motion of the ocular is equivalent to a considerably smaller motion of the screw. Consequently in the ordinary microscope a more exact focussing can be effected by means of the ocular-glass than by the adjusting-screw.

RANVIER [2]) mentions that he has an arrangement fitted to his microscope by which a fine motion can be given to the ocular. This arrangement is used in delicate observations.

Naturally the same principle can be applied to the projecting-microscope, and some preliminary experiments at once gave very satisfactory results.

At first a wooden cross-beam was fixed before the microscope, and entirely free from it. The objective only was left on the microscope, the ocular being fixed on a wooden slide, or sledge, which was so connected to the cross-beam, that it could be moved through about

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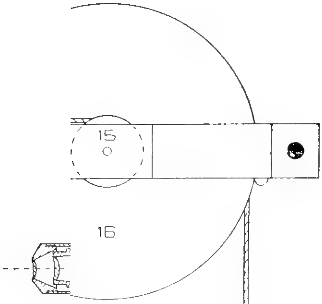
[1]) c. f. LEWIS WRIGHT. Optical Projection. London 1891. This I think is the best work which has been written about methods of projection. The author is also the designer of the microscope by NEWTON & CY.

[2]) RANVIER Technisches Lehrbuch.



G

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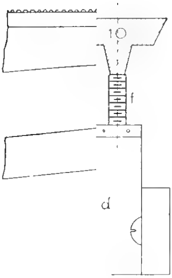

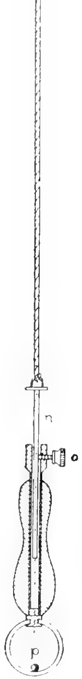

H

5 c. M. in a direction parallel to the axis of the microscope. The distance between the objective- and ocular-glasses could thus be altered by this amount. The motion of the slide towards the screen could be effected by a rope, the other end of which was held by the lecturer at the screen. When the rope was relaxed, the slide was pulled back towards the objective by a weight. The first experiments with this arrangement gave very satisfactory results. The focussing could be effected with at least the same sharpness as in the ordinary microscope, and it was found easy to retain the slide for a considerable length of time in any position, when it was necessary that one particular level should remain in focus for some time.

Thus the principal feature of the arrangement was arrived at, and it was decided to erect a definitive apparatus. In order to secure the necessary stability for the permanent centring of the lenses, this apparatus was made of metal. It is represented in the fig's *A*, *G*, *H* of the plate.

Of course it was also necessary to provide a light-tight but movable connection between the objective- and ocular-glasses, and also a few other details required attention. Firstly it was found possible by a simple device materially to increase the accuracy of the control over the motion of the slide.

The rope runs from the projecting-apparatus upwards, and then along the ceiling of the lecture-room to a place above and beside the screen. There it is fixed to the circumference of a small wooden disc (15 in the figure) of which the diameter in 4,2 c.M. If this disc turns on its axis in such a direction that the rope is wound upon it, the slide consequently is drawn away from the objective. This disc is fixed concentrically to another larger disc (16) of 19,4 c.M. diameter, to the circumference of which the descending rope is so fixed that, when it is pulled, the rope of the smaller disc is wound up[1]). When the rope is relaxed by the demonstrator, the slide is pushed back towards the objective by a pair of strong springs, which replace the weight of the first arrangement. As a consequence of the introduction if this second (large) disc, very little force is required for moving the slide, notwithstanding the very strong counteraction of the steel springs. At the same time the range of motion of the slide, which in the definitive apparatus is 5.5 c.M., is increased

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[1]) Of course it is by no means necessary to place the disc-system exactly above and beside the screen. It can be placed anywhere in the course of the rope, according to local circumstances.

to a range of ± 26 c. M.[1]) for the motion of the hand. Thus a very fine adjustment of the focus is made possible, and it is easy, while pulling the rope, to bring the slide to rest at 20 or more different positions during its motion in one direction. The relative accuracy of focussing is thus here considerably greater than with the ordinary microscope.

Before proceeding to the description of the details of the apparatus I will still call attention to another point.

In the ordinary microscope, before the beginning of the observations, the adjusting screw must occupy a mean position, in order to enable lower as well as higher levels of the object to be brought to focus. Similarly here the slide must occupy a mean position when not in use. The assistant at the microscope focusses on the middle level of the object by means of the adjusting screw, and the demonstrator must then be able by means of the rope to move the slide in either direction. For this purpose there is fixed to the end of the rope a handle (fig. *II*) with a ring, which can be passed on a pin, which has been driven in the wall at a convenient height. The length of the rope must be so adjusted that the slide is in the required mean position when the ring is on the pin. As the discs are high up in the room, the rope in the hand of the demonstrator is long, and he is free to move before the whole of the screen while holding it, in order to point out special details of the image.

It will be seen that the arrangement is very simple and effective. There is however one drawback. When the ocular-glass is moved, the magnification and the illumination of the field are altered, however so slightly that it causes no inconvenience to the audience. As the ocular moves away from the objective the magnification is increased and the illumination is reduced.

Also the changes of level, which are produced by a definite change of position of the ocular, decrease as the ocular moves away from the objective. For this reason the correct optical mean position is not the middle point of the range of motion of the slide, but a point nearer to the objective. The best mean position of the slide has been found by experience to be such that a motion of 2.2 c.M. towards the objective and of 3.3 c.M. towards the screen is possible. In this position of the slide the distance between the objective-

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[1]) Actually the range is somewhat larger (29 c. M.) probably owing to the extensibility of the ropes.

and ocular-glasses is equal to the length of the microscope-tube, which originally belonged to the microscope, *viz.* 15.5 c.M. The design of the lenses did not allow to bring the whole slide with the ocular nearer to the objective, with a view to increasing the illumination.

Having thus explained the principal features I proceed to a detailed description of the apparatus as erected in my laboratory.

As has been said above, the figures *A*, *G*, *H* of the plate represent a side-view of the whole apparatus and a section of some parts of it. On the left-hand side of the figure the front part of the projecting-microscope, is also shown. The ocular (8) is entirely free from the microscope and is mounted on the slide, which rests on a metallic cross-beam (10). This cross-beam is fixed before the microscope to two projecting rafters of the wall of the small room in which the projecting-apparatus is placed. The part of the rope from *A* to *G* is high up in the lecture-room. The two discs are shown at *G*, and *H* represents the handle with the ring, which is kept in its place by the pin *p*.

The slide is shown separately, without the ocular, in fig. *B*, seen from above, and in *C* a part of it is shown in longitudinal section. *D* is a cross-section of the slide together with the cross-beam (10) on which the slide rests and the ring (14), which carries the ocular-tube (7).

Figures *E* and *F* represent some other parts, which will be mentioned further on.

Proceeding now to a more detailed description of the figures, I begin at the left hand side of fig. *A*, which shows that part of the microscope carrying the objective. At 1 we see the head of the bar of the projecting-microscope [1]), which originally carried the whole apparatus. This bar is in my laboratory fixed in a kind of fork (2) of which fig. *E* gives a cross-section. The obliquely descending metal plates (*a*) are firmly connected to the wall of the room. They carry a bent piece of metal (*b*) in which the microscope-bar is fixed by the screw *c* and the plate *d*. In fig. *A* the two adjustable pins (*e*, *e*) are shown, by which the plate *d* is kept in position. This rigid mounting of the microscope has been found

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[1]) The projecting-apparatus of the laboratory at Groningen is NEWTON's new patent triple rotating electric lantern (N⁰. 5345 of the catalogue), which is furnished with the patent electric lantern microscope and micro-polariscope N⁰. 5350, and a BROCKIE-PELL arc-lamp of 40 Ampères. The lantern is placed on an easily movable stand.

necessary independently of the focussing-apparatus described in this communication. During the microscopic projection the automatic centring of the lamp must now and then be corrected by hand and this causes vibrations of the microscope, which are very troublesome.

The block 3 can be moved along the bar 1 by means of a toothed wheel (4). This block 3 originally carried the whole microscope-tube, but now it only supports the tube 9 in which the objective is fitted by means of the tube 6. The rough focussing is done by turning the head 4, while the finer adjustment is effected by means of the screw 5. Every thing that has been described so far belonged to the original apparatus, with the only exception of the objective tube 6 in which the ocular tube 7 can slide in and out, without even at the most outward position admitting any troubling rays of light. Both the objective- and the ocular-tubes have a wider and a narrower part. This is not essential; it was only found necessary to make them thus, in order to be able to use the existing ocular and the existing tube 9, into which originally the whole microscope was fitted. When constructing a new apparatus, it would be better to make both tubes of a uniform width throughout.

I now proceed to describe the details of the new part of the apparatus. The ocular-slide is carried by the cross-beam (10) of cast iron. As it is often necessary in the course of one and the same lecture to use alternately projection of microscopic preparations and other forms of projection, the whole apparatus must be easily removable. For this reason the cross-beam is hinged at one extremity, so that it can be lifted up with the whole apparatus connected with it. In this position the beam is held by a hook, so that the space before the lantern comes free for another front. The other extremity of the beam is in the working position kept down by a catching arrangement, of which a section is shown in fig. *F*. The block *d* is fixed to a projecting rafter of the wall, and carries a catch-spring *b* which keeps the beam in its horizontal position. By the adjustable screw *f* any play which might exist between the beam and the spring, can be abolished. In order to prevent the handle *H* from sliding away from the pin *p* when the beam is lifted up, a rather heavy weight is fixed to the rope immediately above the projecting-apparatus.

The sliding-apparatus itself consists of a base-plate (11) which is cast in one piece with the cross-beam 10. This plate carries two ridges (12) forming a dovetail-guiding for the slide 13, which carries the ocular. This slide carries a ring (14) in which the ocular-tube 7 is fixed, which again carries the ocular 8. The slide 13 has two projections of half-cylindrical form (*g*), each containing a cylindrical hole. Two rods (*h*) occupy the central lines of these holes. These rods are fixed to the base-plate at *i*, *i*. Their other extremities project through openings in the slide, which can thus move freely in either direction. Round these rods are two spiral springs which are compressed when the slide is moved towards the right, and therefore push it back towards the objective when the rope is relaxed. Round the part of each rod, which comes outside the hollow pieces *g*, is a brass tube which guides the spring and prevents it from bending. This detail is well shown in fig. C.

On the slide is fixed a metal cross-piece which carries at its ends two cylindrical blocks *l* in which the ropes are fixed. These ropes run over the pulleys *m*, *m* and are joined together at a short distance above the apparatus. The combined rope then runs upwards and through the lecture-room to the disc 15 of fig. *G*, of which the diameter is 4.2 c. M. The diameter of the larger disc (16) is 19.4 c.M. The rope is made of twisted metallic wire [1]) of 2.1 m. M. diameter, which is very strong and inextensible. This is necessary with a view to the strong counteraction of the springs of the slide.

In fig. *H* the handle for the demonstrator is shown. The required mean position of the slide can be arrived at, as already explained, by adjusting the length of the rope and the height of the pin *p*. But it may naturally be necessary afterwards to alter that adjustment, either because the rope may have been stretched, or if in a special case it is desired to use another mean position than usually. For this reason the handle is made hollow. The rod *n* can be slid in and out, and can be fixed at any required position by the screw *o*.

During the course of lectures given in 1899—1900 the original wooden arrangement was used, and in 1900—1901 the metal apparatus here described. Experience has led me during this time to consider the apparatus as an indispensable auxiliary for the projecting-microscope.

The apparatus was made according to my plans by the firm P. J. Kipp & Sons, J. W. Giltay Opvolger at Delft; the rigid mounting for the microscope bar, the double disc and the adjustable handle were constructed by the amanuensis of the botanical laboratory, Mr. J. Veenhoff.

Groningen, 1901 June 22.

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[1]) "Verzinktes Drahtseil" of C. F. Rochlitz, Berlin.

Mathematics. — *"A definite integral containing Bessel's functions"* by Prof. W. Kapteyn.

If $I_m(t)$ and $I_n(t)$ are two Bessel's functions of the first kind and of orders m and n, then

$$\frac{d^2 I_m}{dt^2} + \frac{1}{t}\,\frac{d\,I_m}{dt} + \left(1 - \frac{m^2}{t^2}\right) I_m = 0$$

and

$$\frac{d^2 I_n}{dt^2} + \frac{1}{t}\,\frac{d\,I_n}{dt} + \left(1 - \frac{n^2}{t^2}\right) I_n = 0.$$

If we multiply the first equation by I_n and the last by I_m, we find by subtraction

$$I_n \frac{d^2 I_m}{dt^2} - I_m \frac{d^2 I_n}{dt} + \frac{1}{t}\left(I_n \frac{d\,I_m}{dt} - I_m \frac{d\,I_n}{dt}\right) = \frac{m^2 - n^2}{t^2} I_m I_n.$$

By putting

$$I_n \frac{d\,I_m}{dt} - I_m \frac{d\,I_n}{dt} = U$$

we obtain

$$\frac{dU}{dt} + \frac{1}{t}\,U = \frac{m^2 - n^2}{t} I_m I_n$$

or

$$\frac{d}{dt}(U\,t) = \frac{m^2 - n^2}{t} I_m I_n\,;$$

and after integration between 0 and ∞

$$(U\,t)_0^\infty = (m^2 - n^2)\int_0^\infty \frac{I_m I_n}{t}\,dt.$$

Now for $t = \infty$ we have

$$I_m = \sqrt{\frac{2}{\pi}}\,\frac{1}{t^{\frac{1}{2}}}\cos(t-\alpha)\,,\quad \alpha = \frac{2m+1}{4}\,\pi$$

$$I_n = \sqrt{\frac{2}{\pi}}\,\frac{1}{t^{\frac{1}{2}}}\cos(t-\beta)\,,\quad \beta = \frac{2n+1}{4}\,\pi$$

$$\frac{d\,I_m}{dt} = -\sqrt{\frac{2}{\pi}}\,\frac{1}{2\,t^{\frac{3}{2}}}\cos(t-\alpha) - \sqrt{\frac{2}{\pi}}\,\frac{1}{t^{\frac{1}{2}}}\sin(t-\alpha)$$

$$\frac{d\,I_n}{dt} = -\sqrt{\frac{2}{\pi}}\,\frac{1}{2\,t^{\frac{3}{2}}}\cos(t-\beta) - \sqrt{\frac{2}{\pi}}\,\frac{1}{t^{\frac{1}{2}}}\sin(t-\beta)\;;$$

hence

$$U\,t = \frac{2}{\pi}\,\sin\,(\alpha-\beta) = \frac{2}{\pi}\,\sin\,\frac{m-n}{2}\,\pi\,,$$

whilst for $t=0$ we arrive at $Ut=0$.

So we find

$$\int_0^\infty \frac{I_m\,I_n}{t}\,dt = \frac{2}{\pi}\,\frac{\sin\frac{m-n}{2}\pi}{m^2-n^2}\,,\qquad m \neq n$$

and as a special case

$$\int^\infty \frac{I_n{}^2}{t}\,dt = \frac{1}{2n}\,.$$

From this formula many others may be deduced important for the theory of Bessel's functions.

Mathematics. — Mr. K. Bes: "*Analytical determination of the ninth point in which two curves of degree three, passing through eight given points, intersect each other.*" (Communicated by Prof. J. Cardinaal).

Let

$$\left.\begin{aligned} a_1x^3+a_2x^2y+a_3x^2z+a_4xy^2+a_5\,xyz+a_6\,xz^2+a_7y^3+a_8\,y^2z+a_9\,yz^2+a_{10}\,z^3=0,\\ b_1x^3+b_2\,x^2y+b_3\,x^2z+b_4\,xy^2+b_5\,xyz+b_6\,xz^2+b_7\,y^3+b_8\,y^2z+b_9\,yz^2+b_{10}\,z^3=0,\end{aligned}\right\}(1)$$

be the equations of two curves of degree three expressed in homogeneous coordinates and let these curves have the eight points in common indicated by x_i, y_i, z_i (i from 1 to 8).

The coefficients of the equations (1) form the assemblant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} \end{vmatrix} \quad . \; . \; . \; . \quad (2)$$

and the eight respectively independent systems of roots satisfying the equation (1) form the assemblant

$$\begin{vmatrix}
x_1^3 & x_1^2y_1 & x_1^2z_1 & x_1y_1^2 & x_1y_1z_1 & x_1z_1^2 & y_1^3 & y_1^2z_1 & y_1z_1^2 & z_1^3 \\
x_2^3 & x_2^2y_2 & x_2^2z_2 & x_2y_2^2 & x_2y_2z_2 & x_2z_2^2 & y_2^3 & y_2^2z_2 & y_2z_2^2 & z_2^3 \\
x_3^3 & x_3^2y_3 & x_3^2z_3 & x_3y_3^2 & x_3y_3z_3 & x_3z_3^2 & y_3^3 & y_3^2z_3 & y_3z_3^2 & z_3^3 \\
x_4^3 & x_4^2y_4 & x_4^2z_4 & x_4y_4^2 & x_4y_4z_4 & x_4z_4^2 & y_4^3 & y_4^2z_4 & y_4z_4^2 & z_4^3 \\
x_5^3 & x_5^2y_5 & x_5^2z_5 & x_5y_5^2 & x_5y_5z_5 & x_5z_5^2 & y_5^3 & y_5^2z_5 & y_5z_5^2 & z_5^3 \\
x_6^3 & x_6^2y_6 & x_6^2z_6 & x_6y_6^2 & x_6y_6z_6 & x_6z_6^2 & y_6^3 & y_6^2z_6 & y_6z_6^2 & z_6^3 \\
x_7^3 & x_7^2y_7 & x_7^2z_7 & x_7y_7^2 & x_7y_7z_7 & x_7z_7^2 & y_7^3 & y_7^2z_7 & y_7z_7^2 & z_7^3 \\
x_8^3 & x_8^2y_8 & x_8^2z_8 & x_8y_8^2 & x_8y_8z_8 & x_8z_8^2 & y_8^3 & y_8^2z_8 & y_8z_8^2 & z_8^3
\end{vmatrix} \quad (3).$$

These assemblants are supplementary.

If the determinants contained in the assemblant (3) are represented in the usual way by $X_{1,2}$, $X_{1,3}$ etc., the property of the supplementary assemblants can be expressed as follows:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} : X_{1,2} = - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} : X_{1,3} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} : X_{2,3} = \text{etc.} \; . \; (4),$$

which constant quotient will be expressed by k.

At the 8th Physical and Medical Congress held at Rotterdam in the month of April last I gave a communication in the subsection for pure and applied mathematics concerning the relations existing between the roots of n homogeneous equations with $n+1$ variables and the coefficients of these equations, and I have pointed out i. a. that for such a system of equations the following property exists: the products of the corresponding elements of the n systems of roots, satisfying this equation, stand in the same proportion as the resultants furnished by the system of equation, when every time one of the variables is made to disappear.

This property provides us with the means of expressing the coordinates of the ninth point in the coordinates of the 8 given points. To do so we apply it to the system of equations (1) as follows:

$$\frac{x_1\, x_2 \;.\;.\; x_8\, x_9}{R_{x=0}} = \frac{y_1\, y_2 \;.\;.\; y_8\, y_9}{R_{y=0}} = \frac{z_1\, z_2 \;.\;.\; z_8\, z_9}{R_{z=0}} \;.\;.\; (5),$$

where x_9, y_9, z_9 represent the coordinates of the ninth point and $R_{x=0}$, $R_{y=0}$, $R_{z=0}$ the indicated resultants.

We now obtain when paying attention to the signs $+$ and $-$ by which the resultants are affected:

$$\frac{x_1\, x_2 \;.\;.\;.\;.\; x_8\, x_9}{\begin{vmatrix} a_7 & b_7 & & & & \\ a_8 & b_8 & a_7 & b_7 & & \\ a_9 & b_9 & a_8 & b_8 & a_7 & b_7 \\ a_{10} & b_{10} & a_9 & b_9 & a_8 & b_8 \\ & & a_{10} & b_{10} & a_9 & b_9 \\ & & & & a_{10} & b_{10} \end{vmatrix}} = \frac{-\, y_1\, y_2 \;.\;.\;.\;.\; y_8\, y_9}{\begin{vmatrix} a_1 & b_1 & & & & \\ a_3 & b_3 & a_1 & b_1 & & \\ a_6 & b_6 & a_3 & b_3 & a_1 & b_1 \\ a_{10} & b_{10} & a_6 & b_6 & a_3 & b_3 \\ & & a_{10} & b_{10} & a_6 & b_6 \\ & & & & a_{10} & b_{10} \end{vmatrix}} = \frac{z_1\, z_2 \;.\;.\;.\;.\; z_8\, z_9}{\begin{vmatrix} a_1 & b_1 & & & & \\ a_2 & b_2 & a_1 & b_1 & & \\ a_4 & b_4 & a_2 & b_2 & a_1 & b_1 \\ a_7 & b_7 & a_4 & b_4 & a_2 & b_2 \\ & & a_7 & b_7 & a_4 & b_4 \\ & & & & a_7 & b_7 \end{vmatrix}} \; (6).$$

The denominators of these fractions can be developed according to products of determinants contained in the assemblant (2) which may be replaced in consequence of the equation (4) by determinants derived from the assemblant (3).

In this way we obtain for the first denominator the following reduction:

$$\begin{vmatrix} a_7 & b_7 & & & & \\ a_8 & b_8 & a_7 & b_7 & & \\ a_9 & b_9 & a_8 & b_8 & a_7 & b_7 \\ a_{10} & b_{10} & a_9 & b_9 & a_8 & b_8 \\ & & a_{10} & b_{10} & a_9 & b_9 \\ & & & & a_{10} & b_{10} \end{vmatrix} = k\,X_{7,8} \begin{vmatrix} a_8 & b_8 & a_7 & b_7 \\ a_9 & b_9 & a_8 & b_8 \\ a_{10} & b_{10} & a_9 & b_9 \\ & & a_{10} & b_{10} \end{vmatrix} - k\,X_{7,9} \begin{vmatrix} a_7 & b_7 & & \\ a_9 & b_9 & a_8 & b_8 \\ a_{10} & b_{10} & a_9 & b_9 \\ & & a_{10} & b_{10} \end{vmatrix}$$

$$+ k\,X_{7,10} \begin{vmatrix} a_7 & b_7 & & \\ a_8 & b_8 & a_7 & b_7 \\ a_{10} & b_{10} & a_9 & b_9 \\ & & a_{10} & b_{10} \end{vmatrix} = k^3 \{ X_{7,8}\ X_{8,9}\ X_{9,10} - X_{7,8}\ X^2_{8,10} +$$

$$+ 2\,X_{7,8}\ X_{9,10}\ X_{7,10} - X^2_{7,9}\ X_{9,10} + X_{7,9}\ X_{7,10}\ X_{8,10} - X^3_{7,10} \} =$$

$$= k^3 \begin{vmatrix} X_{7,8} & X_{7,9} & X_{7,10} \\ X_{7,9} & X_{7,10} & X_{8,10} \\ X_{7,10} & X_{8,10} & X_{9,10} \end{vmatrix} + k^3\,X_{8,9} \begin{vmatrix} X_{7,8} & X_{7,10} \\ X_{7,10} & X_{9,10} \end{vmatrix} \quad \ldots\ldots\ldots\ldots \quad (7),$$

where we have turned to account the relation

$$X_{7,8}\ X_{9,10} - X_{7,9}\ X_{8,10} + X_{7,10}\ X_{8,9} = 0 \quad \ldots\ldots \quad (8),$$

existing between the determinants of the assemblant (3).

By developing the other denominators of the equation (6) in the same way, we arrive, after omitting the factor k^3, at:

$$x_1 x_2 \;.\;.\; x_8 x_9 = \begin{vmatrix} X_{7,8} & X_{7,9} & X_{7,10} \\ X_{7,9} & X_{7,10} & X_{8,10} \\ X_{7,10} & X_{8,10} & X_{9,10} \end{vmatrix} + X_{8,9} \begin{vmatrix} X_{7,8} & X_{7,10} \\ X_{7,10} & X_{9,10} \end{vmatrix},$$

$$y_1 y_2 \;.\;.\; y_8 y_9 = \begin{vmatrix} X_{1,3} & X_{1,6} & X_{1,10} \\ X_{1,6} & X_{1,10} & X_{3,10} \\ X_{1,10} & X_{3,10} & X_{6,10} \end{vmatrix} + X_{3,6} \begin{vmatrix} X_{1,3} & X_{1,10} \\ X_{1,10} & X_{6,10} \end{vmatrix},$$

$$z_1 z_2 \;.\;.\; z_8 z_9 = \begin{vmatrix} X_{1,2} & X_{1,4} & X_{1,7} \\ X_{1,4} & X_{1,7} & X_{2,7} \\ X_{1,7} & X_{2,7} & X_{4,7} \end{vmatrix} + X_{2,4} \begin{vmatrix} X_{1,2} & X_{1,7} \\ X_{1,7} & X_{4,7} \end{vmatrix},$$

by which the coordinates of the ninth point are expressed in the coordinates of the 8 other points of intersection.

The obtained results for the products of the corresponding coordinates of the 9 points are forms of the 72nd degree.

Observation: In quite the same way we can determine the eighth point common to three surfaces of the 2nd degree passing through 7 given points. We then obtain for the products of the corresponding coordinates of the 8 points expressions of the 56th degree.

Physics. — J. C. SCHALKWIJK: "*Precise Isothermals V. The isothermal of hydrogen at 20° C. up to 60 atmospheres.* (Communication N°. 70 (3d continuation) from the Physical Laboratory at Leiden, by Prof. H. KAMERLINGH ONNES).

§ 1. The small number of observations made with hydrogen, of which the most important are those by REGNAULT (Mém. de l'Ac. XXI) going up to 28 Atm. and those by AMAGAT, of which the published results begin only at 100 atm., together with the small reliability of the values, which can be derived from them for VAN DER WAALS' quantities a and b made me resolve to contribute to the investigation of the isothermal of hydrogen, which has for many years been under consideration in the Physical Laboratory at Leiden (comp. Comm. N°. 14 by KAMERLINGH ONNES, Proceedings of December 29th '94) and to do this by again experimenting upon this gas at the ordinary temperature.

The apparatus, methods and investigations discussed in the Proceedings of Oct. 29th '98, June 24th '99, Dec. 29th '00, Jan. 26th '01 and May 25th '01 (Comm. Nos. 44, 50, 67 and 70) allow us to make such a precise determination of pressure and volume that we can a priori count upon the possibility of determining the values to be ascribed to a and b by observations at pressures up to 60 atm.

The hydrogen has been prepared as described in Comm. N°. 27, § 5 (Proceedings of May 30th '96) and N°. 60 § 22 (Proceedings of June 30th '00). The four piezometer tubes were simultaneously filled at the apparatus, and were six times entirely exhausted by means of the mercury pump and heat, and then filled again.

§ 2. *The normal volume.* To calculate each isothermal determined by means of a piezometer it is of the highest importance to know the normal volume. It seems that in most of the measurements of other observers its determination has left much to be desired; but KAMERLINGH ONNES' arrangement of the piezometer tubes allows us, as will be seen from this section, to attain the degree of accuracy desired.

Before being placed into the compression apparatus the piezometer tubes were brought into the water bath (represented Comm. N°. 50, Pl. 2, fig. 5) between the two brass walls of which water at 20° C. from the thermostate flowed during continual stirring. The inner copper vessel was closed at the upper end by means of an india rubber stopper, through which a thermometer had been passed and also an air tight connecting tube[1]) to the barometer, which tube

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[1]) This connecting tube could not easily be made at the lower end, as the inner copper vessel was standing loose on three corks, and a long tube reaching from the stopper to the bottom could not be used, as round the steel flanged tube cemented on the glass tube only half a centimeter space was left. (In order to prevent during the experiments the leakage of mercury, the hard red cement of MENDELEJEFF at the lower end was replaced by soft black cement, which moreover was covered at the inside with an india rubber solution). In order to ascertain, that this connecting tube did not interfere with the attainment of equilibrium of pressure, the measurement

in order to prevent convection currents was bent entirely downwards outside the bath. The standard barometer has been mounted in an other room, and the connecting tube led thither under the floor; it was read by Mr. BOUDIN while I *simultaneously* made a measurement of the difference in level of the mercury in the U-tube of the piezometer. It is this latter determination which most renders uncertain the measurement of pressure. For the diameter of the U-tube is only about 8 m.m., so that with the greatest height of the meniscus that occurs viz. 1.66 m.m. the depression becomes 0.69 m.m. [1]) and a difference in diameter of 0.1 m.m. gives here already a difference of 0.03 m.m., while a difference in height of 0.1 m.m. gives a difference in depression of 0.06 m.m. If moreover we take into account that the height of the mercury in the limbs of the U-tube is read through a water-layer of 6 c.m. thickness, it is obvious that in the measurement mentioned an error may enter, which greatly diminishes the accuracy of the measurement of pressure otherwise to be expected in the normal volume; this may certainly explain the fact that the difference between the largest and the smallest values of the normal volume with tube IV amounts to even $\frac{1}{3.300}$ which under otherwise similar circumstances leads us to expect an accidental difference of 0.22 m.m. in the measurement of pressure. Only from many measurements and repeated mountings — of tube IV 17 were made on 5 different days and every day the apparatus was mounted anew — we can learn the normal volume with sufficient accuracy, for the mean error then appeared to be $\frac{1}{10.000}$.

The volume occupied by the gas during these measurements was measured in entirely the same way as described in the Proceedings of May 25th '01 (Comm. N⁰. 70, 2nd continuation).

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for the determination of the normal volume of one of the piezometer tubes was made in four ways: 1st. with the connecting tube; 2nd. the connecting tube being disconnected from the waterbath; 3rd. the connecting tube being disconnected from the barometer; 4th. without the connecting tube. These four kinds of observations agreed to within $\frac{1}{8000}$.

[1]) In order to judge in how far the depressions occurring in my measurements agree with those given in MENDELEJEFF and GUTKOWSKY's table I have measured directly some depressions for the given width of tube. These gave values which sometimes differed mutually 0.05 m.m., and of which the mean was about 0.03 m.m. higher than in the table. To determine the influence of moisture, I lightly breathed into the tube; this greatly diminished the depression, while the height did not perceptibly decrease.

The temperature within this water bath was not so regular and constant as that of the water in the experiment described in the Proceedings of May 25th '01 (Comm. N^0. 70, 1st continuation) as owing to the much greater height of the water bath, its difference in level with the mixing vessel was much smaller so that the water currents became much weaker. Moreover the piezometer tube was not in the water itself, but in the inner reservoir and it was impossible to stir the air in that enclosed space. To form a judgment of it I give here the temperature readings on April 5th 1900, the temperature of the room being 14° C. The thermometers were suspended against the piezometer tube, N^0. 134 at the higher end near the small reservoir, N^0. 135 in the middle against the steel flanged tube and N^0. 29 at the lower end against the U-tube; the corrections of the thermometers have been applied.

Time.	Thermometer.			Calculated mean temperature.
	No. 134.	No. 135.	No. 29.	
4.10	20°.01	20°.07	20°.01^5	20°.03^5
4.25	20°.01	20°.05	20°.00	20°.02
4.53	19°.99	20°.03	19°.97^5	19°.99^5
5.07	19°.98	20°.01	19°.97^5	19°.98^5

Most days were however more favourable as for instance on May 7th 1900.

Time.	2.06	2.21	2.32	2.43	2.55	3.51	4.06	4.19	4.27	4.42	4.51
Mean temp.	20°.02	20°.02	20°.01	20°.01	20°.01	20°.01	20°.02	20°.02	20°.02	20°.03	20°.04

I give here for one of the tubes viz. N^0. IV, the results of the measurement of the normal volume; the volume is expressed in c.c. and the pressure in c.m. of mercury at Leiden.

Date.	Time.	Volume.	Pressure 20° C.	Product.	Deviation.
4 April.	4.10	174.265	74.071	12908.0	— 1.2
″ ″	4.25	.258	.079	08.8	— 0.4
5 ″	4.10	.141	.138	10.6	+ 1.4
″ ″	4.25	.128	.128	07.8	— 1.4
″ ″	4.53	.121	.142	09.7	+ 0.5
″ ″	5.07	.118	.151	11.0	+ 1.8
6 ″	4.25	.030	.182	09.8	+ 0.6
″ ″	4.39	.027	.174	08.3	— 0.9
″ »	4.53	.026	.176	08.5	— 0.7
» ″	5.17	.024	.172	07.7	— 1.5
7 ″	3.29	.016	.181	08.6	— 0.6
″ ″	3.47	.019	.194	11.2	+ 2.0
″ ″	4.02	.019	.196	11.6	+ 2.4
″ ″	4.33	.019	.182	09.0	— 0.2
9 ″	3.29	.107	.149	09.8	+ 0.6
″ ″	3.50	.106	.140	08.2	— 1.0
″ ″	4.20	.107	.140	08.3	— 0.9

The last column headed "Deviation" gives the difference between the mean value and the observed value; from it we calculate the mean error 1.22 which is $\frac{1}{10.000}$ of the value.

In order to derive the normal volume from the value found I assumed that for the reduction of about 74 c.m. to 75,9467 c.m. (the height of the mercury at Leiden for 1 atm. at 45° northern latitude while the constant of gravitation at Leiden is taken as 981.318 [1]) and at 45° northern latitude as 980.63 [2])) BOYLE's law was sufficient (the deviation is of the order of $\frac{1}{50.000}$) while I have taken $\alpha = 0.0036613$ for the co-efficient of expansion.

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[1] DEFFORGES and BOURGEOIS 1892.

[2] Also accepted in GUILLAUME's "Thermométrie."

In the following table the normal volumes found have been combined and also the relative mean errors have been given

Tube.	I A.	II A.	III.	IV.
Norm. Vol.	162.215	126.025	132.135	158.380
Mean error.	1 : 13000	1 : 11000	1 : 12000	1 : 10000

§ 3. *The measurements at higher pressure.* After the piezometer tubes have been placed in the compression apparatus and this has been connected with the open manometer, the heights of the mercury in them were read by means of a cathetometer of the Société Génevoise. Both rest on the same common foundation, isolated from the floor, and made by bridging over the firm pillars in the observation room with iron rails and stone slabs.

To insure a constant temperature, the apparatus for the regular current of water at constant temperature was put in motion at 10.30 a. m., and the water bath was constantly stirred, the apparatus was put under pressure at 12.30 p. m. and the measurements began at 2.30 p. m. and were continued uninterruptedly till 5 p. m. without anything being changed in the apparatus; we might then be reasonably certain that the temperature measured with the thermometer agreed with that of the hydrogen and that equilibrium of pressure existed between the open and the closed manometers. For these measurements the height of the barometer must also be known, this was read on an aneroïd, of which the correction had been carefully determined. We took into account that the aneroïd had been placed lower than the mercury in the manometer tube A. (Comm. 44 fig. I).

It will hardly ever be possible to avoid very small leakages in the connecting tubes; in the measurements taken into account they were so small that they could not be discovered by means of soap solution. Hence the corrections which must be applied for the motion of the mercury owing to those leakages are very small; with a view to these corrections however all the readings must be made as symmetrically as possible and the time must be always noted for the mean mercury height both in the open and closed manometers. I will give here one observation (comp. the table of readings in the Proceedings of May 25th '01, Comm. N°. 70), of which the calculation will be carried out as an instance.

Reading on Aug. 25th 1900. Tube III.

<table>
<tr><td>Time.</td><td colspan="9">3 52</td><td></td></tr>
<tr><td>Barometer.</td><td colspan="9">75.97^5</td><td></td></tr>
<tr><td></td><td colspan="9">Open Manometer.</td><td></td></tr>
<tr><td>Thermometer
Reading
Corrected.</td><td>t_1
19°.6
19°.47</td><td>t_2
19°.35
19°.22</td><td>t_3
19°.25
19°.07</td><td>t_4
19°.55
19°.45</td><td>t_5
19°.32
19°.24</td><td>t_6</td><td>t_7</td><td>t_8</td><td></td><td>$T_1 = 19°.31$</td></tr>
<tr><td>Manom.tube
above l. to r.</td><td>A
276.53</td><td>B_I
5.02</td><td>B_{II}
6.27</td><td>B_{II}
1.97</td><td>B_{IV}
0.77</td><td>B_V
3.32</td><td>B_{VI}
2.11</td><td>B_{VII}
2.06</td><td>B_{VIII}
2.27</td><td>$\Sigma_1^1 = 300.32$</td></tr>
<tr><td>Time.</td><td colspan="9">4.00</td><td></td></tr>
<tr><td>below r. to l.</td><td>28.26</td><td>5.83</td><td>6.03</td><td>4.15</td><td>3.61</td><td>10.15</td><td>7.38</td><td>8.28</td><td>7.90</td><td>$\Sigma_2^1 = 81.59$</td></tr>
<tr><td></td><td colspan="9">Piezometer.</td><td></td></tr>
<tr><td>Thermometer
Corrected.</td><td colspan="9">$19°.88^5$
$19°.78^5$</td><td></td></tr>
<tr><td>Time.</td><td colspan="9">4.07</td><td></td></tr>
<tr><td></td><td colspan="3">Cathetom. reading</td><td colspan="3">Reading of level</td><td colspan="3">Temperature.</td><td></td></tr>
<tr><td>top men. in tube</td><td colspan="3">60 85[illegible]</td><td colspan="3">0.7</td><td colspan="3">19°.1</td><td></td></tr>
<tr><td>level " " "</td><td colspan="3">60.787</td><td colspan="3">1.35</td><td colspan="3">"</td><td></td></tr>
<tr><td>mark 50.9</td><td colspan="3">60.832</td><td colspan="3">1.5</td><td colspan="3">"</td><td></td></tr>
<tr><td>top men. in measuring glass.</td><td colspan="3">31.422</td><td colspan="3">3.0</td><td colspan="3">"</td><td></td></tr>
<tr><td>level in meas.gl.</td><td colspan="3">31.284</td><td colspan="3">3.15</td><td colspan="3">"</td><td></td></tr>
<tr><td>Time.</td><td colspan="9">4.11^5</td><td></td></tr>
<tr><td>Barometer.</td><td colspan="9">75.98^5</td><td></td></tr>
<tr><td></td><td colspan="9">Open Manometer.</td><td></td></tr>
<tr><td>Thermometer
Reading
Corrected.</td><td>t_1
19°.5
19°.47</td><td>t_2
19°.35
19°.22</td><td>t_3
19°.3
19°.12</td><td>t_4
19°.53
19°.43</td><td>t_5
19°.28
19°.20</td><td>t_6</td><td>t_7</td><td>t_8</td><td></td><td>$T_2 = 19°31$</td></tr>
<tr><td>Manom.tube
below l. to r.</td><td>A
28.27</td><td>B_I
5.86</td><td>B_{II}
6.04</td><td>B_{III}
4.21</td><td>B_{IV}
3.70</td><td>B_V
10.25</td><td>B_{VI}
7.57</td><td>B_{VII}
8.43</td><td>B_{VIII}
8.10</td><td>$\Sigma_2^2 = 82.43$</td></tr>
<tr><td>Time.</td><td colspan="9">4.19</td><td></td></tr>
<tr><td>above r. to l.</td><td>276.52</td><td>5.00</td><td>6.18</td><td>1.89</td><td>0.66</td><td>3.19</td><td>1.91</td><td>1.89</td><td>2.16</td><td>$\Sigma^2 = 299.40$</td></tr>
</table>

For the way in which we derive from this the pressure at 4 p.m and 4.19 p.m. I refer back to the Proceedings of May 25th '01 (Comm. N⁰. 70 § 4); only the barometer height must be added to this, and it must be taken into account that the aneroïd was on the same height as the zero of the measuring rod suspended between the limbs of tube A. The correction for the compression of the mercury was applied to pressures above 32 atm. (Comp. Proceedings of May 25th '01, Comm. N⁰. 70 § 2. G.).

To obtain the pressure at 4.7 p.m. I assumed that the pressure varied proportionally with the time.

The pressure now measured is that of the lower reservoir of the last manometer tube in use; the mercury height in this agreed always within a few centimeters with the height in the level-glass of the piezometers. The correction for the hydrostatical pressure in the gas, which transfers the pressure, may then be neglected.

In measuring the excess of pressure, caused by the difference in mercury level in the piezometer tube and in the level-glass, we must bear in mind that the temperatures of those columns are generally different. However the error is sure to remain within the limits of observation if we assume that the mercury in the steel flanged tube 10 c.m. below the water bath has reached the temper-

August 25th TABLE I.

Time.	Corrected merc. height open manom.	Barom. at level of mercury.	Reading level glass.	Corr. for depress.	Temperature.	Corr. for temp.	Corr. height levelgl.
2.40	2639.47						
2.48		75.91	31.418	0.110	19°.3	— 0.098	32.23
3.00^5	2638.36						
3.09		75.92	31.418	0.110	19°.3	— 0.098	32.24
3.20^5	2636.63						
3.28		75.93	31.427	0.115	19°.2	— 0.098	32.19
3.40	2635.09						
3.48		75.94	31.429	0.114	19°.1	— 0.097	32.21
4.00	2633.59						
4.07		75.95	31.426	0.107	19°.1	— 0.097	32.22
4.19	2631.83						
4.26^5		75.96	31.423	0.114	19°.	— 0.097	32.21
4.38^5	2630.50						

ature of the room; this height co-incided with the zero of the cathetometer scale.

The preceding table gives the mercury heights to be summed for the measurements on Aug. 25th 1900. (Comp. p. 000).

From the corrected mercury position in the open manometer reduced at the time of the observation augmented by the height of the barometer at the level of mercury and the corrected position of the level glass, the corrected height in the piezometer tube must be subtracted and to this we must apply the correction for the compression of the mercury; the pressure then found is that of the hydrogen at the temperature measured; in order to reduce this pressure to 20° C. I have assumed 0.003663 for the co-efficient of expansion of hydrogen, while the influence of the pressure on that co-efficient could be left out of consideration owing to the small deviation of the temperatures measured from 20° C.

The pressure now measured is the one immediately above the mercury and hence the mean pressure of the hydrogen is lower. The greatest difference (when the mercury is at the lower end of the stem) however remains when p is the mean pressure in atm. below 0.000004 p atm., which therefore may be neglected in comparison with p atm.

And hence we obtain for the pressure the following calculation:

August 25th TABLE II.

Time.	Reading of piezom.	Corr. for depr.	Temp. of H.	Corr. for temp.	Corrected reading.	Corr. compr.	Pressure.	Pressure of H at 20°	Pressure at 20° in atm. 45° N L.
2.40									
2.48	60.907	0.115	19.78^5	—0.193	60.829	0.00	2685.55	2687.52	35.387
3.00^5									
3.09	60.897	0.127	19.78^5	—0.193	60.831	0.00	2684.15	2686.12	35.368
3.20^5									
3.28	60.881	0.117	19.78^5	—0.193	60.805	0.00	2682.63	2684.60	35.348
3.40									
3.48	60.868	0.123	19.78^5	—0.193	60.798	0.00	2681.06	2683.03	35.327
4.00									
4.07	60.850	0.125	19.78^5	—0.193	60.782	0.00	2679.55	2681.52	35.307
4.19									
4.26^5	60.825	0.126	19.78^5	—0.193	60.758	0.00	2677.96	2679.93	35.286
4.38^5									

I have purposely given this series of observations as it appears from the foregoing table that there must have been a leakage on that day, which could however not be detected by means of soap solution; the observations on other days show a much smaller decrease of pressure; as for instance on July 11th when the pressure at 2.44 p. m. amounted to 36.898 atm. and at 4.35 p. m. to 36.871 atm. And yet it will appear that the decrease of pressure mentioned has had no disturbing influence on the equilibrium, since the product of volume and pressure, bearing in mind the degree of accuracy attained, may be considered as sufficiently constant.

For the measurement of the volume I refer back to the Proceedings of May 25th 1901 "The calibration of piezometer-tubes". The following table has been calculated in the way described there.

August 25th. TABLE III.

Mark	50.9
Section	0.1264
Correction for elastic expansion of glass	0.0009

Time.	level men. under mark.	Height of men.1)	Mean height men.	Reduced length	Mean section	Volume	Corrected Volume	Specific-volume.
2.40								
2.48	−0.024	0.057	0.029	32.213	0.12690	4.0878	4.0887	0.030944
3.00^5								
3.09	−0.006	0.064	0.033	32.227	〃	4.0896	4.0905	0.030958
3.20^5								
3.28	0.008	0.058	0.030	32.244	〃	4.0918	4.0927	0.030973
3.40								
3.48	0.026	0.061	0.032	32.260	〃	4.0938	4.0947	0.030989
4.00								
4.07	0.046	0.062	0.032	32.280	〃	4.0963	4.0972	0.031008
4.19								
4.26^5	0.064	0.063	0.033	32.297	〃	4.0985	4.0994	0.031024
4.38^5								

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1) In Comm. N°. 67 § 7 I neglected to draw attention to the scale of reciprocals devised by Boys to facilitate the drawing of curves by their curvature.

The correction for the thermal expansion of glass may be neglected on account of the small deviation of the temperature from 20° C.

For the calculation of the product of the specific volume and the pressure in atmospheres at 45° northern latitude the following table is obtained.

August 25th **TABLE IV.**

Time	Pressure Atm. 45° NL.	Specific volume	P × V.
2.40			
2.48	35.386	0.030944	1.0949^{8}
3.00^{5}			
3.09	35.368	0.030958	1.0949^{2}
3.20^{5}			
3.28	35.346	0.030973	1.0948^{3}
3.40			
3.48	35.327	0.030989	1.0947^{6}
4.00			
4.07	35.307	0.031008	1.0948^{2}
4.19			
4.26^{5}	35.286	0.031024	1.0947^{3}
4.38^{5}			
Mean	35.337	0.030983	1.0948^{4}

§ 4. *Results.* In the manner described above I have calculated every time the mean value for one pressure i. e. from 4—8 atm. for every atmosphere, from 8—16 for every 2 atm., from 16—32 for every 4 atm. and from 32—64 atm. for every 8 atm. The values of PV for 4-8 atm. are not given here because when the apparatus was taken to pieces it appeared that the reservoir of tube IA had burst, so that the determination of the normal volume was valueless.

I have tried to express the values found by a formula, and for this I have chosen the following expression:

$$PV = \alpha + \beta d + \gamma d^2, \text{ }^{1})$$

where $d = 1/V$ stands for the density of the gas with regard to that at 0° C. and 1 Atm. at 45° northern latitude. By means of the method of least squares I have calculated α, β and γ from 16 mean values derived from 107 measurements, where the equation which is obtained when $V = 1$ is taken to be absolutely correct and where the weight 1 is given to all measurements, while it was taken into account that the normal volume had been determined by means of REGNAULT's coefficient of expansion $\alpha = 0.0036613$, which does not agree with CHAPPUIS' coefficient of tension $\beta = 0.0036626$. If the latter is taken to be correct, we obtain the following table, for which the mean errors have been derived by means of the weights from the mean errors of the observations to be given subsequently.

	Value.	Weight	Mean error.
α	1.072,58	6,909.	0.000,003
β	$0.000,667^5$	6,914.	0.000,003
γ	0.000,000,98	10,421,200.	0.000,000,08

The following table gives the densities measured at different dates, the product PV corrected for the correct coefficient of expansion and its deviation from the values calculated from α, β and γ.

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1) As appears from a development in series borrowed from AMAGAT's results, the term with d^3 would be 0, while the term with d^1 has little influence below 60 atm.

Date.	Time	Tube.	Density.	P × V.	Deviation.
23 April 1901	3.48	IIA	6.2403	1.0767^3	-0.0000^5
» » »	4.00	»	6.2399	7^9	$+\ 0^1$
» » »	4.10	»	99	7^1	$-\ 0^7$
» » »	4.22	»	91	7^9	$+\ 0^1$
» » »	4.34	»	87	7^2	$-\ 0^6$
On an average.			6.2394	1.07675	-0.0000^3
15 Aug. 1900	2.53	IIA	8.2352	1.0776^8	$-\ 4^7$
» » »	3.03^5	»	45	7^8	$-\ 3^7$
» » »	3.16	»	52	6^8	$-\ 4^7$
» » »	3.26	»	45	7^8	$-\ 3^7$
» » »	3.36	»	38	8^8	$-\ 2^7$
» » »	3.47	»	25	9^8	$-\ 1^7$
» » »	3.57	»	25	9^8	$-\ 1^7$
28 » »	3.25	»	447	8^8	$-\ 2^7$
» » »	3.35	»	47	8^8	$-\ 2^7$
» » »	3.45	»	33	80^8	$-\ 0^7$
» » »	3.55	»	33	0^8	$-\ 0^7$
» » »	4.05	»	27	0^8	$-\ 0^7$
» » »	4.16	»	20	1^8	$+\ 0^3$
» » »	4.26	»	20	1^8	$+\ 0^3$
» » »	4.35	»	13	1^8	$+\ 0^3$
On an average.			8.2385	1.0779^5	-0.0002
16 Aug. 1900	2.57	IIA	10.5775	1.0797^8	$+\ 0^3$
» » »	3.09	»	64	8^3	$+\ 1^3$
» » »	3.20	»	64	7^8	$+\ 0^3$
» » »	3.30	»	64	7^8	$+\ 0^3$
» » »	3.44	»	64	7^8	$+\ 0^3$
» » »	3.55	»	64	7^8	$+\ 0^3$
On an average.			10.5766	1.0798^0	$+0.0000^5$

Date.	Time.	Tube.	Density.	P × V.	Deviation.
17 Aug. 1900	3.04	IIA	12.8353	1.0813^{8}	+ 0^{8}
» » »	3.18	»	20	5^{8}	+ 2^{8}
» » »	3.32	»	287	6^{8}	+ 3^{8}
» » »	3.45	»	71	6^{8}	+ 3^{8}
» » »	3.58	»	54	5^{8}	+ 2^{8}
» » »	4.14	»	22	5^{8}	+ 2^{8}
On an average.			12.8285	1.0815^{8}	$+0.0002^{8}$
21 Aug. 1900	3.45	III	13.0097	1.0814^{7}	+ 0^{4}
» » »	4.00	»	78	5^{1}	+ 0^{8}
» » »	4.14	»	60	5^{8}	+ 1^{5}
» » »	4.27	»	42	6^{0}	+ 1^{7}
» » »	4.41	»	25	6^{8}	+ 2^{5}
» » »	4.57	»	07	7^{0}	+ 2^{7}
27 » »	2.44	»	70	2^{0}	— 2^{8}
» » »	2.56	»	66	1^{8}	— 2^{5}
» » »	3.08	»	66	1^{7}	— 2^{6}
» » »	3.22	»	49	2^{9}	— 1^{4}
» » »	3.36	»	39	2^{2}	— 2^{1}
» » »	3.48	»	37	1^{0}	— 3^{8}
On an average.			13.0053	1.0813^{9}	-0.0000^{4}
18 Aug. 1900	2.45	II A	15.2835	1.0829^{8}	— 0^{3}
» » »	3.00	»	788	30^{8}	+ 0^{7}
» » »	3.21	»	88	29^{8}	— 0^{3}
» » »	3.35	»	65	30^{8}	+ 0^{7}
» » »	3.49	»	18	2^{8}	+ 2^{7}
» » »	4.04	»	695	3^{8}	+ 3^{7}
» » »	4.19	»	48	5^{8}	+ 5^{7}
On an average.			15.2748	1.0831^{9}	$+0.0001^{8}$

Date.	Time.	Tube.	Density.	P × V.	Deviation.
22 Aug. 1900	3.25	III	16.6803	1.0838^2	— 1^7
» » »	3.37	»	775	8^6	— 1^3
» » »	3.49	»	56	8^7	— 1^9
» » »	4.00	»	39	8^3	— 1^6
» » »	4.11	»	31	7^2	— 2^7
» » »	4.24	»	692	7^5	— 2^4
On an average.			16.6750	1.0838^1	-0.0001^8
23 Aug. 1900	3.57	III	21.2988	1.0876^0	+ 3^5
» » »	4.03	»	52	7^0	+ 4^5
» » »	4.26	»	15	6^6	+ 4^1
» » »	4.41	»	866	7^5	+ 5^0
On an average.			21.2930	1.0876^8	$+0.0004^3$
24 Aug. 1900	3.00	III	25.408	1.0908^7	+ 6^9
» » »	3.21	»	398	5^0	+ 3^2
» » »	3.39	»	92	1^2	— 0^5
» » »	3.56	»	87	899^1	— 2^6
» » »	4 12	»	80	9^4	— 2^2
» » »	4.29	»	75	9^1	— 2^5
On an average.			25.390	1.0902^1	$+0.0000^4$
28 June 1900	4.56	IV	26.524	1.0912^0	+ 2^2
29 » »	4.05	»	624	1^4	+ 0^9
» » »	4.51	»	04	3^0	+ 2^6
30 » »	2.46	»	777	0^1	— 1^5
» » »	3.17	»	66	3^0	+ 1^4
» » »	3.54	»	50	4^1	+ 2^7
» » »	4.28	»	37	3^4	+ 2^1
5 July »	4.11	»	86	1^9	+ 0^2
» » »	4.42	»	71	2^4	+ 0^8
On an average.			26.704	1.0912^3	$+0.0001^3$

Date.	Time	Tube.	Density.	P × V.	Deviation.
7 July 1900.	2.29	IV	29.956	1.0931^{7}	— 2^{9}
» » »	2.47	»	41	3^{0}	— 1^{5}
» » »	3.09	»	37	0^{6}	— 3^{8}
» » »	3.24	»	35	0^{9}	— 3^{5}
» » »	3.44	»	20	1^{8}	— 2^{5}
» » »	3.59	»	04	1^{7}	— 2^{5}
» » »	4.16	»	887	1^{9}	— 2^{2}
» » »	4.34	»	73	1^{4}	— 2^{3}
On an average.			29.919	1.0931^{6}	-0.0002^{7}
25 Aug. 1900	2.48	III	32.316	1.0949^{6}	— 2^{3}
» »	3.09	»	01	9^{0}	— 2^{8}
» » »	3.28	»	286	8^{1}	— 3^{6}
» » »	3.48	»	70	7^{4}	— 4^{2}
» » »	4.07	»	50	8^{1}	— 3^{4}
» » »	4.26	»	53	7^{1}	— 4^{2}
On an average.			32.276	1.0948^{2}	-0.0003^{4}
11 July 1900	2.44	IV	33.656	1.0962^{9}	+ 1^{2}
» » »	3.10	»	46	4^{8}	+ 3^{2}
» » »	3.30	»	47	4^{7}	+ 3^{1}
» » »	3.52	»	44	2^{3}	+ 0^{6}
» » »	4.13	»	37	3^{3}	+ 1^{7}
» » »	4.35	»	34	2^{3}	+ 0^{3}
On an average.			33.644	1.0963^{4}	$+0.0001^{8}$
12 July 1900	2.43	IV	40.254	1.1008^{3}	— 2^{3}
» » »	3.05	»	11	9^{8}	— 0^{4}
» » »	3.28	»	167	9^{3}	— 0^{7}
» » »	3.51	»	23	9^{1}	— 0^{5}
On an average.			40.189	1.1009^{1}	-0.0001^{0}

Date.	Time.	Tube.	Density.	P × V.	Deviation.
13 July 1900	4.34	IV	47.218	1.1063^4	+ 0^3
» » »	4 59	»	165	5^6	+ 2^8
On an average.			47.192	1.1064^5	$+0.0001^5$
14 July 1900	3.01	IV	54.127	1.1117^9	+ 1^8
» » »	3.26	»	069	6^8	+ 1^2
» » »	3 56	»	3.987	6^0	+ 1^0
» » »	4 22	»	23	4^8	+ 0^5
» » »	4.55	»	836	1^9	— 1^9
On an average.			53.988	1.1115^5	$+0.0000^5$

The agreement is satisfactory; it may be judged from the following table:

	Number of obser-vations.	Number of positive deviations.	Number of negative deviations.	Sum of the positive deviations.	Sum of the negative deviations.	Mean error in 1 measurement.
Tube II A	39	22	17	0.0034^2	0.0032^9	0.0002^4
» III	34	12	22	0.0036^8	0.0053^4	0.0003^2
» IV	34	20	14	0.0030^9	0.0028^8	0.0002^0
Total	107	54	53	0.0101^9	0.0115^0	0.0002^5

The fourth tube gives the best agreement. With the value found for the mean error in one measurement, to which I had given the weight 1, I have also calculated the mean errors in the table for the values of α, β and γ.

§ 5. We might calculate values of VAN DER WAALS' a and b from the values of α, β and γ, supposing that his original equation of state would hold for the same temperature within the limits of pressure mentioned. It is obvious that then the values of a and b must be corrected, because the terms with higher powers than the second power of density were neglected. If we want to calculate these corrections of a and b by means of the method of least squares, we meet with the difficulty that the terms containing the second powers of those corrections cannot be neglected in comparison with the terms containing the first power, because although each of these are much larger, they yet partially neutralize each other. A calculation in which the terms mentioned were kept, did not give a good result. Therefore by means of the value of b, derived by approximation from α, β and γ, viz. $b = 0.0009$, I have calculated the correction term, which van der Waals' formula requires in addition to the terms used, viz. $RT\,b^3\,d^3\,(1-bd)$, subtracted this value from PV and have equalized the derived value to $\alpha' + \beta' d + \gamma' d^2$, from which is found:

$$\alpha' = 1.07258,$$
$$\beta' = 0.000670,$$
$$\gamma' = 0.00000088.$$

By putting:

$$\alpha' = RT,$$
$$\beta' = RTb - a,$$
$$\gamma' = RTb^2,$$

we find:

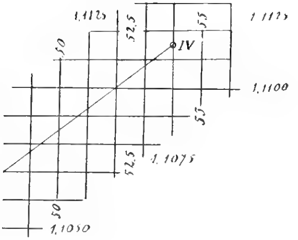
$$a = 0.00030, \qquad m_a = 0.00004^2,$$
$$b = 0.00091, \qquad m_b = 0.00004,$$

Finally let us compare our results with those of Regnault and Amagat. The values determined by me are indicated in the figure by circles, those found by Regnault by squares; in this I have supposed that at the lowest pressure Regnault's result and mine were the same; after this the other points have been drawn.

From a development in series, calculated from Amagat's observations at 0°, 15°.4 and 47.°3 C., we find by means of interpolation:

$$PV_{20^\circ} = 1.07252 + \frac{0.000719}{V} + \frac{0.00000067}{V^2}.$$

If we substitute $V = 0.01129$, Amagat's greatest volume at 15°.5 C. we find $PV_{20^\circ} = 1.1414$ ($PV_{15^\circ.5} = 1.1290$), while from the values of α, β and γ we obtain $PV_{20^\circ} = 1.1394$ and from the original equation of van der Waals with the given values of a and b: $PV_{20^\circ} = 1.1401$.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52,5
1,1075
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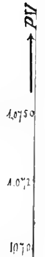
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Physics. — Communication N°. 71 from the Physical Laboratory at Leiden by Prof. H. KAMERLINGH ONNES: "*Expression of the equation of state of gases and liquids by means of series.*"

§ 1. I have followed in this communication a different method in considering the equation of state than has been done up to now. Various methods have been tried to empirically derive functions of v and t for VAN DER WAALS' a and b by means of kinetic or thermodynamic considerations, but without obtaining a good agreement with the observations over the whole range of the equation of state. Neither was I successful in similar attempts which were repeatedly occasioned by my continued research on the corresponding states and other investigations resulting from them at the Leiden laboratory. Whenever I seemed to have found an empirical form, I discovered after having tested it more closely that it appeared useful only within a limited range to complete what had been found in a purely theoretical way by VAN DER WAALS and BOLTZMANN. Hence it appeared to me more and more desirable to combine systematically the entire experimental material on the isothermals of gases and liquids as independently as possible from theoretical considerations and to express them by series. The idea of making this attempt and executing the elaborate calculations required ripened gradually on talking the matter over with my friend Dr. E. F. VAN DE SANDE BAKHUIJZEN, and many thanks are due to him for his advice in arranging and executing all the stages of those calculations.

§ 2. My calculations, in so far as they are given in this paper comprise AMAGAT's observations [1]) relating to hydrogen, oxygen, nitrogen and carbon dioxide.

Although I had in mind the development of the equation of state $p=f(v,T)$ in a convergent double infinite series in terms of the molecular density $\frac{1}{v}$ and the absolute temperature, it follows from the nature of the subject that we can only obtain a representation by a polynomial of a limited number of terms, and we need not wonder that this polynomial does not even converge for all densities. Each co-efficient of such a polynomial can be determined for an individual isotherm only when the polynomial consists of a moderate number of terms. Only in this way we can obtain a good agreement at the first calculation. Therefore the polynomial must be derived

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[1]) Ann. de Ch. et de Phys. 6e Sér. t. XXIX, 1893.

rom the infinite series not only by approximate combination of the remainder to some terms, but also through suitable omission of intermediate terms.

It must be borne in mind, that the co-efficients of each of the remaining terms will have to be also expressed by a contracted series in terms of the temperature and hence it is obvious that we must try in the first place to bring about an agreement with a number of terms as small as possible but equal for all temperatures.

But for this it is not sufficient to pay attention to one substance only. The range of temperatures for which precise observations have been made in the case of single substances, is too limited for each of those substances to derive from them the way in which the coefficients of the terms of an isothermal are dependent on the temperature. We are still far from having realised the idea which has occupied me for many years viz. that of the precise determination of the isothermals of hydrogen at temperatures going down to its boiling point and lower. To a certain extent it is possible to substitute for the investigation of one single substance over the whole range of the equation of state, that of several others within different limits, namely when we combine by means of the law of corresponding states the portions of the ranges of reduced temperature and density given by each of the substances investigated.

It is true that the various substances are not rigidly mechanically similar. Some time ago VAN DER WAALS has especially made clear how the different degrees of compressibility of the molecules (in connection with the number of degrees of freedom) will show themselves by a difference in the equations of state of various substances. And instead of neglecting the deviations in order to arrive at the general equation of state we should be inclined to do the reverse and start from the complete equation of state of each substance, in order to express the deviations from the law of the corresponding states as functions of reduced temperature and pressure, these deviations being small for substances belonging to one group, and somewhat larger for substances belonging to different groups ¹) consisting of mutually almost mechanically similar substances. But so long as the observations have not proceeded further the method described will be the only way to determine as function of the temperature the coefficients of the simple terms in an isothermal developed with regard to density, which I will call *virial co-efficients*, and to form a

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¹) KAMERLINGH ONNES, Proceedings Royal Acad. of Science 1881, p. 11.

representation of the equation of state, given by the observations.

I will not enlarge on the way in which the differences in the combined reduced equations of state will show themselves in the variation of the reduced virial co-efficients with the reduced temperature. We will only remark here that the number of terms in our polynomials must be chosen so that all the isothermals of the different substances which we consider in our calculations can be represented by means of co-efficients, each of which can be taken as derived from the observations.

The more substances are considered the more difficulties arise. Therefore besides hydrogen, oxygen and nitrogen we have taken only carbon dioxide, and so the range covered does not extend far below the critical temperature.

From the standpoint I have chosen the results obtained are to be considered only as the first preliminary data in our method of investigation. A similar calculation will I dare say be required for any investigation. And even if my further calculations would fall short of my expectations the results obtained will be valuable for further investigations by others on the equation of state. They will represent the observations over which they extend in a concise and easily handled form. This is the reason that I no longer delay the publication.

§ 3. For the development in series the following form was chosen:

$$pv = A + \frac{B}{v} + \frac{C}{v^2} + \frac{D}{v^4} + \frac{E}{v^6} + \frac{F}{v^8} \quad . \quad (I)$$

where p represents the pressure in atmospheres (45° northern latitude), v the volume of the molecular weight [1]) the unit being chosen so that v is expressed in the theoretical normal volume [2]) and the value for A at 0° C. is

$$A_0 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(I) can also be considered as the series

$$pv = A + \frac{B}{v} + \frac{C}{v^2} + Z$$

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[1]) Verh. Kon. Akad. v. Wet. 1881, p. 5—7.

[2]) Comp. Communication No. 47 cont § 2 Febr. '99.

with the remainder [1])

$$Z = \frac{D}{v^4} + \frac{E}{v^6} + \frac{F}{v^8}.$$

The case of the densities at which Z may be neglected is sufficiently important to direct special attention to the co-efficients B and C.

With regard to a term $\frac{\Gamma}{v^3}$, it appeared that for the isothermals of 60° C. (and less decisively for those of 10° C.) of carbon dioxide the best agreement was obtained when $\Gamma = 0$. This also decided the choice of even or odd terms either of which could be kept in the remainder. At first it seemed as if with great densities and low temperatures a further addition of $\frac{G}{v^{16}}$ could give a better agreement, but afterwards this was found to be unnecessary.

After

$$pv = A + \frac{B}{v} + \frac{C}{v^2} + \frac{D}{v^3} + \frac{E}{v^4} + \frac{F}{v^5} + \frac{G}{v^6} \quad . \quad . \quad (40^\circ\ V)$$

were investigated a large number of forms at 40° C.

$$pv = A + \frac{B}{v} + \frac{C}{v\,(v-\gamma)} + \frac{D}{v\,(v-\gamma)^2} + \frac{E}{v\,(v-\gamma)^3} + \frac{F}{v\,(v-\gamma)^4} \quad . \quad (40^\circ\ I)$$

$$pv = A + \frac{B}{v} + \frac{C}{v\,(v-\gamma)} + \frac{D}{v\,(v-\gamma)^3} + \frac{E}{v\,(v-\gamma)^5} + \frac{F}{v\,(v-\gamma)^7} \quad . \quad (40^\circ\ II)$$

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[1] It will also be possible to write VAN DER WAALS' original equation of state:

$$pv = RT + \frac{RTb-a}{v} + \frac{RTb^2}{v^2} + Z$$

with $Z = \dfrac{RTb^3}{v^2(v-b)}$.

$$pv = A + \frac{B}{v} + \frac{C}{v\,(v-\gamma)}\, e^{\frac{\gamma}{v-\gamma}} + \frac{D}{v\,(v-\gamma)^2}\, e^{\frac{2\gamma}{v-\gamma}} +$$

$$+ \frac{E}{v\,(v-\gamma)^3}\, e^{\frac{3\gamma}{v-\gamma}} + \frac{F}{v\,(v-\gamma)^4}\, e^{\frac{4\gamma}{v-\gamma}} \quad . \; . \quad (40^\circ\ III)$$

$$pv = A + \frac{B}{v} + \frac{C}{v^3} + \frac{D}{v^5} + \frac{E}{v^7} + \frac{F}{v^9} \quad . \; . \; . \quad (40^\circ\ VI)$$

$$pv = A + \frac{B}{v} + \frac{C}{v^2}\, e^{-\frac{v}{\gamma}} + \frac{D}{v^3}\, e^{-\frac{2v}{\gamma}} + \frac{E}{v^4}\, e^{-\frac{3v}{\gamma}} + \frac{F}{v^5}\, e^{-\frac{4v}{\gamma}} \; . \; (40^\circ\ VII)$$

The reasons for trying all of which in connection with VAN DER WAALS' equation may easily be seen, but they proved less suitable for the calculations than (I) and (40° *V*) while at a nearer investigation (I) appeared to be the most suitable.

It is remarkable that whereas our theoretical representations would lead us to forms as (40° *I, II, III, IV*), the calculation of differences requires a form as (I).

The development in series (I) relates to the theoretical normal volume as unit of volume. AMAGAT gives the volume expressed in the normal volume; if we call this v_N and that given by AMAGAT v_A then

$$v = v_A\, v_N \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

If we apply AMAGAT's results to

$$pv_A = A_A + \frac{B_A}{v_A} + \frac{C_A}{v_A^2} + \frac{D_A}{v_A^4} + \frac{E_A}{v_A^6} + \frac{F_A}{v_A^8} \quad (\mathrm{II})$$

then we have

$$A = A_A\, v_N, \qquad B = B_A\, v_N^2, \qquad C = C_A\, v_N^3, \qquad D = D_A\, v_N^5 \quad . \; . \quad (3)$$

further for $t = 0^\circ$ C.

$$v_N = A_0 + \frac{B_0}{v_N} + \frac{C_0}{v_N} + \ldots = \frac{A_0}{A_{A0}} = \frac{1}{A_{A0}} \quad . \; . \; . \quad (4)$$

and

$$A_{A0} = 1 - (B_{A0} + C_{A0} + D_{A0} + E_{A0} + F_{A0}) \quad . \quad . \quad . \quad (5)$$

We put as condition $\frac{A_T}{A_0} = 1 + 0.0036625\,t$ [1] (where $A_0 = 1$) or in other words we assume that formula (I) would show that at infinite volume the ideal gaseous state would exist. Hence follows with (3) and (1)

$$A_{AT} = A_T \; A_{A0} = A_{A0} \; (1 + 0.0036625\,t) \quad . \quad . \quad . \quad (6)$$

The calculations were begun with preliminary approximations; at the second approximation we worked with

	Carbon dioxide	Oxygen.	Nitrogen.	Hydrogen.
A_{A0} =	1.00706	1.00092	1.00038	0.99932

The co-efficients B_A, C_A, D_A, E_A, F_A, for each isotherm were now found to first approximation by solving the 5 equations following from 5 well-chosen observations, and then by successive calculation of differences without using least squares, were corrected as much as possible. The results obtained in this way individually for all the isothermals investigated are combined in the following table.

The first column gives the substance and the temperature, for which AMAGAT has measured the isothermal. The second column indicates the solution for the five co-efficients, which gave the best agreement with this isothermal. Co-efficients which had to be assumed in those cases where the range of densities was not sufficient to determine 5 co-efficients individually are placed in brackets; this was the case with the higher temperatures for hydrogen, nitrogen, oxygen, where the pressures do not reach 3000 atm. as with the lower temperatures, but go only up to 1000 atm., and with temperatures below the critical temperature of carbon dioxide where the labile region occasions a similar uncertainty.

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[1] In comparing with Comm. N°. 60 on the co-efficient of pressure variation of hydrogen it must be remembered that the calculations were begun before that communication.

The isothermals gave special difficulties below the critical temperature. Here one of the equations was obtained by means of MAXWELL's criterium. Let p_m be the maximum vapour-pressure, v_v the liquid volume and v_d the vapour volume under this pressure then

$$p_m\left(v_d - v_v\right) = A\left(\log v_d - \log v_v\right) - B\left(\frac{1}{v_d} - \frac{1}{v_v}\right)$$

$$-\frac{1}{2}C\left(\frac{1}{v_d^2} - \frac{1}{v_v^2}\right) - \frac{1}{4}D\left(\frac{1}{v_d^4} - \frac{1}{v_v^4}\right)$$

$$-\frac{1}{6}E\left(\frac{1}{v_d^6} - \frac{1}{v_v^6}\right) - \frac{1}{8}F\left(\frac{1}{v_d^8} - \frac{1}{v_v^8}\right) \qquad (7)$$

AMAGAT gives for carbon dioxide the following densities δ (in G. per c.c.) and maximum vapour pressures in atmospheres (Table N°. 28).

	δ_v		δ_d		p_m
0°	0.914	[0.921]	0.096	[0.098]	34.3
10°	0.856		0.133		44.2
20°	0.766		0.190		56.3
30°	0.598	[0.607]	0.334	[0.312]	70.7

In order to arrive at a good representation with a constant p_m it appeared to be desirable in some cases to modify δ_v and δ_d a little; the values used are added in this table between brackets.

The calculations were made by means of the equation obtained from (7) by using v_A; the reduction of δ to v_A was carried out (comp. Communication N°. 47 cont. § 2. Febr. '99) with

$$v_A = \frac{10^{-3}}{\delta}\, 1.9771.$$

Virial coefficients for individual isothermals. First approximation.

		$10^3 . B_A$	$10^6 . C_A$	$10^{12} . D_A$	$10^{18} . E_A$	$10^{24} . F_A$
Hydrogen 200°.25	IV	+ 1.3417	+ 0.80723	+ 2.1460	[− 1.5000]	[+ 0.5000]
99.25	III	+ 1.0572	+ 0.60637	+ 1.9865	[− 1.5000]	[+ 0.5000]
47.3						
15.4	II	+ 0.75371	+ 0.54346	+ 1.8251	− 1.9387	+ 0.86681
0.	IV	+ 0.66815	+ 0.67030	+ 1.2958	− 1.0448	+ 0.36485
Nitrogen 199.5	$\frac{V + VI}{2}$	+ 1.1952	+ 4.5383	+15.318	[+ 2.5000]	[0.0000]
99.45	$\frac{IV + V}{2}$	+ 0.44303	+ 3.5048	+13.955	[+ 2.5000]	[0.0000]
43.6						
16.0	III	− 0.24806	+ 2.7636	+11.090	+ 4.7792	+ 0.28505
0.	II	− 0.37215	+ 2.6217	+ 9.7771	+10.016	− 7.3740
Oxygen 199.5	$\frac{II+III+IV+V}{4}$	+ 0.47351	+ 2.9417	+ 7.3280	[+ 2.5000]	[0.0000]
99.5	$\frac{V + VI}{2}$	− 0.12073	+ 2.4644	+ 5.7202	[+ 2.5000]	[0.0000]
15.6	II	− 0.78278	+ 2.1925	+ 3.3320	+ 7.5866	− 5.2394
0.	III	− 0.92953	+ 2.2931	+ 2.2395	+ 9.1032	− 6.3161
Carb. dioxide 258						
198	II	− 2.9653	+ 8.7762	− 4.4031	+ 125.15	− 117.00
137	4	3.8567	8.7740	7.2828	90.025	12.416
100	1	4.4621	9.4540	15.612	103.27	26.442
90	1	4.6981	10.004	22.078	134.75	94.265
80	2	4.9027	10.338	24.517	136.57	92.657
70	$\frac{I + II}{2}$	5.0997	10.554	24.677	124.37	68.432
60	2	5.3396	11.159	29.716	137.98	85.546
50	2	5.6011	11.747	31.681	124.58	43.496
40	IV 4	5.8506	12.192	33.234	121.52	− 39.039
35						
32						
30	VIII	6.2334	13.712	41.372	[120.00]	+ 25.012
20	IV	6.4155	13.585	38.984	[119.72]	− 30.025
10	VII	6.6896	14.109	45.860	[120.00]	− 57.765
0	X	− 7.0409	−14.904	−43.362	[+ 120.00]	− 32.645

These virial coefficients relate to the normal volume as unit.

In order to judge of the agreement the differences of the observed (O.) and the computed (C.) pv_A are given here in units of the fourth decimal and in percents of pv_A for carbon dioxide at 60° C. and hydrogen at 15°.4 C.

CARBON DIOXIDE 60°.						HYDROGEN 15°.4.					
Tab. N°. 14 AM.			Tab. N°. 13 AM.			Tab. N°. 8 AM.			Tab. N°. 4 AM.		
P.	O.-C.	In pCt.	P.	O.-C.	In pCt.	P.	O.-C.	In pCt.	P.	O.-C.	In pCt.
45	− 8	− 0.1	50	− 9	− 0.1	100	+ 17	+ 0.1	700	+ 27	+ 0.2
48	− 8	− 0.1	75	+ 27	+ 0.3	150	+ 1	0.0	800	+ 24	+ 0.1
50	− 9	− 0.1	100	0	0	200	− 5	0.0	900	− 42	− 0.2
53	− 4	0.0	125	− 1	0	250	+ 11	+ 0.1	1000	− 35	− 0.2
55	− 2	0.0	150	+ 9	+ 0.2	300	− 17	− 0.1	1100	+ 41	+ 0.2
60	− 1	0.0	175	− 25	− 0.5	350	− 12	− 0.1	1200	+ 63	+ 0.3
65	0	0.0	200	+ 2	0	400	− 16	− 0.1	1300	+ 48	+ 0.3
68	+ 2	0.0	225	− 1	0	450	− 8	− 0.1	1400	+ 48	+ 0.2
70	+ 1	0.0	250	+ 10	+ 0.2	500	− 3	0.0	1500	+ 36	+ 0.2
75	− 4	0.0	275	+ 3	0.0	550	+ 1	0.0	1600	+ 19	+ 0.1
80	− 5	− 0.1	300	− 26	− 0.4	600	+ 3	0.0	1700	− 34	− 0.2
85	− 4	− 0.1	350	+ 27	+ 0.4	650	+ 7	0.0	1800	− 26	− 0.1
90	− 2	0.0	400	+ 21	+ 0.2	700	− 5	0.0	1900	+ 10	0.0
95	0	0.0	450	+ 3	0	750	− 8	− 0.1	2000	− 9	0.0
100	0	0.0	500	+ 29	+ 0.3	800	− 26	− 0.2	2100	+ 6	0.0
110	+ 8	+ 0.1	550	0	0	850	− 50	− 0.3	2200	+ 53	+ 0.2
			600	− 40	− 0.3	900	− 40	− 0.2	2300	+ 77	+ 0.3
			650	− 33	− 0.3	950	− 59	− 0.3	2400	+ 113	+ 0.4
			700	− 10	− 0.1	1000	− 65	− 0.4	2500	+ 89	+ 0.3
			750	+ 50	+ 0.4				2600	+ 149	+ 0.5
			800	+ 42	+ 0.3				2700	+ 71	+ 0.3
			850	+ 43	+ 0.3				2800	+ 2	0.0
			900	− 32	− 0.2				2900	− 85	− 0.3
			950	− 194	− 1.1				3000	− 146	− 0.5
			1000	− 444	− 2.4						

There are some cases that are less favourable, but considering the agreement with the isothermals which are represented by the system of co-efficients, B_A, C_A, D_A, E_A, F_A, the regular course of these co-efficients with the temperature may be adduced as a proof that the co-efficients obtained have not only importance for the calculations but have also a physical meaning. Even if some difficulties remain as with the densities of carbon dioxide at pressures above 850 atm. and with the densities in the neighbourhood of saturation, the choice of six terms in the polynomial appears in reasonably good agreement with the nature of the problem and the accuracy of the observations.

§ 4. In order to express the virial co-efficients as functions of temperature we introduce the reduced quantities:

$$\mathfrak{p} = \frac{p}{p_k} \quad , \quad \mathfrak{v} = \frac{v}{v_k} \quad , \quad \mathfrak{t} = \frac{T}{T_k} \quad . \; . \; . \; . \; . \quad (8)$$

where k relates to the critical state.

Then (I) changes into:

$$\lambda \mathfrak{p} \mathfrak{v} = \mathfrak{A} + \frac{\mathfrak{B}}{\lambda \mathfrak{v}} + \frac{\mathfrak{C}}{\lambda^2 \mathfrak{v}^2} + \frac{\mathfrak{D}}{\lambda^4 \mathfrak{v}^4} + \frac{\mathfrak{E}}{\lambda^6 \mathfrak{v}^6} + \frac{\mathfrak{F}}{\lambda^8 \mathfrak{v}^8} \quad (\mathrm{III})$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ and $\mathfrak{F}$ are functions of $\mathfrak{t}$ and

$$\lambda = \frac{p_k v_k}{T_k} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

while

$$\left.\begin{array}{lll} \mathfrak{A} = \dfrac{A}{T_k} \; , & \mathfrak{B} = \dfrac{B}{T_k^2} p_k \; , & \mathfrak{C} = \dfrac{C}{T_k^3} p_k^2 \\ \mathfrak{D} = \dfrac{D}{T_k^5} p_k^4 , & \mathfrak{E} = \dfrac{E}{T_k^7} p_k^6 \; , & \mathfrak{F} = \dfrac{F}{T_k^9} p_k^8 \end{array}\right\} \quad . \; . \; . \quad (10)$$

or with (3)

$$\left.\begin{array}{lll}\mathfrak{A}=\dfrac{A_A}{T_k}v_N\ , & \mathfrak{B}=\dfrac{B_A}{T_k^2}p_k v_N^{\,2}\ , & \mathfrak{C}=\dfrac{C_A}{T_k^3}p_k^2 v_N^{\,3}\\ \mathfrak{D}=\dfrac{D_A}{T_k^5}p_k^4 v_N^{\,5}, & \mathfrak{E}=\dfrac{E_A}{T_k^7}p_k^6 v_N^{\,7}\ , & \mathfrak{F}=\dfrac{F_A}{T_k^9}p_k^8 v_N^{\,9}\end{array}\right\} \quad . \quad . \quad (11)$$

gives on the one hand the connection of the reduced virial co-efficients with the virial co-efficients expressed with regard to the theoretical normal volume as unit, on the other hand with the virial co-efficients expressed with regard to the normal volume as unit.

We attribute to λ the same value for all substances, which gives the advantage that in the reduction calculations the p_k and T_k, which generally are much more accurately known than v_k, can be used.

The calculations are made with the following

Data and reduction factors.

	Hydrogen.	Nitrogen.	Oxygen.	Carbon dioxide.
T_k	38.5 Olsz.	127 Olsz.	154.2 Olsz.	304.45 Am.
p_k	20 Olsz.	35 Olsz.	50.7 Olsz.	72.9 Am.
$\log\mathfrak{B}-\log B_A+5$	3.13070	2.33614	2.32890	1.88988
$\log\mathfrak{C}-\log C_A+5$	2.84657	1.77625	1.84628	1.26619
$\log\mathfrak{D}-\log D_A+6$	3.27830	1.65646	1.88105	1.01880
$\log\mathfrak{E}-\log E_A+7$	3.71003	1.53667	1.91581	0.77141
$\log\mathfrak{F}-\log F_A+8$	4.14176	1.41688	1.95057	0.52402

The reduced virial co-efficients of all the substances together found in this way show already in their relation to the reduced temperature (rounded off in the following table) the same regularity as we found for the non-reduced numbers for a single substance, another proof, that the virial co-efficients have a real meaning.

In the following table, in which the reduced virial co-efficients are combined I have omitted those which would be obtained from the assumed co-efficients [ ] in the table of § 3

Reduced virial co-efficients from individual isotherms to first approximation.

t		$10^8.\mathfrak{B}$	$10^{11}.\mathfrak{C}$	$10^{18}.\mathfrak{D}$	$10.^{25}\mathfrak{E}$	$10^{32}.\mathfrak{F}$
12.29	Hydrogen 200°.25	+ 1812.76	+ 566.98	+ 4073.1		
9.67	99.25	+ 1428.43	+ 425.90	+ 3770.4		
8.32	47.3					
7.49	15.4	+ 1018.38	+ 381.72	+ 3464.1	— 9943	+ 12014
7.09	0	+ 902.78	+ 470.79	+ 2459.5	— 5359	+ 5057
3.72	Nitrogen 199.5	+ 259.16	+ 271.11	+ 694.46		
3.06	Oxygen 199.5	+ 100.98	+ 206.48	+ 557.22		
2.93	Nitrogen 99.45	+ 96.07	+ 209 36	+ 632.69		
2.49	» 43.6					
2.42	Oxygen 99.5	— 25.75	+ 172.98	+ 434.96		
2.28	Nitrogen 16.0	— 53.79	+ 165.09	+ 562.81	+ 164.4	+ 7.4
2.15	» 0	— 80.70	+ 156.61	+ 443.27	+ 344.6	— 192.6
1.87	Oxygen 15.6	— 166.93	+ 153 89	+ 253.36	+ 625.0	— 467.6
1.77	» 0	— 198 23	+ 160.95	+ 170.29	+ 749.9	— 563.7
1.745	Carbon dioxide 258					
1.548	198	— 230.11	+ 162.00	— 45.98	+ 739.4	— 391.0
1.347	137	— 299.30	+ 161.96	— 76.05	+ 531.8	— 41.5
1.226	100	— 346.27	+ 174.51	— 163.02	+ 610.1	— 88.4
1 193	90	— 364.59	+ 184 65	— 230.54	+ 796.1	— 315.0
1.160	80	— 380.46	+ 190.82	— 256.01	+ 806.8	— 309.7
1.127	70	— 395.77	+ 194.81	— 257 69	+ 734.7	— 228.7
1.094	60	414.37	+ 205.97	— 310.30	+ 815.1	— 285.9
1.061	50	— 434.67	+ 216.83	— 330 83	+ 736.0	— 145.4
1.028	40	— 454.02	+ 225.05	— 347.03	+ 717.9	— 130.5
1 012	35					
1 002	32					
0.996	30	— 483.74	+ 253.09	— 432.02		— 83.6
0.963	20	— 497.85	+ 250.76	— 467.07		— 100.3
0.930	10	— 519.13	+ 260.43	— 458.00		— 193.1
0.897	0	— 546.37	+ 275.09	— 452.79		— 109.1

§ 5. Because of the approach to the ideal gas-state at high temperatures, it is desirable in developing the reduced virial co-efficients with regard to t, to start from t and to take polynomials of $\frac{1}{t}$ The linear form in t is insufficient. It is obvious that for instance $\mathfrak{C} = \mathfrak{C}_1\, t + \mathfrak{C}_2$ is altogether unsatisfactory. Some calculations made us suppose that by means of such terms as $t\, e^{\frac{c}{t}}$ some peculiarities in the course of the functions could be represented, and that then c might be chosen nearly equal to 1. Such terms (giving also after the development terms in $^1/_t$) can easily be brought into connection with the idea of the collisions as dissociation phenomenon, and as in a development in series of an equation of state which takes this into account, terms $t\, e^{\frac{2}{t}}$ could also be expected, a term of this form was immediately taken into consideration. Although for the representation of $\mathfrak{B}$ and $\mathfrak{C}$ the three terms considered were sufficient, for $\mathfrak{D}$ it was necessary to introduce still more terms and we returned to $\frac{1}{t}$; this was also used with $\mathfrak{E}$ and $\mathfrak{F}$. So the first approximation for the reduced virial co-efficients was obtained.

$$10^8 . \mathfrak{B} = + 499.79\, t - 297.66\, t\, e^{\frac{1}{t}} - 21.74\, t\, e^{\frac{2}{t}}$$

$$10^{11} . \mathfrak{C} = + 141.725\, t - 149.828\, t\, e^{\frac{1}{t}} + 67.367\, t\, e^{\frac{2}{t}}$$

$$10^{18} . \mathfrak{D} = + 587.74\, t - 313.26\, t\, e^{\frac{1}{t}} + 139.87\, t\, e^{\frac{2}{t}} - 1145.93 \frac{1}{t}$$

$$10^{25} . \mathfrak{E} = - 8448.85\, t + 6594.51\, t\, e^{\frac{1}{t}} - 8793.16 \frac{1}{t}$$

$$10^{32} . \mathfrak{F} = + 9067.14\, t - 7116.56\, t\, e^{\frac{1}{t}} + 10152.19 \frac{1}{t}$$

In the following table are combined the deviations between the virial co-efficients derived from these reduced virial co-efficients with regard to the normal volume and those immediately derived from the observation in the table of § 3.

Deviations of the virial coefficients in first approximation.

	$10^3 \Delta B$	$10^6 \Delta C$	$10^{12} \Delta D$	$10^{18} \Delta E$	$10^{24} \Delta F$
Hydrogen 200°.25	— 0.0348	— 0.2161	— 0.4765	[+ 1.7441]	[— 0.7546]
99.25	+ 0.0343	— 0.19782	— 0.0518	[+ 0.8185]	[— 0.3955]
47.3					
15.4	+ 0.02623	— 0.08029	+ 0.2780	— 0.3770	+ 0.2642
0.	— 0.00466	+ 0.07947	— 0.1601	+ 0.3798	— 0.1849
Nitrogen 199.5	— 0.0591	+ 0.7382	— 12.434	[+ 51.807]	[— 69.735]
99.45	— 0.07318	+ 0.3502	— 4.843	[+ 19.304]	[— 26.879]
43.6					
16.0	— 0.09586	+ 0.0417	+ 0.189	— 0.9606	+ 1.690
0.	— 0.08145	— 0.0398	+ 0 5035	+ 0.713	— 1.799
Oxygen 199.5	— 0.17991	+ 0.1707	— 4.770	[+ 11.6517]	[— 9.806]
99.5	— 0.11754	+ 0.0815	— 1.8276	[+ 1.8991]	[— 1.110]
15.6	— 0.15608	+ 0.0003	+ 0.0504	+ 0.9331	— 1.522
0.	— 0.17291	+ 0.1079	— 0.1803	+ 1.6533	— 2.089
Carbon dioxide 258					
198	— 0.0256	+ 0.2977	— 7 489	+ 3.73	+ 9.81
137	— 0.0115	— 0 2973	+ 3 615	— 37.58	+ 104.08
100	+ 0.0310	— 0.3335	+ 4.141	— 23.11	+ 71.33
90	— 0.0116	— 0.046	+ 0.095	+ 9.33	— 3.29
80	— 0.0133	— 0.014	+ 0 079	+ 12.35	— 9.22
70	+ 0.0029	— 0.144	+ 2.333	+ 1.61	+ 6.78
60	— 0.0113	+ 0.067	— 0.309	+ 16.86	— 19.30
50	— 0.0339	+ 0.202	+ 0.088	+ 5.27	+ 13.21
40	— 0.0286	+ 0.125	+ 0.851	+ 4.11	+ 7.58
35					
32					
30	— 0.1393	+ 1.045	— 5 057	[+ 4.52]	+ 61.13
20	— 0.0279	+ 0.220	— 0 536	[+ 6.07]	— 4.74
10	+ 0.0153	— 0.066	— 3.432	[+ 7.99]	— 43.35
0	+ 0.0093	— 0.219	— 1.162	[+ 9.21]	— 28.90

§ 6. Starting from the first approximation for the virial coefficients given in § 5 we looked now for small variations in the virial coefficients so that the latter while keeping the agreement with the isothermal, which they have to represent, allow of a better development in terms of the reduced temperature. And so the isothermals were mutually smoothed. The weak point in this method, when it is extended over more substances is the imperfect agreement of the reduced equations of state, further increased by uncertainty in the critical data. But it is important to see how far we can advance under these circumstances.

In calculating the variations $\Delta B_A, \Delta C_A, \Delta D_A, \Delta E_A, \Delta F_A$ which had each to satisfy each of the isothermals and which therefore had to be mostly sought out in the same way as B_A, C_A, D_A, E_A, F_A were calculated, it appeared desirable also to modify slightly the form of the development in series with regard to the reduced temperature.

We put:

$$\left.\begin{aligned}
\mathfrak{B} &= \mathfrak{b}_1 \mathfrak{t} + \mathfrak{b}_2 \mathfrak{t} e^{\frac{1}{\mathfrak{t}}} + \mathfrak{b}_2 \mathfrak{t} e^{\frac{2}{\mathfrak{t}}} + \mathfrak{b}_3 \frac{1}{\mathfrak{t}} \\
\mathfrak{C} &= \mathfrak{c}_1 \mathfrak{t} + \mathfrak{c}_2 \mathfrak{t} e^{\frac{1}{\mathfrak{t}}} + \mathfrak{c}_2 \mathfrak{t} e^{\frac{2}{\mathfrak{t}}} + \mathfrak{c}_3 \frac{1}{\mathfrak{t}} \\
\mathfrak{D} &= \mathfrak{d}_1 \mathfrak{t} + \mathfrak{d}_2 \mathfrak{t} e^{\frac{1}{\mathfrak{t}}} + \mathfrak{d}_3 \frac{1}{\mathfrak{t}} + \mathfrak{d}_4 \frac{1}{\mathfrak{t}^2} \\
\mathfrak{E} &= \mathfrak{e}_1 \mathfrak{t} + \mathfrak{e}_2 \mathfrak{t} e^{\frac{1}{\mathfrak{t}}} + \mathfrak{e}_3 \frac{1}{\mathfrak{t}} + \mathfrak{e}_4 \frac{1}{\mathfrak{t}^2} \\
\mathfrak{F} &= \mathfrak{f}_1 \mathfrak{t} + \mathfrak{f}_2 \mathfrak{t} e^{\frac{1}{\mathfrak{t}}} + \mathfrak{f}_3 \frac{1}{\mathfrak{t}} + \mathfrak{f}_4 \frac{1}{\mathfrak{t}^2}
\end{aligned}\right\} \quad \cdots \quad (IV)$$

We find for the 20 coefficients, which cover the range of the 4 substances the following

Coefficients of temperature of the reduced virial coefficients.

	1	2	3	4
$10^{5}\ \mathfrak{b}^{(IV)}$	+ 564.78	— 339.79	— 40.56	+ 186.71
$10^{11}\ \mathfrak{c}^{(IV)}$	+ 197.677	— 190.441	+ 61.328	+ 100.102
$10^{18}\ \mathfrak{d}^{(IV)}$	+ 669.98	— 204.92	— 866.31	+ 391.12
$10^{25}\ \mathfrak{e}^{(IV)}$	— 8546.45	+ 6655.21	— 8628.76	— 128.93
$10^{32}\ \mathfrak{f}^{(IV)}$	+ 9772.71	— 7729.47	+11653.22	— 527 02

From this follow immediately with (IV) and the reduction factors of § 4 the virial coefficients in the development of series (II), which gives the value calculated for the pv_A observed by AMAGAT.

The following tables show the deviations between observation and calculation for all the observations. If we examine them, it appears that the deviations must be chiefly attributed to the difference of the substances which we have combined, and might also be partly due to the uncertainty of the critical data.

O.—C. HYDROGEN (Tab. N⁰. 8 Am. and N⁰. 4 Am.)

P.	0°	15°.4	99°.25	200°.25	P.	0°	15°.4	47°.3
1	0				1	0		
100	+ 0.3	+ 0.5			100	+ 0.3		
150	+ 0.3	+ 0.4	+ 0.2	— 0.1	200	+ 0.4		
200	+ 0.4	+ 0.4	+ 0.2	— 0.3	300	+ 0.4		
250	+ 0.4	+ 0.6	+ 0.1	— 0.5	400	+ 0.6		
300	+ 0.4	+ 0.3	0.0	— 0.7	500	+ 0.6		
350	+ 0.5	+ 0.4	— 0.1	— 0.8	600	+ 0.8		
400	+ 0.6	+ 0.3	— 0.2	— 0.9	700	+ 0.7	+ 0.4	
450	+ 0.7	+ 0.3	— 0.3	— 1.0	800	+ 0.4	+ 0.3	
500	+ 0.6	+ 0.4	— 0.4	— 1.1	900	+ 0.6	— 0.2	
550	+ 0.6	+ 0.4	— 0.5	— 1.3	1000	+ 0.7	— 0.3	— 1.5
600	+ 0.7	+ 0.3	— 0.6	— 1.4	1100	+ 1.2	+ 0.1	— 1.6
650	+ 0.7	+ 0.3	— 0.8	— 1.6	1200	+ 1.0	+ 0.1	— 1.8
700	+ 0.6	+ 0.2	— 1.0	— 1.8	1300	+ 1.0	0.0	— 1.9
750	+ 0.5	+ 0.1	— 1.0	— 2.0	1400	+ 0.8	— 0.1	— 2.0
800	+ 0.5	0.0	— 1.2	— 2.1	1500	+ 0.6	— 0.2	— 2.1
850	+ 0.5	— 0.2	— 1.3	— 2.3	1600	+ 0.5	— 0.4	— 2.1
900	+ 0.4	— 0.2	— 1.5	— 2.5	1700	+ 0.6	— 0.7	— 2.4
950	+ 0.4	— 0.3	— 1.7		1800	+ 0.5	— 0.7	— 2.3
1000	+ 0.2	— 0.4	— 1.7		1900	+ 0.5	— 0.6	— 2.4
					2000	+ 0.5	— 0.7	— 2.6
					2100	+ 0.4	— 0.6	— 2.7
					2200	+ 0.4	— 0.4	— 2.8
					2300	+ 0.5	— 0.3	— 2.6
					2400	+ 0.7	— 0.2	— 2.5
					2500	+ 0.9	— 0.2	— 2.3
					2600	+ 1.0	+ 0.1	— 2.4
					2700	+ 1.1	— 0.1	— 2.2
					2800	+ 0.9	— 0.2	— 2.2
					2900		— 0.4	— 2.1
					3000		— 0.4	

O.—C. NITROGEN (Tab. N°. 9 Am. and N°. 5 Am.)

P.	0°	16°	99°.45	199°.50	P.	0°	16°	45°.6
1	+ 0.0				1	0		
100	+ 0.2	+ 0.0			100	+ 0.2		
150	+ 0.4	+ 0.3	+ 0.7	+ 0.9	200	+ 0.4		
200	+ 0.4	+ 0.4	+ 0.7	+ 0.8	300	+ 0.8		
250	+ 0.5	+ 0.5	+ 1.0	+ 0.9	400	+ 1.1		
300	+ 0.8	+ 0.8	+ 1.3	+ 1.1	500	+ 1.5		
350	+ 1.0	+ 0.9	+ 1.4	+ 1.2	600	+ 1.7		
400	+ 1.1	+ 1.0	+ 1.5	+ 1.3	700	+ 1.8		
450	+ 1.3	+ 1.3	+ 1.5	+ 1.4	800	+ 2.2	+ 0.8	
500	+ 1.5	+ 1.2	+ 1.4	+ 1.5	900	+ 2.3	+ 1.7	+ 1.4
550	+ 1.7	+ 1.3	+ 1.3	+ 1.4	1000	+ 2.3	+ 1.5	+ 0.3
600	+ 1.7	+ 1.4	+ 1.2	+ 1.4	1100	+ 2.5	+ 1.7	+ 0.7
650	+ 1.6	+ 1.4	+ 1.2	+ 1.4	1200	+ 2.6	+ 1.8	+ 1.2
700	+ 1.6	+ 1 3	+ 1.1	+ 1.2	1300	+ 2.7	+ 1.7	+ 1.3
750	+ 1.7	+ 1.4	+ 1.1	+ 1.0	1400	+ 2.8	+ 1.5	+ 1.3
800	+ 1.7	+ 1.5	+ 1.1	+ 0.8	1500	+ 2.8	+ 1.4	+ 1.0
850	+ 1.8	+ 1.5	+ 1.0	+ 0.8	1600	+ 2.8	+ 1.6	+ 1.1
900	+ 1.9	+ 1.6	+ 1.1	+ 0.6	1700	+ 2.5	+ 1.6	+ 1.0
950	+ 2.0	+ 1.7	+ 1.2	+ 0.6	1800	+ 2.2	+ 1.3	+ 0.7
1000	+ 2.1	+ 1.7			1900	+ 2.2	+ 1.4	+ 0.8
					2000	+ 2.1	+ 1.5	+ 0.7
					2100	+ 2.1	+ 1.4	+ 0.6
					2200	+ 2.1	+ 1.4	+ 0.5
					2300	+ 2.2	+ 1.3	+ 0.3
					2400	+ 2.3	+ 1.5	+ 0.5
					2500	+ 2.7	+ 1.6	+ 0.5
					2600	+ 3.0	+ 1.6	+ 0.6
					2700	+ 3.3	+ 1.8	+ 0.4
					2800	+ 3.4	+ 2.0	+ 0.5
					2900	+ 3.6	+ 2.2	+ 0.6
					3000	+ 3.6	+ 2.2	+ 0.7

O.—C. OXYGEN. (Tab. N⁰. 7 Am. and N⁰. 4 Am.)

P.	0°.	15°.6	99°.50	199°.5	P.	0°.	15°.6
1	0				1	0	
100	— 1.2	— 0.5	+ 0.4		100	— 1.3	
150	— 1.3	— 1.1	0.0	+ 0.3	200	— 1.7	
200	— 1.7	— 1.5	0.0	— 0.1	300	— 1.9	
250	— 1.9	— 1.5	0.0	— 0.3	400	— 1.6	
300	— 1.9	— 1.7	— 0.1	— 0.4	500	— 1.3	
350	— 1.7	— 1.7	0.0	— 0.4	600	— 1.0	— 1.4
400	— 1.6	— 1.7	+ 0.2	— 0.5	700	— 0.6	— 0.9
450	— 1.5	— 1.5	+ 0.2	— 0.6	800	— 0.1	— 0.9
500	— 1.5	— 1.5	+ 0.2	— 0.5	900	+ 0.1	— 0.5
550	— 1.3	— 1.4	0.0	— 0.7	1000	+ 0.4	— 0.2
600	— 1.2	— 1.4	0.0	— 0.9	1100	+ 0.6	— 0.1
650	— 0.9	— 1.2	0.0	— 1.0	1200	+ 0.6	— 0.1
700	— 0.8	— 1.1	0.0	— 1.2	1300	+ 0.4	— 0.1
750	— 0.6	— 1.2	0.0	— 1.3	1400	+ 0.1	— 0.4
800	— 0.3	— 1.1	— 0.3	— 1.6	1500	+ 0.2	— 0.4
850	— 0.1	— 1.0	— 0.5	— 1.7	1600	0.0	— 0.5
900	+ 0.1	— 0.8	— 0.7	— 2.0	1700	0.0	— 0.3
950	+ 0.2	— 0.7	— 0.9	— 2.3	1800	— 0.1	— 0.3
1000	+ 0.4	— 0.5	— 1.1		1900	— 0.2	— 0.5
					2000	— 0.4	— 0.5
					2100	+ 0.1	— 0.7
					2200	0.0	— 0.7
					2300	0.0	— 0.6
					2400	0.0	— 0.6
					2500	0.0	— 0.4
					2600	0.0	— 0.4
					2700	0.0	— 0.7
					2800	0.0	— 0.8
					2900	— 0.1	— 1.0
					3000	— 0.4	— 0.9

O.—C. CARBON DIOXIDE. (Tab. N°. 14 AM.)

P.	0	10	20°	30°	32°	35°	40°	50°	60°	70°	80°	90°	100°
31	+ 0.5												
33	+ 0.4	0.0											
34	+ 0.4	0.0											
35	−14.5*	0.0	0.0										
37	18.5*	0.0	0.0	− 0.1									
40		− 0.1	0.0	− 0.1	+ 0.4	0.0	− 0.1						
44		+ 0.1	0.0										
45		+16.5*	− 0.1	− 0.1	+ 0.0	− 0.0	− 0.2	− 0.1	− 0.1	− 0.2			
48			+ 0.1	0.0			− 0.1	− 0.2	− 0.1	− 0.2			
50	−21.6*	+ 6.5*	0.0	0.0	+ 0.0	0.0	− 0.1	− 0.1	− 0.4	− 0.1	− 0.2	0.0	+ 0.4
53			− 0.1	0.0			0.0	− 0.1	− 0.1	− 0.1	− 0.2	0.0	+ 0.4
55			− 0.1	0.0	+ 0.0	0.0	− 0.1	− 0.1	0.0	− 0.1	− 0.2	0.0	+ 0.4
56			− 0.1										
57			+10.1*										
60			+ 5.9*	− 0.1	+ 0.1	0.0	− 0.1	0.0	0.0	− 0.1	− 0.2	− 0.4	+ 0.4
65				− 0.2	− 0.2	− 0.1	− 0.2	0.0	0.0	0.0	− 0.4	− 0.4	+ 0.4
68				− 0.3	− 0.3	− 0.1	− 0.2	− 0.1	0.0	− 0.4	− 0.4	− 0.4	+ 0.4
70				− 0.6	− 0.3	− 0.2	− 0.2	− 0.1	0.0	0.0	− 0.4	− 0.4	+ 0.4
71				+ 4.1									
72				+ 3.6	− 0.3								
73					0.0								
74				+ 2.7	+ 0.3	− 0.2							
74.5					+ 1.8								
75				+ 2.1	+ 3.9	− 0.2	− 0.4	− 0.1	− 0.1	0.0	0.0	0.0	+ 0.2
76					+ 3.8	− 0.4							
78				+ 2.1	+ 3.5	+ 0.4							
80				+ 1.5		+ 2.1	− 0.5	0.0	− 0.1	0.0	+ 0.4	0.0	+ 0.2
82						+ 2.6	− 0.1						
85						+ 2.1	+ 0.2	− 0.1	0.0	+ 0.1	+ 0.4	0.0	+ 0.3
90						+ 1.7	+ 0.9	− 0.1	0.0	0.0	0.0	0.0	+ 0.3
95							+ 0.9	− 0.1	0.0	0.0	+ 0.4	+ 0.1	+ 0.3
100							+ 0.7	− 0.1	0.0	0.0	+ 0.4	+ 0.4	+ 0.3
110							− 0.5	− 0.2	− 0.2	0.0	0.0	+ 0.4	+ 0.4

O.—C. CARBON DIOXIDE. (Tab. N°. 13 Am.)

P.	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	137°	198°	258°
50	−21.6*	+ 6.4*	+ 0.2	0.0	− 0.3	− 0.2	− 0.1	− 0.1	− 0.3	0.0	+ 0.1	− 0.2		
75	−31.4*	−12.7*	+ 0.9	+ 3.0	+ 0.2	+ 0.3	+ 0.3	+ 0.1	0.0	− 0.1	+ 0.2	0.0	+ 0.1	− 0.6
100	−14.1*	− 5.2*	+ 1.5	+ 2.5	+ 0.7	− 0.1	0.0	0.0	+ 0.1	+ 0.1	+ 0.3	+ 0.1	+ 0.3	− 0.5
125	−11.0*	− 0.2	+ 2.7	+ 0.5	+ 0.2	− 0.8	− 0.4	− 0.3	− 0.1	+ 0.1	+ 0.4	+ 0.2	+ 0.5	− 0.3
150	− 9.7*	− 0.2	+ 2.5	− 0.6	+ 0.4	− 0.7	− 0.5	− 0.7	− 0.1	− 0.1	+ 0.3	+ 0.3	+ 0.8	− 0.2
175	− 7.7*	− 0.3	+ 1.9	+ 0.2	+ 0.6	− 0.7	− 1.0	− 0.5	− 0.6	− 0.2	+ 0.2	+ 0.4	+ 0.9	0.0
200	− 7.5*	+ 1.2	+ 2.8	+ 1.5	+ 1.4	− 0.1	− 0.3	− 0.7	− 0.4	− 0.6	+ 0.1	+ 0.4	+ 1.1	+ 0.2
225	− 2.9	+ 0.4	+ 4.5	+ 3.9	+ 1.9	− 0.6	− 0.1	− 0.6	− 1.0	− 0.5	0.0	+ 0.3	+ 1.4	+ 0.4
250	− 3.0	+ 0.9	+ 3.9	+ 4.9	+ 2.0	− 0.5	+ 0.3	− 0.4	− 1.9	− 1.1	0.0	+ 0.1	+ 1.6	+ 0.6
275	− 3.3	+ 1.3	+ 4.1	+ 4.1	+ 3.3	+ 1.3	+ 0.4	− 0.4	− 0.9	− 1.0	− 0.3	− 0.2	+ 1.5	+ 0.7
300	− 3.7	+ 1.5	+ 4.0	+ 4.8	+ 3.4	+ 1.0	+ 0.1	− 0.3	− 1.0	− 1.0	− 0.8	0.0	+ 1.4	+ 0.8
350	− 2.3	+ 2.8	+ 3.8	+ 4.0	+ 3.7	+ 1.9	+ 1.1	− 0.4	− 0.7	− 1.5	− 1.1	− 0.5	+ 1.3	+ 0.9
400	− 1.6	+ 1.8	+ 4.6	+ 4.1	+ 3.9	+ 2.1	+ 1.2	− 0.2	− 0.7	− 1.2	− 1.6	− 1.3	+ 1.0	+ 0.8
450	− 2.8	+ 2.3	+ 4.6	+ 4.3	+ 3.5	+ 1.9	+ 1.1	+ 0.2	− 0.7	− 1.6	− 1.8	− 1.9	+ 0.7	+ 0.6
500	− 2.0	+ 2.4	+ 4.6	+ 3.9	+ 2.6	+ 2.1	+ 1.5	+ 0.3	− 0.6	− 1.5	− 2.0	− 2.3	+ 0.3	
550	− 2.5	+ 1.9	+ 4.7	+ 4.6	+ 3.2	+ 1.7	+ 1.2	+ 0.5	− 0.2	− 1.8	− 2.1	− 2.6	+ 0.1	
600	− 2.1	+ 1.4	+ 3.6	+ 4.2	+ 3.7	+ 2.0	+ 0.9	+ 0.2	− 0.6	− 1.6	− 2.0	− 3.0	− 0.7	
650	− 2.4	+ 2.0	+ 3.9	+ 3.8	+ 3.3	+ 2.0	+ 1.0	+ 0.2	− 0.5	− 1.6	− 2.1	− 3.6	− 1.2	
700	− 2.1	+ 2.2	+ 3.6	+ 4.6	+ 3.9	+ 2.5	+ 1.2	+ 0.1	− 0.7	− 1.7	− 2.3	− 3.9	− 1.9	
750	− 3.0	+ 2.7	+ 4.7	+ 5.5	+ 4.5	+ 3.1	+ 1.6	+ 0.4	− 1.0	− 1.8	− 2.1	− 4.0	− 2.1	
800	− 2.9	+ 2.1	+ 4.4	+ 4.9	+ 4.4	+ 3.0	+ 1.5	+ 0.2	− 0.9	− 1.9	− 2.6	− 4.3	− 2.7	
850	− 2.8	+ 1.3	+ 3.9	+ 4.5	+ 4.2	+ 2.7	+ 1.5	+ 0.2	− 1.2	− 2.4	− 2.8	− 4.5	− 2.9	
900	− 2.9	+ 1.2	+ 3.0	+ 3.6	+ 3.2	+ 2.0	+ 1.0	− 0.3	− 1.3	− 2.5	− 3.2	− 4.8	− 3.7	
950	− 3.4	− 0.1	+ 2.1	+ 2.4	+ 1.9	+ 1.0	+ 0.1	− 1.1	− 2.0	− 2.9	− 3.7	− 5.4	− 4.3	
1000	− 4.2	− 1.0	+ 0.9	+ 1.5	+ 0.8	+ 0.1	− 1.3	− 2.1	− 3.1	− 3.8	− 4.1			

The deviations asterisked in the tables of carbon dioxide are in the neighbourhood of the state of saturation. Owing to the great variation of p with a small variation in v, the agreement cannot be judged from the deviation from pv, and for this we might with more advantage consider the deviation which v would have to undergo in order to yield this variation in pv.

At 0° C. for instance instead of the percent deviation in pv_A at

$P =$	35	37	50	75	100	125	150	175	200 atm.
	—14.5*,	—18.5*,	—21.6*,	—31.4*,	—14.1*,	—11.0*,	—9.7*,	—7.7*,	—7.5*

we have to put

—0.6 ,	—0.8 ,	—1.0 ,	—1.7 ,	—0.9 ,	-0.7^5,	—0.7 ,—	0,6 ,	—0.6

percent variation in v_A.

Also at 10° C. and $P =$	45 ,	50 ,	75 ,	100	atm.
instead of	+16.5*,	+ 6.5*,	—12.7*,	—5.2*	percent of pv_A
	+ 1.5 ,	+ 0.6 ,	— 1.1 ,	—0.5	percent of v_A

and at 20° and $P =$	57 ,	60	atm.
instead of	+19.1*,	+ 5.9*	percent of pv_A
	+ 2.6 ,	+ 1.2	percent of v_A

The criterium of MAXWELL expressed in $\frac{1}{10000}$ parts of the unit of pv_A agrees to

+ 157 at 0°

+ 113 » 10°

+ 71 » 20°

+ 36 » 30°

Both the deviation just considered and this latter, would be considerably smaller if we had only employed carbon dioxide.

§ 7. Finally a similar calculation to that used for form IV was carried out with polynomials

$$\left.\begin{aligned}
\mathfrak{B} &= \mathfrak{b}_1 t + \mathfrak{b}_2 + \mathfrak{b}_3 \frac{1}{t} + \mathfrak{b}_4 \frac{1}{t^3} \\
\mathfrak{C} &= \mathfrak{c}_1 t + \mathfrak{c}_2 + \mathfrak{c}_3 \frac{1}{t} + \mathfrak{c}_4 \frac{1}{t^3} \\
\mathfrak{D} &= \mathfrak{d}_1 t + \mathfrak{d}_2 + \mathfrak{d}_3 \frac{1}{t} + \mathfrak{d}_4 \frac{1}{t^3} \\
\mathfrak{E} &= \mathfrak{e}_1 t + \mathfrak{e}_2 + \mathfrak{e}_3 \frac{1}{t} + \mathfrak{e}_4 \frac{1}{t^3} \\
\mathfrak{F} &= \mathfrak{f}_1 t + \mathfrak{f}_2 + \mathfrak{f}_3 \frac{1}{t} + \mathfrak{f}_4 \frac{1}{t^3}
\end{aligned}\right\} \quad . \; . \; . \quad (V)$$

It gives for the required values

Temperature coefficients of the reduced virialcoefficients.

	1	2	3	4
$10^8 \mathfrak{b}^{(V)}$	+ 183.212	— 405.612	— 127.258	— 122.435
$10^{11} \mathfrak{c}^{(V)}$	+ 67.880	— 62.647	+ 131.275	+ 97.214
$10^{18} \mathfrak{d}^{(V)}$	+ 474.172	— 306.627	— 657.471	+ 126.641
$10^{25} \mathfrak{e}^{(V)}$	— 1871.27	+ 6426.11	— 4651.33	+ 781.52
$10^{32} \mathfrak{f}^{(V)}$	+ 2002.92	— 7272.08	+ 6331.42	— 1170.45

Although I thought at first that a good representation of the observations could best be attained by IV, it appeared that the representation by means of the more simple form V almost entirely corresponds to this. The reduced virial coefficients calculated according to V correspond so closely to those calculated according to IV that it was entirely superfluous to separately calculate another equation for all observations by means of this form (V) as was done for IV (again using the reduction factors in § 4 in order to find the virial coefficients in the development in series II).

Physics. — Mr. Fred. Schuh on: *"Plane waves of light in an homogeneous, electrically and magnetically anisotropic dielectric."* (2nd part).

10. Before examining the wave surface more closely, I shall first show that the ray of light is normal to the electric and the magnetic force, and therefore to the ray-plane. For this purpose we first show that the ray is electrically conjugate to $\mathfrak{D}$ and magnetically to $\mathfrak{B}$, properties which continue to exist after a transformation such as we have used. From equation (59) follows by (56), (57) and (58),

$$\frac{a_x\, l'\, \lambda'}{a_x\, v^2 - s''^2} + \frac{a_y\, m'\, \mu'}{a_y\, v^2 - s''^2} + \frac{a_z\, n'\, \nu'}{a_z\, v^2 - s''^2} = 0,$$

from which by (50), (51) and (52)

$$a_x\, f'\, \lambda' + a_y\, g'\, \mu' + a_z\, h'\, \nu' = 0,$$

which expresses that $\mathfrak{D}'$ and the transformed ray are conjugate diameters of the transformed electric ellipsoid.

If we take (50) into account, we derive from (56),

$$\lambda'\, a' = \frac{s''}{\varrho''}\, l'\, a' - \frac{s''\,(\varrho''^2 - s''^2)}{\varrho''\, v^2\,(a_x\, l'\, f' + a_y\, m'\, g' + a_z\, n'\, h')}\, f'\, a'.$$

Adding to this the corresponding equations for $\mu'\, b'$ and $\nu'\, c'$, we find:

$$\lambda'\, a' + \mu'\, b' + \nu'\, c' = 0,$$

which means that $\mathfrak{B}'$ and the transformed ray are conjugate diameters of the transformed magnetic ellipsoid (sphere with radius 1).

11. The electric force being normal to the magnetic induction and to the ray, we get:

$$\frac{P}{\nu\, b - \mu\, c} = \frac{Q}{\lambda\, c - \nu\, a} = \frac{R}{\mu\, a - \lambda\, b} = \frac{1}{A}.$$

If we substitute this in one of the equations (30), (31) and (32), we find:

$$A = \frac{v}{s}\,(l\,\lambda + m\,\mu + n\,\nu).$$

The expressions $l\,\lambda + m\,\mu + n\,\nu$ and $\frac{s}{\varrho}$ both being the cosine of the angle between the ray and the normal to the wave-front, so that:

$$l\,\lambda + m\,\mu + n\,\nu = \frac{s}{\varrho}, \quad \ldots\ldots\ldots \quad (61)$$

we find $A = \frac{v}{\varrho}$; consequently:

$$P = \frac{\varrho}{v}\,(\nu\,b - \mu\,c), \quad \ldots\ldots\ldots \quad (62)$$

$$Q = \frac{\varrho}{v}\,(\lambda\,c - \nu\,a), \quad \ldots\ldots\ldots \quad (63)$$

$$R = \frac{\varrho}{v}\,(\mu\,a - \lambda\,b), \quad \ldots\ldots\ldots \quad (64)$$

or

$$\mathfrak{E} = \frac{\varrho}{v}\,\underline{sin}_{\pi}\,(\underline{1}_{s}\ \mathfrak{B}). \quad \ldots\ldots\ldots \quad (65)$$

In the same way we deduce:

$$\alpha = \frac{\varrho}{v}\,(\mu\,h - \nu\,g), \quad \ldots\ldots\ldots \quad (66)$$

$$\beta = \frac{\varrho}{v}\,(\nu\,f - \lambda\,h), \quad \ldots\ldots\ldots \quad (67)$$

$$\gamma = \frac{\varrho}{v}\,(\lambda\,g - \mu\,f), \quad \ldots\ldots\ldots \quad (68)$$

or

$$-\,\mathfrak{H} = \frac{\varrho}{v}\,\underline{sin}_{\pi}\,(\underline{1}_{s}\ \mathfrak{D}). \quad \ldots\ldots\ldots \quad (69)$$

These equations are the same as (27) to (32), with the exception that $\mathfrak{D}$ and $\mathfrak{E}$ and in the same way $\mathfrak{B}$ and $\mathfrak{H}$ have been interchanged, and that $\frac{v}{s}$ is replaced by $\frac{\varrho}{v}$. By changing in the same way (39), (40), (41) and (42), we find:

$$\lambda = \frac{v}{\varrho} \, \frac{Q\gamma - R\beta}{4\pi U}, \quad \ldots \ldots \ldots \quad (70)$$

$$\mu = \frac{v}{\varrho} \, \frac{R\alpha - P\gamma}{4\pi U}, \quad \ldots \ldots \ldots \quad (71)$$

$$\nu = \frac{v}{\varrho} \, \frac{P\beta - Q\alpha}{4\pi U}, \quad \ldots \ldots \ldots \quad (72)$$

$$4\pi U \frac{\varrho}{v} \frac{1}{\underline{\ \ }_s} = \frac{sin}{\underline{\ \ }_{-}} (\mathfrak{H}\,\mathfrak{E}), \quad \ldots \ldots \quad (73)$$

from which last equation we further deduce:

$$\varrho^2 = v^2 \frac{\mathfrak{H}^2\,\mathfrak{E}^2\, sin^2\,(\mathfrak{H}\,\mathfrak{E})}{(4\pi U)^2} = v^2\, sin^2\,(\mathfrak{H}\,\mathfrak{E}) \, \frac{\mathfrak{E}}{\mathfrak{D}\, cos\,(\mathfrak{D}\,\mathfrak{E})} \, \frac{\mathfrak{H}}{\mathfrak{B}\, cos\,(\mathfrak{B}\,\mathfrak{H})},$$

and so, according to (9) and (18),

$$\varrho = v\, \underset{\mathfrak{E}}{r'_e} \;\; \underset{\mathfrak{H}}{r'_m} \;\; sin\,(\mathfrak{H}\,\mathfrak{E}). \quad \ldots \ldots \ldots \quad (74)$$

This equation expresses that ϱ is equal to v times the area of the parallelogram, described on the radii vectores of the reciprocal electric and the reciprocal magnetic ellipsoid, resp. in the directions of $\mathfrak{E}$ and $\mathfrak{H}$.

The two values of ϱ for the same ray are equal only when the ray-plane is a section of similitude of the two reciprocal ellipsoids, and the electric and magnetic force are accordingly indeterminate in it.

12. By Poynting's theorem the flow of energy is greatest through a plane through $\mathfrak{H}$ and $\mathfrak{E}$, i.e. through the ray-plane. The amount of energy W, which according to that theorem flows through the ray-plane per unit of time and per unit of area is:

$$W = \frac{v}{4\pi} \mathfrak{H} \mathfrak{E} \sin (\mathfrak{H} \mathfrak{E}),$$

or, by (73)

$$W = U\varrho, \quad \ldots \ldots \ldots \ldots \quad (75)$$

a result which was to be expected.

13. Let us now examine the wave-surface somewhat closer. The section with a plane of coordinates degenerates into 2 ellipses, which are e. g. for $x = 0$:

$$\frac{y^2}{\zeta_e^2} + \frac{z^2}{\eta_e^2} = \frac{v^2}{\varepsilon_m^2 \zeta_e^2 \eta_e^2 w^2}$$

and

$$\frac{y^2}{\zeta_m^2} + \frac{z^2}{\eta_m^2} = \frac{v^2}{\varepsilon_e^2 \zeta_m^2 \eta_m^2 w^2} .$$

The first ellipse is similar to the section of the electric ellipsoid $\frac{x^2}{\varepsilon_e^2} + \frac{y^2}{\zeta_e^2} + \frac{z^2}{\eta_e^2} = 1$ with the plane $x = 0$, and the second ellipse to the section of the magnetic ellipsoid $\frac{x^2}{\varepsilon_m^2} + \frac{y^2}{\zeta_m^2} + \frac{z^2}{\eta_m^2} = 1$ with the plane $x = 0$. I shall call the first ellipse an electric ellipse and the second a magnetic one. The same applies to the sections with the planes $y = 0$ and $z = 0$. It is easy to find that the electric ellipse in one plane of coordinates intersects the magnetic ellipse situated in another plane of coordinates (of course in a point of a coordinate axis). If $b_1 > b_2 > b_3$, the electric ellipse in the YZ-plane lies quite outside, and that in the XY-plane quite inside the magnetic ellipse, while in the XZ-plane the two ellipses intersect in 4 points. These four points are conic points of the wave-surface. (It is easy to find analytically that the wave-surface can only have conic points in the three planes of coordinates and in the plane at infinity, which projectively may also be considered as plane of coordinates. The section of the wave-surface with each of these four planes degenerates into 2 conic sections; so that every plane furnishes four conic points; in all, 16 conic points, of which however only the four lying in the XZ plane are real. The wave-surface intersects the plane at infinity along the sections of the electric and the magnetic ellipsoid with that plane). The wave-surface having a finite number of conic points (16), it cannot degenerate into 2 other surfaces if $b_1 > b_2 > b_3$. The two sheets cohere in the 4 real conic points.

If $b_2 = b_3$, the wave-surface degenerates into two ellipsoids

$$\frac{x^2}{\varepsilon_c^2} + \frac{y^2}{\zeta_c^2} + \frac{z^2}{\eta_c^2} = \frac{v^2}{\varepsilon_c^2\, \zeta_c^2\, \eta_m^2\, w^2}$$

and

$$\frac{x^2}{\varepsilon_m^2} + \frac{y^2}{\zeta_m^2} + \frac{z^2}{\eta_m^2} = \frac{v^2}{\varepsilon_m^2\, \zeta_c^2\, \eta_m^2\, w^2},$$

the first of which is similar to the electric, the second to the magnetic ellipsoid, and which I therefore call the electric and the magnetic part of the wave-surface. The two ellipsoids intersect the X-axis in the same point and touch each other in that point. The electric part of the wave-surface lies inside or outside the magnetic part, according to whether $b_1 > b_2$ or $b_1 < b_2$.

If $b_1 = b_2 = b_3$, the two ellipsoids coincide.

14 Let us return to the case $b_1 > b_2 > b_3$. We have seen that the ray of light is electrically conjugate to $\mathfrak{D}$ and magnetically to $\mathfrak{B}$. The ray of light is therefore the line of intersection of two planes; the first of these is electrically conjugate to $\mathfrak{D}$, and passes through $\mathfrak{B}$ and the line g_e which is electrically conjugate to the wave-front; the second is magnetically conjugate to $\mathfrak{B}$, and passes through $\mathfrak{D}$ and the line g_m, which is magnetically conjugate to the wave-front.

If a point of the wave-surface is given, so that the wave-front and the ray is known, we find $\mathfrak{D}$ and $\mathfrak{B}$ by letting planes pass through the ray and resp. through g_m and g_e, and making these planes intersect with the wave-front. We may also use the planes S_e and S_m which are electrically and magnetically conjugate to the ray and which intersect the wave-front resp. along $\mathfrak{D}$ and $\mathfrak{B}$. If the ray is electrically conjugate to the wave-front, the first construction fails for $\mathfrak{B}$ and the second for $\mathfrak{D}$, so that we can still apply one of the two constructions in order to find $\mathfrak{B}$ and $\mathfrak{D}$. (If the ray is magnetically conjugate to the wave-front, the reverse takes place). $\mathfrak{B}$ is then doubly conjugate to $\mathfrak{D}$ and to the ray and must therefore be a principal direction. The ray will consequently lie in a plane through two principal directions, and so also the point of the wave-surface; since ray and wave-front are electrically conjugate, this point will lie on the electric ellipse. $\mathfrak{B}$ falls now along a principal direction and $\mathfrak{D}$ touches the electric ellipse, as follows from the construction. Both constructions for $\mathfrak{D}$ and $\mathfrak{B}$ fail only when the ray is both electrically and magnetically conjugate to the wave-front, and falls therefore along a principal direction; the wave-front passes then through the two other principal directions. By now paying attention to the adjacent points of the wave surface, specially to points lying in the planes of coordinates, we find that $\mathfrak{D}$ touches the electric ellipse and $\mathfrak{B}$ the magnetic ellipse in the planes of coordinates.

15. To every wave-front belong two rays; if I and II are the directions of the possible electric and magnetic induction in the wave-front, the two rays of light are: the line of intersection of the planes through I and g_e and through II and g_m and the line of intersection of the planes through I and g_m and through II and g_e.

The question might be raised: When do the two rays of light fall in the same direction? Evidently when the ray of light is doubly conjugate both to I and to II and accordingly is a principal direction. We find also that this is the only case in which the two wave-fronts belonging to one direction of the ray coincide. The wave-front passes then through the two other principal directions.

16. Let us now examine the case of the wave-front being a section of similitude of the electric and the magnetic ellipsoid. The two lines g_e and g_m which resp. are electrically and magnetically conjugate to the wave-front, now lie both in the plane through two principal directions, viz. the X-axis and the Z-axis, the wave-front passing through the middle principal direction, the Y-axis; and they do not coincide. The ray being the line of intersection of the plane through $\mathfrak{B}$ and g_e and the plane through $\mathfrak{D}$ and g_m, while $\mathfrak{B}$ and $\mathfrak{D}$ are indeterminate, we get a cone of rays passing through g_e and g_m; for, if $\mathfrak{D}$ falls in the XZ-plane and $\mathfrak{B}$ along the Y-axis, g_e becomes the ray, and if $\mathfrak{B}$ falls in the XZ-plane and $\mathfrak{D}$ along the Y-axis, g_m becomes the ray. Moreover it is easily seen that these are the only rays falling in the XZ-plane and that therefore the cone is quadratic. If inversely the ray is given in this case, $\mathfrak{D}$ and $\mathfrak{B}$ may be found by means of one of the two given constructions. But whatever the course of the ray may be, we have always the same value of s, so that we have to deal with but one tangentplane to the wave-surface. This must touch along a curve (which is of course

an ellipse, as it counts as a double line of intersection and as the whole intersection with the wave-surface is of the 4th degree), which by its radii vectores through O indicates the possible rays of light. The tangent plane just mentioned touches the electric and the magnetic ellipses in the XZ-plane in the points A and B, and is parallel to the Y-axis; the rays g_e and g_m are the radii vectores of these points of contact, so that it is directly to be seen from the wave-surface that they belong to the cone of rays. Let now the ray be given by a point C of the ellipse of contact, then we find $\mathfrak{D}$ as the intersection of the wave-front with the plane through g_m and the ray, so that CB indicates the direction of $\mathfrak{D}$; in the same way CA indicates the direction of $\mathfrak{B}$. $\mathfrak{D}$ and $\mathfrak{B}$ being also conjugate diameters of the section of the wave-front with the electric and the magnetic ellipsoid, it follows directly, that the curve of contact must be similar to these elliptic sections with the same direction of axes. This might be seen, even if we did not yet know that the curve of contact is an ellipse. Further AB is a diameter of the ellipse of contact. (Internal conic refraction).

17. Let now the ray-plane be a section of similitude of the two reciprocal ellipsoids, then it passes through the middle reciprocal principal direction, so that the ray lies in the XZ-plane. Indeed, the ray of light is now the radius vector of one of the conic points of the wave-surface, and these points are only to be found in the XZ-plane. The wave-front is now indeterminate, being a tangent plane to the wave-surface in the conic point. (It is a quadratic conic point; else the line which connects it with a second conic point would have more than four points in common with the wave-surface). In a similar way as in the preceding case we can now show that the planes S_e and S_m which are electrically and magnetically conjugate to the ray, belong to the possible wave-fronts. This is also directly seen from the wave-surface, as the planes S_e en S_m are both parallel to the Y-axis, and, if transferred to the conic point, touch respectively the electric and the magnetic ellipse in the XZ-plane and so also the tangent cone in the conic point. By their intersection with the wave-front these planes S_e and S_m indicate directly the directions of $\mathfrak{D}$ and $\mathfrak{B}$. If, the wave-front coincides with S_e, $\mathfrak{D}$ falls in the XZ-plane and touches the electric ellipse; $\mathfrak{B}$ is then parallel to the Y-axis. Similarly with what we have found, when the wave-front is determinate but the ray indeterminate, we now get what follows. Let from O perpendiculars be drawn to all possible wave-fronts belonging to the radius vector of a conic point as ray. Let this cone of perpendiculars be intersected by a plane normal to the ray, which plane intersects the perpendiculars on S_e and S_m in A and B; let C be a point of that intersection, so that $O\,C$ is the normal to the wave-front, then $C\,B$ indicates the direction of the electric and $C\,A$ the direction of the magnetic force. The intersection is therefore an ellipse, of which AB is a diameter and which is similar to the sections of the ray-plane with the two reciprocal ellipsoids, and has its axes in the same directions as these sections. (External conic refraction).

18. We see directly from the wave-surface that the ray for which the wave-front is indeterminate, and the wave-front for which the ray is indeterminate, do not belong together as ray and wave-front. The phenomena of internal and external conic refraction are therefore wholly separated from each other. The ellipse of contact encloses a conic point of the wave-surface. This ellipse is a spinodal curve of the wave-surface; the tangent-plane in one of its points intersects the surface in a curve with a double point and two coinciding tangents, in such a way however that the curve has not a cusp in the point of contact, as is generally the case on a spinodal curve, but that it consists of two coinciding curves. The surface is everywhere convex-convex, the concave side turned to O, except between the four conic points and the ellipses of contact, where the wave-surface is concave convex.

19. The existence of tangent planes which touch along an ellipse, and which we may call ellipse-tangent planes, can also be directly derived from that of the conic points, if we remember that the surface is dualistic with itself. As we have four conic points in the XZ-plane, so we have four ellipse-tangent planes through the point whose point-coordinates are equal to the plane-coordinates of the XZ-plane, i. e. through the point at infinite distance on the Y-axis; these planes will be parallel to the Y-axis. There are also four ellipse-tangent planes parallel to the X- and four parallel to the Z-axis, which are, however, unreal. Four unreal ellipse tangent planes through O correspond to the four unreal conic points at infinite distance.

20. If $b_1 \lesseqgtr b_2 = b_3$, everything is much simpler. If the motion of light is given e.g. by a point P of the electric part of the wave-surface, which is not the point of intersection with the X-axis, we find that $\mathfrak{D}$ lies in the plane through P and the X-axis, and touches therefore the meridian through P; $\mathfrak{B}$ lies in a plane parallel to the YZ-plane and so touches the parallel-ellipse. The reverse takes place when P lies on the magnetic part of the wave-surface. If however, P lies on the X-axis, i.e. in the point where the two parts of the wave-surface touch each other, the ray is the X-axis and the wave-front the YZ-plane, $\mathfrak{D}$ and $\mathfrak{B}$ being now indeterminate in the wave-front (they must, however, be doubly conjugate to each other).

If finally $b_1 = b_2 = b_3$, the two parts of the wave-surface coincide. In this case to every wave-front belongs one ray and vice versa, $\mathfrak{D}$ and $\mathfrak{B}$ being always indeterminate in the wave-front.

Physics. — Mr. PH. KOHNSTAMM and Mr. B. M. VAN DALFSEN: "*Vapour-tensions of mixtures of ether and chloroform*". (Communicated by Prof. J. D. VAN DER WAALS).

For our determinations of vapour-tensions for mixtures of ether and chloroform we have made use of the dynamical method, i.e. we have determined the boiling-point at a certain pressure. As the methods of the determination of the vapour-tension, and specially the apparatus used by us, will be the subject of an extensive communication by one of us, which will appear before long, it seems superfluous to discuss these two points at length. Yet we will point out, specially to show how far our values are to be trusted, that we found it impossible to attain an accuracy greater than 1 m.m. mercury for dynamical determinations of vapour-tensions of mixtures. The values given are therefore at the utmost only in so far accurate; the errors of some of the observations can even become three or four times the amount. This is specially due to two sources of error, first the hydrostatic pressure of the boiling liquid, the influence of which was already pointed out by Dr. SMITS in the reports of these proceedings ¹), and secondly the superheating. We have tried to annul the disturbing

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¹) Volume II p. 475.

influence of superheating by several methods viz. those proposed by BECKMANN, SMITS and VON ZAWIDZKI, but none of them seemed to warrant greater accuracy. We refer to the fuller discussion which is to follow for the arguments to support this opinion.

The choice of the substances was determined by theoretical considerations. The following rules follow immediately from the formulae given by VAN DER WAALS [1]) for the pressure-curves of mixtures for normal substances if the relation of GALITZINE-BERTHELOT $a_{12} = \sqrt{a_1 a_2}$ holds good, viz. that a minimum pressure cannot occur and that near the border a curve ascending from the border must be concave seen from below. Now it is known of all substances which present a minimum of pressure, that they act chemically upon each other or are anormal in some other respect. Only the mixture ether-chloroform seems to be an exception. But as it is known that these substances may combine by the presence of KOH and that they therefore may exercise some chemical attraction on each other, this contradiction is only apparent. Nor does the literature afford more than one exception to the second rule, viz. the mixture of ether and chloroform examined by GUTHRIE, and for these substances we have most likely not to think of chemical action or anormality from other causes. But the experiments of GUTHRIE do not seem to be very accurate and his curve shows clearly that it was his aim to find so called molecular combinations. On account of the great importance of the matter, for the question is here whether the conduct of a mixture of two substances may be represented only from quantities, characteristic for each of these substances separately, or whether another quantity will appear indicating a mutual influence, it seemed advisable to us, to subject the mixture to a new investigation. We give our results in the table, where x is the molecular proportion of ether, and p the pressure. The temperature is 33°25. (See table p. 158).

These values show clearly that the combination of the two liquids chloroform and ether is not only contrary to the second of the rules given above, as might be expected from the course according to GUTHRIE, but also to the first. For it is evident that this curve shows a minimum, though it is quite on the border. So this curve is remarkable also in this respect, that it adds a second case where $\left[\frac{dp}{dx_1}\right]_{x_1=0} = 0$ to the one found already by CUNAEUS; a possibility which was doubted from different sides.

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[1]) Proceedings Vol. III p. 163.

$t = 33°25$

x	p
0	276 mM.[1]
0.050	276 »
0.080	276 »
0.203	282 »
0.295	294 »
0.500	355 »
0.588	412 »
0.695	500 »
0.898	657 »
0.955	697 »
1.	731 » [2]

It is evident that we shall have to renounce either the relation $a_{12} = \sqrt{a_1 a_2}$ or at least part of the simplifications which lead to the before-mentioned formulae of VAN DER WAALS. The simplifications are: the application 1st of the equation of state in its simplest form, 2nd of the wellknown equations for a_x and b_x, 3rd of the equation $\mu''_x = -\dfrac{d\dfrac{a_x}{b_x}}{dx_1}$. These simplifications are certainly not all quite correct, but it seems nevertheless not probable, that a more accurate formula would give such considerable correction-terms, that through them alone the strongly positive value of $\left[\dfrac{d^2p}{dx_1^2}\right]_{x_1=0}$ might be explained.

For the present it seems therefore most advisable to give up the

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[1] For chloroform we know only the determinations for vapour-tensions of REGNAULT. According to v. ZAWIDZKI however, his values must be inaccurate, as he had no pure chloroform at his disposal.

[2] At 30° RAMSAY and YOUNG found for ether 647.92; BATELLI 648.21. At the same temperature our observations give 649 mm. So the deviations are not greater than our probable error.

Ph. KOHNSTAMM and I

relation of GALITZINE-BERTHELOT, though the matter can really not be decided before the theory has proceeded so far that it will determine the value of the correction-terms more accurately.

Finally it is noteworthy that seen from below the given curve on the right side is concave, though very slightly [1]. So it has an inflection-point.

From the theory of VAN DER WAALS the appearance of such inflection-points under some circumstances was to be expected [2]; OSTWALD, on the other hand, denied their possibility. So our results confirm the before-mentioned theory, as the experiments of RAOULT [3] on mixtures of ether and very slightly volatile substances did before [4].

Yet we wish to state this latter point with some reserve. When determining the points $x = 0.955$ and 0.898, a difficulty was added to those already mentioned viz. that through the very considerable sloping of the p_{x_1}-curve the composition of the mixture when boiling, had a tendency to change in consequence of the extraction of the more volatile ether by distillation; the more so as, these experiments being made only lately, our coolingwater was considerably warmer than in previous experiments made in February and March. The reason why we think that we are justified in considering these two points as accurate is only that they as well as the previously determined point for $x = 0.695$ lead to the same result [5]. We hope, however, to return to this point afterwards.

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[1] This particularity is far less obvious in the subjoined representation on a reduced scale than in the original drawing.

[2] l.c. p. 170.

[3] Zeitschrift f. phys. Chem. 2. p. 353.

[4] Zeitschrift f. phys. Chem. 36, p. 60.

[5] Comp. note on the preceding page.

(**August 6, 1901**).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday September 28, 1901.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 28 September 1901, Dl. X).

The following papers were read:

Chemistry. — "*On the influence of the position of atomgroups in aromatic compounds on the process of the reactions*". By Prof. S. HOOGEWERFF and Dr. W. A. VAN DORP.

Zoology. — "*On the Gastrulation and the formation of the Mesoblast in Mammals*". By Prof. A. A. W. HUBRECHT.

Dr. HUBRECHT circulates and explains eleven plates that have been executed for a memoir on the ontogeny of Tarsius spectrum intended for Vol. VIII of the Transactions of the Academy.

He calls attention to certain theoretical conclusions concerning the gastrulation and the formation of mesoblast in mammals which appear in a different light now that we have become acquainted with the facts recorded in that memoir.

He expresses a wish to add certain woodcuts to the text, but is informed by the President and Secretary that for the present the publication of this memoir will have to be postponed.

Physiology. — "*The negative-inotropic influence of the nervus vagus on the heart*". By Prof. Th. W. Engelmann.

The weakening influence of the vagus on the contractions of the heart, discovered and studied by Nuel in Professor Donders' laboratory, has been more closely examined by the speaker with the aid of the method of suspension and the pantokymographion. Speaker's experiments have mostly been made on the atria of the frog's heart, in which the above-named action can be examined most easily.

It can here be provoked by direct irritation of the vagus-root and the vagus-origines in the brain, as well as through direct irritation of the atria, eventually of the sinus or ventricle-base (irritation of the intracardial vagusbranches), also by reflexaction from different parts of the body, especially so from the intestines. Generally speaking the action is in all cases the same in quality; it is also a case of indifference whether the heart beats spontaneously or whether the ventricle is brought to regular beating, antiperistaltically, by artificial irritation of the ventricle.

The effect is furthermore independent of eventual other, chronotropic, dromotropic or bathmotropic actions of the vagus-irritation, although often combined with it.

It shows itself in the following manner after one single induction-shock causing momentary irritation: After a very short latent stage the diminution in size and length of the contractions commences, which very soon reaches its maximum; after that the systoles gradually become more considerable and of longer duration.

Intensity and duration of the whole process rise within ample limits with the strength of the irritation.

The duration of the period of increasing debility is comparatively constant; during the experiments of the speaker, taken at an ordinary temperature and retained circulation, it proved to be 3—4 sec.; the lowest value was found with the strongest irritation. The latency lasts in the most favourable case less than 0.5″. The stage of the diminishing process can be prolonged for more than one minute. The weakening can lead temporarily to the entire cessation of the contractility.

The *phase* of the period of the heart in which the irritation takes place, has no perceptible influence. A refractory stage is therefore also wanting. Through superposition of cardiograms obtained by frequently repeated vagus-irritation of equal strength and duration, on the same absciss, the turning cylinder of the kymographion having always the same position, this fact can be demonstrated very clearly, as can also the whole process of the negative-inotropic action.

By accumulating the irritation the process may be much increased and lengthened. Fatigue comes readily in consequence. To obtain an equally great effect each time, it is necessary to insert comparatively long intervals between the periods of irritation. The conduction of the negative-inotropic action within the ventricle-wall takes place in a way different from the motory irritation. By compressing the muscular wall by means of a clasp the inotropic conducting power is raised sooner and under lower pressure than the motory one.

Experiments must still be made with regard to the swiftness of the inotropic conduction.

The speaker illustrated his lecture by the aid of a number of cardiograms.

Chemistry. — Dr. A. Smits: "*Investigations with the Micromanometer*". (Communicated by Prof. V. A. Julius).

1. After the investigations, published under this title in Sept. 1899 [1], I have been enabled by the great kindness of Prof. Bakhuis Roozeboom to continue my measurements.

Having made some improvements in the apparatus, which might possibly modify the results, it seemed very desirable to me to proceed with the measurements; moreover I considered the question, whether the decrease of the molecular vapour-tension continues to increase with the concentration also when the solution is still more diluted, as so important, that in my opinion nothing should be neglected, which might contribute to the solution of the question.

The improvements made were the following:

First the manometer was made from chosen tubes, which might be considered as perfectly cylindrical as appeared from the calibration.

Secondly all the taps were done away with, so that there was absolutely no occasion for leaks.

Thirdly the icebath was treated with more care, by excluding it from the surroundings by a thick layer of wool.

The measurements furnished the following result:

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[1] Proc. Royal Acad. at Amsterdam, Vol. II. p. 88.

Na Cl

Concentration in gr. mol. per 1000 gr. H_2O.	p_m in m.m. Hg of 0°.	i
0.0591	0.149	1.79
0.0643	0.146	1.76
0.1077	0.143	1.72
0.1426	0.142	1.71
0.4527	0.141	1.70
0.4976	0.141	1.70
1.0808	0.1432	1.723
1.2521	0.1439	1.730
	H_2SO_4	
0.0951	0.169	2.03
0.1208	0.156	1.87
0.4215	0.1603	1.93
0.9762	0.1716	2.063
	KNO_3	
0.0400	0.153	1.8
0.1450	0.131	1.58
0.5997	0.1156	1.39
0.9288	0.1084	1.304

From this table follows that p_m (decrease of molecular vapour-tension) and i attain a minimum in solutions of NaCl and of H_2SO_4. For Na Cl this minimum lies near 0,45 gr. mol., for H_2SO_4 it appears near ± 0,1 gr. mol. The observations render the existence of these minima very probable, for they appear in a region of concentration, where the accuracy is still great enough to be sure of the course.

As to the absolute value of p_m and i, we must keep in mind that e. g. for NaCl at the concentration 1 gr. mol. the deviation in the manometer was ± 60 m. m., and the uncertainty of the reading was 0.1 m. m. The error in p_m and therefore also in i amounts therefore to ± 0.17 pCt. at the concentration 1 gr. mol. It increases however with the dilution, so that it amounts to ± 1.7 pCt. at 0.1 gr. mol. and to ± 3 pCt. at 0.05 gr. mol. It is obvious that at the small concentrations the absolute value of p_m is comparatively untrustworthy. At the same time it is clear, that a very small difference of temperature between solutions and water or an exceedingly small leakage may conceal the minima from us.

With regard to the measurements with $K\,NO_3$-solutions it is noteworthy that, as was previously found by me, p_m increases with the dilution in opposition to the solutions with NaCl and H_2SO_4. Very remarkable is also the quick decrease of p_m and i at increase of concentration, so that i has reached the abnormally small value 1.3 already at 0,9288 gr. mol.

The results found by means of the boiling-method [1]), are in qualitative concordance with the above mentioned. Owing to the great difference in the temperatures at which the observations are made (100°), a quantitative concordance could not be expected.

At 100° I found namely that the molecular increase of the boiling-point and so also i for the salts NaCl, KCL and $Sr(No_3)_2$ reaches a minimum between the concentrations 0.1 and 0.5 gr. mol., whereas for the salts KNO_3, $NaNo_3$, $Ba(NO_3)_2$, $AgNO_3$ and $Pb(No'_3)_2$ a regular quick decrease of i was observed in case of increase of the concentration [2]).

It occurred to me that possibly also for these latter salts a minimum might appear, but not before the concentrations are much greater; by consulting tables about determinations of the boiling point of concentrated solutions [3]) however, I observed that even for very great concentrations the mol. increase of the boiling-point preserves the same quick decrease.

When considering the here-mentioned results, we should be inclined to generalize and say: all salts which behave abnormally at higher concentrations, as NaCl, KCl, H_2SO_4, $Sr(NO_3)_2$ and probably a great many substances more, will behave normally in very diluted state and show a minimum.

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[1]) Proc. Royal Acad. at Amsterdam. Vol. II p. 469.

[2]) Proc. Royal Acad. at Amsterdam. Vol. III p. 717.

[3]) LEGRAND, Ann. de Chim. et de Phys. T. LIII. Poggend. Ann. Bd. XXXVII.

In my opinion the minimum would indicate the point, where the influence of the deviation from the diluted state in this special case becomes so great, that the direction in which *i* varies is thereby reversed.

A difficulty remains in the difference between what has been observed by means of the method of decrease of the vapour-tension and that of the lowering of the freezing-point. Not only is there e.g. for Na Cl solutions a difference in absolute value of *i* at every concentration, but above the concentration 0,5 gr. mol. there had up to now not been found an increase of *i* with the concentration by means of the freezing-point method, whereas this singular course has been so clearly observed by means of the method of vapour-tension at 0° and that of the boiling-point at 100°.

Though it was formerly found that below the concentration 1 gr. mol. the molecular lowering of the freezing-point and with it *i* pretty quickly decreased with increase of the concentration, this is no longer so. RAOULT [1]) found a very slow decrease with increase of the concentration; afterwards CHROUSTCHOFF [2]) found constant values and a few months ago KAHLENBERG [3]) published the following results, which confirm my suspicion, that possibly also at the determination of the freezing-point soon a minimum may be found.

Concentration in gr. mol. per 1000 gr. H_2O.	Lowering of the freezing-point.	Molecular lowering of the freezing-point.	*i*
0.2043	0.693	3.393	1.82
0.4359	1.512	3.469	1.86
0.5077	1.750	3.447	1.85
0.6713	2.300	3.426	1.84
0.8400	2.866	3.412	1.83
0.9814	3.395	3.459	1.86

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[1]) Zeitschr. f. Phys. chem. 27. S. 617 (1898).

[2]) Comt. Rend. 131. p. 883 (1900).

[3]) Journal of Phys. chem. 5, 339 (1901).

It seems probable to me that, if the experimental methods are continually improved, the results will finally be found to agree, as the theory requires.

3. In a treatise, called: "The theory of electrolytic dissociation as viewed in the light of facts recently ascertained", KAHLENBERG ¹) states results obtained in determining the conductivity, the lowering of the freezing-point and the raising of the boiling-point of not diluted solutions. Here he arrives at the expected result, that in most cases the degree of dissociation indicated by the conductivity, deviates strongly from that, found when following the two other methods. He found i. a. that several salts, which in aqueous solution at a certain concentration are not only not dissociated, but even somewhat polymerized according to the lowering of the freezing-point and the raising of the boiling-point, yet conduct the electric current.

This occurs i.a. with sulphates of Mg, Zn, Cd, Ni, Co, Fe and Cu.

KAHLENBERG, as well as myself found, that the molecular raising of the boiling-point increases with the concentration for $NaCl$, KCl and also for KBr, Kg, $MgCl_2$ and $BaCl_2$, for the molecular lowering of the freezing-point of $NaCl$-solutions he found the course already mentioned, while JONES, CHAMBER and FRAZER ²) discovered minima for $MgCl_2$ and $BaCl_2$ in the region of concentration 0,1—0,6.

The molecular conductivity, on the other hand, increased regularly with the dilution.

I must point out here that KAHLENBERG in his experiments on the raising of the boiling-point of $NaCl$- and KCl-solution did not find a minimum, though he began with the concentration ± 0,2; possibly, however, his method, adapted for more concentrated solutions, was not sensible enough.

Finally the remarkable fact must be mentioned, that it would follow from the raising of the boiling-point of the just mentioned sulphates of Mg, Zn, Cd, Ni, Co, Fe and Cu, that the polymerization is greater at 100° than at 0° and that minima occur both at 100° and at 0°.

As KAHLENBERG has already stated, the investigations on non-aqueous solutions have brought to light, that though a salt in solution shows a normal molecular weight, yet the solution can conduct the electric current. This has been proved for solutions of $AgNO_3$ in

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¹) Journ. of Phys. Chem. 5. p. 339 (1901).

²) Amer. Chem. Journ. 23, p, 89, 512 (1900).

pyridine and benzolnitriel ¹) and also for solutions of $Cd\,J_2$, $Li\,Cl$, $Na\,J$, $Hg\,Cl$ and $NH_4\,CNS$ in acetone ²). It has further appeared, that KJ, NaJ, KbJ, NH_4J and KCNS solved in liquid SO_2 ³), conduct the electric current, notwithstanding their molecular weights are found to be abnormally great.

4. Chiefly on account of what has been mentioned here, KAHLENBERG considers the dissociation theory of ARRHENIUS as untenable. Not sharing this opinion, I have mentioned his principal objections to the dissociation theory.

KAHLENBERG has made experiments on non-diluted solutions, and this is a great impediment for the refutation of the correctness of the dissociation.

For according to theory the degree of dissociation found from the conductivity, can agree with that calculated from the lowering of the freezing-point, the decrease of vapour-tension and the raising of the boiling-point, only in case of exceedingly diluted solutions.

In what way a deviation from the diluted state will be felt in general, is not known, but what I have observed for NaCl, KCl and $H_2\,SO_4$-solutions, where minima of the molecular decrease of vapour-tension and raising of the boiling-point are found at ± 0.5 gr. mol. and below it, gives the conviction, that already at these small concentrations the deviation from the diluted state brings about an influence, which is so great as to even reverse the direction of the course of the molecular decrease of vapour-tension and the raising of the boiling-point.

That this influence is in very close connection with the nature of the salt, follows from the fact that for $NaNo_3$, KNO_3 etc. no minimum appears even at very great concentrations, which as I have already mentioned, may be due to the rapid decrease of the electric dissociation with increase of the concentration ⁴).

It has been shortly mentioned that also non-aqueous solutions make us acquainted with facts, which seem at first sight in opposition to the theory of ARRHENIUS, but to state with certainty that they are really so, seems premature to me.

If for aqueous solutions we should like to have a greater quan-

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¹) WERNER, Zeit. Anorg. Chem. 15, 1 (1897).

²) DUTOIT, FRIDERICH, Bull. Soc. Chim. Paris. (3), 19, 334 (1898).

³) WALDEN, Ber. Chem. Ges. Berlin 32, 2862. (1899).

⁴) Proc. Royal Acad. at Amsterdam, Vol. III. p. 717. (1901).

tity of observations at our disposal, this is of course the case in a much higher degree for non-aqueous solutions.

It is certain that in general the results obtained by the determination of the conductivity, do not agree with those found by a non-electric method, but the question is, what conclusions we have to draw from this.

Have we to conclude from what precedes, that the conductivity does not always indicate the degree of dissociation, or must we say that determinations of the molecular weight in solutions, which are not exceedingly diluted, do not always give information about the degree of dissociation?

It seems to me that there is more to be said in favour of the latter conclusion and that the results obtained indicate that in not very much diluted solutions ions may occur at the same time with products of polymerization or association and so may be in equilibrium with them.

Before the incorrectness of this supposition has been conclusively proved, we should not reject the dissociation theory of ARRHENIUS, which has rendered so many and such important services to chemical science.

Mathematics. — „*On the Theory of the Biquadratic Rest*". By Prof. LEOPOLD GEGENBAUER (extract of a communication to Prof. JAN DE VRIES).

The way followed in the various textbooks of the theory of numbers and even in most lectures on this subject for the generalisation of the quadratic $\left(\frac{m}{n}\right)$, cubic $\left[\frac{m}{n}\right]$ and biquadratic $\left(\left(\frac{m}{n}\right)\right)$ restcharacters, explained at first only for prime denominators, seems to make the introduction of the generalised symbol appear rather arbitrary; hence it does not satisfy the thinking student. The symbol referred to is namely defined either as was already done by JACOBI by the equation

$$\left(\frac{m}{\mu\nu}\right)=\left(\frac{m}{\mu}\right)\left(\frac{m}{\nu}\right) \text{ resp. } \left[\frac{m}{\mu\nu}\right]=\left[\frac{m}{\mu}\right]\left[\frac{m}{\nu}\right] \text{ resp. } \left(\left(\frac{m}{\mu\nu}\right)\right)=\left(\left(\frac{m}{\mu}\right)\right)\left(\left(\frac{m}{\nu}\right)\right)$$

that is, by assuming the existence of the theorem of multiplication for the denominator (e. g. DIRICHLET, „Vorlesungen über Zahlentheorie"; Bachmann, „Die Lehre von der Kreistheilung") or by extension to any uneven n of one of the equations

$$\left(\frac{m}{n}\right) = \prod_{k=1}^{\frac{n-1}{2}} \frac{\sin\frac{2m\,k\pi}{n}}{\sin\frac{2\,k\pi}{n}}$$

$$\left(\frac{m}{n}\right) = \prod_{k=1}^{\frac{n-1}{2}} \frac{\operatorname{tang}\frac{m\,k\pi}{n}}{\operatorname{tang}\frac{k\pi}{n}}$$

proved only for prime denominators, resp. of equations analogous to these of the complex integers formed out of the cubic, resp. biquadratic roots of unity, in which the trigonometric functions have been replaced by the elliptic functions belonging to the curve of Kiepert, that is to the invariants $g_2 = 4$, $g_3 = 0$, resp. by the lemniscatic functions ($g_2 = 0$, $g_3 = 4$), which n in the second particular case is prime to $1 - j$ or $1 + i$ (e. g. Kronecker „Berliner Monatsberichte, 1876").

Out of both definitions the different qualities of the generalised symbol can easily be deduced; and with its help it can also be shown that for the same holds good the corresponding generalisation of the Lemma of Gauss. Now every mathematician will regard as natural, if only on account of the analogy with the procedure followed almost without exception, the definition of the generalised symbol by an extension of the Euler criterium (resp. of the analogon of it in the region under consideration), and, as it does not suffice as is immediately proved, will try to attain it by a corresponding generalisation of the Lemma of Gauss (resp. of its analogon). For the symbol obtained in this way the relations just named are to be pointed out.

In his treatise „Zur Theorie der quadratischen Reste" published in the 1st vol. of the „Acta mathematica" (1882) Schering has applied for the quadratic restcharacters a deduction founded in reality on the ideas developed above; his deductions, however, are rather prolix, not being limited to this subject only, and therefore they have not found a good reception in the books and the students' halls, notwithstanding the simplified representation given to them by **Max Mandl** in his work „Ueber die Verallgemeinerung eines Gaussischen Algorithmus" appearing in the 1st vol. of the „Monatshefte für Mathematik und Physik" (1890). As far as I know no such method was tried for the remaining restcharacters.

Hence the communication will not be quite worthless of the highly simple manner which I generally make use of in my lectures on the introduction of the above named generalised symbol by means of the explained process, the more so, as it is very fit for the transparent treatment of other arithmetic problems, as I wish to prove in the following lines in one example at least. In my developments I shall confine myself to the complex integers formed out of the biquadratic roots of unity which I shall name shortly integers; however, I draw attention to the fact, that this method can be used for all kinds of numbers for which holds good the Euclidian algorithm of determinating the greatest common divisor.

1. As a basis for my developments I make use of the following formula

$$\sum_{x=\{m\}}^{(n)} f(x) = \sum_{d}^{\{m\}} \mu(d) \Big(\sum_{x=\left\{\frac{m}{d}\right\}} f(dx) \Big), \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

for the sum of the values, which are obtained by a given function $f(x)$ when its argument assumes all numbers, prime to a complex integer, of a complete system of restcharacters (except the naught) according to the modulus m, in which formula the summation according to d is to be extended to all the divisors of n lying in the region $\{m\}$ and the numerical function $\mu(x)$ has the value $+1$ when x is a real or a complex unity of the biquadratic cyclotomic body or is compounded of an even number of prime numbers (one or two members) all of which differing mutually, and the value -1 when the argument is a product of an uneven number of different prime factors, and finally the value naught when it is divisible by the square of a prime number. So this function is the function of **Möbius-Mertens** for the region of the complex integers formed out of the biquadratic roots of unity, of which I repeatedly made use in former treatises (see as an example „Zur Theorie der aus den vierten Einheitswurzeln gebildeten complexen Zahlen", Denkschriften der mathematisch-naturwissenschaftlichen Classe der kais. Akademie der Wissenschaften in Wien, 50, Vol.). As is known for these the relation exists

$$\sum_{d} \mu(d) = \begin{cases} 1 \\ 0 \end{cases},$$

according to the norm being equal to or larger than 1, with the help of which one can easily prove the formulae corresponding with each other

$$\sum f(d) = F(n),$$

$$\sum \mu(d)\, F\left(\frac{n}{d}\right) = f(n).$$

To obtain the above mentioned sum we must remove from the sum

$$\sum_{x = \{m\}} f(x)$$

all values of the function whose argument have a common divisor with n. If d_1 is a divisor of n in the region $\{m\}$ with a norm surpassing unity, the sum of the values to be ejected belonging to d_1 is

$$\sum_{x = \left\{\frac{m}{d_1}\right\}} f(d_1\, x).$$

If we were to subtract this sum for every divisor d from the first named one, we should have ejected more than once all values of the function, in which d_1 possesses a quadratic factor and those too in which d_1 is a product of several (k) prime factors; in the latter case this ejection would take place

$$\binom{k}{1} + \binom{k}{2} + \dots + \binom{k}{k} = (2^k - 1)$$

times instead of once, as the respective value appears in each sum corresponding to one of the k prime factors of d_1, to one of the $\binom{k}{2}$ products of any two prime factors of d_1, and so on. So on one hand we have only to take for d_1 those divisors of n built up of prime numbers all differing mutually, and on the other hand on account of $-\binom{k}{1}+\binom{k}{2}-\ldots+(-1)^k\binom{k}{k}=-1$ to take each sum belonging to such a d_1 with the positive or negative sign, according to the number of prime divisors of d_1 being even or uneven. Hence the values really to be subtracted being expressed by the sum

$$\sum_{d_1}^{\{m\}} \mu(d_1)\left(\sum_{x=\left\{\frac{m}{d_1}\right\}} f(d_1 x)\right)$$

extended to the divisors d_1 of n belonging to the region $\{m\}$ with a norm surpassing unity, we then immediately arrive at the formula (1).

For the region of the real numbers the corresponding relation

$$\sum_{x=1}^{x=m}{}^{(n)} f(x) = \sum_{d} \mu(d)\left(\sum_{x=1}^{x=\left[\frac{m}{d}\right]} f(dx)\right)$$

was given as far as I know for the first time by Nasimof in his treatise "Von der Summe der Zahlen, welche theilerfremd zu einer gegebenen Zahl N sind und eine andere gegebene Zahl P nicht überschreiten" published in Russian in the 11th vol. of the Proceedings of the Math. Society in Moskow (1883). This equation of Nasimof, as well as an extension of a special case originating from K. Zsigmondy („Zur Verallgemeinerung der Function $\varphi(m)$ in der Zahlentheorie," Journal für die reine und angewandte Mathematik, 111nd, Vol.) I have considerably generalised in my treatise: "Ueber eine Relation des Herrn Nasimof" contained in the 102nd vol. of the proceedings of the meetings of the Math. Phys. Class of the Imp. Academy of Sciences at Vienna. As I still wish to observe, the equation of Nasimof has moreover been made use of for the case $m=\infty$ in different investigations of which I will point out as an example Séguier's „Formes quadratiques et multiplication complexe" in order to avoid stating a great number of original works.

A special case of formula (1) is the one found by Dirichlet („Recherches sur les formes quadratiques à coefficients et à indéterminées complexes," Journal für die reine und angewandte Mathematik, 24th vol.)

$$\varphi(n) = N\left(p_1^{\alpha_1-1}\right)\left(N(p_1)-1\right) N\left(p_2^{\alpha_2-1}\right)\left(N(p_2)-1\right) \ldots N\left(p_r^{\alpha_r-1}\right).$$

$$.\left(N(p_r)-1\right) = \sum_d N\left(\frac{n}{d}\right)\mu(d) \quad , \quad (n = i^\rho p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r});$$

(p_λ being equal to a prime number of one or two members: $p_\lambda \neq p_k$, $\lambda \neq k$) for the number $\varphi(n)$ of those members of a complete system of rests (with the exception of the naught) according to modulus n which are prime to this.

This formula proves directly that this number is divisible at least by 4 if n is odd and by the 4th power of 2 for any n, when n has at least two mutually differing prime factors; moreover it shows that when m and n are prime to each other we have the relation

$$\varphi(m\,n) = \varphi(m)\,\varphi(n).$$

2. If

$$r_1, r_2, \ldots, r_{\frac{N(n)-1}{4}}$$

are the members of a fourth part of a system of rests according to the modulus n which will henceforth be supposed to be uneven and among these

$$\varrho_1, \varrho_2, \ldots, \varrho_{\frac{\varphi(n)}{4}}$$

prime to n, then there exist for each complex integer m prime to n of the considered region the congruences:

$$r_\lambda\, m \equiv i^{\sigma_\lambda(m,n)}\, r'_\lambda \; (mod.\, n) \; \left(\lambda = 1, 2, \ldots, \frac{N(n)-1}{4}, \sigma_\lambda(m,n) = 0, 1, 2, 3\right). \quad (2)$$

$$\sigma_{\lambda_1} m \equiv i^{\tau_{\lambda_1}(m,n)}\, \varrho'_{\lambda_1} \; (mod.\, n) \; \left(\lambda_1 = 1, 2, \ldots, \frac{\sigma(n)}{4}, \tau_{\lambda_1}(m,n) = 0, 1, 2, 3\right),$$

where the numbers r_λ resp. σ_{λ_1} are distinguished from the numbers r'_λ resp. ϱ'_{λ_1} by the arrangement only. By multiplication of the congruences of the second system we obtain the relation

$$m^{\frac{\varphi(n)}{4}} \equiv i^{\eta(m,n)} \; (mod.\ n),$$

when for brevity is put

$$\sum_{\lambda_1=1}^{\lambda_1=\frac{\varphi(n)}{4}} \tau_{\lambda_1}(m, n) = \eta(m, n).$$

From this follows the generalised theorem of FERMAT of this region of numbers

$$m^{\varphi(n)} \equiv 1 \; (mod.\ n).$$

Now however the congruence exists

$$m^{V\left(\varphi(p_1^{\alpha_1}),\ \varphi(p_2^{\alpha_1}), \ldots,\ \varphi(p_r^{\alpha_r})\right)} \equiv 1 \; (mod.\ n),$$

because the exponent of m is a multiple of each of the arguments of the function V (of the least common multiple of the arguments). Consequently the above mentioned power is congruent to 1 owing to the just named theorem according to each single one of the prime number powers $p_\lambda^{\alpha_\lambda}$ ($\lambda = 1, \ldots, r$) and therefore also according to their product. So if n contains more than one (uneven) prime factor, we certainly have

$$m^{\frac{\varphi(n)}{4}} \equiv 1 \; (mod.\ n)$$

and consequently $\eta(m, n)$ is divisible by 4.

If $n = p^\alpha$ is the power of a prime number (one or two members) then

$$m^{\frac{\varphi(p}{4}} \equiv \left(\left(\frac{m}{p}\right)\right) \; (mod.\ p^\alpha),$$

where the symbol $\left(\left(\frac{m}{p}\right)\right)$ is defined by the equation

$$\left(\left(\frac{m}{p}\right)\right) = i^{\sum\limits_{\lambda=1}^{\lambda=\frac{N(p)-1}{4}} \sigma_\lambda (m, p)} = i^{\sum\limits_{\lambda=1}^{\lambda=\frac{\varphi(p)}{4}} \tau_{\lambda_1} (m, n)} .$$

For $\alpha = 1$ the congruence is proved by the definition of the sign. If it holds good for a definite α, we have the relation

$$m^{\frac{\varphi(p^\alpha)}{4}} = 1 + k\, p^\alpha$$

from which follows by elevation to power $N(p)$

$$m^{\frac{\varphi(p^{\alpha+1})}{4}} = \left(\left(\frac{m}{p}\right)\right)^{N(p)} + \binom{N(p)}{1} \left(\left(\frac{m}{p}\right)\right)^{N(p)-1} k\, p^\alpha + \\ + \frac{N(p)\{N(p)-1\}}{1\,.\,2} \left(\left(\frac{m}{p}\right)\right)^{N(p)-2} k^2\, p^{2\alpha} + \ldots$$

or, $N(p)$ having the form $4s+1$ and all terms appearing after the first term on the right side having at least the factor $p^{\alpha+1}$,

$$m^{\frac{\varphi(p^{\alpha+1})}{4}} \equiv \left(\left(\frac{m}{p}\right)\right) \; (mod.\; p^{\alpha+1}).$$

So

$$\eta\,(m, p^\alpha) \equiv \sum_{\lambda=1}^{\lambda=\frac{N(p)-1}{4}} \sigma_\lambda\,(m, p) = \sum_{\lambda=1}^{\lambda=\frac{\varphi(n)}{4}} \tau_{\lambda_1}\,(m, p) \; (mod.\; 4).$$

3. If we put in (1) $m = n$, and $f(x) = \sigma_\lambda (m, n)$ for $x = r_\lambda$, and in all other cases $f(x) = 0$ then

$$\sum_{x=\{n\}}^{(n)} f(x) = \eta\,(m, n),$$

and

$$\sum_{x=\left\{\frac{n}{d}\right\}} f(dx)$$

is the sum of those numbers $\sigma_\lambda(m,n)$, belonging to the numbers r_λ divisible by d. However these are evidently the terms of a fourth part of a system of rests according to modulus $\frac{n}{d}$ multiplied by d. Now, the numbers $k.dm$ giving after division by n the same rests as the numbers km after division by $\frac{n}{d}$, the last sum is equal to

$$\sum_{\lambda=1}^{\lambda=\frac{N\left(\frac{n}{d}\right)-1}{4}} \sigma_\lambda\left(m,\frac{n}{d}\right) = H\left(m,\frac{n}{d}\right)$$

and so we get the equation

$$\eta(m,n) = \sum_d \mu(d)\, H\left(m,\frac{n}{d}\right),$$

from which ensues reciprocally

$$H(m,n) = \sum_d \eta(m,d).$$

If we define the generalised symbol $\left(\left(\frac{m}{n}\right)\right)$ by the equation

$$\left(\left(\frac{m}{n}\right)\right) = i^{\sum\limits_{\lambda=1}^{\lambda=\frac{N(n)-1}{4}} \sigma_\lambda(m,n)} = i^{H(m,n)}$$

i. e. by a corresponding generalisation of the lemma of Gauss, we have according to previous developments

$$\left(\left(\frac{m}{n}\right)\right) = i^{\sum\limits_d \eta(m,d)}$$

$$= \left(\left(\frac{m}{p_1}\right)\right)^{\alpha_1} \left(\left(\frac{m}{p_2}\right)\right)^{\alpha_2} \cdots \left(\left(\frac{m}{p_r}\right)\right)^{\alpha_r};$$

hence this natural definition coincides with the exposition of Jacobi based on the existence of the multiplication-theorem for the denominator of the sign.

4. To give a further example of the fitness of the method followed in the previous investigations, I will determine according to modulus n the sums of the k^{th} powers of those terms of a system of rests in respect to it belonging to the exponent s. For this I make use of the relation of NASIMOF.

If m belongs according to modulus n to the exponent s which must of course be a divisor of $\varphi(n)$, then *all* terms belonging to s of the system of rests according to modulus n can be represented by the powers m^x in which x assumes all real entire numbers under s prime to s. So the sum s_k of k^{th} powers of this is

$$s_k = \mathop{\Sigma^{(s)}}\limits_{x=1}^{x=s-1} m^{kx}$$

and so according to the relation of NASIMOF equal to the expression

$$\sum_{d} \mu(d) \left(\sum_{x=1}^{x=\left[\frac{s-1}{d}\right]=\frac{s}{d}-1} m^{kdx} \right)$$

in which the summation is to be extended to all divisors d of s and in which $\mu(\alpha)$ denotes the ordinary function of MÖBIUS-MERTENS. So we also get the congruence

$$s_k \equiv \mu(s) + \sum_{d} \frac{1-m^{ks}}{1-m^{kd}}\, m^{kd}\, \mu(d) \;\; (\text{mod. } n) \;\; (d < s), \;\; \ldots\ldots \;\; (3)$$

from which ensues immediately

$$s_k \equiv \mu(s) \;\; (\text{mod. } n.)$$

This congruence furnishes, as we may notice by the way, for $k > 1$ and $Th\,(k, s) = 1$ the relation

$$\sum_{\tau} s_{\frac{k}{\tau}}\, \mu(\tau) \equiv 0 \;\; (\text{mod. } n),$$

in which the summation according to τ is to be extended to all divisors of k.

If σ is the greatest common divisor of k and s, so that

$$k = k_1 \sigma, \; s = \overline{s_1} \sigma, \; Th\,(k_1, \overline{s_1}) = 1\,,$$

then all terms on the right side of the congruence (3) in which d is not a multiple of $\overline{s_1}$ are divisible by n; hence we get

$$s_k \equiv \sum_{\delta} \mu\,(\overline{s_1}\,\delta) \frac{\sigma}{\delta} \quad (\text{mod. } n),$$

where the summation according to δ has to include all divisors of σ, so that specially for

$$Th\left(\sigma, \frac{s}{\sigma}\right) = 1$$

we have

$$s_k \equiv \mu\left(\frac{s}{\sigma}\right) \varphi\,(\sigma) \; (\text{mod. } n)\,,$$

where by $\varphi\,(\sigma)$ we generally denote the ordinary function φ of GAUSS.

If however $\overline{s_1}$ and σ have the greatest common multiple t, so that also

$$\overline{s_1} = t\,.\,\overline{s_2}, \; \sigma = t\,.\,\sigma_1, \; Th\,(\overline{s_2}, \sigma_1) = 1$$

and if farther

$$t = \tau\,.\,\tau_1, \; \sigma_1 = \tau\,.\,\sigma_2, \; Th\,(\tau_1, \sigma_2) = 1,$$

we have the congruence

$$s_k \equiv \sum_{\delta_2} \mu\,(\delta_2) \frac{\sigma_2}{\delta_2}\, \mu\,(\overline{s_1})\, t\, \tau + \sum_{\delta_1} \mu\,(\overline{s_1}\,\delta_1) \frac{t\,\sigma_1}{\delta_1} \; (\text{mod. } n),$$

in which the summation according to δ_2 is to be extended to all divisors of σ_2; the summation according to δ_1, however, to all the remaining divisors of σ. As each one of the numbers δ_1 has a divisor (except 1) in common with $\overline{s_1}$, each argument of the functions $\mu\,(x)$ appearing in the second sum is divisible by a square (except 1) and so this sum has the value naught; the first sum, however, is equal to $\varphi(\sigma_2)$. We have the result including also the previously mentioned special cases:

The sum of the k$^{\text{th}}$ *powers of all terms belonging to the exponent* s *of a system according to modulus* n *in the region of the complex integers formed out of the biquadratic roots of unity is congruent according to modulus* n *with the product*

$$\underline{\mu\left(\frac{s}{\sigma}\right) t\, \tau\, \varphi\left(\frac{\tau}{t\,\tau}\right),}$$

where

$$\underline{\sigma = Th\,(k, s), \quad t = Th\left(\sigma, \frac{s}{\sigma}\right), \quad r = Th\left(t, \frac{\sigma}{t}\right).}$$

5. Finally I wish to make an observation referring to the ordinary function $\mu(x)$. Up till now we have an analytical representation for but an extremely small number of numerical functions; such a one is, however, at least didactically of the greatest value, as it generally causes the student, beginning to occupy himself with higher arithmetic, much trouble to get at home in these functions determined only for integral values of the argument. For the function $\mu(x)$ such a one given is in the cyclotomy, and yet as far as I know no use is made of it in the treatises and textbooks on the theory of numbers. It is

$$\mu(s) = \sum_{x=1}^{x=s-1}{}' e^{\frac{2\pi x i}{s}},$$

where the dash added to the Σ indicates that only the integers prime to s of the interval $1 \ldots s-1$ are to be taken.

From this ensues directly

$$\mu(r)\,\mu(s) = \sum_{x=1}^{x=s-1}{}' \sum_{y=1}^{y=r-1}{}' e^{\frac{2\pi(rx+sy)\pi i}{rs}},$$

or for r and s prime to one another

$$\mu(r)\,\mu(s) = \sum_{z=1}^{z=rs-1}{}' e^{\frac{2\pi z i}{rs}} = \mu(rs) \quad \Big(Th(r,s)=1\Big),$$

as in this case $rx + sy$ assumes all positive integral values prime to and minor to them. So we have but to determine the value of $\mu(s)$ for an argument which is a prime number power (p^α). For $\alpha = 1$ we have

$$\mu(p) = \sum_{x=1}^{x=p-1} e^{\frac{2x\pi i}{p}} = -1,$$

whilst for $\alpha > 1$, we get

$$\mu(p^\alpha) = \sum_{x=1}^{x=p^{\alpha-1}-1}{}' e^{\frac{2x\pi i}{p^\alpha}} + \sum_{x=p^{\alpha-1}+1}^{x=2p^{\alpha-1}-1}{}' e^{\frac{2x\pi i}{p^\alpha}} + \ldots + \sum_{x=(p-1)p^{\alpha-1}+1}^{x=p^\alpha-1}{}' e^{\frac{2x\pi i}{p^\alpha}}$$

$$= \sum_{x=1}^{x=p^{\alpha-1}-1}{}' e^{\frac{2x\pi i}{p^\alpha}} \left(1 + e^{\frac{2\pi i}{p}} + e^{\frac{4\pi i}{p}} + \ldots + e^{\frac{2(p-1)\pi i}{p}}\right)$$

$$= 0.$$

The application of the just-named definition of $\mu(x)$ is to be recommended for lecturing purposes.

Mathematics. — "*The number of conics intersecting eight given right lines*". By Prof. JAN DE VRIES.

1. The number of conics resting on eight right lines given arbitrarily can be defined by the direct application of the principle of the conservation of the number.

For this we begin by searching for the number of conics through the points P_1 and P_2 intersecting four given right lines l_1, l_2 l_3, l_4.

If we suppose l_1, l_2, l_3 to lie in a plane φ, then the conic degenerated into the right line $P_1 P_2$ and the right line connecting its point of intersection on φ with that of l_4, satisfies the condition. Proper conics answering the question are obtained in the planes connecting P_1 and P_2 with one of the points of intersection of l_1, l_2, l_3.

So the demanded number is *four*: in the notation of SCHUBERT $P^2 \nu^4 = 4$.

2. To find the number of conics passing through a given point P and resting on six right lines we again consider the case of l_1, l_2, l_3 lying in a plane φ. Proper conics can only then satisfy, when they pass through P and a point of intersection of two of those right lines and intersect the remaining four lines; according to § 1 this number amounts to twelve.

If a pair of lines are to satisfy, one of the right lines must pass through P and intersect one or two of the lines l_4, l_5, l_6.

When it meets l_4 it intersects φ on the line connecting the traces of l_5 and l_6, which is then the second right line of the pair.

If P is situated on a line resting on l_4 and l_5, the second right line connects the trace of that line on φ with the trace of l_6.

Six degenerated conics being found in this manner, the indicated number amounts to *eighteen*; $P\nu^6 = 18$.

3. We can now easily determine the number of conics resting on the eight right lines $l_i (i = 1$ to $8)$.

If again l_1, l_2, l_3 are lying in a plane φ, we obtain a first conic passing through the traces of l_4, l_5, l_6, l_7, l_8 with φ; meeting each of l_1, l_2, l_3 twice, it must be accounted for eight times.

Moreover each conic passing through one of the points of intersection of l_1, l_2, l_3, and reposing on each of the remaining six lines, satisfies the conditions; according to § 2 their number is equal to 54.

The line connecting the traces L_4, L_5 of l_4, l_5 on φ forms with each line resting on $L_4\, L_5$, l_6, l_7, l_8 a conic satisfying the proposed conditions. This consideration evidently furnishes $10 \times 2 = 20$ new answers.

If at last T is the trace of a transversal of l_4, l_5, l_6, l_7, the line TL_8 forms with that transversal a conic of the indicated system. This gives rise to $5 \times 2 = 10$ new answers.

So we have $\nu^8 = 8 + 54 + 20 + 10 = 92$; in accordance with LÜROTH (*Journal für Mathematik*, Bd. 68) we have thus found that *eight given right lines are intersected by* 92 *conics*.

Our considerations cannot be extended to cubic curves; for if to find ν^{12} we suppose four right lines to lie in a plane φ, the traces of the remaining eight lines furnish an infinite number of cubic curves satisfying the question.

4. The conics resting on seven right lines evidently form a surface of order 92. Each transversal of four of those right lines being completed to a conic by each right line cutting it and the remaining three given lines, this surface certainly contains 140 right lines.

The locus of the conics through P, meeting five right lines l, is according to § 2 a surface of order 18.

Each one of those right lines is a quadruple line, because according to § 1 four conics of the system can be made to pass through each point lying on it.

Each line of intersection of two of the lines l through P is a double line, it being completed to conics of the system by two transversals of the remaining three right lines l; so there are ten double lines.

In addition to the 20 indicated single right lines of the surface there are 20 more, originating from the pairs of lines of which one right line intersects four lines l, the other passing through P and meeting the fifth line l.

The section of the surface with the plane (Pl_1) contains the quadruple right line l_1, four of the double lines, two single right lines drawn through P and finally the conic through P and the traces of the remaining four lines l; this has to be counted double, as it cuts the lines l_1 twice; in this way we also find that the surface is of order 18.

By considering a right line of the pencil of rays Pl_1, we can easily understand that P is a twelvefold point.

On the biquadratic surface of the conics through P_1 and P_2 and resting on four given lines, P_1 and P_2 are triple points whilst $P_1 P_2$ is a double line.

5. The number of conics in the planes passing through a given right line a and cutting six given lines l, is easily found by regarding l_1, l_2, l_3 as resting on a. In each of the planes (al_1), (al_2), (al_3) a proper conic must necessarily lie and each of these conics is then to be accounted for twice, as it meets one of the lines l twice; it is evidently determined by the traces of the remaining five. As moreover a forms with two transversals of l_4, l_5, l_6 conics satisfying the question, there are in general eight conics, meeting the six lines l and intersecting a twice. In the notation of SCHUBERT $\mu^2\nu^6 = 8$.

6. Let us finally determine the number of conics intersecting seven right lines whilst their planes pass through a given point M.

If again we suppose l_1, l_2 and l_3 to lie in a plane φ then in the first place we have to deal with all the conics through one of the points of intersection P of those three lines resting on the remaining five lines, whilst their planes pass through M.

Now, according to § 4, P is a twelvefold point of the locus of the conics through P resting on five given lines; this locus proving to be of degree 18, the right line PM is intersected twice by six conics of the surface. In this way we find 18 proper conics satisfying the given conditions.

The line connecting the traces of l_4, l_5 on φ determines with M a plane on which l_6, l_7 furnish two points of the second right line of a degenerated conic. This consideration furnishes 6 conics.

If the line connecting M with the trace L_4 is intersected by the transversal t of l_5, l_6, l_7, the line of intersection of φ with the plane (Mt) completes the line t to a degenerated conic answering the question. In this way 8 new solutions are found.

Finally we find 2 more answers by connecting each of the transversals of l_4, l_5, l_6, l_7 by a plane with M; each of the traces of those planes on φ is then the second right line of the degenerated conic.

Thus we arrive at the conclusion, that $\mu \nu^4 = 34$.

Botany. — Dr. W. Burck, presents a paper: „*On the irritable stigmas of Torenia Fournieri and Mimulus luteus and on means to prevent the germination of foreign pollen on the stigma.*"

While occupied last year in Batavia with an investigation about the question whether or no there exist contrivances on the stigma to prevent the germination of foreign pollen, concerning which I presented a paper in the Meeting of September 29, 1900, my attention was drawn by a biological particularity of the irritable stigmas of *Torenia Fournieri.* This induced me to a nearer research, which I have been enabled to continue in this country on another plant with irritable stigmas: *Mimulus luteus.*

Irritable stigmas have been known for a long time. Already Linnaeus and Koelreuter mentioned them, and later they have been repeatedly discussed, in particular for the various species of *Mimulus* [1]).

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[1]) See on irritable stigmas: Delpino, Bot. Zeit. 1867. pag. 284.

Batalin, Bot. Zeit. 1870 pag. 53.

Ch. Darwin, Cross- and self-fertilisation. Chapt. 3. Mimulus luteus.

A. Hansgirg, Phytodynamische Untersuchungen in Sitzungsberichte der Königl. Böhmischen Gesellschaft der Wissensch. 1889 II, p. 308.

Kerner von Marilaun, Pflanzenleben II. p. 260.

Personally I observed them, besides in the above named plants, in *Mimulus Tillingii*, *M. hybridus*, *Incarvillea Delavayi*, *Martynia fragrans*, *M. formosa*, and *M. proboscidea.*

In all these plants the stigma consists of two broad lips which, in normal state diverging under a considerable angle, on being touched approach each other and close.

In nature this closing of the lobes of the stigma is caused by some insect penetrating the flower in order to get to the nectar. The structure of the flower does not allow the insect to reach the honey without touching the lips. So if an insect withdraws from the flower, the before widely opened stigma is seen to have closed during the visit.

Furthermore it has been observed that if the stigma is only touched by the insect, the lobes will soon open again, but that they remain shut when the insect has at the same time rubbed off on them some of the pollen it carried along.

In my experiments on fertilisation with *Torenia Fournieri* my attention was roused by the different behaviour of the lobes of the stigma, according as the pollen was taken either from the two shorter or from the two longer of the four didynamous stamens.

If the pollen was taken from the *shorter* stamens, the stigma reopened after a few minutes, but the lips remained closed when the pollen had been taken from one of the *longer* ones.

It should now be remarked that the anthers of the two longer stamens of this *Torenia* burst during the flowering and bring their pollen to the surface; those of the short ones, however, don't open. As a rule we find after the fertilisation the anthers of the short stamens still closed in the fallen corolla.

For the rest they are completely developed and their remaining closed is the only difference to be observed in these anthers.

A soft pressure of the anthers with a piece of glass will suffice to collect on it all the contents of the two cells, and now it also appears that the pollen freed in this way does not differ from that of the longer stamens. As a proof of its complete maturity may be added that just as well as that of the longer stamens it begins to germinate in a drop of distilled water, after 2 hours, if the precaution has been taken to put at the same time a stigma of *Torenia* in the drop of water.

I have moreover found that fecundation is as well effected by it as by the other pollen.

So the only difference, between the freely expelled pollen from the longer stamens and that enclosed in the shorter ones is found,

— save in what will presently be said, — in their relation to the lobes of the stigma.

In order to see how the stigmas behaved toward the pollen of another origin, they were covered with that of *Cassia florida*, *Morinda citrifolia*, *Begonia spec. div.*, *Canna indica*, *Calonyction speciosum*, *Argyreia speciosa*, *Impatiens sultani*, and of many other plants selected at will, whereby I found that whichsoever pollen might be used, the stigma always reopens within 10 to 15 minutes and thus behaves towards it in the same way as to the pollen from the shorter stamens.

Only then when the pollen from the long stamens has been deposed on the stigma its lobes remain closed, no matter whether the stigma, in previous experiments, has already been dusted with the pollen of one or more other plants.

I have now found of late that the irritable stigmas of *Mimulus luteus* display the same particularity with this difference, however, that the stigma of *Mimulus always* remains closed when dusted with its own pollen, no matter whether it is taken from the shorter or the longer anthers.

Dusted with pollen of another origin the closing was always of short duration. Experiments were made with the pollen of *Hemerocallis fulva*, *Digitalis purpurea*, *Epilobium angustifolium*, *Tropaeolum majus*, *Torenia Fournieri*, *Pisum sativum*, *Datura Stramonium*, *Ononis spinosa*, *Maurandia erubescens*, *Lathyrus odoratus*, *Impatiens noli tangere*, and other plants.

A nearer examination of the stigmas of these two plants proves that the inner surface of the lips is highly irritable. The slightest touch makes them close directly. The outer surface, on the other hand, can bear considerable irritation without this causing the closing of the lobes. Few minutes, — in the mean a quarter of an hour, — after the lobes have closed in consequence of the stimulus, they open again and may then anew be induced to close in the same way. This can be repeated many times consecutively without the stigmas losing their irritability. The stimulus acts locally, that is, touching the inner surface of one of the lips does not affect the other but only makes the directly touched one move. If the style is cut through this changes nothing in the sensibility of the lobes, which are neither thereby brought to a temporary closing. Moreover, either of the lips may be cut away wholly or partly; the remaining portion continues irritable in the same degree. In consequence of the wound the stigma will shut, but within the determined time it is open again.

The closing of the lobes when pollen is deposed on them is exclusively due to the thereby nearly unavoidable touching of the irritable inner surface.

If very cautiously a quantity of pollen is shedded on the stigma, so that the grains adhere to the stigmatic hairs, the lobes remain open; if it is effected more or less rudely, so that the cellular tissue is touched, they close.

What has been stated here about the closing in consequence of some mechanic stimulus, and the again opening of the stigmatic lobes when that stimulus has ceased to act, clearly points to the fact that the irritable stigmas, in their movements, show many points of accordance with what has been observed in other irritable organs, for instance, in the articulations of the leaves of *Mimosa pudica*, the stamens of *Centaurea jacea*, and other *Cynareae*, so that by analogy it may be admitted, that touching the inner stigmatic surface is accompanied by a loss of water in the turgescent cells at that place, in consequence of which the cell-layers at that side lose their tension, whilst that at the outside increases by the absorption of a part of the expelled water. Hence, the tension of the outer side becomes greater than that of the inner one which explains the closing of the lobes.

When now the stimulus has ceased to act the flaccid cells at the inner face again absorb water, by which the turgor is restored and the lobes reopen again.

Hence it follows that the reopening of the stigmatic lobes, after they have been temporarily closed by only touching or by covering them with pollen, may be referred to well-known phenomena, but that the *not* opening of the lobes when pollen is shedded on them, of the same species (*Mimulus*), or from special stamens of the same species (*Torenia*), requires some explanation. It is clear that here, in dusting a stigma with this special pollen, a factor appears which prevents the restoring of the turgor.

In order te explain this phenomenon it should be observed that the stigmatic fluid of different plants not only varies in qualitative composition, concerning which some particulars were given in the account of the Meeting of Sept. 29, 1900, but that, besides, to all probability, the different constituents of this fluid can occur in varying proportions; that the concentration of the stigmatic fluid can vary very much, and that pollen-grains of distinct origin diverge considerably in their power of drawing water from one and the same stigmatic fluid. The latter fact may at once be seen by putting pollen of distinct plants in a solution of saccharose of a certain degree of concentration, which solution might be called a stigmatic fluid of the simplest composition.

It is likely that among these different kinds of pollen there may occur some, — if at least concentration be not too high — which by a too energetic absorption of water burst at once, and whose contents stream out into the liquid in the same way as is seen with many species of pollen brought into pure water. Among the species of pollen which do not burst there are some which increase in size under absorption of water.

Pollen-grains which in the dry state are elliptical, such as those of *Hemerocallis fulva*, *Torenia Fournieri*, *Digitalis purpurea*, *Maurandia erubescens*, are rounded thereby into balls.

A few species form their germinal tubes ¹) in the fluid, and finally, there are some which not only don't germinate but neither suck water from the liquid, and retain the shape and size which they possessed in the dry state. The higher the degree of concentration, the less the pollen-grains are able to draw water from the saccharose solution.

Also pollen-grains of plants of the same genus frequently possess in a very different degree the power to absorb water from a solution of saccharose.

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¹) It is known that many species of pollen which do not germinate in water, may be induced to do so in solutions of sugar, agar-agar, gum, dextrine, or in mixtures of these substances in a certain degree of concentration for each species. Hereabout informations have been given by Van Tieghem *a*), Kny *b*), Strasburger *c*) and Molisch *d*).

Molisch determined for 60 plants the degree of concentration which should be given to the canesugar solution in order to call forth the germinal process. From his statement it appears that there are pollen species which no more germinate when the saccharose solution is higher than 2 pCt. (*Platanthera bifolia*) or 5 pCt. (*Allium ursinum*), whilst some can still germinate in solutions of 40 pCt. a few even in solutions of 50 pCt. (*Epipactis latifolia, Lilium Martagon*). It is now supposed that this proves that special relations of nutrition govern the germination, and that distinct species of pollen have in this respect distinct requirements. However, considering the fact that a great many pollen-grains want no food at all to germinate, and that others form their germinal tubes in water with addition of a special chemical substance, which can serve as a stimulus, it is my opinion that we should rather think here of an adjustment for the absorption of water required for the germination.

a) Annales des s. c. nat. Bot. 5me Série, tome XII, 1872.

b.) Sitzungsber. des bot. Vereins d. Provinz. Brandenburg XXXIII, 1881.

c.) Strasburger, Neuere Untersuchungen ueber den Befrüchtungsvorgang bei den Phanerogamen, etc. Jena, 1884, Pringsheim's Jahrb. Bd. XVIII, 1886.

d.) H. Molisch, Zur Physiologie des Pollens. Sitzungsber. der math. naturw. Classe der K. Akademie der Wissensch. Wien. Bd. CII, Abth. I, 1893.

Impatiens sultani, for instance, can do this from solutions of 0—20 pCt., *J. balsamina* of 0—10 pCt. and *J. latifolia* of 0—8 pCt.

Beneath the limit mentioned for each species they are all three able to form their germinal tubes; above that limit the germination ceases, together with the absorption of water and the accompanying increase of volume.

In the same way as the pollen behaves towards saccharose solutions, it behaves towards *the fluid of the stigma*, and now it is my opinion, that as a rule it may be admitted that pollen, which does not besides put special requirements to the qualitative composition of the stigmatic fluid (of which my paper in the Meeting of September 29, 1900 treats), can only then germinate on a stigma when the concentration of the stigmatic liquid does not exceed a certain maximum, varying for each species of pollen.

The fact, now, that the stigmas of *Torenia* and *Mimulus* open no more after being dusted with their own pollen (*Torenia* with the pollen from the longer stamens) should be ascribed to the faculty of these kinds of pollen to withdraw considerable quantities of water from the stigmatic fluid of those two plants.

This withdrawing of water is the factor which counteracts the restoring of the turgor.

Direct observation taught me that none of the other pollen species with which experiments on fertilisation had been made possess the same property.

If the elliptical pollen of *Hemerocallis fulva*, *Maurandia erubescens*, *Digitalis purpurea*, or *Lupinus Cruykshanksii*, is put on the stigmas of *Mimulus* or *Torenia*, and if it is again examined after the stigmas have reopened, it is seen to have retained the shape which it possessed in the dry state; none of the grains has been able to become globular.

The pollen of *Torenia*, on the other hand, also elliptical as long as it is dry, and that of *Mimulus*, which on the optical section shows an oblong square, is directly after the stigma has closed found back between the lobes strongly swollen and rounded into balls.

That this is indeed the explanation of the observed phenomena is shown by control experiments. In the first place we see that when the stigma of *Torenia* or *Mimulus*, is covered with pollen which beforehand has been enabled to absorb water and become globular, — simply breathing over the pollen will suffice to this end, — this pollen acts in the same way on the stigma as foreign pollen, namely, as concerns the reopening of the lobes. Just the same is seen to occur when the stigma, previously to the fertilisation, is moistened by means of a pulverisator. Furthermore the fact that, when using the pollen from the shorter stamens of *Torenia*, the stigma opens again, and not when that from the longer is taken, may likewise be referred to the difference in the ratio of water of these grains.

If the pollen from the closed shorter stamens is collected on an piece of glass and exposed to the air for some time until, by loss of water, it has assumed the elliptical shape, it acts in the same way as that of the longer, and if, inversely, that of the longer stamens is used when these have not yet opened, it behaves on the stigma like the pollen from the shorter.

Finally, to determine whether both species of pollen possess the faculty of absorbing water in an equal degree, pollen of *Torenia* was put on the stigma of *Mimulus*, and inversely, that of *Mimulus* on the stigma of *Torenia*. It appeared now that the stigma of *Torenia* remained closed when dusted with pollen of *Mimulus*; inversely, however, the stigma of *Mimulus* opened again after being dusted with pollen of *Torenia*, whence it is evident that the power of absorbing water is greater in *Mimulus* than in *Torenia*.

In reference to an earlier paper on contrivances on the stigma to prevent the germination of foreign pollen, in which it was inferred that for some plants the pollen-grains want a special chemical stimulus in order to form their germinal tubes, I will now call to mind, in accordance with what has been mentioned above, how also the concentration of the stigmatic fluid should be considered as a means to prevent the germination of foreign pollen.

Torenia and *Mimulus* have hardly any chance of foreign pollen-tubes developing on their stigmas. The composition of their stigmatic liquid warrants them from it.

What has been stated here is by no means an isolated fact but should be considered as a special case in the appearance of a frequently occurring means to prevent the germination of foreign pollen. Accordingly, in many cases it explains the phenomenon mentioned by STRASBURGER [1]) that very often the pollen of closely allied plants cannot germinate on each other's stigma, whilst foreign pollen does.

The description of a few more experiments will nearer elucidate this point.

The pollen of *Impatiens sultani*, *I. Balsamina*, and *I. latifolia*, belong to those kinds which easily germinate in distilled water.

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[1]) STRASBURGER, Ueber fremdartige Bestäubung. Pringsheim's Jahrb. für w. Botanik, Bd. XVIII, 1886.

As a rule, it begins to form the germinal tubes directly after being laid in water, whilst other species only after 2 or 3 hours show the first signs of germination. In a quarter of an hour already the pollen-tubes have got a considerable length. Hence, that pollen does not want a special chemical stimulus, nor a germinal fluid of a higher degree of concentration.

In dilute solutions of saccharose, levulose, dextrose, and mannite, it germinates as well as in water.

How indifferent this pollen may appear, it is still far from being able to form germinal tubes on all possible stigmas.

If the pollen of *Impatiens sultani* is introduced into the thick viscous liquid which covers the stigmas of *Uvaria purpurea*, it does not try to form germinal tubes, and when, after residing for several hours in this fluid, it is again put in water, the germination leaves nothing to desire. If then again a portion of the viscid matter is put on a piece of glass and diluted with water, the pollen-grains of *Impatiens* directly begin to germinate, whence may be inferred that only the higher concentration prevented the germination on the stigma.

If now the experiment is inverted by covering the stigma of *Impatiens sultani* with pollen of *Uvaria*, then, after about 24 hours, the germinal tubes are seen to appear. Such like cases, where the pollen of a plant *A* germinates on the stigma of *B*, but inversely, not that of *B* on the stigma of *A*, STRASBURGER has frequently called attention to. The above observations give a plain explanation of the phenomenon, though, of course, for other plants other causes may be active too.

As little as on the stigmas of *Uvaria* does the pollen of *Impatiens* germinate on the stigma of *Pentas carnea*, *Begonia goegoeënsis* and *Torènia Fournieri*. It can remain for days on these stigmas without any change being observed. If then the stigmas of *Begonia* and *Torenia* with the pollen of *Impatiens* upon them are put in a drop of water, they will begin to form germinal tubes within a few minutes. For *Pentas carnea*, however, the high concentration of the stigmatic liquid is not the only cause that the pollen-grains do not germinate in it. After having been on the stigma of *Pentas carnea*, the pollen of *Impatiens* is dead, no matter whether it has been for some days, or only for a few hours in contact with the stigmatic fluid. This evidently poisonous influence on the pollen of *Impatiens* is, however, only exerted by the concentrated liquid of the stigma. If a stigma of *Pentas carnea* is placed in a drop of water in which pollen of *Impatiens* has been sown the germination is quite satisfactory.

Inversely, the pollen of *Pentas* germinates on the stigma of *Impatiens sultani.*

Finally I wish to observe that the 3 mentioned species of *Impatiens* can form germinal tubes on one another's stigmas.

Returning to *Torenia* and *Mimulus* we have to pause a moment at the question what use those plants can draw from the possession of an irritable stigma. This question should be considered regardless of the advantage which many plants, — albeit not all[1]), — can have in possessing a stigmatic fluid of such a composition that not each kind of pollen can develop in it.

That advantage the plants would likewise enjoy if their stigmas had not the power of shutting.

It is the general opinion that the movements of the stigmatic lobes of *Mimulus* will prevent the self-fecundation of the flowers.

It is i. a asserted by BATALIN [2]), that when a bee without pollen on its back penetrates a flower, it touches the stigma and when then the bee laden with pollen flies away, it cannot rub off the pollen on the stigma of the same flower. When entering *another* flower, however, the pollen is brushed off on the stigma, by which cross-fertilisation is effected. I doubt, however, whether the insect can actually contribute to the fecundation of *Mimulus* by pollen of another individual.

Mimulus being a profusely flowering plant, the *other* flower referred to: the one visited after the first, is all but always a flower of the same plant. That flower gets pollen from the first, the third from the second, the fourth from the third, and so on. Finally the bee, still laden with pollen of *Mimulus*, leaves the plant and may carry this pollen to the first flower of another *Mimulus*; but the chance that it will directly return again to a *Mimulus* does not appear greater to me than its visiting quite another plant.

But let this be as it may, albeit that the structure of the flowers of *Mimulus* has given rise to the opinion that the irritable stigmas prevent self-fertilisation, because first the stigma is touched before the insect comes in contact with the anthers, this holds only good

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[1]) From STRASBURGER's experiments may be inferred that often germination of foreign pollen on the stigma and the entering of foreign pollen-tubes into the style-canal and the ovarium do not prevent the development and growth of the plant's own pollen-tubes and the normal course of the fertilisation.

[2]) A. BATALIN. Beobachtungen ueber die Bestäubung einiger Pflanzen. Bot. Zeit 1870. p. 53.

for *Mimulus* and not for *Torenia*, as for the latter the relation is just the reverse. In *Torenia* the two longer stamens are placed in such a way with regard to the stigmas, that a bee first loads its back with pollen, in order to rub it off on the stigma on its farther penetrating the flower.

Hence, if the irritable stigmas of *Torenia* had first been examined instead of those of *Mimulus*, the opinion would never have prevailed that they should serve to prevent self- and promote cross-fertilisation.

The view of KERNER VON MARILAUN [1]) that the movement of the stigmatic lips should serve to carry the pollen to a spot of the stigma where it can further germinate is based on the double supposition that the pollen on the stigma changes its place by the movement and that not each part of the stigma is fit for its germination. Neither the one nor the other assertion I have found confirmed.

In my opinion, therefore, there has hitherto not been given a right explanation of the advantage which a plant may draw from the possession of irritable stigmas.

The closing of the stigma after fertilisation with the plant's own pollen is undeniably accompanied by the advantage that not on each consecutive visit of insects it runs the risk of being rubbed off the stigma to be replaced by pollen of perhaps quite another origin. But this advantage is counter-balanced by the drawback that the inferior pollen from a flower of the same plant, can neither be replaced by pollen of another individual. If the stigmas did not close, then, with frequent visits of insects and after its own stamens had been emptied, many a flower of *Torenia* might be crossed and for *Mimulus* the same might take place still before its own stamens had been brushed out.

To this, however the way is closed, and the said consideration leads to the conclusion that the advantage can in no case be of great importance.

Physics. — "*On the origin of double lines in the spectrum of the chromosphere, due to anomalous dispersion of the light from the photosphere*". By Prof. W. H. JULIUS.

(Will be published in the Proceedings of the next Meeting).

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[1]) KERNER VON MARILAUN, Pflanzenleben II, p. 260.

(October 23, 1901).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday October 26, 1901.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 26 October 1901, Dl. X).

The following papers were read:

Physics. — "*On the origin of double lines in the spectrum of the chromosphere, due to anomalous dispersion of the light from the photosphere*". By Prof. W. H. JULIUS.

A peculiarity, appearing in the photographs which Prof. A. A. NYLAND obtained with the prismatic camera during the total eclipse of 18 May 1901 [1]), caused me to investigate more closely, in the line of my former paper on solar phenomena, what characteristics

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[1]) With the consent of Messrs NYLAND and WILTERDINK (the members of our expedition who were most concerned with the spectrographic researches) only this special feature of the photographs will be shortly referred to in this paper. The report, containing a full account of the various observations made by our party, will be published afterwards.

the chromosphere lines must show, when they really derive their light from the photosphere [1]).

At the meeting of 24 Febr. 1900 I developed considerations which lead us to expect, that the light of the chromosphere is to a large extent composed of photosphere light, which has undergone anomalous dispersion in the absorbing vapours of the sun. The wave-length of the bright lines in the spectrum of the prominences, chromosphere and flash cannot, according to this hypothesis, be exactly the same as the wave-length of the corresponding absorption lines of the ordinary solar spectrum. For every bright line belonging to an absorption line of wave-length λ, was supposed to be produced by two groups of radiations, whose wave-lengths are respectively all smaller and all larger than λ. The light on the red side of the absorption lines will perhaps in most cases be a little more intense than that on the violet side, because, however variable as to place and time the density relations of the solar gases may be, it is always a little more probable that the average density of the layers which are penetrated by the light that reaches us, increases towards the sun's centre, than otherwise [2]). Where powerful "Schlieren" occur, however, the wave groups on the violet side may be the stronger ones.

Further it is clear that from each group, those rays, whose wave-lengths differ much from λ, can in general only be seen close to the sun's edge, for there only a small abnormality in the refractive index is necessary to deflect photosphere rays to our eyes. Light whose wave-length differs less from λ can reach us from a broader strip of the chromosphere; and far from the sun's edge, as a rule, we may expect to see only rays, whose wave-lengths differ very little from λ [3]).

To this rule too exceptions may be found at places, where mighty prominences show us the presence of great irregularities in the density distribution of the sun's gases.

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[1]) I shall frequently make use of the terms photosphere and chromosphere, but I wish to state emphatically that I mean by them only the white disk of the sun and the more or less coloured edge or light ring, as they appear to our eyes. I do not imply the idea of a sharply limited ball, emitting white light and surrounded by a translucent shell, which itself emits coloured light.

[2]) W. H. JULIUS, Proc. Royal Acad. Amsterdam, Vol. II, p. 581 and p. 585, Astron. Nachr Bd. 153, S. 439.

[3]) Proc. Royal Acad. Amsterdam, Vol. II, p. 581.

Let us now consider what under ordinary circumstances the light distribution in a chromosphere line would be, if we were only concerned with refracted photosphere light, unmixed with any appreciable radiation emitted by the absorbing gas.

In Fig. 1 is given a representation of the form, which the dispersion curve of the absorbing gas will assume in the neighbourhood of one of the absorption lines. The line XX' be the axis of wave-lengths with the value λ at the point O, and let an ordinate zero represent that the refractive index is equal to unity. If no absorption line existed in this part of the spectrum, the dispersion curve would be a nearly straight line NN' at a small distance above XX' and almost parallel to it. But if rays of wave-length λ are strongly absorbed, then the curve consists of two branches of the form represented.

Light with a wave-length λ cannot now occur in the chromosphere spectrum. Rays $\lambda \pm \delta$, in the normal spectrum belonging to positions a and a', will reach us from a chromosphere ring of relatively great width, but naturally with greater intensity from the inner than from the outer parts of the ring. Rays $\lambda \pm 2\,\delta$, belonging to places b and b', come only from a smaller chromosphere ring etc. All these rings have the photosphere for their inner limit. The breadth of the rings from which we can receive light of wave-lengths $\lambda \pm \delta$, $\lambda \pm 2\,\delta$ etc. will depend upon the ordinates of the dispersion curve at the points given by a, a', b, b' etc. We can, as a first approximation, put these widths proportional to the quantities $a_1\,a_2$, $a_1'\,a_2'$, $b_1\,b_2$, $b_1'\,b_2'$, etc. by which these ordinates differ from the ordinates of the normal dispersion curve $N\,N'$.

In recent eclipse work both the slit spectrograph and the prismatic camera (or the objective grating) have been used; up to this time most results have been obtained by the latter. We shall, therefore, investigate the character of a chromosphere line as it must show itself in ordinary circumstances in the prismatic camera.

The prismatic camera gives for every monochromatic radiation, coming from the chromosphere, an image of the crescent, ranging these images according to the wave-lengths. The light distribution in such an image shows us the intensity with which the corresponding radiation comes out of the various parts of the crescent. Consequently a pure monochromatic image will, as a rule, possess the greater intensity on the concave side, where it is limited by the moon's edge, and will gradually fade away on the convex side.

The images due to neighbouring rays will, however, partially overlap. This will be especially noticed with the two ray groups which together form a chromosphere line; in this combination of arc images we may expect a quite different distribution of the light than would be found in an image, formed either by monochromatic light or by one simple ray group, such as a more or less rarefied gas would show us in its emission lines.

Let Z (fig. 2) be a portion of the moon's edge at the instant of the second or third contact of a total eclipse. We may now consider the compound light, arising from a small column $Z\alpha$ of the chromosphere, dispersed into a horizontal spectrum parallel to the line PP'. In order to obtain more easily an idea of the share which the various rays contribute to the light distribution in the band, we separate the various rays from one another and represent on distinct lines PP', QQ', RR' .... those parts of the spectrum, where chromosphere light is found of wave-lengths equal respectively to $\lambda, \lambda \pm \delta$, $\lambda \pm 2\delta$, etc.

The point O may indicate the place, where the moon's edge would be seen if absolutely monochromatic light of wave-length λ appeared on its left.

The rays of wave-length λ are, however, completely absorbed, so that nothing need be represented on the line PP'.

On the line QQ' we find first the light of wave-length $\lambda - \delta$, which projects the sharp edge of the moon at a and reaches (with decreasing intensity) from there to α, and secondly the light of wave-length $\lambda + \delta$, which reaches from a' to α'.

In the same way we find on RR' the rays $\lambda - 2\delta$ and $\lambda + 2\delta$, corresponding respectively to the sections $b\beta$ and $b'\beta'$; on SS' the rays $\lambda - 3\delta$ and $\lambda + 3\delta$ at the sections $c\gamma$ and $c'\gamma'$, etc.

Because the sections $a\alpha$, $a'\alpha'$, $b\beta$, $b'\beta'$... represent the width of the chromosphere rings corresponding to the various sorts of rays, we have considered them proportional to the lengths $a_1 a_2$, $a_1' a_2'$, $b_1 b_2$, $b_1' b'_2$ of fig. 1. Hence the extremities $\alpha, \beta, \ldots$ and $\alpha', \beta', \ldots$ etc. lie on two curves, whose shape is closely related to that of the dispersion curve. The share which all intermediate waves bear in the light distribution is thus shown, if we only notice that for each kind of light the intensity decreases from right to left. This is represented by shading in the upper part of fig. 3. Finally to obtain the light distribution in the chromosphere line, we only need suppose that the figure is compressed in the vertical direction and that thus the light intensities are added together. The resulting intensity is then found to be approximately distributed as is shown by the shading in the spectrum given below. Hence a *double line* is produced, each of the components of which shades off gradually

on each side, so that there is still light of a somewhat considerable intensity in the intervening space.

If the rays whose wave-lengths are less than λ are on the average of the same intensity as those with wave-lengths greater than λ (this case is shown in the figure), the „centre of gravity" of the chromosphere line is shifted a little to the convex side of the image with respect to the place belonging to the absorption line of wavelength λ. If, on the contrary, we consider the inner limit of the crescent, it appears that the line has shifted to the other side. This must involve us in difficulties when trying to find the exact wave-length of a chromosphere line.

Moreover, all kinds of variations may be expected in the intensity distribution. The ray group whose wave-lengths are greater than λ, may be intenser or vice versa. In such a case the displacements of the chromosphere line, both with regard to limit and to position of centre of gravity, may assume quite other values. Such displacements of variable character are actually often observed (by CAMPBELL, FROST, LORD i. a.).

The figure represents a case where on the convex side of the crescent the intensity of the system decreases faster than on the concave side (just otherwise than we should expect from a cursory examination; indeed the chromosphere crescent, observed without a spectroscope, is sharply limited on the concave side). This peculiarity too has been often seen in the chromosphere spectrum (cf. FROST, Astroph. Journ. XII, p. 315, Dec. 1900). In general, many of the irregularities in the form of the lines of the chromosphere and the flash, as given by MASCARI [1]), CAMPBELL [2]), BROWN [3]), LORD [4]), FROST [5]), and also the principal features of the chromosheric spectrum, recently once more discussed by Sir NORMAN LOCKYER [6]), can be easily explained if we suppose the lines to be produced by anomalous dispersion.

A convincing argument for the correctness of our explanation would be obtained if it appeared, that all chromosphere lines were really double lines of the above described character.

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[1]) MASCARI, Mem. Spettr. 27, p. 83—89; Ref. Naturw. Rundsch. 13, S. 618.

[2]) CAMPBELL, Astroph. Journ. XI, p. 226—233.

[3]) BROWN, Astroph. Journ. XII, p. 61—63.

[4]) LORD, Astroph. Journ. XII, p. 66—67.

[5]) FROST, Astroph. Journ. XII, p. 307—351.

[6]) LOCKYER, Recent and coming Eclipses, Chapter X and XVIII, London 1900.

Hence I have repeatedly sought for dark cores in the chromospheric arcs on photographs taken during former eclipses, and have indeed found several indications of them; but a plate where this peculiarity was the rule, where all the chromosphere lines were double, has certainly never before been obtained, for if so, the phenomenon could not have escaped notice.

The Dutch expedition had the fortune to get the first plates which quite clearly show all the chromosphere and flash lines, visible on them, to be *double lines.*

This important result is in the first place due to the great care with which the whole plan of observation with the beautiful prismatic camera of COOKE was designed and elaborated by Prof. NYLAND, and not less to the extraordinary exactness, with which both before and during the eclipse he has performed all necessary manipulations. But besides, it is not impossible that the result was favourably influenced by the in other respects very unfortunate cloudiness of the sky. For if the light had not been considerably weakened, the chromosphere lines would have been found on the plate both broader and in greater number, and the doubling would have been perhaps as little marked as on the plates, obtained on former occasions.

Shortly after the second contact five exposures were made on one plate, each of them during about $^3/_4$ sec. They show each only 9 lines, all double. On the four plates, prepared for the corona spectrum, some of the stronger chromospheric lines are represented by often interrupted arcs. The light of these evidently comes from prominences which project rather far beyond the photosphere. Here it appears not so easy to distinguish the duplication, just as we might expect by our theory; but still it is visible at many places.

On the sixth plate another set of five exposures, of $^3/_4$ sec. each, were taken a little after the third contact. In the first of the spectra thus obtained (reaching from λ 3880 to λ 5000) 150 double chromosphere lines can be counted between λ 3889 and λ 4600, these being also visible in the other four spectra, as far as the increasing scattered light permits [1]).

A little below the continuous spectrum, due to the just appearing edge of the sun, the double lines are most conspicuous. We find there, parallel to the spectrum, a bright narrow streak which appears broader

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[1]) On the original negatives the duplication can only be distinguished with a magnifying glass. Enlargements (which were shown in the meeting) will soon be reproduced and published.

in the following exposures and which is probably owing to a small depression in the moon's edge or to a projecting part of the apparent sun's edge. In the fifth exposure, below the light band so produced there appears a similar streak. These bands give so to say repeated spectra of the flash (a fortunate circumstance, for the totality was over sooner than was calculated and the exposures were thus a little later than was intended) so that we obtain at one and the same exposure both the pure flash spectrum and the continuous spectrum of the sun's limb.

Prof. **Nyland** and I have discussed together the possibility of ascribing the origin of double lines to disturbing circumstances, such as irregular motion of the siderostat, vibrations of the prismatic camera, light reflections etc. [1]), but we were not able to find any disturbance which could account for the observed phenomena and we must conclude that here we really have a property of the chromospheric lines.

The Fraunhofer lines in the continuous spectrum are weak. This may in part be due to the diffusion of light by the clouds. For the just appeared edge of the photosphere, which plays the same part with the prismatic camera as the illuminated slit with an ordinary spectroscope, was not darkly limited, but surrounded by a marked aureole (this can be seen in some of our corona photographs). The clouds, however, cannot have been the only cause of the faintness of the absorption lines in the first stage after totality, this phenomenon having been also observed in a clear sky [2]). There must therefore be another reason for the partial absence of the lines. Our theory gives such a reason immediately. For the chromosphere spectrum will at the end of totality become more and more like a continuous spectrum, because more bright lines will continually appear, each of which, according to our hypothesis, forms a double band in which the absence of the absorbed waves is not easily perceived. But as soon as a portion of the photosphere appears, the already existing, apparently continuous spectrum will be dominated by a more really continuous spectrum, the source of light of which is limited by two nearly sharp edges (those of the photosphere and of the moon).

In this spectrum the absence of absorbed rays must show in the usual way as Fraunhofer lines. The light of the chromospheric arcs will, of course, partially overlap those lines, but compared with

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[1]) The mounting of the instruments will be fully discussed in the report of the expedition.

[2]) **Campbell**, Astroph Journ. XI. p. 228, April 1900.

the direct photospheric light it is weak enough for the dark lines to be visible. Thus, not considering the presence of clouds, the absorption lines must yet, during the transition from the flash spectrum to the Faunhofer spectrum, at first show very faint and with abnormal relative intensities, then grow stronger, intensities appearing normal.

Because the double lines are not sharply defined objects, it is difficult to give the width of these systems. But we can make settings on the brightest parts of the components and measure their distance with a reading microscope. It differs for the different double lines, still it generally lies between 0.7 and 1.3 ÅNGSTRÖM's units. Wider and narrower systems follow each other in irregular succession, but on an average the distance of the components appears to decrease as we proceed from the green to the violet. Perhaps this fact may be important for dispersion theories.

With some lines the stronger component is that which has the greater, in others that which has the smaller wave-length. It happens that even in the same line (e. g. in the arcs of H_γ and H_δ on our plate) the two cases occur close by one another, which means that in neighbouring places of the sun's atmosphere the density distribution of the absorbing gas is different in this, that at one place the average density along the path of the ray increases, at another decreases towards the sun's centre.

CAMPBELL states [1]) that in some cases where dark and bright lines are to be found together, they are separated from one another by a distance of from 0.4 to 0.5 ÅNGSTRÖM's units. This is about the half of the distance between the components of our double lines. We may here reasonably suppose that CAMPBELL was concerned with cases, where one of the components was strongly marked. A similar case is found on our photograph in H_β, where the component with the greater wave-length is stronger over nearly its whole length than that with the smaller wave-length, and such is the case not only at the third contact but also during the second and even on the four plates, prepared for the corona spectrum, which were exposed for 5, 20, 190 and 60 sec. respectively.

I have not found until now in any chromosphere line a peculiarity in the distribution of the light, which would make it necessary to ascribe even a part of this light to radiations, emitted by incandescent

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1) CAMPBELL, Astroph. Journ. XI, p. 229.

chromosphere gases. Now we can hardly assume that these gases really do not emit any light; the question is only, in what cases and how far the intensity of the true chromospheric emission is comparable with the intensity of the abnormally refracted photosphere light.

Perhaps the photographs obtained by our expedition are accidentally so extremely fit to show the part played by anomalous dispersion in causing chromosphere light, that they induce one to overestimate the importance of the new principle.

It would therefore be very interesting if the plates of other expeditions were also studied from this point of view.

Mathematics. — "*Considerations in reference to a configuration of* SEGRE". By Prof. P. H. SCHOUTE (first part).

1. In a treatise published in 1888 "Sulle varietà cubiche dello spazio a quattro dimensioni, ecc" (*Memorie di Torino*) Dr. C. SEGRE proved the following remarkable theorem:

The locus of the right lines cutting any four planes assumed in the space S_4 is a curved space of order three containing besides these four planes eleven planes more; one of these eleven new planes is intersected by all the right lines cutting the four given planes. The fifteen planes pass six by six through one of ten points, which are double points of the cubic locus.

If we call the four given planes $\alpha, \beta, \gamma, \delta$ and if we denote by α' the plane through the three points of intersection $(\gamma\delta)$, $(\delta\beta)$, $(\beta\gamma)$, by β' the plane through the three points of intersection $(\delta\alpha)$, $(\alpha\gamma)$, $(\gamma\delta)$, etc., then the four points of intersection lie in one and the same space ε and the five planes form such a quintuple, that each right line cutting four of these planes, also cuts the fifth.

In a study also published in 1888 "Alcune considerazioni elementari sull' incidenza di rette e piani nello spazio a quattro dimensioni" (*Rendiconti del circolo matematico di Palermo*, vol. 2, pages 45—52) the same writer gives a rather simple geometrical proof of the second part of this theorem, and then ascends to the configuration mentioned in the first part by the indication that the ten points spoken of are the points of intersection of the five planes $\alpha, \beta, \gamma, \delta, \varepsilon$ two by two and the ten new planes are deduced out of the triplets of these five planes as α' out of $(\beta\gamma\delta)$, etc.

In the following pages we will submit the configuration of SEGRE to a simple analytical investigation. For this let us consider the dualistically opposite figure of fifteen lines and ten three-dimensional spaces.

2. If we begin with the second part of the theorem, then we have to deal with the figure consisting of eight lines

$$\left|\begin{array}{cccc} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \end{array}\right|$$

corresponding in this respect with the wellknown double six of SCHLÄFLI, that each of these eight lines intersects only those three of the remaining ones, corresponding with them neither in letter nor in index. We suppose the four given right lines a_1, a_2, a_3, a_4 to be given in S_4 in such a way that among the six connecting spaces $(a_1\,a_2)$, . . . $(a_3\,a_4)$ there are not three having a plane in common. And b_1 is then again the line of intersection of the three spaces $(a_3\,a_4)$, $(a_4\,a_2)$, $(a_2\,a_3)$, etc. To this figure which in a previous study we considered as the basis of a particular net of quadratic curved spaces we therefore gave the name of "double four" ("Ein besonderer Bündel von dreidimensionalen Räumen zweiter Ordnung im Raum von vier Dimensionen", *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 9, pages 103—114).

If we consider the spaces $(a_1\,a_3)$ and $(a_2\,a_4)$, it is immediately evident that each of these spaces contains four of the eight lines of the double four and that this is therefore broken up into the two skew quadrilaterals

$$(a_1\ b_2\ a_3\ b_4),$$

$$(b_3\ a_4\ b_1\ a_2),$$

of which the sides written here under each other meet each other in four points of the plane of intersection of the spaces $(a_1\,a_3)$ and $(a_2\,a_4)$. If to begin with we draw only the first of those skew quadrilaterals (fig. 1), then it is clear that the lines $P_1\,P_2$ and $P_3\,P_4$, connecting those points in which the pairs of sides $(a_1,\ b_2)$ and $(a_3,\ b_4)$

are intersected by this plane, will meet each other on the line of intersection l of the planes $(a_1 b_2)$ and $(a_3 b_4)$. In like manner the line of intersection m (fig. 2) of the planes $(b_1 a_2)$ and $(b_3 a_4)$ passes through the point of intersection O of $P_1 P_2$ and $P_3 P_4$.

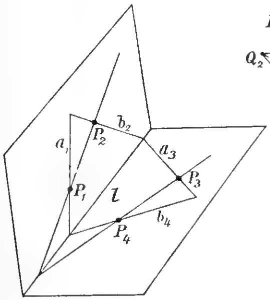

Fig. 1.

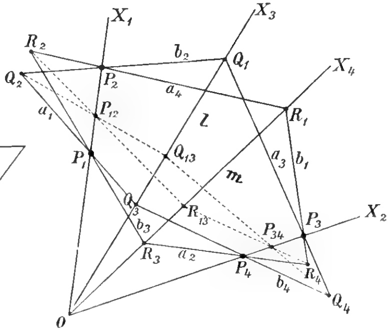

Fig 2.

If we now indicate the two skew quadrilaterals according to the vertices in the way pointed out in fig. 2 by $Q_1\, Q_2\, Q_3\, Q_4$ and $R_1\, R_2\, R_3\, R_4$ and if P_{12}, P_{34}, Q_{13}, R_{13} represent the points separating O harmonically from the pairs of points (P_1, P_2), (P_3, P_4), (Q_1, Q_3), (R_1, R_3), it is easy to see that

$$(Q_2\, P_{12}\, Q_{13}),\ (Q_4\, P_{34}\, Q_{13}),\ (R_2\, P_{12}\, R_{13}),\ (R_4\, P_{34}\, R_{13})$$

are four triplets of points on a right line,

$$(P_{12}\, P_{34}\, Q_2\, Q_4\, Q_{13}),\ (P_{12}\, P_{34}\, R_2\, R_4\, R_{13})$$

two quintuples of points in a plane and that

$$P_{12}\, P_{34}\, Q_2\, Q_4\, Q_{13}\, R_2\, R_4\, R_{13}$$

are eight points of a same space. We now choose the five-cell of which O is a vertex, the four lines $P_1 P_2$, $P_3 P_4$, $Q_1 Q_3$, $R_1 R_3$ are the edges passing through this point and the space just found is the side-space situated opposite O, as five-cell of coordinates; this has the five points O, P_{12}, P_{34}, Q_{13}, R_{13} as vertices. By assuming the point of intersection of the four spaces

$$(P_1\, P_{34}\, Q_{13}\, R_{13}),\ (P_4\, P_{12}\, Q_{13}\, R_{13}),\ (Q_1\, P_{12}\, P_{34}\, R_{13}),\ (R_1\, P_{12}\, P_{34}\, Q_{13})$$

as point of unity of the homogeneous system of coordinates and by following for the rest the notation of fig. 2 we obtain the following table of proportions of coordinates of the points the knowledge of which is sufficient for the determination of the equations of the lines of the double four:

$$\begin{array}{llll}
P_1 \ldots (\ \ 1, \ 0, 0, 0, 1), & & Q_1 \ldots (0, 0, \ \ 1, \ \ 0, 1), \\
P_2 \ldots (-1, \ 0, 0, 0, 1), & & Q_3 \ldots (0, 0, -1, \ \ 0, 1), \\
P_3 \ldots (\ \ 0, -1, 0, 0, 1), & & R_1 \ldots (0, 0, \ \ 0, \ \ 1, 1), \\
P_4 \ldots (\ \ 0, \ \ 1, 0, 0, 1), & & R_3 \ldots (0, 0, \ \ 0, -1, 1).
\end{array}$$

So the equations of the two quadruples of lines are

$$\left.\begin{array}{llll}
a_1 \ldots\ldots & x_1 - x_3 = x_5 & , \ x_2 = 0 & , \ x_4 = 0 \\
a_2 \ldots\ldots & x_2 - x_4 = x_5 & , \ x_1 = 0 & , \ x_3 = 0 \\
a_3 \ldots\ldots & x_3 - x_2 = x_5 & , \ x_1 = 0 & , \ x_4 = 0 \\
a_4 \ldots\ldots & x_4 - x_1 = x_5 & , \ x_2 = 0 & , \ x_3 = 0
\end{array}\right\} ,$$

$$\left.\begin{array}{llll}
b_1 \ldots\ldots & x_4 - x_2 = x_5 & , \ x_1 = 0 & , \ x_3 = 0 \\
b_2 \ldots\ldots & x_3 - x_1 = x_5 & , \ x_2 = 0 & , \ x_4 = 0 \\
b_3 \ldots\ldots & x_1 - x_4 = x_5 & , \ x_2 = 0 & , \ x_3 = 0 \\
b_4 \ldots\ldots & x_2 - x_3 = x_5 & , \ x_1 = 0 & , \ x_4 = 0
\end{array}\right\} .$$

So the equations of the spaces $(a_i \, b_i)$, $i = 1, 2, 3, 4$, through the opposite sides of the double four are

$$\left.\begin{array}{ll}
(a_1 \, b_1) \ldots\ldots & -x_1 + x_2 + x_3 - x_4 + x_5 = 0 \\
(a_2 \, b_2) \ldots\ldots & x_1 - x_2 - x_3 + x_4 + x_5 = 0 \\
(a_3 \, b_3) \ldots\ldots & -x_1 + x_2 - x_3 + x_4 + x_5 = 0 \\
(a_4 \, b_4) \ldots\ldots & x_1 - x_2 + x_3 - x_4 + x_5 = 0
\end{array}\right\}$$

and these four spaces have evidently the right line

$$x_1 = x_2 \quad , \quad x_3 = x_4 \quad , \quad x_5 = 0$$

in common.

It remains to be proved that the relation between the five lines, obtained by adding this new line a_5 to the four given lines a is mutual in such a sense, that all quadruples of which a_5 is a line lead back in the indicated way to the fifth line.

We prove this for the quadruple $a_1 a_2 a_3 a_5$, and for this complete this to the double four

$$\left|\begin{array}{cccc} a_1\ , & a_2\ , & a_3\ , & a_5 \\ c_1\ , & c_2\ , & c_3\ , & b_4 \end{array}\right|$$

and then verify that the four spaces $(a_1 c_1)$, $(a_2 c_2)$, $(a_3 c_3)$, $(a_5 b_4)$ have the line a_4 in common.

The lines c_1, c_2, c_3 as the lines of intersection of the triplets of spaces

$$\left.\begin{array}{ll} (a_3\ a_5)\ \ldots\ldots & -x_1+x_2-x_3+x_4+x_5=0 \\ (a_2\ a_5)\ \ldots\ldots & x_1-x_2-x_3+x_4+x_5=0 \\ (a_2\ a_3)\ \ldots\ldots & x_1 \hphantom{{}-x_2-x_3+x_4+x_5} =0 \end{array}\right\}\ \ldots\ldots c_1\,,$$

$$\left.\begin{array}{ll} (a_3\ a_5)\ \ldots\ldots & -x_1+x_2-x_3+x_4+x_5=0 \\ (a_1\ a_5)\ \ldots\ldots & -x_1+x_2+x_3-x_4+x_5=0 \\ (a_1\ a_3)\ \ldots\ldots & \hphantom{-x_1+x_2+x_3-{}} x_4 \hphantom{{}+x_5} =0 \end{array}\right\}\ \ldots\ldots c_2\,,$$

$$\left.\begin{array}{ll} (a_1\ a_5)\ \ldots\ldots & -x_1+x_2+x_3-x_4+x_5=0 \\ (a_2\ a_5)\ \ldots\ldots & x_1-x_2-x_3+x_4+x_5=0 \\ (a_1\ a_2)\ \ldots\ldots & -x_1-x_2+x_3+x_4+x_5=0 \end{array}\right\}\ \ldots\ldots c_3$$

are then represented by the equations

$$\left.\begin{array}{llll} c_1\ \ldots\ldots & x_3-x_4=x_5\,, & x_2=0\,, & x_1=0 \\ c_2\ \ldots\ldots & x_1-x_2=x_5\,, & x_4=0\,, & x_3=0 \\ c_3\ \ldots\ldots & x_1=x_3\ \ , & x_2=x_4\,, & x_5=0 \end{array}\right\};$$

so the four spaces $(a_1\ c_1)$, $(a_2\ c_2)$, $(a_3\ c_3)$, $(a_5\ b_4)$ with the equations

$$\left.\begin{array}{llr}
(a_1\ c_1) \ldots\ldots & x_2 & = 0 \\
(a_2\ c_2) \ldots\ldots & x_3 & = 0 \\
(a_3\ c_3) \ldots\ldots & x_1 + x_2 - x_3 - x_4 + x_5 & = 0 \\
(a_5\ b_4) \ldots\ldots & x_1 - x_2 + x_3 - x_4 + x_5 & = 0
\end{array}\right\}$$

really pass through the line a_4 with the equations

$$x_4 - x_1 = x_5, \quad x_2 = 0, \quad x_3 = 0.$$

If we complete in the same way the remaining quadruples to double fours out of the five lines a according to the notation

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 & a_3 & a_5 \\ c_1 & c_2 & c_3 & b_4 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 & a_4 & a_5 \\ d_1 & d_2 & c_3 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & a_4 & a_5 \\ e_1 & d_2 & c_2 & b_2 \end{vmatrix}, \begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ e_1 & d_1 & c_1 & b_1 \end{vmatrix},$$

then it is also evident that the spaces

$(a_1\ d_1)$, $(a_2\ d_2)$, $(a_4\ c_3)$, $(a_5\ b_3)$ pass through a_3,
$(a_1\ e_1)$, $(a_3\ d_2)$, $(a_4\ c_2)$, $(a_5\ b_2)$ through a_2
and $(a_2\ e_1)$, $(a_3\ d_1)$, $(a_4\ c_1)$, $(a_5\ b_1)$ „ a_1.

We then find for the right lines d_1, d_2, e_1 the equations

$$\left.\begin{array}{llll}
d_1 \ldots\ldots & x_2 - x_1 = x_5, & x_3 = 0, & x_4 = 0 \\
d_2 \ldots\ldots & x_4 - x_3 = x_5, & x_2 = 0, & x_1 = 0 \\
e_1 \ldots\ldots & x_1 = x_4, & x_2 = x_3, & x_5 = 0
\end{array}\right\},$$

by which the equations of the fifteen lines have been indicated.

By this we are now able to point out which of the fifteen lines intersect each other; it is evident that each of the fifteen lines cuts six of the remaining fourteen according to the following table:

a_1	$b_2\ b_3\ b_4\ c_2\ c_3\ d_2$	b_1	$a_2\ a_3\ a_4\ c_2\ c_3\ d_2$	c_2	$a_1\ a_3\ a_5\ b_1\ b_3\ d_1$
a_2	$b_1\ b_3\ b_4\ c_1\ c_3\ d_1$	b_2	$a_1\ a_3\ a_4\ c_1\ c_3\ d_1$	c_3	$a_1\ a_2\ a_5\ b_1\ b_2\ e_1$
a_3	$b_1\ b_2\ b_4\ c_1\ c_2\ e_1$	b_3	$a_1\ a_2\ a_4\ c_1\ c_2\ e_1$	d_1	$a_2\ a_4\ a_5\ b_2\ b_4\ c_2$
a_4	$b_1\ b_2\ b_3\ d_1\ d_2\ e_1$	b_4	$a_1\ a_2\ a_3\ d_1\ d_2\ e_1$	d_2	$a_1\ a_4\ a_5\ b_1\ b_4\ c_1$
a_5	$c_1\ c_2\ c_3\ d_1\ d_2\ e_1$	c_1	$a_2\ a_3\ a_5\ b_2\ b_3\ d_2$	e_1	$a_3\ a_4\ a_5\ b_3\ b_4\ c_3$

From this it is easy to deduce that the fifteen lines are situated six by six in ten spaces of which four pass through each line, according to the following table:

$a_1\ a_2$	$b_3\ b_4\ c_3\ e_1$	$a_3\ a_4$	$b_1\ b_2\ c_3\ c_1$
$a_1\ a_3$	$b_2\ b_4\ c_2\ d_1$	$a_1\ a_5$	$b_1\ c_2\ d_1\ d_2$
$a_1\ a_4$	$b_2\ b_3\ c_1\ d_2$	$a_2\ a_5$	$b_2\ c_1\ d_1\ d_2$
$a_2\ a_3$	$b_1\ b_4\ c_1\ d_2$	$a_3\ a_5$	$b_3\ c_1\ c_2\ e_1$
$a_2\ a_4$	$b_1\ b_3\ c_2\ d_1$	$a_4\ a_5$	$b_4\ d_1\ d_2\ e_1$

As follows from the first table each of these sextuples of right lines consists of two triplets of generatrices of the two systems of rays lying on the same quadratic surface.

With this the configuration of SEGRE is found back analytically.

We now point to another particularity which will soon be of service. Out of the first table follows that the fifteen triplets

$$a_1\ b_2\ c_3\,,\quad a_2\ b_1\ c_3\,,\quad a_3\ b_1\ c_2\,,\quad a_4\ b_1\ d_2\,,\quad a_5\ c_1\ d_2\,,$$

$$a_1\ b_3\ c_2\,,\quad a_2\ b_3\ c_1\,,\quad a_3\ b_2\ c_1\,,\quad a_4\ b_2\ d_1\,,\quad a_5\ c_2\ d_1\,,$$

$$a_1\ b_4\ d_2\,,\quad a_2\ b_4\ d_1\,,\quad a_3\ b_4\ e_1\,,\quad a_4\ b_3\ e_1\,,\quad a_5\ c_3\ e_1$$

consist of three lines intersecting each other two by two. The question whether the three lines of a triplet lie in a same plane, pass through a same point or show both particularities of situation is analytically easy to answer. We immediately find that the lines of a triplet always pass through a same point but never lie in a same plane. So we find fifteen new points connected with the configuration and in accordance with this fifteen "spaces through three lines."

If the 15 "points in three lines" and the 45 "planes through two lines" are considered as parts of the configuration, we then find that each of the points lies in three lines, in fifteen planes and in seven spaces, each of the lines lies in six planes and in three spaces and each of the planes lies in three spaces, whilst reversely each of the lines passes through three points, each of the planes passes through five points and through two lines, each of the spaces passes through seven points, through three lines and through nine planes. This all is given summarily in the symbol

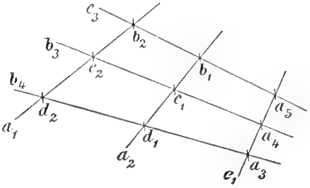

Fig. 3.

$$Cf.\ (15,\ 3,\ 15,\ 7 \mid 3,\ 15,\ 6,\ 3 \mid 5,\ 2,\ 45,\ 3 \mid 7,\ 3,\ 9,\ 15),$$

by which we represent the configuration extended in this way.

For clearness' sake we assemble in fig. 3 the elements of the configuration lying in the space $(a_1\, a_2)$; the nine lines not lying in it are represented by their points of intersection.

3. The fifteen found lines are written as follows in the form of a determinant which with respect to a diagonal, here the diagonal of the missing elements, is symmetric

a_1	a_2	a_3	a_4	a_5	
b_1	b_2	b_3	b_4		a_5
c_1	c_2	c_3		b_4	a_4
d_1	d_2		c_3	b_3	a_3
e_1		d_2	c_2	b_2	a_2
	e_1	d_1	c_1	b_1	a_1

and the following laws are proved:

1⁰. If we state that the five lines among which the relation of the five lines a exists, are conjugate to each other, then each row or each column of the determinant contains five conjugate lines.

2⁰. Each of these six quintuples of conjugate lines leads back to the fifteen lines, if we search for the lines of intersection of the ten spaces through the lines of the quintuples two by two.

3⁰. Every two rows or two columns of the determinant furnish a double four after omission of the two elements of which the corresponding ones are wanting.

The proof for the first law is immediately given. If we connect with the first row one of the others, say the fourth, it follows from the definition of the last merely, that

$$\begin{vmatrix} d_1 & , & d_2 & , & c_3 & , & b_3 \\ a_1 & , & a_2 & , & a_4 & , & a_5 \end{vmatrix}$$

is a double four and that the four spaces $(d_1\, a_1)$, $(d_2\, a_2)$, $(c_3\, a_4)$, $(b_3\, a_5)$ intersect each other according to a right line, which must be a_3, the five lines a being conjugate; so the five lines of the fourth row are also conjugate.

The second law is immediately proved out of the second of the two tables. And the third follows out of the first table, when in connection with the above-named determinant we reduce this to the observation that two of the fifteen lines intersect each other when they belong nowhere in the determinant to the same row or the same column. The word "nowhere" inserted here refers to the circumstance, that each line appears twice. So taken all together the system of the fifteen lines contains fifteen double fours; each of these we can call in the configuration opposite to the right line, which is the section of the four spaces passing through the opposite elements of the double four.

The analytical representation of the fifteen right lines as well as the notation of the determinant leaves still something to be desired. In one as well as the other the circumstance that four of the fifteen lines a are brought to the foreground harms the regularity. Firstly we shall now try to improve the notation of the determinant.

Starting from the five lines a the other ten lines are found as common transversals of the triplets to be formed out of those five lines. This leads to the idea of representing all lines by a, where the original lines a retain their index for the present, each of the remaining lines however is to be indicated by an a with two indices derived from those two of the original quintuple which does not cut it. Then the ten lines

$$b_1 \;,\; b_2 \;,\; b_3 \;,\; b_4 \;,\; c_1 \;,\; c_2 \;,\; c_3 \;,\; d_1 \;,\; d_2 \;,\; e_1$$

become

$$a_{15} \;,\; a_{25} \;,\; a_{35} \;,\; a_{45} \;,\; a_{14} \;,\; a_{24} \;,\; a_{34} \;,\; a_{13} \;,\; a_{23} \;,\; a_{12} \,.$$

If moreover we add to each of the lines of the original quintuple the index 6 and if we allow $a_{i,k}$ to be written $a_{k,i}$, then the previous determinant passes after a quarter of rotation into the entirely regular form:

	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
a_{21}		a_{23}	a_{24}	a_{25}	a_{26}
a_{31}	a_{32}		a_{34}	a_{35}	a_{36}
a_{41}	a_{42}	a_{43}		a_{45}	a_{46}
a_{51}	a_{52}	a_{53}	a_{54}		a_{56}
a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	

In this entirely regular notation two different lines a cross or intersect each other according to their having an index in common or not. In the first place we of course think here of the fifteen lines of the three-dimensional space which are left when from the 27 right lines of a cubic surface a double six is set aside; for, these lines behave in regard to the crossing and cutting in quite the same way with corresponding notation. However, we must not lose sight of the fact, that in the configuration three lines cutting each other two by two pass through a point without lying in a plane, whilst with an arbitrary surface of order three they lie in a plane without passing through a point. If however we polarize the surface of order three to a surface of class three, for instance with respect to a sphere lying in the same space, then the 27 right lines, bearers of points and of double tangent planes, pass into 27 right lines, bearers of tangent planes and double points, whilst each double six of the surface of order three transforms itself into a double six of the surface of class three. In the second place — and with more right — we see therefore in the notation with two indices applied above to the lines of the configuration the notation of the fifteen lines of three-dimensional space forming with a double six the **27** right lines of a surface of class three.

If we project the fifteen right lines of the configuration out of any point P, neither with two crossing lines of the fifteen in the same space nor with two cutting lines of the fifteen in the same plane, on any space S_3 not containing this point P, we then have in S_3 fifteen lines a'_{kl}, which are really bearers of tangent planes of a single surface of class three. By polarization of a wellknown proof we find namely, that the lines

$$\begin{vmatrix} a'_{12} & a'_{34} & a'_{56} \\ a'_{45} & a'_{61} & a'_{23} \\ a'_{36} & a'_{25} & a'_{14} \end{vmatrix}$$

as the lines connecting two triplets of points representing degenerated surfaces of class three form the developable surface degenerated into nine pencils of planes enveloping a tangential pencil of surfaces of class three. And now the surface belonging to that tangential pencil touching at the same time the plane passing through a'_{13} and a'_{24} has with each of the six pencils of planes round the remaining lines

$$a'_{13} \quad , \quad a'_{15} \quad , \quad a'_{24} \quad , \quad a'_{26} \quad , \quad a'_{35} \quad , \quad a'_{46}$$

four planes in common, so that this surface has all the fifteen lines a'_{kl} as bearers of tangent planes.

But this same surface of class three is moreover connected in a simple manner with the sextuples of the conjugate lines which in future we shall briefly indicate by the sign (v_i), where i stands for the common index of the lines. For, this surface contains besides the fifteen lines a' still twelve lines forming a double six. And if we indicate this double six in connection with these lines a' in the manner customary with the surface of order or class three by

$$\begin{vmatrix} b_1 , & b_2 , & b_3 , & b_4 , & b_5 , & b_6 \\ c_1 , & c_2 , & c_3 , & c_4 , & c_5 , & c_6 \end{vmatrix}$$

then each pair of opposite elements b_i, c_i meets the lines of the conjugate quintuple (v_i), from which ensues that the pairs of planes (Pb_i), (Pc_i) do so likewise. So we have proved the following theorem:

Through any point P two planes β_i, γ_i pass cutting the lines of the conjugate quintuple (v_i). The six pairs of planes β_i, γ_i, ($i = 1, 2, \ldots 6$), and the fifteen planes connecting P with the lines of the configuration are cut by an arbitrary space not passing through P according to the 27 right lines of a same surface of class three.

Of the lines cutting four arbitrary planes $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ given in S_4 two lie in an arbitrary space S'_3; for this space cuts the four given planes according to four crossing lines, which admit of two common transversals. By dualistic reversion it ensues from this, that through an arbitrary point P two planes pass cutting any four given lines a_1, a_2, a_3, a_4. And then the above mentioned theorem teaches us that the connection among the five conjugate lines a_i is also expressed by the circumstance, that each plane cutting four of the five lines also cuts the fifth. This characteristic property forms the starting point of Dr. SEGRE's considerations.

In the second part of this communication will be indicated how the analytical representation of the fifteen planes can be simplified; so I will have occasion to show the relation of my results to two studies of Dr. G. CASTELNUOVO, of the existence of which I was not aware in the moment this first part was passing through the press.

Chemistry. — Professor LOBRY DE BRUYN presents a dissertation from Dr. N. SCHOORL and a communication on: "*Urea derivatives (carbamides) of sugars*". II.

In continuation of the first communication (see report of 29 Dec. 1900) the following has been taken from the said dissertation.

The method of determining the *molecular weight* by means of the increase in the boiling point gave unsatisfactory results with glucose-ureïde because, as was shown afterwards, this is decomposed by hydrolysis into glucose and urea and the latter again into carbon dioxide and ammonia.

The lowering of the freezing point gave an average molecular weight of 216, the calculated figure for $C_6 H_{12} O_5 . N . CO . NH_2$ being 222.

Of the physical properties the following are of importance.

The specific gravity was determined at 25° by the method of floating in a dense liquid using mixtures of amylalcohol and ethylene bromide. That of glucose-ureïde was found to be 1.48 and that of glucoseanhydride 1.544. Assuming that of urea to be 1.31 being the average of 1.3 (Boedeker) and 1.323 (Schröder) we find the molecular size of:

urea	glucose	glucose-ureïde
$\frac{60}{1.31} = 46$	$\frac{180}{1.54} = 117$	$\frac{222}{1.48} = 150$

A slight expansion, therefore, takes place during the formation of the ureïde from its components with liberation of water.

$$\underset{(46)}{CO(NH_2)_2} + \underset{(117)}{C_6 H_{12} O_6} = \underset{(150)}{C_6 H_{12} O_5 . N . CO . NH_2} + \underset{(18)}{H_2 O}$$

The *heat of combustion* as determined with Berthelot's calorimetric bomb was found to be 3742 calories per gram or 8307 K. per gram-molecule. The heat of combustion of glucose amounts to 6736 K. per gram-molecule, that of urea 1525 K., so that the reaction is accompanied by a heat absorption of 49 K.

$$C_6 H_{12} O_6 + CO(NH_2)_2 = C_6 H_{12} O_5 . N . CO . NH_2 + H_2 O - 49 C.$$

Chemical properties. The *reducing power* towards Fehling's solution is maintained, but the time of boiling being equal it was less pronounced than in the case of glucose. It is in fact a secondary phenomenon which is preceded by the dissociation of the glucose-ureïde by aqueous soda at the boiling temperature. Then if the solution of glucose-ureïde is reduced alkaline with aqueous soda and mixed in the cold with copper sulphate, the copper hydroxide is dissolved with a dark blue colour. Unlike glucose this solution is not reduced at the temperature of the room but only after a few moments boiling. A slightly acid solution of copper acetate (Barfoed's reagent) is not reduced by glucose-ureïde, a proof that the aldehyde group is no longer present.

In order to form *osazone*, glucose-ureïde requires a more prolonged heating with an acetic acid solution of phenylhydrazine than glucose. On adding sodium acetate, the formation of osazone is still more retarded.

By oxidation with *sodiumhypobromide* the ureïde is very slowly attached at the temperature of the room and it takes hours before the evolution of nitrogen has quite ceased.

On heating to 50°, it ceases in a few minutes. The total quantity of nitrogen evolved amounts in both cases to nearly the theoretical quantity. Both atoms of nitrogen are therefore liberated.

By the action of *nitrous acid* the urea-group is completely eliminated from the glucose-ureïde molecule (partly as nitrogen and partly as ammonia) but at the ordinary, or a more elevated temperature, the liberated glucose is also simultaneously attached. The decomposition proceeds, however, normally and almost quantitatively by passing gaseous N_2O_3 into the aqueous solution of the ureïde cooled to 0°. This is, therefore, a convenient way for recovering sugar from its combination with urea.

In the first communication it has already been suggested that the reaction of glucose and urea might be a reaction of limits. This has now been confirmed by the *hydrolytic dissociation* of glucose-ureïde by dilute acid into glucose and urea. It appeared that this reaction does not proceed to the end but that an equilibrium limit is reached which at the same temperature and the same concentration of the acid equals the equilibrium limit at which the condensation of urea and sugar remains stationary.

The *action of alkali* at an increased temperature was finally found to also cause a hydrolytic dissociation. At 25° this is, however, not the case or only in a very small degree, but still the solution at that temperature undergoes a rapid and strong change in rotatory power. This change seemed to be dependent on the temperature but not on the concentration or nature of the alkali. The amount of alkali after a few days action at 25° was found to be unchanged. It was, therefore, suspected that a partial shifting of atoms of the glucose-ureïde had taken place.

This was confirmed as fully 75 percent of glucose-ureïde was recovered unchanged from the solution and also a syrupy substance which appeared to have nearly the same percentage of nitrogen (11.5) as glucose-ureïde but a much higher specific rotation ($[\alpha]_D =$ about $+ 39°$). It is, therefore, highly probable that it is a mixture of ureïdes of other aldoses formed from glucose by the shifting of atoms.

By *acetylation* of glucose-ureide with acetic anhydride and a trace of zinc chloride a fine crystalline acetyl derivate was obtained (m.p. 200°) which on analysis was found to contain five acetyl groups. As very probably one of the acetyl groups in placed near the urea-group (the substance does not react with alkaline solution of bromine or nitrous acid) four of the acetyl groups must be distributed over the five hydroxyl groups of the glucose. The substance, however, could not be higher acetylated either by acetic anhydride or acetyl cloride.

The derivative obtained by benzoylation of glucose-ureide proved to be the *tetrabenzoyl compound* (m.p. 117°).

Of the *substituted ureas* and *other amides* the following have been investigated as to their behaviour with glucose in dilute sulphuric acid solution and found to give:

change in the rotation.	*no change in the rotation.*
methylurea	symm. dimethylurea
asymm. dimethylurea	„ diethylurea
phenyl urea	„ diphenylurea
benzyl urea	(asymm. diphenylurea)
thio urea	acetamide
biuret	benzamide
urethane	glycocol
	guanidine
	alanine, leucine, taurine.

Some of the compounds glucose with substituted ureas have been isolated and their properties appeared to be quite analogous to those of glucose-ureide.

Of *the different sugars* whose behaviour towards urea has been investigated, only those appeared to react which possess an open carbonyl group such as the aldopentoses and aldohexoses; of the bihexoses: lactose and maltose. Ketoses, however, as shown by experiments made with fructose and sorbose do *not* react with urea.

The ureides of *galactose*, *mannose* and *lactose* have been isolated.

Mannose-ureide has a different composition, as shown by the amount of nitrogen $C_6H_{12}O_5 . N . CO . NH_2 + C_6H_{12}O_6$. It reduces not only Fehling's solution but also Barfoed's reagent but does not yield with phenylhydrazine acetate the phenylhydrazone of mannose.

Lactose ureide has a much greater crystallising power than the ureides of the hexoses and could, therefore, be readily isolated from the products of the reaction without removing the excess of lactose by fermentation. The composition of lactose-ureide is

$$C_{12}H_{22}O_{11} . N . CO . NH_2 + H_2O$$

and its properties are analogous to those of glucose-ureide.

A full report of this investigation which is being continued in different directions will appear in the "Recueil".

Mathematics. — "*The elementary motion of space* S_4", by Dr. S. L. van Oss. (Communicated by Prof. Jan de Vries).

In this paper the writer, starting from the wellknown properties of the elementary motion in S_3 and from a principle already formerly applied in his dissertation, intends to effectuate in a purely geometrical way the reduction of the elementary motion in S_4 to a simultaneous rotation about two (perfectly) normal planes, or, as we shall call it, to a "double rotation".

Theorem I. $\mathfrak{A}$ and $\mathfrak{A}'$ being two congruent systems in S_4, $\mathfrak{A}$ can be made to coincide with $\mathfrak{A}'$ by two successive simple rotations.

Let $\mathbf{A}$ and $\mathbf{A}'$ be a pair of homologous S_3 in $\mathfrak{A}$ and $\mathfrak{A}'$, intersecting in the plane $\alpha \equiv \beta'$; then, if we determine the planes α' and β, the lines of intersection $\alpha // \beta$ and $\alpha' // \beta'$ are a pair of homologous rays lying in the plane $\alpha \equiv \beta'$, and consequently have a centre of rotation R in this plane. If now through R we bring the normal plane ϱ to α, a rotation about ϱ will cause $\mathfrak{A}$ to obtain a double line d_1 with $\mathfrak{A}'$ ($d_1 = \alpha' // \beta'$). This rotation being effectuated, we choose d_1 as the common axis of two homologous pencils of planes in $\mathfrak{A}$ and $\mathfrak{A}'$ and bring through an arbitrary point D in this axis a S_3 cutting it normally. This S_3 being homologous to itself, intersects the pencils of planes in two homologous pencils of rays, which being congruent and possessing a double point, also have a double ray d_2. So the rotation about ϱ has caused the plane $(d_1\, d_2)$ to become a double plane of $\mathfrak{A}$ and $\mathfrak{A}'$. A following rotation about this double plane brings $\mathfrak{A}$ into coincidence with $\mathfrak{A}'$.

Corollary. The elementary motion in S_4 can be represented by a simultaneous rotation about two planes having a point in common.

Theorem II. A rotation in S_4 about a plane can be resolved only in one way into two simple rotations about planes one of which lies in a given S_3 the other passing through a given point or being normal to this S_3.

Definition. A rotation about a plane being resolved into two components the planes of which are respectively in and normal to a given S_3, the rotation of this S_3 caused by the normal component shall be called the *section* of the given rotation with the S_3.

Theorem III. If two systems of rotation in S_4 are equivalent, then also their sections with any S_3 are equivalent.

Theorem IV. If the sections with an arbitrary S_3 of two systems of rotations in S_4 about planes *all passing through one point* are equivalent, then these systems themselves are equivalent.

Corollary. The elementary rotation in S_4 can be reduced in an infinite number of ways to a pair of conjugate rotations corresponding to the conjugate rotations of its section with an arbitrary S_3.

The reduction of a pair of (perfectly) normal planes comes to the same thing as the

Problem: To indicate among the pairs of conjugate axes of the elementary motion in S_3 the pair which determines with a given point in S_4 a pair of normal planes.

It is a matter of course that we have but to search among those conjugate axes that cross each other normally.

Lemma. The locus of the point of intersection of the normal planes through a pair of lines crossing each other normally is the circle having the common normal as a diameter and the plane of which is normal to the S_3 through those lines.

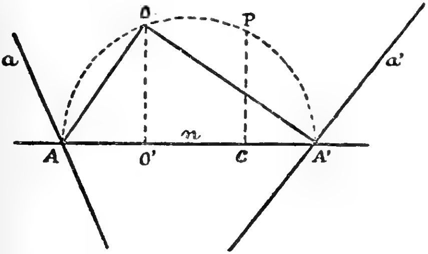

Fig. 1.

If namely in fig. 1 a and a' are the given rectangular lines, n their common normal, O the point of intersection of a pair of normal planes through a and a', the plane (On), must be normal to the two lines a and a' (to a because it contains n and OA', to a' because it contains n and OA). The plane (On) is therefore normal to the S_3 (aa').

Hence for the proposed reduction, only those conjugate normal axes of the section are to be taken into account whose common normal passes through the projection O' of the centre of motion O; this is according to a wellknown property the normal from O' on the central axis c of the sectional motion.

When in the above figure the lines a and a' represent a pair of such axes, their common normal cuts the central axis in the point C in such a way that $AC \times CA' = p^2$, p being the speed of the screw motion about c. The locus of the point of intersection of the normal planes through a and a' consequently passes through a fixed point P, in the normal of S_3 through C and at a distance

p from this point. If the locus passes through O, a and a' are the pair of axes we looked for in our problem.

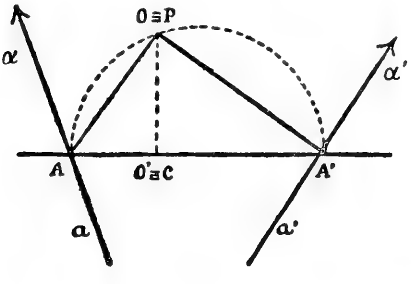

Fig. 2.

In general there is but *one* circle through O and P having its centre in n; then the reduction is in general possible only in one way.

However, the case may be, that O and P coincide. Then the numberof solutions is ∞^2.

If in this case ω and ω' are the components about the planes Oa and Oa', α and α' the rotations about a and a', then

$$\left.\begin{aligned} \alpha &= \omega \sin OAA' = \omega \frac{p}{OA} \\ \alpha' &= \omega' \sin OA'A = \omega' \frac{p}{OA'} \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

Now by the theory of the rotation in S_3:

$$\alpha \times O'A = p\alpha',$$

$$\alpha' \times O'A' = p\alpha,$$

from which ensues:

$$\frac{\alpha^2}{\alpha'^2} = \frac{O'A'}{O'A} \quad \ldots\ldots\ldots \quad (2)$$

whilst from (1) follows:

$$\frac{\alpha^2}{\alpha'^2} = \frac{\omega^2}{\omega'^2} \times \frac{OA'^2}{OA^2} \quad \ldots\ldots\ldots \quad (3)$$

As $\frac{OA'^2}{OA^2} = \frac{O'A'}{O'A}$, it follows from (2) and (3) that $\omega^2 = \omega'^2$. (4)

So we see that the indefiniteness of the reduction coincides with the circumstance, that the components of the double rotation are equal to one another.

The relations (1) show a remarkable analogy to those existing between the values of the rotation α and α' about a and a' and the rotation round the central axis, the latter evidently amounting to ω.

We have namely:

$$\alpha = \omega \sin (ca'),$$

$$\alpha' = \omega \sin (ca).$$

Comparing these relations to (1), we see that $\angle OAA' = \angle (ca')$.

Now the plane (Aa') being the polar plane of A, it follows that the rotation of this plane, when A describes the line $O'A$, is equal to the rotation of OA: in other words, the rotation of a is equal to that of OA.

If now we imagine a line passing through O parallel to a, we immediately see that when A describes the line $O'A$, the plane (Oa) assumes a motion originating from a double rotation with equal components about the plane OAA' and the plane normal to it through O.

If finally we make the thus generated system of planes turn about the plane through O and the central axis c, we obtain the complete image of the reduction.

The results arrived at here entirely agree with those of Dr. W. A. WYTHOFF in his dissertation: „De Biquaternion als bewerking in de ruimte van vier afmetingen."

Astronomy. — "*On* J. C. KAPTEYN's *criticism of* AIRY's *method to determine the Apex of the solar motion,*" By J. STEIN S.J. (Communicated by Prof. H. G. VAN DE SANDE BAKHUYZEN).

At the meeting of the Section of Sciences of Jan. 27th 1900, Prof. J. C. KAPTEYN has given some critical remarks on the methods followed till now to determine the co-ordinates of the Apex of the solar motion. In his paper the writer would point out: first, that neither AIRY's nor ARGELANDER's method is based on the known hypothesis on the proper motions: "the peculiar proper motions of the fixed stars have no preference for any particular direction." Secondly he has tried to develop a method satisfying this condition. (Proceedings Vol. II, pag. 353).

It seems to me that this charge against AIRY's method is unjust; and I hold that this method, even when the equations of condition are treated with least squares, remains in perfect harmony with the hypothesis mentioned.

For a better understanding of the question it may perhaps be useful to give here in short AIRY's reasoning.

AIRY resolves the apparent proper motion into two axes at right angles to each other, and represents the components by the sum of the components of the parallactic motion of the sun, of the error of observation, of the error in the precessional constants and of the motus peculiaris.

Let T and U be the directions of those axes, M that of the motus peculiaris, H that of the Antapex, τ_0 and v_0 the components of the proper motion of a star, t and u the components of the errors of observation, m the linear motus peculiaris, h the linear motus parallacticus and ϱ the distance from the sun, then we have, omitting the correction of the precessional constants, the equations:

$$\left.\begin{aligned} \tau_0 &= \frac{h}{\varrho} \cos (H, T) + \frac{m}{\varrho} \cos (M, T) + t \\ v_0 &= \frac{h}{\varrho} \cos (H, U) + \frac{m}{\varrho} \cos (M, U) + u \end{aligned}\right\} \quad . \; . \; . \; . \quad (A)$$

If we resolve the parallactic motion of the sun into three directions at right angles to each other, and we substitute

$$X \cos (X, U) + Y \cos (Y, U) + Z \cos (Z, U) \text{ for } h \cos (H, U)$$

$$X \cos (X, T) + Y \cos (Y, T) + Z \cos (Z, T) \text{ for } h \cos (H, T)$$

then each star will give two equations for the determination of X, Y and Z.

As however the relations between the error of observation and the motus peculiaris are not known, AIRY proposes two different solutions of these equations: 1°. on the supposition that the irregularities of proper motion are entirely due to errors of observation; 2°. that they are entirely due to peculiar motions of the stars. In either solution he supposes that the errors of observation or the motus peculiares respectively may be considered as chance-errors, and hence he solves the equations in both cases so, that either the sum of the squares of the errors of observation or the sum of the squares of the motus peculiares is a minimum.

We confine ourselves to the second supposition, and therefore give to each equation — ceteris paribus — the same weight. In the first supposition the weights ought to be proportional to ϱ^2. As we suppose with KAPTEYN (Proceedings p. 357) that the distances from the sun to the stars, whose motions are considered, as equal, the two sets of normal equations are identical on both the suppositions.

The three normal equations for X, Y and Z, proposed by AIRY, are derived from the equations for the two components τ and v. If however for some stars one of the components is unknown, we can deduce three normal equations from the other component, or if both components are known for all stars, we can, starting from each of the components separately, construct two sets each of three normal equations. KAPTEYN follows the last method, and so shall we.

It is of course immaterial what are the directions T and U of the components of the proper motion and we may choose those that are the most appropriate. AIRY uses the direction towards the north pole of the equator and the direction of the parallel; KAPTEYN chooses the direction towards a point near the Antapex and the direction at right angles to it; we shall also follow the latter method.

2. Meaning of the symbols according to KAPTEYN:

A_0 and D_0 right ascension and declination of the assumed Antapex ;
λ_0 the distance from the star to this point;
χ_0 the angle made by the declination circle with the direction towards this point;
v_0 the component of the total proper motion μ according to the latter direction;
τ_0 the component perpendicular to the preceding;
p_0 the angle, made by the total proper motion with the parallactic proper motion.

The symbols without index ($_0$) will be used when the real, instead of the assumed, Antapex is meant.

Let ε be the angle made by the directions of the star towards the assumed and the real Antapex, then we can put the equations (A) in this form :

$$\tau_0 = \mu \sin p_0 = \frac{h}{\varrho} \sin \lambda \sin \varepsilon \, ; \quad v_0 = \mu \cos p_0 = \frac{h}{\varrho} \sin \lambda \cos \varepsilon \quad . \quad . \quad (A')$$

The two last terms of the equations for τ_0 and v_0 are both considered as chance-errors of observation, and hence are left out of consideration.

Now

$$\varepsilon = - d\chi_0 = - \left(\frac{\partial \chi}{\partial A}\right)_0 dA - \left(\frac{\partial \chi}{\partial D}\right)_0 dD,$$

$$\lambda = \lambda_0 + \left(\frac{\partial \lambda}{\partial A}\right)_0 dA + \left(\frac{\partial \lambda}{\partial D}\right)_0 dD,$$

and neglecting small quantities of higher order we can put:

$$\tau_0 = \frac{h}{\varrho} \sin \lambda_0 \left\{ - \left(\frac{\partial \chi}{\partial A}\right)_0 dA - \left(\frac{\partial \chi}{\partial D}\right)_0 dD \right\};$$

$$v_0 = \frac{h}{\varrho} \sin \lambda_0 + \frac{h}{\varrho} \cos \lambda_0 \left\{ \left(\frac{\partial \lambda}{\partial A}\right)_0 dA + \left(\frac{\partial \lambda}{\partial D}\right)_0 dD \right\}.$$

Here we assume as unknown quantities $\frac{h}{\varrho}$, $\frac{h}{\varrho} dA$ and $\frac{h}{\varrho} dD$, and obtain from the equations for τ_0 the two following normal equations:

$$\left.\begin{aligned} \left[-\tau_0 \left(\frac{\partial \chi}{\partial A}\right)_0 \sin \lambda_0\right] &= \left[\left(\frac{\partial \chi}{\partial A}\right)_0^2 \sin^2 \lambda_0\right] \frac{h}{\varrho} dA + \left[\left(\frac{\partial \chi}{\partial A}\right)_0 \left(\frac{\partial \chi}{\partial D}\right)_0 \sin^2 \lambda_0\right] \frac{h}{\varrho} dD \\ \left[-\tau_0 \left(\frac{\partial \chi}{\partial D}\right)_0 \sin \lambda_0\right] &= \left[\left(\frac{\partial \chi}{\partial A}\right)_0 \left(\frac{\partial \chi}{\partial D}\right)_0 \sin^2 \lambda_0\right] \frac{h}{\varrho} dA + \left[\left(\frac{\partial \chi}{\partial D}\right)_0^2 \sin^2 \lambda_0\right] \frac{h}{\varrho} dD \end{aligned}\right\} (B)$$

From the equation for v_0 we derive three normal equations, of which the first is:

$$\left[v_0 \sin \lambda_0\right] = \left[\sin^2 \lambda_0\right] \frac{h}{\varrho} + \left[\sin \lambda_0 \cos \lambda_0 \left(\frac{\partial \lambda}{\partial A}\right)_0\right] \frac{h}{\varrho} dA +$$

$$+ \left[\sin \lambda_0 \cos \lambda_0 \left(\frac{\partial \lambda}{\partial D}\right)_0\right] \frac{h}{\varrho} dD$$

The two other equations are left out of consideration on account of their small weight.

For stars, distributed symmetrically with regard to the Apex and the Antapex, both $\left[\sin \lambda_0 \cos \lambda_0 \left(\frac{\partial \lambda}{\partial A}\right)_0\right]$ and $\left[\sin \lambda_0 \cos \lambda_0 \left(\frac{\partial \lambda}{\partial D}\right)_0\right]$

are equal to zero. The same holds for stars in the same great circle passing through the assumed Antapex at distances of λ_0 and $180—\lambda_0$ from that point. Hence, when the stars are equally scattered over the heavens

$$\left[\sin\lambda_0\cos\lambda_0\left(\frac{\partial\lambda}{\partial A}\right)_0\right] \text{ and } \left[\sin\lambda_0\cos\lambda_0\left(\frac{\partial\lambda}{\partial D}\right)_0\right]=0.$$

When the stars are unequally distributed, these two values will yet be small with regard to $\left[\sin^2\lambda_0\right]$. Moreover $\frac{h}{\varrho}dA$ and $\frac{h}{\varrho}dD$ are small quantities with regard to $\frac{h}{\varrho}$, when the error in the assumed Apex is small. If however after a first calculation it would appear that dA and dD were not so small that we could neglect the quantities of the second order, the calculation must be repeated with more accurate values of A_0 and D_0; in this case the two last terms of the normal equations may be neglected with regard to $\left[\sin^2\lambda_0\right]\frac{h}{\varrho}$.

We then obtain:

$$[v_0\sin\lambda_0]=[\sin^2\lambda_0]\frac{h}{\varrho} \text{ or } \frac{h}{\varrho}=\frac{[v_0\sin\lambda_0]}{[\sin^2\lambda_0]}.$$

If this value of $\frac{h}{\varrho}$ *differs from zero*, it may be substituted in the equations (B), and then the determination of dA and dD depends on the solution of:

$$\left.\begin{aligned}\left[-\tau_0\left(\frac{\partial\chi}{\partial A}\right)_0\sin\lambda_0\right]&=\left[\left(\frac{\partial\chi}{\partial A}\right)_0^2\sin^2\lambda_0\right]\frac{[v_0\sin\lambda_0]}{[\sin^2\lambda_0]}dA+\\&\quad+\left[\left(\frac{\partial\chi}{\partial A}\right)_0\left(\frac{\partial\chi}{dD}\right)_0\sin^2\lambda_0\right]\frac{[v_0\sin\lambda_0]}{[\sin^2\lambda_0]}dD\\\left[-\tau_0\left(\frac{\partial\chi}{\partial D}\right)_0\sin\lambda_0\right]&=\left[\left(\frac{\partial\chi}{\partial A}\right)_0\left(\frac{\partial\chi}{\partial D}\right)_0\sin^2\lambda_0\right]\frac{[v_0\sin\lambda_0]}{[\sin^2\lambda_0]}dA+\\&\quad+\left[\left(\frac{\partial\chi}{\partial D}\right)_0^2\sin^2\lambda_0\right]\frac{[v_0\sin\lambda_0]}{[\sin^2\lambda_0]}dD\end{aligned}\right\}\quad(B')$$

If we have started from the correct Apex, dA and dD are both equal to zero; therefore if the Apex is determined according to AIRY's method, the conditions:

$$\left[\tau \frac{\partial \chi}{\partial A} \sin \lambda\right] = 0 \text{ and } \left[\tau \frac{\partial \chi}{\partial D} \sin \lambda\right] = 0 \quad . \quad . \quad . \quad . \quad (C)$$

must be satisfied. The same conditions have been deduced by KAPTEYN from his fundamental hypothesis (p. 359).

If however $\frac{h}{\varrho} = 0$, the coefficients of dA and dD in the equations (B) are zero, and further Apex-determination is out of the question. As a first objection against KAPTEYN's normal equations may be mentioned that it is not self-evident that a solution of his equations is impossible in this case; on the contrary, with a given combination of τ and v, the position of the non-existing Apex may be arrived at.

3. We shall now try to prove that the conditions which, according to KAPTEYN (Proceedings p. 362), may be derived from AIRY's method are not correct.

When the position of the Apex and the amount of the solar motion have been found, and the apparent proper motion is resolved into the peculiar proper motion and the parallactic one, the sum of the squares of the components of the peculiar proper motion, according to AIRY, must be a minimum. As the component of the parallactic solar motion perpendicular to the direction of the true Apex is zero, the place of the Apex and the amount of the solar motion must be determined so, that:

$$[\tau^2] = \text{ minimum and } \left[\left(v - \frac{h}{\varrho} \sin \lambda\right)^2\right] = \text{minimum}$$

(see KAPTEYN l. c.)

Let q be the angle made by the motus peculiaris m with the direction of the star towards the true Antapex, whose Right Asc. and Decl. are A and D then:

$$\tau = \frac{m}{\varrho} \sin q.$$

If, however, we resolve the proper motion into two components, one in a direction towards a point outside the Apex (Right Asc.

$A + dA$ Decl. $D + dD$), and one in a direction at right angles to it, the latter will be

$$\tau_0 = \frac{m}{\varrho} \sin (q + \varepsilon) + \frac{h}{\varrho} \sin (\lambda + d\lambda) \sin \varepsilon,$$

when ε represents the small angle made by the direction towards the true Apex with the direction towards $A + dA$, $D + dD$, or neglecting small quantities:

$$\tau_0 = \tau + \frac{m}{\varrho} \cos q . \varepsilon + \frac{h}{\varrho} \sin \lambda . \varepsilon .$$

In order that $[\tau^2]$ may be a minimum

$$\left[\tau \frac{m}{\varrho} \cos q . \varepsilon + \tau \frac{h}{\varrho} \sin \lambda . \varepsilon \right] \text{ must be zero.}$$

If we substitute for ε its value $-\frac{\partial \chi}{\partial A} dA - \frac{\partial \chi}{\partial D} dD$, we obtain ($dA$ and dD being independent quantities):

$$\left[\tau \frac{m}{\varrho} \cos q \frac{\partial \chi}{\partial A} + \tau \frac{h}{\varrho} \sin \lambda \frac{\partial \chi}{\partial A}\right] = 0 \text{ and} \left[\tau \frac{m}{\varrho} \cos q \frac{\partial \chi}{\partial D} + \tau \frac{h}{\varrho} \sin \lambda \frac{\partial \chi}{\partial D}\right] = 0$$

As, however, the motus peculiaris may be considered as an error of observation, which does not enter into the equations (A'), the equations of condition are reduced to:

$$\frac{h}{\varrho} \left[\tau \sin \lambda \frac{\partial \chi}{\partial A}\right] = 0 \text{ and } \frac{h}{\varrho} \left[\tau \sin \lambda \frac{\partial \chi}{\partial D}\right] = 0, \quad . \quad . \quad . \quad (C')$$

which shows that also the equation $[\tau^2] =$ minimum leads to the right result; for, if $\frac{h}{\varrho}$ differs from zero, the conditions (C) and (C') are identical; if $\frac{h}{\varrho} = 0$, (C') assumes the indefinite form

$$\left[\tau \sin \lambda \frac{\partial \chi}{\partial A}\right] = \frac{0}{0}, \left[\tau \sin \lambda \frac{\partial \chi}{\partial D}\right] = \frac{0}{0}.$$

4. The reasoning which leads **Kapteyn** to reject the condition $[\tau^2] =$ minimum, is as follows:

"$[\tau^2]$ is a minimum for

$$\left[\tau \frac{\partial \tau}{\partial A}\right] = 0 \text{ and } \left[\tau \frac{\partial \tau}{\partial D}\right] = 0,$$

and if we put

$$\frac{\partial \tau}{\partial A} = v \frac{\partial \chi}{\partial A}, \; \frac{\partial \tau}{\partial D} = v \frac{\partial \chi}{\partial D},$$

the minimum conditions are:

$$\left[\tau \, v \frac{\partial \chi}{\partial A}\right] = 0 \text{ and } \left[\tau \, v \frac{\partial \chi}{\partial D}\right] = 0. \quad . \quad . \quad . \quad . \quad (D)$$

which differ from the right ones (C)."

It will be seen immediately, that the set (D) corresponds to the solution of the equations (one for each star):

$$\tau_0 = - v_0 \left(\frac{\partial \chi}{\partial A}\right)_0 dA - v_0 \left(\frac{\partial \chi}{\partial D}\right)_0 dD \quad . \quad . \quad . \quad . \quad (E)$$

It is therefore perfectly consequent to his reasoning, when **Kapteyn** puts **Airy**'s relation in this form (Proceedings p. 369), differing from the form given by me

$$\tau_0 = \frac{h}{\varrho} \sin \lambda \left\{ - \left(\frac{\partial \chi}{\partial A}\right)_0 dA - \left(\frac{\partial \chi}{\partial D}\right)_0 dD \right\}.$$

Kapteyn's equation (E) would be the right one, if **Airy** had formulated his question thus: *To find a point so, that if it is connected with all the stars by means of great circles, the sum of the squares of the components of the proper motion, perpendicular to those circles, is a minimum — without considering the question whether a parallactic solar motion exists or not.* But this not being the principle of **Airy**'s method **Kapteyn**'s criticism of that method is incorrect.

5. The difference between the two considerations may also be put thus:

let A_0 and D_0 be the co-ordinates of a given point at the heavens; (if there is a parallactic solar motion, that point may be the assumed Antapex).

1⁰. If there existed only a parallactic solar motion, the proper motion for each separate star would be represented exactly by the formulae:

$$\tau_0 = \frac{h}{\varrho} \sin\lambda \sin\varepsilon ; \quad v_0 = \frac{h}{\varrho} \sin\lambda \cos\varepsilon ;$$

hence:

$$\frac{\partial \tau_0}{\partial A_0} = \frac{h}{\varrho} \sin\lambda \cos\varepsilon \frac{\partial \chi_0}{\partial A_0}, \quad \frac{\partial \tau_0}{\partial D_0} = \frac{h}{\varrho} \sin\lambda \cos\varepsilon \frac{\partial \chi_0}{\partial D_0} \text{ etc.}$$

2⁰. Even if the proper motions are distributed arbitrarily, without being influenced by any parallactic motion

$$\tau_0 = \mu \sin p_0, \quad v_0 = \mu \cos p_0,$$

$$\frac{\partial \tau_0}{\partial A_0} = v_0 \frac{\partial \chi_0}{\partial A_0}, \quad \frac{\partial \tau_0}{\partial D_0} = v_0 \frac{\partial \chi_0}{\partial D_0} \text{ etc.}$$

hold for each star separately.

AIRY starts from the first set, KAPTEYN from the second.

6. If we substitute in the conditions (C)

$$\tau = \tau_0 + \frac{h}{\varrho} \sin\lambda \left\{ - \left(\frac{\partial \chi}{\partial A}\right)_0 dA - \left(\frac{\partial \chi}{\partial D}\right)_0 dD \right\}$$

$$\lambda = \lambda_0 + \left(\frac{\partial \lambda}{\partial A}\right)_0 dA + \left(\frac{\partial \lambda}{\partial D}\right)_0 dD \text{ and } \frac{\partial \chi}{\partial A} = \left(\frac{\partial \chi}{\partial A}\right)_0 +$$

$$+ \left(\frac{\partial^2 \chi}{\partial A^2}\right)_0 dA + \left(\frac{\partial^2 \chi}{\partial A \, \partial D}\right)_0 dD \text{ etc.}$$

and neglect the small quantities, we obtain, the equations found before (B).

The equations which KAPTEYN (l.c.p.360) deduces from the same two conditions (C) differ from ours, because also in this case he uses the development

$$\tau = \tau_0 + v_0 \left(\frac{\partial \chi}{\partial A}\right)_0 dA + v_0 \left(\frac{\partial \chi}{\partial D}\right)_0 dD$$

which equation, in contradiction to that used by AIRY, is independent from the existence of a parallactic solar motion; therefore I hold myself authorized to consider AIRY's transformed equations (B) as corresponding more closely to the fundamental hypothesis than those of KAPTEYN.

7. The condition $\left[\left(v - \frac{h}{\varrho} \sin \lambda\right)^2\right] =$ minimum, may again serve to eliminate $\frac{h}{\varrho}$ from the equations (B).

As the position of the Apex and the amount of the solar motion are mutually independent, we consider:

1⁰. the relation which exists between $\left[\left(v - \frac{h}{\varrho} \sin \lambda\right)^2\right]$ and the position of the Apex.

If we augment the right ascension and the declination with dA and dD, we get v_0 for v, and $\lambda_0 = \lambda + d\lambda$ for λ.

Now

$$v = \frac{m}{\varrho} \cos q + \frac{h}{\varrho} \sin \lambda \text{ and } v_0 = \frac{m}{\varrho} \cos (q + \varepsilon) + \frac{h}{\varrho} \sin \lambda \cos \varepsilon$$

or:

$$v_0 = \frac{m}{\varrho} \cos q - \frac{m}{\varrho} \sin q \, . \, \varepsilon + \frac{h}{\varrho} \sin \lambda ,$$

$$\text{while } \frac{h}{\varrho} \sin \lambda_0 = \frac{h}{\varrho} \sin \lambda + \frac{h}{\varrho} \cos \lambda \; d\lambda .$$

Hence:

$$\left[\left(v_0 - \frac{h}{\varrho} \sin \lambda_0\right)^2\right] = \left[\left(\frac{m}{\varrho} \cos q - \frac{m}{\varrho} \sin q \, . \, \varepsilon - \frac{h}{\varrho} \cos \lambda \; d\lambda\right)^2\right].$$

In order that $\left[\left(v - \frac{h}{\varrho} \sin \lambda\right)^2\right]$ or $\left[\left(\frac{m}{\varrho} \cos q\right)^2\right]$ may really be a minimum

$$\left[\frac{m}{\varrho} \cos q \, . \, \frac{m}{\varrho} \sin q \, . \, \varepsilon - \frac{m}{\varrho} \cos q \, . \, \frac{h}{\varrho} \cos \lambda \; d\lambda\right] \text{ must be } 0.$$

But as in the case in hand the peculiar proper motion m is left out of consideration, this condition does not teach us anything about the position of the Apex.

2⁰. the relation which exists between $\left[\left(v - \frac{h}{\varrho} \sin\lambda\right)^2\right]$ and the amount of the solar motion $\frac{h}{\varrho}$.

$\left[\left(v - \frac{h}{\varrho} \sin\lambda\right)^2\right]$ must be always smaller than $\left[\left(v - \frac{h+dh}{\varrho} \sin\lambda\right)^2\right]$.

In order that it may be so,

$$\left[-v\frac{dh}{\varrho}\sin\lambda + \frac{h\,dh}{\varrho^2}\sin^2\lambda\right] \text{ or } \left[-v\sin\lambda + \frac{h}{\varrho}\sin^2\lambda\right] \text{ must be } 0,$$

whence follows

$$\frac{h}{\varrho} = \frac{[v\sin\lambda]}{[\sin^2\lambda]}.$$

Thus, after the substitution of this value, we again arrive at the same normal equations (B') for the determination of dA and dD.

8. To conclude I remark that the equations derived in this paper become identical with those of KAPTEYN as soon as we confine ourselves to stars in one direction only. But even when we apply our theory to a great number of stars scattered over the heavens, the two sets will yield little differing results. For if we resolve v_0 into two parts $v_1 + v_2$, where $v_1 = \frac{h}{\varrho}\sin\lambda =$ the component of the parallactic solar motion, and $v_2 =$ the component of the peculiar proper motion, the coefficient of dA in the first of KAPTEYN'S equations becomes

$$\left[v_0\sin\lambda_0\left(\frac{\partial\chi}{\partial A}\right)_0^2\right] = \left[\frac{h}{\varrho}\sin^2\lambda_0\left(\frac{\partial\chi}{\partial A}\right)_0^2\right] + \left[v_2\sin\lambda_0\left(\frac{\partial\chi}{\partial A}\right)_0^2\right].$$

As according to the hypothesis there is an equal number of positive and negative values of v_2, the second term may be neglected, by which the coefficient becomes identical to the corresponding one in our set of equations (B). The same holds for the other coefficients.

It is superfluous to refute at large the objections against AIRY'S method derived by KAPTEYN from a few particular cases of proper motion, because it seems to me that conclusions deduced from the consideration of only a few proper motions, chosen quite systematically, can hardly serve as criteria of a method which, as a matter of course, presupposes as data a great number of proper motions chosen at random. Finally attention must be drawn to an important point. In this paper (comp. § 1), following the method of KAPTEYN and others, I have considered separately the equations for τ and v.

Also in this modified form, as I have proved, AIRY's method leads to the right result. In AIRY's original method however, the three normal equations are composed from the equations for the two components τ and v. In this case there is but one equation of condition, viz.:

$$[m^2] \text{ or } [\tau^2] + \left[\left(v - \frac{h}{\varrho} \sin \lambda\right)^2\right] = \text{minimum},$$

i. e. "the direction and the amount of the parallactic motion must be chosen so, that *the sum of the squares of the* TOTAL *motus peculiares becomes a minimum.*" If this condition is applied to the instances given by KAPTEYN, it immediately becomes evident, that we arrive at the same Apex as KAPTEYN determines by applying the condition $[\tau = 0]$.

Astronomy. — *Reply to the criticism of* Dr. J. STEIN S.J. *by* J. C. KAPTEIJN.

It appears to be very probable that Dr. STEIN has not completely understood my paper in the proceedings of the February meeting of last year. This fact, and the fear that on the other hand I may also have misunderstood STEIN's reasoning (for one part at least of his paper this is certain) have led me to make my reply more circumstantial and elementary than might otherwise seem necessary.

With a view to the importance of the application of the method of least squares for the whole problem, it seems desirable to recall to mind the following elementary points relating to that method.

a). Let a system of equations of condition be given, thus:

$$\left.\begin{aligned} a_1\,x + b_1\,y &= n_1 \\ a_2\,x + b_2\,y &= n_2 \\ a_3\,x + b_3\,y &= n_3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$\ldots\ldots\ldots$$

where the n's represent observed quantities all having the same weight, and let the number of equations exceed the number of unknowns.

If now we take arbitrary values for x and y, and substitute these in the equations (1), there will remain certain residuals (Δ). A second set of values of x and y will be regarded as more probable than the first set, if it gives rise to a smaller value of $\Sigma\,\Delta^2$.

Therefore, in order to find the most probable values of x and y, the adopted values of these quantities must be made to vary until $\Sigma\,\Delta^2$, or, as is commonly said, the sum of the squares of the errors of observation [1]) reaches its minimum value.

b). The equations of condition must *not* be regarded as ordinary algebraical equations. It is not allowed to eliminate unknowns from them, to multiply some of them by a constant factor, etc.

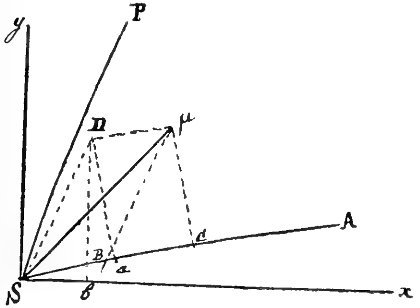

Fig. 1.

Let S be an arbitrarily chosen star, of which the observed proper motion is $S\mu = \mu$;

P the North pole of the heavens;

A (coordinates A and D, at distance λ from the star S) an arbitrarily adopted position of the Antapex;

$\frac{h}{\varrho} \sin \lambda$ an arbitrarily adopted value of the parallactic motion;

The position of A and the value of $\frac{h}{\varrho}$ are variable; they coincide with the *most probable* position of the Antapex and the *most probable* value of the parallactic motion, if certain minimum-conditions are satisfied.

---

[1]) The expression is, of course, not literally correct. The true errors of observation, and consequently also the sum of their squares, are *constant* quantities, which can have neither maximum nor minimum. Exactly in the same way the expression $\Sigma\tau^2 =$ minimum, is not literally correct when τ is defined as the projection of the p. m. on the line perpendicular to the direction of the Antapex. The true meaning of the first expression is explained above. That of the latter is entirely analogous.

υ the projection SC of μ on SA;

τ the projection μC of μ perpendicular to the former;

The values of τ and υ evidently vary with the position of the point A.

Let further x be an arbitrary *fixed* point on the celestial sphere; Sx the great circle joining S to that point and Sy the great circle perpendicular to Sx and let

$\chi =$ the angle PSA;

$\Gamma =$ „ „ PSx. Then, according to STEIN's notation,

$\varepsilon = \Gamma - \chi$.

υ_0, $\tau_0 =$ the projections of μ on Sx and Sy respectively.

These projections do *not* vary with the position of A.

For the present purpose we can confine ourselves to the case that all the stars which are considered are at the same distance from the sun, so that $\frac{h}{\varrho}$ is constant.

We can then say that AIRY's method is based on the hypothesis (*Hyp. A*): that the projections of the *motus peculiares* may be treated as errors of observation, and that the most probable values of A, D, $\frac{h}{\varrho}$ are those for which the sum of the squares of the projections of the motus peculiares on two mutually perpendicular directions is a minimum.

2. This being premised, it is easy to form a judgment about the value of Dr. STEIN's criticism.

Everything depends on the choice of the directions on which the *motus peculiares* are projected.

AIRY takes the parallel and the declinationcircles;

STEIN takes the great circles through the *fixed* point x and the circles perpendicular thereto. He takes the point x in the neighbourhood of the most probable Antapex.

KAPTEYN takes the circles through the point A and those perpendicular to them.

In the first part of STEIN's criticism his own decomposition and mine are confounded. (Further on, e. g. in the enunciation of the problem: "if Airy had formulated his question thus: . . .", there seems to be no such confusion). He says: "Kapteyn chooses the directions towards a point near the Antapex and the direction at right angles to it."

The words relating to this point are however (Proceedings 1900, Febr. p. 362). "The direction from the star towards the Antapex and the great circle through the star at right angles to the former" while it is moreover abundantly clear from the contents of that paper generally, what the meaning is.

Reading Stein's paper one gets the impression, that the author considers my decomposition either as impossible, or as identical to his own, if only care is taken to choose for the point x a point which coincides (or even approximately coincides) with the definitive most probable position of the Antapex. Neither of the two is true.

It might perhaps be considered a sufficient reply to the principal point of Stein's criticism, to have pointed out this confusion. I think however that by going into somewhat fuller details the question as a whole will be more easily understood.

3. If the *total* sum of the squares of the projections of the motus peculiares on the two directions is considered, then the methods of Stein and of Kapteyn lead of course to the same result. For in that case, amongst all the different positions which can be given to the point A, that one will be (according to Airy) considered as the most probable position of the Antapex for which 1)

according to Kapteyn $\Sigma\,(\overline{Da}^2 + \overline{Sa}^2)$ minimum . . . (2)

according to Stein 1) $\Sigma\,(\overline{Db}^2 + \overline{Sb}^2)$ minimum . . . (3)

or

$$\Sigma \left\{ \tau^2 + \left(v - \frac{h}{\varrho} \sin\lambda \right)^2 \right\} \text{ minimum (Kapteyn)} \quad . \quad . \quad (4)$$

and

$$\Sigma \left\{ \left(\tau_0 - \frac{h}{\varrho} \sin\lambda \sin(\Gamma - \chi) \right)^2 + \left(v_0 - \frac{h}{\varrho} \sin\lambda \cos(\Gamma - \chi) \right)^2 \right\} \quad . \quad (5)$$

minimum (Stein)

---

1) I have supposed that by Stein's „*real*" Antapex is meant what has been defined above as the point A, and which might be called the *variable* Antapex. If this is *not* the case, then his reasoning seems to me unintelligible.

respectively, both of which evidently may be reduced to

$$\Sigma\, \overline{SD}^2 \text{ minimum}\ [1]) \ldots \ldots \quad (6)$$

However, if it is true that the two components of the *motus peculiaris* may be treated as errors of observation, then there is evidently no reason why it should not be allowed, so far as possible, to base the determination of the elements of the solar motion on *one* of these components only. That this is the generally adopted view, is apparent from the fact that generally a result is derived *both* from the right ascensions and the declinations separately.

AIRY himself says expressly: (*Mem. of the Roy. Astr. Soc. XXVIII)* „we must consider *the elements of motion* of the different stars as „being, to all intents, chance quantities, to be treated in the same „way as chance errors of observation," and a little earlier (which also shows clearly which meaming is attached by AIRY to the words *elements of motion*): „in the instances in which evidence as to proper „motion in one element fails, it enables us to take account of .... the „evidence derived from the proper motion in the other element alone."

4. To this, however, STEIN does not object. The real point at issue is this, that, according to my contention, such a determination from *one* component alone may lead to an unacceptable solution, a solution which, for the particular case of stars situated at

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[1]) To avoid all mistakes, even at the risk of falling into repetitions I give here *in extenso* the reasoning by which the formulae (2) and (3) are derived. The reasoning is essentially the same as that which was used in § 1 (*a*). Suppose for a moment that the Antapex is at the arbitrarily chosen point A (fig, 1), at a distance λ from S, and take the arbitrary quantity $\frac{h}{\rho} \sin \lambda = \overline{SB}$ for the parallactic motion. In the supposition that the Antapex is at A, this parallactic motion is in the direction SA and the motus peculiaris $\overline{SD}$ is such that the resultant of $\overline{SD}$ and $\overline{SB}$ is μ. The projections of this motus peculiaris are, always in the same supposition,

for my decomposition Da and Sa,
for STEIN's decomposition Db and Sb.

If now we successively take other points for the Antapex and other values for $\frac{h}{\rho}$, then also the direction of the parallactic motion and the amount of the motus peculiaris and its components will change. According to AIRY that point A will be regarded as the most probable position of the Antapex, for which the minimum conditions (2) and (3) respectively are fulfilled.

one and the same point of the sky, does not generally fulfill the condition

$$\Sigma \tau = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

while STEIN asserts that this condition will be satisfied.[1].

To refute my contention he shows that a determination from the components y, i. e. from the condition

$$\Sigma \left(\tau_0 - \frac{h}{\varrho} \sin \lambda \sin (\Gamma - \chi) \right)^2 \text{ minimum} \quad . \quad . \quad . \quad . \quad (8)$$

for stars at one and the same part of the heavens, does indeed lead to the condition (7).

I never denied this.

---

[1] The determination from both components *together* does satisfy the condition (7). For if the derivatives of the expression (4) with respect to $\frac{h}{\rho}$, A and D are equalled to zero, and if the values

$$\frac{\partial \tau}{\partial A} = v \frac{\partial \chi}{\partial A}, \quad \frac{\partial \tau}{\partial D} = v \frac{\partial \chi}{\partial D}, \quad \frac{\partial v}{\partial A} = - \tau \frac{\partial \chi}{\partial A}, \quad \frac{\partial v}{\partial D} = - \tau \frac{\partial \chi}{\partial D}$$

are introduced, we have for the determination of the three unknowns the three normal equations

$$\left[\left(\frac{h}{\varrho} \sin \lambda - v \right) \sin \lambda \right] = 0$$

$$\left[\tau v \frac{\partial \chi}{\partial A} + \left(\frac{h}{\varrho} \sin \lambda - v \right) \left(\frac{h}{\varrho} \cos \lambda \frac{\partial \lambda}{\partial A} + \tau \frac{\partial \chi}{\partial A} \right) \right] = 0$$

$$\left[\tau v \frac{\partial \chi}{\partial D} + \left(\frac{h}{\varrho} \sin \lambda - v \right) \left(\frac{h}{\varrho} \cos \lambda \frac{\partial \lambda}{\partial D} + \tau \frac{\partial \chi}{\partial D} \right) \right] = 0.$$

For a group of stars, situated at one and the same point of the heavens, these are reduced to the following:

$$\frac{h}{\varrho} \sin \lambda = \bar{v} \qquad (\bar{v} = \text{mean value of } v)$$

$$[\tau] = 0.$$

I only showed that a determination from the components τ, i. e. from the condition

$$\Sigma\, \tau^2 \text{ minimum}, \quad \ldots\ldots\ldots \quad (9)$$

does *not* fulfill this condition.

STEIN's reasoning thus misses its aim.

As has been remarked above it seems to me that, in STEIN's opinion, the conditions (8) and (9) are identical, if only care is taken to choose the point x near enough to the position which will ultimately be found to be the most probable position of the Antapex. Perhaps the great difference between the two conditions is best seen by comparing the results which they give for the position of the Apex in the example which I used formerly to show the inadequacy of AIRY's principle.

Suppose, therefore, two stars at one and the same point of the sky, having equal proper motions $S\mu$ and $S\mu'$ forming an obtuse angle. We suppose (which can safely be admitted for the present purpose) that the magnitude of the parallactic motion is equal for the two stars (say $= SA = SA'$).

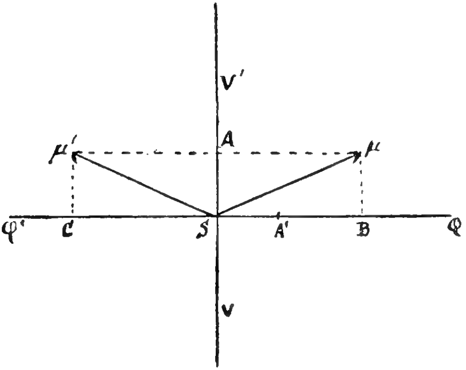

Fig. 2.

It is easily seen that in consequence of the conditions imposed the Apex must lie either on the line QQ' (in which case the parallactic motion is SA') or on the line VV' (in which case the parallactic motion is SA). The question is only on which of these two lines the Apex must be sought.

Now according to my contention, the condition (9) gives the line QQ', because:

$\Sigma\, \tau^2$ for Apex at $Q' < \Sigma\, \tau^2$ for Apex at V, i. e. because:

$$\overline{\mu B}^2 + \overline{\mu' C}^2 < \overline{\mu A}^2 + \overline{\mu' A}^2.$$

According to STEIN's contention the condition (8) gives the line

VV', because, all the projections being taken on the *same* line QQ', Σ proj.^{2} mot. pec. for Apex at $V < \Sigma$ proj.^{2} mot. pec. for Apex at Q', i. e. because

$$\overline{\mu A}^2 + \overline{\mu' A}^2 < (\overline{\mu A} - \overline{SA'})^2 + (\overline{\mu' A} + \overline{SA'})^2$$

The two contentions are nowise contradictory. They are in fact *both* true. They have nothing whatever to do with each other.

5. STEIN does not however confine himself to the assertion that for stars at one and the same point of the heavens (8) leads to (7), he also tries to prove directly, in two different ways, that (9) gives a determination which is identical to that by (8). In fact he tries to prove that both of these conditions lead to his conditions C.

It must already be evident from the above, that this proof must be impossible, and that consequently there must be an error in STEIN's argument.

The condition (9) evidently gives rise to the two equations

$$\left[\frac{\partial \tau}{\partial A}\, dA\right] = 0 \qquad \left[\frac{\partial \tau}{\partial D}\, dD\right] = 0.$$

In STEIN's *second* proof he simply derives the values of the derivatives $\frac{\partial \tau}{\partial D}$ and $\frac{\partial \tau}{\partial A}$ from the *equations of condition*

$$\tau_0 = \frac{h}{\varrho} \sin \lambda \sin \varepsilon \qquad v_0 = \frac{h}{\varrho} \sin \lambda \cos \varepsilon.$$

Here, therefore, he falls into the error, against which was warned in § 1 (*b*).

In the *first* proof of STEIN I cannot point out the main error, as I have been unable to follow the author's reasoning. [1]).

---

[1]) Still I will remark that if, as is done by STEIN, in the equation

$$\left[\tau \frac{m}{\varrho} \cos q \frac{\partial \chi}{\partial A} + \tau \frac{h}{\varrho} \sin \lambda \frac{\partial \chi}{\partial A}\right] = 0$$

the first term may simply be left out of account, because "the motus peculiaris may be considered as an error of observation", then with the same right the other component $v - \frac{h}{\varrho} \sin \lambda$ of the motus peculiaris may be neglected, so that in the second term we may write v for $\frac{h}{\varrho} \sin \lambda$. The equation thus becomes no other than the *contested* one:

$$\left[\tau v \frac{\partial \chi}{\partial A}\right] = 0.$$

6. The above will be a sufficient refutation of STEIN's criticism. I think I have shown:

a. That STEIN's equation B cannot prove anything against my contentions, because it is not to the point.

b. That his direct criticism of my treatment of the question is erroneous.

I may however be permitted to make the following remark, which will perhaps explain how STEIN was led to his erroneous views in the matter. He asserts: the second component (i. e. in his decomposition the component v_0, in my decomposition the component v) "does not teach us anything about the position of the Apex", or again "The two other equations" (the normal equation by which the position of the Antapex would be determined) "are left out of consideration on account of their small weight."

If this was really true, the determination of the position of the Apex would, as well for my treatment of the problem as for his, depend *solely* on the components τ and $\tau_0 - \frac{h}{\varrho} \sin\lambda \sin(\Gamma - \chi)$ respectively and it would thus be possible, merely by a different way of decomposing the *motus peculiares*, to derive from *the same* equation (6) (for this is, as already remarked, equivalent to both (4) and (5)) *different* results for the position of the Apex, which is absurd. From this point of view there was thus every reason for the belief that either my treatment or his own was affected by some error, and it is explicable that the author, not finding this error in his own work, looked for it in mine.

As a matter of fact however the error is neither in STEIN's determination based on (8), nor in my own based on (9) but only in his contention about what can be derived from the second component. This component does actually give a determination of the Apex of which the weight, compared to that of the determination from the first component, is generally not at all small.

To show this it is sufficient to write down the neglected normal equations. They are:

$$\left[\sin\lambda_0 \cos\lambda_0 \left(\frac{\partial\lambda}{\partial A}\right)_0\right]\frac{h}{\varrho} + \left[\cos^2\lambda_0 \left(\frac{\partial\lambda}{\partial A}\right)_0^2\right]\frac{h}{\varrho}\,dA + \\ + \left[\cos^2\lambda_0 \left(\frac{\partial\lambda}{\partial A}\right)_0 \left(\frac{\partial\lambda}{\partial D}\right)_0\right]\frac{h}{\varrho}\,dD = \left[v_0 \cos\lambda_0 \left(\frac{\partial\lambda}{\partial A}\right)_0\right]$$

$$\left[\sin\lambda_0 \cos\lambda_0 \left(\frac{\partial\lambda}{\partial D}\right)_0\right]\frac{h}{\varrho} + \left[\cos^2\lambda_0 \left(\frac{\partial\lambda}{\partial A}\right)_0 \left(\frac{\partial\lambda}{\partial D}\right)_0\right]\frac{h}{\varrho}\,dA + \\ + \left[\cos^2\lambda_0 \left(\frac{\partial\lambda}{\partial D}\right)_0^2\right]\frac{h}{\varrho}\,dD = \left[v_0 \cos\lambda_0 \left(\frac{\partial\lambda}{\partial D}\right)_0\right].$$

For brevity's sake I will suppose that we consider a group of stars distributed uniformly over the whole sky. In that case we find at once that $\frac{h}{\varrho} dA$ and $\frac{h}{\varrho}\, dD$ are determined with the weights

$$\left[\cos^2 \lambda_0 \left(\frac{\partial \lambda}{\partial A}\right)_0^2\right] \quad \text{and} \quad \left[\cos^2 \lambda_0 \left(\frac{\partial \lambda}{\partial D}\right)_0^2\right]$$

If the same quantities are determined from the first component (see STEIN's equations B) they have, in the case here considered, the weights

$$\left[\sin^2 \lambda_0 \left(\frac{\partial \chi}{\partial A}\right)_0^2\right] \quad \text{and} \quad \left[\sin^2 \lambda_0 \left(\frac{\partial \chi}{\partial D}\right)_0^2\right].$$

Now

$$\frac{\partial \lambda}{\partial A} = -\sin \lambda \cos D \frac{\partial \chi}{\partial D} \qquad \frac{\partial \lambda}{\partial D} = \frac{\sin \lambda}{\cos D} \frac{\partial \chi}{\partial A}$$

It is thus clear at once that the weights of the two determinations are entirely of the same order of magnitude.

The same thing is true of the analogous component in my treatment. A determination of the position of the Apex actually derived from this component was published by me some time ago. (*Astr. Nachr.* N^o. 3721 *Meth. IV*).

7. It may be urged that the objections made by me would not apply to AIRY's method if it were understood as is implied by the equation (4), i. e. if the sum of the squares of *both* components of the *motus peculiares together* were made minimum.

This is quite true. I now [1]) go even further, and express as my opinion that the equations, which would be derived in this way for the determination of the Apex, must be considered as very acceptable. The confidence which they would deserve could however *not* be derived from the hypothesis (*Hyp. A*) of AIRY. For, if it is admitted that STEIN's criticism is erroneous, it was shown by me that a legitimate application of this hypothesis may lead to unacceptable results.

This confidence must therefore rest on quite another basis. For me this basis is the following: the equations which are derived

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[1]) *Now*, because at the time when I wrote my communication of Febr. 1900, my attention had not yet been drawn te BRAVAIS' method.

from (4) are identically the same as those which **Bravais** derived, long before **Airy**, from a quite different, mechanical, principle. (*Journal de Liouville* 8; 1843, p. 435).

8. With a view to the two last breaks of **Stein**'s communication, the wording of § 3 and the note to § 4 may seem somethat strange. The reason of this is that these two paragraphs failed in the M. S. which the author kindly permitted me to use.

The reply to the *last* part of this addition is however already contained in the above. The *first* part strikes us as very peculiar. The author thinks "it is superfluous to refute at large the objections against **Airy**'s method derived by **Kapteyn** from a few particular cases of proper motion," *not*, as might be expected, because he has proved before that the treatment of these cases is *erroneous* [1]), but because "conclusions deduced from the consideration of only a few proper motions chosen quite systematically can hardly serve as criteria of a method . . .".

It looks as if the author is not very strongly convinced of the stringency of his own proof. Moreover the facts are not quite fairly represented.

The special cases to which the author refers were treated, not as *proof* of any particular thesis, but only as *examples* to illustrate the different conclusions to which the conditions $[\tau] = 0$ and $[\tau v] = 0$ can lead. As such I do not think they are badly chosen. That, in my opinion, especially the first example incidentally has some considerable direct power of proof, I will certainly not deny. The absurdity to which a treatment of this example by **Airy**'s method leads is very much of the same sort as that which would be involved [2]), if from the indication of two clocks showing 8 minutes to and 8 past twelve respectively it was concluded that most probably the time is either a quarter to or a quarter past twelve, but on no account twelve o'clock.

Does Dr. **Stein** really mean to say that there is nothing in such a result that throws any doubt on the method by which it was obtained?

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[1]) For **Stein** thinks he has shown that the condition $[\tau^2] = 0$ for stars at one and the same part of the sky is equivalent, *not* to $[\tau v] = 0$, but to $[\tau] = 0$, which both according to *his* and to *my* opinion, is the correct solution.

[2]) In order that my reply might still be printed in the Proceedings of October, the dutch text had to be written within a few hours after I got sight of the last paragraphs in **Stein**'s criticism. Owing to this haste the illustration contained in the last lines was not so well chosen as I could wish. I have taken the liberty to remedy to this defect in the translation.

Botany. — "*On the Hydrosimeter*". By Prof. J. W. Moll.

Many years ago I published the results of an investigation on the exudation of drops and injection of leaves [1]. The greater part of this investigation was made with cut branches into which water was driven by means of pressure of mercury. The instrument used for this purpose was as simple as possible and consistedof nothing but a U-shaped tube with a short and a long leg. The branch was fixed on the short leg of the tube filled with water, after which mercury was poured into the other leg until the desired pressure was obtained. This apparatus was rather imperfect, the pressure rapidly changing on account of the falling of the mercury in the long leg and the rising in the short one, and never remaining constant for a moment; so that in the experiments taken with this apparatus after some hours the pressure was generally for the greater part or entirely neutralized. The measuring of the quantity of water driven in could take place but imperfectly; it would have given rather much difficulty to arrive at a greater precision.

In the meanwhile the apparatus was sufficient for the purpose I then had in view. Yet the question arose in my mind whether it would not be possible to construct a better one, with which the driving in of the water could take place under constant pressure, whilst at the same time there would be a possibility of measuring accurately at any moment of the experiment the quantity of water driven in. Such an apparatus being of use for various botanical investigations, I have thought it worth while to solve this problem, and in this manner the apparatus has been made which I will now describe under the name of Hydrosimeter [2].

In this apparatus constant pressure has been obtained in the following manner.

The mercury causing the pressure is in a flask of Mariotte, to which an india-ruber tube is attached, so that it can be placed higher and lower. In this way a constant higher level is obtained. This tube conveys the mercury finally into a U-shaped tube, placed upright with the curve downwards and sealed into in a glass-reservoir filled with water, to which also the plant under observation has been attached. In this manner the constant lower level is

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[1] Untersuchungen über Tropfenausscheidung und Injection bei Blättern. Proc. and Communic. of the Royal Acad. of Sciences, Sect. Physics, 2nd Series, Vol. XV.

[2] From ὕδωρ = water and ὦσις = thrust, so gauge of waterpressure.

formed, for if liquid is driven from the reservoir into the plant, it is replaced by the mercury which flows off over the open leg of the U-shaped tube. So in this apparatus there is neither a falling nor a rising column of mercury; the lower level of the column of mercury being always equal with the upper edge of the open leg of the U-shaped tube. The volume of the overflowing mercury is equal to that of the water driven into the plant; this mercury can be drawn off and measured at any time. These are the principles on which the apparatus is founded.

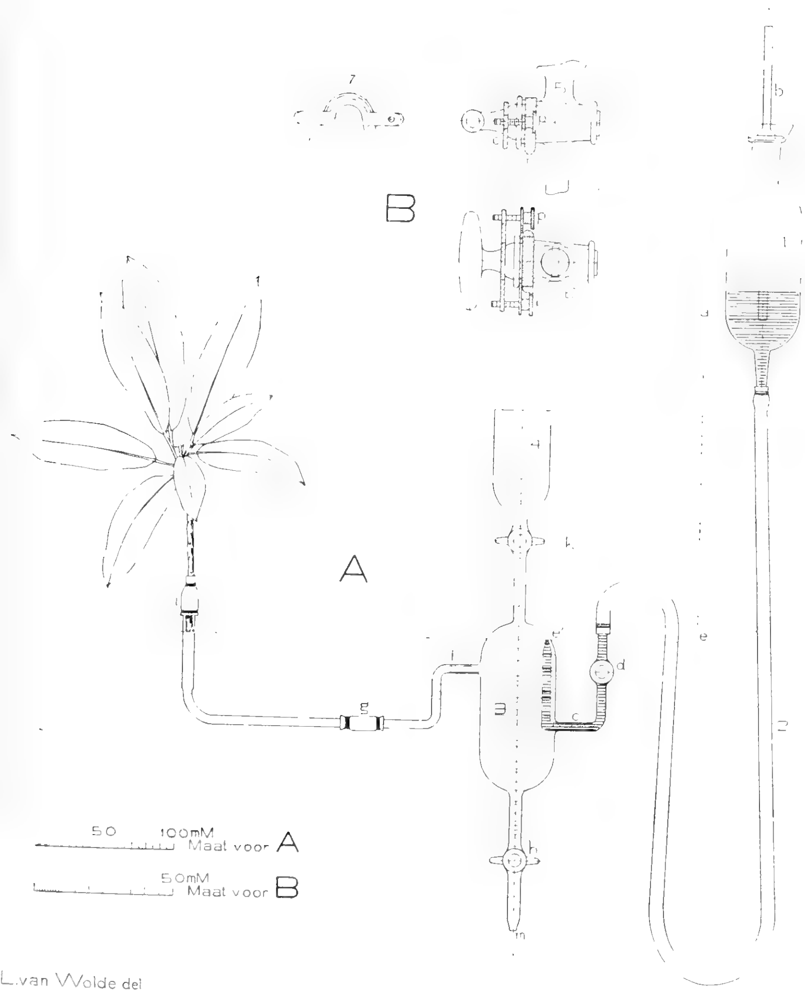
I shall now pass on to a brief description of the various parts, for which I wish to refer the reader to the plate added to this paper, and in the first place to ask his attention for fig. *A*. At the top on the right side at 1 is the mercury reservoir which can be placed at different heights. It is arranged as will be seen like a flask of MARIOTTE, so that on the level *a* there is always atmospherical pressure. The tube *b* fits on the neck of the reservoir as a stopper and to secure it better is closed wich mercury.

The mercury flowing from the reservoir passes through an india-rubber tube (2), with a wall-thickness of 2,5 mm. and a lumen of the same diameter, to the water-reservoir 3, to which also the plant is attached. Into this the mercury enters through the U-shaped tube *c* which possesses a glass tap *d* outside the reservoir. When the plant takes in water out of the reservoir 3, an equal volume of mercury will flow at *e'* over the edge of the open leg of the U-shaped tube, and collect at the bottom of the reservoir 3. So the lower level of the pressing column of mercury is always in the plane *ee'* and the distance *a e*, to be regulated arbitrarily, indicates the constant pressure under which water or any other liquid is driven into the plant.

Three tubes more are attached to the water-reservoir 3:

1°. the tube *f* leading to the plant; this tube is fixed high on to the reservoir, to let this contain as much mercury as possible thus the necessity of frequently drawing off the mercury being avoided. However, the tube soon bends downwards, passing into an open horizontal end. By this arrangement joining-pieces of simple shape can be used at *g* into which nevertheless the plant can be easily fixed with its plane of section in the level *ee'*. The connection at *g* is formed by means of the same thick-walled kind of india-rubber tube, which draws the mercury from the upper reservoir. This tube is so firm, and the surface upon which the pressure operates is so small, that even with a pressure of a full atmosphere the expansion is not of the slightest importance. The joining-pieces may have any shape,

J. W. MOLL. The Hydrosimeter.



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so that branches of any size or even entire plants with their roots can be connected with the apparatus, or pieces of wood through which fluid must be driven downwards. By using long joining-pieces it is also possible to place the plant in a glass case or any other apparatus, which if necessary can be put up at a tolerable distance from the hydrosimeter, whilst the reading of the result on the latter remains possible. On the plate the apparatus is shown as it is used for the injection of the leaves of Rhododendron.

2°. a lower tube with a glass tap (*h*); through this the mercury can be drawn off, the volume of which is to be determined.

3°. a topmost tube, likewise with a glass tap *k*, which tube ends in a reservoir 4 filled with water. This is necessary to replace the mercury which is tapped through tube *h*.

After this description of the apparatus shown in fig. *A* I wish to draw the attention to fig. *B* giving a representation of a metal clip, one attached to each of the three glass taps. The hydrosimeter is arranged for the use of one atmosphere as maximum pressure, this being at least for botanical investigations more than sufficient, whilst for the rest there are no obstacles to the use of a higher pressure, though it might make some further precautions necessary. But already with pressures of less than one atmosphere we meet with the drawback that not only do the taps leak, but that even the whole tap-stopper is pressed out. This is indeed not to be wondered at, considering the stopper being always more or less conically shaped and especially the holes being never entirely perfect at the edges. To overcome this drawback each of the taps is furnished with a clip which makes the leaking and the pressing-out of the stopper quite impossible. Fig. *B* gives at 5, 6 and 7 three different views of such a clip. It consists of two brass rings in the shape of a horse-shoe and acting like a spring, so that they can be easily fixed round the neck of the tap to fit well. The ring *o* fits round the thinnest part of the tap-stopper, the ring *n* on the contrary round the tube in which the stopper turns, the thick edge of this preventing the sliding off.

As is seen, both rings are provided with two projecting metal lips. These lips of the two rings correspond in place and size to each other, and are connected by two screws (*p*), so that they are easily adjustable at different distances from each other. It is clear that when such a clip is placed on a tap, the stopper can be fixed very firmly by means of the screws into the tap, whilst the latter, when well greased, will remain easy to turn. As was said before, each of the three taps is provided with such a clip.

To put the apparatus into operation first of all the necessary amount of mercury must be brought into the reservoir 1, the india-rubber tube 2 and the **U**-shaped tube *c*, after which the remaining part is filled with water. Then the taps have to be tried, in a way I shall not describe here in details, whether they close well, and finally the plant is connected with the apparatus in the manner shown in fig. *A*.

It goes without saying that during the experiment one can measure as often as one likes the water driven into the plant during a certain time. To do so tap *d* is closed and the pressure for a moment neutralized, to which in most cases there will be no objection. The mercury is drawn off and replaced by water, when for a moment the taps *h* and *k* are opened simultaneously. The experiment can be immediately continued by causing the pressure to operate again.

In order to make the tapping of the mercury possible the tube leading off the mercury may not be too narrow at *m* and its lumen must at least amount to 3 mM. The drawn off mercury can now and again be poured into the reservoir 1, so that for lengthy experiments a limited amount of mercury is sufficient.

The size of the drops overflowing at *e′* the edge of the **U**-shaped tube depends on the width of the tube and can thus be regulated within certain limits according to special wants. In the apparatus I am using at present, the top of the **U**-shaped tube measured on the outside has a thickness of about 2 mM. The size of the drops, which is very constant, amounts to 0.02 cM^3. So the apparatus enables the experimentist to make very accurate observations. Moreover in such cases when one wishes to take measurements at not too long intervals without neutralizing the pressure, one can easily attain one's aim by counting the drops flowing over.

The apparatus has been made according to my directions by the firm J. C. TH. MARIUS at Utrecht and is brought by the same into the trade.

(November 21, 1901).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday November 30, 1901.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 30 November 1901, Dl. X).

The following papers were read:

Physics. — "*Some observations on the resolving power of the* MICHELSON *echelon spectroscope*", by Prof. P. ZEEMAN.

§ 1. On a recent occasion [1]) I have given a few observations on this subject. The acquiring of some new data induces me to return to it in this place.

In his "Investigations in optics" Lord RAYLEIGH [2]) expressed the wish that spectroscopists in possession of powerful instruments would compare the actual resolving power with that of which they are theoretically capable and remarked that a carefully arranged suc-

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[1]) BOSSCHA Collection of Memoirs. Archiv. Néerl. sér. II. T. 6, p. 319. 1901.

[2]) Phil. Mag. 1879, 1880.

cession of tests of gradually increasing difficulty would be of especial value.

I remembered these remarks as I tested the very original echelon, invented by MICHELSON.

The echelon at my disposition, made by HILGER, London, consists of thirty plates each about 7,8 m.m. thick, made of light flint-glas, set with 1 m.m. steps. A clear aperture of 1 m.m. is left beyond the width of the largest glassplate. The number of apertures n, operative in the formation of the spectrum is hereby one more than the number of plates. The mounting was somewhat improvised. Telescope and collimator belonging to a KIRCHHOFF spectroscope were employed. The telescopes had object-glasses of 50 cm. focus and 38 m.m. aperture. It is evident that in order to get greater intensity, glasses of shorter focus would have been preferable.

Denoting by $d\lambda_1$ the difference of wave-length of spectral lines when they are just distinguishable as separate in the spectroscope, by t the thickness of the plates of glass, and by n the above mentioned number then, we know

$$q_t = \frac{d\lambda_1}{\lambda} = \frac{\lambda}{knt} \quad \dots\dots\dots \quad (1)$$

if

$$k = (\mu - 1) - \lambda \frac{d\mu}{d\lambda},$$

The resolving power is given by

$$r = \frac{\lambda}{d\lambda_1} = \frac{knt}{\lambda} \quad \dots\dots\dots \quad (2)$$

For the green line $\lambda = 5460$ A. M. we obtain in the case of our echelon $r = \dfrac{0{,}63.\ 31.\ 7{,}8}{5460.\ 10^{-7}} = 280000$ and $q_t = \dfrac{d\lambda_1}{\lambda} = 3{,}6.\ 10^{-6}$.

In the calculation of k I used the following values of the refractive indices given to me by HILGER

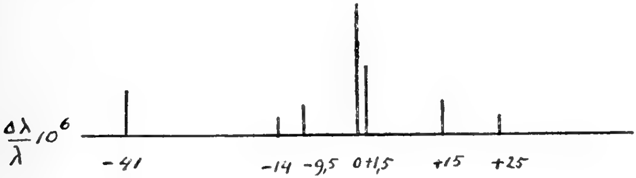
$\mu_C = 1.5713$
$\mu_D = 1.5753$
$\mu_F = 1.5853$
$\mu_{G'} = 1.5936$

Henceforth I will denote by q_t the theoretical value of the limit of resolution calculated according to (1), by q_e the experimental value. By means of a HOFFMANN direct vision spectroscope the light of the vacuum tubes (driven by a RUHMKORFF) undergoes the necessary preliminary analysis. In some cases absorbing media were therefore sufficient. In some experiments the mercury arc-lamp of FABRY and PEROT was used.

§ 2. The very intense *green* (5460) line of *mercury* was investigated first. Using the echelon in a position in which two strong lines of equal intensity corresponding to successive orders of the radiation were visible, I could distinguish also 5 faint, very narrow lines between the principal ones. The distance between two pairs of these lines was very small.

As I could not find a table of the wave-lengths of these feeble radiations, I addressed myself to Messrs FABRY and PEROT. I am very much obliged to Messrs PEROT and FABRY for their kindness to investigate for me anew the green radiation of the mercury arc in vacuo.

The following scheme represents the constitution of this very complex radiation according to their observations. The ordinates are *approximately* proportional to the intensities.



The given numbers are only approximate, especially (— 14) and (— 9,5).

The radiation (+ 1,5) was observed by FABRY and PEROT only in the radiation of a MICHELSON tube; it is too approximate to the principal radiation to be seen separately in the arc light. In the photographical reproduction in the Astrophysical Journal [1]) of the interference fringes of the green mercury line the radiation (— 41) coincides with the radiation (+ 15) and is therefore invisible.

I could distinguish very clearly the radiations (— 9,5) and (— 14)

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[1]) FABRY and PEROT Astrophysical Journal. Vol. 13. p. 272. 1901.

as separate lines. For these radiations $q = \frac{d\lambda}{\lambda} = 4{,}5.10^{-6}$ or $r = 222000$ and hence q_e rather smaller; calculation gave $q_t = 3{,}6.10^{-6}$.

Using the *green* line of *thallium*[1] I extremely easily distinguished the faint radiation at a distance $\frac{d\lambda}{\lambda} = 21.10^{-6}$ from the principal radiation, but I could not see as a separate line the one determined by $\frac{d\lambda}{\lambda} = 3.10^{-6}$.

Hence q_e exceeds 3.10^{-6} but is smaller than 21.10^{-6}.

Indeed for the thallium radiation (5440)

$$q_t = \frac{5440.10^{-7}}{0{,}63.31.7{,}8} = 3{,}6.10^{-6}\ .$$

For the *green* (5086) line of *cadmium* it was just possible to see that this line is a double one. The distance of the components is according to FABRY and PEROT $\frac{d\lambda}{\lambda} = 5.10^{-6}$.[2]) For $\lambda = 5086$ I calculate $q_t = 3{,}2.10^{-6}$. Hence with the mentioned echelon it is possible to almost reach the limit of the theoretical resolving power.

§ 3. Perhaps the best series of tests of gradually increasing difficulty can be obtained by observation of the change of spectral lines in magnetic fields of gradually increasing intensities, a Nicol between source and apparatus being used in order to reduce the complexity of the radiation. In this manner all values between e.g. 0.001 A. U. to about 1 A. U. can be obtained. Corresponding herewith are the values $q_t = 0{,}2.10^{-6}$ and $r = 5\,000\,000$ resp. $q_t = 200.10^{-6}$ and $r = 5000$. The performances of echelons and interferometers and of ordinary spectroscopes with a few glass prisms lie between the limits indicated. This test I have not yet applied systematically to the mentioned echelon.

In order however to show its fitness I will use some observations of Lord BLYTHSWOOD and Dr. MARCHANT[3]). In their § 6 „Results obtained of the ZEEMAN Effect on the Chief Lines of the Mercury Spectrum" p. 397 these authors communicate observations with an

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1) FABRY et PEROT. Ann. de Chim. et de Phys. (7) **16**. p. 134. 1899.

2) l. c. p. 137.

3) Phil. Mag. Vol. 49. p. 384. 1900.

echelon spectroscope concerning the difference in wave-length between the components of the outer components of the sextet of the blue (4358) line of mercury. The following table is an extract ($\delta\lambda_3$ in A.U.)

H	$\delta\lambda_3$
5.000	.....
12.100	.....
12.900	0.052
20.000	0.098?
21.300	0.09
23.400	0.098

For a value of the field between 12.100 and 12.900 the splitting up of the lines becomes sufficient to make them appear as separate lines *on a photograph* (upon which the measurements were taken). Two lines can of course be *seen* separated at a considerably smaller distance.

Thus now $q = \frac{0{,}052}{4358} = 11{,}9.10^{-6}$ and q_e considerably smaller. For the echelons of these observers we have $t = 7{,}5$, $n = 15$.

With these data I calculate $q_t = 5{,}3.10^{-6}$.

Thus it appears from the data given in this paper that it is possible to manufacture echelons, performing nearly as well as they are theoretically capable.

Mathematics. — "*Considerations in reference to a configuration of* SEGRE". By Prof. P. H. SCHOUTE. (Second part).

5. We have already remarked that the form of the equations of the fifteen lines obtained in the first part of this communication was not yet a quite regular one. If we shorten $x_1 - x_3 = x_5$ into $(1 - 3)$, $x_1 = x_2$ into 12 and if everywhere we omit the equations $x_1 = 0$, $x_2 = 0, \ldots x_5 = 0$, then the following table gives the obtained result in the form of the determinant repeatedly used

(1—3)	(2—4)	(3—2)	(4—1)	(12, 34)	
(4—2)	(3—1)	(1—4)	(2—3)		(12, 34)
(3—4)	(1—2)	(13, 24)		(2—3)	(4—1)
(2—1)	(4—3)		(13, 24)	(1—4)	(3—2)
(14, 23)		(4—3)	(1—2)	(3—1)	(2—4)
	(14, 23)	(2—1)	(3—4)	(4—2)	(1—3)

This table shows that so far there is regularity in the irregularity, that each conjugate quintuple as to this irregularity corresponds with the quintuple of the lines a_i.

Before we pass to an entirely regular form of the equations of the fifteen lines, we determine, also to show the fitness of the system of equations, the locus of the planes cutting the four lines a given originally. For this we search for the conditions, under which the space

$$p_x \equiv p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4 + p_5 x_5 = 0$$

contains such a plane. The number of planes cutting four lines given arbitrarily in S_4 being twofold infinite, this investigation must lead us to a homogeneous equation $f(p) = 0$ in the five spacial coordinates p_i, the tangential equation of the curved space enveloped by the spaces $p_x = 0$.

The coordinates of the points of intersection of the space $p_x = 0$ with the four lines a_1, a_2, a_3, a_4 are the elements of the four rows of the matrix

$$\left| \begin{array}{ccccc} p_3 - p_5 \,, & 0 \,, & -(p_1 + p_5), & 0 \,, & p_1 + p_3 \\ 0 \,, & p_4 - p_5 \,, & 0 \,, & -(p_2 + p_5), & p_2 + p_4 \\ 0 \,, & -(p_3 + p_5), & p_2 - p_5 \,, & 0 \,, & p_2 + p_3 \\ -(p_4 + p_5), & 0 \,, & 0 \,, & p_1 - p_5 \,, & p_1 + p_4 \end{array} \right| ;$$

by putting this matrix equal to naught, we let the space $p_x = 0$ satisfy the given condition. Here, however, an obstacle seems to present itself. For the five equations obtained by putting the determinants comprised in the matrix equal to naught, furnish in general two respectively independent relations, which cannot be the case here. However, as is immediately evident after development, each of those five determinants consists really of the form

$$(p_1+p_2+p_3+p_4)p_5^2+p_2p_3p_4+p_1p_3p_4+p_1p_2p_4+p_1p_2p_3$$

every time multiplied by another linear form, and we find the wanted equation of the enveloped surface by putting this common factor equal to naught.

The same obstacle seems to appear when we make use of the following method to determine the equation of the enveloped space. If $p_x = 0$, $q_x = 0$ represent an arbitrary plane, it intersects the four lines a_1, a_2, a_3, a_4 under the conditions

$$\begin{vmatrix} p_1 & p_3 & p_5 \\ q_1 & q_3 & q_5 \\ -1 & 1 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} p_2 & p_4 & p_5 \\ q_2 & q_4 & q_5 \\ -1 & 1 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} p_3 & p_2 & p_5 \\ q_3 & q_2 & q_5 \\ -1 & 1 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} p_4 & p_1 & p_5 \\ q_4 & q_1 & q_5 \\ -1 & 1 & 1 \end{vmatrix} = 0$$

and by eliminating q_1, q_2, q_3 we arrive at the equation

$$\begin{vmatrix} p_3-p_5, & 0, & -(p_1+p_5), & (p_1+p_3)q_5 \\ 0, & p_4-p_5, & 0, & -(p_2+p_5)q_4+(p_2+p_4)q_5 \\ 0, & -(p_3+p_5), & p_2-p_5, & (p_2+p_3)q_5 \\ -(p_4+p_5), & 0, & 0, & (p_1-p_5)q_4+(p_1+p_4)q_5 \end{vmatrix} = 0,$$

which also furnishes two equations, as it must hold good for all values of the quotient $\dfrac{q_4}{q_5}$. We recognise in these two equations immediately those which are obtained by omitting from the matrix found above respectively the last column and the last but one.

By the way we notice that the second method can prove in a simple way how each plane intersecting a_1, a_2, a_3, a_4 also cuts a_5. If we introduce for the determinant $p_k q_l - q_l p_k$ the notation (kl) the four conditions can be written in the form

$$\left.\begin{array}{l}(53)+(51)+(13)=0\\(54)+(52)+(24)=0\\(25)+(35)+(23)=0\\(15)+(45)+(14)=0\end{array}\right\}.$$

So addition gives

$$(13)+(14)+(23)+(24)=0,$$

which is the condition that the plane cuts the line a_5. For substituting

$$x_1=x_2 \quad , \quad x_3=x_4 \quad , \quad x_5=0$$

into $p_x=0$ and $q_x=0$ we find

$$(p_1+p_2)\,x_2+(p_3+p_4)\,x_4=0,$$
$$(q_1+q_2)\,x_2+(q_3+q_4)\,x_4=0;$$

by eliminating the quotient $\frac{x_2}{x_4}$ we get

$$\begin{vmatrix} p_1+p_2 \ , & p_3+p_4 \\ q_1+q_2 \ , & q_3+q_4 \end{vmatrix}=0,$$

which can be immediately developed into

$$(13)+(14)+(23)+(24)=0.$$

6. Now that we have found the equation of the enveloped space the full investigation of it may be omitted. We shall confine ourselves to some ready observations.

In the first place it is evident that the ten "spaces through six lines" with the spacial coordinates

$(a_1\, a_2)$	$(-1, \; -1, \; 1, \; 1, \; 1)$
$(a_1\, a_3)$	$(\;0, \; 0, \; 0, \; 1, \; 0)$
$(a_1\, a_4)$	$(\;0, \; 1, \; 0, \; 0, \; 0)$
$(a_1\, a_5)$	$(-1, \; 1, \; 1, \; -1, \; 1)$
$(a_2\, a_3)$	$(\;1, \; 0, \; 0, \; 0, \; 0)$
$(a_2\, a_4)$	$(\;0, \; 0, \; 1, \; 0, \; 0)$
$(a_2\, a_5)$	$(\;1, \; -1, \; -1, \; 1, \; 1)$
$(a_3\, a_4)$	$(\;1, \; 1, \; -1, \; -1, \; 1)$
$(a_3\, a_5)$	$(-1, \; 1, \; -1, \; 1, \; 1)$
$(a_4\, a_5)$	$(\;1, \; -1, \; 1, \; -1, \; 1)$

are double spaces of the enveloped space, so that this must be of order four and cannot admit of an eleventh double space, because a space of order two cannot be of class three.

Moreover it is evident, that the fifteen "points in three lines" are points for which the tangent planes of the curved space, passing through it, envelope a conic space degenerated into three nets of planes (all planes through a line). So the three lines a_5, c_3, e_1 cut each other in the point

$$x_1 = x_2 = x_3 = x_4, \quad x_5 = 0$$

with the equation

$$p_1 + p_2 + p_3 + p_4 = 0$$

and the combination of this with the equation of the cubic envelope causes the latter to be transformed by elimination of p_4 into

$$(p_2 + p_3)\,(p_3 + p_1)\,(p_1 + p_2) = 0,$$

which in connection with the first furnishes the planes

$$\left.\begin{array}{l} p_2 + p_3 = 0 \\ p_1 + p_4 = 0 \end{array}\right\}, \qquad \left.\begin{array}{l} p_3 + p_1 = 0 \\ p_2 + p_4 = 0 \end{array}\right\}, \qquad \left.\begin{array}{l} p_1 + p_2 = 0 \\ p_3 + p_4 = 0 \end{array}\right\}$$

with the axes e_1, c_3, a_5.

But the following must be pointed out particularly: the surface of the third class out of n⁰. 4 is connected in a simple way with the enveloped space. In the cone by which this surface is projected from the point P taken there, we have namely before us the envelope of all the tangent planes of the cubic space, passing through this point P. This will be clear if we resume in the following form the dualistically opposite results forming an extension of the theorem of SEGRE mentioned in n⁰. 1:

If we take quite arbitrarily in S_4 four planes α_{13}, α_{14}, α_{15}, α_{16}, and if we determine the planes α_{23}, α_{24}. α_{25}, α_{26}, forming with the former one a double four

$$\left|\begin{array}{cccc} \alpha_{13}\ , & \alpha_{14}\ , & \alpha_{15}\ , & \alpha_{16} \\ \alpha_{23}\ , & \alpha_{24}\ , & \alpha_{25}\ , & \alpha_{26} \end{array}\right|$$

— where two planes have a point or a line in common according to their symbols having a common index or not —, then the four points of intersection of the opposite elements of the double four — placed here under each other — lie in a same plane α_{12}.

If we add this plane α_{12} to the assumed planes, we obtain a quintuple of "conjugate planes" with the remarkable property, that each of those planes plays the same part in reference to the double four, of which the four remaining planes form one of the two quadruples, as α_{12} in reference to the above mentioned double four.

If we complete all quadruples to be formed out of this quintuple to double fours, we find fifteen planes in all, which can be characterised by symbols $\alpha_{k,l}$ in such a way, that two planes have a line or point in common according to their symbols having a common index or not. We then find that ordering of those symbols in form of determinants — as the corresponding $a_{k,l}$ in n⁰. 4 — gives six rows or six columns of conjugate quintuples $(\varphi_i), i=1,2,...6$. Each line intersecting four planes of a conjugate quintuple also cuts the fifth.

Each triplet of planes $(\alpha_{12}, \alpha_{34}, \alpha_{56})$, cutting each other two by two in a line, lie in a same space indicated by $S_{12,\,34,\,56}$; in this space they pass through a same point $P_{12,\,34,\,56}$. There are fifteen of such spaces and points.

Each sextuple of points as $(\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{45}, \alpha_{46}, \alpha_{56})$, built up of two triplets $(\alpha_{12}, \alpha_{13}, \alpha_{23})$ and $(\alpha_{45}, \alpha_{46}, \alpha_{56})$, with the property that each plane of one triplet cuts each plane of the other in a line, pass through a same point $P_{123,\,456}$. There are ten such points.

A threedimensional space S_3 taken arbitrarily cuts each of the six conjugate quintuples (φ_i) in five lines, which admit of two common transversals (b_i, c_i); these six pairs of lines (b_i, c_i) are opposite elements of a double six of a surface F^3 of order three of which the 27 right lines consist of these twelve lines and the fifteen lines of intersection of S_3 with the planes $\alpha^{k,l}$.

The locus of the lines intersecting four planes belonging to a same conjugate quintuple — and so also the fifth — is always the same curved space $S^{3,4}$ of order three and class four through the fifteen planes $\alpha_{k,l}$, whichever of the six quintuples (φ_i) are taken; so this space $S^{3,4}$ contains six different twofold infinite systems of right lines. It has the ten points $P_{123,\,456}$ as double points, the quadratic conic spaces of contact of which contain the sextuples of planes passing through those points; it is cut into three planes by each of the fifteen spaces $S_{12,\,34,\,56}$. Its section with the above introduced arbitrary space S_3 must contain the double six of the pairs of lines (b_i, c_i) as well as the fifteen lines of intersection of S_3 with the planes $\alpha_{k,l}$ and so it must coincide with the surface F^3 found there, of which the points lying outside these 27 lines are points of intersection of S_3 with lines of the locus $S^{3,4}$ not situated in S_3.

7. If we apply to the equations

$$\left.\begin{array}{ccc}
x_1 - x_3 = x_5 \quad , & x_2 = 0 \quad , & x_4 = 0 \\
x_2 - x_4 = x_5 \quad , & x_1 = 0 \quad , & x_3 = 0 \\
x_3 - x_2 = x_5 \quad , & x_1 = 0 \quad , & x_4 = 0 \\
x_4 - x_1 = x_5 \quad , & x_2 = 0 \quad , & x_3 = 0 \\
x_1 = x_2 \quad , & x_3 = x_4 \quad , & x_5 = 0
\end{array}\right\}$$

of the lines a_1, a_2, ... a_5 the transformation

$$\left.\begin{array}{r} -x_1+x_2+x_3+x_4-x_5=y_1 \\ x_1-x_2+x_3+x_4-x_5=y_2 \\ x_1+x_2-x_3+x_4-x_5=y_3 \\ x^1+x_2+x_3-x_4-x_5=y_4 \\ -2\,x_5=y_5 \end{array}\right\},$$

which it is possible to write in the reversed form

$$\left.\begin{array}{rl} 4\,x_1= & -y_1+y_2+y_3+y_4-y_5 \\ 4\,x_2= & y_1-y_2+y_3+y_4-y_5 \\ 4\,x_3= & y_1+y_2-y_3+y_4-y_5 \\ 4\,x_4= & y_1+y_2+y_3-y_4-y_5 \\ 2\,x_5= & \hfill -y_5 \end{array}\right\},$$

these equations pass into

$$\left.\begin{array}{lll} y_1=y_5\ , & y_2=y_4\ , & y_3=0 \\ y_2=y_5\ , & y_1=y_3\ , & y_4=0 \\ y_3=y_5\ , & y_1=y_4\ , & y_2=0 \\ y_4=y_5\ , & y_2=y_3\ , & y_1=0 \\ y_1=y_2\ , & y_3=y_4\ , & y_5=0 \end{array}\right\}.$$

Moreover the equations of all the fifteen lines of the configuration present themselves in the collective formula

$$y_p=y_q\,,\quad y_r=y_s\,,\quad y_t=0\,,$$

where p, q, r, s, t indicate one of the permutations of the five indices 1, 2, . . 5.

The regular representation of the lines obtained in this way allows of a very simple geometrical realization. If we suppose for simplicity's sake that the homogeneous system of coordinates y_i with respect to the five-cell of the spaces $y_i=0$ is a system of normal distance coordinates, then it is evident that the fifteen lines of the configuration are the lines connecting the mid-points of the pairs of edges of that five-cell crossing each other. So, if we suppose (fig. 4)

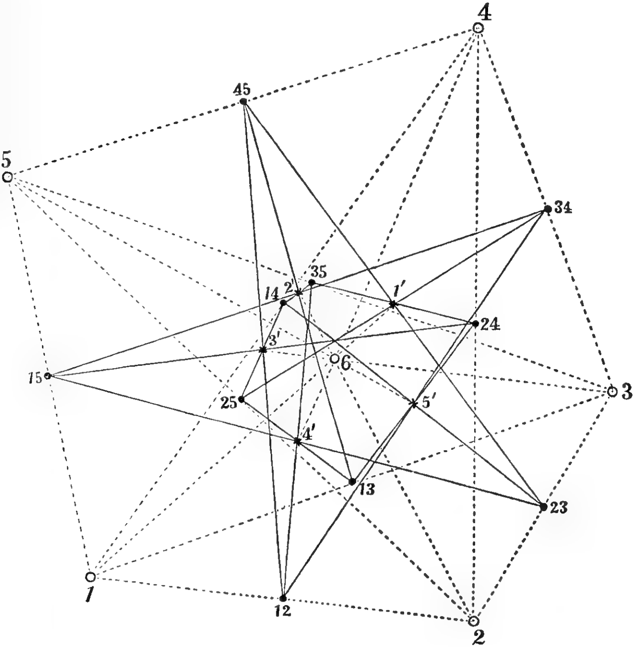

Fig. 4.

five points 1, 2, 3, 4, 5 not lying in a three-dimensional space to be the bearers of equal masses and if we determine the barycentres 12, 13, . . . 45 of the ten pairs of these masses, then the lines (12, 34), . . . , (23, 45) connecting two of these points belonging to four different masses will form the fifteen lines of a configuration of Segre. These lines pass through one of the barycentres 1', 2', 3', 4', 5' of four of the five masses; moreover the five lines (1, 1'), (2, 2'), . . , (5, 5') pass through the barycentre 6 of the five masses. Out of this figure we easily find the remaining elements of the configuration

Cf. (15, 3, 15, 7 | 3, 15, 6, 3 | 5, 2, 45, 3 | 7, 3, 9, 15).

For we recognize in the ten barycentres 12, 13, . . , 45 of two masses and the five barycentres 1′, 2′, . . , 5′ of four of the five masses the fifteen points, in the thirty planes (12, 34, 35, 4′, 5′) and the fifteen planes (1′, 24, 35, 23, 45) the forty-five planes, in the ten spaces (12, 34, 35, 45, 3′, 4′, 5′) and the five limiting spaces of the five-cell the fifteen spaces of the configuration. And if we assume in the points 1, 2, 3, 4, 5 entirely arbitrary different masses instead of equal ones, then the special case represented in fig. 4 passes into the general one. If we then also call the barycentre of those unequal masses the point 6, we arrive at the following simple representation of the configuration:

If in space S_4 six points 1, 2, . . , 6 are taken in such a way that no five of these points lie in a three-dimensional space, we can find fifteen three-dimensional spaces $R_3^{(1,2)}=(3456)$, $R_3^{(1,3)}=(2456)$, . . , $R_3^{(5,6)}=(1234)$ each one of which contains four of the six points. Of these spaces $S_3^{(i,k)}$ any three, the indices i, k of which complete each other to 1, 2, . . , 6, pass through a same line furnishing in all fifteen lines. And these same lines forming with each other the chief part of a configuration of SEGRE are also found if each of the lines $l_{12}, l_{13} \, . \, . \, , l_{56}$ connecting the six points two by two is cut by the opposite space $R_3^{(12)}$, $R_3^{(13)}$, . . , $R_3^{(56)}$ by which operation we obtain fifteen points 12, 13, . . , 56 characterized by the property that any three points the indices of which complete each other to 1, 2, . . , 6 are lying on a right line. Every five lines containing together the fifteen points 12, 13, . . , 56 form a quintuple of conjugate lines.

8. The simple representation we have now given of SEGRE's configuration is closely related to results published already in 1888 by Dr. G. CASTELNUOVO in his treatise „Sulle congruenze del terzo ordine dello spazio a quattro dimensioni" (*Atti del R. Istituto Veneto*, serie 6, vol. 6). If namely we assume the five points 1, 2, 3, 4, 5 as vertices $p_i = 0$, $i = 1, 2, 3, 4, 5$, of the five-cell of coordinates and the point 6 as point of unity $p_6 \equiv (p_1 + p_2 + p_3 + p_4 + p_5) = 0$, then the pair of equations

$$p_1 + p_2 = 0, \quad p_3 + p_4 = 0,$$

into which the system

$$x_1 = x_2, \quad x_3 = x_4, \quad x_5 = 0$$

passes by means of the relation

$$p_x \equiv p_1 x + p_2 x_2 + p_3 x_2 + p_4 x_4 + p_5 x_5 = 0$$

between the coordinates x_i of points and the coordinates p_i, of spaces, can be completed by

$$p_5 + p_6 = 0$$

and from this is evident, that each of the fifteen lines of SEGRE's configuration is represented by a triplet of equations of the form

$$p_1 + p_2 = 0, \quad p_3 + p_4 = 0, \quad p_5 + p_6 = 0$$

and that the lines are situated six by six in ten spaces with the coordinates

$(1, 1, 1, -1, -1, -1)$, $(1, 1, -1, 1, -1, -1)$, ... $(-1, -1, -1, 1, 1, 1)$.

So the six lines

(14, 25, 36), (15, 24, 36), (16, 23, 45)

(14, 26, 35), (15, 23, 46), (16, 24, 35)

are situated in the first of those ten spaces, etc.

In the quoted treatise CASTELNUOVO has represented the fifteen planes of the dualistically opposite figure by the system of the fifteen corresponding triplets of equations

$$x_1 + x_2 = 0, \quad x_3 + x_4 = 0, \quad x_5 + x_6 = 0.$$

Though the connection between the above-mentioned treatise and these considerations have already hereby been indicated, yet I desire to acknowledge that not until the existence of that treatise had been brought to my notice by SEGRE did I succeed in deducing from my considerations founded upon the double four the above-given simple representation.

Out of the fifteen triplets of equations

$$p_1 + p_2 = 0, \quad p_3 + p_4 = 0, \quad p_5 + p_6 = 0$$

it directly ensues that the curved space of class three enveloped by the pencils of planes with one of the lines of the configuration as axis must have the equation

$$p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3 = 0,$$

the first member disappearing for each of those triplets.

So the six points $p_i = 0$ are for this curved space of class three what the pentaeder of SYLVESTER is for the surface of order three.

9. In a second treatise published in 1891 and entitled: „Ricerche di geometria delle rette nello spazio à quattro dimensioni" (*Atti del. R. Istituto Veneto*, series 7, vol. 2) CASTELNUOVO has represented the curved space

$$\varrho^3 \equiv x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 = 0$$

on our space S_3 in such a way, that the spacial sections of $\varrho^3 = 0$ correspond with quadratic surfaces passing through five fixed points. In that case the fifteen planes correspond to the five vertices and to the ten faces of a complete quintangle in S_3, whilst the ten double points of $\varrho^3 = 0$ correspond to the ten edges of this quintangle. Instead of continuing these researches we put the question in how far the configuration of fifteen lines is unique in its kind.

Of course it is not difficult to point out in the poly-dimensional spaces configurations having characteristics in common with the configuration of SEGRE. So we find one in each group of $n + 2$ points taken arbitrarily in S_n when $n + 2$ is not a prime number. Let us take as an example nine arbitrary points 1, 2, 3, . . , 9 in S_7 and let us represent the point of intersection of the plane (1, 2, 3) with the space S_5 through the six remaining points by the symbol P_{123}; then each three points P_{123}, P_{456}, P_{789}, whose indices complete each other to 1, 2, 3. . . , 9, will be situated on a right line. It may be, that the easiest way of proving this is by the aid of a system of parallel forces in equilibrium applied to the points 1, 2, 3, . . , 9. As is known the parallel forces applied to the points 1, 2, 3, . . , 8 may be chosen in such a way that the resultant acts on point 9; if we add to these eight forces a force applying in point 9 equal and opposite to this resultant, then such a system of forces in equilibrium has been obtained. It is now evident that point P_{123} is the point of application of the resultant of the three forces working at the forces 1, 2, 3 as well as that of the resultant of the six remaining ones. For, these points of application must coincide, on account of the equilibrium, in a point situated in the plane (1, 2, 3) as well as in the space S_5 through the six other points. If we reduce the nine forces to three by compounding those operating in 1, 2, 3 and those operating in 4, 5, 6 and those operating in 7, 8, 9, we obtain those parallel forces applied in P_{123}, P_{456}, P_{789} and these three forces can only then be in equilibrium when the three points of application lie on a right line. So the figure of the nine points S_7 leads to $(9)_3 = 84$ points P_{123}, situated three by three on $\frac{1}{6}\,(9)_3 . (6)_3 = 280$ lines, whilst reversely ten of those 280 lines pass through each of those 84 points. So we can deduce out of 12 arbitrary points in S_{10}, performing the decomposition of 12 into two factors in different ways, 66 points P_{12}, 220 points P_{123} or 495 points P_{1234} and remark that the points P_{12} are situated six by six in 10395 spaces S_4, the points P_{123} four by four in 15400 planes and the points P_{1234} three by three in 13305600 lines, etc.

Although the configuration of SEGRE is undoubtedly a part of the general group indicated here, it is certainly distinguished from most of them and probably from all of them by the property that it is determined by a quadruple of crossing lines taken arbitrarily and these lines fix in a narrower sense a fifth of the fifteen lines, which group of conjugate lines then bear together the fifteen points of the configuration. Indeed, in S_4 the system of six points as well as that of the four lines is dependent on 24 parameters and so these figures agree in number of constants, according to an expression of SCHUBERT. If on this point we examine the configuration deduced from the nine points of S_7, it is even evident by merely consulting the numbers of constants 63 and 12 of the nine points and of a line in S_7, that it is impossible to determine the 280 right lines (123, 456, 789) by some of them crossing each other;

for 63 is not divisible by 12. In the second example of the twelve points in S_{10} this is likewise proved for the configuration of the 13305600 lines from the fact, that 18 is not a factor of 120. However, the numbers of constants of plane and four-dimensional space in S_{10}, namely 24 and 30, being factors of 120, considerations of another kind only can teach us that the 15400 planes and 10395 spaces S_4 cannot be determined by some of them crossing each other. So a characteristic difference between these two examples in S_{10} and the configuration of SEGRE would already appear if it was proved that five planes in a narrower sense do not lead to 55 planes through the 220 points, and four spaces S_4 in a narrower sense not to eleven spaces S_4 through the 66 points. And should this prove to be the case in one of the two, there still remains the difference that in S_4 five lines are found intersecting six arbitrary planes and that these lines are related in such a way that each line cutting four of the five lines also cuts the fifth; whilst according to a general formula of SCHUBERT (*Mitt. der math. Gesellschaft in Hamburg*, vol. 2, pag. 87, 1883) in S_{10} are found not only 55 but 116848170 planes and not only 11 but 689289872070 spaces S_4, which have respectively a point in common with each of the 24 arbitrary spaces S_7 and with each of the 30 arbitrary spaces S_5.

Finally we must stated that H. W. RICHMOND has made the figure of the six arbitrary points in the space S_4, called by him "hexastigm", the subject of two papers (*Quarterly Journal of Math.*, vol. 31, pag. 125—160, 1899 and *Math. Ann.*, vol. 53, pag. 161—176, 1900). In these important studies the configuration of SEGRE and its simplest analytical representation is brought into close connection with G. VERONESE's theorems about the PASCAL hexagram; but, comparatively spoken, the curved space of SEGRE is only cursorily mentioned.

Chemistry. — Professor LOBRY DE BRUYN presents a communication from Dr. J. J. BLANKSMA: "*On the influence of different atoms and atomic groups on the conversion of aromatic sulphides into sulphones.*"

It is known that organic sulphides may be converted first into sulphoxides and then into sulphones by means of oxidising agents, nitric acid for instance. From the following observations it appears to what extent atoms or groups which occupy an ortho-position in regard to the sulphur atom, render the latter unoxidisable.

If picrylsulphide is heated on the waterbath with nitric acid of 1.52 sp. gr. the unchanged substance is obtained on diluting with water, while tetranitrodiphenylsulphide (1. 3. 4) when treated in the same manner, is entirely converted into the corresponding sulphone. Picrylsulphide may even be heated with the same nitric acid for some hours at 200° in a sealed tube without undergoing any change.

If we look into the formula of picrylsulphide we notice that four nitro-groups occupy an ortho-position in regard to the sulphur atom. It was therefore probable that the sulphur atom is protected by these nitro-groups against the oxidising action of the nitric acid; a case, therefore, of so-called sterical obstacle (hindrance).

After this had been confirmed the following questions arose: 1°. Are perhaps two nitro-groups, when occupying an ortho-position in regard to the sulphur atom in one of the benzene nucleï, alone sufficient to prevent the oxidising action of the nitric acid? This was not quite improbable after VICTOR MEYER's researches on esterification and those of HOLLEMAN on the protective action of ortho-placed nitro groups during the action of Cl and Br on 2.6. dinitrotoluene etc. [1]. 2°. Are there other atoms or atomic groups which exercise the same influence?

To answer the first question trinitrothioanisol, prepared by the action of methyliodide on sodium thopicrate, was treated with nitric acid. The trinitrothioanisol m.p. 98° was converted into a substance with m.p. 210° whilst the analysis showed that oxygen had indeed been absorbed; the two ortho-placed nitro-groups alone were consequently insufficient to protect the sulphur atom against the action of the nitric acid.

To answer the second question two of the ortho-placed nitro-groups in the picrylsulphide were replaced by cyanogen in such a manner that a CN-group was present in each of the benzene nucleï. This compound seemed also to be stable towards nitric acid. Afterwards the compound was tested in which two nitro-groups of the picrylsulphide were replaced by CH_3 This compound also was not changed by heating on the waterbath with nitric acid of 1.52 sp.gr. It therefore appears that the cyanogen and the methyl-group as well as the nitro-group can protect the sulphur atom against the oxidising action of the nitric acid.

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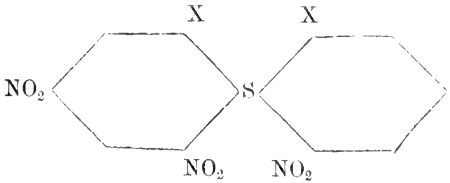
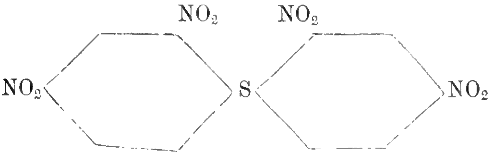
[1] Compare SCHOLTZ, Raumerfüllung der Atomgruppen.

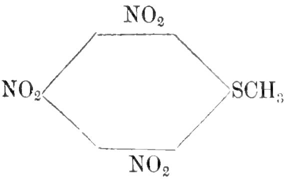
Formation of sulphone does therefore not occur with:



in which X may be NO_2, CN or CH_3; it takes place however, with:



and with:



The influence of other atoms and atomic groups such as Cl, Br, COOH and also of three ortho-placed groups will be further investigated.

Amsterdam, Nov. 1901. Org. Chem. Lab. Univ.

Physiology. — *"On function and structure of the trunkdermatoma."*

By Prof. C. Winkler from researches made in connection with Dr. G. van Rijnberk.

The basis on which is founded the actual physiological knowledge of the innervation of the skin by the posterior roots, through which we are taught in what manner the area's of those roots are ranged on trunk and extremities, is to be sought in most ingeniously conceived and eminently well-executed vivisections, made by Ludwig

TÜRCK [1]) in 1856 on dogs, and by SHERRINGTON [2]) from 1893—1900 on monkeys.

Careful dissections of the nerves of man and monkey, have led BOLK [3]) to represent a schema of the arrangement of the area's of the posterior roots — the dermatomata — which, arrived at by anatomical proceedings, offers great analogies with the results of the above-mentioned physiological researches.

Clinical experience too, registering methodically the sensory troubles, found in individuals with organical lesions of the posterior roots, set up a topography of the dermatomata, (by the researches of ROSS [4]), ALLEN STARR [5]), THORBURN [6]), KOCHER [7]), HEAD [8]), WICHMANN [9]), agreeing well enough with the representations of TÜRCK, SHERRINGTON and BOLK.

All these researches consider the dermatomata as unities and agree to a certain degree with one another as to the way in which they are ranged on trunk and extremities, but they hardly teach us anything about the constitution of that unity itself or about the manner of its functioning.

And yet the supposition that the posterior root represents in itself a unity, is by no means proved beforehand, because it is composed of different bundles (in the animals we experimented upon, from

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[1]) C. WEDL. Ueber die Haut-Sensibilitäts Bezirke der einzelnen Rückenmarkspaare von weilen Prof. Dr. L. TÜRCK. Aus dessen litterarischen Nachlässen zusammengestellt. Denkschriften der Wiener Akademie 1869. Vol. 29. S. 299.

[2]) CHARLES S. SHERRINGTON. Experiments in examination of the peripherical distribution of the fibres of posterior roots of some spinal nerves. Phil. Trans. R. S. 1893. B. Vol. 184.

CHARLES S. SHERRINGTON. Experiments in examination of the peripherical distribution of the fibres of the posterior roots of some spinal nerves. Part. II. Phil. Trans. R. S. 1898. B. Vol. 190. p. 45.

[3]) LOUIS BOLK. Een en ander uit de segmentaal-anatomie van het menschelijk lichaam Ned. Tijdschr. voor geneesk. 1897. 12 Juni en 4 Sept. I. pag. 982. II. p. 365, voorts LOUIS BOLK in Morph. Jahrb. XXII, XXIII, XXV, XXVI, XXVII, XXVIII.

[4]) JAMES ROSS. Distribution of anaesthesia in cases of disease of the branches and of the roots of the bracchial plexus. Brain. Vol. 7. 1885. p. 66.

[5]) ALLEN STARR. The surgery of the spinal Cord. 1889.

ALLEN STARR. Local anaesthesia as a guide in the diagnosis of lesions of the upper portion of the spinal cord. Brain. Vol. 17. 1894. p. 483.

[6]) WILLIAM THORBURN. On injuries of the cauda equina. Brain. Vol. 10. 1888. p. 381.

[7]) KOCHER. Grenzgebiete der Medicin.

[8]) HENRY HEAD. On disturbances of sensation with especial reference to the pain of visceral disease. Brain. 1893. Prt. I and II. p 1 etc.

[9]) R. WICHMANN. Die Rückenmarks-Nerven und ihre Segmentbezüge. Wiesbaden.

4—7) and the question arises whether the root-bundles, whose common area of innervation is the dermatoma, do not possess as just a claim to be called unities. At any rate we desire to know how each root-bundle separately behaves itself in reference to the whole of the dermatoma, the more so now one of us [1]) has demonstrated that clinical experience demands a still further analysis of the dermatomata. The question whether a separate area of innervation within the dermatoma ought to be attributed to each root-bundle, and if so, whether these area's of root-bundles ought to be ranged therein cranio-caudally or dorso-ventrally, has to be solved experimentally.

For this reason there have been made in the laboratory for neurology at Amsterdam, a great number of experiments on dogs, in view of solving this question for the trunk-dermatomata of that species.

First of all we must observe that the form of the trunk-dermatoma, as it has been deduced from the experiments of Türck and Sherrington, and from the dissections of Bolk ought to remind one of a trapezium stretched longitudinally. Because the dorsal part of the trunk from the 1st to the 12th vertebra of the chest, is much shorter than that part of the trunk situated ventrally between manubrium sterni and symphysis pubis, the base of the trapezium lying adjacent to the mid-ventral-line must be stretched and thus made longer than the side lying adjacent to the mid-dorsal-line.

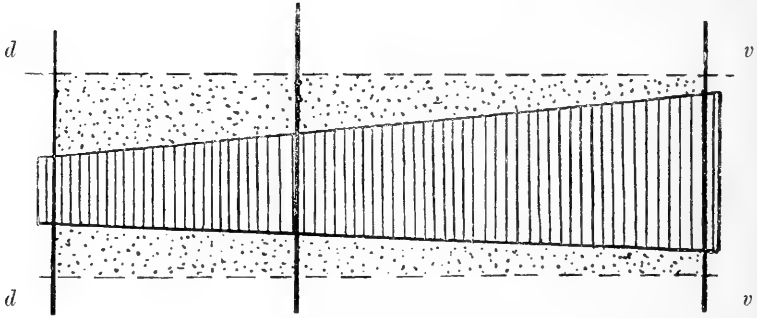

fig. 1.

Ideal form of the (arched) dermatoma, between two (dotted) insensible area's [2]).

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[1]) C. Winkler. Feestbundel Dr. Sape Talma aangeboden. Klinische bijdragen tot de kennis der wortelinnervatie van de huid.

[2]) In all these figures arbitrary demarcation-lines have been traced, limiting the sensible area above and below. The exact situation of these demarcations will occupy us further on.

Supposing the animal to be cut open along the mid-ventral-line and the skin from the right half of the body bent to a level, in such a manner that the mid-ventral line is situated to the right, then the form of the ideal dermatoma isolated by SHERRINGTON may be represented in the manner indicated by fig. 1.

What exceeds the mid-dorsal (*d*) and mid ventral (*v*) lines then signifies SHERRINGTON's crossed overlapses. This ideal form however is never to be met with in dogs, when a trunk-dermatoma is isolated in the manner indicated by SHERRINGTON, by cutting through two or more roots above and beneath the root which is left intact.

The results obtained in this way were not always the same. What follows here gives an account of them.

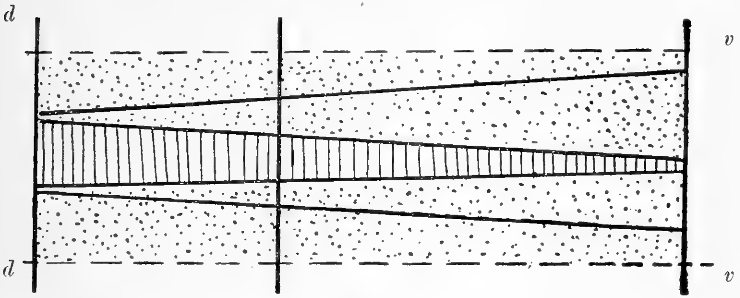

fig. 2.
The isolated sensible (arched) between two insensible (dotted) area's showing an important narrowing of the ideal dermatoma-area.

1. Between two insensible area's, there may continue to exist a sensible area, broadest towards the mid-dorsal-line and narrowest at the mid-ventral-line and therefore much smaller than the ideal dermatoma-area, and different in form. (fig. 2).

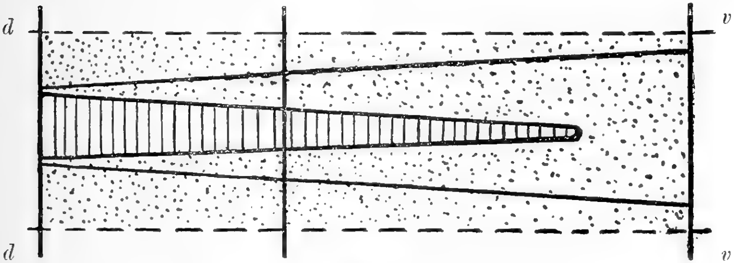

fig. 3.
Further narrowing of the sensible dermatoma-area.

2. Or the sensible area situated between two insensible area's may take the shape of a stretched triangle, whose base is lying adjacent to the mid-dorsal-line and whose top does not reach the mid-ventral-line (fig. 3).

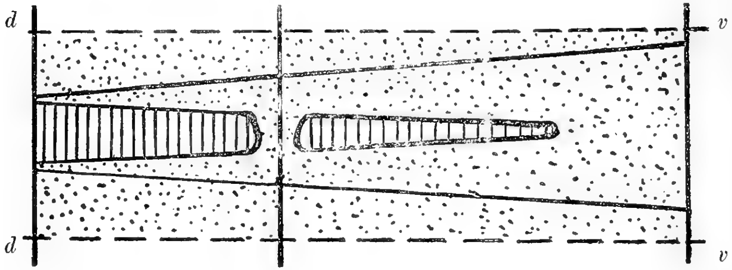

fig. 4.

Narrowing of the sensible dermatoma-area with transversal division into a dorsal and a ventral piece of the dermatoma.

3. Or the sensible area situated between two insensible area's not only does not reach the mid-ventral-line, but forms even no longer an uninterrupted area.

It is divided into two parts by a transversal insensible zone. The isolated ventral piece of dermatoma may in this way have become either greater or smaller (fig. 4).

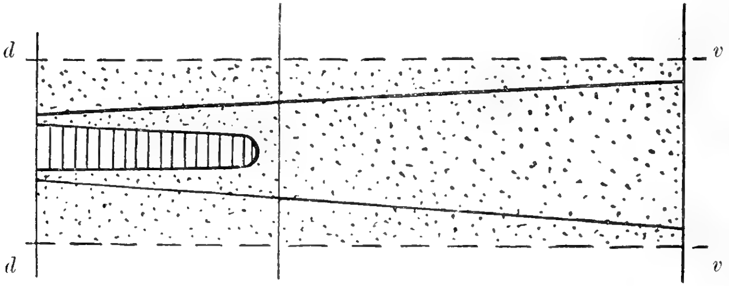

fig. 5.

Dorsal piece of the dermatoma.

4. Or the sensible area situated between two insensible zones, above and beneath it, is found only next to the mid-dorsal-line in the form of a short triangle. Of a ventral piece of dermatoma there is no longer any trace to be found. (fig. 5).

5. Lastly it may sometimes be stated, that only next to the mid-dorsal-line some remnant of sensation is left, localised in single spots as indicated by fig. 6, surrounded by a broad insensible area.

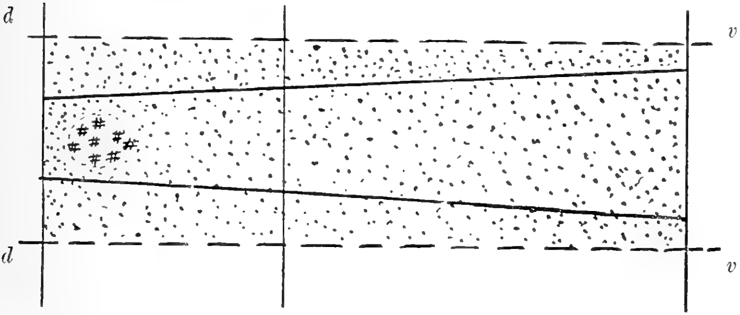

fig. 6.
The part of the dermatoma retaining sensation longest.

As soon as in the posterior root of the dermatoma, isolated in the manner indicated before, one or more bundles are cut through, results are obtained similar to those mentioned above and represented in fig. 2, 3, 4, 5 and 6.

The narrowings of the dermatoma-area's to triangular zones with or without transversal division, are also found, if any bundles of the intact root are sectioned, it making no difference whether caudal or cranial bundles of that root have been cut through. It must however be understood, that in proportion as greater experimental ability is acquired in isolating the trunk-dermatomata, a simple isolation will in most cases offer the sensible dermatema-area described sub. 1 (fig. 2), whilst the cutting through of bundles in the posterior root of this dermatoma generally will present one of the area's described sub 3—5.

It may thus be rightly supposed that the more or less important narrowings, or even shortenings with transversal division of the dermatomal area's, often to be found after simply isolating the dermatoma must be accounted for by unintentional lesions of the root left intact or of the medulla (profuse hemorraghe, pressure of coagulated blood etc.). There exists only a difference in degree with the results obtained by the intentional lesion, effected by partly cutting through the root-bundles.

One single spot of the dermatoma, lying adjacent to the mid-dorsal-line, preserves sensation longest, in whatever way the operation may be made.

This spot justly deserves to be called the "ultimum moriens" of the dermatoma.

These facts may induce us firstly to distinguish in the dermatoma between a central area, having the form of a knotted triangle the base of which is lying adjacent to the mid-dorsal-line and the narrow side directed towards the mid-ventral-line, and the remainder of the dermatoma, extending on both sides (cranio-caudally) of the first area, and which may be called the marginal area. The latter may be represented by two triangles, whose bases are lying next to the mid-ventral-line, and whose tops next to the mid-dorsal-line, whilst the central area is enclosed between them (fig. 7).

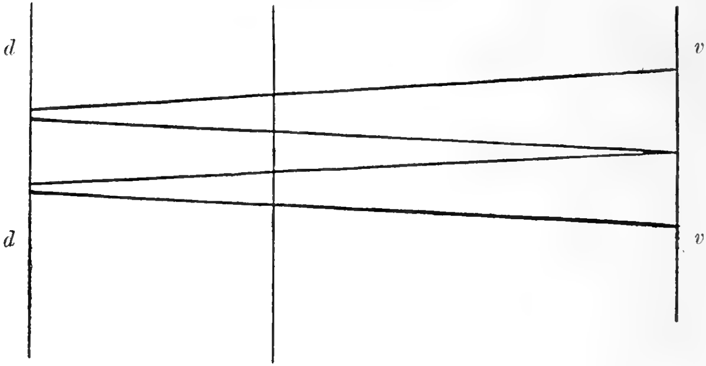

Fig. 7.

This distinction is necessary. The central area, as it is clearly demonstrated in favourable cases by the isolation of the dermatoma, is by itself sufficient for sensation, without assistance from the surrounding dermatomata. The marginal area by itself is insufficient for sensation. It can only be made sensible by means of the central or marginal area's of neighbouring dermatomata, by which it may be overlapsed.

Evidently the marginal area is not to be identified with SHERRINGTON's antero-posterior overlapses; for the central area's too overlap one another partly. Still there may be found in SHERRINGTON's researches an indication pointing towards the schema demonstrated here: where he states that the periphery of the dermatomata often possesses only a very imperfect sensation, and sensation diminishes rapidly from the centrum to the periphery.

The marginal area possesses too great a threshold-value to preserve any perceptible sensation when isolated. Because this part does not possess any sensation, as long as it is functioning separately (without assistance from neighbouring central or marginal area's) it cannot be demonstrated by isolation.

Secondly, the transversal division that may be observed and the manner in which the dorsal piece of dermatoma presents itself as "ultimum moriens", compels us to accept for the central area at least two maxima of sensation, one situated next to the mid-dorsal line, the other next to a lateral line of demarcation. If therefore the intensity of sensation is expressed in ordinates, that will rise higher, in proportion as sensation is longer preserved, on abscisses, taken on axes parallel to the mid dorsal and the mid-ventral lines, then those thus formed curves situated to the left (*a*, figure 8) represent the sensation in the central- and marginal area's. If taking the axisline of the central area for absciss, the intensity of sensation is put thereon as ordinate, then the two-topped curve *b*, will represent the sensation in this axisline. Both groups of lines then represent a schema of the distribution of sensation in the dermatoma.

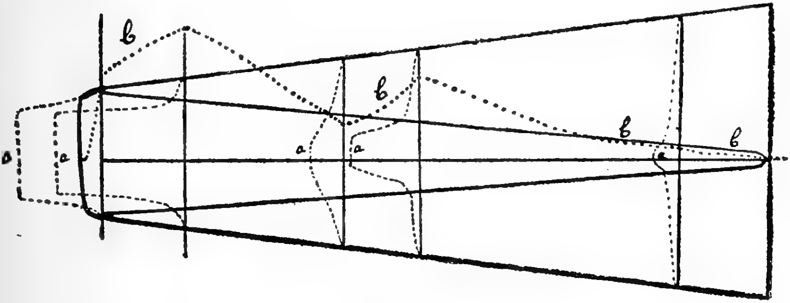

Fig. 8.

This unequal distribution of sensation in the dermatoma is the more remarkable, because already Voigt believed to have demonstrated by means of anatomical proceedings a lateral minimum of innervation.

Finally we are bound to conclude that every bundle of roots exerts an influence on the whole central area, as this area can preserve sensation over its whole extension, it making no difference whether root-bundles situated cranially or caudally have been sectinoned. That in reality every bundle of roots exerts its influence on *the* *whole of the dermatoma*, will be made clear by the two following experiments.

Experiment of September 22th on a brown male dog. The medulla is laid bare, and the dura mater opened. The originating bundles of the 19th, 20th, 21th and 22th posterior roots then were distinctly visible. On both sides a few caudally situated bundles of each of these roots are cut through, whilst the bundles situated cranially are left intact.

The next day there is found on abdomen and back a not very strictly defined hyperalgetic area, corresponding to the extension-area's of the four partly injured roots. Nowhere however, analgesia can be found.

As counterpart to this experiment, another is made Oct. 22th on a white female dog. The cranial bundles of four posterior roots (the 17th, 18th, 19th and 20th) are cut through, whilst the cadual ones remain intact.

The result on the next day is again a hyperalgetic area, now perfectly well defined, roundabout the trunk, with strong hyperreflectory reaction, nowhere however in that area analgesia can be pointed out.

A few bundles only, may they be caudal or cranial (provided the normal conditions of overlapping of the dermatomata do exist), are sufficient for the innervation of the whole dermatoma, and make sensation possible everywhere. If they are cut through in an isolated dermatoma, then sometimes the whole central area may appear, but in most cases those narrowings, resp. transversally divided shortenings of that central area, represented in fig. 3, 4, 5 and 6. It must be understood withal that as long as one bundle remains intact, at least in that part of the dermatoma lying adjacent to the back, sensation is preserved

Astronomy. — "*Contributions to the determination of geographical positions on the West-coast of Africa.*" By C. SANDERS. (Communicated by Dr. E. F. VAN DE SANDE BAKHUYZEN).

I. *The instruments used. Determinations of the corrections and the rates of the chronometer.*

The hope expressed in my paper, read at the meeting of the Academy of January 1900, of soon having at my disposal a small universal instrument, by means of which a much higher degree of accuracy could be given to my future observations, has been fulfilled. In August 1900, at Chiloango, my present station, situated on the mouth of the Chiloango-river in Portuguese West-Africa I received a universal instrument made by G. Heyde at Dresden, and in this paper I take the liberty of communicating to the Academy the results of the observations for the determination of latitude and longitude of Chiloango made till now with this instrument.

The instrument has circles of 14 c.m. in diameter, which both are movable and hence different parts of the limb may be used. The circles are divided to 10′ and each of them is read by two microscopes. The distance of the threads of the micrometer screws is such that two revolutions correspond with an interval of 10′ and each micrometer head is divided into 60 parts so that one division represents 5″, of which tenth parts may be estimated. The telescope placed at the end of the horizontal axis is 28 m.m. in aperture and 25 c.m. in focal length and has a magnifying power of 30 times. The reticule consists of 2 horizontal threads at a distance of about 50″ and 7 vertical ones, it may however be turned 90°.

The value of a division of the two levels is 5″ according to the instrumentmaker, and as till now the instrument has been used almost exclusively for observations of zenith distances, I have for the present investigated the alidade-level only. This level is graduated from 0 to 25. For readings for the middle of the bubble between 9 and 16 (the length of the bubble being here always equal to 9—10 parts) I have found for the value of a part on 3 days:

$$\begin{array}{rr} 1\text{ p.} = & 5''.14 \\ & 5\ .50 \\ & 5\ .57 \\ \hline \text{mean result } 1\text{ p.} = & 5''.40 \end{array}$$

Towards the ends on the contrary the value of a part was found to be about 3″.5.

As the soil about Chiloango is so elastic as to render the observations very difficult, if the instrument were mounted on a wooden or iron tripod, I had a stone pier made. This pier is surrounded by an insulated floor, and is protected by a zinc roof which can be opened in two directions and can moreover be moved, so that all parts of the heavens may be observed without much difficulty.

That in this way I have succeeded in rendering the stability of the instrument very satisfactory, may be seen from the results of the transit observations in the meridian which I have lately made for finding the time.

The observations of 6 October may serve as an instance. Within a space of 1 hour and 15 min. 9 complete levelings of the horizontal axis were made and the greatest difference between the results obtained amounts to $3''.0$; as, however, no regular variation can be seen, it is probable that these differences are due for the greater part to errors of observation.

To facilitate the mounting of the instrument in the required position, I determined the azimuth of the harbour-light situated 4 kilometers to the south and found therefor $1^\circ 23' 7''$.

Before the arrival of the universal instrument I used a sextant and an artificial horizon for finding the time and the latitude. It is obvious that these earlier observations are of no value for determining the latitude which has been found much more accurately by means of the universal instrument. Yet from another point of view they deserve notice. The sextant used was a different instrument from the one with which I formerly observed at Ambriz and San Salvador. Its errors have been investigated in 1899 by Mr. L. ROOSENBURG, director of the Amsterdam branch of the Netherland Meteorological Institute; they were found to be lying between $+ 7''$ and $- 6''$.

The chronometer used was one of THOMAS HEWITT; its rate was tested at regular intervals by determinations of time.

Except of late these determinations were made by means of observations of zenith distances at great hour angles, first from March till August 1900 with the sextant, after that time with the universal instrument. With the latter instrument I always observed the zenith-distances of stars. The way in which I made these observations will be described in the next section. Here I only give the formulae borrowed from "CASPARI *Astronomie pratique*", by means of which the mean hour angle was derived from each pair of observations made in the two positions of the instrument, using only an approximate value of the zenith-point.

Let z and z' be the two zenith distances, t and t' the corresponding hour angles; let

$$\cos Z = \cos \tfrac{1}{2}(z - z') \cos \tfrac{1}{2}(z + z')$$

and T the hour angle corresponding to the zenith distance Z, then we have finally:

$$\cos \tfrac{1}{2}(t + t') = \frac{\cos T}{\cos \tfrac{1}{2}(t - t')}$$

Whereas the observations were always made at large zenith distances, $z - z''$ is small, therefore we see that an approximate knowledge of this difference and accordingly also of the zenith-point will be sufficient [1]).

I always made several pairs of pointings; sometimes two stars have been employed.

Here follow the results obtained for the correction of my chronometer and the daily rates derived from them for the period including my observations for latitude and longitude discussed in the following sections.

The corrections of the chronometer as given here are corrections on the mean time of Chiloango. For converting sideral into mean time the longitude of Chiloango was taken to be: 48 min. 32 sec. east of Greenwich. In January 1901 I began noting down the temperature once a day, namely at 9 o'clock in the morning; therefrom the mean temperatures have been derived for the periods between the time determinations. They are given in the last column of the table.

If we leave out one interval of only one day, the agreement of the daily rates appears to be very satisfactory. The fact that here temperature varies little and only very slowly, will certainly have contributed to this result. Yet it is clearly visible that smaller daily rates correspond to lower temperatures and this is confirmed by the observations before October 1900; for between 15 August and 5 September the mean daily rate was $+ 0.^{s}01$, and between 5 September and 5 October $+ 0.^{s}26$.

Afterwards it will be possible to apply small corrections to the results given here, especially for the division errors of the vertical circle. For the determination of the latitude the errors left as yet in the chronometer corrections are of no consequence and for the approximate reduction of the observations for longitude, which is as yet

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[1]) As $\cos \frac{1}{2}(z - z')$ and $\cos \frac{1}{2}(t - t')$ differ little from unity we may also in a known way compute the differences between Z and $\frac{1}{2}(z + z')$ and between $\frac{1}{2}(t + t')$ and T directly by means of approximate formulae.

the only one possible, they are of little moment. At a final computation of the latter observations, however, it will be necessary to account for them.

DATE.	M. T.	CHRON. CORR.	D. R.	TEMP.
1900 Oct. 5	6h 8	+ 54 m 41s 33		
			+0s 48	
» 16	6.3	46.58		
			+0.50	
Nov. 28	7.9	55 7.98		
			+0.18	
» 30	8.8	8.34		
			+0.58	
Dec. 8	8.3	12.98		
			+0.64	25.0
1901 Jan. 9	6.6	33.34		
			+0.64	24.8
» 18	8.2	39.12		
			+0.83	26.0
» 25	8.8	44.94		
			+0.84	25.8
Febr. 6	8.7	54.98		
			+0.91	26.5
» 12	8.4	56 0.43		
			+1.04	26.0
» 19	8.2	7.74		
			+0.87	26.2
» 28	8.4	15.53		
			+0.94	25.8
April 17	6.2	57 0.84		
			+0.75	25.2
» 19	7.7	2.34		
			+0.85	26.2
» 25	9.2	7.41		
			+0.99	26.1
» 28	8.0	10.38		
			+0.86	25.7
May 9	8.1	19.84		
			+0.48	25.1
» 12	8.4	21.27		
			+0.68	24.7
» 21	8.2	27.42		
			+1.48	25.6
» 22	8.6	28.92		
			+0.73	24.7
» 30	8.3	34.74		
			+0.60	24.1
June 10	7.6	41.34		
			+0.53	23.6
» 17	7.2	45.03		
			+0.29	21.8
July 5	7.7	50.16		
			+0.18	21.0
» 16	7.8	52.16		
			+0.24	21.0
» 25	8.3	54.35		
			+0.09	21.0
Aug. 7	8.4	55.54		

II. *Determination of the latitude of Chiloango from circummeridian altitudes.*

To determine the latitude of Chiloango a great number of observations of circummeridian altitudes of stars have been made. For these observations the vertical circle was turned successively in different positions and in each position one or more pairs of stars were observed, each consisting of a northern and a southern star culminating as near as possible (within a few degrees) at the same zenith distance. In this way we already eliminate in each position of the circle besides the flexure of the telescope a great part of the periodic division errors, while the influence of the latter, especially to be feared with a small instrument, is further diminished by the combination of the different positions.

At first observations were made in the 4 positions, indicated by zenith-point 0°, 90°, 180° and 270°; of these however the 1st and 3rd, and also the 2nd and 4th bring the same divisions under the two microscopes. After this the two positions 225° and 135° were added. But before the observations in the last position had completely succeeded, I resolved to start an entirely new series of observations, consisting of the 4 positions 0°, 45°, 90° and 135°, which moreover allowed me to improve upon the arrangement of the observations themselves.

Hence my observations may be divided into two series. The second series by its arrangement would seem to be the more accurate; it was found however, that the weight of the first series was not so much less. Therefore I have only rejected my very first observations, and of the later ones only those where the pointings had succeeded in one position of the instrument only.

I now proceed to describe the arrangement of my observations. The star was always bisected by the horizontal middle thread, or in the position the reticule had before February 1901, alternately by the two horizontal threads. Near the meridian the pointings were made by means of the screw for slow motion; but whenever possible I let the star come on the thread by its own motion and determined the moment of bisection. The readings were always made with each of the two microscopes on the preceding and on the following division. Care has always been taken that the bubble of the alidade level did not deviate more than three divisions from the middle; in the second series the deviations were generally even smaller and were equalised as much as possible in the two positions of the instrument. This was much facilitated by the stability of the mounting.

As already stated before, the pointings have always been made in the two positions of the instrument, and though the zenith-point has also been determined a few times by means of the harbour-light, the results thus obtained have not been used in the computation. On an average a complete observation consists of 5 pointings in each position; a few times clouds prevented me from making more than two. In the first series the instrument was usually reversed only once in the middle of the observation, in the second series the pointings were made alternately in both positions. Symmetry with respect to the meridian was secured as much as possible and I never observed in hour angles greater than 30 min.

For the reduction of my observations all microscope readings were first corrected for the error of runs. The amount of this error was always derived from all the observations of the night, and the required corrections were computed by means of small tables. For the level correction, the value of a division could always be taken = 5″.40.

The reduction to the meridian was computed by means of the usual development in series; it was always sufficient to retain only the terms depending on $sin^2 \frac{1}{2} t$ and $sin^4 \frac{1}{2} t$. The refractions were computed from CAILLET's tables in the *Connaissance des Temps*; afterwards small corrections have been applied to reduce them to the tables of RADAU. The barometric pressure was read on an aneroid, of which the correction for a temperature of + 25° amounts to — 1.5 mM. as derived from a comparison with a barometer of FUESS. Except in one case ($z = 62°$) the zenith distances at which the observations were made lie between 40° and 60°. The star places are taken from the Nautical Almanac and therefore rest on the catalogue of NEWCOMB.

As an example I give here the original data for one observation and choose therefor out of the second series the first at which six pairs of pointings were made, viz. that of β Crucis on May 31. The first column of the following table gives the positions of the instrument designated by R and L as the telescope and circle were either to the right or to the left of the observer at the ocular, the second column gives the chronometer times of the pointings; the third and fourth give the readings of the two divisions at microscope A, the fifth and sixth give the same for microscope B, the seventh gives the readings of the level.

POS.	CHRON.	MICROSC. A		M.B		LEV.	HOUR A.	LATIT.
R	$6^h\ 43^m\ 32^s$	189° 10′ 55″	59″	56″	54″	14^p1	$26^m\ 35^s8$	— 5°12′
L	49 34	80 55 37	39	57	58	12.4	20 32.8	4″23
R	54 42	189 1 27	25	27	25	11.7	15 23.9	
L	59 50	81 2 8	8	29	27	12.2	10 15.1	4.10
R	7 4 42	188 57 6	11	12	12	11.5	5 22.3	
L	9 25	81 4 13	18	36	33	12.0	0 38.5	1.25
R	15 1	188 56 54	58	56	55	13.1	4 58.4	
L	19 22	81 2 29	31	49	51	11.7	9 20.1	2.25
R	23 41	189 0 17	19	19	16	13.1	13 39.8	
L	28 13	80 57 30	29	54	50	11.7	18 12.6	3.34
R	32 34	189 6 59	60	62	60	12.3	22 34.3	
L	37 32	80 48 29	33	54	50	13.7	27 33.6	3.26

From this observation and from one of η Ursae Majoris on the same night I derived as value of the run correction for 10′ and for the mean of the two microscopes + 0″.68. The temperature and the barometric pressure were found to be 25°.3 C. and 760.7 respectively, while for the chronometer correction $+ 57^m\ 35^s.5$ was accepted.

These data allow us entirely to reduce the observations. In the table above I have added in the eighth column for each pointing the computed hour angle, and in the ninth the value for the latitude derived from each pair of pointings, from which we obtain as mean result — 5°12′ 3.07″.

Leaving out of consideration the observations discarded on the principles stated above, we retain 23 observations, which reduced in the manner explained yield the mean results given in the table below. In this table the first column shows the position of the circle indicated by the reading for the zenith, the 2d and 3d give the date of the observation and the star observed, the 4th and 5th the readings in the two positions *R* and *L* in full degrees, the difference of which is twice the zenith distance, the 6th the number of double pointings made and the 7th the mean result for the latitude.

ZENITH P.	DATE	STAR	R	L	P	LATITUDE.
1st Series.						
0°	1900 Dec. 2	α Eridani	53°	307°	6	— 5°11′55″88
	1901 Jan. 11	ε Persei	45	315	3	68 77
	Febr. 11	β Aurigae	50	310	5	67.77
	〃 12	Canopus	47	313	4	55.87
90	Apr. 29	θ Argus	148	31	8	11 69.05
	〃 〃	γ Ursae maj.	149	30	4	60.36
180	〃 25	μ Ursae maj.	227	132	2	11 68.72
	〃 〃	μ Argus	223	136	4	56.90
270	1900 Oct. 11	α Gruis	312	228	4	11 68.65
	1901 Apr. 19	μ Argus	313	226	8	68.00
	〃 20	λ Ursae maj.	318	221	5	60.79
	May 2	β Ursae maj.	332	208	6	59.54
225	〃 13	γ Ursae maj.	284	166	5	11 64.78
	〃 〃	γ Crucis	276	174	5	65.43
2d Series.						
0°	1901 June 12	η Ursae maj.	55°	305°	4	— 5°11′68″36
	〃 15	β Centauri	55	305	6	59.04
	〃 〃	α_2 Centauri	55	305	2	58.86
45	May 9	γ Crucis	96	353	3	11 65.34
	June 10	η Ursae maj.	100	350	5	63.28
90	〃 5	β Crucis	144	36	4	11 67.29
	〃 6	η Ursae maj.	145	35	6	61.15
135	May 31	β Crucis	189	81	6	11 63.07
	〃 〃	η Ursae maj.	190	80	5	64.90

From these results mean values must be computed for each position of the circle (i. e. combining 0° and 180°, and 90° and 270°), separately for the northern and the southern stars.

Before combining the two series for this purpose, I have derived for each the mean error of the result of a double pointing by comparing these separate results with the mean result for the whole observation. In the first series I have combined for this purpose the first pointing with the last, the second with the last but one etc. A regular variation of the zenith-point during the observation will then reveal itself in the differences between the separate results. Contrary to the expectation an almost equal value was found for the mean error in the two series, namely:

1st series Mean error of 2 pointings ± 1″.51

2d „ „ „ „ „ „ ± 1″.59

Hence the influence of a variation of the zenith-point is not at all perceptible. Yet after all I thought better to assign a smaller weight to the observations of the 1st series, considering that for this series the uncertainty originating from the level may be greater, and that less attention had been paid to the equality of the zenith distances for the different observations. Thus I assigned to the mean result of the positions 0° and 180° together in the 1st series, the same weight as to the position 0° in the 2d, to 90° and 270° in the 1st series the same weight as to 90° in the 2d, and lastly to 225° in the first series half of that of 45° in the 2d. When two observations made under the same circumstances had to be combined, the number of pointings was not considered, except with the observations of β and α_2 Centauri on June 15, of which the results were combined with weights 2 and 1. Lastly the observation of β Ursae majoris on May 2 was rejected owing to the diverging zenith distance.

In this manner the results, given in the following table, were obtained. In this table column 2 and 3 give the mean zenith distance for the northern and the southern stars, column 4 and 5 the results derived for the latitude from these two groups, column 6 their difference and column 7 their half sum.

ZEN. P.	z. N.	z. S.	NORTH	SOUTH	N − S	$\frac{N+S}{2}$
			− 5°11′	− 5°11′		− 5°
0°	51°	51°	68″43	57″68	− 10″75	12′ 3″06
45	56	52	63.78	65.37	+ 1.59	4.58
90	54	52	60.86	67.99	+ 7.43	4.42
135	55	54	64.90	63.07	− 1.83	3.98

From the mean error of 2 pointings ± 1″.55 follows for that of the result of a star with 5 pairs of pointings, apart from the systematical errors, ± 0″.69, and for that of the mean result from a northern and a southern star ± 0″.49. Hence we see that there are considerable systematical differences between the results from the northern and the southern stars, which points towards considerable values for the systematical division errors, but at the same time we find confirmed that their influence is eliminated already for a great part in the mean value $\frac{1}{2}(N + S)$ for each position of the circle.

If once more we take the mean of the results of the different positions, we finally arrive at:

$$\varphi = -5^\circ\ 12'\ 4''.01.$$

If we had kept the two series separated until the end, we had found in the same way:

1st series — 5° 12′ 3″.85

2d „ — 5° 12′ 4″.05

or if we had used in the first series only the 4 positions 0°, 90°, 180° and 270°

— 5° 12′ 3″.54.

Hence it appears that after all it does not matter much which relative weights we assign to the two series and as final result for the latitude of my place of observation I consider the value:

$$\boldsymbol{\varphi = -5^\circ\ 12'\ 4''.0}$$

a value which probably will be not more than 1″ in error.

Although we have thus obtained a result as free as possible from the influence of division errors and flexure, I wanted also to know the value of these errors themselves, especially in order to be able to determine their influence on the chronometer corrections derived from my observations.

Let p be the zenith-point and z the zenith distance of a star; we can derive from each observation an equation in which occur the difference between the division errors for the points $p+z$ and $p-z$ and the flexure for the zenith distance z. But much simpler and with a sufficient degree of accuracy we may proceed as follows.

Let the correction for the periodical error of division in a reading α, be

$$a \sin 2\,\alpha + b \cos 2\,\alpha$$

as more than 2 terms cannot well be determined.

If we assume that in each position the mean zenith distance is 53° the influence of the errors of division on the arcs of 106° measured is:

$$a\,\{\sin (2\,p + 106^\circ) - \sin (2\,p - 106^\circ)\} + b\,\{\cos (2p + 106^\circ) - \cos (2p - 106^\circ)\}$$

$$= 2\,a \sin 106^\circ \cos 2\,p - 2\,b \sin 106^\circ \sin 2\,p$$

while the influence of the flexure of the telescope on the zenith distance will be $c \sin 53^\circ$. Hence we may represent the differences N—S found above, by a formula:

$$N - S = c' + a' \cos 2\,p + b' \sin 2\,p$$

and we find

$$c' = 2\,c \sin 53^\circ = -0''.96$$

$$a' = 2\,a \sin 106^\circ = -8''.94$$

$$b' = -2\,b \sin 106^\circ = +1''.71$$

whence:

$$c = -0''60$$

$$a = -4.65$$

$$b = -0.89$$

Finally we find:

Corr. for Division errors $= +4''.74 \sin (2\,\alpha - 169^\circ.\,2)$

Corr. for Flexure $= -0''.60 \sin z$

and it is found that after having applyied these corrections, the mean results for the different positions are in good agreement.

The correctness of these formulae may be tested by the results of an observation of γ Crucis on May 15, not considered before, in which the circle was brought successively in 6 positions differing 30° inter se. The following table gives the 6 results obtained directly as well as those after correction for division errors and flexure.

Zen. P.	φ uncorr.	φ corr.
0°	59″04	64″03
330	58.64	62.07
300	63.03	61.97
270	67.70	63.62
240	61.93	59.44
210	58.85	60.84

The corrected values agree as well as might be expected. The mean of them is 2.″00

Finally I have also corrected all the observations individually and these are the corrected results:

1st SERIES	
α Eridani	0″83
ε Persei	3.70
β Aurigae	2.72
Canopus	0.95
θ Argus	5.40
γ Ursae maj.	3.94
μ Ursae maj.	3.63
μ Argus	1.97
α Gruis	4.42
μ Argus	3.75
λ Ursae maj.	4.97
β Ursae maj.	2.87
γ Ursae maj.	5.05
γ Crucis	5.03
γ Crucis	2.00

2nd SERIES	
η Ursae maj.	3″50
β Centauri	3.90
α_2 Centauri	3.72
γ Crucis	5.02
η Ursae maj.	3.62
β Crucis	3.35
η Ursae maj.	5.03
β Crucis	4.39
η Ursae maj.	3.58

It would be possible to combine these results in various ways, yet I will only give the mean values resulting from the two series if we assign equal weight to each observation. We obtain in this way:

1st series 3″.42 ± 0″.38

2nd series 4 .01 ± 0 .22

By means of a simple triangulation I have connected my observation pier with the harbour-light and with the flagstaff of the residence at Landana. We derive from it the following differences of latitude:

Harbour-light—Obs. Pier	— 2′13″.2
Residence—Obs. Pier	— 1 14 .3

whence:

Latitude Harbour-light	— 5°14′17″.2
„ Residence	— 5 13 18 .3

The last value is in good harmony with that found by Mr. NUNEZ Portuguese Commissioner for the demarcation of the boundary between French and Portuguese Congo. According to a photography kept at Landana of the map drawn at that occasion, the latitude of the Residence is:

— 5° 13′ 17″.

On the English Admiralty-chart „Landana-bay from a sketch-survey of Sub Lieut. Robbins H.M.S. Widgeon 1897" the latitude of the harbour-light, which here is wrongly placed just in front of the residence, is — 5° 11′ 15″, hence 3′ too much to the north.

III. *Corrections to the latitude of Ambriz and of San Salvador.*

Among the observations with the sextant and artificial horizon, made during the first months of my residence at Chiloango, were also some of circummeridian altitudes of stars, and from them I derived the following values for the latitude:

from α Crucis	20	obs.	— 5° 12′ 19″.4 ± 2″.8
„ α Ursae maj.	20	„	— 5° 11′ 55″.2 ± 3″.2

The mean of these values is

$$-5^\circ\,12'\,7''.3 \pm 2''.1$$

in sufficient agreement with the result obtained above. On the other hand we find for the difference of the two:

$$\varphi \text{ Northern star} - \varphi \text{ Southern star} = +24''.2 \pm 4''.2$$

which result points to the existence of considerable systematical errors.

As the errors of my sextant had been investigated before (comp. above) and as the adjustment of the mirrors and the telescope was always tested, while moreover the readings for the two stars differed only 20°, I was led to conclude that the difference between the two values results from a personal error in making the two images of the star coincide. The value of this personal error would then be 12″.

If I conclude that this is the case and assume that such a personal error already existed in my observations in 1894 and 1895, then I have to apply corrections to a few results obtained from the zenith distances of α Crucis for the latitude of Ambriz and of San Salvador [1]). By doing so the agreement between these results and those derived from observations of the sun would much improve.

Therefore I hold this supposition not improbable and I will indicate how, if it is accepted, the values formerly adopted for the latitude of the two places must be changed.

For Ambriz we find (see l.c. pag. (405) 8) as the result of the meridian altitude of α Crucis: $-7^\circ\,49'\,57''.5$, while 6 observations of the meridian altitude of the sun give 57″.2 and the mean result becomes 57″.3 instead of 59″.0.

The mean result from the 3 series becomes then:

$$-7^\circ\,49'\,59''.6$$

instead of 60″.2, or it remains:

$$-7^\circ\,50'\,0''.$$

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[1]) C. Sanders, Determination of the latitude of Ambriz and of San Salvador (Portuguese West-Africa). Proc. Acad. Amsterdam, vol. II, p. 398.

For San Salvador the 3 corrected values (l. c. p. (406) 9) become:

Sun	— 6° 15′ 12″.9
„	10.0
α Crucis	14.0
Mean value	— 6° 15′ 12″

Thus the mean result for San Salvador changes 4″ and seems at the same time more reliable than before.

IV. *Preliminary determination of the longitude of Chiloango.*

To determine the longitude of Chiloango I applied the method recommended by KAISER [1]) for use in the Dutch Indies as it has been developed with regard to the reduction of the observations by Prof. J. A. C. OUDEMANS [2]), who himself has reduced in this way the observations of Messrs DE LANGE for determining the longitude of Batavia [3]).

I commenced my observations on January 22d of this year and since that date I have obtained on 9 nights 19 observations of equal altitudes of the moon and a star. All these times however I observed the first limb of the moon, as with the limited time I had at my disposal observations after the full moon did not yet succeed. But I hope to continue my work and then to be able to obtain also observations of the second limb.

In the first 7 observations, made in January, the times of transit were observed over the 2 horizontal threads; after the reticule had been turned 90°, the 7 horizontal threads were used. Therefore the first observations will be less accurate. All the observations were made with the telescope to the right, as a reversal of the instrument was of no use here. Care was always taken to keep the deviations of the level bubble as small as possible; the difference between the readings for the corresponding observations of the moon and the star was always less than one division, and generally much less.

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[1]) F. KAISER. De sterrekundige plaatsbepaling in den Indischen Archipel enz. Amsterdam 1851.

[2]) J. A. C. OUDEMANS. Over de bepaling der geografische lengte door de waarneming van gelijke hoogten der maan en eener ster. (*Versl. en Med. Akad. Amst.* 6e deel, pag. 25. Amsterdam 1857).

[3]) J. A. C. OUDEMANS. Verslag van den geographischen dienst in Nederlandsch-Indië van Januari 1858 tot en met April 1859. Batavia 1860.

Besides that the number of the observations obtained leaves still to be desired and especially that they are deficient in that they all relate to the same limb, owing to which circumstance the resulting value for the longitude may be considerably vitiated by my mode of observing the limb, there is still another reason why the results given here must be considered as only provisional. This reason is that not until later we shall be able to use in the computation places of the moon which are as accurate as possible. The provisional reduction was made by means of the data given in the Nautical Almanac, but the coefficients of the differential variations of these data were computed after the formulae of Prof. OUDEMANS, so that afterwards it will be easy to apply the required corrections. For the moment I have only applied provisional corrections for the mean error of the moon's Right Ascension, as found at Greenwich for the year 1900 [1]); from these observations we get: Corr. R.A. N. A. = + $0^s.13$.

In my reduction I started from the value for the longitude of Chiloango — $48^m 32^s$ i. e. $48^m 32^s$ east of Greenwich and until now my observations have yielded the following provisional corrections for this value:

			dL	Daily Means.
1901	Jan.	22	— $16^s.6$	
	»	»	+ 2.7	— $6^s.9$
	»	23	+ 5.2	+ 5.2
	»	25	+ 4.2	
	»	»	+ 3.5	
	»	»	+ 4 6	
	»	»	— 10.7	+ 0.4
	Febr.	25	+ 1.4	+ 1.4
	March	3	— 14.3	
	»	»	— 13.6	
	»	»	— 17.0	— 15.0
	May	1	— 13.9	— 13.9
	»	21	+ 5.4	
	»	»	— 5.0	+ 0.2
	»	23	— 3.1	
	»	»	+ 1.7	— 0.7
	July	22	+ 12.9	
	»	»	+ 13.5	
	»	»	+ 9.1	+ 11.8
Mean value			— $1^s.6$	— $1^s.9$

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[1]) Monthly Notices R. A. S. Vol. 61. p. 202.

Besides the individual results I have given in the table above the daily means and either the former or the latter may be combined with equal weights. Probably, as the results of one day agree better with each other than with those of other days, the latter method will be the best, but we find that here this makes little difference.

Hence I accept as provisional result of my observations of the moon:

Longitude of Chiloango — $48^{\mathrm{m}}\ 33.^{\mathrm{s}}9$

The differences between the daily means are still rather large, but we may expect that a final reduction will improve this.

Finally mention must be made that three times I had an opportunity of comparing my chronometer with Greenwich time as shown by the chronometers on board the ships that touched at this port. In that way I found for the longitude of my place of observation:

— $48^{\mathrm{m}}\ 37^{\mathrm{s}}7$
8.0
14.7
Mean value — $48^{\mathrm{m}}\ 20^{\mathrm{s}}1$

It will be seen that as yet no great weight can be assigned to the result of this mode of determining the longitude.

(December 24, 1901).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday December 28, 1901.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 28 December 1901, Dl. X).

The following papers were read:

Physics. — W. H. KEESOM: "*Contributions to the knowledge of* VAN DER WAALS' *ψ-surface. V. The dependence of the plait-point constants on the composition in binary mixtures with small proportions of one of the components.*" (Communication N⁰. 75 from the Physical Laboratory at Leiden by Prof. H. KAMERLINGH ONNES).

§ 1. In § 2 of Communication N⁰. 59*a* (Proceedings June 1900 p. 276) KAMERLINGH ONNES has drawn attention to the question as to how far, for the study of VAN DER WAALS' ψ-surfaces, we can avail ourselves of the law of corresponding states by applying this to the homogeneous phases (stable and unstable) of mixtures of two normal substances, especially, in how far from an empirically correct representation of the isothermals (and thus of the ψ-lines), at different temperatures for a simple substance we can find the ψ-lines for mixtures of different composition x at one temperature, and hence also the unstable part of the ψ-surface. In Communication N^0. 59*b* (in collaboration with REINGANUM) this method appeared permissible even with entirely arbitrary mixtures of carbon dioxide and methylchloride; also when the mixtures under observation differ only a little in composition, as when investigating the critical phenomena in a normal substance with small impurities, it promised to be very useful. For a long time (comp. Communication N^0. 68, Proceedings March 1901 p. 630) investigations for this work have been made at the Leiden Laboratory, in which I have taken part by making some measurements. As the first part of them we may consider VERSCHAFFELT's measurements on the conduct of mixtures of carbon dioxide and hydrogen, in so far as they concern mixtures with a small proportion of the latter substance. They show, as VERSCHAFFELT [1]) has demonstrated in Communication N^0. 65 that the law of corresponding states indeed is nearly applicable to these mixtures, and that by means of the said law, the critical temperature and pressure of the homogeneous mixtures can be determined.

In the following pages we will examine in the first place what may be derived from VAN DER WAALS' investigation of the mixtures, by the aid of the law of corresponding states, about the course in the pT-diagram of the plait-point curve of binary mixtures in the case that the proportion of one of the components of the mixture is very small. Then the formulae obtained are compared with VERSCHAFFELT's observations. Lastly by means of the law of corresponding states I have derived an expression for the variation of the plait-point volume with the composition.

§ 2. As a starting point some formulae given by VAN DER WAALS in the Proceedings May and June 1895, Arch. Néerl. t. XXX p. 266 and 278 are used, which formulae may be also derived using the property that in the plait-point:

$$\left(\frac{\partial^2 \zeta}{\partial x^2}\right)_{pT} = 0,$$

$$\left(\frac{\partial^3 \zeta}{\partial x^3}\right)_{pT} = 0,$$

where $\zeta = \psi + pv$, if ψ represents the free energy.

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[1]) Arch. d. Sc. Néerl. (2) t. V p. 644. Comm. Phys. Lab. Leiden n^0. 65.

The first of these equations states that the plait-point is situated on the spinodal line, the second that the isobar on the ψ-surface drawn through the plait-point, does not enter the plait.

VAN DER WAALS finds (Proceedings May 1895) for $x = 0$, a formula (5), which we write:

$$\left(\frac{dp}{dT}\right)_{pl} = \left(\frac{\partial p}{\partial T}\right)_v - \frac{MRT_k\left(\frac{\partial^2 p}{\partial v\, \partial T}\right)\left(\frac{\partial p}{\partial x}\right)_{vT}}{\left(\frac{\partial p}{\partial x}\right)_{vT}^2 + MRT_k\left(\frac{\partial^2 p}{\partial x\, \partial v}\right)_T} \quad . \quad . \quad . \quad (1a)$$

where it is taken into account that at the critical point of a simple substance

$$\left(\frac{\partial^2 \varepsilon}{\partial v^2}\right)_T = T_k\left(\frac{\partial^2 p}{\partial v\, \partial T}\right).$$

Besides VAN DER WAALS finds an equation (9) (Proceedings June 1895) which we write:

$$\frac{dT_{pl}}{dx} = -\frac{\left(\frac{\partial p}{\partial x}\right)_{vT}^2 + MRT_k\left(\frac{\partial^2 p}{\partial x\, \partial v}\right)_T}{MRT_k\left(\frac{\partial^2 p}{\partial v\, \partial T}\right)} \quad . \quad . \quad . \quad . \quad (1b)$$

for x equal to zero. From these two equations follows:

$$\frac{dp_{pl}}{dx} = \left(\frac{\partial p}{\partial x}\right)_{vT} - \left(\frac{\partial p}{\partial T}\right)_v \frac{\left(\frac{\partial p}{\partial x}\right)_{vT}^2 + MRT_k\left(\frac{\partial^2 p}{\partial x\, \partial v}\right)_T}{MRT_k\left(\frac{\partial^2 p}{\partial v\, \partial T}\right)} \quad . \quad . \quad (1c)$$

Here $\frac{dT_{pl}}{dx}$ and $\frac{dp_{pl}}{dx}$ indicate the variation of the critical plait-point temperature and pressure with small admixtures, T_k is the critical temperature of the pure substance.

We may now naturally introduce the law of corresponding states into the equations given above. By means of

$$p = \pi\, p_{xk}\,; \qquad v = \omega\, v_{xk}\,; \qquad T = \tau\, T_{xk}$$

where p_{xk}, v_{xk} and T_{xk} respectively represent: critical pressure, volume and temperature of the mixture taken as homogeneous, and therefore:

$$\pi = f(\omega, \tau),$$

we find

$$\left(\frac{\partial p}{\partial x}\right)_{vT} = \pi \frac{dp_{xk}}{dx} - \frac{p_{xk}}{v_{xk}} \frac{dv_{xk}}{dx} \, \omega \left(\frac{\partial \pi}{\partial \omega}\right) - \frac{p_{xk}}{T_{xk}} \frac{dT_{xk}}{dx} \, \tau \left(\frac{\partial \pi}{\partial \tau}\right),$$

$$\left(\frac{\partial^2 p}{\partial x \, \partial v}\right)_T = \frac{1}{v_{xk}} \frac{dp_{xk}}{dx} \left(\frac{\partial \pi}{\partial \omega}\right) - \frac{p_{xk}}{v^2_{xk}} \frac{dv_{xk}}{dx} \left(\frac{\partial \pi}{\partial \omega}\right) - \frac{p_{xk}}{v^2_{xk}} \frac{dv_{xk}}{dx} \, \omega \left(\frac{\partial^2 \pi}{\partial \omega^2}\right) -$$

$$- \frac{p_{xk}}{T_{xk} \, v_{xk}} \frac{dT_{xk}}{dx} \, \tau \left(\frac{\partial^2 \pi}{\partial \omega \, \partial \tau}\right).$$

This becomes for $x = 0$, as then $\pi = \omega = \tau = 1$,

$$\left(\frac{\partial \pi}{\partial \omega}\right) = 0 \text{ and } \left(\frac{\partial^2 \pi}{\partial \omega^2}\right) = 0 \ :$$

$$\left(\frac{\partial p}{\partial x}\right)_{vT} = p_k \left\{ \beta - \alpha \left(\frac{\partial \pi}{\partial \tau}\right) \right\}$$

$$\left(\frac{\partial^2 p}{\partial x \, dv}\right)_T = - \frac{p_k}{v_k} \, \alpha \left(\frac{\partial^2 \pi}{\partial \omega \, \partial \tau}\right)$$

in which we have called:

$$\frac{1}{T_k} \frac{dT_{xk}}{dx} = \alpha \ ; \quad \frac{1}{p_k} \frac{dp_{xk}}{dx} = \beta.$$

With these we have for equations ($1b$) and ($1c$):

$$\frac{1}{T_k} \frac{dT_{pl}}{dx} = \alpha - \frac{\left\{ \beta - \alpha \left(\frac{\partial \pi}{\partial \tau}\right) \right\}^2}{C_4 \left(\frac{\partial^2 \pi}{\partial \omega \, \partial \tau}\right)} \quad . \quad . \quad . \quad . \quad (2a)$$

$$\frac{1}{p_k} \frac{dp_{pl}}{dx} = \beta - \left(\frac{\partial \pi}{\partial \tau}\right) \frac{\left\{ \beta - \alpha \left(\frac{\partial \pi}{\partial \tau}\right) \right\}^2}{C_4 \left(\frac{\partial^2 \pi}{\partial \omega \, \partial \tau}\right)} \quad . \quad . \quad . \quad . \quad (2b)$$

if

$$C_4 = \frac{MRT_k}{p_k v_k},$$

which two equations also give immediately the slope of the plait-point curve in the Tp-diagram at the critical point of the pure substance. Hence we see that the variation of critical plait-point temperature and pressure due to the presence of small admixtures is entirely determined by the two quantities α and β.

From these two equations follows also that for very small values of x:

$$\frac{p_{pl} - p_{xk}}{T_{pl} - T_{xk}} = \left(\frac{dp}{dT}\right)_k \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

if $\left(\frac{dp}{dT}\right)_k$ is the $\frac{dp}{dT}$ for the saturated-vapourpressure curve of the simple substance at the critical point. In this way we again find the relation, given by VAN DER WAALS Proceedings Nov. '97, p. 298.

If therefore we connect in the pT-diagram the critical points of the homogeneous mixtures with the plait-points of these mixtures, these connecting lines at the critical point of the pure substance are parallel to the vapourpressure curve of the latter.

§ 3. In order to be able to compare these formulae with observations on mixtures $\left(\frac{\partial \pi}{\partial \tau}\right)$ and $C_4 \left(\frac{\partial^2 \pi}{\partial \omega \, \partial \tau}\right)$ are required.

$\left(\frac{\partial \pi}{\partial \tau}\right)$ can be determined in two ways. For according to a thesis of VAN DER WAALS $\left(\frac{\partial \pi}{\partial \tau}\right)$ at the critical point $= \left(\frac{\partial \pi}{\partial \tau}\right)_{coëx}$, where $\pi_{coëx}$ represents the maximum vapourpressure. Prof. VAN DER WAALS was so kind as to communicate to me the following proofs for this thesis, as developed many years ago in his lectures.

1. MAXWELL's criterium at a coëxistence pressure independent of the volume is given by

$$p_{coëx} \, (v_d - v_v) = \int_{v_v}^{v_d} p \, dv,$$

where v_v and v_d refer respectively to liquid and vapour. By differentiating this with regard to T, we have

$$\left(\frac{dp}{dT}\right)_{co\ddot{e}x}(v_d-v_v)+p_{co\ddot{e}x}\left(\frac{dv_d}{dT}-\frac{dv_v}{dT}\right)=\int\limits_{v_v}^{v_d}\left(\frac{\partial p}{\partial T}\right)_v dv+p_{co\ddot{e}x}\left(\frac{dv_d}{dT}-\frac{dv_v}{dT}\right)$$

or

$$\left(\frac{dp}{dT}\right)_{co\ddot{e}x}=\frac{1}{v_d-v_v}\int\limits_{v_v}^{v_d}\left(\frac{\partial p}{\partial T}\right)_v dv$$

and in words: $\left(\frac{dp}{dT}\right)_{co\ddot{e}x}$ is the mean value of $\left(\frac{\partial p}{\partial T}\right)_v$ between the volumes v_v and v_d. At the critical temperature the mean value is the pressure-variation co-efficiënt itself or

$$\left\{\left(\frac{dp}{dT}\right)_{co\ddot{e}x}\right\}_{crit}=\left\{\left(\frac{\partial p}{\partial T}\right)_v\right\}_{crit}.$$

2. By integrating the isentropic relation

$$\left(\frac{\partial \eta}{\partial v}\right)_T=\left(\frac{\partial p}{\partial T}\right)_v$$

and by putting $(v_d - v_v)\left(\frac{dp}{dT}\right)_{co\ddot{e}x}$ instead of $\eta_d-\eta_v$ and proceeding as in 1°.

3. By using the pvT-surface.

At the critical point a tangent to this surface may be drawn parallel to the v-axis. The tangent-plane at the critical point is thus at right angles to the pT-plane, and its section is a straight line. Each curve on the pvT-surface which passes through the critical point and whose osculating plane at the critical point, while cutting the tangent plane to the pvT-surface at an angle, is not perpendicular to the pT-plane as is the case with the tangent plane to the pvT-surface, is projected on the pT-plane as a curve touching the above named section. Hence 1. the border curve, of which the osculating plane co-incides with the tangent plane to the pvT-surface at the critical point, 2. the section of the pvT-surface by a plane at right angles to the v-axis, 3. the isentropic line for the critical point, etc. All these lines have projections on the pT-plane, touching each other in the projection of the critical point and hence give $\left(\frac{dp}{dT}\right)$ values with different indices v, η, coëx., etc. which are equal to one another at the critical point.

Analytically we bring this into the following form. As for each point of the pvT-surface

$$dp = \left(\frac{\partial p}{\partial v}\right)_T dv + \left(\frac{\partial p}{\partial T}\right)_v dT$$

and as at the critical point $\left(\frac{\partial p}{\partial v}\right)_T$ is equal to zero, we obtain for each curve passing through that point

$$\frac{dp}{dT} = \left(\frac{\partial p}{\partial T}\right)_v .$$

We are uncertain of the equality only when the curve has an element in common with the isotherm. In this case the value of $\left(\frac{\partial p}{\partial v}\right)_T \frac{dv}{dT}$ must be more closely investigated. This strict investigation would therefore be also required for the border curve were this not rendered superfluous by the proofs 1 and 2. That the relation $\left(\frac{dp}{dT}\right)_{coëx} = \left(\frac{\partial p}{\partial T}\right)_v$ holds for the border curve must be ascribed to the circumstance that the latter lies in a cylinder surface, as mentioned above, which touches the pvT-surface at the critical point.

From a graphical representation of p as function of v in the neighbourhood of the critical point according to AMAGAT's data for the isothermals of carbon dioxide I found:

$$\left(\frac{\partial \pi}{\partial \tau}\right) = 7.3 \text{ and } C_4 \left(\frac{\partial^2 \pi}{\partial \omega \, \partial \tau}\right) = -32.2$$

putting the critical volume 0.00426 and $C_4 = 3.45$ while

$$\left(\frac{\partial \pi}{\partial \tau}\right) = 6.5$$

would follow from an extrapolation of AMAGAT's observations for the vapour pressures.

With this uncertainty in $\left(\frac{\partial \pi}{\partial \tau}\right)$ it may be well to investigate what would be yielded by different equations of state.

VAN DER WAALS' equation:

$$\left(p + \frac{a e^{\frac{T_k - T}{T_k}}}{v^2}\right)(v - b) = MRT$$

as well as CLAUSIUS':

$$\left\{p + \frac{K}{T\,(v + nb)^2}\right\}(v - b) = MRT,$$

give:

$$\left(\frac{\partial \pi}{\partial \tau}\right) = 7, \qquad C_4 \left(\frac{\partial^2 \pi}{\partial \omega\, \partial \tau}\right) = -\,32.$$

I would also have availed myself of the equation of state given by DIETERICI (DRUDE's Annalen Bd. 5 p. 51—88, May 1901), but found that there has crept an error into the derivation of this to which I draw attention in the following section.

§ 4. DIETERICI puts $p = \frac{RT}{v_a - b_a}$ in which v_a is derived from v by means of the equation:

$$\int_v^{v_a} \frac{dv}{v - b} = a \,\frac{v_k}{v}\, \frac{T_k^{\,3/2}}{T^{\,3/2}},$$

$$b_a = b_c - \frac{1}{4}\,\frac{b_c^2}{v}, \qquad b_c = \frac{2}{3}\, v_k,$$

where a is a constant to be calculated for each substance from the critical data, for instance for isopentane $a = 2.116$.

From these equations we find:

$$\frac{T}{p}\left(\frac{\partial p}{\partial T}\right)_v = 1 + \frac{3}{2}\, a \,\frac{v_k}{v}\, \frac{T_k^{\,3/2}}{T^{\,3/2}} \left\{1 - \frac{1}{9}\,\frac{v_k^2}{v_a^2}\right\}$$

which for the critical point becomes

$$\frac{T_k}{p_k}\left(\frac{\partial p}{\partial T}\right)_v = 1 + \frac{3}{2}\, a \left\{1 - \frac{1}{9}\,\frac{v_k^2}{v_a^2}\right\},$$

hence for isopentane $\frac{T_k}{p_k}\left(\frac{\partial p}{\partial T}\right)_v < 4.174$, which does not agree with the observations. Indeed DIETERICI wrongly supposes that his formulae also represent the correct isothermals for volumes smaller than the critical. [1])

He derives (l.c. p. 74) the following formula for $\frac{dp_s}{dT}$:

$$v_1\, v_2\, T^{3/2} \frac{dp_s}{dT} = 2.116\, v_k\, R T_k^{3/2},$$

where p_s represents the saturated vapourpressure and v_1 and v_2 the volumes of saturated liquid and vapour.

This formula gives at the critical point:

$$\frac{T_k}{p_k}\, \frac{dp_s}{dT} = 7.9.$$

According to the thermo-dynamic thesis laid down in § 3

$$\frac{dp_s}{dT} = \left(\frac{\partial p}{\partial T}\right)_v$$

at the critical point. Hence the two results contradict each other. The explanation may be found in the circumstance that DIETERICI has assumed that

$$\int_{v_1}^{v_2} \pi\, dv = J\,.\,\varrho$$

if

$$p + \pi = \frac{RT}{v-b}$$

and if ϱ is the internal latent heat of evaporation. This would be true if π were not a function of the temperature; as this is however the case with DIETERICI the internal latent heat of evaporation must be found from

$$J\,.\,\varrho = \int_{v_1}^{v_2} \left[T\left(\frac{\partial p}{\partial T}\right)_v - p\right] dv.$$

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[1]) See DANIEL BERTHELOT, Arch. Néerl. d. Sc. (2) t. V. p. 441.

Hence the circumstance that DIETERICI's formula for $\frac{dp_s}{dT}$ gives a good representation of the observations proves that his original equations do not do so. Hence his equation of state must be left out of consideration.

§ 5. The existing uncertainty about $\left(\frac{\partial \pi}{\partial \tau}\right)$ and $C_4\left(\frac{\partial^2 \pi}{\partial \omega\, \partial \tau}\right)$, for a simple substance (when $\left(\frac{\partial \pi}{\partial \tau}\right)$ diminishes, probably $C_4\left(\frac{\partial^2 \pi}{\partial \omega \partial \tau}\right)$ must also be diminished) is a great impediment in the application of our considerations to mixtures; we may, however, accept that we are not far wrong, when with VAN DER WAALS we put $\left(\frac{\partial \pi}{\partial \tau}\right) = 6.7$ and $C_4\left(\frac{\partial^2 \pi}{\partial \omega\, \partial \tau}\right) = -\,32.2$.

§ 6. We will apply the equations (2) and (3) to the observations of VERSCHAFFELT. He gives (Communication n°. 47, Proceedings Febr. '99 and Communication n°. 65, Arch. Néerl. série II t. V p. 646) for two mixtures of carbon dioxide and hydrogen T_{pl} and p_{pl}, for three mixtures T_{xk} and p_{xk} as he has derived them by superimposing the logarithmic isothermal-systems, and also for carbon dioxide T_k and p_k as calculated by him from AMAGAT's isothermals

x	T_{pl}	p_{pl}	T_{xk}	p_{xk}
0	304.7	73.6	304.7	73.6
0.0494	300.4	91.85	287.8	68.1
0.0995	297.5	114.2	273.6	63.5
0.1990			248.7	54.8

To derive from this $\alpha = \frac{1}{T_k}\left(\frac{dT_{xk}}{dx}\right)_{x=0}$ when x is small, we may put $T_{xk} = T_k\,(1 + \alpha x + \alpha' x^2)$ and compute the co-efficients α and α' from the observations for $x = 0.0494$ and $x = 0.0995$. Thus we find $\alpha = -\,1.219$ $\beta = -\,1.645$. If on the contrary we compute α and β from the formulae accepted by VERSCHAFFELT (Communication n°. 65, Arch. Néerl. t. V p. 649) to represent T_{xk} and p_{xk} as functions of x:

$$T_{xk} = C'_1 \sqrt{\frac{K_x}{b_x}}\ ,\ p_{xk} = C'_2 \sqrt{\frac{K_x}{b_x^3}}$$

where

$$\frac{K_x}{K_{22}} = 1-1.40\,x + 0.18\,x^2\ ,\quad \frac{b_x}{b_{22}} = 1 + 0.526\,x - 0.035\,x^2,$$

then we find :

$$\alpha = -\ 0.963 \qquad \beta = -\ 1.489.$$

It appears therefore that much uncertainty exists about the values of α and β. This must partly be ascribed to the uncertainty in the determination of T_{xk} and p_{xk} following the method of RAVEAU (from VERSCHAFFELT's quadratic formulae I calculate for $x = 0.0494$: $T_{xk} = 290.3°$ C., $p_{xk} = 68.35$ atm., while VERSCHAFFELT found $T_{xk} = 287.8°$ C. and $p_{xk} = 68.1$ atm.), and partly to the small number of observations, from which the variation of T_{xk} and p_{xk} with x must be derived.

However this may be, where the uncertainties in α and β are added to those in $\left(\frac{\partial\pi}{\partial\tau}\right)$ and $C_4\left(\frac{\partial^2\pi}{\partial\omega\,\partial\tau}\right)$ it is obvious that nothing can be derived from the comparison of the observations with the formulae (2*a*) and (2*b*). It is even easy to choose within the limits of uncertainty values for α, β, $\left(\frac{\partial\pi}{\partial\tau}\right)$, $C_4\left(\frac{\partial^2\pi}{\partial\omega\,\partial\tau}\right)$, that lead to results for $\frac{dT_{pl}}{dx}$ and $\frac{dp_{pl}}{dx}$, which are entirely at variance with the observations. With the selection we made, as stated above, of $\left(\frac{\partial\pi}{\partial\tau}\right) = 6.7$ and $C_4\left(\frac{\partial^2\pi}{\partial\omega\,\partial\tau}\right) = -\ 32.2$ we should derive by means of the values of α and β, found from the quadratic formulae, in which VERSCHAFFELT expressed his observations:

$$\alpha = -\ 0.963\ ,\quad \beta = -\ 1.49:$$

$$\frac{1}{T_k}\frac{dT_{pl}}{dx} = -\ 0.20\ ,\qquad \frac{1}{p_k}\frac{dp_{pl}}{dx} = 3.63,$$

which values might be made to agree with the form of the plait-point curve according to VERSCHAFFELT.

Of greater importance is the test of the relation (3), which not only for hydrogen but for every substance mixed with carbon dioxide will hold with x exceedingly small:

$$\frac{p_{pl} - p_{xk}}{T_{pl} - T_{xk}} = 1.61.$$

For we find for this proportion with

$$x = 0.0494 \quad : \quad 1.88\,,$$

$$x = 0.0995 \quad : \quad 2.12\,,$$

so that the value 1.61 for $x = 0$ will come near to the truth and therefore in this respect theory and observations confirm each other sufficiently.

§ 7. In the same way as we have considered $\frac{dp_{pl}}{dx}$ and $\frac{dT_{pl}}{dx}$, we can also consider $\frac{dv_{pl}}{dx}$ for $x = 0$, a quantity important with a view to the determination of the critical volume of a simple substance.

In the equation:

$$\frac{dv_{pl}}{dx} = \left(\frac{\partial v}{\partial p}\right)_{xT} \frac{dp_{pl}}{dx} + \left(\frac{\partial v}{\partial T}\right)_{px} \frac{dT_{pl}}{dx} + \left(\frac{\partial v}{\partial x}\right)_{pT} \quad . \quad . \quad . \quad (4)$$

the co-efficients $\left(\frac{\partial v}{\partial p}\right)_{xT}$, $\left(\frac{\partial v}{\partial T}\right)_{px}$ and $\left(\frac{\partial v}{\partial x}\right)_{pT}$ are all infinitely large for $x = 0$ and such that in the development their terms cancel one another. Therefore in the development of $\left(\frac{\partial v}{\partial p}\right)_{xT}$ etc. the finite terms, in that of $\frac{dp_{pl}}{dx}$ and $\frac{dT_{pl}}{dx}$ the terms with the first power of x must be kept.

As a starting point I have chosen the relations

$$\left(\frac{\partial^2 \zeta}{\partial x^2}\right)_{pT} = 0\ , \quad \left(\frac{\partial^3 \zeta}{\partial x^3}\right)_{pT} = 0$$

of § 2, which hold for the plait-point.

If in equation (4) we substitute the values $\frac{dp_{pl}}{dx}$ and $\frac{dT_{pl}}{dx}$ which we obtain by the differentiation of the above relations:

$$0 = \left(\frac{\partial^3\zeta}{\partial x^2\,\partial p}\right)_T \frac{dp_{pl}}{dx} + \left(\frac{\partial^3\zeta}{\partial x^2\,\partial T}\right)_p \frac{dT_{pl}}{dx},$$

$$0 = \left(\frac{\partial^4\zeta}{\partial x^3\,\partial p}\right)_T \frac{dp_{pl}}{dx} + \left(\frac{\partial^4\zeta}{\partial x^3\,\partial T}\right)_p \frac{dT_{pl}}{dx} + \left(\frac{\partial^4\zeta}{\partial x^4}\right)_{pT},$$

while we write

$$\left(\frac{\partial\zeta}{\partial p}\right)_{xT} = v, \qquad \left(\frac{\partial\zeta}{\partial T}\right)_{vx} = -\,\eta\,,$$

we find:

$$\frac{dv_{pl}}{dx} = \left(\frac{\partial v}{\partial x}\right)_{pT} + \left(\frac{\partial^4\zeta}{\partial x^4}\right)_{pT} \frac{\left(\frac{\partial v}{\partial p}\right)_{xT}\left(\frac{\partial^2\eta}{\partial x^2}\right)_{pT} + \left(\frac{\partial v}{\partial T}\right)_{px}\left(\frac{\partial^2 v}{\partial x^2}\right)_{pT}}{\left(\frac{\partial^2 v}{\partial x^2}\right)_{pT}\left(\frac{\partial^3\eta}{\partial x^3}\right)_{pT} - \left(\frac{\partial^2\eta}{\partial x^2}\right)_{pT}\left(\frac{\partial^3 v}{\partial x^3}\right)_{pT}} \,. \quad (5)$$

To determine $\left(\frac{\partial v}{\partial x}\right)_{pT}$ with x small we may avail ourselves of the equation

$$\left(\frac{\partial^2\zeta}{\partial x^2}\right)_{pT} = \left(\frac{\partial^2\psi}{\partial x^2}\right)_{vT} + \left(\frac{\partial^2\psi}{\partial x\,\partial v}\right)_T \left(\frac{\partial v}{\partial x}\right)_{pT} = 0.$$

We put

$$\psi = MRT\{(1-x)\,l\,(1-x) + x\,l\,x\} + \varphi$$

where therefore

$$\varphi = -\int^v p\,dv.$$

If we call

$$\left(\frac{\partial\varphi}{\partial x}\right)_{vT} = \varphi', \; \left(\frac{\partial^2\varphi}{\partial x^2}\right)_{vT} = \varphi'',$$

then

$$\left(\frac{\partial^2\psi}{\partial x^2}\right)_{vT} = \frac{MRT}{x\,(1-x)} + \varphi''.$$

Hence

$$\left(\frac{\partial v}{\partial x}\right)_{pT} = \frac{MRT}{x\,(1-x)\left(\frac{\partial p}{\partial x}\right)_{vT}} + \frac{\varphi''}{\left(\frac{\partial p}{\partial x}\right)_{vT}} = \frac{MRT}{x\left(\frac{\partial p}{\partial x}\right)_{vT}} + \frac{MRT+\varphi''}{\left(\frac{\partial p}{\partial x}\right)_{vT}} +$$

+ terms with x in the denominator.

In a similar way we develop each of the differential quotients in (5), always retaining the terms of the highest order but one. We shall not repeat here the rather lengthy computations; it must be remarked that in the reduction of the differential quotients of η we have started from the fundamental form

$$d\varepsilon = Td\eta - pdv + \left(\frac{\partial \zeta}{\partial x}\right)_{pT} dx.$$

We obtain

$$T\left(\frac{\partial^2 \eta}{\partial x^2}\right)_{pT} = T\left(\frac{\partial p}{\partial T}\right)_{vx}\left(\frac{\partial^2 v}{\partial x^2}\right)_{pT} + \left(\frac{\partial^2 \varepsilon}{\partial x^2}\right)_{vT} +$$

$$+ 2\left(\frac{\partial^2 \varepsilon}{\partial x \partial v}\right)_T \left(\frac{\partial v}{\partial x}\right)_{pT} + \left(\frac{\partial^2 \varepsilon}{\partial v^2}\right)_{xT}\left(\frac{\partial v}{\partial x}\right)_{pT}^2$$ [1]

In the same way $\left(\frac{\partial^3 \eta}{\partial x^3}\right)_{pT}$ is determined by means of ε.

$\left(\frac{\partial^4 \zeta}{\partial x^4}\right)$ is expressed by means of differential quotients of ψ, where among others $\left(\frac{\partial^2 v}{\partial x^2}\right)_{pT}$ and $\left(\frac{\partial^3 v}{\partial x^3}\right)_{pT}$ occur. The equation $\left(\frac{\partial^3 \zeta}{\partial x^3}\right)_{pT} = 0$ gives a value for $\left(\frac{\partial^2 v}{\partial x^2}\right)_{pT}$ after which $\left(\frac{\partial^3 v}{\partial x^3}\right)_{pT}$ can be determined from the equations obtained by differentiating

$$p = f\,(v\;x\;T)$$

three times with regard to x, keeping T and p constant.

All this performed, we finally obtain:

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[1] Comp. VAN DER WAALS, Continuität II pag. 125.

$$\frac{dv_{pl}}{dx} = -\frac{MRT_k}{\left(\frac{\partial^3 p}{\partial v^3}\right)_T}\left[\frac{\left(\frac{\partial p}{\partial x}\right)^3_{vT}}{(MRT_k)^3} + \frac{\left(\frac{\partial^3 p}{\partial x\,\partial v^2}\right)_T}{MRT_k} + 3\,\frac{\left(\frac{\partial^2 p}{\partial x\,\partial v}\right)_T\left(\frac{\partial p}{\partial x}\right)_{vT}}{(MRT_k)^2} - \right.$$

$$\left. - \frac{\left(\frac{\partial^3 p}{\partial v^2 \partial T}\right)}{\left(\frac{\partial^2 p}{\partial v\,\partial T}\right)\left(MRT_k\right)^2}\left\{\left(\frac{\partial p}{\partial x}\right)^2_v + MRT_k\left(\frac{\partial^2 p}{\partial x\,\partial v}\right)_T\right\}\right]$$

If once more as in § 2 we introduce the law of corresponding states, this form may be reduced to:

$$\frac{1}{C_4}\,\frac{1}{v_k}\,\frac{dv_{pl}}{dx} = \frac{1}{C_4}(\alpha-\beta) - \frac{\beta - \alpha\left(\frac{\partial \pi}{\partial \tau}\right)}{C_4^3\left(\frac{\partial^3 \pi}{\partial \omega^3}\right).C_4\left(\frac{\partial^2 \pi}{\partial \omega\,\partial \tau}\right)} \times$$

$$\times \left[C_4\left(\frac{\partial^2 \pi}{\partial \omega\,\partial \tau}\right)\left\{\beta - \alpha\left(\frac{\partial \pi}{\partial \tau}\right)\right\}^2 - \right.$$

$$\left. - C_4^2\left(\frac{\partial^3 \pi}{\partial \omega^2\,\partial \tau}\right)\left\{\beta - \alpha\left(\frac{\partial \pi}{\partial \tau}\right)\right\} - 3C_4\left(\frac{\partial^2 \pi}{\partial \omega\,\partial \tau}\right)\alpha\right], \quad . \; . \; . \quad (2c)$$

where for $\alpha - \beta$ we may write: $\frac{1}{v_k}\;\frac{dv_{xk}}{dx}$.

In this formula two new constants occur, to be derived from the equation of state, viz.

$$C_4^3\left(\frac{\partial^3 \pi}{\partial \omega^3}\right) \text{ and } C_4^2\left(\frac{\partial^3 \pi}{\partial \omega^2 \partial T}\right).$$

According to VAN DER WAALS' formula with $ae^{\frac{T_k - T}{T_k}}$ for the molecular attraction (as well as for CLAUSIUS' formula) at the critical point

$$C_4^2\left(\frac{\partial^3 \pi}{\partial \omega^2\,\partial \tau}\right) = 256, \quad C_4^3\left(\frac{\partial^3 \pi}{\partial \omega^3}\right) = -\frac{512}{3}.$$

In consequence of the results found during the testing of the formulae (2*a*) and (2*b*) it is obvious that the observations do not yield sufficient data to test formula (2*c*), the more as higher differential quotients of π occur. For the comparison of observation and calculation formula (3) is for the present the most important.

Physiology. — "*On function and structure of the trunk dermatoma.*"

II. Communication made by Prof. C. WINKLER from researches in connection with Dr. G. VAN RIJNBERK.

In our last meeting a communication was made about experiments tending chiefly to elucidate in what manner the function of the trunkdermatoma was destroyed.

These experiments led to the conclusion that in the dermatoma a distinction ought to be established between a central area, being by itself capable of sensation, and a marginal area, owing its value for sensation only to the assistance of neighbouring dermatomata.

At present the results will be given of a few other experiments, that are not only perfectly in accordance with this conception, but would even become unintelligible without it; they will supply further informations as to the ranging of the trunkdermatomata, which is by no means so simple as might generally be supposed.

If in experimenting on a dog *one posterior root* or one pair of roots (an anterior and a posterior root) of the chest-medulla are cut through, without any more, a research into the sensibility of the animal made the same or the next day, gives the following result: Along the mid-dorsal line sensation has been preserved. Along the mid-ventral line an insensible spot is to be found, shaped like a triangle. The basis is lying adjacent to the mid-ventral line and is rather broad. The top usually does not reach the lateral boundary-line. The triangle is encompassed by a narrow hyperalgetic area, which in most cases may be pursued until the mid-dorsal line (fig. 9).

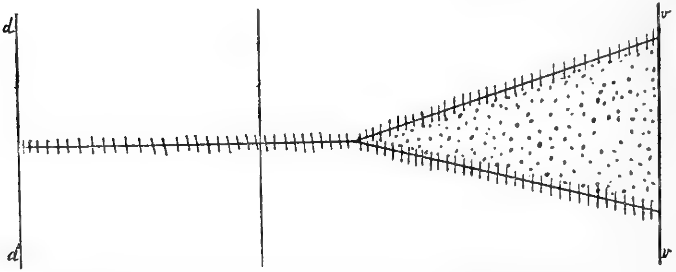

Fig. 9. Insensible triangle, lying adjacent to the mid-dorsal (dotted area) after cutting through one pair of roots.

If, in experimenting on a dog, *two succeeding pairs of roots*, without any more, are carefully cut through, the result is not — as might be expected — an uninterrupted analgetic area all round the trunk, similar to that found often after isolation of a dermatoma above and beneath the sensible area, but two analgetic triangles, separated by a highly sensible, usually hyperalgetic area.

In this case too, a hyperalgetic area encompasses on both sides the analgetic areas. (See fig. 10).

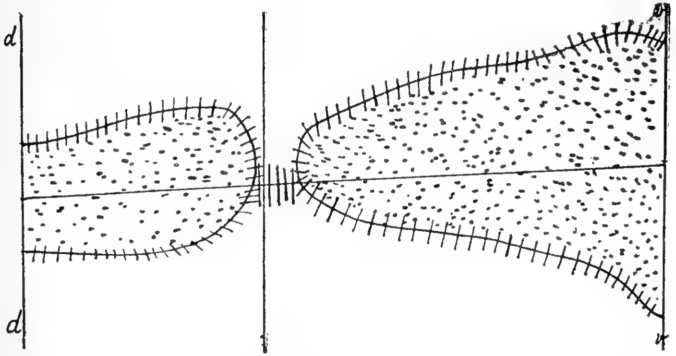

Fig. 10. Consequences of cutting through two succeeding pairs of roots of the chest The insensible (dotted) area has been divided into two triangles by a hyperalgetic band on the lateral boundary-line. Towards the mid-ventral line the insensible area is much broader than towards the mid-dorsal line.

The first fact cannot possibly be made intelligible — and this communication is only aiming thereat — without taking note of other experiments, made on carefully isolated dermatomata, whose central areas must be as large as possible, and without some knowledge at least of the ranging of the trunkdermatomata.

In the first place the analgetic band, that has been found after the isolation of one trunkdermatoma above and beneath the sensible area, demands a closer examination.

If the boundaries of the analgetic bands on the animal under experiment are designated by black or white stripes on the skin, and if then a photograph be taken from the animal (after measuring the breadth of the sensible and insensible areas on those spots that are judged of some importance), the thus obtained photographs will procure a lasting image of the changes found on the day after the operation.

Reproductions of such photos may not always be of use for subsequent measurings. Therefore the simpler method may be sometimes to surround the trunk of the animal with transparent paper, to transfer the stripes on this paper and to preserve them.

Still similar photographs as those directly counter-drawn from the negative in Fig. 11 and 12, reproduce with sufficient clearness an image of the fact, that towards the mid-dorsal line the analgetic areas (occasioned by the sectioning of *two* roots) above and beneath the sensible area (dependent on *one* root left intact) are both of nearly equal breadth, whilst each of them is as broad as the sensible area.

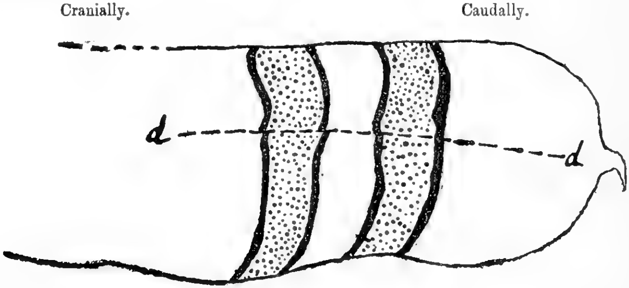

Fig. 11. Dog XV.

On the 15th of Sept. 1901 the 18th dermatoma is isolated on both sides by extradural sectioning of the 16th, 17th, 19th and 20th pair of roots. On Sept. 16th, 1901 sensation is determinated. On Sept. 17th the photograph is taken. The insensible (dotted) area is on both sides adjacent to the mid-dorsal line, and is equivalent to the sensible area cranially as well as caudally. *dd* = mid-dorsal line.

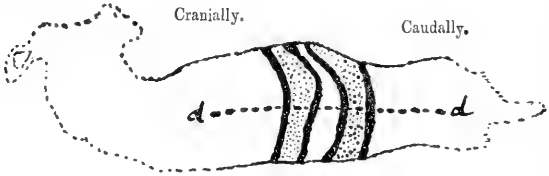

Fig. 12. Dog VII.

Aug. 8th, 1901. The 16th, 17th, 19th and 20th pair of roots are cut through on both sides extradural.

Of the 18th to the right, the first most cranially situated rootbundles are sectioned intradural in the posterior root.

Of the 18th to the left, the most caudally situated rootbundles are sectioned intradural in the posterior root.

On Aug. 9th sensation is determinated. On Aug. 13th the photograph is taken. Autopsy on Aug. 15th confirms that the four caudally situated bundels to the right and the three most cranially situated bundles to the left are intact.

Notwithstanding this, the insensible (dotted) area is nearly equivalent to the sensible area towards the mid-dorsal line.

This is not quite accurate. The analgetic area situated most caudally is always a little broader than that situated cranially, whilst the sensible area is midling in breadth between those two.

Many more or less important deviations from this rule may occur besides[1]; it may safely be stated however that, proceeding carefully, the analgetic areas will always be much narrower than double, and much broader than half the breadth of the sensible area.

In most cases their relation is such that, taking the sensible area for unit, the analgetic area situated cranially is from 0,8—1,1, and the one situated caudally from 1,1—1,4.

Consequently the breadth of the sensible area towards the mid-dorsal line is nearly equal to that of each analgetic area, though the latter represents two sectionned roots, the former only one root left intact.

These two facts:

1st. One sectioned root does not cause analgesy towards the mid-dorsal line.

2nd. One root left intact commands a central area, which towards the mid-dorsal line is nearly as broad, as each analgetic area (dependent on two sectioned roots) is towards that region,

may teach us something about the manner in which the dermatomata are ranged on the mid-dorsal line.

Since the researches of SHERRINGTON it is known that the dermatomata overlap one another in antero-posterior direction, it is known too that these overlappings must be rather important, but the two facts mentioned above, connected with the facts already previously communicated, teach us something more about them.

If, notwithstanding the lesion of the root on which it is dependent, the central area preserves a considerable breadth towards the mid-dorsal line, the marginal areas cannot be very broad towards that region. During the whole of the following demonstration the lawful supposition has been taken for granted, that towards the mid-dorsal line each marginal area can be equivalent at the utmost to half of the central area only.

This supposition needs not to be proved here, and it simplifies all calculations in a most important way.

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1) As to the trunkdermatomata, situated cranially and influenced by the growth the upper extremity, or caudally and influenced by that of the lower extremity, the conditions are very much more complicated.

The first thesis, then following directly from the experimental facts, is this:

1st. The mutual overlapping of the dermatomata, must extend farther than half-way.

Two cases may present themselves here.

a. Either the central areas don't touch one another, or

b. They do touch one another.
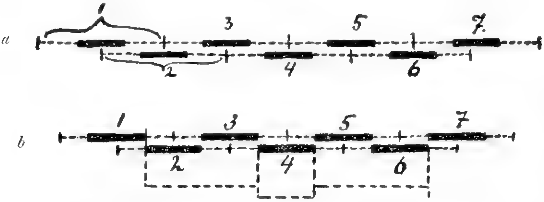
Fig. 13. The ranks of the dermatomata towards the mid-dorsal line if they overlap one another halfway.

a. if the central areas don't touch one another.

b. if they do touch one another.

▬ central areas --- marginal areas.

In both cases (see fig. 13*a* and 13*b*) the destroying of a dermatoma denudates the non-overlapped marginal areas of the precedent and of the following one. Analgesia must be the consequence and would be contradictory therefore to the first fact.

If on the contrary one dermatoma was isolated, e.g. the fourth, by taking away the 2nd, 3rd, 5th and 6th the gap, made analgetic by non-overlapped marginal areas became twice as large as the isolated central area (fig. 13*b*) or even larger (13 *a*).

The second thesis which follows immediately is:

2nd. Even if the overlapping extends farther than halfway, the ranks of dermatomata must be filled more thickly, than would be consistent with mutual contact of the central areas, without their overlapping one another too.

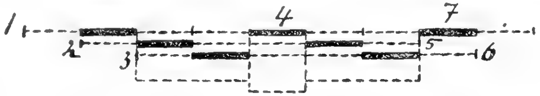

Fig. 14. The ranks of the dermatomata towards the mid–dorsal line, if the central areas touch one another, the overlapping extending farther than halfway. In isolating the 4th dermatoma, the analgetic area becomes equivalent to double the sensible area.

▬ central areas --- marginal areas.

For if the marginal area's are supposed to be larger — even until their overlapping the whole of the nearest central area —, still in isolating the 4th dermatoma (see fig. 14) the analgetic areas (always dependent on two dermatomata) remain twice as large as the sensible central area, which they encompass. In other words: the central areas necessarily must always overlap one another.

The third thesis which follows immediately is:

3rd. The ranks of dermatomata are not filled so thickly, that each central area overlaps one half of the next. The case is represented by fig. 15.

Fig. 15. The ranks of the dermatomata towards the mid-dorsal line, if the central areas overlap one another halfway. In isolating the 4th dermatoma the analgetic area becomes equivalent to $\frac{1}{2}$ of the sensible area.

▬ Central areas. - - - Marginal areas.

Truly, this being the case, after taking away one dermatoma, the uninjured central areas of the neighbouring ones would undertake sensation, which would be in accordance with the first fact. But the isolation of one dermatoma is only possible between analgetic bands half as broad as the sensible area (or even narrower, if the marginal areas exert any influence).

Consequently each central area must possess a part, not overlapped by *one* neighbouring central area, but by *two* marginal areas of neighbouring dermatomata.

If both pre-supposed experimental facts may be taken for granted, it would follow thence:

4th. The central areas overlap one another each for one third, the remaining third, situated in the midst is overlapped by two marginal areas.

For if, as in fig. 16, that part of the central area, overlapped by a neighbouring central area, is called y, and the part, of the central area overlapped by marginal areas x, then the central area, isolated as sensible area is $x + 2y$.

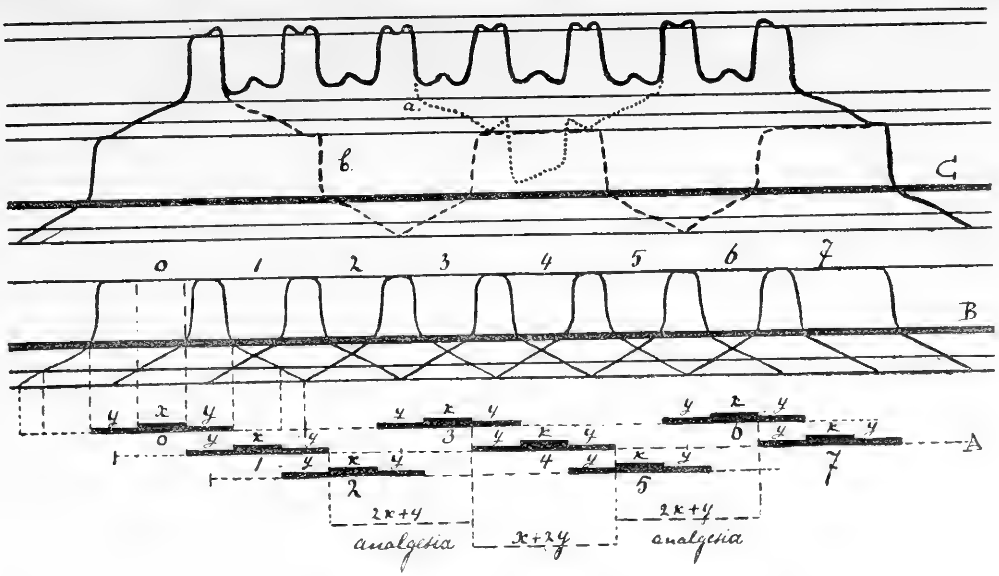

Fig. 16. A. The ranks of the dermatomata towards the mid dorsal line. y is representing that part of the *central area* overlapped by a neighbouring central area, and x that part of the *central area* overlapped by two marginal areas (Drawn for the special case of Dog XV, see fig. 16).

If in isolating one dermatoma the sensible area is equivalent to the analgetic areas of two dermatomata, then it follows:

$$x + 2y = 2x + y \quad \text{or} \quad x = y$$

or in isolating two dermatomata, $2x + 3y$ is equivalent to the analgetic areas of three dermatomata $3x + 2y$ etc., provided the marginal areas remain smaller than the central area.

B. Above each dermatoma the curve of sensibility has been traced. The thick tracing represents the threshold that has to be passed, to make any "sensation" possible.

C. The addition of the curves in B presenting a schema of sensation along the mid-dorsal line. From this schema may be deduced directly the consequences for sensation, e.g. curve a if one root is cut through, b if one root left intact is situated between two sectioned roots above and two sectioned roots beneath it etc.

Accordant to the 1st fact mentioned above, each marginal area must at least possess a breadth $= x$; as long however as the marginal area does not reach a larger breadth than is expressed by $x + y$, the following view may be accepted:

The insensible area (e.g. in fig. 11 and fig. 12) is the total of both central areas of the sectioned roots, but because they overlap, the total must be diminished by three times the central-area-overlapping y, which they give one another reciprocally, and receive from the two neighbouring central areas.

Thence the insensible zone produced by sectioning two roots is $2 \times (x + 2y) - 3y = 2x + y$, and if it is proved experimentally that the sensible area is equivalent to the insensible area, then $x + 2y$ must be $= 2x + y$ or $x = y$.

This holds good only, if the marginal area does not exert a narrowing influence on the insensible area, which may only occur when the former is larger or equivalent to $2x + y$ — in our case equal to the central area. The marginal area however, must possess a minimum breadth x. For our purpose the marginal area hitherto possesses only any value, because it must supply, together with another neighbouring marginal area, the sensation for the band x. For the rest it may even extend $^1/_2 y$ farther, but there it even may have a sensation-value so low under the threshold, that even the assistance of a second marginal area cannot enable it to be sensible.

If the marginal area extends only until x is overlapped, the dermatoma overlap one another for $^3/_5$, if however it extends to a breadth of $x + {}^1/_2 y$, then they overlap one another for $^2/_3$. The central areas always overlap one another for $^1/_3$, if the analgetic area is equivalent to the sensible area. In this case the dermatomata are ranged in such a manner, that the first does rest against the fourth, the 2nd against the 5th, the 3rd against the 8th. This is represented in Fig. 16, and above each dermatoma its curve of sensibility has been indicated.

The thick line represents the threshold that must be passed to make any sensation possible. (fig. 16 B).

By addition of these curves, as is done in fig. 16 C, the curve of sensibility along the mid-dorsal line has been expressed, representing a schema of sensation, by means of which may be calculated directly, what breadth sensible and insensible areas ought to possess with different sectionings. That the destroying of one dermatoma does not necessarily cause loss of sensation, is shown by the dotted line a, remaining above the threshold-line; whilst by the interrupted

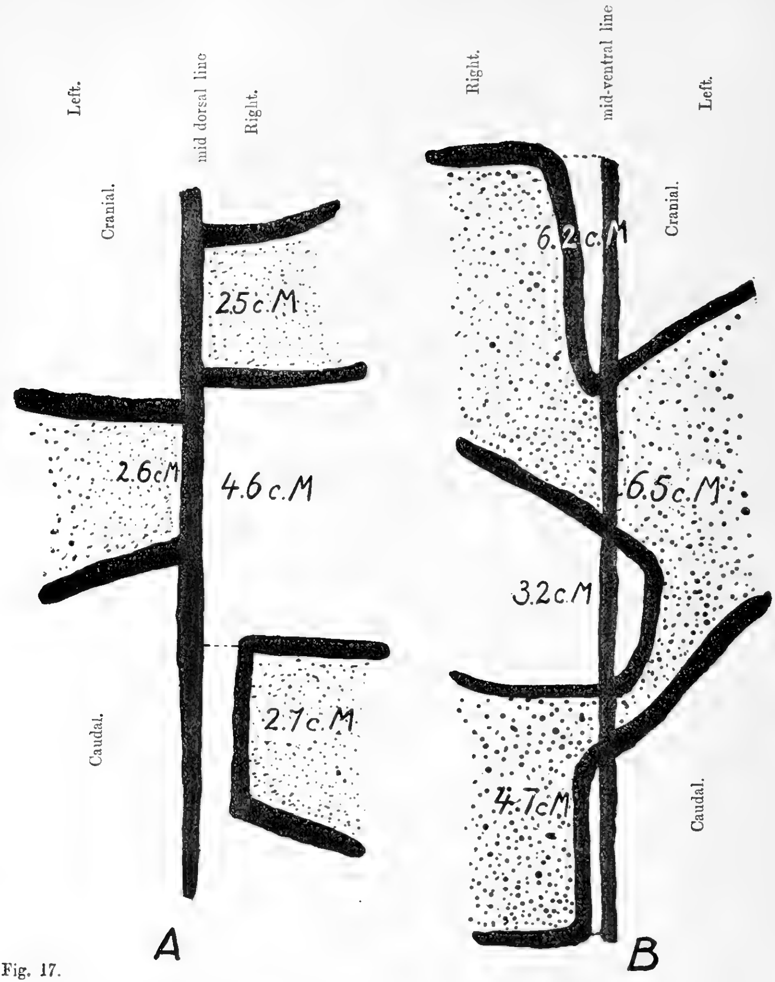

Fig. 17.

Dog 1. On Dec. 27th 1901 the following sectionings were made: to the *left*, sectioning of two pairs of roots (extradural) 16th and 17th,
and to the *right*:
Sectioning of two pairs of roots, (extradural) situated more cranially, 14th and 15th root. Two succeeding pairs of roots (extradural), 16th and 17th, remain intact. Sectioning of the two succeeding pairs of roots (extradural), 18th and 19th root.

A Three (dotted) areas, insensible, one sensible, towards the mid-dorsal line Their relation stands in reason of 2.5, 2.6 or 2.7 : 4.6, or as 0.58 : 1. (Calculation was in reason of 0.6 : 1).

B Three (dotted) insensible areas, one sensible, towards the mid-ventral line. Their relation stands in reason of 6.2, 6.5 or 4.7 : 3.2.

line it is shown that isolation of one dermatoma leads to equivalent bands.

This schema is pretty well in accordance with the facts, as may be seen from the following instance.

In experimenting on a dog the 16th and 17th root to the left, the 14th, 15th, 18th and 19th root to the right, are cut through, Dec. 17th 1901. The 16th and 17th root to the right remain intact.

Along the mid-dorsal line there are found therefore two analgetic areas on the right, one on the left. From the animal's back these are directly counterdrawn on transparent paper (see fig. 16 *A*).

According to the requisites of the schema the sensible area on the right ought to stand towards the analgetic areas in reason as 5 : 3 or 1 : 0.6. The relation found in reality is: Analgetic areas on the right 2 5 and 2.7, on the left 2.6 c.M.; sensible area 4.6 c.M., consequently as 1 : 0.58.

Towards the mid-ventral line other conditions prevail.

As has been made evident in a previous communication, the central areas towards that region are placed so far between, that mutual overlapping is impossible.. Here too the ranging of the dermatomata may be found, if the supposition be taken for granted, that the marginal area, however much it may have been enlarged at the cost of the central area, can never become so large that it overlaps the neighbouring central area [1]).

Starting e.g. from the relations found on dog I (fig. 17 B) towards the mid-ventral line, the sensible area in that region has a breadth of 3.2 c.M., the opposite (somewhat prefixed?) insensible area of 6.5 cM., the upper insensible area of 6.2 cM. and the lower one of 4.7 cM. only.

Supposing that a relation may be stated between the insensible and sensible area at the mid-ventral line as 6.5 : 3.2, or that the sensible area being $= 1$, the insensible area is $= 2$, then fig. 18 represents the calculation.

Calling in this case the central area k, and the marginal area, inasmuch as it overlaps another marginal area y, inasmuch as it overlaps a central area x, it will be made clear by a simple reasoning that the two isolated dermatomata must cover an area equivalent to $2k + y$ (for the marginal area does not extend beyond the neighbouring central area). It is clear too that the analgetic

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[1]) Afterwards, when treating about the middle of the dermatomata, it will perhaps become clear why these suppositions concerning mid-dorsal and mid-ventral line ought to be taken for granted.

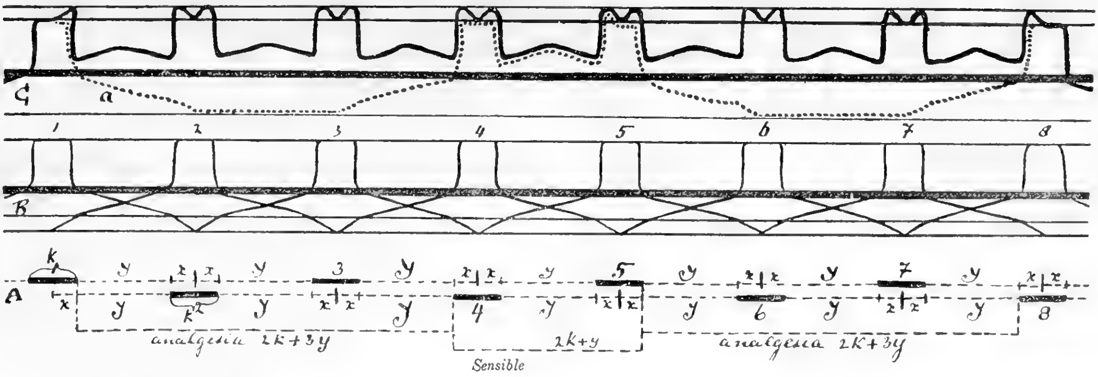

Fig. 18. A. The ranging of the dermatomata along the mid-ventral line. That part of the *marginal area* overlapping another central area, being called x, and that part of the marginal area, overlapping another marginal area, being called y. (Represented for the special case of Dog 1 fig. 17). The central area is called k.

The sensible area from two dermatomata then is = two central areas $+ y = 2k + y$.

The analgetic area from two dermatomata = 2 central areas $+ 3y = 2k + 3y$.

If therefore on dog I the sensible area (3.2 cM.) is $= \frac{1}{2}$ of the insensible area (6.5 cM.), then $2 \times (2k + y) = (2k + 3y)$ or $y = 2k$. Supposing always x not becoming larger than k.

B. Curve of sensibility traced above each dermatoma. The thick line represents the threshold line.

C. The addition of the curves in B presenting a schema of sensibility along the mid-ventral line. From this schema the consequences of each sectioning of roots for the abdomen may be directly read; e.g. a when, as in the case of Dog I, two succeeding roots are cut through, the two following left intact, and then again two cut through.

area must be equivalant to $2\,k + 3\,y$. As however in this special case experiment teaches that $2\,(2\,k + y) = 2\,k + 3\,y$, it follows that $y = 2\,k$.

In fig. 18 this has been represented, and in the same way as for fig. 16 the curves of sensibility are traced above each dermatoma. From the addition of these curves the curve of sensation along the mid-ventral line has been deduced (fig. 18 C). Here again the upper curve presents a schema, that may be further controlled by experiments.

Towards the mid-ventral line therefore the ranging of the dermatomata is different from their ranging towards the mid-dorsal line. There the first is placed against the 3rd, the 2nd against the 4th etc.

Dog I however teaches us something more.

The sensible area on the back has its origin in two central areas it is therefore $^{5}/_{3}$ central area $= 4.6$. The dorsal central area is therefore $= 2.7$ c m.

The sensible area on the abdomen is $2\,k + y = 4\,k = 3.2$ c.m. The ventral central area therefore is 0.8 c.m. The insensible area is $= 2\,k + 3\,y = 8\,k = 6.5$ c.m., the central area thus being again $= 0.8$ c.m.

The ventral central area is therefore midling between $^{1}/_{3}$ and $^{1}/_{4}$ of the dorsal central area. On dog I the situation of central and marginal areas may thus be easily described for a special case. Fig 19 represents the ranging of the central areas.

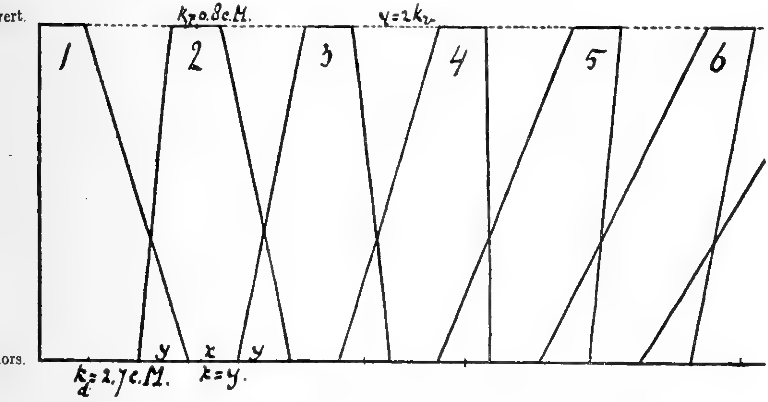

Fig. 19. Ranging of the central areas in the special case of Dog I (fig. 16).
Ventral central area $= 0.8$ c.M $= k_v$ $2\,k_v + y = 3.2$, $2k_v + 3y = 6{,}5$ cM.
$y = 1.6$ cM.
Dorsal central area $= 2.7$ c.M $= k_d$ $2x + 3y = 4.6$ $2x + y = 2.8$ cM.
$x = 0.9$ cM. $y = 0.9$ cM. (See text)

By the three figures 16, 18 and 19, the fact is made intelligible, that sensation is lost towards the mid-ventral, not towards the mid-dorsal line. For this fact again the contrast between central and marginal areas of the dermatoma is necessary. Only by a quite separate series of experiments however, concerning sensibility in the middle of the dermatomata, (on the spot where the central area possesses a minimum of innervation) can it be made clear where the top of the insensible area must be situated, and in what manner is caused the interruption in the analgetic zone when two roots are sectioned.

Physics. — Dr. Ph. A. Kohnstamm on: "*The shape of an empiric isothermal of a binary mixture.*" (Communicated by Prof. J. D. van der Waals).

In § 8—§ 10 of my thesis for the doctorate I have discussed the shape of the empiric [1]) isothermal of a binary mixture. Without writing it down, I start there [2]) from the equation:

$$v = v_{vl} + v_d$$

where v_{vl} and v_d represent the actually measured volumes of liquid and vapour, and v the total volume. If we assume that we have to deal with a molecular quantity, v is at the same time the molecular volume of the mixture; v_{vl} and v_d, however, are no molecular volumes. Now I have shown, that the shape of the course of v as function of p, so also the shape of the empiric isothermal cannot differ sensibly from the shape of the course of v_d and after having drawn up a formula for v_d, I derive by differentiating this formula the shape of the empiric isothermal.

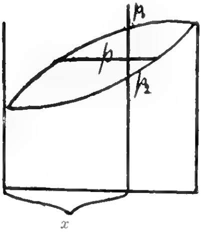

Fig 1.

Perhaps we can arrive at this fundamental formula in a simpler way by not starting from the really measured vapour- and liquid volumes, but from the molecular volumes. We have then (see fig. 1):

$$v = \frac{x_2 - x'}{x_2 - x_1} v_{vl} + \frac{x' - x_1}{x_2 - x_1} v_d$$

---

[1]) I speak there of "derived" isothermal. The name chosen here is perhaps clearer.

[2]) p. 141.

Here v is the same quantity as above; v_{vl} and v_d, however, are now the molecular liquid and vapour volumes; x', x_1 and x_2 are respectively the composition of mixture, liquid and vapour. We may write this equation:

$$v = v_{vl} + \frac{x' - x_1}{x_2 - x_1} (v_d - v_{vl}).$$

Here we must not neglect the second term with regard to the first, as the factor $x' - x_1$ approaches 0, but in the second term we may neglect v_{vl} with respect to v_d and put for the latter $\frac{MRT}{p}$. By differentiation we get now:

$$dv = dv_{vl} + d\left(\frac{MRT}{p} \frac{x' - x_1}{x_2 - x_1}\right)$$

or, if we take the molecular liquid volume as invariable:

$$dv = d\left(\frac{MRT}{p} \frac{x' - x_1}{x_2 - x_1}\right)$$

and so proceeding in this way we arrive also at the general formula for the shape of the empiric isothermal given on p. 140 of my thesis.

Let us apply this formula in the case that the px_1 line is a straight one, what comes to about the same thing as equal critical pressure for the two components. [1]).

In this case the px_2 line is a hyperbola [2]), so if p_A and p_B [3]) represent the vapour-tension of the components:

$$p = p_A(1 - x_1) + p_B x_1$$

and

$$p = \frac{p_A\ p_B}{p_B(1 - x_2) + p_A x_2}$$

If we now take p_1 and p_2 for the greatest and the smallest co-existing pressure which can occur with the composition x', so

---

[1]) See Zeitschrift für phys. Chemie **36**, p. 52.

[2]) Van der Waals. Proc. IX p. 172.

[3]) We shall allways assume $p_B > p_A$.

the pressure when the whole is vapour or the whole is liquid, we get also:

$$p_1 = p_A(1-x') + p_B\, x'$$

$$p_2 = \frac{p_A\ p_B}{p_B(1-x') + p_A\, x'}$$

If we substitute these values in our differential equation, we find:

$$(p_B - p)^2\,(p - p_A)^2\,\frac{dv}{dp} = -\,v_2\,p_2\,[(p_B - p)\,(p_1 - p_A) - (p_1 - p)\,(p - p_A)].$$

So we find that $\frac{dv}{dp}$ is always < 0, as it ought to be, for

$$p_B - p > p_1 - p \text{ and } p_1 - p_A > p - p_A.$$

For the second differential coefficient we get

$$(p_B - p)^2\,(p - p_A)^2\,\frac{d^2v}{dp^2} = \frac{2v_2p_2}{(p_B - p)\,p - p_A)} \times$$

$$\times\,[(p_A + p_B - 2p)\,(p_B - p)\,(p_1 - p_A) + (p - p_A)^2\,(p_1 - p)].$$

Now $\frac{d^2p}{dv^2} = -\left(\frac{dp}{dv}\right)^3 \frac{d^2v}{dp^2}$ and as $\frac{dp}{dv}$ is always negative $\frac{d^2p}{dv^2}$ has the same sign as $\frac{d^2v}{dp^2}$.

Hence it is clear, that when $2p < p_A + p_B$, so $x < {}^1\!/_2$, the convex side of the curve is always turned downward (fig. 2 I). If on the other hand $x > {}^1\!/_2$, then the whole factor is negative for $p = p_1$; it is therefore clear that on the side of the greatest pressure the empiric isothermal for those mixtures must begin with having its convex side turned upwards. But there can be but one inflection point, if any. For:

$$F(p) = (p_A + p_B - 2p)\,(p_B - p)\,(p_1 - p_A) + (p - p_A)^2\,(p_1 - p)$$

$$F'(p) = -\,3\,[(p_1 - p)^2 + (p_1 - p_2)\,(p_A + p_B - p)]$$

$F'(p)$ is therefore always negative; the further we therefore get from p_1, in other words the smaller p becomes, the greater becomes $F(p)$. So if the isothermal is first concave, it can become convex further on, but it cannot be convex at a certain value of p and become concave at a smaller value.

Let us now examine for which mixtures the inflection point, which has entered the isothermal at $x = \frac{1}{2}$, has reached the other end and we accordingly pass from the case of fig. 2 II to that of fig. 2 III.

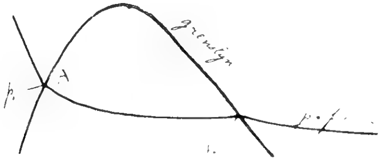

Fig. 2 I.

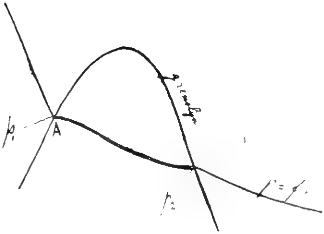

Fig. 2 II.

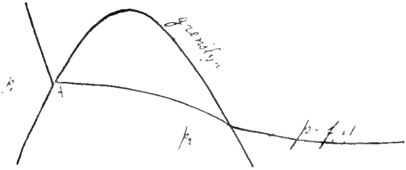

Fig. 2 III.

It is obvious that the p_2 for this point must be greater than ${}^1/_2 (p_A + p_B)$, for then the first term of $F(p)$ is positive, so a fortiori the whole expression. In order to find the exact point where the inflection point makes its appearance we substitute in:

$$F(p) = 0 \text{ the value } p = p_2 = \frac{p_A p_B}{p_A + p_B - p_1}$$

and solve p_1.

This furnishes:

$$p_1 = p_B - \frac{1}{p^2_A + p^2_B + p_A p_B} \left[\pm \sqrt{p^3_A p^3_B} - p^3_A \right]$$

Now it is clear that only the positive sign is possible, else we could get a $p_1 > p_B$, and as $p_1 = p_A + x'(p_B - p_A)$ we get for the composition of the mixture that separates case II from case III:

$$_1x' = 1 - \frac{\sqrt{p^3_B p^3_A} - p^3_A}{p^3_B - p^3_A}.$$

Therefore the inflection-point will appear for every mixture, for which $1/2 < x' < {}_1x'$. Hence we see that the region which remains for the entirely concave curve, depends only on p_A / p_B.

In order to examine this dependence more closely, I have inserted in my thesis a needlessly elaborate calculation, which I shall not repeat here, the more, because it contains an error of calculation which I could not correct anymore. In the formula for $\varphi''(y)$ on pag. 153 $9y^6$ ought to be substituted for $10y^6$. In consequence of this fig. 16 is not quite accurate. Therefore I should like to substitute what follows for the passage of my thesis which applies to this.

If we introduce in the formula $_1x'$:

$$y = \frac{p_A}{p_B}$$

we get:

$$_1x' = \frac{1 - \sqrt{y^3}}{1 - y^3} = \frac{1 - \sqrt{y^3}}{(1 - \sqrt{y^3})(1 + \sqrt{y^3})} = \frac{1}{1 + \sqrt{y^3}}$$

$$\frac{d_1x'}{dy} = -\frac{3}{2} \frac{\sqrt{y}}{(1 + \sqrt{y^3})^2}$$

From this formula follows that for $y = 0$, so $p_A = 0$, $_1x' = 1$. So isothermals which are concave throughout their course do not exist. As soon, however, as p_A gets a value, they appear and the region of the isothermals with an inflection-point decreases regularly, as appears from the differential coefficient which is always negative, till for $y = 1$, so $p_A = p_B$, ${}_1x' = {}^1/_2$. At this limit the region for isothermals with an inflection-point has quite disappeared, and only entirely concave or entirely convex isothermals exist, but it is clear that for $p_A = p_B$ also all other pressures become equal in this case.

In order to solve the problem to its full extent, we have still to examine, at what pressure the inflection-point eventually appears in the derived isothermal. It is clear that in order to find that dependence between p and x', we have simply to substitute in the equation $F(p) = 0$:

$$p_1 = p_A + x'(p_B - p_A).$$

In order to determine the pressure at the inflection-point belonging to the mixture x' we get therefore:

$$(p_A + p_B - 2p)(p_B - p)x'(p_B - p_A) + (p_1 - p_A)^2 [p_A - p + x'(p_B - p_A)] = 0$$

$$x' = \frac{(p - p_A)^3}{(p_B - p_A)[(p_A + p_B - 2p)(p_B - p) + (p - p_A)^2]}$$

The condition that x' must be positive, is fulfilled, for:

$$(p_A + p_B - p)(p_B - p) + (p - p_A)^2 = (p_B + p_A - 2p)^2 + (p - p_A)(p_B - p)$$

So we have for the locus of the inflection-points a cubic curve of which we saw before that in the heterogeneous region it cuts all the coordinates but once.

We get further:

$$(p_B - p_A)\frac{dx'}{dp} = \frac{3(p - p_A)^2 (p_B - p)^2}{[(p_A + p_B - 2p)(p_B - p) + (p - p_A)^2]^2}$$

so $\dfrac{dp}{dx'}$ is always > 0, and has in the beginning at $2p = p_A + p_B$ the value $\dfrac{p_B - p_A}{3}$, which disappears for $p_A = p_B$, as it ought to do.

And finally we get:

$$\frac{3}{(p_B-p_A)^3}\,\frac{d^2p}{dx'^2}=\frac{dp}{dx'}\,\frac{2\,(2\,p-p_A-p_B)}{(p-p_A)^3\,(p_B-p)^3}\times$$
$$\times\,[(p_B+p_A-2\,p)^2+(p-p_A)\,(p_B-p)]$$

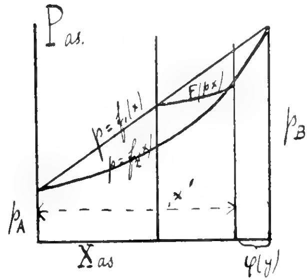

Fig. 3.

so that the locus is everywhere convex in the heterogeneous region with an inflection-point at the beginning, and we get a curve as drawn in fig. 3, which clearly indicates the different possibilities for the empiric isothermal. The point where the locus cuts the curve $p=f(x_2)$ is of course determined by the formula for ${}_1x'$.

Mathematics. — “*Factorisation of large numbers*”, by Mr. F. J. Vaes, Mechanical Ingeneer at Rotterdam. (Communicated by Prof. P. H. Schoute).

Introduction.

The history of the research about the divisibility of large numbers is very simple.

Eratosthenes (275—194 b. C.) is said to have invented the method of the sieve (determination of the prime numbers under a given limit by removing from the series of odd numbers those divisible by 3, 5, 7, etc.).

In 1643 Fermat decomposed a number proposed to him by Mersenne. In a letter dated “Toulouse le 7 Avril 1643” we find: “Vous me demandez si le nombre 100895598169 est premier ou “non, et une méthode pour découvrir, dans l'espace d'un jour, s'il “est premier ou composé. A cette question, je réponds que le “nombre est composé et se fait du produit de ces deux: 898423 et “112303, qui sont premiers.”

The method of Fermat has never been published [1]).

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[1]) In 1640 Fermat believed $2^{2^n}+1$ gives prime numbers for all values of n. Afterwards Euler found that $2^{2^5}+1$ (a number of ten figures) is the product of 641 and 6700417. The author is inclined to ask: If in 1643 Fermat really could factorise

In the preceding century the theory of numbers was built up by Gauss, Legendre, Lejeune-Dirichlet, Riemann, Tschebicheff and others who obtained very important results. But the value of these results with reference to the factorisation of large numbers can be derived sufficiently from the following two citations: . . . "dans "l'état actuel de la théorie des nombres on ne connaît aucun procédé "direct pour la recherche des diviseurs des nombres ayant plus de "dix chiffres dans le système décimal" (Edouard Lucas, *Théorie des Nombres*, Tome premier, p. 333) and "Les méthodes de Gauss "seraient impuissantes à résoudre le problème proposé par Mersenne à Fermat" (Edouard Lucas, *Récréations Mathématiques*, Tome II, p. 231).

Any number N can be factorised, if it can be thrown into the form $a^2 - b^2$, under the condition $a > b + 1$. For then we have $N = (a + b)(a - b)$, which proves that the decomposition can be performed by seeking a square b^2 that, added to N, furnishes a new square a^2. This simple property is also mentioned by Fermat, but apparently not made use of to factorise large numbers either by him or by another.

By means of a table of squares the factorisation of any number N can be performed.

However it is more convenient to determine the square root of N and to increase the last figure of this root by unity.

Example $N = 1073$.

$$\sqrt{10 \mid 73} = 33, \quad \text{so } 1073 = 33^2 - 16 = 33^2 - 4^2 = 37 \times 29.$$

$$\begin{array}{r} 3^2 = 9 \\ \overline{173} \\ 63.3 = 189 \\ \overline{-16} \end{array}$$

As a rule the required result will not be obtained so soon; rather we shall find in general $N = a_1^2 - b_1$.

By adding n to a_1 we find

$$N = (a_1 + n)^2 - (b_1 + 2a_1 n + n^2);$$

so we can reach our aim by choosing n in such a manner that $b_1 + 2a_1 n + n^2$ is a square.

---

a number of twelve figures, wherefore did he not apply his method to this number of ten figures? May we not conclude from this that Fermat was in possession of a special method for special numbers, and that he had dictated to Mersenne a condition to which the proposed number had to satisfy? The correspondence between F. and M. may enlighten this point. In that case also could be decided if any of the methods given here be related to the method of Fermat.

The easiest way to obtain such an n consists in increasing a_1 several times by unity and continuing this operation

$$\begin{aligned} N &= a_1^2 - b_1 \\ &= (a_1+1)^2 - (b_1 + 2a_1 + 1) \text{ or } = a_2^2 - b_2, \\ &= (a_2+1)^2 - (b_2 + 2a_2 + 1) \text{ or } = a_3^2 - b_3, \text{ etc.} \end{aligned}$$

until the number b in the last line has become a square.

Example $N = 57$.

$$\begin{aligned} \text{We write } N &= 8^2 - 7 \\ &= 9^2 - (7 + 2\times 8 + 1) = 9^2 - 24 \\ &= 10^2 - (24 + 2\times 9 + 1) = 10^2 - 43 \\ &= 11^2 - (43 + 2\times 10 + 1) = 11^2 - 64 \\ &= 11^2 - 8^2 = (11+8)(11-8) = 19 \times 3. \end{aligned}$$

For shortness' sake we make use of the algorithm

$$\begin{array}{rr} 57 = 8^2 - & 7 \\ 2\times 8 + 1 = & 17 \\ \cline{2-2} & 24 \\ & 19 \\ \cline{2-2} & 43 \\ & 21 \\ \cline{2-2} & 64 = 8^2, \end{array}$$

so $57 = (8 + \text{number of additions})^2 - 8^2$, where only the first additional number 17 has to be calculated.

An important abbreviation can be obtained by paying attention to the terminal figures, as is shown by the following example. Here the reckoning was to be:

$$\begin{array}{lr} N = 513667. & \\ N = 717^2 \quad - & 422 \\ 2\times 717 + 1 = & 1435 \\ \cline{2-2} & 1857 \\ & 1437 \\ \cline{2-2} & 3294 \\ \qquad\qquad \text{etc.} & \end{array}$$

However as a^2 can terminate in 0, 1, 4, 5, 6 or 9 only and the last figure of N is 7, $a^2 - N$ can terminate in 3, 4, 7, 8, 9 or 2 only. But $a^2 - N$ being also a square (b^2), it can only terminate in 4 of 9. So it is unnecessary to perform all the additions and the abbreviated algorithm comes to this:

$$\begin{array}{rr}
N = 717^2 & -\quad 422 \\
2 \times 717 + 1 = & 1435 \\
 & 37 \\ \cline{2-2}
 & 3294 \\
 & 1439 \\
 & 41 \\ \cline{2-2}
 & 6174 \\
 & 1443 \\
 & 5 \\
 & 7 \\ \cline{2-2}
 & 10509 \\
 & 1449 \\
 & 51 \\ \cline{2-2}
 & 13409 \\
 & 1453 \\
 & 5 \\
 & 7 \\ \cline{2-2}
 & 17774 \\
 & 1459 \\
 & 61 \\ \cline{2-2}
 & 20694 \\
 & 1463 \\
 & 5 \\
 & 7 \\ \cline{2-2}
 & 25089 \\
 & 1469 \\
 & 71 \\ \cline{2-2}
 & 28029 \\
 & 1473 \\
 & 5 \\
 & 7 \\ \cline{2-2}
 & 32454 \\
 & 1479 \\
 & 81 \\ \cline{2-2}
 & 35414 \\
 & 1483 \\
 & 5 \\
 & 7 \\ \cline{2-2}
 & 39869 \\
 & 1489 \\
 & 91 \\ \cline{2-2}
 & 42849 = 207^2 = b^2
\end{array}$$

As soon as $b^2 = 207^2$ has been found, the value of a^2 can be calculated in different manners. In the first place by adding N to the result 42849 we find

$$a^2 = 556516 = 746^2.$$

Secondly one can remark, that from 1435 unto 1491 (the first and last of the added numbers) a number of

$$\frac{1491-1435}{2} + 1, \text{ i. e. } 29$$

odd numbers have been added; this gives

$$a = 717 + 29 = 746.$$

Thirdly — and this method is the shortest — one can observe that by continuing the operation the number 1493 had to be added and that this number is equal to $2a + 1$; so

$$a = \frac{1493-1}{2} \text{ or}$$

$$\frac{1491+1}{2} = 746.$$

So we have

$$N = 746^2 - 207^2 = 953 \times 539.$$

After two additions the factor 539 appears to be equal to $30^2 - 19^2$, i.e. equal to 49 × 11. On the other hand the factor 953 gives $477^2 - 476^2$, i.e. 953 × 1 and appears to be a prime number.

The last operation is rather long, a number of 446 numbers having to be added; as for larger numbers the number of additions nearly increases proportionally it is peremptorily necessary to seek for a shorter method.

Such an abbreviation can be derived from the consideration of last *two* figures of the number to be factorised.

For a square must terminate in one of the following pairs of figures: 00; 01, 21, 41, 61, 81; 04, 24, 44, 64, 84; 25; 16, 36, 56, 76, 96; 09, 29, 49, 69, 89.

The last pair of figures of the number N being 53 we have to determine out of the given pairs those pairs which, by addition of 53, furnish another pair.

So one sees immediately that b^2 only can terminate in one of the pairs 16, 36, 56, 76, 96, in which cases a^2 terminates in 69, 89, 09, 29, 49 respectively.

So we can shorten our algorithm to:

$$\begin{array}{lr}
953 = 31^2 \quad - & 8 \\
2 \times 31 + 1 = & 63 \\
 & 65 \\ \cline{2-2}
 & 136 \\
 & 67 \\
 & 69 \\
 & 71 \\
 & 73 \\ \cline{2-2}
 & 416 \\
 & 75 \\
 & 77 \\
 & 79 \\
 & 81 \\
 & 83 \\
 & 85 \\ \cline{2-2}
 & 896
\end{array}$$

etc.

Now, as is immediately evident $67 + 69 + 71 + 73 = 4 \times 70$, and $75 + 77 + 79 + 81 + 83 + 85 = 6 \times 80$; as similar groups present themselves over and over, we can shorten still more as follows:

$$\begin{array}{lrcr}
 & 953 = 31^2 & - & 8\\
 & 2 \times 31+1 & = & 63\\
 & & & 65\\ \cline{4-4}
a^2 = 33^2 & & & 136\\
 & 4 \times 70 & = & 280\\ \cline{4-4}
37^2 & & & 416\\
 & 6 \times 80 & = & 480\\ \cline{4-4}
43^2 & & & 896\\
 & 4 \times 90 & = & 360\\ \cline{4-4}
47^2 & & & 1256\\
 & 6 \times 100 & = & 600\\ \cline{4-4}
53^2 & & & 1856\\
 & \text{etc.} & &
\end{array}$$

The importance of this last abbreviation is self-evident.

The numbers obtained by addition must be looked for in a table of squares. For shortness' sake one can make use of the following table (see next page) representing all the possible groups of four figures, in which a square can end[1]).

For if this table shows that the four terminal figures of a number cannot occur in a square, it is unnecessary to use the table of squares.

II. *Classification of the row of natural numbers according to their divisibility.*

By diminishing any square, e.g. 13^2 by the squares 1^2, 2^2, 3^2, etc. we obtain the composed numbers 168, 165, 160, 153, etc. or if we pay attention to the odd numbers only: 165, 153, 133, 105, 69, 25.

It is immediately evident that in the factorisation of any of these numbers 13^2 *may*, but not that 13^2 necessarily *must* present itself as a^2.

For shortness' sake we will say that 13^2 *dominates* these numbers. So a number admitting of more than two factors is dominated by more than one square, e.g. 273 by 137^2, 47^2, 23^2, 17^2.

In following the method developed in § I one always finds the *least* dominating square.

---

[1]) In his "Théorie des Nombres" LUCAS states that PRESTET has published a table for the same purpose in his "Nouveaux Eléments de Mathématiques, 1689. It has been impossible for us to make out if this table was constructed in an analogous manner.

TABLE

CONTAINING ALL THE GROUPS OF FOUR FIGURES IN WHICH A SQUARE CAN END.

NUMBER FORMED BY THE HUNDREDS AND THOUSANDS.																									NUMBER FORMED BY THE UNITIES AND TEN																		
																									01	21	41	61	81	04	24	44	64	84	16	36	56	76	96	09	29	49	69
00	04	08	**12**	**16**	**20**	24	28	**32**	**36**	**40**	44	48	**52**	**56**	**60**	64	68	**72**	**76**	**80**	84	88	**92**	**96**	×		×		×	×			×	×	×	×			×	×		×	
01	05	09	13	17	21	25	29	33	37	41	45	49	53	57	61	65	69	73	77	81	85	89	93	97		×		×				×	×		×			×	×		×		×
02	**06**	**10**	14	18	**22**	**26**	**30**	34	38	**42**	**46**	**50**	54	58	**62**	**66**	**70**	74	78	**82**	**86**	**90**	94	98	×		×		×		×	×					×	×		×		×	
03	07	11	15	19	23	27	31	35	39	43	47	51	55	59	63	67	71	75	79	83	87	91	95	99		×		×		×	×			×		×	×				×		×

The numbers printed in fat type can terminate in 25.

Example: A square may terminate in 4164; for the intersection of the row of 41 with the column of 64 is marked by a cross (×). A square cannot terminate in 4156; for the intersection of the row of 41 with the column of 56 has not been marked.

In the following table are indicated under each square the numbers dominated by it:

TABLE I.

1^2	2^2	3^2	4^2	5^2	6^2	7^2	8^2	9^2	10^2	11^2	12^2 etc.
1	4	9	16	25	36	49	64	81	100	121	144
	3	8	15	24	35	48	63	80	99	120	143
		5	12	21	32	45	60	77	96	117	140
			7	16	27	40	55	72	91	112	135
				9	20	33	48	65	84	105	128
					11	24	39	56	75	96	119
						13	28	45	64	85	108
							15	32	51	72	95
								17	36	57	78
									19	40	63
										21	44
											23

The classification of the numbers in this table is very remarkable:

1°. In the columns as well as in the rows the successive differences are 1, 3, 5, 7, etc.

2°. Parallel to the odd numbers in the hypothenusa of the triangle we find the fourfolds 4, 8 12, 16 etc.

3°. Ascending from one of the odd numbers of the hypothenusa in a direction perpendicular to it one finds 3, 5, 7, etc. times this number, e. g. starting from 9 one finds 27, 45, 63, 81.

Now descending from 81 in the direction of the hypothenusa we find the continuation 11, 13, 15 etc. times 9 or 99, 117, 135, etc.

So all the odd multiples of 9 are to be found in two lines passing through 81 and inclined under 45°.

4°. Ascending from one of the fourfolds, e. g. 16, in a direction perpendicular to the hypothenusa we find 2×16, 3×16, $4 \times 16 = 8^2$ and from this point parallel to the hypothenusa 5×16, 6×16, etc. The proof of all these properties is easily given.

From the remark sub 3⁰ ensues that the number 7×11 will be found in the line passing through 7^2 parallel to the hypothenusa and also in the line passing through 11^2 perpendicular to it.

So the factors of a number prove to be the roots of the squares that can be reached by proceeding from this number in the two directions inclined under 45°.

The prime numbers present themselves only once (in the hypothenusa), the composed odd numbers present themselves still one time or several times more.

As it is not very convenient to make out if a given number is contained in the table, if this table is continued much farther than here, the classification of Table II recommends itself more. Here equal numbers are placed in the same row, the head of each column bearing the dominating square. The prime numbers appear only in the inclined line at the right side; their rows are denoted by horizontal lines.

Beneath any square a^2 are arranged the numbers $a^2 - 1$, $a^2 - 4$, $a^2 - 9$, etc. with the differences 1, 3, 5, 7, etc. The last number of each column is always *zero*, the last but one $a^2 - (a-1)^2 = 2a - 1$, and this number appears always in the inclined line at the right side.

If we assume any number of this line, e.g. 19, then we find above it: $19+17=36$, $19+17+15=51$, $19+17+15+13=64$, $64+11=75$, $75+9=84$, $84+7=91$, $91+5=96$, $96+3=99$, $99+1=100$.

Now 19 can be called the *base* of the numbers 36, 51, 64 etc. Above the base $2a-1$ we then find:

$$(2a-1)+(2a-3)=4(a-1),\ (2a-1)+(2a-3)+(2a-5)=3(2a-3),$$

$$3(2a-3)+(2a-7)=8(a-2),\ 8(a-2)+(2a-9)=5(2a-5),$$

$$5(2a-5)+(2a-11)=12(a-3),\ 12(a-3)+(2a-13)=7(2a-7),\ \text{etc.}$$

Therefore the 3-folds, 5-folds, etc. of the odd numbers are situated on oblique lines passing through the numbers 3, 5, 7, etc., whilst the 4-folds, 8-folds, etc. are situated on intermediate oblique lines commencing at 4, 8, 12, etc.

Twofolds of prime numbers do not present themselves.

For immediate application this table has the inconvenience, that it cannot be continued far enough without becoming unmanageable.

However it leads to an important abbreviation of the method given in § I by means of the simple remark that between the oblique lines no numbers can present themselves. For illustration a small number is chosen; the application to a large number will be evident.

Example $N = 83 = 10^2 - 17$.

To 17 we must successively add $2 \times 10 + 1 = 21, 23, 25$, etc.

until a square is obtained. As 83 is a primenumber, one will be forced to continue until 83 has been added; so the number of additions amounts to $\frac{83-21}{2}+1=32$. Now if in Table II we proceed horizontally from the base 83, this number is not found in the oblique line $4(a-1)$; so we can pass to the line $3(2a-3)$. This means however that it is not necessary to continue the additional operations until $\left(\frac{83+1}{2}\right)^2=42^2$, but only until $\left(\frac{31+1}{2}\right)^2=16^2$, the base of which is 31, as this number is the base of the larger one of the two numbers 81 and 87 between which 83 is situated.

If it is found that 83 is not a threefold, we can pass from the line $3(2a-3)$ over the lines $8(a-2)$, $5(2a-5)$ and $12(a-3)$, to the line $7(2a-7)$. So the operation can be stopped at $\left(\frac{19+1}{2}\right)^2=10^2$.

In the case of the number $N=112303$ (see the introduction) one would have to perform nearly 398 additions, (if the numbers to be added were combined in groups of 4 and 6). If however division proves that none of the numbers 3, 7, 11, 13, 17, 19, 23 is a factor, only 211 additions are necessary.

III. *Determination of non-divisors.*

If we put $N=ab+c$, any codivisor of a and c or of b and c will be divisor of N, whilst a divisor of c relative prime to a and b cannot be a divisor of N.

Example $N=73489207$ [1]).

We put $N=8573^2-7122$

$$\text{or} = 8573^2-1^2-7121 = 8574\times 8572-7121$$

$$=2\times 3\times 1429\times 4\times 2143-7121;$$

this proves that 3, 1429, 2143 and 7121 are non-divisors of N.

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[1]) This number was not obtained by multiplication of smaller numbers but chosen arbitrarily. Likewise all the other numbers of five and more figures decomposed in this study were chosen at random, the number mentioned in the introduction excepted.

If on the other hand we put

$$N = 8573^2 - 2^2 - 7118 \quad = 8575 \times 8573 - 7118$$

$$= 5^2 \times 7^3 \times 8573 - 2 \times 3559,$$

$$N = 8573^2 - 3^2 - 7113 \quad = 8576 \times 8570 - 7113$$

$$= 2^7 \times 67 \times 10 \times 857 - 3 \times 2371,$$

$$N = 8577 \times 8569 - 7106 \ = 9 \times 953 \times \mathbf{11} \times \mathbf{19} \times 41 - 2 \times \mathbf{11} \times 17 \times \mathbf{19},$$

then N appears to be divisible by 11 and 19, but not by 8573, 3559, 67, 857, 2371, 953, 41, 17.

The series 7121, 7118, 7113, 7106, etc. of the numbers c shows the differences 3, 5, 7, etc. if a is increased and b diminished by unity.

The division having been achieved, we find as quotient $N_1 = 351623 = 593^2 - 26$, with 13 (factor of 26) as non-divisor.

By means of 14 operations already 23 of the 106 prime numbers minor to $\sqrt{N}$ can be declared to be non-divisors; by testing 47 and 61 by direct division all divisors minor to 71 are shut out.

If we put $N = \left(\frac{N+1}{2}\right)^2 - \left(\frac{N-1}{2}\right)^2$, at least one of the two successive numbers $\frac{N+1}{2}$ and $\frac{N-1}{2}$ is divisible by 2 or 3 or 2×3.

The divisors of the quotient so obtained are non-divisors of N.

IV. *Determination of the difference of the factors.*

We put $a = b + m$ and therefore $N = (b + m)^2 - b^2 = m\,(m + 2b)$. Now we try to determine b and m by assuming for m a value near to $\sqrt{N}$, calculating $N^2 - m^2$ and then $2\,b = \frac{N^2 - m^2}{m}$. If this quotient be not an entire number we repeat this calculation with an m smaller by two (m being odd with N).

(To be continued.)

Mathematics. — "*A formula for the volume of the prismoid.*"

By Prof. JAN DE VRIES.

As is known the volume of the prismoid is given by the expression

$$I = \frac{1}{6} h (P + Q + 4 M),$$

where h denotes the distance of the parallel faces, P and Q represent the areas of these faces and M stands for the area of the section with the plane bisecting the distance of the parallel faces.

We wish to show here, that this formula forms a special case of a more general one, in which M has been replaced by the area D_{q}^{p} of the section with the plane dividing the distance h between P and Q in the ratio of p to q.

By joining any point O of the face P with all the vertices of the prismoid and combining these lines two by two by the necessary planes the prismoid is divided into: 1°. a pyramid with vertex O and base Q, 2°. a number of tetrahedrons with three vertices in P and one vertex in Q, 3°. a number of tetrahedrons with two vertices in each of the faces P, Q.

Let us first consider a tetrahedron $P_1 P_2 Q_3 Q_4$ of the last group and suppose that the edges $P_1 Q_3$, $P_1 Q_4$, $P_2 Q_3$, $P_2 Q_4$ meet any plane parallel to $P_1 P_2$ and $Q_3 Q_4$ in the points D_{13}, D_{14}, D_{23}, D_{24}. Then we have

$$D_{13} D_{23} = D_{14} D_{24} = \frac{q}{p+q} P_1 P_2 \text{ and } D_{13} D_{14} = D_{23} D_{24} = \frac{p}{p+q} Q_3 Q_4.$$

Now this tetrahedron can be considered as the second of three tetrahedrons, into which a certain prism with three side-faces is divided by two diagonal planes. Of any of the two triangles in the parallel planes limiting this prism two of the three sides are parallel to sides of the parallelogram $D_{13} D_{23} D_{24} D_{14}$; so the areas of these figures are in the ratio of $P_1 P_2 \times Q_3 Q_4$ to 2 times $D_{13} D_{23} \times D_{13} D_{14}$, i. e. as $(p+q)^2$ to $2pq$. So the volume of the tetrahedron is equal to

$$\frac{1}{3} h \,.\, \frac{(p+q)^2}{2pq} D',$$

D' representing the area of the parallelogram.

The pyramid with vertex O and base Q determines in the plane

D^{p}_{q} an area D'' equal to $\frac{p^2}{(p+q)^2}\,Q$. So the volume of this pyramid is equal to

$$\frac{1}{3}\,h\,.\,\frac{(p+q)^2}{2\,pq}\,D'' + \frac{1}{3}\,h\,.\,\frac{2\,q-p}{2\,q}\,Q.$$

In the same manner we find that the volume of a tetrahedron of the second group can be represented by

$$\frac{1}{3}\,h\,.\,\frac{(p+q)^2}{2\,pq}\,D''' + \frac{1}{3}\,h\,.\,\frac{2\,p-q}{2\,p}\,P.$$

Moreover we have $\Sigma\,D' + D'' + \Sigma\,D''' = D^{p}_{q}$; so the volume of the entire prismoid is given by the formula

$$I = \frac{1}{6}\,h\left[\frac{2\,p-q}{p}\,P + \frac{2\,q-p}{q}\,Q + \frac{(p+q)^2}{pq}\,D^{p}_{q}\right] \quad . \quad . \quad (1)$$

For $p = q = 1$ we reobtain the result $I = \frac{1}{6}\,h\,(P + Q + 4\,M)$, as it ought to be. From the two formulae we deduce

$$(p+q)^2\,D^{p}_{q} = q\,(q-p)\,P + p\,(p-q)\,Q + 4\,pq\,M \quad . \quad . \quad (2)$$

For $p = 1$, $q = 2$ we find the remarkably simple relation

$$I = \frac{1}{4}\,h\,(Q + 3\,D^{1}_{2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Still in another manner the volume can be expressed by means of *two* parallel sections. By interchanging p and q in (1) we get

$$I = \frac{h}{6\,pq}\left[(2\,pq - p^2)\,P + (2\,pq - q^2)\,Q + (p+q)^2\,D^{q}_{p}\right].$$

By addition of this equation to (1) we find

$$I = \frac{h}{12\,pq}\left[(4\,pq - p^2 - q^2)\,(P+Q) + (p+q)^2\,(D^{p}_{q} + D^{q}_{p})\right] . \quad . \quad (4)$$

For $p = \sqrt{3} + 1$ and $q = \sqrt{3} - 1$ this relation gives finally

$$I = \frac{1}{2}\,h\left(D^{\sqrt{3}+1}_{\sqrt{3}-1} + D^{\sqrt{3}-1}_{\sqrt{3}+1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Physics. — Dr. L. H. Siertsema: "*The dispersion of the magnetic rotation of the plane of polarisation in negatively rotating salt-solutions.* II. *Further measurements with potassium ferricyanide.*" Communication N°. 76 from the Physical Laboratory at Leiden. (Communicated by Prof. H. Kamerlingh Onnes.)

In the Lorentz jubilee volume [1]) I have discussed some measurements on the negative rotation of solutions of potassium ferricyanide, where the rotation of the salt was found by letting the pencil of light traverse the solution and water alternately and by measuring each time the position of the dark band in the spectrum.

These measurements have now been continued with the same salt, in the first place to investigate the influence of some small improvements made in the apparatus and also in order to investigate whether the rotation constants, as they have been calculated for the different wave-lengths are really independent of the concentration of the solution.

The chief alteration in the apparatus is that now the two adjoining tubes, one for the solution and one for water are closed by means of the same glass plates. While formerly, when the two tubes were interchanged the spectrum was considerably displaced, because the glass plates were not perfectly parallel to each other, this displacement has now been reduced to a small quantity, and hence more certainty has been obtained in the determination of the wave-length. Moreover I could make use of sunlight which enabled me to determine the wave length still more accurately.

For the rest the observations were made exactly in the same way as before, with solutions of 1, ½ and 2 per cent. For the calculation I have availed myself of the measurements of the magnetic rotation in water, published in the Bosscha Jubilee volume [2]).

The results are given in the following table. The symbols used are the same as in the previous Communication [3]) viz.:

λ_1	the wave length for the dark band in the solution.	
λ_2	" " " " " " " " water.	
φ_{λ_1}	the rotation constant of the solution	for the wave-length λ_1
$\varphi_{w\lambda_1}$	" " " " water	
$\varphi_{s\lambda_1}$	" " " " the salt	

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[1]) Arch. Néerl. (2) 5 p. 447; Comm. Phys. Lab. Leiden N° 62.

[2]) Arch. Néerl. (2) 6 p. 825; Comm. Phys. Lab. Leiden N° 73.

[3]) " " (2) 5 p. 457; " " " " N° 62 p. 451.

Solution $1\,^0/_0$.

λ_1	λ_2	φ_{λ_1}	$\varphi_{w\lambda_1}$	$\varphi_{s\lambda_1}$
512	546	1.182	1.352	— 16.2
524	556	1.137	1.288	— 14.4
512	549	1.167	1.352	— 17.7
525	556	1.137	1.283	— 13.9
517	548	1.172	1.325	— 14.6
543	572	1.068	1.195	— 12.1
545	573	1.064	1.186	— 11.6
542	570	1.076	1.200	— 11.8
566	590	0.996	1.093	— 9.1
578	603	0.947	1.043	— 9.1
611	635	0.842	0.919	— 7.3

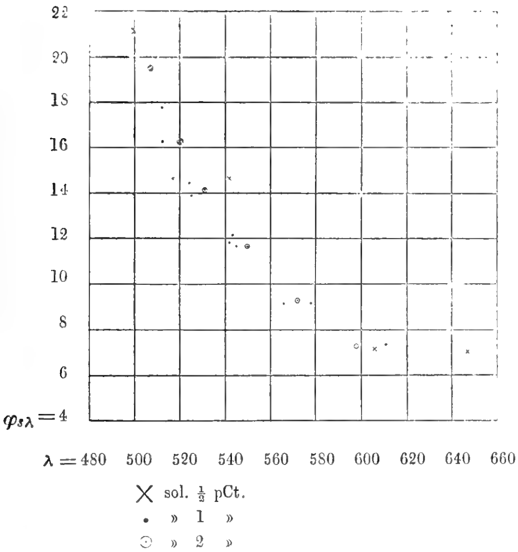
Solution $\frac{1}{2}\,^0/_0$.

500	519	1.314	1.423	— 21.2
647	661	0.770	0.807	— 7.0
606	617	0.899	0.936	— 7.1
541	558	1.128	1.204	— 14.6

Solution $2\,^0/_0$.

598	646	0.810	0.966	— 7.2
520	597	0.969	1.309	— 16.2
550	612	0.916	1.163	— 11.6
572	625	0.872	1.068	— 9.2
507	595	0.976	1.380	— 19.3
531	601	0.954	1.252	— 14.1

The results have been combined in the annexed figure, in which the rotation constant $\varphi_{s\lambda_1}$ is represented as a function of the wave-length λ_1. If we take into consideration that with the solutions of $\frac{1}{2}$ percent the accuracy becomes smaller, on account of which we must not attach too much importance to the two deviating results for $\lambda_1 = 541$ and 647, it appears that no distinct variation of the rotation constant $\varphi_{s\lambda}$ with the concentration can be demon-



strated. The strong increase of $\varphi_{s\lambda}$ when approaching the limit of absorption ($\lambda_1 = 490$) is again shown here very distinctly.

The measurements will be continued with other salts. Possibly also the important theoretical results on the magnetic rotation in absorbing solutions found by VOIGT[1]) can experimentally be tested by means of this method.

Physiology. — Dr. C. WINKLER presents in the name of Dr. J. K. A. WERTHEIM SALOMONSON an essay, entitled: "*A new law concerning the relation of stimulus and effect.*"

Whenever a stimulus, is applied to a biological element this fact will generally be followed by a stimulation-effect. In cases where a contractile protaplasma is concerned, the effect will consist in contraction of the protoplasma. With other elements, like neurones the stimulation is transformed into a potential-wave, extending itself with moderate rapidity, some dozens of meters per second, along the nerve-

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[1]) Drude's Ann. 6 p. 734.

fibre, and in this way conducted towards the end arborization, where a new nervecell is exposed to a temporary electrical *P. D.* procuring an adequate stimulus for that cell; here the same fact will repeat itself, we obtain again a potential-wave, extending along the second neurone. If a lower motor neurone were stimulated, then too a potential-wave would extend itself along the nerve-fibre, to pass finally on to the muscle-protoplasma, where the stimulus is transformed into potential energy, which in this case however would be a contraction.

It is sufficiently known, that by increasing the stimulus the effect increases too, but as to the exact relation between the two magnitudes — cause and effect — we are in ignorance as yet. A law does exist concerning a definite kind of stimuli, acting upon some of our senses; a law indicating approximately in what manner the intensity of sensation increases with the magnitude of sense-stimulation — the well-known psycho-physical law of WEBER—FECHNER. This law however decidedly does not prevail for the stimulation of contractile substance: the endeavour of PREYER to establish a *myophysical* law, quite analogous to the psycho-physical law is to be considered as a failure [1]).

Apart from the above mentioned treatise of PREYER, which may be safely left out of consideration, together with the objections raised against it by LUCHSINGER [2]) and BERNSTEIN [3]), a few other communications have been published concerning the subject occupying us here.

Among the first publications on this matter ranks that of HERMANN [4]), who in 1861 loaded a muscle with different weights and measured the intensity of the smallest stimulus, sufficient to cause a minimum contraction. The results of these researches are rendered clearly enough by his words: „Es ergab sich, dasz bei gleichmässigem „Wachsthum der verlangten Energie die Reizgrössen anfangs langsam, „dann schneller wachsen und bald nicht mehr erreicht werden. Mit „anderen Worten: bei gleichmässig zunehmenden Reizgrössen wachsen „die Energien zuerst schnell, dann immer langsamer und erreichen „bald ein Maximum (die sog. „absolute Kraft")."

I wish to claim attention for the fact that the conclusion thus put before us is substantially a right one, but — as has been demonstrated

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[1]) W. PREYER, Das myophysische Gesetz. Jena 1879.

[2]) LUCHSINGER, Archiv f. Physiologie. VI. S. 295, 642. VIII. S. 538.

[3]) BERNSTEIN, " " " VI. S. 403. VII. S. 90.

[4]) HERMANN, Archiv f. Anatomie u. Physiologie. 1861. S. 369—396. Id., Handbuch d. Physiologie. I. 1. S. 108. 1879.

already by TIGERSTEDT — it cannot be based solely upon the single series of experiments published by HERMANN, but has probably been deduced from several non-published series of experiments.

Among later investigators I will only mention A. FICK [1]), TIEGEL [2]), LAMANSKY [3]), A. B. MEYER [4]). FICK arrives at the conclusion that with small magnitudes of stimulus, rising only very little above the thresholdvalue, the effect measured by the intensity of contraction, the load remaining constant, is pretty well proportional to the stimulus, but increases less rapidly afterwards. At last a maximum is obtained, increasing no more by any increment of stimulus. This result is confirmed by A. B. MEYER.

First of all among the latest investigators, the results of whose researches demand a closer examination, TIGERSTEDT [5]) ought to be mentioned, who arrives by a series of most carefully executed experiments with the aid of an unimpeachable instrumentarium, to the conclusion that „bei gleichförmigem Zuwachs der Stärke des electrischen Reizes die Muskelzuckungen zuerst schnell, dann immer langsamer zunehmen, um schlieszlich sich einem Maximum asymptotisch zu nähern. Dieses Gesetz gilt, sowohl bei directer Muskelreizung als bei indirecter von Nerven aus, für constante und inducierte Ströme, in welcher Richtung sie den Nerven oder den Muskel durchflieszen mögen."

Finally we ought to remember A. WALLER [6]), who also published a series of experiments. Although starting from the unexact premises, that the phsychological law of WEBER—FECHNER prevails here too — with which statement however the obtained results are in absolute disaccordance — I believe his series to possess great importance from different points of view. Furtheron we will repeatedly have to quote and to examine them more closely.

FICK [7]) accepted a lineal relation between feeble stimuli and the effect, whilst by increased stimuli no further increment of effect was

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[1]) A. FICK, Untersuchungen über elektrische Nervenreizung. 1864. Id., "Studien über elektrische Nervenreizung" in Festschrift f. E. H. WEBER. 1871.

[2]) TIEGEL, Arbeiten aus der physiologischen Anstalt in Leipzig. X. 1875. Bericht der Sächsischen Gesellschaft f. Wissenschaften, Math.-Phys. Classe. 1875.

[3]) LAMANSKY., Studien des physiologischen Instituts zu Breslau. IV. S. 220. 1868.

[4]) A. B. MEYER, Untersuchungen aus dem physiologischen Laboratorium der Züricher Hochschule. I. pag. 36—50. 1869.

[5]) R. TIGERSTEDT, Mittheilungen vom physiologischen Laboratorium des Carolinischen Instituts in Stockholm. 3es Heft. 1884.

[6]) A. WALLER, Points relating to the WEBER—FECHNER Law. Brain 1895, p. 200.

[7]) l. c. pag. 14.

to be obtained; a mathemical expression or a graphic schema of this law was not in accordance with his results, as was demonstrated afterwards by TIGERSTEDT [1]).

Neither does the psycho-physical law fit into the framework of muscle-stimulation.

TIGERSTEDT's hypothesis that the relation between stimulus and effect might possibly be expressed by a hyperbola, has already been rejected by himself.

Hitherto then there does not exist a law expressing this relation with sufficient accuracy.

In the following essay an endeavour is made to fill up this deficiency. I believe indeed to have succeeded, starting from a few definite premises, to establish a law indicating with great accuracy the relation between intensity of stimulus and effect.

To begin with I intend giving the deduction of the law, to proceed afterwards to test this law to the results, obtained hitherto by experiments about muscle-contractions. In a following essay we shall try to examine whether this law holds good in other cases than those where muscle-contraction is concerned.

As soon as a stimulus is applied to the muscle-protosplasma this last suffers a change. A part of it is transformed, and potential energy is developed by means of which labour may be performed. If K be the force of tension, then $\int K\,dt$ represents the whole of the potential energy. On this sum of the potential energy depends the performed external labour.

If we call $\int K\,dt$ the effect or the quantity of muscular power made free, then we may deem this to be ultimately dependent on a quantity of changed chemical substance, on a number of transformed molecules.

Supposing a stimulus R to be the cause of the transformation, by which a quantity E is transformed, we will see what is going on during a small particle of time. In a very small particle of time dt the stimulus may change for a certain amount dR, in consequence of which the quantity of matter is changed for a certain amount $-dE$. Under the restriction, that these increments are extremely small, we may admit proportionality between increment of stimulus

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[1]) l. c. pag. 5.

and increment of effect, we may therefore state provisionally:

$$\frac{-dE}{dt} = a \frac{dR}{dt}$$

Physical chemistry however has made us acquainted with a law of mass-action, GULDBERG and WAAGE's law, stating that in a small particle of time the quantity of transformed matter is proportional to the quantity of transformable substance. Applying this law here, and calling the extant quantity of transformable substance E_r, we obtain:

$$\frac{-dE_r}{dt} = B\ E_r \frac{dR}{dt} \quad \ldots\ldots\ldots \quad (1)$$

wherein B represents a constant. Multiplying with dt and then separating the variables we obtain:

$$\frac{-dE_r}{E_r} = B\, dR \quad \ldots\ldots\ldots \quad (2)$$

and after integrating:

$$\log \text{nat } E_r = -BR + \text{constant} \quad \ldots\ldots \quad (3)$$

We may express this formula still in a somewhat different way, for which I choose from several motives this formula:

$$\log \text{nat } \frac{E_r}{A} = -BR \quad \ldots\ldots\ldots \quad (4)$$

which may be written:

$$E_r = A\varepsilon^{-BR} \quad \ldots\ldots\ldots \quad (5)$$

A and B representing constants and ε the base of the Napierian logarithms.

This formula represents the quantity of transformable matter still extant after the action of the stimulus R. If the quantity of substance originally extant — before the action of the stimulus — may have been E_0, then the quantity of transformed substance, in other words the effect, amounts to:

$$E = E_0 - A\varepsilon^{-BR} \quad \ldots\ldots\ldots \quad (6)$$

In direct consequence of the initial condition, i. e. that R being $= 0$, E too must be $= 0$, it follows that $E_0 = A$, therefore (6) becomes

$$E = A\,(1 - \varepsilon^{-RB})\ .\ .\ .\ .\ .\ .\ .\ .\ . \quad (7)$$

For practical use we ought still to add a restriction to this expression. We know that a stimulus operates an effect only then, when its intensity has risen above a certain initial value, the so-called threshold-value, „Schwellenwerth". Therefore we are not allowed to count the full value R, but only the remainder obtained by substracting the threshold-value; the latter being expressed by C, our formula finally then will be found to stand thus:

$$E = A\,\{1 - \varepsilon^{-B(R-C)}\} \text{ for } R > C \quad .\ \ .\ \ . \quad (8)$$

In this way we have established a formula, expressing a relation between magnitude of stimulus and effect, based on a few suppositions initially taken for granted. These premises were:

1^0. that the increment of stimulus being very small, the increment of effect was proportionate to it,

2^0. that the quantity of transformed substance in a very small particle of time was proportionate to the extant quantity of transformable substance (Law of mass-action).

3^0. that the effect was proportionate to the quantity of transformed substance.

For the present I do not intend testing the appropriateness of these premises. If the suppositions are inexact, then the law too shall prove infallibly inexact. If on the other hand the law is proved to be in accordance with the results of experiment, this sole fact will be sufficient proof for the probability of our premises. Undoubtedly however we will have even then to examine more closely the significance of the premises, from which we started.

What now is the real meaning of the formula (8). In the first place I wish to lay some stress on its analogy with several well-known physical laws.

For the charging of a condensator through a non-inductive resistance we possess a similar formula, wherein however the quantity of electricity extant is substituted for E, and the time t for the stimulus $R-C$. For the monomolecular reactions too we find a similar expression, indicating the quantity of substance transformed after a definite

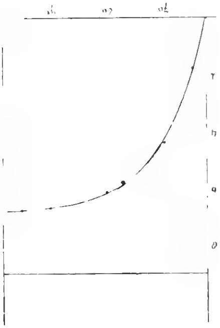


Fig. 1.

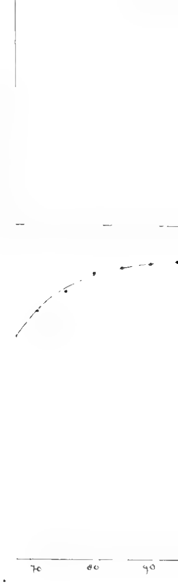


Fig. 5.

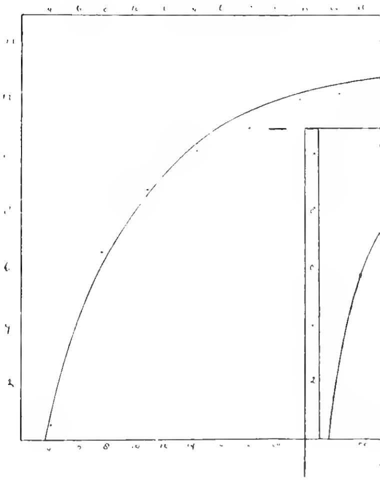
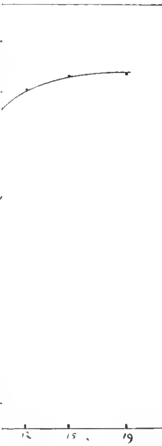


Fig. 6 and 7.



g. 10.

lapse of time since the beginning of the reaction. This conformity however is hardly astonishing, when we consider the genesis of the formula. Furtheron we may perhaps be enabled to demonstrate that in the domain of muscle-function phenomena may occur that are time-functions and proceed in absolutely the same manner as the laws for the discharge of a condensator or the velocity of monomolecular reactions, as e.g. the process indicating recovery from fatigue.

The formula (8) represents an exponential asymptotic curve, viz. with the increment of the variable R, E too increases, but gradually more slowly, till at last it approaches a definite maximum, never quite attainable. Whilst the increment of E is strongest, when the magnitude of R is very small, the increment diminishes gradually, as R becomes greater.

This relation will appear more clearly by the aid of a graphical expression of this function. If in a rectangular system of coördinates we take the different values of R as abscissae and after assuming definite values for the constant quantities A, B and C, calculate the magnitude of the E, belonging to each value of R, and take them as ordinates then we are enabled to plot the curve expressed by (8). (See figures 1—10).

The zeropoint on the x-axis is determined by the quantity C. The factor B determines the steepness of the curve. By a very simple method this steepness may be indicated in a still more striking manner. Therefore we consider firstly that the curve shows a marked tendency to attain a definite maximum, indicated by the constant A, which fact is evident by a single glance at the figure as well as at the formula.

The value $\varepsilon^{-B(R-C)}$ grows smaller in reason of the increase of R, until R being $=\infty$, $\varepsilon^{-B(R-C)}$ will have become $=0$, thus reducing the formula to $E_{R=\infty}=A$.

We now may express the steepness of curve by the magnitude of stimulus necessary to make $B(R-C)=1$. The formula then will stand thus.

$$E_{B(R-C)=1} = A\left(1-\frac{1}{\varepsilon}\right) = 0{,}632\ A$$

or about $^2/_3\,A$. From this condition, follows:

$$R-C=\frac{1}{B}\cdot \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

viz. we must give to the $R-C$ a magnitude making it equal to $\frac{1}{B}$ to obtain a stimulus operating an effect that will amount to $^2/_3$ of the maximal effect. We will call this value $\frac{1}{B}$ the *stimulus constant*. The knowledge of this constant, combined with the knowledge of the *thresholdvalue* of the stimulus and the *maximum-value* A, supplies us with a complete image, of the magnitude of effect in relation to the stimulus, the two former ones giving moreover an image of the rapidity with which our curve rises.

We will now try to test our law to the facts, furnished by the literature on this subject. For several reasons I have not deemed it necessary to add my own results: the facts, supplied by former publications appear to me quite sufficient.

As the older series, published by HERMANN, VOLKMANN i. a., are obtained almost without exception by the „Ueberlastungsverfahren," we cannot make use of these for our purpose. Never yet has it been proved that the maximum force of a muscle during the contraction may be considered as representing the total effect: generally this will not even approximately be the case.

We may however safely assume this to be the case for the lifting-height of an isotonic contraction, provided the tension be excessively small.

This last restriction made, we may assume the lifting-height to be proportionate to the force-integral, to $\int K\,dt$.

In the litterature I found communications by two investigators, who by the acknowledged accuracy of their work may be said a priori to offer reliable results. Therefore I have restricted myself to the results of these two physiologists, R. TIGERSTEDT and A. WALLER (l. c.)

I will now proceed to present a few series, calculated from the results of their experiments after the above-mentioned formula.

$$E = A\{1 - \varepsilon^{-B(R-C)}\},$$

Here E represents the amount of contraction R the magnitude of the stimulus applied. If in this formule the value R is determined, whilst to A, B and C the value, mentioned at the head of each series is allotted, then we may from all these calculate E (2^d column). The E's measured by TIGERSTEDT and WALLER, are indicated in the third column, whilst the fourth column ϱ gives the differences between calculated and measured values. Underneath the series are placed $\Sigma\varrho^2$, the sum of the squares of the errors and also the mean error of a single observation.

TABLE I, calculated from the numbers of A WALLER. (Brain 1895, pag. 209).

A = 12.838

B = 6.4029

C = 1.59284

R	E calc.	E meas.	ϱ
1.65	3.935	4.	— 0.065
1.79	9.205	9.	+ 0.205
1.87	11.692	12.	— 0.310
2.13	12.426	12.5	— 0.174
2.32	12.716	12.5	+ 0.216

$\Sigma\varrho^2. = 0.2193$

$\varrho_m = 0.3310$

TABLE II, calculated from' the numbers of R. TIGERSTEDT. Versuch 6. 1 Indirecte Reizung vom Nerven aus; absteigende Schliessungsinductionsschläge.

Fig. 1

A = 6.87456

B = 0.13016

C = 62.59855

R.	E calc.	E meas	ϱ.
65.	1.8454	1.8	+ 0.0454
70.	4.2512	4.4	— 0.1488
75.	5.5061	5.4	+ 0.1061
80.	6.1619	6.1	+ 0.0619
85.	6.5022	6.6	— 0.0978
90.	6.6803	6.65	+ 0.0303
95.	6.7732	6.75	+ 0.0232

$\Sigma\varrho^2. = 0.050318$

$\varrho_m = 0.11216$

TABLE III. TIGERSTEDT Versuch 7, II pag. 11. Indirecte Reizung von Nerven aus. Absteigende Schlieszungsinductionsschläge.

Fig. 2.

$A = 11.72188$

$B = 0.35383$

$C = 41.63615$

R	E calc.	E meas.	ϱ
44	6.6431	6.7	+ 0.0569
46	9.2190	9.4	+ 0.1810
48	10.4890	10.4	— 0.0890
50	11.1144	11.15	+ 0.0356
55	11.6182	11.4	— 0.2186
60	11.7043	11.55	— 0.1543
65	11.7190	11.7	— 0.0150
70	11.7213	11.8	+ 0.0787
75	11.7218	11.75	+ 0.0282
80	11.7219	11.8	+ 0.0781
85	11.7219	11.7	— 0.0219
90	11.7219	11.75	+ 0.0281
95	11.7219	11.8	+ 0.0781

$\Sigma \varrho^2 = 0.137594$

$\varrho_m = 0.1173$

TABLE IV. TIGERSTEDT. Versuch 20 II a, pag. 13. Indirecte Reizung vom Nerven aus. Absteigende Oeffnungsinductionsschläge.

Fig. 3.

$A = 13.62872$

$B = 0.17218$

$C = 35.05147$

R	E calc.	E meas.	ϱ
40	7.81520	7.8	— 0.01520
45	11.17069	11.2	+ 0.02931
50	12.58954	12.75	+ 0.16046
55	13.18925	13.05	— 0.13925
60	13.44281	13.3	— 0.14281
65	13.55002	13.5	— 0.05002
70	13.59534	13.5	— 0.09534
75	13.61545	13.45	— 0.16445
80	13.62260	13.55	— 0.07260
85	13.62605	13.7	+ 0.14395
90	13.62762	13.85	+ 0.22238
95	13.62825	13.8	+ 0.17175

$\Sigma \varrho^2 = 0.210204$

$\varrho_m = 0.15283$

TABLE V. Tigerstedt Versuch 27 a I pag. 14. Indirecte Reizung des Muskels Aufsteigende Oeffnungsinductionsschläge

Fig. 4.

A = 10.93776

B = 0.090832

C = 53.20286

R	E calc.	E meas.	ϱ
60	5.0385	5.3	+ 0.0615
65	7.1910	7.0	— 0.1910
70	8.5593	8.9	+ 0.3417
75	9.4275	9.6	+ 0.1725
80	9.9788	10.3	+ 0.3212
85	10.3288	10.5	+ 0.1712
90	10.5512	10.6	+ 0.0488
95	10.6923	10.7	+ 0.0077

$\Sigma \varrho^2 = 0.321598$

$\varrho_m = 0.25361$

TABLE V bis. Tigerstedt 27 a I, the second series of numbers being neglected.

Fig 5.

A = 10.827

B = 0.1112

C = 54.19

R	E calc.	E meas.	ϱ
60	5.1523	5.3	+ 0.1477
[65]	[7.5713]	[7.0]	[— 0.5713]
70	8.9604	8.9	— 0.0604
75	9.7562	9.6	— 0.1562
80	10.2129	10.3	+ 0.0871
85	10.4747	10.5	+ 0.0253
90	10.6250	10.6	— 0.0250
95	10.7110	10.7	— 0.0110

$\Sigma \varrho^2 = 0.058833$

$\varrho_m = 0.12128$

Much better values are obtained by neglecting the second number of Table V in making the calculation. We then obtain Table V bis, in which the mean error is diminished by half.

TABLE VI. TIGERSTEDT Versuch 28 I, pag. 14. Indirecte Reizung vom Nerven aus. Aufsteigende Oeffnungsinductionsschläge.

A = 10.873652

B = 0.149257

C = 39.70838

R	E calc.	E meas	ϱ
40	0.4630	0.95	+ 0.4870
42	3.1481	2.6	− 0.5481
44	5.1431	5.15	+ 0.0069
46	6.5303	6.65	+ 0.1197
48	7.7193	7.8	+ 0.0807
50	8.5330	8.9	+ 0.3670
52	9.1374	9.45	+ 0.3126
54	9.5852	9.8	+ 0.2148
56	9.9180	10.15	+ 0.2320
58	10.1649	10.4	+ 0.2351
60	10.3476	10.45	+ 0.1024
62	10.4833	10.5	+ 0.0167
64	10.5841	10.6	+ 0.0159
66	10.6588	10.65	− 0.0088
68	10.7145	10.65	− 0.0645
70	10.7555	10.7	− 0.0555
75	10.8178	10.7	− 0.1178
80	10.8473	10.7	− 0.1473
85	10.8612	10.75	− 0.1112

$$\Sigma \varrho^2 = 1.012387$$

$$\varrho_m = 0.25154$$

TABLE VII. TIGERSTEDT Versuch 32. Directe Reizung des curarisierten Muskels mit absteigenden Oeffnungsinductionsschlägen.

Fig 6.

A = 13.189552

B = 0.146606

C = 3.65321

R	E calc.	E meas.	ϱ
4.4	0.83633	0.5	− 0.33633
7.7	6.20483	6.65	+ 0.44517
10.9	8.63098	8.85	+ 0.21902
14.4	10.46045	10.2	− 0.26045
18.1	11.60301	11.0	− 0.60301
21.6	12.23968	12.0	− 0.23968
25.2	12.62932	12.2	− 0.42932
28.8	12.85908	13.0	+ 0.14092
32.8	13.00559	13.4	+ 0.39441
36.4	13.08097	13.5	+ 0.41903

$$\Sigma \varrho^2 = 1.38348$$

$$\varrho_m = 0.4446$$

TABLE VIII. Tigerstedt Versuch 38.
Indirecte Reizung des Muskels; aufsteigende Oeffnungsinductionsschläge.

Fig. 7.

$A = 9.974127$

$B = 0.1287685$

$C = 48.25489$

R	E calc.	E meas.	ϱ
55	5.7901	5.8	+ 0.0099
60	7.7766	7.8	+ 0,0234
65	8.8200	8.8	— 0.0200
70	9.3690	9.4	+ 0.0310
75	9.6558	9.65	— 0.0058

$\Sigma \varrho^2 = 0.00204021$

$\varrho_m = 0.03194$

In reviewing the series here communicated, we cannot but remark the singularly fine agreement between the calculated lifting-height and the measured numbers; an accordance such as is indeed hardly ever to be met with in physiology. The errors remain generally within limits, not exceeding the errors of observation. Usually the measurings are given with an accuracy of 0,1 m.m.; sometimes even 0,05 m.m. is noted. We may therefore safely assume that the observation error amounts to 0,1 m.m.; the mean error of a single observation in each series amounts from 0,1 to 0,15 m.m.; the smallest value amounts only to 0,03 m.m. and was found in Tigerstedt's series N⁰. 38. In the other series we find 0,08, 0,10, 0,11, 0,14, 0,15 and 0,25 m.m., whilst one of Waller's series offers a larger error, viz. of 0,33 m.m., just like Tigerstedt's series N⁰. 32, where the mean error of each observation even rises to 0,44 m.m.

For both errors, that in Waller's series and that in table VI, a very good reason seems to exist. We will begin with Tigerstedt's series.

A closer examination of the curve shows that probably the first observation of the series has caused this abnormality. Therefore I give for series 28 I a new approximation, in which the first value has been left out. We remark directly that the mean error is greatly diminished and has fallen from 0,25 to 0,08.

TABLE IX. Tigerstedt Versuch 28. I

(The first observation has been rejected).

Fig. 8

$A = 10.7698$

$B = 0.17784$

$C = 40.463$

R	E calc.	E meas.	ϱ
42	2.5756	2.6	— 0.0244
44	5.0279	5.15	— 0.1221
46	6.7462	6.65	+ 0.0962
48	7.9498	7.8	+ 0.1498
50	8.7940	8.9	— 0.1060
52	9.3854	9.45	— 0.0646
54	9.7994	9.8	— 0.0006
56	10.0859	10.15	— 0.0605
58	10.2940	10.4	— 0.1060
60	10.4359	10.5	— 0.0641
62	10.5354	10.6	— 0.0606
64	10.6054	10.65	— 0.0446
66	10.6544	10.65	+ 0.0044
68	10.6888	10.7	— 0.0112
70	10.7130	10.7	+ 0.0130
75	10.7461	10.7	+ 0.0461
80	10.7597	10.7	+ 0.0597
85	10.7654	10.75	+ 0.0654

$\Sigma \varrho^2 = 0.09878$

$\varrho_m = 0.08115$

Calculation shows that, in the case of an exceedingly feeble stimulus the response does not quite agree with the formula, though for all other stimuli, excepting only the minimum ones, this formula expresses perfectly in what manner the effect changes with the changing of the stimulus. When examining some of the curves added by TIGERSTEDT to his publication we will observe directly the same deviation. In his figures 2, 3, 4, 6, 9, 13 we see that the curves, instead of being invariably concave to the x-axis, show in the beginning a marked tendency to become slightly convex. I believe a good reason to exist for this fact, which, though not expressed in my formula, may easily find expression in it by a small alteration. I trust to be able to explain this question in a later communication.

We have now still to examine more closely WALLER's series. WALLER gives with his experiments not only the value of the lifting height for each magnitude of stimulation, but adds also the magnitude of the negative variation responding to each nerve-stimulation. In fig. 9 of his publication he presents a continuation of the same experiment, in which only the negative variation is recorded, and showing that the negative variation increases regularly with increasing strength of stimulation. In his figure 8 (l. c.) the graphic representation of this fact is given, reproduced by me in fig. 9; when stimulation is increased, the negative variation continues as a perfectly straight line, at any rate within the limits of the intensities used for excitation.

We may represent this relation by the simple mathematical expression $y = P + Qx$, y representing the negative variation, x the magnitude of stimulation and P and Q two constants. Inverting this, we may therefore also represent the magnitude of stimulation as a function of the negative variation:

$$x = L + Sy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

wherein L and S are new constants. At any rate we may assume here the negative variation to be a reliable indicator of the magnitude of stimulation used, and we may therefore, with the aid of the formula (10) correct the numbers stated for the value of stimulation by means of the numbers found for the negative variation, by putting the values

$$L = 1{,}41989 \text{ and } S = 0{,}04811$$

in that formula. We then obtain for R new values, communicated in the next Table X: (p. 356)

When from these new values for R and a few new constants E is again calculated, it will be seen that the agreement is as fine as we can desire, and that the mean error has fallen from 0,331 to 0,109.

Something more still is proved by these numbers of WALLER. In the first place that the magnitude of stimulus possesses another thresholdvalue for the negative variation than for the muscle-

TABLE X. Table calculated from the numbers given by A. WALLER, the Negative variation having been taken as measure for the magnitude of stimulation.

Fig. 10.

$R = 1.40989 + 0.04811$ Neg. var.

$A = 12.65056$

$B = 7.25272$

$C = 1.5971165$

Neg. var.	R meas.	λ	R (calc.)	E (calc.)	E meas.	ϱ
2.5	1.52	+ 0.01016	1.53016			
5	1.65	+ 0.00044	1.65044	4.0573	4	— 0.0573
7.5	1.79	— 0.01922	1.77078	9.0578	9	— 0.0578
12	1.97	+ 0.01721	1.98721	11.9033	12	+ 0.0967
15	2.13	+ 0.00154	2.13154	12.3886	12.5	+ 0.1114
19	2.32	+ 0.00398	2.32398	12.5857	12.5	— 0.0857

$\Sigma \lambda^2 = 0.00078722$ $\Sigma \varrho^2 = 0.035729$

λ middl. $= 0.01493$ ϱ middl. $= 0.1091$

contraction. Whilst for the former the thresholdvalue, expressed in the units chosen by WALLER, amounts to 1,40989, for the muscle-stimulation it amounts to 1,5971165. From this we may conclude to the important fact that this latter thresholdvalue consists of the total of two threshold-value's, that therefore the threshold from nerve to muscle is passed only then, when the stimulus acting on the nerve is increased with 0,18722 of WALLER's units.

We may easily calculate by the aid of the formula (10) that the negative variation, in the moment that the magnitude of stimulation reaches the threshold from nerve to muscle, must amount to 3,894 mM.; as WALLER has stated further that each mM. of galvanometric deflection responding to the negative variation, represents a potential difference of 0.0001 Volt, we know that on reaching the threshold for the stimulation passing from the nerve upon the muscle, there is a potentialdifference of 0,0003894 Volt.

I believe to have proved by the foregoing the correctness of the law expressing the relation between stimulus and effect.

I wish to add the remark that I believe this law to prevail also for other excitable organs, especially the senses.

I intend making further researches on this subject and communicating their results.

(January 23, 1902).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday January 25, 1902.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 25 Januari 1902, Dl. X).

The following papers were read:

Physics. — "*The effect of the induction coil in telephonic apparatus*" (1st part). By J. W. GILTAY. (Communicated by Prof. P. ZEEMAN.)

(Communicated in the meeting of November 30, 1901.)

When HUGHES some twenty years ago announced his invention of the microphone, this simple apparatus at once drew the general attention. Everywhere experiments were made with it. The microphone was connected with a couple of Leclanché-cells and a telephone, and the sound of a ticking watch, etc. was transferred microphonically to the telephone.

When shortly after attempts were made to use the microphone in practical telephony, it was soon made evident that on lines of some resistance the sound emitted was a great deal too faint. The slight changes of resistance taking place in the microphone were of so little importance when compared to the great resistance of the line, that only a slight, practically useless variation of the current appeared in the line.

It was then that EDISON, and at about the same time HOORWEG, found a simple means to overcome this obstacle: the microphone was no longer placed in the line but, with the battery, in the primary circuit of a small induction-coil. The ends of the secondary wire of that coil were connected with the two lines or with line and earth. By an exact selection of the number of secondary turns the undulating current of the battery was transformed into an alternating current of much higher potential, and telephonic messages could be sent over lines of much higher resistance than formerly.

Except for the use of the telephone at distances of but a few hundred metres, not a single telephonic apparatus nowadays is found without induction-coil.

The induction-coils generally used for telephonic purposes, are as a rule provided wich four layers of primary wire, each layer consisting of about 90 turns; the thickness of the wire generally is 0.5 mM. The secondary is usually made of 0.12 mM. thickness and consists of about 3000 turns. In the coil is an iron core.

Practice has shown that such a coil in most cases gives the best results.

The literature about the choice of the induction-coil is very scanty, very few experiments seem to have been made on this subject. In the well-known book of PREECE and STUBBS "a Manual of Telephony" is a table which gives the results of some comparative experiments made by Mr. ABREZOL for the Swiss Telegraphic-Department. The microphone used by him was that of Blake, then in general use; it was connected respectively with ten different induction-coils and then the observed intensity and „clearness" was noted down. The word "clearness" probably denotes the degree of articulation.

The intensity and "clearness" of one of the coils were both indicated by 1; for another coil, e.g. 0.3 was found for the intensity and 0.9 for the "clearness".

The small amount of what is communicated about these experiments in the book of PREECE & STUBBS makes me think, that the figures of the above-named table have not the slightest value.[1] Estimating the relation of the intensities of 2 sounds by hearing is rather precarious, but how it is possible to express the "clearness" in figures I cannot understand.

In the following lines I wish to give a description of the manner in which I have endeavoured experimentally to investigate whether in reality an increase of the number of primary turns over and above the usual number does not strengthen or improve the telephonic sound. And if that does *not* take place, what may be the reason.

To do so I made 10 different induction coils, described in Table I (page 360). The coils marked *A* are provided with an iron core, those marked *B* are not. For the rest, the coils bearing the same number are perfectly alike; so for instance, the only difference between 6*A* and 6*B* is that 6*A* has an iron core and 6*B* has none. When in future I speak of 2 coils, one of which has more primary layers than the other, I shall call the former for simplicity's sake the coil of higher order.

The microphone I used for these experiments was the "Hunnings-Cone", a modification of the Hunnings-microphone. To measure the resistance of this microphone, twelve of them were put in one circuit and the resistance was determined. Then I tapped on all the microphones and the total resistance was again determined and this was repeatedly done. The mean of all these measurements, divided by 12, amounted to about $3^1/_2$ Ohm.

From this microphone the ebonite mouthpiece was screwed off and a resonator, $Fa_3 = 682.6$ s. v., placed before the opening in such a manner that the narrower opening of the resonator was as close as possible to the vibrating plate of the microphone. Before the wider opening of the resonator an electro-magnetic tuning-fork Fa_3 was placed, kept in motion by a small accumulator. Resonator and tuning-fork were both made by KOENIG. With my first experiments the resonator was fastened in a cork ring, fitting in the aperture of the microphone. It was then however evident that the microphone vibrated very irregularly, as regards the intensity. Then the cork ring was removed and the resonator, clasped in a separate stand, was placed before the microphone, so that resonator and microphone-box could not touch each other; I then found that

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[1] I asked Mr. PREECE to inform me where the original description of these experiments was to be found, whereupon he answered me that he was sorry to say he did not remember whence he had the figures.

TABLE I.

Name of coil.	Number of layers of prim. wire.	Number of turns of prim. wire.	Resistance of prim. wire	Thickness of prim. wire.	Number of turns of sec. wire.	Resistance of sec. wire.	Thickness of sec. wire.	Length of coil between flanges.	External φ of wooden tube on which prim wire is wound.	Number of wires of which iron core consists.	Length of ironwires.	Thickness of ironwires.
3 A	3	270	0.8	0.5 m.m.	3050	216	0.12	57 m.m.	10.5 m.m.	75	60 m.m.	0.6 m.m.
3 B	3	270	0.8	〃	〃	214	〃	〃	〃	0		
6 A	6	540	1.85	〃	〃	247	〃	〃	〃	75	〃	〃
6 B	6	540	1.9	〃	〃	253	〃	〃	〃	0		
9 A	9	810	3.2	〃	〃	316	〃	〃	〃	75	〃	〃
9 B	9	810	3.2	〃	〃	295	〃	〃	〃	0		
12 A	12	1080	4.9	〃	〃	353	〃	〃	〃	75	〃	〃
12 B	12	1080	4.9	〃	〃	510	〃	〃	〃	0		
15 A	15	1350	6.5	〃	〃	363	〃	〃	〃	75	〃	〃
15 B	15	1350	6.4	〃	〃	373	〃	〃	〃	0		

the irregularity of the motion of the microphone had much decreased.

The battery used on the microphone generally consisted of one Leclanché-cell.

My original plan was to compare by hearing the effect of my different induction-coils. I therefore placed the vibrating tuning-fork before the microphone and connected by turns one of the two coils under comparison with the microphone and the telephone; of course in such a way that the primary wire was connected with the microphone and the battery, the secondary with the telephone.

However, I soon found that nothing was to be expected of such an investigation, the comparison by hearing of the intensity of 2 tones being exceedingly uncertain, unless the difference of intensity be very great. This is well known from the experiments of VOLKMANN and others [1]), which proved, that the ear does not hear any difference in the intensity of 2 tones, when it is smaller than about 30 per cent.

So I was obliged to compare the induced currents, given by the different coils, to each other in another way, and for this purpose I made use of BELLATI's electro-dynamometer [2]). It is true, that even in this way we cannot yet obtain very accurate figures on account of the great irregularity with which such a microphone and such a tuning-fork work; yet in the following pages it will be seen, that by combining a sufficient number of readings the figures finally speak for themselves.

The electro-dynamometer was provided with 2400 turns, thickness of wire 0.1 m.m., resistance 460 Ohm. The damping was brought about by means of a small circular piece of platinum in concentrated sulphuric acid. To determine the coefficient of damping the instrument was con-

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[1]) WUNDT, Physiologische Psychologie, 3rd edition Vol. I p. 364. This is also easily shown with a simple apparatus, indicated by HEYMANS, "Zeitschrift für Psychologie und Physiologie der Sinnesorgane", Vol. XXI, p. 351, sub. 1.

It has often astonished me that the innumerable improvements or rather modifications applied to the construction of our magneto-telephones (especially in the first years after BELL's invention) have scarcely any of them found their way. Although many of them were quite impossible, yet there were a few modifications from which something might have been expected. The cause of this is probably to be found in the above-mentioned fact: the reinforcement of the sound of the telephone arising from those improvements, will have been smaller than that minimum-limit.

[2]) Wied. Ann. Neue Folge Vol. XXV, 1885, p. 325. As the instrument nowadays differs in shape so widely from the sketch, given with the above mentioned description, I hereby give the drawing of the apparatus as it is made at present. The iron protecting-ring was not used by me.

nected with the secondary wire of an induction-coil, the primary wire of which being connected with a Leclanché-cell and with that microphone whose tuning-fork had been made to vibrate. As soon as the needle of the electro-dynamometer had undergone a deviation and was nearly in rest in that position, the connection with the secondary wire was broken; the needle oscillated around its original zeropoint and by reading the turning-points the coefficient of damping was determined in the usual manner. It was not possible to determine this coefficient whilst the alternating currents were passing through the instrument on account of the irregular variations in the intensity of these currents. The reader can convince himself of this by calculating K in one of the following tables from p_1 and p_2 or from p_2 and p_3.

The electro-dynamometer was provided with a concave mirror of 50 cm. focal distance. The lamp-stand and scale differed somewhat from that generally used, as is shown in fig. 1. At *c* an electric lamp was placed in the focus of a convex lens, on which a blackened vertical diameter had been drawn. The light of the small lamp was cast upon the concave mirror *a* of the electro-dynamometer and thence reflected to *d*. On the scale *cd* was in the middle the figure 0; the part of the scale on the right hand of 0 I called +, the other —. *ab* is a normal to the scale passing through the centre of the mirror. The scale was not divided into mm. but into smaller parts, (a very inconvenient thing in reading) and in such a way that 1.59 divisions of the scale were equal to 1 m.m.[1]).

During these investigations the electro-dynamometer was 3 times removed from its place and every time when again placed provided with fresh sulphuric acid. The coefficient of damping, the figure where *ab* cuts the scale and the length of *ab* (fig. 1) are therefore different for the different measurements. They were:

for the 1st time $K = 3.3$ $ab = 1830$ $b = -220$
„ „ 2nd „ $K = 4.5$ $ab = 1700$ $b = -260$
„ „ 3th „ $K = 5.3$ $ab = 1750$ $b = -202$

The first thing I had to do was to investigate whether I might take for granted that with alternate currents of that intensity as

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[1]) This remarkable lamp-stand was made by Messrs. **Nalder Bros** & Co. London. „Theuer und schlecht", to use the wellknown verdict of Prof. **Reuleaux**, with a slight modification.

they were used by me, the deviations of the electro-dynamometer were proportional to the square of the mean intensity of the current. I set to work as indicated in fig. 2.

The microphone was connected with an accumulator and the primary wire of coil 3 A. As is the case with all the experiments described in this paper, the resonator Fa 3 and the tuning-fork Fa 3 were placed before the microphone. One of the ends of the secondary wire of 3 A was connected with the electro-dynamometer and moreover with the pivot of a Morse-key; the other end of the secondary wire with the pivot of a 2nd Morse-key. The contacts of rest of the keys were connected with each other; the working contacts likewise, but with *that* wire were connected: 1°. a coil of wire, *a*, in which a bundle of ironwires was placed under an angle of 45 degrees to the plane of winding, quite similar to the coil and the bundle of ironwires of the electro-dynamometer; 2°. the secondary wire, *b*, of coil 3 B, into which however for this experiment an iron core had been put.

If the Morse-keys were not pressed down, the current induced in 3 A passed to the electro-dynamometer and thence along 9, 5, 6, 10 back to the secondary wire. If however both keys were pressed down, the current had to pass along 9, 1, *a*, *b*, 2, 10 and so passed through a 2nd Bellati-coil and a 2nd secondary wire. In that case the resistance as well as the self-induction of the secondary circuit was doubled and consequently the strength of the current reduced to half its former intensity.

The relation between current-strength and deviation I have determined for three different currents. In the first case the deviations with the currents i and $2\,i$ were respectively 118'.9 and 481'.7; in the second case those deviations were 93'.1 and 350'.4 and in the third case 21'.9 and 85'.1.

Three successive turning-points were always observed; the position of rest was calculated according to the well known formula $p_0 = p_2 + \frac{p_1 - p_2}{1 + k}$, from p_1 and p_2 as well as from p_2 and p_3. A further description is superfluous on account of Table II (p. 364), giving the entire measurement of the 2nd of the above-mentioned cases.

In this way I found in the three different measurements the numbers 4.05, 3.76 and 3.89 for the relation between the deviations brought about by the currents $2\,i$ and i. The mean of these three values is 3.9. So for our further measurements we can assume

TABLE II.

Zeropoint before the experiment + 311 } mean + 312.
" after " " + 312 }

Strength of current.	p_1	p_2	p_3	*a* calculated position of equilibrium from p_1 and p_2.	*b* id. from p_2 and p_3.	*c* $\frac{a+b}{2}$	*d* deviation for 2 *i* (from *c*).	*e* mean of 2 successive *d*.	*f* deviation for *i* (from *c*).	*g* mean *e*	*h* mean *f*	*i* *g* in minutes.	*k* *h* in minutes.	$\frac{i}{k}$
2 *i*	— 137	— 17	— 56	— 45	— 47	— 46	358							
i	+ 256	+ 184	+ 211	+ 201	+ 205	+ 203		386	109					
2 *i*	— 223	— 75	— 100	— 109	— 94	— 102	414							
i	+ 252	+ 181	+ 218	+ 198	+ 209	+ 203		415	109					
2 *i*	— 181	— 76	— 121	— 100	— 111	— 105	417							
i	+ 286	+ 186	+ 199	+ 209	+ 196	+ 202		404	110	387	106	350′.4	93′.1	3.76
2 *i*	— 130	— 63	—	— 79	—	— 79	391							
i	+ 268	+ 185	+ 201	+ 204	+ 197	+ 200		386	112					
2 *i*	— 99	— 43	— 94	— 56	— 82	— 69	381							
i	+ 224	+ 215	+ 226	+ 217	+ 223	+ 220		346	92					
2 *i*	— 93	+ 28	— 5	0	+ 3	+ 1	311							

Damping = 3.3, *ab* (fig. 1) = 1830, Point of intersection *b* (fig. 1) = — 220.

with sufficient accuracy that the deviations of the electro-dynamometer are proportional to the square of the mean current.

To investigate which of the induction-coils named in Table I, connected in the usual way to a telephonic apparatus, would give the strongest induction-current I set to work in the following way:

In the first place only coils with iron were compared with one another. In fig. 3, K and K 1 are the coils to be compared. I, II, III and IV are Morse-keys which by means of a simple arrangement can be all pressed down at the same time. In the position of rest the keys themselves are connected with the upper contacts; *cc* is a double switch by which the current flowing to the electro-dynamometer can be opened or shut. R is a rheostat, shunted to the coil of the electro-dynamometer to reduce the deviations to a suitable size. As is easily seen from the figure, coil K1 is connected with the electro-dynamometer and with the microphone, as long as the keys are *not* pressed down; when the keys *are* pressed down, coil K is connected with the electro-dynamometer and with the microphone. All contacts not being broken or made by the four keys exactly at the same time when the coils were interchanged, care was taken that the commutator *cc* was always opened a little time before the keys were pressed down and before they were relinquished again, as otherwise strong induction-currents might find their way to the electro-dynamometer, which would give rise to remanent magnetism in the bundle of iron wires and a great modification of the zero-point.

In the first place I have compared coil 3 A with 6 A, 9 A, 12 A and 15 A and after that again 6 A with 9 A, 9 A with 12 A and 12 A with 15 A.

Table III (p. 366) shows how the experiment to compare 3 A with 6 A was made; in quite the same way the six other measurements were made. From the first four measurements I got for the proportion of the intensity of the induced currents given by our five coils:

Coil:	3 A	6 A	9 A	12 A	15 A
Intensity of the induced current:	1	1.507	1.429	1.114	0.818

TABLE III.

Zeropoint before the experiment + 325 } mean + 328.
" after " " + 330 }

Coil.	p_1	p_2	p_3	*a* calculated position of equilibrium from p_1 and p_2.	*b* id. from p_2 and p_3.	*c* $\frac{a+b}{2}$	*d* deviation for 3 A (from *c*).	*e* mean of 2 successive *d*.	*f* deviation for 6 A (from *c*).	*g* mean *e*	*h* mean *f*	*i* *g* in minutes.	*k* *h* in minutes.	$\sqrt{\frac{k}{i}}$
3 A	+ 44	+ 107	+ 92	+ 92	+ 96	+ 94	234							
6 A	— 259	— 115	— 158	— 148	— 148	— 148		223	476					
3	+ 67	+ 131	+ 113	+ 116	+ 117	+ 116	212							
6 A	— 300	— 127	— 187	— 167	— 173	— 170		211	498					
3 A	+ 36	+ 138	+ 113	+ 114	+ 119	+ 117	211							
6 A	— 256	— 95	— 139	— 133	— 129	— 131		209	459	212	470	188′	427′	1.507
3 A	+ 43	+ 141	+ 115	+ 120	+ 122	+ 121	207							
6 A	— 244	— 86	— 140	— 123	— 127	— 125		209	453					
3 A	+ 42	+ 137	+ 112	+ 115	+ 118	+ 117	211							
6 A	— 255	— 99	— 145	— 136	— 134	— 135		210	463					
3 A	+ 35	+ 139	+ 113	+ 115	+ 119	+ 117	211							

Damping = 3,3, *ab* (fig 1) = 1830, point of intersection *b* (fig. 1) = — 220.

From this we find:

$$\frac{9\ \mathrm{A}}{6\ \mathrm{A}} = 0.95 \qquad \frac{12\ \mathrm{A}}{9\ \mathrm{A}} = 0.78 \qquad \frac{15\ \mathrm{A}}{12\ \mathrm{A}} = 0.74.$$

And from the 3 last measurements I got:

$$\frac{9\ \mathrm{A}}{6\ \mathrm{A}} = 0.782 \qquad \frac{12\ \mathrm{A}}{9\ \mathrm{A}} = 0.761 \qquad \frac{15\ \mathrm{A}}{12\ \mathrm{A}} = 0.775.$$

For the value of $\frac{9\ \mathrm{A}}{6\ \mathrm{A}}$ the two results do not agree very nicely; as for $\frac{12\ \mathrm{A}}{9\ \mathrm{A}}$ and $\frac{15\ \mathrm{A}}{12\ \mathrm{A}}$, the correspondence is very good.

So now we know, that by increasing the number of primary turns of the induction-coil we soon reach a maximum for the intensity of the induced current and that by continuing to increase the number of turns we arrive at a decrease in the strength of the induced current.

It was obvious to think, while searching for an explanation of this phenomenon, of the possibility that the iron in the coils of higher order would be much nearer to its magnetic saturation point than in the coils of lower order. In that case the undulations of the microphone-current would bring about, in coil 15 A for instance, smaller variations in the magnetism of the iron than would be the case in coil 3 A.

This current of thoughts led me to compare the coils with iron with those without iron as regards the intensity of the induced currents furnished by them. The method I followed for this was just the same as the one I had followed for getting Table III.

The result of this investigation is found below. If we call A the current induced by a coil with iron and B the current induced by a coil of the same order but without iron, we then find:

For coil:	3	6	9	12	15
The value for $\frac{A}{B}$:	5.56	3.95	2.59	1.95	1.48
or for the efficiency of the iron, if we take that for coil 3 A to be equal to 1:	1	0.71	0.47	0.35	0.27

We see from these numbers that the influence of the iron on the intensity of the induced current is much less in the coils of higher order than in those of lower order. And comparing these numbers with those we gave on page 11 for the value of A we see that the decrease of the effect of the iron between 3 A and 6 A is still compensated by the increase of the number of primary turns, but that higher than 6 A the decrease of the effect of the iron predominates and that the increase of the number of turns is then no longer sufficient to make good that decrease.

To investigate whether in reality an approach to saturation of the iron may be regarded as the cause of this phenomenon, the magnetism of the iron core of coil 15 A was investigated as follows for different currents: In fig. 4, M is a bell-magnet hanging on a fibre in a copper damper. In the line *ab* perpendicular to the magnetic axis the coils 15 A and 15 B were placed. First the iron core was taken out of coil 15 A, then the two coils, 15 A (without iron) and 15 B, were placed at such distances from the magnet, that a current of 1 Ampère, traversing both in opposite directions, did not give a deviation to the magnet. Now the iron core was again put into coil 15 A and the latter laid in exactly the same place as before. The distance from the centre of the bell-magnet to the middle of the iron core was 307 mm.

Now the Morse-key was pressed down ; as soon as the index attached to the magnet stopped, the position was read, and at the same time the Ampère-meter was read by a second observer. This was a necessary precaution, because, at least when using strong currents, the index of the magnet as well as that of the Ampere-meter slowly went back to zero in consequence of the increase of resistance of the coils caused by the heating of the wires. To be quite sure that the coils had not been damaged by the strong currents, the iron was removed out of 15 A after the measurements were made and a current of 1.5 Ampère led through the two coils in opposite direction: the magnet remained in its position of equilibrium just as at the beginning of the experiment before the iron was put into 15 A.

Fig. 5 represents the results of these measurements in the form of a curve; the ordinates are proportional to the magnetic intensity, the abscissae to the strength of the current. The figure shows, by the small circles, that I have taken fifteen readings with a strength of current, increasing from 0.11 to 1.53 Ampère. α is the angle between the magnetic axis of the magnet and the magnetic meridian.

Now the current in the primary wire of 15 A, connected with the microphone (in rest) and a Leclanché-cell amounts to about 0.08 Ampère. The curve of fig. 5 shows clearly that there is not the slightest question about the iron being saturated when the current has this intensity.

I then considered whether the iron might not become *slower* in changing its magnetic condition if the magnetism reached a higher degree, even though there were no question about an approaching saturation. That in other words a piece of iron would need more time to change its magnetism from 20 to 18 or 22 than from 6 to 8 or 4.

To investigate this the experiment shown in fig. 6 was made.

T is a translator without iron core, consisting of 2 × 1500 turns; thickness of wire 0.4 mM., resistance of each of the two series of turns about 21.5 Ohm. The battery consists of a number of secondary cells, of which I used respectively 1, 4, 8 and 11 cells in the four different sets of experiments I made with this apparatus. For the rest the sketch speaks for itself. If the Morse-key was *not* pressed down, a telephonic alternating current passed through the primary wire of 15 A; if on the contrary the key *was* pressed down, a constant current also passed through it.

For the proportion between the strength of the current, induced in the secondary wire, if only an alternate current passed through the primary wire, to that of the induced current when also a constant current passed through the primary wire, I found:

for the currents:	0.08	0.35	0.66	0.88 Ampère,
the proportion:	0.997	0.981	1.01	1.149.

The number of readings and the calculation of the mean value were for these measurements quite similar to those given in Tables II and III.

From the above it is evident that the constant currents 0.08, 0.35 and 0.66 Ampère have no perceptible influence on the intensity of the induced current. Not until the strength of the current was 0.88, did this influence become perceptible: there the induced current becomes distinctly feebler when the constant current flows through the coil. But with that strength of current the magnetism is no more proportional to the intensity of the current, as the curve of fig. 5 shows. The slowness of the iron cannot now be regarded as the cause of the induced current becoming feebler, but, with such a current, this can be explained from the saturation of the iron. But currents of such intensity are not used in our microphones.

So the fact that the influence of the iron diminishes if the number of primary turns is increased, *cannot* be explained by slowness or saturation of the iron.

It seemed of some importance to me to investigate experimentally whether the bundle of iron wires giving so much less effect in coil 15A than in 3A, is also less subject to magnetic changes in 15A than in 3A.

For that purpose two apparatus were made, one of which is shown in fig. 7. It is an induction-coil of exactly the same dimensions as the coils used until now; but one of the two wooden flanges is transformed into a vibrating-box for an iron diaphragm, covered with and held in position by a mouth piece provided with a circular aperture. So the whole forms a telephone, in which the bundle of iron wires plays the part of the permanent magnet with attached iron core or armature in the ordinary telephone. One of the two apparatus was wound with wire quite similar to coil 15A, the other one like 3A. Care had been taken that the end of the bundle of iron wires was in both coils as accurately as possible at the same distance from the diaphragm.

The experiment was now arranged as indicated in fig. 8. *B* is a storage-cell, *M* the microphone. The primary wire of 3A is connected with the contacts of rest of the two keys, that of 15A with the working contacts. The secondary wires are both shunted by a coil of wire (*a*) quite similar to that of our electro-dynamometer and in which also a bundle of iron wires under an angle of 45° to the winding-plane has been put. Before the microphone were the tuning-fork and the resonator as usual. If both keys were pressed down, telephone 15 spoke; if they were *not* pressed down, it was the turn of 3.

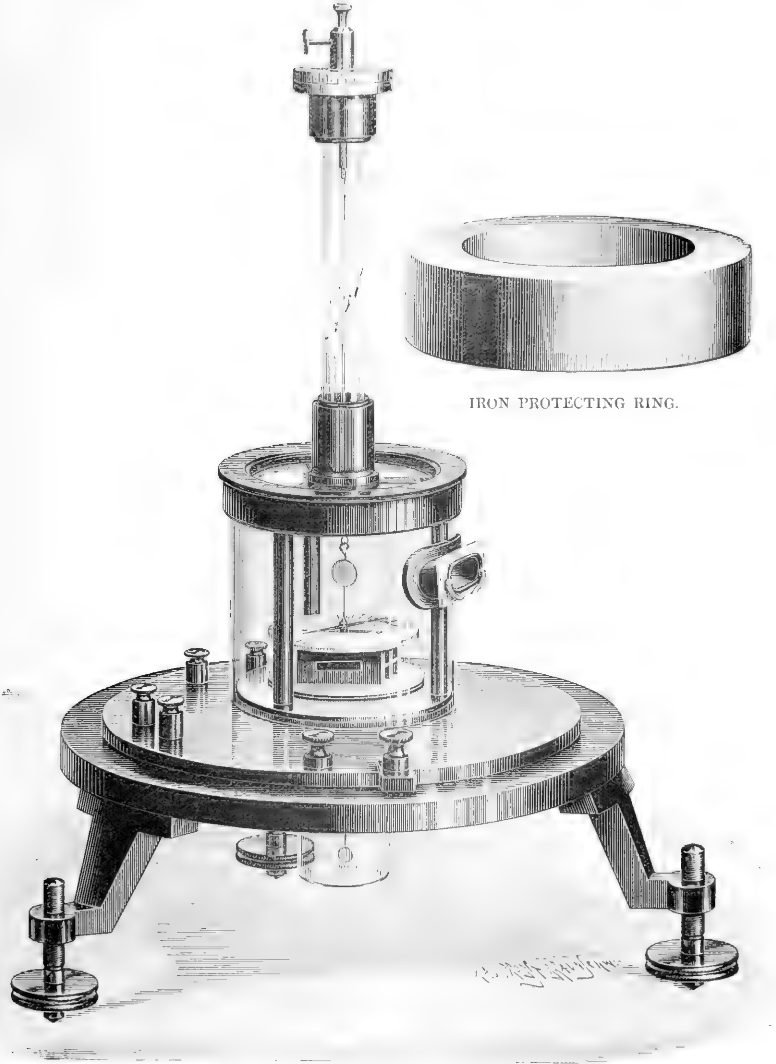
The experiment showed, that 15 gave a distinctly stronger sound than 3.

Then I exchanged the storage-cell acting on the microphone for a Leclanché-cell and placed the tuning-fork a little farther from the resonator, which caused the undulations of the primary current to become feebler. Now again telephone 15 gave a stronger sound than telephone 3.

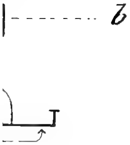
So this experiment does not give us any light for the problem we wished to solve. We should be more inclined to conclude from it, that the iron in the coils of higher order ought to have *more* efficiency than in those of lower order!

PLATE I.

J. W. GILTAY „The effect of the induction-coil in telephonic apparatus." (I).



SCALE 1 : 2.



F°

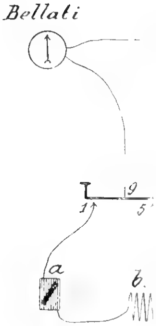
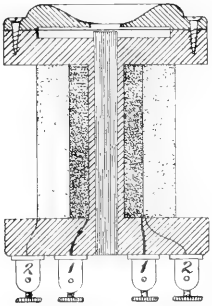
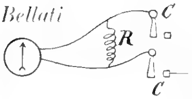


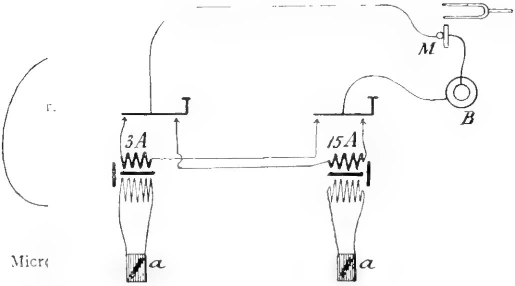
Fig. 7.





Amp.

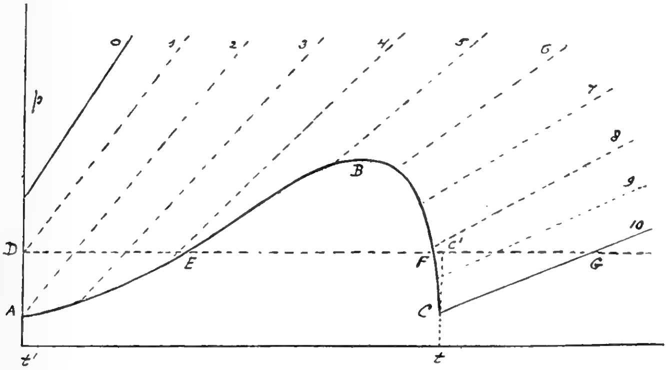
Fig. 8.



Chemistry. — *"Saline solutions with two boiling points and phenomena connected therewith."* By Prof. H. W. Bakhuis Roozeboom.

(Communicated in the meeting of December 28, 1901.)

The present communication is connected with my researches on the vapour pressure curves of saturated solutions which were completed as far back as 1889. These researches related to systems in which gases and water, water and salts, or ammonia and salts were the components. They led to the result that in all cases the vapour pressure curves of saturated solutions, that is of solutions which are in contact with a solid phase, have a similar course, it being immaterial whether the solid phase is one of the two components or a combination of them. In the case of water and salt we start at lower temperatures with less concentrated solutions. Generally the concentration of the saturated solution increases on raising the temperature, so that in the ideal case we finally arrive at the melting point of the salt which forms the solid phase, which may be either a hydrated or anhydrous salt. In all cases in which that melting point is attainable, the vapour pressure curve shows the form indicated in the annexed figure *A B C*, from which it



is seen that the vapour pressure of the saturated solution increases first from *A* to *B*, reaches a maximum in *B* and then decreases from *B* to the melting point, *C*, of the salt.

In order to indicate the increasing concentration of the solution the lines 0—10 have also been drawn, these lines representing the vapour pressures of water and of solutions containing 10, 20...100 mol. per cent of salt, so that 10 represents the vapour pressure of the fused salt.

On the line *AB*, the evaporation of the saturated solution takes place when heat is supplied. If we disregard the small quantity of salt in the vapour the following process takes place:

Saturated solution → solid salt + water vapour.

As long as the solution is very rich in water, the evaporation of the water absorbs more heat than is yielded by the solidification of the salt dissolved therein. Consequently the vapour pressure increases with the elevation of the temperature. With the elevation of the temperature, however, the concentration of the solution is increased and a point *B* will be reached where these two quantities of heat become equal; here the thermic effect becomes *nil* and this point is the maximum.

At still higher temperatures and still more concentrated solutions on the part *BC* of the curve, the evaporation of the small quantity of water would on the other hand absorb less heat than that yielded by the crystallisation of the large amount of salt dissolved therein, consequently the evaporation of the saturated solution would evolve heat; therefore the vapour pressure now decreases.

If the evaporation of the salt is disregarded the line *BC* ought to end vertically in *C* [1]).

Up to now this peculiar course of the line *ABC* was only observed by me in the case of solutions saturated with $Ca\,Cl_2 . 6\,H_2O$ [2]). Although quantitative agreement was found with the course calculated from the quantities of heat, the pressures were here so small that the example was little suited to bring out the great significance of the matter.

As other phenomena outside the region of salts with water could only find their explanation in the course indicated by the vapour curve, I thought it desirable to further examine this curve and the phenomena connected with it by means of other examples.

Dr. SMITS, to whom I wish to express here my thanks, at my request took charge of the experiments. Our choice fell on some

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[1]) VAN 'T HOFF. Vorlesungen I. 35 (1898).

[2]) Recueil trav. Chim. Pays-Bas. 8. 100 (1889).

anhydrous salts. Among these many are known whose curve of solubility may be continued up to their melting point *C*. If this melting point is situated far enough above 100°, the solubility at this temperature is still small enough that the saturated solutions belong to the part *AB* of the vapour pressure curve. If we now draw a line *DEFG* parallel to the temperature axis at a height of $p = 1$ atm. this will intersect the vapour pressure curve first in a point *E* which is situated above 100° and indicates the temperature at which the saturated solution has a vapour pressure of 1 atm.

If, therefore, we start with a saturated solution at a lower temperature, the vapour pressure will rise from *A* to *E* owing to the continuous solution of solid salt caused by the heating, but in an open vessel it will not be able to get above that point, for, on further heating, the solution will evaporate to dryness while the temperature remains constant. For we have got here three phases, solid salt, solution and vapour at a constant pressure of 1 atm.

The same result is obtained in a still more striking manner when we start with a dilute solution boiling at *t'*. Owing to the boiling its concentration will increase and so we travel along all the points of the line *DE* until the saturation point *E* is reached. The boiling point does not rise any further because in proportion as the water evaporates solid salt is deposited and the concentration remains unchanged.

This state of affairs has been repeatedly observed by different investigators. We were satisfied with convincing ourselves that the temperature in *E* remains very constant if the regularity of the boiling is promoted by passing steam, and the source of heat is an oilbath and not a naked flame. We will call the point *E* the first boiling point.

As soon as the solution is quite evaporated, the temperature may rise and so we further proceed along the line *EF* while the salt remains exposed to water vapour of 1 atm. This line *EF* now intersects the vapour pressure curve of the saturated solution for the second time at *F*. As soon as this point was exceeded we should arrive, along *FG*, in the region of the unsaturated solutions. Consequently a new solution must form in *F*.

In *F*, therefore, a solution is formed from solid salt and aqueous vapour of 1 atm. pressure, which being just the reverse of the evaporating of the saturated solution on branch *BC*, will absorb heat. Consequently, on supplying heat the temperature in *F* remains again constant until all the salt has been solved for which generally but very little water is required.

From here we proceed with continued heating and constant loss of water along the line *FG* to solutions gradually getting poorer in water until we should have reached in *G* the boiling point of the salt itself.

I will now call the point *F* the second boiling point of the saturated solution. This solution, however, will not show the phenomenon of boiling on heating but on cooling. We may for that object proceed along the reverse way, melt the salt, bring it in contact with water vapour which would yield the solution corresponding with *C* and cool the same. If we continue passing H_2O over or through the liquid this will absorb more of it as t decreases until the point *F* is reached. On further cooling, solid salt would crystallise and the pressure consequently increase along *FB*; this not being possible at 1 atm. pressure the liquid in *F* boils to dryness on cooling — the temperature remaining again constant.

It is a very remarkable sight to see the boiling start as soon as the hot liquid commences to deposit solid salt and the phenomenon may be accelerated by taking the vessel out of the oilbath in which the melting of the salt and its saturation with aqueous vapour took place. The quicker the vessel cools the more rapid the boiling takes place. On continuing to cool, absorption of aqueous vapour ought of course, to take place again at *E*.

The following table contains the observations collected by Dr. SMITS.

Salt.	1st Boiling point.	2nd Boiling point.	Melting point.
$K\,NO_3$	115	331	334
$Na\,NO_3$	120	310	313
$Na\,Cl\,O_3$	126	255	261
$Ag\,NO_3$	133	191	208
$Tl\,NO_3$	105	196	205

From this list it appears that the second boiling points are generally situated close to the melting points. This follows from the small value of the temperature interval which will generally exist in systems containing salts and water between the points *B* and *C* on account of the great heat of evaporation of the water, causing the point *B*, and even more so *F*, to correspond to solutions containing very little water.

For salts with low melting points the 1st boiling point will generally be situated at higher temperatures (Tl NO_3 does not conform to this rule) because the solubility at lower temperatures is generally greater than with salts having a high melting point. In such a case the distance from *E* to *C* and therefore also from *E* to *F* becomes smaller. This will as a rule be coupled with the fact that the maximum pressure at *B* is lower. Of the examples cited, Ag NO_3 has been more closely investigated.

Drs. Cohen and Smits, in a preliminary experiment, found the following vapour pressures of the saturated solution.

t	*p*	*t*	*p*
133°	760 mM.	170°	1010 mM.
135°	800 »	185°	900 »
150°	960 »	191°	760 »
160°	1000 »		

From this we derive a maximum at 167° and 1015 m.m. The lower the maximum point of the curve, the greater is the chance that the points *F* and *C* will lie at widely different temperatures because the part of the curve *BC* must now bend round more suddenly in order to end vertically in *C*. The greatest difference between the second boiling point and the melting point has been found in the case of Ag NO_3. If we take salts with a still lower melting point, the maximum pressure may then fall below 1 atm. and the line *DEFG* will, therefore, lie entirely in the region of the unsaturated solutions above the curve.

Such salts will therefore, completely liquefy, at lower temperatures, in aqueous vapour of 1 atm. pressure and this solution will never deposit solid salt when the temperature is raised.

Such a case occurs for instance with NH_4 NO_3 (m. p. 164°) and of course always with salts melting below 100° as was shown in the case of ammonium acetate (m. p. 89°).

In the case of such salts the two boiling points of the saturated solutions can only appear at a lower pressure or they may appear at 1 atm. if a more volatile solvent is taken which boils below the melting point and in which the solubility of the salt at lower temperatures is not too large. Dr. Smits succesfully used alcohol in the case of NH_4 NO_3. The alcohol boiled at 78°.4, the first boiling point of the saturated solution was situated at 83°, the second at 161°, therefore 3° below the melting point.

Consequently there can be no doubt about the generality of the phenomenon. A single remark may show its importance in explaining several widely varying chemical processes. First of all, attention may be called to the occurrence of water in plutonic rocks. **Bunsen**[1]) when carrying on his investigations on the Icelandic rocks suggested that liquefied rocks might have absorbed water. Since **Gautier** has lately demonstrated the presence of 1 to 2 per cent of water in all kinds of igneous rocks, this possibility has well nigh become a certainty. **Bunsen**, however, could only account for the absorption by assuming the existence of enormous pressures. The investigations and views now communicated lead to the belief that the absorption may possibly take place at a very low pressure[2]); and it is not even precluded that afterwards on cooling and solidification, the dissolved aqueous vapour did not escape owing to supersaturation phenomena assisted perhaps by rapid congealing, or because, on cooling, hydrated silicates are deposited first. Owing to our complete ignorance as to the situation of the vapour pressure curves of solutions of anhydrous and hydrated silicates when close to their melting points, it is not possible to form a decided opinion.

But we have already got a better insight in a totally different sphere of phenomena, namely the oxidation of melted metals. The system water + salt is indeed quite comparable with the system oxygen + metal and therefore a hydrated salt with an oxide. Of late it has been abundantly proved that all kinds of melted metals dissolve the gases of the air, oxygen in particular. Whether we must consider this to be a simple solution or a solution of the oxide makes no difference for in either case the quantity of oxygen dissolved is dependent on p and t. The melting point of the metals is lowered thereby even much more than was formerly believed, for instance 20° for silver and 16° for copper. With a free supply of air the absorption therefore takes place at $^1/_5$ atm. oxygen pressure.

A line similar to $AEBFC$ necessarily exists for the oxygen containing liquids saturated with solid metal. If now a horizontal line drawn for $^1/_5$ atm. intersects this line below the maximum it means that on cooling to the point F the metal will crystallise and all the dissolved oxygen will be expelled. This phenomenon is shown for instance by silver; it is known under the name of spitting.

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[1]) Lieb. Ann. 61. 271. (1847).
Pogg. Ann. 83. 237. 240. (1851).

[2]) It is no objection that the melting temperatures are situated very far above the critical temperature of water, as we are dealing here with solutions containing very little water whose critical temperature is therefore considerably raised.

This spitting therefore takes place at the point called above the second boiling point. A similar phenomenon is shown by lead oxide which on melting also absorbs excess of oxygen but releases it on cooling.

On the other hand it is not shown by copper, either because the line *CFB* cannot be continued up to the maximum, or because this maximum is situated below $^1/_5$ atm. so that in the presence of air the absorption of oxygen will cause the formation of Cu_2O.

In all these cases, a point corresponding with the first boiling point of salt solutions does not appear to exist because before that time other vapour pressure lines appear in connection with the fact that at lower temperatures the lower or higher oxides are stable. As to the exact conditions of their formation much might be deduced from existing investigations, which I must, however, pass over here.

The foregoing may also be applied to the absorption and loss of H_2, CO_2 and other gases, and to the formation of chemical compounds with those gases.

Chemistry. — "*The Enantiotropy of Tin.*" (VII). By Prof. ERNST COHEN. (Communicated by Prof. H. W. BAKHUIS ROOZEBOOM).

(Communicated in the meeting of December 28, 1901.)

1. Through the kindness of Prof. PAUL of Tübingen who informed me that a very striking case of *tin plague* had occured at Ohlau in Silesië, I am in a position to communicate some particulars about this case.

The case owes its importance to the dimensions of the corrosion. At my request Dr. HAMBERGER of Ohlau gave me some further information about the matter.

The catholic church in the said village possesses an organ which was repaired in the year 1833. While of the newly fitted pipes no less then 28 became corroded in a very high degree, the old pipes were not affected. The phenomenon was again noticed for the first time 18 years ago (1883). The two accompanying illustrations taken from photographs show to what extent the pipes have been attacked. According to Dr. HAMBERGER the corrosion is still going on.

It is worthy of notice that the wooden roof of the church is situated immediately above the pipes, causing the church to be very warm in summer but very cold in winter time.

The average winter temperature at Ohlau is — 1°,64 C., the average yearly temperature + 7°,97 C.

The above case is one of the most striking of those which have as yet, come to my knowledge.

2. In my previous communications on this subject the purity of material used in my investigations was but little commented on. As from various quarters the question was put to me in how far any impurities contained in the tin might influence the observed phenomena, I will now give, some particulars as to the purity of the materials used, laying stress on the fact that I had paid the necessary attention to this matter from the beginning.

My material was derived from a block of Banca tin which had been procured some 15 years ago for the collection of the laboratory and which at the commencement of my investigations was still quite intact in the well-known boat-shape.

How little the composition of Banca tin changes in the course of years is proved by the following figures from G. J. MULDER's [1]) analyses from the year 1851, who tested blocks sent by 20 different ships, and from those of J. D. VAN DER PLAATS [2]) from the year 1885.

	MULDER (1851)	v. D. PLAATS (1885)
Fe	0,019	0,015
Pb	0,014	0,017
Cu	0,006	0.005
Si	—	0,010
Sn	99,96	99,95

The figures represent percentages.

I have submitted 100 grams of my tin to a microchemical analysis in which work I had the valuable assistance of Dr. SCHOORL. The result of the investigation was that minute traces of lead could be detected. Only doubtful reactions could be obtained for the other above mentioned elements so if they were present at all, their amount must have been particularly small.

We may, therefore, safely conclude that phenomena previously described relate indeed to pure tin.

Amsterdam, Chemical Laboratory of the University, Dec. 1901.

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[1]) Scheikundige Onderzoekingen. 5, 529.

[2]) C. R. **100**, 14 (1885).

Fig. 1.

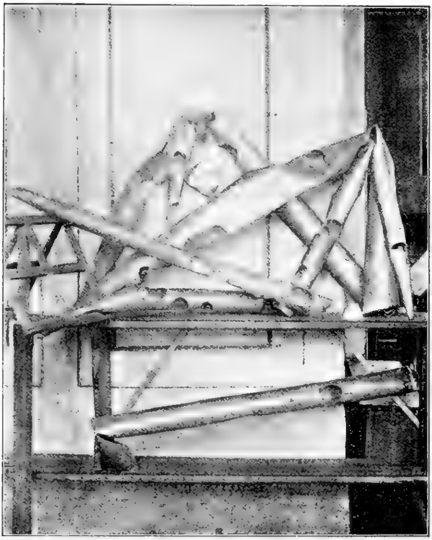

Fig. 2.

Zoölogy. — "*The unfavourable condition of which complain the oyster cultivators on the Eastern-Schelde.*" By Dr. P. P. C. Hoek.

(Communicated in the meeting of December 28, 1901.)

Dr. Hoek discusses the unfavourable condition of which the oyster-cultivators on the Eastern-Schelde complain and explains that the reason of this is; 1. that for some years the oysters have not thriven so well nor so quickly, 2. because of great mortality in certain years, 3. of a much greater mortality than formerly in all the later years. The speaker was commissioned to investigate this matter. A detailed report on the results of these researches was now in the press. With the permission of the Minister of Public Works, by whose order the investigation had taken place, the following summary was given of what these researches had brought to light.

There were three possibilities, which were successively taken under consideration, viz.

1st that the physical circumstances under which the cultivation takes place, had undergone changes;

2nd that the oyster itself had altered;

3rd that the less favourable condition, in which the oyster-culture finds itself at present, was to be imputed to that culture itself.

As regards the physical circumstances, it would not seem probable *a priori* that meteorological factors have played a prominent part in this decline, considering the period of 1870 – 85 as a time of high prosperity of the oyster-culture in Zeeland and the years 1885—1900 as a period of decline. Some of those interested in the culture, are of opinion, that the laying of the dyke at Woensdrecht has raised the salinity of the water in the eastern part of the Western-Schelde and that the consequences have become fatal for the oysters, which are cultivated there.

Considering however that this dyke has been built in 1867, that the oysterculture only began after 1870 and from the beginning attained to a period of prosperity, this supposition must also be rejected as extremely improbable. From investigations made in '81—82 and from a comparison of those with similar ones of later years, it is moreover evident that the salinity has not been raised in any way in the last twenty years.

It would appear from informations supplied by the "Waterstaat" that the quantity of seawater, which at every tide restocks the Eastern-Schelde-basin, has not diminished to any amount, since the oyster-culture there has appeared to be in a less favourable condition. And in regard to the nature of the soil the hydrographic survey has indeed brought to light that here and there, locally, shallow places have lately arisen by accumulation of sand. If it must therefore be considered as proved, that places which had formerly value for the oyster-culture, have lost in quality, the evidence has however not been given that by such influence the good qualities of the oysterbeds in general should have been lost.

The assertion, as if a change, a degeneration of the oyster itself, were the ground of the less favourable results of the culture in the later years, is founded on the supposition that an injurious influence is still exercised by the French oysters which were many years ago imported in the Eastern-Schelde, and that therefore the Zeeland-oysters should have lost of their good qualities by interbreeding. This idea finds a slight affirmation in the extraordinary rich brood-production of these later years. But really not more than the slightest; for in the first place it is not at all sure that the French oysters produce a more numerous posterity than the original Zeeland oysters. And in the second place the abundance of births can very well be explained by the great mortality and the unusual numerous population of oysters, which for many years have been found in the Zeeland-oyster-beds; abundance of births, bad growth and great mortality are symptons connected with one another; which combined, give proofs of overproduction and of insufficient nourishment of the separate individuals caused thereby.

Proofs can easily be furnished, that the very first young Zeeland oyster taken, still exhibits the same excellent disposition to grow to be an extremely suitable shell-fish fit for human food. This proof is given by the excellent results, which again for instance in this year, have been obtained with young Zeeland oysters transported to other waters; to de Grevelingen, near Bruinisse; to some places on the Western-Schelde; to places in the Zuiderzee, near the coast of Texel.

Not the oyster itself is therefore to be blamed for the decline of which the cultivators complain, but the oyster-culture as such. The circumstances, the favourable results in the first years, competition, which raised the leases tremendously, have occasioned heavier claims to have been put to the oyster-producing territory. Considering the thorough renewal of water of some hundred millions cubic Meters of water with every tide, and taking for granted that the oyster feeds itself with the small organisms, which are carried along with the tide and form the so-called plankton, the quantity of oysters to be obtained seemed indeed unlimited. From investigations made in Zeeland for some years, it has however become evident that the oysters do not chiefly feed on plankton but on small vegetable organisms abiding near the bottom, Diatomaceae, of the benthos.

These bottom-Diatomaceae however are found not to be equally abundant in all parts of the Eastern-Schelde-bed, but their appearance is very markedly connected with the nature of the sea-bottom. The latest investigations of G. KARSTEN, who has made the bottom-Diatomaceae in the Gulf of Kiel a subject of his studies, have now made it very probable that different nutritive salts which are present in the bottom of the sea, have an active part in the development of those Diatomaceae; a part comparable to that which they play in the development of agriculture. Whether certain bacteria will prove to render the same important services here as in the growth of our agricultural plants, can as yet only be supposed, however probable this may be. It is already obvious however that exhaustion of the soil, in consequence of overpopulation in the oyster-culture, is equally imaginable as in any other culture.

The speaker hopes soon to be able to present to the Academy a copy of the report on his investigations, for which he has obtained considerable co-operation from different sides.

Physiology. — "*A new law concerning the relation of stimulus and effect.*" By Dr. J. K. A. WERTHEIM SALOMONSON. (2nd Communication). (Communicated by Dr. C. WINKLER).

In a former essay I have formulated a law expressing the relation between the magnitude of a stimulus and the effect operated by it. Our deductions were based exclusively on wellknown physical principles, viz. on the fact that an infinitely small increment of stimulus will cause an infinitely small but proportional increment of consumption of the substance by the transformation of which the effect is occasioned, whilst we admitted finally with GULDBERG and WAAGE that the magnitude of this transformation, must be proportionate to the quantity of transformable substance. Starting from these perfectly admissible premises, we finally found a law, expressing the relation we were in search of, in the following formula:

$$E = A \{ 1 - \varepsilon^{-B(R-C)} \}$$

We tested this law in the first place to the relation of stimulus and *muscle*-effect, and arrived at the conclusion that on this point there existed a perfect accordance between my law and the results, obtained experimentally by other investigators.

In this way we proved the validity of the law for the muscles.

We will try now to examine whether our law prevails also in regard to other organisms or parts of organs.

Our first researches, following here, were made concerning the result of the stimulation of the nervefibre taken as such.

In the work of A. D. WALLER (Points relating to the WEBER-FECHNER LAW. Brain 1895, pag. 200), we again find the necessary data in two Tables, completing each other and representing one stimulation-series of one definite nerve (L. c., page 209 and 214).

From the second of these series we again calculate the most probable values for the constants A, B and C, obtaining the results communicated in our Table I. — That we did not make use of all numbers of the first and second parts of the series together, may be explained by the fact that with the first numbers (magnitude of stimulus 1.52—2.32) the galvanometer had no shunt, whilst such was the case with the higher magnitudes of stimulus. For this reason the strength of the current had evidently risen a little, which becomes visible also in the curve, this latter showing a slight bent upward just at the spot where the shunt begins to act.

TABLE I.

A. WALLER. Brain 1895, pag. 214
Fig 1.

$A = 109.1756$

$B = 0.16336$

$C = 0.61809$

R	E calc.	E meas.	ϱ
3.02	35.434	35	$-$ 0.434
3.33	39.075	40	$+$ 0.925
4.25	48.857	50	$+$ 1.143
5.50	59.997	60	$+$ 0.003
7.00	70.685	70	$-$ 0.685
9.60	84.006	85	$+$ 0.994
28.75	108.074	105	$-$ 3.074
38.30	108.946	110	$+$ 1.054
47.90	109.129	110	$+$ 0.871
57.50	109.169	110	$+$ 0.831

$\Sigma \varrho^2 = 15.8172$ $\varrho_m = 1.5032$

It will be seen that the accordance between the calculated and the measured effects leaves nothing to be desired.

We find in WALLER's work still another series of numbers (*Philosophical Transactions* of the R. S. Vol. 188 p. 60, 1897).

From this series the following Table may be calculated.

TABLE II,

A. WALLER. Philos. Trans. Roy. Soc. 1897 p. 60.

$A = 20.1779$

$B = 0.117467$

$C = 2.4972$

R	E calc.	E meas.	ϱ
5	5.140	5.5	$+ 0.360$
10	11.820	11.	$- 0.820$
15	15.533	16.5	$+ 0.967$
20	17.596	18.	$+ 0.404$
30	19.380	19.5	$+ 0.220$
50	20.102	20.	$- 0.102$
100	20.177	20.	$- 0.177$

$\Sigma \varrho^2. = 1.9904 \qquad \varrho_m = 0.705$

In this series there occurs again one single value, that causes a great deviation. If the most probable values were calculated with rejection of this most probably erroneous observation, the mean observation-error was reduced from 0,705 to 0,164.

Fig. 1.

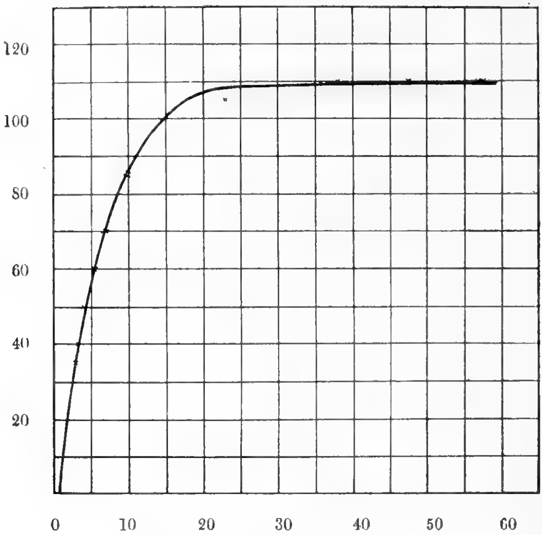
120
100
80
60
40
20
0
10
20
30
40
50
60

Fig. 2.

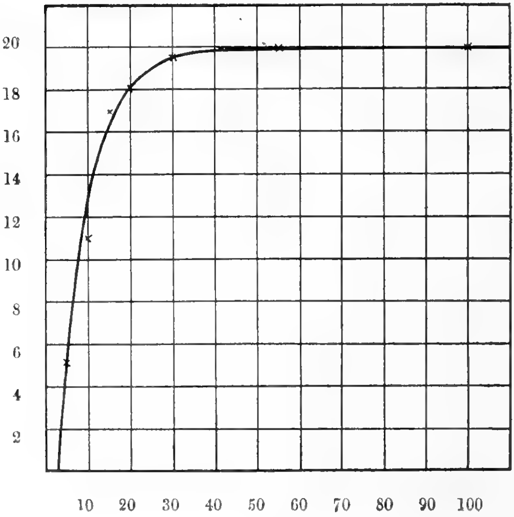
20
18
16
14
12
10
8
6
4
2
10
20
30
40
50
60
70
80
90
100

TABLE III.

Id. with rejection of the second observation.

Fig. 2.

$A = 19.96176$

$B = 0.14011$

$C = 2.71794$

R	E calc.	E meas.	ϱ
5	5.463	5.5	+ 0.037
[10]	[12.765]	[11.]	[— 1.765]
15	16.307	16.5	+ 0.193
20	18.189	18.	— 0.189
30	19.525	19.5	— 0.025
50	19.935	20.	+ 0.065
100	19.962	20.	+ 0.038

$\Sigma \varrho^2 = 0.0806$ $\varrho_m = 0.16395$

From the foregoing we may conclude that our law prevails also for the nerve as an excitable organ.

Already in this place it is of importance however, to point out the fact that in nature generally only the first ascending part of the curve may be considered normal. This fact is made evident by the last of the tables published in my former essay, showing that with magnitudes of stimulus, sufficient to make the muscle contract and even to compel it to the maximum-contraction, the manner in which stimulus and effect were related for the nerve could be represented with absolute accuracy by a linear function of the first degree.

We may admit therefore that practically the nerve is a purely transmissive organ, giving within the limits of practical use a faithful transmission of the stimulus as *perceived* by it.

I only have to recall to mind the fact that the minimum perception of stimulus by the nerve in Table X of my former essay began with 1.40989 units. Only when the stimulus was increased unto 1.59712 units, it was transmitted to the muscle. With a stimulus of 1.98 units we practically reached already a maximum contraction. For the nerve the transmissible maximum was only reached with about 33 units (in Table I in the foregoing).

We simply recall to mind this fact here, with the intention of making its application afterwards in due season and place.

Another organ for which accurate numbers are to be found in WALLER is the eye.

For this organ I found the following series of numbers, calculated in the tables inserted below.

Tables IV and V both relate to the excitation of the retina by light of growing intensity.

TABLE IV.

A. WALLER. Brain 1900. Pag. 25.

$A = 111.702$

$B = 0.324965$

$C = -0.14981$

R.	E calc.	E meas.	ρ.
1	34.826	34	-0.826
2	56.154	58	$+1.846$
3	71.565	72	$+0.435$
4	82.632	82	-0.632
5	90.746	90	-0.746

$\Sigma \rho^2 = 5.2351$ $\rho_m = 1.6179$

TABLE V.

A. WALLER. 1900. Pag. 26.

(Fig. 22 l. c.).

$A = 73.919$

$B = 0.6382$

$C = -0.44257$

R	E calc.	E meas.	ρ
1	44.480	44	-0.480
2	58.367	60	$+1.633$
4	69.580	68	-1.580
6	72.708	72	-0.708
8	73.580	75	$+1.420$

$\Sigma \rho^2 = 7.9097$ $\rho_m = 1.7189$

Finally Table VI offers us the results obtained by electrical excitation of the retina. The magnitudes of stimulus are expressed in ergs, whilst the effect is measured again by determining the strength of the action current.

TABLE VI,

A. Waller. Brain 1900. Pag. 26.

(l. c. fig. 23).

$A = 44.3478$

$B = 0.00856403$

$C = 7.25434$

R.	E calc.	E meas.	ϱ
20	4.583	4	− 0.583
80	20.563	22	+ 1.437
180	34.247	34	− 0.247
320	41.302	40	− 1.302
500	43.696	45	+ 1.304

$\Sigma\varrho^2 = 1.8615$ $\varrho_m = 1.712$

Tables IV and V give still rise to another remark, because in these tables we obtain a negative value of C. This might be explained in the first place by supposing that with every illumination of the eye — in this case the applying of the stimulus — the real illumination has been stronger than is stated by the numbers for the intensity of light. I believe however that this supposition may be directly excluded or at least may be considered highly improbable, as with Mr. Waller's well known accuracy in experimenting, sufficient precautions will have been taken against the occurring of such inaccuracies.

We ought therefore to consider another possibility, viz. the fact that *addition of stimulus* may have been caused by the rather long duration of each illumination. In that case the effects caused by small stimuli probably have undergone this influence in a larger measure than those caused by more intensive ones. The effect, both in the case of smaller or greater magnitudes of stimulus, will consequently always be augmented with an amount, dependent chiefly on the length of time during which a stimulus has acted. As the time for each stimulation experiment of the series has been the same, each effect is augmented with a constant amount, or in other words: the x-axis is pushed downward a little, or, what again remains the same: the constant C is diminished by a certain definite amount. I hope afterwards to be able to treat this question more fully.

I believe to have proved by the foregoing that the law concerning the relation of stimulus and effect prevails also for other organs than the muscle, more especially for the nerve-fibre and for the eye.

This law is therefore to all probability *a general law*, prevailing for every excitable organism.

Geology. — "*On the Supply of Sodium and Chlorine by the Rivers to the Sea.*" By Prof. EUG. DUBOIS. (Communicated by Prof. H. W. BAKHUIS ROOZEBOOM).

Fifteen years ago Sir JOHN MURRAY has, at the end of a now well known paper on the total annual discharge of rivers, given a small table showing the amount of dissolved matter in average river water [1]). He states to have taken the analyses of 19 rivers "as representing on an average the composition of river water" and promises a subsequent paper, in which the amount of matter in solution would be discussed in some detail [2]). This, as far as I know, has not yet appeared; but it is possible, by comparing the figures, to find those 19 rivers which most probably have been chosen by MURRAY, from the analyses known till 1886, for his calculations of averages. Those averages then appear to have been arrived at by joining the quantities such as they are stated in the analyses.

The mentioned table of 14 salts, for the first time giving averages of the existing data concerning the amount of solid matter conveyed annually in solution in the river water from the land to the sea, has ever since its appearance been dear to geologists, as it seemed to enable them in some degree to give a more concrete form to the notions they had of the process of denudation lying at the base of geological knowledge.

BISCHOF and ROTH [3]) had already compiled a great number of analyses of river water in their text books of chemical geology

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[1]) JOHN MURRAY, On the total annual Rainfall on the Land of the Globe, and the Relation of Rainfall to the annual Discharge of Rivers. Scottish Geographical Magazine. Vol. III. February 1887, p. 71 en 76.

[2]) Sir JOHN MURRAY does not tell us the names of those 19 rivers, nor that many of them are "principal rivers of the world." The nineteen rivers in his Table VI are not those of which the analyses have been taken for the estimate of the average composition of river water.

[3]) G. BISCHOF, Lehrbuch der chemischen und physikalischen Geologie. Zweite Aufl., Bd. I, p. 269—279. Bonn 1863. — J. ROTH, Allgemeine und chemische Geologie, Bd. I, p. 454—462. Berlin 1879.

and Mellard Reade 1) since 1876 indeed "opened out a novel line of investigation" by applying many other such analyses in an attempt to estimate geological time from chemical denudation. But only the estimate of averages by Murray seems to have put the new modulus of past time ready to every geologist's hand. The reputation of Sir John Murray contributed to consider that estimate as of great value, probably greater than he himself has wished; perhaps too it has been the motive of taking those averages without further control as starting point even for geological problems of far reaching tendency.

Lately Prof. Joly, in an important essay, has estimated the age of the Earth by comparing the annual supply of sodium by the rivers to the ocean, according to Murray's estimate, to the quantity which the ocean contains of that element, assuming that the primeval ocean only contained a small amount of sodium, assuming moreover that but very little is again withdrawn from the ocean when once conveyed to it, and accepting uniformity of the removal of the land surface by solution since the earliest sediments were laid down 2). The quotient of the amount of sodium in the ocean and that annual supply, corrected for the sodium obtained by the ocean by means of a primeval accelerated denudation and as regards the amount of that element which only circulates between the ocean and the land, gives the age of the Earth in years. Those corrections are not very considerable, but as Joly does not base them upon determined data one can estimate their amount as widely differing from his. O. Fisher has explained this in an able review of Joly's essay, to which it may be permitted to refer here 3).

There has not yet been much doubt expressed concerning the exactness of the basis of Joly's estimate 4). As for the one leading factor in it, the quantity of sodium in the sea, it seems unnecessary to doubt its accuracy. But is it the same with the quantity of sodium annually supplied by the rivers, namely with the proportion in which sodium is admitted by him to take part in the dissolved matter in average river water? This question has such an important

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1) T. Mellard Reade, Chemical Denudation in relation to Geological Time. Londen 1879.

2) J. Joly, An Estimate of the Geological Age of the Earth. Scientific Transactions of the Royal Dublin Society. Vol 7. (Series 2), p. 23—66. Dublin 1899.

3) O. Fisher, Geological Magazine. New Series. Decade 4. Vol. 7. (1900), p. 124—132.

4) Prof Sollas, in his very interesting address to the British Association of 1900, discusses the quantity of sodium present in river water. In his opinion the proportion in Joly's average river water is probably too small, as much sodium is supplied by rivers draining volcanous regions. He believes, moreover, that the supply of sodium to the ocean has proceeded in a gradually diminishing rate with diminishing temperature of the earth's crust. (Nature, Vol. 62, p. 485).

bearing on great geological problems that it seems indeed not superfluous to ascertain that quantity. At the same time apparently some information about the quantity of chlorine in river water is necessary, for in the sea sodium appears exclusively in chloride of sodium.

When comparing the numerous analyses of river water now known, we observe that one fact is most salient. Of all the salts in solution, the carbonate of lime only, which is by far in the greatest quantity, is found in a somewhat constant proportion to the dissolving water. This salt, especially in the form of bicarbonate, the solubility of which is determined by the pressure of the carbonic acid of the atmosphere, is dissolved in it in about the same proportion as in pure water, which has been a long time in contact with a surplus of solid carbonate of lime and carbonic acid of the tension which it possesses in the atmosphere; evidently because river water has nearly always had plenty of opportunity to come in contact with limestone.

Where in river water a surplus of carbonic acid is developed from the pollution with organic matter (as mostly is the case) the quantity of carbonate of lime increases somewhat. It remains on the contrary under the normal quantity in those cases, rare with large rivers, where the contact of the water with limestone was greatly limited or entirely wanting. It therefore may be accepted, that the water discharged by the rivers in the ocean may under the given atmospheric pressure of carbonic acid be considered as on an average saturated or a little more than saturated with carbonate of lime. Knowing the total annual discharge of water by the rivers, the annual supply of that salt to the ocean can be estimated. As for the other salts the case is quite different. Their solution remains far under the saturation point and they are found in greatly varying and often in very small quantities in the river water. As long as the quantity of them is only known from a limited number of rivers it is impossible to make anything approaching an accurate estimate of the amount which the ocean receives annually. This is especially the case with the sodium salts. Sodium belongs moreover to matters which it is difficult to determinate correctly in quantitive analysis, so that a number of analyses, especially such of former time, are not entirely to be relied upon.

Repeated attempts of calculations like those of JOLY had already been made from the quantity of chloride of sodium in the river water and the ocean water. According to MURRAY's estimate the river water would contain per liter on an average 4.06 mgrm of it. MELLARD READE had estimated the average quantity from the analyses of upland waters in England and Wales to be 16.6. QUENSTEDT and DITTMAR had even admitted 1 gram per liter. Since the appearance of Joly's paper, von Romer and Loziński have still taken the average quantity of chloride of sodium in river water as the basis of those calculations. Von Romer, according to the analyses cited by Bischof and Roth, had calculated 9 mgrm, Loziński, from 8 rivers by Roth, 7.97 mgrm per liter of river water.

In the following table, which makes no pretention of completeness, in which however most known analyses that can serve our purpose are made use of, as many new and better analyses as were disposable have been selected. Besides rivers some lakes with an outlet, properly river enlargements, the composition of whose waters, relatively speaking, is little variable, have also been taken. A number of smaller rivers, which possess a very deviating high amount of chloride or sodium or of both, in consequence of quite local causes, sometimes due to man, are left out. As extremes I name, among the former, the Chélif in Algeria, which contains at Orléansville 200 mgrm of sodium and 316 mgrm of chlorine, higher up at Ksar-Boghari, even 1296 mgrm of sodium and 2163 mgrm of chlorine, being a surplus of chlorine of 171 mgrm per liter, and the Cettinje in Dalmatia, which has at Podgaraje a quantity of 38.8 mgrm of sodium and 175 mgrm of chlorine, with a surplus of 115.3 mgrm of chlorine per liter of water, on account of the abundance of salt in the soil of the drainage area. The Gergogne and the Beuvronne near Paris, owe their chlorine surplus of 10 and 8 mgrm quite certainly to the pollution by manufactories. When calculating the average quantity of sodium and chlorine in river water these analyses cannot be made use of. Such circumstances affecting the quantity of sodium and chlorine in river water independently of true chemical denudation play also their part in most other rivers, howbeit in a smaller degree.

RIVERS and RIVER-LAKES	Milligrams per liter (parts in 1000.000 parts) of water			
	Na	Cl	Cl that could be combined with the Na	Deficit of Cl to combine with the Na
1. Meuse (Liege)	3.44	5.89	5.29	-0.60
2. Elbe (Tetschen near the N. fronteer Bohemia)	5.46	8.81	8.40	-0.41
3. Danube (above Vienna).	2.82	2.40	4.34	1.94
4. Lake of Geneva (Geneva)	4.12	1.20	6.34	5.24
5. Moldau (before Prague and about 10 K.M. upwards).	5.98	6.87	9.20	2.33
6. Thames (Londen, at 6 localities, from Ditton to Lambeth)	11.23	14.06	17.96	3.90
7. Rhine (Mayence)	2.18	3.47	3.35	-0.12
8. „ (Bonn).	6.23	8.79	9.58	0.79
9. „ (above Cologne)	4.00	6.15	6.15	0

RIVERS and RIVER-LAKES	Milligrams per liter (parts in 1000.000 parts) of water			
	Na	Cl	Cl that could be combined with the Na	Deficit of Cl to combine with the Na
10. Rhine (before Arnhem)	9.78	11.09	15.50	4.01
11. Main (above Offenbach)	3.99	4.20	6.14	1.94
12. Nahe (Bingen)	4.00	14.35	6.15	-8.20
13. Weser (Bremen)	22.10	41.50	34.00	-7.50
14. Dwina (above Archangel)	16.78	33 06	25.80	-7.26
15. Spree (Berlin and Charlottenburg-Spandau)	15.25	18.16	23.44	5.28
16. Vistula (Culm)	2.76	4.87	4.24	-0.63
17. Welikaja (Pskow)	4.56	9.22	7.01	-2.21
18. Embach (above Dorpat)	2.73	4.03	4.20	0.17
19. Lake Peipus	2.91	3.90	4.47	1.57
20. Lake Onega	6.58	6.39	10.12	3.73
21. Lake Baikal	4.95	1.68	7.61	5.73
22. Seine (Bercy, down Paris)	4.85	——	7.46	——
23. Loire (Orleans)	3.00	2.91	4.61	1.70
24. Doubs (Riotte)	2.55	1.76	3 92	2.16
25. Garonne (Toulouse)	3.00	1.94	4.61	2.67
26. Lake of Zürich	2.23	0.97	3.43	2.46
27. Würmsee or Lake of Starnberg	1.34	2.10	2.05	-0.05
28. Walchensee	1.41	1.00	2.17	1.17
29. Königssee	0.52	0.60	0.80	0.20
30. Schliersee	1.63	1.00	2.51	1.51
31. Traunsee (Lake of Gmunden)	3.49	2.42	5.37	2.95
32. Lough Neagh	24.00	14.75	36.90	22.15
33. Lake of Gérardmer (in granite)	2.02	traces	3.11	2.50(approx.)
34. » » Issarlès (in granite)	4.23	1.70	6.51	4.81
35. » » Gaube (in granite)	2.02	very faint traces	3.11	3.00(approx.)
36. » » Bouchet (in basalt)	4.97	traces	6.74	6.50 »
37. » » Pavin (in basalt)	7.94	——	12.21	12.00 »
38. Rachelsee (in Cordiëritegneiss)	5.12	0.90	7.87	6.97
39. Torrents of the Pyrenees	0.35	0.55	0.55	——
40. Nile (Caïro)	25.12	7.36	38.61	31.25
41. White Nile (Omdurman and Fashoda)	7.92	——	12.18	——
42. Blue Nile (Khartoum)	4.86	——	7.47	——
43. St. Lawrence (Pointe des Cascades, S. E. of Vaudreuil)	5.13	2.42	7.89	5.47
44. Ottawa (St. Ann's Lock, Montreal, Canada)	2.39	0.76	3.67	2.91
45. Lake Superior (Grandmarais, Minnesota)	1.24	1.27	1.91	0.64
46. Mississippi (Hydrant, City Water Works, New-Orleans)	31.00	48.00	47.68	-0.32
47. Amazonas (between the narrows and Santarem)	1.00	1.08	1.54	0.46
48. » (Obidos)	1.60	3.95	2.46	-1.49
49. La Plata (8 K.M. above Buenos Aires)	15.70	11.40	24.14	12.74
50 » » (Buenos Aires)	51.30	37.30	78.75	41.45
51. Parana (8 K.M. above mouth)	14.03	14 90	21.58	6.68
52. Uruguay (Salto)	1.48	0.25	2.28	2.03
53. » (3 K.M. down Fray Bentos)	3.93	3.40	6.04	2.64
54. Rio Negro (above Mercedes)	13.48	5.83	20.65	14.82
55. Bear-river (Evanston, Wyoming)	8.20	4.90	12.60	7.70
56. Croton » (Reservoir for the city of New-York)	2.98	2 13	4.58	2.45
57. Cumberland-river (Reservoir at Nashville, Tennessee)	10.32	2.99	15.87	12.88
58. Delaware (Reservoir at Trenton, New Jersey)	0.72	1.21	1.11	-0.10
59. Hudson-river (probably at the town of Hudson, New-York)	2.43	5.81	3.74	-2.07
60. James-river (Richmond, Water Works, Virginia)	2.44	1.05	3.75	2.70
61. Los Angeles (hydrant at Los Angeles, Californië)	29.68	10.44	45.63	35.19
62. Maumee (probably at the mouth in Ohio-Lake at Maumee)	1.62	2.50	2.50	0.

RIVERS and RIVERS-LAKES.	Milligrams per liter (parts in 1000.000 parts) of water			
	Na	Cl	Cl that could be combined with the Na	Deficit of Cl to combine with the Na
63. Passaic (4 miles above Newark, New Jersey)	23.57	31.92	36.25	4.33
64. Rio Grande del Norte (Fort Craig, New Mexico. Upper course)	32.20	36.04	49.50	13.46
65. Sacramento (hydrant at Sacramento, Californië)	2.00	——	3.08	——
66. Humboldt-river (Battle-Mountain, Nevada)	46.70	7.50	71.80	64.30
67. Truckee » (Lake Tahoe, Nevada)	7.30	2.30	11.22	8.92
68. Walker » (Mason Valley, Nevada)	31.80	13.10	48.90	35.80
69. Jordan (Utah Lake)	17.80	12.40	27.37	14.97
70. Mohawk (Utica, New York St.)	3.60	2.30	5.54	3.24
71. Genesee (Rochester, New York St.)	4.40	2.40	6.77	4.37
72. Red river (Fergus Falls)	4.00	1.39	6.15	4.76
73. Assiniboine (above junction with Red river)	56.34	28.36	86.66	58.30
74. Big Stone Lake	36.96	9.16	56.82	47.66
75. Mississippi (Brainerd, upper course)	10.00	0.91	15.38	14.47

QUOTATIONS TO THE TABLE OF ANALYSES.

1. W. Spring et E. Prost, Annales de la Société Géologique de Belgique. Tome 11. Liège (1884), p. 123—220. (Mean of the analyses during a year).
2. F. Ullik, Abhandlungen der Kön. böhmischen Gesellschaft der Wissenschaften. VI Folge, Bd 10, Math.-Naturw.-Classe. Prag. (1880), p. 51—53. (Mean of the analyses during a year).
3. J. W. Wollfbauer, Sitzungsberichte der Math.-Naturw.-Classe der Kais. Akademie der Wissenschaften. Wien 1879. Bd. 87, Erste Abtheilung, p. 404—424. (20 KM. above Vienna. 23 analyses spread over a year).
4. F. A. Forel, Le Léman. Lausanne 1895. Tome II, p. 586. (Mean of 11 analyses).
5. A. Bělohoubek, Sitzungsber. k. Böhm. Gesellsch. d. Wissensch. Prag (1877), p. 27—42. (6 analyses, of which 3 at Modřan, about 10 KM. above Prague, 3 before Prague).
6. Graham, Miller and Hoffmann, Report by the Government Commission on the chemical Quality of the Supply of Water to the Metropolis. London 1851, en H. M. Witt, Philosophical Magazine. Vol. 12 (4th Series), p. 115. London 1856.
7. E. Egger, Chemisches Centralblatt 1888, p. 1131. (Mean of summer and winter).
8. G. Bischof, Lehrbuch der chemischen und physikalischen Geologie. Zweite Auflage, p. 271. Bonn 1863. (Very low water, March 1852).
9. H. Vohl, Dingler's Polytechnisches Journal 1871. Bd. 199, p. 315 en 317. (Two analyses).
10. J. W. Gunning, Onderzoek naar den oorsprong en de scheikundige natuur van eenige Nederlandsche wateren. Utrecht 1853, p. 66. (February 1852).
11. Merz, Jahresber. Chemie für 1866, p. 987.
12. E. Egger, l. c. (10 Oct. 1886).
13. R. Kissling, Abhandlungen Naturw. Verein zu Bremen. Bd. X, p. 141, Bremen 1889. (Mean from a series of analyses in winter and summer).
14. C. Schmidt, Bulletin Académie imp. des Sciences, St. Pétersbourg, 1875, T. 20, p. 154. (24 Juli 1873).
15. Finkener in Veitmeyer's, Vorarbeiten zu einer künftigen Wasserversorgung der Stadt Berlin, 1871, p. 347 en 349. Agreeing with these are the analyses by B. Proskauer, of the chlorides in the water of the Spree and of the Tegeler See in Chem. Centralblatt, 1890 b, p. 824.

16. G. Bischof, l. c. p. 275 (4 Maart 1853).
17—21. C. Schmidt, l. c. and Tome 16 (1871), p. 191; Tome 28 (1883), p. 248.
22—25. Ste. Cl. Deville, Annales de Chémie et de Physique, 1848, (3) T. 23, p. 42. (17 Juni 1846).
26. Moldenhauer, Jahresber. Chemie für 1857, p. 724. (Very low water; January 1857).
27—30. W. Ule, Der Würmsee (Starnbergersee) in Oberbayern, Leipzig 1901, p. 198. (Analyses of the water of 4 lakes in Upper-Bavaria by A. Schwager).
31. R. Godeffroy, Zeitschr. d. allgem. österr. Apothekervereins. Ref. in Jahresber. Chemie für 1882, p. 1623.
32. Hodges, Chemical News, 1874, Vol. 30, p. 133.
33—37. A. Delebecque, Les lacs français. Paris 1898, p. 202, 203, 213, 275, 285, 292.
38. H. L. Johnson, in Liebigs Annalen d. Chemie 1855, Bd. 95, p. 230.
39. A. Muntz, Comptes rendus Acad. des Sciences. T. 112, (1891), p. 449.
40. A. Chélu, De l'Équateur à la Méditerrannée. Le Nil, le Soudan, l'Egypte, p. 177. Paris 1891. (Mean of 12 monthly analyses by Mathey).
41. Chélu, Ibid. p. 19.
42. Chélu, Ibid. p. 25.
43—44. Geology of Canada. Geological Survey of Canada. Report of Progress from its Commencement to 1863, p. 565—566.
45. Geological and Natural History Survey of Minnesota. Eleventh Annual Report, p. 175.
46. Report Louisiana State Board of Health 1882, p. 370. (Cl and Na not determined independently).
47. T. Mellard Reade, American Journal of Science. (3). Vol. 29, p. 295.
48. F. Katzer, Sitzungsberichte der Kön. böhmischen Gesellschaft der Wissenschaften. Math.-Nat. Classe. Jahrgang 1897. p. 3—6. Prag. 1898. (Mean of 2 analyses: deep and superficial water. Na not directly determined).
49, 51, 52. J. J. J. Kyle, Chemical News. Vol. 38. (1878), p. 28.
50, 53, 54. R. Schoeller, Berichte der deutschen Chemischen Gesellschaft. 26 Jahrgang. (1887), p. 1784—1788.
55—71. I. C. Russell, Geological History of Lake Lahontan. Table A, p. 176. Washington 1885.
72—75. Warren Upham, The glacial Lake Agassiz, p. 540. Washington 1895.

According to Murray's estimate a cubic mile of average river water contains 31805 Engl. tons of sodium sulphate, 26800 Engl. tons of sodium nitrate and 16657 Engl. tons of sodium chloride. From this we compute 5.88 mgrm of sodium per liter of water.

The deviation of the real values from this average is very striking, even if in the drainage area there exists, certainly, nothing like rock salt beds, brines etc. as on the contrary is indeed the case with the Weser and the Dwina. Compare e. g. the Amazonas to the La Plata, two of the greatest rivers of the Earth. Neithers hould in the latter case, of a large quantity of sodium, this merely be considered to be derived from the sea; as is apparent from the fact that there is far more sodium in the water of the La Plata than could combine with the chlorine. In the case of the Mississippi-water taken from the water works at New-Orleans, however, we have to think of the derivation of an important part of the sodium chloride from the sea, most likely by means of the rain, perhaps too, to some amount through the soil.

Generally the quantity of sodium is much less than the estimated average of MURRAY. On nearer consideration of the results of analyses of some river waters this fact is still more clearly brought to evidence. Especially this appears in the cases of the Meuse and the Elbe. In both cases the stated values are not only of especial value on account of the large number of reliable analyses on which they are based, so that we may admit them as indicating the true average composition during a year, but also because the local circumstances are somewhat better known. Considered together they moreover have a particular importance from the fact that in the drainage area of the Elbe crystalline rocks predominate, whereas in that ot the Meuse the soil is composed of clastic rocks belonging to the most different formations.

In both cases a part of the sodium and a still more considerable part of the chlorine is surely not to be accounted for as resulting from chemical denudation, but as derived from the sea and carried back to it by the rivers. SPRING and PROST calculated that the principal manufactories of chemical products between Namur and Liege use so much sea salt, that they bring yearly 17.5 million kilograms of chlorine, i.e. about 45 pCt of the whole amount stated in the analyses, into the Meuse. Certainly 40 pCt of it is not combined with sodium. Instead of the small surplus of 0.60 mgrm chlorine (above the quantity which could combine with all the sodium) we should, without this pollution of the water of the Meuse, undoubtedly have a considerable deficit, of about 1.75 mgrm of chlorine per liter. Moreover great quantities of sodium and chlorine were supplied to the river in consequence of the physiological consumption of salt by the population of the basin of the Meuse up to Liege, at that time at least 3.000.000. Stating this supply of chlorine with SPRING at about 7 mgrm per head and per day we find more then 7.6 million kilograms of chlorine per annum. According to the statistics for the German empire the annual consumption of salt for the preparation of food was 7.6 kilograms per head, i. e. 4.6 kilograms of chlorine and 3 kilograms of sodium. Supposing an equal consumption in the basin of the Meuse, we would have from this source an annual contribution to the river of 13.8 million kilograms of chlorine and 9 million kilograms of sodium.

No insignificant quantities of chloride of sodium in the rivers are, moreover, supplied by the rain water and therefore simply return to the ocean, whence they were derived by the wind. For Great-Britain this quantity is so important, that JOLY admits that rivers, free from pollution, may owe their amount of salt to that source. In fact the water of the Dee near Aberdeen contains 9.6 mgrm, that of the Don 18.9 mgrm of chlorides, whereas the average quantity of it in rain water on the western and eastern coasts of Scotland is 11.9 and 12.6 mgrm; at greater distance from the sea it is less, but that river water represents a concentration to one-third of the rainfall. BOBIERRE had found at Nantes, likewise not far from the sea, during 12 months, an average of 14 mgrm of chloride of sodium per liter of rain water. Our Dutch down water, which also may be considered as rain water concentrated to about one-third, contains 40 till 70 mgrm of chloride of sodium, according to its being taken farther from or nearer to the sea. Direct mixing with sea water is, as has long been known, totally excluded here; the part of the rain water sinking in the dunes flows partly to the sea, partly landward. The rain water may therefore have a quantity of about 13 till 23 mgrm of sodium chloride per liter there.

At greater distance from the sea the quantity of sodium chloride in the rain water is much smaller. At Ootacamund, 120 KM. from the coast, in South India, it contains only 0.4 mgrm per liter. The evaporating quotient there is 5, according to MURRAY[1]), so that on this account the river water can still have a quantity of 2 mgrm of sodium chloride per liter. MUNTZ[2]) found in low regions, at Bergerac (Dordogne), at a distance of 136 KM. from the coast, on an average 2,50 mgrm, and at Joinville-le-Pont near Paris, 190 KM. from the sea, on an average 7.60 mgrm, on the Pic du Midi (2877 M. above the sea) however only 0.34 mgrm of chloride of sodium per liter of rain water. According to these data an estimate of 0.4 mgrm of sodium chloride in the rain water falling on the drainage area of the Meuse, is certainly not too high. Now considering that the evaporating-quotient for the area of the Meuse is 2.6, according to SPRING and PROST, we find that 1 mgrm at least of the sodium is owing to the rain. According to what is known of the domestic consumption of salt and other human sources supplying sodium to the river we probably have to add to this at least an equal quantity. So we arrive at the conclusion that certainly not more than 1,5 mgr. of the sodium, found by the analyses, is due to chemical denudation, and that this amount is perhaps not at all combined with chlorine.

Still we did not charge our account for the possible source of „fossil sea water", indicated by STERRY HUNT, O. FISHER and especially JOHANNES WALTHER[3]), but which is wholly uncontrollable.

The Elbe carries, in absolute quantities, more than twice as much

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[1]) L. c., p. 76.

[2]) A. MUNTZ, Sur la répartition du sel marin suivant les altitudes. Comptes rendus de l'Académie des Sciences. T. 112 (1891), p. 447—449.

[3]) JOHANNES WALTHER. Das Gesetz der Wüstenbildung, p. 143 sqq. Berlin 1900.

chlorine and sodium as the Meuse, the population on its drainage area however amounts to 5.000.000, and Bohemia is, like Belgium, one of the first manufacturing countries of Europe. The evaporating quotient is in the area of this river at least 4, according to an estimate by HARLACHER[1]). Although detailed facts are not at hand it may be accepted that the supply of sodium by chemical denudation is but little larger here than that of the Meuse and, moreover, that without the afore mentioned pollution there would be a deficit of chlorine to combine with the sodium.

In the area of the Danube above Vienna, the population is not nearly so dense and the consumption of sea salt in manufactories much less important. We therefore find in the water of this river a smaller quantity of sodium and a much smaller quantity of chlorine than in the Meuse and the Elbe.

From the consideration of those three most reliable and valuable analyses of river water the average quantity of sodium admitted by JOLY according to MURRAY's estimate proves far too high.

Amongst others this appears also in the St. Lawrence, which river (important not only as a large stream, but also because of its being the outlet of a number of immense lakes) drains at its origin in Lake Superior a region of old sandstones and crystalline rocks, but afterwards passes through lakes receiving water from a region of palaeozoic strata, containing much rock salt. The ratio of rainfall to the discharge of water is 3.9. Here too a large reduction of the figures found in the analysis ought to be made.

What is most apparent in our table, besides the greatly varying quantity of sodium in the river water, is the almost general deficit of chlorine to combine with the existing quantity of that metallic element. There is, almost always more sodium than could combine with the chlorine in the same river water. In such rivers where this is not the case we can account for this deviation by the presence of considerable amounts of other chlorides, refuse from manufactories, as is the case with the Meuse and the Elbe, or we can attribute it to accidental circumstances, where the difference is small and where only one analysis has been taken (and perhaps even no separate determination of sodium, as in the case of the Amazonas at Obidos). The water of the Dwina, the Weser and the Nahe has had the opportunity of taking from the soil in the drainage area considerable quantities of chloride of potassium.

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[1]) Cited in: F. POŠEPNÝ, Zur Genesis der Salzablagerungen, besonders jener im nordamerikanischen Westen. Sitzb. d. math.-naturw. Cl. Akad. d. Wiss. Wien, 1877, Bd. 76, Abth. I, p. 193.

In the water of the lakes named under N^0. 33 to 38, which are entirely surrounded by granite, basalt or gneiss, there are on the contrary no small quantities of sodium, with only traces of or very little chlorine.

So it appears that the process of chemical denudation, by which sodium is dissolved from silicates and conveyed by the rivers to the sea, is *not* going on, for the greater part, through the agency of chlorine.

Now, in the sea water all the sodium being combined with chlorine and there having been since innumerable centuries a supply of sodium from the rivers with a deficit of chlorine, this deficit must have been provided for. It is known that continually there is hydrochloric acid gas poured in the atmosphere through volcanism. Considering that at least three-fourths of the rain falls into the ocean and only one fourth on the land it is easy to understand in what manner this deficit of chlorine is made up.

Were it possible to find the annual absolute value of this deficit, did we know the average of it in river water, then we could, when starting from the uniformitarian point of view, and accepting that the ocean has acquired all its choride of sodium only in the actual way, estimate the geological age of the Earth, or better the length of past time of the process of denudation.

The ocean contains, according to DITTMAR's analyses and the newest estimates concerning the volume of the water in the ocean, 21400×10^{12} tons of chlorine combined with sodium. According to MURRAY's estimate [1]) the rivers annually discharge somewhat more then 27×10^{12} tons of water. Supposing a deficit of $^1/_{1\,000\,000}$ or 1 mgrm of chlorine per liter and admitting that on the land there took place $^1/_4$ of the total production of sodium chloride, those processes of chemical denudation ought to have proceeded with the same intensity during 590 million years (without this contribution of the land during 787 million years). The deficit however is on an average decidedly larger than a millionth part of the water. Were it nearly 25 mgrm per liter and did we again admit that on the land there is produced one-third as much chloride of sodium as in the ocean, then the results of our computation would agree with that of Lord KELVIN who estimated the age of the Earth to be about 24 million years. The latter supposed deficit of chlorine seems however to be far above the real average.

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[1]) L c., p. 70. As stated by MURRAY 6524 cubic miles. These are equal to 27192 KM^3.

In the Nile, the La Plata and the Rio Negro and also in Lough Neagh this deficit of chlorine, as shown in our table, is extraordinarily large. But considering that, according to MURRAY, the evaporation of the rain water in the area of the Nile is more than 14 times as high as that observed in the basin of the Meuse (in most of the other European rivers a similar proportion is found) the solution of salts represented by the water of the Nile at Caïro can be considered as a far more concentrated one. If the evaporation were not stronger there than in the area of the Meuse, the deficit of chloride would only be 2.2 mgrm. For the La Plata the proportion between the rainfall and the discharge of the river is only 2.34 times as large as that for the Meuse; under similar proportions of evaporation as in Europe, the deficit of chlorine would therefore be 5.44 mgrm according to one analysis, 17.71 mgrm according to the other. Perhaps the deviation is there too only a seeming one, caused by stronger evaporation. Not taking into consideration the rivers draining soils rich in salt, as many of the smaller North-American rivers, N°. 55—75 of our table — and in the case of the Rio Negro and Lough Neagh something analogous may be accepted — the exceptions, with a great deficit of chlorine are certainly so rare, that they cannot alter our conclusion based on what we observe in allmost all other cases.

Most likely only $^1/_4$ of the quantity of sodium in the annual discharge of the rivers stated by JOLY, according to MURRAY's estimate, is really to be attributed to chemical denudation; we then have to multiply the (not corrected) period of 99.4 million years, calculated, for the geological denudation by the first named author, with 4 and arrive at the result of about 400 million years.

The existing data concerning the quantity of sodium and chlorine in river water are therefore not appropriated to use in the problem treated by JOLY, if at least the result arrived at by Lord KELVIN, or even a period of one hundred million years, is approximatively right.

Thus the opinion, to which other geological facts had led, that the greater part of the chloride of sodium in the ocean must have been produced, by a process far more rapid than the present process of denudation is confirmed.

Physics. — "*The effect of the induction coil in telephonic apparatus*" (2nd part). By J. W. GILTAY. (Communicated by Prof. P. ZEEMAN.)

(Communicated in the meeting of December 28, 1901.)

To arrive at our purpose we shall now try another method. Let us suppose an induction-coil without iron; in the primary wire is an undulatory current, in the secondary wire an alternating current is induced. If we now slide an iron core into that coil, the coefficient of mutual induction will get $\frac{A}{B}$ times greater, *and if we take care that the strength of the current in the primary wire rests at the same value, notwithstanding the introduction of the iron*, the induced current will also be $\frac{A}{B}$ times stronger than it was before. We shall now investigate whether this $\frac{A}{B}$ has the same value for all our coils.

As, however, it would be difficult to make the intensity of the primary current after the introduction of the iron core equal to what it was before, we shall try to reach our aim in an indirect way.

Fig. 9 shows how we set about for this experiment. To find the value of $\frac{A}{B}$ for a coil with 3 primary layers, the primaries of 3 A and 3 B were connected with microphone and battery in one circuit. *a* is a small coil with wire and an iron core in it, quite similar to that of the electro-dynamometer. Now the secondary of 3 A was connected with the coil *a* and the secondary of 3 B with the electro-dynamometer. So now we measured the current induced by 3 B. By means of a commutator, left out of Fig. 9 for clearness' sake, the electro-dynamometer was made to change places with coil *a*, so that 3 A was connected with the measuring instrument and 3 B with coil *a*. So now the current induced by 3 A was measured, and the strength of the current in the primary circuit had necessarily remained unaltered in both cases.

Fig. 10 shows more elaborately how this experiment was arranged. If the 4 Morse keys are pressed down, 3 A is connected with the electro-dynamometer and 3 B with coil *a*. If the keys are in rest, 3 A is connected with coil *a* and 3 B with the measuring instrument.

The result of these measurements are given in Table IV. For every measurement or comparison of 2 coils with each other 33 turning points were read as formerly. As is seen from this table

$\frac{A}{B}$ has a smaller value for the coils of higher order than for those of lower order.

TABLE IV.

Coils:	i Mean deviation in minutes.	k Mean deviation in minutes.	$\sqrt{\frac{i}{k}}$
3A	526.5		6.52
3B		12.4	
6A	337.5		5.9
6B		9.7	
9A	410.3		
9B		13.5	5.51
12A	186.1		4.78
12B		8.15	
15A	501.1		3.94
15B		32.3	

The explanation of this phenomenon we shall have to look for in the fact that for the coils of higher order the secondary turns are farther removed from the iron core than for those of lower order. On account of this, for the former coils many of the lines of force, originating from the iron and being closed curves, will cut the cylindric space in the secondary coil twice, in 2 opposite directions. So these curves of force are entirely without value for producing induced currents in the secondary.

If the induction coils were arranged in such a way that the iron core formed a ring closed in itself, the above mentioned phenomenon could not take place, as then each line of force would cut the space in the secondary coil only in *one* direction.

If we write down the values for $\frac{A}{B}$, found according to the two different methods represented respectively in Fig. 3 and Fig. 9:

Coil	3	6	9	12	15
$\frac{A}{B}$ found according to fig. 9	6.52	5.9	5.51	4.78	3.94
$\frac{A}{B}$ found according to fig. 3	5.56	3.95	2.59	1.95	1.48
Weakening of the primary current by the iron	1.17	1.49	2.13	2.45	2.66

we see that the figures of the 2nd row and those of the 3rd row differ pretty much.

In measuring according to Fig. 9 we have taken care that the primary strength of the current always had the same value, whether the induced current of coil A or that of coil B was measured. When measuring according to fig. 3 we have *not* heeded that. So if the two methods give different numbers, the reason can be looked for only in the change which the strength of the primary current suffers in consequence of the iron.

Comparing the numbers of the 2nd with those of the 3rd horizontal row of the table given above, we see that the latter are all smaller than those of the 2nd row. This must evidently be explained from the fact, that in measuring according to fig. 9, only the useful factor of the influence of the iron — the increasing of the coefficient of mutual induction — is prominent, whilst if we act according to fig. 3, the disadvantageous factor of the influence of the iron also has a part in it: the increasing of the self-induction of the primary and the weakening of the primary current caused by it.

Besides the fact that the numbers in the 3rd row of the above table are smaller than those in the 2nd row, we also see that the 3rd row converges much more. This is due to the weakening of the primary current by the introduction of the iron core being greater for the coils of higher order than for those of lower order. So the disadvantageous factor of the iron is the greatest for coils of higher order. This weakening can be calculated by dividing the numbers of the 2nd row by those of the 3rd; the numbers obtained in this way I placed in the 4th horizontal row of the table.

Although it is sufficiently known that this weakening of the current by the iron increases with the number of primary turns, yet I have tried to show it experimentally by the measurements of which table V gives the results.

TABLE V

Coils	*i.* Mean deviation in minutes.	*k.* Mean deviation in minutes.	$\sqrt{\frac{i}{k}}$
3A	470		1.036
3B		504.5	
15A	111		1.539
15B		263	

The method followed for getting these results is shown in Fig. 11. T is the translator without iron, already mentioned on page 369. G is the electro-dynamometer, which for this experiment however is provided with thicker wire. The thickness of this wire is 0,2 mm., the resistance 42 Ohm. *a* is the coil with the iron core in it, which we used before. Before the microphone are the resonator and the tuning fork as before. If the 4 keys are pressed down, the primary of 3 A is connected with the electro-dynamometer and the translator; and the secondary of 3 A is connected with coil *a*. If the keys are on the contact of rest, 3 B has changed places with 3 A. These comparative experiments were made with the coils 3 and the coils 15; it is true, that table V shows but a very trifling influence of the iron, but yet it is clear that the influence is greater for 15 than for 3. These numbers cannot have an absolute value, the introduction of the translator and of the electro-dynamometer in the primary circuit greatly increasing the self-induction. On purpose to make this increase as small as possible, we provided the electro-dynamometer with thicker wire. If this experiment could be made with a measuring-instrument and a translator without self-induction we would of course have got 1,17 for coil 3 and. at 2,66 for coil 15 (see row 4, table page 402). The proportion of those numbers, $\frac{2,66}{1,17} = 2,3$, should correspond with the proportion of the final numbers of table V. This gives however for that proportion only 1,5.

The cause of this far from favourable result was due to the fact, as I afterwards found, that the microphone was not in good order when I made these measurements (they were the last I made with it). When it was connected with a telephone and a battery the former made a creaking sound, whilst no sound at all was made in the vicinity of the microphone.

Now we know that the phenomenon, that the iron in our coils of higher order has smaller efficiency, than in those of lower order, is due to 2 facts:

1. to the fact, that for the coils of higher order the secondary wire is wound on a wider cylinder, which causes many of the lines of force to cut twice the hollow space of that cylinder.

2. to the weakening of the primary current when the iron is introduced, by the increase of the self-induction, which weakening is greater for the coils of higher order than for those of lower order.

As is seen from the numbers of the 2nd and 3rd rows on page 402, the cause mentioned sub 2 is the most important. This is also clear if we note that the primary coil is the only object with self-induction in the primary circuit, as the self-induction of the microphone, the microphone-battery and the very short connecting wires are practically equal to 0.

So we have determined the influence of the iron on the intensity of the induced current in two ways: 1. by letting the harmful as well as the favourable influence of the iron act freely and 2. by bringing out only the favourable influence.

I have determined the influence of the iron still in a third way, standing midway between the two above-mentioned methods. This method is represented in fig. 12 and fig. 13; the arrangement of the experiment is shown a little more in details in fig. 14. As fig. 12 shows, the two primary wires of coils A and B, which are to be compared, are connected in one circuit with the microphone and the microphone battery. The secondary of A is connected with the electro-dynamometer, the secondary of B remains open. The electro-dynamometer is now again the same as the one used for all the other experiments except for those of table V. In this way the current induced by A is measured. Now the secondary of coil B is connected with the electro-dynamometer, the secondary of A remaining open, as fig. 13 indicates. This change was made by means of the 2 Morse keys, as is shown in fig. 14: when the keys were pressed down, 3 B was connected with the measuring-instrument; if the keys were in rest, 3 A was connected with it.

The values for $\frac{A}{B}$ found according to this method are placed in the 3rd horizontal row of the following table. The measurements were taken in quite the same way as indicated in tables II and III.

Coil	3	6	9	12	15
$\frac{A}{B}$ found according to fig. 9	6.52	5.9	5.51	4.78	3.94
$\frac{A}{B}$ found according to fig. 12, 13 and 14	4.94	5.03	4.91	4.25	3.92

As will be seen, $\frac{A}{B}$ is smaller in the 3rd row than in the 2nd row, for all coils. This is made clear by the following consideration:

In fig. 13 the current of B is measured whilst the secondary wire of A is open. On the other hand, in fig. 9 the secondary of A was closed when the current of B was measured. This closing of the secondary wire of A weakens the variations of the magnetism of the iron core, and by this weakening the current in the primary is strengthened; so the deviation given by coil B to the electro-dynamometer will be greater in the case of fig. 9 than in that of fig. 13.

In fig. 12 the current of A is measured whilst the secondary wire of B is open. In fig. 9 the secondary of B was closed whilst A was being measured. The opening or closing of the secondary of B will however give rise to only a trifling difference in the strength of the primary current, and the deviations given by A to the electro-dynamometer will have about the same value whether the measurements are made according to fig. 12 or to fig. 9.

The result is that $\frac{A}{B}$, determined according to fig. 9, must be greater than $\frac{A}{B}$ measured according to fig. 14.

We see that the difference between the numbers of the 2nd row and those of the 3rd row is smaller for the coils of higher order; for coil 3 the difference is rather great, whereas for coil 15 it has almost disappeared. This is owing to the fact, that the closing of the secondary in a coil of higher order causes a *smaller* decrease in the magnetic changes of the iron core than in coils of lower order.

The reason for this is 1. that the current induced in the secondary is weaker for 15 A (for instance) than for 3 A, and 2^0. that the secondary turns are farther removed from the iron in 15 A than in 3 A. This is also proved by the following experiment: I took the coils 3 A and 15 A of fig. 7 (changed into telephones) and again arranged the experiment as in fig. 8. The tuning-fork was placed near the resonator, the microphone-battery consisted of a storage-cell. Both telephones, the secondaries being open, gave a strong sound. If the secondary of 3 A was closed, the sound produced by this telephone, became perceptibly weaker; if the secondary of 15 A was closed, there was no difference perceptible in the intensity (though there was in the quality). The same experiment was repeated after the storage-cell had been exchanged for a Leclanché-cell, i.e. with a weaker undulating current. With coil 3 the closing of the secondary produced a very perceptible weakening of the sound; with 15 now a slight weakening of the sound was noticed when the secondary wire was closed but it was extremely small.[1])

When we divide the numbers, found on page 367 for the effect of the various A coils, by the values of $\frac{A}{B}$, found on page 402, we obtain for the effect of the various B coils:

Coil:	3	6	9	12	15
Strength of the induced current of the A coils:	1	1.507	1.429	1.114	0.818
$\frac{A}{B}$:	5.56	3.95	2.95	1.95	1.18
Strength of the induced current of the B coils;	0.18	0.38	0.48	0.57	0.55

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[1]) At first it was my intention in writing this paper not to mention the experiment of fig. 14, the question concerning the influence of the iron being to my idea sufficiently answered. But on the other hand it seemed to me that these numbers could serve to heighten the trustworthiness of the other results arrived at by me, as the reason of the differences between the numbers of the two rows on page 405 was, to my idea, perfectly explained. And where quantitative investigations are made with such capricious apparatus as microphones and electro-magnetic tuningforks, an indirect confirmation seemed not superfluous.

From this we see, that the decrease of the intensity of the induced current by the increase of the number of primary turns is much smaller for the B coils than for the A coils. The intensity increases as we see, from 3 B to 12 B, but that increase becomes less and less and is at last negative from 12 B to 15 B. The reason for 15 B giving a weaker induced current than 12 B can be found in the increase of resistance of the primary circuit, without reckoning with the self-induction:

The resistance of the microphone is 3.5 Ohm, that of the coil 12 B is 4.9 Ohm and that of 15 B 6.4 Ohm; the resistance of the microphone can be neglected. Then in the first case the whole resistance of the primary circuit is 8.4 and in the second case 9.9 Ohm. The proportion of those resistances is 1.18; the proportion of the number of turns in both cases is $\frac{15}{12} = 1,25$.

If we introduce into a circuit, in which a microphone and a battery have been placed, a dead resistance causing the entire resistance (when the microphone is at rest) to become n times greater, the strength of the current will be reduced to $\frac{1}{n}$ of its former value and the change of resistance in the microphone will also retain but $\frac{1}{n}$ of its former value. The undulations of the primary current will be reduced in this case to $\frac{1}{n^2}$ of their former value.

So in the above-mentioned case, by substituting coil 15 for coil 12, the undulations of the primary current become $1,18^2 = 1,39$ times smaller. On the other hand the number of primary turns becomes 1,25 times greater. So according to this calculation, the induced current given by 12 B would have to be $\frac{1,39}{1,25} = 1,11$ times stronger than those given by 15 B. The numbers in the table on page 406 give for that proportion $\frac{57}{55} = 1,04$. If we take into consideration that the measurement of the resistance of a microphone at rest often gives very different values, then we may consider the correspondence between these two proportions to be quite sufficient.

In the table on page 406 we see, that when 6 A connected with microphone and battery gives an induced current 1,507, 15 B will induce in the same circumstances a current 0,55. That proportion

$\frac{1,507}{0,55} = 2,73$ shows us, that with coil 15 B in our microphone-circuit, (i. e. with a coil without iron), we shall hear as much as with 6 A (the best of our coils, as far as intensity goes) if only we use 2,73, let us say 3 times, the number of cells as for 6 A. The resistance of the battery is neglected and moreover this calculation holds good only for the tone Fa 3.

The experiment with the telephone confirmed this entirely. Before the microphone the tuning-fork and resonator were placed as usual; by means of 4 Morse-keys we could bring into the circuit 6 A with 1 Leclanché-cell as well as 15 B with 3 cells It was found that not the slightest difference in the intensity of the sound was to be heard whether 6 A or 15 B was used. In the same way I compared 15 A with 2 Leclanché-cells with 15 B with 3 such cells. In both cases the intensity was exactly the same.

Sharper tones are weakened in a greater degree by the self-induction of the primary coil than lower ones. If I had made these experiments with a tuning-fork of a sharper note, I would have obtained other numbers; the series I found on page 367 for $\frac{A}{B}$ would have converged much more.

We have till now occupied ourselves only with the *intensity* of the currents (or of the telephonic sound) induced by our various coils. We shall now try to investigate which coils are best adapted for a pure articulation.

In general we can assume that for our purpose those coils in the first place come under consideration, which render a simple tone, produced before the microphone, also as a simple sound in the telephone; and which will reproduce a compound tone in such a way that the mutual relation of the intensities of the simple tones out of which the compound consists, is the same for the reproduced sound as it was for the original sound. Theoretically the induction coils with iron core must necessarily be inferior to those without iron:

1. because by the self-induction of the primary coil with iron the sharper tones weaken more than the lower ones, so the quality of the reproduced sound will not be the same as that of the original. This is in less degree the case in the coils without iron.

2. because a simple sound does not induce one sinusoidical current in the secondary wire of a coil with iron core, but two such currents, which are somewhat shifted in respect to each other. One of the sine currents is induced by the primary current, the other

by the magnetism of the core. For the sake of brevity we shall call the former the galvanic, the latter the electro-magnetic induction current. Now as the strengthening and weakening of the magnetism of the iron core, caused by the changes of the primary current, requires a certain time, the induced currents, produced by those changes of magnetism will appear and disappear' later than the currents induced directly by the primary current. The tables given on pages 402 and 406 enable us to calculate about how many times the electro-magnetic induction current is stronger for the different coils than the galvanic. So e.g. for coil 3 A:

The currents induced by 3 A and 3 B are in the proportion 1 : 0.18 (table page 406). The primary current of coil 3 B becomes 1.17 times weaker by the introduction of the iron (table page 402). So the galvanic induction current produced by coil 3 when iron is introduced into it (in other words the galvanic induced current given by coil 3 A) has the intensity $\frac{0.18}{0.17} = 0.15$ if 1 is the whole current induced by 3 A. Now the latter current is the sum of the galvanic and of the electro-magnetic current; the galvanic being $= 0.15$, the electro-magnetic will be equal to 0.85. It follows from this, that the electro-magnetic induced current for coil 3 A is $\frac{85}{15} = 5.7$ times stronger than the galvanic.

In this manner we find for the coils 3 A, 6 A, 9 A, 12 A, 15 A
the proportions 5.7, 4.9, 5.25, 3.8, 3.0.

With 3 A the electro-magnetic induced current will be so much stronger than the galvanic, that the latter will not be able to give any change to the quality of a simple sound. If we suppose the amplitude of the diaphragm of the telephone to be proportional to the strength of the induced currents, the intensity of the sound will be proportional to the square of the strength of the current. For 3 A the electro-magnetic tone will be $5.7^2 = 32$ times stronger than the galvanic: so it is not probable that the latter will have any perceptible effect.

In this respect 3 A will probably be the best coil as far as articulation goes. But also in respect to the 2nd condition, named on page 9 for a good articulation, 3 A will be the best of our A-coils, the self-induction being less than in any of the 4 other A-coils.

With 15 A the proportion of the electro-magnetic induced current to the galvanic $= 3$, so the proportion of the two tones $= 9$. In using this coil there will be the greatest chance that a simple tone made before the microphone will be reproduced with changed timbre by the telephone. And with 15 A the self-induction is also greater than with the other four A-coils, so in this respect also 15 A will be the least suitable for pure articulation.

With 15 B of course the double curves do not appear at all and the self-induction is much less than with 15 A. So we can expect 15 B to articulate better than 15 A.

In order to investigate in how far difference of articulation was perceptible with the different coils, I compared 3 A with 15 B, likewise 6 A with 15 B and finally also 15 A with 15 B. Of course the battery was chosen in such a way that the intensity of the sound remained the same with the two coils under comparison. So for instance in using 15 B (see table on pag. 406) 3 times more elements had to be taken than for 6 A. It was however evident, that the microphone got too much current with 3 Leclanché-cells and 15 B, as, even though no sound was made in its vicinity, it began to vibrate and to make a noise in the telephone. I therefore made use for these experiments of a thermo-electric battery of GÜLCHER consisting in all of 66 couples. In comparing 6 A with 15 B I made the former coil act with 7 couples, the other with 21 couples; in comparing 15 A with 15 B I used respectively 15 and 21 couples, etc. Now an article of a newspaper was read before the microphone and by pressing down or releasing 4 Morse-keys the 2 coils under comparison were exchanged. It appeared *that not the slightest difference in articulation was perceptible*, either when comparing 3 A with 15 B, 6 A with 15 B, or 15 A with 15 B. Women's voices generally sounding clearer out of our small telephones (with small, thin diaphragms) than men's voices, the experiments were also made with these, but with the same negative result. The telephone spoke equally clear in all the different cases.

To investigate whether in a musical sound change of quality would be perceptible when the coils were exchanged, the experiment was made with a musical box and also with the tuning-fork Fa 3, but without any result. To avoid the influence of the proper tones of the telephone-diaphragm, the telephone was substituted by a condensor with a permanent charge of about 32 volts, but the result was the same.

As in many cases, also here practice has found the right way. The induction coils with iron and 4 or 6 primary layers of wire in

J. W. GILTAY. „The effect of the induction coil in telephonic apparatus." (II.)

Fig. 9.

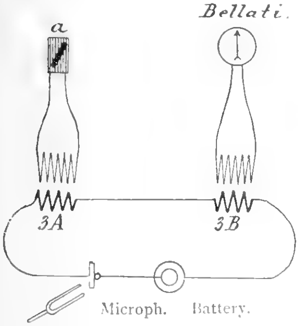


Fig. 10.

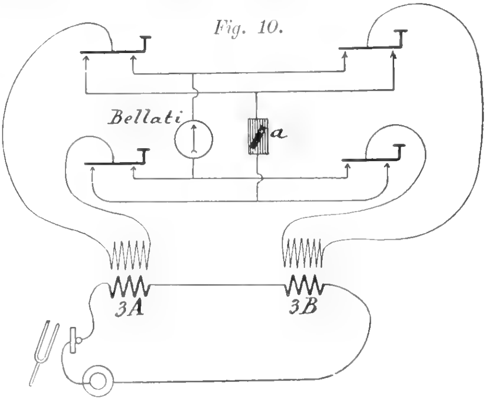


Fig. 11.

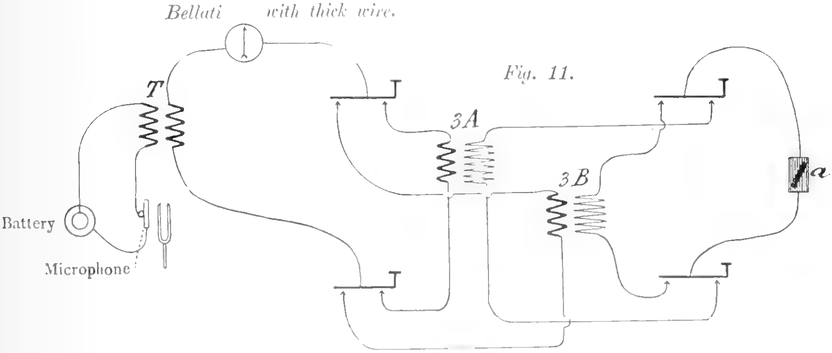


Fig. 12.

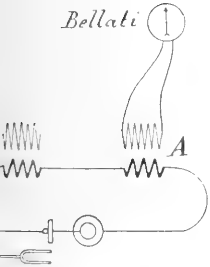


Fig. 13.

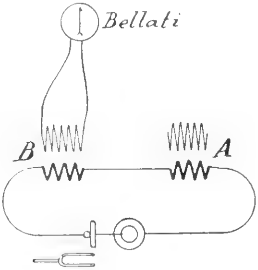
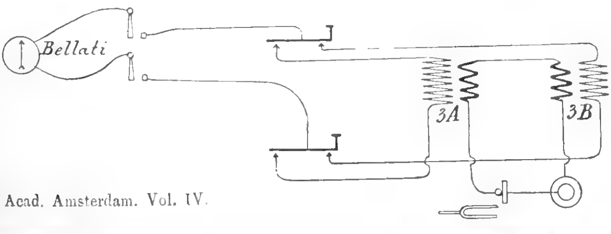


Fig. 14.



general use in practice, give the strongest sound and although theoretically they ought to articulate less accurately than coils with more primary wire and no iron, in practice this is not at all perceptible, not even in laboratory-practice, which is the only practice I have a right to speak of. Our sense of hearing is evidently so accustomed to content itself with defective sounds and to understand them that we cannot at all observe the small differences in the accuracy of the reproduction, which must undoubtedly exist when using the different coils.

I cannot omit quoting some lines of **Heaviside**, [1]) who expresses himself as follows, after having enumerated the different distortions to which telephonic transmission is exposed before the sound of the telephone is observed by us:

"And yet, after all these transformations and distortions, practical telephony is possible. The real explanation is, I think, to be found in the human mind, which has been continuously trained during a lifetime (assisted by inherited capacity) to interpret the indistinct indications impressed upon the human ear; of which some remarkable examples may be found among partially deaf persons, who seem to hear very well when all they have to go by (which practice makes sufficient) is as like articulate speech as a man's shadow is like the man."

As respects practice it is evident that nothing is to be learnt from my paper. The only thing deducible from it, is perhaps the following: On very long telephonic cables where all slight influences which might weaken the transmission of the sound, must be avoided, it is the custom that each station shunts the secondary of its induction-coil during the time that it takes a message, by pressing down a button. The telephonic currents coming from the sending station need not in this way pass through the secondary of the receiving station and are not needlessly weakened by the self-induction of that secondary with iron core. If now we were to take a coil with more primary layers and no iron, the self induction of the secondary would be much smaller and the troublesome shunting during the listening might perhaps be avoided. But no doubt the shunting of the coil is the more efficacious means to prevent the weakening of the telephonic currents.

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[1]) Oliver Heaviside, Electrical Papers, Vol II, Page 348.

Physiology. — Prof. PEKELHARING makes a communication concerning „*pepsin*".

Some years ago I communicated to this meeting a new method for the preparation of pepsin. In artificial gastric juice, obtained by digesting the mucous membrane of the stomach with 0.5% HCl for some days, I had found a substance, which is little soluble in water, containing about 0.02% HCl, but which becomes more easily soluble as well by a higher as by a lower amount of hydrochloric acid in the fluid. This substance appeared to be an extremely complicated proteid, which possessed to a very high degree the power of digesting proteid matter in acid solution. I supposed this substance might be the enzyme itself and not a proteid matter mixed with the enzyme and my grounds for this supposition were in the first place the extraordinary amount to which this substance was able to digest proteid matter and secondly the observation that the coagulation-temperature of this proteid matter, dissolved in hydrochloric acid, is just the same as the temperature, which makes the enzyme inactive. I had however to confine myself to a supposition, as my material was not of a sufficient degree of purity. I found that it always contained phosphorus, but the amount of phosphorus, though generally about 1%, was variable.

On account of the great importance of the question concerning the nature of the enzymes, I have continued my efforts to obtain the substance in a purer state. At first I used for that purpose the mucous membrane of the stomach of the pig. The method of preparation was altered in some respects in order to improve the purifying of the substance and to increase the amount of pepsin.

I can add to what was stated before, about the nature of the substance, that from its solutions it can also be precipitated by ammoniumsulfate.

After treating the extract of the mucous membrane in the above mentioned way, with basic plumbic-acetate and ammonia and decomposing the precipitate with oxalic acid and dialysing and filtering the thus obtained concentrated solution, there could be obtained out of the filtrate, by saturating with ammonium sulfate, still a considerable quantity of pepsin, which when purified by dissolving in HCl 0.2% and dialysis, showed exactly the same qualities as the pepsin, prepared directly from the artificial gastric juice and from the lead-precipitate, it especially possessed as great a

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[1]) Account of the Meeting on May 30, 1896.

digesting power. Whereas from the solution that was greatly contaminated by products of digestion, a precipitate fit for being filtered was only produced by complete saturation with ammonium sulfate, the pepsin will completely separate from solutions containing but few products of digestion, by half saturating the fluid with this salt.

If the substance is slowly precipitated either by dialysis into water or under the influence of ammoniumsulfate, the precipitate is not amorph, but has the shape of small globules, resembling the globulites of albumen, which after HOFMEISTER's method can be obtained from egg-albumen. The little globules of pepsin are however smaller, the largest have a diameter of about 15 à 20 μ.

When kept under ammoniumsulfate, pepsin can remain unaltered for a very long time. A preparation, entirely consisting of small globules, repeatedly washed out by decantation with half saturated ammoniumsulfate solution and finally kept in this fluid in an apartment, in which the temperature undergoes great variations, has already remained unaltered for four years. It shows not only no alteration when investigated with the microscope, but when freed from ammoniumsulfate and dissolved in hydrochloric acid, it also digests proteid matter very powerfully.

Though I have treated some hundreds of mucous membranes of pigs' stomachs I have not succeeded in preparing from them a pepsin of constant composition. By analysing some five preparations, purified as much as possible, the values found for the various elements varied between the following numbers:

	C	H	N	P	S
maximum	50.77	7.27	15.06	0.75	1.6
minimum	48.18	6.72	14.02	0.425	1.45

I thus became convinced, that I could not hope for a better result if I continued the purification of the substance by dissolving it again and again in HCl 0.2% and then precipitating it by dialysis into water, on account of the great variability of the substance, the perseverance with which it retains contaminations and the great loss of substance, which is inevitable when purifying it and which would make the preparation of quantities such as are required for analysis, practically impossible.

I therefore resolved to try whether the gastric juice of the dog, which now, owing to PAWLOW's brilliant researches, can be obtained in unlimited quantity and without admixtures, should perhaps give more satisfactory results.

My request to Prof. Pawlow for some further information about the method of operating, was answered with the greatest kindness by a very detailed and accurate account by Dr. Walther about everything that must be taken into consideration when applying the gastral fistula and oesophageal fistula, according to Pawlow's method. Prof. Narath then kindly applied for me in a dog of 25 KG., first a gastral fistula and when the wound was completely healed and the silver cannula was fixed in it, also an oesophageal fistula. Besides to the gentlemen already mentioned, I am also greatly obliged to Messrs. de Bruin, Schimmel and Thomassen, lecturers at the State Veterinary School at Utrecht, who when the good healing of the wound in the stomach was in some danger, repeatedly gave me excellent assistance.

Throughout the whole of the year 1901 the dog, who was always in an excellent state of health, was regularly used, generally three times a week, for the producing of gastric juice. By a pseudo-feeding with meat there was obtained in the first hour an average of 200 cc., in the second hour still an average of 100 cc. of gastric juice. Each time the fluid obtained in a quarter of an hour, was filtered in order to free it from little flakes of mucus. The filtrate was then perfectly colourless and clear. Only occasionally when the amount of pepsin was great and the temperature of the room was low, it showed some opalescence, which however, when heated to the temperature of the body, disappeared at once without leaving a trace, to reappear again when cooled. The amount of acid was on an average 0.16 norm. H Cl.

As Nencki and Sieber already stated a short time ago, the pepsin is precipitated for the greater part from the pure gastric juice of the dog, in the same way as from the extract of the mucous membrane of the stomach, prepared with hydrochloric acid, as soon as the amount of acid is decreased to about 0.02 $^0/_0$: the best by dialysis. In the dialysator the substance is then deposited just like the pepsin out of the mucous membrane of the pig, in the shape of transparent small globules. While however the pepsin of the pig, at least when dried, was always somewhat coloured, even when it was separated out of a solution, in which no decided colour could be seen, the pepsin of the dog was always,

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1) Zeitschr. f. Physiol. Chemie, Bd. XXXII, S. 291.

even after drying, of a pure white colour. This was only then not the case, when, what occurred a few times, the gastric juice was mixed with bile. Gastric juice which, by admixture with bile, was distinctly coloured yellow, was never used for the preparation of pepsin. But it could be shown that even the smallest traces of bile-pigment, were retained by the pepsin. When the gastric juice contained but so little of it, that the yellow colour could not clearly be distinguished, the pepsin, precipitated from it by dialysis, yet showed a yellow tint which became greenish when dried. For analysis I have exclusively used entirely colourless pepsin.

The fluid, separated from pepsin which had been precipitated by dialysis, was half saturated with ammoniumsulfate. Thus a not unimportant precipitate was again obtained. This was dissolved, when liberated from ammoniumsulfate by dialysis into 0.2 % H Cl, at 37° C. in as little hydrochloric acid as possible of the same strength, filtered and subjected to dialysis into distilled water. The substance thus precipitated was dried and separately collected for analysis.

When the pure gastric juice was half saturated with ammonium-sulfate, in the filtrate of this no perceptible precipitate could any more be produced by complete saturation with this salt. The fluid, dialysed for 24 hours, filtered and then half saturated with ammonium sulfate, still after total saturation showed a certain amount of turbidness. When dialysed at a low temperature, a very small portion of the substance is always decomposed, whereby albumose is liberated.

In the first place I have investigated the amount of phosphorus. The continued investigation of the pepsin, obtained out of pigs' stomachs, had already made me doubtful, whether the phosphorus I found in it, really owed its origin to the highly complicated proteid, of which this pepsin principally consists. As I have formerly stated the amount of P of my first preparations was about 1%, but already then I observed that a more or less considerable contamination with substances containing phosphorus, was likely. The more the pepsin was purified, the smaller the amount of phosphorus was generally found to be, though I found the pepsin out of the pig's stomach never free from it. Moreover I had become convinced that I had erroneously placed the coagulation-product, which is obtained by heating of the acid solution of pepsin, on one line with the well-known nucleo-proteids. Later on I shall again refer to that.

I now found the pepsin of the dog free from phosphorus. The substance separated by dialysis from the gastric juice, still contained a trace of it, but such a small quantity (about 0.01 %) as can only be attributed to impurity.

This impurity was apparently quite removed together with the precipitate, which was formed when dialysing. For when afterwards the pepsin which was still in solution, was precipitated by half saturating with ammoniumsulfate and purified by dissolving it in hydrochloric acid and dialysis, there was not found in it the slightest trace of phosphorus. Nevertheless this pepsin was equally able to digest proteid matter as was that which was separated directly by dialysis out of the gastric juice.

Nencki and Sieber did find phosphorus in the pepsin, prepared by them out of the gastric juice of the dog, even when the pepsin was washed out with alcohol and had thus been altered. However they found the amount of P not only small, but also in various preparations very different. In the pepsin, precipitated by dialysis and not washed out, it varied between 0.073% and 0.148%, in the preparations washed out with alcohol between 0.045% and 0.091%.

They take for granted that pepsin contains lecithine; partly at least, not as an impurity, but in a combination to be compared with the compounds of lecithine with glycose, morphin, etc., especially studied by Bing.

I am not prepared to deny the possibility of the presence of a compound of pepsin with lecithine in the gastric juice, nevertheless I wish to lay stress on this : that the existence of the enzyme should not be considered as being connected with the presence of lecithine or any other P-containing group in the molecule of the pepsin, now that I have succeeded in preparing very powerful pepsin, in which either no phosphorus at all, or only a very insignificant trace of it could be shown.

Contamination of the pepsin with phosphorus can, besides by lecithine, also be caused by other substances. In the gastric juice, obtained by pseudo-feeding, there always occurs, at least in the dog used by me, some mucus, which can easily be removed by filtration. This consists of a P-containing proteid matter. When taken from the filter and washed out with water and alcohol, it scarcely dissolves at all in diluted hydrochloric acid, but by digesting with pepsin and hydrochloric acid, it gradually loses the gelatinous character and dissolves for the greater part, while a sediment, easily soluble in alkali, is formed.

This mucine is probably a nucleo-proteid and it is certainly possible, that when kept in contact with the gastric juice for some time and partly digested, it may yield *P*-containing decomposition-products to the solution. I have therefore immediately filtered the gastric juice, obtained in each quarter of an hour.

As to the amount of chlorine my statements agree with those of NENCKI and SIEBER. While the latter found in five determinations, either 0.47 $^0/_0$ or 0.48 $_0/^0$, I found 0.49 $^0/_0$. To the reasons, given by these investigators, for the opinion, that chlorine forms an element of the molecule of pepsin I may add another. The pepsin that was separated either directly by dialysis, or first by ammonium sulfate, was then dissolved, in as great a concentration as possible, at 37° C. in 1 $^0/_0$ oxalic acid and precipitated from this solution by dialysis into distilled water. This precipitate rapidly digested fibrin, when dissolved in oxalic acid. If now the chlorine found in the pepsin, which was separated out of the solution in hydrochloric acid, did owe its origin to hydrochloric acid that was not sufficiently washed out, then we might expect, that in the substance, separated out of oxalic acid, the clorine could no more be found. Nevertheless chlorine could repeatedly be shown very distinctly in this.

The results of the analysis of six preparations follow here.

	C	H	N	S	
I.	52.13 52.06 51.81	7.06 7.19 7.09	14.13 14.33	1.66	Pepsin precipitated by dialysis of the gastric suice.
II.	51.92	7.14	14.50		
III.	52.13	7.01	14.58	1.63	
IV.	51.61	6.93	14.57	1.61	
V.	52.32	7.16	14.75	1.83	Pepsin precipitated by half saturating the dialysed gastric suice with ammoniumsulfate.
VI.	52.01	7.02	14.65		

That the amount of sulfur in V was found a little higher than in the other preparations, may perhaps be attributed to contamination with ammonium sulfate. This supposition is perhaps supported by the amount of nitrogen. With 0.2 $^0/_0$ S derived from $(NH_4)_2SO_4$ corresponds 0.175 $^0/_0$ N. So the amount of N whould become 14.575 $^0/_0$ and would better agree with the amount of nitrogen of the other preparations.

The quantity of preparation VI was not sufficient also to determine the amount of sulfur. The amount of ash is not, as with all the other preparations, taken into account here. Herein lies however no objection of any significance, because the amount of ash was always found to be very slight. In the pepsin precipitated by dialysis it was ± $0.1^0/_0$, in that which was first precipitated by ammonium sulfate ± $0.2^0/_0$.

M^me^. SCHOUMOW—SIMANOWSKI [1]) found for the pepsin precipitated by cooling out of the gastric juice of the dog and washed with alcohol:

C	H
50.71	7.17

and for pepsin precipitated from the gastric juice, by saturating with ammoniumsulfate:

C	H	N	S
50.37	6.88	14.55	1.35
		15.0	1.24

where the amount of ash was not taken into account, whereas NENCKI and SIEBER found, when analysing the substance precipitated by dialysis:

	C	H	N	S
	51.26	6.74	14.33	1.5
and				
	51.99	7.07	14.44	1.63

is the average result of the four preparations analysed by me, of pepsin precipitated by dialysis.

The difference concerns especially the amount of carbon, which was found by me in all my preparations higher than by the Russian investigators, though my figures are already very near those of NENCKI and SIEBER. On account of the absence of phosphorus in my preparations and also of the smaller amount of ash (N and S found $0.57^0/_0$) I think I may consider the substance prepared by me, as being better purified. Also in the pepsin out of the mucous membrane of the pig's stomach, which I did not succeed in liberating from phosphorus, I found a lower amount of carbon, varying between 48.18 and $50.77^0/_0$.

In the ash I could detect iron in accordance with NENCKI and SIEBER. I have not made determinations of the amount of iron.

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[1]) Arch. des Sc. biol. T. II, p. 463.

From the cleavage-products, into which the substance is split by heating, I have up till now only studied the coagulation-product more accurately, which is precipitated when the heating is done rapidly. Formerly I already stated that by boiling with mineral acids, purinebases can be obtained out of this. To this I can only add that I have been able to prepare out of the pepsin of the pig a basis which was found to be xanthine. At first I found no reducing substance after boiling the coagulation-product with mineral acid. Further investigation however taught me, as is already stated by FRIEDENTHAL [1]) and also by NENCKI and SIEBER, that when so treated a reducing substance is indeed liberated, which shows the qualities of a pentose. Nevertheless, although this proteid matter contains purinebases and a carbohydrate group, it will not do to continue to group it any longer among the nucleo-proteids, as it contains no phosphorus. That no phosphorus was found in it, when it was prepared out of the pure pepsin of the dog, is a matter of course. But also when it was precipitated from the fresh gastric juice by boiling and then carefully washed, successively with water, alcohol and ether, I could not find a trace of phosphorus in it. When investigating the coagulation-product out of the pepsin of the mucous membrane of the pig I had already been doubtful, whether this substance might indeed be considered to be a nucleo-proteid, on account of the failing of all my efforts, to prepare a nucleic acid out of it.

It did produce an acid, when treated with alkali, but this acid was a proteid matter, which proved to be little soluble in water, insoluble in diluted acid and easily soluble in warm alcohol.

This acid is best prepared in the following way:

The coagulation product is dissolved in 1% hydrate of potassium and is boiled with this in the water-bath for five minutes. The fluid which was first entirely colourless, then acquires a light yellow tint. It is now made acid with hydrochloric acid. Hereby a considerable precipitate is formed under development of sulfureted hydrogen, which, when cooled, is filtered off. It is of a pure white colour. The filtrate is also colourless, but this becomes yellow again with alcalic reaction. The filtrate gives biuret reaction and produces, after boiling with hydrochloric acid, pentose.

The precipitate is washed out with 0.5% HCl, dissolved in boiling alcohol of 85% and filtered, when hot. On cooling it, the

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[1]) ENGELMANN's Archiv. f. Physiol. 1900, S. 189.

substance is precipitated as a gelatinous mass, at the surface of which a clear layer of alcohol gradually separates, which retains but little of the substance in solution. The precipitate, was first washed by decantation with 96 % alcohol and then mixed with equal volumes of alcohol and ether, when it becomes flaky and deposits well. It is at last brought on the filter, washed with pure ether and dried. When drying the precipitate which was first of a pure white, sometimes acquires a light yellow tint.

This substance has the qualities of an acid. Brought into water, it is dissolved, with an acid reaction if an alkali be carefully added. When this acid reacting solution is subjected to electrolysis, under a tension of ± 50 Volt, in the way lately described by HUISKAMP [1]) after being freed as much as possible from other salts, by dialysing into repeatedly renewed distilled water for a few days, then the proteid is transported to the anode and is there deposited as a gelatinous lump, while at the negative pole alkali is accumulated.

I propose to give to this substance the name of *pepsinic acid.*

Out of the alcoholic solution the substance separates by evaporation of the alcohol, as a varnish-like substance. It gives biuret and xanthoprotein-reaction and the reactions of ADAMKIEWICZ and of MILLON. As might be expected it does not yield sulfur, when boiled with hydrate of potassium. By boiling for a long time with alkali it is changed further. Addition of hydrochloric acid then no more causes any precipitate.

Out of pepsin solutions, which are changed by heating slowly without becoming turbid by this, this substance cannot be obtained. I first became acquainted with the pepsinic acid as a decomposition product of the pepsin, prepared out of the mucous membrane of the pig's stomach and I afterwards obtained it in exactly the same way out of the pepsin of the dog. However I never succeeded in preparing the pepsinic acid of the pig entirely colourless. Therefore I did not use this for elementary analysis.

The analysis of the pepsinic acid of the dog gave the following results:

C	H	N	S
50.79	7.02	14.44	1.08

whereas for the coagulation-product, out of which this acid is prepared and which may be considered as an acid proteid matter itself, was found:

C	H	N	S
50.35	6.98	14.90	1.64

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[1]) Zeitschr. f. Physiol. Chemie, Bd. XXXIV, S. 32.

The highly complicated proteid matter, which besides mucine, is the only albuminous element of the gastric juice of the dog, could only in so far be said to differ from the substance, prepared out of the mucous membrane of the pig's stomach, by the fact that the latter was not sufficiently pure. As to the qualities and the decomposition-products, as far as the investigation of these was possible, they were exactly the same. As I mentioned before, such a substance may also be prepared out of the mucous membrane of the stomach of the dog and of the calf.

This substance is, like other proteid substances, levogyr. I have not been able to detect any influence of the reaction of the solution upon the degree of rotation.

When it is now taken into consideration that the substance obtained out of the gastric juice of the dog can be made to acquire a degree of purity that is satisfactory for proteid matters, the hypothesis that this substance is the enzyme itself and does not own its activity to admixtures, does not appear to be a very bold one.

The substance loses the action of pepsin by heating, at the exact temperature which decomposes it.

When gastric juice, by half saturating it with ammoniumsulfate, is liberated from this substance, it loses its capacity to digest proteid. With respect to this it should be taken into consideration that the presence of ammoniumsulfate is highly detrimental to the action of pepsin. But I have repeatedly convinced myself, that the fluid, also when the salt was removed to a trace by dialysis, was quite unable to digest fibrin. If we now consider that $^1/_{1000}$ mgr. of the substance prepared by me still shows a distinct though weak action on fibrin in 6 ccm. HCl 0.2 $^0/_0$, then we may rightly conclude, that the filtrate freed from ammoniumsulfate by dialysis, does not contain any pepsin, all the enzyme thus being precipitated by the salt out of the gastric juice. And in this precipitate we find nothing but the proteid matter, which shows the action of the enzyme as strongly as possible.

I find another argument for my view in the observation, that the digesting power of the gastric juice keeps pace with the quantity of the coagulation-product precipitated from it by heating. This is especially clear, when in the dog the secretion of gastric juice is increased by injection of diluted alcohol into the rectum during the pseudo-feeding.

The gastric juice secreted under the influence of alcohol, is more considerable in quantity and surely as rich, but generally a little richer in acid than that secreted before; but poorer in pepsin. The coagulation-product, which doubtless owes its origin to the substance which I consider to be the enzyme, is then always precipitated in a perceptibly smaller quantity. The determination of the relative quantity of pepsine in the gastric juice took place after the method of METT, whereas the quantity of the coagulation-product was determined by boiling 50 cc. of the gastric juice over the flame and bringing the precipitate, after cooling, on a weighed filter, washing it out successively with water, alcohol and ether, drying it at 110° C. and weighing.

That solutions of pepsine can be prepared, which act powerfully and yet show no proteid-reactions is, as I pointed out before and as has also been stated lately by NENCKI and SIEBER, no reason to deny to pepsin the nature of a proteid matter.

By BLISS and NOVY an observation is mentioned, which might raise a doubt, as to whether pepsin can indeed be considered to be a proteid matter [1]). They found namely, that pepsin was not changed at all by formaldehyde, although this substance affects various proteid matters and makes them insoluble. I convinced myself of the justness of the observation. Solutions of pepsin in hydrochloric acid, to which formol is added to an amount of 2 to 3 $^0/_0$, may be kept for days, without perceptibly losing any digesting power. Of course the amount of formaldehyde must be considerably decreased, either by diluting or by dialysis, before the solution is brought in contact with fibrin, because otherwise the fibrin itself would be affected by formol and made unsusceptible for digestion.

It can not be maintained, however, that all proteidmatters are changed by formaldehyde. It is especially the proteidmatter here treated, which is not affected by this. I have dissolved the purified matter in 0.2 $^0/_0$ H Cl, while adding formol, and been able to precipitate it again from this solution, as well by dialysis, as by ammoniumsulfate, without it having lost any of the qualities of pepsin.

The substance possesses not only the power of digesting albumen in an acid solution, but it also causes milk to coagulate, as I stated before. In accordance with NENCKI and SIEBER I also found that it forms "plastein" out of albumose.

NENCKI and SIEBER have argued, that there is no serious objection to the supposition, that one and the same molecule may have various enzym-actions. With that argument I quite agree. To stick

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[1]) Journal of exp. med. Vol. IV. p. 47.

to the well-known analogy, given by E. FISCHER, there are keys, which consist of a ring with different pieces attached to it, which each fit into a different lock. Even when one or more of those pieces are bent or made useless in some other way, those that are left can still be used.

While the gastric juice of the dog used by me, showed very clearly the fat-decomposing action, described by VOLLHARD[1]), the pure pepsin, prepared out of it, had not the slightest action on fat, neither by neutral nor by acid reaction.

Mathematics. — "*The differential equation of* MONGE." By Prof. W. KAPTEYN.

In our communication of June 6th 1901 we gave the results of our investigation of the differential equation

$$r - \lambda^2 t + \mu = 0,$$

in which λ and μ were supposed to be dependent only on p and q or on x, y and z.

If we now assume no limiting conditions with respect to λ and μ, we shall find that the above mentioned equation can possess two intermediate integrals in the following case only.

Suppose c, h, v and ϱ to be any functions respectively of x, of y, of x, y, z and of v; then putting

$$G = \frac{\partial}{dx} + p\frac{\partial}{\partial z} \quad , \qquad H = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z},$$

λ must be equal to $\frac{P}{Q}$ and μ equal to $\lambda^2 Q_1 + P_1$,

where

$$P = \frac{c^2 G^2(v) - 1}{c\, v_3{}^2} \qquad Q = \frac{h^2 H^2(v) - 1}{h\, v_3{}^2}$$

$$P_1 = \frac{\varrho'}{\varrho}\frac{v_3}{c} P + \frac{c_1}{c\, v_3} G(v) + \frac{1}{v_3} G\, G(v)$$

$$-Q_1 = \frac{\varrho'}{\varrho}\frac{v_3}{h} Q + \frac{h_2}{h\, v_3} H(v) + \frac{1}{v_3} H\, H(v),$$

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[2]) Münch. med. Wochenschr., 1901, S. 141 and Zeitschr. f. klin. Med. Bd. XLII, S. 414.

whilst the indices 1, 2, 3 refer to differentiations according to x, y or z of the functions, with which they are connected.

If these conditions are satisfied the first intermediate integral is found by connecting the two common integrals of the system

$$\frac{\partial\varphi}{\partial x}+\left(\frac{u+1}{c\,v_3\,(u-1)}-\frac{v_1}{v_3}\right)\frac{\partial\varphi}{\partial z}+\frac{2\,u}{c}\,\frac{\varrho'}{\varrho}\,\frac{\partial\varphi}{\partial u}=0,$$

$$\frac{\partial\varphi}{\partial y}+\left(\frac{u+1}{h\,v_3\,(u-1)}-\frac{v_2}{v_3}\right)\frac{\partial\varphi}{\partial z}+\frac{2\,u}{h}\,\frac{\varrho'}{\varrho}\,\frac{\partial\varphi}{\partial u}=0,$$

where

$$u=\frac{1+c\,G\,(v)}{1-c\,G\,(v)}\cdot\frac{1+h\,H\,(v)}{1-h\,H\,(v)}.$$

A first common integral $\varphi_1=\varrho\,\frac{\sqrt{u}}{u-1}$ of this system is easy to find, the second φ_2 however cannot be found without v and $\varrho=\varrho\,(v)$ being known.

In the same way the second intermediate integral is deduced by connecting the two common integrals of the system

$$\frac{\partial\psi}{\partial x}+\left(\frac{w+1}{c\,v_3\,(w-1)}-\frac{v_1}{v_3}\right)\frac{\partial\psi}{\partial z}+\frac{2\,w}{c}\,\frac{\varrho'}{\varrho}\,\frac{\partial\psi}{\partial w}=0,$$

$$\frac{\partial\psi}{\partial y}-\left(\frac{w+1}{h\,v_3\,(w-1)}+\frac{v_2}{v_3}\right)\frac{\partial\psi}{\partial z}-\frac{2\,w}{h}\,\frac{\varrho'}{\varrho}\,\frac{\partial\psi}{\partial w}=0,$$

where

$$w=\frac{1+c\,G\,(v)}{1-c\,G\,(v)}\cdot\frac{1-h\,H\,(v)}{1+h\,H\,(v)}.$$

Here too a common integral $\psi_1=\frac{\varrho\sqrt{w}}{w-1}$ is directly known, whilst the second ψ_2 requires the functions v and ϱ to be known.

Mathematics. — "*Factorisation of large numbers*" (2nd part). By F. J. VAES. (Communicated by Prof. P. H. SCHOUTE).

V. *Method of remainders.*

If $G = a_1^2 - b_1$ (or $= a^2 - b^2$) is divisible by p, then the difference of the remainders left by a_1^2 and b_1 after division by p must be a p-fold.

If we write $G = \left(\frac{G+1}{2}\right)^2 - \left(\frac{G-1}{2}\right)^2$ and $\frac{G-1}{2}$ gives after division by p a remainder r, then it is evident that $\frac{G+1}{2}$ must give a remainder $r+1$.

So $G = (p\text{-fold} + r + 1)^2 - (p\text{-fold} + r)^2 = p\text{-fold} + 2r + 1$.

If p is a factor of G, then $2r+1$ must be a p-fold.

Example:

$$G = 80047 = (40024)^2 - (40023)^2.$$

$$G_1 = 40023 = 200^2 + 23.$$

So we may write

$$40023 = 200^2 - 1^2 + 24 \quad \text{or} \quad = 201 \times 199 + 24.$$

Each of the divisors 199 or 201 will leave the remainder 24. As $2r+1=49$ is not a 199- or 201-fold, those two numbers cannot be factors of G. We now find successively:

	r	$2r+1$
$G_1 = 201 \times 199 +$	24	49
$202 \times 198 +$	27	55
$203 \times 197 +$	32	65
$204 \times 196 +$	39	79
$205 \times 195 +$	48	97
$206 \times 194 +$	59	119
$207 \times 193 +$	72	145
$208 \times 192 +$	87	175
$209 \times 191 +$	104	**209**

The remainders 23, 24, 27, 32, etc. ascend with 1, 3, 5, 7, etc.

Evidently 209 is a factor.

The other factor 383 can be found by direct division or as follows:

$$G_1 = 209 \times 191 + 104, \quad \text{so}$$

$$\begin{aligned} G &= (G_1+1)^2 - G_1^2 = 2\,G_1 + 1 \\ &= 209 \times 382 + 208 + 1 \\ &= 209 \times 383. \end{aligned}$$

After two operations the factor 209 gives 11 × 19.

With some attention the factor 11 might have been found before (besides by the well-known property).

For as

$$G_1 = 202 \times 198 + 27 = 202 \times 11 \times 18 \quad \text{and} \quad 2 \times 27 + 1 = 55$$

the factor 11 might already have been pointed out in that place.

It is evident from this, that it is desirable to investigate the factors of $2\,r + 1$ and of the numbers multiplied by each other. At the same time however non-divisors can be determined, and in this way the indivisibility of 383 can be proved after 6 operations, because all prime numbers under $\sqrt{383}$ have then already disappeared. Without omission of the non-divisors, we should have 13 operations to deal with.

The number $G = 100895598169$ mentioned in the preface is immediately factorised after application of the method of remainders. For $G = 50447799085^2 - 50447799084^2$, and the root G_1 of the subtrahend is equal to:

$$224605^2 + 393059$$

$$\text{or } 224606 \times 224605 + 168454, \text{ where } 2\,r + 1 = 336909.$$

Evidently $G_2 = 112303$ is a factor and

$$G = 2\ G_1 + 1 = 2 \times 2\ G_2 \times 224605 + 3\ G_2 = G_2 \times 898423.$$

Each of the factors $G_2 = 112303$, and $G_3 = 898423$ must still be dealt with.

$$G_2 = 56152^2 - 56151^2$$

r	$2\,r + 1$	(non-divisors minor to $\sqrt{G_2}$ or 335)
$56151 = 236^2 + 455$	910	7, 13, 59
$236 \times 237 + 219$	439	3, 79,
$235 \times 238 + 221$	443	47, 17
$234 \times 239 + 225$	450	13, 239
$233 \times 240 + 231$	463	233
$232 \times 241 + 239$	479	29, 241
$231 \times 242 + 249$	499	11

A remainder has been found greater than the smallest factor, namely $249 = 231 + 18$. So we can write $231 \times 243 + 18$, which causes the remainders to have smaller values.

After a total of 82 operations all prime numbers $\sqrt{G_2}$ have disappeared and G_2 proves to be indivisible. Without the pointing out of the non-divisors more than 260 operations would have been necessary.

The factor G_3 requires about three times the number of operations as G_2.

As the calculation is performed with relatively small numbers, the method of remainders is to be most recommended for the investigation of great numbers.

VI. *Testing of divisors.*

Example $G_3 = 898423 = 948^2 - 281$.

To find out whether 7 is a divisor the remainders must be determined that are left by 948 and 281 after division by 7. If the difference of the (remainder)^2 of 948 and the remainder of 281 is a sevenfold then 7 is a factor.

So we must write:

divisor	quotient	remainder	(remainder)^2	remainder of 281	
7	135	3	2	1	
11	86	2	4	6	
13	73	—1	1	8	etc.

To obtain smaller numbers the remainders can be taken negatively without any inconvenience.

It is at once evident, that the divisor 73 will give a quotient 13 with a remainder —1; likewise that the divisor 43 (half of 86) will give a quotient 2×11 with remainder 2; etc., so that often the result can be written down for two divisors at once.

For larger numbers the operation would be:

divisor	quotient	remainder	(remainder)^2	remainder of 281
509	always 1	439	192721 or 319	always 281
511		437	190969 or 366	

VII. *Calculation of the number of operations.*

If the operation in § V for the number 56151 is performed as follows: $236^2 + 455$, $235 \times 237 + 456$, $234 \times 238 + 459$, etc., then the numbers 455, 456, 459, etc. differ 1, 3, 5, etc.; so they form a series of order two, of which the $(n-1)^{\text{th}}$ term is:

$$n^2 + 455.$$

So this is the value of r, whilst (after n operations) the factors have become: $236 - n$ and $236 + n$.

One of these will be a factor of $G_2 = 112303$ if $2r + 1$ is divisible by that factor, that is for

$$2\,n^2 + 911 = p\,(236 + n), \quad \text{or} \quad = p\,(236 - n).$$

The first of these equations gives

$$n = \frac{p}{4} \pm \frac{1}{4}\sqrt{p^2 + 1888\,p - 7288}.$$

The form under the root sign must of necessity be a square and p at least 4.

To investigate which value must be taken for p it may be noticed that if p increases with 1, the whole form grows with $2p + 1 + 1888$.

So the algorithm of § I gives:

p		$p^2 + 1888\,p - 7288$
4		280
	$2 \times 4 + 1889 =$	1897
5		2177
	$2 \times 5 + 1889 =$	1899
		4076
		1901
		3
		5
		7
		9
6		13601
	$2 \times 1912 =$	3824
		17425
	$1 \times 1915 =$	1915
		19340
	$2 \times 1918 =$	3836
		23176
	$5 \times 1925 =$	9625
		32801
	etc.,	

to which the same abbreviations as formerly can be applied by paying attention to the two last figures of the numbers.

The smallest factor being 1, p can be at most 235 and on account of the combining of the additional numbers into groups of 2, 1, 2 and 5 the number of operations is about 92.

The advantage of this method above the direct method of § I is, that the numbers are smaller and the number of additions is considerably less.

The number 513667 dealt with in § I gives after 3 additions the value of n.

If in general $G = a_0^2 + b_0$, then after n operations we have $G = (a_0 + n)(a_0 - n) + b_0 + n^2$, so that $r = b + n^2$. The factor $a + n_0$

put equal to $p(2r+1)$ gives $n = \frac{p}{4} \pm \frac{1}{4}\sqrt{p^2 + 8a_0 p - 8b_0}$, so that p must be greater than $\frac{b_0}{a_0}$.

The other factor gives

$$n = -\frac{p}{4} \pm \frac{1}{4}\sqrt{p^2 + 8a_0 p - 8b_0},$$

so that the form under the root sign proves to be in both cases the same.

VIII. *Use of the series* $1+3+5+$ *etc.*

A square a^2 is equal to $1+3+5+\ldots(2a-3)+(2a-1)$.

To denote a non-square, e.g. $953\ (=30^2+53=31^2-8)$ in this manner, let us write that series in the form

$$1+3+5+\ldots 57+59+(53+8)+63+65\ldots$$

According to the method of § I we write:

$$953 = 31^2 - 8 = (31^2+63) - (8+63) = 32^2 - 71$$
$$= (32^2+65) - (71+65), \text{ etc.},$$

until the subtrahend has become a square.

The series gives a clear representation of that operation.

But if $8+63+65+\ldots$ is a square, then it is possible to build it up out of $1+3+5+$ etc. and this gives the following operation:

subtracted	remainder
	8
1	7
3	4 + 63
5	62
7	55
9	46
11	35
13	22
15	7 + 65
17 to 21 or 3 × 19	15 + 67
3 × 25	7 + 69
2 × 30	16 + 71

etc.

The method is laborious, whilst there is a great chance for errors. However, it gives a clear insight into the composition of the numbers and the writer is or opinion that probably from this method later investigations will start.

IX. *Another use of the series* $1 + 3 + 5 +$ *etc.*

Each divisible number consists of the sum of a few successive terms of the series.

Example $57 = 3 \times 19 = 17 + 19 + 21$, where 17 is the base of 57.

So we can write:

$$57 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 8 = 3 + 5 + 7 + 9 + 11 + 13 + (8 + 1) =$$

$$5 + 7 + 9 + 11 + 13 + (8 + 1 + 3) = 7 + 9 + 11 + 13 + (8 + 1 + 3 + 5) =$$

$$7 + 9 + 11 + 13 + 15 + 2 = 9 + 11 + 13 + 15 + (2 + 7) = 11 + 13 + 15 + (9 + 9) =$$

$$11 + 13 + 15 + 17 + 1 = 13 + 15 + 17 + (1 + 11) = 15 + 17 + (1 + 11 + 13) =$$

$$15 + 17 + 19 + 6 = 17 + 19 + 21$$

Or abbreviated:

added	remainder
	8
1	9
3	12
5	$17 - 15 = 2$
7	9
9	$18 - 17 = 1$
11	12
13	$25 - 19 = 6$
15	$21 - 21 = 0$

For large numbers with factors greatly differing, this method is also very laborious.

The possibility, however, is not excluded that later on from the succession of some remainders we shall be able to decide upon the divisibility or non-divisibility of a given number.

The decomposition $G = a \times b + c$ discussed in § III finds an explanation in the series too.

Example: $G = 57 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 8.$

Seven terms precede 8, of which 7 is the middle one, so $G = 7 \times 7 + 8$.

If we add 1 to 8, the 6 remaining terms have a mean 8, so that $G = 6 \times 8 + (8 + 1)$.

So we have successively:

$G = 7^2 + 8$ namely $1+3+5+7+9+11+13+8$

$6 \times 8 + 9$,, $3+5+7+9+11+13+(8+1)$

$5 \times 9 + 12$,, $5+7+9+11+13+(8+1+3)$

$4 \times 10 + 17$,, $7+9+11+13+(8+1+3+5)$

$3 \times 11 + 24$,, $9+11+13+(8+1+3+5+7)$

X. *Abbreviation of the method of § I in special cases.*

In the series $1 + 3 + 5$ etc. each term is a 4-fold $+ 1$ or a 4-fold $- 1$, so that the sum of two successive terms is always a 4-fold. If we leave out the first term, then the sum of 2 successive terms $3 + 5$, $7 + 9$, etc. is always two times their mean, i. e. an 8-fold. This leads to an abbreviation.

Example: $G = 953 = 31^2 - 8$.

In § I is given that to 8 the following had to be added:

$$\underbrace{63 + 65}_{\text{2 terms}} + \underbrace{67 + 69 + 71 + 73}_{\text{4 terms}} + \underbrace{75 + 77 + 79 + 81 + 83 + 85}_{\text{6 terms}} + \underbrace{\ldots\ldots\ldots}_{\text{4 terms}} + \text{etc.};$$

or $16 \times 8 + 35 \times 8 + 60 \times 8$, etc.

The groups of 4 terms give: 35×8, 45×8, 55×8, etc.

The groups of 6 terms give: 60×8, 75×8, 90×8, etc.

The entire sum must be a square and so necessarily divisible by 16 and as the first addition would give $8 + 16 \times 8$ or 17×8, the numbers 17, 35, 60, 45, 75, 55, 90, etc. must be taken in such wise together, that they form even numbers.

It is evident that to 8 we can first add $2 + 4$ terms, then 6 terms, then 10, 14, 10, 6, 10, 14, 10, 6, 10, 14 etc. terms at once. This gives rise to a considerable decrease in the number of additions; however, the method can be applied only when the (negative) remainder of the extraction of the root is an 8-fold.

XI. *The added factor.*

If a number can be written down in more than one manner as a product of two factors, the method of § I always gives the factors closest to the root, thus those differing the least.

For $273 = 3 \times 7 \times 13$ we find 13×21, for $1155 = 3 \times 5 \times 7 \times 11$ likewise 33×35.

This observation immediately furnishes us with an abbreviation as to the decomposition of one of the factors of a number, when the other factor has been decomposed into its factors.

Example: In § I is found $G = 513667 = 539 \times 953$, and $539 = 7 \times 7 \times 11$.

Now, 953 cannot possibly contain a factor, which multiplied by one or more of the factors 7, 7 or 11 of 539, furnishes a number closer to $\sqrt{G}$, namely 716, than 539. For, if that were the case then by the operation the *other* number would have been found, and not 539.

As

$$\frac{716}{7}=102,\ldots,\quad \frac{716}{11}=65,\ldots,\quad \frac{716}{7\times 11}=9,\ldots,\quad \frac{716}{7\times 7}=14,\ldots$$

it is immediately evident, that 953 cannot contain a factor smaller than 14, or between 49 and 65, or between 77 and 102.

The first limit 14 shows that 13 cannot be a factor.

From the last limits would ensue, that there are no factors between $\frac{953}{65}=14,\ldots$, and $\frac{953}{102}=9,\ldots$

The possibility that 953 is divisible by 7 or 11 (which would bring about no change in the result 539×953) is excluded with respect to 11, because 11 lies between 9 and 14; so that if by direct division 7 proves to be a non-divisor of 953, the factors of 7, 11 and 13 are excluded, and so 17 would have to come under consideration as lowest factor.

So the last operation in § I (pag. 331) has been continued already too far. For after the second addition we get:

$$953=37^2-416=37^2-(20,\ldots)^2,$$

so that the difference 17 is already exceeded.

The preceding can be applied only to one of the factors if the other one has been factorized. The following extension can however be given:

Example: $G=8695261$ is obtained of 9803×887, thus equal to 5345^2-4458^2.

As $\sqrt{G}$ is equal to 2948, ... the operation according to the method of § I would be rather long.

Suppose we know that one factor is more than 10 times but less than 14 times the other factor, then by multiplying G by 11, 12 or 13 we can obtain a number that can be decomposed into two factors differing but slightly.

In this way $11\times G$ would give: $G_1=95647871=9803\times 9757=9780^2-46^2$, and the extraction of the root would immediately furnish a result.

Furthermore $13\,G=9803\times 11531=10667^2-1728^2$ would require a somewhat longer operation.

The factor 11 or 13 can be called the *added factor*. A danger is attached to this method, in the case of multiplication by an *even* factor.

Example: $G = 57 = 3 \times 19$. The factor 6 would give:

$$G_1 = 342 = 18 \times 19 = \left(\frac{37}{2}\right)^2 - \left(\frac{1}{2}\right)^2.$$

To avoid fractions we can in such a case *moreover* multiply by 4 and so write

$$G_2 = (2 \times 18) \times (2 \times 19) = 37^2 - 1^2.$$

Example: $G = 100895598169$ (see introduction) if we know, that the factors contain an equal number of figures.

The number 1008 formed by the 4 first figures of G can have originated from 11×99, 12×84, 13×77 . . . . . 13×31.

The greatest factor is thus at most 9 times the other. So we might try whether $3\,G$, $5\,G$, $7\,G$ or $9\,G$ after extraction of the root would immediately give the difference of two squares. If this is not the case, we might do the same for

$$4 \times 2\,G, \quad 4 \times 4\,G, \quad 4 \times 6\,G \quad \text{and} \quad 4 \times 8\,G.$$

The number $4 \times 8\,G$ immediately gives a result [1]).

XII. *Application of the results of the theory of numbers.*

A few properties out of the theory of numbers will be discussed here briefly:

1. *Every odd number G whose factors are 4-folds* **+ 1** *can be decomposed into the sum of two squares in 2^{n-1} ways, if n represents the number of different prime factors.*

For this the squares are supposed to be respectively indivisible.

Example: $325 = 5^2 \times 13 = 1^2 + 18^2 = 6^2 + 17^2$ (but the decomposition $10^2 + 15^2$ is to be left out of consideration).

The above mentioned decomposition which is rather prolix, if accomplished by the operations of the theory of numbers, becomes pretty simple when the reverse of the method of § I is applied to it.

Example: $G = 953 = 30^2 + 53$ or

$$953 = 1 + 3 + 5 + \ldots . + 57 + 59 + 53.$$

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[1]) See the note in the introduction.

One square must be

$$1 + 3 + 5 + \ldots\ldots + x,$$

the other

$$53 + 59 + 57 + \ldots\ldots (x + 2).$$

So we have:

$$\begin{array}{rr}
G = 30^2 + & 53 \\
2 \times 30 - 1 = & 59 \\ \hline
29^2 + & 112 \\
 & 57 \\ \hline
28^2 + & 169 \\
 & 55 \\ \hline
 & 224 \\
 & 53 \\
 & 51 \\
 & 49 \\
 & 47 \\ \hline
 & 424 \\
 & 45 \\ \hline
 & 469 \\
 & 43 \\ \hline
 & 512.
\end{array}$$

So $G = 28^2 + 13^2$.

It is unnecessary to continue farther when half of G is attained.

A square is always a 4-fold or a 4-fold + 1, so the sum of two squares is a 4-fold or a 4-fold + 1 or a 4-fold + 2.

So a 4-fold — 1 can never be decomposed into the sum of two squares.

As a special case of the property sub. 1 we have:

2. *Every indivisible 4-fold + 1 can be decomposed in one way into the sum of two squares.*

If a 4-fold + 1 contains factors which are 4-fold — 1, the decomposition is not always possible.

Example: $G = 957 = 3 \times 11 \times 29$. The factors 3 and 11 are an objection to the decomposition.

From this follows immediately:

3. *If a 4-fold + 1 cannot be decomposed into the sum of two squares, it is divisible.*

The number of factors, 4-folds — 1, must necessarily be even, as only an even number of 4-folds — 1 can produce a product 4-fold + 1.

4. *If a 4-fold + 1 can be decomposed only in one way it is not necessarily indivisible, but it may contain factors which are 4-folds —1; the number of those factors amounts to two less than the number of factors which are 4-folds + 1.*

Of the above-mentioned properties we can make use in the following way:

Example: $G = 898423$, 4-fold — 1, so it is not to be decomposed.

In the first place let us determine the greatest possible number of factors in the following manner:

The product $3 \times 7 \times 11 \times 13 \times 17 \times 19 = 969969$, thus more than G, so that the number of different factors of G cannot amount to more than 6.

If we leave out 3 and 7, it is evident that $11 \times 13 \times 17 \times 19 \times 23$ is also $> G$. Likewise $13 \times 17 \times 19 \times 23 \times \frac{29}{2}$ is $> G$, so that G can contain at most 4 different factors. These might be: one 4-fold — 1 and three 4-folds + 1, or three 4-folds — 1 and one 4-fold + 1.

By leaving out 13, 17, 19 and 23 we find $29 \times 31 \times 37 \times \frac{41}{1.5} > G$, so that G can have at most three different factors, namely one 4-fold — 1 and two 4-folds + 1, or three 4-folds — 1.

We have supposed here that the numbers left out are not divisors of G.

We now multiply G by 3 to obtain a 4-fold + 1, and by two prime numbers more, 4-folds + 1, non-divisors of G, for instance 5 and 13, and we test whether $13 \times 5 \times 3G$ can be decomposed. If this should prove possible G would really possess three factors: one 4-fold — 1 and two 4-folds + 1, which number has been increased by the added factors to two 4-folds — 1 and four 4-folds + 1.

If $13 \times 5 \times 3G$ cannot be decomposed, there can still be three factors 4-folds — 1, and we should have to try $17 \times 13 \times 5 \times 3G$.

If this can be decomposed, G is indivisible, for only in that case the number of factors 4-folds + 1 (namely 17, 13 and 5) is two more than the factors 4-folds — 1 (namely 3 and G).

The last number to be calculated being rather great, we can follow another method by remarking that $\sqrt[3]{G}$ is nearly 96; so G is smaller than $97 \times 101 \times$ a number smaller than 103; so that if we have found by direct division or by the method of § III, that the prime numbers under 97 are non-divisors of G, we can immediately draw the conclusion that G possesses at most two factors, a 4-fold $+ 1$ and a 4-fold $- 1$.

Then it is only necessary to apply the operation to $13 \times 5 \times 3G$.

Such like methods as the one under consideration could be applied to the other data of the theory of numbers: the decomposition of a number into the sum of a square and a twofold, threefold or fivefold of a square.

(February 19, 1902).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday February 22, 1902.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 22 Februari 1902, Dl. X).

The following papers were read:

Chemistry. — *"On pentanitrophenylmethylnitramine and tetra- and pentanitrophenol."* By Dr. J. J. BLANKSMA. (Communicated by Prof. C. A. LOBRY DE BRUYN).

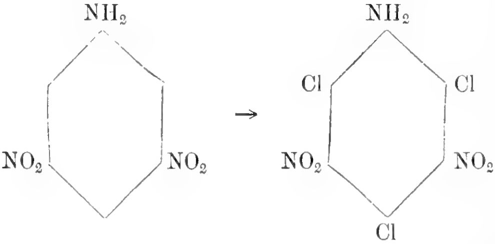
(Communicated in the meeting of December 28, 1901).

It has been known for a long time that aniline and phenol are very readily converted into trihalogen substitutionproducts by treatment with chlorine- and bromine-water and also that trinitrophenol (picric acid) may be readily obtained from phenol by the action of nitric acid. The three halogen atoms and the three nitro-groups then always jointly occupy the three free meta-places and, therefore, take up in regard to NH_2 and OH the ortho- and para-positions, but never the two remaining meta-places. It is a remarkable fact that these substitutions take place so much more readily than they do in the case of other benzene derivatives.

Some time ago LANGER[1]) has considerably supplemented some existing observations by a systematic investigation of aniline-derivatives and demonstrated that even if one or both meta-places are occupied in regard to NH_2, the substitution of the three remaining places by halogen proceeds quite as readily as in the case of free aniline.

No similar systematic investigation has been made in the case of phenol; on consulting the literature we find that a number of observations made at the halogenation and nitration of some substituted phenols point to the existence of a similar regularity.

In order to obtain additional confirmatory evidence in the case of the derivatives of aniline and phenol Dr. BLANKSMA has in the first place investigated the behaviour of symmetric dinitro-aniline towards halogens. Trichloro- and tribromodinitro-analine were very readily formed :



It further appeared that symmetric dinitrophenol was readily converted by bromine water into the already known but differently obtained tribromodinitrophenol; in the case of *m*-nitrophenol the conversion into a tribromoderivative had already been proved.

Dr. BLANKSMA secondly put the question how nitric acid will behave towards nitrated meta-derivatives of aniline and phenol. It is known that by direct nitration no more than three nitro-groups can be introduced into benzene and its derivatives; even picric acid

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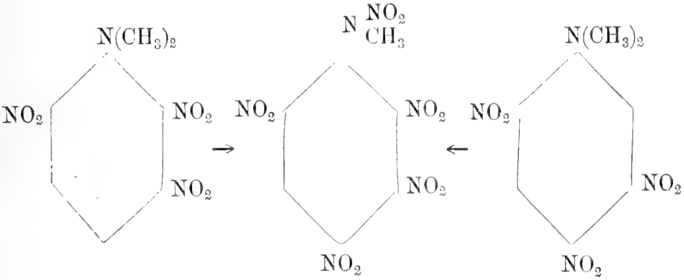
[1]) B. 15. 1061. 1328.

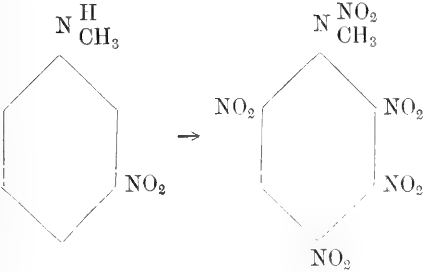
is not capable of taking up a fourth nitro-group. The only exception as yet known is tetranitrophenylmethylnitramine discovered by VAN ROMBURGH in 1889 and obtained by the action of fuming nitric acid on two different trinitrophenyldimethylanilines. [1])



VAN ROMBURGH has proved that the nitro-group placed between two other nitro-groups is very mobile and readily liable to substitution (by OH, OCH_3, etc.)

Of late years, NIETZKI has prepared a tetranitrophenol and a tetranitrobenzene, partly in an indirect way, by oxidation of oximes; the first one by oxidation of a trioxime (in this case a NO_2-group also enters), the second one by oxidation of a dinitrodinitrosobenzene. [2])

Dr. BLANKSMA now subjected *m.* nitro- and *m.m.* dinitromethylaniline to the action of concentrated nitric acid and proved that the first one yielded VAN ROMBURGH's tetranitrophenylmethylnitramine but that the second one gave the corresponding pentanitroderivative:



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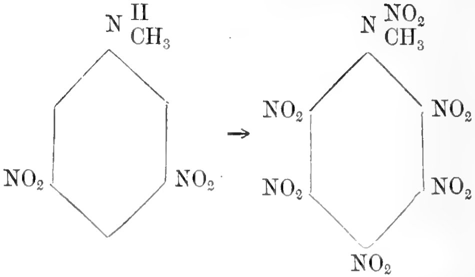
[1]) Recueil 8. 973. It will be seen that the formation of the tetranitroproduct is accompanied by a substitution of one of the CH_3-groups by NO_2.

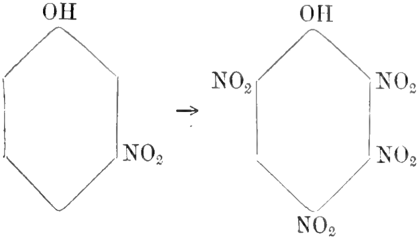
[2]) B. 30. 181, 34. 55.

and



The last substance is a yellow, properly crystallised substance (m. p. 132°) which explodes when heated on platinum foil. Its properties will be further investigated.

Dr. Blanksma next investigated the behaviour of *m.* nitro- and *m.m.* dinitrophenol towards concentrated nitric acid. The result obtained thoroughly agreed with the facts just mentioned as there had been an abundant formation of tetra- and pentanitrophenol, melting respectively at 140° and 190°.



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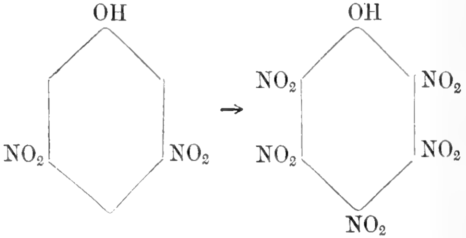

When preparing these two substances Dr. Blanksma confirmed his previous idea that a precaution should be taken, which other chemists (Bantlin, Henriques), having occupied themselves with the study of the behaviour of nitrated phenols towards nitric acid, have disregarded. The nitration must be done with nitric and sulphuric acids and not by boiling with nitric acid alone; the crystalline mass must than be drained and recrystallised from chloroform. If it is boiled with water, one, respectively two, of the nitrogroups are replaced by hydroxyl and instead of tetra-, respectively pentanitrophenol, trinitroresorcinol and trinitrophloroglucinol are obtained. The appearance of trinitroresorcinol so often noticed during the nitration of phenolic bodies has now been explained by BLANKSMA and the previous opinion expressed by BANTLIN and by VAN ROMBURGH that the formation of trinitroresorcinol from ordinary phenol derivatives is due to the intermediate formation of tetranitrophenol has now been confirmed by experiment.

NIETZKI's tetranitrophenol is evidently not identical with the one described above.

Dr. BLANKSMA has also proved that one nitro-group in tetranitrophenol and two nitro-groups in pentanitrophenol may be readily replaced by OH, OCH_3, $OC_2 H_5$, NH_2 and $NHC_6 H_5$.

In getting the above mentioned results (the particulars of which will be published in the „Recueil") Dr. BLANKSMA has followed up the same idea which in the case of aniline derivatives had already been successfully tried with an extensive material and which after his experiments may now be considered to apply also to phenol. Briefly summarising, the matter amounts to this (when other unmentioned facts are also taken into consideration): that phenol and aniline (methyl aniline etc.) are very readily halogenated, nitrated and sulphonated, much more readily than benzene and the homologues of other benzene derivatives and further that the presence of one atom or group occupying the meta-position in regard to OH or NH_2 ($NH CH_3$) does not offer a sterical obstacle, whilst the entering atoms or groups exclusively select the para-place and the two ortho-places.

If we want to explain these phenomena we must begin by assuming (which has already been proved in several other cases) that in the first stage of the action intermediate products are formed which contain the group to be introduced attached to the nitrogen or the oxygen and that these products by intramolecular change are more or less readily converted into the direct benzene derivatives.

If the various observations which have already been made in this matter are collected and arranged, 15 to 20 different cases may be distinguished, many of which may be illustrated by a great number of special examples and which all amount to this: that an atom or a group linked to N or O shows an inclination to pass from this atom to the nucleus and then always (or nearly so) changes place with the H atoms occupying para- and ortho-places. Whether the meta-places are occupied or not does not affect this isomerisation process.

If we now limit ourselves to nitric acid it may be observed that the idea that in the nitration of aniline derivatives (best in the case of those containing NHX instead of NH_2) the nitro-group should always combine first with the nitrogen forming nitramines, before passing on to the nucleus, has already been suggested by BAMBERGER and confirmed by experiment. After Dr. BLANKSMA's experiments we arrive at a similar conclusion as regards the formation of nitrated phenols. This is confirmed by another observation of Dr. BLANKSMA that if the phenol-hydrogen of symmetric dinitrophenol is replaced by methyl, the ready bromination or nitration is no longer possible. Efforts will be made to prepare the as yet unknown nitrates of the phenols and, if successful, their behaviour will be closely studied.

Physiology. — "*On the development of the entoderm, of* KUPFFER's *vesicle, of the mesoderm of the head and of the infundibulum in Muraenoïds*" (preliminary paper), by Dr. J. BOEKE. (Communicated by Prof. T. PLACE).

(Communicated in the meeting of January 25, 1902).

In his well-known paper on pelagic eggs of the Gulf of Naples, RAFFAELE described five species of big pelagic eggs, which were found in the plankton of the Gulf during the months of August and September, which he suggested that might belong to different members of the Muraenoïd group.

In 1893 and 1896 GRASSI and CALANDRUCCIO confirmed this theory, but they did not study the eggs closer.

During the summer of 1900 and 1901 I had the good fortune to secure several hundreds of these eggs during a stay at the Stazione Zoologica at Naples, and was enabled to study all the stages of development of the embryos until the critical period. The description and suggestion of RAFFAELE I found to be perfectly true, and moreover I collected three other species of eggs, which although undoubtedly belonging to Muraenoïd species, could be distinguished sharply from the other five spp., described by RAFFAELE. From these they differed in the dimensions of the yolksphere and the perivitelline space, in the number, and the distribution of the oil-drops in the yolk, in the character of the egg-capsule and in different characters of the more developed embryos. These eggs however were very rare and could only be secured in small numbers.

The processes of development which I am about to describe in this paper, were studied therefore exclusively on the eggs of the spp. RAFFAELE already described. My efforts to keep the fry alive more than a few days after the yolk had been absorbed, being unsuccessful, I could not identify distinct species of larvae with distinct spp. of Muraenoïds.

I therefore will follow the example of RAFFAELE, and write Muraenoïd N°. 1, Mur. N°. 2 etc. Mur. N°. 1 is his spec. 6, Mur. N°. 2 his spec. 7 etc.

In describing the development of the entoderm and KUPFFER's vesicle, I will begin by calling attention to the paper by F. B. SUMNER on KUPFFER's vesicle and its relation to gastrulation and concrescense, published last year.

SUMNER maintains that KUPFFER's vesicle in Teleosts and Ganoïds (Amia) is formed by a solid or hollow invagination of the superficial layer („Deckschicht"). Before the closure of the blastopore the superficial layer by proliferation of its cells forms a thickening at the hind part of the embryo, that SUMNER called the prostomal thickening in connection with KUPFFER's theory of gastrulation in Teleosts. In Muraenoïd eggs, of which he could study some stages, he found a material in which the process was to be followed „with almost diagrammatic distinctness". But being short of Muraenoïd material he could not determine accurately the relations between the prostomal thickening and the entoderm. From one of his drawings (page 60 cross section of a young Noturus embryo) seems to follow, that he thinks it probable, that chorda and mesoderm are derived from the entoderm and that the entoderm is formed at least partially from the proliferation of the superficial layer.

Having an abundance of Muraenoïd eggs to study, I was able to follow the entire process and obtained the following results: as soon as — the cleavage-process being ended — the cells at the blastodermmargin begin to invaginate inward, the cells of the superficial layer, which everywhere else are flat and do not partake of the invagination process, begin to increase in size at one point. This increase of size of the cells of the „Deckschicht" is limited to a small area at the hind end of the thickening of the blastoderm, which is the first indication of the embryonic shield. There, at the edge of the blastodermring, the cells of the superficial layer thicken, become rather cilindrical, and divide parallel to the surface of the egg, so that one of the daughtercells travels inward.

At a somewhat later stage of development, when the blastoderm has spread further over the surface of the yolk, and the embryonic shield is to be seen clearly, a median section gives the following view: the invagination of the blastoderm is sharply defined. At the hind end of the embryonic shield the cells of the superficial layer, which everywhere else are flat and separated by a sharp line from the periblast, are cilindrical with the long axe at right angles to the surface of the egg, and send a tongue of cells inward between the invaginated layer of the blastoderm and the periblast. This is to be concluded from the direction of cell division in the projecting layer of cells. This tongue of cells consists of loosely packed cells, reaches inward almost as far as the invaginated layer of the biastoderm and is distinctly separated from it.

The thickened part of the superficial layer SUMNER called „prostomal thickening". Beneath it are to be seen many nuclei in the periblast and often one gets the impression as if cells from the periblast partake of the process and travel inward with the other cells of the prostomal thickening. However I could not state it with a sufficient amount of certainty. On cross sections (parallel to the blastodermring) the inward proliferation of the superficial layer may also be sharply separated from the other blastodermcells. The cells of the invaginated layer of the blastoderm form in the median line the chorda. The sideparts become the mesodermic plates. The proliferation of the superficial layer (with cells from the periblast ?) form the gut-entoderm (some cells are separated in the course of development, and seem to form primary blood-corpuscles. At least, they do not take a part in the forming of any particular layer).

At the closure of the blastopore the superficial layer forms an invagination and in this manner KUPFFER's vesicle is formed, just as SUMNER described it in his paper. [1])

In different Muraenoïd spp. however the forming of KUPFFER's vesicle does not take place in the same way. In some spp. (viz. Mur. N⁰. 2) the invagination of the „Deckschicht" (or the overgrowing of the prostomal thickening by the tail-knob) begins some time before the closure of the blastopore, in other spp. (viz. Mur. N⁰. 1, N⁰. 3) it begins much later, as the blastopore is nearly closed and

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[1]) RAFFAELE too saw a transitory communication of KUPFFER's vesicle with the exterior in the Muraenoïds.

the periblast almost entirely overgrown. The first mode of development gives rise to a KUPFFER's vesicle, bounded on the dorsal side with epithelium, on the ventral side with periblast, the second to a KUPFFER's vesicle almost entirely bounded with epithelium.

The wall of KUPFFER's vesicle is continuous with the wall of the gut. The dorsal row of cells forms the hypochorda.

It would lead us too far to describe the later development of the gut, the concentration of cells towards the median line, the folding and forming of the gut-tube. I will here restrain myself to show, how in these Teleosts the secondary entoderm — the gut-entoderm — is formed independently of the chorda and the mesodermic plates, and how the chorda is differentiated out of the median part of the invaginated layer of the blastoderm. For that seems to me to be of great value to understand rightly the processes of development that take place in the head-part of the embryo.

In following the development of the head on median sagittal sections, as are lying before me in sufficient quantities and of different stages of development, the observer sees, that in the stage in which the central nervous system forms only a solid keel and there is as yet nothing to be seen of the optic vesicles, the chorda, until the point where in later stages it ends, is composed of flat disk-like cells, but then grows thinner and can be traced as two rows of cells dorsally of the entoderm to the foremost part of the head, where they form a big mass of cells lying in front of the brain and beneath the point where the anterior neuroporus is formed. The cells of the entoderm can everywhere be distinguished sharply from the prolongation of the chorda, except in the foremost part, where in median sections it was difficult to distinguish them from each other. As soon as the optic vesicles are formed, the infundibulum develops and pushes the mesodermcells away. On median sections the prolongation of the chorda is now to be followed until it reaches the infundibulum. In front of the infundibulum the anterior mesodermmass is then to be seen. The ectoderm grows inward beneath this mass of cells and unites with the entoderm. But the cells of the entoderm not containing any particles of yolk here, and being only distinguishable by their position from those of the other layers, I cannot yet fix accurately the point where the entoderm and the ectoderm fuse. The ectoderm seems always to be sharply separated from the anterior mesodermmass, that in later stages forms the sceletogenous tissue of the fore-head.

In following the further development of the mesoderm of the head in median sections, the chorda is seen to be rounded off and to be separated from the anterior row of cells; by the growth in length of the region of the fourth ventricle the cleft between the top of the chorda and the anterior row of cells widens. This anterior row of cells is then to be seen to extend to the infundibulum, and is at both sides connected with the head-mesoderm.

At first solid, it becomes hollow, is shortened and lays itself closely at the back of the infundibulum; finally it disappears altogether. It seems to be the homologon of the connection-piece of the head-mesoderm described for selachians by BALFOUR and MARSHALL, closely studied by VAN WIJHE, the „Sclerotomcommissur'' of KILLIAN.

In now examining cross-sections through the head, we see that the mesoderm on both sides of the chorda does not break up (as is the case with the other Teleosts) into mesenchym, without being segmented, but that the somites of the trunk are continued without break as far as the auditory and farther on as far as the optic vesicles. These somites are, it is true, smaller and not as regular as the somites of the trunk, but everywhere myotomes and side-plates are to be distinguished very sharply. There is no trace of mesenchym until a late period.

There where the chorda is rounded off as a distinct rod, the myotomes on both sides are separated from the chorda and from each other, more in front behind (and in young stages beneath) the infundibulum they are united by the connection-piece mentioned above. The somites are to be traced up to the foremost part of the head, where the optic vesicles are formed. They do not seem to be connected with the anterior mesoderm mass.

In later stages of development the posterior somites of the head become hollow. The lumina fuse with each other, and the somites become small epithelial vesicles, which enlarge and form the head cavities; by the connection-piece they are connected with each other. The anterior walls of the head cavities and the solid somites lying in front of the head cavities form the eye muscles, but at present I cannot tell from which somites the different eye-muscles are formed. For the same reason I cannot fix the exact number of the head-somites, for, although the segmentation of the head-mesoderm into somites is to be seen in longitudinal sections as clearly as the peculiar form of the somites (as real myotomes) in cross-sections, I could not until now follow the different somites in their entire development. To make the necessary plastic reconstructions of the section-series time failed me and I am not sure if it will be possible at all, for especially the foremost somites break up after a time into mesenchym and are to such a degree compressed by the hind wall of the optic vesicles, that it is difficult to recognise them as different somites. Before long I hope in a full paper to work out this theme with drawings and plastic reconstructions. In this preliminary communication I will confine myself to describing the forming of distinct somites in the head of Teleosts, independent of the entoderm and entirely analogue with the myotomes of the trunk, and to showing the signification of the connection-piece as a simple prolongation of the chorda.

Before ending I beg to be allowed to give some more data on the structure and the development of the infundibulum in the Muraenoïds. The description of the structure of the infundibular organ I gave in the „Anatomischer Anzeiger", after further examination appeared to be perfectly true. The big cells I described in the infundibulum of Muraenoïd embryos, with a protoplasmic conical protuberance and a crown of small vesicles on it, I could distinguish with great clearness in the living larvae. The crown of small vesicles showed itself with the same regularity as in the stained sections, they were standing on a distinct protoplasmic conus and the whole complex was sharply bounded off from the other parts of the brain. What I had concluded already from peculiar alterations of the form of the cells, viz. that the protoplasmic conus was able to alter its form, I could confirm by studying the living embryos. Several times I saw one of the big sensory cells prolong its protoplasmic conus and draw it back again. In no case I saw one of the small vesicles fall off or lying free in the lumen of the infundibulum.

In my first communication I had overlooked the description of the infundibular gland by STUDNICKA in his paper on the ependym, published last year. He described the same crown of small vesicles on the cells of the infundibular gland in different fishes (Selachians, Teleosts, Ganoids), and takes them for a product of the secretion of the gland cells. From my description and from the fact, that they develop out of cilia, which I could state by studying the intermediate stages, it seems to me to follow, that this conclusion cannot be the right one, and that we have to see in the saccus vasculosus not a gland, but a sense-organ. For adult anguillae I could confirm the statements of STUDNICKA. But here too, the small vesicles are sitting on distinct „Basalkörperchen", and continue in the cell as a bundle of thin but distinct fibres. In all the embryos of Teleosts I studied in this direction (Hippocampus, Syngnathus acus, Clupea spp., Uranoscopus, Mullus barbatus, Lepidopus caudatus, Scorpaena scrofa, Fierasfer acus) the same structure of the saccus vasculosus was to be seen.

As to the function of the infundibular organ, it seems to me that the abundance of blood capillaries around the organ seems to point to a connection with the regulation of the blood pressure in the cerebrum or the pressure in the ventricles, as von Cyon maintained to be the case for the entire hypophysis. Some experiments I made on the live embryos, seemed to confirm this conclusion.

Kolozsvár, January 1902.

Physics. — "*Ternary systems.*" (1^{st} part). By Prof. J. D. van der Waals.

THE PRINCIPLE OF CONTINUITY FOR A TERNARY SYSTEM.

The equilibrium phenomena of a binary mixture at a given temperature may be illustrated geometrically by a surface, where volume and composition serve as abscissas and the free energy as ordinate. In the second part of my Continuity I have discussed the shape of such a surface and I have demonstrated what conclusions may be drawn from the properties of such a surface $\psi = f(x, v)$.

If we have a ternary system, for which two quantities x and y are required for the determination of the composition, then

$$\psi = f(x, y, v)$$

and so such a geometrical representation cannot be used. Though the geometrical representation is not necessary for the deduction of the conditions of equilibrium, and though it is not even possible to use it with increasing number of the components, yet for a binary mixture the great advantages of the graphical treatment have been sufficiently proved to continue employing it as long as we can.

For a ternary system we find the means of effecting this in the properties of the ζ-function, according to the rule of equilibrium given by Gibbs, that at given T and p the substance arranges itself in such a way that the value of ζ is as small as possible. At given T and p the value of ζ is only dependent on x and y, and so the geometrical representation can again be used.

If we think a ternary mixture as composed of $1 - x - y$ molecules of the first substance, of x molecules of the second substance and y molecules of the third substance in homogeneous phase, we obtain the value of ζ, as I have given it. (Verslag van 25 Sept.

1897. pag. 212 and Arch. Néerl. Série II Tome II pag. 71), viz.

$$\zeta = M R T \left\{ (1 - x - y) \log (1 - x - y) + x \log x + y \log y \right\} +$$

$$+ \left\{ pv - M R T \log (v - b_{xy}) - \frac{a_{xy}}{v} \right\} + Ax + By + C.$$

In this equation we must think v as being eliminated by means of the equation of state, and therefore determined by p, T, x and y. I have previously pointed out l. c. pag. 69, that such a ζ-surface can generally consist of three sheets. These three sheets might be distinguished by calling them: liquid sheet, vapour sheet and sheet for the unstable state.

Strictly it is not necessary from an experimental point of view to know all these sheets and the way in which they cohere, for only the conditions represented by the lowest sheet are stable. The others are unstable or metastable, and can therefore not be realized or only as phenomena of retardation. But it has been proved even for a simple substance that for the laws of coexistence the knowledge of unstable conditions is required. Think e.g. of the criterium of MAXWELL for the determination of the pressure of coëxistence. For a binary mixture the knowledge of the plait on the ψ-surface is necessary for the deduction of the critical phenomena. In all these cases the connection of what may be realized on one side of a certain limit and that which may be realized on the other side of another limit can only be fully grasped when also the conditions that cannot be realized are known — so when we assume continuity.

So we can only make the full use of the function ζ for a binary mixture, when we know the connection of the three before mentioned sheets and the shape of the ζ-surface also for the metastable and unstable phases. For the ψ-surface of a binary mixture the unstable and metastable part appeared to form a plait in the surface, which was for the rest convex-convex (seen from below). A plane section through this plait brought a convex part of a curve in connection with a convex part lying on the other side of the plait through a curve whose course was continuous, and which did not show any complication except two points of inflection. That this will not generally be the case for the ζ-sheets, and that much greater complications may be expected there, might already be anticipated by the shape of ζ of a simple substance, as occurs in fig. (1) p. 4 of

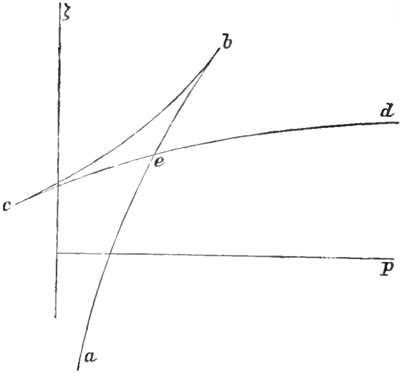

Fig. 1.

the second part of the Continuity and which is here reproduced. Between two values of p, viz. the maximum pressure and the minimum pressure of the isothermal, ζ is trivalent and the transition of the vapour sheet to the liquid sheet takes place through a curve which has two cusps. For the rest the fact that the ζ-surface thought as function of x and y consists of three sheets in some cases and of one sheet in other cases and that ζ in the neighbourhood of the critical circumstances for the same value of p and T, will have three values above one part of the values of x and y, and will have one value above the remaining values of x and y, shows that the metastable and unstable part of the ζ-surface will show a more intricate configuration than what we call a plait.

Let us begin with examining for a simple substance the value of ζ for a molecular quantity as function of p, assuming the temperature to be constant. From the differential equation $d\zeta = vdp$ we could derive ζ, if we could give v as function of p. For the rarefied gaseous state we may put $v = \frac{MRT}{p}$, from which follows:

$$\zeta = f(T) + MRT \log p.$$

If we write

$$\int vdp = pv - \int p\, dv ,$$

we get

$$\zeta = MRT - MRT \log v ,$$

or

$$\zeta = MRT + MRT \log \frac{p}{MRT}$$

By combining these equations we get $f(T) = MRT - MRT \log MRT$.

The vapour branch of the ζ-curve appears therefore to rise with increasing value of p starting from $-\infty$ for $p = 0$. In fig. (1) this is represented by the lowest part of the branch $a e b$. As for coexisting phases of a simple substance the value of ζ per unity of weight (the thermodynamic potential) must have the same value, the vapour branch and the liquid branch have a point in common. In fig. (1) this point is e. The further course of the liquid branch may be easily derived from $d\zeta = v\,dp$. For the experiment we only want to know the branches $a e$ and $e d$. But the place of e is only found as double point of the intersecting branches of the complete curve.

By applying the principle of continuity we find:

1st. the continuation of the vapour branch, rising till the pressure has become equal to the maximum pressure of the isothermal;

2nd. the unstable branch, for which the pressure decreases to the value of the minimum pressure of the isothermal;

3rd. the beginning of the liquid branch (the portion $c e$).

The direction of the curve at any point is determined by the value of the volume. That the point b is a cusp follows from the fact that immediately before and immediately after that point the direction of the tangent is given by the same value of v. But $\frac{d^2\zeta}{dp^2} = \frac{dv}{dp}$ has opposite sign immediately before and immediately after the point b. On the unstable branch this quantity is positive, on the other parts of the curve negative. What applies to b, is also of force for c.

When the pressure lies between the two limits mentioned, which I shall indicate by p_M and p_m (maximum pressure and minimum pressure of the isothermal), the value of ζ is trivalent. If p is smaller than the pressure of the double point, the vapour branch is the lowest, and if on the contrary p is greater, then the liquid branch is below. For a simple substance the pressure of the double point is the pressure of coexistence (maximum pressure).

To conclude to this form of the ζ-curve, the real form of the isothermal need not be known. The principle of continuity suffices.

We conclude to the same shape of the ζ-curve for a homogeneous mixture. It is true that in the equation of state given by me, the values of a and b are dependent on the nature and the concentration of the components; but the shape of the equation of state, for which the substance is always assumed to fill the given volume homogeneously, remains the same. But if we should doubt this, the assumption of the principle of continuity would enable us again to conclude to a similar form.

This is the simplest way of uniting the two branches which may be experimentally realized, in the same way as for the isothermal the usual way of uniting the gas- and liquid branch is the simplest. And they are actually equivalent — one is the mathematical consequence of the other.

So if T and p are given, three values of ζ belong to any mixture when p remains between the values of p_M and p_m, which values belong to that mixture, which is always assumed to be homogeneous, at this value of T. As soon therefore as T is above what might be considered as the critical temperature of such a mixture, these three values are reduced to one. In this that value of T is considered as critical temperature, for which the isothermal can show only one horizontal tangent for homogeneous phases.

But for every mixture these three values of ζ or that one value of ζ depend on the composition and in general they will be different and that for two reasons 1st because of the fact that in the value of ζ occurs the pure function of x and y, which indicates increase of entropy in the mixing; viz.:

$$-\ MR\left\{(1-x-y)\,log\,(1-x-y)+x\,log\,x+y\,log\,y\right\}$$

and 2nd because also the second part of ζ, viz.:

$$pv-\int pdv=pv-MRT\,log\,(v-b_{xy})-\frac{a_{xy}}{v}$$

differs for the different mixtures, when they are all taken at the same value of p and T. It is the value of this expression, which is represented as ordinate in fig. 1. When v can be calculated from $v=\dfrac{MRT}{p}$, this ordinate is independent of the nature of the substance, and so has the same value for all mixtures in the rarefied gaseous state. The gas branches can therefore always coincide, at least as long as p is exceedingly small. But as soon as the degree of density is such that they are no longer perfect gases, these lines deviate — and in the liquid state the difference of the ordinates for two substances can become so great, that the difference mentioned sub 1 is quite insignificant compared to it. Therefore it does not seen exaggerated to me, if I call a calculation, in which the change mentioned sub 1 is taken into account, but that mentioned sub 2 neglected, absurd.

Let us now determine the connection between the different sheets of the ζ-surface of a ternary mixture at given T and p, by applying the principle of continuity. For this it will be sufficient to show the connection between the three curves which are found in a section normal to the xy-plane and for simplicity 's sake we shall begin with the three curves lying in the ζ-surface in the coordinate plane, for which $y = 0$. This is really the same thing as if we said that

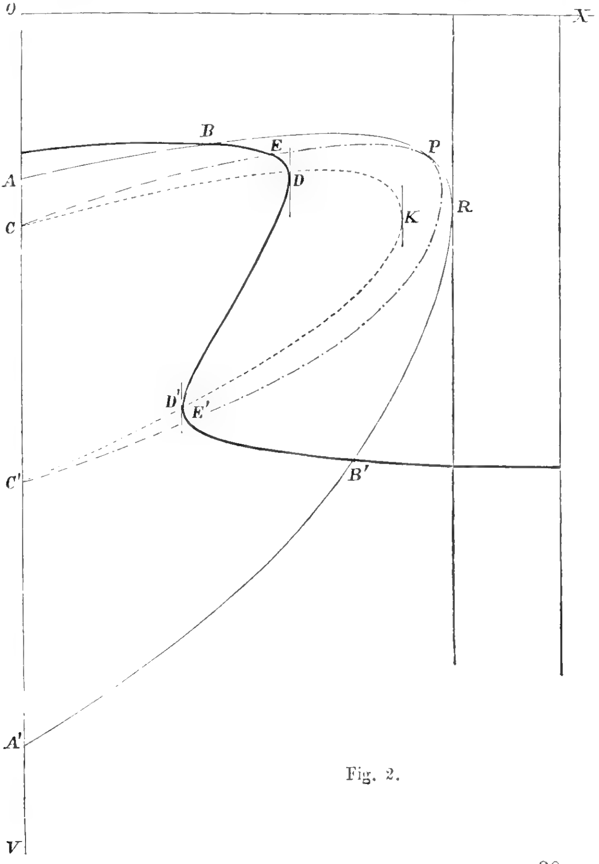

Fig. 2.

we seek the relation of the three values of ζ for a binary mixture.

As it is our purpose to examine how the three curves unite to a single curve, we shall assume values of p and T, at which the quantity ζ has three values for certain values of x, whereas it has only one value for other values of x. Let in fig. 2 the curve $ABPB'A'$ represent the connodal curve of the binary mixture in an x,v diagram. It is assumed in the figure that for $x > x_R$ at the given value of T no separation into two phases can occur, however great the value of p be. The point R represents therefore the critical point of contact, and P the plaitpoint. Going from A to P and also from A' to P, the pressure of the coexisting phases of the binary mixture increases. Hence for $x_R > x > x_P$ there is retrograde condensation of the first kind. Let B and B' represent a pair of nodes, for which $x_{B'} > x_B$. Then it must be possible to draw an isobar passing through B and B', because at coexistence the pressure must be the same.

Besides the connodal curve, we have also drawn the curve CPC', which indicates the limit between the stable and unstable homogeneous phases. For the points of this line $\frac{\partial^2\psi}{\partial x^2} - \frac{\left(\frac{\partial^2\psi}{\partial x\,\partial v}\right)^2}{\frac{\partial^2\psi}{\partial v^2}}$ or $\frac{\partial^2\zeta}{\partial x^2}_p$ is equal to 0. I may assume as being known that for a binary system this curve coincides for $x=0$ with the point for which $\frac{\partial p}{\partial v}$ of the first component is equal to 0.

In the third place the locus of the points for which $-\frac{dp}{dv} = \frac{\partial^2\psi}{\partial v^2}$ is equal to 0, is given. This locus lies entirely within the region of the unstable phases. For the spinodal curve viz. $\frac{\partial^2\psi}{\partial x^2}$ and $\frac{\partial^2\psi}{\partial v^2}$ must both be positive and their product must be $\left(\frac{\partial^2\psi}{\partial x\partial v}\right)^2$. From $\frac{\partial^2\psi}{\partial v^2} = -\frac{dp}{dv}$ positive, follows $\frac{dp}{dv}$ negative, hence the points of the spinodal curve lie outside those for which $\frac{dp}{dv} = 0$. Only in special cases the spinodal curve and the locus CKC' will have points in common, viz, for $x=0$ or $x=1$, or for the special point for which $\frac{\partial^2\psi}{\partial x\partial v} = -\left(\frac{\partial p}{\partial x}\right)_v$ is equal to 0. The last mentioned case will be disregarded for the present. The point K, where a tangent might be drawn at the third locus, parallel to the V-axis, represents a point for which two values of v, making $\frac{\partial p}{\partial v}=0$, have coincided, and might therefore be considered as critical point of the mixture, when it behaved as a simple substance.

Let us now examine the course of the isobar, passing through B and B'. On the left side of B it must indicate volumes smaller than that of the connodal curve, because the pressure ot B is greater than that of A. In D it is supposed to pass through the minimum pressure of the mixture, whose $x=x_D$. That this will take place on the right side of B agrees again with the fact that the pressure on the connodal curve increases from A to P. In D the isobar under consideration must have an element in common with the isothermal of the concentration x_D and from that point it will go back to smaller concentrations. It is supposed to meet in D' for the second time the locus going through the points for which $\frac{dp}{dv}=0$; viz. the branch where the pressure on the isothermal is maximum. So the point D' must lie on the left side of B. From D' the isobar moves on to greater values of x.

At the chosen value of p therefore, there is a continuous series of phases of the binary mixture. They are liquid phases on the left side of B, gas phases on the right side of B'. Between B and D the isobar cuts the spinodal curve, also between D' and B'. If we indicate these points of intersection by E and E', the metastable phases are to be found between B and E. In the same way between E' and B'; whereas all the phases between E and E' are unstable.

A line parallel to the volume axis and for which $x_D' < x < x_D$, cuts the isobar in three points. For all these values of x there are therefore three different phases which have the chosen value of p as pressure, and therefore ζ will have three values. For all values of x outside x_D and x_D' there is only one volume that has p as pressure, and therefore ζ will have one value. It is viz. easy to see that no other points than those marked can have p as pressure.

So if we have to draw ζ as function of x at this pressure and at this temperature, we get fig. (3). To conclude to this form, we have to take into account: 1st that $\frac{\partial \zeta}{\partial x}$ at $x=0$ is equal to $-\infty$ and for $x=1$ to $+\infty$, which follows from the pure function of x. From this we derive viz. $\frac{d\zeta}{dx_{pT}}=MRT\,log\,\frac{x}{1-x}+\ldots$ 2nd that in the stable

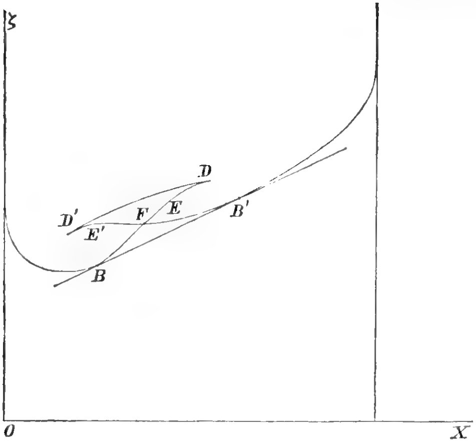

Fig. 3.

region $\frac{d^2\zeta}{dx^2}$ is positive, and in the unstable region negative.[1]) Between E and E' the ζ-curve must turn its concave side downwards, and outside E and E' its convex side. 3rd that between D and D' the value of x decreases. The place of the points B and B' is found by tracing the double tangent of the ζ-curve, as follows from the conditions of the equilibrium. According to these conditions p and T are in the first place equal for the pair of coexisting phases, and further $\frac{\partial \zeta}{\partial x_{pT}} = M_2\,\mu_2 - M_1\,\mu_1$ equal and $\zeta - x\frac{\partial \zeta}{\partial x_{pT}}$ equal. The double tangent therefore cuts off from the vertical axis ($x = 0$) a portion equal to the molecular potential of the first substance, and a portion from the vertical axis at $x = 1$ equal to the molecular potential of the second substance.

If we now increase the pressure, so that it varies in the direction towards the plaitpoint pressure, we could find from

$$d\,\zeta = v\,d\,p$$

the modification in the traced curve for every value of x.

In our figure this could only lead to a correct result, if it were not schematical, but if it were numerically accurate down to the smallest peculiarities.

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[1]) In the unstable region $\frac{d^2\zeta}{dx^2}$ is negative, as long as $\frac{\partial^2\psi}{\partial v^2}$ is positive; so between the points D and E and D' and E'. If $\frac{\partial^2\psi}{\partial v^2}$ is negative, as will be the case along the curve DD', then $\frac{d^2\zeta}{dx^2}$ is positive. In fig. 3 the branch DD' has wrongly been drawn as being concave. The points D and D' must therefore be ordinary cusps.

We shall therefore not avail ourselves of this means for the determination of the modification of the ζ-curve which is the consequence of increase of pressure. We shall go back to fig. 2 and through the projection of the connodal curve of the ψ-surface drawn there we shall draw a curve for which p is greater. Without actually tracing the line it is easy to see that as long as p has a value below that of the isobar which would pass through K, the retrogression of x will continue, and so also the branch DD' in fig. 3. Only its limits and so the values x_D and $x_{D'}$ will approach each other. For the isobar of K the retrogression has ceased. This curve of constant pressure touches the locus CKC' and the common tangent is parallel to the v-axis. As a point of inflection must occur between D and D', we conclude that the isobar of K must have a point of inflection in the point K. This isobar still cuts the spinodal curve twice and the ζ-curve for that pressure will retain its two points of inflection. Consequently the *great* complication will not disappear from the ζ-curve till we get this pressure. We shall henceforth use the name of crest for the configuration which lies above the double point. So we may say that the crest has disappeared for pressures greater than K, whereas it remains for pressures smaller than K. If we now trace the ζ-curve again, there is only in so far a complication in this line, that it contains a concave part with two points of inflection, and so that there will still be a double tangent. It is proved here in another way than follows from the theory of the ψ-surface that the critical phenomena will make their appearance in a mixture only at pressures and temperatures, which are higher than when we had to deal with a simple substance. But paying attention to the way in which we have arrived at this result, we see that we have only been able to derive all this by means of the knowledge of the ψ-surface. And this is the reason why I have formerly made use only of the ψ-surface and why I have considered the ζ-curve not suitable for leading to the knowledge of the critical phenomena.

If the pressure is still higher, the two points of inflection of the ζ-curve draw nearer to each other; and when the pressure reaches the value of the plaitpoint-pressure, the concave part of the curve disappears and the ζ-curve has turned its convex part downward everywhere.

In what precedes we have discussed the way in which an existing complication in the ζ-curve disappears. Let us now examine what happens when such a complication extends.

Let us for this purpose examine, what will be the consequence, when the pressure decreases. The isobar traced in fig. 2 remained entirely within the limits of $x=0$ and $x=1$, and was therefore a continuous curve without interruption. If the pressure is decreased, the point D' will move to the left, and will have reached the side of our diagram at a certain value of p, so that for that pressure $x_{D'}=0$. Then the pressure must have the value of the maximum pressure of the isothermal of the first component. This pressure is of course much greater than the pressure for A or for A'. This isobar cuts the connodal curve somewhere between A and B or between A' and B'. Nor has the isobar any interruption then. The modification of the ζ-curve consists then in the following two points 1st the crest has become broader, 2nd the crest has become higher, so that the point D' has reached the $O\zeta$-axis. The point which it has in common with this axis must then lie higher than the point where the liquid branch in the ζ-axis begins, as appears from fig. 1. This case is represented in fig. 4. If the pressure descended still lower, the pressure curve would lie partly outside the oxv diagram, and we should get the case represented in fig. 5. Then we should have to add a part on the left side of the ζ-axis, in order to be able to consider the ζ-curve as one coherent curve.[1])

Fig. 4.

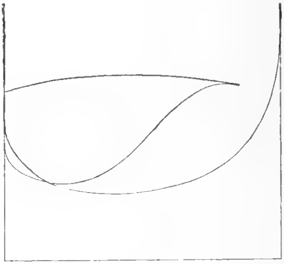

Fig. 5.

If the pressure should have come below that of A and A', then the vapour branch has become the lowest branch all over its breadth — and the remaining part of the crest has entirely separated from the vapour branch. Then there is no longer question of tracing a double tangent and so there are no longer coexisting phases.

A few points are still to be discussed, viz. the signification of the double point of the ζ-curve, and the displacement of the points of

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[1]) See for fig. 4 and fig. 5 the note on page 456.

inflection. The double point is always a point of intersection of the vapour branch and of the liquid branch; so it is never found on the branch of the unstable conditions. [1] For a simple substance it occurs, when the pressure is equal to the coexistence pressure at the chosen temperature. For a mixture it indicates the composition, for which at a chosen temperature the pressure which would be found by the criterium of MAXWELL if applied to the isothermal of the homogeneous phases, would be exactly equal to the pressure for which the ζ-curve has been drawn, or in other words: the place of the double point determines the mixture, for which the chosen pressure would be the maximum tension, if such a mixture continued to behave as a simple substance. I have proposed before (Arch. Néerl. Série II Tome II, pag. 69) to call such a pressure coincidence pressure. When, therefore, the pressure, for which the ζ-curve has been drawn, is either smaller or greater than the coincidence pressure of any mixture at the assumed temperature, no point of intersection will occur. In such a case the ζ-curve consists of three branches, which keep quite separate and for which, when p is smaller than the smallest of the coincidence pressures would be, the vapour branch lies lowest. Above it we get the liquid branch, and still higher the unstable branch. If on the other hand p is greater than the highest of the coincidence pressures, the liquid branch lies lowest.

As to the place of the points of inflection we have to observe that the relative position of E and E' (fig. 3) may be different. In the figure drawn E' lies on the side of B and E on the side of B'. If, however, with increase of pressure the crest should disappear altogether, then E' lies on the side of B' and E on the side of B. In a special case the transition of these two cases might take place, if the points E and E' have coincided in the point of intersection of the two branches. It appears from fig. 2 that for such a transition it is necessary that the two points E and E' are to be found at the same value of x. If E lies on the side of B, and E' on the side of B', we shall call this the normal position. To determine the value of the plaitpoint-pressure and of the plaitpoint composition at a given temperature by means of the ζ-curve is: to seek for what value of p the two points of inflection, after they have assumed the normal position, coincide, and at what value of x this happens. To determine the composition and the pressure of the criticaltangent state is to seek at what pressure the point B' begins to retrograde and what the value of $x_{B'}$ is at the moment.

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[1] In some complicated cases an apparent exception may be found.

Now we may proceed to investigate what geometrical configuration the ζ-surface will show for a ternary system, specially at the critical circumstances of the gas- and liquid state. Let us assume an x and an y-axis, in order to be able to indicate by means of the values of x and y the composition of the mixture, which consists of $1-x-y$ molecules of the first substance, x molecules of the second substance and y molecules of the third substance. The ζ-surface can only extend over points lying within the right-angled triangle of which the sides containing the right angle are lying on the x and y axes and have a length equal to unity. For x negative, for y negative or for $1-x-y$ negative, the first part of ζ, viz. the pure function of x and y, is imaginary. Let us now think the conditions of fig. 2 satisfied for the x component so that the temperature lies between the critical temperatures of the first and the second component, and the ζ-curve as having the shape of fig. 3 above $o\,x$ for a certain pressure below the plaitpoint pressure. Let us assume the same of the y component.

The temperature is chosen in such a way, that (see fig. 6) $(T_{cr})_O < T$, but $(T_{cr})_A > T$ and also $(T_{cr})_B > T$. If the two components x and y were identical, we could construe in the OXY-plane within the triangle $O\,A\,B$, the following rectiliniar projections, parallel to the hypothenuse. 1st The projection of the double points. 2nd The projections of the points of contact B and B', which lie on the double tangent. 3rd The projections of the points of inflection E and E' and 4th. The projections of the cusps D and D'. In this case, however, the system is only seemingly ternary, but in reality it is a binary mixture with $x+y$ molecules of a second component. A pair of coexisting phases are then indicated by two points of the projection mentioned sub 2, chosen in such a way that the line which connects them, passes through the origin O. But if the third component is made to differ from the second component, so that the ζ-curve over OB, though it has in its main features the form of fig. 3, yet in details deviates from it, then we get four curvilinear projections instead of the four rectiliniar projections. Then we get again 1st the locus of the projection of the double points. The existence of such a locus may be derived from the following considerations. We wish to represent the value of ζ for homogeneous phases, and we have had to assume for the isothermal for homogeneous phases according to the principle of continuity that below a certain temperature, the pressure will have a maximum value and a minimum value and that it will therefore be possible to draw a straight line in accordance with the criterium of MAXWELL.

For the pressure of that straight line we should get according to the law of the corresponding states the relation

$$\frac{p}{p_{cr}} = \varphi\left(\frac{T}{T_{cr}}\right)$$

If the law of the corresponding states should not be applicable, the shape of the function φ would be variable with x and y, and p_{cr} and T_{cr} also being variable with x and y, there will be only one relation between x and y at given T and p. This relation given under the form

$$y = F(x)$$

gives the equation of the locus discussed. For our purpose we shall suppose, that it is represented by a single continuous curve, running from a point of the x-axis to a point on the y-axis. 2nd the projection of the points, representing coexisting phases, and consisting of two branches one on either side of the curve mentioned sub 1st. This projection is not the projection of the points B and B'. It is namely obtained not by tracing a double tangent at the ζ-curves, but by construing a double tangent-plane at the two sheets of the ζ-surface. In general this locus lies outside the projection of the points B and B' except in the axes when it coincides with it. 3rd. The projection of the points indicating the limit between the metastable and unstable phases, so of the points for which $\frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2} - \left(\frac{\partial^2\zeta}{\partial x \partial y}\right)^2 = 0$. This locus does not coincide with the projection of the points E and E'; only in the axes it coincides. 4th. The projection of the points D and D', so of the points for which at the chosen temperature the pressure chosen is equal either to the maximum pressure, or to the minimum pressure of the isothermal drawn for homogeneous phases.

So with this configuration we can speak of a connodal curve and also of a spinodal curve; but the spinodal curve need not lie between the connodal curve.

Let us now increase the pressure, then the crest, lying above the line of the double-points, will decrease, and of course it will change its place, and let us assume the pressure to be greater than that of the point K (fig. 2) for one of the pairs of components. Let us choose the pair represented by the y-axis, then the line of the double points has retired, so that is ceases to exist somewhere in the xy-plane. If moreover the pressure is higher than the plaitpoint-curve of the binary mixture represented by the y-axis, then the binodal curve too has got detached from the y-axis, and the two branches of the binodal curve have united to a single curve. In fig. 6 the drawn curve KC represents the projection of the double-points and the curve DPE the binodal curve.

From what has been observed for a binary mixture appears that the projection of the spinodal curve, touching the binodal curve in P, must have the shape as indicated by the dotted line. So the projection of the binodal curve must have a double point lying on the projection of the double-points or near it, and so further from P the two branches of the spinodal curve have interchanged their relative position.

What is such a configuration to be called? At the final point, so in the neighbourhood of P it has entirely the properties of a plait. There is a plaitpoint, a connodal and a spinodal curve, which are placed in the usual way with regard to each other. Every plane section between the points P and K cuts the ζ-plane along a curve which has two points of inflection. But so great a modification occurs at a great distance from P, that we can only recognise a plait in it by comparison with the parts near P. This plait has then been subjected to a transformation. We could viz. in a plait make the two binodal branches approach each other, so that the convex-convex parts preserve their dimensions nearly or quite up to a certain distance from the spinodal curve, but the whole convex-concave

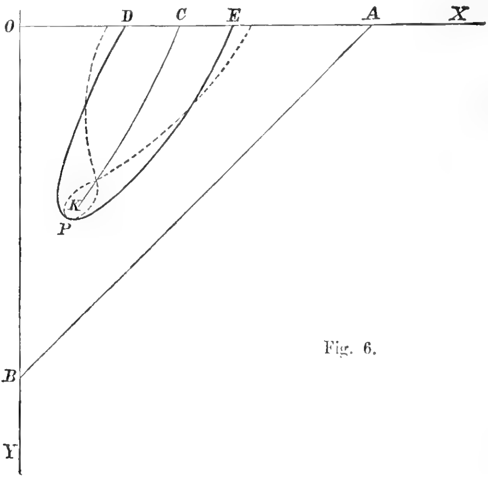


Fig. 6.

part has been covered by the sides which approach each other. If we now think this deformation exceedingly slight at the top, but strongly increasing when moving from the top, then the character of the plaitpoint remains almost unaltered at the plaitpoint, but far from the top a sharp crevice is substituted for the gently sloping part between the two side-walls. In this we have only wished to represent the shape for that part of the ζ-surface which is visible to an eye placed under it.

If we want to imagine a deformation of the plait, which leaves the other parts also in existence, we should have to apply four folds originating in the point K, of which the two outer folds brought together would have to represent the line of the coincidence pressures, and the two others would form the series of cusps. But in this way we do not account for the fact, that (fig. 3) the left-side points E' and D' belong to the right-side point B' and vice versa.

At all events it is clear that the name "plait" for a such a configuration might give rise to a great deal of misunderstanding, unless we take care to distinguish it by an adequate addition. We might, e.g. speak of a plait of three sheets.

If we pass now from the geometrical treatment to the question what the science of physics may derive from it, we can summarize the answer in the thesis: the critical phenomena of a ternary system are equal to those of a binary system. At the chosen temperature all the mixtures indicated by the course of P are under the plaitpointcircumstance. (fig. 6). The plaitpoint pressure alone varies for all those different mixtures. The limiting value of $\frac{y_2 - y_1}{x_2 - x_1}$ is indicated by the direction of the tangent in P at the connodal curve.

The mixtures which are in the point-of-contact circumstance at the chosen temperature, are found by means of the envelope of the different binodal curves, and that of those branches of these curves which lie on the side of the hypothenuse of $\triangle\ OAB$. The mixtures indicated by points lying between the locus of P and that of the before-mentioned envelope have retrograde condensation, and in accordance with the suppositions made in fig. 2, retrograde condensation of the first kind.

It has not been my purpose in the preceding pages to examine the different cases which may occur for a ternary system. But it has been my purpose to demonstrate in what way they may be explained by means of the ζ-function, when for some cause or other they will have come more to the front.

Geology. — "*Examination of specimens of sand from borings done at the works of the outer harbour at Scheveningen*". By J. A. GRUTTERINK. (Communicated by Prof. J. L. C. SCHROEDER VAN DER KOLK).

H. WORTMAN Esq. engineer at those works, was kind enough to send six specimens of sand to the Geological and Mineralogical department of the Polytechnical Institute at Delft.

Those specimens had been taken, respectively at depths of 26.50 M., 27.50 M.; 28.50 M.; 29.50 M.; 3150 M. and 32.50 M. under New Amsterdam level. It could however not be guaranteed with perfect certainty that the exact order had been observed.

Stated was that sand, up to 26.50 M., being ordinary dunesand, had not been collected.

The fact that all the specimens contained gross-grained matter rendered it impossible to use at once the separatory-funnel. So they were sieved; the sieve having openings of 0.5 cM.

The examination of the gross-grained matter yielded not much of a result.

In it were found pieces of colourless and of white quartz, up to the size of 1 cM., pieces of slate and fragments of shells. Granite was not found.

The percentage of heavy minerals in the finer matter was fixed by separating it in a separatory funnel, with the help of Bromoform. For the purpose two samples of 5 Gr. of each specimen were taken. The percentage was very low:

	I.	II.	III.	IV.	V.	VI.
a	0.10	0.13	0.25	0.07	0.15	0.24 %
b	0.09	0.10	0.22	0.10	0.10	0.11 %

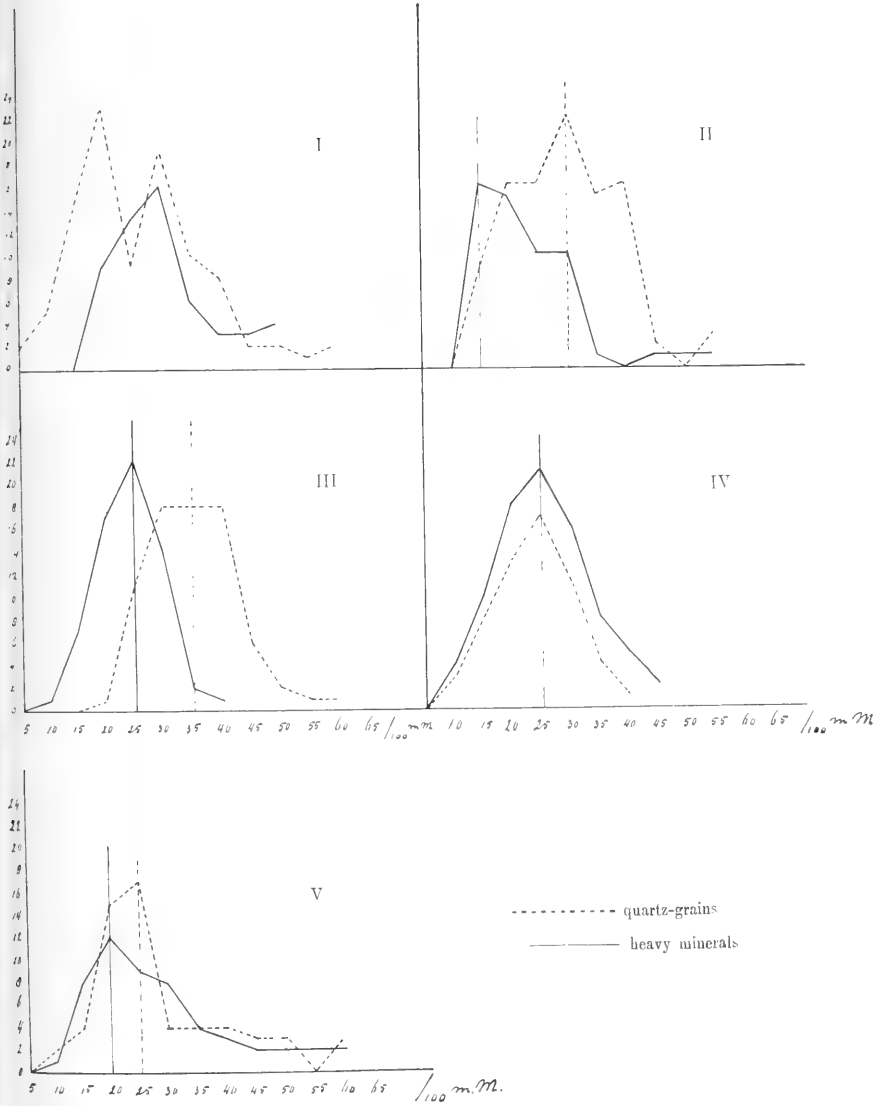
Microscopic examination showed the grains to be rather sharp-edged.

Amphibole was found in all specimens, even more abundant than garnet; only in specimen IV, amphibole was not predominating.

The size of the grains was also settled microscopically, by moving the object-glass parallel with itself across the stage and accepting as the size of the grains, not its greatest dimension, but its projection on the micrometer.

So as to avoid passing over a disproportionate number of big grains, part of which easily remain out of the field of observation,

J. A. GRUTTERINK. „Examination of specimens of sand from borings done at the works of the outer harbour at Scheveningen.”



only those grains were measured which were passing through the very middle of the field.

Of specimen VI the size of the grains could not be fixed, the matter for the greater part (90%) being too coarse to be treated in that way; a proper set of sieves not being at my disposal.

The result of the examination as to the size of the grains is graphically shown in the additional table.

Grouped together were grains up to the size of 0.05 mM., those from 0.05 to 0.10 mM. etc. From this table appears that in specimen II the average size of the quartz-grains considerably exceeds that of the heavy minerals, although, in a smaller degree, specimen III shows forth the same, whereas in specimen IV there appears to be no difference in the average size of the grains; also in specimen V we can hardly speak of a moving apart of the two lines. Specimen I is left out of consideration, because, on account of the irregular quartz-line, no conclusion can be drawn as to its average size.

Judging from the abundant quantity of Amphibole; from the low percentage of heavy minerals and from the sharp edges of the grains it seems not presuming too far to consider the examined sands, a deposit of the ice. The influence of water however cannot be denied for how otherwise can be explained the presence of so many fragments of shells and the washed character of specimens II and III. From the graphical table, one feels inclined to conclude that the influence of the water has been greater on the surface than deeper down.

It is to be regretted that no specimen has been kept of the upper sand. Now we miss for our comparison, a purely alluvial sand formed on that spot.

's-Gravenhage, 21 Februar 1902.

Astronomy. — "*The relation between the brightness of a luminous point and the moments at which we observe its sudden appearance or disappearance.*" By Prof. H. G. VAN DE SANDE BAKHUYZEN.

At the meeting of the Royal Academy May 29th 1886 I read a paper on researches made to determine the influence of the brightness of a luminous point on the moments at which we observe its sudden appearance. These researches have later been continued and in the "Archives Néerlandaises" Serie II, Vol. VI, p. 727 the results of all these observations have been published, together with an explanation of the fact that a great many astronomers observe the transits of smaller stars over a thread in the field of the telescope later than those of bright ones.

I have now extended these researches by observations on the sudden disappearance of a luminous point at different degrees of brightness.

The luminous point was produced by placing a copper plate with a small opening before the flame of a paraffin lamp. Its brightness can be diminished by a wedge of coloured glass movable in a frame before the opening. Moreover a light metal screen at the further end of a lever, which can be moved quickly by means of a strong spring, was placed in front of the opening and in the researches on the appearance of the artificial star it uncovers the opening suddenly. At the same moment a metal point fastened to the screen dips into mercury and closes a current, thereby registering a mark on a chronograph. The observer at a distance of about 25 meters from the apparatus, observes the appearance of the point by means of a telescope with a terrestrial eye-piece and at the moment when he perceives it, he closes with a key a second current which produces a mark on the same chronograph. The distance between the two marks will give the sum of the three following periods: 1st the time required to receive a perception of the luminous point; 2nd the time elapsing between the perception and the closing of the current; 3° the difference between the times which elapse between the closing of the currents by the apparatus and by the observer, and the registering of the marks on the chronograph.

For observations on the disappearance of the luminous point the arrangement remains the same except for the screen, which is replaced by one of a different shape which suddenly covers the opening, when the lever is worked.

Two Nicols of which one is fixed and the other can be turned, so that its rotation is measured by a graduated circle, are placed between the observer's eye and the artificial star; they enable us to diminish its brightness in known proportions. If we determine at what reading of the graduated circle the brightness of the artificial star corresponds to the brightness of a star of known magnitude, seen through a telescope of given dimensions, we can express the different degrees of brightness in the corresponding star magnitudes. These determinations have been made by different astronomers of the observatory at Leyden who have taken part in the observations with the meridiancircle; they compared, from memory, the brightness of the luminous point at different positions of the Nicol with that of stars in the meridiancircle at Leyden in the illuminated field with the usual magnifying power of 200. Availing myself of these determinations I have, in the following accounts, expressed the different degrees of brightness in star magnitudes.

During several days observations have been made with this apparatus in order to determine the personal error in the observation of the sudden appearance or disappearance of the artificial star in different positions of the Nicol, and the first question was: how are we to deduce from such a series the most probable result for that personal error? The result from each observation, obviously, depends on a great number of different quantities, among others on: A, the attentiveness of the observer; B, the sensitiveness of his eye, C, his tiredness, etc. which all will be different in different observations. If we assume that for a given series of observations it is as probable that A, B, C etc. are larger than A_0, B_0, C_0, as that they are smaller than these values, the personal error which belongs to the quantities A_0, B_0, C_0 etc. will be called mean personal error T.

It is obvious that this value T cannot be found by forming the arithmetical mean of the several values t, the distances of the two corresponding marks on the chronograph. With a small degree of attentiveness A, for instance, t may become very large and $t—T$ may increase indefinitely, whereas with a high degree of attentiveness t will become smaller, but not indefinitely, as the appearance of the luminous point can never be registered earlier (errors excepted) than the moment at which it has actually appeared. Hence the positive errors are sure to be larger than the negative ones. An approximate value of that mean personal error may be found by arranging the measured distances of the two marks in a series in the order of their values and by taking as the most probable value the middle quantity of this series. This may also be found following a more accurate method. I have found from a very long series of observations that the values $log\,(t-t_0)$, where t_0 is a constant to be determined for each series, follow the ordinary exponential law of errors; the mean value of $log\,(t—t_0)$, $log\,(T—t_0)$ will then yield the most probable value T of the mean personal error. The constant quantity t_0 introduced into this computation, represents the smallest possible value of the personal error with the maximum of attentiveness and sensitiveness, the minimum of tiredness etc.

From the series of observations of the sudden appearance and disappearance of the artificial star, according to the latter method, the following values were found:

Magnitude of the artificial star	$5^m,7$	$7^m,2$	$8^m,6$	$9^m,5$
Personal error at the appearance	$0^s,275$	$0^s,316$	$0^s,413$	$0^s,530$
„ „ „ „ disappearance	$0^s,314$	$0^s,329$	$0^s,387$	$0^s,489$

It is obvious that the absolute value of this personal error depends not only on the observer, but also on the apparatus and on the observing method. The differences of the personal error resulting from the different degrees of brightness are neither influenced by the apparatus nor by the observing method. As care has been taken that the observations with different magnitudes should be made in rapid succession and that they should be distributed symmetrically, the variations of the personal error with brightness may be deduced from the numbers given above with tolerable accuracy.

In the "Archives Néerlandaises" I have derived from these results an explanation of the relation between the personal error in the transit observations of stars and their brightness. Here I shall only refer to the influence of this error on astronomical observations in which the sudden appearance or disappearance of a star must be determined, as for instance with a ring or bar micrometer. In determining the differences of right ascension of small and bright stars systematical errors of more than $0^s.2$ may occur; a good example is seen in the series of observations of O. A. L. Pihl to determine the positions of stars in the stellar cluster of χ Persei, (The stellar cluster χ Persei micrometrically surveyed, 1891) where the errors amount to more than a second of time.

This personal error will have a great influence on the position of the moon derived from the observations of occultations. The amount of the error with different methods of observation (registering method, eye and ear method) may obviously vary; it seems, however, that this error does not differ much for the appearance and the disappearance, so that the influence on the moon's diameter will not be great, wheras the moon's longitude, derived from occultations especially of smaller stars will be too large and its error may amount to $0''.2$ and more; therefore it is necessary to determine this error and to account for it.

Physiology. — Dr. C. WINKLER presents in the name of Prof. J. K. A. WERTHEIM SALOMONSON an essay, entitled: "*A new law concerning the relation of stimulus and effect.*" (III).

In our two former essays we have demonstrated the relation, existing between magnitude of stimulus and effect. We proceeded to test the law, by which this relation is expressed, to the results obtained experimentally by some physiologists, and found this law prevailing for all direct and indirect stimulation of the muscles, for galvanic as well as for faradic stimulation, for stimulation with ascending or descending make or break shocks. The law was further proved to prevail too for the electrical stimulation of nerve-fibres. Lastly we found the facts, kown to us about the relation between stimulus and effect for the eye, to be in accordance with our law, for all stimulation either by light or by electricity.

Our present essay intends to examine if our law may be applied to sense-perception; in how much therefore it might be called a psychophysical law (in the biological sense of course). At the same time we will try to investigate its connection with the well known psychophysical law of WEBER-FECHNER.

Whilst for the nerve-muscle preparations the relations are very transparent — there we have only one conducting organ: the nerve fibre, and one organ receiving and transforming the stimulus: the muscle — in the case of the senses the relation is infinitely more complicated.

Here in the first place there does exist a peripherical organ, receiving the stimulus, e. g. the retina-elements, the Cortian organ, the tactile corpuscles etc. These are directly connected to the primary sensible neuron or form part of it. From this primary neuron a stimulus is transferred to a secondary neuron ending in the thalamus opticus; thence the stimulus passes on to a third sensible neuron, its endarborisation being placed in the cortex cerebri. To reach the sensible cortical centrum therefore a stimulus must pass at least three neura. And even after this having happened, we know to a certainty that only a first step has been taken towards the bringing about of a "sensation". We have only succeeded in raising the general impression of light, touch, hearing or smell, without a single indication as to the nature of that sensation. This is achieved only when still higher neura — associative neura — have been brought into action and when the stimulus has been conducted from the projection-systems to the associationpaths.

Whilst in the case of the passage from the secondary motor neuron to the primary neuron we have to consider only the passage from one single neuron to another single neuron, (this latter however receiving probably also stimuli from other secondary neura: commissural cells, cerebellar and sensible neura etc.) we know for certain that in the case of the sensible neura every peripheric sensible neuron is related to numerous secondary neura. An elementary stimulus, applied to one single sensible neuron of the first order, will therefore needs rouse to action several secondary neura Furtheron we shall have occasion of referring again to this fact.

Meanwhile we may state with absolute certainty that no sensation is possible after a peripherical stimulus, unless at least three successive neura have been roused to action, and also that, if we desire a sensation explaining the cause of the stimulation, — in other words an associative sensation — at the very least four successive neura must be in function, but most probably a great many more.

First of all we will examine what becomes of our law in the case of a stimulus being transmitted to several excitable organs, in such a manner that the effect of the stimulus on the most peripheric neuron is supposed to form the stimulus for the following neuron. We take thus for granted here that the effect of the original stimulus e. g. the potential wave extending itself along the peripheral neuron forms the adequate stimulus for the next neuron.

Very strong arguments may be forwarded in favour of this view, whilst a combating of it, as took place only very recently on the physiological congres at Turin, does not imply the whole of the conclusion from the there alleged experiments. For the moment I think our view may safely be considered as an orthodox one.

The stimulus R, applied to a peripheric neuron, operates an effect that, as has been proved elsewhere, may be represented by:

$$E_1 = A_1 \left\{ 1 - \varepsilon^{-B_1(R-C_1)} \right\} \quad . \; . \; . \; . \; . \; . \quad (1)$$

This formula may also be written in a slightly altered form:

$$E_1 = a_1 - c_1 \varepsilon^{-b_1 R} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

wherein:

$$a_1 = A_1 \quad b_1 = B_1 \text{ en } c_1 = \frac{\varepsilon^{B_1 C_1}}{A_1}$$

consequently only one of the constants having been changed.

The stimulus, acting on the *second* neuron is formed now by this effect E_1.

The effect of the stimulus on the second neuron may be represented by

$$E_2 = a_2 - c_2\,\varepsilon^{-b_2\,\rho} \quad \ldots\ldots\ldots\ldots \quad (3)$$

Or, if we give to ϱ the value from (2) this becomes:

$$E_2 = a_2 - c_2\,\varepsilon^{-b_2\left(a_1 - c_1\,\varepsilon^{-b_1 R}\right)} \quad \ldots\ldots\ldots \quad (4)$$

In this form the formula (4) is unfit for use. But we may first alter it a little to:

$$E_2 = a_2 - c_2\,\varepsilon^{-a_1 b_2} \,.\, \varepsilon^{b_1 c_1 \varepsilon^{-b_1 R}}$$

and finally to:

$$E_2 = a_2 - k\,\varepsilon^{\beta \varepsilon^{-b_1 R}} \quad \ldots\ldots\ldots\ldots \quad (5)$$

in which

$$k = c_2\,\varepsilon^{-a_1 b_2} \quad \text{and} \quad \beta = b_2\,c_1$$

Expanding the exponential form in an infinite convergent series, we obtain:

$$E_2 = a_2 - k\left(1 + \beta\varepsilon^{-b_1 R} + \frac{1}{2}\beta^2\varepsilon^{-2\,b_1 R} + \frac{1}{6}\beta^3\varepsilon^{-3\,b_1 R} + \text{etc}\right) \;.\; (6)$$

Neglecting the higher terms, and retaining only the two first, this expression is simplified to:

$$E_2 = a_3 - \beta_1\varepsilon^{-b_1 R} \quad \ldots\ldots\ldots\ldots \quad (7)$$

wherein:

$$a_3 = a_2 - k \quad \text{en} \quad \beta_1 = k\,\beta.$$

Which last expression (7) is perfectly identical to (1) as regards the form, only the constants having been changed.

We still have to account for the possible consequences of neglecting the 3rd and following terms in the series of formula (6).

A simple consideration shows that in so doing we put a lineal function of the first degree in the place of the exponential function, in a case where the action of the stimulus on a nerve-fibre was concerned. Now in our two former essays we proved precisely that it is allowed to do so where the nerve is concerned, because this behaved as a purely transmissive organ, and because only in the case of infinitely stronger stimuli than those that practically occur, a lineal function of a higher degree or even an exponential function may be needed to explain in what manner the nerve-fibre itself behaves under the influence of stimuli.

Consequently we obtain for the transmission of a stimulus from a first to a second excitable element — this last element being a *nerve-fibre* — the same expression, as the one originally established for the law expressing the relation between stimulus and effect in the case of one single organism. The above made demonstration may of course be extended to a third and a fourth element, the law always remaining unaltered and only the constants being changed. From the foregoing we deduce the consequence that a priori it may be deemed very probable that for sense-stimulation too the effect is expressed by our law, though we must instantly add the restriction, that this may be expected to hold true only where momentary stimuli are concerned. For in establishing the law the time-function was neglected. From this we may not conclude that time does not exert any influence at all, but simply that the law prevails only in cases where that influence may be excluded. This is possible only in cases where the space of time, during which a stimulus is acting, is extraordinarily short [1]), so that in reality the stimulus has finished acting before the effect reveals itself, consequently only for momentaneous stimuli.

Within my knowledge there have been made only a very few, and rather insufficient researches about the magnitude of the sense-perception in the case of momentary stimuli. For the present therefore the efficacy of the law cannot be tested to these.

It is possible however to investigate in how far this law is connected with the well-known psycho-physical law of WEBER-FECHNER.

Previously to this we will show, how, guided by our former demonstrations, we may arrive at a biological explanation for the

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[1]) I have succeeded in finding an expression for the addition of stimuli, opening quite novel views on some as yet insufficiently explained physiological facts.

fact of the „Unterschiedschwelle". In what manner, relying on the neuron-theory, can we understand the process of the originating of a sensation? In the first place a stimulation of the direct sensible neuron is needed. This causes a potential wave to travel along the nerve and to act on a series of secondary neura, in case it has reached a certain magnitude. As soon as here too the stimulation-effect has passed a certain threshold-value, tertiary neura are excited. In the same way the stimulus is transmitted to neura of still higher order. In what now may consist a *difference* of sensation?

Firstly in the fact that a neuron of higher order is alternately more or less strongly stimulated. This view is highly improbable, as it would postulate a localisation for every elementary sensation; each neuron, each nervecell would serve for one particular sensation. Still another view may be taken however, one that appears to me far more probable. We know that a whole system of neura of a higher order is excited, whenever an external stimulus is applied. The excitation of this complex of neura occasions a sensation; to intensify this sensation it is necessary that the *number* of excited neura should be augmented. And this will be the case only then, when the external stimulus is increased to such an amount, that the threshold-value of those several following neura is passed.

The Unterschiedschwelle in a sensation is formed by the threshold-value of a following system or number of neura.

About the manner in which by a stronger stimulus a higher system of neura or perhaps more neura may be excited, different views may of course be taken into consideration. One of these views I think especially plausible: viz. that the potential wave, in proportion to its centripetal extension along the primary neuron, decreases in amplitude. Consequently this stimulus will become feebler in centripetal direction, and operate an ever diminishing effect on the successive secondary neura, until at last the amplitude has become so small, that it remains under the threshold-value of a following secondary neuron.

In accepting this view we suppose the conditions for the originating of a more or less intensive sensation to be determined already in the spinal chord; the facts known to us about the skin-reflexes make this supposition very allowable.

What is the relation existing between our law and the psychophysical law? As we know, the mathematical formulation of the latter is the following:

$$E = p \log (R - s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which p and s represent constants, R the stimulus and E the effect. The curve expressed by (8) is a logarithmic curve showing great resemblance to the curve representing our law, but, unlike ours, possessing no asymptote. This fact is one reason amongst others why the psychophysical cannot express the relation between stimulus and effect for single neura or for the muscle (myophysical law). In psychophysics, where it is never allowed to make use of stimuli, capable of damaging the sense-organ, scarcely ever an effect is reached that may be considered a maximum, consequently the possibility of examining the upper part of the curve is excluded.

As far however as my law may be represented with sufficient accuracy by a lineal function of the 2nd degree (because we are not allowed to introduce more than three constants), so far it is found to be in perfect accordance with the psychophysical law, as this latter too may be represented in the same manner by expanding in a series.

And I wish to point out the fact that this part is relatively a rather large one: for a height of curve reaching more than one half of the maximum, we might with sufficient accuracy for practical aims, make use of a lineal function with three constants or of the law of WEBER-FECHNER.

In all experiments that have hitherto been taken to confirm or to combat the law of WEBER-FECHNER, generally not even approximately so large a part of the curve has been examined.

Now does there exist no single fact, capable of procuring us a decisive answer on the question whether for the sense-perception the possibility or the certainty of a maximal sensation may be taken for granted; a fact capable of deciding whether or not a psychophysical curve possesses an asymptote? I believe such a fact to exist for one sense-perception, viz. for the *sense of weight*. For the asymptote may be said to have been reached at the point where the action of the muscle proves insufficient to lift a burden: at that moment we have the maximum sensation of weight that may be obtained with the organism employed, and that cannot be any more surpassed. Probably therefore for the sense of weight there exists an asymptote, approached by the curve expressing the relation between the magnitude of burden and the sensation of weight.

How now for the other senses? A palpable fact tending to prove as for the sense of weight, the existence of an asymptote, is not known hitherto. Here thus we shall have to try to find an indirect proof. This latter too cannot easily be furnished. A few indications however we may be enabled to find. To that purpose we will consider in the first place in what manner we may deduce from our law an expression for the absolute and relative „Unterschiedschwelle". We proceed therefore in forming from:

$$E = A \{1 - \varepsilon^{-B(R-c)}\}$$

the differential quotient:

$$\frac{dE}{dR} = A\,B\,\varepsilon^{-B(R-c)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or

$$dR = \frac{1}{AB} \,.\, \varepsilon^{B(R-c)}\,dE. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Passing from the differential to the difference we obtain:

$$\triangle R = \frac{1}{AB} \,.\, \varepsilon^{B(R-c)} \triangle E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and as, according to Fechner, the value $\triangle E$, the so-called minimum differential sensation, is constant, the formula stands:

$$\triangle R = k_1 \,.\, \varepsilon^{B(R-c)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and putting $k_1 \,.\, \varepsilon^{-Bc} = k$:

$$\triangle R = k\,\varepsilon^{BR} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

By this process we have calculated the value of the absolute differential threshold-value for each value of R.

For the relative differential threshold-value we find:

$$\frac{\triangle R}{R} = k \,.\, \frac{\varepsilon^{BR}}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

In the psychophysical law of WEBER-FECHNER we find for the absolute and relative differential threshold-value:

$$\Delta R = const. \times R \qquad \text{and} \qquad \frac{\Delta R}{R} = const. \quad . \quad . \quad . \quad (15)$$

i. e. the relative threshold-value is constant.

Almost all experiments concerning the question of the correctness of the WEBER-FECHNER law, have tended to prove this last condition.

The results obtained by them are well-known. Neither FECHNER, nor HELMHOLTZ, nor KÖNIG, nor BRODHUN, nor MÜLLER—LYER, nor the English, American and French investigators have been able to afford definite proofs for the absolute constancy of the differential threshold-value.

Most of them have been led by their experiments to the conclusion that over a definite area the relative threshold-value was smallest, and that it became greater as well above as below that area. This is beautifully demonstrated by the experiments of STANLEY HALL, YUZERA MOTORA, AUBERT, HELMHOLTZ, KÖNIG—BRODHUN and many others, who all found an „optimum" in the curve of intensity above and below which the relative threshold-values rose.

Considering now our formula (16) for the relative threshold-value, deduced from our law for the relation between stimulus and effect, we find:

$$\frac{\Delta R}{R} = k \, \frac{\varepsilon^{BR}}{R}$$

This formula may again be represented by a curve. In what manner will that curve progress? To know this we ask in the first place whether there is a maximum or a minimum. This last is proved to be the case. *There is a minimum for:*

$$R = \frac{1}{B}.$$

If however we compare the curve represented by (14) with curves that may be obtained from the results of the above mentioned experiments, then it becomes clear that only for the sense of weight a somewhat sufficient accordance is obtained.

By the experiment of KÖNIG and BRODHUN is shown clearly, that in (14) the curve makes a much more acute bent, descends with far greater rapidity to the minimum and ascends far more rapidly too, than the curve that may be deduced from their experiments.

Whilst the possibility is not excluded that by our law the relation between magnitude of stimulus and effect is rendered with some accuracy for the sense of weight, such is not the case for the senses of sight and hearing, within the limits of observation at least.

I trust soon to be able to show, that this fact may be the consequence of a biological correction, quite independent from accommodation or pupil-alteration, caused solely by addition of stimuli, and accompanying every stimulation of sense.

Bacteriology. — "*The Physiological Bacteriology of the intestinal canal (2nd abridged paper: The Bacteriological relations in the intestinal canal of the rabbit).*" By Dr. ALEX KLEIN. (Communicated by Prof. PLACE).

Already PASTEUR fixed the attention on the great signification of the question in how far the numerous lower organisms, present in the intestinal canal of man and animals, play a part in the digestion. Since that time (1885) an extensive literature on the bacteriology of the intestinal canal has appeared; the results of these researches are in brief as follows:

In the first part of the small intestine of most animals, if no ingesta are present, no, or at least very few bacteria are found, (auto-sterilisation of KOHLBRUGGE); downward in the small intestine the number increases (NENCKI, GILBERT-DOMINICI, BROTZU, ESCHERICH i. a.). A very considerable increase is observed in the Coecum (ALAPY, ESCHERICH, KOHLBRUGGE i. a.), whilst in the rest of the large intestine now an increase, then a diminution of the number of lower organisms may be stated. Basing on these observations and at the same time in consequence of direct experiments conducted in that direction (BIENSTOCK, VINCENZI, SCHÜTZ), the existence of an anti-bacterial action in the small intestine is admitted. This anti-bacterial action, by killing the "wild germs", introduced together with the food, is cause of the restriction of the putrefaction processes in the intestinal canal. In the Coecum there is no such an anti-bacterial action; it has a flora of its own, consisting of the "obligative intestinal bacteria" (chiefly B. coli and allied organisms), of which a symbiose with the mucous membrane of this portion of the intestinal canal must be accepted (SCHOTTELIUS, KOHLBRUGGE); this flora is of importance for the digestion. Moreover, in consequence of the results obtained by painstaking and interesting experiments on animals, bred sterile from their birth, the influence of the intestinal bacteria on digestion, has been denied by some (NUTTALL and THIERFELDER), confirmed by later researches (SCHOTTELIUS).

The foundation of these numerous investigations, the determination of the number of bacteria in the different parts of the intestinal canal, must be called an erroneous one, in as much as all these experiments relate only to the living bacteria present in the intestinal canal. And as little as the sanitary condition of a population can be judged only after the number of living individuals present at a given moment, or the murderousness of a battle exclusively after the number of remaining soldiers, without taking into consideration the killed, as little is a right insight to be acquired into the sanitary condition of the bacterial population of the intestinal canal and the battles fought there between these lower beings and the living animal organism (anti-bacterial influences), by only looking at the living individuals, leaving the dead ones out of account.

The proportion existing between the number of living and of dead individuals I call the *sterility-index*; this proportion thus indicates the degree of sterility reached by a determined population of bacteria.

The sterility-index of a bacterial population is estimated from the relation of two data: 1st The difference in number of this population found between the culture-method and the microscopic counting-method (proportional number), and, 2nd the determination of that fraction of the microscopically counted organisms, which are still able to propagate.

The latter determination is effected *biologically*: the living organisms in the bacterial population are allowed to propagate and after a certain time the number is again determined, as well microscopically as by culture. In order to accomplish this propagation the same medium is by preference used in which the original bacterial population developed, as the lower organisms also have in the beginning been able to increase in this medium. Only it will be necessary by dilution sufficiently to remove an eventually active anti-bacterial action, which originates from the fluids of the human or animal body. The moment for the second determination should in each special case be fixed experimentally. The difference in time must not be too short, else the increase of living organisms is too slight to produce a distinct augmentation of the number of microscopically counted bacteria; and, again, the quickness of increase of the living individuals corresponds with the nature of the food, the presence of anti-bacterial influences, etc. Nor must this period be taken too long because finally the dead individuals decompose and vanish.

If the excess of microscopically counted bacteria prove to have completely died, the sterility-index is directly found by diminishing the original proportional number with 1; this is for instance the case in human faeces [1]. But when only a fraction died, so that part of the microscopically counted organisms are alive but do not develop on our usual culture-media, then, for the estimation of the sterility-index from the data found, it is admitted that the living but not cultivable individuals have multiplied during the period of observation in the same measure as the cultivable ones.

The determination of the sterility-index might also directly be made *microscopically*, if the bacteria that have died already before, could be distinguished by microscopically perceptible changes from those organisms which in the beginning were alive. Indeed, after death there occur modifications in the bacterial bodies, at least in the intestinal canal of the rabbit, which considerably alter the pigment-absorbing faculty of these organisms. In general three stadia of decomposition may be distinguished:

1st. The *granule-stadium*, where one or two (seldom more) very darkly coloured granules may be observed in a for the rest lightly coloured stroma of the bacterial body;

2nd. The *shade-stadium*, the dark granules are no more present and the bacterial stroma is still lighter coloured; and

3rd. The *membrane-stadium* in which the bacterial stroma absorbs no pigment at all; at least there remains nothing but a fine coloured line, which still very markedly and sharply indicates the margin of the original organism.

The greater part of these decomposing bacteria are so fragile that they can but be observed by the use of the delicate treatment of the „moist staining"; when applying KOCH's staining-method they mostly fall asunder at the drying and flaming of the preparations and then form what has been by some described as bacterial detritus.

These post-mortem bacterial phenomena cannot however at first be used for the determination of the index of sterility, as they do not appear directly, but only some time after the death of the

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[1] See my Paper „Bacteriological investigations of human faeces", Proceedings Royal Acad. of Sciences, Amsterdam. Vol. IX, p. 57.

bacteria; hence a greater number of dark and evenly coloured organisms are microscopically detected than corresponds with the number of living bacteria.

Perfectly healthy rabbits were killed by the blow on the neck; directly after death the abdomen was opened, the bowels were bound off with sterilised silk threads and the contents of the different parts of the intestine were introduced into sterilised mortars under aseptic precautions. By means of sterilised pestles the substance was evenly mixed in the mortar; when making the dilutions the intestinal contents were for some time thoroughly shaken in sterile flasks with porcelain balls. In particular for the contents of the small intestine which mostly consist of a viscid, sticky mass, in which the lower organisms are most unevenly distributed, such a laborious manipulation cannot be dispensed with in order to get a homogeneous emulsion. Nearly always all the contents of each portion of the intestine were used for the research, with exception only of the Coecum, of which after previous emulsion, 10 grs. at least were used.

In the different parts of the intestinal canal there now appeared to be a great disproportion between the number of cultivated and that of microscopically counted bacteria; the culture under other circumstances or on other nutrient media had no perceptible influence on the surplus of microscopically countable bacteria.

TABLE I.

Parts of the intestine of rabbit nº 2.	Times of the research.	Number of bact. found by micr. counting in 1 mgr. of contents.	Increase with	Numbers of bacteria found by culture in 1 mgr. of contents.	Increase with
Small intestine...........	directly	685.000	206.000	[1 organism on 5 mgrs.]	275.000
	24 hours at 37° C.	891.000		275.000	
Coec. and Proc. vermiform.	directly	26.975.300	more than 12½ millions	136	more than 12½ millions
	24 hours at 37° C.	39.792.000		12.558.700	
Large intestine and Rectum	directly	8.565.000	nearly 9½ millions	67	nearly 9½ millions
	24 hours at 37° C	17.937.000		9.432.000	

When the sterility-index of the different parts of the intestinal canal is determined it appears that the great excess of microscopically countable bacteria *as a whole* consists of dead organisms; hence, the sterility-index can be directly found by diminishing the proportional number with 1.

TABLE II.

Rabbit n° 5.

Parts of the intestine.	Contents (in grams).	Solid subst. in pCt.	Total numbers.		In 1 mgr. of contents.		Sterility-index.	
			Microscop. counted.	Cultiv.	Microscop.	Cultiv.		
Small intestine (upper portion)	18	7.89	9.550.967.000	[5.940]	530.600	[about **1** on **3** mgrs.]	1.607.905	169.942
Id. (downmost port. **1** M.)	3.5	8.49	2.870.175.000	67.150	820.000	19	42.741	
Coec, Pr. vermif. and Col. adsc.	111	15.61	7.555.215 000.000	7.770.000	68.065.000	70	972.356	
Large intestine and Rectum . .	10	34.26	176.070 000.000	255.800	17.607.000	25	688 310	

When considering the column „cultivated" organisms, quite the same relations are met with as described till now: very few bacteria in the first portion of the small intestine; the number increases in the end of it, but a vigorous augmentation appears in the Coecum, Processus vermiformis, and Colon adschendens, whilst in the rest of large intestine and Rectum again a decrease may be observed.

Quite otherwise, however, the image becomes when viewing at the last column, representing the sterility-indices of the different portions of the intestine: whilst the sterility-index of the whole small intestine is 169942, that index mounts in the Coecum to 972356; the increase in number of living organisms in the Coecum is accordingly only an apparent one; on the contrary, instead of an increase we find in the Coecum, with a more than five times greater index of sterility, that no less than 80 pCt. of the original number of bacteria which arrived living in the Coecum are dead.

In the rest of the large intestine and in the Rectum the sterility-index (688310) has become a little lower than in the Coecum. It may become lower by an increase of the number of living individuals, but also by the decrease of dead organisms; the latter being the case here. The proportion of solid substance in large intestine and Rectum has mounted to 34.26 pCt., thus more than two times that of the Coecum, hence, we might be expect here per mgr. at least twice as many microscopically countable organisms. Instead of 2 × 68.065000, however, only 17.607000, are found, which consequently proves that a large number of dead bacteria are decomposed and have disappeared, yet a considerable number of bacteria that had remained in the Coecum, have died, as this number should at least have mounted to 2 × 70 per mgr. whilst only 25 per mgr. were found.

In the whole small intestine no ingesta were present. The number of microscopically counted bacteria in the hindmost part of the small intestine, is in accordance with the higher rate of solid substance somewhat greater per mgr. than in the superior part; still the sterility-index of the former is much lower than in the latter. Hence, in that first portion there must have been a considerable dying of bacteria, whilst, as a matter of course, the 2nd part has only for a shorter time been free from ingesta and accordingly also contains a greater number of living organisms. After the passing of the ingesta the living organisms thus die off largely in the small intestine, so that the number of living individuals there may finally grow very small; and at the irregular distribution of the bacteria in the viscid-mucous substance, there may then be found relatively large sterile quantities (e. g. loops of some mgrs.). Still the contents of the small intestine never prove quite sterile; an absolute sterility of the small intestine (so-called auto-sterilisation) is not attained in any case. To be convinced of this it is only necessary to use a more refined method

TABLE III.

Rabbit n° 8.

Parts of the intestine.	Contents [in grams].	Solid substance in pCt.	Total numbers.		In 1 mgr. of contents.		Sterility-index.	
			Microscopically counted.	Cultiv.	Counted.	Cultiv.		
Duodenum and Jejunum	6.250	6.07	4.969.593.750	1.581	795.135	1 organism on 4 mgrs.	3.140.791	2.586.626
Ileum....................	11 500	8.37	13.517.031.000	5.566	1.175.394	1 organism on 2 mgrs	2.428.499	
Coecum, Pr. vermif. and Col. adsc.	105.500	13.84	5.977.793.525.000	632.472	56.661.550	6	9.451.466	
Large intestine and Rectum	5.500	28.33	190.329 521 250	84.095	34.605.367	15	2.263.266	

TABLE IV.

Rabbit n° 4.

Parts of the intestine.	Contents [in grams].	Solid substance in pCt	Total numbers		In 1 mgr. of contents.		Sterility-index.	
			Microscopically counted.	Cultiv.	Counted.	Cultiv.		
Small intestine (upper part)..	12.250 (ingesta)	13.60	8.189.500.000	1.447.674	668.500	118	5.656	8.180
id. (downmost 75 cm.)	4.750	15.31	3.696.385.000	5.058	778.000	1	730.798	
Coecum, Pr. vermif. and Col. adsc.	127.200	22.73	607.316.400.000	3.816.000	4.774.000	30	159.149	
Large intestine and Rectum.	7.500	40.23	50.599 000.000	468.000	6.746.000	62	108.116	

of examination. If all the contents of the small intestine in which there are but few living bacteria are placed at 37° C., in dilute state, in order to remove an eventually present anti-bacterial action, so that those few bacteria are allowed to propagate, then, after a

TABLE V.

Rabbit n° 3

Parts of the intestine.	Contents [in grams].	Solid substance in pCt.	Total numbers.		In 1 mgr. of contents.		Sterility-index.	
			Microscopic. counted.	Cultiv.	Counted.	Cultiv.		
Small intestine (Superior part)	12	8.93	2.509.125.000	—	209.000	—	—	63.392
Small intestine (Inferior 75 cms.)	4 (ingesta)	11.63	1.317 900 000	60.465	329.000	15	21.795	
Coecum, Pr. vermif. and Col. adsc.	82	24.55	5.191.109 162.000	10.297.560	63.306.209	126	504.109	
Large intestine and Rectum	6	40	34.189.921.000	1.416.000	5.698.320	236	24.144	

TABLE VI.

Rabbit n° 7.

Parts of the intestine.	Contents [in grams].	Solid substance in pCt.	Total numbers.		In 1 mgr. of contents.		Sterility-index.	
			Microscopic. counted.	Cultiv.	Counted.	Cultiv.		
Duodenum and Jejunum.....	27.5	8.17	19.963.000.000	15.125	726.000	about **1** on **2** mgrs	1.320.000	465.847
Ileum................	39 (ingesta	11.19	48.537.060.000	131.923	1.244.540	3	368.931	
Coecum, Pr. vermif. and Col. adsc	109.3	22 70	895.524.135.000	1.789.360	8.193.267	16	499 893	
Large intestine and Rectum .	25.2	35.71	140.260.428.000	263.340	5.565.890	10	532.620	

certain number of hours, a great many organisms are found. Whilst at first in the superior part of the small intestine on each quantity of 3 mgrs. only one living bacterium was to be found, the same quantity, placed in a dilution of 1 : 11 at 37° C., after 21 hours, and estim-

TABLE VII.

Rabbit n° 6.

Parts of the intestine.	Contents (in grams).	Solid substance in pCt.	Total numbers.		In 1 mgr. of contents.		Sterility-index.	
			Microscopic. counted.	Cultiv.	Counted.	Cultiv.		
Duodenum and Jejunum. . .	12.5	9.53	24.199.725.000	—	1 936 000	not **1** on **6** mgrs.	—	2.2120533
Ileum.	30 (ingesta)	12.96	280.021.500.000	137.499	9.334.000	4	2 036.534	
Coecum, Pr. vermif. and Col adsc.	81	22 03	6.098.909.256.000	1.113.750	75.295 176	13	5.476.011	
Large intestine and Rectum .	9.5	39.45	124.577.055.000	136 875	13.133.374	14	910.151	

ated for 1 mgr. of the original contents, showed by means of culture-method 2.660.000 living organisms. If the small number of living bacteria present in the small intestine in absence of ingesta, is taken into consideration it is commonly possible, either by plate culture or by different dilutions in bouillon, to determine the number of these living bacteria.

In rabbit n⁰. 8 (Tab. III), without ingesta in the small intestine, quite the same relations are observed. Probably the Ileum here already for a long time contained no ingesta; hence the sterility-index is higher than in the former case. The Duodenum and Jejunum which have undoubtedly already for a longer time been free from ingesta possess a still higher sterility-index. In the Coecum, Processus vermiformis, and Colon adschendens the sterility-index again rises much, whilst in the rest of the large intestine and the Rectum again decomposition of dead organisms has taken place, and, in relation to the rate of solid substance, neither increase nor diminution of living bacteria has occurred. (Table IV, p. 483).

If ingesta are present in the superior portion of the small intestine, the sterility-index is of course very low; the second part of the small intestine which for a longer time already contained no ingesta, shows a very high index. In the Coecum the index is again many times higher than in the whole small intestine.

If ingesta are present in the Ileum, whilst Duodenum and Jejunum are devoid of them, (Tab. V, VI and VII), these ingesta prove still to contain a great number of living organisms; the sterility-index is then low there; in the Coecum, Proc. vermiformis, and Colon adschendens it is, however, always higher than in the whole small intestine. As the ingesta from the second part of the small intestine arrive at the Coecum with a relatively low index, the dying of living bacteria there is evidently much more considerable than might be supposed from the comparison of the sterility-index of the Coecum with that of the whole small intestine, especially if the latter contain no ingesta at all.

Accordingly the bacterial population in the intestinal canal of the rabbit has the following course. With the ingesta large numbers of living and dead bacteria come from the stomach into the small intestine. As they move on, a number of living bacteria that have remained in the parts of the small intestine devoid of ingesta, die off, without these parts, however, becoming altogether sterile. When the ingesta have entered the large intestine the same process takes place in the inferior part of the small intestine. Furthermore, in the Coecum, Processus vermiformis, and Colon adschendens there is a dying on great scale of living lower organisms. In the rest of the large intestine and the Rectum in not one case increase is observed, in most cases on the contrary, a continuous dying away. From the Duodenum down to the Rectum nowhere an increase of bacteria; mostly from beginning to end throughout the intestinal canal a continuous annihilation of the living bacteria. Bacteria are in the intestines of a rabbit as an army passing through the country of the enemy, being continually decimated.

In the Coecum, Processus vermiformis, and Colon adschendens the greatest mortality by far is observed. This may be nearer pointed out if, using one and the same dilution, the living organisms present in the different parts of the intestinal canal are compared. The after-action of the influences which during the life of the animal provide for the extermination of the bacteria in the intestinal canal, also proves to be most vigorous in the Coecum, Processus vermiformis, and Colon adschendens.

TABLE VIII.

N⁰. of the rabbit.	Parts of the intestine.	Time at 37⁰ C.	Degree of dilution.	Original number in 1 mgr. of contents.	Final number in 1 mgr. of contents.	Number of times that the original number has increased.
8.	Duodenum and Jejun.	$9^1/_2$ hours	1 : 33	1on4mgrs	96	384
	Ileum.	id.	1 : 33	1on2mgrs	673	1.346
	Coecum, etc.	id.	1 : 33	6	137	23
	Large intest. and Rect.	id.	1 : 33	15	1.195	79
7.	Duod. and Jejunum.	20 hours	1 : 11	1on2mgrs	21.000	42.000
	Ileum.	id.	1 : 11	3	243.000	81.000
	Coecum, etc.	id.	1 : 11	16	108.000	6.750
	Large intest. and Rect.	id.	1 : 11	10	130.000	13.000
5.	Duod. and Jejunum.	21 hours	1 : 11	1on3mgrs	2.660.000	7.980.000
	Ileum.	id.	1 : 11	19	1.543.000	81.000
	Coecum, etc.	id.	1 : 11	70	726.000	10.300
	Large intest. and Rect.	id.	1 : 11	25	1.831.000	73.000

Notwithstanding in most cases more living organisms per mgr. are originally to be found in the Coecum than in the other parts of the intestinal canal, they multiply in equal times, much less vigorously in the former than in the latter.

If thus we arrive at the conclusion that in not one locality of the intestinal canal an increase of lower organisms is seen, there can be no question of that canal having "a flora of its own", nor of a distinction between "obligative" and "facultative" intestinal bacteria. Indeed, we find in the whole intestinal canal the same species of organisms, chiefly Coli and Coliform bacteria, probably because these organisms are so universally distributed in nature and, accordingly, with the food get in large numbers into the intestinal canal. Furthermore, because by their great resistance, — which is also shown by their ubiquity in nature, — they are able longest to resist the anti-bacterial influences of the intestinal canal. At those places of the small intestine where ingesta are present, there are commonly found a number of other bacteria too. On the culture-plates these species, either by the quickness with which they liquefy the gelatin, or by the great number in which they are eventually present, may prevent the appearance of Coliform bacteria. The greater part of these organisms possess, however, a much slighter power of resistance than the bacteria of the Coli-group, and thus die first, so that in the Coecum, Processus vermiformis, and Colon adschendens chiefly part of these Coliform bacteria alone are found in living state.

Putrefaction is excluded from the intestinal canal of the rabbit, bacteria not increasing there at all.

TABLE IX.

Rabbit	Contents in grams.	Total numbers in the whole intestinal canal.		Sterility-index of all the contents of the intestine.	On 1 million of dead organisms are alive.
		Microscopic. counted.	Cultiv.		
N° 1.	70	945.727.500.000	14.651.000	64.549	15
» 2.	125.250	2.784.775.050.000	14.298.700	194.756	5
» 3.	104	5.229.126.108.000	11 774.025	444.123	2
» 4.	151.700	669.801.285.000	5.736.732	116.755	8
» 5.	142.500	7.743.706.142.000	8.100.375	955.967	1
» 6.	133	6.527.707.536.000	1.388.124	4.702.538	0.2
» 7.	201	1.104.286.623.000	2.199.748	502.004	2
» 8.	128.750	6.186.609.671.000	723.714	8.548.417	0.1

Finaliy we come to the last question, of great signification for the physiological bacteriology of the intestinal canal: Have the micro-organisms of the intestinal canal of the rabbit to play a part in the digestion?

If attention is paid to the following facts:

1°. The very small number of living bacteria with respect to the number of grams of intestinal contents;

2°. The very small number of living bacteria with regard to the number of dead ones, in particular perceptible from the high sterility-indices of the whole intestinal canal, and from the slight number of living bacteria found on 1 million of dead organisms, and

3°. That at no single place there is a multiplication, on the contrary, that nearly in the whole intestinal canal there is a mortality on large scale of living bacteria, we are obliged to deny the bacteria playing any part in the digestion in the intestinal canal of the rabbit.

Mathematics. — „*On the motion of variable systems*" by Prof. Cardinaal.

1. With considerations relative to the theory of motion, we generally start from the principle that two phases of the system are congruent. If the two systems considered in this way are situated in a plane the pole of the motion is the only real point of coincidence of the two systems; if they are situated in space the principal axis of the motion is their line of coincidence. If we suppose the second system to have approached the first at infinitesimal distance, the rays connecting the homologous points are directions of velocities and one of the principal problems of motion consists of the construction of the directions of these velocities. Special constructions exist for this, the second system not being suitable for use.

2. In the plane the construction of the direction of velocities is a simple matter, the polar rays being normals to the orbits and the velocities touching them. In space the construction becomes already more elaborate; however, we can notice that the directions of velocities are the rays of a tetraedral complex; to this complex belongs moreover a focal system, the properties of which enable us to find the points belonging to these rays. This paper now purposes to investigate this same subject for systems of points in space changing projectively during their motion. The investigation is independent of the length of the velocities; however it is necessary to investigate more closely the above-mentioned tetraedral complex.

3. When a system remaining congruent to itself is displaced, two opposite edges of the tetraeder of coincidence (principal tetraeder) are real: the principal axis l and the line l_∞ at infinity common to the planes normal to l. Now there must be on l as well as on l_∞ two real or imaginary vertices; if we call the first pair P, Q and the second R,S, then P,Q are the double points of two congruent ranges of points on l and therefore united in *one* point at infinity; R, S are the double points of two congruent systems in the plane λ_∞ at infinity, therefore two cyclic points of any plane normal to l.

From this ensues:

Of the four vertices of the principal tetraeder of the complex two coincide in λ_∞, the two other being the cyclic points of a plane normal to l; of the faces of this tetraeder two likewise coincide in λ_∞.

Suppose the direction of a velocity v_a is given and we wish to construct the point A possessing this direction of velocity; then we have to construct the line d of shortest distance from l and v_a; this cuts v_a in A. If we bring a plane α through d normal to v_a then this plane is normal in A to the locus of A; the rays through A in α are the normals to the orbits and at the same time rays of a focal system having l and l_∞ as conjugate polars. Thus the focal system is connected with the tetraedral complex.

4. Let us now suppose, that the system in motion changes projectively. If we imagine two positions of the system, then the points of coincidence are the vertices of the principal tetraeder $PQRS$, which tetraeder we suppose for our further consideration to be constructed and for the present entirely real. The tetraedral complex is determined by the principal tetraeder and the line connecting one pair of homologous points. This ray, however, is not only the line connecting two homologous points; the same complex appears, when it is regarded as the bearer of ∞^2 pairs of homologous points.

If now this ray is the direction of velocity v_a of a point A of this system, it is evident that for the determination of this point further conditions must be introduced, namely such as permit the construction of the point A as well as of the plane α.

5. In the first place a pair of opposite edges of the tetraeder must be conjugate polars of a focal system; PQ and RS to be taken for these. This condition, however, is not yet sufficient, as in the

preceding problem d was not only normal to l but also to v_a. The plane ld constructed according to the supposition made there and a plane through l parallel to v_a are normal to each other and are thus conjugate in respect to the planes through l and the cyclic points in the plane normal to l; by applying this last named principle to the case of projectively altering systems, we obtain the following construction for the point A, whose direction of velocity is v_a.

Suppose the given direction of velocity v_a cuts the plane PRS in the point L; construct in this plane the ray PL' harmonically conjugate to PL with respect to PR and PS, which cuts RS in L'; bring through L' a ray cutting PQ and v_a; then this ray will cut v_a in the point A possessing the given direction of velocity.

For the determination of the focal system the construction of the ray $L'A$ is not sufficient, as we know of the focal plane α belonging to A only that it passes through $L'A$. To determine α entirely we have to notice that with the motion of congruent systems α cuts the plane λ_∞ in the polar of the point of intersection L of v_a and λ_∞ in respect to the imaginary circle in λ_∞. This circle passes through the imaginary vertices in λ_∞ of the principal tetraeder and the point of intersection of l and λ_∞ is the centre of it. By applying these principles to the case of systems changing projectively, we obtain the following construction.

We assume a conic K^2, touching PR and PS in R and S; we construct the polar p of L through L' in reference to K^2 and we bring the plane α through A and p; now α is the focal plane of A. So with the motion of systems projectively varying the complex of rays and the focal system are connected with each other.

6. The edges PQ and RS determine with v_a a hyperboloid H^2, on which the polar of v_a in reference to the focal system is also situated. This cuts the plane PRS besides in RS also in PL; so from this ensues that the polar of v_a relatively to the focal system cuts the plane PRS in a point of PL. It is then easy to see, that PL is the polar of L' relatively to K^2.

7. Not until the inverse problems are solved, are the constructions complete, thus (a) when for each point the direction of velocity and the focal plane are constructed, (b) when for each plane the focus and the direction of velocity of this focus are constructed. We suppose in these constructions the complex of rays to be determined and K^2 moreover constructed.

a. Given point A. Draw through A the line cutting PQ and

RS, the latter in L'; construct PL harmonically conjugate to PL' relatively to PR and PS; according to the preceding L must be determined on PL. When the complex of rays is determined the ray through A must cut not only PL, but also a ray in the plane QRS belonging to the pencil of rays with centre Q, projective to the pencil P/RSL; this ray QL'' corresponds to PL. So we have to determine this ray QL'' according to the known projectivity of the pencils with the centres P and Q and to construct the line through A cutting PL and QL''; by this the plane α is at the same time known.

b. Given α; the connection of the point of intersection of α and PQ with the point of intersection L' of α and RS produces a focal ray. We determine farthermore the pole L of the line of intersection of α and PRS relatively to K^2, construct the ray QL'' corresponding to the ray PL in the two projective pencils P/RSL, Q/RSL'' and bring through L a right line, cutting QL'' and the constructed focal ray. This right line is the direction v_a and its point of intersection with the focal ray is A.

8. The preceding considerations point to a connection existing between the investigations of A. SCHOENFLIES "Geometrie der Bewegung" pages 79—129 and those of L. BURMESTER "Kinematisch geometrische Untersuchung der gesetzmässig veränderlichen Systeme", Zeitschrift für Mathematik und Physik, vol. 20, pag. 395—405. The former treats very completely of the constructions ensuing from the focal system and the tetraedral complex belonging to it, when the system remains congruent to itself during its motion; the latter assumes the projective variability of the moving systems, but does not make use of the focal system.

It would not be difficult to give a more general form to most constructions appearing in the former consideration; this would however give rise to unnecessary repetitions; so it will be sufficient if this is shown in a single example.

9. To do so we take the construction corresponding to that of the characteristic of invariable systems; so the question is to determine according to the foregoing principles in the plane α the right line a containing the points the directions of velocities of which lie in α. For this a must be the line of intersection of two homologous planes of the two systems at infinitesimal distance of each other; so if we think on v_a the point A to be determined and the point A' at an infinitesimal distance of it, and the planes α and α' to be

constructed, then a proves to be the polar of v_a relatively to the focal system.

According to (6) a passes through a point of PL, moreover a lies in α, so it has a point in common with the line of intersection of the planes α and PRS which is the polar P of L relatively to K^2; from this ensues:

The polar a of v_a relatively to the focal system cuts the plane PRS in the pole of LL' relatively to K^2 and lies on a hyperboloid of which PQ, RS and v_a are three generators.

10. Up till now we have supposed the four vertices of the principal tetraeder to be real. The constructions, however, can still be performed if we assume that the two vertices RS are conjugate imaginary. The two edges PQ and RS namely remain real, also the planes PRS and QRS, the imaginary edges PR and RS are represented as imaginary double rays of an elliptic involution of rays with the centre P; the construction of the conic K^2 and of the polar p remains however possible and therefore also the remaining constructions of the complex of rays and of the focal system.

If, however, the four vertices of the principal tetraeder are imaginary, then the construction can no longer be performed, because according to the preceding it ought to take place in a plane (PRS), which becomes imaginary itself. As the constructions treated of here will also be considered from another point of view, this case shall for the present remain unnoticed.

11. In the theory of the motion of an invariable system we imagine cylinders of revolution to be described round the principal axis. If one of these cylinders is constructed, the velocities touching these cross the principal axis under the same angle, so that they are tangents to helices of definite inclination. Let us now find out the analogon of these cylinders in the motion of projectively changing systems and let us to do so return to the formerly (5) constructed rays PL and PL' which are harmonically conjugate with respect to PR and PS.

We suppose furthermore a quadratic cone C^2 to be constructed, the vertex of which is P which touches the planes PQR and PQS of the tetraeder according to the edges PR and PS; then the planes PQL and PQL' are conjugate polar planes of C^2. If we now bring a tangent plane to C^2 through PL, this touches the cone according to a generator lying in the plane PQL'; from this ensues:

The right line d through L', cutting PQ and v_a, also cuts the generator according to which the tangent plane through PL touches C^2. If C^2 is constructed in such a way that the plane through PL and v_a touches it, d cuts v_a in the point of contact with C^2. We can thus make a geometric image of all directions of velocities namely in the following way:

Given a ray v_a of the complex, cutting plane PRS in L; construct a cone C^2, having P as vertex, touching the planes PQR and PQS according to PR and PS and touching moreover v_a. Construct the harmonic ray PL'; then the rays through L' cutting PQ, also cut the complex rays through L in the points of contact with C^2. If we construct all the rays of the complex, a pencil of cones is formed; to each tangent plane PLv_a belongs a cone and a pencil of complex rays through a point of PL.

So whilst to each tangent plane a cone belongs, two tangent planes PLv_a, PLv'_a belong to each cone; the ray d through L' cutting PQ and v_a, also cuts v'_a in a point of contact with C^2.

The planes dPQ, dRS, dv_a, dv'_a form a harmonic pencil.

12. Finally a few general observations may be in their place at the conclusion of this communication.

a. It is clear, that if we consider the four vertices P,Q,R,S as the points of coincidence of two projective systems, each of these points plays the same part; by regarding, as was done in the beginning, the edges PQ and RS as conjugate polars of a focal system a limiting condition has been introduced.

And the introduction of this condition is allowed as the principal tetraeder and one direction of a velocity do not determine, the position of the homologous points of two projective systems though they determine the complex of rays. By the second assumption, that of the conic K^2, the focal system is determined. As it is possible to choose in three different ways a pair of edges as conjugate polars and moreover the point of intersection L can be assumed in two different planes, the point A can be determined in twelve different ways on a direction of velocity v_a.

b. The number of solutions for the determination of the point A on the direction of velocity v_a diminishes, when two of the vertices, say R, S are imaginary. So PQ and RS form the only possible pair of opposite edges. The point of intersection L can be determined in two faces (PRS and QRS). If now also the points P and Q coincide as is the case for the motion of invariable systems, only one solution is possible.

c. The entire preceding consideration is independent of the length of the velocities. It is also possible to find constructions for which use is made of that length. This will be done in a following communication.

Physics. — „*On the measurement of very low temperatures IV. Comparison of the platinum thermometer with the hydrogen thermometer*," by B. MEILINK. (Communication n°. 77 from the Physical Laboratory at Leiden by Prof. H. KAMERLINGH ONNES).

§ 1. The research on the relation between the electric resistance of platinum and the temperature (below zero) considered in this paper forms a part of a more general investigation including some other metals, which will be treated in following papers. It seemed advisable to divide this entire investigation into two parts, viz.

1st to express as accurately as possible the resistance as a function of the temperature for a standard metal, for which purpose platinum offers great advantages;

2nd to express as accurately as possible the variation due to temperature, in the ratio between the resistances of several perfectly pure metals or metals with certain known small impurities and the resistance of the standard metal (platinum).

The absolute determinations mentioned under 1 consist of a comparison of the platinum thermometer with the hydrogen thermometer, a subject with which OLSZEWSKI [1]), HOLBORN and WIEN [2]), DEWAR and FLEMING and lately HOLBORN [3]) again have been occupied.

§ 2. *Arrangement of the resistance thermometer.*

The hydrogen thermometer with which my resistances have been compared, is described in Communication n° 27, viz. the one with the reservoir of 90 cc. In the main the observations are made as related in n°. 60. From the agreement of the pressure coefficient of hydrogen, deduced from the investigations described there, with that of CHAPPUIS, it appears that a high degree of accuracy can be attained in this way. Under favourable circumstances the temperature can be determined to within 0.02 deg. C.; even for the boiling-point of oxygen the uncertainty remains less than 0.03 deg. C. In terms of resistance this corresponds to about $^1/_{30}$ % at —180° C. and $^1/_{100}$ % at 0° C. Therefore the determinations of resistance were arranged so that this accuracy could also be attained with the absolute determinations.

Even if we increase without special arrangements the accuracy of the determinations of resistance, and perhaps also that of the

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[1]) Akademie der Wissenschaften in Krakau. Juni 1895.

[2]) Wied Ann. 59. S. 222. 1896.

[3]) Ann. der Physik 6. S. 252. 1901.

measurements of temperature by the hydrogen thermometer, this is not of much use as long as we are not correspondingly certain of the equality of the temperature in the resistance and the hydrogen thermometer. This equality was obtained by immersing the hydrogen thermometer together with the resistance to be compared into the bath of liquified gas in the cryostate (described in Comm. 54). Therefore in the construction of the resistance thermometer and in the arrangement of the measurements of resistance the circumstances under which the resistance is employed are taken into account.

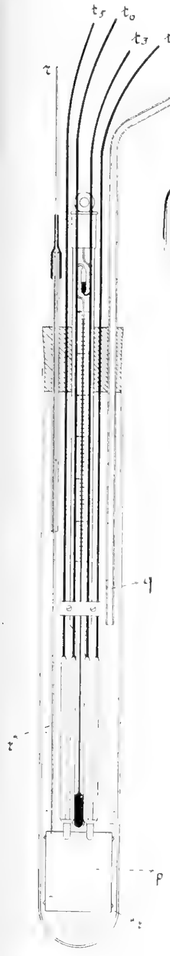
In order to bring the platinum wire and the hydrogen thermometer as much as possible under the same conditions, the former was wound round a hollow cylinder, into which the reservoir of the hydrogen thermometer was placed. This form of the resistance is also a very appropriate one to measure the temperature of the same vessel, when other apparatus than the hydrogen thermometer occupy the place usually occupied by the latter. Even with a high resistance a thermometer of this form occupies a very small space, and the remainder of the vessel is thus left very satisfactorily free for other experiments.

In order to obtain reliable measurements with the hydrogen thermometer, the latter is kept for a long time at a constant temperature and the operations required for this are simplified when the wire follows directly fluctuations or variations of the temperature of the vessel, even when they are very small, and indicates them in the reading of the galvanometer. This is obtained by bringing the wire into immediate contact with the liquefied gas. Also the heat due to the Joule effect and — a question to which the proper attention has not always been paid — the influence of radiation and of conduction of heat along the leads are diminished by this as much as possible. There is no objection to using a naked wire in the case of platinum, with regard to any action on the substance itself. But the difficulty remains that in using naked wires (comp. Commun. n⁰. 27 § 10) care must be taken that the bath into which the platinum resistance is immersed should be a better insulator than otherwise. Therefore in the construction of the resistance thermometer it was desirable to avoid the use of ebonite or similar substances liable to be attacked more or less by liquefied gases or by the liquid bath used in the determination of the zero.

This has been obtained very satisfactorily by constructing a resistance, exclusively of glass and metal, which even in boiling oxygen amounts to 30 Ohms.

On a glass cylinder (see fig. 1) height 50 m.m., outer diameter

B. MEILINK. „On the measurement of very low temperatures IV. Comparison of the platinum thermometer with the hydrogen thermometer." Plate I.



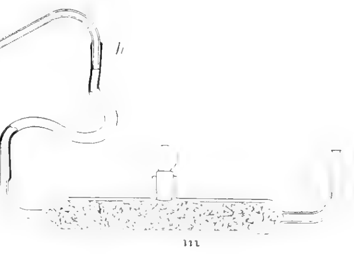


Fig. 4.

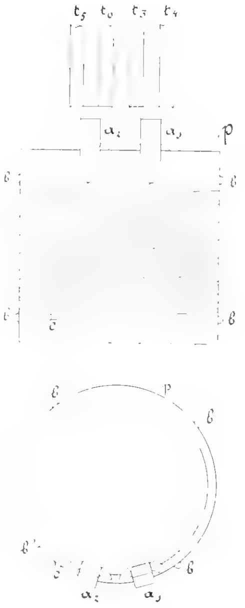


Fig. 1.

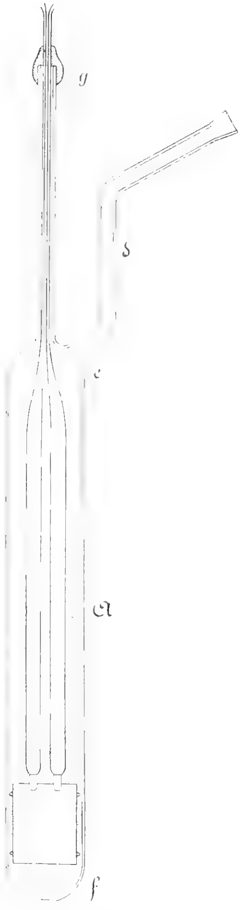


Fig. 3.

48 m.m. a screw thread was etched as deep as possible [1]). On the upper end, at a distance of 1 c.m., two copper blocks a_1 and a_2 are fastened to the glass by means of screws passing through holes. In order to avoid uneven tension some leaves of tin-foil were laid between the blocks and the glass. To prevent the blocks from turning round the screws they rest on the rim by means of two wings. Finally to prevent the wire from rubbing against the wall of the vessel (see fig. 3) protruding points of glass *b* were made at the lower and the upper end of the cylinder. The wire was wound double, for which purpose another projecting point of glass was sealed on at *c*, round which the wire was turned, the ends of this wire were soldered on to the blocks, where also the leads were fastened. Hence the wire was free from induction which was a great advantage in the measurements with the WHEATSTONE's bridge although on the contrary the conduction through the liquid has now a much greater influence and hence the bath had to be kept very pure. If the wire is wound on two concentric cylinders, the thermometer occupies more space, but then the insulating power of the liquid need not be so great as was required for this thermometer. We tried to make the resistance so high that the measurements could be sufficiently accurate even at the lowest temperature observed; this was attained both by taking a long wire and by choosing it as thin (0.1 m.m.) as could be done with advantange without making the surface too large in proportion to the resistance of the wire. As the numerous (50) turns are distributed over the whole height of the cylinder, their resistance expresses as well as possible the mean temperature of the bath over a large range, if this should not be the same everywhere.

Four copper leads are used, 2 at each end. This number is required in the measurements with the WHEATSTONE's bridge for the entire elimination of the resistance of the leads. Moreover this gives an opportunity for measurements by means of the differential galvanometer and similar apparatus.

For the first experiments the leads were 0.5 m.m. in diameter. But the variation of resistance of these relatively thin wires sometimes proved too great in the measurements, so that later copper wires of 2 m.m. diameter were taken. The ends of these in the bath had been flattened over a length of 11 c.m. to 5 m.m.'s width and 0.15 m.m.'s thickness, which prevented the increased conduction of heat along them from being of any influence.

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[1]) For the resistances used later, similar screw threads were ground in the glass.

Fig. 3 represents the resistance mounted in the bath for the determination of the zero. (comp. § 4).

§ 3. *Measurements with the* Wheatstone's *bridge.*

The resistance was measured by means of a Wheatstone's bridge, of which two branches were formed by coils of manganin wire. For this purpose pairs of about 7, 14 and 21 Ohms are wound, which may be introduced separately or together. The ratio of these resistances has been several times determined at different periods. Values were found for it, which differed by less than $^1/_{100}$ $^0/_0$.

The third branch was a resistance box of Hartmann and Braun, also of manganin wire. This has been tested at the Reichsanstalt, where 0.00001 was given as the co-efficient of temperature, so that no correction for the temperature had to be applied. A resistance box of Siemens and Halske was arranged in parellel in order to apply variations of resistance smaller than 0.1 Ohm.

In determining the dependance of the resistance on the temperature the absolute values of the resistances are of less moment than their ratios. The latter have been determined a few times for the resistance box in the ordinary way and corrections have been applied wherever necessary.

In the measurements with the Wheatstone's bridge the contact resistances, if the resistance to be measured is small, may become of great influence. Even when all possible care is taken to keep the plugs clean and to tighten them well, yet with ordinary stops the resistance is not certain to less than 0.0005 Ohm. Therefore it was important that the platinum wire even at —180° C. should still have a resistance of about 30 Ohms, as in comparison with this resistance the connecting resistances may be entirely neglected.

The connecting resistances in other parts of the circuit which may vary considerably from day tot day, were always eliminated as indicated by Callendar. I took 4 wires instead of 3 in order to render the elimination more complete. The connections required were made possible by a commutator, made of copper in order to avoid thermo-electric effects — they occur for instance to a high degree with contacts in mercury cups. The annexed fig. 2 represents diagrammatically the commutator with its connections.

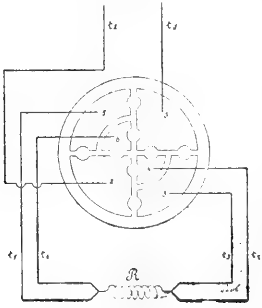

Fig. 2.

By means of the blocks 1 and 2 the resistance to be determined is included in one branch of the **Wheatstone**'s bridge, 3 and 4 are connected to one pair of leads t_3 and t_4, 5 and 6 to the other t_5 and t_6.

By a proper arrangement of the plugs we measure successively 1st $t_3 + t_5 + R.$, 2nd $t_4 + t_6 + R$, 3rd $t_3 + t_4$, 4°. $t_5 + t_6$.

It is clear how the value of R will follow from this.

No uncertainty remains here but that caused by the differences in the connecting resistances of the plugs in the commutator and of the resistance box used in the measurement.

The galvanometer was a thick wire **Thomson** with a resistance of about 1.2 Ohm, and a sensitiveness of 1 m.m. at a distance of 3 M. for 2.10^{-8} Ampère. Period of oscillation about 10″. The galvanometer was suspended after the manner of **Julius** in a stand on a fixed pillar. So the zero was very quiet. Thermo-currents were so small that they could be neglected. They have been prevented as much as possible by making contacts of copper on copper and by bringing the soldering places of the platinum wire as close as possible together. In the branch of the galvanometer was a commutator with mercury contacts as described in Communication N°. 27.

§ 4. *The determination of the zero.*

The determination of the resistance at 0° C. offered at first many difficulties.

It is absolutely necessary to enclose the wire in a glass vessel containing a liquid, as otherwise we have no certainty that it really becomes 0° C., when this vessel is placed in ice. Paraffin oil or petroleum as mentioned in § 10 of Communication N°. 27, proved unsuitable, because it was difficult to clean the wire again. It is a good expedient to use a liquid which boils at a temperature a little above zero, and which by moderate heating under diminished pressure can be removed altogether. As such isopentane and amylene were used.

In order that this liquid should leave nothing behind on the wire after evaporation, it is necessary to transfer it by distillation, the air being excluded, into a vessel in which the resistance is to be brought to 0° C.

When we used thin conducting wires the arrangement of this vessel differed a little from that used with thick and hence stiff wires.

Fig. 3 represents the apparatus as used with thin wires. The tube A, after being narrowed at the top and provided with a side tube d and ground joint, was carefully cleaned. Then the resistance was put in at the wide end, pushed up as far as e and the tube sealed off at f. During this operation care was taken that the interior of the vessel remained perfectly dry. The opening g was closed with sealing wax. The wire was then annealed by keeping it for several hours at 100° C.

The amylene or the isopentane were distilled into it through the side-tube d by means of a flask connected to it by a ground joint. After the distillation a tube with phosphorus pentoxide was connected to the joint, lest moist air should come into contact with the liquid with time and should be absorbed by it. If this had been neglected the liquid after some time no longer insulated sufficiently. The smallest quantities of water caused short circuiting in the resistance which owing to the magnitude of the latter, soon rendered the measurements inaccurate. This secondary circuit revealed itself in the variation of the resistance and in the appearance of polarisation current on breaking the principal current. The glass vessel was placed in ice, planed as described in Communication N^{o}. 27. This was heaped up till high above the narrowed part. Heaping the ice still higher round the neck of the vessel appeared to have no influence. Hence we might accept that the resistance came to 0° C.

When afterwards as leads copper wires of 2 m.m. were used it appeared that the operations, required to bring the resistance into a zero-vessel, as described above, would, owing to the stiffness of the copper wire, prove too dangerous for the cylinder and resistance. Therefore (c. f. fig. 2a, Pl. III, Communication N^{o} 27) a tube was taken (see fig. 4), closed nearly hermetically at the upper end by an india rubber stopper, through which passed the 4 leads $t_3, \ldots, t_6$, a tube to distil the liquid into it q, a sensitive thermometer (from a BECKMANN apparatus), and a stirring apparatus r.

The thermometer and the stirring apparatus were introduced because now the ice cannot be heaped up above the india rubber stopper: for fear 1st of introducing moisture, 2nd of conduction between the leads through the ice. Then we cannot be certain that the liquid takes the temperature of the ice, the more so as the leads are so much thicker than in the case considered above. However, when the vessel was brought into ice the temperature appeared to differ not more than 0.02 deg. C. from 0° C.

Again a tube with phosphorus pentoxide was connected to the joint h, as long as the resistance was kept in the vessel.

Mathematics. — "*Factorisation of large numbers.*" (3rd part).
By F. J. Vaes. (Communicated by Prof. P. H. Schoute).

XIII. *Abbreviated method of* § I.

The determination of the differences of squares allows of a very considerable abbreviation.

According to that method we write $G = a_1^2 - b_1$, and to b_1 we add $2a_1 + 1$, $2a_1 + 3$ etc. until a square b^2 is arrived at.

If to reach this p additions are necessary, then

$$b^2 = b_1 + p \times 2a_1 + \underbrace{(1 + 3 + 5 + \ldots\ldots)}_{p \text{ terms}}$$

$$= b_1 + p \times 2a_1 + p^2,$$

$$\text{so } b^2 = p\,(2a_1 + p) + b_1.$$

Example: $G = 513667 = 717^2 - 422$. (See pag. 328).

We test for p successively the values 1, 2, 3, etc. as follows

$$1 \times (1434 + 1) + 422 = 1435 + 422 = 1857$$
$$2 \times (1434 + 2) + 422 = 3294$$
$$3 \times (1434 + 3) + 422 = 4733$$
$$4 \times (1434 + 4) + 422 = 6174$$

etc.,

and we see whether the result is a square. It is immediately evident, that in this way the same numbers are obtained as for the additions on pages 328 and 329, but at the same time that it is not necessary to take all values 1, 2, 3, etc. for p.

For b^2 must terminate in 4 or 9 (pag. 328), so that the product (before 422 is added) can have only 2 or 7 as final figure. So we have but to calculate:

$2 \times (1434 + 2) + 422 =$	$2872 + 422 =$	3294
4×1438	5752	6174
7×1441	10087	10509
9×1443	12987	13409

etc.,

where we obtain exactly the sums, found on page 329 by adding 2 or 3 numbers at the same time.

So the operation gives a good insight into the reason of the addition of the numbers in groups of 2 or 3; it admits however still of a simplification if we pay attention to the last *two* figures of the sums.

The terminal figure 4 or 9 of b^2 must be preceded by an *even* figure, so the terminal figure 2 or 7 of the product must likewise be preceded by an *even* figure. If on the place of the tens an *odd* figure appears, we need not continue.

We directly see that this is the case for 2×1436, and 4×1438, but then also for 12×1446, 14×1448, 22×1456, 24×1458, in general for $(2 + 10\,n)\,(1436 + 10\,n)$ and $(4 + 10\,n)\,(1438 + 10\,n)$.

So there remains:

$$\begin{array}{rcrcr}
7 \times 1441 + 422 & = & 10087 + 422 & = & 10509 \\
9 \times 1443 & & 12987 & & 13409 \\
17 \times 1451 & & 24667 & & 25089 \\
19 \times 1453 & & 27607 & & 28029 \\
 & & \text{etc.} & &
\end{array}$$

Only the first two multiplications need be executed; after that additions suffice.

For $(7 + 10\,n) \times (1441 + 10\,n) = 7 \times 1441 + 14480\,n + 100\,n^2$, so that we can arrive at 17×1451 by adding $14480 + 100$ or 14580 to 10087; in the same way we find 27×1452 by adding to the obtained result

$$14480 + 300 \text{ or } 14780,$$

etc.; each following number to be added is 200 more than the preceding.

This holds good for the products with factors 9, 19, 29 etc. and the operation becomes

$$\begin{array}{r|r}
7 \times 1441 + 422 = 10509 & 9 \times 1443 + 422 = 13409 \\
(7 + 1441) \times 10 + 100 = \underline{14580} & (9+1443)\times 10+100 = \underline{14620} \\
25089 & 28029 \\
\underline{14780} & 14820 \\
39869 & \overline{42849} = 207^2.
\end{array}$$

In both columns the addition must be performed at the same time; the entire number of additions is only 4.

Example $G = 1677803 = 1296^2 - 1813$.

Here b^2 can end only in 1, so $b^2 - 1813$ or $p\,(2a_1 + p)$ only in 8.

Of 1×2593, 2×2594, etc. we need but take those, whose first factor ends in 2 or 6.

This gives the operation (at the same time in two columns):

$2 \times 2594 + 1813 = 6901$	$6 \times 2598 + 1813 = 17401$
$(2 + 2594) \times 10 + 100 = 26060$	$(6+2598) \times 10 + 100 = 26140$
32961	43541
2626	2634
5922	6988
2646	2654
8568	9642
2666	2674
11234	12316
2686	2694
13920	15010
2706	2714
166261	$177241 = 421^2$

$$\text{from which ensues } G = 1362^2 - 421^2$$

$$= 1783 \times 941.$$

According to the common method of § I

$$1362 - 1296 = 66$$

numbers ought to have been added; now their number is only 12.

The additions of the first column are useless and the question might be put whether this was not discernible beforehand.

This is really the case, if we make use of the table of the 4 last figures of a square (page 332).

For b^2 can terminate only in 1, so a^2 in 04, 24, 44, 64, 84.

According to the table a number terminating in 04 can be square only when the number formed by thousands und hundreds is a 4-fold or a 4-fold — 1. For shortness' sake we shall indicate this by: $\binom{4\,v}{4\,v - 1}04$. If we subtract this number 7803 formed by the

4 last figures of G, there remains $\binom{4v+2}{4v+1}01$, for the 4 last figures of b^2, because 78 is a 4-fold + 2.

According to the table the terminal figures 01 in a square can be preceded only by a 4-fold, or a 4-fold + 2, so that for b^2 we can only have: $(4v+1)01$ and for a^2: $(4v-1)04$.

If we apply the same to the terminal figures 24, 44, 64, 84 of a^2, we arrive at:

$$\binom{4v}{4v-1}04, \binom{4v+2}{4v-1}24, \binom{4v+1}{4v+2}44, \binom{4v}{4v+1}64, \binom{4v}{4v-1}84$$

to be subtracted 7803

$$\text{remainder } \binom{4v+2}{4v+1}01, \binom{4v}{4v+1}21, \binom{4v-1}{4v}41, \binom{4v+2}{4v-1}61, \binom{4v+2}{4v+1}81,$$

of which are only possible the cases:

$$(4v+2)01,\ (4v+1)21,\ (4v)41,\ (4v-1)61,\ (4v+2)81.$$

Now the first column of the additions begins with 6901, and therefore with a $(4v+1)01$, to which is added 26060, that is

$$\begin{array}{l} (4v \quad\ )60 \\ \hline \end{array}$$

which gives $(4v+1)61$ and therefore never a square.

To this is added 26260, a $(4v+2)60$, together $(4v)21$, which can neither be a square.

And in succession we shall have:

$$\begin{array}{l}
(4v \quad\ )21 \\
(4v \quad\ )60 \\ \hline
(4v \quad\ )81 \\
(4v+2)60 \\ \hline
(4v-1)41 \\
(4v \quad\ )60 \\ \hline
(4v \quad\ )01 \\
(4v+2)60 \\ \hline
(4v+2)61 \\
(4v \quad\ )60 \\ \hline
(4v-1)21 \\
(4v+2)60 \\ \hline
(4v+1)81 \\
(4v \quad\ )60 \\ \hline
(4v+2)41 \\
(4v+2)60 \\ \hline
(4v+1)01 \\
\text{etc.,}
\end{array}$$

from which is evident, that a square can never be obtained.

So it is only necessary to calculate the second column, which however still admits of a simplification.

For 17401 is a $(4v+2)01$, and 26140 a $(4v+1)40$, so that we obtain in succession:

$$\begin{array}{rl}
(4v+2)\,01 & \\
(4v+1)\,40 & \\ \hline
(4v-1)\,41 & \\
(4v-1)\,40 & \\ \hline
(4v+2)\,81 & * \\
(4v+1)\,40 & \\ \hline
(4v)\,21 & \\
(4v-1)\,40 & \\ \hline
(4v-1)\,61 & * \\
(4v+1)\,40 & \\ \hline
(4v+1)\,01 & \\
(4v-1)\,40 & \\ \hline
(4v)\,41 & * \\
(4v+1)\,40 & \\ \hline
(4v+1)\,81 & \\
(4v-1)\,40 & \\ \hline
(4v+1)\,21 & * \\
(4v+1)\,41 & \\ \hline
(4v+2)\,61 & \\
(4v-1)\,40 & \\ \hline
(4v+2)\,01. & *
\end{array}$$

Only the numbers marked with an asterisk can be squares and the operation can become:

$$\begin{array}{rr}
6\times 2598+1813= & 17401 \\
26140+26340= & 52480 \\ \cline{2-2}
 & 69881 \\
 & 53280 \\ \cline{2-2}
 & 123161 \\
 & 54080 \\ \cline{2-2}
 & 177241 = 421^2
\end{array}$$

with only *three* additions.

Example: $G = 33379631 = 5778^2 - 5653$.

Now a^2 can terminate only in 00 or 56, b^2 in 69 or 25, and $p(2a_1+p)$ in 56 or 92.

The terminal figure of p can be only 2, 6 or 8 and we have

$$\begin{array}{rr|rr}
2\times 11558+5653= & 28769 & 6\times 11562+5653= & 75025 \\
(2+11558)\times 10+100= & 115700 & (6+12562)\times 10+100= & 115780 \\ \cline{2-2}
 & 144469 & &
\end{array}$$

$$\begin{array}{rr}
8\times 11564+5653= & 98165 \\
(8+11564)\times 10+100= & 115820 \\
 & 116020 \\
 & 116220 \\ \cline{2-2}
 & 446225
\end{array}$$

The sums of the second and third columns must always have 25 as terminal figures, so that we shall be able to add 5 numbers at a time.

So we must add to the second column:

5×116180, 5×117180, 5×118180, etc.,

or 580900, 585900, 590900, that is a $\binom{4v+1}{4v-1}00$, and as 75025 is a $(4\,v + 2)\,25$, we shall have:

$$\begin{array}{l} (4\,v + 2)\,25 \\ (4\,v + 1)\,00 \\ \hline (4\,v - 1)\,25 \\ (4\,v - 1)\,00 \\ \hline (4\,v + 2)\,25. \end{array}$$

According to the table 25 can be preceded only by a 4-fold or a 4-fold + 2; consequently the second column can produce a square only after addition of successively 10 numbers.

To the third column we shall have to add successively:

$$5 \times 116820,\ 5 \times 117820,\ 5 \times 118820,\ \text{etc.},$$

or 584100, 589100, 594100, that is a $\begin{pmatrix} 4\,v + 1 \\ 4\,v - 1 \end{pmatrix} 00$,

and as 446225 is a $(4\,v + 2)\,25$, also this column can produce a square only after addition of every time 10 numbers.

As 28769 of the first column is a $(4\,v - 1)\,69$ and 115700 a $(4\,v + 1)\,00$, we shall have successively

$$\begin{array}{l} (4\,v - 1)\,69 \\ (4\,v + 1)\,00 \\ \hline (4\,v \quad\ \ )\,69 \\ (4\,v - 1)\,00 \\ \hline (4\,v - 1)\,69, \end{array}$$

from which in connection with the table is evident, that only the 2nd, 4th, 6th etc. additions can produce squares, so that the operation becomes:

$$\begin{array}{rl} & 2 \times 11558 + 5653 = 28769 \\ (2 + 11558) \times 10 + 100 = & \left.\begin{array}{r} 115700 \\ 115900 \end{array}\right\} 231600 \end{array}$$

$$\begin{array}{r} \hline 260369 \\ 2324 \\ \hline 4927 \\ 2332 \\ \hline 7259 \\ 2340 \\ \hline 9599 \\ 2348 \\ \hline 11947 \\ 2356 \\ \hline 14303 \\ 2364 \\ \hline 16667 \\ \text{etc.} \end{array}$$

The numbers 2316, 2324, etc. ascend with 4. Another small abbreviation can be brought about by remarking that b^2 must be a $(4v-1)\,69$, so a^2 $(=b^2+G)$ a $(4v)\,00$, so that a^2 can terminate only in .400 or .600, and by this b^2 only in .769 or .969. So we could have added the numbers 2316 and 2324 at once, the numbers 2356 and 2364 likewise, etc. Of every 4 additions one is dropped.

After adding 44 numbers, so after about 36 additions, we find

$$b^2 = 10975969 = 3313^2,$$

so that

$$G = 6660^2 - 3313^2 = 9973 \times 3347.$$

The second and third columns were not continued for a reason to be mentioned in the following §.

According to the common method of § I 6669 — 5778 or 882 numbers would have to be added, which can be taken in groups, but which would require a considerably larger number than 36 additions.

Also the method of § VII would require a greater number of operations.

A table showing the 6 or 8 last figures which may appear in a square, would undoubtedly lead to further abbreviations.

§ XIV. *Property of a^2 and b^2.*

If for shortness' sake we call $\left(\frac{G+1}{2}\right)^2$ and $\left(\frac{G-1}{2}\right)^2$ c^2 and d^2, then $G=c^2-d^2$. If moreover $G=a^2-b^2$, we have $a^2-b^2=c^2-d^2$.

Now b^2 and d^2 can never have the two terminal figures alike; neither can a^2 and c^2.

To show this we consider the table, giving the four last figures, which can appear in a square; immediately the following theorem strikes us: *in a column under an even number and in a column under an odd number, there is always one of the two* ×× , *but never more than one at the same height.*

Or: *In the columns under an even number the two* ×× *are placed immediately under each other, in the other columns there is every time a space of a line.* We must moreover fancy that under the fourth line of the table the two first lines have been repeated.

The consequence is evident from an example:

On pag. 504 a number a^2 of the form $\begin{pmatrix}4v\\4v-1\end{pmatrix}04$ is given, from which 7803, a $(4v+2)\,03$ is subtracted.

The remainder was $\begin{pmatrix}4v+2\\4v+1\end{pmatrix}01$; by the subtraction we arrive from the lines of the 4-folds and the 4-folds — 1 on two lower lines, namely those on which for instance the ×× of column 44 are to be found.

As b^2 is odd, only one of those two ×× can be used, namely on the line of the 4-folds+2. Consequently b^2 can be but a $(4v+2)\,01$, and therefore a^2 only a $(4v)\,04$.

The same consideration holds good for c^2 and d^2.

If now b^2 and d^2 were to have the same two terminal figures, then also d^2 must be a $(4v+2)\,01$, and c^2 $(4v)\,04$.

Now $c^2-a^2=d^2-b^2$; and the second member will be a $(4v+2)\,00$, the first a $(4v)\,00$.

So d^2 and b^2 can never have the terminal figures alike.

In the last example in § XIII d must end in 5, so d^2 in 25; so the second and third columns will lead to $\left(\frac{G-1}{2}\right)^2$.

Apparently there is an exception to the property mentioned here, namely in the case, that the number formed by thousands and hundreds of G are just a $\begin{matrix}\text{4-fold}\\\text{4-fold}-1\end{matrix}$ and the two terminal figures form a number $\begin{matrix}\text{smaller}\\\text{greater}\end{matrix}$ than the number formed by the two terminal figures of a^2.

For if in the example under consideration (page 505).

$$G=\ldots.\mathbf{76}\,03$$

we should have:

$$\begin{array}{lr} & a^2\begin{pmatrix}4v\\4v+3\end{pmatrix}04 \\ \text{to be subtracted} & \underline{76\quad 03} \\ \text{remainder} & \begin{pmatrix}4v\\4v+3\end{pmatrix}01, \end{array}$$

of which is possible only:
for b^2 (and so also for d^2) $(4v)\,01$
„ a^2 („ „ „ „ c^2) $(4v)\,04$.
Then $d^2-b^2=(4v)\,00$
and $c^2-a^2=(4v)\,00$

Evidently the 4-folds preceding the terminal noughts must however be the same for both remainders, so that G would have to be **0003**. But then we shall make use of the same consideration as above for the number formed by the 5th and 6th figure of G (reckoned from the right side).

(March 20, 1902).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday March 29, 1902.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 29 Maart 1902, Dl. X).

The following papers were read:

Physiology. — "*Structure and function of the trunk-dermatoma.*" III. By Prof. C. WINKLER and Dr. G. VAN RIJNBERK.

(Communicated in the meeting of February 22, 1902).

Before treating of that part of the dermatoma, situated between the mid-dorsal and mid-ventral lines, which may be called the "lateral part" of the dermatoma, we will proceed with the communication of a few chosen experiments from the extensive series at our disposition.

OBSERVATION I. Young black female dog. (Series I Nr. XVII). On Aug. 9th 1901, under rigorously aseptic conditions, the 19th pair of roots is sectioned extradural on both sides. On the next day the sensibility to pain is determined. by pinching with a fine artery-pincet.

Fig. 20. The 19th pair of roots 1) sectioned on both sides.

On both sides an insensible triangle is found, lying adjacent to the mid-ventral line (see fig. 20). The insensible area did not suffer any notable change during the next four weeks. Autopsy confirms the section of the 19th pair of roots.

Observation II. Young male dog. On December 27th 1901, under rigorously aseptic conditions, the 15th, 16th, 18th and 19th pair of roots to the right, the 17th pair of roots to the left, are cut through extradural. The autopsy on Jan. 13th 1902 confirms that the 17th root is intact to the right and sectioned to the left. The determination of sensibility on Dec. 28th shows the following : to the right, a sensible band between two insensible continuous bands ; to the left an insensible triangle lying adjacent to the mid-ventral line, including that part of the central area of the opposite dermatoma exceeding this line.

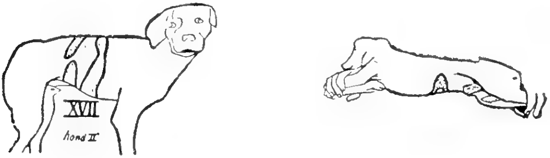

Fig. 21. The 17th dermatoma. To the right, isolated between two sectioned roots, to the left cut through.

To the right the caudal analgetic area is no longer a continuous band. Besides, it does not reach the mid-dorsal line because the sensible left half of the body, situated behind the caudal end of the section through the skin made at 27 Dec., exceeds there the mid-dorsal line.

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1) This figure has been obtained by counter-drawing the original photo on transparent paper in a few crude outlines, very carefully designing the boundary-lines, and afterwards reducing this figure, consisting only of a few simple lines, by photographic proceeding. The images, obtained in this manner, are highly demonstrative, if the analgetic areas are dotted, as has been done for all our figures.

RIGHT SIDE.

Measurings.	Mid-dorsal line.	Largest breadth of central area.	Mid-ventral line.
Cranial analgetic area (2 roots)	2.7 c.m.	2 c.m.	5.3 c.m.
Sensible area	3.1 „	± 7.5 „	1.5 „
Caudal analgetic area (2 roots)	3 „	interrupted (0 or —)	4.5 „

The interruption of the right caudal analgetic area occurs on 8.7 c.m. distance from the mid-dorsal and on 12 c.m. distance from the mid-ventral line, and possesses a breadth of 15 c.m.

LEFT SIDE.

The analgetic triangle, adjacent to the mid-ventral line has a breadth of 3.2 c.m. Its top is situated at 3.8 c.m. distance from the mid-ventral line.

Observation III. Young black male dog. (Series II, No. 1). On Dec. 17th 1901 under rigorously aseptic conditions the 14th, 15th, 18th and 19th pair of roots to the right and the 16th and 17th pair of roots to the left, are sectioned extradural. The autopsy on Jan. 2d 1902 confirms that this has been the case.

The results of the determination of sensibility on Dec. 19th are somewhat different from those on Dec. 22th, the latter being rendered in fig. 22.

Fig. 22. The 16th and 17th dermatoma to the right isolated between two sectioned roots.

Here too the caudal analgetic area is interrupted, it does not reach the mid dorsal line, because the sensible left half of the body exceeds this line below the caudal end of the wound. The right sensible zone exceeds the mid-ventral line, where it is encompassed by the analgetic interrupted band on the left side.

RIGHT SIDE.

Measurings.	Mid-dorsal line.	Largest breadth of central area.	Mid-ventral line.
Cranial analgetic area (2 roots)	2.5 c.m.	1.2 c.m.	6.2 c.m.
Sensible area „	4.6 „	± 7 „	3.2 „
Caudal analgetic area „	2.7 „	interrupted (0 or —)	4.7 „

The place of interruption is situated at 8.5 c.m. distance from the mid-dorsal line and at 13.6 c.m. distance from the mid-ventral line, and has a breadth of 2 c.m.

LEFT SIDE.

Measurings on Jan. 2d:	Mid-dorsal line.	Largest breadth of central area.	Mid-ventral line.
Analgetic area (2 roots)	2.6 c.m.	Interrupted.	6.5 c.m.

The place of interruption is situated at 7 c.m. distance from the mid-dorsal and at 10 c.m. from the mid-ventral line, and possesses a breadth of 3.6 c.m.

Observation IV. A spotted female dog. (Series II, No. IV). On Jan. 7th 1902 under aseptic conditions are cut through extradural: to the right the 13th, 14th, 15th, 17th, 18th and 19th pair of roots; to the left the 15th, 16th and 17th. The autopsy on Jan. 22th confirms that this has been the case.

The first determination of sensibility is made on Jan. 8th: the hind-legs of the dog are lamed. Fig. 21 represents the areas found to the right on Jan. 8th.

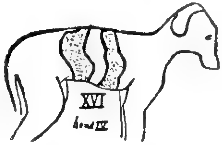

Fig. 23. The 16th dermatoma.
Isolated to the right between 3 roots on both sides.

RIGHT SIDE.

The measurings on Jan. 8th are the following:

	Mid-dorsal line.	Largest breadth of central area.	Mid-ventral line.
Cranial analgetic area (3 roots)	3.5 c.m.	4.8 c.m.	± 7.2 c.m.
Sensible area (1 root)	3.1 „	3.2 „	is not reached (distance 0.7 c.m.)
Caudal analgetic area (3 roots)	6.1 „	8.6 „	± 6.8 c.m.

LEFT SIDE.

The measurings on Jan. 8th are the following:

	Mid-dorsal line.	Largest breadth of central area.	Mid-ventral line.
Analgetic area (3 roots)	6.9 c.m.	5 c.m.	7 c.m.

On Jan. 22th the dog is again able to walk normally. Sensibility is determined once more. Death by chloroform.

RIGHT SIDE.

Measurings on Jan. 22th are the following:

	Mid-dorsal line.	Largest breadth of central area.	Mid-ventral line.
Cranial analgetic area (3 roots)	3.2 c.m.	2.2 c.m.	7.3 c.m.
Sensible area (1 root)	3.9 „	5.5 „	2.1 „
Caudal analgetic area (3 roots)	6.3 „	4 „	6.8 „

LEFT SIDE.

The measurings on Jan. 22th are the following:

	Mid-dorsal line.	Largest breadth of central area.	Mid-ventral line.
Analgetic area (3 roots)	4.6 c.m.	2.4 c.m.	7 c.m.

The distance from the mid-dorsal to the mid-ventral line is: to the right in the cranial area 21.5 c.m., in the caudal area 21 c.m. and to the left in the analgetic area 22 c.m.

The most important narrowing in the cranial analgetic area to the right is at 11 c.m. distance from the mid-dorsal, at 10.5 c.m. distance from the mid-ventral line.

The most important narrowing in the caudal analgetic area to the right is at 9 c.m. distance from the mid-dorsal, at 13 c.m. distance from the mid-ventral line.

The most important narrowing in the analgetic area to the left is at 9 c.m. distance from the mid-dorsal, at 13 c.m. distance from the mid-ventral line.

Observation VI. A black male dog. (Series II, dog VI) On Jan. 1st 1902 under rigorously aseptic conditions, are cut through extradurally: to the right the 11th, 12th, 13th, 15th, 16th and 17th pair of roots; to the left the 12th, 13th, 15th and 16th.

On both sides the 14th remains intact, as is confirmed by the autopsy on Jan. 19th. The sensibility, determined and noted on Jan. 18th, presents the images, rendered by fig. 24.

Fig. 24. The 14th dermatoma,
isolated to the right between three sectioned roots on each side, to the left between two sectioned roots on each side.

RIGHT SIDE.

The measurings on Jan. 18th are the following:

	Mid-dorsal line.	Largest breadth of central area.	Mid-ventral line.
Cranial analgetic area (3 roots)	2.7 c.m.	2.4 c.m.	7.6 c.m.
Sensible area (1 root)	4.2 „	6 „	is not reached (distance 0.4 c.m.) breadth of sensible area at that point 1.9 c.m.
Caudal analgetic area (3 roots)	5.2 c.m.	4.6 c.m.	7.6 c.m.

LEFT SIDE.

Measurings on Jan. 18th are the following:

	Mid-dorsal line.	Largest breadth of central area.	Mid-ventral line.
Cranial analgetic area (2 roots)	1.8 c.m.	Interrupted (0 or —)	6.4 c.m.
Sensible area (1 root)	3.4 „	± 6.5 c.m.	is not reached (distance 0.3 c.m.) breadth of sensible area at that point 1.1 c.m.
Caudal analgetic area (2 roots)	3.4 c.m.	Interrupted (0 or —)	6.4 c.m.

The distance from the mid-dorsal to the mid-ventral line is in the cranial analgetic area to the right 24.5 c.m., to the left 26 c.m.; and in the caudal analgetic area to the right 23.5 c.m.. to the left 24.5 c.m.

The narrowing of the right caudal area begins at 9 c.m. distance from the mid-dorsal line.

The top of the dorsal triangle of the left cranial area is situated at 11 c.m. distance from the mid-dorsal line, the top of the ventral triangle at 13 c.m. distance from the mid-ventral line. The breadth of the interrupting band is 1.5 c.m. For the left caudal area these distances become relatively 10 c.m. and 11 c.m., whilst here the interrupting band has a breadth of 3 c.m.

Observation V. A large brown dog. (Series II Nr. VII). On Jan. 26th 1902 are cut through extradural under aseptic conditions: to the right the 12th, 13th, 14th, 16th, 17th and 18th pair of roots, to the left the 13th, 14th, 16th and 17th pair of roots. The autopsy on Febr. 4th confirms the intactness of the 15th pair of roots on both sides.

The determination of sensibility on Jan. 27th proves the extension of the 15th dermatoma to be such as is represented by fig. 25.

Fig. 25. The 15th dermatoma, isolated to the right between three sectioned roots on each side, to the left between two sectioned roots on each side.

RIGHT SIDE.

The measurings on Jan. 27th are the following:

	Mid-dorsal line.	Largest breadth of central area.	Mid-ventral line.
Cranial analgetic area (3 roots)	3.5 c.m	1.5 c.m.	6.2 c.m.
Sensible area (1 root)	4.4 „	8.2 „	2 „
Caudal analgetic area (3 roots)	6 „	2.2 „	6.9 „

The distance from the mid-dorsal to the mid-ventral line is in the cranial area 26.4 c.m., in the caudal area 27 c.m. In the cranial area the most important narrowing is situated at 12 c.m. distance from the mid-dorsal and at 15 c.m. distance from the mid-ventral line. In the caudal area it is situated at 13 c.m. distance from the mid-ventral line.

LEFT SIDE.

The measurings on Jan. 27th are the following:

	Mid-dorsal line.	Largest breadth of central area.	Mid-ventral line.
Cranial analgetic area (2 roots)	2.6 c.m.	Interrupted (0 or —)	5 c.m.
Sensible area (1 root)	4.8 „	± 9 c.m.	2 „
Caudal analgetic area (2 roots)	4.5 „	Interrupted (0 or —)	5 „

The distance from the mid-ventral to the mid-dorsal line is 28 c.m as well in the cranial as in the caudal area. The top of the anterior dorsal triangle is situated at 9 c.m. distance from the mid-dorsal line, the top of the anterior ventral triangle at 16 c.m. distance from the mid-dorsal line and at 9 c.m. distance from the mid-ventral line. The breadth of the interrupting band is 7 c.m.

The top of the posterior dorsal triangle is situated at 8 c.m. distance from the mid-dorsal line, the top of the posterior ventral triangle at 16 c.m. distance from the mid-dorsal line. The breadth of the interrupting band is 8 c.m.

Observation VI. A brown and white spotted male dog (Series II, Nr. V). On Jan. 14th 1902 are cut through under aseptic conditions extradural; to the right the 10th, 11th, 12th, 14th, 15th and 16th pair of roots and to the left the 11th, 12th, 14th and 15th pair of roots. The autopsy on Febr. 3rd confirms that the 13th has been left intact on both sides.

The sensibility, determined on Jan. 15th and Febr. 2nd, offers only slight differences, and the relation between analgetic and sensible areas is represented in fig. 26 as it has been found existing on Jan. 15th.

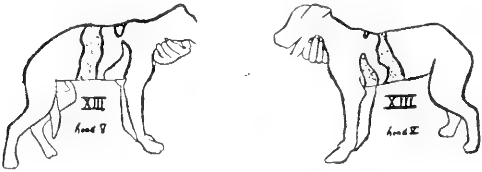

Fig. 26. The 13th dermatoma
to the right between three sectioned roots on each side, to the left between two sectioned roots on each side.

RIGHT SIDE.

Measurings.	Mid-dorsal line.		Largest breadth of central area.		Mid-ventral line.	
Measurings on	Jan. 15th	Febr. 2d	Jan. 15th	Febr. 2d	Jan. 15th	Febr. 2d
Cranial analgetic area (3 roots)	3.7 c.M.	3 c.m.	Interrupted (over1.5c.m)	Interrupted (over2c.m.)	6.5 c.m.	6.4 c.m.
sensible area (1 root)	3.5 „	4.2 „	4.7 „	6 „	2 „	2 „
Caudal analgetic area (3 roots)	5.4 „	4.5 „	6 „	5 „	10 „	10.5 „

On Febr. 2d the distance from the mid-dorsal to the mid-ventral line is 22.1 c.m. in the cranial, 21.2 c.m. in the caudal area.

The top of the dorsal triangle is situated at 5.5 c.m. distance from the mid-dorsal line; the top of the ventral triangle at 13.5 c.m. distance from the mid-ventral line; the breadth of the interrupting band is 2 c.m.

On Febr. 2d the most important narrowing in the caudal area is found at 9 c.m. distance from the mid-dorsal, at 12 c.m. distance from the mid-ventral line.

LEFT SIDE.

Measurings.	Mid-dorsal line.		Largest breadth of central area.		Mid-ventral line.	
Measurings on	Jan. 15th	Febr. 2d	Jan. 15th	Febr. 2d	Jan. 15th	Febr. 2d
Cranial analgetic area (2 roots)	1.9 c.m.	1.4 c.m.	Interrupted (over 6 c.m.)	Interrupted (over9c.m.)	4.8 c.m.	5.2 c.m
sensible area (1 root)	3.8 „	4 „	6 „	6 „	1.5 „	1.8 „
Caudal analgetic area (2 roots)	3.7 „	3 „	2 „	6 „	6.5 „	7 „

On Febr. 2d the distance from the mid-dorsal to the mid-ventral line is 23 c m. in the cranial, 24 c.m. in the caudal area.

The top of the anterior dorsal triangle is lying at 3.2 c.m. distance from the mid-dorsal line, the top of the anterior ventral triangle at 14 c.m. distance from the mid-ventral line. The breadth of the interrupting band is 9 c.m.

The top of the posterior dorsal triangle is situated at 55 c.m. distance from the mid-dorsal line, the top of the posterior ventral triangle at 16 c.m. distance from the mid-ventral line. The breadth of the caudal interrupting band is 2 c.m.

Observation VII. A black male dog (Series II, No. III). On Dec. 31th 1901, under aseptic conditions, are cut through extradural: to the right the 11th, 12th, 14th, 15th, 17th and 18th pair of roots, to the left the 11th, 12th, 16th and 17th pair of roots. The autopsy on Jan. 2d confirms that this has been the case. On Jan. 1st the hindlegs of the animal are paralysed.

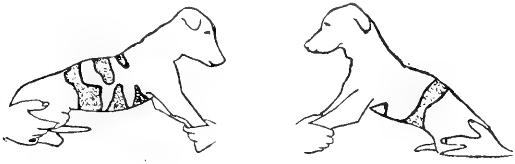

Fig. 27. To the right the 13th and 16th roots are isolated, both between two sectioned roots on each side. To the left the 13th, 14th and 15th roots have been preserved intact between two sectioned roots above and below these three.

On Jan. 17st the sensibility is determined. To the left, although two roots have been cut through, there is only found cranially a ventral analgetic spot, offering again a triangular form. To the right the most cranially situated sensible area, has pierced on both sides the neighbouring analgetic areas. On the contrary the most caudally situated sensible area is very considerably narrowed, even half of it being interrupted by the caudal analgetic area.

From these observations it becomes clear, that under favourable conditions, *the isolated central areas not only reach their largest* *breadth in the lateral part of the dermatoma, but are also considerably broader there than at any other place.*

This fact, considered in its connexion with our former communications, is of the highest importance.

Indeed, when we first found it necessary to distinguish in the dermatoma between a central and a marginal area, we were at the same time obliged to consider these two as two variable quantities, suffering mutual interchange of action.

As soon as the dermatoma is damaged, the marginal area becomes larger according to definite rules, whilst the central area, also according to definite rules, is diminished first to a triangular figure (fig. 2—4)[1]) which afterwards in the lateral part is interrupted by an insensible band (fig. 5).

In this place is located the relative minimum for the sensibility of *the central area*,[1]) but at the same time, as has been made clear by the foregoing, it is at this place too that the central area is broadest.

Every serious operative lesion, and more especially a partly sectioning of root-bundles in the posterior root, will create the possibility of central areas being formed, not properly fulfilling their functions.

By not properly fulfilled functions is meant here: the threshold-value for the sensibility being raised, or the marginal area being enlarged at the expense of the central area.

This diminished action evidently will become most obvious at the place where a very broad central area possesses a relative minimum of sensibility.

By this fact an apparent contradiction is solved. The obtained results were viz.: in the case of one or two roots being simply sectioned without any more, interrupted analgetic areas, [2]) and in the case of one root being left intact, between two sectioned roots on either side of it, continuous analgetic bands.

In the latter case the larger bone-wound, the greater hemorrhage, the increased cooling of the spinal medulla, furnish so many unfavourable conditions, by means of which, during the first days at least, are originated central areas of the isolated dermatoma, not properly fulfilling their functions.

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[1]) See: Proceedings of the Royal Acad. of Sciences. Meeting of Nov. 30th 1901, pag. 281, Fig. 8.

[2]) See: Proceedings of the Royal Acad. of Sciences. Meeting of Dec. 28th 1901, pag. 361, Fig. 10.

These central areas will necessarily suffer their most important reduction in that place where, their breadth being largest, already under normal circumstances the sensibility is raised relatively only a little above the threshold-value, in the lateral part therefore. Consequently there is formed in this case a continuous insensible band, denoting the destruction of two roots. This band is found to be interrupted in the case of these same roots being simply cut through without any more. Still there may be noted instances (in Fig. 24 and 25, left side) where after a few days or under especially favourable conditions, even directly after the operation, both analgetic bands correspondent with two sectioned roots, are interrupted in the lateral part by the sensible central area. There the accordance with the simple sectioning of two roots is found to be perfect.

It may happen however that the central area, either at the caudal or at the cranial side, has been damaged by the isolation, and that therefore (as is the case in fig. 21 and in fig. 22 for the analgetic area correspondent with the 18th and 19th pair of roots), there is found a one-sided interruption of the central area by the neighbouring analgetic area, whilst it continues to fulfill its functions properly on the other side.

Indeed, with deeply destroying operations it may even happen that one of the analgetic areas encompassing a central area, and correspondent each with two sectioned roots, begins to interrupt the central area — generally a caudally situated analgetic area may do this — whilst at the same time on the cranial side the central area, properly fulfilling its functions there, may be found interrupting the cranial analgetic area.

This is the case e.g. in fig. 25, right side, where two dermatomata have been isolated by sectioning six roots, and where the anterior (13th) central area shows on both sides the lateral interruption, whilst the posterior (16th) central area, probably because the medulla has been damaged (hindlegs are paralysed), is threatened to be interrupted by the analgetic area of the 17th and 18th root, which latter interruption however is only half achieved.

The lateral part of the central area and to all probability of the whole dermatoma is its broadest, but at the same time its feeblest part.

The first influence, exerted by the out-growth of the upper extremities on the higher trunk-dermatomata, is their being pressed into one another. The consequence is such as represented in fig. 27, left side.

The sectioning of two roots (in this case the 10th and 11th) entails here the same consequences as the sectioning of one root for the lower trunk-dermatomata. Towards the mid-dorsal line sensibility is preserved, towards the mid-ventral line an insensible triangle is formed. Even by sectioning three roots in this region, as has been done in fig. 26 (10th, 11th and 12th), we only succeed in achieving here what is achieved for the lower dermatomata by the sectioning of two roots, i.e.: an upper ventral insensible triangle with broad basis, and, separated from this by a sensible lateral band, a lower ventral insensible triangle with narrow basis. Therefore the statement does hold no longer good here, that the insensible band of two sectioned roots cannot be smaller at the mid-dorsal line than half of the isolated central area. Even on the 11th and 12th this influence of the extremities is still exerted (see Fig. 26). The insensible dorsal triangle, dependent on the section of these two roots, is very narrow there.

Meanwhile we are enabled also to survey in a simple manner the different relations, existing at the mid-dorsal line, in the lateral part (largest breadth of central area), and at the mid-ventral line.

When we conceive a ranging of successive central areas, whether overlapping one another or not, supposing them (in as far as the middle trunk-dermatomata are concerned) to be all of the same size, and, if overlapping one another, to do so to the same extent, then the differences in these relations may be expressed by a formula in the following manner.

Starting from a lower central area, we will call the distance between its cranial end and the caudal end of the next central area y, and the distance between its cranial end and the caudal end of the second following central area x. Then it will follow clearly (for at the mid-dorsal line only a transposition was performed, as y represents in that case the overlapping of the central area, x the part of the central area not overlapped by central areas, that the breadth of the central area $= x + 2y$.[1]). The breadth of two central areas lying adjacent to one another, is thus represented by $2x + 3y$, three by $3x + 4y$ etc. In the same manner the breadth of the analgetic bands will be represented: for one sectioned root by x, for two roots by $2x + y$, for three roots by $2x + 3y$ etc.

But this formula is a general one, in as much as it holds good not only here, but also for the mid-ventral line and the lateral parts.

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[1] See: Proceedings of the Roy. Acad. of Sciences. Meeting of Dec. 28th 1901, p. 360.

At the mid-dorsal line (Fig. 28A) x is represented in caudo-cranial, y in cranio-caudal direction.

If this is supposed to be the positive direction, then in the case of central areas being ranged in wide intervals from one another, as they are at the mid-ventral line, there will only be this difference, that y will be set like x in caudo-cranial direction, consequently y will be set in negative direction. (Fig. 28 B) The breadth of the ventral central area thus becomes $x—2y$, that of two succeeding central areas $2x—3y$, three $3x—4y$ etc., whilst the breadth of the insensible bands may be represented by x, $2x—y$, $3x—2y$ etc. according to the number of sectioned roots.

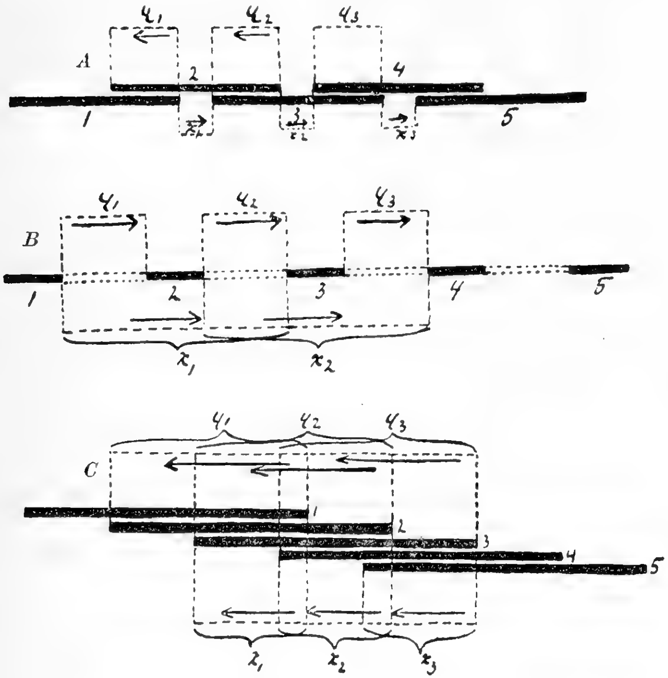

Fig. 28. *A.* General representations of the ranks of the central areas at the mid-dorsal line, $k = x + 2y$. *B.* of the ranks of the central areas at the mid-ventral line, $k = x - 2y$. *C.* of the ranks of the central areas at the lateral part of the body, $k = 2y - x$.

Besides the inverted direction of y, there exists still another difference between the ranks of the dorsal and those of the ventral central areas, because experiment shows, that by sectioning one root insensibility will be roused at the mid-ventral, but not at the mid-dorsal line. At the mid-dorsal line therefore the first x does not exist. Insensibility in that part is counteracted by marginal areas overlapping one another, which is not the case at the mid-ventral line.

In the lateral part on the contrary the central areas are ranged more closely than at the mid-dorsal line. They overlap one another farther, even until more than midway. Still the central area may be here again expressed by $x + 2y$, it being understood however that in this case too x and y are set in the same direction. But now x will be counted in cranio-caudal direction, and obtain consequently a negative value. Therefore the value of the central area will be in the lateral part $2y - x$ etc.

The results, obtained by applying these formulas, are very simple.

They may be demonstrated by means of Observation III (Series II, Nr. 1), the same observation put to use already in our former essay, where the sensible area corresponds with two roots left intact, whilst three analgetic areas (two to the right and one to the left) in their turn correspond each with two sectioned roots.

The central area in this case is:

Mid-dorsal line. Largest breadth of the central area. Mid-ventral line.

$$2x + 3y = 4.6 \quad 2x + y = \begin{cases} 2.5 \\ 2.6 \\ 2.7 \end{cases} \quad 2x + 3y = 7.5 \quad 2x + y = \begin{cases} 1.2 \\ 0 \text{ or} - \end{cases} \quad 2x + 3y = 3.2 \quad 2x + y = \begin{cases} 6.2 \\ 4.7 \\ 6.5 \end{cases}$$

thence

$$x = \begin{cases} 0.725 \\ 0.8 \\ 0.875 \end{cases} \quad y = \begin{cases} 1.05 \\ 1 \\ 0.9 \end{cases} \qquad x = \begin{cases} -0.975 \\ -1.875 \end{cases} \quad y = \begin{cases} 3.15 \\ 3.7 \end{cases} \qquad x = \begin{cases} 3.8 \\ 2.7 \\ 4.1 \end{cases} \quad y = \begin{cases} -1.2 \\ -0.7 \\ -1.6 \end{cases}$$

or

$$k = \begin{cases} 2.825 \\ 2.8 \\ 2.775 \end{cases} \qquad k = \begin{cases} 5.3 \\ 5.6 \end{cases} \qquad k = \begin{cases} 0.8 \\ 1.2 \\ 0.9 \end{cases}$$

$$k = \pm 2.8 \quad \begin{matrix} x = \pm 0.8 \\ y = \pm 1.- \end{matrix} \qquad\qquad k = \pm 0.8 \quad \begin{matrix} x = \pm 3.8 \\ y = \pm (-1.5) \end{matrix}$$

By reconstructing now the ranging of the central areas on this dog, a ranging communicated [1]) already formerly, without however any mention then being made of their enlargement in the lateral part, fig. 29 is obtained.

In this figure moreover the curve of sensibility on the central area has been designed each time. At the mid-ventral and at the mid-dorsal line are designed the addition-curves of sensibility, deduced from those curves, and represented in our former communication for the same dog by fig. 16 and 18. In the lateral part too has been designed on each central area, at the place where its breadth is found to be largest. a curve of sensibility (B), rising only a little above the threshold-line A, because in this place is located the relative minimum of the central area.

The slight rise above the threshold-line and the large basis of this curve, combined with the mutual overlapping of 4 or 5 of these curves, are cause that in the lateral part the tops and valleys of their addition-curves offer differences of far less importance than those found at the mid-dorsal and especially at the mid-ventral line.

This latter fact promises to become of great interest clinically. Already the existence of hyperalgetic bands next to bands of diminished sensibility had been demonstrated by Dr. LANGELAAN [2]) on normal persons and by Dr. BEYERMAN [3]) on patients, suffering from tabes. Especially with tabes, when in consequence of the diminished sensibility the threshold-value rises, and only the tops of the addition-curves have preserved sensibility intact (or sometimes increased it) the contrast can be striking, on chest and ventral part, between the valleys of the curve of sensibility, perceiving pain hardly or not at all, and the hyperalgetic tops. In the lateral parts of the thorax this difference either does not exist or is only slightly pronounced.

But apart from the differences in sensibility, existing between the dorsal part, the lateral part and the ventral part of the trunk, by this drawing is shown directly what will happen in the case of one or more dermatomata being destroyed.

When one dermatoma is destroyed, then the dotted area disappears and the ventral insensible triangle is formed. This single circumstance is in itself sufficient to prove the necessity of admitting the existence of

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[1]) See: Proceedings of the Roy. Acad. of Sciences. Meeting of Dec. 28th 1901. Fig. 19.

[2]) See: Proceedings of the Royal Acad. of Sciences. Meeting of Sept. 29th, 1900.

[3]) See: Proceedings of the Royal Acad. of Sciences. Meeting of Sept. 29th, 1900.

marginal areas in the structure of the dermatomata, for under normal conditions the ventral part is sensible over its whole extension. But the smaller dorsal analgetic triangle, which should be formed at the mid-dorsal line is never observed. This again argues the existence of marginal areas — or, if one prefers, parts of the dermatoma, whose sensibility remains below the threshold-value — of the neighbouring dermatomata, being able of providing *together* in the sensibility of this band, as soon as one dermatoma has been taken away, though unable of doing so alone.

As soon however as two dermatomata (4 and 5) are taken away, there is formed a double triangle; marked with ++. In the lateral part an interruption will necessarily occur, because in that part the overlapping of the central areas is so great, that the 3th and 6th dermatoma are situated in very close proximity, even touching one another.

The relative minimum of the central area, here drawn out to its largest breadth, is cause of different facts: the contrast between central and marginal area being at one time very slight, and then again very striking. If the central area has been isolated under very favourable circumstances, performing its functions in the best possible way, there will be hardly any question here about a marginal area. If on the contrary it has been isolated, not fulfilling its functions so well, then this part of the central area falls wholly or almost wholly below the threshold-value. It then becomes totally or nearly totally marginal area, and the lateral insensible interruption of the central area is brought about.

We conclude from the foregoing that the improvement of sensibility in the first 4 or 5 days after the operation, is achieved solely by parts of marginal areas being joined to the central area. According to our belief, in no other manner any improvement of sensibility may be achieved, at least during some months. The results, observed consequently to similar operations after the first fortnight, are found unaltered after six weeks or three months.

Mathematics. — "*Series of Polynomials*". (1st part). By Prof. J. C. Kluyver.

(Communicated in the meeting of February 22, 1902.)

Starting from a given power series representing within a definite circle of convergence an analytical function $F(x)$, investigators have endeavoured to deduce for this function other developments with a different region of convergence. In the first place another power series will present itself as a new development; however this has the drawback, that each coefficient of whatever order is dependent on all coefficients without exception of the given power series. We can require the n^{th} term in the new development to be completely determined by the first n terms of the given power series.

This demand is fulfilled by the development of $F(x)$ in a series of polynomials according to the idea of Mittag-Leffler, and now it is known that for each function such like developments can be found in infinite variety.

The purpose of this paper is in the first place to give a simple deduction for such like series of polynomials, in the second place to treat of a couple of simple examples which can serve as an explanation of the peculiar conditions of existence of the polynomials.

1. When the power series

$$F(x) = F(0) + \sum_{h=1}^{h=\infty} \frac{x^h}{h!} F^{(h)}(0)$$

must be continued by a series of polynomials we need some auxiliar function $g(u)$, of which we suppose the following:

1. $g(u)$ is holomorphic in $u = 0$ and uniform for $|u| < k$, where k is greater than 1.
2. If $g(u)$ is not everywhere finite within the circle $|u| = k$, the function is for $|u|$ increasing from nought for the first time finite in a point a.
3. $g(0) = 0$, $g(1) = 1$.

By these three suppositions one is but slightly limited in the choice of the function $g(u)$. From any function $f(u)$ holomorphic in $u = 0$, a function $g(u)$ can be deduced; we can take

$$g(u) = \frac{f(\lambda u) - f(0)}{f(\lambda) - f(0)}.$$

We now consider $F(xg(u))$ first as a function of u, then as a function of $xg(u)$. So we arrive at the two developments

$$F(xg(u)) = F(0) + \sum_{m=1}^{m=\infty} \frac{u^m}{m!} \overset{m}{D}_{u=0} F(xg(u))$$

and

$$F(xg(u)) = F(0) + \sum_{h=1}^{h=\infty} \frac{x^h\, g(u)^h}{h!} F^{(h)}(0),$$

from which last equation ensues

$$\underset{u=0}{\overset{m}{D}} F(xg(u)) = \sum_{h=1}^{h=m} x^h \frac{F^{(h)}(0)}{h!} \overset{m}{D}_{u=0}\, g(u)^h .$$

So we have

$$F(xg(u)) = F(0) + \sum_{m=1}^{m=\infty} u^m \sum_{h=1}^{h=m} x^h \frac{F^{(h)}(0)}{h!} \cdot \frac{\overset{m}{D}_{u=0}\, g(u)^h}{m!}, \quad . \quad . \quad (\mathrm{I})$$

or

$$F(xg(u)) = F(0) + \sum_{m=1}^{m=\infty} u^m\, T_m(x).$$

The coefficients $T_m(x)$ of this power series in u are polynomials in x and the coefficients of the polynomial $T_m(x)$ contain of the originally given power series only the m first coefficients

$$F^{(1)}(0),\ F^{(2)}(0),\ \ldots\ F^{(m)}(0).$$

The power series (I) has when x is given a definite radius of convergence, which may not, as is easy to understand, exceed a limit independent of x, i. e. $|u|$ may never become greater than k, because for $|u| > k$, $g(u)$ and with it $F(xg(u))$ cannot have a fixed value.

For the rest with each x the series is convergent for very small values of u approaching nought, and its sum is that value of $F(xg(u))$ which transforms itself for $u = 0$ into the given constant $F(0)$. With a steady increase of $|u|$ the series remains convergent and its sum is to be stated at $F(xg(u))$ if only the argument $z = xg(u)$ continues to indicate a point within a region, extending around $z = 0$, in which $F(z)$ is holomorphic. Such a region is for instance the star of MITTAG-LEFFLER although the rays of this star need not be right lines. If $F(z)$ is uniform in the whole plane and holomorphic in $z = \infty$, this region can even surround the point $z = \infty$. In every other case $z = \infty$ must remain outside this region and the series (I) must certainly diverge if $|u|$ has reached the limit $|a|$, because then $g(u)$ and also $z = xg(u)$ might become infinite.

The region in which $z = x\,g(u)$ is moving, must never contain the singular points of the function $F(z)$. So if we call these singular points

$$Aj = rj\,e^{i\alpha}j,$$

$$(j = 1, 2, 3, \ldots)$$

then, when $|u|$ increases fluently, we may never have

$$x\,g(u) = Aj \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(II)}$$

We suppose that for given x from this equation (II) u is deduced. Perhaps that solution is impossible, and not to be satisfied by any value of u with a modulus smaller than k. In that case we shall decide, that for the point x under consideration the radius of convergence of the series (I) is equal to k. If there are solutions with a modulus smaller than k, we then call $R_{x,j}$ the smallest of the moduli of the obtained solutions and R_x the smallest of all moduli $R_{x,j}$ $(j = 1, 2, 3, \ldots)$. For $|u| < R_x$ now $z = x\,g(u)$ will not be able to reach any of the singular points A_j, $F(xg(u))$ will have a finite and definite value and the series (I) will converge if only the nature of the function $F(z)$ in $z = \infty$ does not limit $|u|$ more closely.

So the result of the foregoing considerations respecting the radius of convergence of the series (I) is the following. For a multiform function $F(x)$ the radius of convergence when x is given is equal to the smallest of the two quantities R_x or $|a|$. If there are no singular points in the finite part of the plane the radius of convergence is equal to $|a|$. On the other hand for a uniform function $F(x)$, holomorphic in $x = \infty$, the radius of convergence is always R_x. In some cases, where the limit R_x is wanting, the radius of convergence can increase to k.

In the series (I) we now substitute $u = 1$ and we obtain in this way formally the development of $F(x)$ in a series of polynomials.

We find

$$F(x) = F(0) + \sum_{m=1}^{m=\infty} \sum_{h=1}^{h=\infty} x^h \frac{F^{(h)}(0)}{h!} \cdot \frac{\overset{m}{D}_{u=0}\, g(u)^h}{m!} \quad F(0) + \sum_{m=1}^{m=\infty} T_m(x), \; . \quad \text{(III)}$$

and now the question arises whether this series converges or diverges in a given point x. If it converges it will produce that value of $F(x)$ which is deduced out of $F(xg(u))$ when $|u|$ gradually increases from 0 to 1. To judge its nature we must arrange the points x of the plane according to the values of R_x belonging to it. In the first place we find a region G_2 simply connected or of higher connectivity within which everywhere $R_x < 1$. That region G_2 will include $x = \infty$ and the singular points A_j, or one or more of these points lie on the boundary of this region, and in each point of G_2 there is divergence. The value of R_x gives a notion of the rate of this divergence in a definite point x, for we have

$$\overline{Lim}_{m=\infty} \sqrt[m]{\mid T_m(x) \mid} = \frac{1}{R_x}{}^{1)} \quad . \quad . \quad . \quad . \quad . \qquad (IV)$$

By one or more closed curves G_2 is separated from the region G_1 containing the points x, where R_x is > 1. So this region G_1 is the region of convergence of the series of polynomials. But not everywhere in G_1 is the convergence equally strong. With the aid of the value of the limit (IV) the rate of convergence can be judged. For shortness' sake we shall call that value "mark" of the series of polynomials in the point x and in general the convergence will be better the lower this mark falls below unity.

On the boundary of G_2 and G_1, which latter region may consist of different parts separated from one another, the mark is constant and equal to 1. If we move away from this boundary the mark falls. However, it never falls below $1 : k$, for k was the uttermost limit for the radius of convergence of the power series (I). Perhaps too, that $F(\infty)$ is infinite or indefinite, whilst $g(u)$ has a pole a inside the circle $\mid u \mid = k$. Then convergence is possible only for $\mid a \mid > 1$ and the mark of the series (III) is at least $1 : \mid a \mid$.

So we shall often be able to distinguish two different parts in the region G_1. In the outer part G_1'' bordering everywhere on G_2 the mark of the series is variable. It varies either between 1 and $1 : k$, or between 1 and $1 : \mid a \mid$. In the inner part G_1' however, the mark is constant; it is continually equal to $1 : k$ or equal to $1 : \mid a \mid$ according to the nature of $F(x)$. The shape and size of these considered regions is entirely dependent on the singular points of $F(x)$ and of the choice of $g(u)$. It is very well possible that G_2, G_1'' or G_1' are wanting. Thus for instance the series will converge everywhere independently of x with a constant mark, if in the finite part of the plane there are no singular points A_j and when $\mid a \mid$ is > 1. On the other hand it may happen that everywhere for R_x a value smaller than 1 will be found out

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1) $\overline{Lim}$ denotes here „la plus grande des limites" of CAUCHY.

of the equations (II) and that although $F(x)$ at infinity is completely regular, the region of divergence G_2 covers the whole x-plane. Finally there are cases in which a region of convergence G_1 can be found, but in which the mark of the series never falls to the lowest limit $1:k$ or $1:|a|$, so that there is no question about a division of G_1 into the regions G_1' and G_1''.

Beforehand we can take care to let the series of polynomials possess by all means a region of convergence G_1. This takes place when we choose for $g(u)$ a function always remaining finite within the circle $|u| = 1$. For then for a point x in the immediate vicinity of the origin $x = 0$ the equations (II)

$$x\,g\,(u) = A_j$$

will furnish for R_x a value greater than 1 and the series (III) will of necessity converge in the vicinity of the origin $x = 0$. And in the same way, if $g(u)$ remains finite inside the circle $|u| = k$ and the point a is lacking, the origin $x = 0$ will also be a point of the region G_1' inside which region the series converges everywhere with the mark $1:k$.

2. After having pointed out in general the possibility of the existence of the regions G_2, G_1'', G_1', we must find out how to construct these regions. A point x belongs to G_2 when one of the equations

$$x\,g\,(u) = A_j$$

admits of a solution u of which the modulus is smaller than 1. If thus by the above-mentioned equation we map the u-circle with radius unity and centre at the origin on the x-plane, we obtain a region $G_{2,j}$ where the series diverges. It is very well possible that a part of the x-plane is covered bij that representation not only once but more times and that the representation of the circumference of the circle is a closed curve, cutting itself several times. Any region that is covered can be considered to belong to $G_{2,j}$, no matter how often this takes place. All the regions $G_{2,j}$ together form the region of divergence G_2 of the series. What remains of the x-plane is to be regarded as the region of convergence G_1, which we can now divide with the aid of a second conform representation into the two parts G_1'' and G_1'. Again with the aid of the equation

$$x\,g\,(u) = A$$

we determine in the x-plane the region $G_{2,j} + G_1{}'',_j$ as the conform representation of the u-circle $| u | = k$ or in other circumstances of the u-circle $| u | = | a |$. Together these regions give after subtraction of the region G_2 already found the region G_1''; the remainder of the region G_1 will be the region G_1'.

It is a fact that according to the nature of the given function $F(x)$ the regions G_1 and G_2 can assume forms widely differing, but the manner is most simple in which the function exercises its influence on the form of the series of polynomials (III).

As long as $g(u)$ is retained, it is sufficient to replace in this series everywhere $F^{(h)}(0)$ by $\varphi^{(h)}(0)$, to change the development of $F(x)$ into the development of another function $\varphi(x)$. So it is to be recommended if we search for the development of $F(x)$ to regard beforehand the simplest function $1 : 1-x$, which according to (III) assumes the following form

$$\frac{1}{1-x} = 1 + \sum_{m=1}^{m=\infty} \sum_{h=1}^{h=m} x^h \frac{\overset{m}{D}_{u=0}\, g(u)^h}{m!} \quad . \quad . \quad . \quad . \quad (\mathrm{V})$$

If we write the terms $T_m(x)$ of this series in the form

$$T_m(x) = \sum_{h=1}^{h=m} B_{m,h}\, x^h,$$

it follows that the first coefficient $B_{m,1}$ is always equal to

$$\frac{1}{m!}\, g^{(m)}(0),$$

and so equal to the coefficient of u^m in the power series into which $g(u)$ can be developed.

To determine the general coefficient $B^{m,h}$ [1]) we can generally

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[1]) By ARBOGAST was given in his "Calcul des Dérivations (1800)" the development of $F(a_0 + a_1 y + a_2 y^2 + \ldots)$ according to powers of y. If we put $a_0 = 0$, $a_j = b_j\, x$ and if we replace y by u we have the development of $F[x(b_1 u + b_2 u^2 + \ldots)] = F(x\, g(u))$ according to ascending powers of u and to determine the coefficients we can follow the rule found by ARBOGAST and called by CAYLEY "the rule of the last and the last but one".

apply successfully the identity

$$B_{m,h} = \sum_{k=j}^{k=m-h+j} B_{k,j}\, B_{m-k,h-j}\,.$$

Here j is rather arbitrary, only it must be taken $\geqq h-1$.

The function $1:1-x$ is uniform with the single pole $x=1$, so the regions G_2 and G_1 are determined with the aid of the single equation

$$x\, g(u) = 1$$

and inside G_1 the mark of the series is always indicated by $1:R_x$. We need not look about for possible infinities a of $g(u)$.

By the introduction of the coefficients $\dfrac{F^{(h)}(0)}{h!}$ we obtain out of the series (V) a series for $F(x)$ and it is apparent that we shall now be able to find the various parts $G_{2,j}$ of the region of divergence of this new series by applying successively to the region of divergence of series (V) the transformations

$$(x,\ A_j\, x).$$

In a similar way the regions G_1' and G_2' can be determined.

However we must keep in mind that in accordance with the nature of the function $F(x)$ the mark for the new series will perhaps be in some regions no longer $1:R_x$ but $1:\mid a\mid$. The case might present itself that notwithstanding the existence of a region of convergence for the series (V), the deduced series for $F(x)$ might diverge everywhere.

The preceding more general considerations on series of polynomials we do not desire to continue before we have treated some simple examples. It seems difficult to make a simple calculation of the coefficients $B_{m,h}$ agree with a large extension of the region of convergence. But if on this last point we are not too exacting we can obtain rather useful developments which are suitable for application.

Supposing that a is a given constant, we put

$$g(u) = \frac{(a-1)\,u}{a-u}\,.$$

The function $g(u)$ is everywhere uniform, so in this case $k=\infty$; inside the circle $\mid u\mid = k$ is the single pole $u=a$.

For the coefficients $B_{m,h}$ we find

$$B_{m,h} = \frac{1}{m!} D^{m}_{u=0} \frac{(a-1)^h u^h}{(a-u)^h} = \frac{1}{m!}\left(\frac{a-1}{a}\right)^h D^{m}_{u=0} u^h \left(1-\frac{u}{a}\right)^{-h},$$

$$B_{m,h} = (m-1)_{h-1} \frac{(a-1)^h}{a^m}.$$

So the series of polynomials (V) becomes

$$\frac{1}{1-x} = 1 + \sum_{m=1}^{m=\infty} \frac{1}{a^m} \sum_{h=1}^{h=m} (m-1)_{h-1}\, x^h (a-1)^h, \quad . \quad . \quad . \quad \text{(VI)}$$

or if we like

$$\frac{1}{1-x} = 1 + \frac{x(a-1)}{a} \sum_{m=1}^{m=\infty} \left[\frac{1+x(a-1)}{a}\right]^{m-1}.$$

The series of polynomials has again become a geometrical series with the ratio $[1+x(a-1)] : a$, and the region of convergence must be a circle. We do not pay attention to this casual circumstance, but we determine G_2 according to the general rule, according to which G_2 is made in the x-plane the representation ot the circle unity of the u-plane with the aid of the relation

$$x\frac{(a-1)u}{a-u} = 1,$$

or

$$\frac{x+\frac{1}{a-1}}{1+\frac{1}{a-1}} = \frac{1}{u}.$$

From this is evident that the region of divergence G_2 encloses that part of the x-plane which is situated outside the circle with centre $-\frac{1}{a-1}$ and radius $\left|\frac{a}{a-1}\right|$. The interior of this circle is the region of convergence (fig. 1); on the boundary of G_1 lies the singular point $x = 1$.

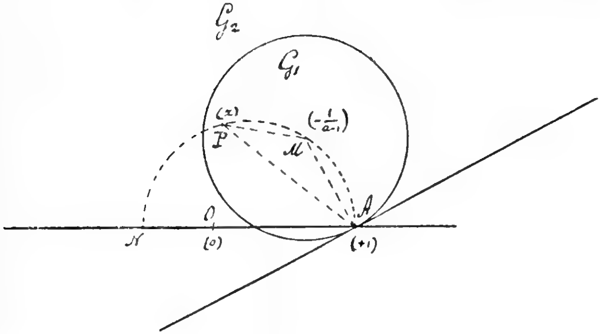

Fig. 1.

If R_x is the modulus of u deduced from the above-mentioned equation, then $1 : R_x$ is the mark of the series in the point x. We find

$$\frac{1}{R_x}=\frac{\left|x+\frac{1}{a-1}\right|}{\left|1+\frac{1}{a-1}\right|}=\frac{PM}{AM};$$

then the mark is constant on circumferences of circles concentric with G_1 and it can vary on each radius between 0 and 1.

Just as this will present itself in more general cases, the auxiliar function $g(u)$ contains here a parameter a, which can be varied arbitrarily either for a definite point x to lower the mark of the series i. e. to strengthen its convergence, either to enlarge or to reduce the region of convergence G_1. In the considered very simple case all this is easy to see. Considering in the first place the transformation and also the situation of the region of convergence, we see that for G_1 $a=\infty$ transforms itself into the unity circle and that the series (VI) again becomes the ordinary power series.

When $| a-1 |$ decreases the circle G_1 becomes larger and larger, at last when a approaches to unity G_1 encloses half of the x-plane, it has become the half plane containing the point $-\frac{1}{a-1}$ (M) and limited by the normal, which can be erected in the point $x=1$ (A) to AM (fig. 1). But this is a limiting case, for then in G_1 the mark of the series is everywhere 1; independent of x the terms of the series

(VI) become infinitely small and the convergence of the series becomes infinitely slow.

It will be noticed that only for $| a | > 1$ the origin will be inside G_1. If thus we deduce in the indicated way a development of polynomials for an arbitrary function $F(x)$ out of the series (VI), we have no certainty, if $| a |$ is assumed < 2, that there exists for this development a region of convergence G_1. And very surely if $| a | < 1$ the series deduced for $F(x)$ would diverge everywhere if this function were not holomorphic in $x = \infty$ or not uniform in the entire x-plane. In the second place the question rises: what is for a given point x the value of a which causes in x the mark of the series to fall most?

It is evidently always most advisable to bring the centre of G_1 in the point x, that is by taking

$$1 - \frac{1}{x} = a.$$

In that case the mark of the series will be 0. One may wish a to be real in the series (VI). To obtain the lowest mark with this limitation we must bring a circle through P and A with the centre of the axis of the real quantities. For the centre of the circle G_1 we must take the second point of intersection N of that circle and of that axis (fig. 1). In the point x the mark of the series will be equal to $\sin \angle P A N$.

We will now deduce out of the series of polynomials the development of another function, in the first place of $\log(1-x)$. In the finite part of the plane there is only the singular point $x = 1$, for the rest the function is multiform and not holomorphic in $x = \infty$.

Therefore we must keep the pole a of $g(u)$ outside the circle $| u | = 1$. In that supposition the region of convergence of the new series is identical with that of the series (VI). The series itself will be

$$\log(1-x) = -\sum_{m=1}^{m=\infty} \frac{1}{a^m} \sum_{h=1}^{h=m} (m-1)_{h-1} \frac{x^h}{h} (a-1)^h =$$

$$= -\sum_{m=1}^{m=\infty} \frac{1}{m\, a^m} \left[\Big\} 1 + x(a-1) \Big\{^m - 1 \right].$$

Again in the point a the mark of the series is generally determined by the equation

$$\frac{1}{R_x}=\frac{\left|x+\frac{1}{a-1}\right|}{\left|1+\frac{1}{a-1}\right|}=\frac{PM}{AM},$$

but now that $g(u)$ has the pole a and $log\ (1-x)$ is multiform, the mark can not fall below $1:|\,a\,|$. The circle G_1 can here really be divided into two parts G''_1 and G'_1. In the outer rim G''_1 (fig. 2) the mark varies between 1 and $1:|\,a\,|$; in the inner part G'_1, a circle with centre $-\frac{1}{a-1}$ and radius $\frac{1}{|a-1|}$, the mark is everywhere $1:|\,a\,|$.

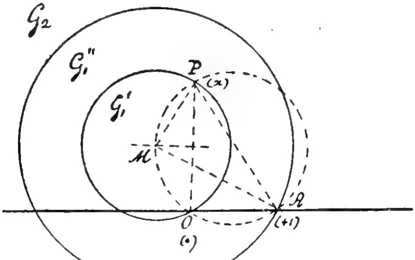

Fig. 2.

If we wish to calculate $log\ (1-x)$ for $x=\varrho^{i\theta}$, we must bring a circle through the points $x(P)$, $0(O)$ and $+\ 1(A)$ and assume the centre M of the arc OP as the centre of the region G_1. The mark of the series is then as low as possible and equal to

$$\frac{MP}{MA}=\frac{\varrho}{1+\sqrt{\varrho^2+1-2\varrho\cos\theta}}.$$

The point $x(P)$ is now situated just on the boundary of G''_1 and G'_1. So for instance for $x=-2$ we shall assume the centre M of G_1 in $x=-1$, i.e. we shall put $a=2$ and the mark of the series wil be $\frac{1}{2}$. Indeed we find in this supposition

$$log\ 3=-\sum_{m=1}^{m=\infty}\frac{1}{m}\cdot\frac{(-1)^m-1}{2^m}=2\sum_{k=0}^{k=\infty}\frac{1}{(2k+1)2^{2k+1}}=log\,\frac{1+\frac{1}{2}}{1-\frac{1}{2}},$$

a series, bearing distinctly the mark $\frac{1}{2}$.

As second application we take the function $tan^{-1}x$. Again the function is not uniform in the whole plane; now there are two singular points $+i$ and $-i$ at final distance. So convergence is again only possible for $|a| > 1$ and the mark of the series can fall at most to $1 : |a|$. The region of convergence is found by applying to circle G_1 of fig. 1 successively the transformations (x, ix) and $(x - ix)$. The new region G_1 is the part of the x-plane common to the two transformed circles. It is a double segment, bounded by the circles with the radius $\left|\frac{a}{a-1}\right|$, having the centres $+\frac{i}{a-1}$ and $-\frac{i}{a-1}$ (fig. 3). If we describe out of these centres circles with the radius $\left|\frac{1}{a-1}\right|$

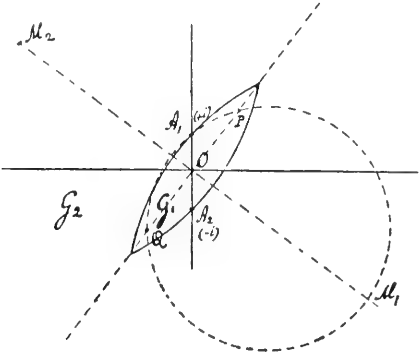

Fig. 3.

these circles touch each other externally. So here is no region G_1 within which the mark of the series is constantly equal to $1 : |a|$. For a given point x inside G_1 the mark $1 : R_x$ is found by solving according to u the two equations

$$x\,\frac{(a-1)\,u}{a-u} = \pm i.$$

The smallest value of $|u|$ is R_x. To obtain the lowest mark for the series we must bring a circle through $x\,(P), +i\,(A_1)$ and $-x\,(Q)$ and determine the midpoint M_1 of the arc PQ. If we now describe the circle with M_1 as centre and $A_1 M_1$ as radius and symmetrically to the former the circle through A_2 with radius $M_2 A_2$, then we obtain for the region of convergence G_1 a double segment, inside which lies the point $x(P)$, where the mark of the series will be $M_1 P : M_1 A_1 = M_2 P : M_2 A_2$.

Out of (VI) we find for function $tan^{-1} x$ the following series:

$$tan^{-1} x = \sum_{m=1}^{m=\infty} \frac{1}{a^m} \sum_{h=0}^{h=\left[\frac{m-1}{2}\right]} (m-1)_{2h} \frac{(-1)^h}{2\,h+1} x^{2h+1} (a-1)^{2h+1}, \quad . \; . \; . \qquad \text{(VII)}$$

or

$$tan^{-1} x = \frac{1}{a}\left[\frac{x\,(a-1)}{1}\right] + \frac{1}{a^2}\left[\frac{x\,(a-1)}{1}\right] + \frac{1}{a^3}\left[\frac{x\,(a-1)}{1} - \frac{x^3\,(a-1)^3}{3}\right] +$$

$$+ \frac{1}{a^4}\left[\frac{x\,(a-1)}{1} - 3.\frac{x^3\,(a-1)^3}{3}\right] + \frac{1}{a^5}\left[\frac{x\,(a-1)}{1} - 6.\frac{x^3\,(a-1)^3}{3} + \frac{x^5\,(a-1)^5}{5}\right] +$$

$$+ \frac{1}{a^6}\left[\frac{x\,(a-1)}{1} - 10\,.\frac{x^3\,(a-1)^3}{3} + 5.\frac{x^5\,(a-1)^5}{5}\right] + \, . \; . \; . \; . \; . \; .$$

Who wishes to calculate $tan^{-1}\, 1 = \frac{\pi}{4}$ with the aid of this series will arrive at the best convergence for $a=2$, i.e. by placing M_1 in the point $-i$, and according to the above-mentioned the mark will be $1 : \sqrt{2}$.

We shall find

$$tan^{-1}\, 1 = \frac{1}{2}\left[\frac{1}{1}\right] + \frac{1}{2^2}\left[\frac{1}{1}\right] + \frac{1}{2^3}\left[\frac{1}{1} - \frac{1}{3}\right] + \frac{1}{2^4}\left[\frac{1}{1} - 3.\frac{1}{3}\right] +$$

$$+ \frac{1}{2^5}\left[\frac{1}{1} - 6.\frac{1}{3} + \frac{1}{5}\right] + \frac{1}{2^6}\left[\frac{1}{1} - 10\,.\frac{1}{3} + 5.\frac{1}{5}\right] +$$

$$+ \frac{1}{2^7}\left[\frac{1}{1} - 15\,.\frac{1}{3} + 15\,.\frac{1}{5} - \frac{1}{7}\right] + \frac{1}{2^8}\left[\frac{1}{1} - 21\,.\frac{1}{3} + 35\,.\frac{1}{5} - 7.\frac{1}{7}\right] +$$

$$+ \frac{1}{2^9}\left[\frac{1}{1} - 28\,.\frac{1}{3} + 70\,.\frac{1}{5} - 28\,.\frac{1}{7} + \frac{1}{9}\right] + \, . \; . \; . \; . \, ;$$

The first nine terms contain of the power series for $tan^{-1} x$ only the first five coefficients. Together they give: $tan^{-1} 1 = 0.7821$ instead of 0.7854.

Out of the first five terms of the power series we should find 0.8349. It is not difficult to transform the terms of the series of polynomials into a simpler form.

After some deduction we get

$$\frac{\pi}{4} = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{6}\right) - \frac{1}{8}\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{11}\right) + \frac{1}{32}\left(\frac{1}{9} + \frac{1}{10} + \frac{1}{22}\right) + \dots$$

or

$$\frac{\pi}{2} = \sum_{n=0}^{n=\infty} \left(-\frac{1}{4}\right)^n \left[\frac{1}{4\,n+1} + \frac{1}{4\,n+2} + \frac{1}{8\,n+6}\right],$$

and by a similar notation it is evident that really $1 : \sqrt{2}$ is the mark of the series of polynomials.

The example treated of here can serve to notice a phenomenon which will undoubtedly make its appearance in more intricate cases.

In the series (VII) all kinds of values of the parameter a might have served to calculate $tan^{-1} 1$. If for simplicity's sake we consider only real values of a, then the following is to be noticed.

For $a = \infty$ we should have obtained the narrowest region of convergence G_1, which encloses just the point $x = 1$; the mark of the series would have been 1. A wide region of convergence we should have had for instance for $a = \frac{3}{2}$. In the direction of the axis of reality it would have extended to $x = \sqrt{5}$, and in $x = 1$ the mark of the series would have amounted to $3 : \sqrt{13} = 0.83$. The smallest mark 0.70 was obtained for $a = 2$, that is with a region of convergence reaching to $a = \sqrt{3}$.

Something similar will always take place as soon as we can enlarge or reduce the region of convergence by changing a parameter. If by means of a series of polynomials we wish to find the value of $F(x)$ in a given point x, we must avoid an unnecessary extension of the region of convergence, but on the other hand we must not allow the boundary of this region to approach the point x too much. A region of convergence enclosing x not too narrowly and yet not too widely furnishes the best converging series of polynomials.

Physics. — Prof. J. D. VAN DER WAALS on: "*Ternary Systems.*"
II. (Continued from page 463).

It occurs frequently in a binary mixture that two phases coexist which have the same concentration. In this case the pressure when we move along the connodal curve is either maximum or minimum. An isobar may then be drawn, touching both the liquid- and the vapour branch, and the isobars of other values of p cut then both branches twice. The ζ-curves for such a binary mixture will have a much more complicated shape than I have given up to now, and if we should now have a ternary system in which one or two or the three pairs which may be formed from it show this particularity then the ζ-surface will also show particularities whose main features we shall examine. As a *minimum* pressure has not yet been observed for normal substances, we shall only discuss the case of *maximum* pressure.

The property that for a binary system the concentration of liquid and vapour are the same, coheres with another property for such systems, which I have discussed, Cont. II, p. 86, though I have neglected to point out the relation between these two properties. This second property and the relation is found from the following formula. One of the conditions for coexistence is:

$$\left(\frac{\partial\psi}{\partial x_1}\right)_{v_1T} = \left(\frac{\partial\psi}{\partial x_2}\right)_{v_2T}.$$

We write

$$\psi = f(x) - \int p\,dv,$$

and so

$$\left(\frac{\partial\psi}{\partial x}\right)_{vT} = f'(x) - \int\left(\frac{\partial p}{\partial x}\right)_{vT} dv.$$

For the case that $x_1 = x_2$ we derive from the above formulae:

$$\int_{v_2}^{v_1}\left(\frac{\partial p}{\partial x}\right)_{vT} dv = 0$$

which equation occurs already in Théor. Mol. Arch. Néerl. XXIV.

In order that $\left(\frac{\partial p}{\partial x}\right)_{vT} dv$, integrated between the liquid and the vapour volume, be equal to 0, $\left(\frac{\partial p}{\partial x}\right)_{vT}$ must be equal to zero somewhere between these two volumes. In consequence the particularity that mixtures may be formed, for which $x_1 = x_2$ occurs only when a locus exists in the xv diagram, along which $\left(\frac{\partial p}{\partial x}\right)_{vT} = 0$. Accordingly QUINT has observed the circumstance that, keeping T constant the curve $p = f(x, v)$ in the mixture of HCl and C_2H_6 shows a maximum. In Cont. II p. 86. I have discussed such a locus, and proved that for great volumes it has an asymptote parallel to the volume-axis, and that for small volumes it moves to the side of the component for which b is greater. In fig. 7 the dotted line passing through P and Q, represents this locus. On the left of this curve $\left(\frac{\partial p}{\partial x}\right)_{vT}$ is positive, and on the right negative. All the isobars must then possess a tangent parallel to the x-axis in the points where they cut this locus. In fig. 7 the course of some curves of equal pressure has been traced. The temperature is assumed to be so low that the plait on the ψ-surface stretches over the whole breadth of the diagram, and so the curve, for which $\left(\frac{\partial p}{\partial v}\right)_{xT} = 0$, continues to consist of two isolated branches. The curves $L\,P\,M$ and $L'\,Q\,M'$ represent these branches, viz. the dotted ones.

The limits of the unstable region are somewhat wider, and they are also indicated as passing through L, P and M, or L', Q and M'; in the figure they are indicated by lines of alternately larger and smaller dots. That these limits of the unstable region must pass through P and Q, follows from the property, that if $\left(\frac{\partial p}{\partial x}\right)_{vT}$ is equal to 0, the condition:

$$\frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial v^2} - \left(\frac{\partial^2 \psi}{\partial x\, \partial v}\right)^2 = 0$$

is satisfied in the points, in which $\frac{\partial^2 \psi}{\partial v^2} = -\left(\frac{\partial p}{\partial v}\right)_x = 0$ is.

If we closely examine the character of the points P and Q, we conclude that p in the point Q is really a maximum. The point Q,

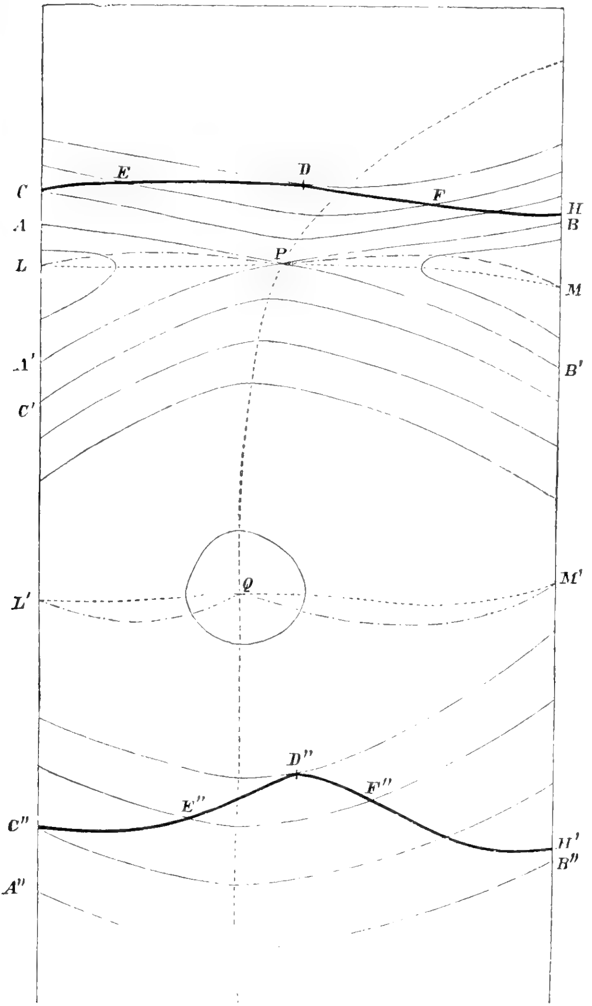

Fig 7.

namely, lies on the isothermal of the concentration x_Q and on that isothermal it is the point where the pressure is maximum while on a section parallel to the x-axis, the value of p is also maximum. The point P on the other hand, lying on the isothermal of the concentration x_P, represents a phasis for which the pressure on the isothermal is minimum, while p along a line parallel to the x-axis in P, is maximum. From this follows 1st. that the isobar passing through P has a double point in the point P, and 2nd that the lines of equal pressure enclose the point Q, — and in such a way that as the pressure approaches that of Q, the closed curves get narrower, and keep also entirely within the limits of $x = 0$ and $x = 1$ and are completely enclosed in the xv diagram.

The isobar through P had a shape which may be considered as transition form between the shapes for pressures which are greater, and pressures which are smaller than p_P. For pressures which are greater the isobar consists of two separate branches, viz. 1st. a closed curve round Q and 2nd a branch lying above P and which therefore, remaining on the liquid sheet, belongs to a volume smaller than that of P. In fig. 7 we can take for it e. g. the branch passing through C, which must show a maximum-volume on the dotted line. Then the closed curve of Q, belonging to the same isobar, is indicated by the curves passing through C' and C''. For the isobar of P these two separate branches have drawn so near each other, that they have met in P. Accordingly this isobar has the following shape, indicated by $APB'B''A''A'PB$. The points B' and B'' must be thought connected by a piece lying outside $x = 1$ and in the same way the points A'' and A' by a piece outside $x = 0$. The lines of equal pressure for $p < p_P$ must fill up the space lying within APA' and BPB' and that below $A''B''$. A similar curve, provided $p > p_L$, begins below A on the curve $x = 0$, passes on to greater values of x, has a tangent parallel to the v-axis on the locus for which $\left(\frac{\partial p}{\partial v}\right)_x = 0$, and then returns to smaller values of x. It is continued for vapour volumes greater than that of $A''B''$, and appears again between BPB' provided p be also greater than P_M. The shape between BPB' is analogous to that of APA'.

In order to find the mixture for which liquid and vapour concentration is equal, we must determine on a line, parallel to the v-axis two points chosen on the same isobar, in such a way that $\int_{v_2}^{v_1} \left(\frac{\partial p}{\partial x}\right) dv = 0$. In the figure the points D' and D'' have been chosen for this. Of course one point D had to be taken on the left of the locus PQ, the other on the right.

The pressure on the connodal curve being maximum for that mixture, the connodal curve must touch a curve of equal pressure both in D and in D'. On the left and on the right of D and also of D' the connodal curve must pass on to isobars of lower pressure. If we assume the maximum pressure of the first component at the chosen temperature to be equal to p_C, and that of the second component to be lower e.g. p_B, the connodal curve has a shape as is represented by the somewhat heavier curves $CEDFB$ and $C''E''D''F''B''$. But we must take care that $x_{E''} > x_E$ and $x_{F''} < x_F$.

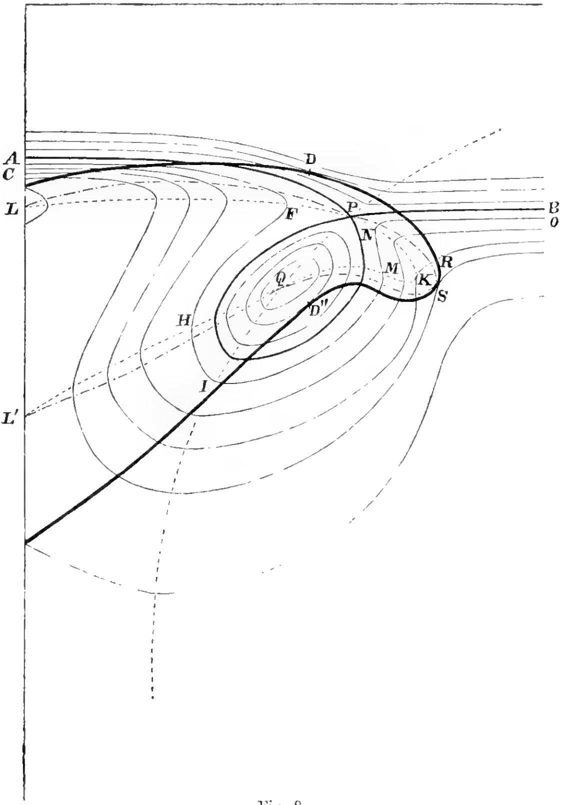

Fig. 8.

The accurate knowledge of the course of the isobars is required for indicating the value of ζ at every value of p for the binary mixture and so also for the description of the shape of the ζ-surface for a ternary mixture, if the here discussed peculiarity occurs for at least one of the pairs. And in order to be able to do this also at higher temperatures, at which the critical phenomena occur, and because the course of the isobars at such temperatures is greatly modified, I have represented this modification in Fig. 8 (pag. 543).

The principal modification, which is to be introduced in fig. 7, and which leads to fig. 8 is this, that the isobar through P presents its whole closed curve within the limits of $x=0$ and $x=1$. The curve PQ has slightly changed its place and its shape (see Cont. II pag. 88), but the change is comparatively small. That the whole curve keeps within the xv-diagram follows from two circumstances. 1st. The pressure of P, which is a minimum pressure on the isothermal of x_P, lies but little below that which can be realised as liquid phasis, if the temperature is close to that at which both minimum- and maximum pressure disappear and 2nd the difference of pressure along a connodal curve generally increases at higher temperatures. The whole curve showing itself, the course of isobars for which $p < p_P$ can also take place without interruption within the whole xv-diagram — at least for values of p lying above a certain limit which may be derived from the preceding considerations without nearer indication. If we follow such an isobar, e.g. $CFHIMNO$, it must be possible to draw a tangent parallel to the v-axis in F and H and also in M and N, in which four points the curve for which $\left(\frac{\partial p}{\partial v}\right)_x = 0$ is cut; in point I, in which the curve for which $\left(\frac{\partial p}{\partial x}\right)_v = 0$, is cut, the tangent must be parallel to the x-axis Between F and H, and also between M and N the value of x retrogrades. For the isobar passing through K the retrogression on the left side of the figure continues, but it has just ceased on the right side.

In fig. 8 point S indicates the plaitpoint and the isobar passing through S must therefore touch the connodal curve in that point. In the same way the connodal curve must touch a curve of equal pressure in the points D and D'. All this proves that the temperature is thought to be so high, that there is still question of a maximum pressure on the connodal curve. (Consult the observations of KUENEN and those of QUINT for mixtures which have minimum-critical temperature). Between S and R is retrograde condensation of the second kind and the fact that the values of x_R and x_S differ so little is in accordance with the fact that it is very difficult to prove r. c. II experimentally.

Let us now proceed to describe the shape of ζ for the binary mixture, in the first place according to fig. 7, so at lower temperatures. Let us begin with $p < p_L$, so p smaller than the minimum pressure of the isothermal of the first component. We assume this value of p_L to be greater than 0. In this case has ζ one value, at least on the side of $x = 0$. As soon as p is chosen somewhat greater than p_L, there are three values of the volume for small values of x and so also for ζ. If we apply the same considerations to values of x near $x = 1$, p_M must be substituted for p_L. The whole curve consists then, for p somewhat larger than p_L and p_M, first of a continuous curve (vapour branch), and further of two separate parts lying on the right and the left, each terminating in a cusp (see fig. 9).

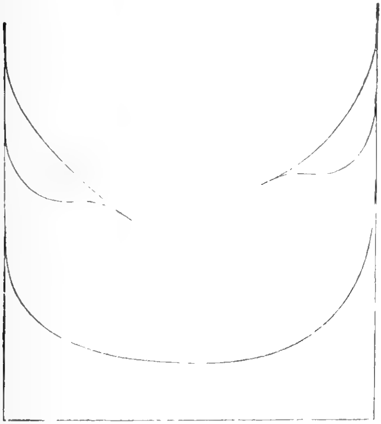

Fig. 9.

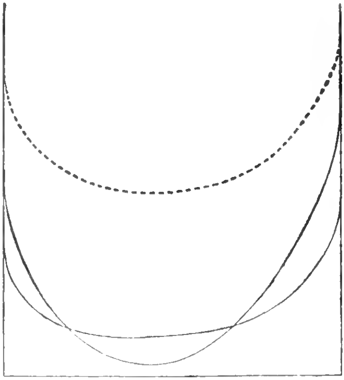

Fig 10.

If p has increased to the value which the pressure has on the line with the loop, the two cusps in fig. 9 have met, and the upper branches on the right have coincided with those on the left and form two curves with a double point. This is the case which I mentioned in note (1) on p. 459. In this case the vapour branch lies still lowest, then follows the liquid branch which shows a discontinuity, and above it again the branch of unstable conditions, also showing a discontinuity.

As soon as the pressure has increased to p_C (the maximum tension of the first component) the vapour branch has moved so far upward that it has reached the liquid branch on the left side of the figure. For the right side this would take place for $p = p_H$ (maximum tension of the second component). And for pressures between p_C and p_D the gas- and liquid branches have a double point on the left of x_D — in the same way for pressures between p_H and p_D a double point on the right of x .

For pressures above P_D the gas-branch has moved above the liquid branch; for $p = P_D$ these two branches touched each other. If the pressure is made to draw near to p_Q, the gas-branch and the branch of unstable phases form a closed curve, which has a cusp right and left, which curve is reduced to a point for $p = p_Q$, and for still higher values of p also this point has disappeared, and only the liquid branch remains.

We shall be brief in the discussion of the value of ζ at different pressure at the temperature assumed in fig. 8. The ζ-curve for the pressure $p = p_C$ is represented in fig. 11; the four cusps lie at x_F, x_H, x_M and x_N. For a somewhat lower pressure $p = p_K$ the right crest has disappeared, and for a still lower pressure $p = p_S$ (plaitpoint pressure) the right part of ζ is curved continuously. So we have here between S and R retrograde condensation of the second kind. I shall leave the modification of ζ for pressures greater than p_C undiscussed.

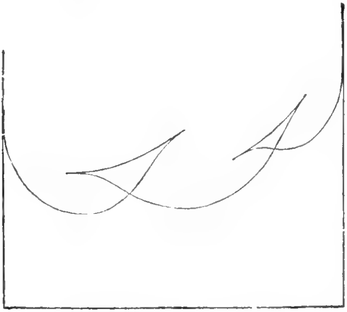

Fig. 11.

For a binary mixture I have pointed out that there is a connection between the circumstance that two phases of equal concentration can coexist, and the circumstance that for a mixture of the two components of that system a minimum critical temperature occurs. The existence of such a connection having since been perfectly confirmed by the experiments of KUENEN and QUINT, we are naturally led to investigate whether a minimum-critical temperature can occur also for a mixture of three substances — and what are the conditions for the existence of such a minimum critical temperature. That connection could however not follow and could not be derived simply from the principle of continuity, but considerations of a molecular-theoretical nature were required to conclude to the existence of such a connection. Therefore I shall at the moment, as we consider it only our task to examine what follows for a ternary system from the assumption of continuity, refrain from seeking the conditions which the components must satisfy in order to be able to form a mixture which possesses maximum pressure and assume only that a mixture can really be formed from the three chosen components for which liquid and vapour are composed in the same way and whose coexistence pressure is therefore maximum.

If we take the pressure somewhat smaller than that maximum pressure, so that we get a section of vapour and liquid sheet as drawn in fig. 10 for every section normal to the xy-surface passing through the point representing that special mixture, the connodal curve will consist of two closed curves, of which the inner curve indicates the vapour phasis. If p is equal to that maximum pressure, the two closed curves have been reduced to one point, the point where the two sheets touch each other. Under a still greater pressure the vapour sheet will have risen quite above the liquid sheet. With decreasing pressure the two closed curves extend, and if we took only the principle of continuity into account, a great many cases would be possible. For instance the extending closed curves might reach the sides of the triangle which represent the pairs of which the ternary system consists, and cut them in two points, either one side or two sides, or all three the sides. In the last case the three pairs which compose the ternary system, would possess all three the properties of maximum pressure. But an extension is also possible, at which the second and the third side is never cut twice — and even one at which none of the sides is cut twice, and at which therefore the closed curve which extends and which is changing its shape, reaches the sides of the triangle for the first time in one of the angles. In this case the ternary system would have maximum pressure, without this being the case with any of the pairs of which it consists. The investigation of conditions which are required for a minimum critical temperature will probably be able to decide the question as to the possibility of these cases. But I shall not enter into this subject at least for the present.

Between the two closed parts of which the projection of the connodal curve on the xy-surface consists, lies another closed curve, the projection of the double points. The curve, which consisted of one branch in the case discussed on page 460 within the triangle OAB, consists in this case of two branches lying within the triangle. At any rate it will always consist of two branches theoretically; but for the ζ-surface we need only to know that part which lies within the triangle. When p has the value of the before-mentioned maximum pressure, the closed curve of double points too contracts to one point. This point is the same as that to which the two closed parts of the connodal curve contract.

If for a moment we take recourse to molecular-theoretical considerations to derive properties of the locus of the double points, we should write down the equation of p. 461 in this way:

$$log \frac{p_{cr}}{p} = f\left(\frac{T_{cr}}{T} - 1\right).$$

In this equation which holds good at least as an approximation, we put $f =$ constant and $p_{cr} = \frac{1}{27}\frac{a}{b^2}$ and $\frac{T_{cr}}{273} = \frac{8}{27}\frac{a}{b}$.

Keeping T constant, we get by differentiation:

$$\frac{dp}{p} = \left(\frac{da}{a} - 2\cdot\frac{db}{b}\right) - f\frac{T_{cr}}{T}\left(\frac{da}{a} - \frac{db}{b}\right)$$

or

$$\frac{dp}{p} = \left(f\frac{T_{cr}}{T} - 2\right)\frac{db}{b} - \left(f\frac{T_{cr}}{T} - 1\right)\frac{da}{a}.$$

Keeping p constant we find the condition:

$$\frac{\frac{da}{a}}{\frac{db}{b}} = \frac{f\frac{T_{cr}}{T} - 2}{f\frac{T_{cr}}{T} - 1}.$$

In the limiting case, for continually decreasing values of T, the value of the second member $= 1$, and so:

$$\frac{1}{a}\left(\frac{\partial a}{\partial x} + \frac{dy}{dx}\frac{\partial a}{\partial y}\right) = \frac{1}{b}\left(\frac{\partial b}{\partial x} + \frac{dy}{dx}\frac{\partial b}{\partial y}\right),$$

from which for the projection of the curve of the pressures of coincidence at very low temperatures follows:

$$\frac{dy}{dx} = -\frac{\frac{1}{a}\frac{\partial a}{\partial x} - \frac{1}{b}\frac{\partial b}{\partial x}}{\frac{1}{a}\frac{\partial a}{\partial y} - \frac{1}{b}\frac{\partial b}{\partial y}}$$

or

$$\frac{dy}{dx} = -\frac{\frac{\partial \frac{a}{b}}{\partial x}}{\frac{\partial \frac{a}{b}}{\partial y}} .$$

If $\frac{dy}{dx}$ has an arbitrary value, as is the case when the locus has been reduced to a point, then $\frac{\partial \frac{a}{b}}{\partial x} = 0$, and in the same way $\frac{\partial \frac{a}{b}}{\partial y} = 0$, i.e. $\frac{a}{b}$ and so also T_{cr} must be capable of having a minimum. In this way we arrive therefore for a ternary system at the same result, as I had formerly obtained for a binary system, also for the limiting case of low temperatures.

If we do not consider the limiting case, which would correspond with $T=0$, but if we give T a definite value, we find:

$$d\frac{a}{b} = -\frac{1}{f\frac{T_{cr}}{T} - 1}\,\frac{a}{b}\,\frac{db}{b},$$

from which follows that for the maximum pressure at temperature $= T$ the values of x and y are found from the two following equations

$$\frac{\partial \frac{a}{b}}{\partial x} = -\frac{1}{f\frac{T_{cr}}{T} - 1}\,\frac{a}{b}\,\frac{1}{b}\,\frac{\partial b}{\partial x}$$

and

$$\frac{\partial \frac{a}{b}}{\partial y} = - \frac{1}{f\frac{T_{cr}}{T} - 1} \frac{a}{b} \frac{1}{b} \frac{\partial b}{\partial y}.$$

If we assume that b decreases with x, $\frac{a}{b}$ must increase with x and in the case of a minimum it must have passed that minimum. So the point for which the coincidence pressure is maximum, and for which also there is equality of composition of liquid- and vapour phasis, lies more to the side of the components with the smallest molecules than the point indicating the mixture with minimum-critical-temperature — and this will be the more so in proportion as T is higher.

For the case that T_{cr} as function of x and y might be represented by approximation as a plane (see Cont. II, p. 153) we find $\frac{dy}{dx}$ = constant, and so the line of the double points under a constant pressure is a straight one.

RELATION OF VOLUME, CONCENTRATION AND TEMPERATURE FOR COEXISTING PHASES OF A TERNARY SYSTEM.

In fig. 2 of the preceding communication the curve $ABPRB'A$ represents the projection of the connodal curve on the ψ-surface of a binary system in the xv-diagram at a constant temperature. This line may also be taken as representing the relation between molecular volume and concentration of a binary mixture at given temperature. If the temperature was put lower, this curve would have consisted of two isolated branches, one representing the liquid volumes, and the other the gas-volumes. Let us now think as third axis, a y-axis, and let us think also in the oyv-plane a similar curve drawn for a binary mixture that consists of the first and the third substance. If we further draw for every point of the right-angled triangle of the oxy-plane the volume at which a mixture represented by that point loses or resumes its homogeneity at increasing pressure, we get a surface which consists of two isolated sheets at low temperatures, and which at higher temperatures e. g. when T is above T_{cr} of one of the components, is contracted to one sheet.

If T is increased the form of the surface is modified in that sense that the new surface lies quite within that of lower temperature. At least for substances which do not enter into chemical combinations with each other, and which continue to consist in themselves of invariable molecules — so which do not associate to more complicate atom-groups.

For a binary mixture I have (Cont. II p. 101) derived the differential-equation for the relation between v, x and T. We shall be able to find in the same way the differential equation for the relation between v, x, y and T.

For coexistence of two phases of a ternary system, distinguishing the phases by the indices 1 and 2, the following equations must be satisfied

$$\left(\frac{\partial\psi}{\partial v}\right)_1 = \left(\frac{\partial\psi}{\partial v}\right)_2$$

$$\left(\frac{\partial\psi}{\partial x}\right)_1 = \left(\frac{\partial\psi}{\partial x}\right)_2$$

$$\left(\frac{\partial\psi}{\partial y}\right)_1 = \left(\frac{\partial\psi}{\partial y}\right)_2$$

and

$$\psi_1 - v_1\left(\frac{\partial\psi}{\partial v}\right)_1 - x_1\left(\frac{\partial\psi}{\partial x}\right)_1 - y_1\left(\frac{\partial\psi}{\partial y}\right)_1 =$$

$$= \psi_2 - v_2\left(\frac{\partial\psi}{\partial v}\right)_2 - x_2\left(\frac{\partial\psi}{\partial x}\right)_2 - y_2\left(\frac{\partial\psi}{\partial y}\right)_2$$

in which $\frac{\partial\psi}{\partial v}$ represents $\left(\frac{\partial\psi}{\partial v}\right)_{xyT}$ etc.

If the concentration for the first phasis is given, and so x_1 and y_1, then the quantities v_1, x_2, y_2 and v_2 are determined by the four above equations, and so the properties of the coexisting phasis. But in order to calculate them all the equations would have to be known, for which the knowledge of the equation of state is required. Even if we make use of them, the intricacy of these forms does not admit of the solution of the unknown quantities. Results, however, can be derived from the differential equation, even if we do not know these quantities accurately, and these results are not without interest. In the same way as is followed in Cont. I page 102 for a binary mixture, we find for a ternary system:

$$(v_2-v_1)\left\{\frac{\partial^2\psi}{\partial v_1^{\,2}}dv_1+\frac{\partial^2\psi}{\partial v_1\,\partial x_1}dx_1+\frac{\partial^2\psi}{\partial v_1\,\partial y_1}dy_1\right\}+$$

$$+(x_2-x_1)\left\{\frac{\partial^2\psi}{\partial v_1\,\partial x_1}dv_1+\frac{\partial^2\psi}{\partial x_1^{\,2}}dx_1+\frac{\partial^2\psi}{\partial y_1\,\partial x_1}dy_1\right\}+$$

$$+(y_2-y_1)\left\{\frac{\partial^2\psi}{\partial v_1\,\partial y_1}dv_1+\frac{\partial^2\psi}{\partial x_1\,\partial y_1}dx_1+\frac{\partial^2\psi}{\partial y_1^{\,2}}\right\}+(\varepsilon_{21})_v\frac{dT}{T}=0\;.\;.\;.\;.\;.\;(1)$$

The quantity $(\varepsilon_{21})_v$ is, see l.c. p. 104, for normal substances a negative quantity.

If we keep T constant for the moment, so if we inquire into properties of one of the before-mentioned surfaces, we can derive the following rule for the position of the line that connects the two coexisting phases. If we for instance imagine on the liquid sheet a point determined by v_1, x_1 and y_1 and if we inquire into the direction of the line connecting the coexisting phasis with the chosen liquid phasis, so into the quantities proportionate to v_2-v_1, x_2-x_1 and y_2-y_1, we bring in point 1 as center a quadric surface:

$$\frac{\partial^2\psi}{\partial v_1^{\,2}}v^2+\frac{\partial^2\psi}{\partial x_1^{\,2}}x^2+\frac{\partial^2\psi}{\partial y_1^{\,2}}y^2+2\frac{\partial^2\psi}{\partial x_1\,\partial v_1}xv+2\frac{\partial^2\psi}{\partial y_1\,\partial v_1}yv+2\frac{\partial^2\psi}{\partial x_1\,\partial y_1}xy=C. \quad (2)$$

We cut that surface through the tangent plane at the liquid sheet, then the direction of the line connecting the two nodes, will be conjugate to the section of tangent plane and quadric surface cosines. The locus of the middle of the chords, whose cosines are equal to λ, μ and ν, is given by:

$$\lambda\frac{\partial f}{\partial v}+\mu\frac{\partial f}{\partial x}+\nu\frac{\partial f}{\partial y}=0$$

and this equation leads to (1), when in equation (1) dT is put equal to 0 and when dv, dx and dy, are substituted for v, x and y and so when this middle plane is tangent plane to the v, x, y-surface under consideration.

On account of the importance which the surface represented by (2) has for the equilibrium of the ternary systems, it deserves a closer examination.

If a definite quantity of a substance, which is ternary composed, is to be in equilibrium at given temperature in a given volume then $\frac{\partial\psi}{\partial v}$, $\frac{\partial\psi}{\partial x}$, $\frac{\partial\psi}{\partial y}$ and $\psi-v\frac{\partial\psi}{\partial v}-x\frac{\partial\psi}{\partial x}-y\frac{\partial\psi}{\partial y}$ must have an invariable value throughout the space. For a homogeneous phasis this condition is satisfied. And when therefore the given quantity of substance is homogeneously distributed through the space, we have a state of equilibrium. But if that state is to be realised the condition of stability must also be satisfied. From the principle that ψ must be a minimum, we derive for the condition of stability: ¹)

$$\frac{\partial^2\psi}{\partial v^2}dv^2+\frac{\partial^2\psi}{\partial x^2}dx^2+\frac{\partial^2\psi}{\partial y^2}dy^2+2\frac{\partial^2\psi}{\partial x\,\partial v}dx\,dv+$$
$$+2\frac{\partial^2\psi}{\partial y\,\partial v}dy\,dv+2\frac{\partial^2\psi}{\partial x\,\partial y}dx\,dy>0\,.$$

This condition can be brought under the following form: ²)

$$\frac{1}{\dfrac{\partial^2\psi}{\partial v^2}}\left\{\frac{\partial^2\psi}{\partial v^2}dv+\frac{\partial^2\psi}{\partial x\,\partial v}dx+\frac{\partial^2\psi}{\partial y\,\partial v}dy\right\}^2+\left\{\frac{\partial^2\psi}{\partial x^2}-\frac{\left(\dfrac{\partial^2\psi}{\partial x\,\partial v}\right)}{\dfrac{\partial^2\psi}{\partial v^2}}\right\}dx^2+$$
$$+\left\{\frac{\partial^2\psi}{\partial y^2}-\frac{\left(\dfrac{\partial^2\psi}{\partial y\,\partial v}\right)}{\dfrac{\partial^2\psi}{\partial v^2}}\right\}dy^2+2\left\{\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\dfrac{\partial^2\psi}{\partial x\,\partial v}\dfrac{\partial^2\psi}{\partial y\,\partial v}}{\dfrac{\partial^2\psi}{\partial v^2}}\right\}dx\,dy>0\quad.\ (3)$$

Now follows from:

$$\left(\frac{\partial\zeta}{\partial x}\right)_{pTy}=\left(\frac{\partial\psi}{\partial x}\right)_{vTy}$$

$$\left(\frac{\partial^2\zeta}{\partial x^2}\right)_{pTy}=\left(\frac{\partial^2\psi}{\partial x^2}\right)_{vTy}+\left(\frac{\partial^2\psi}{\partial x\,\partial v}\right)_{Ty}\left(\frac{dv}{dx}\right)_{pTy}$$

and from:

$$\left(\frac{\partial\psi}{\partial v}\right)=-p$$

follows

$$\left(\frac{\partial^2\psi}{\partial v\,\partial x}+\frac{\partial^2\psi}{\partial v^2}\frac{dv}{dx}\right)_p=0$$

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¹) For a binary system the derivation of the condition of stability is given Cont. II p. 8. Before that time in Théor. Mol. Arch. Néerl. XXIV.

²) See Arch. Néerl. Série II Tome II page 73.

Hence

$$\left(\frac{\partial^2\zeta}{\partial x^2}\right)_{pTy} = \frac{\partial^2\psi}{\partial x^2} - \frac{\left(\dfrac{\partial^2\psi}{\partial x\,\partial v}\right)^2}{\dfrac{\partial^2\psi}{\partial v^2}}.$$

In the same way we find

$$\left(\frac{\partial^2\zeta}{\partial y^2}\right)_{pTx} = \frac{\partial^2\psi}{\partial y^2} - \frac{\left(\dfrac{\partial^2\psi}{\partial y\,\partial v}\right)^2}{\dfrac{\partial^2\psi}{\partial v^2}}$$

and

$$\left(\frac{\partial^2\zeta}{\partial x\,\partial y}\right)_{pT} = \frac{\partial^2\psi}{\partial x\,dy} - \frac{\dfrac{\partial^2\psi}{\partial x\,\partial v}\dfrac{\partial^2\psi}{\partial y\,\partial v}}{\dfrac{\partial^2\psi}{\partial v^2}}$$

So we can write (3) also under the following form:

$$\frac{1}{\dfrac{\partial^2\psi}{\partial v^2}}\left\{\frac{\partial^2\psi}{\partial v^2}dv + \frac{\partial^2\psi}{\partial x\,\partial v}dx + \frac{\partial^2\psi}{\partial y\,\partial v}dy\right\}^2 + \frac{\partial^2\zeta}{\partial x^2}dx^2 + \frac{\partial^2\zeta}{\partial y^2}dy^2 + 2\frac{\partial^2\zeta}{\partial x\,\partial y}dx\,dy > 0$$

or

$$\frac{1}{\dfrac{\partial^2\psi}{\partial v^2}}\left\{\frac{\partial^2\psi}{\partial v^2}dv + \frac{\partial^2\psi}{\partial x\,\partial v}dx + \frac{\partial^2\psi}{\partial y\,\partial v}dy\right\}^2 + \frac{1}{\dfrac{\partial^2\zeta}{\partial x^2}}\left\{\frac{\partial^2\zeta}{\partial x^2}dx + \frac{\partial^2\zeta}{\partial x\,\partial y}dy\right\}^2 +$$

$$+\,dy^2\left\{\frac{\partial^2\zeta}{\partial y^2} - \frac{\left(\dfrac{\partial^2\zeta}{\partial x\,\partial y}\right)^2}{\dfrac{\partial^2\zeta}{\partial x^2}}\right\} > 0$$

In order to satisfy this last equation for every arbitrary value of dv, dx and dy, the following equation must be satisfied:

$$1^{\text{st}}\ \frac{\partial^2\psi}{\partial v^2} > 0\ ,\quad 2^{\text{nd}}\ \frac{\partial^2\zeta}{\partial x^2} > 0 \quad\text{and}\quad 3^{\text{rd}}\ \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2} - \left(\frac{\partial^2\zeta}{\partial x\,\partial y}\right)^2 > 0.$$

The form sub. 3rd or

$$\left\{\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial v^2} - \left(\frac{\partial^2\psi}{\partial x\,\partial v}\right)^2\right\}\left\{\frac{\partial^2\psi}{\partial y^2}\frac{\partial^2\psi}{\partial v^2} - \left(\frac{\partial^2\psi}{\partial y\,\partial v}\right)^2\right\} - \left\{\frac{\partial^2\psi}{\partial x\,\partial y}\frac{\partial^2\psi}{\partial v^2} - \frac{\partial^2\psi}{\partial x\,\partial v}\frac{\partial^2\psi}{\partial y\,\partial v}\right\}^2 > 0$$

can be brought under the form:

$$\frac{\partial^2 \psi}{\partial v^2} \times \begin{vmatrix} \frac{\partial^2 \psi}{\partial v^2} & \frac{\partial^2 \psi}{\partial x \, \partial v} & \frac{\partial^2 \psi}{\partial y \, \partial v} \\ \frac{\partial^2 \psi}{\partial x \, \partial v} & \frac{\partial^2 \psi}{\partial x^2} & \frac{\partial^2 \psi}{\partial x \, \partial y} \\ \frac{\partial^2 \psi}{\partial y \, \partial v} & \frac{\partial^2 \psi}{\partial x \, \partial y} & \frac{\partial^2 \psi}{\partial y^2} \end{vmatrix} > 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

And we know from the theory of the quadric surfaces that if a relation exists between the coefficients as is indicated by the equation (4), such a surface is an ellipsoid. Coexisting phases being stable phases, the surface is a real ellipsoid, if C is positive.

If we bring through the line connecting the coexisting phases a plane cutting the tangent plane at the v, x, y-surface along a straight line and the surface of stability along an ellipse, then the directions of the nodal line and the before-mentioned straight line are conjugate directions for those elliptic sections. In the same way conjugate directions are the projection of these two lines on an arbitrary plane for the elliptic projection on that plane. If we give to the plane such a position that:

$$\frac{\partial^2 \psi}{\partial v^2} dv + \frac{\partial^2 \psi}{\partial x \, dv} dx + \frac{\partial^2 \psi}{\partial y \, dv} dy = 0$$

or what is the same $p =$ constant, then the factor of $v_2 - v_1$ is equal to zero, and we get after having eliminated dv_1:

$$(x_2 - x_1) \left\{ \frac{\partial^2 \zeta}{\partial x_1^{\,2}} dx_1 + \frac{\partial^2 \zeta}{\partial x_1 \, dy_1} dy_1 \right\} + (y_2 - y_1) \left\{ \frac{\partial^2 \zeta}{\partial x_1 \, \partial y_1} dx_1 + \frac{\partial^2 \zeta}{\partial y_1^{\,2}} dy_1 \right\} = 0 \ (5)$$

The projection on the xy-plane of the line connecting the coexisting phases, is therefore conjugate to the projection on that plane of the section of the tangent plane, indicated by $p =$ constant, with respect to the elliptic projection of the section of the surface of stability. This is the theorem which has been proved under another form Arch. Néerl. p. 76.

By giving such a position to the plane that

$$\frac{\partial^2 \psi}{\partial x \, \partial v} dv + \frac{\partial^2 \psi}{\partial x^2} dx + \frac{\partial^2 \psi}{\partial y \, \partial x} dy = 0$$

or $\frac{\partial \psi}{\partial x}$ = constant, we might derive an equation analogous to (5) for the y, v-surface; also for the x, v-surface by choosing the plane in such a way that:

$$\frac{\partial^2 \psi}{\partial y\, \partial v} dv + \frac{\partial^2 \psi}{\partial y\, \partial x} dx + \frac{\partial^2 \psi}{\partial y^2} dy = 0$$

or $\frac{\partial \psi}{\partial y}$ = is constant.

If we take volumes lying within the limits of the x, v, y-surface under consideration, which we shall henceforth call surface of coexistence, then the homogeneous phase thought in such a volume, will be stable, as long as

$$\frac{\partial^2 \psi}{\partial v^2} > 0 \; , \; \frac{\partial^2 \zeta}{\partial x^2} = \frac{\partial^2 \psi}{\partial x^2} - \frac{\left(\frac{\partial^2 \psi}{\partial x \partial v}\right)^2}{\frac{\partial^2 \psi}{\partial v^2}} > 0 \text{ and } \frac{\partial^2 \zeta}{\partial y^2} - \frac{\left(\frac{\partial^2 \zeta}{\partial x \partial y}\right)^2}{\frac{\partial^2 \zeta}{\partial x^2}} > 0$$

In proportion as we move further from the sides of the surface of coexistence, we approach the volumes, for which $\frac{\partial p}{\partial v} = \frac{\partial^2 \psi}{\partial v^2} = 0$.

The surface, for which $\frac{\partial^2 \psi}{\partial v^2} = 0$, will for a ternary system take the place of the curve which we have represented by CKC' for a binary mixture in fig. 2 (previous communication).

But the stability will have ceased long before we have reached the volumes for which $\frac{\partial^2 \psi}{\partial v^2} = 0$. For such volumes $\frac{\partial^2 \zeta}{\partial x^2} = -\infty$, whereas the condition of the stability is that this quantity be positive. Also $\frac{\partial^2 \zeta}{\partial y^2}$, which is equal to $\frac{\partial^2 \psi}{\partial y^2} - \frac{\left(\frac{\partial^2 \psi}{\partial y \partial v}\right)^2}{\frac{\partial^2 \psi}{\partial v^2}}$, would be $-\infty$ for such volumes, whereas the condition of stability is not only that this quantity be positive, but even that it have a value such that:

$$\frac{\partial^2 \zeta}{\partial y^2} \; \frac{\partial^2 \zeta}{\partial x^2} > \left(\frac{\partial^2 \zeta}{\partial x \partial y}\right)^2 .$$

The conditions for stability increase therefore with the number of components. For a simple substance a phasis is stable as long as

$$\frac{\partial^2\psi}{\partial v^2} > 0 \, .$$

For a binary mixture the following condition must be satisfied:

$$\frac{\partial^2\psi}{\partial v^2} > \frac{\left(\dfrac{\partial^2\psi}{\partial x\,\partial v}\right)^2}{\dfrac{\partial^2\psi}{\partial x^2}}$$

for a ternary mixture the following condition must be satisfied:

$$\left\{\frac{\partial^2\psi}{\partial v^2} - \frac{\left(\dfrac{\partial^2\psi}{\partial x\,\partial v}\right)^2}{\dfrac{\partial^2\psi}{\partial x^2}}\right\} \left\{\frac{\partial^2\psi}{\partial v^2} - \frac{\left(\dfrac{\partial^2\psi}{\partial y\,\partial v}\right)^2}{\dfrac{\partial^2\psi}{\partial y^2}}\right\} > \left\{\frac{\partial^2\psi}{\partial v^2} - \frac{\dfrac{\partial^2\psi}{\partial x\partial v}\;\dfrac{\partial^2\psi}{\partial y\partial v}}{\dfrac{\partial^2\psi}{\partial x\partial y}}\right\}^2 \frac{\left(\dfrac{\partial^2\psi}{\partial x\,\partial y}\right)^2}{\dfrac{\partial^2\psi}{\partial x^2}\;\dfrac{\partial^2\psi}{\partial y^2}}$$

The transition of the stable and unstable phases takes therefore place at

$$\frac{\partial^2\psi}{\partial v^2} > 0 \, , \; \frac{\partial^2\psi}{\partial v^2} - \frac{\left(\dfrac{\partial^2\psi}{\partial x\,\partial v}\right)^2}{\dfrac{\partial^2\psi}{\partial x^2}} > 0 \, , \; \frac{\partial^2\psi}{\partial v^2} - \frac{\left(\dfrac{\partial^2\psi}{\partial y\,\partial v}\right)^2}{\dfrac{\partial^2\psi}{\partial y^2}} > 0$$

and

$$\begin{vmatrix} \dfrac{\partial^2\psi}{\partial v^2} \, , & \dfrac{\partial^2\psi}{\partial x\partial v} \, , & \dfrac{\partial^2\psi}{\partial y\partial v} \\[2ex] \dfrac{\partial^2\psi}{\partial v\partial x} \, , & \dfrac{\partial^2\psi}{\partial x^2} \, , & \dfrac{\partial^2\psi}{\partial y\partial x} \\[2ex] \dfrac{\partial^2\psi}{\partial v\partial y} \, , & \dfrac{\partial^2\psi}{\partial x\partial y} \, , & \dfrac{\partial^2\psi}{\partial y^2} \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

What the spinodal curve is for the binary mixture (see fig. 2 the curve *CEPE'C'*) the surface represented by (6) is for the ternary mixture, viz. the limit between the stable and unstable phases. In the sides of the prism described on the triangle *oxy*, this surface must therefore pass through the spinodal curves of the pairs of which the ternary system is composed. So we find from equation (6) which may be written:

$$\frac{\partial^2\psi}{\partial v^2}\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial v^2}\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2-\frac{\partial^2\psi}{\partial x^2}\left(\frac{\partial^2\psi}{\partial y\partial v}\right)^2-\frac{\partial^2\psi}{\partial y^2}\left(\frac{\partial^2\psi}{\partial x\partial v}\right)^2+$$

$$+2\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\psi}{\partial x\partial v}\frac{\partial^2\psi}{\partial y\partial v}=0,$$

putting $\frac{\partial^2\psi}{\partial x^2}=\infty$ as it must be in the *ovy* coordinate plane:

$$\frac{\partial^2\psi}{\partial v^2}\frac{\partial^2\psi}{\partial y^2}-\left(\frac{\partial^2\psi}{\partial y\partial v}\right)^2=0.$$

But just as the spinodal curve for a binary mixture (fig. 7 and fig. 8) can have points in common with the curve $\frac{\partial^2\psi}{\partial v^2}=0$, so it can also happen that the two surfaces corresponding to these curves have points in common for a ternary system. First of all they touch if $\frac{\partial^2\psi}{\partial v\,\partial x}$ and $\frac{\partial^2\psi}{\partial v\,\partial y}$ are both equal to zero — and in the second place they have points in common in the edges of the prism, so for the simple substances. And finally just as the spinodal curve and the connodal curve can have a point in common (the plaitpoint) for a binary system, in the same way the corresponding surfaces can have points in common for the ternary system, and touch each other in these points. For if for the second phasis we have:

$$v_2=v_1+dv_1\;x_2=x_1+dx_1\text{ and }y_2=y_1+dy_1$$

the equation (1) becomes:

$$\frac{\partial^2\psi}{\partial v_1^2}dv_1^2+\frac{\partial^2\psi}{\partial x_1^2}dx_1^2+\frac{\partial^2\psi}{\partial y_1^2}dy_1^2+2\frac{\partial^2\psi}{\partial v_1\,\partial x_1}dv_1\,dx_1+$$

$$+2\frac{\partial^2\psi}{\partial v_1\,\partial y_1}dv_1\,dy_1+2\frac{\partial^2\psi}{\partial x_1\,\partial y_1}dx_1\,dy_1=0$$

or

$$\frac{\left\{\frac{\partial^2\psi}{\partial v_1^2}dv_1+\frac{\partial^2\psi}{\partial x_1\,\partial v_1}dx_1+\frac{\partial^2\psi}{\partial y_1\,\partial v_1}dy_1\right\}^2}{\frac{\partial^2\psi}{\partial v_1^2}}+$$

$$+\frac{\left\{\left[\frac{\partial^2\psi}{\partial x_1^2}-\frac{\left(\frac{\partial^2\psi}{\partial x_1\,\partial v_1}\right)^2}{\frac{\partial^2\psi}{\partial v_1^2}}\right]dx_1+\left[\frac{\partial^2\psi}{\partial x_1\,\partial y_1}-\frac{\frac{\partial^2\psi}{\partial x_1\,\partial v_1}\,\frac{\partial^2\psi}{\partial y_1\,\partial v_1}}{\frac{\partial^2\psi}{\partial v_1^2}}\right]dy_1\right\}^2}{\frac{\partial^2\psi}{\partial x_1^2}-\frac{\left(\frac{\partial^2\psi}{\partial x_1\,\partial v_1}\right)^2}{\frac{\partial^2\psi}{\partial v_1^2}}}+$$

$$+\,dy_1^2\left\{\left[\frac{\partial^2\psi}{\partial y_1^2}-\frac{\left(\frac{\partial^2\psi}{\partial y_1\,\partial v_1}\right)^2}{\frac{\partial^2\psi}{\partial v_1^2}}\right]-\frac{\left[\frac{\partial^2\psi}{\partial x_1\partial y_1}-\frac{\frac{\partial^2\psi}{\partial v_1\partial x_1}\,\frac{\partial^2\psi}{\partial v_1\partial y_1}}{\frac{\partial^2\psi}{\partial v_1^2}}\right]^2}{\frac{\partial^2\psi}{\partial x_1^2}-\frac{\left(\frac{\partial\psi}{\partial x_1\,\partial v_1}\right)^2}{\frac{\partial^2\psi}{\partial v_1^2}}}\right\}=0$$

$\frac{\partial^2\psi}{\partial v_1^2}dv_1+\frac{\partial^2\psi}{\partial v_1\,\partial x_1}dx_1+\frac{\partial^2\psi}{\partial v_1\,\partial y_1}dy_1$ being equal to $-\,dp$, and the numerator of the second term being equal to $d\left(\frac{\partial\zeta}{\partial x}\right)_p$, and as p and $\frac{\partial\zeta}{\partial x}$ must be equally great for coexisting phases, the above equation cannot be satisfied without the factor of dy_1^2 being 0. As we saw before, we reduce this factor to the criterion for the limit of the unstable and stable phases; and the surface of coexistence and the spinodal surface have therefore an element in common.

As a rule these two surfaces will not only touch each other in one point, but we shall be able to give a continuous series of points of contact, so a curve along which the surface of coexistence envelopes the spinodal surface. The latter case has already been discussed in our former communication, when at equal temperature and variable pressure every time another mixture was in plaitpoint circumstance. The case that they touch each other only in one point occurs when we can form a mixture of the three components for which T_{cr} is a minimum. But it may be advisable to wait with the discussion of this and similar cases till an experimental investigation has brought them to light.

If by increase of temperature the surface of coexistence has so far contracted that it no longer covers the whole triangle oxy, a tangent cylindre may be drawn normal to the xy-surface. All the points, in which the tangent cylindre and the surface of coexistence touch, represent mixtures which are in critical tangent-point circumstance. A plaitpoint can never lie on this apparent circumference of the surface of coexistence, except in some special cases. For as the generatrices of this tangent cylindre are parallel to the volume-axis and p must have the same value for the pair of phases coinciding in a plaitpoint, we have

$$\frac{\partial^2 \psi}{\partial v^2} = 0$$

for such a special case. In order not to have $\frac{\partial^2 \zeta}{\partial x_1^2}$ or $\frac{\partial^2 \zeta}{\partial y_1^2}$ negative, $\frac{\partial^2 \psi}{\partial x \, \partial v}$ and $\frac{\partial^2 \psi}{\partial y \, \partial v}$ must be equal to 0. Such a mixture behaves as a simple substance even under critical circumstances. See for a similar circumstance with a binary system Cont. II page 116. So the plaitpoints lie either on the liquid sheet, or on the vapour sheet of the surface of coexistence. In the first case all mixtures, indicated by points of the xy-surface, lying between the section of the tangent cylindre and the projection of the curve on which the plaitpoints are situated, have retograde condensation of the first kind. If the plaitpoints lie on the vapour sheet, then such mixtures have r. c. II.

(To be continued).

Physics. — "*On the asymmetry of the electro-capillary-curve.*" By Dr. J. J. VAN LAAR (communicated by Prof. VAN DER WAALS).

1\. We may suppose, that it is well known, that the new theory of the so called *Capillary-Electrometer* of LIPPMANN may be described as follows.

Two mercury surfaces, one large (A), the other small (B) — this latter in the so called capillary — are separated by a conductive liquid C, diluted H_2SO_4, a solution of KCl, or any other solution. In all these cases some mercury is solved and a saturated though very diluted solution of $\mathrm{Hg_2\,SO_4}$ or $\mathrm{Hg_2\,Cl_2}$ [1]) is formed.

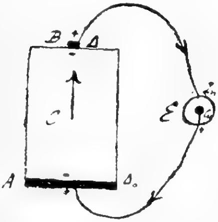

Fig. 1.

The mercury and the solution assume therefore a different potential. The difference between those potentials I call „natural difference of potential”. The concentration of the solved $\overset{++}{\mathrm{Hg}_2}$-ions being greater than that, at which the Hg-electrode and these ions are in equilibrium, some $\overset{++}{\mathrm{Hg}_2}$-ions are deposited on the mercury, where they are discharged immediately. So at the separating surface between the mercury and the solution is established a so called electric double-layer: on the side of the mercury + electrons, on the side of the solution — ions $\mathrm{SO_4}$ or Cl. The thus established difference of potential be $V_2 - V_1 = \Delta_0$. (I always indicate the solution by means of the index 2). In the normal case this quantity is negative.

We know however, that the absolute value of this difference of potential will decrease, when the *concentration* of the $\overset{++}{\mathrm{Hg}_2}$-ions in the solution decreases. When we extend the dilution beyond a certain point, the difference reverses its sign, the potential of the mercury becoming —, that of the solution +, so the electric behaviour of mercury becomes comparable to that of zinc.

How can this dilution at one of the electrodes be brought about?

To that purpose we apply an electromotive force E in a manner as is indicated by the figure. In consequence of the transport of ions, the concentration of the $\overset{++}{\mathrm{Hg}_2}$-ions in the solution near the large mercury surface will become greater, that near the small surface will decrease. At this latter surface namely, $\overset{++}{\mathrm{Hg}_2}$ (and $\overset{+}{\mathrm{H}}$ or $\overset{+}{\mathrm{K}}$) will be continually deposited in consequence of the transport of $\overset{-}{\mathrm{SO}_4}$ or Cl towards the large surface; but when the current of the inserted cell has only passed for a very short time, the concentration of the $\overset{++}{\mathrm{Hg}_2}$-ions near the large surface will practically not have varied, near the small surface however it will have varied considerably in consequence of the much greater density of the current.

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[1]) Not HgCl. The mercuro-ion being proved to be $\overset{++}{\mathrm{Hg}_2}$ and not $\overset{+}{\mathrm{Hg}}$. [See e. g OGG, Zeitschr. f. Ph. Ch. **27**, 298 (1898)].

The more so, as the solubility of $Hg_2\,SO_4$ (or $Hg_2\,Cl_2$) is so small, that the number of $\overset{++}{Hg_2}$-ions present in the solution was already very small from the beginning.

Because of this considerable change in concentration near the small surface, the difference of potential soon varies at that place, and it is easy to see, that this „kathodical'' polarisation causes the current to cease quite, or nearly quite [1]). Be E namely the electromotive force of the inserted cell, then the intensity of the current has become zero, as soon as the original difference of potential Δ_0 at the small surface has varied so much, that the new value Δ satisfies the equation

$$E = \Delta - \Delta_0,$$

from which follows:

$$\underline{\Delta = \Delta_0 + E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When this value is reached, the concentration ceases to vary and a stationary state is established.

Δ_0 being negative, Δ will, when E increases, first reach a value zero and afterwards when E increases still more, Δ will become positive. For every electromotive force E, which we may apply, we get a definite value of Δ, and so — measuring every time the surface-tension γ of the mercury in the capillary — we get a series of values, which form together a curve

$$\gamma = f(\Delta) \quad \text{or} \quad = F(E),$$

which is called the electro-capillary-curve. This is the curve whose properties we will discuss in this paper.

On purpose I have given the above explanation rather elaborately, because for the theoretical considerations which follow, a clear insight is required in what takes place in the capillary-electrometer according to the new electrochemical theories of NERNST, PLANCK and others. These theories are confirmed brilliantly by the experi-

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[1]) We shall leave out of account the so called „residual current'', which will be caused by the slow depolarisation by diffusion, the intensity of that current being extremely small.

ments of PALMAÈR[1]), SMITH[2]) and many others, and so the old theory of VON HELMHOLTZ, that of the so called "charging current" has been overthrown. The electromotive forces, calculated according to the old theory, do not agree at all with the experimental data, whereas those, calculated according to the new theory, agree quite well.

II. We will now deduce two relations. First one, giving Δ as a function of the concentration of the $\overset{++}{Hg_2}$-ions in the solution; in the second place one, giving γ as a function of Δ.

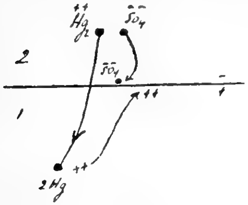

Fig. 2.

In order to find the conditions of equilibrium between a metal (mercury) and a solution (in which the ions of the metal must occur), we imagine a $\overset{++}{Hg_2}$-ion to pass the separating surface in the direction from the solution towards the mercury. The mercury is thought $+$, the solution $-$. The molecular *thermodynamic* potential of $\overset{++}{Hg_2}$ in the solution be μ_2, and that of 2 Hg in the mercury be μ_1, then the change of the thermodynamic potential per gram-ion will be:

$$\mu_1 - \mu_2.$$

As in every gram-ion $2\,\varepsilon_0$ electric units are stored, the change of the thermodynamic potential for the passage of a quantity $\overset{++}{Hg_2}$, corresponding with *de* electric units, will be:

$$\frac{\mu_1 - \mu_2}{2\,\varepsilon_0}\, de \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

The *electric* potential of the liquid being represented by V_2, and that of the mercury by V_1, the passage of *de* electric units through the bordering layer will work a change of the electric energy equal to

$$(V_1 - V_2)\, de \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

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[1]) Z. f. Ph. Ch. **25**, 265 (1898); **28**, 257 (1899); **36**, 664 (1901).

[2]) Id. **32**, 433 (1900).

Finally an accessory process takes place — and to this no sufficient attention has been paid as yet. As soon as a $\overset{++}{Hg_2}$-ion has passed from the solution into the mercury, a superfluous $\overset{--}{SO_4}$-ion (or $\overset{-}{Cl}$-ions) will go to the bordering layer, and in the mercury the + electron, which is liberated, will also move to the bordering layer. In the bordering layer therefore *changes* take place. Let φs be the so called *capillary energy* (s representing the surface). This quantity changes, when the $\overset{--}{SO_4}$- (or $\overset{-}{Cl}$-)ions go from the interior to the bordering layer. For the transport of a quantity $\overset{--}{SO_4}$, corresponding with de electric units, this change will be:

$$\frac{\partial(\varphi s)}{\partial e} de \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

In combining (a), (b) and (c), we get the following condition of equilibrium:

$$\frac{\mu_1 - \mu_2}{2\,\varepsilon_0} + (V_1 - V_2) + \frac{\partial(\varphi s)}{\partial e} = 0,$$

or when we call $\mu_2 - \mu_1 = \mu_{12}$, and $V_2 - V_1 = \Delta$, as we have already done:

$$\frac{\mu_{12}}{2\,\varepsilon_0} + \Delta - \frac{\partial \rho}{\partial \omega} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\omega = \frac{e}{s}$ represents the *surface density* of the charge of the bordering layer.

If we had made the supposition, that the mercury is negative, the solution positive (as is the case when the concentration of the $\overset{++}{Hg_2}$-ions is exceedingly small), the electrons in the mercury at the bordering layer would have been negative, and in the solution positive $\overset{++}{Hg_2}$-ions would have occurred instead of the $\overset{--}{SO_4}$- or $\overset{-}{Cl}$-ions. In that case we should have deduced the conditions of equilibrium, by imagining 2 Hg + positive electrons to pass from the mercury into the solution, where they would have formed $\overset{++}{Hg_2}$. The superfluous negative electrons in the mercury would then go to the bordering layer, while in the solution the $\overset{++}{Hg_2}$-ions go thence. In this case we should have got:

$$\frac{\mu_2 - \mu_1}{2\varepsilon_0} + (V_2 - V_1) + \frac{\partial \varphi}{\partial \omega} = 0,$$

where $\frac{\partial \varphi}{\partial \omega}$ is related to the change of the number of $\overset{++}{Hg_2}$-ions in the bordering layer of the solution. So equation (2) would have been :

$$\underline{\frac{\mu_{12}}{2\,\varepsilon_0} + \Delta + \frac{\partial \varphi}{\partial \omega} = 0} \quad . \; . \; . \; . \; . \; . \; . \quad (2bis)$$

Let us pay attention to the fact, that in (2) and also in (2*bis*) the surface-density of the charge ω is always taken positive ; Δ can be + or —, but ω is always +.

Formula (2) has already been found by PLANCK [1]), though in another form and deduced in a somewhat different manner. We shall see how great the importance of the supplementary term $\frac{\partial \varphi}{\partial \omega}$ is for the explanation of the asymmetry of the capillary-curve.

Before we proceed to express γ as a function of Δ, we will show how the usual expression of NERNST may be deduced from equation (2). To that purpose the term $\frac{\partial \varphi}{\partial \omega}$, whose value is small, compared with the two other terms, is neglected. So we find :

$$\Delta = -\frac{\mu_{12}}{2\,\varepsilon_0}.$$

But for μ_{12} we may write :

$$\mu_{12} = \mu_2 - \mu_1 = (\mu_2' + RT \log c) - \mu_1 ,$$

where, when the solutions are diluted, μ_2' will be independent of the concentration of the $\overset{++}{Hg_2}$-ions. [As we mentioned above, the solubility of $Hg_2\,SO_4$ (or $Hg_2\,Cl_2$) is so small, that the solutions will always be extremely diluted]. If we write :

$$\mu_1 - \mu_2' = RT \log C,$$

then

$$\mu_{12} = RT \log \frac{c}{C} ,$$

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[1]) WIEDEMANN's Annalen **44**, 385 (1891).

and therefore:

$$\Delta = \frac{RT}{2\,\varepsilon_0} \log \frac{C}{c}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (2a)$$

which is the well known formula of NERNST. This formula represents Δ as dependent on the concentration c of the $\overset{++}{Hg_2}$-ions in the solution. C is a constant.

If we express Δ in Volts, and introduce Briggian logarithms, we get as a factor 0,0002, and the equation becomes:

$$\Delta = 0{,}0001 \; T \log^{10} \frac{C}{c}. \quad . \; . \; . \; . \; . \; . \; . \; . \qquad (2b)$$

For mercury at 18° the quantity C is $10^{-33,7}$, when we namely put $c = 1$ for *normal* concentration of the $\overset{++}{Hg_2}$-ions. For *normal* solutions we get therefore as the value of the difference of potential ($T = 291°,2$):

$$\Delta = 0{,}0291 \times -33{,}7 = -0{,}980 \;\; Volts,$$

as also NEUMANN and others have found [with the exceedingly small concentration of $Hg_2\,Cl_2$ in $\frac{1}{10}$ normal KCl-solutions, $\Delta = -0{,}616$ Volts (OSTWALD)].

From formula (2*a*) it is clear, that Δ will reverse its sign, when $c = C$, i.e. 10^{-34}-normal.

III. Let us now calculate the surface-tension γ as a function of Δ or ω.

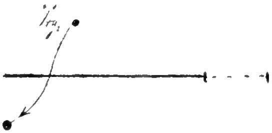

Fig. 3.

To that purpose we increase the surface s virtually with an element ds. In that element a new state of the surface must be established. We have called φ the *total* increase of the thermodynamic energy per unit of surface, accompanying this change; therefore an increase of the surface with ds will involve an increase in energy with

$$\varphi ds \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (a)$$

But this quantity φds, including the formation of a new double-layer, this formation must have been rendered possible by transition of the required number of $\overset{++}{Hg_2}$-ions from the solution into the mercury. [We will namely first discuss the case that the mercury is +, the solution —]. Then in the solution SO_4-or $\overset{-}{Cl}$-ions are left free, and in the mercury + electrons, which may move to the new separating surface. The total number $\overset{++}{Hg_2}$-ions required, is obviously that, corresponding with ωds electric units. But we saw in II, that the change of energy, accompanying the passing of de electric units, is (only (a) and (b) have to be added):

$$\frac{\mu_1 - \mu_2}{2\,\varepsilon_0}\, de + (V_1 - V_2)\, de\,.$$

When in consequence ωds units pass, this quantity will amount to

$$-\,\omega\left(\frac{\mu_{12}}{2\,\varepsilon_0} + \Delta\right) ds\,.$$

According to (2), we may write this:

$$-\,\omega\,\frac{\partial\varphi}{\partial\omega}\, ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Moreover for the formation of a new electric double-layer (mercury +, solution —) a (negative) electric energy $\omega ds\,(V_2 - V_1)$ is required, viz.

$$\omega ds\,.\,\Delta\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Adding (a), (b) and (c), and representing the mecanical energy, which must counterbalance these changes, by

$$\gamma ds,$$

we get finally:

$$\gamma = \varphi - \omega\,\frac{\partial\varphi}{\partial\omega} + \omega\,\Delta.$$

ω being always +, we have in the case we are treating of (mercury +, solution —):

$$\Delta = -\,k\,\omega,$$

and finally we get the equation

$$\underline{\gamma = \varphi - \omega\,\frac{\partial\varphi}{\partial\omega} - k\omega^2,} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and not simply $\gamma = \varphi - k\omega^2$, as was assumed in the old theory, in which the changes of the bordering layer were neglected.

Had we started from the supposition, that the mercury was —, the solution +, a transition of 2 Hg + positive elections from the mercury into the solution — where they would have formed $\overset{++}{Hg_2}$, — would have been required, in order to render the formation of a new double layer possible; and that would have caused the above deduction to be modified as follows.

The part (*b*) would have been derived from

$$\omega\left(\frac{\mu_{12}}{2\,\epsilon_0} + \Delta\right) ds.$$

But, according to (2*bis*), this would still have yielded $= -\,\omega\frac{\partial\varphi}{\partial\omega}\,ds$, so this part is not modified. The change of the (negative) electric energy on the other hand becomes now $\omega\, ds\,(V_1 - V_2)$, i. e.

$$-\,\omega\, ds\,.\,\Delta\,,$$

where now Δ is positive $= k\omega$; so finally equation (3) is yielded quite unmodified.

This equation therefore is of general application, as well in the case, that in the solution at the bordering layer negative $\overset{-}{SO_4}$- or $\overset{-}{Cl}$-ions occur, as in the case that there occur positive $\overset{++}{Hg_2}$-ions.

But — and this is a circumstance of great importance — the term $\omega\frac{\partial\varphi}{\partial\omega}$ will in the two cases *not* have the same value for equal values of ω. For it would be a curious coincidence, that the change of energy in the bordering layer, occasioned by adding an infinitely small quantity of $\overset{++}{Hg_2}$, would be the same as that, occasioned by adding an equal quantity of $\overset{--}{SO_4}$ or $\overset{-}{Cl}$. In fact this does not happen. The experiments show clearly, that the curve represented by (3) is *not* symmetrical on the two sides of the point, where ω is zero; and that the curve does *not* consist of one continuous parabola, *but of two parts of quite different parabolae,* which meet in the point where $\omega = 0$. *Only one* of them, namely the *ascending branch* (mercury +, solution —), presents a *maximum near* the point where $\omega = 0$ (so not exactly *at* that point.)

In order to enter into more details, we must first examine what is the form of φ as a function of the quantity ω.

IV. Let us to this purpose state the fact, that φ represents the excess of the thermodynamic potential in the bordering layer above that in the mercury and the solution. Therefore we may write in any case:

$$\varphi = (\varphi_0 + \alpha\,\omega + \beta\,\omega^2 + \ldots) + A\,\omega\,log\,c,$$

where c represents the concentration of the $\overset{-}{S}\overset{-}{O}_4$- or $\overline{Cl}$-ions in the *bordering layer*, or — when the sign of Δ is reversed — that of the $\overset{++}{Hg_2}$-ions. The constant A may have the positive, as well as the negative sign. When the charge spreads in such a way, that it penetrates rather deeply into the bordering layer — as the experiments seem to prove for the case that the $\overset{++}{Hg_2}$-ions form the + charge (the mercury being negative) — then A will be positive. So this is the case for the descending branch of the electro-capillary-curve. But when the charge remains more at the surface of the bordering layer, as is the case, when $S\overline{O}_4$- or $\overline{Cl}$-ions form the negative charge in the solution (the mercury being positive), then A is negative. We find this realised in the ascending branch of the curve.

Writing $a\omega$ for c, we get:

$$\omega\frac{\partial\varphi}{\partial\omega} = (\alpha\,\omega + 2\,\beta\,\omega^2 + \ldots) + A\omega\,log\,a\omega + A\omega\,,$$

and equation (3) takes the following form:

$$\underline{\gamma = \varphi_0 - A\omega - (k+\beta)\,\omega^2\,{}^{1})} \quad \ldots\ldots\ldots \quad (4)$$

This is the accurate equation of the electro-capillary-curve, and in what follows we will determine the value of φ_0, A and $k+\beta$ for the two parts of the curve — on the left and on the right of the point, where $\omega = 0$.

The maximum is obviously to be found in one of the branches, when

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[1] φ_0 is in this equation still a function of the concentration of the electrolyte, as appears from the experiments of SMITH. See i.a. OSTWALD, Lehrbuch I, 531 ff.; EULER, Z. f. Ph. Ch. **28**, 625 (1899); **39**, 564 (1901).

$$\omega_m = -\frac{A}{2(k+\beta)}.$$

As $k+\beta$ is always positive, and ω must be positive, the maximum can only be found in a branch where A is negative, i.e. in the ascending branch. The value of the maximum is given by the equation

$$\gamma_m = \varphi_0 + \frac{A}{4(k+\beta)}.$$

Before we pass to the calculation of a series of experiments of SMITH, we will give to equation (4) another form, where not ω, but the electromotive force E of the inserted cell is used as argument. According to equation (1), we have

$$\Delta = \Delta_0 + E.$$

For the *descending* branch Δ is positive, namely

$$\Delta = k\omega.$$

So we may write for (4):

$$\gamma = \varphi_0 - A\cdot\frac{\Delta}{k} - (k+\beta)\frac{\Delta^2}{k^2},$$

or

$$\gamma = \varphi_0 - \frac{A}{k}(\Delta_0 + E) - \frac{k+\beta}{k^2}(\Delta_0+E)^2 \quad . \quad . \quad . \quad (a)$$

For the *ascending* branch Δ is negative, namely

$$\Delta = -k\omega.$$

So we get:

$$= \varphi_0 + \frac{A}{k}(\Delta_0 + E) - \frac{k+\beta}{k^2}(\Delta_0+E)^2. \quad . \quad . \quad . \quad (b)$$

When developing, we find:

$$\gamma = \left[\varphi_0 \mp \frac{A}{k}\Delta_0 - \frac{k+\beta}{k^2}\Delta_0^2\right] + \left[-2\frac{k+\beta}{k^2}\Delta_0 \mp \frac{A}{k}\right]E - \frac{k+\beta}{k^2}E^2, \quad (5)$$

where the higher sign relates to the descending branch, the lower sign to the ascending branch. This equation may be represented by

$$\gamma = a + b\,E - c\,E^2 \,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

of which we will determine the coefficients a, b and c.

I have chosen to this purpose a series of experiments of SMITH [1]) with $^1/_{10}$-normal KCl as electrolyte. The concentration of the solved $Hg_2\,Cl_2$ is here exceedingly small, and the difference of potential between the solution and the mercury is for this "normal-electrode" accurately known. (OSTWALD) [2]), namely:

$$\Delta_0 = -0{,}616 \;\; Volts.$$

The place, where ω (or Δ) becomes zero, may be determined without difficulty. For from $\Delta = \Delta_0 + E$ follows, that if $\Delta = 0$:

$$E = -\Delta_0 = 0{,}616 \;\; Volts.$$

Now in the experiments of SMITH E is expressed in such units that $E = 500$ corresponds to 0,102 Volts. The value

$$0{,}616 \text{ Volts} = 6{,}04 \times 0{,}102 \text{ Volts}$$

corresponds therefore in the units of SMITH with

$$E = 6{,}04 \times 500 = 3020.$$

V. For the calculation of the *descending* branch we have therefore to take into account only such values of E as are greater than 3020. From this I calculated:

$$a = 29{,}766, \quad b = \frac{0{,}8318}{0{,}102}, \quad c = \frac{0{,}090}{(0{,}102)^2}.$$

The following table shows, that these values represent the descending branch in fact with great accuracy.

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[1]) Zeitschr. f. Ph. Ch. **32**, 460 and 467 (1900).

[2]) Zeitschr. f. Ph. Ch. **35**, 335 (1900).

	γ calculated.	Found.	Δ
$E = 3000 = 6 \times 0.102$ V.	$\gamma = 29.77 + 4.99 - 3.24 = 31.52$	31.41	+ 0.11
3500 = 7 »	$5.82 - 4.41 = 31.48$	31.20	− 0.02
4000 = 8 »	$6.65 - 5.76 = 30.66$	30.70	− 0.04
4500 = 9 »	$7.48 - 7.29 = 29.97$	29.99	− 0.02
5000 = 10 »	$8.32 - 9.00 = 29.09$	29.10	− 0.01
5500 = 11 »	$9.45 - 10.89 = 28.03$	28.00	+ 0.03
6000 = 12 »	$9.98 - 12.96 = 26.79$	26.72	+ 0.07
6500 = 13 »	$10.81 - 15.21 = 25.37$	25.33	+ 0.04
7000 = 14 »	$11.65 - 17.64 = 23.78$	23.79	− 0.01
7500 = 15 »	$12.48 - 20.25 = 22.00$	22.00	± 0.00
8000 = 16 »	$13.31 - 23.04 = 20.04$	20.01	+ 0.03
8500 = 17 »	$14.14 - 26.01 = 17.90$	17.90	± 0.00
9000 = 18 »	$14.97 - 29.16 = 15.58$	15.60	− 0.02

As we see, formula (6) with these values for a, b and c represents the descending branch with extraordinary accuracy. If we leave out of account the value for $E = 3000$, which no longer belongs to the descending branch, as I have shown above, the difference between the calculated and the observed value surpasses nowhere $\frac{1}{6}$ %; only once (at $E = 6000$) the difference is $\frac{1}{4}$ %.

If in (6) we substitute for E the value $6{,}04 \times 0{,}102$, we get γ for $\omega = 0$, i. e. φ_0. So we find

$$\varphi_0 = \mathbf{31{,}508}.$$

In order to calculate $\frac{A}{k}$ and $\frac{k+\beta}{k^2}$, we combine (5) and (6):

$$\frac{k+\beta}{k^2} = c = \frac{0{,}090}{(0{,}102)^2} = 8{,}651.$$

$$-2\,\frac{k+\beta}{k^2}\,\Delta_0 - \frac{A}{k} = b = \frac{0{,}8318}{0{,}102}.$$

Taking into account, that $\Delta_0 = -0,616$, the latter equation yields

$$\frac{A}{k} = 2 \times \frac{0,090}{(0,102)^2} \times 0,616 - \frac{0,8318}{0,102} = 10,658 - 8,155 = 2,503.$$

It is superfluous to mention, that the manner in which φ_0 is calculated, involves that equating the first term of (5) with $a = 29,766$, an identical value $\frac{A}{k}$ is found.

For the *descending* branch we may therefore write either (according to (6))

$$\left.\begin{array}{l} \gamma = 29,766 + \dfrac{0,8318}{0,102} E - \dfrac{0,090}{(0,102)^2} E^2 \\ \text{or, according to (4) } \gamma = \mathbf{31,508 - 2,503\,\Delta - 8,651\,\Delta^2}, \end{array}\right\} \ldots\ldots (7)$$

putting again Δ for $k\omega$ (Δ positive).

For the calculation of the *ascending* branch we have to make use of the values of E between 0 and 3000. From these I calculated as the most probable values:

$$a = 25,456, \quad b = \frac{2,153}{0,102}, \quad c = \frac{0,1906}{(0,102)^2}.$$

At once we see, that we have to deal here with a branch of another parabola than in the descending branch of the electro-capillary-curve; b being nearly three times, c more than twice as great. The slope of the ascending branch is therefore, as all experiments show, steeper than that of the descending one.

The following table may serve to verify the values, found for a, b and c, by means of the experiments. We notice, that the experimental data for the ascending branch are few in number, and moreover are considered as *unreliable* by the experimentators. $^1)$

Notwithstanding the agreement may be considered to be satisfactory.

$^1)$ See i. e. SMITH, l. c. pag 455.

	γ calculated.	Found.	Δ
$E = \quad 0 = 0 \times 0.102$ V.	$\gamma = 25.46 + 0.00 - 0.00 = 25.46$	24.78 ?	$+ 0.68$?
$500 = 1$ »	$2.15 - 0.19 = 27.42$	27.40	$+ 0.02$
$1000 = 2$ »	$4.31 - 0.76 = 29.00$	29.00	± 0.00
$1500 = 3$ »	$6.46 - 1.72 = 30.20$	30.20	± 0.00
$2000 = 4$ »	$8.61 - 3.05 = 31.02$	30.65 ?	$+ 0.37$?
$2500 = 5$ »	$10.77 - 4.77 = 31.46$	31.33	$+ 0.13$
$3000 = 6$ »	$12.92 - 6.86 = 31.51$	31.41	$+ 0.10$

The values for $E = 0$ and $E = 2000$, namely $\gamma = 24{,}78$ and $\gamma = 30{,}65$, have been marked with a note of interrogation by the experimentator himself; so these values may be left out of consideration (l. c. page 460 and 467). The greatest difference is then about ½ %. We have still to mention that for

$$E = 3020 = 6{,}04 \times 0{,}102 \text{ Volts}$$

the values, calculated for a, b and c, yield $\gamma = \varphi_0 = 31{,}51$, as they ought to do. The two parabolae meet there.

For $\dfrac{k+\beta}{k^2}$ and $\dfrac{A}{k}$ we find in the same way as above:

$$\frac{k+\beta}{k^2} = c = \frac{0{,}1906}{(0{,}102)^2} = 18{,}320 .$$

$$-2 \frac{k+\beta}{k} \Delta_0 + \frac{A}{k} = b = \frac{2{,}153}{0{,}102} ,$$

so putting $\Delta_0 = -0{,}616$:

$$\frac{A}{k} = -2 \times \frac{0{,}1906}{(0{,}102)^2} \times 0{,}616 + \frac{2{,}153}{0{,}102} = -22{,}570 + 21{,}108 = \mathbf{-1{,}462}.$$

$\dfrac{A}{k}$ appears to be negative, i. e. the negative charge ($\overset{-}{\text{Cl}}$-ions) is situated in the solution at the surface of the bordering layer, and not — as the positive charge of the $\overset{++}{\text{Hg}_2}$-ions — deeper in that layer.

For the *ascending* branch we may write for this case either

$$\left.\begin{aligned}\gamma &= 25{,}456 + \frac{2{,}153}{0{,}102}E - \frac{0{,}1906}{(0{,}102)^2}E^2 \\ \text{or} \quad \gamma &= \mathbf{31{,}508 - 1{,}462\,\Delta - 18{,}320\,\Delta^2},\end{aligned}\right\} \quad \ldots\ldots \quad (8)$$

putting $-\Delta$ for $k\omega$ (Δ being negative).

The maximum, which is to be found in the ascending branch, may be calculated from the first of these equations $\left(E_m = \frac{b}{2c}\right)$:

$$E_m = \frac{2{,}153}{2 \times 0{,}1906} \times 0{,}102 = 5{,}65 \times 0{,}102 \text{ Volts},$$

i. e.

$$E_m = 5{,}65 \times 500 = 2820.$$

Further we find for

$$\gamma_m = a + \frac{b^2}{4c}:$$

$$\gamma_m = 25{,}456 + \frac{(2{,}153)^2}{4 \times 0{,}1906} = 25{,}456 + 6{,}080 = 31{,}54.$$

In the descending branch no maximum is to be found, because a maximum requires there a negative value of Δ, and Δ is here positive.

We see, that the maximum ($E = 2820$) does *not* coincide with the point, where $\omega = 0$ ($E = 3020$, as we found above). The difference is not great, but still $(6{,}04—5{,}65) \times 0{,}102$ Volts $= 40$ millivolts. And in other instances it may be greater of course. It depends wholly on the value of A.

The figure represents the accurate course of the two parts of parabolae. The dotted curves indicate, how the course should have been, if the branches had been continued on the other side of $\Delta = 0$.

The abscissae are the electromotive forces E of the inserted cell, and increase with $500 = 0{,}102$ Volts; so they are respectively 1×500, 2×500, 3×500 etc. The ascending branch is AP, and would have been continued in PA', if the coefficients remained the same after $\Delta = 0$. The maximum is to be found at M, somewhat to the left of P, the point which separates the two different parabolae. The descending branch is PB, and would be continued along PB'. Its maximum is to be found at M'. So the curve, *really* passed through, is APB. The experimental values agree perfectly with the calculated values, now that the figure is made on this scale; only those found for $E = 0$ and $E = 4 \times 1{,}02$ (those with ?) do not agree (as is indicated in the figure by the sign ×).

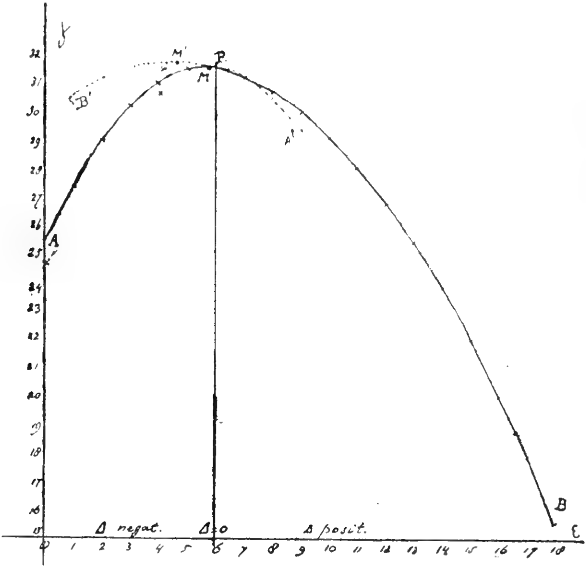

Fig. 4.

We may therefore summarise the above conclusions as follows.

1st. An accurate theoretical investigation of the capillary-electrometer shows, that the surface-tension as a function of the charge of the double-layer is not to be expressed by the simple equation $\gamma = \varphi_0 - k\omega^2$, but by the relation

$$\gamma = \varphi_0 - A\omega - (k + \beta)\,\omega^2.$$

2nd. The coefficients A and β differ as to whether the charge of the solution is negative (Δ negative) or positive. For a $^1/_{10}$-normal solution of KCl $k + \beta$ is in the first case twice as great as in the second, and A is in the former case negative, in the latter positive.

3rd. The electro-capillary-curve consists of two parts of different parabolae, which meet each other at $\Delta = 0$, and of which the course of the ascending branch is much deeper than that of the descending branch.

4th. The ascending branch presents a maximum ,and this maximum need *not* coincide with the point, where $\Delta = 0$. In our example this difference amounts to 40 millivolts.

5th. The hitherto unexplained peculiarities of the electro-capillary-curve are in this way fully explained.

6th. The capillary-electrometer of LIPPMANN is according to 4th not at all reliable for an accurate measuring of the differences of potential between metal and electrolyte.

March 1902.

Mathematics. „*Right lines on surfaces with multiple right lines*". by Prof. JAN DE VRIES.

§ 1. If a surface S^n of order n possesses a line l of multiplicity $n-2$, it is cut in a conic by any plane passing through l. In order to find the locus of the centre of these conics we consider the section C^n_∞ of S^n by the plane at infinity. The point L_∞ on l at infinity is a point of multiplicity $n-2$ on this curve; so C^n_∞ is of class $(4n-6)$ and admits of $2(n-1)$ tangents passing through L_∞ and touching it elsewhere. Each of these tangents determines a plane through l cutting S^n in a parabola; so the locus of the center is a curve of order $2(n-1)$, of deficiency zero, cutting l $2n-3$-times. This curve meets S^n a number of $(2n-3)(n-2)$-times on l and $2(n-1)$-times at infinity; the remaining points of intersection are double points of degenerated conics. From this ensues the known property that the line l of multiplicity $(n-2)$ is met by $(3n-4)$ pairs of single lines.[1]).

§ 2. If l is chosen for the axis OZ of a right-angular system of coordinates the surface S^n can be represented by an equation of the form

$$A_n(x,y) + A_{n-1}(x,y)\,z + B_{n-1}(x,y) +$$
$$+ A_{n-2}(x,y)\,z^2 + B_{n-2}(x,y)\,z + C_{n-2}(x,y) = 0,$$

the indices n, $(n-1)$, $(n-2)$ denoting the order of the corresponding functions A,B,C.

From this is evident that an S^n with given $(n-2)$-fold line l can be made to pass through $(6n-3)$ more points chosen at random. As we have $6n-3 = 5(n+1) + (n-8)$ it seems that for $n > 7$ we

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[1]) See e. g. R. STURM, Math. Annalen, vol. IV, p. 249.

can imply the condition that the surface S_n shall contain five arbitrarily chosen lines. That this conclusion is to be rejected is clearly shown in the following.

The locus of the conics cutting twice a given line l and once each of five given lines c_k $(k = 1, 2, 3, 4, 5)$ is a surface S^8 with sixfold line l.)[1]. So for $n > 8$ only four lines of S^n crossing the $(n-2)$-fold line l can be chosen arbitrarily.

The mentioned surface S^8 contains the common transversals a_{klm}, b_{klm} of the quadruples l, c_k, c_l, c_m. The planes $(a_{klm}l)$, $(b_{klm}l)$ cut S^8 in right lines which can be denoted by b_{np} and a_{np}. So the 20 pairs of lines, lying on S^8 according to § 1, can be represented by (a_{klm}, b_{np}) and (b_{klm}, a_{np}).

§ 3. A twisted curve R^p of order p cutting the sixfold line l of S^8 in $p-1$ points, has still $2p+6$ more points in common with the surface. So the conics in planes through l meeting R^p and the four lines c_1, c_2, c_3, c_4 generate a S^{2p+6} with $(2p+4)$-fold line l. If R^p cuts c_1 in P, the plane (Pl) contains an infinite number of conics of the locus; in that case S^{2p+6} breaks up into this plane and an S^{2p+5}.

Now we are able to construct an S^n with $(n-2)$-fold line l passing through four lines c_k crossing this line. For $n = 2m$ any twisted curve R^{m-3}, for $n = 2m + 1$ any twisted curve R^{m-2} may figure as a fifth directing line.

More generally we can suppose R^p to have successively in common with c_1, c_2, c_3, c_4 a number of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ points. Then the conics cutting l twice and each of the five directors c_1, c_2, c_3, c_4, R^p once generate a surface of order $2p + 6 - (\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)$.

In order to obtain e. g. a surface S^4 with double line l and four simple lines c_k crossing it, we put $2p = \Sigma \gamma_k - 2$; for $\gamma_k = 1$ we find $p = 1$. So we can assume a fifth line c crossing l under the condition that it meets each of the four lines c_k; then the five lines c represent together 21 points[2]. (§ 2).

§ 4. The surface S^8 also breaks up if c_5 cuts l. For the plane (lc_5) contains an infinite number of conics each of which represents

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[1]) See e. g. my communication in these Proceedings, Sept. 28, 1901, p. 183.

[2]) It is also possible to choose for c_1, c_2, c_3, c_4 the four sides of a skew quadrilateral in which case c_5 can be taken arbitrarily; then S^8 breaks up into S^4 and four planes. The five lines c_1, c_2, c_3, c_4, c_5 may form also a simple broken line.

two conics of the locus on account of its cutting c_5 twice; so S^8 breaks up into this double plane and an S^6 with fourfold line l and fivefold point $S \equiv lc_5$.

This surface S^6 contains the common transversals a_{klm}, b_{klm} of l, c_k, c_l, c_m, completed to pairs of lines by the lines b_n, a_n through S meeting c_n. Moreover it passes through the lines a_k through S meeting c_k and c_l, completed by the lines b_{mn} cutting c_m, c_n. So the 14 pairs of lines situated in S^6 (§ 1) have been indicated.

If S^6 still admits of a fourfold line l and four simple lines c_1, c_2, c_3, c_4 crossing it, but not of a fivefold point S, the 14 pairs of lines are enumerated as follows. Firstly we indicate the 8 transversals a_{klm}, b_{klm} and the corresponding lines b_n, a_n. By remarking that the quadric determined by l, c_1, c_2 cuts S^6 still in an R^6 breaking up into six lines, and that a_{123}, b_{123}, a_{124}, b_{124} are amongst these, it is found that the missing pairs of lines can be denoted as a_{kl}, b_{mn}.

§ 5. The conics through the given points S, S' meeting the lines c_1, c_2, c_3 generate an S^4 with double line $l \equiv SS'$ and triple points S, S', derived from the S^8 by supposing that c_4 and c_5 meet l in S and S'. Of the eight pairs of lines cutting l six can be represented as a_{kl}, b_m and b_{kl}, a_m, the lines a of which pass through S and the lines b through S'; the missing pairs are formed by l and one of the common transversals of l and the lines c_k.

By the intersection of this particular surface S^4 by a twisted curve R^p meeting l a number of $(p-1)$ times it is found that the conics through S, S' and meeting R^p, c_1, c_2 generate an S^{2p+2} with two $(2p+1)$-fold points S, S' and a $2p$-fold line l. The $2p$ lines forming with l degenerated conics of the system can be indicated as follows: the quadric $(l\, c_1\, c_2)$ cuts R^p in still $(p+1)$ more points, each of which gives rise to a common transversal of l, c_1, c_2 and R^p lying on S^{2p+2}; the remaining $(p-1)$ transversals of l, c_1, c_2, R^p lie in the planes through l touching R^p in its points situated on l.

The cone (S, R^p) determines with c_1 (and with c_2) p transversals of R^p and c_1 (c_2). The same holds good for the cone (S', R^p). Adding to these $2p$ transversals the lines through S (and S') cutting c_1 and c_2, we get the $(4p+2)$ pairs of lines, forming with the $2p$ pairs of lines amongst which figures l the $(6p+2)$ pairs of lines S^{2p+2} must contain.

§ 6. If a surface $S^{\mu+\nu+1}$ contains a μ-fold line m and a ν-fold line n crossing it, then the skew surface generated by the lines meeting m, n and any given plane section $C^{\mu+\nu+1}$ of $S^{\mu+\nu+1}$ admits only of one generator situated in the plane of this section, i. e. of the line joining the μ-fold point and the ν-fold point of the section; the section $C^{\mu+\ +1}$ being itself a simple curve of the surface, the order of this skew surface is $\mu+\nu+2$; and m and n are $(\mu+1)$-fold and $(\nu+1)$-fold lines of it. By another plane section $D^{\mu+\nu+1}$ of $S^{\mu+\nu+1}$ it is met in $2\mu\nu+\mu+\nu+1$ points belonging neither to m, nor to n, nor to $C^{\mu+\nu+1}$. So the multiple lines m and n cut $2\mu\nu+\mu+\nu+1$ simple lines of $S^{\mu+\nu+1}$.

Evidently the last number is modified if a third multiple line cutting m and n presents itself. So a λ-fold line l reduces this number of single lines by λ^2, as it is also a λ-fold line of the indicated skew surface.

§ 7. If e. g. the surface S^5 has two double lines m and n, we find 13 single lines cutting these. Each of the lines m, n counting for 16 points in the determination of the surface, still 23 points can be chosen at random; so we can make S^5 to pass through three lines c_1, c_2, c_3 crossing m, n [1]) Then the relation of position between the different lines can be indicated by the following table:

c_1	meets	d_1	d_2	d_3	d_4			d_7						
c_2	„	d_1	d_2			d_5	d_6		d_8					
c_3	„			d_3	d_4	d_5	d_6			d_9				
m n	meet	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}.

If S^5 still admits of a threefold line l, this line replaces 9 of the 13 lines d. But then each of the lines m, n counts for a couple of lines cutting the threefold line d. Then the curve of double curvature considered in § 1, by the occurrence of the double points M_∞, N_∞ of the plane curve $C^5{}_\infty$, is of order four; as it meets m, n the surface S^5 can only have three more pairs of lines meeting l. Their position with respect to the lines d is indicated by the table:

d_1	rests on	a_{12}, a_{13}, a_{14};
d_2	„	a_{12}, b_{24}, b_{23};
d_3	„	b_{34}, a_{13}, b_{23};
d_4	„	b_{34}, b_{24}, a_{14}.

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[1]) Also four lines c_k forming a skew quadrilateral can be assumed.

§ 8. By supposing the threefold line (l) and the double line (m) of S^5 to coincide with the axes of Z and Y, it is immediately seen that these lines count for 28 and 19 points in the determination of the surface and that still 11 points more can be chosen arbitrarily. Instead of a second double line n the four lines c_1, c_2, c_3, c_4 resting on m can be assumed at random. Then we find that l is cut by seven pairs of lines, the position of which with respect to the lines c_k is as follows:

c_1 cuts $a_{123}, a_{124}, a_{134}, b_1, a_{12}, a_{13}, a_{14}$;
c_2 " $a_{123}, a_{124}, b_2, a_{234}, a_{12}, b_{24}, b_{23}$;
c_3 " $a_{123}, b_3, a_{134}, a_{234}, b_{34}, a_{13}, b_{23}$;
c_4 " $b_4, a_{124}, a_{134}, a_{234}, b_{34}, b_{24}, a_{14}$.

§ 9. In order to obtain a group of surfaces with two multiple lines we first consider the locus of a plane curve C^n cutting n-times the right line l and once each line of a given set of $\frac{1}{2}\, n\, (n+3)$ lines c_k crossing l. The order of this surface is equal to the number of curves C^n lying in planes through l and resting on $\frac{1}{2}\, (n+1)\, (n+2)$ lines c. To find this number $f(n)$ we suppose that $(n+1)$ of these lines c meet l; then the plane passing through l and any of these $(n+1)$ lines contains a single curve C^n satisfying the conditions and is evidently to be counted n-times. All the remaining curves C^n we are in search of break up into l and a curve C^{n-1} resting on $\frac{1}{2}\, n\, (n+1)$ lines c. So the relation $f(n) = n\, (n+1) + f(n-1)$ holds; moreover $f(1)$ representing the number two of the common transversals of four crossing lines, we find immediately $f(n) = \frac{1}{3}\, n\, (n+1)(n+2)$ for the order of the locus of the curve C^n meeting n times the line l and once each of the $\frac{1}{2}\, n\, (n+3)$ given lines c_k.

Considering curves C^n in planes through l which pass μ-times through the point of intersection of the plane with m and once through each of the points of intersection of the plane with $\frac{1}{2}\, n\, (n+3) - \frac{1}{2}\, \mu\, (\mu+1)$ lines c, we find for the order $\varphi(n, \mu)$ of the locus the relations $\varphi(n, \mu) = n\, (n+1) + \varphi(n-1, \mu)$, etc. until we get $\varphi(\mu+2, \mu) = (\mu+2)(\mu+3) + \varphi(\mu+1, \mu)$.

Here $\varphi(\mu+1, \mu)$ is the number of plane curves $C^{\mu+1}$ with a μ-fold point on m, cutting $(2\mu+3)$ given lines c. By supposing that $\mu+2$ of these lines rest on l the curves $C^{\mu+1}$ break up into l and a C^{μ} with μ-fold point on m cutting $(\mu+1)$ lines c, i. e.

into l and a star of μ rays passing through this m-fold point. So one of the curves C^{μ} in question is found by uniting a common transversal of l, m and any pair of the $(\mu+1)$ lines c by a plane and joining the point of intersection of this plane with m to the points where this plane meets the remaining $(\mu-1)$ lines c. So $\varphi(\mu+1, \mu) = (\mu+1)(\mu+2) + (\mu+1)\mu = 2(\mu+1)^2$. And this number being inferior to $f(\mu+1)$ by $\frac{1}{3}(\mu+1)\mu(\mu-1)$, we find at last $\varphi(n, \mu) = \frac{1}{3} n(n+1)(n+2) - \frac{1}{3}(\mu+1)\mu(\mu-1)$. Therefore the locus under investigation is a surface of order $\varphi(n, \mu)$ with a μ-fold line m, a $[\varphi(n, \mu) - n]$-fold line l and $\frac{1}{2} n(n+3) - \frac{1}{2}\mu(\mu+1)$ single lines.

§ 10. For $n=3$, $\mu=2$, the obtained surface is an S^{18} with a 15-fold line l, a double line m and six single lines c_k.

By cutting it by a curve of double curvature R^3 of which l, m, c_1, c_2, c_3 are double secants it is seen that a surface S^{14} is obtained if the director line c_6 is replaced by the curve of double curvature R^3.

Furthermore a surface S^{10} and a surface S^6 are found if c_5 is also replaced by a curve of space $R_1{}^3$, or if c_5 and c_4 are replaced by curves of space $R_1{}^3$, $R_2{}^3$ having l, m, c_1, c_2, c_3 for double secants. In the supposition that these two curves $R_1{}^3$, $R_2{}^3$ have a point in common the obtained surface is S^5 with two double lines l, m and three single lines c_1, c_2, c_3 (compare § 7). It is clear that surfaces with two multiple lines l, m and any number of single lines c alternating from unity to $\frac{1}{2}(n^2+3n-2) - \frac{1}{2}\mu(\mu+1)$ can be obtained in the same way.

§ 11. By choosing the μ-fold line m of a surface S^n for axis of z it is easily seen that the assumption of a μ-fold line counts for $\frac{1}{6}\mu(\mu+1)(3n-2\mu+5)$ points. So a surface S^n admitting of a μ-fold line m and a ν-fold line n can still be made to pass through:

$$\frac{1}{6}(\mu+\nu+2)(\mu+\nu+3)(\mu+\nu+4) - 1 - \frac{1}{6}\mu(\mu+1)(\mu+3\nu+8) -$$

$$- \frac{1}{6}\nu(\nu+1)(3\mu+\nu+8) = 2\mu\nu + 3(\mu+\nu+1).$$

points. For $\nu=\mu$ this is $2\mu^2 + 6\mu + 3 = (\mu+1)(2\mu+2) + (2\mu+1)$, So a surface $S^{2\mu+1}$ with two μ-fold lines can admit of still $(\mu+1)$ simple crossing lines, or of $(\mu+2)$ lines two of which intersect each other.

In the first case the $(\mu+1)$ lines c taken two by two determine with m and m' a number of $(\mu+1)\mu$ lines a situated on $S^{2\mu+1}$. Of these lines evidently 2μ resting on m, m', $c_{\mu+1}$ form with these three lines the intersection of $S^{2\mu+1}$ with the quadric skew surface $(c_{\mu+1}, m, m')$; so the latter contains still one line a cutting m, m'. So in all $(\mu+1)^2$ lines a cut at least one of the lines c, the remaining μ^2 lines a have no point in common with a line c. By choosing for $(\mu+2)$ lines c the sides of a skew polygon of $(\mu+2)$ sides it is seen in a corresponding manner that $\mu^2-\mu-1$ lines a have no point in common with any line c.

The condition that a surface $S^{\mu+\nu+1}$ with a μ-fold line m and a ν-fold line n contains $\nu+2$ given right lines c counts for $(\mu+\nu+2)(\nu+2)$ points. Now

$$2\mu\nu+3(\mu+\nu+1)-(\mu+\nu+2)(\nu+2)=\mu\nu+\mu-\nu^2-\nu-1$$

and this number is positive for $\mu>\nu$. So at least $(\nu+2)$ lines c can be chosen for $\mu>\nu$; it depends on the value of $(\mu-\nu)$ whether it be possible to choose arbitrarily a greater number or not.

§ 12. After the assumption of three multiple lines l, m, n of the multiplicity λ, μ, ν respectively of a surface $S^{\mu+\mu+\nu}$ we can dispose of $(\lambda+1)(\mu+1)(\nu+1)-1$ more points. Probably a general rule referring to the maximum number of lines c does not exist. In the case $\lambda=\mu=\nu$ on a $S^{3\lambda}$ still $(\lambda+1)$ right lines can be assumed for

$$(\lambda+1)^3-1-(\lambda+1)(3\lambda+1)=\lambda^3-\lambda-1$$

and this is positive for $\lambda>1$.

Then the hyperboloid (l_1, l_2, l_3) contains 3λ right lines a situated on $S^{3\lambda}$; on each of the $(\lambda+1)$ lines c rest two of these.

In the preceding lines only a few fundaments for a general treatment of surfaces with a limited number of right lines have been indicated; a more complete study about the subject will be published elsewhere.

Mathematics. — *„On Integrals containing functions of Bessel"*: An extract out of a letter of Prof. L. GEGENBAUER at Vienna communicated by Prof. W. KAPTEYN.

In your interesting paper „Sur quelques intégrales définies contenant des fonctions de Bessel" in the „Archives Néerlandaises des sciences exactes et naturelles" you have determined the value of the integrals

$$\int\limits_0^\infty \frac{I^n(t)\cos(t\sin\varphi)}{t}\,dt\,, \qquad \int\limits_0^\infty \frac{I^n(t)\sin(t\sin\varphi)}{t}\,dt$$

for entire positive values of n. I shall take the liberty to communicate to you in the following lines a generalisation of your noteworthy results.

If we multiply the relation

$$e^{i\,t\cos\varphi\cos\psi}\,I^{\frac{2\nu-1}{2}}(t\sin\varphi\sin\psi) = 2^{2\nu-1}\,[\Pi(\nu-1)]^2(\sin\varphi\sin\psi)^{\frac{2\nu-1}{2}}$$

$$\sqrt{\frac{2}{\pi t}}\sum_{\lambda=0}^{\lambda=\infty} i^\lambda\,\frac{(\lambda+\nu)\,\Pi(\lambda)}{\Pi(\lambda+2\nu-1)}\,I^{\lambda+\nu}(t)\;C_\lambda^\nu(\cos\varphi)\;C_\lambda^\nu(\cos\psi)\,(\nu \geqq 0),$$

given by me in the tenth volume of the „Monatshefte für Mathematik und Physik" („Notiz über die BESSEL'schen Functionen erster Art."), in which relation $C_n^\nu(x)$ is the coefficient of α^n in the development of

$$(1-2\alpha x+\alpha^2)^{-\nu}$$

according to ascending powers of α, with

$$\frac{I^{\nu+m}(t)\,dt}{\sqrt{t}}\;(m>0)$$

and if we integrate according to t from 0 to ∞, we obtain by making use of the well-known value of the integral

$$\int\limits_0^\infty \frac{I^{\nu+\lambda}(t)\,I^{\nu+m}(t)\,dt}{t}$$

(thus in the manner followed by yourself), by separating of the real and the imaginary parts of the formula thus arrived at, the two equations

$$\int_0^\infty \frac{\cos(t\cos\varphi\cos\psi)\; I^{\frac{2\nu-1}{2}}(t\sin\varphi\sin\psi)\; I^{\nu+m}(t)\,dt}{\sqrt{t}} =$$

$$= [2^\nu\, \Pi(\nu-1)]^2 (\sin\varphi\sin\psi)^{\frac{2\nu-1}{2}} \sin\frac{m\pi}{2}\,\frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sum_{\lambda=0}^{\lambda=\infty}$$

$$\frac{(2\lambda+\nu)\,\Pi(2\lambda)}{\Pi(2\lambda+2\nu-1)}\;\frac{C^\nu_{2\lambda}(\cos\varphi)\;C^\nu_{2\lambda}(\cos\psi)}{(\nu-m)^2-(\nu+2\lambda)^2},$$

$$\int_0^\infty \frac{\sin(t\cos\varphi\cos\psi)\; I^{\frac{2\nu-1}{2}}(t\sin\varphi\sin\psi)\; I^{\nu+m}(t)\,dt}{\sqrt{t}} =$$

$$= [2^\nu\, \Pi(\nu-1)]^2 (\sin\varphi\sin\psi)^{\frac{2\nu-1}{2}} \cos\frac{m\pi}{2}\,\frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sum_{\lambda=0}^{\lambda=\infty}$$

$$\frac{(2\lambda+\nu+1)\,\Pi(2\lambda+1)}{\Pi(2\lambda+2\nu)}\;\frac{C^\nu_{2\lambda+1}(\cos\varphi)\;C^\nu_{2\lambda+1}(\cos\psi)}{(\nu+2\lambda+1)^2-(\nu+m)^2}.$$

First suppose $m = 2r$ = even; then

$$\int_0^\infty \frac{\cos(t\cos\varphi\cos\psi)\; I^{\frac{2\nu-1}{2}}(t\sin\varphi\sin\psi)\; I^{\nu+2r+1}(t)\,dt}{\sqrt{t}} =$$

$$= (-1)^r\, [2^{\nu-1}\, \Pi(\nu-1)]^2 (\sin\varphi\sin\psi)^{\frac{2\nu-1}{2}}\; \frac{\Pi(2r)}{\Pi(2r+2\nu-1)}$$

$$\sqrt{\frac{2}{\pi}}\; C^\nu_{2r}(\cos\varphi)\; C^\nu_{2r}(\cos\psi),$$

$$\int_0^\infty \frac{\sin\,(t\cos\varphi\cos\psi)\; I^{\frac{2\nu-1}{2}}\,(t\sin\varphi\sin\psi)\; I^{\nu+2r}\,(t)\,dt}{\sqrt{t}} =$$

$$= (-1)^{r-1}\,[2^\nu\,\Pi\,(\nu-1)]^2\,(\sin\varphi\sin\psi)^{\frac{2\nu-1}{2}}\,\frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sum_{\lambda=0}^{\lambda=8}$$

$$\frac{(2\lambda+\nu+1)\,\Pi\,(2\lambda+1)}{\Pi\,(2\lambda+2\nu)}\;\frac{C^\nu_{2\lambda+1}\,(\cos\varphi)\,C^\nu_{2\lambda+1}\,(\cos\psi)}{(\nu+2r)^2-(\nu+2\lambda+1)^3}\,.$$

Next suppose $n = 2r + 1 =$ odd; then

$$\int_0^\infty \frac{\sin\,(t\cos\varphi\cos\psi)\; I^{\frac{2\nu-1}{2}}\,(t\sin\varphi\sin\psi)\; I^{\nu+2r+1}\,(t)\,dt}{\sqrt{t}} =$$

$$= (-1)^r\,[2^\nu\,\Pi\,(\nu-1)]^2\,(\sin\varphi\sin\psi)^{\frac{2\nu-1}{2}}\,\frac{1}{\pi}\sqrt{\frac{2}{\pi}}\sum_{\lambda=0}^{\lambda=\infty}$$

$$\frac{(2\lambda+\nu)\,\Pi\,(2\lambda)}{\Pi\,(2\lambda+2\nu-1)}\;\frac{C^\nu_{2\lambda}\,(\cos\varphi)\;C^\nu_{2\lambda}\,(\cos\psi)}{(\nu+2r+1)^2-(\nu+2\lambda)^2},$$

$$\int_0^\infty \frac{\cos\,(t\cos\varphi\cos\psi)\; I^{\frac{2\nu-1}{2}}\,(t\sin\varphi\sin\psi)\; I^{\nu+2r+1}(t)\,dt}{\sqrt{t}} =$$

$$= (-1)^r\,[2^{\nu-1}\,\Pi\,(\nu-1)]^2\,(\sin\varphi\sin\psi)^{\frac{2\nu-1}{2}}\sqrt{\frac{2}{\pi}}\;\frac{\Pi\,(2r+1)}{\Pi\,(2r+2\nu)}$$

$$C^\nu_{2r+1}\,(\cos\varphi)\quad C^\nu_{2r+1}\,(\cos\psi)\,.$$

If we put in the last four formulae $\nu = 0$ and if we observe that

$$\sqrt{\frac{\pi x}{2}}\,\overset{-\frac{1}{2}}{I}(x) = \cos x$$

$$\sqrt{\frac{\pi x}{2}}\,\overset{\frac{1}{2}}{I}(x) = \sin x$$

and $\left[\frac{1}{\nu} C^{\nu}_{n}(\cos \vartheta)\right]_{\nu=0}$ is the coefficient of α^n in the development of $-\,l\,(1 - 2\,\alpha \cos \vartheta + \alpha^2)$ according to ascending powers of α, thus 1 for $n = 0$ and $\frac{2}{n}\cos n\,\vartheta$ for $n > 0$, we find the special relations:

$$\int_0^{\infty} \frac{\cos(t \cos\varphi \cos\psi)\cos(t \sin\varphi \sin\psi)\,\overset{2r}{I}(t)\,dt}{t} = (-1)^r \frac{\cos(2r\varphi)\cos(2r\psi)}{2r}\,, \quad (1)$$

$$\int_0^{\infty} \frac{\sin(t \cos\varphi \cos\psi)\cos(t \sin\varphi \sin\psi)\,\overset{2r}{I}(t)\,dt}{t} =$$

$$= \frac{(-1)^{r-1}\,4}{\pi} \sum_{\lambda=0}^{\lambda=\infty} \frac{\cos(2\,\lambda + 1)\,\varphi \cos(2\,\lambda + 1)\,\psi}{4\,r^2 - (2\,\lambda + 1)^2}\,, \quad (2)$$

$$\int_0^{\infty} \frac{\cos(t \cos\varphi \cos\psi)\cos(t \sin\varphi \sin\psi)\,\overset{2r+1}{I}(t)\,dt}{t} =$$

$$= \frac{(-1)^r\,4}{\pi} \sum_{\lambda=0}^{\lambda=\infty} \frac{\varepsilon_\lambda \cos(2\,\lambda\,\varphi)\cos(2\,\lambda\,\psi)}{(2\,r + 1)^2 - 4\,\lambda^2} \quad (\varepsilon_0 = \tfrac{1}{2}\,;\ \varepsilon_\lambda = 1,\ \lambda > 0)\,, \quad (3)$$

$$\int_0^{\infty} \frac{\sin(t \cos\varphi \cos\psi)\cos(t \sin\varphi \sin\psi)\,\overset{2r+1}{I}(t)\,dt}{t} =$$

$$= (-1)^r \frac{\cos(2\,r + 1)\,\varphi \cos(2\,r + 1)\,\psi}{2\,r + 1}\,. \quad (4)$$

From these your formulae arise immediately when we put in (1) and (3) $\psi = \frac{\pi}{2}$, in (2) and (4) $\psi = \pi$ and interchange moreover in the two last named φ and $\varphi - \frac{\pi}{2}$.

If we put in the four general formulae $\nu = 1$ and if we remark that

$$C_n^{\nu}(\cos\vartheta) = \frac{\sin(n+1)\vartheta}{\sin\vartheta}$$

there appear besides two relations really alike to the formulae (2) and (4) the two new ones :

$$\int_0^{\infty} \frac{\sin(t\cos\varphi\cos\psi)\sin(t\sin\varphi\sin\psi)\,I^{2r+1}(t)\,dt}{t} =$$

$$= \frac{(-1)^{r-1}\,4}{\pi}\sum_{\lambda=0}^{\lambda=\infty}\frac{\sin(2\lambda+2)\varphi\,\sin(2\lambda+2)\psi}{(2r+1)^2-(2\lambda+2)^2},$$

$$\int_0^{\infty} \frac{\sin(t\cos\varphi\cos\psi)\sin(t\sin\varphi\sin\psi)\,I^{2r+2}(t)\,dt}{t} =$$

$$= (-1)^r\,\frac{\sin(2r+2)\varphi\,\sin(2r+2)\psi}{2r+2}.$$

Kinematics. — *"The motion of variable systems"* (2nd part). by Prof. J. CARDINAAL.

1. In the previous communication the construction of the direction of velocity was our subject, no mention being made of the length of velocity ; in the following some relations will be deduced in which these lengths appear.

As is known, the rule that the extremities of the velocities form a system similar to the original one holds good for the motion of a plane system remaining congruent or similar to itself. If the system remains affine to itself, the extremities of the velocities form an affine system.

In each of these cases the points of coincidence of the original system (Σ) and the system formed bij the extremities of the velocities (Σ_v) are the same as those of the system Σ and the one lying at infinitesimal distance. We shall now consider, which theorems hold good for the motion of spacial systems.[1]

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[1] Whilst writing this paper I got to hand number 1—2 of the 47th. vol. of the "Zeitschrift der Mathematik und Physik", in which L. BURMESTER treats of a subject closely allied to the paper, cited in the previous communication. As however the proofs given there differ from those given here, though the results agree, I have made no change in this text.

2. We again start from the point of the motion of a system Σ, remaining congruent to itself; we construct the direction of velocity v_a of a point A and mark off on v_a a distance AA_v representing the velocity of A. When Σ moves in such a way that A follows the direction $A\,A_v$, then A comes to A_v; however we can suppose the motion to be different but yet A reaching A_v.

3. For this purpose we construct the rectangular triangle $A\,A_x A_v$, of which $A_x A_v$ is parallel to the principal axis l and AA_x is normal to it. We then bring the plane ξ through AA_x normal to l; let X be the point of intersection of ξ and l. If we now consider the points of Σ situated in ξ and if we compare the position of A_x to that of A, we then see that A and A_x can be regarded as homologous points of two similar systems in ξ. The same reasoning can be applied to each plane normal to l brought in a similar way through points $B\,B_x$, $C\,C_x\,\ldots$; from this ensues:

The spacial systems $ABC..$, $A_x\,B_x\,C_x$ are projective; the tetrahedron of coincidence has as opposite edges l and l_∞; all points of l are points of coincidence, the two other points of coincidence are the cyclic points of any plane cutting the principal axis l at right angles. Each point of l is the pole of similitude of two similar plane systems situated in a plane normal to l.

Now to the system $A_x\,B_x\,C_x\,..$ is communicated a translation parallel to the principal axis the amount of which is $A_x\,A_v$; in this motion this system remains congruent to itself; the tetrahedron of coincidence of the systems $A_x\,B_x\,C_x\,\ldots$, and $A_v\,B_v\,C_v$ has further as opposite edges l and l_∞, but now all points of l_∞ are points of coincidence and the points of coincidence on l coincide in the point of intersection of l and λ_∞.

4. By the preceding considerations the connection between the systems Σ and Σ_v has become evident. If the intermediate system of the points $A_x\,B_x\,C_x\,\ldots$ be denoted by Σ_x, we then have the following relations:

a. The systems Σ and Σ_x are projective; the points of coincidence are all points of l and the cyclic points on l_∞.

b. The systems Σ_x and Σ_v are projective; the points of coincidence are all points of l_∞ and the point of intersection of l and λ_∞ counting double.

c. The systems Σ and Σ_v are projective; the points of coincidence are the cyclic points on l_∞ and the point of intersection of l and λ_∞ counting double.

So in space the rules for projectivity as found in the plane do not hold good unconditionally; the system Σ_v is not similar to Σ; for though in the sections normal to l two similar systems are situated, those which are parallel to l contain congruent systems; as however λ_∞ is a face of the tetrahedron of coincidence and the condition of affinity is satisfied, the theorem ensues: The extremities of the velocities of the points of a spacial system Σ which remains in its motion congruent with itself, form a system Σ_v affine with Σ. The tetrahedron of coincidence of Σ and Σ_v agrees with that of Σ and the system Σ' situated at infinitesimal distance, with which Σ will coincide in its motion in the direction of the velocities.

5. As little as in the preceding paper is it necessary to give applications of all relations existing between two affine spacial systems; now too we shall content ourselves with a single one. It is well kwown that the directions of velocity of the points of a right line are generatrices of a hyperbolical paraboloid; so the extremities of these velocities are also situated on this paraboloid and the systems Σ and Σ_v being affine, they lie on a right line, on which they generate a range of points similar to that on the given right line.

6. From the preceding ensues that with given direction of velocity v_a we can assume a point A_v on it, representing the extremity of the velocity of a point A. According to the preceding paper however A cannot be taken arbitrarily on v_a but is constructed on it as foot of the distance between e and v_a; reversely the direction A_x, when A is given, will be completely determined as tangent to the circle having AX as radius, whilst $A_x A_v$ must be parallel to l; so the direction of velocity AA_v must be situated in a definite plane.

7. We now pass to the consideration of the velocities of a system Σ changing projectively and we retain the annotations used in the preceding communication. After a closer analysis the point A_v homologous to A proves to have been found by first assuming l as axis of rotation and then by letting the points of Σ perform a motion round this axis, which we might call a "rotation of similitude"; after this the points of Σ_x have performed a rotation round l_∞. With these successive motions the points of coincidence were retained. Now we again assume the tetrahedron of coincidence $PQRS$. And we construct, taking the found principles into consideration, the point A_v homologous to A.

The motions now to be performed we call "projective rotations."

8. We bring through A and RS a plane cutting the planes PQR and PQS in P_1R and P_1S; we construct the conic C_1^2 passing through A and touching P_1R and P_1S in the points R and S. The tangent through A to C_1^2 indicates the direction, in which A must move when the system of points in the plane P_1RS performs a projective rotation; if we assume a point A_x on this direction of velocity, the length of the velocity is determined. We have now to prove that the points of Σ_x lying in the plane P_1RS as well as other points are determined.

9. The points of Σ in the plane P_1RS can be instantly found by making use of two principles holding good for the projective rotation in a plane and which we may suppose to be known:

a. The extremities of the velocities of all points of C_1^2 also lie on a conic, touching P_1R and P_1S in R and S; a point A_x is sufficient to determine this conic.

b. The extremities of the velocities of all points of a right line through P_1 also lie on a right line through P_1.

If we assume a point B outside the plane P_1RS, we construct the PQ in P_2. With the projective rotation the axis PQ being in all plane BRS cutting its points a line of coincidence, a projective rotation is also made in P_2RS, having P_2 as pole and where we can imagine the points to move in the direction of tangents to conics, touching P_2R and P_2S in R and S. So we can determine the direction of velocity of B as tangent to the conic constructed with the aid of B. So of each point in the plane P_2RS the direction of velocity is situated in this same plane and according to the preceding perfectly determined.

10. The relations just found are sufficient to determine to the points ABC . . . . , the directions of velocity AA_x, BB_x, CC_x . . . . . , but of the system of points Σ_x itself only A_x is assumed; B_x, C_x . . . . have not yet been constructed. For these the following principles can be maintained:

a. To be constructed the directions of the velocities of the points of the right line AB; these form a system of generators of a hyperboloid, to the second system of which AB and RS belong; the right line through A_x and belonging with the two last named to one and the same system contains the points B_x . . . . .

b. The generator of the system Σ_x homologous to a right line a of Σ, cutting RS, is situated together with a and RS in *one* plane.

c. The generator of the system Σ_x homologous to a right line b, cutting PQ, cuts PQ in the same point.

11. Let us now suppose the point of Σ_x to be determined; we let this system also rotate projectively, allowing P, Q and all points of RS to be points of coincidence. If we now bring, in a corresponding way as was done when Σ was made to pass into Σ_x, planes through PQ cutting RS in R_1, R_2. . . . , we arrive at the construction of Σ_v and we can easily state rules for the connection between Σ_x and Σ_v corresponding to those for Σ and Σ_x; at the same time the general theorem ensues:

The extremities of the velocities of the points of a spacial system Σ, changing projectively in its motion, form a system Σ_v projective to Σ. The tetrahedron of coincidence of Σ and Σ_v agrees with that of Σ and the system Σ' at infinitesimal distance from it, with which Σ will coincide in its motion in the direction of the velocities [1]).

12. The construction of A_v belonging to A remains possible when the points P and Q as well as R and S are conjugate imaginary. We suppose on PQ as well as on RS an elliptic involution of points to be given of which P,Q and R,S are to be the imaginary double points. Now we can construct as in the entirely real tetrahedron of coincidence a plane ARS cutting PQ in P_1.

In this plane the conic G_1^2 is determined by the following data: the point A; pole and polar P_1 and RS; the condition that each pair of rays of the involution P_1/RS is conjugate polar with respect to C_1^2. So the system Σ_x as well as Σ_v can be constructed.

13. The comparison of the constructions of this communication with those of the preceding one finally gives rise to some observations.

a. The focal system formerly applied does not appear in this paper; it has been replaced by other assumptions. This focal system served as a help for the determination of the directions of velocity of spacial systems at infinitesimal distance from each other; we now see that the length of the velocities could be used for that purpose. The binding conditions for the construction of the directions of velocity rest on the identity of the tetrahedron of coincidence between Σ and Σ_v and that between Σ and the system Σ' lying at infinitesimal distance.

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[1]) It is to be noticed that for accelerations theorems can be found corresponding to those now treated. For similar and affine plane systems **Burmester** did so in his "Kinematik". For spacial systems remaining congruent Dr. P. **Zeeman Gzn.** gave a deduction in Problem 132 of the "Wisk. opgaven van het Wisk. Genootschap" Vol VIII with statement of literature. Note the last already quoted paper of **Burmester.**

b. Just as in the preceding communication the first choice of the opposite edges PQ and RS is arbitrary; so here too the observations made formerly on the multiplicity of the solutions hold good.

c. It has been proved that the construction of the direction of velocity v_a is possible when the vertices of the tetrahedron are imaginary. The cones, treated of in the first communication, having P as vertex and with tangential velocities, disappear however in this case, their existence resting upon the reality of the vertices and the faces.

Astronomy. Prof. H. G. VAN DE SANDE BAKHUYZEN presents, on behalf of the Eclipse-Committee, the *„preliminary report of the Dutch expedition to Karang Sago (Sumatra) for the observation of the total Solar Eclipse of May* 1901."

Before giving the preliminary account of our work, we shall recall in a few words the history of and the preparations for the expedition, as well in Holland as in the Dutch Indies.

The first step towards the preparation of the expedition was taken on the 14th of October 1898, when, at the house of Prof. J. A. C. OUDEMANS and in the presence of Professors H. G. V. D. SANDE BAKHUYZEN and J. C. KAPTEYN, Prof. A. A. NIJLAND called attention to the matter and at the same time intimated that already a considerable sum had been promised him for the purpose.

As a consequence of this conference two meetings were held in the Trippenhuis on Jan. 28 and April 22 1899, which were attended by the same four gentlemen. The plan of a Dutch eclipse expedition appeared to be capable of execution; a rough estimate of the costs was made (*f* 50000 [1]) and the means to get the necessary money were discussed. Prof. V. D. SANDE BAKHUYZEN had already requested Dr. J. P. V. D. STOK, Director of the Royal Magnetical and Meteorological Observatory at Batavia and Major J. J. A. MULLER, Commander of the Triangulation-brigade of the topographical service, also at Batavia, to provide information about the localities best situated for the observations and about the climatological conditions prevalent in these places.

He had also addressed himself to His Excellency the Colonial Minister with the request to support the necessary investigations, and to promote the interests of the expeditions to be sent out, both from the Netherlands and from foreign countries.

At the request of Mr. NIJLAND it was further decided to call in

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[1] *f* 300 = 120 \$ = 25 £ = 600 = 500 Mk.

the aid of the Academy of Sciences. Accordingly at the meeting of the Section of Sciences of May 1899, on the proposal of Mr. v. d. Sande Bakhuyzen, an „Eclipse-Committee" was nominated, consisting of the astronomical members of the Academy, Messrs. H. G. v. d. Sande Bakhuyzen, J. A. C. Oudemans, J. C. Kapteyn and E. F. v. d. Sande Bakhuyzen, and, as the physicist of the Committee, Prof. W. H. Julius. The Committee, excercising a granted ,right, assumed as further members Messrs. J. H. Wilterdink and A. A. Nijland, and (on his arrival in Holland in June 1899) Mr. J. P. v. d. Stok.

The Committee called Mr. H. G. v. d. Sande Bakhuijzen to the chair and elected Mr. A. A. Nijland as their secretary. In the years 1899, 1900 and 1901 nine meetings were held. At the request of the Committee Messrs. Julius, Wilterdink and Nijland were appointed by the Government to take charge of the observations in India, and Major J. J. A. Muller undertook to act as chief of the expedition in India. A detailed programme of the observations was next drawn up and the necessary instruments were planned and bought. (See Proceedings Royal Academy Amsterdam, Vol. III pages 529—543).

In the meantime the Eclipse-fund had increased to *f* 42000, of which no less than *f* 24000 had been contributed by private persons who took an interest in the expedition. Further *f* 3000 was received from various scientific Societies; the Netherlands Government gave a subsidy of *f* 5000 and the Indian Government promised *f* 10000.

It must also be mentioned that Messrs. Julius, Kapteyn, van der Stok and Nijland composed a short pamphlet entitled „Directions for amateur observers", which was printed at Batavia and of which nearly 500 copies were distributed amongst the officials and inhabitants of the path of totality.

In the meantime preparations for the expedition had been made in India by the Royal Physical Society of Batavia, at the request of the Government. This Society nominated a Committee, consisting of Major Muller, Dr. S. Figee and Mr. A. C. Zeeman, inspector of the „Gouvernementsmarine" etc., in order to continue in a more official manner the work begun by Messrs. Muller and van der Stok. This Committee published in 1900 a pamphlet under the title: „Informations for observing parties and climatological conditions along the track of the moon's shadow", containing, as the result of the investigations which had also partly been made by Dr. van der Stok, the data about the climate, the suitability and the accessibility of many places near the central line of the eclipse, and of which 200 copies were distributed.

The Committee received much support from the part of the Government of the Dutch Indies. Not only was great pecuniary and other material aid provided, but the authorities also did their utmost to promote the interests of the expedition in every possible manner. Thus, to mention only the most prominent matters, the co-operation of several officers was granted *viz.*: captain WACKERS for making determinations of time and latitude in the eclipse-camp, lieutenant DE ROCHEMONT for the building of the camp, captain KERKHOFF for the observations of the flash-spectrum near the northern limit of totality; further the promise was given that a man-of-war would be stationed near the place of observation, so that the officers and crew might assist in the observations.

Messrs. MULLER and FIGEE personally inspected in two voyages (May and September 1900) the localities which appeared suitable for observations generally and for the erection of a Dutch eclipse-camp specially. On their advice the Dutch expedition chose as observing station a locality near the kampong *Karang Sago* on the western coast of Sumatra.

In January 1901 Dr. FIGEE and Dr. VAN BEMMELEN were authorised to take part in the expedition at the expense of the Observatory at Batavia, in order to make the necessary magnetical and meteorological observations during the eclipse.

In the middle of March Mr. DE ROCHEMONT arrived at the locality which was chosen for the erection of the eclipse-camp; in the time of three weeks he established there a very comfortably arranged camp. Not to mention many coolies, the expedition disposed, for all kinds of preparatory work, of the services of twelve men of the corps of military engineers with two European and two Amboinese sergeants, two artisans of the same corps, a warrant-officer and a mandoer of the triangulation, a photographer and a mechanician. There also was a post, and telegraph office on the spot.

The members of the expedition appointed by the Eclipse-Committee, Messrs. JULIUS, WILTERDINK and NIJLAND started on their voyage about the beginning of March. At Genoa they met Mr. J. B. HUBRECHT phil. nat. cand., who had been permitted to assist the expedition as a volunteer, and on board the „Koningin Regentes'' they met eight American and English astronomers, who also intended to observe the eclipse in Sumatra. We are specially indebted to the S.S. Company „Nederland'' for the liberal arrangements as to the conveyance of the observers and their instruments.

On the sixth of April we arrived at Padang, whence, after some official calls, we started, on the 10th of April, together with Mr. MULLER on our voyage to the camp by Government steamer. The English expedition, which was going to *Aoer Gadang*, was also on board.

On the 11th a start was made by the unpacking of the instruments (which appeared almost without exception to be in good condition), the surveying of the locality and the building of several (22) brick piers for the astronomical, physical, magnetical and meteorological instruments.

Mr. WACKERS had already started the determinations of time and latitude. The latitude of the pillar of the Universal-instrument was found to be 1° 19′ 27″.5 South; the longitude was approximately determined at 100° 33′.3 East of Greenwich.

On the 23rd of April the last pillar was ready; the greater number of the instruments was already in position, protected by huts or sheds of bamboo and atap. The long light-tight tube and hut for the 40-feet coronagraph were only ready on the 30th of April.

The accompanying plan of part of the camp (Plate I) gives an idea of the relative positions of the instruments. (See also Proceedings, Vol III, pages 529—543).

The adjustment of the instruments occupied the members of the expedition up to the last day. The task of the other members was made much more difficult by the accident which happened to Mr. WILTERDINK on the 3rd of May: while occupied with Mr. NIJLAND with the adjustment of the 10-inch coronagraph, he fell from a wooden scaffolding and broke his right radius. Not only was the use of his hand strictly forbidden him, but he had to be transported to Padang for a few days, and though after his return he constantly assisted the other members with his advice, he was not able during the eclipse to take charge of one of the principal instruments.

The erection of the magnetical and meteorological instruments was started on the 27th of April, under the direction of Mr. VAN BEMMELEN, who arrived in the camp the day before. On the 7th of May Dr. S. FIGEE arrived, accompanied by his son Mr. TH. FIGEE, volunteer-assistant. On the same day also Her Majesty's ironclad *Sumatra*, under temporary command of the lieutenant first class GILDERMAN, arrived on the road. At the same time captain KERKHOFF visited the camp in order to study, under the direction of Messrs.

JULIUS and NIJLAND, the objective grating spectrograph to be erected near the northern limit of totality.

Messrs. MULLER, WILTERDINK and NIJLAND had in the meantime drawn up a programme for the astronomical observations while Mr. JULIUS did the same for the physical observations, so that on the 9th of May the practice drills with the officers and men of the *Sumatra* could begin.

The work was divided as follows:

Chronometers and *Siderostats*: 5 assistants;
Large spectrograph: Mr. S. FIGEE with 2 assistants;
Small spectrograph: Mr. DE ROCHEMONT with 6 assistants;
Spectroscope: Mr. HUBRECHT with 3 assistants;
Prismatic camera: Mr. NIJLAND with 4 assistants;
„40-*feet*" *coronagraph*: Mr. MULLER with 3 assistants;
„10-*inch*" *coronagraph*: Mr. WACKERS with 5 assistants;
Coronagraphs on polar axis: midshipman BRANDT with 10 assistants;
Photometer: Mr. VAN BEMMELEN and midshipman BARON MACKAY with 2 assistants;
Actinometer: Mr. JULIUS with 4 assistants;
Polarimeter: lieutenant V. D. ESCH with 3 assistants;
Pyrheliometer: 2 assistants;
Cloud-theodolite: lieutenant DE BRUYNE with 1 assistant;
Observations of wind: 2 assistants;
Thermometers: 4 assistants;
Atmospheric electricity: Mr. TH. FIGEE;
Declinometer: 2 assistants;
Drawings of corona: midshipman BALSEM with 4 assistants;
Shadowbands and *further observations*: midshipmen LANGELAAN and BUDDING with 14 assistants;
Look-out: one assistant;

The total number of observers was thus 93, of whom 7 were officers and 68 non-commissioned officers and men of the *Sumatra*.

The uncertainty of the longitude of the place of observation and the experiences from former eclipses pointed to the desirability of deriving the last warning sign of „ready" not from the computation, but from the observations of the eclipse itself. It was arranged that Mr. MULLER should give this sign, when in the dark hut of the 40-feet coronagraph the crescent of the sun measured 45°. According to computation this would occur 16^s before the beginning of totality. Moreover, the look-out was to watch the search-light of the English man-of-war *Pigmy*, stationed at *Aoer Gadang*, which light would, at our request, be screened off at the moment when totality began at that station.

Computation had shown that the shadow would reach the Dutch eclipse-camp about 15 seconds later.

In the beginning of May the English astronomer NEWALL, who was stationed at *Sawah Loentoh*, sent notice that he had been informed by the *Nautical Almanac Office* at London that the duration of totality at *Karang Sago* would be $6^m\ 22^s$ instead of $6^m\ 32^s$. It was decided, to be on the safe side, to shut all cameras on the 380th second after the beginning of totality.

The programme was for the greater part promptly executed on the 18th of May. The sky was however rather heavily clouded, and the various observations and exposures have all experienced more or less the disturbing influence of the clouds. Unfortunately, moreover, the nervousness of one of the assistants has made the results of the *small spectrograph* entirely useless.

Owing to the clouds it was not possible to observe the sun's crescent of 45°; the last warning sign was given by the look-out. Totality commenced $0^h 19^m 55^s$ (local time) and ended $0^h 26^m 16^s$; the duration therefore was $6^m 21^s$. The computation according to the Nautical Almanac gave $0^h\ 19^m 58^s$ for the beginning and $0^h 26^m 30^s$ for the end of totality, and therefore $6^m\ 32^s$ for the duration.

Shadowbands have not been observed; besides *Venus* and *Mercury* only *Aldebaran* and a few stars in *Perseus* were visible. As a consequence of the rather heavy clouds it was not nearly so dark during the eclipse as had been expected, so that the lamps which were kept ready have not been used.

Immediately after the eclipse it was known that the *spectroscope* and the *small spectrograph* had given no results. On developing the plates it also appeared that the exposure with the *large spectrograph* was an entire failure.

Before mentioning the results of the astronomical and physical observations — the magnetical and meteorological observations are being discussed at Batavia — we shall finish the history of the expedition.

On the 22nd of May Mr. KERKHOFF arrived at the camp with the plate exposed by him at the northern limit of totality: unfortunately this plate too brought a disappointment. Apparently the instrument had been adjusted correctly, but by a misunderstanding, and in consequence of the fact that the northern limit was actually further North than had been expected according to the computation, the moment of observation was not well chosen: the observer failed to get the phenomenon of the flash and photographed the spectrum of the corona instead, for which the lens however was not rapid enough.

On the 24th of May all instruments were packed and the expedition left the camp with the last of the 54 cases. Mr. MULLER undertook to have diapositives made and to send the cases to Holland. Thanks to his care everything has arrived in good order.

At Padang and Batavia Messrs. MULLER, JULIUS, WILTERDINK and NIJLAND called on some high officials to express their thanks for the good care and many facilities which had been given to the expedition; on the 6th of June they had an audience of His Excellency the Governor-General at Buitenzorg, and were invited to dinner in the evening.

On the 26th of June Mr. JULIUS returned to Holland; at *Padang* he took charge of the negatives. Messrs. WILTERDINK and NIJLAND left Batavia on the 29th of June to return home via Japan and America. Before they left they had an opportunity to inspect, at the Batavia Observatory, many drawings and descriptions which had been sent in by amateur observers, who had followed up the summoning contained in the above mentioned „Directions for amateur observers."

We now come to the principal contents of this report, *viz.*: the scientific results. We will treat them in the following order:

A. The coronagraphs;
B. The spectrographs;
C. The physical observations;
D. The amateur observations.

A. Coronagraphs.

With regard to the photographs which were obtained of the corona, we are sorry to report that only the short exposures with slow instruments have given any decent result. The long exposures and the exposures with rapid lenses have too much suffered from the general illumination of the sky and the illuminated clouds. Moreover several plates are fogged.

I. Polar axis with 4 coronagraphs.

These four coronagraphs were:

a VOIGTLÄNDER *collinear* (the travelling camera R; equivalent focal length $f = 87$ cm. $= 49\ a$);

a DALLMEYER *lens* (D; $f = 153$ cm. $= 15\ a$);

a VOIGTLÄNDER *portrait-objective* (V_1; $f = 38$ cm. $= 3.6\ a$);

a VOIGTLÄNDER *euryscope* (V_2; $f = 85$ cm. $= 8.1\ a$).

According to the programme six exposures were made, three of 1, 2 and 55 seconds respectively on plates „*Lumière bleue*", and three of 178, 55 and 1 seconds on plates „*Lumière jaune*". During the long exposures a yellow light-screen was used with the objectives V_1 en V_2.

Exposure Nr. 1 is useless: owing to the strong vibration or jolting of the apparatus the image has been displaced on some plates by as much as 4′.

Nr. 6 has also failed owing to direct sunlight. The long exposed plates all show the edge of the moon diffused, chiefly through irradiation and through the rather considerable motion of the moon (which amounts to $^1/_{110}$ diameter in 55^s and $^1/_{34}$ diameter in 178^s).

The camera *R*, which was only added at the last moment, evidently was not sharply focussed. The only occasion to focus it was on the 17th of May, about eclipse-time.

The *camera D* shows clouds on all plates; Plate Nr. 2 however is good, giving polar streamers to a distance of 10′, equatorial ones to 15′ from the moon's limb; Nrs. 3 and 4 are entirely fogged.

The *camera* V_1 shows only clouds and fogging.

The *camera* V_2 also shows clouds on all plates; the images are diffused by irradiation; Plate Nr. 3 is entirely fogged.

The following summary gives for each of the 12 partly successful exposures the distance from the moon's limb to which the corona is visible, expressed in minutes of arc.

	Nr. 2.	Nr. 3.	Nr. 4.	Nr. 5.
R	10	25	30	45
D	45	25	30	45
V_2	20	—	30	45

II. „*10-inch*" *coronagraph* (*Objective by* STEINHEIL, $f = 342$ *cm.* $= 13.3$ *a*).

Three exposures (of 2, 1 and 1 seconds respectively) have been taken on plates „*Lumière jaune*", six (of 50, 5, 216, 5, 1 and 1 seconds respectively) on plates „*Lumière bleue*".

The plates 1, 2 and 3 are fairly good though not perfectly sharp. The corona can be traced to distances of 10′, 7′.5 and 5′.0 respectively from the moon's limb, with prominences and structural detail of inner corona. Nr. 7 also shows, besides many clouds, a few details of the polar streamers. N^rs^. 8 and 9 however are entirely spoiled by clouds and by direct sunlight.

The plates 4, 5 and 6 were fitted with rotating aluminium screens (BURCKHALTER apparatus) with a view to obtaining faint and bright parts of the corona on the same plate. Three screens of different shapes had been selected. In consequence of the cloudiness, it has unfortunately not been possible to bring the moon exactly on the centre of the plate, and to keep it there, so that the axis of rotation of the screens did not coincide with the centre of the moon's image.

N^r^. 4 on one side shows details of the structure of the corona; N^r^. 6 shows the corona to a distance of 20′, N^r^. 5 to 15′ from the moon's limb; the rotating screen of this plate evidently was too small, so that the corona-streamers are lost in the completely fogged outer parts of the plate.

III. „40-feet" coronagraph ($f = 11.77\ m. = 91\,a$).

This lens was most kindly lent us by the Superintendent of the U. S. Naval Observatory, Washington D. C., to whom we tender most cordial thanks.

The original size of the plates was 50 × 60. Having satisfied ourselves that no parts of the corona, nor any stars would be lost by the proceeding, we cut them down to 24 × 30 in order to ensure a safer transport. During this operation n^r^. 2 was broken into three pieces, without spoiling, however, the essential parts of the image.

On N^r^. 1 ($^2/_3$ sec. *Lumière bleue*) accidently three exposures of different durations have been impressed; some of the prominences have consequently gained in details, while others are smudged together and unfit for measurement.

N^r^. 2 (20 sec. *Lumière jaune*) shows very fine prominences and complicated details of the inner corona, which forms dome-shaped arcs round the prominences.

N^r^. 3 (40 sec. *Lumière bleue*) is nearly as good as the former.

N^r^. 4 (180 sec. *Lumière bleue*). This plate was fitted with a large rotating screen. According to the programme a short time before the beginning of totality the mirror of the *coelostat* should be definitively adjusted so as to throw the image of the sun exactly on the centre of the plate. Notwithstanding the aperture was greatly reduced, the sunlight was still so strong that Mr. MULLER, fearing a fogging of the freely exposed plates, dared not admit the sunlight for more than a single moment. The image of the sun apparently has fallen much too high on the plate, so that the centre of rotation of the wing did not coincide with the centre of the moon's disc but with a point nearly on the moon's limb.

Consequently this plate shows details of the corona on only one side, moreover the image is diffused in the direction from East to West, probably owing to an irregularity in the motion of the driving clock of the coelostat.

Nr. 5 (120 sec. *Lumière jaune*) shows the same want of sharpness, which here also exists in the direction from North to South. Prominences and „*Bailey's beads*" are elongated and useless.

Nr. 6 has little value owing to direct sunlight.

B. Spectrographs.

I. Three Prism Spectrograph (f = 7.7 a).

An exposure of 340 seconds on a panchromatic plate of LUMIÈRE has not given the slightest trace of the spectrum of the corona. The width of the slit was $^1/_{40}$ mm. On this plate and on the one exposed immediately afterwards, air-spectra of 20 to 60 sec. exposure are very well measurable, some of them have even been over-exposed. Also the iron-spectrum given by a small coil and Leyden battery in 3.5 minutes is well measurable. The plate is entirely fogged, which is evidently due to the development, since the edges, which were not exposed, are also fogged. The plate cannot have been spoiled or stale, since proof-plates taken from the same batch and exposed on the preceding days are very clearly developed, and are entirely clear up to the edges. The matter is at present entirely inexplicable, since at all events the clouds should have given a spectrum.

II. Two Prism Spectrograph (f = 3.5 a).

The exposure with this instrument has entirely failed, the assistant charged with the control of the running plate having lost his presence of mind. The plate shows very strongly the spectrum of the last crescent of the sun (exposure 5^s on moving plate). It has however been stopped exactly in the moment when the first flash spectrum was to be impressed on it, though according to the programme it ought to have moved on for another 5 seconds. The plate has then remained unmoved for some time and shows, adjacent to the above mentioned spectrum, a strongly over-exposed spectrum, by which everything in the neighbourhood is fogged. Next the plate ought to have been brought to its middle position for the exposure of the corona. In this operation the lines of H and Ca have just been impressed, and are visible on the plate. On the middle of the plate however there is no trace of a spectrum; probably the plate has not been sufficiently displaced, and the corona has impressed itself on the strongly fogged part. In this part there are a couple of continuous spectra, in which G is shown as a dark line, not however at the proper distance (the moon's diameter) from each other. Of the second flash spectrum there is no trace.

This plate also is not well developed. The width of the slit was $^1/_{50}$ mm.

III. Spectroscope.

In consequence of the cloudiness no trace of the spectrum of the corona could be observed with this instrument.

IV. Objective Grating Spectrograph.

As has already been mentioned, the grating-camera ($f = 16\,a$) which was mounted at the northern limit of the path of totality has given no results. This is specially to be regretted since this locality was favoured by a very clear sky, and since, judging by the few crescents of the chromosphere which are shown on the spectrograms, the instrument has been correctly adjusted.

V. Prismatic Camera.

This instrument (Cooke's triplet, $f = 260$ cm. $= 17\,a$; two prisms of 45° with a total length of bases = 31.2 cm.) was fed by a siderostat. The plates (*Lumière panchromatique*) measured 16 × 16 cm, but might with advantage have been longer in the direction of the dispersion. A solar spectrum, photographed by means of a temporarily mounted collimator, showed the lines sharp along the whole length of the spectrum. The focussing was done the night before the eclipse by visual observation of the spectrum of *Arcturus*.

According to the programme 5 exposures have been made of the first flash (plate 1), each of $^3/_4$ seconds by estimation, *viz.* at the times $-\,2^s$, 0^s, $+\,2^s$, $+\,5^s$ and $+\,7^s$, where 0 is the zero of counting. (It has appeared afterwards that the second contact, which was hardly visible through the clouds, has happened at the time $-\,3^s$). After this four long exposures were taken (plates 2—5) by which it was hoped corona-rings would be photographed; these exposures lasted 5, 20, 190 and 60 seconds respectively. Finally (plate 6) again 5 exposures of the flash were made, each of about $^3/_4$ seconds, at the times 2^s, 4^s, 6^s, 8^s and 10^s after third contact.

On plate 1 only a small number (9) of chromosphere-crescents are visible; the second contact is badly shown owing to the clouds.

On two of the four corona plates (Nrs. 2 and 3) the ring λ 3987 is but faintly visible, these exposures naturally being much impaired by the disturbing influence of the clouds.

The third contact appears to have occurred at a relatively clear moment. Plate 6 shows a large number (150) of chromosphere-crescents between λ 3880 and λ 5000, in different phases of the flash phenomenon. Especially the first exposure on this plate (2^s after third contact) is very rich in details. The spectrum appears to be well focussed over its entire length.

Prof. W. H. JULIUS has already pointed out[1]) the very remarkable fact that every one of these chromosphere-crescents is double; the distance of a pair varies from 0.7 to 1.6 *t. m.*

Though we have given much thought to the question, we are unable to ascribe this doubling to instrumental causes. The distance of the components of a pair is in the mean 0.13 mm., i. e. about 11″ of arc. The doubling, though different for different crescents and for different parts of the same crescent, has a very decided preference for one direction, inclining 9° (counted from N. through E.) to the parallel, or 15° to the horizontal, 22° to the ecliptic and 29° to the sun's equator.

Though we do not wish to go too much into detail in this preliminary report, we may be permitted to mention that the most important instrumental causes which offer themselves for explanation of the observed doubling, are the following three:

1st irregularities in the rate of the driving clock of the siderostat,
2nd a vibration of the prismatic camera or of its parts, and
3rd defective focussing.

On closer examination it appears to us to be very improbable that any of these causes has been effective. It is true, during the last exposures with the 40-feet coronagraph the clock of the siderostat seems to have performed badly, but *all* plates of the prismatic camera show the same doubling, as well the exposures of $^3/_4$ sec., as those of 60^s and 190^s, and both at the first and at the last flash. Moreover there is no question of a diffused appearance of the images, but of a doubling, in a direction differing from that of the parallel.

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[1]) W. H. JULIUS, *On the origin of double lines in the spectrum of the chromosphere, due to anomalous dispersion of the light from the photosphere.* Proceedings Royal Ac. Amsterdam, Vol. IV, p. 195—203.

The opening and closing of the shutter of the prismatic camera might have caused a vibration, probably chiefly in an horizontal plane; the fact however that the doubling is not constant in one and the same exposure, and on the other hand shows itself in the same manner both in the long and in the short exposures, makes also this explanation entirely unacceptable.

No more can a considerable error of focussing be admitted, since on plate 6 lines are shown parallel to the direction of dispersion — Bailey's beads drawn out into lines by the prisms — which have a breadth of only 0.06 mm.

Finally, as far as we can see, every instrumental explanation must fail through the fact that the distance of the components of a double crescent is very different for different crescents, while also the direction of the doubling and the relative intensity of the components is variable. Moreover the corona-ring λ 3987 is, on plate 3, *sharply outlined* on the concave side. (On plate 2 this ring is too faint to state the same fact with certainty). On comparing this outline with that of the neighbouring crescents of calcium (H and K), which are very distinctly double on the same plate, there can hardly remain any doubt as to the reality of the doubling of the chromosphere-lines. The light of the corona-rings on the other hand appears to be more purely monochromatic.

A reproduction of part of the first exposure on plate 6 is appended to this preliminary report (Plate II, enlargement about 4 times) in order to enable the reader to form an approximate idea of the character of the double lines. The original negatives of course show many more details, such as Fraunhofer lines, etc., which would also be visible on a darker copy. *This* copy has however purposely been kept fainter, in order to bring out clearly the almost linear flash spectrum formed by the lowest of „Bailey's beads". The wave-lengths given in the figure are relative to this spectrum.

Enlargements of the other exposures will be published afterwards.

The doubling which was observed by us is also shown on previous flash-spectrograms; it appears however not to have drawn the attention of observers. See e. g. the crescents of λ 4584, 4572, 4564, 4554, 4550, 4534, 4501, 4472, 4341 etc. in the spectrograms which were obtained by Fowler with the „6 inch Prismatic Camera" during the eclipse of Jan. 22 1898.

When once the attention has been directed to this point, on these photographs too nearly all the crescents appear double, though less distinctly.

The appended diagram (Plate III) gives a summary of the times of exposure of the coronagraphs and the prismatic camera.

C. Physical observations.

I. *The polarization of the light of the corona.*

The aim of the observations of polarization was to measure the proportion of polarized light in as many well determined points of the corona as possible.

The observations have been made in an excellent manner by lieutenant T. J. van der Esch and four assistents.

Before the day of the eclipse the observers had repeatedly practised on an artificial corona, under circumstances as similar as possible to those which were expected during the eclipse. This artificial corona gave in the telescope an image of about the same size as that of the real corona; its light was partly polarized in radial planes, while the proportion of polarized light could be altered at will, so that control was possible. The observers got so well in practice that they were able to measure the proportion of polarized light on 12 arbitrarily selected points within 6 minutes, and to read off the coordinates of these points.

A few minutes before the beginning of totality, as well as a few minutes after the end, the diffused light of the sky round the sun (at a distance of about one diameter from the sun's limb in the directions North, East, South and West) was analyzed to decide whether

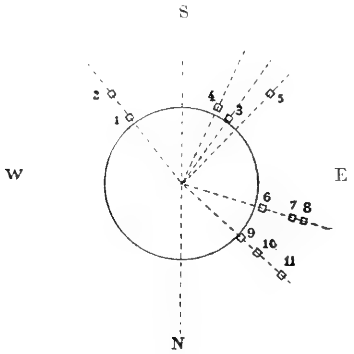


Fig. 1.

any atmospherical polarization was shown. This was however not found to any appreciable amount.

Mr. VAN DER ESCH has succeeded during totality in making complete measures on 11 points of the corona. The result is given in the following table. The position angles are counted from North through East; the radius of the moon is 13,5 mm. Fig. 1 shows the positions of the points chosen. The image of the moon and the aperture of the diaphragm, (1.1 mm)^2, are drawn at their real size. The light admitted through this aperture was analyzed by a modified polarimeter of CORNU.

Nr.	Position-angle.	Radius-vector.	Reading of analyzer α	Proportion of polarized light $\cos 2\alpha$
1	218,°5	15 mm	38°	0,242
2	218,5	20	34	0,375
3	145	14	37,5	0,259
4	155,5	15	43	0,070
5	136,5	22,5	45	0,000
6	73,5	15	38	0,242
7	73,5	20,5	42,2	0,098
8	73,5	22,5	45	0,000
9	48	14	38,2	0,237
10	48	18	34	0,375
11	48	24	44	0,035

It appears from these numbers that the light of the corona at a certain distance from the sun's limb is more strongly polarized than in the neighbourhood of that limb. At still greater distances the polarization decreases again. That the observation Nr. 4 has given a smaller proportion of polarized light than Nr. 3 must probably be explained by the fact that the pointing was made on one of the dark rifts.

All the observations having been made through clouds the question naturally arises how far the result of the observations can have been disturbed by this circumstance. It might be that a cloud or haze, like a piece of frosted glass, had an appreciable depolarizing influence on the transmitted light.

An investigation about the influence of haze on polarized light has been taken in hand in the physical laboratory at Utrecht, but has not yet been finished. As the result of preliminary experiments we can only mention here that most probably the measures of the polarization of the light of the corona have only slightly been disturbed by the presence of clouds. If the sky had been clear, the main results would have been the same.

II. Heat-radiation of the corona.

In a much higher degree than the light-rays the dark heat-rays are intercepted by clouds and haze. The conditions were thus very unfavourable for the measures of radiation, and this circumstance is so much the more to be regretted since the observations have shown that in clear weather the ratio between the total radiation of the corona and the total radiation of the uneclipsed sun would undoubtedly have been successfully determined.

The method of observation was very simple. A very sensitive small thermopile (consisting of 8 couples of alloys of bismuth whose total exposed surface occupied a circle of 5 mm. diameter) was exposed without the intervention of either lenses or mirrors to the radiation of a circular patch of sky of 3° diameter.

The apparatus was mounted parallactically and could easily be kept in such a direction that the eclipsed (or non-eclipsed) sun remained in the centre of the field of 3°.

If we take as unit the radiation which will give a deviation of one scale-division, when the resistance in the circuit and the sensibility of the galvanometer have the same values which they actually had during the pointings on the corona, then the strongest radiation we observed during our stay in the camp is represented by the number 1941000. On clear days (e. g. on the 6th of May) the observations ran very regularly. At $7^h 58^m$ a. m. (in this section of the report we use *civil* time) of that day the radiation was 1457000; it increased with small oscillations caused by invisible haze to 1902000 at $12^h 0^m$.

Not quite so regular but still very satisfactory were the readings on the fairly clear days of the 14th, 15th and 16th of May. On the 17th no observations were made on account of the heavy clouds, and on the morning of the 18th very discordant deviations of the galvanometer were obtained, clouds continually passing before the sun.

During the time between the first and second contacts only 26 determinations of radiation were made at moments when the clouds were not too thick. The direct radiation of the sun on the little thermopile at $10^h\ 46^m\ 10^s$, (i. e. about 20^s before first contact) amounted to 1762000. The diminution which in the case of clear weather *could have been followed step by step*, now of course took place very irregularly. At $12^h\ 18^m\ 54^s$, i. e. 61^s before second contact we observed through the clouds an intensity of radiation of 604.

After the beginning of totality the thermopile was directed in succession to four points of the sky, situated to the North, East, South and West of the sun, each at a distance of 3° from the disc, and between each of these exposures it was exposed to the corona. All these fields were filled by clouds. The readings of the galvanometer for the four fields round the corona showed irregular differences (varying between — 9 and + 1 scale-divisions) compared with the readings for the central field. It was not possible to derive from the obervations an estimation of the radiation of the corona, but the numbers obtained seem to point to the conclusion that the radiation of the corona cannot be much larger than the observed differences between the radiation of neighbouring groups of clouds. The full moon would, in a clear sky, have given a deviation of 22 or 23 scale-divisions; the radiation from the corona seems to be of the same order of magnitude.

One thing is certain, viz. that our results cannot be made to agree with the opinion of DESLANDRES that it is possible to measure the heat-radiation of the corona without an eclipse (C. R. CXXXI p. 660).

After third contact the surroundings of the sun were comparatively clear for a few minutes, but soon afterwards the clouds came on again. The observations were continued at intervals to $1^h\ 12^m$ when they were discontinued, the results being considered untrustworthy.

The large influence of the clouds on the heat-radiation is shown by the fact that 8 minutes before 2nd contact the transmitted radiation was 455, while 8 minutes after 3rd contact it amounted to 74300.

Every precaution had of course been taken to make the instrument practically insensible for alterations of temperature in the neighbourhood. Alterations within the above mentioned field of 3° diameter were the only ones that were recorded by the galvanometer.

III. Absolute amount of solar radiation.

A number of observations were made with the electrical compensation-pyrheliometer of Ångström on different dates and also on the date of the eclipse, but the great variability of the weather and the continual high percentage of aqueous vapour in the atmosphere are the reasons why the western coast of Sumatra is a very unfavourable region for obtaining good series of observations of absolute solar radiation.

The number of gram-calories per minute per square cm. on May 18 at 10^h 45^m a. m. was

$$Q = 1{,}23.$$

Very seldom a larger value than this was found during any of the days preceding the eclipse; only once we found 1,35.

At Teneriffe, at a small elevation above the level of the sea Ångström found $Q = 1{,}37$, the sun's altitude being 70°. It appears therefore that even on clear days in the coast-region where we stayed the heat-radiation is to a considerable extent absorbed by the atmosphere.

We intended to make observations with the pyrheliometer during the partial phase of the eclipse so long as the heat-radiation would be sufficiently intense, in order to check the simultaneous results of the thermopile, and to convert these latter into absolute measure. This end has however not been attained; owing to the exceeding variability of the radiation the pyrheliometer ceased to give reliable results soon after first contract.

D. Amateur observations.

The „Directions for amateur observers" contained a recommendation for the following operations:

I drawings of the corona;
II photographs of the corona;
III photographs of the landscape during the different phases of the eclipse;
IV observations of „shadowbands";
V meteorological observations;
VI further observations about:
  a the colour of the corona;
  b the visibility of the corona before and after totality;
  c the colour of landscape and sky;
  d the visibility of stars.

Limiting ourselves in this preliminary report to those points from which possibly a result of some value can be derived, we will omit the observations mentioned sub III and VI.

I. The drawings of the corona, of which a fairly large number (39) has been sent in, show much smaller discrepancies than have generally been recorded before. (See e. g. ABBE's report of the total eclipse of July 1878). This fact must be ascribed to the care with which evidently the observations have been prepared.

Probably the policy recommended in the „Directions", and which has been followed up by several observers, of making one person responsible for only one quadrant, or one half of the corona, has done much towards securing a good result, while also the long duration of totality (maximum $6^{m}.4$) naturally was a very favourable circumstance.

In the following table the localities have been arranged from West to East.

LOCALITY.	Number of drawings of corona.	Quality.	Remarks.
Fort de Kock..........	1 whole and 8 quadr.	fair	
Padang Pandjang.......	2 〃 〃 8 〃	good	
Karang Sago	1 〃 〃 4 〃	fair	partly clouded.
Pajacombo	5 〃	fair	
Moeara Tambesi........	1 〃	useless	
Blinjoe................	1 〃 and 2 halves	very good	
Pontianak	1 〃 〃 4 quadr.	very good	
Moeara Djawa	〃	fair	
Samarinda.............	9 〃	good	cirrus clouds. A screen has been used to cover the bright parts of the corona.
Saparoea	2 〃 and 8 quadr.	very good	
Banda	1 〃 〃 8 〃	fair	
Gisser.................	3 〃	fair	
Total	39 drawings.		

These 39 drawings of the corona provide, in our opinion, very useful data for a definitive picture of the corona independent of the photographs, and which may be a real contribution to our knowledge of the corona of 1901.

II. Of the 69 photographs of the corona which have been sent in, made on all sorts of plates with all sorts of cameras, 15 appear to be practically useless. The quality of the others is given in the following table, where

1 means all but useless
2 „ fair
3 „ good
4 „ very good
5 „ excellent.

The second column gives the diameter of the image of the sun in millimeters.

LOCALITY.	☉	Quality.					Total.	Remarks.
		1	2	3	4	5		
Padang	5		2	3	1		6	
Fort de Kock	1	1	1				2	
„	$1^1/_4$	1	1				2	
„	$3^1/_2$				1	5	6	
Djambi	2	1					1	
Blinjoe	$2^1/_2$		1			5	6	
„	$2^1/_2$		2		4		6	film
Pontianak	2				1		1	
Samarinda	$1^1/_2$	1			1		2	
Donggala	$1^1/_2$	3	2	1				
Saparoea	$3^1/_2$			2	3	1	6	
Banda	$1^1/_2$					4	4	
„	$2^1/_2$		1			2	3	
?	4				3		3	
Total		7	10	6	14	17	54	

Further informations must still be provided about the name of the locality denoted by ?. The three photographs sent in from that locality would have been excellent if not by an unjudicious treatment the sensitive film had been damaged.

IV. Reports about observations of shadowbands have come in from the following localities, again arranged from West to East. A + sign in the 2nd, 3rd, 4th, 5th and 6th columns respectively means that the shadowbands have been observed on an horizontal plane, on a vertical plane, before totality, after totality, and that details are given about their motion. The 7th column gives some remarks about the cloudiness.

LOCALITY.	hor.	vert.	before	after	motion	Cloudiness.
Fort de Kock	+	+	+	+	+	
Padang Pandjang	+	—	+	+	+	
Karang Sago	—	—	—	—	—	partly clouded.
Pajacombo	+	+	+	—	—	
Moeara Tambesi	—	—	—	—	—	lightly clouded.
Djambi	—	—	—	—	—	half ″
Blinjoe	+	+	+	+	+	
Pontianak	+	—	+	—	—	
Boentok	—	—	—	—	—	
Moeara Teweh	+	—	+	—	+	
Moeara Djawa	—	—	—	—	—	lightly clouded.
Samarinda	+	+	+	+	+	″ ″
Donggala	+	—	+	+	+	
Posso	—	—	—	—	—	cloudy.
Saparoea	+	+	+	+	+	
Banda	+	—	+	+	+	
Gisser	—	—	—	—	—	clear.

We further append a list of the localities mentioned in this account, their co-ordinates and the duration of totality.

LOCALITY.	Long. E	Lat. S.	Duration.	LOCALITY.	Long. E.	Lat. S.	Duration.
Banda..........	129°.9	4°.5	$3^{m}.0$	Moeara Tambesi.	103°.1	1°.4	$5^{m}.1$
Blinjoe..........	105°.7	1°.6	$2^{m}.0$	» Teweh..	114°.9	1°.0	$5^{m}.7$
Boentok.........	114°.8	1°.1	$5^{m}.7$	Padang.........	100°.3	1°.3	$6^{m}.2$
Djambi..........	103°.6	1°.6	$4^{m}.3$	Padang Pandjang	100°.4	0°.5	$4^{m}.3$
Donggala........	119°.7	0°.7	$3^{m}.6$	Pajacombo......	100°.6	0°.2	$2^{m}.6$
Fort de Kock....	100°.4	0°.3	$2^{m}.8$	Pontianak......	109°.3	0°.0	$5^{m}.7$
Gisser..........	130°.9	3°.9	$4^{m}.4$	Posso..........	120°.8	1°.3	$5^{m}.1$
Karang Sago.....	100°.5	1°.3	$6^{m}.4$	Samarinda......	117°.3	0°.5	$4^{m}.8$
Moeara Djawa....	117°.2	0°.9	$5^{m}.5$	Saparoea........	128°.6	3°.6	$4^{m}.4$

V. The meteorological observations which have been sent in, will be discussed at Batavia.

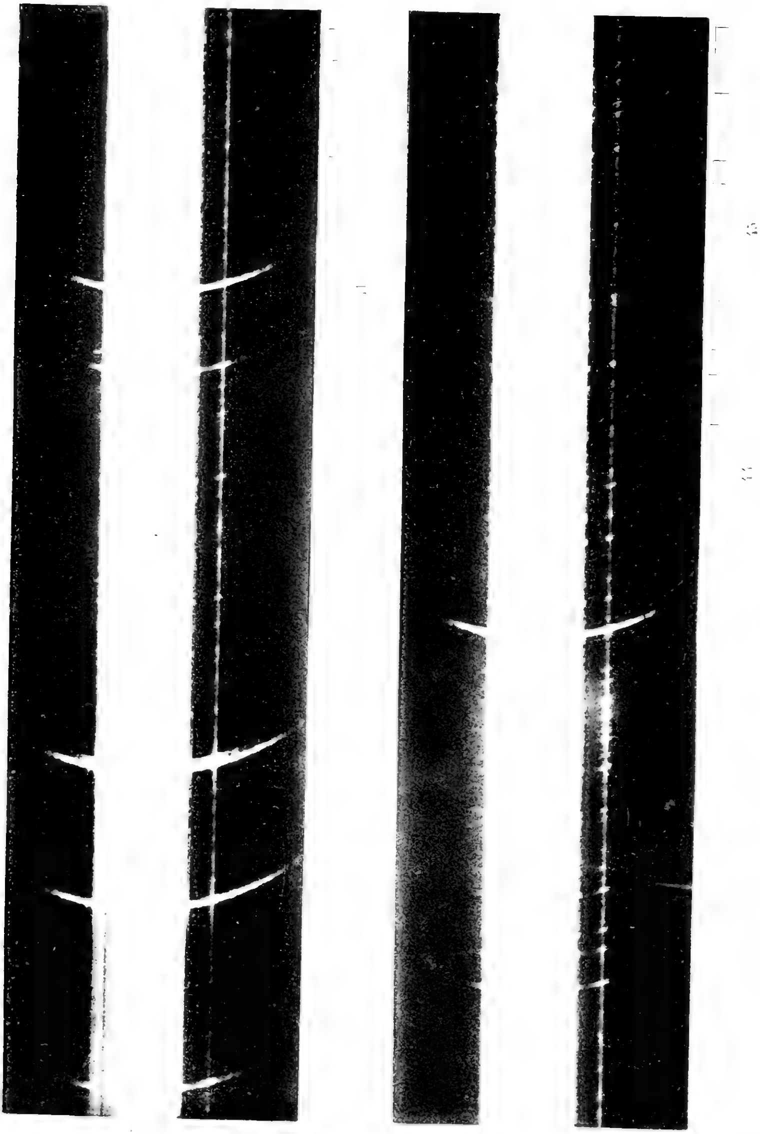
Utrecht,
Leiden, March 1902.

W. H. JULIUS.
J. H. WILTERDINK.
A. A. NIJLAND.

(April 28, 1902.)

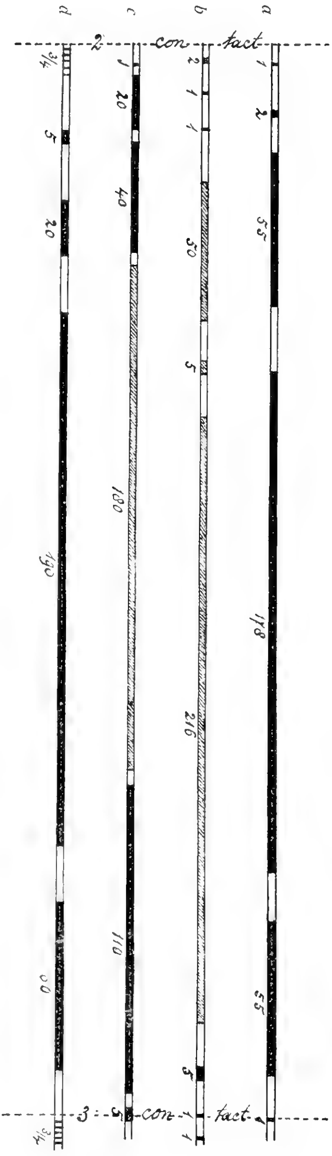
Total Solar Eclipse, 1901. Dutch Observations.

Plate II.



[illegible] AMSTERDAM

DIAGRAM OF EXPOSURES.



a. Small coronagraphs. — *b.* coronagraph f = 3.42 M. — *c.* coronagraph f = 11,77 M. — *d.* prismatic camera.
The hatched spaces denote exposures in which BURCKHALTER screens were used.

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING
of Saturday April 19, 1902.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 19 April 1902, Dl. X).

The following papers were read:

Chemistry. — Dr. P. VAN ROMBURGH: *„On the action of nitric acid on alkylated amides of p.-toluenesulphonic acid"*. (Communicated by Prof. A. P. N. FRANCHIMONT).

(Communicated in the meeting of February 22, 1902).

Many years ago [1]), I communicated the results of an investigation of the action of nitric acid on substituted amides of benzenesulphonic acid which not only revealed an easy way of preparing dimethylnitramide but also the peculiar fact that one ot the ethyl groups of diethylamide became replaced by NO_2. FRANCHIMONT and KLOBBIE have afterwards demonstrated that the substituted amides of ethylsulphonic acid behave in quite an analogous manner. During the action of nitric acid on the amides of benzenesulphonic acid the benzene group remains intact. I now wanted to ascertain in how far the reaction would be affected by the introduction of a methyl group into the nucleus, just as I had done before in the case of substituted anilines, and I studied the reaction first of all with the amides of p.-toluenesulphonic acid.

These amides, like those of benzenesulphonic acid, are readily prepared by acting on the chloride with an aqueous solution of the amines.

Monomethyl- and ethylamides of p.-toluenesulphonic acid have already been described by REMSEN and PALMER, the first compound melting at 75° and the other at 58°. I found the respective melting points to be 76—77° and 64°.

I obtained dimethylamide as a beautifully crystallised substance melting at 76° while diethylamide melts at 59—60°.

If monomethylamide is dissolved in fuming nitric acid of sp. gr. 1.48 two products are formed which may be separated, although with difficulty, by recrystallisation from alcohol. The one melting at 116° proved to be a nitro p.-toluenesulphomethylnitramide, while the other melting at 60° was p.-toluenesulphomethylnitramide. When using a very concentrated acid of sp. gr. 1.52 we get exclusively nitramide nitrated in the nucleus; with an acid of a sp. gr. below 1.48 more of the readily melting substance is formed but the formation of the substance with the higher melting point is not entirely prevented. Nitric acid of sp. gr. 1.4 seems to be inert.

Monoethyl-p-toluenesulphamide behaves quite analogous towards concentrated nitric acid; with an acid of sp. gr. 1.52 a nitro-

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[1]) Rec. Trav. Chim. Pays-Bas 3, 7.

p-toluenesulphoethylnitramide (m. p. 76°) is exclusively obtained. Strong sulphuric acid decomposes it with evolution of N_2O and a nitro-p.-toluenesulphonic acid is formed, the barium salt of which crystallises with 2 mols. of water.

The dimethylamide of p.-toluenesulphonic acid yields with nitric acid of sp. gr. 1.48 dimethylamide melting at 57° while diethylamide yields with an acid of sp. gr. 1.5, amidst oxydation phenomena, a product which is liquid at first but solidifies after some time. When recrystallised from alcohol a product is obtained which melts at 76° and appears to be identical with the above mentioned nitramide nitrated in the nucleus, as obtained from the monoethyl compound.

Nitric acid of sp. gr. 1.52 has scarcely any oxydising action on the diethylamide, the product being again nitrated nitramide. After pouring the mixture into water and distilling a liquid product is obtained which sinks in water (ethyl nitrate).

The properties of the said barium salt rendered it probable that the nitrogroup had occupied the ortho-position in regard to the CH_3 group. To make sure of this, I have endeavoured to prepare the nitrated nitramines by a method which showed with certainty the position of the nitrogroup in the nucleus. I first of all prepared a nitrotoluenesulphochloride by acting on nitrotoluene with $HO\text{-}SO_2Cl$; the product treated with ammonia yielded an amide melting at 140°, but the amount obtained left much to be desired. Even the action of PCl_5 on the potassium salt of a sulphonic acid prepared by acting with fuming sulphuric acid on nitrotoluene (which yielded two isomers) did not produce the chloride in sufficient quantity to prepare the substituted amides. The action of ammonia produced two amides one of which melted at 141°.

By acting on p.-toluenesulphochloride with absolute nitric acid a nitroderivative was obtained which could be recrystallised from petroleum ether when the temperature of the surroundings was low, and melted at 45°. 1) With ammonia it yielded an amide melting at 143°.

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1) Fr. Réverdin and P. Crépieux BB. 34 (1901) 2993 state that they have obtained the same substance by nitrating *p*.-toluenesulphochloride with a mixture of nitric and sulphuric acids. It melted at 36° but could not be recrystallised. They found the melting point of its amide to be 144°, this confirming the result obtained by Anna Wolkow. In the literature at my disposal mention is only made of the wrong statements of Otto and Grüber who found 128°. I had already obtained my results when the paper of Réverdin and Crépieux appeared.

In this nitrated chloride the position of the nitrogroup is known, as it is also formed from o.-nitrotoluene.

I have prepared the corresponding amides by acting with monomethyl- and monoethylamine on this. 2. nitrotoluene. 4. sulphochloride. By treatment with nitric acid the methylamide gave the above mentioned nitroparatoluenesulphomethylnitramide melting at 116° and the ethylamide gave the analogous ethyl compound melting at 76°.

During the action of nitric acid on the alkylated amides of paratoluenesulphonic acid, the reactions in regard to the amido-group are analogous to those observed in the case of benzenesulphonic acid; a nitrogroup, however, readily enters the nucleus and occupies the ortho position in regard to the CH_3. Consequently 2. nitrotoluene. 4. alkylnitramide sulphonates are also formed.

An extensive communication on the numerous compounds obtained during this investigation will shortly appear elsewhere. I intend to state on a future occasion the results of the action of nitric acid on the two other isomers of toluenesulphonic acid.

Chemistry. — Dr. P. van Romburgh: "*On some further constituents of the essential oil of Kaempferia Galanga L.*" (Communicated by Prof. Franchimont).

(Communicated in the meeting of February 22, 1902.)

On May 26th 1900, I had the honour to present to the Academy a communication on the ethyl ester of p.-methoxycinnamic acid, the chief constituent of the essential oil of *Kaempferia Galanga L.* (Kentjoer, Mal.). Since then, I have continued the investigation of the liquid portion of that oil and tried in the first place to identify the nature of the second acid which I had found in it. Its purification, when contaminated with p.-methoxycinnamic acid, is not an easy matter and it must be recrystallised many times from water and alcohol before obtaining a product with a constant melting point (133°) which proved to be cinnamic acid. The acid may be more readily obtained pure by distilling the liquid essential oil in vacuum and saponifying the fraction boiling at 155°—165° (at 30 m.m.). The acid thus obtained at once showed the right melting point.

The alcohol formed during the saponification of that fraction proved to be ethyl alcohol so that the Kentjoer oil consists to a large extent (about a quarter of its volume) of ethyl cinnamate. VON MILLER[1]) has found the same ester in liquid storax.

The isolation of the pure ester from Kentjoer oil is very difficult as it is accompanied by a few other products one of which is very indifferent and has nearly the same boiling point. An effort to isolate the ethyl cinnamate (m.p. 12°) by cooling failed, for when placed in ice nearly the whole of the mass congealed which is no matter for astonishment, as it afterwards appeared that the indifferent admixture melts at 10°.

If the fraction boiling at 155°—165° (at 30 m.m.) is treated with 80 percent alcohol, this dissolves the ester while the other compound floats on the alcoholic solution. By repeating this treatment a few times, I succeeded in obtaining an already fairly pure ester which was still further purified by partial freezing. A quantitative saponification and an elementary analysis showed, however, that it was still contaminated with a compound richer in carbon and hydrogen which also somewhat decreased the specific gravity. The odour and the boiling point, however, were the same as those of a specially prepared specimen of ethyl cinnamate. The compound readily absorbed bromine but before the required amount has been taken up the liquid assumes a splendid violet colour[2]) which on further addition of bromine becomes green.

The portion of the essential oil insoluble in alcohol which also turns violet with bromine was freed from small quantities of ethyl cinnamate by heating with alcoholic potassium hydroxide which became dark coloured and, consequently, the treatment was repeated a few times. After distillation of the purified oil a colourless liquid was obtained which turned the plane of polarisation 7° to the left, when using a tube of 200 m.m. Placed in ice, it solidified to leaf-shaped crystals melting at about 8° which, however occluded some liquid product. By draining the crystals and remelting them a liquid was obtained which turned only — 5.5°. The further purification would not have given much trouble on a cold winter day in Europe but in the vapour-saturated air of the tropical climate it has taken much patience and labour to finally obtain by the freezing process a liquid which only feebly turned the plane of polarisation and hardly turned green with bromine. To obtain it still purer it was treated with a solution of bromine in chloroform

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[1]) Ann. d. Chemie 188, 203; VAN ITALLIE, Ned. Tijdschr. v. Pharm. 13, 103.

[2]) This reaction reminds of that for myrrh.

until coloured yellow; after removal of the chloroform it was washed with alcohol and aqueous potassium hydroxide and finally distilled in vacuum. The distillate showed a feebly green colour which disappeared by shaking the liquid with a little concentrated sulphuric acid which became dark in colour. In this way was finally obtained an inactive, colourless and odourless liquid which boils at 267.5° (738 mm), solidifies completely when placed in ice and melts at 10°. Its sp. gr. taken with the Westphal balance at 26° was 0.766. The elementary analysis and cryoscopic determination (in phenol) agree with that of a hydrocarbon of the composition $C_{15}H_{32}$.

Of hydrocarbons with this empirical formula only one is known, namely normal pentadecane prepared by KRAFFT [1]) the physical constants and properties of which agree in a remarkable manner with those of the described substance and I, therefore, do not hesitate to declare it identical with the hydrocarbon isolated from the essential oil of *Kaempferia Galanga*. It constitutes more than one-half of the liquid portion of the oil.

I hope to communicate within a reasonable time the results of the investigation of the compound which turns violet with bromine, of the terpene present in the essential oil and of a substance reminding of borneol, after I shall have again at my disposal larger quantities of the essential oil which is now being prepared.

I have to thank Dr. LONG who has assisted me in this investigation with much zeal and patience.

Mathematics: — "*On series of Polynomials*" 2nd part. By PROF. J. C. KLUYVER.

(Communicated in the meeting of March 29, 1902).

4. To the region of convergence of series VII for $tang^{-1}x$ the greatest possible extension can be given by making a approach unity. Then the region G_1 is transformed into a band limited by two parallel lines through the two singular points $+i$ and $-i$, the common direction of which is entirely determined by the argument of $a-1$. Now we will examine how, still by application of the auxiliar function

$$g(u)=\frac{(a-1)\,a}{a-u},$$

the region G_1 is formed, when an arbitrary function $F(x)$ characterised

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[1]) Berl. Ber. 15, 1700.

by any number of singular points A_j is to be developed into a series of polynomials.

As region of convergence G_1 is to be considered the portion of the x-plane situated within the circumferences of circles with radii $\left|\frac{a\,A_j}{a-1}\right|$ and centers $-\frac{Ai}{a-1}$ (fig. 4). The development into a series will be

$$F(x) = F(0) + \sum_{m=1}^{m=\infty} \frac{1}{a^m} \sum_{h=1}^{h=m} (m-1)_{h-1} \frac{F_{(h)}(0)}{h\,!}\, x^h (a-1)^h \, , \quad \text{(VIII)}$$

and, as in the case of series VII, the mark will always be greater than $1 : a$. Only in $x = 0$ this minimum value will appear. So it

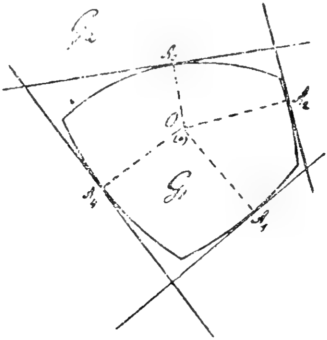

Fig. 4

is necessary to suppose $|a| > 1$. A division of G_1 into the two regions G_1' and G_1'' does not present itself.

The form of G_2 for a approaching to 1 is remarkable; the circles enclosing G_1 at the same time transform themselves into right lines, making in the singular points A_j with the radii vectores of these points in the same direction of rotation an angle $\varepsilon = arg\,\frac{a}{a-1} + \frac{\pi}{2}$. These right lines enclose a polygon, coinciding with MR. BOREL's "polygon of summatility" in the case of $a-1$ positive and real, where the angle ε is a right one. At the limit such a polygon, whether that one of BOREL or another the sides of which form an acute angle ε with the radii vectores of the corresponding singular points A_j, forms the region of convergence. But, if $lim.\ a = 1$, for any x all the terms of the series VIII become infinitesimal; then the series converges with the mark unity i. e. infinitely slow.

Now here the question presents itself, if it be possible to bring a given point x within the region of convergence by an appropriate choice of the parameter a. It is immediately seen that this question is to be answered in the negative, when x is a point of the prolongation of the radius vector of one of the singular points A_j, i. e. of one of the rays of MITTAG-LEFFLER's star. Furthermore a geometric construction can give a complete answer. It is necessary to construct arcs of circles passing through any pair A_j, A_k of singular points, the prolongation of which passes through the origin. By considering this figure it will be seen that a development into a series convergent for a given point x is possible or not, according to the radius vector of this point intersecting none, or one or more of these arcs A_j, A_k.

In such a manner it can be proved that the development (VIII) is to be made convergent in any point x, if all the singular points A_j lie on a right line through the origin. In particular this result can be applied to the series (VII) representing $tang^{-1}\, x$. The parameter a can be determined in such a manner that the double circle-segment of fig. 3 includes any point x.

5. We have but to make a slight modification in the development (VIII) in order to find certain other regions of convergence, which presented themselves in the investigations of BOREL.

Given $F(x)$ by means of a power series, which can be decomposed into n various power series in x^n. We put

$$F(x) = \sum_{k=0}^{k=n-1} x^k \varphi_k(x).$$

These new functions $\varphi_k(x)$ have not only the singular points A_j of $F(x)$ in common, now the new singular points

$$A_j\, e^{\frac{2\pi i}{n}},\quad A_j\, e^{\frac{4\pi i}{n}},\quad A_j\, e^{\frac{6\pi i}{n}},\quad \ldots\ldots$$

also make their appearance. Other singular points there are none. We again take the function

$$g(u) = \frac{(a-1)\,u}{a-u},$$

and consider $\varphi_k(x\, g(u)^{\frac{1}{n}})$ as before we did $F(x\, g(u))$. It is only seemingly that $\varphi_k(x\, g(u)^{\frac{1}{n}})$ is not holomorphic in $u = 0$; $\varphi_k(x)$ being in reality a function of x^n we can treat $\varphi_k(x\, g(u)^{\frac{1}{n}})$ as we formerly did $F(x\, g(u))$.

For $\varphi_k(x)$ we now find instead of the development (VIII)

$$\varphi_k(x) = \varphi_k(0) + \sum_{m=1}^{m=\infty} \frac{1}{a^m} \sum_{h=1}^{h=m} (m-1)_{h-1} \frac{\varphi_k^{(hn)}(0)}{hn\,!}\, x^{hn}\, (a-1)^h,$$

so that we can really represent $F(x)$ as the sum of n series of polynomials. We shall arrive at

$$F(x) = \sum_{k=0}^{k=n-1} x^k \frac{F^{(k)}(0)}{k!} +$$
$$+ \sum_{m=1}^{m=\infty} \frac{1}{a^m} \sum_{h=1}^{h=m} (m-1)_{h-1}\, x^{hn} (a-1)^h \sum_{k=0}^{k=n-1} x^k \frac{F^{(hn+k)}(0)}{(hn+k)!}, \quad \text{(IX)}$$

which result may also be written as follows:

$$F(x) = \sum_{h=0}^{h=n-1} x^h \frac{F^{(h)}(0)}{h!} +$$
$$+ \sum_{m=1}^{m=\infty} \frac{1}{a^m} \sum_{h=n}^{h=mn+n-1} (m-1)_{\left[\frac{h}{n}\right]-1} \frac{F^{(h)}(0)}{h!}\, x^h (a-1)^{\left[\frac{h}{n}\right]}.$$

The sign $\left[\frac{h}{n}\right]$ indicates here the integers of $h:n$.

The region of divergence G_2 of this series consists of the complex of the images of the circle unity in the u-plane made with the aid of the successive equations

$$x^n \frac{(a-1)\,u}{a-u} = A_j^n,$$

and for the convergence in a point x it is necessary that the condition

$$\left| x^n + \frac{A_j^n}{a-1} \right| < \left| \frac{a\,A_j^n}{a-1} \right|$$

is satisfied.

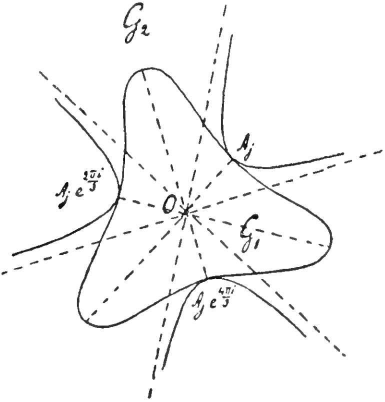

Fig. 5.

For a definite singular point A_j on account of this inequality x

is limited to a closed figure, containing the origin and the boundary of which passes through the points A_j, $A_j e^{\frac{2\pi i}{n}}$, $A_j e^{\frac{4\pi i}{n}}$, ... In the supposition $n = 3$ and that a has a fixed, real and positive value that closed figure is given in fig. 5. The part common to all regions constructed in this way is the region of convergence G_1 of the series (1X). What becomes of G_1, if a is allowed to approach 1 in some way or other? The closed figure in fig. 5 expands, the boundary opens and, a being real, passes finally into the curve

$$\varrho^n \cos n\,(\theta - \alpha_j) = r_j^n.$$

So we see, that the greatest extension which the region of convergence G_1 of the series (IX) with a real a can ever attain is the inside of the curved polygon of summability of Borel, whose sides are all arcs of some or other of the curves

$$\varrho^n \cos n\,(\theta - \alpha_j) = r_j^n.$$

By allowing complex values also for a, the region of convergence can undergo here again certain modifications. In fig. 5 we can neutralize by this the symmetry in respect to the radius vector of A_j, and speaking generally we can cause the region of convergence G_1 to expand more or less in a fixed direction. Such a modification will also present itself for a complex a if $| a |$ approaches to 1; besides the curved polygon of Borel there are others of which the sides are not symmetrically inclined with respect to the corresponding radii vectores of the singular points A_j.

Of the development (IX) we will again give a simple example, for which we again take $tang^{-1} x$ as the function to be developed.

We obtain, if n is taken equal to 2,

$$tang^{-1}\, x = 2\, \varphi_1(x),$$

and the function $\varphi_0(x)$ is missing.

The development (IX) here becomes

$$tang^{-1}\, x = x + \sum_{m=1}^{m=\infty} \frac{1}{a^m} \sum_{h=1}^{h=m} (m-1)_{-1} (-1)^h \frac{x^{2h+1}}{2h+1} (a-1)^h\,, \qquad \text{(X)}$$

or

$$tang^{-1}\, x = x + \frac{1}{a}\left[-\frac{x^3(a-1)}{3}\right] + \frac{1}{a^2}\left[-\frac{x^3(a-1)}{3} + \frac{x^5(a-1)^2}{5}\right] +$$

$$+ \frac{1}{a^3}\left[-\frac{x^3(a-1)}{3} + 2.\frac{x^5(a-1)^2}{5} - \frac{x^7(a-1)^3}{7}\right] +$$

$$+ \frac{1}{a^4}\left[-\frac{x^3(a-1)}{3} + 3.\frac{x^5(a-1)^2}{5} - 3.\frac{x^7(a-1)^3}{7} + \frac{x^9(a-1)^4}{9}\right] +$$

$$+ \frac{1}{a^5}\left[-\frac{x^3(a-1)}{3} + 4.\frac{x^5(a-1)^2}{5} - 6.\frac{x^7(a-1)^3}{7} + 4.\frac{x^9(a-1)^4}{9} - \frac{x^{11}(a-1)^5}{11}\right] + ..$$

The development has to be compared to development (VII). The region of convergence G_1 is in this case easy to construct. The function $tang^{-1}x$ has in the finite part of the plane the two singular points $\pm i$ which are situated already symmetrically with respect to the origin. Therefore only 'one condition of convergence has to be considered here, namely

$$\left| x^2 - \frac{1}{a-1} \right| < \left| \frac{a}{a-1} \right| .$$

For our case we shall consider only real values of a. The mark of the series $1 : R_x$ we can then find by solving R_x from the equation

$$\left| x^2 - \frac{1}{a-1} \right| = \frac{a}{a-1} \cdot \frac{1}{R_x} .$$

According to the mark of the series the points x are arranged in G_1 on confocal lemniscates with the foci $\pm \sqrt{\frac{1}{a-1}}$. The region G_1 is closed by the lemniscate

$$\left| x^2 - \frac{1}{a-1} \right| = \frac{a}{a-1} ,$$

passing through the singular points $\pm i$. By taking R_x equal to a we find the lemniscate

$$\left| x^2 - \frac{1}{a-1} \right| = \frac{1}{a-1} ,$$

having a node in the origin. This lemniscate divides G_1 into two parts G_1' and G_1'', and within the two loops the mark of the series is fixed and always equal to $1 : a$. Outside them (fig. 6) the mark rises from $1 : a$ to 1.

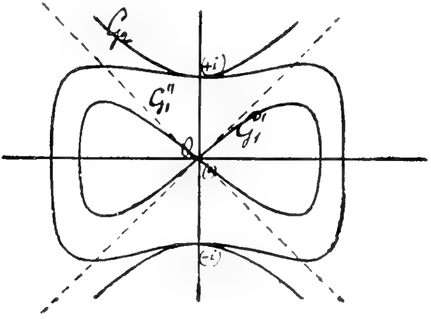

Fig. 6.

If the parameter a approaches 1 the lemniscate by which G_1 is bounded opens, to pass finally into the equilateral hyperbola.

$$\rho^2 \cos 2\,\theta = -1.$$

In the same moment all terms in (X) become infinitesimal and the development of the series loses its significance.

We will once more calculate $tang^{-1}\,x$ and inquire after the best value of a, always supposing that a must remain real. In the point $x = 1$ we have

$$\frac{1}{R_x} = \frac{a-2}{a},$$

and this mark not being able to fall below $1:a$, the best development will be had for $a = 3$. That series will be

$$tang^{-1}\,1 = 1 + \frac{1}{3}\left[-\frac{2}{3}\right] + \frac{1}{3^2}\left[-\frac{2}{3} + \frac{4}{5}\right] +$$
$$+ \frac{1}{3^3}\left[-\frac{2}{3} + 2.\frac{4}{5} - \frac{8}{7}\right] + \frac{1}{3^4}\left[-\frac{2}{3} + 3.\frac{4}{5} - 3.\frac{8}{7} + \frac{16}{9}\right] +$$
$$+ \frac{1}{3^5}\left[-\frac{2}{3} + 4.\frac{4}{5} - 6.\frac{8}{7} + 4.\frac{16}{9} - \frac{32}{11}\right] + \ldots\ldots$$

By writing it

$$tang^{-1}\,1 = 1 - 2\sum_{m=1}^{m=\infty} \frac{1}{3^m}\int_0^1 z^2\,(1-2\,z^2)^{m-1}\,dz,$$

we easily get convinced of its accuracy. The terms written out give together 0.7854, that is in 4 decimals the value of $tang^{-1}\,1$.

The six first terms of the power series for $tang^{-1}\,x$, i. e. the only ones which were used here, would produce together 0.7440.

The function $log\,(1+x^2)$ has the same singular points as $tang^{-1}\,x$. The region of convergence for the series of polynomials deduced from (IX) agrees with the one just considered. We have

$$log\,(1+x^2) = -\sum_{m=1}^{m=\infty} \frac{1}{m\,a^m}\left[\{1-x^2\,(a-1)\}^m - 1\right],$$

and for $x = 1$ and $a = 3$ we get

$$log\,2 = 2\sum_{h=0}^{h=\infty} \frac{1}{(2h+1)\,3^{2h+1}} = log\,\frac{1+\frac{1}{3}}{1-\frac{1}{3}},$$

that is a series which really bears the mark $\dfrac{1}{3}$.

6. Though after application of the simple auxiliary function

$$g\,(u) = \frac{(a-1)\,u}{a-u}$$

in many cases extensive regions of convergence can be obtained, as is evident from the preceding, after development of an arbitrary function $F(x)$ there will generally be a limit to the extension of G_1.

There exist, as Mittag-Lefler has shown us, series of polynomials, which have, whatever $F(x)$ may be, an almost unlimited region of convergence and for which the region of divergence shrivels up to a group of right lines, namely the rays of the star belonging to $F(x)$. Such series of polynomials are not to be constructed by making use of an auxiliary function $g(u)$ everywhere uniform. We must choose for $g(u)$ a function holomorphic within the circle $|u| = k$, which shows somewhere on the circumference of that circle an algebraical or logarithmical singularity fit for that purpose.

Each singular point A_j causes a certain part $G_{2,j}$ of the region of divergence G_2 to appear, namely the image made in the x-plane of the circle unity in the u-plane with the aid of the equation

$$x\,g(u) = A_j.$$

We have now to act in such a way that this image can be reduced to an exceedingly narrow loop drawn from the point $x = \infty$ round the ray of the star of Mittag-Leffler passing through A_j.

By this demand we are more limited in our choice of $g(u)$, but yet undoubtedly auxiliary functions of a most divergent nature are still possible. A very simple auxiliary function $g(u)$ which allows of the observation of all important phenomena, even though it does not furnish perhaps in all circumstances the most converging series of polynomials, is the function

$$g(u) = \frac{1-(1-\mu u)^\beta}{1-(1-\mu)^\beta} = \frac{1-(1-\mu u)^\beta}{N},$$

μ and β being here positive real fractions. The function is uniform within the circle $|u| = \frac{1}{\mu} > 1$ and the conditions $g(0) = 0$, $g(1) = 1$ are satisfied, if the power $(1-\mu u)^\beta$ for $|u| < \frac{1}{\mu}$ is defined in such a way that its argument is naught with real u. The quantity $\frac{1}{\mu}$ takes here the place of the quantity k.

The selected function $g(u)$ we substitute into (V) and we obtain

$$\frac{1}{1-x} = 1 + \sum_{m=1}^{m=\infty} \sum_{h=1}^{h=m} B_{m,h}\, x^h\,,$$

$$B_{m,h} = \frac{1}{m!} D^m_{u=0}\, g\,(u)^h = \frac{(-\mu)^m}{N^h} \cdot \frac{1}{m!} D^m_{t=1}\,(1-t^\beta)^h\,.$$

It is perhaps simpler to write here

$$\frac{1}{1-x} = 1 + \sum_{m=1}^{m=\infty} (-\mu)^m \sum_{h=1}^{h=m} C_{m,h} \left(\frac{x}{N}\right)^h \quad . \quad . \quad . \quad . \quad . \quad \text{(XI)}$$

Then

$$C_{m,1} = -(\beta)_m ,$$

$$C_{m,h} = -(h)_1(\beta)_m + (h)_2(2\beta)_m - (h)_3(3\beta)_m + \ldots + (-1)^h (h\beta)_m ;$$

whilst we can calculate successively these coefficients, with the aid of the general identity

$$C_{m,h} = \sum_{k=j}^{k=m-h+j} C_{k.j}\, C_{m-k,\,h-j} .$$

The coefficients, appearing in the first seven terms of the series of polynomials, follow here.

$m=1$. $C_{1,1} = -\beta$,

$m=2$. $C_{2,1} = -\frac{1}{2}\beta(\beta-1)$, $C_{2,2} = \beta^2$,

$m=3$. $C_{3,1} = -\frac{1}{6}\beta(\beta-1)(\beta-2)$, $C_{3,2} = \beta^2(\beta-1)$, $C_{3,3} = -\beta^3$,

$m=4$. $C_{4,1} = -\frac{1}{24}\beta(\beta-1)(\beta-2)(\beta-3)$, $C_{4,2} = \frac{1}{12}\beta^2(\beta-1)(7\beta-11)$,

$C_{4,3} = -\frac{3}{2}\beta^3(\beta-1)$, $C_{4,4} = \beta^4$,

$m=5$. $C_{5,1} = -\frac{1}{120}\beta(\beta-1)(\beta-2)(\beta-3)(\beta-4)$,

$C_{5,2} = \frac{1}{12}\beta^2(\beta-1)(\beta-2)(3\beta-5)$,

$C_{5,3} = -\frac{1}{4}\beta^3(\beta-1)(5\beta-7)$, $C_{5,4} = 2\beta^2(\beta-1)$,

$C_{5,5} = -\beta^5$,

$m=6$. $C_{6,1} = -\frac{1}{720}\beta(\beta-1)(\beta-2)(\beta-3)(\beta-4)(\beta-5)$,

$C_{6,2} = \frac{1}{360}\beta^2(\beta-1)(\beta-2)(31\beta^2-132\beta+137)$,

$C_{6,3} = -\frac{1}{8}\beta^3(\beta-1)(3\beta-5)(2\beta-3)$,

$C_{6,4} = \frac{1}{6}\beta^4(\beta-1)(13\beta-17)$, $C_{6,5} = -\frac{5}{2}\beta^5(\beta-1)$,

$C_{6,6} = \beta^6$,

$m=7$. $C_{7,1} = -\frac{1}{5040}\beta(\beta-1)(\beta-2)(\beta-3), (\beta-4)(\beta-5)(\beta-6)$,

$C_{7,2} = \frac{1}{120}\beta^2(\beta-1)(\beta-2)^2(\beta-3)(3\beta-7)$,

$C_{7,3} = -\frac{1}{120}\beta^3(\beta-1)(\beta-2)(43\beta^2-141\beta+116)$,

$C_{7,4} = \frac{1}{6}\beta^4(\beta-1)(10\beta^2-29\beta+21)$,

$C_{7,5} = -\frac{5}{6}\beta^5(\beta-1)(4\beta-5)$, $C_{7,6} = 3\beta^6(\beta-1)$,

$C_{7,7} = -\beta^7$.

If we consider how the region of divergence G_2 is formed, that is in what way the u-circle is represented in the x-plane by the equation

$$x \frac{1-(1-\mu u)^\beta}{N} = 1 ,$$

we then come to the conclusion that for each point u within the circle or on the circumference of the circle only *one* point x of G_2 is found and reversely. So G_2 is a region simply connected, enclosing $x=\infty$ and divided from the region of convergence G_1 by a closed curve not cutting itself, passing through $x=1$ and enclosing the origin. Evidently the curve lies symmetrically with respect to the axis of the real quantities. We now extend the circle unity in the x-plane and consider how on account of that the image of this circle extends in the x-plane,

As long as $|u|$ is $< \frac{1}{\mu}$ still a single point x remains belonging to one point u and reversely. So the rim in the u-plane between the circles with the radii 1 and $\frac{1}{\mu}$ is represented point for point on an annular region G_1'' surrounded entirely on the outer side by G_2. On the inner side G_1'' is closed in by the representation of the circle $|u| = \frac{1}{\mu}$, a curve still passing round the origin and with an angular point in the point $x=k$. The angle is directed towards $x=0$ and has the value of $\beta\pi$ (fig. 7). So of the x-plane remains after deduction of the regions G_2 and G_1'' another part, the region G_1' containing the origin and extending in the direction of the real axis from

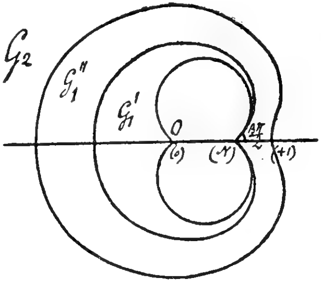

Fig. 7.

$$x = -\frac{N}{2^\beta - 1} \quad \text{to} \quad x = N.$$

For the points x inside this region G_1' the equation

$$x\,g(u) = 1$$

can be satisfied only by values of u with modulus greater than $\frac{1}{\mu}$ and as $g(u)$ becomes multiform for $|u| < \frac{1}{\mu}$ the series of polynomials (XI) will have the mark μ in each point of G_1'.

By the following consideration we can form an idea of the extension of this region G_1' for different values of β and μ. On the line $\overline{oN}$ as chord we can describe in the x-plane on either side two circle segments, each of which containing the angle $\frac{\beta\pi}{2}$. The common radius of the segments is $N : 2 \sin \frac{\beta\pi}{2}$ and each point x in one of the segments has the property

$$\pm\, arg.\, \frac{x-N}{x} = \pm\, arg.\left(1 - \frac{N}{x}\right) > \frac{\beta\pi}{2}\,.$$

As the argument of $(1-\mu u)^\beta$ for every value of u within the circle with radius $\frac{1}{\mu}$ can vary but between $-\frac{\beta\pi}{2}$ and $+\frac{\beta\pi}{2}$, the relation

$$1 - \frac{N}{x} = (1-\mu u)^\beta$$

or

$$x\,g(u) = 1$$

is not satisfied by a single point x within the segments for $|u| > \frac{1}{\mu}$.

From this ensues that these two segments will always form part of the region G_1' and this allows us to easily oversee the extension of the region G_1'.

If we wish to deduce from (XI) the development of a given function $F(x)$ with the singular points $A_j = r_j\, e^{i\alpha j}$ we must then successively apply the transformations $(x, A_j\, x)$ to the region of convergence $G_1 = G_1' + G_1''$ just found (i. e. we must let those regions rotate round $x = 0$ through an angle α_j and we must let the radii vectores increase in a ratio of $r_j : 1$).

That which is common to the transformations $G_{1,j}''$ of G_1' or to the transformations $G_{1,j}''$ of G_1'' forms for $F(x)$ the region G_1' or G_1''. As $g(u)$ has no points of infinity within the circle with a radius $1 : \mu$, we need not take heed of a singular point at infinity if this makes its appearance in $F(x)$.

We will now first of all indicate how by a modification of the parameters μ and β the region of convergence G_1 of the series (XI) can be extended indefinitely. For that purpose we put $\mu = 1$ and use the auxiliary function

$$g(u) = 1 - (1-u)^\beta .$$

The region of the rim G_1'' has disappeared. The region G_1' covers the region G_1 entirely and bounds immediately on G_2. The angle $\beta\pi$ in the limitation is found in $x = 1$ (fig. 8). Within G_1' the mark of the series

$$\frac{1}{1-x} = 1 + \sum_{m=1}^{m=\infty} (-1)^m \sum_{h=1}^{h=m} C_{m,h}\, x^h$$

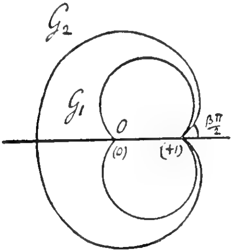

Fig. 8.

is everywhere equal to unity and the region of convergence G'_1 contains two circle segments, now described on the line $\overline{01}$ as chord on either side of that line, each of them containing an angle $\frac{\beta\pi}{2}$.

It is clear what happens if we allow β to approach to naught. The region G'_1 expands and if we but notice the segments becoming larger and larger, we shall see that finally every point x can be brought inside G'_1 except those points situated on the line of $+1$ to $+\infty$.

Passing on to the development of an arbitrary function $F(x)$ we come to the conclusion: By taking β small enough the development

$$F(x) = F(0) + \sum_{m=1}^{m=\infty} (-1)^m \sum_{h=1}^{h=m} C_{m,h} \frac{F^{(h)}(0)}{h!} x^h , \quad . \; . \; . \quad \text{(XII)}$$

or written in full

$$F(x) = F(0) + \sum_{m=1}^{m=\infty} (-1)^m \sum_{h=1}^{h=m} x^h \frac{F^{(h)}(0)}{h!} \frac{D^m_{t=1} (1-t^\beta)^h}{m!} ,$$

can always be made to converge with the mark unity in every point of the k-plane, if only this point is not situated on one of the prolongations of the radii vectores drawn from the origin to the singular points.

If we wish to include in the region of convergence all the points of MITTAG-LEFFLER's star, we must allow β to approach to naught; in this case however we see immediately out of (XII) that all terms of this series become infinitesimal, so that in each point x the convergence with the mark unity becomes infinitely slow.

7. To this result we will add another and no less important one. The supposition $\mu = 1$ could furnish series only, which were convergent with the mark unity; we shall now show that series can be formed which converge with a lower mark in a given point $x = \varphi\, e\, \theta$, if only G is finite and the argument θ gives with each of the arguments $A_j = r_j\, e^{i\alpha j}$ a difference with a finite sinus. For this purpose we make β approach to naught and obtain in this way the auxiliary function

$$g(u) = \frac{log\,(1-\mu\, u)}{log\,(1-u)}\,.$$

If it regarded the development of $1:1-x$, then the form of the regions G_2, G''_1 and G'_1 would still be about the same as was given in fig. 7. In the meanwhile together with β the original quantity $N = 1-(1-\mu)^\beta$ has disappeared. The limitation of G'_1 has no longer an angular point but now a cusp in $x = 0$. The state is that of fig. 9.

Inside G'_1 are now two entire circles with the radius $\frac{1}{\pi}\, log\, \frac{1}{1-\mu}$, and the series (X1) has passed into

$$\frac{1}{1-x} = 1 + \sum_{m=1}^{m=\infty} (-\mu)^m \sum_{h=1}^{h=m} \left(\frac{x}{log\,(1-\mu)}\right)^h \frac{D^m_{t=1}\, log^h t}{m\,!} \qquad \text{(XIII)}$$

Putting

$$\frac{D^m_{t=1}\, log^h t}{m\,!} = E_{m,h}\,,$$

then

$$E_{m,1} \text{ becomes } \frac{(-1)^{m-1}}{m}\,,$$

and we have again

$$E_{m,h} = \sum_{k=j}^{k=m-h+j} E_{k,j}\ E_{m-k,\,h-j}\ .$$

Here follow the first terms of this development. For $1 : log\,(1-\mu)$ we have put c.

$$\frac{1}{1-x} = 1 - \mu\left(x\,c\right) - \mu^2\left(\frac{1}{2}\,xc - x^2c^2\right) - \mu^3\left(\frac{1}{3}\,xc - x^2\,c^2 + x^3\,c^3\right) -$$

$$-\mu^4\left(\frac{1}{4}\,xc - \frac{11}{12}\,x^2\,c^2 + \frac{3}{2}\,x^3\,c^3 - x^4c^4\right) -$$

$$-\mu^5\left(\frac{1}{5}\,xc - \frac{10}{12}\,x^2c^2 + \frac{7}{4}\,x^3\,c^3 - 2\,x^4\,c^4 + x^5\,c^5\right)$$

$$-\mu^6\left(\frac{1}{6}\,xc - \frac{137}{180}\,x^2c^2 + \frac{45}{24}\,x^3c^3 - \frac{17}{6}\,x^4c^4 + \frac{5}{2}\,x^5\,c^5 - x^6\,c^6\right) -$$

$$-\mu^7\left(\frac{1}{7}\,xc - \frac{7}{10}\,x^2c^2 + \frac{29}{15}\,x^3c^3 -\right.$$

$$\left. - \frac{7}{2}\,x^4c^4 + \frac{25}{6}\,x^5c^5 - 3\,x^6c^6 + x^7c^7\right) \quad \ldots\ldots\ldots\ldots \quad \text{(XIV)}$$

If we will now bring a given point $x = \varrho\, e^{i\theta}$ inside the region G'_1 of fig. 9, then this is always possible, if only θ is not naught.

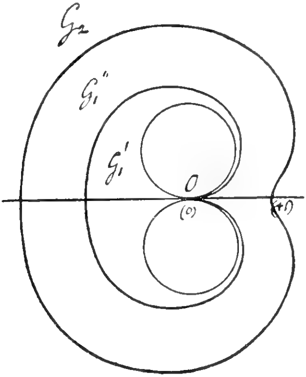

Fig. 9.

For by making μ increase, the circles inside G'_1 always become larger and it will even be possible to make x lie on the circumference of one of these circles or inside it. For this it is only necessary, that

$$\varrho \leqq \frac{2}{\pi}\,log\,\frac{1}{1-\mu} \times \left|\ sin\,\theta\ \right|,$$

or that we assume

$$\mu > 1 - e^{-\frac{\pi\rho}{|\sin\theta|}}.$$

For a finite ρ and $\sin\theta$ not approaching to naught, we can always have a series (XIII), converging in the point x with a mark, giving perhaps a very small but yet certainly a finite difference with unity. The demonstration would not hold good for $\theta=\pi$, that is for negative real x. Meanwhile in order that in a certain point x the series may converge with the marks μ, it is no necessity that one of the circles of fig. 9 includes the point; it suffices if x lies in G'_1 and as now the boundary of G'_1 cuts the real axis on the left side of $x=0$ at a distance

$$\frac{\log\frac{1}{1-\mu}}{\log 2},$$

it is clear that also for negative but finite x there are series (XIII) which converge with a mark decidedly smaller than unity.

Suppose we have $x=-1$, we then find from

$$\frac{\log\frac{1}{1-\mu}}{\log 2} = 1,$$

that the series (XIV) converges still just with the mark $\frac{1}{2}$ for $\mu=\frac{1}{2}$, $c=-\frac{1}{\log 2}$ in the point $x=-1$. By verification we find

$$\frac{1}{2}=1-0.7187+0.3407-0.1752+0.0861-0.0439+0.0216-0\ 0107+.$$

The quotient of the successive terms is already immediately fairly equal to $\frac{1}{2}$ and the sum of the calculated terms is 0.5009.

If we now pass to the development of an arbitrary function $F(x)$ on the ground of the development XIII we have but a slight difference to bring about in the reasoning. For a point $x=\rho e^{i\theta}$ the successive values

$$\mu_j = 1 - e^{\frac{\pi\rho}{r_j\,|\sin(\theta-\alpha_j)|}}$$

have to be calculated when $\theta-x_j$ differs from π. Should θ become equal to $\alpha_j+\pi$ the value μ_j must be replaced by

$$\mu_j = 1-2^{-\frac{\rho}{r_j}}.$$

The greatest of all these values μ_j is assigned to the parameter μ of the development (XIII) and in the point x the series for $F(x)$ will have the mark μ.

In nowise will the series obtained in this way have the lowest possible mark; the reasoning served only to show that for finite ϱ and for $\theta \neq \alpha_j$ there are series, converging in x with a mark decidedly lower than unity.

This argument does not hold good for those points x, situated exactly between the origin and one of the singular points, let us say A_k.

For that case we must return to the development of the series (XI). We have now still at our disposal the two parameters μ and β, and we begin to arrange these in such a manner, that the given point x lies inside the region $G'_{1,k}$, which by the transformation

$$(x,\, A_k\, x)$$

is deduced from the region G'_1 of fig. 7. For this it is necessary to have

$$N = 1-(1-\mu) \geqq \frac{\varrho}{r_k} \, .$$

Further we allow β to decrease in such a way that the point x arrives inside the segments into which the segments of fig. 7 by the remaining transformations

$$(x,\, A_j\, x)$$

pass. In other words we put for all singular points, except the one point A_k, the inequality

$$\varrho < \frac{N\, r_j}{\sin \frac{\beta\pi}{2}} \mid \sin (\theta - \alpha_j) \mid .$$

Very decidedly this inequality is satisfied if β satifies the condition

$$\sin \frac{\beta\pi}{2} \leqq \frac{r_j}{r_k} \mid \sin (\theta - \alpha_j) \mid .$$

A finite value of x differing from naught is found from all these inequalities; after that we take

$$\mu = 1 - \left(1 - \frac{\varrho}{r_k}\right)^{\frac{1}{\beta}} ,$$

and we may be quite assured that the series deduced for $F(x)$ out of (XI) shall converge in the points x with the mark μ.

So finally it is possible to construct a really convergent series of polynomials in each point x of the star. Only on the rays of the star the convergence becomes infinity slow and so the development becomes worthless.

Chemistry. — „*On the melting of binary solid mixtures by cooling*".

By Prof. H. W. Bakhuis Roozeboom.

(Communicated in the meeting of March 29, 1902).

The phenomena which may occur when mixtures pass from one state of aggregation into another are as a rule of a much more complex nature than those which take place with a single substance. For this reason a number of cases have been discovered in which the succession of the states of aggregation is wholly or partially reversed.

The earliest known instance is met with in the phenomenon of retrograde condensation, where a binary liquid is first formed from vapour by increase of pressure at a constant temperature and then passes again into vapour.

A similar reversion occurs with ternary liquid-mixtures when these on evaporation first deposit a solid substance and then redissolve the same. [1])

We may also consider in this light my recent communications on saline solutions with two boiling points. [2]) Here, the possibility of a solution commencing to boil by cooling was shown to exist.

I will now mention a case where a binary mixture first solidifies on cooling but then again partly liquefies by further cooling.

The possibility of such a phenomenon first occurred to me during my theoretical study of the changes of binary solid mixtures. [3])

Among the large number of types which I then distinguished were found a few instances (l.c. fig. 15 and 20) where mixed crystals, on cooling, undergo a transformation and break up into another solid phase and liquid. For want of some detailed example of transformation of mixed crystals, I did not further investigate these special cases although it seemed to me that one of these might possibly be found during the study of the so-called liquid crystals. A case where mixed crystals occurred in the liquid crystalline condition had been investigated shortly before by Schenck who had found such a mixing between azoxyanisol and benzophenone. My surmise has now been completely confirmed and Mr. de Kock who is engaged with a dissertation on this subject has had the good fortune to meet with a very striking example in mixtures of azoxyanisol and hydroquinone.

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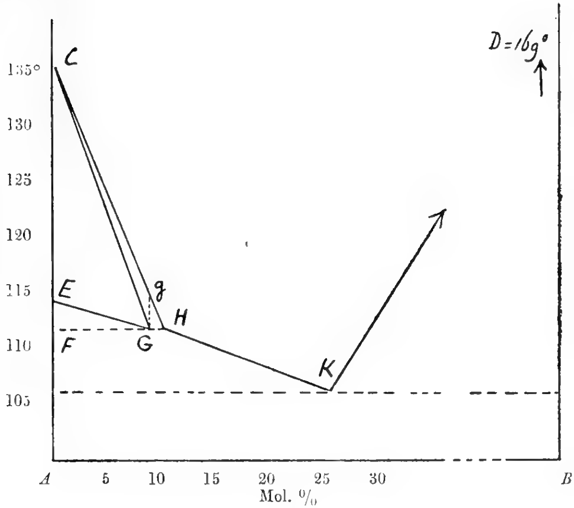
[1]) Schreinemakers, Zeit. phys. Chem. 10.467 and Mohr ibid 27.214. A good many allied examples are found in binary and ternary systems with two or three liquid-layers.

[2]) Rep. Meeting 28 Dec. 1901.

[3]) Zeits. phys. Chem. 30. 413 (1899).



In the above figure *AB* represents the proportions of azoxyanisol (*A*) and hydroquinone (*B*); the vertical axis represents the temperature. Substance *B* has a melting point at *D* : 169°. Substance *A*, however, which is solid below *E* (condition A_2) passes above $E = 114°$ into the condition of liquid crystals A_1 which may be recognised by the eye as an apparently turbid liquid which passes at $C = 135°$ into an isotropic clear yellow liquid.

If the liquid crystals are really looked upon as *crystals* for which there are sound reasons, then *C* is the melting point and *E* a transition point from A_1 into A_2.

In the liquid condition *A* and *B* are completely miscible above *C* and *D*. Below these temperatures, the mixing-limits for the different concentrations are *CH*, *HK* and *KD*.

On the line *DK* commences the separation of *B*, on *HK* of solid A_2 ($K = 106°$ and 25 mol. %), but on the line *CH* liquid mixed crystals are deposited. Consequently, to *CH* a second line *CG* must be added which shows the composition of the adjoining liquid mixed crystals. This line is the key to the important phenomenon which I have in view. The formation of the liquid mixed crystals takes place regularly on cooling any liquid-mixture below *CH* and is completed below *CG*. The clear liquid has then completely disappeared and has made room for the liquid crystals.

The region in which these mixtures exist does not however extend below the line *EG*. For in *E* the transformation takes place of pure azoxyanisol from A_1 to A_2. In the mixed crystals this temperature is lowered in an increasing degree with the concentration of hydroquinone; this causes the fall of the line *EG*. From the liquid mixed crystals, azoxyanisol is separated in the ordinary solid condition A_2 by cooling below *EG*; the remaining mixed crystals successively move towards the point *G*. This, however, is the lowest point of the triangle *CEG* within which mixed crystals are possible.

At this temperature, 111.6°, they already exist in presence of solid A_2 and can also exist in presence of the liquid *H*. As below the line *FGH*, solid A_2 can only exist with liquids *HK*, the mixed crystal *G* must, on further cooling, be transformed into solid A_2 and liquid *H*; this indeed happens. The phenomenon is the most striking if one starts from a concentration corresponding with *G*. The formation of liquid mixed crystals then takes place from *g* to *G* and on further cooling the transformation: liquid crystal $G \rightarrow$ solid F + liquid *H* immediately takes place. The liquid again appears, not however *G* but *H*, therefore of modified concentration and also in a more limited degree because some solid A_2 is deposited at the same time.

The relation of liquid to solid is as *FG*: *GH* and as *G* and *H* respectively belong to 8.75 and 10 mol. % of hydroquinone the calculation shows that 87 % by weight of the total mass returns to the state of liquid *H*. The phenomenon is the more striking because the formation of the liquid crystals from *g*—*G* took place within a small temperature-interval (3.5°). The great ease with which mixed crystals undergo change when in the liquid crystalline condition renders it probable that the phenomenon will often present itself with mixtures in which substances of this kind occur.

A condition is, of course, that the line *EG* must intersect the line *CG*, or expressed in the language of the theory of the dilute solutions, that the depression of the melting point *C* by the admixture of the second substance, is sufficiently larger than the depression of the transition point *E*. Since, as far as is known, the heat of transformation at *C* is many times smaller than that at *E* (about 40 times in the case of azoxyanisol), therefore the molecular depression

$$\frac{0.02\,T^2}{q}$$

in which *q* represents that heat, will be much greater for point *C* than for point *E*, even when taking into consideration that in the calculation the concentration *GH* must be taken in the first and *FH* in the second case. The necessary condition for the appearance of the phenomenon will, therefore, easily be satisfied.

The above phenomenon is one of retrograde solidification if we consider the liquid crystalline condition as a solid.

Fortunately another instance has been found recently with mixed crystals which are solid in the ordinary meaning of the word.

With the aid of my theory, HEYCOCK and NEVILLE when interpreting the results of their studies on the solidification phenomena of alloys of copper and tin, have come to the conclusion that a mixture containing 27 atom $^0/_0$ of tin which solidifies between 720°—633° to homogenous mixed crystals, breaks up when the lower temperature is passed into the solid compound Cu_3 Sn (25 at. $^0/_0$ of Sn) and a liquid containing 42 at $^0/_0$ of Sn. Calculation shows that a maximum of 6.5 $^0/_0$ by weight may again become liquid below 633°. Whether this quantity of liquid metal would be visible between the crystals is perhaps doubtful. As the interpretation in question was only brought to light after the experiments were finished, it could only be proved by a microscopical examination of the completely solidified mass, that at 633° a partial liquefaction of the previously solid mixed crystals had taken place.

The theory of phases has here again explained the exact connection and there is no doubt that the liquefaction of binary mixtures by cooling is not an extremely rare phenomenon.

Geology. — "STARING *and the coal-question of Southern Limburg.*"

By Prof. J. L. C. SCHROEDER VAN DER KOLK.

(Communicated in the meeting of March 29, 1902).

Lately the coal-fields of Southern Limburg have repeatedly been discussed, also in the report of the Royal mining commission, instituted April 17th 1899.

The name of STARING is however invariably omitted, and the result of his researches simply laid aside. Therefore I wish to point out, in this paper, the extent of all that was known to STARING, already full forty years ago. I wish to do so, not only from a feeling of justice towards STARING, for much that now seems new to us, was already known to him, but also because to my opinion, his knowledge may be of use to us, even now.

The principal object of this paper is, clearly to bring out STARING's standard of knowledge, at the time, so I shall almost exlusively make use of his statements. I borrow those statements from: Plate

26 Geological Atlas, and from the second volume of STARING's well known: "Bodem van Nederland."

To give a clear survey of the whole, I have drawn out a map, of what he states, and made a profile of it.

On this map and likewise in mentioned plate, of STARING's Atlas, we find in the south-east of the Netherlands, a very old formation "devone" indicated by the letter D: This devone runs in an almost straight line parallel with carbon, bordering on it, as is marked in our map. The direction of the border-line is almost north-east, a very important one in the Limburg coal-fields.

A look at the map, will show at once, the importance of that direction. For everywhere in this carbon, we find narrow seams of devone, and all those seams run in mentioned direction. The border-line between devone and the younger formations, cretaceous and the tertiary ones, is very irregular. On the stretch of land the cretacious mapped out, three tongues stand out clearly, that of Kohlscheid, that of Sippenaken and that of Aubel. Those three tongues stand almost straight on mentioned devone seams.

Of the two features, the seams and the tongues, STARING furnishes us with the explanation.

We shall begin with that of the seams, running north-east, as we saw.

Those seams are the result of folds, through which the devone forms anticlinal folds, the productive carbon being formed in the synclinal folds. So roughly speaking we may say that, when we are somewhere in the productive carbon, we shall continue being there, as long as we move north-east or south-west; we shall however soon get into other layers, when we move north-west or south-east. The other feature, that of the tongues, is not quite so easily explained.

In vol. II of his "Bodem van Nederland" STARING however mentions a number of faults which run through our coal-fields. Those faults just as the longitudinal axis of the tongues, stand as a rule, at right angles, on mentioned seams. The direction they have in common, already points towards the possibility of a common origin. Those faults I have marked in our map, according to the indications, given by STARING and numbered from I—VIII. Of the five first, the spot is more or less accurately known; with number VI, there exists some uncertainty, as to the spot, whereas the numbers VII and VIII are not directly mentioned. On the whole may be said that those faults are the better known, the more east they are.

Let us now see what STARING knew of those faults and how far they influence the position of the coal-beds. To give a distinct survey of STARING's statements, I have united them in a sort of "profile". One must however bear in mind, that this "profile", does not give what really is, only what STARING thought it to be as we know from his "Bodem van Nederland". So e. g. in the "profile", the faults are drawn vertically, which is certainly contrary to fact (whereas the coal-fields are drawn horizontally; the older anthracite in thin, the younger, bituminous-coal in thick-lines).

So we have to deal with the following faults:

I. In the neighbourhood of Eschweiler, we find a big fault, the so called Sandgewand, which runs parallel with the Münstergewand.

II. Very remarkable is the presence of bituminous coal, so of that of younger strata, east of Kohlscheid and Kerkrade, which are known for their anthracite, a product of older strata; the former are, at the very least, lying 40 ell deeper than the latter.

At the time of their formation, the position of the beds was of course horizontal, the younger covering the older.

The cause of the present strange position is, the fact that the beds, which lie east, have sunk, and those which lie west, have risen, in a line, running parallel, with the one we just mentioned i. e. that of Kerkrade to Corneli-Münster. This fault, called Münstergewand, commences in the coal-valley of Eschweiler and Stolberg and runs on as Feldbiss, (a few hundred ells) north of the domain-mines of Kerkrade. At the top of the slope, to the right, of the valley of the Worm near Pley, south of Kerkrade, the outcrop of the fault lies bare.

According to what we now know of the direction of the Feldbiss, the coal-bed which has been found in the Berenbosch, near Kerkrade, lies east.

This Feldbiss evidently forms the east border of the Kohlscheid tongue.

III. Another fault must be somewhere near the valley of the Molenbeek, near Kerkrade. The coal found, by boring west of that valley, is truly anthracite, just as that of Kerkrade, but it seems to belong to quite different strata and not to those of Kerkrade, although one might presume them to be a continuation of those. Not impossible however it is that they are the outrunners of those of Kerkrade.

IV. Further there is a considerable fault more or less in the direction of the road between Heerlen and Aix-la-Chapelle. At the one side of it at Spekholzerheide, the level of the coal-beds, covered by tertiary layers, lies (one hundred and eighty ells) higher than it does a few minutes farther, west of the road, near Frosschenhäuschen (Vrusschehueske), where cretaceous layers cover it.

This latter fault (IV), probably forms the west border of the Kohlscheid tongue. Its continuation towards the north, appears from the sharp, straight line, on STARING's map separating the cretaceous from the tertiary formations. The northern continuation of this tongue, constitutes the most valuable coal-beds of Dutch Limburg; therefore it is advisable, not to omit trying to get from STARING as much information as we can, on the subject of the other tongues.

V. More westward still, at Bochholtz, we find the coal-strata some hundred ells, or thereabout, higher than at Frosschenhäuschen.

VI. Also the valley of the Zelzerbeek or the heights between Bocholtz and Bommerig, must hide a great alteration in the level of the coal-strata, the lowest layers of the coal formations lying more than sixty ells higher than the younger layers of the coal-formation at Bocholtz.

The exact spot of this latter fault, can consequently only vaguely be indicated; it is even very well possible that in the stretch of land, between Bocholtz and Bommerig, there is more than one fault, I confine myself however to what STARING says, and in order not to make our "profile", too complicated, I have marked only one fault. If we wish to limit ourselves to but one, the most probable spot seems to me the outrunner of the east border of the tongue of Sippenaken, and for more than one reason. The line n.l. runs close to Bommerig of which place STARING tells us the following:

"Of importance also may become to us, the metal ore, which we know to occur beyond our frontiers in the coal-strata and between them, likewise, between those of the Devonian groups, those traces of ore, already having been found near Bommerig. In 1856 the Dutch mining-company found galena in the neighbourhood of Bommerig at a depth of sixty-five ells".

That ore, in Belgium, principally occurring in the faults, the presence of ore, near Bommerig, makes the existence of a fault there probable. When moreover, we lengthen out that line, farther north-west, it crosses the environs of Valkenburg of which STARING says:

A considerable fault, presumingly of fifty to sixty ells has been found in the southern part of the quarries of Valkenburg.

VII. There is every chance that also the west-border of the tongue, at Sippenaken, will prove to be the result of a fault.

VIII. The east-border of the tongue of Aubel has probably also something to do with a fault. This opinion is at any rate supported by STARING, who states the following:


crank

At Mesch, at a depth of thirty three ells and at Bannet, near Mheer, at a depth of one hundred and nine ells, sand-stone and coal-slate are found.

This considerable difference in level would be explained by fault VIII.

From the information we have gained from STARING, two conclusions may be drawn: In the first place, it will be advisable not to limit further researches in Limburg, to the tongue of Kohlscheid and its immediate surroundings, but to extend them to the other two tongues, especially to the one of Sippenaken. It is true there does not seem to be much chance of finding coal at the latitude on which Bommerig is situated but the very absence of productive carbon increases the chance of finding ore, whereas the seamlike construction and also the comparison with other spots add to the chance of there being more north, perhaps on a latitude with Valkenburg, a fresh synclinal fold, containing productive carbon.

The tongue of Aubel, on the other hand is probably of less importance to us, since, on that latitude we are on Belgian territory.

The second conclusion relates to a border-line, mentioned in the Report of the Royal-mining-Commission, instituted on April 17th 1899.

In this Report mention is made of "a probable border-line of the productive coal-beds", a line which from Bocholtz bends to Valkenburg and from there runs north. As will be seen, that line touches the tongue of Sippenaken, and it is very well possible and according to my opinion even probable, that west of that line coal-beds will be found.

The Hague, March 29th 1902.

Chemistry. — „*Bromination and nitration in the aromatic series*". By Dr. J. J. BLANKSMA. (Communicated by Prof. LOBRY DE BRUYN).

(Communicated in the meeting of March 29, 1902).

In a previous article [1]) a few particulars were communicated on the bromination and nitration of some derivatives of phenol and aniline. Attention was then called to the fact that the nitration of aniline derivatives containing the group NH_2 presents difficulties

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[1]) Rep. Meeting 25 Jan. 1902.

but that this readily takes place with derivatives of methylaniline; for instance *m*.-nitro-methylaniline was readily converted into van Romburgh's tetranitro-phenylmethylnitramine. Several observations have already been made as to the bromination and nitration of methylamine derivatives and it seemed of importance to further extend the same.

Vaubel 1) has shown that methylaniline readily yields a tribromo-derivative, while van Romburgh 2) has obtained trinitro-phenylmethylnitramine by nitration of methylaniline. Van Romburgh 3) has also proved that if the *o*.- and *p*.-positions are occupied by a CH_3-group only two NO_2-groups are introduced. The same has been demonstrated by Grimaux en Lefèvre 4) for the group OCH_3.

The object of my research was to ascertain the influence exercised on the further substitution by the occupation of ortho- and para-positions and also of the meta-positions of methyl and ethyl derivatives. The first substance examined was *p*.-nitro-methylaniline obtained by heating *p*.-chloronitrobenzene with methylamine in alcoholic solution for 4 hours at 160°. By treatment with nitric acid of sp. gr. 1.52 it was directly converted into trinitro-phenylmethylnitramine. A treatment with 1 mol. of bromine in glacial acetic acid gave 2-bromo-4-nitro-methylaniline, m. p. 118°; with 2 mols. of bromine 2.6-dibromo-4-nitro-methylamine, m. p. 113° is obtained.

By the action of nitric acid the first compound gave 2-bromo-4.6-dinitro-phenylmethylnitramine. m. p. 125°, the last one gave 2.6-dibromo-4-nitro-phenylmethylnitramine, m. p. 84°.

o.-Nitro-methylaniline, obtained like the para-compound from *o*.-chloronitrobenzene is converted by the action of nitric acid into trinitro-phenylmethylnitramine. Bromine converts this substance first into 4-bromo-2-nitro-methylaniline, m. p. 101° and then into 4-6-dibromo-2-nitro-methylaniline, m. p. 100°. Treatment with HNO_3 gives 4-bromo-2.6-nitro-phenylmethylnitramine, m. p. 109° and 2-4-dibromo-6-nitro-phenylmethylnitramine, m. p. 90°.

On treating oxyethyl-dinitrobenzonitrile (1.2.4.6) in alcoholic solution with methylamine or ethylamine the OC_2H_5-group is replaced by $NHCH_3$ or NHC_2H_5. By the action of nitric acid of sp. gr. 1.52 on these substances, the hydrogen combined with the nitrogen is replaced by NO_2 and consequently the corresponding nitramines are formed. In the same way it was shown that when a Cl-atom

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1) Journ. pract. Chem. (2) **48**. 315.

2) Rec. **2**. 108. 3) Rec. **3**. 392. 4) C. R. **112**. 727.

occupies the para-position, the as yet free ortho-positions get occupied by NO_2 and the hydrogen in combination with the N is replaced by NO_2. If, however, a COOH-group occupies an ortho- or para-position in regard to $NHCH_3$ this will then be replaced by Br or NO_2. If a chlorine or bromine atom occupies the meta-position in regard to the $NHCH_3$ (obtained by treating chloro- or bromodinitrobenzene with methylamine) these groups will have no influence on the further introduction of bromine atoms or nitrogroups.

We therefore, see:

1⁰. That on bromination of methyl-(ethyl) aniline derivatives, the free ortho- and para-positions are always occupied by bromine, while the hydrogen atom linked at the nitrogen is not attacked. The same happens during the nitration but there the hydrogen-atom of the $NHCH_3$-group gets replaced by NO_2.

2⁰. that a carboxyl group in an *o.*- or *p.*-position is replaced by Br or NO_2 with evolution of CO_2.

3⁰. that the occupation of the meta-position by other groups (NO_2, Cl, Br, CH_3) has no influence on the introduction of bromine atoms and nitrogroups in ortho- and para-positions.

In the previous communication it has already been observed that during the bromination and nitration of phenol and aniline derivatives, the NO_2-groups and halogen atoms probably first enter into the side chain to shift later on towards the nucleus. [1]) One might call this substitution of the hydrogen atoms of the nucleus an indirect substitution in contrast to the direct substitution which takes place with benzene and substituted benzenes containing the groups COOH, Cl, Br, NO_2, SO_3H, etc.

We then see that:

1⁰. indirect substitution as is well known, takes place very readily. Phenol and aniline, for instance, are readily nitrated, or brominated by dilute nitric acid or bromine water; the direct substitution is much more difficult (nitration of benzene, nitrobenzene, chlorobenzene etc.)

2⁰. during the indirect substitution, the substituents always occupy the ortho- and para-positions in regard to the side chain; during the direct substitution meta-substitution products are also formed, sometimes by preference.

3⁰. during the indirect substitution it is possible to replace some

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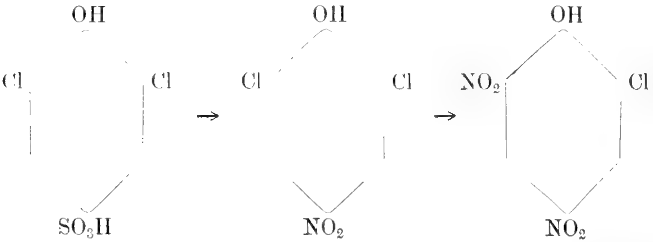
[1]) Bunge, Ann. Supp. 7. 117. (1869). ARMSTRONG, Soc, 77. 1047. CHATTAWAY and ORTON, Soc. 79, 274.

groups by others. COOH,[1] CH_3[2] CH_2OH[3] and SO_3H[4] when in ortho- or para-position in regard to OH may be replaced by bromine with formation of tribromophenol. A chlorine atom may, by indirect substitution enter in the place of bromine (HANTSCH,[5] CHATTAWAY and ORTON[6]) while chlorine may again in turn be replaced by NO_2 (ARMSTRONG[7])



A group COOH may further be replaced by C_6H_5NN and also by NO.

We, therefore, see that during the indirect substitution the following groups may successively replace each other: COOH, SO_3H, Br, Cl, NO_2; the nitro-derivatives are, therefore, the most stable so that the phenol derivatives finally produce picric acid and the methylaniline derivatives trinitro-phenylmethylnitramine. In the direct nitration a similar substitution is not noticed; benzoic acid and benzenesulphonic acid are nitrated and halogenated without elimination of the COOH or SO_3H-group.

In the same manner, tribromobenzene (1.3.5) yields on nitration successively mono- di- and trinitro-tribromobenzene without any substitution of bromine atoms taking place; a behaviour totally different from that of tribromophenol.

The group CH_3 behaves in many cases analogous to OH, NH_2 and NHX. It is not impossible that we meet also here with indirect substitution, witness the researches of AUWERS, ZINCKE and

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1) BENEDIKT, Ann. **199**. 128.

2) BAUMANN and BRIEGER, Ber. **12**. 804.

3) AUWERS and BÜTTNER, Ann. **302**. 133.

4) *p.*-Potassium phenolsulphonate readily assimilates three mols. of bromine, the liquid remains clear and slowly deposits tribromophenol (more rapidly on heating).

5) Ber. **30**. 2334, **33**. 505.

6) Journ. Chem. Soc. **1901**. 823.

7) Zeitschrift 1871, 516.

BAMBERGER where the investigated substances generally contained one or more CH_3 groups next to OH.

It should also be pointed out that in tribromo-mesitylene the Br atoms may be replaced by NO_2, while dimethylmesitylene yields dinitro-mesitylenenitramine with NO_2 groups in the meta-positions. (KLOBBIE [1]).

The particulars of this research will be published in the „Recueil".

Physiology. — "*On the effect as a time-function.*" By Dr. J. K. A. WERTHEIM SALOMONSON. (Communicated by Prof. C. WINKLER).

(Communicated in the meeting of March 29, 1902).

In the course of three former communications we have considered the relation between the magnitude of effect and stimuli of different intensity. In its strictest sense the law there formulated by us, is valid only in cases of momentaneous stimuli, where the effect appears after the stimulus has ceased to act.

The following essay will consider the manner in which an effect proceeds — the way it changes perpetually from one moment to another, whether under the influence of momentaneous stimuli or of lasting constant stimuli.

Considering the effect as being due to the transformation of chemical substance, we may admit with GULDBERG and WAAGE that the rapidity of this transformation will be proportional to the available quantity of transformable substance. If the amount of the latter may be represented by W, then:

$$-\frac{dW}{dt} = \alpha W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now let us examine what will be the amount of this quantity W in general. Firstly we may admit that at the moment t there is in store a certain quantity of transformable substance P.

But we are dealing with living protoplasma.

After activity of the protoplasma we know that products of fatigue will have been formed. These will gradually disappear and be replaced by fresh substance. In general it is a most plausible supposition that the quantity of new transformable substance sup-

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[1] Rec. 6. 31.

plied at each moment, is proportional to the quantity of fatigue-substance in store. As the latter has been originated by the transformation of transformable "energy-substance", the available quantity of "fatigue-substance" may be supposed to be proportional to the whole of the hitherto operated effect, the final consequence being that the supply of fresh "energy-substance" at each moment is proportional to the hitherto transformed quantity. This may be represented by $\beta \int P dt$.

Finally there exists still a third cause for the continual change undergone by P, i. e. the influence of the stimulus. This change may be considered to be a time-function, and represented by $f(t)$.

Thence we abtain:

$$W = P + \beta \int P\, dt + f(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

By transposing this value of W in the equation (1), we obtain:

$$-\left(\frac{dP}{dt} + \beta\, P + f'(t)\right) = \alpha\, P + \alpha\, \beta \int P\, dt + \alpha\, f(t) \quad . \quad . \qquad (3)$$

which produces after differentiating:

$$-\left(\frac{d^2P}{dt^2} + \beta\, \frac{dP}{dt} + f''(t)\right) = \alpha\, \frac{dP}{dt} + \alpha\, \beta\, P + \alpha\, f'(t)$$

or after a slight transposition:

$$\frac{d^2P}{dt^2} + (\alpha + \beta)\frac{dP}{dt} + \alpha\, \beta\, P = \alpha\, f'(t) + f''(t). \quad . \quad . \quad . \qquad (4)$$

The solution of this equation is:

$$P = A\varepsilon^{-\alpha t} + B\varepsilon^{-\beta t} + \frac{\alpha\, f'(t) + f''(t)}{\alpha\, \beta + (\alpha + \beta)\, D + D^2} \quad . \quad . \quad . \qquad (5)$$

A and B representing constants and D and D^2 the operators $\left(\frac{d}{dt}\right)$ and $\left(\frac{d^2}{dt^2}\right)$.

The last term of the second member can be solved only when the numerator is known, viz. when we know what form of stimulation has been applied.

I. In the very first place let us consider what will happen in the case of the stimulus having acted only for a very brief moment, and then ceasing. This implies the existence of a certain quantity of stimulation-substance no longer altered by a fresh supply; our $f(t)$ thus being constant, whilst $f'(t)$ and $f''(t) = 0$. In this case the last term may be omitted, the solution of the equation becoming consequently:

$$P = A\varepsilon^{-\alpha t} + B\varepsilon^{-\beta t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

From the initial condition, that t being $= 0$, P too we will be $= 0$, it follows that $A = - B$, the formula therefore will stand finally:

$$P = A\,(\varepsilon^{-\alpha t} - \varepsilon^{-\beta t}) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

II. In the case of our making use of a constant stimulus, the quantity of stimulation-substance supplied in each particle of time will be the same, consequently:

$$f(t) = C_{11} t \quad \text{and} \quad f'(t) = C_{11} \quad \text{and} \quad f''(t) = 0.$$

In this case the third term of (5) becomes: $= \dfrac{C_{11}}{\beta} = C$, the formula therefore standing as follows:

$$P = A\varepsilon^{-\alpha t} + B\varepsilon^{-\beta t} + C$$

in which, keeping in view the individual significance of both exponential functions, we must again take B as a negative value, thus obtaining finally:

$$P = A\varepsilon^{-\alpha t} - B\varepsilon^{-\beta t} + C \quad . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

If once it has been proved that the expression of an effect caused by a stimulus may be represented by a differential equation of the second degree we cannot allow any restriction about the coefficients $\alpha + \beta$ and $\alpha\,\beta$, as we may easily imagine circumstances, occasioning such alterations in the values of these coefficients, as would make impossible the solution of (4) in the form of (5). This will be the case if we take for our coefficients the arbitrary values $2q$ and k^2. Anticipating therefore on this right we alluded to before, we will now introduce these new constants. The equation in its most general form then will stand thus:

$$\frac{d^2P}{dt^2} + 2q\,\frac{dP}{dt} + k^2\,P = \varphi\,(t) \quad . \; . \; . \; . \; . \; . \; . \quad (9)$$

In this formula we have also put $\varphi\,(t)$ directly for the stimulus, and will distinguish again between the two cases of $\varphi\,(t) = 0$ and $\varphi\,(t) =$ constant.

III. In the case of $\varphi\,(t) = 0$, (9) will become:

$$\frac{d^2P}{dt^2} + 2q\,\frac{dP}{dt} + k^2\,P = 0$$

If here $k^2 < q^2$, the solution (6) will follow from this differential equation.

If on the contrary $k^2 > q^2$, we may write the solution in this form:

$$P = \varepsilon^{-qt}\{A\,cos\,at + B\,sin\,at\}. \quad . \; . \; . \; . \; . \; . \quad (11)$$

As it follows from the initial condition that t being $= 0$, $P = 0$, we obtain:

$$P = B\varepsilon^{-qt}\,sin\,at. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

in which:

$$a = \sqrt{k^2 - q^2}.$$

IV. Lastly, if $\varphi(t) = C_1$, the differential equation becomes:

$$\frac{d^2P}{dt^2} + 2q\frac{dP}{dt} + k^2 P = C_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12a)$$

of which the solution is:

$$P = \varepsilon^{-qt}\{A \sin \alpha t + B \cos \alpha t\} + C. \quad . \quad . \quad . \quad . \qquad (13)$$

In the foregoing we have deduced a few relations, to the number of four, all of which may be maintained on theoretical grounds. We next intend trying to investigate whether there exist any facts, likely to lend some support to our views. To this purpose we will examine the facts concerning each formula separately. As a matter of course however, we ought to state beforehand, that in these brief and „provisory" communications, opening a new and most extensive area of research, it will be possible only to give superficial indications, complete proofs being wholly out of the question.

Ad. I. The course of an isotonic muscle-contraction shows a striking resemblance to the curve represented by (7). Of course this does not imply that practically the course of an isotonic muscle-contraction may be expressed by our formula (7) as it stands here. Still there do exist a few points indicating the possibility of deducing from our formula an appropriate expression. We know that a contraction does not begin simultaneously at all parts of the muscle; it begins at one point, whence it extends itself over the muscle like a wave. During this process some parts of the muscle are slightly stretched before contracting. Consequently a certain space of time shall necessarily elapse before the shortening commences, or at any rate before it may be observed at the end of the muscle. Starting now from the supposition that on a definite point of a muscle an effect appears in the form indicated by our formula, subsequently extending itself gradually over the whole of the muscle, we have got a few indications enabling us to proceed afterwards to a theoretical treatment resulting in an expression, rendering with sufficient accuracy the course of the contraction, as observed practically: an expression therefore, in which the fact is taken into consideration of a certain space of time being required for putting into motion the mass of muscle and the writing-apparatus fastened to it.

Literature offers much information on this subject which we might quote in corroboration of our theory. I have only to recall to mind the views on the nature of muscle-contraction, put forward by FICK, HERING, SCHENK, GAD and others.

At any rate the question possesses the merit of actuality, and I intend trying to elucidate it more fully afterwards.

All that has been said here about the muscle-contraction, may be said to apply also, with a slight alteration, to the course of the one-phased action-current of muscle and nerve, which possesses likewise a latent period.

Ad. II. With formula (8) the case is far less doubtful. In this formula is expressed the effect of a biological constant stimulus.

The effect is formed by three components. Of these three the first term expresses the ever-diminishing magnitude of the effect, if no assimilation takes place. The initial magnitude therefore is A. As the final magnitude is represented by C, it follows $B = - C$, and the formula is altered to:

$$P = A\,\varepsilon^{-\alpha t} + B(1-\varepsilon^{-\beta t}) \quad . \; . \; . \; . \; . \; . \; . \quad (14)$$

In this formula the first term may be considered as *dissimilation* or *exhaustion-term*, whilst the second represents the *assimilation-* or *restoration-term*.

Therefore we may call α the dissimilation and β the assimilation-modulus.

Both these moduli are constants only in a mathematical sense, i. e. for each special case of stimulation. They may be different however for each succeeding case of stimulation.

The whole of the effect is moreover still dependent on the magnitude of stimulus, in the manner explained in our former communications.

We may therefore add still another factor to formula (14), obtaining then finally, as a complete representation of the relation between magnitude of stimulus, time and effect:

$$P = \{1-\varepsilon^{-p(R-c)}\}\{A\,\varepsilon^{-\alpha t} + B(1-\varepsilon^{-\beta t})\} \quad . \; . \; . \quad (15)$$

For very small particles of time the second factor in (15) is transformed into A, and we again obtain our original law about the relation of stimulus and effect, which indeed, as is proved now, prevails only where instantaneous stimuli are concerned. Otherwise after a brief space of time the influence of assimilation and dissimilation will make itself felt.

As we remarked before, the moduli α and β are magnitudes, differing for each particular case. The more or less rapid appearance of fatigue will be dependent, amongst other causes, on the effect or perhaps directly on the magnitude of stimulus. Very often however there will occur circumstances, exerting their influence on the magnitude of effect, independent of the magnitude of stimulus, e. g. the temperature of or the weight lifted by a muscle.

We ought therefore still to consider what will happen in the case of α and β being altered.

To that purpose we will examine the curve represented by the second factor of (15).

In the first place we find the curve to possess a maximum for:

$$t = \frac{1}{\alpha - \beta} . \ln \frac{\alpha A}{\beta B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

This maximum will be reached the sooner in proportion as α is greater and β smaller. Consequently in the case of a great dissimilation the maximum will soon appear. It even may appear, t being $= 0$; in other words: the curve in this case descends from the very first, whilst in the case of α being small, it ascends at first, to descend subsequently. Furthermore the curve possesses a point of inflexion at:

$$t = \frac{1}{\alpha - \beta} . \ln \frac{\alpha^2 A}{\beta^2 B} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

The curve first turns its concave side to the axis, to continue convex to the abscissa.

Finally the curve possesses an asymptote B, where the assimilation is therefore in perfect equilibrium with the dissimilation.

For the particular case $\alpha = \beta$ and $A = B$, the curve is transformed into a straight line.

Constant stimulation of muscle. When a muscle is stimulated directly or indirectly by a faradic current of constant rhythmus and constant intensity, we may speak of a constant stimulus, and the views we put forward just now, may find their application. Indeed the tetanuscurve shows in its course all particulars that are to be deduced from (15). This fact may be observed still more clearly, if the rhythmus is protracted until it becomes so slow that the single contractions appear without uniting themselves to tetanus. We then perceive the ascending period of the curve (the staircase), reaching rapidly a maximum, and subsequently descending.

Circumstances, dependent on the magnitude of the constants, may sometimes occasion alterations in the course of the curve. Sometimes the curve may be seen to descend immediately, at first slowly, then more rapidly, and at last again slowly. At the end of the curve, when the stimulus ceases, we always, if the muscles be not too much exhausted, observe a line answering very nearly to the formula:

$$V = m\,\varepsilon^{-nt} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

This follows directly from (7).

If the restorative power of the muscle may not be called infinite, e.g. in the case of the dissected muscle of a frog, B is no longer a constant, but an ever diminishing magnitude. For in this case the quantity of protoplasma, destined to replace the fatigue-products, is no longer maintained by the supply of blood, keeping the freshly supplied substance in equilibrium with that consumed, but the quantity is limited and continually diminishes. Here therefore we shall have to alter the assimilation-term in such a manner that for the constant B is substituted a term steadily decreasing in magnitude, e.g. $B\,\varepsilon^{-gt}$.

Probably in so doing we shall obtain an expression offering a more accurate represention of what practically occurs.

The formula then will become:

$$P = (1 - \varepsilon^{-pR})\,\{A\,\varepsilon_{-\alpha t} + B\varepsilon^{-gt}\,(1 - \varepsilon_{-\beta t})\} \quad . \quad . \quad . \quad (19)$$

In the case too of a muscle receiving a regular fresh supply of new substance, it may occur that this supply is conveyed less promptly than is necessary for even a minimum of regularly performed labour. In such cases there will come a moment, when no more work can be done. Yet fresh substance is continually supplied, and after some lapse of time the moment may come that the stimulus is enabled anew to transform a sufficient quantity of protoplasma to occasion one or more contractions, which however will soon cease again. In this way we obtain finally *rhythmic effects*, to be classified partly under the range of groups III and IV.

In formula (15) we have given an expression for the action of a *biological* constant stimulus. We may likewise make use of this formula in the case of the stimulus being only *physically* constant. Such is the case e.g. with the constant galvanic current. As we know, this stimulus possesses only an initial effect, soon descending again. This denotes α and A being very great, whilst B is small.

In this case we may without any inconvenience simplify the formula by neglecting the term containing B. If furthermore we restrict ourselves to a small intensity of current, and may consequently take R and P to be proportional to the intensity of current, this latter being represented by i, the formula will finally stand as follows:

$$P = A\,i\,\varepsilon^{-\alpha t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

This is the law established and proved by HOORWEG. As however for currents of great intensity we are not allowed to neglect the B-term, (15) offers a complete and more correct image of the action of a constant current upon nerve or muscle, explaining at the same time the possibility of tetanus in the case of stimulation by stronger constant currents; it is not necessary therefore to regard this tetanus as caused by polarisation.

Constant stimulation of sense-organs. For stimulation of muscle, the views expounded just now, may be proved right in a very simple manner. For that purpose we only need to measure a number of tetanus-curves, taken under different circumstances, and to test the results to our formula. Indeed I intend this to be my first task, immediately after publishing this very succinct, theoretical paper.

But the matter becomes far less simple whenever sense-organs are concerned, because in that case it is impossible to give an objective representation of the intensity of a stimulation-effect. We cannot here do more than search for facts, to support the once established theory. For the moment I will confine myself to the following facts.

Ear and eye. According to our theory in the case of very feeble stimuli being used, an increment of effect will be observed when the stimulus begins to act. Practically this is an acknowledged fact. It has been discovered repeatedly, and is known well enough especially to those who have occupied themselves much with X-rays. If a thoroughly reposed eye begins to regard a radioscopic image on a fluorescent screen, it will see indeed the whole of the screen illuminated, but only after a few seconds it will begin to distinguish the image more clearly. (This fact must be distinguished from the improvement of sensation acquired by the *non*-reposed eye in regarding a fluorescent screen, this improvement being achieved far less rapidly).

URBANTSCHITSCH found also that feeble tones of a tuning fork were not immediately heard in full intensity, but that it needed therefor a certain space of time, dependent on the intensity of the sound. This space of time may amount to 1 or 2 seconds. In the Röntgenroom I observed repeatedly that a space of time from 1—3 seconds was needed to obtain with the well-reposed eye a good sensation of a feeble fluorescent image. These facts were regarded hitherto as "Bahnungserscheinungen." Probably it is partly through this summation too that must be explained the long latent period in the case of feeble stimuli of temperature acting upon the skin. And in all probability in the same way the summation of feeble stimuli of touch causes the sensation of titillation and may lead to a tickling-reflex. The feebler the stimuli the more *time* will be the prominent feature.

Finally we can deduce from our formula an expression for *restoration of fatigue.* Researches concerning this point have been made for the sense of sight by **Aubert** and lately by means of radioscopic experiments by **Béclère**. One glance at the curves of **Aubert** will show us, that in all probability these curves may be expressed with very sufficient accuracy by the term of restoration in our formula (4).

Formula (15) offered us a complete image of the course of the magnitude of effect. For very feeble stimuli this formula can be greatly simplified. We know that for minimum-stimuli magnitude of stimulus and effect are nearly proportional, consequently we may, without committing a mathematical error, apply the time-factor to the stimulus instead of applying it to the effect. Moreover with such small stimuli the dissimilation may safely be neglected. We may then write for the effect:

$$P = pR\,[A + B\,(1 - \varepsilon^{-\beta t})] - pC. \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

Where subminimum-stimuli are concerned, if $t = 0$, $A\,p\,R < p\,C$. Through the influence of $B\,(1 - \varepsilon^{-\beta t})$, after a sufficient time having elapsed, the formula will be transformed to:

$$P = pR\,(A + B) - p\,C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

in which it is possible that $p\,(A + B)\,R > p\,C$. In other words: circumstances may occur, rendering it possible that subminimum-stimuli still cause an effect, if only the duration of their time of action be long enough. The truth of this fact is known so well, that it would be useless to give instances here.

Now we are enabled also to explain the initial deviation from our law, mentioned in the three former communications. **Waller** demonstrated in his essay that, concerning the sense of sight, the graphical representation of the relation between stimulus and effect resembled somewhat the shape of an S, stretched longitudinally.

This lower curve, placed convex to the x-axis, is perfectly explainable by the fact, that we were dealing with time-stimuli, causing the subminimum stimuli still to operate an effect.

I think I may refrain here from an alteration in a formula expressing this curve. This much only I wish to indicate: that this might be possible e.g. by substituting for the constant c in our stimulation-law an exponential function of R; the latter then would stand thus:

$$P = A \left\{ 1 - \varepsilon^{-BR - \alpha\left(1 - \varepsilon^{-\beta R}\right)} \right\} \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

I give this formula here because it may be maintained on purely physical grounds, because at the same time by means of it we may get indeed some conception about the nature of the constant c, about the threshold-value; and this conception shows us the threshold-value behaving entirely like a counter E. M. F of polarisation, which as we know, according to BARTOLI may be represented in the manner indicated here.

In a former essay we have demonstrated the addition of stimuli to be a corrective, by means of which feeble stimuli were observed to act with greater intensity than in the case of their being only of short duration. Meanwhile we have seen that with stronger stimuli the effect diminishes very rapidly. In this way the possibility occurs, that the relative differential threshold-value will remain fairly constant over a comparatively large part of the curve, and that for the sharp bend, mentioned in my last communication, will be substituted a slower and more regular course of the curve. For the moment I cannot enter into details concerning this point.

Ad. III and IV. As a necessary consequence of the manner of deduction, chosen for these equations, we ought to postulate the existence of *rhythmical effects*, under the influence of instantaneous and constant stimuli. The occurrence of similar effects in physiology is beyond any doubt. Among the most convincing proofs on this point we may rank the beautiful curves published by S. GARTEN, about the course of the potential oscillations of the action current in a muscle, in which suddenly a transverse section is made, or in the case of stimulation by a strong battery-current. In both cases he demonstrated the existence of damped vibrations, possessing an extraordinarily regular course, and answering wholly to formula (2) or formula (3).

But in the animal organism there may occur other series of rhythmic, nearly sinusoïdal oscillations, being however not damped. I believe these too might be mathematically treated. One might be induced to think that some of these oscillations will appear as soon as the coefficient q in the equation (9) is so small that practically the first differential quotient is omitted from the equation. For in this case undamped, sinusoïdal effects would be originated. It is not impossible indeed, that perhaps for certain effects the damping may be so slight, that sinusoïdal effects of constant amplitude are achieved. I have in view, amongst other facts, the rhythmic innervation passing from the secondary to the primary neuron under the influence of impulses of will, or of cerebral or spinal excitation; cases of pathological and of physiological tremor. Meantime I believe these phenomena generally to present some particulars, denoting that we ought to think here of the well known „forced" vibrations under the influence of a constant impulse.

Finally we have to take into consideration the fact that sometimes only very intensive stimuli cause rhythmic effects, often accompanied by a rather important deformation of the one-termed sinusform. This might tend to indicate that the simple proportionality of the deviation and the stimulus no longer existed, and that we ought therefore to add higher terms. The phenomena, possibly belonging to this series, are besides the different tremores the rhythmical contractions of the apex of the heart in absence of nervous cells, rhythmical muscle-contractions by galvanic stimulation and many others.

All these questions will be made the object of later researches.

Though in the foregoing only the crude outlines of a provisory sketch have been given, I yet believe myself justified in publishing them. For by means of my theory a great number of facts are combined to one group, and it tends to show how these many different functions, apparently so widely divergent, may still be considered from one point of view; how notwithstanding their apparent complete incongruity, there still exists one common ground, one and the same basis on which they all together are founded.

Chemistry. — „*On the decomposition of mercury nitrates by heating.*" By DR. J. MIJERS of Hoorn (Communicated by Prof. BAKHUIS ROOZEBOOM).

(Communicated in the meeting of March 29, 1902).

When crystallised mercurous nitrate, either the normal compound or one of the basic salts, is heated in a retort which is connected with suitable apparatus to collect or condense the gaseous products, the apparatus nearest to the retort will be found to contain coloured HNO_3, the condensing tube will contain an indigo-blue liquid consisting of N_2O_3 and NO_2, while on continuing the decomposition NO is given off. If the anhydrous salts are taken, nearly pure NO_2 is condensed, but as it is difficult to expel the last traces of water a little coloured HNO_3 is also formed.

This result is in contradiction with that of **Gerhardt**[1]) who states that on melting $Hg_2(NO_3)_2$, nitric oxide is merely given off; he does not say whether he means the anhydrous or the crystallised salt.

I think the following result an important one. During the said decomposition a yellow compound sublimes in the upper side and neck of the retort; in the lowest and consequently the hottest part it has fused and turned to a yellow crust, but in the neck it has sublimed in yellow needles. Its composition appeared to agree with that of a compound obtained by the moist process by **Rose**, [2]) **Gerhardt** [3]) and **Brooks** [4]) and considered by them to be basic mercuro-mercuric nitrate, $2\, Hg_2O, N_2O_5\; 4\, HgO\, N_2O_5$, while it might also be considered as mercuro-mercuric ortho-nitrate, $(Hg_2)\, Hg_2(NO_4)_2$, derived from an acid H_3NO_4.

The analysis of that substance gave the following results:

	Hg_2Cl_2	HgS	$\%\ Hg^0$	$\%\ Hg^i$	theory for both	theory $\%NO_3$	$\%NO_4$
1) 0.4038 gr.	0.2054	0.1980	43.21	42.25	83.68 %	12.98	16.32
2) 0.6381 „	0.3139	0.3007	42.2	43.--			
3) 0.8505 „	gave 23.5 cc. of N, temp. 20° press. 763 mm = 0.02685 gr. of N. or					14.08	17.94
4) 1.1643 »	» 28.5 „ „ » » 19° „ 757 » = 0.03226 » „ » »					12.31	15.68
5)	To 0.6895 gram containing 0.0085 gr. of free mercury were added 128.7 cc. of $\frac{N}{10}$ HNO_3, then 157.7 cc. of $\frac{N}{10}$ KHO; after diluting to 250 cc. the excess of KHO was determined in half of the filtrate. Used 7.25 cc. of $\frac{N}{10}$ HNO_3.					13.11	16.7

The excess of mercury found in my determinations is due to the presence of a little free metallic mercury as shown in experiment 5. When dissolving the compound in HNO_3 the amount of both the mercurous and mercuric nitrates is consequently increased.

Of this yellow salt a not inconsiderable quantity is formed. 96.4 grams of anhydrous mercurous nitrate yielded me 18.3 grams or nearly 19 % of yellow crusts and needles, also 12.2 grams or 12.8 % of reddish-brown crusts formed from the at first sublimed and then greatly decomposed orthonitrate. It might, therefore, be suspected that the preparation of red HgO according to the Dutch Pharmacopœa should be attended with a serious loss of mercury. In one experiment which I made in this direction, the loss amounted to $12^1/_2$ %.

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[1]) Ann. Chem. u. Pharm. LXXII. p. 74.

[2]) Poggend. Ann. LIV. p. 121.

[3]) l. c.

[4]) Poggend. Ann, LXVI. p. 63.

The said yellow salt has been investigated by BROOKS as to its behaviour at higher temperatures. He found that it had lost hardly any weight at 220°, and not until 260° brown vapours were evolved. This behaviour, I think, explains the formation of that salt on heating $Hg_2(NO_3)_2$. At the then reigning temperature it is formed because it can then exist and at the same time volatilise. For the greater portion of the salt the opportunity for sublimation does not exist and the increased temperature decomposes it completely.

It is still more interesting to know that this salt is also formed by heating mercuric nitrate. Absolutely mercurous-free $Hg(NO_3)_2$ heated in the above-mentioned apparatus yields the same yellow sublimate. It has, therefore, been formed in a strongly oxidising surrounding containing oxygen and NO_2. If pure mercuric nitrate is heated in a porcelain dish until the decomposition is fairly started, the still undecomposed white crust which covers the red powder of nitrate-containing HgO (free from Hg_2O) will be found covered on the inner surface with a yellow substance which on analysis is found to contain mercurous-mercuric nitrate. The white crust, however, does not contain any mercurous compound.

The formation of this compound, particularly during the decomposition of mercuric nitrate, leads me to suppose that during the heating complex molecules of $Hg_4(NO_3)_8$ or (in the case of mercurous nitrate) complex molecules of $Hg_4(NO_3)_4$ are subjected to decomposition. The latter then yield $(Hg_2)Hg_2(NO_4)_2$ with elimination of $2\,NO_2$, while the former do so with elimination of $6\,NO_2$ and $2\,O_2$.

I must also state that I have heated anhydrous $Hg_2(NO_3)_2$ in vacuum tubes at different temperatures and have always noticed that the salt turned yellow and was converted to a larger or smaller extent into the said orthonitrate with evolution of a brown gas which behaved in every respect like NO_2.

Physics. — W. H. KEESOM. "*Contributions to the knowledge of* VAN DER WAALS' *ψ-surface.*" VI. *The increase of pressure at condensation of a substance with small admixtures.*" (Communication N°. 79 from the Physical Laboratory at Leiden, by Prof. H. KAMERLINGH ONNES.)

(Communicated in the meeting of March 29, 1902).

§ 1. In Communication N°. 75 I have expressed by the two constants

$$\alpha = \frac{1}{T_k}\left(\frac{dT_{xk}}{dx}\right)_{x=0} \quad \text{and} \quad \beta = \frac{1}{p_k}\left(\frac{dp_{xk}}{dx}\right)_{x=0},$$

the corrections by means of which we can derive the critical data for the pure substance from the observed plaitpoint phenomena T_{pl}, p_{pl} and v_{pl}, in a substance with small impurities (in a molecular proportion x). These constants were thought by KAMERLINGH ONNES especially suitable to determine the phenomena in the case of small admixtures. It is also desirable to represent the differences in composition of the co-existing phases, the variations in pressure at the condensation of a mixture and the further deviations in properties of such a mixture from the pure (normal) substance by means of the same constants and the reduced quantities of a simple (normal) substance. For in this way we can avail ourselves of the empirically reduced equation of state for simple (normal) substances and of the empirical relations connected with it for the application of VAN DER WAALS' theory to the calculation of the relation between the deviations mentioned. If a comparison of the calculated with the observed deviations should confirm the validity of the law of corresponding states for the mixtures considered here, then from an experimental point of view it would be important to reduce the deviations caused by admixture in a whole series of observations to the two constants mentioned. These latter can be derived from observations on the mixture itself, and we might especially avail ourselves of this in calculating observations on substances which can be purified only with great difficulty or not at all.

In this communication I will develop in the first place by means of the law of corresponding states applied to mixtures, the relation for the proportion of the compositions of the co-existing phases when small admixtures occur. This relation will then be tested by HARTMAN's observations on the composition of co-existing phases in mixtures of methyl chloride and carbon dioxide. Lastly formulae will be derived for the difference of the saturation pressure of the mixture and that of the pure substance together with the increase of pressure at condensation of the mixture. These formulae also enable us to derive the proportion of admixture from the increase of pressure observed, after which the saturated vapour-pressure of the pure substance can be derived from the observed beginning- or end-condensation pressure of the impure substance. The latter formula is applied to KUENEN's observations on impure ethane.

The result is satisfactory in different respects. For the proportion of admixture in KUENEN's experiments corresponding values have indeed been found from his different measurements and, moreover, from the experiments with the impure substance values for the vapour-pressures and the critical data of the pure substance are found which can deviate only a little from the real values.

§ 2. In order to learn the relation between the compositions at co-existence of two phases, I start from the well-known property that in this case:

$$\left(\frac{\partial\psi_1}{\partial x_1}\right)_{vT} = \left(\frac{\partial\psi_2}{\partial x_2}\right)_{vT}, \quad \dots\dots\dots \quad (1)$$

where ψ represents the free energy, while in the following the index 1 will always refer to the liquid, the index 2 to the vapour phase.

As in Communication N⁰ 75 we put:

$$\psi = MRT\{(1-x)\,l\,(1-x) + x\,l\,x\} + \varphi \text{ [1]},$$

where therefore $\varphi = -\int^{v} p\,dv$.

If again we write $\left(\frac{\partial\varphi_1}{\partial x_1}\right)_{vT} = \varphi_1'$, (1) passes into:

$$MRT\,l\left(\frac{x_1}{1-x_1}\right) + \varphi_1' = MRT\,l\left(\frac{x_2}{1-x_2}\right) + \varphi_2'.$$

It follows from this that for x small we have

$$x_1 = x_2\, e^{\frac{\varphi_2'-\varphi_1'}{MRT}} \text{ [2]}. \quad \dots\dots\dots \quad (2)$$

Now

$$\varphi_1' = -\int^{v_{1_{x_1}}} \left(\frac{\partial p}{\partial x}\right)_{\substack{vT\\x=x_1}} dv,$$

or by MACLAURIN's theorem:

$$\varphi_1' = -\int^{v_1} \left(\frac{\partial p}{\partial x}\right)_{\substack{vT\\x=0}} dv - x_1 \int^{v_1} \left(\frac{\partial^2 p}{\partial x^2}\right)_{\substack{vT\\x=0}} dv - x_1 \left(\frac{\partial p}{\partial x}\right)_{\substack{vT\\x=0}} \left(\frac{dv_{1_x}}{dx}\right)_{x=0} \quad \text{etc.}$$

For small x we may write:

$$\varphi_2' - \varphi_1' = -\int_{v_1}^{v_2} \left(\frac{\partial p}{\partial x}\right)_{vT} dv,$$

if we are not too near the critical temperature of the pure substance, as in that case $\frac{dv_{1_x}}{dx}$ and $\frac{dv_{2_x}}{dx}$ become infinite. Here v_1 and v_2 represent the molecular volume of saturated liquid and vapour of the pure substance. If now we introduce the law of corresponding states by means of the relation mentioned in Communication n⁰. 75:

$$\left(\frac{\partial p}{\partial x}\right)_{vT} = \pi\frac{dp_{xk}}{dx} - \frac{p_{xk}}{v_{xk}}\frac{dv_{xk}}{dx}\,\omega\left(\frac{\partial\pi}{\partial\omega}\right) - \frac{p_{xk}}{T_{xk}}\frac{dT_{xk}}{dx}\,\tau\left(\frac{\partial\pi}{\partial\tau}\right),$$

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[1] Comp. VAN DER WAALS, Contin. II, p. 147.

[2] See VAN DER WAALS, Arch. Néerl. XXVI, p. 96, Contin. II p. 148.

and using **Maxwell**'s theorem:

$$\int_{\omega_1}^{\omega_2} \pi\, d\omega = \pi_m (\omega_2-\omega_1),$$

and:

$$\int_{\omega_1}^{\omega_2} \left(\frac{\partial \pi}{\partial \tau}\right) d\omega = \frac{d\pi_m}{d\tau} (\omega_2-\omega_1) \ ^{1)},$$

where π_m represents the reduced co-existence pressure, ω_2 and ω_1 reduced volumes of saturated vapour and liquid, then, if α and β have the value mentioned before, and $C_4 = \frac{MRT_k}{p_k v_k}$, we find

$$\frac{\varphi_2'-\varphi_1'}{MRT} = \frac{1}{C_4}\left\{\alpha \frac{\tau}{\pi_m} \frac{d\pi_m}{d\tau} - \beta\right\} \frac{\pi_m(\omega_2-\omega_1)}{\tau} \quad . \quad . \quad . \quad (3)$$

or

$$\frac{\varphi_2'-\varphi_1'}{MRT} = \left\{\alpha \frac{T}{p_m} \frac{dp_m}{dT} - \beta\right\} \frac{p_m(v_2-v_1)}{MRT} \quad . \quad . \quad . \quad . \quad (4)$$

§ 3. This relation can be tested by **Hartman**'s 2) observations

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1) Comp. Communication N°. 75, Proceedings Dec. 1901, p. 298.

2) **Hartman**, Thesis for the Doctorate, Leiden. p. 104.

As an addition to the graphic representation given in Comm. 43, Proceedings June 1898, we give here the numbers found by **Hartman**, where x = the ratio of carbon dioxide:

Co-existing phases at 9.5° C.

p	x_1	x_2	v_1	v_2
3.58	0	0	0.00256	0.173
5	0.021	0.242	257	145
10	136	640	262	0813
15	267	745	252	0562
20	394	810	248	0454
25	534	861	246	0356
30	675	908	251	0269
35	800	947	248	0209
40	0.910	0.980	245	0166
43.64	1	1	0.00243	0.0145

The volumes of the vapour-phase are very uncertain; probably the blowing off, which was necessary before the analysis, produced a systematical error. **Hartman** in connection with KUENEN's [1]) observations on mixtures of methyl chloride and carbon dioxide.

No determinations exist for the plait point constants of carbon dioxide and methyl chloride or for isothermals of mixtures in the case that the quantity of one of the substances is very small. To calculate α and β I shall therefore start from the constants K_{11}, etc. derived by KAMERLINGH ONNES and REINGANUM [2]) for these substances from KUENEN's observations. They represented the critical temperature and pressure of the mixtures, considered as homogeneous by:

$$T_{xk} = C_1 \sqrt{\frac{K_x}{b_x}} \quad , \quad p_{xk} = C_2 \sqrt{\frac{K_x}{b_x^3}} \, ,$$

while:

$$K_x = K_{11}(1-x)^2 + 2\, K_{12}\, x\, (1-x) + K_{22}\, x^2,$$
$$b_x \ = b_{11}\ (1-x)^2 + 2\, b_{12}\, x\ \ (1-x) + b_{22}\, x^2,$$

in which, therefore, as x is always supposed to be small, the index 1 refers to the substance present in quantity, the index 2 to the admixture. Then:

$$\left.\begin{aligned} \alpha &= \frac{K_{12}}{K_{11}} - \frac{b_{12}}{b_{11}} \\ \beta &= \frac{K_{12}}{K_{11}} - 3\,\frac{b_{12}}{b_{11}} + 2.\ ^{3)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

For carbon dioxide with methyl chloride as admixture [4])
$K_{11} = 2.176$; $K_{12} = 3.314$; $b_{11} = 0.000780$; $b_{12} = 0.000893$.
Then:

$$\alpha = 0.378 \, , \quad \beta = 0.088.$$

This gives for $T = 273.0 + 9.5 = 282.5$, at which temperature according to AMAGAT for carbon dioxide: $p_m = 43.64$, $\frac{dp_m}{dT} = 1.088$, and the specific volumes of saturated vapour and liquid are also according to AMAGAT: $\frac{v_2}{M} = 0.0151$ and $\frac{v_1}{M} = 0.00230$, while I put

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himself wanted to repeat his experiments with a compound pump (described in Comm. N° 39, Proceedings May 1897) so as to avoid this. In the calculation therefore I have used for these volumes AMAGAT's values for carbon dioxide, and for methyl chloride the values derived by means of the law of corresponding states from YOUNG's data.

[1]) KUENEN, Thesis for the Doctorate, Communication N°. 4. Proceedings April 1892.

[2]) KAMERLINGH ONNES and REINGANUM, Comm. N°. 59*b*, Proceedings September 1900, p. 295.

[3]) Comp. VAN DER WAALS, Contin. II, p. 174.

[4]) KAMERLINGH ONNES and REINGANUM, l.c.

$$R = 1.00706 \times 0.0036625 = 0.003688 : {}^{1})$$

hence
$$\frac{x_1}{x_2} = 3.98.$$

HARTMAN finds at $x_2 = 0.020$: $x_1 = 0.090$, therefore:

$$\frac{x_1}{x_2} = 4.50.$$

I find for methyl chloride, with carbon dioxide as admixture, from: $K_{11} = 6.276$, $K_{12} = 3.314$, $b_{11} = 0.001193$, $b_{12} = 0.000893$:

$$\alpha = -0.221, \qquad \beta = 0.281.$$

From YOUNG's observations on normal pentane [2] I derive at

$$\tau = \frac{282.5}{416.0} = 0.679:$$

$$\frac{\tau}{\pi_m}\frac{d\pi_m}{d\tau} = 10.26 \qquad \text{and} \qquad \frac{1}{C_4}\frac{\pi_m(\omega_2-\omega_1)}{\tau} = 0.931.$$

With this I find:

$$\frac{x_2}{x_1} = 10.73,$$

while HARTMAN finds at $x_1 = 0.021 : x_2 = 0.242$, therefore $\dfrac{x_2}{x_1} = 11.5$. The agreement between observation and calculation may be considered as satisfactory.

§ 4. Now that we have a relation between x_1 and x_2 we may derive from the equation

$$\left\{v_2 - v_1 - (x_2 - x_1)\left(\frac{\partial v_1}{\partial x_1}\right)_{pT}\right\}\frac{dp}{dx_1} = (x_2 - x_1)\left(\frac{\partial^2 \zeta_1}{\partial x_1^2}\right)_{pT} {}^{3)} \quad . \quad . \quad (6)$$

how the pressure of saturated liquid and vapour varies by adding a small quantity of a second substance. For small x_1 and x_2 we may put for this:

$$(v_2 - v_1)(p_1 - p) = MRT \, . \, x\,(e^{-k} - 1), \quad . \quad . \quad . \quad . \quad (7)$$

if we are not too near the critical temperature of the pure substance for then $\left(\dfrac{\partial v_1}{\partial x_1}\right)_{pT}$ becomes very large.

Here v_2 and v_1 represent the molecular volumes of saturated vapour and liquid, p the saturated vapour pressure of the pure substance, p_1 the pressure above the liquid with the composition x, while

$$k = \frac{\varphi'_2 - \varphi'_1}{MRT} = \left\{\alpha\frac{T}{p}\frac{dp}{dT} - \beta\right\}\frac{p\,(v_2 - v_1)}{MRT}.$$

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[1] Comp. KAMERLINGH ONNES, Comm. no. 71. Proceedings June 1901, p. 130.

[2] S. YOUNG, Trans. Chem. Soc. 1897, p. 452.

[3] VAN DER WAALS, Continuität II, p. 108.

In the same way is

$$(v_2 - v_1)(p_2 - p) = MRT : x\,(1 - e^k), \quad . \quad . \quad . \quad . \quad . \quad (8)$$

if p_2 is the pressure of the saturated vapour, of the composition x. The increase of pressure at the condensation of vapour of the composition x into liquid of the same composition follows therefore from

$$(v_2 - v_1)(p_1 - p_2) = MRT\,x\,(e^k + e^{-k} - 2) \quad . \quad . \quad . \quad (9)$$

§ 5. Equation (9) can serve to determine the composition of the substance under examination from measurements on the increase of pressure.

Such measurements have been made by **Kuenen** [1]) on the condensation at different temperatures of ethane, with a small admixture, probably of butane. Although these observations are very accurate, they are not sufficient to test my formulae with certainty. The degree of purity of the ethane itself was not so high that we can derive the constants α and β accurately from the critical data found for this substance with those for the mixture of 4 % butane, while the mixture of 9 % differs too much from the pure substance for us to derive these quantities from the two mixtures. I have tried to form an idea of the usefulness of the formulae by deriving from these observations the quantity of the admixture, by means of values assumed for $\frac{K_{12}}{K_{11}}$ and $\frac{b_{12}}{b_{11}}$.

These might have been determined from observations on mixtures of the two substances in known proportions. But as such observations on these mixtures are not at hand and as for our purpose we do not require the very greatest accuracy I have put:

$$K_{12} = \sqrt{K_{11}.\,K_{22}} \quad \text{and} \quad b_{12} = \frac{b_{11} + b_{22}}{2} \ ,$$

while I calculate K_{11}, K_{22}, b_{11}, b_{22} from the critical constants of the substances, according to the relations mentioned in § 3, borrowed from **Clausius**' equation of state:

$$T_k = C_1 \sqrt{\frac{K}{b}} \, , \quad p_k = C_2 \sqrt{\frac{K}{b^3}} \, ,$$

where C_1 and C_2 have the same value for all substances.

Then

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[1]) **Kuenen**, Proc. Roy. Soc. Edinburgh 1897, p. 433; Zeitschrift für comprimirte und flüssige Gase I, p. 153.

$$\alpha = \sqrt{\frac{K_{22}}{K_{11}}} - \frac{1}{2}\frac{b_{22}}{b_{11}} - \frac{1}{2},$$

$$\beta = \sqrt{\frac{K_{22}}{K_{11}}} - \frac{3}{2}\frac{b_{22}}{b_{11}} + \frac{1}{2},$$

while : $$\frac{K_{22}}{K_{11}} = \left(\frac{T_{k_2}}{T_{k_1}}\right)^3 \frac{p_{k_1}}{p_{k_2}}\ , \quad \frac{b_{22}}{b_{11}} = \frac{T_{k_2}}{T_{k_1}}\frac{p_{k_1}}{p_{k_2}}.$$

In this way I find for methyl chloride with carbon dioxide as admixture :

$$\alpha = -0.236, \quad \beta = 0.113,$$

and for carbon dioxide with methyl chloride as admixture:

$$\alpha = 0.426, \quad \beta = -0.108,$$

which values, if we take into consideration that α has by far the greater influence, agree sufficiently with the values derived directly from the observations in § 3, to enable us to apply these formulae also to ethane and butane. For butane no critical data are known to me. From the regular succession of the critical constants determined by YOUNG for the members of the paraffin-series following butane : viz.

	t_k	Difference.	p_k	Difference.
Normal pentane	197.2		25100 m.m.	
		37.6		2590
» hexane	234.8		22510	
		32.1		2095
» heptane	266.9		20415	
		29.3		1685
» octane	296.2		18730	

we are justified in concluding that we may assume for butane :

$$t_k = 152, \quad p_k = 28300 \text{ m.m.} = 37.2 \text{ atm.}$$

With the values for ethane :

$$t_k = 32.0, \quad p_k = 48.8$$ [1]),

this yields :

$$\alpha = 0.47, \quad \beta = -0.36.$$

The values of π_m, $\frac{d\pi_m}{d\tau}$, ω_2 and ω_1 were borrowed from YOUNG's observations on pentane ;

$$log\ \pi_m = -2.980\frac{1-\tau}{\tau}$$ [2])

and

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[1]) KUENEN, l. c.

[2]) Comp. VAN DER WAALS, Contin. I. p. 158.

$$\delta = 1 + 0.9280\,(1-\tau) \pm 1.8893\,(1-\tau)^{0.3327} \quad ^{1})$$

were used as interpolation formulae.

In the last formula δ represents the reduced density and the sign $+$ gives the liquid, the sign $-$ the vapourdensity. The first formula gives for the region which we want (from $\tau = 0.944$ to $\tau = 0.995$) π_m with an accuracy of 0.5%, the second gives over the whole range on which YOUNG observed volumes of saturated liquid and vapour (from $\tau = 0.6824$ to $\tau = 0.9963$) : $\delta_1 + \delta_2$ accurately to within 0.2%, $\delta_1 - \delta_2$ to within 1.5%, δ_1 to within 0.5%; with the lower τ's we find for δ_2 important deviations, but for the region that we want the deviations are not more than 2%, an accuracy more than sufficient for our purpose. In consequence of the value found by YOUNG ²) for C_4 for the substances: pentane, hexane, heptane and octane, I put for butane $C_4 = 3.6$. By means of these data we can compute the composition of the mixture of ethane with (according to KUENEN) 4% butane ³). I find from the values given by KUENEN for p_2 and p_1 the pressure at the beginning and at the end of the condensation the following values for x:

t	p_2	p_1	x	p
14.95	29.02	32.91	0.0132	33.62
20.15	33.65	36.86	0.0121	37.73
24.85	37.84	40.62	0.0126	41.60
30.35	43.15	45.35	0.0174	

It is to be expected that the last value for x deviates from the preceding, if we bear in mind what has been said in §§ 2 and 4, as we are getting very near to the critical temperature of the pure substance (32.0° C.) The satisfactory agreement of the other values of x, shows that formula (9) represents well the relation of the increase of pressure to the temperature. Moreover I should derive from this that the mixture had the composition $x = 0.0126$, and therefore contained: 2.3% of its mass of butane.

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¹) Comp. VERSCHAFFELT, Proc. June 1896. Comm. N⁰. 28 p. 12 and Comm. N⁰. 55 Proc. April 1900, p. 588. According to VAN DER WAALS, Thermodynamical theory of Capillarity, the index for $\tau = 1$ must approach to 0.5

²) S. Young, Phil. Mag. Sept. 1900, p. 303.

³) KUENEN, Proc. Roy. Soc. Edinburgh, 1897, p, 442; Zeitschrift für comprimirte und flüssige Gase I p. 157.

With this value for x I have calculated according to formula (7) the corrections to be applied to p_1 in order to arrive at the saturated vapour pressure of the pure substance. The values of p found thus are given in column p of the table. They agree well with the pressures given by KUENEN[1]) for almost pure ethane. Further according to formulae (2*a*) and (2*b*) of communication n⁰ 75:

$$\frac{1}{T_k}\frac{dT_{pl}}{dx}=\alpha-\frac{\left\{\beta-\alpha\left(\frac{\partial\pi}{\partial\tau}\right)\right\}^2}{C_4\left(\frac{\partial^2\pi}{\partial\omega\,\partial\tau}\right)},$$

$$\frac{1}{p_k}\frac{dp_{pl}}{dx}=\beta-\left(\frac{\partial\pi}{\partial\tau}\right)\frac{\left\{\beta-\alpha\left(\frac{\partial\pi}{\partial\tau}\right)\right\}^2}{C_4\left(\frac{\partial^2\pi}{\partial\omega\,\partial\tau}\right)},$$

where I have put $\left(\frac{\partial\pi}{\partial\tau}\right)=6.86$ and for $C_4\left(\frac{\partial^2\pi}{\partial\omega\,\partial\tau}\right)$ the value -32.2 found there for carbon dioxide, I have calculated the corrections to be applied to the plaitpoint temperature and pressure observed in order to obtain the critical temperature and pressure of pure ethane.

So I found:

$$t_k=32.0^\circ\ C. \quad \text{and} \quad p_k=48.2\ \text{atm.},$$

which values according to those given by KUENEN for almost pure ethane, cannot differ much from the real values.

One would now be induced to calculate from the increases of vapour pressure for almost pure ethane given by KUENEN l.c. the composition and then the corrected values for the pure substance. As beforehand nothing is known with certainty about the character of admixture, one might wish to calculate α and β from the observations on the increase of pressure at different temperatures, but it then appears that the temperature range of the observations is too small to allow us to calculate α and β separately. If however the observations do not allow us to make this calculation, it appears sufficiently from the preceding that the formulae communicated are confirmed by KUENEN's observations, and that they may have a practical value in the reduction of similar series of observations with a not altogether pure substance.

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[1]) Comp. KUENEN, Zeitschrift für comprimirte und flüssige Gase, I, p. 159, table 5.

Physics. — H. A. LORENTZ. "*The rotation of the plane of polarization in moving media.*"

(Communicated in the meeting of March 29, 1902).

§ 1. In my „Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern" (Leiden, 1895) I examined the propagation of light in transparent bodies having a constant translation with velocity $\mathfrak{p}$, the aether being supposed to remain at rest, and tried to find, in how far optical phenomena may be affected by this motion. In the case of the rotation of the plane of polarization in optically active substances, I had to leave the question undecided. Indeed, the relation between the electric force $\mathfrak{E}$ and the electric moment $\mathfrak{M}$, to which I was led by certain general principles (linear form of the equations, isotropy of structure and reversibility of the motions) does not only contain the coefficient j, which determines the rotation in the quiescent medium; there is besides a second coefficient k, which is multiplied by the velocity p, and whose ratio to j I could not determine, because I wished to refrain from special hypotheses as to the mechanism of the phenomenon.

The equation in question is [1])

$$\mathfrak{E} = \sigma\, \mathfrak{M} + j\, Rot\, \mathfrak{M} + k\, [\dot{\mathfrak{M}} \,.\, \mathfrak{p}]\,, \quad \ldots \ldots \quad (1)$$

and the rotation for unit length was found to be [2])

$$\omega = \frac{2\,\pi}{\sigma^2}\, n'^2\, j,$$

if the body is at rest, and

$$\omega = \frac{2\,\pi}{\sigma^2}\, n'^2 \left(1 + \frac{W \mathfrak{p}_x}{c^2}\right) j + \frac{2\,\pi}{\sigma^2}\, n'^2\, W \mathfrak{p}_x\, k, \quad \ldots \ldots \quad (2)$$

if it has a translation along the axis of x, the light travelling in the same direction. In these formulae n' is the frequency, i. e. the number of vibrations in the time 2π, for an observer, moving with the medium, W the mean of the velocities of right-handed and left-handed circularly polarized rays in the medium at rest, and c the velocity of light in the aether.

The two terms with $\mathfrak{p}_x$ would annul each other, if

$$k = -\frac{j}{c^2} \quad \ldots \ldots \ldots \ldots \quad (3)$$

I saw however no reason to admit this relation.

§ 2. Mr. LARMOR has published [3]) some objections to my in-

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[1]) l. c., p. 80. σ is the coefficient which determines the index of refraction.

[2]) l. c., p. 118.

[3]) J. LARMOR. Aether and Matter, Cambridge, 1900.

vestigation. According to him we may infer from theory that a translation has *no* influence on the rotation. Mr. LARMOR believes the contradiction between our results to be due to an error on my side 1), consisting in an oversight which he points out p p. 214 and 215 of his work.

A new examination of the problem has convinced me that LARMOR must be wrong in this assertion, the formula (2) following undoubtedly from my fundamental equations. I also found that the equations of LARMOR are the same as mine, if in these one puts $k = 0$, and that it is only in consequence of a mistake that his analysis does not lead him to an expression agreeing with the first term in (2). I shall now show that, whereas my equations leave room for a compensation, just because they contain the second coefficient k, LARMOR, treating only the particular case $k = 0$, ought to have arrived at a rotation, different for a moving and for a quiescent body.

§ 3. In my calculations I used the equations

$$\left.\begin{aligned} & Div\, \mathfrak{D} = 0\,, \\ & Div\, \mathfrak{H} = 0\,, \\ & Rot\, \mathfrak{H}' = 4\,\pi\, \dot{\mathfrak{D}}\,, \\ & Rot\, \mathfrak{E} = -\,\dot{\mathfrak{H}}\,, \\ & \mathfrak{E} = 4\,\pi\, c^2\, \mathfrak{d} + [\mathfrak{p} \cdot \mathfrak{H}]\,, \\ & \mathfrak{H}' = \mathfrak{H} - 4\,\pi\, [\mathfrak{p} \cdot \mathfrak{d}]\,, \\ & \mathfrak{D} = \mathfrak{d} + \mathfrak{M}\,, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4)$$

to which is to be added the relation (1).

The meaning of $\mathfrak{E}$ and $\mathfrak{M}$ has already been mentioned; $\mathfrak{H}$ is the magnetic force and the remaining vectors are defined by the equations themselves. The components of the vectors are regarded as functions of the time and of the coordinates x, y, z, referred to axes, fixed to the moving medium; the time-rates of variation for constant values of these coordinates are denoted by $\dot{\mathfrak{D}}$ and $\dot{\mathfrak{H}}$.

We may omit the first and second formulae, these being implied, in the cases to be considered, in the third and fourth equations. Moreover, just like LARMOR, we shall restrict the investigation to bodies, moving parallel to the axis of x, and traversed by rays of light of this same direction. Then, the only independent variables are x and t, and the equations (4) become

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1) l.c., p. 62.

$$\left.\begin{array}{ll} -\frac{\partial \mathfrak{H}'_z}{\partial x} = 4\pi \dot{\mathfrak{D}}_y , & \frac{\partial \mathfrak{H}'_y}{\partial x} = 4\pi \dot{\mathfrak{D}}_z , \\ -\frac{\partial \mathfrak{E}_z}{\partial x} = -\dot{\mathfrak{H}}_y , & \frac{\partial \mathfrak{E}_y}{\partial x} = -\dot{\mathfrak{H}}_z , \\ \mathfrak{E}_y = 4\pi c^2 \mathfrak{d}_y - \mathfrak{p}_x \mathfrak{H}_z , & \mathfrak{E}_z = 4\pi c^2 \mathfrak{d}_z + \mathfrak{p}_x \mathfrak{H}_y , \\ \mathfrak{H}'_y = \mathfrak{H}_y + 4\pi \mathfrak{p}_x \mathfrak{d}_z , & \mathfrak{H}'_z = \mathfrak{H}_z - 4\pi \mathfrak{p}_x \mathfrak{d}_y , \\ \mathfrak{D}_y = \mathfrak{d}_y + \mathfrak{M}_y , & \mathfrak{D}_z = \mathfrak{d}_z + \mathfrak{M}_z . \end{array}\right\} \quad . \quad . \quad (5)$$

§ 4. In LARMOR's equations [1]) the velocity of translation $\mathfrak{p}_x$ is represented by v, the sign $\frac{\delta}{dt}$ is used for those time-rates of variation, which I have indicated by a dot, and the sign $\frac{d}{dt}$ for the differential coefficients relating to a fixed point of space. Hence, in his notation,

$$\frac{\delta}{dt} = \frac{d}{dt} + v \frac{d}{dx} .$$

If now, we write $\dot{\varphi}$ instead of $\frac{\delta \varphi}{dt}$, and $\varphi - v \frac{\partial \varphi}{\partial x}$ instead of $\frac{d\varphi}{dt}$ (φ being any quantity, depending on place and time), and if besides we suppose the substance to be unmagnetizable, so that $\mu = 1$, the equations of LARMOR become

$$\left.\begin{array}{l} -\frac{\partial \gamma}{\partial x} = 4\pi (\dot{g} + \dot{g}') - 4\pi v \frac{\partial (g + g')}{\partial x} , \\ \frac{\partial \beta}{\partial x} = 4\pi (\dot{h} + \dot{h}') - 4\pi v \frac{\partial (h + h')}{\partial x} , \end{array}\right\}$$

$$-\frac{\partial R}{\partial x} = -\dot{\mathrm{b}} , \qquad \frac{\partial Q}{\partial x} = -\dot{\mathrm{c}} ,$$

$$Q = 4\pi c^2 g - v\,\mathrm{c} \quad , \quad R = 4\pi c^2 h + v\,\mathrm{b} ,$$
$$\beta = \mathrm{b} - 4\pi v h' \quad , \quad \gamma = \mathrm{c} + 4\pi v g' .$$

These are the same as (5), as will be seen, if we replace

$$Q, \quad R, \quad g, \quad h, \quad g', \quad h'$$

by

$$\mathfrak{E}_y, \quad \mathfrak{E}_z, \quad \mathfrak{d}_y, \quad \mathfrak{d}_z, \quad \mathfrak{M}_y, \quad \mathfrak{M}_z$$

and

$$\mathrm{b}, \quad \mathrm{c}, \quad \beta + 4\pi v (h + h'), \quad \gamma - 4\pi v (g + g')$$

by

$$\mathfrak{H}_y, \quad \mathfrak{H}_z, \quad \mathfrak{H}'_y, \quad \mathfrak{H}'_z .$$

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[1]) l. c., p. 212.

§ 5. As to the relation between electric polarization and electric force, this is given by LARMOR in the form [1])

$$g' = \frac{K-1}{4\pi c^2} Q + \frac{\varepsilon_2}{4\pi c^2} \frac{\partial R}{\partial x},$$

$$h' = \frac{K-1}{4\pi c^2} R - \frac{\varepsilon_2}{4\pi c^2} \frac{\partial Q}{\partial x},$$

or

$$\left.\begin{aligned} \mathfrak{M}_y &= \frac{K-1}{4\pi c^2} \mathfrak{E}_y + \frac{\varepsilon_2}{4\pi c^2} \frac{\partial \mathfrak{E}_z}{\partial x}, \\ \mathfrak{M}_z &= \frac{K-1}{4\pi c^2} \mathfrak{E}_z - \frac{\varepsilon_2}{4\pi c^2} \frac{\partial \mathfrak{E}_y}{\partial x}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now, in my formula (1) the rotational terms are very much smaller than the first term $\sigma\, \mathfrak{M}$. We may therefore, in those terms, replace $\mathfrak{M}$ by $\frac{1}{\sigma}\, \mathfrak{E}$. Hence

$$\mathfrak{M} = \frac{1}{\sigma} \mathfrak{E} - \frac{j}{\sigma^2}\, Rot\, \mathfrak{E} - \frac{k}{\sigma^2} [\dot{\mathfrak{E}} \,.\, \mathfrak{p}], \quad . \quad . \quad . \quad . \quad (7)$$

and, in the case under consideration,

$$\left.\begin{aligned} \mathfrak{M}_y &= \frac{1}{\sigma} \mathfrak{E}_y + \frac{j}{\sigma^2} \frac{\partial \mathfrak{E}_z}{\partial x} - \frac{k}{\sigma^2} \mathfrak{p}_x \dot{\mathfrak{E}}_z, \\ \mathfrak{M}_z &= \frac{1}{\sigma} \mathfrak{E}_z - \frac{j}{\sigma^2} \frac{\partial \mathfrak{E}_y}{\partial x} + \frac{k}{\sigma^2} \mathfrak{p}_x \dot{\mathfrak{E}}_y. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If this is compared with (6), it appears that the formulae of LARMOR agree with the particular case $k = 0$ of my theory, and that the coefficients we have introduced are related to each other as follows:

$$\frac{K-1}{4\pi c^2} = \frac{1}{\sigma}, \quad \frac{\varepsilon_2}{4\pi c^2} = \frac{j}{\sigma^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

§ 6. For $k = 0$ my formula (2) gives

$$\omega = \frac{2\pi}{\sigma^2} n'^2 \left(1 + \frac{W \mathfrak{p}_x}{c^2}\right) j, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

a value depending on $\mathfrak{p}_x$. On the contrary, LARMOR's result does not contain the velocity of translation, but this is only so, because his calculation of the angle of rotation is not quite exact.

As is well known, this angle may be expressed in the velocities of propagation of right- and left-handed circularly polarized rays. In doing this, we have first of all to assign to the period of vibration, taken with reference to a fixed point of the substance, a

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[1] l.c., p. 211. As I shall not consider the magnetic rotation, I have put $\varepsilon_1 = 0$.

definite value τ, the same for the two kinds of rays. If then, V_1' und V_2' are the velocities of propagation, taken relatively to the moving ponderable matter, we shall have

$$\omega = \frac{\pi}{\tau}\left(\frac{1}{V_1'} - \frac{1}{V_2'}\right). \quad \ldots\ldots\ldots \quad (11)$$

For the velocity of one of the circularly polarized rays, LARMOR finds (p. 214)

$$V_1' = \frac{c}{K_1^{1/2}} - \frac{v}{K_1}, \quad \ldots\ldots\ldots \quad (12)$$

where

$$K_1 = K + \frac{2\pi\varepsilon_2}{\lambda}, \quad \ldots\ldots\ldots \quad (13)$$

λ being the wave-length. By substituting this value in (12), putting at the same time

$$\lambda = V_1'\tau, \quad \ldots\ldots\ldots \quad (14)$$

we might obtain an equation, by means of which V_1' could be determined in function of τ. We may however simplify by observing that ε_2 has a very small value and that in (13) λ occurs only in a term, containing this factor ε_2. For this reason, it is allowed to substitute for λ the value corresponding to $\varepsilon_2 = 0$. Thus, by (12), (13) and (14)

$$\lambda = \left(\frac{c}{K^{1/2}} - \frac{v}{K}\right)\tau \quad \ldots\ldots\ldots \quad (15)$$

Let us now put

$$\frac{c}{K_1^{1/2}} = U_1,$$

i. e., on account of (13), if we neglect the square of ε_2,

$$U_1 = \frac{c}{K^{1/2}}\left(1 - \frac{\pi\varepsilon_2}{K\lambda}\right); \quad \ldots\ldots\ldots \quad (16)$$

then (12) takes the form

$$V_1' = U_1 - \frac{U_1^2}{c^2}v \quad \ldots\ldots\ldots \quad (17)$$

In order to obtain the velocity of the other circularly polarized ray, we have only to change the sign of ε_2, so that we may write

$$V_2' = U_2 - \frac{U_2^2}{c^2}v, \quad \ldots\ldots\ldots \quad (18)$$

where

$$U_2 = \frac{c}{K^{1/2}}\left(1 + \frac{\pi\varepsilon_2}{K\lambda}\right) \quad \ldots\ldots\ldots \quad (19)$$

It is to be remarked, that in this equation, as well as in (16), λ has the value (15).

Now, if we neglect terms containing v^2, as we shall always do, the formulae (17) and (18) are the same as the two first equations, given by LARMOR on p. 215. Further, it is there pointed out that in the result for the angle of rotation the quantities depending on the last terms of (17) and (18) disappear. Indeed

$$\frac{1}{V_1'} = \frac{1}{U_1}\left(1 + \frac{U_1}{c^2}\, v\right) = \frac{1}{U_1} + \frac{v}{c^2},$$

$$\frac{1}{V_2'} = \frac{1}{U_2} + \frac{v}{c^2},$$

whence

$$\omega = \frac{\pi}{\tau}\left(\frac{1}{U_1} - \frac{1}{U_2}\right).$$

So far, I agree with LARMOR's calculation. But, in coming to his conclusion, he has overlooked that the value of ω still contains the velocity of translation. This is seen by referring to (16) and (19). Using these, we find

$$\frac{1}{U_1} = \frac{K^{1/2}}{c}\left(1 + \frac{\pi\varepsilon_2}{K\lambda}\right),\quad \frac{1}{U_2} = \frac{K^{1/2}}{c}\left(1 - \frac{\pi\varepsilon_2}{K\lambda}\right),$$

$$\frac{1}{U_1} - \frac{1}{U_2} = \frac{2\,\pi\,\varepsilon_2}{K^{1/2}\,c\lambda},$$

and, taking from (15)

$$\frac{1}{\lambda} = \frac{K^{1/2}}{c\tau}\left(1 + \frac{v}{cK^{1/2}}\right),$$

$$\omega = \frac{2\,\pi^2\,\varepsilon_2}{c^2\,\tau^2}\left(1 + \frac{v}{cK^{1/2}}\right) \quad \ldots\ldots\ldots \quad (20)$$

If the body were at rest, the velocities of the circularly polarized rays would be $\frac{c}{K_1^{1/2}}$ and $\frac{c}{K_2^{1/2}}$, if $K_2 = K - \frac{2\,\pi\,\varepsilon_2}{\lambda}$. The mean of these values, up to the first power of ε_2, is

$$W = \frac{c}{K^{1/2}}.$$

If we also take into account the relation (9) and the value

$$n' = \frac{2\pi}{\tau}$$

of the frequency, we find that (20) does not differ from my result, expressed in the equation (10).

§ 7. In order to show that the rotation must be independent of the motion of the earth, LARMOR adduces also the general considerations that are to be found in Chapter X of his work; from these the proposition may really be inferred, though not without an auxiliary hypothesis. As is well known, the theory of optical phenomena in moving bodies is simplified very much by the introduction, instead of the time t, of the so-called "local" time t' as an independent variable, the equation

$$t' = t - \frac{1}{c^2}\left(\mathfrak{p}_x\, x + \mathfrak{p}_y\, y + \mathfrak{p}_z\, z\right)$$

serving to define this quantity in terms of t and the coordinates x, y, z with respect to axes fixed in the body. By means of this contrivance the electric force, exerted by a small electrically polarized particle P on an electron Q, situated at some distance, is made to be determined by equations of the same form, whether there be or not a common translation of P and Q.

Let [1]) $\mathfrak{m}$ be the electric moment, varying with the time, of P, x, y, z the coordinates of the kind just mentioned in the surrounding field, r the distance to P; then for any point in the field, at its own local time t', the components of the said electric force will be

$$c^2\left\{\frac{\partial^2}{\partial x\,\partial y}\left(\frac{\mathfrak{m}_y}{r}\right)+\frac{\partial^2}{\partial x\,\partial z}\left(\frac{\mathfrak{m}_z}{r}\right)-\frac{\partial^2}{\partial y^2}\left(\frac{\mathfrak{m}_x}{r}\right)-\frac{\partial^2}{\partial z^2}\left(\frac{\mathfrak{m}_x}{r}\right)\right\},$$

etc.,

provided we take for $\mathfrak{m}_x$, $\mathfrak{m}_y$, $\mathfrak{m}_z$ the values corresponding to the instant at which the local time in P is $t' - \frac{r}{c}$, so that the numerators in the expressions $\frac{\mathfrak{m}_x}{r}$, $\frac{\mathfrak{m}_y}{r}$, $\frac{\mathfrak{m}_z}{r}$ depend on t', x, y, z. The differentiations must be performed for a constant t'.

§ 8. We shall now suppose that a dielectric contains a very large number of particles, in which electric moments $\mathfrak{m}$ can be excited, that the sole interaction between these consists in the above mentioned electric forces, and that for each particle the connexion between its moment and the electric force is not altered by a translation. If then, in the absence of such a motion, $\mathfrak{m}_x$, $\mathfrak{m}_y$, $\mathfrak{m}_z$ for the different particles of the body can be certain functions of the time t, we shall obtain a state that is possible in the moving body, by supposing these moments to be exactly the same functions of the local time t'. This follows at once from what has been said in the last §. It is also easily seen that in a point fixed to the ponderable matter, the time of vibration will be the same in the two states, and that, if the first of these states consists in a propagation of light with rotation of the plane of polarization, we shall have in the second state a similar propagation, the angle between the vibrations in any

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[1]) See my "Versuch u. s. w.", § 33.

two points of the body being the same in the two cases. The rotation would therefore be independent of the translation, always provided we compare cases in which the frequency in a point of the body has a definite value.

§ 9. What precedes calls forth two questions. In the first place: can a substance, like the one we have supposed, really have the rotatory property? And, secondly, if this be so, is the picture we have formed of the substance, the only one that agrees with the phenomena, or are there others, equally satisfying?

The answer to the first question must undoubtedly be affirmative. Within the limits of the hypotheses of § 8 there is room for a large variety of optical properties, which may depend either on the form of the connexion between the electric force and the moment of a single particle, or on the relative position of the different particles, and a peculiar arrangement may very well produce a rotation of the plane of polarization. For this it is only necessary that the structure of the system should be asymmetric, i.e. that the system should not be in every respect equal to its reflected image. If, in such a case, we consider the electric interaction between neighbouring particles, we shall have to introduce into the equations certain terms of a rotational character. As a simple example of the required structure we may take a molecule containing 4 unequal particles situated at the angles of an asymmetric tetrahedron, and each of which may be electrically polarized.

As to the second question, it is clear that in real bodies there may very well be circumstances, differing from those we have supposed in § 8. We may e. g. conceive a movable electron, situated at one angle of the asymmetric tetrahedron, to be subject not only to the electric action of a moment, situated at one of the other angles, but also to a force of some other kind ("molecular" force), issuing from that angle. If, in such a case, the action between two elements of matter A and B were such that the action on A at the local time t' were determined by the state of B *at the same local time*, what has been said about two corresponding states might still be true. But this need no longer be so, if the action on A at the time t depends on the state of B *at that same instant*.

However this may be, it must certainly be deemed possible that after all the rotation is not altered by a uniform motion of the active substance; this possibility would however be excluded if we began by omitting in the equation (1) the term with k.

§ 10. The necessity of retaining this term may also be seen in the following way. In the fundamental equations (4) the coordinates are already taken relative to axes, moving with the medium, but the local time has not yet been introduced. We shall now do this, so that our independent variables become x, y, z and t'. We shall distinguish by accents the differential coefficients with respect to x, y, z, for a constant t' from the corresponding differential coefficients, taken for a constant t. We shall likewise denote by *Div'* and *Rot'* operations, in which the new differentiations occur in the same way as the original ones in the operations, represented by *Div* and *Rot*.

The formulae of transformation are

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x}\right)' - \frac{\mathfrak{p}_x}{c^2}\frac{\partial}{\partial t'}\,,\ \text{etc.}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'}\,,$$

and, if $\mathfrak{A}$ be any vector,

$$Rot\ \mathfrak{A} = Rot'\ \mathfrak{A} + \frac{1}{c^2}\,[\dot{\mathfrak{A}}\cdot\mathfrak{p}]\ \ldots\ldots\ldots \quad (21)$$

Using these, and introducing instead of $\mathfrak{D}$ the new vector

$$\mathfrak{D}' = \mathfrak{D} + \frac{1}{4\,\pi\,c^2}\,[\mathfrak{p}\cdot\mathfrak{H}],\ \ldots\ldots\ldots \quad (22)$$

we may write for the first four of the equations (4)

$$Div'\ \mathfrak{D}' = 0,$$
$$Div'\ \mathfrak{H}' = 0,$$
$$Rot'\ \mathfrak{H}' = 4\,\pi\,\dot{\mathfrak{D}}',$$
$$Rot'\ \mathfrak{E} = -\,\dot{\mathfrak{H}}'.$$

These formulae have the same form as those which, for a body at rest, determine $\mathfrak{E}$, $\mathfrak{D}$ and $\mathfrak{H}$, as functions of x, y, z and t; the rotation of the plane of polarization will therefore be independent of the translation, if the connexion between $\mathfrak{D}'$ and $\mathfrak{E}$ in one, and that between $\mathfrak{D}$ and $\mathfrak{E}$ in the other case correspond to each other in the same way. Now, if, according to (7), we put for the body at rest

$$\mathfrak{M} = \frac{1}{\sigma}\,\mathfrak{E} - \frac{j}{\sigma^2}\,Rot\ \mathfrak{E}\,,$$

or

$$\mathfrak{D} - \frac{1}{4\,\pi\,c^2}\,\mathfrak{E} = \frac{1}{\sigma}\,\mathfrak{E} - \frac{j}{\sigma}\,Rot\ \mathfrak{E},$$

the said agreement requires for the moving system

$$\mathfrak{D}' - \frac{1}{4\,\pi\,c^2}\,\mathfrak{E} = \frac{1}{\sigma}\,\mathfrak{E} - \frac{j}{\sigma^2}\,Rot'\ \mathfrak{E}.$$

But, for this system, by (22), joined to the 5th and 7th of the equations (4),

$$\mathfrak{D}' - \frac{1}{4\pi c^2}\mathfrak{E} = \mathfrak{M} ;$$

so that we find by using (21)

$$\mathfrak{M} = \frac{1}{\sigma}\mathfrak{E} - \frac{j}{\sigma^2} Rot\, \mathfrak{E} + \frac{j}{c^2 \sigma^2}[\dot{\mathfrak{E}} . \mathfrak{p}].$$

This is precisely the formula (7), if for k we take the value (3).

Physics. — H. A. Lorentz. "*The intensity of radiation and the motion of the earth*".

(Communicated in the meeting of March 29, 1902).

Many years ago Fizeau[1]) remarked that, if the aether does not follow the earth in its annual motion, the radiation, emitted by a terrestrial source of light or heat L, might possibly have unequal intensities in different directions. Let A be a point that is likewise fixed to the earth, and whose distance from L we shall denote by l. Then, if LA have the direction of the earth's velocity v, a vibration produced by L will have to travel over a length

$$l . \frac{c}{c - v}$$

(c velocity of light), before it reaches A. On the contrary, its course will be

$$l . \frac{c}{c + v},$$

if LA has the opposite direction. Fizeau expected that the intensities received by A in the two cases would be inversely as the squares of these expressions, so that there would be a difference which one might hope to detect by means of suitable experiments with a thermo-electric battery.

From our present views regarding electric and optical phenomena in moving bodies it may be inferred that the experiment, proposed by Fizeau would have a negative result, the amount of heat which is imparted to an absorbing body being independent of the earth's motion.

It will suffice to consider a simple case, omitting all terms depending

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[1]) Pogg. Ann., Bd. 92, p. 652, 1854.

on the square of v. Let there be a single radiating particle in the origin of coordinates, and let it have an electric moment

$$\mathfrak{m}_y = a \cos nt,$$

in the direction of OY. In order to find the dielectric displacement $\mathfrak{d}$ and the magnetic force $\mathfrak{H}$ in the surrounding field, we may start from the formulae, I have developed in § 33 of my „Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern". Let the velocity v of the earth be in the direction of OX, let r be the distance to O,

$$t' = t - \frac{v}{c^2} x \,, \quad \ldots\ldots\ldots\ldots \quad (1)$$

and

$$\psi = \frac{a}{r} \cos n \left(t' - \frac{r}{c}\right).$$

Then

$$\mathfrak{d}_x = \frac{1}{4\pi} \frac{\partial^2 \psi}{\partial x \, \partial y},$$

$$\mathfrak{d}_y = -\frac{1}{4\pi}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2}\right) + \frac{v}{4\pi c^2} \frac{\partial^2 \psi}{\partial t' \, \partial x},$$

$$\mathfrak{d}_z = \frac{1}{4\pi} \frac{\partial^2 \psi}{\partial z \, \partial y},$$

$$\mathfrak{H}_x = -\frac{\partial^2 \psi}{\partial t' \, \partial z},$$

$$\mathfrak{H}_y = - v \frac{\partial^2 \psi}{\partial z \partial y},$$

$$\mathfrak{H}_z = \frac{\partial^2 \psi}{\partial t' \partial x} - v \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2}\right).$$

The auxiliary quantity ψ is to be regarded as a function of x, y, z, t', and it is only after the differentiations have been performed, that the value (1) must be substituted.

We may further confine ourselves to values of r, very much larger than the wave-length λ. In this case we have only to retain the terms whose denominator has the first power of r, all other terms being, with respect to these, of the order $\frac{\lambda}{r}$ or $\frac{\lambda^2}{r^2}$. For points situated on the positive axis of x, we find

$$\mathfrak{d}_x = 0, \; \mathfrak{d}_z = 0,$$

$$\mathfrak{d}_y = \frac{n^2}{4\pi c^2}\left(1 + \frac{v}{c}\right) \frac{a}{r} \cos n \left\{ t - \left(1 + \frac{v}{c}\right) \frac{x}{c} \right\},$$

$$\mathfrak{H}_x = 0 \,, \; \mathfrak{H}_y = 0,$$

$$\mathfrak{H}_z = \frac{n^2}{c}\left(1 + \frac{v}{c}\right) \frac{a}{r} \cos n \left\{ t - \left(1 + \frac{v}{c}\right) \frac{x}{c} \right\},$$

so that

$$\mathfrak{d}_y = \frac{1}{4 \pi c} \mathfrak{H}_z \, .$$

The corresponding energy per unit of space is

$$2 \pi c^2 \mathfrak{d}_y^2 + \frac{1}{8\pi} \mathfrak{H}_z^2 = \frac{1}{4\pi} \mathfrak{H}_z^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, according to POYNTING's theorem, there is a flow of energy along OX

$$c^2 \mathfrak{d}_y \mathfrak{H}_z = \frac{c}{4\pi} \mathfrak{H}_z^2,$$

this quantity being the amount of energy per unit of time and unit of area, which traverses an element of surface, perpendicular to OX, and *not* moving with the earth.

In what follows we have only to attend to the mean values of the energy and its flow, taken for a full period or for a lapse of time, embracing a large number of periods. The mean value of (2) is

$$U = \frac{n^4}{8\pi c^2}\left(1 + \frac{2v}{c}\right)\frac{a^2}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and for that of the energy-current we may write

$$c\,U \, .$$

Since v may be negative as well as positive, the above formulae apply not only to the vibrations, sent out in the direction of the earth's motion, but equally to those which go forth in opposite direction. The factor $1 + \frac{v}{c}$ in the expressions for $\mathfrak{d}_y$ and $\mathfrak{H}_z$ is different in the two cases; it would however be rash, to conclude from this, without closer examination, that the difference will make itself felt in measurements on the heating of a body exposed to the rays.

Let there be, in any point of the positive axis of x, and placed perpendicular to it, a disk of infinitely small area ω, and composed of a perfectly black material, so that it reflects no part of the incident radiation. This disk will be supposed to be fixed to the earth, and we shall deduce the amount of heating from the law of conservation of energy, taking into account that the rays exert on the disk a certain normal pressure, the amount of which per unit area is given precisely by U [1]).

Imagine a right cylinder C, having ω for its base and turned towards the source of heat, and suppose the face of it, that is opposite

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[1]) See e. g. my „Versuch u. s. w.", §§ 16 and 17.

to ω — we shall call this ω' — to have a fixed position in space. Let θ be a lapse of time, consisting of a large number of periods, and consider, for this interval, the change of the amount of energy, contained within C. Let the cylinder be of so great a length h, that, if v should be negative, the disk ω cannot reach the plane ω', before the end of the time θ, and let h at the same time be so small in comparison with the distance r, that terms which are of the order $\frac{h}{r}$ with respect to the quantities we are considering may be neglected. Then we need not trouble ourselves about the difference between the values of U for ω and ω'; neither will it be necessary to attend to the flow of energy through the cylindrical surface of C.

If, for the time θ, e_1 is the amount of energy, by which the plane ω' is traversed, e_2 the increment of the energy, contained within the cylinder, and e_3 the work done by the pressure exerted on ω, the absorption is evidently given by

$$e = e_1 - e_2 - e_3.$$

Now:

$$e_1 = c\, U \omega \theta,$$

and, the volume of the cylinder being increased by $v \omega \theta$,

$$e_2 = v\, U \omega \theta.$$

Finally we have, since the displacement of the disk is $v\, \theta$,

$$e_3 = v\, U \omega \theta.$$

The result is therefore

$$e = (c - 2\, v)\, U \omega \theta,$$

or, by (3), if we continue to neglect terms in v^2,

$$e = \frac{n^4}{8\pi c} \frac{a^2}{r^2} \omega \theta ,$$

independent of the velocity of the earth.

Physics. — "*Ternary systems*". III. By Prof J. D. van der Waals. (Continued from page 560).

The quantity $(\varepsilon_{21})_v$ occurring in equation (1) as a factor of $\frac{dT}{T}$, is negative for normal substances. It represents (Cont. II, pag. 101 and following pages) the decrease of energy per molecule, when we have a finite quantity of the first phasis and an infinitely small quantity of the second phasis, and when we then make the substance fill the volume homogeneously, keeping volume and temperature constant; so we may also say that it represents the heat which in this process is given out per molecular quantity of the second phasis. Or, if one prefers to reverse the process, it is the heat absorbed per molecular quantity of the second phasis if a homogeneous phasis is splitted up into two phases, — volume and temperature being kept constant, — provided the quantity of the second phasis is infinitely small compared with the quantity of the first phasis.

We have proved l. c., that for substances, for which the attraction may be reduced to a molecular pressure of the form $\frac{a}{v^2}$, as well in the case that a is a function of the temperature, as in the case that a does not depend on the temperature, $(\varepsilon_{21})_v$ is certainly negative, and so that the energy increases, if in the process mentioned we make two separate phases form one homogeneous phasis. In the volume of the Arch. Néerl. dedicated to H. A. LORENTZ I have shown, by means of the calculation of the coefficient of compressibility, that the conclusions, deduced from the calculated value for $(\varepsilon_{21})_v$ for a simple substance (Cont. II, pag. 104) namely

$$(\varepsilon_{21})_v = - v_2 (K_1 + K_2 - 2 K_{12})$$

hold good.

The conclusion, that the surface representing coexisting phases contracts at increasing temperature, which is only true for the case that $(\varepsilon_{21})_v$ is negative, is only in contradiction with the results of experiments for water at a temperature below 4°, nor will it hold good for diluted aqueous solutions at less than 4 degrees. For other substances, even for such as are known to be abnormal, as yet no deviations have been proved with certainty. I have pointed out l. c. that for associating substances a reason may be found tending to make the value of $(\varepsilon_{21})_v$ positive. Yet the scarcity of the exception to the rule that $(\varepsilon_{21})_v$ is negative, has afterwards made me consider the question once more, whether the fact, that the mixture may be considered as a solution in a substance, consisting in the liquid state of molecules of greater complexity than in the gaseous state, is sufficient to account for the fact that:

$$(\varepsilon_{21})_v > 0,$$

and the result of these more accurate considerations is, that in the case of the exception, — water below 4°, — other reasons as yet unknown must be present, and that the fact, that the solvent is an associating substance is not sufficient, to make $(\varepsilon_{21})_v$ from a negative to a positive quantity.

Let us examine the process, for which $(\varepsilon_{21})_v$ represents the decrease of the energy more closely. For a simple substance it may be thought to take place in two operations. In a certain volume we have a phasis, say a liquid phasis, at constant temperature. This phasis fills the whole volume, except a very small part, which is filled with the coexisting vapour phasis. Now we make this gaseous part condense. For a simple substance we may imagine the condensed part to be in exactly the same state as the liquid phasis. The volume now ceases to be totally filled. As second operation we imagine the substance to extend till the whole volume is filled homogeneously. In the first part of the process the system loses a certain positive quantity of energy, namely m times the amount of the internal latent heat of evaporation, if m represents the number of molecules in the vapour phasis. But in the second part the system gains energy. The loss is equal to $m(\varepsilon_2-\varepsilon_1)$ and the gain may be represented by:

$$\left(\frac{\partial\varepsilon}{\partial v_1}\right)\times m\,(v_2-v_1).$$

So the total loss is:

$$m\left\{(\varepsilon_2-\varepsilon_1)-(v_2-v_1)\left(\frac{\partial\varepsilon}{\partial v_1}\right)\right\};$$

which agrees with the general expression for $(\varepsilon_{21})_v$

$$\varepsilon_2-\varepsilon_1-(v_2-v_1)\left(\frac{\partial\varepsilon}{\partial v_1}\right)-(x_2-x_1)\left(\frac{\partial\varepsilon}{\partial x_1}\right)-(y_2-y_1)\left(\frac{\partial\varepsilon}{\partial y_1}\right),$$

if we put the quantities x_1, x_2, y_2 and y_1 equal to zero.

For substances, for which the cohesion may be represented by a molecular pressure, we have:

$$\varepsilon_2-\varepsilon_1=a\left(\frac{1}{v_1}-\frac{1}{v_2}\right)\text{ and }(v_2-v_1)\left(\frac{\partial\varepsilon}{\partial v_1}\right)=a\frac{v_2-v_1}{v_1^2}.$$

From this we deduce, that $(v_2-v_1)\left(\frac{\partial\varepsilon}{\partial v_1}\right)=\frac{v_2}{v_1}(\varepsilon_2-\varepsilon_1)$, so that the gain of energy in the second part is much greater than the loss in the first part.

In the supposition, that the substance associates, $\varepsilon_2-\varepsilon_1$ would be greater than $a\left(\frac{1}{v_1}-\frac{1}{v_2}\right)$, but also in this case this quantity is equal to the internal latent heat. And even for water the amount of this latent heat does not differ so much from the value, which it would have for a normal substance, that this might account for the reversal of the sign of $(\varepsilon_{21})_v$. So the positive amount of $(\varepsilon_{21})_v$ for water below 4° is not to be ascribed to the value of $\varepsilon_2-\varepsilon_1$, but to that of the second part, namely to $\left(\frac{\partial\varepsilon}{\partial v_1}\right)(v_2-v_1)$. As $\left(\frac{\partial\varepsilon}{\partial v_1}\right)$ is equal to $T\left(\frac{\partial p}{\partial T}\right)_v-p$, and $\left(\frac{\partial p}{\partial T}\right)_v$ is in this case negative, $(\varepsilon_{21})_v$ consists of two parts, both of them being positive. Water below 4° is therefore a substance, whose energy decreases, if it extends at constant temperature. Though the molecules of water cohere strongly, it behaves in this respect as if a repulsion between the particles existed. As this repulsion is, no doubt, only apparent, it appears to me that this loss of energy cannot be explained otherwise, than by assuming that in this case extension causes the complexity to increase; and this again makes us suppose that the volume of water molecules increases, when they associate to more complex systems.

For a mixture, either binary or ternary, the process, for which $(\varepsilon_{21})_v$ represents the loss of energy, might be divided into three parts. Between the two operations, mentioned in the case of a simple substance, we have to insert here the mixing of the first phasis with the condensed second phasis; but the change of energy arising from this mixing, may be considered to be small, even compared with $\varepsilon_2-\varepsilon_1$. I have discussed this point rather elaborately in order to strengthen the conviction that $(\varepsilon_{21})_v < 0$ is the general rule.

Yet it remains of course the task of the experiment to inquire into those cases for which this rule fails.

From equation (1) of the preceding communication we find the connection between the sign of $(\varepsilon_{21})_v$ and the way in which the coexistence surface changes its place at increasing temperature. This connection is expressed by the following equation:

$$
\begin{aligned}
&(v_2-v_1)\left\{\frac{\partial^2\psi}{\partial v_1^2}\,dv_1+\frac{\partial^2\psi}{\partial v_1\,\partial x_1}\,dx_1+\frac{\partial^2\psi}{\partial v_1\,\partial y_1}\,dy_1\right\}+\\
&+(x_2-x_1)\left\{\frac{\partial^2\psi}{\partial v_1\,\partial x_1}\,dv_1+\frac{\partial^2\psi}{\partial x_1^2}\,dx_1+\frac{\partial^2\psi}{\partial y_1\,\partial x_1}\,dy_1\right\}+\\
&+(y_2-y_1)\left\{\frac{\partial^2\psi}{\partial v_1\,\partial y_1}\,dv_1+\frac{\partial^2\psi}{\partial x_1\,\partial y_1}\,dx_1+\frac{\partial^2\psi}{\partial y_1^2}\,dy_1\right\}=-(\varepsilon_{21})_v\frac{dT}{T}.
\end{aligned}
$$

If T is kept constant, dx_1, dy_1 and dv_1 represent the projections of an element situated in the coexistence surface. But if dT differs from zero these quantities represent projections of a small line, connecting a point of the second surface (that for the temperature $T+dT$) with a point chosen on the first surface (that for the temperature T). We choose the point on the second surface such that it lies between the nodes, on the right line connecting them, then:

$$
\frac{dv_1}{v_2-v_1}=\frac{dx_1}{x_2-x_1}=\frac{dy_1}{y_2-y_1}=\frac{1}{\lambda}
$$

λ being positive. The first member is positive in consequence of the stability of the phases of the coexistence surface. If $(\varepsilon_{21})_v$ is negative

dT is positive, and the second surface, which is quite enclosed in the first represents the surface at higher temperature. If $(\varepsilon_{21})_v$ should vanish for the chosen point, then that point would not change its place when the temperature is changed. If $(\varepsilon_{21})_v$ should be positive for certain points, then that point would move away from the second phasis when T is increased. As a special case the well known properties of the bordering curve for a simple substance are of course implied in this. If we have a ternary system, one of the components of which is water, at a temperature below 4°, then near the point representing water on the coexistence surface a curve is to be found where the peculiarity to contract when heated ceases to exist.

We can easily form an idea of gradual change of the coexistence surfaces, and the other surfaces under consideration at increasing temperature, if the critical temperature of the mixtures of the three components always changes in the same direction, when the quantity of the second, or that of the third component is augmented. According to our equation of state this would signify, that $\dfrac{d\dfrac{a}{b}}{dx}$ has always the same sign, — and also that $\dfrac{d\dfrac{a}{b}}{dy}$ has always the same sign. If we put $(T_{cr})_y < (T_{cr})_x < (T_{cr})_o$, then the three surfaces, coexistence surface, spinodal surface, and surface for which $\dfrac{dp}{dv} = 0$, will consist of two separated sheets as long as $T < (T_{cr})_y$. If T has a value between $(T_{cr})_y$ and $(T_{cr})_x$, the liquid sheet and the vapour sheet have met for mixtures, which consist chiefly of the third component, and in triangle OXY a curve may be drawn, which indicates the limit between mixtures, which yet admit of coexisting phases at the given value of T, and those, which continue to fill the volume homogeneously, however large or small the pressure may be. This limit connects in this case a point of OY with a point of the hypothenuse. If T has risen above $(T_{cr})_x$, then the limit connects a point of OX with a point of OY, and if T has reached the value $(T_{cr})_o$ the limit has contracted to the point 0. This limit is the projection of the points which the coexistence surface has in common with the tangent cylinder parallel to the v-axis; it represents the mixtures for which the chosen temperature is that of the critical tangent point. In order to find a property of these tangent points, we put the values of dT, dx_1 and dy_1 equal to zero in the differential equation of the coexistence surface. So we get:

$$(v_2-v_1)\frac{\partial^2\psi}{\partial v_1^2}+(x_2-x_1)\frac{\partial^2\psi}{\partial x_1\,\partial v_1}+(y_2-y_1)\frac{\partial^2\psi}{\partial y_1\,\partial v_1}=0\ .\ \ .\ \ .\ \ (\alpha)$$

we may write this equation in the following form:

$$(v_2-v_1)-(x_2-x_1)\left(\frac{dv_1}{dx_1}\right)_{p,y,T}-(y_2-y_1)\left(\frac{dv_1}{dy_1}\right)_{p,x,T}=0.\ \ .\ \ .\ \ (\beta)$$

The equation is reduced to this form, if after division by $\frac{\partial^2\psi}{\partial v_1^2}$ we take into account, that for constant values of y and T:

$$dp=\frac{\partial p}{\partial v}dv+\frac{\partial p}{\partial x}dx$$

or

$$0=\frac{\partial p}{\partial v}\left(\frac{dv}{dx}\right)_p+\frac{\partial p}{\partial x}$$

or

$$0=\frac{\partial^2\psi}{\partial v^2}\left(\frac{dv}{dx}\right)_p+\frac{\partial^2\psi}{\partial x\partial v}.$$

And also that

$$0=\frac{\partial^2\psi}{\partial v^2}\left(\frac{dv}{dy}\right)_p+\frac{\partial^2\psi}{\partial y\partial v}.$$

Written in the form (β) a property of the tangent point phases proves to be (see for a binary system Cont. II, pag. 109) that the mixing of a finite quantity of this phasis with an infinitely small quantity of the coexisting phasis to one homogeneous phasis, T and p being constant, involves decrease of volume, which is infinitely small even compared with the infinitely small quantity of the second phasis.

If we write (α) in the following form:

$$(v_2-v_1)\frac{\partial p}{\partial v_1}+(x_2-x_1)\frac{\partial p}{\partial x_1}+(y_2-y_1)\frac{\partial p}{\partial y_1}=0.\ \ .\ \ .\ \ .\ \ (\gamma)$$

and if we take for the length of the line, connecting the two phases, the positive quantity L, then no difference of pressure will exist between the tangent point phasis and a phasis whose difference from it is given by the quantities dv_1, dx_1 and dy_1, such that:

$$\frac{dv_1}{v_2-v_1}=\frac{dx_1}{x_2-x_1}=\frac{dy_1}{y_2-y_1}=\frac{dl}{L}\ \text{is,}$$

equation (γ) namely may be represented in the following simple shape:

$$L\frac{dp}{dl}=0\ .$$

If we construct a surface of constant pressure through a tangent point phasis, the line joining the nodes will be a tangent to that surface. (See for a binary system: HARTMAN, Proceedings III, p. 66).

We have to distinguish different cases, if the critical temperatures do not agree with the condition we have mentioned. If T_{cr} has a minimum value for a certain mixture, i.e. according to our equation of state, if a value of x and y is to be found between 0 and 1, for which

$$\frac{d\frac{a}{b}}{dx} = 0$$

and

$$\frac{d\frac{a}{b}}{dy} = 0$$

then the connection between liquid- and vapour sheet is established in the middle of the triangle OXY.

If T_{cr} has reached its minimum value, the sheets of the coexistence surface have not yet coincided but the two sheets of the surface, for which $\frac{\partial p}{\partial v} = 0$, have. For the point at which this takes place, the following conditions must be satisfied:

$$\frac{\partial^2 p}{\partial v^2} = 0 \, , \; \frac{\partial^2 p}{\partial x \partial v} = 0 \text{ and } \frac{\partial^2 p}{\partial y \partial v} = 0.$$

We deduce from the circumstance that $\frac{\partial p}{\partial v}$ and $\frac{\partial^2 p}{\partial v^2}$ vanish at the same time, that this coincidence of the two sheets occurs at the ordinary critical circumstances. The two other equations $\frac{\partial^2 p}{\partial x \partial v} = 0$ and $\frac{\partial^2 p}{\partial y \partial v} = 0$ must be equivalent with $\frac{\partial T_{cr}}{\partial x} = 0$ and $\frac{\partial T_{cr}}{\partial y} = 0$. This equivalence follows immediately from our equation of state.

Only at a somewhat higher temperature the sheets of the coexistence surface coincide in one point, — and at the same time in the same point the two sheets of the spinodal surface; but we will not deduce the equations for the moment. At a still higher temperature a closed curve is to be found within the triangle OXY, the points inside which represent mixtures, which do not admit of coexisting phases at this value of T. This closed curve extends if T increases, which extension may take place in different ways, as we have mentioned before.

Before leaving off the discussion of the general properties of these surfaces, we will still make a remark of general character.

Let us imagine a point on the liquid sheet of the surface representing the limit between stable and unstable phases. The equation of that surface may be written as follows:

$$f = \left(\frac{\partial^2 \zeta}{\partial x^2}\right)_p \left(\frac{\partial^2 \zeta}{\partial y^2}\right)_p - \left(\frac{\partial^2 \zeta}{\partial x \partial y}\right)_p^2 = 0 \, .$$

We get into the unstable region when keeping x and y constant, and making v increase. For such a point is in consequence $\frac{\partial f}{\partial v} < 0$. Differentiating f according to p we get:

$$\frac{\partial v}{\partial p}\frac{\partial f}{\partial v} = \frac{\partial^2\zeta}{\partial y^2}\frac{\partial^3\zeta}{\partial p\partial x^2} + \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^3\zeta}{\partial p\partial y^2} - 2\frac{\partial^2\zeta}{\partial x\partial y}\frac{\partial^3\zeta}{\partial p\partial x\partial y};$$

and as $\frac{\partial f}{\partial v}$ is negative, and $\frac{\partial v}{\partial p}$ is also negative, we may write:

$$\frac{\partial^2\zeta}{\partial y^2}\left(\frac{d^2v}{dx^2}\right)_p + \frac{\partial^2\zeta}{\partial x^2}\left(\frac{d^2v}{dy^2}\right)_p - 2\frac{\partial^2\zeta}{\partial x\partial y}\left(\frac{d^2v}{dxdy}\right)_p > 0 \quad . \quad . \quad . \quad (\delta)$$

Here we have to take into account, that $\frac{\partial\zeta}{\partial p} = v$.

If we have such a point in the limiting surface between the unstable and stable phases, one set of variations of v, x and y exists which is of special importance, viz. that for which dp and $d\frac{\partial\zeta}{\partial x}$ are equal to zero; so that, for which:

$$\frac{\partial^2\psi}{\partial v^2}dv + \frac{\partial^2\psi}{\partial x\partial v}dx + \frac{\partial^2\psi}{\partial y\partial v}dy = 0$$

and

$$\frac{\partial^2\zeta}{\partial x^2}dx + \frac{\partial^2\zeta}{\partial x\partial y}dy = 0.$$

Then also

$$\frac{\partial^2\zeta}{\partial x\partial y}dx + \frac{\partial^2\zeta}{\partial y^2}dy = 0$$

$$\text{or } d\frac{\partial\zeta}{\partial y} = 0,$$

as follows from the condition $f = 0$.

For that set of variations we have:

$$\frac{\partial^2\psi}{\partial v^2}dv^2 + \frac{\partial^2\psi}{\partial x^2}dx^2 + \frac{\partial^2\psi}{\partial y^2}dy^2 + 2\frac{\partial^2\psi}{\partial x\partial v}dx\,dv + 2\frac{\partial^2\psi}{\partial y\partial v}dy\,dv + 2\frac{\partial^2\psi}{\partial x\partial y}dx\,dy = 0.$$

For every other set the value of the first member of this equation is positive. This set of variations of v and x is for a binary mixture given by a line on the ψ-surface, which has the tangent plane in a point of the spinodal curve in common with the ψ-surface at least for an element. There it is indicated by an isobar, i. e. by a curve $p = C$. For we have in this case:

$$\frac{\partial^2\psi}{\partial v^2}dv + \frac{\partial^2\psi}{\partial x\partial v}dx = 0.$$

From this and from the equation of the spinodal curve follows

$$\frac{\partial^2\psi}{\partial x\partial v}dv + \frac{\partial^2\psi}{\partial x^2}dx = 0.$$

We will call the direction of the line, for which dv, dx and dy are proportional to the cosines of the angles with the axes, the spinodal direction or direction of unstability.

Applying equation (δ) to the direction of unstability we may write it as follows:

$$\frac{\partial^2\zeta}{\partial y^2}\frac{\left(\frac{d^2v}{dx^2}\right)_p dx^2 + 2\left(\frac{d^2v}{dxdy}\right)_p dx\, dy + \left(\frac{d^2v}{dy^2}\right)_p dy^2}{dx^2} > 0 \,.$$

The signification of the positive sign of the numerator, occurring in the first member, is as follows. If we imagine a surface $p =$ constant to be constructed in the point in question, the curve situated in that surface and following the spinodal direction turns its convex side towards the plane $v = 0$. For a binary system this signifies that the p-curve in a point of the liquid branch of the spinodal line turns its convex side towards the axis of x (see i. a. fig. 8 of our previous communication). From this follows as a special case for a plaitpoint, situated on the liquid sheet both of the coexistence surface and of the spinodal surface, that the curve for which $p =$ constant, and which has moreover the direction determined by the limiting composition of the coexisting phases, is convex towards the side of the xy-plane.

If for our investigation we had chosen a point on the vapour sheet of the spinodal surface, $\frac{\partial f}{\partial v}$ would have been positive, we should have had to substitute the sign $<$ for the sign $>$ and we should have had to read "concave side" instead of "convex side" (see also fig. 8 of our previous communication). In some special cases the point on the spinodal surface may be chosen in such a way, that $\frac{\partial f}{\partial v} = 0$. This may be done: first in those points for which the tangent cylinder normal to the xy plane, touches that plane; — in this case the p-line, which follows the spinodal direction has a point of inflexion; — and in the second place in the point, in which the two sheets of the coexistence surface and also those of the spinodal surface coincide, and for which at the same time $\frac{\partial f}{\partial v}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are equal to zero. Therefore we have also for that point:

$$\left(\frac{d^2v}{dx^2}\right)_p dx^2 + 2\left(\frac{d^2v}{dxdy}\right)_p dx\, dy + \left(\frac{d^2v}{dy^2}\right)_p dy^2 = 0.$$

This signifies for a binary system, that the isobar passing through the point in which the plait splits up into separate parts has there a point of inflexion.

RELATION BETWEEN PRESSURE, COMPOSITION AND TEMPERATURE FOR COEXISTING PHASES OF A TERNARY SYSTEM.

I have deduced the differential equation, which represents the relation between and dp, dx_1, dy_1 and dT for the coexisting phases of a ternary system (Arch. Néerl. Série II, Tome II, pag. 74). This equation has the following form:

$$v_{21}\,dp = W_{21}\frac{dT}{T} + \left\{(x_2-x_1)\frac{\partial^2\zeta}{\partial x^2} + (y_2-y_1)\frac{\partial^2\zeta}{\partial x_1\,\partial y_1}\right\}dx_1 + \\ + \left\{(x_2-x_1)\frac{\partial^2\zeta}{\partial x_1\,\partial y_1} + (y_2-y_1)\frac{\partial^2\zeta}{\partial y_1^2}\right\}dy_1 \quad . \; . \; . \quad \text{(I)}$$

This equation may be derived from the differential equation of the coexistence surface (equation 1 of our previous communication) when we substitute in it the value of dv_1 derived from:

$$-dp = \frac{\partial^2\psi}{\partial v_1^2}\,dv_1 + \frac{\partial^2\psi}{\partial x_1\,\partial v_1}\,dx_1 + \frac{\partial^2\psi}{\partial y_1\,\partial v_1}\,dy_1 - \left(\frac{\partial p}{\partial T}\right)_{v,x,y} d\,T$$

Substituting this we get as factor of dp.

$$-\frac{(v_2-v_1)\dfrac{\partial^2\psi}{\partial v_1^2} + (x_2-x_1)\dfrac{\partial^2\psi}{\partial x_1\,\partial v_1} + (y_2-y_1)\dfrac{\partial^2\psi}{\partial y_1\,\partial v_1}}{\dfrac{\partial^2\psi}{\partial v_1^2}}$$

and as factor of $\frac{dT}{T}$:

$$(\varepsilon_{21})_v + T\left(\frac{\partial p}{\partial T}\right)_v \frac{(v_2-v_1)\dfrac{\partial^2\psi}{\partial v_1^2} + (x_2-x_1)\dfrac{\partial^2\psi}{\partial x_1\,\partial v_1} + (y_2-y_1)\dfrac{\partial^2\psi}{\partial y_1\,\partial v_1}}{\dfrac{\partial^2\psi}{\partial v_1^2}} .$$

If we disregard the sign we may write the factor of dp as follows:

$$(v_2-v_1) - (x_2-x_1)\left(\frac{dv_1}{dx_1}\right)_p - (y_2-y_1)\left(\frac{dv_1}{dy_1}\right)_p .$$

In the same way as we have done for a binary mixture (Cont. II, pag. 109), we may show also for a ternary mixture, that this quantity represents the amount with which the volume decreases per molecule of the second phasis if we mix a quantity of the second phasis with the first and afterwards reduce pressure and temperature to their original value, provided we take the limiting value of this decrease of volume for the case that the quantity of the second phasis is infinitely small compared with that of the first. The sign v_{21} represents this limiting value.

The factor of $\frac{dT}{T}$ may be written as follows:

$$(\varepsilon_{21})_v + T\left(\frac{\partial p}{\partial T}\right)_v v_{21} = (\varepsilon_{21})_v + \left[p + \left(\frac{\partial \varepsilon}{\partial v}\right)\right] v_{21} .$$

We may prove, — in the same way as we have done for a binary mixture (Cont. II, pag. 110) — that this quantity represents also for a ternary mixture and even for a mixture with an arbitrary number of components, the heat given out per molecule of the second phasis in the above mentioned process. We represent this quantity by the sign W_{21}; so we put

$$m\, W_{21} = m\,(\varepsilon_{21})_v + \left(\frac{\partial p}{\partial v}\right)_T m\, v_{21} + pmv_{21} .$$

This formula is simply an application of the principal formula of thermodynamics:

$$dQ = d\varepsilon + pdv.$$

For the application of equation (I) knowledge of the signs of v_{21} and W_{21} is required in the first place.

The rule for the sign of v_{21} is very simple. If the first phasis is represented by a point on the liquid sheet of the coexistence surface, then v_{21} is positive. If on the other hand that point is to be found on the vapour sheet, v_{21} is negative; and, as we saw already on pag. 686, $v_{21} = 0$ if the first phasis is represented by a point on the contour of the coexistence surface, so on the limit between vapour and liquid sheet. (See for a binary system Cont. II, pag. 126).

In this the transition of the sign of v_{21} from positive to negative takes place. It is true that the value of v_{21} is also zero in the plaitpoint, but there the value zero is not a transition from positive to negative; at both sides of the plaitpoint v_{21} has the same sign; positive if the plaitpoint lies on the liquid sheet and vice versa. In the plaitpoint v_{21} may be written:

$$\tfrac{1}{2}\left\{ (x_2 - x_1)^2 \frac{d^2v}{dx^2}_p + 2\,(x_2 - x_1)\,(y_2 - y_1)\left(\frac{d^2v}{dxdy}\right)_p + (y_2 - y_1)^2 \left(\frac{d^2v}{dy^2}\right)_p \right\}$$

and at very little distance from the plaitpoint the value of v_{21} will not perceptibly differ from this value. At the end of our previous chapter we have discussed the sign of this expression.

If we put

$$v_{21} = \frac{(v_2 - v_1)\dfrac{\partial^2\psi}{\partial v_1^2} + (x_2 - x_1)\dfrac{\partial^2\psi}{\partial x_1 \partial v_1} + (y_2 - y_1)\dfrac{\partial^2\psi}{\partial y_1 \partial v_1}}{\dfrac{\partial^2\psi}{\partial v_1^2}} ,$$

and if we represent an element of the right line, connecting the first phasis with the second, by dl, then we get (pag. 686)

$$v_{21} = -L \frac{\frac{dp}{dl}}{\frac{\partial^2 \psi}{\partial v_1{}^2}}.$$

Therefore if the first phasis is a liquid phasis, [1]) p decreases if we move in a straight line towards the coexisting phasis, and vice versa. (See i. a. fig. 2 of our previous communication). If both coexisting phases are situated on the liquid sheet, this applies as well to phasis 1 as to phasis 2. So in this case we have $v_{21} = v_{12} > 0$. And if both coexisting phases are to be found on the vapour sheet, then $v_{12} = v_{21} < 0$.

As to the sign of W_{21}, this quantity is negative, if the first phasis lies on the vapour sheet. It consists in this case of two parts, namely $(\varepsilon_{21})_v$ and $T\left(\frac{\partial p}{\partial T}\right)_v v_{21}$, which are both negative. Also W_{21} is negative, if the first phasis is represented by a point on the contour (critical tangent point). In this case $v_{21} = 0$ and $W_{21} = (\varepsilon_{21})_v$ so it is negative. If the plaitpoint is to be found on the vapour sheet, then for that point $W_{21} = 0$, but at this value again no transition from negative to positive values takes place. For points on the liquid sheet W_{21} may be either negative or positive. W_{21} consists then of two parts, one of which is negative, the other positive. The value of W_{21} may reverse its sign either between the plaitpoint and the tangent point or beyond those points. So we have, if T is chosen so high that it does not cover any more the whole triangle OXY_1 the following loci on the coexistence surface: 1st contour for the points of which v_{21} is equal to zero; 2nd the series of plaitpoints, which lies either exclusively on the liquid sheet or on the vapour sheet. For the moment we will not discuss the case, that this series of plaitpoints passes through the contour. For the plaitpoints v_{21} as well as W_{21} vanishes, but the ratio $\frac{W_{21}}{v_{21}}$ has a finite value, which we will determine presently; 3rd the series of points for which W_{21} is equal to zero. If the temperature rises, the coexistence surface contracts, and the three loci mentioned are displaced, so a point whose projections x_1 and y_1 are given lies successively either on the second or third locus and on the contour. If T rises still

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[1]) For simplicity's sake I call "liquid phasis" each phasis, which is represented by a point on the sheet of the small volumes. In the same way I call vapour phasis each phasis, which is represented by a point on the sheet of the large volumes.

more, the coexistence surface contains no longer a point with these values for x_1 and y_1, and a mixture of this composition has become a permanent gas.

After these preliminary considerations we may begin the discussion of equation (I).

A. We find, if we keep x_1 and y_1 constant:

$$T\left(\frac{dp}{dT}\right)_{x_1, y_1} = \frac{W_{21}}{v_{21}}.$$

The quantity p in this equation represents the limiting value of the pressure, for which a mixture with the given composition x_1 and y_1 is still homogeneous. If the phasis is a gas phasis, an increase of the pressure, T being kept constant, would cause condensation. If it is a liquid phase, decrease of the pressure would make part of it evaporate. This equation, which for the components themselves is nothing but the equation of CLAPEYRON, has for a ternary system the same form as for a binary one, and would also have the same form for a system of still more components. I suppose the shape of such a p,T curve to be known from the knowledge of the properties of a binary system; here I only point out that the peculiar points of this curve are found, if W_{21} or v_{21} or both these quantities are zero. If $v_{21} = 0$, then $\left(\frac{dp}{dT}\right) = \infty$; if $W_{21} = 0$, then p is a maximum. If both values are zero, as is the case in the plaitpoint, we get the value of $\frac{dp}{dT}$ in an indefinite form, yet from the position of the plaitpoint we may conclude the real value of $\left(\frac{dp}{dT}\right)_{Pl}$. If it lies on the vapour sheet, v_{21} and W_{21} are both negative, therefore the value will be positive, and the greater as it gets nearer to the contour. If it lies on the liquid sheet, between the border and the locus for which $W_{21} = 0$, then $\left(\frac{dp}{dT}\right)_{Pl}$ is negative. If we want to bring $\left(\frac{dp}{dT}\right)_{Pl}$ in an analytical form, we have to write:

$$T\left(\frac{dp}{dT}\right)_{Pl} = p + \frac{\partial \varepsilon}{\partial v_T} + \frac{(\varepsilon_{21})_v}{v_{21}}.$$

For

$$(\varepsilon_{21})_v = (\varepsilon_2 - \varepsilon_1) - (v_2 - v_1)\frac{\partial \varepsilon}{\partial v_1} - (x_2 - x_1)\frac{\partial \varepsilon}{\partial x_1} - (y_2 - y_1)\frac{d\varepsilon}{dy_1},$$

we put

$$2\,(\varepsilon_{21})_v = \frac{\partial^2\varepsilon}{\partial v_1{}^2}\,dv_1{}^2 + \frac{\partial^2\varepsilon}{\partial x_1{}^2}\,dx_1{}^2 + \frac{\partial^2\varepsilon}{\partial y_1{}^2}\,dy_1{}^2 +$$

$$+ 2\,\frac{\partial^2\varepsilon}{\partial x_1\partial v_1}\,dx_1\,dv_1 + 2\,\frac{\partial^2\varepsilon}{\partial y_1\partial v_1}\,dy_1\,dv_1 + 2\,\frac{\partial^2\varepsilon}{\partial x_1\partial y_1}\,dx_1\,dy_1$$

and further

$$v_{21} = \left(\frac{d^2v}{dx_1{}^2}\right)dx_1{}^2 + 2\left(\frac{d^2v}{dx_1dy_1}\right)dx_1\,dy_1 + \left(\frac{d^2v}{dy_1{}^2}\right)dy_1{}^2$$

The quantities dv_1, dx_1 and dy_1 are determined by the properties of the plaitpoint as limiting values of $v_2 - v_1$, $x_2 - x_1$ and $y_2 - y_1$. If we divide numerator as well as denominator of $\frac{(\varepsilon_{21})_v}{v_{21}}$ by $dx_1{}^2$, the cause why this quotient is indefinite is removed. I have already pointed out that $(\varepsilon_{21})_v$ is always negative, and this sign is not reversed by division by $dx_1{}^2$. If the plaitpoint lies on the vapour sheet, then the denominator is also negative, as has been shown pag. 691. In that case we have therefore:

$$\left(\frac{dp}{dT}\right)_{Pl} > \left(\frac{\partial p}{\partial T}\right)_v .$$

If the plaitpoint lies on the liquid sheet, then we have:

$$\left(\frac{dp}{dT}\right)_{Pl} < \left(\frac{\partial p}{\partial T}\right)_v .$$

Between the two branches of the p, T curve we might trace the p, T curve of the coincidence pressures. It represents the ordinary pressure curve for the saturated vapour pressure of a simple substance. It does not reach so far upwards as to meet the curve in which it is enclosed; but it ends in the critical point, if we thought the mixture to behave like a simple substance. Only in one case the three curves, which lie one above the other have an element in common, namely if the ternary system admits of two coexisting phases of equal composition. If this might still be the case at the critical temperature, then the transition from the lower branch of the p, T curve to the higher one does not take place fluently, but both branches end in a point, where their tangents have the same direction. According to the law of the corresponding states we have in this case $\frac{T}{p}\frac{dp}{dT} = f = 7$. For a binary system this occurred frequently in the experiments of KUENEN and QUINT.

(To be continued.)

Physiology. — "*On the infundibular region of the brain of Amphioxus lanceolatus*". By Dr. J. Boeke. (Communicated by Prof. T. Place).

In connection with the hypothesis that the part of the saccus vasculosus in Ichthyopsida described as Glandula Infundibuli, is not a gland but a sensory organ reacting on a distinct stimulus, it seemed to be of importance to study Amphioxus lanc. in this direction.

The brain cavity of Amphioxus used to be figured as a pear-shaped enlargement of the central canal, provided only with the hollow prolongation beneath the bottom of the olfactory groove, discovered by Langerhans.

In the first part of his "Studien zur Entwicklungsgeschichte des Kopfes der Kranioten"[1] Kupffer for the first time gave a detailed discription of this cavity and pronounced a distinct homology, based on anatomical facts, between the brain of thə higher vertebrates and the brain cavity of Amphioxus. He came to the conclusion, that the ventricle of the brain of Amphioxus is homologous, not with the fore-brain ("Vorderhirn") of the Craniata in its definitive form, but with the primary anterior part ("Vorhirn") of the two parts in which the brain was seen to be divided in young Acipenser and Petromyzon embryos.

According to Kupffer the brain cavity of Amphioxus (l.c. Fig. 22) is bounded posteriorly by a distinct folding or flexure of the ventral wall of the central canal; in front of this flexure there is to be seen a thickening of the wall, homologised with the tuberculum posterius; in front and ventrally of it the floor of the brain cavity is thinned out very much and forms a funnel-shaped cavity „der gebogen sich nach hinten unter dem Tuberculum posterius vorschiebt" (l.c. Page 75). This funnelshaped cavity Kupffer homologises with the infundibulum of the higher vertebrates. In his somewhat schematic drawings no celldifferentiation whatever is to be seen; in front of the rather deep infundibular cavity and behind it the cells are somewhat spindleshaped and arranged in several layers, at the bottom of the infundibular cavity one layer of very flat cells may be distinguished.

The results of my researches on adult and larval amphioxus did not coincide with Kupffers account mentioned above.

On median sections through large adult amphioxus specimens (fig. 1) just

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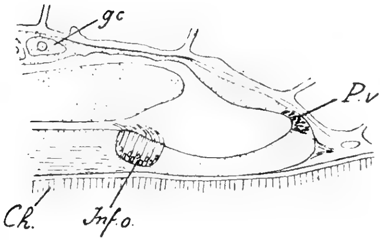
[1] Heft 1. Die Entwicklung des Kopfes von Acipenser sturio an Medianschnitten untersucht. München 1893.

Fig. 1.

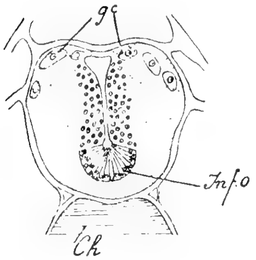


Median section through the head of Amphioxus lanceolatus 4,6 cM. long.

Inf. O. = differentiated epithelium. Pv. = pigmented spot.
Ch. = chorda. g.c. = large ganglion-cells.

in the region of the infundibular cavity of KUPPFFER'S figure, a sharply marked off part of the floor of the brain cavity (Inf. o.) was to be seen, consisting of long cylindrical cells, the whole in the form of a cup. The inner boundary of the brain cavity did not show even the faintest hollowing out, but only went sloping down gradually until somewhat behind the eye-spot. The sharply defined differentiated part of the floor projected somewhat (but only very little) into the brain cavity. It consisted of very long cylindrical cells with a very clear protoplasm and a small round nucleus lying at the base of the cell. On their free surface these cells carried thin but long cilia, bent backwards and forming a bundle of thin fibres that could be traced as far as the beginning of the central canal. Each cell seemed to be provided with one hair.

Fig. 2.



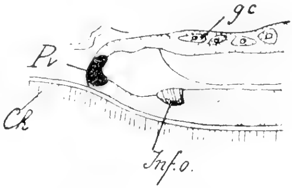
In sections stained with chloride of gold (after Apáthy) in each cell was to be seen a rather thick "Neurofibrille", entering the cell at its base, forming a network round the nucleus and finally ascending through the cell and ending just beneath the place of attachment of the cilia with a small knob, lying close against the intracellular prolongation of the cilia. By means of these „Neurofibrillen" the cells seemed to be connected with other nerve-cells lying behind the differentiated epithelium. However I could not until now state that with a sufficient amount of certainty.

On cross-sections the differentiated part of the floor appeared as sharply defined as on median longitudinal sections. It was then seen to be shaped somewhat like a fan. (fig. 2).

Now the figures drawn by KUPFFER are taken from specimens of amphioxus of 4 resp. 2 cM. in length, and in older individuals KUPFFER himself could not find the infundibular cavity.[1])

But in examining younger Amphioxus of about 1,5 to 4 c.M. in length no trace whatever was to be found of an infundibular cavity; in every section-series the differentiated part of the ventral wall of the brain could be distinguished with the same clearness as in the large specimens I had first studied (fig. 3).

Fig. 3.



Median section through the head of Amphioxus, 1,9 .cM. long.

In young as well as in older forms there is to be seen a large mass of nuclei just behind the differentiated cells, that take a deeper stain than the clear cylindrical cells in front of them. When the protoplasm of the cylindrical cells is shrunken a little, which is not easily to avoid in a material so difficult to fix as Amphioxus, this mass of nuclei seems to project into the brain cavity. In one section-series, in other respects too less satisfactorily fixed than the other specimens I examined, something like this was to be seen. Examined under a high power it appeared to be an artifact, caused by the shrinking of the long cylindrical cells.

Different individuals even of the same age show a great amount of variation in the median sections through the brain. It is therefore possible that in other forms than I was able to study an infundibular cavity as the one described by KUPFFER, may exist.

I can only say, that in no case I could find it (in specimens of about 1,5 c.M. to 4,8 c.M. in length) and therefore must draw the conclusion that where it is found, it can be only of a secondary importance. In all cases however there appeared the sharply defined differentiated part of the epithelium described above.

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[1]) On a drawing after a cross-section Prof. KUPFFER had the kindness of showing to me, the long cylindrical cells were clearly to be seen.

Place and form of this structure seem to me to justify the conclusion, that in these cells is to be seen the homologon of the infundibular organ of the higher vertebrates.

But is there any ground to see in it a sensory organ, and must we not attribute to it a glandular function?

The clear aspect of the cells, even in sections stained with protoplasmic dyes, could lead us to put it down as a gland, secreting a clear mucous substance. But the structure of the cells, the absence of any coagulated secretion on the free surface of the cells, the long cilia and the neurofibrillar structure inside the cells, the absence of any line of separation between secretion products and protoplasm inside the cells, is not favourable to that conclusion and seems to point at a nervous sensory function.

If we then consider it to be a sensory organ reacting on a certain stimulus, the question arises: what is the stimulus. Could it be the variation of pressure inside the brain ventricle, as it may perhaps be the case with the infundibular organ of the higher vertebrates? As yet there is no answer to the question.

By VAN WIJHE the groove of Hatschek is homologised with the hypophysis of the higher vertebrates. If this be true, there is no primary connection between the infundibular organ and the hypophysis, for the chorda extends between them.

In the higher vertebrates too this connection does not exist in all cases. For in some Teleostean larvae I could see the chorda separating the inf. organ and the pituitary body, by extending dorsally of the hypophysis and ventrally of the infundibulum.

Amsterdam, April 1902.

Physiology. — "*The principle of entropy in physiology*". By Dr. J. W. LANGELAAN. 1st part. (Communicated by Prof. T. PLACE).

Presented in the meeting of March 29th 1902.

CLAUSIUS deduced the two principal laws of the mechanical theory of heat from the premise that heat consists in a motion of smallest particles, and that the quantity of heat is a measure of the kinetic energy of this motion [1]).

The first of these two principal laws states the equivalence of work

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[1]) CLAUSIUS, Mechanische Wärmetheorie. 1887 1ster Bd. p. 23.

and heat [1]). It follows directly from the application to this form of motion of the mechanical principle which has been exactly formulated by HELMHOLTZ, according to which the variation of the energy of an isolated system can only vanish [2]).

According to the second principal law the variation of the entropy for an isolated system, which passes through a cyclical process, can only vanish or be positive [3]). The generality of this thesis is founded upon the additional hypothesis, that heat cannot of itself pass from a colder to a warmer body [4]).

If we call ε the energy of the system and η the entropy, then this thesis may be analytically represented by:

$$(\delta\,\eta)_\varepsilon \geqq 0$$

where the subscript letter after a variation indicates the quantity of which the value is not to be varied.

According to CLAUSIUS a cyclical process is defined as a series of changes through which the system passes, and through which it returns to its initial state [5]). This series of changes can be reversible, in which case the sign of equality is to be applied, or quite or partly irreversible in which case the sign of inequality must be used. As perfectly reversible processes do not occur in nature, the sign of equality applies only to an ideal case to which the actual occurring cases more or less approach [6]).

But when for an isolated system which passes through a cyclical process, the variation of the entropy can only vanish or be positive, then such a system is in equilibrium when

$$(\delta\,\eta)_\varepsilon \leqq 0$$

for all possible variations of the state of the system which do not alter its energy [7]).

GIBBS has applied this criterion, which is necessary and sufficient for equilibrium to a thermally, mechanically and chemically isolated system. In this he excluded the action of gravity, of capillarity and of electrical influences [8]). The general condition of equilibrium leads to the following more special conditions of equilibrium [9]).

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[1]) CLAUSIUS, l.c. p. 24.

[2]) HELMHOLTZ, Ueber die Erhaltung der Kraft. Klass. d. exakten Wiss. N°. 1. p. 14.

[3]) CLAUSIUS, l.c. p. 93. 222.

[4]) CLAUSIUS, l.c. p. 81.

[5]) CLAUSIUS, l.c., p. 35.

[6]) PLANCK, Thermodynamik. 1897, p. 79.

[7]) GIBBS, Transact. Conn. Acad. 1874—78, vol. III. p. 109.

[8]) The electrical phenomena in physiological systems require a separate treatment, which rests, however, on the same basis.

[9]) GIBBS l. c., p. 115 e.sq.

1st. The temperature must be constant throughout the whole system. This is the necessary and sufficient condition for the thermal equilibrium.

2nd. The pressure must be constant throughout the whole system. This condition is necessary and sufficient for the mechanical equilibrium.

3rd. The thermodynamic potential must be constant for every independent variable component of the system throughout the whole system. This is the characteristic condition for the chemical equilibrium.

Equilibrium can only exist, when these three conditions are simultaneously satisfied. The applicability of the two first conditions of equilibrium to physiological systems is not open to doubt, the application, however, of the third condition of equilibrium forms a necessary hypothesis for the application of the more special results of thermodynamics to physiological systems.

In order to apply the results of thermodynamics, we imagine a reflex-apparatus, which is thermally, mechanically and chemically isolated. The reflex apparatus itself is thought as consisting of a (generally peripherically situated) receptive-apparatus going over in or connected with an afferent nerve, which is connected in the central organ with the cellbody of an efferent nerve. This efferent nerve carries as endorgan a transformer.

This transformer can be a muscle, in which case it chiefly transforms potential energy of the chemical system into mechanical work and heat. The nature of the transformer and the external circumstances determine how the distribution in the two forms of energy will take place. If the transformer is a gland, a chemical system is transformed into another and into heat. Here it is chiefly the nature of the transformer which determines the final result. If we apply the name of transformer to the muscle and to the gland, we do so because in these organs the transformation is most preponderant. In reality the whole reflex-apparatus transforms energy, but the part which the before-mentioned organs take in it is so prevailing, that no appreciable fault is made when we neglect the quantities of energy transformed in the rest of the reflex-apparatus. If the receptive-apparatus is a large senseorgan this neglection is no longer admissible.

In this reflex-apparatus we imagine two chemical systems in equilibrium. According to Hering [1]) one system may be called the system of the products of asssimilation, the second that of the products

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[1]) Hering, Lotos, Neue Folge, 1889 Bd. IX p. 35.

of dissimilation. As in the circumstances, under which we consider the physiological systems, a slight exchange of matter is always taking place, a part of the first system passes continually into the second. In this case we have therefore to deal with a stationary state founded on a chemical equilibrium. The third condition of equilibrium is, therefore, not exactly satisfied, and it is only by approximation, that we deal with these systems as if they were in a state of equilibrium.

In the first place I shall discuss the results which may be derived from the principle of equilibrium for the reflex-apparatus; and in the second place the law according to which the passage of the products of assimilation to the products of dissimilation takes place.

When discussing states of equilibrium we have in the first place to know, what the nature and the causes of the equilibrium are. As to the nature of the equilibrium in physiological systems, we must consider it as stable, because all physiological experiments have proved, that a stimulus of a finite value is required to bring about a finite effect. No infinitely small cause, therefore, is capable of bringing about a finite change of state in these systems [1]).

As cause of the equilibrium we must regard in the first place the active tendencies of the system. If this were the only cause of the equilibrium, external influences or changes in the initial state of the system, infinitesimal in amount, should be sufficient to produce changes in this system [2]). The method of the just perceptible differences which is also physiologically applicable, shows clearly that this is not the case. Beside active tendencies also passive resistances which prevent changes must exist in physiological systems.

When the reflex-apparatus is subjected to the action of a stimulus of short duration, it undergoes a series of changes, through which it finally returns to its initial state. We may therefore say that a cyclical process takes place. As far as I know, there are no measurements about the duration of this cyclical metabolic process for a simple reflex-apparatus. For the intact organism we may learn it approximately from ergographical curves. If the cyclical metabolic process shall be complete, then for not too short a period every successive elevation must be equal to the preceding and must not differ considerably from the very first elevation. It appears from the ergograms communicated by LEHMANN [3]), that this is the case for a

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[1]) GIBBS, l. c. p. 112.

[2]) GIBBS, l. c. p. 111.

[3]) LEHMANN. Körperlichen Ausserungen psychischer Zustände. 1901, plate X and XIX.

tact of 6 a minute; for a tact of 10 a minute the successive elevations decrease distinctly in height. The duration of the cycle of metabolism must therefore be longer than 6 seconds, while 10 seconds is sufficient for it under the given circumstances. When the cycle is complete, the reflex apparatus has returned to its previous state and the relation between the masses, the energy and the entropy of the system of the assimilation products to those of the system of the dissimilation products has returned to its former constant value.

If we now drop our premise of thermal, mechanical and chemical isolation of the reflex-apparatus, which state is never realized in nature, the nature and the state of the surrounding medium, in regard to which the isolation is imperfect, are to be taken into account. For the higher animals the blood may be considered as the surrounding medium, whereas for the lower animals the haemolymph fluid takes this place. For plants where a surrounding fluid medium fails and the cells are for the greater part separated by walls of cellulose, the degree of isolation of every tissue-element is very complete. Both, with regard to its chemical nature and to its physical condition, the surrounding medium has for all animals the property to be variable only within very narrow limits. For approximation I shall neglect these variations. Further we introduce the passive resistances of the physiological systems in our consideration. It is a general characteristic of these passive resistances that they prevent a certain kind of motion or change, in whatever way the initial state of the system may be modified and to whatever external agencies of force, and heat, it may be subjected. Possibly these impediments exist only within certain limits, but always within such limits, which allow finite variations in the values of all the quantities which express the initial state of the system or the mechanical or thermal influences acting on it, without producing the change in question [1]).

As a proof of the existence of these passive resistances in physiological systems we can adduce the very great number of experiments on the influence of water upon the metabolism in plants. Perfectly dry seeds, mosses and lichens [2]), which do not respire, can be exposed to variations of temperature of more than 100° C., without producing any modification in these systems, either temporal or lasting, notwithstanding these tissues in turgescent state are not distinguished by a high power of resistance against variations of temperature [3]). We know of these passive resistances that they

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[1]) Gibbs, l. c., p. 111.

[2]) Pfeffer. Pflanzenphysiologie 1ster Bd. 1897. p. 576.

[3]) Pfeffer. l. c. 2ter Bd. 1901. p. 288.

are characteristic of chemical systems and that they are of general occurrence in nature. Later investigations suggest that an area, sharply defined by pressure and temperature, may exist for every substance or system of substances, within which it can be in the state of false equilibrium in consequence of these passive resistances [1]). Though these passive resistances probably decrease rapidly on the limits, yet there is no reason to suppose, that they should decrease to zero beyond this area.

If we now stimulate a reflex-apparatus as it occurs in nature, the stimulus increase must in the first place reach a definite finite value, large enough to overcome the passive resistances in the system, before this stimulus is capable to bring about a change in this system. In consequence of the imperfect isolation, the increase of the stimulus per unit of time must in the second place exceed a definite finite value in order to bring about a measurable change in the system. If we have to deal with a thermal effect, then by conduction, heat will pass from the reflex-apparatus to the surrounding medium and be lost as measurable heat in the reflex-apparatus. The same holds good for the pressure, as only semipermeable walls occur in nature. In physiology the phenomenon that the increment of the stimulus, both as regards its absolute value as its velocity, must exceed a finite value in order to cause a measurable effect, is known under the name of the "threshold".

The elements, which compose the threshold value, belong therefore to two groups, such as arise from passive resistances in the chemical system, and such as originate in the imperfect isolation of the reflex-apparatus from the surrounding medium.

For the higher plants, where this latter group of elements fails in the threshold value, the latter is remarkably small as compared to the whole interval over which the stimulus can be extended. From the experiments of geotropic stimulation of CZAPEK [2]), where a centrifugal force expressed in grams is considered as measure of the stimulus, appears that while the whole interval of stimulation extends over 40 grams, the threshold value lies near 1 milligram.

The refractory period is due to the existence of a threshold in a system which is in a modified physical and chemical state in consequence of an immediately preceding stimulus. Since this state is partly determined by the stimulus which preceded, the refractory period is also partly determined by this stimulus in its characteristic

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[1]) VAN 'T HOFF, Vorlesungen. 1898. Heft I. p. 206.

[2]) CZAPEK, Jahrb. f. Wiss. Botanik. 1898. Bd. 32. p. 193.

quantities. The latter fact appears very clearly from the experiments of ZWAARDEMAKER and LANS [1]).

The existence of a threshold has as a second necessary consequence the existence of a latent period, for as long as the increase of the stimulus takes place with finite velocity, a definite interval of time will always elapse, before the stimulus has reached the value requisite to exceed the threshold value.

When in a system with passive resistances a cyclical process takes place, this process is quite or partly irreversible. In this case the variation of the entropy of the system is greater than zero and the value of this variation is a measure of the irreversibility [2]).

If we apply this to the isolated reflex-apparatus, the cyclical metabolic process must be quite or partly irreversible in consequence of the passive resistances in the chemical system. This is in perfect concordance with what we know of metabolism. If the reflex-apparatus has, therefore, passed through a cyclical process, the entropy of the system must have increased. As soon as the thermal equilibrium is reestablished in the reflex-apparatus, the temperature of the system has increased. Probably a similar consideration applies also to the pressure, so that at the end of the process, when the mechanical equilibrium has been reestablished in the reflex-apparatus, the pressure in the system will have risen.

When we inquire what influence this will have on the two chemical systems which we suppose as being in equilibrium, we find that in general increasing temperature favours the system which is formed under absorption of heat [3]). Now it is beyond doubt that in the formation of the system of the products of assimilation the endothermic reactions are preponderant. Increase of pressure favours the system with the smallest volume [4]), and it is very probable that this will be again the system of the products of assimilation.

If therefore an isolated reflex-apparatus passes through a cyclical metobolic process, this is not complete, while the change that sets in is such, that the equilibrium is displaced to the side of the system of the products of assimilation. If on such a changed system a second stimulus acts of the same extent, then the displacement of the equilibrium and together with it the external measurable effect must be greater.

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[1]) ZWAARDEMAKER, Separatabdruck. Sitz.ber. IX. Intern. Opth. Congr. Utrecht 1899.

[2]) CLAUSIUS l.c. p. 223.

[3]) VAN 'T HOFF. l. c. p. 158.

[4]) VAN 'T HOFF. l. c. p. 156.

This general property of the reflex-apparatus is called „augmentation''. The myothermal investigations of FICK [1]) prove this very clearly for the production of heat in the muscle, and the staircase phenomenon proves the same for the mechanical effect. If a stimulus acts on a reflex-apparatus great enough to overcome the passive resistances, but whose increase per unit of time is too small to bring about a perceptible effect, this stimulus will produce a change in the reflex-apparatus, every successive stimulus will do the same in a more and more changed reflex-apparatus. At last the stimulus, which at first seemed too weak, will cause an external measurable effect. This phenomenon is called the "summation of stimulus". It would be more accurate to speak of the summation of the effect of stimulation.

If we again drop our premise of perfect isolation, no equilibrium can exist between the surrounding medium and the reflex-apparatus, unless the latter has perfectly returned to its initial state, as far as temperature and pressure are concerned. If this has happened, the chemical system returns also to its original state. In this case the cycle is complete. This latter part of the cycle is of course irreversible and proceeds at first very fast [2]), at the end very slowly. For the muscle-reflex-apparatus upon which the tonus depends, an interval lying between 3 and 5 minutes seems requisite for that part of the process [3]). For plants, with their more perfect isolation, this interval can exceed 30 minutes [4]).

Following the example set by MARSHALL HALL, GRAINGER, SHERRINGTON [5]) and many others I have considered the reflex-apparatus as a unit, because the normal course of the process of assimilation is bound to the integrity of the reflex-apparatus. If we interrupt the reflex-arc anywhere a lasting change takes place at once. As a proof of this we can adduce, that the tonicity of a muscle changes the moment the reflex-arc on which the tonus rests, is interrupted [6]). In the second place the atrophy which the muscle and the gland show some time after their motor or secretory nerves has been cut through. As soon as the reflex-arc is interrupted, the reflex-apparatus passes still through a cycle as regards its physical state, but the chemical cycle is no longer complete. This incompleteness of

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[1]) FICK. Myothermische Untersuchungen. 1889 p. 120, 121.

[2]) WARD. Archiv. f. Physiologie. 1880. p. 87.

[3]) LANGELAAN. Proceedings Royal Acad. Amsterdam May 25th 1901. p. 50.

[4]) CZAPEK. l.c. p. 207.

[5]) SHERRINGTON. On the spinal animal. 1898 Repr. Thomas Yates Lab. Report p. 27.

[6]) LANGELAAN. Archiv. f. Physiologie Jahrgang 1901. p. 119.

the chemical cycle may be slight at first and by approximation all the considerations remain of force for this short period, also for a part of a reflex-apparatus, but after a longer period atrophy necessarily sets in. As soon, however, as the chemical cycle is no longer complete, the reflex-apparatus will be in a changed state after each stimulation. This interrupts the regularity of the phenomena.

Some botanists have tried to introduce into botanical physiology the notion of a reflex-apparatus built up by a concatenation of cells. There is here, however, no question of a differentiated reflex-apparatus, and a great many investigations have proved, that the same cell which receives the stimulus serves at the same time as transformer. Evidently the whole cycle of metabolism takes place in a single cell. It is therefore more accurate to consider the cell as a physiological unit for plants and probably also for the lowest animals.

In a second paper I hope to discuss the law according to which the transition of the system of the products of assimilation to the products of dissimilation takes place and the results which ensue from this for the reflex-apparatus.

Physiology. — "*The principle of entropy in physiology.*" By Dr. J. W. LANGELAAN. 2nd part (Communicated by Prof. T. PLACE).

In this paper I shall try to discuss the law according to which the system of the products of assimilation passes over into that of the products of dissimilation. Our knowledge of the nature of these two systems is very limited. We scarcely know, which are the principal components originally present and which are the later appearing. The equations of condition existing between these components, are also only partly known to us as regards their form and their number. The number of independent variable components is therefore not known to us. The only thing we know with certainty is that we have to deal with condensed systems.

When the application of thermodynamics in its simplest form is correct, the same law will hold true for the transition of the system of the products of assimilation to that of the products of dissimilation as that according to which two condensed systems pass into one another.

Supported by a considerable experimental material VAN 'T HOFF has applied here the following formula [1]:

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[1] VAN 'T HOFF. Vorlesungen Iles Heft 1898 p. 175.

$$dE = q\frac{dT}{T}$$

In this formula T represents the absolute temperature, E the quantity of energy, transformed in consequence of the transition of the two systems into each other, and q a quantity of heat. If this transition is accompanied by chemical changes, then q is the quantity of heat which can be developed in maximo, per unity of mass, by this transition.

As far as the present investigations reach this formula seems independent of the number of the component substances and independent of the configuration of the two systems which pass into one another. The form into which the chemical energy is transformed, is determined by the nature of the transformer.

VAN 'T HOFF himself considers this formula as the expression ot the second principal law of the mechanical theory of heat [1]). From this point of view its generality is clear, but it is no less clear that we have here an application of the law of entropy far beyond the limits for which the deduction of CLAUSIUS holds. In its application this formula has been chiefly restricted to approximately reversible processes in systems characterized by a point of transition. Therefore the limits within which this formula can be applied are not to be determined for the present.

If the transitions are quite or partly irreversible, the sign of inequality takes the place of the sign of equality. This makes the formula assume its more general form:

$$dE \leqq q\frac{dT}{T}$$

If this transition takes place in consequence of variations of pressure, the same relation exists approximately between the quantity of transformed energy and the variation of pressure for the interval over which this change can take place.

If on a receptive-organ of a reflex-apparatus a continuous stimulus acts, this stimulus will cause a change in the variables which determine the state of the chemical system. In consequence of these variations a displacement of equilibrium takes place. Equilibrium in the reflex-apparatus is only possible, when throughout the chemical system the pressure, the temperature and the thermodynamic potential for each of the independent variable components is uniform. Under influence, therefore, of the strain applied on a part of the system

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[1]) VAN 'T HOFF. l. c. pag. 174.

the whole system must pass into a changed state of equilibrium. If the other circumstances remain unchanged, the new state of equilibrium will be determined by the stimulus measured in physical units [1]). The real relation between the stimulus measured in physical measure and the change in the system not being known, we must for the present represent it by an implicite function. The same holds good for the quantity of transformed energy, as we generally measure only part of it in our experiments.

If we now apply the before-mentioned formula to the transition of the system of the products of assimilation to that of the products of dissimilation, we find for a reflex-apparatus:

$$\triangle F(E) \leqq K \frac{\triangle \varphi (R)}{\varphi (R)} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (I)$$

In this formula the quantity of transformed energy is represented by $F(E)$ and the value of the stimulus in physical units by R. For a reflex-apparatus with passive resistances, as it only occurs in nature, only the sign of inequality has sense.

As in our experiments of stimulation never one single receptive organ is stimulated, but always a great number, we must summate the above expression between 0 and n, n being the number of stimulated elements.

$$\Sigma \triangle F(E) \leqq K \Sigma \frac{\triangle \varphi (R)}{\varphi R}.$$

If we do not understand by $\triangle \varphi (R)$ the change as it really occurs in a certain receptive-organ, but the average change, we may write:

$$\Sigma \triangle \varphi (R) = n \triangle \varphi (R)$$

As the number of stimulated receptive-organs is always a function of the extent of the stimulated area, we are led to introduce, as HELMHOLTZ did [2]), this quantity in the formula of stimulation. The representation of the effect of stimulation as a sum has no sense, as it is shown by the researches of RAMÓN Y CAJAL [3]) and many others, how great the number of connections is between the afferent and the efferent nerves. In the second place this decomposition has no sense as by its construction the transformer is to be considered as a whole. By introducing a new proportionality constant the above formula goes over in:

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[1]) The way by which this changed state is reached, however, is not determined by the stimulus, and this enables us to determine or modify by external circumstances the distribution of the transformed energy over different forms.

[2]) HELMHOLTZ. Handb. physiol. Optik. 1896. p. 409.

[3]) RAMÓN Y CAJAL. Nuevo Concepto de la Histologia de los Centros nerviosos 1893.

$$\Delta F(E) \leqq C \frac{\Delta \varphi(R)}{\varphi(R)} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (Ia)$$

In this formula $F(E)$ represents the total quantity of transformed energy and R again the stimulus in physical units. The transition of the system of the products of assimilation to that of the products of dissimilation does not depend on the intactness of the reflex-apparatus, but takes place as long as the tissue is alive. The given formula is therefore the representation of a general law of metabolism.

The special forms under which this formula has been brought in physiology and experimental psychology, may be all derived from the more general formula by means of the same method. The function which occurs in the right-hand side of the equation must be very complicated and it is therefore improbable, that we shall soon be able to put an explicite function in its place. This function has been generally developed into a series by means of the theorem of MACLAURIN. The term at which the series is interrupted in connection with the value of the coefficients determine the degree of approximation. The formula obtained in this way is naturally a formula of interpolation.

The implicite function occurring in the left-hand side of the equation, cannot be treated in this way. This function is always decomposed in a sum of several terms agreeing with the number of forms into which the chemical energy of the system is transformed.

If we apply this to formula (Ia), it goes over in:

$$\Delta E_1 + \Delta E_2 + \Delta E_3 + \ldots \leqq C \frac{c_1 \Delta R + 2c_2 R \Delta R + 3c_3 R^2 \Delta R + \ldots + \Sigma(c\varepsilon)}{c_0 + c_1 R + c_2 R^2 + c_3 R^3 + \ldots .}$$

In this formula the quantities which occur in $\Sigma(c\varepsilon)$ are very small and have the property to approach to zero in the limit at the same time with ΔR. According to the first principal law $\Sigma(\Delta E)$ will always be equal to the total quantity of transformed energy.

When we, as a first approximation, break off the series in which $\varphi(R)$ is developed at the first powers of R and put at the same time $c_0 = 0$, the general interpolation formula assumes the following form:

$$\Delta E_1 + \Delta E_2 + \Delta E_3 + \ldots . \leqq C \, \frac{\Delta R + \varepsilon}{R}$$

The introduction of the special value $c_0 = 0$ means physiologically that we neglect the existence of the threshold value. Under these circumstances the simplified interpolation formula will only apply to a perfectly isolated reflex-apparatus without passive resistances. Under these circumstances, however, the sign of inequality has no sense. If, moreover, we neglect for the small interval for which the formula holds, the variations of the quantities E_2, E_3 etc. with regard to the variation of E_1 and also neglect the very small quantity ε, the formula goes over in the very special interpolation formula which we know as the expression of the law of FECHNER:

$$\Delta E = C \frac{\Delta R}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

When applying this formula to physiological problems we generally do not measure the energy quantity E itself, but another quantity which (at least by approximation) is connected with that quantity by a linear relation.

In order to make this formula which applies only for a very small increase of the stimulus, available for a greater interval, FECHNER has summated this expression over a larger interval. When executing this summation he neglected the small quantity ε and replaced in this way the summation by an integration. This gives formula (II) the following form:

$$E = C \, lgn \, R + C' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III)}$$

If this summation shall be correct, the same thing must take place from moment to moment. For the left-hand side of the equation this involves the special law, according to which the distribution of the transformed energy over its different forms is independent of the value of the increment of the stimulus. As the state of the system, the other circumstances remaining unchanged, is determined by the value of this increment, we can express this law also as follows: the distribution of the transformed energy over its different forms is independent of the state of the transformer. It is clear that this simple law of distribution, which we might call the law of the constant proportions, can only be correct by approximation. By means of ergographical investigations LEHMANN [1]) has tried to deduce this same law, for the contracting muscle, from the first principal law. This however, is only possible by means of the second principal law, which is itself a law of distribution. The summation of the energy quantities is beyond doubt.

For the right-hand side of the equation this summation involves that C be constant throughout the interval over which the summation is extended. This constant contains the constant K of formule (I). The value of this constant is determined by the nature of the transformer. Accordingly the summation involves this law: the

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[1]) LEHMANN, Körperl. Aüss. psych. Zustände, 1901. p. 191.

nature of the transformer is not changed in consequence of the stimulation. The elastic after-phenomena in muscles show most clearly, that also this law which we might call the law of the invariability of the transformer, holds only by approximation. Led by my own experiments I hope to demonstrate, that besides these small continuous variations the constant K can also show discontinuities. In consequence of these two laws of approximation involved in the summation, the interpolation formula of **Fechner** is only applicable for a limited interval, within which no discontinuities occur.

The method used by **Fechner** for the determination of the constant of integration which occurs in formula (III), is erroneous for physiological systems, as it is based on the existence of a threshold value, while the formula, in virtue of its deduction, premises an ideal reflex-apparatus. In accordance with its sense this formula of interpolation has proved to be of general application in physiology. To confirm this I shall only refer to the investigations of **Dewar** and **M'Kendrick** [1]), of **Waller** [2]), of **Winkler** and **Van Wayenburg** [3]), of **Langelaan** [4]) and so many others. Also botanists have found it applicable, as appears from the experiments of **Pfeffer** [5]), **Massart** [6]), **Van Rysselberghe** [7]).

The apparent uniformity of the natural phenomena to which this formula leads, is not founded on the nature of these phenomena, but is due to the character of interpolation formula of this form.

Experimental psychology which has discovered this interpolation formula through the investigations of **Weber** and **Fechner**, substituted the value of the sensation for E in formula (III). If we consider the physiological law as being given first and the conformity with the psychological law as not accidental, then we may say that experimental psychology extended the simple law of distribution also to that quantity, whose variation appears to our consciousness as a change in our sensation.

Looked upon from this point of view, **Fechner** has come in contradiction with his theory, which explained the logarithmical relation from the form of the distribution law. For the case that this psy-

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[1]) Trans. Roy. Soc. Edinb. 1876, vol. 27. p. 141.

[2]) **Brain**, 1895, vol. 18. p. 200.

[3]) **Van Wayenburg**, Dissertatie. 1897.

[4]) Archiv f. Physiol. 1901. p. 106.

[5]) **Pfeffer**, Unters. Bot. Inst. Tübingen, 1884, 1ter Bd. 3tes Heft, p. 395.

[6]) **Massart**, Bull. Acad. roy. Belgique, 1888. 3me Série T. 16, p. 590.

[7]) **Van Rysselberghe**, Extrait Mém. couronnés. Acad. R. de Belgique, T. LVIII.

chophysiological law of distribution might be really represented by a logarithmic function, we should be led to formulae of interpolation as have been proposed by PLATEAU, BRENTANO, FULLERTON and CATTELL.

If we want to make the formula of interpolation applicable to a greater interval and in this way account to a certain extent for the deviations from the law of FECHNER, we must break off the series in which $\varphi(R)$ was developed at a higher power of R. If we break off this series at the second powers of R and introduce the approximations already mentioned, a formula is obtained which accounts for the deviations of the law of FECHNER at the under limit. This formula is identical with the formula, which VON HELMHOLTZ [1]) has found for it. If we break off the series at the third powers of R, a formula is formed which is analagous to that of VON HELMHOLTZ [2]) and which, like the former, partly accounts for the deviations at the upper limit.

As yet we have only considered states of equilibrium. In this way we obtained the simplification that our considerations were independent of the time as a factor.

The laws which determine the transition from one state of equilibrium to another, are very complicated, because the new state of equilibrium is not reached at once, but after some oscillations round this new state. As the physiological systems show passive resistances the laws of the oscillating systems in a resisting medium will apply here. HERING [3]) has come to the same conclusion, and has rightly applied this idea to the physiology of the heart. These oscillations are of general occurrence and the clinic furnishes several examples of it.

In a third communication I hope to elucidate some of these points of view led by my own experiments.

Conclusions.

1. For the higher animals the reflex-arc is the morphological unit within which the cycle of metabolism takes place. For plants and probably also for the lowest animals this is the cell.

2. Between the system of the products of assimilation and that of the products of dissimilation exists a stable equilibrium.

3. The cyclical metabolic process is partly irreversible.

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[1]) HELMHOLTZ l. c. p. 411 the reduced formula of form 2 f.

[2]) HELMHOLTZ l. c. p. 413 form. 3.

[3]) HERING. Lotos. Neue Folge IX Bd. 1889, p. 69.

4. Threshold value and refractory period are complex quantities which originate in the imperfect isolation of the reflex-arc from the surrounding medium and in the passive resistances of the chemical system.

5· Augmentation and summation of the effect of stimulation are the consequence of not compensated changes (in the sense of CLAUSIUS).

6. The form which expresses the law of WEBER-FECHNER is a formula of interpolation deduced from the principle of entropy.

Dynamics. — H. A. LORENTZ. "*Some considerations on the principles of dynamics, in connexion with* HERTZ'S "*Prinzipien der Mechanik*".

In his last work HERTZ has founded the whole science of dynamics on a single fundamental principle, which by the simplicity of its form recalls NEWTON's first law of motion, being expressed in the words that a material system moves with constant velocity in a path of least curvature ("geradeste Bahn"). By means of the hypothesis that in many cases the bodies whose motion is studied are connected to an invisible material system, moving with them, and by the aid of a terminology akin to that of more-dimensional geometry, HERTZ was able to show that all natural motions that may be described by the rules of dynamics in their usual form, may be made to fall under his law.

From a physical point of view it is of the utmost interest to examine in how far the hypothesis of a hidden system, connected with the visible and tangible bodies, leads to a clear and satisfactory view of natural phenomena, a question which demands scrupulous examination and on which physicists may in many cases disagree. On the contrary, it seems hardly possible to doubt the great advantage in conciseness and clearness of expression that is gained by the mathematical form HERTZ has chosen for his statements. I have therefore thought it advisable to consider in how far these advantages still exist, if, leaving aside the hypothesis of hidden motions, and without departing from the general use in dynamical investigations, one considers the motion of a system as governed by "forces" in the usual sense of the word.

In what follows there is much that may also be found in the book of HERTZ. This seemed necessary in order to present the subject in a connected form.

As to the authors who have, before HERTZ, published similar investigations, I need only mention BELTRAMI, LIPSCHITZ and DARBOUX.

§ 1. We shall consider a system, consisting of n material points and we shall determine its position by the rectangular coordinates of all these points. The coordinates of the first point will be represented by x_1, x_2, x_3, those of the second point by x_4, x_5, x_6, etc., and any one of the coordinates by x_ν, the index ν varying from 1 to $3n$. We shall write m for the mass of the system, and m_ν for that of the point to which the index ν belongs. This implies that any one of these quantities m_ν has the same meaning as two other ones.

§ 2. We shall determine an infinitely small *displacement* of the system by the increments dx_ν (or, as we shall write in some cases, δx_ν) of the several rectangular coordinates. We shall ascribe to such a displacement a definite *length*, to be denoted by ds, and defined as the positive value that satisfies the equation

$$m\, d\, s^2 = \sum_1^{3n}{}_\nu\, m_\nu\, dx_\nu^2 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

The displacement of the system may be considered as the *complex* of the displacements of the individual points, and the rectangular components of these last displacements, i. e. the differentials dx_ν, may be called the *elements* of the displacement of the system. We shall also call ds the *distance* between the positions of the system before and after the infinitely small displacement.

§ 3. Let P, P', P'' be three positions, infinitely near each other, ds, ds', ds'' the lengths of the displacements $P \rightarrow P'$, $P \rightarrow P''$, $P' \rightarrow P''$. It may be shown by (1) that any of these lengths can never be greater than the sum of the other two, so that we may construct a triangle, having ds, ds', ds'' for its sides. By *the angle between the displacements* $P \rightarrow P'$ and $P \rightarrow P''$ we shall understand the angle between the sides ds and ds' of this triangle. If we denote it by (s, s'), the elements of the first displacement by dx_ν and those of the second b dx'_ν, we shall have

$$m\, ds\, ds' \cos (s, s') = \sum_1^{3n}{}_\nu\, m_\nu\, dx_\nu\, dx_\nu' \quad . \; . \; . \; . \; . \; . \quad (2)$$

In special cases the angles of the triangle may be on a straight line, so that $(s, s') = 0$ or 180°.

The above may be extended to two displacements, having the elements dx_ν and dx'_ν, the lengths ds and ds', whose initial positions do not coincide. In this case, just like in the former one, we calculate the angle between the displacements by the formula (2).

§ 4. If we have to do with a set of vector-quantities of one kind or another — but all of the same kind — each belonging to one of the material points, we shall call the complex of all these quantities a *vector in the system* or simply a *vector*. The rectangular components of the several vector quantities will be called the *elements of the vector in the system.*

From this it follows that an infinitely small displacement is itself a vector in the system, and that any vector may be geometrically represented on an infinitely small scale by such a displacement. The *length* or *value of a vector* and the *angle between two vectors* may be defined in a similar way as the corresponding quantities in the case of infinitely small displacements.

We shall often denote a vector by the letter $\mathfrak{S}$, its value by S, its elements by X_ν. Accents or other suffixes will serve to distinguish one vector from another. Other Gothic letters for vectors, and the corresponding Latin ones for their values will likewise be used. If an infinitely small displacement is to be regarded as a vector, we shall denote it by $d\mathfrak{s}$ or $\delta\mathfrak{s}$.

The value S of a vector, considered in most cases as a positive quantity, is given by the formula

$$m\,S^2 = \sum_{1}^{3n}{}_{\nu}\, m_\nu\, X_\nu^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the angle $(\mathfrak{S}, \mathfrak{S}')$ between two vectors by

$$m\,S\,S' \cos(\mathfrak{S}, \mathfrak{S}') = \sum_{1}^{3n}{}_{\nu}\, m_\nu\, X_\nu\, X_\nu' . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If $(\mathfrak{S}, \mathfrak{S}') = 0$, the vectors are said to have the same direction. For this it is necessary and sufficient that the ratios between the elements X_ν should be the same as those between the elements X_ν'. The ratios between the elements and the length will then likewise be the same for the two vectors. It is natural to call these last ratios the *direction-constants.* If these are α_ν, so that

$$\alpha_\nu = \frac{X_\nu}{S} ,$$

the equation (4) becomes

$$m \cos(\mathfrak{S}, \mathfrak{S}') = \sum_{1}^{3n}{}_{\nu}\, m_\nu\, \alpha_\nu\, \alpha_\nu' . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The angle between two vectors depends therefore on their direction-constants, or, as we may say, on their directions.

The direction-constants of a vector may not be chosen independently from each another, the relation

$$\sum_{1}^{3n}{}_{\nu}\; m_\nu \, \alpha_\nu^2 = m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

having always to be satisfied.

Two vectors are said to be perpendicular to each other, if $(\mathfrak{S}, \mathfrak{S}') = 90°$.

If the angle is 180°, the vectors have opposite directions. This will sometimes be expressed by saying that the two have the same direction-constants, but that one value is positive and the other negative.

§ 5. Multiplying a vector $\mathfrak{S}$ by a positive or negative number k means, that each element is multiplied by k, and that the products are taken as the elements of a new vector, which we shall indicate by $k\mathfrak{S}$.

Two vectors $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are said to be compounded with each other, if any two corresponding elements are added algebraically, and the sums thus obtained are taken as the elements of a new vector. This is called the *resultant* or the *sum of the two vectors*, and represented by $\mathfrak{S}_1 + \mathfrak{S}_2$; it may again be decomposed into the *components* $\mathfrak{S}_1$ and $\mathfrak{S}_2$.

There are a number of theorems, closely corresponding to those in the theory of ordinary vectors. We need only mention some of them.

If

$$\mathfrak{S}_1 + \mathfrak{S}_2 = \mathfrak{S}_3 \, , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and if h be an arbitrarily chosen direction in the system, i. e. the direction of some vector in the system, we shall have

$$S_1 \cos(\mathfrak{S}_1 \, , h) + S_2 \cos(\mathfrak{S}_2 \, , h) = S_3 \cos(\mathfrak{S}_3 \, , h).$$

From this it appears that, as soon as two of the vectors $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$ are perpendicular to the direction h, the third will likewise be so.

It may further be shown that a given vector $\mathfrak{S}$ may always be decomposed into one component having a given direction h (or precisely the opposite direction) and a second component, perpendicular to h. This decomposition can be effected in only one way, the value of the first component being $S \cos(\mathfrak{S} \, , h)$. This may be positive or negative; in one case the component has the direction h, in the other it has the opposite direction.

The value of the component along h is also called the *projection* of $\mathfrak{S}$ on the direction h.

By the *scalar product* of the vectors $\mathfrak{S}_1$ and $\mathfrak{S}_2$ we understand the expression

$$S_1 \, S_2 \cos(\mathfrak{S}_1, \mathfrak{S}_2) \, ,$$

for which the sign $(\mathfrak{S}_1 \, . \, \mathfrak{S}_2)$ will be used.

It is also to be remarked that, in the case of (7),

$$S_3^2 = S_1^2 + S_2^2 + 2\,(\mathfrak{S}_1 \, . \, \mathfrak{S}_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and that we may regard the formula

$$\mathfrak{S}_1 = \mathfrak{S}_3 - \mathfrak{S}_2 \,,$$

as expressing the same relation as (7). In this way the *difference* of two vectors is defined.

We shall speak of the sum and the difference of two vectors not only if these relate to the same position of the system, but likewise if they are given for different positions.

§ 6. The material points of the system are said to be *connected* with one another, if the system is, by its nature, only capable of such infinitely small displacements as satisfy certain conditions. We shall suppose that these may be expressed by i equations of the form

$$\sum_{1}^{3n}{}_{\nu}\, x_{\iota\nu}\, dx_\nu = 0, \;\; (\iota = 1, 2, \ldots . \, i), \ldots \ldots \qquad (9)$$

in which the coefficients $x_{\iota\nu}$ are functions of the rectangular coordinates, but do not explicitly contain the time. Displacements agreeing with (9) are called *possible* displacements; displacements which violate the conditions are however equally *imaginable.*

A position of the system and a vector in it being given, we may examine if the vector have or not the direction of a possible displacement. If will have such a direction if its elements or its direction-constants obey i equations, similar to (9).

If two of the three vectors in (7) have the direction of a possible displacement, the third will have the same property.

§ 7. There are *directions perpendicular to all possible displacements.* If a vector $\mathfrak{S}$ is to have such a direction, it must be possible to express its elements X_ν in i quantities Ξ_ι by means of the equations

$$m_\nu X_\nu = \sum_{1}^{i}{}_{\iota}\, x_{\iota\nu}\, \Xi_\iota \quad \ldots \ldots \ldots \qquad (10)$$

Any system of values for Ξ_ι will give a vector that has the property in question.

If, among the vectors occurring in (7), there are two that are perpendicular to all possible displacements, the same will be the case with the third vector.

A given vector may be decomposed into two components of which the one is perpendicular to all possible displacements, and the other has the direction of a possible displacement. There is only one such decomposition.

In order to show this, we may regard as unknown quantities the $3n$ elements of the second component and i quantities Ξ_ι, in which, by (10), the elements of the first component may be expressed. There is an equal number of linear equations, i of them expressing (§ 6) that the first component has the direction of a possible displacement, and the remaining $3n$ equations, that the elements of the given vector are the sums of the corresponding elements of the two components.

§ 8. The *path* of a moving system is determined by the positions it occupies one after the other. It may be considered as a succession of infinitely small displacements, which we shall call the *elements of the path*. The *length of any part of the path* is defined as the sum

$$\int ds$$

of the lengths ds of the elements of which it consists.

The *direction of a path* in one of its positions is given by the direction of an element.

We shall always think of the system as moving along a path in a definite direction. Then the coordinates x_ν, and all other quantities that have determinate values for every position in the path, may be regarded as functions of the length s of the path, reckoned from some fixed position. Accents will serve to indicate differentiation of such quantities with respect to s.

From what has been said in § 4 it follows that the quantities x'_ν are the direction-constants of the path; they will always satisfy the relation

$$\sum_1^{3n}{}_\nu\, m_\nu\, x'^2_\nu = m \,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

as appears from (6). Using (3), we see that a vector whose elements are x'_ν has the value 1. This vector of value 1, in the direction of the path, may be called the *direction-vector*. We shall represent it by $\mathfrak{D}$.

§ 9. We define the *curvature of a path* as the vector $\mathfrak{c}$, given by

$$\mathfrak{c} = \frac{d\mathfrak{D}}{ds} \,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

the numerator being the difference between the vectors $\mathfrak{D}$ at the beginning and at the end of an element of the path of length ds. The elements of $\mathfrak{D}$ being x'_ν, we see at once that those of $\mathfrak{c}$ are x''_ν; accordingly, in virtue of (3), the value c of the curvature is given by

$$m\, c^2 = \sum_1^{3n}{}_\nu\, m_\nu\, x''^2_\nu . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

By differentiating (11) one finds

$$\sum_{1}^{3n}{}_{\nu}\, m_\nu\, x'_\nu\, x''_\nu = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

the meaning of which is that the curvature is perpendicular to the path.

Let P_1 and P_2 be two paths, having in common a position A and the direction in this position, so that the direction-vector $\mathfrak{D}$, in the position A, is likewise the same for the two paths, or

$$\mathfrak{D}_{1(a)} = \mathfrak{D}_{2(a)}.$$

Let us consider elements of the two paths, beginning in A, and of *equal* lengths ds. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are the direction-vectors at the ends of these elements, the vector

$$\mathfrak{c}_r = \frac{\mathfrak{D}_1 - \mathfrak{D}_2}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

may appropriately be called the *relative curvature* of the path P_1 with respect to the path P_2. Now, we may replace the numerator in (15) by $[\mathfrak{D}_1 - \mathfrak{D}_{1(a)}] - [\mathfrak{D}_2 - \mathfrak{D}_{2(a)}]$; the relative curvature is therefore related as follows to the curvatures $\mathfrak{c}_1$ and $\mathfrak{c}_2$ of the two paths:

$$\mathfrak{c}_r = \mathfrak{c}_1 - \mathfrak{c}_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Like $\mathfrak{c}_1$ and $\mathfrak{c}_2$, the relative curvature is perpendicular to both paths.

§ 10. What has been said thus far holds for every imaginable path. We shall now consider *possible paths*, i. e. such as are composed of possible infinitely small displacements. The direction-constants of such a path satisfy the i conditions

$$\sum_{1}^{3n}{}_{\nu}\, x_{\iota\nu}\, x'_\nu = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

as may be deduced from (9).

Let there be given a position A and a direction in this position, so that the values of x_ν and x'_ν are known, and let us seek the values of x''_ν, which make the curvature $\mathfrak{c}$ a minimum.

In solving this problem, we have to take into account equation (14) and the conditions

$$\sum_{1}^{3n}{}_{\nu}\, x_{\iota\nu}\, x''_\nu + \sum_{1}^{3n}{}_{\nu}\sum_{1}^{3n}{}_{\mu}\, \frac{\partial x_{\iota\nu}}{\partial x_\mu}\, x'_\mu\, x'_\nu = 0, \quad . \quad . \quad . \quad . \quad (18)$$

which are got by differentiating (17). We may therefore write for the values of x''_ν that make (13) a minimum

$$m_\nu\, x''_\nu = \sum_{1}^{i}{}_{\iota}\, x_{\iota\nu}\, P_\iota + m_\nu\, x'_\nu\, Q\,,$$

P_ι and Q being $i+1$ quantities whose values can be determined by means of (14) and (18). The first of these equations, combined with (17) and (11), gives $Q = 0$. The solution becomes therefore

$$m_\nu x''_\nu = \sum_1^i {}_\iota x_{\iota\nu} P_\iota , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and the formulae (18) will serve to determine the quantities P_ι.

A possible path which, in each of the positions belonging to it, is less curved than any other possible path of the same direction may be called a *path of least curvature.* In every position through which it passes it has the property expressed by (19) or, as we may also say, its curvature is perpendicular to all possible displacements.

A path of least curvature is determined by one position, and the direction in that position.

§ 11. We shall next consider a possible path P and the path P_o of least curvature, having in common with P one position A and the direction in that position. Let, in the position A, $\mathfrak{c}_o$ be the curvature of P_o, $x''_{\nu(o)}$ the elements of this curvature, $\mathfrak{c}$ and x''_ν the corresponding quantities for P, and let us fix our attention on the relative curvature of P, with respect to the least curved path P_o. We shall denote this relative curvature by $\mathfrak{c}_f$, and call it the *free curvature* of the possible path P. It may be shown to have the direction of a possible displacement.

Indeed, we have by definition

$$\mathfrak{c}_f = \mathfrak{c} - \mathfrak{c}_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

so that the elements of $\mathfrak{c}_f$ are $x''_\nu - x''_{\nu(o)}$. Now, if we write down two times the equations (18), first for P_o and then for P, we find by subtraction

$$\sum_1^{3n} {}_\nu x_{\iota\nu} \left[x''_\nu - x''_{\nu(o)}\right] = 0,$$

which proves the proposition. We may add that $\mathfrak{c}_o$ is perpendicular to $\mathfrak{c}_f$, being perpendicular to all possible displacements, and that therefore by (8)

$$c^2 = c_o^2 + c_f^2.$$

This confirms the inequality $c_o < c$.

It is easily seen that a possible path is wholly determined if one knows one position belonging to it, the direction in that position and the free curvature in all positions.

§ 12. Let P be a possible path. From every position A lying in it we make the system pass to a *varied position* A', by giving to it an infinitely small displacement $\delta\mathfrak{s}$, for whose elements we write δx_ν, these elements being supposed to be continuous functions of the length s of the path, reckoned along P.

The path P' that is determined by the succession of the new positions A' will be called the *varied path* and the letter δ will serve to indicate the difference between quantities relating to this path and the corresponding ones relating to the original path. We shall use the sign d, if we compare the values of some quantity at the beginning and the end of an element ds of the path P.

It is easy to obtain an expression for the variation in the length of an element. Starting from (1), we find

$$m\,\delta\,d\,s = \sum_{1}^{3n}{}_{\nu}\, m_\nu\, x'_\nu\, \delta\, d\, x_\nu = \sum_{1}^{3n}{}_{\nu}\, m_\nu\, x'_\nu\, d\, \delta\, x_\nu =$$

$$= d \sum_{1}^{3n}{}_{\nu}\, m_\nu\, x'_\nu\, \delta\, x_\nu - d\,s\,.\sum_{1}^{3n}{}_{\nu}\, m_\nu\, x''_\nu\, \delta\, x_\nu,$$

and we may simplify this by introducing the notation for the scalar product of two vectors, and writing $(\delta\,\mathfrak{s})_s$ for the projection of $\delta\,\mathfrak{s}$ on the direction of the path. The sums on the right-hand side may then be replaced by

$$m\,(\mathfrak{D}\,.\,\delta\,\mathfrak{s}) = m\,(\delta\,\mathfrak{s})_s$$

and

$$m\,(\mathfrak{c}\,.\,\delta\,\mathfrak{s})\,.$$

In the last expression, in virtue of (20)

$$(\mathfrak{c}\,.\,\delta\,\mathfrak{s}) = (\mathfrak{c}_o\,.\,\delta\,\mathfrak{s}) + (\mathfrak{c}_f\,.\,\delta\,\mathfrak{s})\,,$$

and this is reduced to the last term, if we confine ourselves to *possible* virtual displacements $\delta\,\mathfrak{s}$, these being perpendicular to $\mathfrak{c}_o$. Finally

$$\delta\,d\,s = d\,(\delta\,\mathfrak{s})_s - (\mathfrak{c}_f\,.\,\delta\,\mathfrak{s})\,d\,s\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

a result, which can be illustrated by simple geometrical examples.

§ 13. It is to be remarked that the varied path of which we have spoken in the last § is not in general a possible path. This will however be the case, if the i equations (9) admit of complete integration, i. e. if the connexions may be expressed by i equations between the coordinates.

Systems having this peculiar property are called by HERTZ *holonomic.* For these, the equation (21) gives the variation which arises if one possible path is changed into another, infinitely near it, and *likewise possible.*

If now the original path were one of least curvature, we should have $\mathfrak{c}_f = 0$, and by integration over some part of the original path, in the supposition that the initial and final positions are not varied,

$$\delta \int_1^2 d\,s = 0\,.$$

This shows that for holonomic systems the paths of least curvature are at the same time *geodetic* paths.

§ 14. In considering the motion in relation to the time t, we shall indicate differentiations with respect to this variable either by the ordinary sign or by a dot. If some quantity φ may be conceived as a function of t and likewise as one of the length of path s, we have the relation

$$\frac{d\varphi}{dt} = \frac{d\varphi}{ds} \cdot \frac{ds}{dt}, \qquad \dot{\varphi} = \varphi' \frac{ds}{dt}.$$

We shall define the *velocity* $\mathfrak{v}$ of the system as the complex of the velocities of the individual points. Its elements are $\dot{x}_\nu = x'_\nu \cdot \frac{ds}{dt}$ and the vector itself is

$$\mathfrak{v} = \frac{ds}{dt} \mathfrak{D}.$$

The *direction of the velocity* is that of the path, so that we may write

$$\mathfrak{v} = v\, \mathfrak{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and the *value* is

$$v = \frac{ds}{dt}.$$

If the value is determined by (3), the *kinetic energy* is easily found to be

$$T = \tfrac{1}{2}\, m\, v^2 .$$

By the *acceleration* $\mathfrak{f}$ of the system we understand the complex of the accelerations of all the material points. Thus the elements of $\mathfrak{f}$ are $\ddot{x}_\nu$, and

$$\mathfrak{f} = \dot{\mathfrak{v}},$$

An interesting result is obtained if in this equation we use (22), (12) and (20). We are then led to the following decomposition of the acceleration into three components:

$$\mathfrak{f} = v\dot{\mathfrak{D}} + \dot{v}\mathfrak{D} = v^2 \frac{d\mathfrak{D}}{d\,s} + \dot{v}\mathfrak{D} = v^2\, \mathfrak{c} + \dot{v}\mathfrak{D} = v^2\, \mathfrak{c}_o + v^2\, \mathfrak{c}_f + \dot{v}\, \mathfrak{D} \quad . \quad (23)$$

The first component is perpendicular to all possible displacements, the second has the direction of the free curvature and the third that of the path.

It is easily seen that a possible motion will be quite determined, if we know one position, the velocity in that position and, for every instant, the second and the third component of the acceleration. Indeed, the second component determines the free curvature, and by this the change in the direction of the path, and the third component determines the change in the value of the velocity.

§ 15. Let the material points of the system be acted on by forces, in the usual sense of the word, and let X_ν be the rectangular components of these. We shall take together all these forces, so that we may speak of them all as of one thing, but in doing so we shall slightly depart from the way in which we have defined the velocity and the acceleration. We begin by multiplying each individual force by $\frac{m}{m'}$, m' being the mass of the point on which it acts, and m the mass of the whole system, and we understand by *the force* $\mathfrak{F}$ *acting on the system* the complex of these new vectors.

The elements of $\mathfrak{F}$ are therefore $\frac{m}{m_\nu} X_\nu$.

Assigning to $\mathfrak{F}$ a definite direction and a definite value will of course imply that all the forces acting on the material points of the system are given in direction and magnitude.

The definition of the force $\mathfrak{F}$ has been so chosen that the work of the forces in an infinitely small displacement, i. e. the expression

$$\sum_{1}^{3n}{}_{\nu}\, X_\nu \, d\, x_\nu,$$

becomes equal to the scalar product $(\mathfrak{F}.\ d\,\mathfrak{s})$.

§ 16. Every force $\mathfrak{F}$ may be decomposed into one component $\mathfrak{F}_0$, perpendicular to all possible displacements, a second component $\mathfrak{F}_1$, having the direction of a possible displacement and perpendicular to the path, and a third component $\mathfrak{F}_2$, in the direction of the path.

One can conduct this operation in two steps. Replace first (§ 7) $\mathfrak{F}$ by $\mathfrak{F}_0$, perpendicular to all possible displacements, and $\mathfrak{F}'$, in the direction of such a displacement. This being done, we have to decompose (§ 5) $\mathfrak{F}'$ into a force $\mathfrak{F}_2$, along the path, and a force $\mathfrak{F}_1$, perpendicular to it. The latter component will have the direction of a possible displacement, because $\mathfrak{F}'$ and $\mathfrak{F}_2$ have such directions.

For a given force the three components are wholly determinate.

§ 17. We may imagine each material point to be acted on by a force in the direction of the acceleration of the point and equal to the product of the acceleration and the mass. We shall denote by $\mathfrak{S}$ the force acting on the system in this special case, by $\mathfrak{S}_0$, $\mathfrak{S}_1$, $\mathfrak{S}_2$ its components in the above mentioned directions.

Now we have in the supposition just made $X_\nu = m_\nu \ddot{x}_\nu$, from which we find $m\,\ddot{x}_\nu$ for the elements of $\mathfrak{S}$, $\mathfrak{S} = m\,\mathfrak{f}$ for the force itself, and, by (23),

$$\mathfrak{G}_0 = m\, v^2\, \mathfrak{c}_o, \quad \mathfrak{G}_1 = m\, v^2\, \mathfrak{c}_f, \quad \mathfrak{G}_2 = m\, \dot{v}\, \mathfrak{D}$$

for the three components.

§ 18. What precedes has prepared us for the consideration of the fundamental principles by which the motion of the system under the action of given forces is to be determined. We may in the first place start from the following assumptions:

a. The system will have the acceleration $\mathfrak{f}$, if the force is precisely $\mathfrak{G} = m\mathfrak{f}$.

b. Two forces $\mathfrak{F}_a$ and $\mathfrak{F}_b$ may have the same influence on the motion. For this it is necessary and sufficient that the force

$$\mathfrak{F}_a - \mathfrak{F}_b$$

should be perpendicular to all possible displacements.

Let the system be subject to the force $\mathfrak{F}$ with the components $\mathfrak{F}_0$, $\mathfrak{F}_1$, $\mathfrak{F}_2$, and let the acceleration be $\mathfrak{f}$. Then, by the first principle, $\mathfrak{F}$ has the same influence as $\mathfrak{G} = m\mathfrak{f}$, and by the second principle $\mathfrak{F} - \mathfrak{G}$ must be perpendicular to all possible displacements. This amounts to the same thing as $\mathfrak{F}_1 = \mathfrak{G}_1$, $\mathfrak{F}_2 = \mathfrak{G}_2$, or

$$\mathfrak{F}_1 = m\, v^2\, \mathfrak{c}_f, \quad \mathfrak{F}_2 = m\, \dot{v}\, \mathfrak{D}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

It will be immediately seen that the above assumptions are equivalent to D'ALEMBERT's principle. We might also have replaced them by the following rule:

Decompose the acceleration into two components $\mathfrak{f}_0$ and $\mathfrak{f}'$, the first perpendicular to all possible displacements, and the second in the direction of such a displacement. Decompose the force $\mathfrak{F}$ in the same way into the components $\mathfrak{F}_0$ and $\mathfrak{F}'$. Then the equation of motion will be

$$\mathfrak{F}' = m\mathfrak{f}'.$$

This leads directly to the equations (24), by which it is clearly seen that the change in direction of the path is determined by the component $\mathfrak{F}_1$, and the change in the value of the velocity by the component $\mathfrak{F}_2$. It is to be kept in mind that the first of the formulae (24) is a *vector-equation*. In general the free curvature, as well as the force $\mathfrak{F}_1$, may have different directions, in some cases a great many of them. The equation does not only show us *to what amount* the path deviates from one of least curvature, but also *to which side* the deviation takes place.

If $\mathfrak{F} = 0$, we have $\mathfrak{c}_f = 0$ and $\dot{v} = 0$; we are then led back to the fundamental law of HERTZ.

§ 19. Let us now return to the equation (21), taking for the original path one that is described under the action of the existing forces. Attending to (24), we may write in (21)

$$(\mathfrak{c}_f \,.\, \delta\,\mathfrak{s}) = \frac{1}{mv^2}\,(\mathfrak{F}_1 \,.\, \delta\,\mathfrak{s}) = \frac{1}{mv^2}\,[(\mathfrak{F} \,.\, \delta\,\mathfrak{s}) - (\mathfrak{F}_0 \,.\, \delta\,\mathfrak{s}) - (\mathfrak{F}_2 \,.\, \delta\,\mathfrak{s})].$$

Now we have $(\mathfrak{F}_0 \,.\, \delta\,\mathfrak{s}) = 0$, because $\mathfrak{F}_0$ is perpendicular to the virtual displacement. Further:

$$(\mathfrak{F}_2 \,.\, \delta\,\mathfrak{s}) = m\,\dot{v}\,(\mathfrak{D} \,.\, \delta\,\mathfrak{s}) = m\,\dot{v}\,(\delta\,\mathfrak{s})_s = m\,v'\,v\,(\delta\,\mathfrak{s})_s = m\,v\,\frac{dv}{ds}\,(\delta\,\mathfrak{s})_s,$$

so that (21) becomes

$$\delta\,d\,s = d\,(\delta\,\mathfrak{s})_s + \frac{1}{v}\,d\,v\,(\delta\,\mathfrak{s})_s - \frac{1}{mv^2}\,(\mathfrak{F} \,.\, \delta\,\mathfrak{s})\,d\,s,$$

or, multiplied by $m\,v$,

$$m\,v\,\delta\,d\,s + \frac{1}{v}\,(\mathfrak{F} \,.\, \delta\,\mathfrak{s})\,d\,s = m\,d\,[v\,(\delta\,\mathfrak{s})_s\,].$$

The scalar product $(\mathfrak{F} \,.\, \delta\,\mathfrak{s})$ on the left is the work of the force for the virtual displacement; in the case of a *conservative* system with potential energy U, it may be denoted by $-\,\delta\,U$. The result therefore takes the form

$$m\,v\,\delta\,d\,s - \frac{1}{v}\,\delta\,U\,d\,s = m\,d\,[v\,(\,\delta\,\mathfrak{s})_s\,]\,. \quad . \quad . \quad . \quad (25)$$

§ 20. Thus far, we have spoken only of a varied path, but not of a *varied motion*; we have said nothing about the instants at which we imagine the varied positions to be reached. In this respect we may make different assumptions, and among these there are two, which lead to a simple result of the equation (25), if integrated over a part of the path.

a. Let the varied positions A' be reached *at the same moments* as the corresponding positions A in the original motion. Then

$$m\,v\,\delta\,d\,s = m\,v\,\delta\,v\,.\,d\,t = \delta\,T\,d\,t\,;$$

(25) becomes

$$\delta\,(T - U)\,d\,t = m\,d\,[v\,(\delta\,\mathfrak{s})_s]\,,$$

and, if integrated along the path which the system travels over between the instants t_1 and t_2, in the supposition that $\delta\,\mathfrak{s} = 0$ for $t = t_1$ and $t = t_2$,

$$\delta\int(T - U)\,d\,t = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

This is HAMILTON's principle, which is in itself sufficient for the determination of the motion really taking place under the action of given forces, and from which we may infer e. g. that in the course of this motion

$$T + U = E$$

remains constant.

b. In the second place we shall assume that *in the varied motion the energy $T + U$ has the same constant value E as in the original motion.* This value E having been chosen, and U being known for every position, the value of the velocity is given by

$$v = \sqrt{\frac{2}{m}(E - U)}.$$

This second assumption therefore, as well as the first, leaves no doubt as to the velocity with which the system is supposed to travel along its varied path.

The total energy remaining constant, we have now

$$\delta U = - \delta T = - m v \delta v,$$

and (25) becomes

$$\delta (v \, d s) = d [v (\delta \mathfrak{s})_s],$$

or

$$\delta (\sqrt{E - U} \, d s) = \sqrt{\tfrac{1}{2} m} \, d [v (\delta \mathfrak{s})_s] . \quad . \quad . \quad . \quad . \quad (27)$$

Hence, if we integrate along a certain part of the path, supposing again the extreme positions to remain unchanged,

$$\delta \int \sqrt{E - U} \, d s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

This is the principle of least action in the form that has been given to it by JACOBI. Indeed we may define the action along a path of the system as the integral that occurs in the equation (28)[1]. Its value may be calculated for any path $A_1 A_2$ whatsoever. For the sake of brevity we shall denote it by $V_{A_1}^{A_2}$.

Both the principle of HAMILTON and that of JACOBI have been here obtained by the consideration of the variation in the length of an element of a *curved* path that is caused by virtual displacements of the system. It is clear that both principles hold for every system, be it *holonomic* or not, the only condition being that the virtual displacements do not violate the connexions. We must however keep in mind that it is only in the case of holonomic systems that the varied motion may be said to be, as well as the original one, a *possible* motion. Hence, if we wish to compare the motion taking place under the action of the given forces with another motion, differing infinitely little from it, and such that it is not excluded by the connexions, the two principles will only be true for holonomic

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[1] The action is usually defined as the integral, multipied by $\sqrt{2m}$.

systems [1]). The variations of the integrals occurring in (26) and (28) will be 0, if the original motion is not only a possible one, but such that it may really take place under the influence of the acting forces. We shall call such a motion a *real* or a *natural* one.

§ 21. We shall conclude by briefly showing how some well known propositions may be presented in a form, agreeing with what precedes. These propositions relate to holonomic systems. Let us therefore assume that the connexions are expressed by i equations which must be satisfied, independently of the time, by the coordinates x_ν, and let us confine ourselves to *possible positions*, i. e. such as agree with these i conditions. We might determine these positions by $3n-i$ "free" coordinates, but in what follows, it is not necessary to do so.

If, in addition to the equations expressing the connexions, we assume still one other equation between the coordinates, we shall call the totality of positions satisfying that equation a *surface of positions*.

In case one of these positions A is reached by a certain path — the other positions in this path not all of them belonging to the surface — the path may be said to *cut* the surface in the position A. For simplicity's sake it will be supposed in such a case that the surface and the path have only that one position in common.

Starting from a position A, which belongs to, or lies in a surface of positions S, we may give to the system infinitely small displacements, in such directions that by them the position does not cease to belong to the surface.

Another infinitely small possible displacement $d\mathfrak{s}$ whose direction is perpendicular to all those *displacements in the surface* may be said to be *perpendicular to the surface.* It is easily shown that a displacement of the latter kind may always be found and that its direction is entirely determinate.

Let S be a surface of positions, A a position that does not belong to it but is infinitely near others that do, B a position in the surface, such that the infinitely small displacement $A \rightarrow B$ is perpendicular to S, and C any position in S infinitely near B. Let ϑ be the angle between the displacements $A \rightarrow B$ and $A \rightarrow C$, and let us denote by $\overline{AB}$ and $\overline{AC}$ the lengths of these displacements. Then

$$\overline{AB} = \overline{AC} \cos \vartheta .$$

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[1]) See HÖLDER, Gött. Nachr., 1896, p. 122.

This follows from what has been said in § 5, if we consider that $A \rightarrow C$ is the resultant of the displacements $A \rightarrow B$ and $B \rightarrow C$.

§ 22. Henceforth we shall treat only of *natural* motions, taking place with a fixed value E of the total energy, which we choose once for all. We shall suppose that, if O and A are any two positions, there is one and only one such a natural motion which leads from O to A. The action along the path of this motion,

$$V_O^A = \int_O^A \sqrt{(E-U)}\, ds,$$

will have a definite value, depending on the coordinates of O and A, and we shall examine the variations of this action, if we change the final position A, the initial one being fixed.

In the first place it is clear that, if we move A along a path issuing from O, V_O^A will be the greater, the farther A recedes from O. Indeed, V_O^A presents a certain analogy with the length of the path, the difference being that, in calculating the action, we must multiply each element ds by the factor $\sqrt{E-U}$, which changes with the position.

The increment of the action, corresponding to an element of the path, is obviously

$$\sqrt{E-U}\, d s.$$

In the second place we compare two paths, both issuing from O, but in directions that differ infinitely little from each other. We shall proceed along these so far, say till we have reached the positions A and A', that the action is equal in the two cases, i. e.

$$V_O^A = V_O^{A'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Now, the motion $O \rightarrow A'$ may be conceived as the result of an infinitely small variation of the motion $O \rightarrow A$; we may therefore apply the equation we deduce from (27), if we integrate from O to A. On account of (29), we get 0 on the left-hand side, hence, the projection $(\delta \mathfrak{s})_s$, which vanishes for the position O, must likewise be 0 for the position A, and the infinitely small displacement $A \rightarrow A'$ is found to be perpendicular to the path OA.

We may next fix our attention on all paths that issue from a definite position O. In each of these we choose a position A at such a distance from O, that the action V_O between O and these positions has the same value for all of them. The positions A will belong to a certain surface and this will be cut under right angles by all the paths. We may therefore call these latter the orthogonal trajectories of the surfaces

$$V_O = const. \qquad (30)$$

Let S be that one of these surfaces, to which a certain position A belongs, and let B be some position, infinitely near A, and further from O than the surface S. In order to find an expression for the difference $\overset{B}{V}_O - \overset{A}{V}_O$, we consider also the surface S', likewise belonging to the group (30), and containing the position B; this surface will be cut in a certain position C by the path OA prolonged. If ϑ is the angle between $A \rightarrow B$ and the direction of the path in A, we shall have

$$V_O^B - V_O^A = V_O^C - V_O^A = \sqrt{E-U}.\overline{AC} = \sqrt{E-U}.\overline{AB}\cos\vartheta \quad . \ (31)$$

§ 23. Instead of considering the paths issuing from one and the same position O, we may also begin by choosing a surface of positions S_o, and think of all the motions in which the system starts from a position belonging to this surface, in a direction perpendicular to it. We shall suppose that any given position A may be reached by one and only one of these motions, and we shall write V_{So}^A for the action along the path leading from S_o to A.

This function has properties similar to those of the function we have studied in the preceding §. The paths are the orthogonal trajectories of the surfaces

$$V_{So}^A = const.,$$

and the change in the action, caused by an infinitesimal variation of A is given by a formula of the same form as (31).

§ 24. The values of V_O^A and V_{So}^A may depend in many different ways on the coordinates of the variable position A, according to the choice of the initial position O or the surface S_o from which we start. All these different functions have however one common property, which follows immediately from what has been said, and for which a concise form of expression is obtained in the following way.

If Q is a function of the coordinates, we may, for every infinitely small possible displacement ds, beginning in a position A, calculate the ratio

$$\frac{dQ}{ds} \qquad (32)$$

The value of this will of course depend on the direction of $d\mathfrak{s}$, and the position A being chosen, there will be one definite direction (perpendicular to the surface $Q = \text{const.}$), for which the ratio takes the largest positive value. Now, if we denote this maximum value of (32) by DQ, the property in question of the functions V may be expressed by

$$DV_O = \sqrt{E-U} \text{ and } DV_{S_0} = \sqrt{E-U}.$$

These formulae may be written in the form of a partial differential equation which is satisfied by V_O and V_{S_0}.

§ 25. Let R be any solution of this differential equation, i. e. a function of the coordinates, such that

$$DR = \sqrt{E-U}; \quad \ldots \ldots \ldots \quad (33)$$

then the orthogonal trajectories of the surfaces

$$R = const. \quad \ldots \ldots \ldots \ldots \quad (34)$$

are natural paths of the system.

The proof of this is as follows. Imagine the infinitely small displacements $d\mathfrak{s}$, lying between the two consecutive surfaces

$$R = C \text{ and } R = C + dC,$$

that is to say, the displacements whose initial position belongs to the first and whose final position belongs to the second surface, and subject them to the further condition that they are to be perpendicular to the first surface. Then we have by (33)

$$\frac{dC}{ds} = \sqrt{E-U}, \qquad \sqrt{E-U}\,ds = dC,$$

so that the action is the same for all these elements, whichever be the position in the surface from which they start.

On the contrary, if $d\mathfrak{s}'$ is another element of path between the two surfaces, not perpendicular to them, but making an angle ϑ with one of the first-named displacements $d\mathfrak{s}$ in the immediate vicinity, we shall have

$$d\,s' = \frac{d\,s}{\cos\vartheta},$$

and for the action along $d\mathfrak{s}'$

$$\frac{d\,C}{\cos\vartheta}.$$

It appears from this that

$$\delta\{\sqrt{E-U}\,d\,s\} = 0, \quad \ldots \ldots \ldots \quad (35)$$

if we pass from an element $d\mathfrak{s}$, perpendicular to $R = C$, to an element $d\mathfrak{s}'$, lying between the same two surfaces, and of such a direction that ϑ is infinitely small.

This being established, we may take an arbitrarily chosen part $M N$ of an orthogonal trajectory of the surfaces (34), we may divide it into elements by means of surfaces belonging to the group, and infinitely near each other, and we may give to $M N$ an infinitely small variation, without however changing the positions M and N. Then, applying what has just been said to every element of $M N$, and integrating, we find

$$\delta \int \sqrt{E-U}\, ds = 0,$$

showing that $M N$ is a natural path.

At the same time the meaning of the function R becomes apparent. Its value in a certain position A is the action along a trajectory, ending in A and beginning at the surface

$$R = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

§ 26. We shall now assume that we know a function $R(c)$ of the coordinates, satisfying the differential equation (33) and containing an arbitrary constant c.

Then $\frac{\partial R}{\partial c}$, which is itself a function of the coordinates, will have the same value for all positions lying on a path P, perpendicular to (36).

To show this, we consider the consecutive surfaces whose equations are

$$R\,(c) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

and

$$R\,(c + d\,c) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Let A_0 be the position in the first surface where the path P begins, A any position belonging to P. We shall suppose this path $A_0 A$ to cut the surface (38) in a position A_1, which is, of course, infinitely near A_0. We shall finally think of the path, such as there certainly is one, leaving the surface (38) in a perpendicular direction and terminating in A. It has a definite initial position in (38), infinitely near A_1, and which we shall call A'.

Since

$$V^A_A = V^A_{A'},$$

as may be deduced from (27), we have

$$V^A_{A'} - V^A_{A_0} = V^A_{A_1} - V^A_{A_0} = - V^{A_1}_{A_0} \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Now, $V^A_{A_0}$ and $V^A_{A'}$ are the values of $R(c)$ and $R(c + dc)$ for the position A. Consequently, the first member of (39) is the value of $\frac{\partial R}{\partial c} dc$ for this position, and

$$\frac{\partial R}{\partial c} = -\frac{1}{dc} V_{A_0}^{A_1} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (40)$$

The right-hand member of this formula remaining the same if we move the system along the path P, the proposition is proved.

§ 27. A remarkable and well known theorem of JACOBI is a direct consequence of our last proposition. If we have found a function R of the coordinates, satisfying the differential equation (33) and containing, besides an additive constant, $3n-i-1$ other arbitrary constants c_1, c_2, etc., the values of $\frac{\partial R}{\partial c_1}, \frac{\partial R}{\partial c_2}$ etc. will not change, while the system describes a path, perpendicular to the surface

$$R(c_1, c_2, c_3, \ldots) = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (41)$$

The $3n-i-1$ equations of such a path will therefore be of the form

$$\frac{\partial R}{\partial c_1} = \gamma_1, \frac{\partial R}{\partial c_2} = \gamma_2, \text{ etc.} \quad . \; . \; . \; . \; . \; . \quad (42)$$

where γ_1, γ_2, etc. are constants.

The total number of the constants c and γ is $2(3n-i-1)$; this is just sufficient for the representation of every path that may be described with the chosen value of the energy. If we confine ourselves to fixed values of the constants c, and change those of the constants γ, we shall obtain all paths that are perpendicular to one and the same surface (41). One of these paths will be distinguished from the other by the value of the action along the parts of the path, lying between the surface (41) on one, and each of the surfaces

$$R(c_1 + d\,c_1, c_2, c_3, \ldots) = 0$$
$$R(c_1, c_2 + d\,c_2, c_3, \ldots) = 0$$
$$R(c_1, c_2, c_3 + d\,c_3, \ldots) = 0,$$

on the other side. Indeed, by (42) the action along these parts is $-\gamma_1 dc_1$, $-\gamma_2 dc_2$ etc.

By giving other values to the constants c, we shall change the surface (41) and we shall find the paths that are perpendicular to the new surface.

Physiology. — "*On the intestinal juice of man*". Communication made by Prof. H. J. Hamburger from researches carried out conjointly with Dr. E. Hekma.

(Communicated in the meeting of March 29, 1902).

Up to the present time only three investigations on the intestinal juice of man have been recorded; namely: those of Demant [1]) in 1879, of Tubby and Manning [2]) in 1892, and of Nagano [3]) in 1902.

Upon the whole these investigations have yielded few satisfactory results; they have failed to answer the question, what is the significance of the juice secreted by those hundreds of thousands of Lieberkühn's glands, which everywhere line the inner surface of the intestine.

When therefore Prof. Koch directed our attention to the flow of intestinal juice, obtained from a lady with an intestinal fistula produced by an operation he had performed upon her, we did not hesitate for a moment to examine the juice secreted. We are extremely indebted to Prof. Koch, for his cooperation, and also for the interest with which he has followed our researches and for his readiness to comply with our wishes, after consulting the patient.

In this respect we owe much also to Dr. P. M. E. Roessingh, who always took the interests of the investigations to heart, besides those of his patient.

The patient in question, was unmarried and 43 years of age; in March 1901 she underwent a supravaginal amputation of the uterus, rendered necessary by a myoma uteri. When she was discharged from the Infirmary on the 27th of April, a suppurating fistula in the abdominal scar remained still open but it gradually closed. It reopened on the 5th July, discharging the liquid contents of the intestines. Soon afterwards, a similar discharge took place through the vagina. The patient became considerably weaker so that Prof. Koch, as consulting physician, advised another operation. This took place on the 15th of July.

When the abdomen was opened a loop of intestine, grown together, was found, and fixed to this was a stercoral abscess. It seemed dangerous to clean this out. It was therefore decided, simply to separate the loop with the abscess from the rest of the intestinal canal; this was effected by cutting through the intestine at its entrance into the loop and at the other end where it came out; the ends were then united. The separated piece was ligatured at both extremities. Notwithstanding this the loop continued to discharge its contents in two places, namely, through the scar in the abdomen and through the vagina. At first the discharge

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[1]) B. Demant. Virchow's Archiv. **75**, 1879. S. 419.

[2]) Tubby and Manning. Guy's Hospital's reports **48**, 1892. p. 271.

[3]) Nagano. Mittheilungen aus den Grenzgebieten der Medicin und Chirurgie **9**, 1902. S. 293.

evidently consisted of what was still left in the loop. When this was removed the opening in the vagina closed, but juice still continued to flow from the abdominal opening. This juice was, as will appear later on, nothing else but intestinal juice.

Relying on his experience, Prof. Koch anticipated that this part of the intestine would ultimately become spontaneously occluded, after which the patient would be entirely cured; after the operation, she regained strength and felt in perfect health. To allow the juice to flow freely from the fistulous opening in the belly, a drainage tube was placed in the wound; the tube led to a little bottle which was fixed to the thigh in a slanting position; in this way the juice could be collected without inconvenience to the patient.

This is the juice the first sample of which we received on the 14th of December 1901, and with which we have carried out our researches. — We will first take up the composition of this juice, then its action and finally give some particulars with regard to its secretion.

I. Composition of the juice.

On superficial examination the product was seen to consist of a liquid, in which a not inconsiderable amount of morphological elements were present. When these were examined microscopically, it was found that they consisted of sound and phantom white blood-corpuscles, and a few red blood corpuscles; epithelium cells, bacteria and some fine crystalline needles were also present. Most probably the latter were fat crystals, for they dissolved in aether and became black where treated with osmic acid.

By the use of the centrifuge, it was determined that the volume of morphological elements varied between 6 and 12 per cent of the liquid.

After full subsidence without centrifugalising, the percentage volume of the deposit varies between 10 and 20. Mucus was not present in the juice.

The presence of a large number of leucocytes and bacteria suggested inflammation, but as the liquid contained only a trace of albumin which was difficult to detect, this could not have been the case.

The presence of bacteria can be explained because during and after the operation no rinsing of the intestinal loop was performed; this process was omitted, because on therapeutic grounds there was no indication for it. We shall speak later on of the bacteria.

After centrifugalizing, the liquid was opalescent, had an alkaline reaction and on the addition of sulphuric acid carbonic acid was given off; Na_2CO_3 was therefore present. To determine this amount quantitatively titration was performed with $^1/_{20}$ normal acid and lakmoid-paper as indicator. The quantity of $Na_2 CO_3$ appeared to range between 0.17 % and 0.24 %.

The proportion of the chlorine was determined by titration according to **Volhard**'s process. 10 CC of the juice were equal to 11.4 CC of decinormal silver nitrate solution and therefore contained a quantity of chlorine, such as is present in a 0.96 % solution of sodium chloride. The total quantity of solid substances in the opalescent liquid amounted to 1.066 %.

The freezingpoint-reduction, determined with **Beckmann**'s apparatus, amounted to 0.620°. This is greater than that of normal human serum, which averages 0.57°.

II. Action of the intestinal juice.

If we compare the results of **Demant**, **Tubby** and **Manning** and **Nagano** concerning the action of the juice examined by them, it appears that all agree that its action on nutritive matter is insignificant. Its action on proteids like fibrin and boiled white of egg is doubtful; so also is its action on cane sugar and fat; it sometimes has a feeble digestive action on starch.

Investigations made on dogs during the last few years, in **Pawlow**'s laboratory, have however cast a new light on the significance of the intestinal juice. It has been shown in these researches that the juice has little or no ability by itself to digest nutritive matter but that it possesses a powerful action when it coöperates with pancreatic juice. According to **Pawlow**, the intestinal juice of the dog possesses the power of increasing considerably the action of the ferments of the pancreatic juice, and this is specially the case in regard to the proteolytic ferment, trypsin. We have therefore to deal with a ferment in the intestinal juice, which supports or increases the action of other ferments, „a ferment of the ferments", which **Pawlow** has named „enterokinase" [1]).

Our aim was now in the first place to ascertain, whether "enterokinase", found in dogs is also present in the intestinal juice of man and if this should be the case, to investigate further the nature of its action.

The juice used in this work was freed from morphological elements by centrifugalizing.

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[1]) **Pawlow**. Das Experiment als zeitgemässe und einheitliche Methode medizinischer Forschung. Dargestellt am Beispiele der Verdauungslehre, Wiesbaden 1900, S. 15.

We have already mentioned that this was opalescent. It is rendered as clear as water by filtration through a Chamberland bougie, but the filtered and unfiltered juice have the same actions.

Let us begin by examining the

a. Behaviour of intestinal juice in reference to the action of trypsin on white of egg.

For these researches white of hen's eggs was used, which according to METTE's method, was coagulated in narrow glass tubes by heating to 95°. The digestion was determined quantitatively by measuring on both sides the length of the little column which had disappeared. Two little tubes were used in each experiment and in the following tables the lengths of the four consumed little columns are added together.

It was now seen that the intestinal juice alone had effected no visible change, after having been allowed to act on the tubes with coagulated white of egg for 48 hours or longer.

Along with these experiments with intestinal juice, other experiments were performed in exactly the same way with a $1^0/_0$-solution of trypsin (MERCK) in a $0.3^0/_0$-solution of Na_2CO_3 and at the same time with a trypsin-solution, to which intestinal juice was added.

It was now evident that both the liquids containing trypsin had digested white of egg, the latter mixture however being by far the more powerful.

Intestinal juice had therefore in this case exercised a furthering influence on the action of the trypsin in digesting white of egg.

The following figures will make this clear.

Furthering influence of intestinal juice on the digestive action of trypsin on white of egg.

	Millimeters of white of egg consumed in	
	24 hours.	2 × 24 hours.
5 cc of the trypsin sol. + 5 cc water	4.50	9
5 cc of the trypsin sol. + 5 cc $Na_2 Co_3$ 0.3 pCt.	5	11
5 cc of the trypsin sol + 5 cc intestinal juice	**7.50**	**20**
Intestinal juice alone........................	0	0

Not *all* commercial preparations of trypsin, however, gave such decided results; it even happened sometimes that no reinforcing action could be detected.

The same result was obtained, when instead of a trypsin sol. in Na_2CO_3 0.3 %, a glycerine-extract of pancreas was employed; sometimes an evident action, at other times none at all. Sometimes even it would seem as if boiled intestinal juice, in which any ferment should have been annihilated, operated actively, whereas the unboiled juice was inactive. Looking at these contradictory results, it appeared possible that they might be explained on the hypothesis that trypsin was liberated from a combination (zymogen) in which it is present in the pancreatic juice.

A communication recently made at Turin at the physiological Congress, by one of PAWLOW's collaborators, Dr. WALTHER 1)) lends support to this supposition.

In opposition to PAWLOW's opinion, this investigator considers that the action of enterokinase is due to a transformation of zymogen, which has no influence on white of egg, into the active trypsin. In his dissertation, written in 1899 in the laboratory of Petersburgh, SCHEPOWALNIKOW 2)) appears to be of the same opinion.

In this way it can be explained that there were trypsin-preparations, which were not, and others which were rendered active by intestinal juice. It, however, remained to be proved that the first preparations still contained a considerable quantity of zymogen and the others none at all.

It was therefore thought desirable to make experiments with a zymogen, free from trypsin. To this end a pig's pancreas was taken, as soon as possible after the animal had been killed (one or two hours), cut into small pieces and subjected to pressure. The thick juice obtained in this way, did not show the least action on coagulated white of egg. When however intestinal juice (which in itself is equally unable to digest white of egg) was added to it, the coagulated white of egg rapidly disappeared.

Evidently therefore the intestinal juice had liberated trypsin from its zymogen.

The following table gives the details of one experiment, selected from many others which all gave the same result. It also shows that the intestinal juice is rendered inactive by boiling.

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1) **WALTHER**, Archives Italiennes de Biologie, 1901.

2) We have been enabled to read this dissertation which is written in Russian through the kindly aid of Dr. **SALTYKOW**, assistant in pathological anatomy in Groningen.

	Millimeters of white of egg consumed		
	After 17 hours.	After 45 hours.	After 88 hours.
2 drops of pancr. juice + 3 cc. water.........	0	0.7	7
2 » » » » + 3 cc. Na_2CO_3 0.15 pCt.	0	0	10
2 » » » » + » 0.3 »	0	0	12
2 » » » » + » 1.5 »	0	0	0
2 » » » » + » 3 »	0	0	0
2 » » » » + unboiled int. juice	**2.5**	**11**	Tubes empty, therefore **24** at the least.
2 » » » » + boiled » »	0	0	2

There is no need to heat the intestinal juice to boiling to make it inactive; the same result can be obtained by heating for 3 hours to 67°.

In our next series of investigations, we ought to ascertain quantitatively the power of intestinal juice in setting free trypsin from different quantities of zymogen. To this end, increasing quantities of fresh pancreatic juice were every time mixed with 5 cc. of intestinal juice, and the power of these mixtures to digest white of egg was examined.

	Millimeters of white of egg consumed			
	After 18 hours.	After 41 hours.	After 65 hours.	After 99 hours.
1 drop of pancreatic juice + 5 cc intest. juice	0	1.5	4	4.5
2 drops ,, ,, + ,, ,, ,,	3	7.5	10	13
3 ,, ,, ,, + ,, ,, ,,	4	8.5	13	20
4 ,, ,, ,, + ,, ,, ,,	4	8.5	12.5	20
5 ,, ,, ,, + ,, ,, ,,	4	8.5	12.5	20

These experiments show that a definite quantity of intestinal juice can only render a limited quantity of zymogen active for if 5 cc. intestinal juice are added to more than 2 drops of pancreatic juice, the digestive power is not thereby increased.

This result does not confirm the opinion of PAWLOW and his collaborators, that the active substance of the intestinal juice is to be regarded as a ferment. If this were the case, then the digestive power ought to rise, when larger quantities of pancreatic juice are used. For, it is well known that a characteristic property of ferments is to reinforce chemical changes in an *indefinite* amount of the substances upon which they act (catalysis).

These results rather lead us to suppose, that this is a case of chemical action between "enterokinase" and zymogen, in stoechiometrical proportions.

If this supposition is right, then after a certain quantity of intestinal juice has been added sufficient to transform all the zymogen of a definite quantity of pancreatic juice into the enzyme, an addition of more intestinal juice would not only not increase the transformation, but would have the opposite effect; because in that case the surplus of intestinal juice, which in itself does not digest white of egg, would act as a diluent. This in fact is the case.

The name "enterokinase", which corresponds with "lipase", "maltase", etc. suggests a ferment and is therefore not appropriate. The name "zymolysine" appears to us to be preferable.

We have already mentioned that the intestinal juice contains micro-organisms. The question arises whether these, or the products of their metabolism furnished the active substance. To answer the question whether the bacteria as such are responsible for the action, the intestinal juice was filtered through a Chamberland's bougie and the experiments with the clear filtrate, were repeated. The results were the same; although the first filtrate showed a somewhat less powerful action than the unfiltered fluid, this was not the case with the last filtrate. This must be attributed to the circumstance, that the bougie retains active substances in the beginning of the filtration as was found by SAMOÏLOW, in his experiments with solutions of pepsin and diphtheria-toxin.

The bacteria as such are therefore not the active agents, but the question still remains whether their metabolic products have any such action.

To investigate this the intestinal juice was again filtered through a Chamberland's bougie and the filtrate boiled. By this process the active substance is destroyed. If this had formerly originated from bacteria, then, after bringing the same bacteria into the boiled liquid and heating to 38°, another culture was bound to produce the same products as before. But after the bacteria had full scope for developement in the previously boiled intestinal juice, the juice obtained in this manner, mixed with fresh pancreatic juice, did not show the slightest activity on white of egg.

b. Action on gelatine.

Following the same method as was applied to white of egg, the action on gelatine was examined next. The gelatine which we used, was prepared by dissolving 10 gr. in 100 cc. of warm water; the free acid was then neutralised and a little thymol, with a little gentian-violet-solution added. The little pieces of glass tube, placed in the liquid mass, filled of their own accord. After the mass had gelatinised around and in the tubes, the latter were lifted out, cleaned and if needful the protuding gelatine cut off at the ends. ¹) As the gelatine is liquid at 38° and would consequently run from the tubes, the experiments were performed at room-temperature.

In the following summary the results are given.

The actual figures are omitted for want of room.

1st. that intestinal juice alone digests neither gelatine nor white of egg;

2nd. that intestinal juice freed from bacteria and afterwards mixed with the bacteria, originally present in the juice, is also not capable of digesting gelatine.

3rd. that pancreatic juice, mixed with water, only becomes active after a considerable time; this may be ascribed to the circumstance that the zymogen is gradually transformed into trypsin;

4th. that after 18 hours, within which time neither pancreatic juice alone, nor the intestinal juice alone is able to dissolve gelatine, the mixture of both had caused a considerable amount of solution.

5th. that this is equally the case whether the intestinal juice is filtered or not.

6th. that pancreatic juice, mixed with boiled intestinal juice, shows no activity after 18 hours, but it becomes active after 48 hours, probably for the reasons stated under 3°.

7th. that the action cannot in any way be attributed to bacteria. We have therefore shown that intestinal juice with pancreatic juice behaves precisely in the same way in relation to gelatine as in relation to coagulated white of egg.

c. Action on starch.

The digestion of starch was studied in the same way as that of white of egg and gelatine. Arrowroot starch was chosen, as in

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¹) Care should be taken when preparing the gelatine mass, not to heat too strongly and for as short a time as possible; otherwise gelatinisation does not readily take place.

LINTWAREW's work [1]), but the process of preparing it, and transferring it into the glass tubes, was a difficult one.

First the intestinal juice itself was examined. It appeared that in most cases this caused some digestive action, but this was extremely trifling compared with the powerful action which fresh pancreatic juice exerts.

	Room-temperature	
	Dissolved after 18 hours.	Cons. after 48 hours.
Intestinal juice alone................	0	3
Boiled intest. juice..................	0	1.50
1 drop pancreatic juice + 2 cc. water	9	11

In contrast with white of egg, starch is therefore immediately affected by fresh pancreatic juice; in other words, the diastatic ferment as such is already present in the pancreatic juice, which is not the case with the tryptic ferment. It might have been possible however that the pancreatic juice, used in these experiments, had already undergone a transformation for some cause or other, when the starch tubes were brought into contact with it. That this was not the case was proved from the fact that coagulated white of egg was not affected at all by the same juice. The question next arose whether intestinal juice has the power of increasing the activity of the diastatic ferment of the pancreatic juice, either by liberating an enzyme from a zymogen (in which form it might still be partly present) or by supporting the action, according to PAWLOW's hypothesis.

It was however found that it made no difference whether 2cc. water or 2cc. of intestinal juice are added to the pancreatic juice.

The possibility still remained that the quantity of intestinal juice in relation to that of pancreatic juice, had been too large and that the surplus had acted as a diluent of the amylopsine of the pancreas, in the same way as it may do in relation to trypsin.

Therefore mixtures of pancreatic juice were made, with relatively smaller quantities of intestinal juice, but the amylolytic power was never increased in any sample.

d. Action on fat.

Here the question had to be answered whether intestinal juice as such was able to split fat into fatty acid and glycerine, and further

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[1]) J. J. LINTWAREW, Dissert. Petersburg 1901 (Russian).

whether the intestinal juice possessed the power of increasing the fat-splitting action of the pancreatic juice.

It was possible to solve both questions at the same time.

The following mixtures were prepared:

(1) 2cc. intestinal juice + 0.25cc. monobutyrin.

(2) 2cc. boiled intestinal juice + 0.25cc. monobutyrin.

(3) 2cc intestinal juice + 0.01cc fresh pancreatic juice + 0.25cc. monobutyrin.

(4) 2cc. boiled intestinal juice + 0.01cc. fresh pancreatic juice + 0.25cc. monobutyrin.

(5) 2cc. intestinal juice + 0.01cc. fresh pancreatic juice + 0 25 almond-oil.

(6) 2cc. boiled intestinal juice + 0.01cc. fresh pancreatic juice + 0.25 almond-oil.

After these liquids had been allowed to stand for 24 hours at room-temperature, they were titrated with decinormal potash, phenolphtaleïn being employed as indicator. The results were as follows:

For (1) were necessary 0.3cc. $^1/_{10}$n. KOH for saturation
„ (2) „ „ 0.3 „ „ „
„ (3) „ „ 5.4 „ „ „
„ (4) „ „ 5.3 „ „ „
„ (5) „ „ 2.4 „ „ „
„ (6) „ „ 2.4 „ „ „

From these numbers the following conclusions may be drawn:

(1) and (2) show that whether boiled or unboiled intestinal juice is used, acid is present and that its proportions are equal; this leads us to suppose that the monobutyrin was already acid and that the unboiled intestinal juice had not exercised any fat-splitting action. This supposition was confirmed by titration of the monobutyrin itself.

(3) and (4) show that the mixture of pancreatic juice and intestinal juice brings about a considerable amount of splitting but that the intestinal juice is not concerned in this action since the same result is obtained whether the intestinal juice is boiled or unboiled.

It is therefore the pancreatic juice which has here effected the divison.

(5) and (6) show that the same holds true for almond-oil, except that this fat is not as quickly decomposed as monobutyrin.

The intestinal juice has therefore neither the power to split fats *independently, nor that of increasing the fat-splitting action of the pancreatic juice.*

e. Action on albumose.

So far, our investigations regarding the action of intestinal juice on proteids had been carried out with coagulated white of egg, though we also performed a few confirmatory experiments with fibrin. No attention had however been paid to liquid white of egg. Therefore we have also made experiments with that substance, but as this investigation is at present incomplete we postpone any further description of it for a later communication. We will, however describe in a few words, some experiments in reference to the influence of intestinal juice on albumose. The results obtained were striking; namely that albumose (the so-called peptonum siccum of commerce) was wholly transformed in a powerful way by intestinal juice into a substance or substances which give no biuret reaction, and are not precipitable by saturation with ammonium sulphate.

In order to ascertain whether the agent reponsible for this action is the same as or different from zymolysine, an attempt was made to separate the two actions by fractional heating, and with perfect success. After heating for 2 hours to 59°, the power of the intestinal juice to transform albumose had disappeared, but that of liberating trypsin from zymogen was still present nearly unimpaired.

By heating the intestinal juice first to 59° and then to 67°, the latter action was also lost and by heating at once to 67°, both disappeared at the same time.

After having ascertained these facts, a paper written by O. Cohnheim[1]) and published quite recently came into our hands. This writer had stated that a substance exists in the intestinal mucous membrane of dogs and cats, which is able to decompose peptone and also albumose and which he has named "erepsin". At the end of his paper Cohnheim points out, that he obtained this ferment from the finely-divided intestinal mucous membrane, but that the question is still open whether the substance is also present in the intestinal juice; in other words, whether the erepsine-action takes place intracellularly or extracellularly. *Our researches prove undoubtedly that the erepsine is secreted and contained in the intestinal juice.*

What the substances are which are produced by the influence

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[1]) O. Cohnheim. Zeitschr. f. physiol. Chemie **33.** 1901. S. 452.

of erepsin on albumose and peptone, COHNHEIM has not been able to determine, and we also have refrained from such investigations at present, not knowing for how long the juice of the patient would be at our disposal, and to be able to answer other questions, which demanded less time and did not seem less important.

III. Secretion of the intestinal juice.

The *quantity* of intestinal juice from the loop varied under normal circumstances from 50 to 125 cc., averaging for the 66 days during which our observations were made 88 cc. in the 24 hours, the morphological elements included. This quantity is much larger than that which was obtained by previous investigators. To know whether there were variations in the course of the day, the patient was requested to collect the juice for a given length of time every three hours and each time to put it into a different bottle. This method of collection could not however be carried out during the night (8 p.m. to 8 a.m.).

This was continued for 19 days, and when the volumes obtained in the same periods were added together it was found that the largest quantity was collected in the evening between 5 and 8 o'clock.

If we compare further what was collected during the day between 8 a. m. and 8 p. m. and that which was obtained during the same period of time in the night, then much more was obtained during the night (1027) than during the day ($766^3/_4$).

We cannot draw any very definite conclusions from these figures because it was impossible to ascertain exactly how the intestinal loop was situated; it is not improbable that, in a lying position, a certain quantity could flow down, which was retained in a standing, sitting and walking condition.

The difference between the periods 2—5 p. m. and 5—8 p. m. is striking. This difference indicates that there are besides the intestinal lumen, still other factors which inflnence the quantity of the secretion.

SCHEPOWALNIKOFF found that, with dogs, the secretion of the intestinal juice was of a purely local nature. To excite secretion a caoutchouc tube had to be inserted for a considerable distance into the intestine through the fistula, otherwise not a drop of intestinal juice was excreted; a glass tube similarly inserted was not a sufficient stimulus.

In a later communication we hope to speak of the influence of different physiological conditions on the quantity, composition and action of the juice. At present we only wish to state that when the drainage-tube was lying in the intestine for a length of 12 c.M., 70 c.c. were secreted every 24 hours; but when the tube was pushed in further to a length of 22 c.M., the quantity rose to 170 c.c.; and the next day, when the tube was again pushed in only a small distance into the intestine, the quantity sank to 70 c.c.

There is therefore no doubt that in man also a local excitation has a powerful influence on the quantity of the secretion. This is what one would expect, and no doubt under physiological circumstances, the amount secreted is regulated by the necessities of the case.

Summary.

The following are the principal results obtained from the researches described in the foregoing pages :

1. The intestinal juice examined by us, does not possess the power in itself of digesting white of egg or fat; it is however, in a small degree, able to hydrolyse starch.

2. Whereas the intestinal juice exercises an extremely small influence on the ordinary constituents of food, *it is of great importance when it acts in conjunction with pancreatic juice.*

When a freshly excised pancreas is cut into small pieces, pressed out and the expressed juice is brought into contact with coagulated white of egg no digestion of that substance occurs; if however the pancreatic juice is mixed with intestinal juice, then a considerable amount of digestion takes place.

This is best explained on the hypothesis that the proteolytic enzyme, trypsin, is not present as such in the juice, but in the form of its precursor (trypsinogen); this zymogen is in itself inactive. By means of *the intestinal juice trypsin is liberated from its zymogen.*

3. This character is lost when the intestinal juice is boiled and also when it is heated for 3 hours to 67°.

4. Pawlow and his fellow-workers, who first drew attention to the co-operative action of the two juices in dogs, regard the active substance of the intestinal juice as a ferment and have named it enterokinase. We do not agree with this, but are of the opinion that it is no ferment, rather a compound, which causes a transformation of the zymogen into the enzyme, trypsin, in stoechiometrical proportion.

The name "zymolysine" seems to us more appropriate for the substance under consideration.

5. The combination of pancreatic and intestinal juice acts on gelatine in exactly the same way as on white of egg.

6. The other pancreatic ferments, the diastatic and the fat-splitting, are present in the pressed out juice in a fully formed condition; the intestinal juice examined by us does not exercise the least influence on either of them.

7. In addition to zymolysine there appear to be present in the intestinal juice, a substance, probably a ferment, which possesses in a high degree the power of breaking up albumoses into simple substances which have not yet been fully identified.

This is the same substance, which O. COHNHEIM obtained a short time ago, in a different way, from the intestinal mucous membrane of dogs and cats and to which he has given the name of "erepsine".

COHNHEIM leaves it an open question whether erepsine only acts intracellularly or whether it is also secreted in the juice and therefore acts in the intestinal lumen. Our investigations clearly show that erepsine is present in man, and that it is contained *in the intestinal juice.*

8. Zymolysine and erepsine are two different substances. This is proved by the fact that after heating the intestinal juice to 59° for 2 hours, the erepsine-action is lost, whereas that of the zymolysine continues to exist in nearly the same condition. If the juice so treated is subsequently heated for 3 hours to 67°, then it also loses its power of rendering fresh pancreatic juice active upon white of egg and gelatine; in other words, then also the zymolysine is destroyed.

9. By local, mechanical excitation of the intestinal mucous membrane, the secretion is considerably augmented.

Physiology. — „*On the influence of feeding on the composition of the fat of milk*". By DR. B. SJOLLEMA (Communicated by Prof. H. J. HAMBURGER).

(Communicated in the meeting of February 22, 1902).

As is universally known, the fat of milk is distinguished from nearly all other kinds of fat, because apart from glycerides of non-volatile fatty acids other glycerides are present, which contain both non-volatile fatty acids and fatty acids which are volatile with water vapour such as butyric acid. This has been the means to distinguish the fat of milk from other kinds of fat.

As it was generally supposed that the amount of volatile fatty acids in the fat of cow's milk does not vary considerably, the determination of the amount of volatile fatty acids has been applied as a means to detect the adulteration of butter with other kinds of fat (margarine) and this is still done at the present time.

In this investigation a conventional method is usually adopted, which does not reveal the total quantity of the volatile fatty acids, but gives a number which serves as a measure for the amount of volatile fatty acids and at the same time enables us to form an opinion as to the genuineness of the butter.

According to this conventional method 5 grams of filtered butterfat are saponified with 2 cc. of very strong soda-lye, with addition of 20 cc. of glycerin. After addition of 90 cc. of hot water the soap is decomposed with 50 cc. of dilute sulphuric acid. With due regards to certain rules the liquid is distilled until 110 cc. are collected. The distillate is titrated, after filtration, with $n/_{10}$ baryta water (indicator phenolphthaleïne). The number of cc. of baryta water required for the neutralisation, gives the figure which represents the amount of volatile fatty acids and is called the REICHERT-MEISSL-WOLLNY-number, (R.M.W.-number).

For the examination of milkfat, the determination of the index of refraction has come into use of late. It has been ascertained that this physical constant is smaller with fat of milk than with nearly all other kinds of fat.

For the determination of the index of refraction of milkfat, the oleorefractometer of ZEISS is always used. Instead of the real index of refraction the so-called refraction-number is generally given. This is the number of the empirical division of the scale of the instrument, which coincides with the line of demarcation. (The refraction-number 40 corresponds with the index of refraction 1.4524; 60 with 1.4659.)

It is remarkable, that the butter, which in certain months of the year is being produced in the northern provinces of the Netherlands, shows a considerable lower R.M.W.-number and a considerable higher refraction-number than butter, produced in other parts of the country in the same season. From an economical, as well as from a scientific point of view, it is of importance to trace the reason of this.

With regard to the economical side of the problem, which has caused this investigation, it must be mentioned that the butter exported from the Netherlands with a low R.M.W.-number and a high refraction-number, has, in other countries, been repeatedly charged with containing foreign fat. Latterly it has become known that the fat of our milk possesses an abnormal composition during certain months of the year, but owing to the difficulty of distinguishing it from adulterated butter, this fact is not taken into account.

On the contrary in some countries it has been considered necessary to prohibit the sale of such butter.

Enabled by the Government to find the means of raising the amount of volatile fatty acids of the fat of milk, I have during my experiments, which are in progress since the middle of last year, arrived at results, which also from a purely scientific point of view, are not to be considered unimportant, as I expect they will throw some light on the still dark problem of the process of the secretion of milk.

In the first place we may mention here the experiments, which have been made to ascertain the influence of feeding with sugar-beet-foliage, on the R.M.W.-number and the refraction-number of the butter. It is unnecessary to state here the motive which led to making experiments with beet-foliage.

The cattle, selected for this experiment, grazed in old meadow-land, had taken no additional food when the experiment commenced, and fed exclusively on grass.

In order to ascertain the R.M.W.-number and the refraction-number of the fat of the milk of each of the 4 cows, samples of milk were taken from each cow three days in succession, namely on the 19th, 20th and 21st of October, and the fat obtained therefrom by churning was examined.

These and the samples of milk taken later on, have been obtained by mixing equal quantities of morning- and evening-milk of the same day. Each sample contained $1^1/_2$ litre and was always sent without delay to the Experiment-station, where they were made sour with buttermilk and churned as soon as they were ready for it [1]). Milk of different cows was never mixed.

On the 22nd of October the 4 cows were divided into two groups. Until that day they had grazed in the same meadows, but had to be separated as two of them, (Nos 1 and 2), were to be fed on beet-foliage [2]).

To prevent any change which might be caused by removing them

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[1]) From experiments made on purpose it has appeared to me, that the means of souring and the degree of sourness of the milk ready for churning, have no influence on the R.M.W.-number and the refraction-number.

[2]) It must be understood that beet-foliage not only consists of the leaves of the beet, but of the heads of the beet-roots as well. These adhere to the foliage at harvest time.

It may be taken for granted that 20 to 25 percent of the weight of the so-called beet-foliage consists of the tops of beet-roots.

to other meadows, the cows which received the beet-foliage were left in the same place.

The two others (N^os^. 3 and 4) were removed to meadows of the same quality. When the additional feeding of the two cows (N^os^. 1 and 2) with beet-foliage had lasted some time (7 days) and the effects were very clearly visible, a commencement was made, namely on the 29th of October, with the feeding of the other cows (N^os^. 3 and 4) with beet-foliage.

On the 7th of November the 4 cows were brought to the stables. They were fed until the 19th of November with beet-foliage and in addition with clover-hay, straw, ½ kilo of linseed-cake and 2 litres of oats.

In Table I are recorded the way of feeding with beet-foliage, the % of fat in the milk, the refraction-numbers and the R.M.W.-numbers of the milkfat.

TABLE I.

Date of sample taking	Way of feeding.	% of Fat in the milk.				Refraction-numbers at 25° C.				R. M. W-numbers.			
		Nº 1	2	3	4	Nº 1	2	3	4	Nº 1	2	3	4
19 Oct.	All without beet-foliage.	3.18	3 4	3.45	3.7	55.4	55.9	56.1	55.5	20.4	19.7	13.8	18.8
20 »		2.85	3.35	3.72	4.03	55.1	55.9	56.1	55.5	20.2	19.5	14.35	19
21 »		2 8	3.7	3.6	3.5	55.4	56.5	55.2	55.2	20.65	18.7	—	—
23 »	N 1 a. 2 with Nº 3 a. 4 without beet-foliage.	2.45	3.63	3.75	4.05	53.1	53.3	56.3	56.1	22.4	22	13.6	—
26 »		3.35	3.55	4.—	4.55	49.6	50.8	56.3	56.2	25.7	27.9	14.7	19
28 »		3.2	3.25	4.—	4.25	50	50.5	56.3	56.2	25.3	28.4	15.05	19
2 Nov.	All with beet-foliage.	3.7	4.5	3.6	3.48	50.3	49.5	50.4	50.2	25.9	30.6	24.3	26
13 »	In the stables with beet-foliage	3.7	4.-	3.82	3.95	51.7	52.8	52.8	52.3	26.4	25.4	23.8	22.4
23 »		3.45	4.05	4.—	4.05	50.8	50.6	51.5	51.1	27.8	29.1	26.55	26.8

In Table II are given some particulars of the 4 cows, the average R.M.W.-numbers and the average % of fat in the milk for the different periods.

TABLE II.

N°.	Age in years.	Number of times of calving	Calved in 1901 on	Conceived in 1901 on	Yield of milk per diem.	R.M.W.-number.			% of Fat in the milk.	
						Average without foliage.	Average with foliage in the meadows.	Difference.	Average without fol.	Average with fol. in the meadows.
1	4	3	20 May	20 July	± 10 L	20.4	25.6[1])	5.2	2.94	3.18
2	2	1	16 March	19 June	± 5 »	19.3	29.—[1])	9.7	3.45	3.73
3	3	2	7 April	5 July	± 6 »	14.3	24.3[2])	10.—	3.6	3.84
4	3	2	8 »	15 »	± 5 »	19.—	26.—[2])	7.—	3.74	4.08
					Average	18.25	26.2	8 —		

The above-mentioned figures show, that a considerable increase of the R.M.W.-number *average* 8, *that is about* 44 %, *has to be recorded after feeding with beet-foliage.* With the two cows, which served for control from 22—29 October, the R.M.W.-numbers remained pretty constant during those 7 days. The figures given above also show, that the great influence of feeding with beet-foliage, which has been noticed with the cows N^os^ 1 and 2, between 22 and 29 October, was confirmed with the cows N^os^ 3 and 4 [3]). Among the 4 cows there was one the milk of which yielded a fat containing but a small quantity of volatile fatty acids [4]). With this cow the R.M.W.-numbers ranged between 13.6 and 15.05 before the feeding with beet-foliage.

Of the results contained in Table I a graphic representation is given; in fig. I, the dates when the samples were taken, are put down on the *X*-axis and the R.M.W.-numbers on the *Y*-axis; in fig. II likewise the dates on the *X*-axis and the refraction-numbers on the *Y*-axis.

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[1]) This is the averagefigure calculated from the R.M.W.-numbers of the butter of 26 October, 28 October and 2 November.

[2]) This is the figure of only one sample.

[3]) It must be observed that the % of fat in the milk rose after feeding with beet-foliage and that the amount of milk yielded by Nos 2 and 4 remained almost the same, but increased greatly with Nos 1 and 3.

[4]) It is evident from these as well as from several others of our experiments, that the amount of volatile fatty acids in milkfat is greatly dependent on the nature of the individual cow and that under unfavourable circumstances this is probably the case in a stronger measure than under favourable ones.

It is remarkable how steep the curves of the R.M.W.-numbers ascend, how quickly therefore the influence of the beet-foliage is felt.

On the 22nd of October a commencement was made with feeding Nos 1 and 2 with beet-foliage and on the 23d of October the R.M.W.-numbers of the fat, obtained from the milk, which these cows had, therefore, yielded about 18 and 30 hours after the commencement of feeding with beet-foliage, showed already a considerable increase. The increase amounted to 1.75 and 4.7. After 5 days the increase was 5.05 and 9.2, therefore about the same as the average increase during the whole period. With the cows Nos 3 and 4 the R.M.W.-numbers increased in 3 days with 9.25 and 7.0.

As appears from the above-mentioned numbers and the curves, the feeding with beet-foliage has not only very favourably affected the R.M.W.-numbers, but likewise the refraction-numbers; these have decreased considerably; namely about 6.

In the second place we may mention here the experiments, which have been undertaken to solve the question as to which ingredient of the beet-foliage the above-named results must be attributed.

Owing to the results of one of my investigations, where the influence of beet-foliage without the tops of the beets was compared with that of beet-foliage from which the tops were not severed, it was probable that the favourable effects stated above were due to the sugar contained in the beet-root-tops.

During the above-mentioned experiment the animals took each about 1.5 kilo of sugar per diem. One of the next experiments was made with sugar only, whilst in a second experiment, with other cows, first the effects of molasses and next those of sugar were tried. For the experiment with sugar alone, three cows were taken which were kept in a stable and fed with 6 litres of barleymeal, $^1/_2$ kilo of linseed-cake and also with some hay and straw. One of the cows served as controlling-cow, the two others, Nos 2 and 3, received, in addition to the above-named ration, also sugar. The sugar consisted of the raw, unrefined article. The daily allowance was gradually raised from 0.4 to 2 kilos per cow and then again reduced to 1 kilo.

Before commencing the feeding with sugar, the fat of the milk of the three cows taken on 6th, 8th and 10th January was examined.

In Table III are recorded the quantities of sugar wich were administered, the R.M.W.-numbers of the milkfat and the $^0/_0$ of fat in the milks.

TABLE III.

Date of sample-taking.	Quantity of sugar administered to N⁰. 2 a. 3	R.M.W-numbers.			% of fat in the milk.		
		N⁰ 1	N⁰ 2	N⁰ 3	N⁰ 1	N⁰ 2	N⁰ 3
6 January	None.	30.9	28.6	24.6	3.25	4.35	3.35
8 »		32.5	27.5	24.6	3.05	3.9	2.9
10 »		31.5	27.4	25.3	3.05	3.85	3.05
15 »	Increasing quantities	30.6	31.2	27.—	3.—	3.65	2.75
20 »	2 kilos	30.6	28.95	26.3	3.35	3.5	2.6
23 »		30.1	29.4	27.65	3.—	3.45	2.55
27 »	Decreasing quantities 1.5 and 1 kilo	28.3	29.9	29.—	3.1	3.45	2.8
30 »		29.6	31.—	29.3	3.—	3.75	2.85
3 February.		28.9	28.9	28.8	2.9	3.7	2.9

In Table IV are recorded, besides some particulars of the cattle, the average R.M.W.-numbers and the average percentage of fat in the milk of the different periods.

TABLE IV.

N⁰.	Born	Number of times of calving	Calved 1901.	Conceived in 1901.	Yield of milk p. diem in Litres, from 6—30 Jan.	R.M.W.-numbers.				% of Fat in the milk.	
						Average from 6-10 Jan.	On 15 Jan.	Average from 15 Jan.—3 Febr.	On 3 Febr.	Average from 6-10 Jan.	Average from 15 Jan.—3 Febr.
1	Spring 1898.	2	27 May	Not	6.5	31.6	30.6	29.7	28.9	3.12%	3.06%
2	Spring 1898.	2	13 March	Not	5.5	27.8	31.2	29.9	28.9	4.03 „	3.63
3	August 1898.	2	27 „	In June	6.5	24.8	27.—	28.—	28.8	3.1	2.74

From the above-given figures it appears that after 4 days (on 15th Jan.) of sugar-feeding, the R.M.W.-numbers of N^os^. 2 and 3 had risen respectively 3.4 and 2.2, whereas with N⁰. 1, the controlling-cow, a decrease of 1.0 was observed.

During the whole period of sugar-feeding the R.M.W.-number was with N⁰. 2, 2.1 and with N⁰. 3, 3.2 higher than the average before the 10th of January. With the controlling-cow on the other hand, a decline was recorded to the amount of 1.9.

In order to be able to judge correctly of the effect of the sugar-feeding, it must be observed, that during the feeding with 2 kilos of sugar, a tendency to decrease was noticed; evidently this quantity was too much [1]).

When the daily allowance was diminished an actual rise was again recorded. Omitting the figures of the fat of the milk during the days when 2 kilos of sugar were given (20th and 23d Jan.), the average rise of the R.M.W.-numbers of Nos. 2 and 3 is 2.45 and 3.7. The decrease with the controlling cow amounted on an average to 2.25, if the figures of the fat of the milk of No. 1 of 20th and 23d January are also omitted.

If it be taken for granted, that the R.M.M.-numbers of the fat of the milk of the cows Nos. 2 and 3 would have decreased like those of No. 1, if these had not been fed on sugar, then the influence of the sugar-feeding is found, by adding the decrease which was noticed with the controlling-cow to the increase found with each of the cows fed on sugar, *so that favorable influence of the sugar with No.* 2 *amounts to* 4.7 *and with No.* 3 *to* 5.95.

Of the results, recorded in Table III concerning the R.M.W.-numbers, fig. III gives a graphic representation. The dates are put down on the X-axis, the R.M.W.-numbers on the Y-axis.

For the experiment, in which first molasses and afterwards sugar was given, three of the cows, namely Nos. 1, 2 and 3 were used, which had also served for the experiment with beet-foliage. They were in the stables and were fed with oats, broken barley, linseed-cake, hay and straw. The quantities were different, considering the unequal age and the difference in the yield of milk.

One of the cows, namely No. 2 was here the controlling-cow. The molasses used were absorbed in peat and contained about 35 % of sugar. The daily allowance of molasses was gradually raised — from 24 Dec.—3 Jan. — to 1.5 kilo and lasted from 3—14 Jan. As the cows were highly pregnant, I did not venture to give larger quantities. The quantity of sugar, which the animals took daily, amounted therefore only to about 0.5 kilo. From 14th Jan.

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[1]) That no other cause, except the sugar-feeding, is responsible for these decreases would appear from the fact, that, during these days the R.M.W.-numbers were not lower than otherwise for this controlling-cow, on the contrary rather higher.

raw sugar was administered to the same cows, at first 1 kilo for some days, afterwards 1.5 kilo each cow, per diem.

For this experiment the same sugar was used, which served for the preceding experiment.

Before commencing the feeding with molasses, the fat of the milk taken on 17, 20 and 23 Dec. from the 3 cows was examined.

In Table V are stated the molasses- and sugar-rations, the R.M.W.-numbers of the milkfat and the % of fat in the milk.

TABLE V.

Date of sample-taking.	Quantity of molasses or sugar administered to No. 1 and 3.	R.M.W.-numbers.			% Fat in the milk		
		N⁰. 1.	N⁰. 2.	N⁰. 3.	N⁰. 1.	N⁰. 2.	N⁰. 3.
17 Dec.	No molasses nor sugar.	24.8	26.3	27.2	3.45	4.10	3.50
20 »		25.3	27.2	26.6	3.55	4.15	3.60
23 »		24.2	27.—	26.4	3.10	4.10	3.20
28 »	1 kilo of molasses.	26.8	27.6	27.—	3.45	4.05	3.60
3 Jan.		26.6	27.2	26.3	3.20	3.90	4.00
7 »	1.5 kilo of molasses.	27.7	27.2	27.8	3.53	4.20	3.2
10 »		27.—	26.7	28.4	3.30	4.15	3.35
13 »		27.65	26.55	27.7	3.40	4.15	3.—
17 »	1.— or 1.5 kilo of sugar.	28.55	25.1	28.1	3.10	3.85	2.8
19 »		28.1	24.1	28.2	3.30	4.6	3.2
20 »		27.5	23.7	27.9	3.47	4.35	3.5

In Table VI are given the average R.M.W.-numbers of the milkfat and the average % of fat in the milk for the different periods. [1])

From the above-mentioned figures it appears, that in the period of feeding with molasses (from 3—13 *Jan.), the average R.M.W.-number had risen* 2.45 *with N⁰.* 1, *and* 0.85 *with N⁰.* 3. *The R.M.W.-number of the controlling-cow remained pretty constant, it rose from* 26.8 *to* 26.9.

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[1]) The particulars of the cattle have already been stated in Table II.

TABLE VI.

N°. of the cow.	Yield of Milk.	R.M.W. Numbers.				% of Fat in the Milk.	
		Average 17–23 Dec.	on 28 Dec.	Average. 3-13 Jan	Average 17–20 Jan.	Average 17–23 Dec.	Aver. 28 Dec.-20 Jan.
1	± 10 L.	24.8	26.8	27.25	28.05	3.37	3.34
2	± 5 L.	26.8	27.6	26.9	24.3	4.12	4.16
3	± 5 L.	26.7	27.–	27.55	28.05	3.43	3.33

By changing from molasses to sugar another rise occurred with both cows; with No. 1 to the amount of 0.8, with No. 3 to 0.5, consequently the rise in regard to the period of 17—23 Dec. amounted to 3.25 and 1.35. Taking into consideration that the R.M.W.-number of the controlling-cow came down to the average of 24.3 in the latter period, therefore to the amount of 2.5 the influence of the feeding with sugar appears very clearly; for again it may be taken here for granted that, with No. 1 and 3, a decrease would have had to be recorded as well, if they had not received sugar, which can be easily conjectured, because at the end of the lactation-period a decrease usually occurs in the R.M.W.-number.

Taking for granted, that with N^{os}. 1 and 3 a decrease, amounting to that of No. 2, namely of 2.5 would have taken place without sugar-feeding, *then the rise caused by sugar, amounted with No.* 1 *and* 3, *resp. to* 5.75 *and* 3.85.

Of the results, which are recorded in Table V concerning the R.M.W.-numbers, fig. IV gives a graphic representation. The dates are put down on the X-axis, the R.M.W.-numbers on the Y-axis.

It should here be observed, that with cattle kept in stables, such a considerable influence cannot be expected, because in the stables the R.M.W.-numbers are higher, even without artificial means, than in the autumn when the cattle are grazing in meadow-land. Compare figures of Table V with those of Table I.

From the results of the two last series of experiments the conclusion may therefore be drawn, that sugar exercises a considerable influence on the increase of the volatile fatty acids in milkfat. The question, as to how this process takes place, can not as yet be explained. Investigations into this matter are being made by me at present.

Dr. J. E. Tulleken, assistant at the Experiment-station, has assisted me in carrying out the above-mentioned experiments, and

Mr. J. J. HUISMAN, dairy-inspector took part in the experiments with the beet-foliage.

Government-Agricultural-Experiment-Station
Groningen.

EXPLANATION OF THE GRAPHIC REPRESENTATIONS.

Fig. I. Influence of feeding with beet-foliage on the R.M.W.-numbers (amount of volatile fatty acids) of milkfat.

The dates of sample-taking are put down on the X-axis, the R.M.W.-numbers on the Y-axis.

Fig. II. Influence of feeding with beet-foliage on the refractionnumbers of milkfat.

On the X-axis the dates, on the Y-axis the refractionnumbers have been put down.

Fig. III. Influence of feeding with sugar on the R.M.W.-numbers of milkfat.

On the X-axis the dates, on the Y-axis the R.M.W.-numbers have been put down.

Fig. IV. Influence of feeding with molasses and with sugar on the R.M.W.-numbers of milkfat.

On the X-axis the dates have been put down; on the Y-axis the R.M.W.-numbers.

Chemistry. — "*The densities of mixtures of hydrazine and water.*" By Mr. J. W. DITO. (Communicated by Prof. C. A. LOBRY DE BRUYN).

Some years ago I was able to prepare hydrazine, a substance previously unknown in the free state (discovered in the form of its compounds by CURTIUS) and to study many of its properties [1]). Several physical constants [melting point, boiling points at different pressures, specific gravity, index of refraction [2]), critical temperature and heat of dilution [3])] were then determined.

The study of hydrazine has not since been continued. Still it was deemed of importance to again take up the study of a substance which belongs to a class of liquids of comparatively simply constitution: $H_2N.NH_2$ closely related to ammonia, miscible with water with considerable evolution of heat and according to several of my experiments, a good solvent for many salts.

Mr. DITO has now in the first place commenced with the determination of the densities of mixtures of water and hydrazine. This was of all the greater importance because CURTIUS had stated that

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[1]) Versl. Akad. Juni 1895. Recueil, **13**. 433, **14**. 88, **15**. 174.

[2]) By Prof. J. F. EYKMAN.

[3]) With Dr. REICHER.

if one tries to liberate hydrazine from the aqueous solutions of its salts by bases, it is always combined with water and yields on distillation a hydrate of the composition: N_2H_4-H_2O which boils at 119° and in which the water is very firmly united with the hydrazine.

Since free hydrazine (b.p. 113°.5) had become accessible it was possible to investigate the whole system hydrazine + water and to see whether the existence of CURTIUS's constant-boiling hydrate would show itself in the curve of the specific gravities of the mixtures. It was *a priori* not improbable that this should be the case and experiment has confirmed this idea.

From the subjoined table in which each figure is the result of two or more concordant determinations and from the curve constructed with the aid of these figures, it appears that on mixing hydrazine with water a contraction takes place. It will be noticed that the maximum density corresponds with the formula $N_2H_4.H_2O$.

%N_2H_4	mol. N_2H_4 on 100	spec. weight $d\frac{15}{4}$
100.0	100	1.0114
90.8	84.1	1.0300
84.0	74.7	1.0358
80.0	69.25	1.0379
78.5	67.25	1.0400
74.9	62.65	1.0421
72.0	59.15.	1.0440
67.4	53.75	1.0464
64.1 [1]	50 15	**1.0470**
59.9	45.7	1.0464
55.3	41.0	1.0461
46.4	32.75	1.0425
40.85	28.0	1.0389
34.25	22.65	1.0340
26.45	16.85	1.0272
14.0	8.4	1.0142

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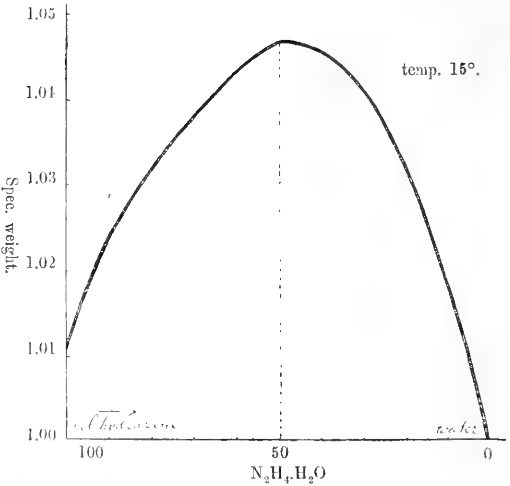
[1] Nearly N_2H_4 H_2O.



It will be seen that the system $N_2H_4+H_2O$ behaves in an analogous manner to the system SO_3+H_2O which has been recently studied by KNIETZSCH in his well-known research on sulphuric acid.

Here we also meet with contraction; there does not, however, exist a maximum density for $SO_3.H_2O$, (but for 2 $SO_3.H_2O$) which phenomenon may be explained by the great difference in density of the two components.

Efforts will be made to determine the boiling point curve for the system $N_2H_4+H_2O$ (or at least for $N_2H_4.H_2O+H_2O$).

Chemistry. — "*A method for separating crystals from alloys*". By Dr. C. VAN EYK. (Communicated by Prof. H. W. BAKHUIS ROOZEBOOM).

The following methods have, up to now, been applied in investigating the constitution and structure of metallic alloys.

1. Microscopical examination of polished surfaces so as to distinguish the several crystalline elements often rendered more conspicuous by etching (BEHRENS, OSMOND and others).

Differentiation of crystalline elements presents as a rule no great difficulties but it is difficult to come to a proper conclusion as a quantitative analysis is seldom possible. In some cases it is possible to approximately determine the composition of the crystals and motherliquor by testing the etchingliquor and the grindings of the protruding crystals (BEHRENS). An accurate quantitative analysis by this process is, however, an impossibility.

2. Determination of the solidifying curve (LE CHATELIER, ROBERTS-AUSTEN, i. a.). From the course of this curve, the existence of chemical compounds may often be traced and indications found as to the crystallising of mixed crystals. This determination by itself is, however, rarely conclusive. Guided by the modern ideas about the equilibria of phases (BAKHUIS ROOZEBOOM, Zeit. phys. Chem. **30**, 385) which occur in alloys, HEYCOCK and NEVILLE have succeeded, by combining the two above-mentioned methods, in obtaining almost positive certainty about the very complicated relations which occur in alloys of copper and tin. The circumstances that the crystals which separate on solidification may be readily fixed by chilling and that the heat-effects which occur on solidification and transformation are large have materially contributed to make this possible.

3. The separation of the crystals which are deposited from the alloy on cooling.

The methods used for this purpose are:

a. The draining of the motherliquor when solidification sets in; the amount of motherliquor adhering to the crystalline mass is then determined by adding a third metal (VAN BYLERT, Zeit. phys. Chem. 8, 358);

b. by pressure between hot plates; the still adhering motherliquor is dissolved in an acid which does not attack the crystals (BEHRENS, Verslag Kon. Akad. **1898**, 58);

c. filtration (KEMP on amalgams, Zeit. anorg. Chem. **17**, 284 and **25**, 1).

The last method can only be applied to alloys which have a very low melting point and gives a very incomplete separation of crystals and motherliquor.

In cases where different chemical compounds or mixed crystals are likely to be deposited from an alloy during its solidification, the first two methods must be objected to because the temperature is not constant. They are, moreover, only applicable to alloys which have a comparatively low melting point and oxidise but little in contact with the air; it is also by no means sure that on dissolving the motherliquor in an acid the crystals are not attacked. In addition to the determination of the solidifying curve and the microscopic test, the isolation of the crystals from the partly solidified

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alloy will be of great importance in the investigation of metallic alloys.

The following method is to a large extent free from the objections mentioned.

After having first determined the temperature at which crystals commence to deposit from the molten alloy, about 30 grams of the metal are introduced into a testtube of difficultly fusible glass which has in the centre a capillary constriction; if necessary a small plug of asbestos is inserted above this. The open end of the tube is now drawn out after which it is exhausted and sealed. The tube thus filled is placed in a small electric furnace of such dimensions that the tube quite fills it. The furnace consists of an unglazed porcelain tube wound with platinum wire and is surrounded by a wider tube, the space between being filled in with fireclay. This furnace surrounded by asbestos is now placed in one of HUGERSHOFF's centrifugal-machines such as are used for the estimation of fat in milk but provided with a higher rim so as to have more space at disposal.

Two holes are bored in the lid of the centrifugalmachine to admit the conducting wires to the furnace.

These wires are connected at *A* and *B* (fig. 1) with a set of copper rings *C* and *D* which may be screwed on to the axis of the centrifugalmachine. The figure gives sufficient information as to the working arrangement. The two copper rings are isolated by ebonite placed on the screw and isolated from the latter, also by ebonite. By attaching spring contacts to the rings *C* and *D*, an arrangement was obtained for admitting the current and heating the furnace, during the rotation.

That the temperature during the operation remains constant could be ascertained by having the lid of the centrifugalmachine fitted with a slide valve at the place where the furnace was fixed. Before and after the whirling the valve was opened and the thermometer, inserted in a testtube filled with asbestos which was placed in the furnace, was read off. For temperatures up to 300° no greater difference than 2° has been observed before and after the whirling.

To ascertain in how far this method can give good results, a few alloys which yield crystals of known composition have been subjected to whirling, namely $SbSn_2$ (BEHRENS, Verslag Kon. Akad. Wetensch. **1898** 58) and Cu Sn (HEYCOCK and NEVILLE Proc. Roy. Soc. **69** 320).

The experiments have, therefore, provisionally been made at temperatures which may be measured with the mercury thermometer. Trials with alloys of high melting points are being made in porcelain tubes.

The crystals which are obtained from an alloy of 15 % Sb and

Dr. C. VAN EYK. A method for separating crystals from alloys.

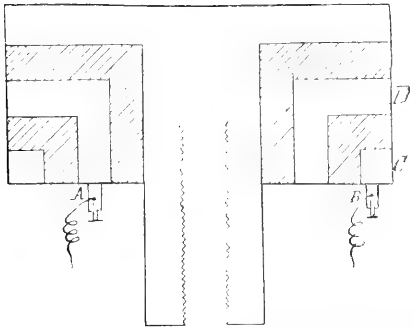

Fig. 1.

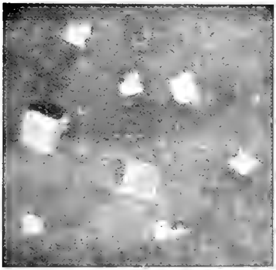

Fig. 2.

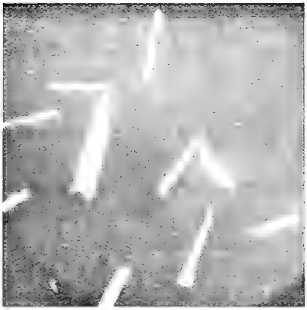

Fig. 3.

85 % Sn or 16 % Sb and 84 % Sn by whirling at a constant temperature of 270° (the commencing solidifying points of the alloys are 284° and 290°) are partly found loose (fig. 2); near the narrowing of the tube they are partly united to a loose cake by the adhering motherliquor.

By mixing with 1 % of Ag it could be ascertained how much motherliquor had been retained by the crystals (VAN BYLERT, l. c.).

The amount of Ag in the loose crystals was so small that it could not be estimated. In the loose crystalline cake 8 % of motherliquor was found.

The crystals contain 32.5 % of Sb, $SbSn_2$ contains 33,6 % of Sb. In the case of SnCu crystals (fig. 3) which were obtained by whirling at 300° an alloy containing 4.5 % of Cu, the amount of loose crystals is much smaller and insufficient for analysis.

Several alloys containing 4.5 % of Cu were therefore whirled and the loose crystals (fig. 3) collected as far as possible and analysed. Found 67,9 % of Sn, calculated 65 % of Sn.

The crystals cannot therefore be completely freed from motherliquor by whirling, but by operating in this manner it will be as a rule easy to ascertain whether pure metals, compounds or mixed crystals are deposited from the alloy.

If a change takes place in the composition of the alloy, the separated crystals in the first two cases will remain of constant composition within a few per cent, while in the case of mixed crystals a gradual change will be observed.

Physics. — Communication n⁰ 78ᵃ from the Physical Laboratory at Leiden: "*Isotherms of diatomic gases and their binary mixtures. II. The determination of density with the piezometer of variable volume for low temperatures*". (By Prof. H. KAMERLINGH ONNES and H. H. FRANCIS HYNDMAN).

§ 7. *The measurements.*

These are of two kinds. 1st. The density being given in terms of the normal density d_N (at 0.0° C. and 1 At.) the quantity of gas contained in the piezometer tube is determined by measurements of the normal volume V_N. 2nd. The volume of the compressed gas is measured under known conditions of pressure and temperature. The measurements of V_N are made both before and after the measurements at higher pressure. In general this is very desirable and to make it possible the closure of the piezometer tube by a *U*-tube with mercury was devised (see Comm. n⁰ 50). In our case however it is absolutely necessary because the screw joints g_2. g_3 (Comm. n⁰ 69) can develope a leak.

1. The normal volume is in principle determined in the manner employed by SCHALKWIJK Comm. n⁰ 70 V.

The bath used by him was not large enough to enclose the whole piezometer, but only tbe whole of the large volume. The graduated stem b_4 and the piezometer bulb were thus outside the bath and the constancy of the temperature of these parts had to be otherwise provided for. Had we sought the same accuracy as required by SCHALKWIJK, a new waterbath of the total length of the piezometer would have been required.

However as we could not from the other measurements expect such a high accuracy, we were satisfied here also with a somewhat less degree and found that the temperatures of the divided stem b_4 and of the piezometer bulb f_1 could be sufficiently determined by the use of simple water baths. Owing to the great length of the large reservoir and to its near equality with that of the bath for constant temperature, it was particularly necessary to avoid real differences in temperature between the bottom and top of the bath. To attain this it was necessary to arrange that the temperature of the room should not differ more than a few degrees from that of the bath. The latter was thus set at a temperature that could be reasonably attained in the room, and the room kept as near as possible to this temperature.

A small change was also made in the xylene regulator. The tube t (see Plates, Comm. n⁰. 70, III) which was formerly of glass was now brass above and steel where it entered the mercury, the requisite form could thus be given to the end and narrower tubes employed. In this way a greater sensitiveness was attained combined with an adjustability of some five degrees by merely moving the upper brass tube up or down in its brass support, without the troublesome removal or addition of mercury which was necessary before and which would also be required for greater changes.

The thermostat was also raised some 1.5 M. to obtain a quicker flow, which was shown to be desirable before (Comm. n°. 70, V).

In the earlier measurements the pressure was read by a controlled aneroïd at intervals. When however the barometric height is changing rapidly a larger error may arise from this than is allowable. In the later measurements the arrangement of Comm. n°. 60 was employed, the joint b (Plate II, Comm. n⁰. 69) being connected by glass tubes to an air reservoir at 0.0° C. and a barometer, the whole forming a perfectly closed system at constant pressure. The barometer was placed on a screw stand of the type of Comm. n⁰. 60 and was read by the cathetometer used to read the mercury menisci in the *U*-tube of the piezometer.

The normal volume was ordinarily determined at least twice both before and after the measurements of compressed volume.

2. When the normal volume has been determined the piezometer tube is removed to the pressure cylinder *A* (Plate I, Comm. n⁰. 69) previously filled with mercury, it is then carefully warmed so that the air shall be completely expelled from the outer leg of the *U*-tube. When this operation is finished the small piezometer reservoir is brought into the bath necessary for the particular constant temperature required for the isotherm; the construction of this bath and the constancy of the temperature will be considered with the various isotherms.

The graduated tube b_4 is kept at a constant temperature by a stream of water through the water bath b_8 from a similar thermostat to the one above mentioned. Although this must be at a distance of 10 M. the connecting tube is so well wrapped with wool that the temperature at b_4 is constant to 0.02 deg C. It was set at either 15.6° C. or 20.0° C. and determined to the above accuracy.

Finally we must consider the temperature of the steel capillary which sometimes is 130 cm. long and owing to its exposed position usually shows considerable differences of temperature although it was carefully wrapped in wool and enclosed in paper. In many cases the difference at different parts reaches 1 deg. C. As however the influence of this uncertainty depends upon the relative volumes of the capillary and the piezometer, this question will be further considered with the measurements.

§ 8. *The calculation of the measurements.* 1. *The normal volume.*

For this calculation we divide the volume of gas in the piezometer tube into 4 parts (Comm. N⁰. 69) at the four temperatures t'_1, t'_3, t'_4, t_5. The constants are as follows:

V'_1 the volume of the small piezometer reservoir f_1 with the glass capillary f_2

V'_3 of the steel capillary from f_2 to a mark on the top of the glass tube b_5

V'_4 from this mark to the zero point of b_4

V'_5 from this zero point to the zero point of the *U*-tube b_2

v' from the zero point to the surface of the meniscus in the graduated leg of the U-tube [1])

B the pressure on the outer meniscus of U-tube

H the height of the top of the meniscus in the outer above the top of that in the graduated limb
expressed in cms of mercury at 0.0° C. at Leiden.

h, h' the capillary depressions of the menisci in the graduated and outer limbs of the U-tube

A the pressure of 1 atmosphere.

Then neglecting the differences in PV_A due to the difference between the pressure at the mercury meniscus in the graduated limb of the U-tube and A, and also the difference between this pressure and the mean pressure of the gas together with the change in the mean expansion coefficient α_N from 0.0° C to the various temperatures we have

$$V_N = \frac{B+H-h+h'}{A}\left\{\frac{V'_1}{1+\alpha_N t'_1}+\frac{V'_3}{1+\alpha_N t'_3}+\frac{V'_4}{1+\alpha_N t'_4}+\frac{V'_5+v'}{1+\alpha_N t_5}\right\}$$

which is more convenient for calculation when written

$$V_N = \frac{B+H-h+h'}{A}\left\{V_1+V_3+V_4-v_1-v_3-v_4+\frac{V_5+v'+v''}{1+\alpha_N t_5}\right\}. \quad (1)$$

where V_1, V_3, V_4, V_5 are the volumes of the various parts at 17° C. found by calibration and

$$v_1 = V_1 \alpha_N t'_1,\ v_3 = 17\, V_3 \alpha_N\,,\ v_4 = V_4 \alpha_N t'_4, \quad v'' = V_5\, k\,(t_5-17). \quad (2)$$

where k is the expansion coefficient of the glass and

$$\alpha_N = \frac{1}{v_0}\left[\left(\frac{\overline{dv}}{dt}\right)_p\right]_0^{17} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (3)$$

The last corrections can be conveniently tabulated.

2. *The measurements at higher pressure.* In this calculation we may neglect the corrections for the capillary depression in the graduated tube b_4 (the inner diameter is about 0.6 c.m.), and for the pressure of the gas due to its own weight, which corrections can easily be applied to the result if necessary.

The volume of gas under the pressure p at the surface of the meniscus in b_4 is divided into four parts each at its own temperature

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[1]) The volume of the supplements of the menisci used are all determined after Comm. N° 67, I.

t_1, t_2, t_3, t_4, which parts do not correspond exactly with those of the former calculation. Let (cf. Plate II, Conn. N⁰. 69) U_1 be the volume of the small piezometer reservoir with x cm. of glass capillary, U_2 that of the remaining glass capillary, $U_3 = V_3$ again that of the steel capillary with the joints g_2 and g_3 to a mark on b_5, U_4 the remaining volume from this to the mean mercury meniscus.

The volumes are all calculated to 17° C. and so with sufficient accuracy we may put the volumes U_1', U_2', U_3', U_4', of these parts at the temperatures $t_1, t_2 \cdot t_3, t_4$ under the pressure p as

$$\left.\begin{array}{l} U_1' = U_1 + w_1 + w_1',\ U_2' = U_2,\ U_3' = U_3,\ U_4' = U_4 + w_4 + w_4' \\ w_1 = k\,(t_1 - 17)\,U_1,\ w_4 = k\,(t_4 - 17)\,U_4,\ w'_1 = \beta_1\, p\, U_1,\ w_4' = \beta_4\, p U_4 \end{array}\right\} \quad . \ (4)$$

where β_1 and β_4 are the coefficients for the elastic deformation of the glass walls calculated after the formulae of **Clebsch** and **Lamé** which agreed satisfactorily with some direct determinations.

If we call V_{N1} the normal volume of the gas in the piezometer volume U_1' and V_{N2} that of the gas in U_2, U_3, U_4 together then is

$$V_{N1} = V_N - V_{N2} = V_N - (U_2\, d_{Apt2} + U_3\, d_{Apt3} + U_4\, d_{Apt4}) \quad . \ . \quad (5)$$

where d_{Apt2} etc. are the densities at p and t_2 etc. in terms of the normal density.

If further we call τ_2 etc. the difference $t_\alpha - t_2$ of the temperatures t_2 etc. from a given temperature t_α not far removed from them at which the values of d_{Apt} are known.

Let U_2'', U_3'', U_4'' be such quantities that $U_2''\, d_{Apt\alpha} = U_2'\, d_{Apt2}$, etc. and put

$$U_2'' = U_2' + u_2 \text{ etc.}, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (6)$$

If then we replace d_{Apt} by (pv_A), where the volume v_A of the gas is expressed in terms of the normal volume, we have with (5)

$$(pv_A)_{t1} = \frac{p}{d_{Apt1}} = \frac{pU_1'}{V_N - \dfrac{p}{(pv_A)_{t\alpha}} \left\{ U_2'' + U_3'' + U_4'' \right\}} \quad . \ . \ . \quad (7)$$

The quantities u_2 etc. in (6) are calculated by

$$u_2 = \alpha_{p2}\ U_2\ \tau_2\,,\quad u_3 = \alpha_{p3}\ U_3\ \tau_3\,,\quad u_4 = \alpha_{p4}\ U_4\ \tau_4 \,. \quad . \quad (8)$$

where owing to the small difference of t_4 from t_α, either at 15.°6 C. or 20.°0 C.

$$\alpha_{p4} = \left[\frac{1}{v_A} \left(\frac{dv_A}{dt} \right)_p \right]_{t_\alpha} \quad . \ . \ . \ . \ . \ . \ . \quad (9)$$

as t_3 lies between 15.°6 and 20° C. we put

$$\alpha_{p3} = \left[\frac{1}{v_A} \left(\frac{dv_A}{dt} \right)_p \right]_{15 \cdot 6}^{20} \quad . \ . \ . \ . \ . \ . \quad (10)$$

assuming the same mean expansion coefficient for all cases, but

$$\alpha_{p2} = \frac{1}{v_{t_2}} \left[\left(\overline{\frac{dv_A}{dt}}\right)_p\right]_{t_2}^{t_\alpha} \quad \ldots\ldots\ldots \quad (11)$$

must be determined for each value of t_2.

After determining $(pv_A)_{t_1}$ it is necessary to reduce it from its value at t_1 to that of the required isotherm t_s not much removed from it, this reduction is only possible when approximate values of the change in $(pv_A)_{t_1}$ due to temperature are known.

If we put $\tau_1 = t_s - t_1$ we have

$$(pv_A)_{t_s} = (pv_A)_{t_1}\,(1 + u_1) \quad \ldots\ldots\ldots \quad (12)$$

$$\text{where } u_1 = \alpha_{p1}\,\tau_1 = \left[\frac{1}{V_A}\left(\frac{dv_A}{dt}\right)\right]_{t_1 p} \tau_1, \quad \ldots\ldots\ldots \quad (13)$$

for the reduction at constant pressure, and correspondingly

$$(pv_A)_{t_s} = (pv_A)_{t_1}\,(1 + u'_1) \quad \ldots\ldots\ldots \quad (14)$$

$$\text{where } u'_1 = \alpha_{v_1}\,\tau_1 = \left[\frac{1}{p}\left(\frac{dp}{dt}\right)\right]_{t_1 V_A} \tau_1 \quad \ldots\ldots \quad (15)$$

for the reduction at constant volume, and thus to another pressure.

§ 9. *The constants necessary for the calculations.*

$A = 75{,}9467$ cm. after Comm. no. 60.

k was assumed at the ordinary temperature to be equal to 0,000024 as for the kind of glass used it cannot differ much from that of Jena glass.

The values of α_v and α_p are calculated for each gas from the series of Comm N⁰. 70 when no better data are at hand.

In the calibrations of graduated tubes and determinations of the volumes V and U no very special precautions where taken beyond those ordinarily employed in accurate calibration. U_1 was determined by treating the piezometer as a barometer, pumping free of gas and filling in vacuum, it was then placed in ice. U_2 was calibrated by a mercury thread. U_3 can be directly determined by screwing the end g_1 of the steel capillary onto the end of b_5 and providing the other end of the capillary with a steel connecting tube carrying a glass cock. By connecting b_2 to the air pump the whole can be then well evacuated and the mercury slowly admitted through the above mentioned cock, until it reaches above the top divisions of the tube b_4. When dry air has been admitted the tube is placed vertically and the steel connecting tube carefully removed. By adjustment the mercury is brought just to the end of the capillary and its position in the tube b_4 read by a cathetometer, the temperature being also observed. To check these determinations the volume is calibrated in parts, the steel capillary after the method of Comm. N⁰. 60 and the glass either by a mercury thread or by screwing a nut carrying a glass cock unto b_5, evacuating and reading as before In the same way the area of the graduated tube b_4 was determined but with more precautions as to constancy of temperature, and of the large volume V_5 with the area of b_2. The precautions as to constancy of temperature and accuracy of reading and weighing were always referred to a higher accuracy than the one actually sought.

Physics. — Communication n^0. 78^b from the Physical Laboratory at Leiden: "*Isotherms of diatomic gases and their binary mixtures. III. The isotherms of Oxygen at* 20.°0 *C.* 15.°6 *C.* 0°.0 *C.*" (By Prof. H. KAMERLINGH ONNES and H. H. FRANCIS HYNDMAN).

(Communicated to the meeting of March 29, 1902).

§ 10. *The purpose of the research at ordinary temperatures.*

Measurements were made in this region for two purposes.

1^0. In order to calculate our observation at low temperatures by equation (7) it is necessary to know the values of pv_A at the given pressures to at least the same accuracy as that sought for in the measurements at low temperatures. In Comm. n^0. 69, § 6 we have given $\frac{1}{1000}$ as the accuracy aimed at and to this accuracy there are no data at present available. The most trustworthy measurements of AMAGAT [1]) commence at 100 At. and even should they have no constant error they do not admit of interpolation for the values of pv_A to this accuracy.

2^0. We wished to so test our apparatus at ordinary temperatures in a region where the results could be compared with and linked into the above mentioned measurements of AMAGAT, that we could satisfy ourselves of the accuracy of the method at low temperatures.

Measurements were made at 15°.6 C. to compare directly with AMAGAT, but the majority at 20° C. because it is a more generally useful temperature and can always be obtained. At the same time as the measurements at ordinary temperature we give a series at 0.0 C. as this can also be directly compared with AMAGAT.

With the help of these measurements we can improve the coefficients in the series for pv_A given in Comm. n^0. 71 and can also obtain the expansion coefficients required for the small temperature differences τ_3 and τ_4, more accurately than heretofore.

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1) Ann. de Chim. et de Phys. 1893.

§ 11. *The measurements.*

The oxygen was prepared by heating potassium permanganate in a tube which was connected by a ground joint with a purifying apparatus consisting of washbottles filled with water, potassium hydroxide solution and strong sulphuric acid respectively, and then of a tube containing layers of glass wool covered with phosphorous pentoxide. This apparatus was preliminarily filled with oxygen. The washbottles are of the type shown in Comm. n⁰. 27 fig. 3 and there is a short cut between the tube containing the permanganate and the end connected to the air-pump. In this way the purifying apparatus can be kept in permanent order, it only being necessary to recharge the tube with permanganate and to completely evacuate it. The whole system of purifying apparatus both for oxygen and hydrogen (Comm. n⁰. 27), the mercurypump and the connecting tubes for filling the piezometers or thermometers are permanently connected together so that they remain dry.

The purity of the oxygen used was tested by its critical phenomena in a tube provided with a magnetic stirrer. The values of the critical constants will be considered in a later communication.

For the reduction of the Normal volume we use

$$\alpha_N = \frac{1}{v_0}\left[\left(\frac{dv}{dt}\right)_p\right]_0^{17} = 0{,}00368 \text{ for } p = 1\,.$$

For the reduction of the measurements at higher pressure we have derived αp_4, αp_3 and the αp from which αp_2 is calculated, from a graphical representation of our measurements. The values are given in the following table.

TABLE I. **O_2 , α_p Comm. n° 78.**

	Temperatures.	$p = 20$ At.	$p = 40$ At.	$p = 55$ At.
$\alpha p_4 = \alpha p_3$	15.6	0.0038	0.0041	0.0043
	20.0	38	40	42
$\frac{1}{vA_0}\left[\left(\frac{dv_A}{dt}\right)_p\right]_0^{20}$	0 — 20	0.0040	0.0043	0.0045

The constants of the piezometers in O_2 Series V for which the results are given in §§ 13 and 14 are given in the following table.

When V_4 is the volume at 17° C. between the above mentioned mark on b_5 and the zero of b_4, and U_5 the volume at the same temperature from this zero to the meniscus which bounds the gas, (see Pl. II, Comm. 69) we have

$$U_4 = V_4 - U_5 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

If also we call the number of centimeters at which the mercury stands above the zero point at the reduced reading y, and note that

$$V_1 = U_1 + U_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

we have

TABLE II. **Constants O_2 Series *V***

$U_1 = 10{,}3676$ cc $\qquad \beta_1 = 7{,}3.10^{-6}$
$U_2 = 0{,}0380$ cc $\qquad \beta_4 = 4{,}7.10^{-6}$
$V_3 = U_3 = 0{,}6075$ cc
$V_4 = 2{,}0156$ cc
$U_5 = (0{,}29286\ y + 0{,}0005\ y^2)$ cc
$V_5 = 613{,}450$ cc
$v' = 0{,}637$ cc per cm.

The small values of V_4 in this series was due to the breakage of the graduated tube b_4. In Series II V_4 was about 6.2 cc. in Series III and IV about 4.8 cc.

In Series II as in Series V the small piezometer was about 10 cc., in Series III about 20 cc. and in Series IV about 5 cc.

The thermometers are all compared from time to time with a Standard corrected at the Reichsanstalt.

§ 12. *The determination of the normal volume.*

The temperature t_5 of the piezometer in the constant temperature bath was determined as a mean of three read on different thermometers placed at various heights, their corrected difference was not allowed to exceed 0.05 deg. C. The temperature of the steel capillary was determined by a thermometer placed at about its middle point. These thermometers together with those used for the temperatures of the baths surrounding b_4 and f_1 were all divided into 20ths of degrees. In the following table the readings for one determination are given, in column A the reading of the level of the cathetometer, B that of the scale, C the temperature of the barometer, D that of the large water bath at the top the middle a the bottom, E and F those of the water baths round b_4 and f_1, G that of the steel capillary (in this example it was not read, the temperature taken being that of the top thermometer of the barometer).

TABLE III. **Normal volume for the measurements O_2, Series V_1, n° 4.**

			A.	B.	C.	D.	E.	F.	G.
TEMPERATURES..........						19.56 19.52 19.51		19.45	
U-tube	meniscus inner	top	6.2	51.854					
		rim	6.6	51.706					
	Division 0.8..		6.5	51.782					
	meniscus outer	top	4.4	51.972					
		rim	4.0	51.828					
BAROMETER	meniscus above	top	4.4	98.630	19.5				
		rim	4.2	98.512					
	below...	top	5.0	21.612	17.8				
		rim	5.0	21.512					
TEMPERATURES						19.57 19.53 19.52	20.05		(19.5)
U-tube	meniscus inner	top	5.2	51.838					
		rim	5.8	51.700					
	Division 0.8..		5.5	51.774					
	meniscus outer	top	3.7	51.968					
		rim	5.0	51.828					
TEMPERATURES..........						19.58 19.54 19.52		19.48	

From this Table IV is derived, in column A is given the corrected readings of the top of the menisci, in *B* the heights of the menisci, in *C*, *D*, *E*, *F*, *G* the corrected temperatures from the corresponding columns of Table III.

TABLE IV. **O_2 Series V n° 4.**

		A	B	C	D	E	F	G
TEMPERATURES................					19.48 19.44 19.48		19.09	
U-tube	inner meniscus.	0.724	0.149					
	outer meniscus.........	0.604	0.142					
BAROMETER	above meniscus ...	98.628	0.114	18.7				
	below meniscus....	21.612	0.110	17.4				
TEMPERATURES................					19.49 19.47 19.49	19.96		(18.7)
U-tube	inner meniscus..........	0.735	0.140					
	outer meniscus	0.600	0.144					
TEMPERATURES................					19.50 19.49 19.49		19.12	

In Table V are given in column A the readings of the top of the menisci in the graduated limb of the *U* tube (as in Table IV), in *B* the height of the cylinder equal to the supplement of the meniscus which must be subtracted from *A* to give *C* the corrected position of the meniscus, *D* the mean of the latter, *E* the height and *F* the capillary corrections *h* and *h'* for these menisci, *G* the uncorrected height of the connected barometer.

TABLE V. **O_2 Series *V* n° 4.**

	A	B	C	D	E	F	G
U-tube	0.724	0.071	0.795	0.798	+0.120	0.042 0.040	
	0.735	0.067	0.802		+0.135	0.039 0.041	
BAROMETER							77.016

Finally in Table VI are found the only volume which could not be taken directly from Table II i. e. v', the mean corrected temperatures and the mean corrected values of $H - h + h'$, and B that is to say the values of the determination required for direct insertion in formula (1).

TABLE VI. **Normal volume O_2 Series *V*. n° 4.**

VOLUMES.	TEMPERATURES.	$H - h + h'$	B
$v' = 0.502$	$t_5 = 19.48$ $t_4 = 19.96$ $t_1 = 19.10$	0.127	76.789
$V_N = 592.49$			

The collected determinations of the same normal volume are given in Table VII.

TABLE VII. **Normal volume O_2 Series *V*.**

N°.	Volume.	Pressure.	pv_A.	Mean.	Mean.	Diff.
1	589.47	76.306	592.26	592.40	592.25	+ 0.01
2	589.60	76.308	.41			.16
3	584.93	76.917	.42			.17
4	585.01	76.916	.49			.24
24	590.60	76.148	592.16	592.06		— 0.09
25	590.45	76.041	1.96			0.29
26	590.53	76.045	2.07			0.18

In Table VIII are the normal volumes of the various series with the numbers of the determinations in chronological order in each series including the measurements at higher pressure.

TABLE VIII. **Normal volumes O_2, Comm. 78*b*.**

Determination.		Before.	After.	Mean.	Diff.
Ser. *II*	n°. 1 2 25	602.24	602.80	602.61	— 0.37 + 0.36 + 0.02
Ser. *III*	1 2 8 9	596.62	596.70	596.66	0.00 — 0.08 + 0.05 + 0.03
Ser. *IV*	1 2 8 9 10	585.99	586.14	586.08	— 0.11 — 0.08 — 0.05 + 0.08 + 0.15
Ser. *V*		592.40	592.06	592.25	See Tab. VII.

§ 13. *Measurements with higher pressure at* 20 *and* 15.6.

The constant temperature water from the thermostate is equally divided between the water baths round b_4 and f_1, the difference in temperature of the baths thus remaining usually below 0.02 deg. The pressure is gradually raised to the required value and the pressure and temperature kept constant for at least an hour before the beginning of the measurements.

The steel capillary is protected in the manner described in § 8 and its temperature t_3 determined by three thermometers. If we consider that an uncertainty may remain in the mean temperature equal to the maximum difference found between any two temperatures i. e. 1 deg. this still influences the results with the smallest piezometer (5 cc.) and the highest pressure 60 atm. by hardly $\frac{1}{2800}$.

The thermometers were the same as these used for the normal volume.

In the following table are given all the readings for a single measurement. Column *A* gives the reading of the level of the cathetometer, *B* the scale reading, *C* the temperature of the small piezometer reservoir, and the temperature at some known point of the stem, *D* the temperature of the waterbath round the tube b_4, *E* the temperature of the capillary by three thermometers, *F* the reading of the level glass C_3 by the reservoir C_1 on its scale (Pl. I Comm. n°. 69) and of the level glass of the manometer on its scale, *G* the reading on the divided stem of the manometer, *H* the temperature of the manometer bath above and below.

TABLE IX. **O_2 Series *V*. n°. 6.**

	A.	B.	C.	D.	E.	F.	G.	H.
Manometer						28.15	80.07	
Temperatures			20.00	20.11				19.85
								19.91
					18.00			
					18.32			
					19.36			
Glass capillary			19.00					
Piez.-meniscus top	4.0	30.120				24.2		
Piez.-meniscus rim	3.8	30.076						
Division 0.6	4.0	30.022						
Temperatures			20.03	20.14				19.88
								19.90
Manometer						28.15	80.05	

From this Table X containing the corrected values is derived, in which column *A* gives the reading of the top of the meniscus referred to the tube, *B* the height of the meniscus, *C*, *D*, *E* the corrected and mean temperatures of the corresponding columns in Table IX, *F* the height of the menisci in the manometer and piezometer above these in the level tubes, *G* the mean manometer reading, *H* the mean corrected temperature of the manometer.

TABLE X. **O_2 Series *V*. n°. 6.**

	A	B	C	D	E	F	G	H
Manometer mean........						104.3	80.06	
Temperatures t_1, t_4, t_3 ..			19.94	20.04	17.9			19.83
t_2			19.00					
Piezometer	0.698	0.043						
						54.3		

Finally we obtain Table XI which contains the quantities obtained by the measurements and which are required together with Table II for the calculation by formula (7). The pressure of the gas in the manometer is given by p_m and the pressure difference between this and the meniscus in the piezometer corrected for capillary depression (see also § 5) by H_p.

For the measurements at ordinary temperature i. e. at 15.°6 and 20°.0 C. we may put $t_a = t_1$, when (7) is simplified and reduces to

$$(pv_{\mathcal{A}})_{t_1} = \frac{p}{V_N} \left\{ U_1' + U_2'' + U_3'' + U_4'' \right\} \quad \ldots (18)$$

The corrections w_1, w_4, w_1', w_4', u_2, u_3, u_4 with which U_1', U_2'', U_3'', U_4'' are obtained by (4) and (8) are given in Table XI.

TABLE XI. **O_2 Series *V*. n° 6.**

Volumes U_4 = 1.8168 cc	$p_m = 47.360$ At.[1]
w_4 = + 0.0001 ″	$Hp =$ 0.663 ″
w'_4 = 0.0004 ″	
u_4 = − 0.0003 ″	
u_3 = + 0.0050 ″	
u_2 = 0.0001 ″	
w_1 = 0.0007 ″	
w_1 = 0.0036 ″	t_1, t_2, t_3, t_4
$(pv_{\mathcal{A}})_{t_1} = 1.0417$	$t_1 = 19.94$
$(pv_{\mathcal{A}})_{20} = 1.0419$ at $p = 48.023$ At.	

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[1] The difference between the values of the pressure now given and those in the Dutch text, is due to improved calculations of the calibrations and corrections see Comm. n⁰ 78*c*.

The value of pv_A obtained in this manner for 20°0 C. and 15°6 C. are given in the following tables.

TABLE XII. O_2 at 20°.00 C.

Determinations.	p	pv_A
Series *III.* N° 3	23.713	1.0549
4	26.701	1.0533
5	29.901	1.0519
Series *V.* N° 5	45.911	1.0431
6	48.023	1.0419
7	48.165	1.0419
8	48.174	1.0417
11	50.882	1.0410
12	50.919	1.0409
13	54.142	1.0402
Series *IV.* N° 3	56.927	1.0395
4	65.396	1.0362

TABLE XIII. O_2 at 15°.60 C.

Determinations.	p	pv_A
Series *II.* N° 7	36.208	1.0303
8	36.643	1.0298
19	38.635	1.0286
20	39.749	1.0283
21	42.910	1.0266
22	47.683	1.0244
24	49.385	1.0231
23	51.464	1.0224

§ 14. *Measurements under higher pressure at* 0°0 C.

For these measurements the small piezometer reservoir is placed in ice in the same manner and with the same precautions as the thermometer reservoir in Comm. n°. 27, Plate III, fig. 2*b*. The uncertainty in the temperature of the steel capillary was of the same magnitude and has the same influence here as in the former section. The values of pv_A used for the calculations in this section were obtained by interpolating graphically the values given above.

The following determinations were made.

TABLE XIV. **O_2 at 0°.0 C.**

Determinations.	p	pv_A
Series *III* n°. 6	22.401	0.9762
7	27.746	0.9731
Series *II* n°. 3	34.498	0.9672
6	38.144	0.9656
Series *V* n°. 9	43.686	0.9623
10	43.756	0.9608
Series *II.* n°. 8	45.238	0.9612
10	48.691	0.9584
Series *IV* n°. 5	55.017	0.9533
6	60.301	0.9508
7	66.787	0.9477

Physics. — Communication N°. 78^c from the Physical Laboratory at Leiden: "*Isotherms of diatomic gases and their binary mixtures. IV. The compressibility of Hydrogen at* 0°.0 *C. and* 20°.0 *C. determined by the piezometers with variable volume for low temperatures.* By Prof. H. KAMERLINGH ONNES and H. H. FRANCIS HYNDMAN.

§ 15. The object of the determinations at 20°.0 is the direct comparison of an isotherm determined by our method with an if possible more carefully determined one to serve as a check. The isotherm given by SCHALKWIJK in Comm. N° 70 V was chosen as the most accurate known to us and we treat it as our standard of reference in these measurements.

Determination of compressibility were also made at 0°.0 C. to compare directly with AMAGAT's determinations at the same temperature.

§ 16. *Measurements with Hydrogen.* The piezometer used was the same as that given in Comm. No. 78^b O^2 Series V, the only difference being that the capillary volume is here somewhat larger.

The hydrogen was prepared in the manner described in Comms. N° 29 and N° 60, and the piezometer filled in the same way as for oxygen.

The measurements were also carried out in precisely the same way as described above for oxygen. After the measurements given here were finished we had intended to proceed directly to low temperatures, but during the insertion of the piezometer into the cryostate the steel capillary was broken so that these measurements had be postponed and moreover we were unable to redetermine the normal volume after the measurements. Fortunately the measurements at 0°.0 C. were included between two sets at 20°.0 C. which agreed well among themselves, so that we could be reasonably certain that no leakage had occurred.

For at the calculation of the normal volume we have used

$$\alpha_{pN} = \frac{1}{v_0}\left[\left(\frac{dv_A}{dt}\right)_p\right]_{0^0.0,\ \text{for}\ p\,=\,1}^{17^0} = 0{,}00366$$

and for the measurements at higher pressure

$$\alpha_p = \frac{1}{v_0}\left[\left(\frac{dv_A}{dt}\right)_p\right]_{0^0.0,\ \text{for}\ p\,=\,40}^{20^0} = 0{,}00357$$

The measurements of the normal volume gave

TABLE XV. **Normal volume H_2 Series II.**

	Mean	Diff.
Series *II.* N° 1	582.74	+ 0.24
2		— 0.24

and the measurements at higher pressure the following values. The measurements at 20°.0 C. are directly compared with those of **Schalkwijk** (loc. cit.) at the same temperature calculated by means of a slightly improved equation to the one given before

$$pv_A = 1.07258 + 0.000667\,d + 0.00000099\,d$$

where d is expressed in terms of the normal volumes,

TABLE XVI. **H_2 at 20°.0 C.**

	p	pv_A	Diff. S
Series *II.* N° 3	47.837	1.1040	+ .0007
4	48.231	1.1042	+
5	52.222	1.1067	+ 5
6	56.372	1.1089	— 0001
12	56.432	1.1091	+ 1
13	56.447	1.1095	+ 4

The difference of these determinations from those of SCHALKWIJK is thus small and falls within the limits of our accuracy, so that we may assume with some confidence that our measurements on oxygen have the same accuracy.

TABLE XVII. **H_2 at 0°.0 C.**

	p	pv_A
Series *II.* N° 7	44.996	1.0293
9	49.376	1.0316
8	49.422	1.0319
11	53.203	1.0342
10	53.249	1.0346

§ 17. *The standardization of the working manometer by the open and closed standards.* In our measurements we do not require a manometer of quite the same accuracy as the standards but rather one of somewhat simple construction which can be more rapidly and easily read, yet the accuracy should be of at least the same degree as arranged for in the construction of the piezometers.

As mentioned before in § 5, Comm. No 69 this accuracy has been obtained by a combination of the preliminary mercury calibration with many direct comparisons with the standard manometers at various points of the scale.

These are given in the following table. In column A are the calculated values of the pressure [1]) at the various readings obtained from 1st the volume determined by the previous mercury calibration, 2nd a mean value of the normal volume of the enclosed gas calculated from the comparison with the standard manometers, the calibrations being all reduced to 20°.0 C. so that the isotherm determined by SCHALKWIJK could be directly employed. As the deviations in temperature were some times considerable the reductions were made with the value for $\alpha_p = 40$ given above.

Column B gives the mark of the standard manometer used. *AIII* and *AIV* are the tubes *III* and *IV* of Comm. N°. 70, *O* the open manometer of Comm. N°. 44.

As the temperature differences were here greater than with the other determinations the numbers have not the same relative value as these of the other series.

Column C gives the pressure on the gas in the manometer obtained by applying all corrections for capillary depression level tubes etc. to the readings of the standards.

Column D gives the differences between columns A and C and represents some error in calibrations or standardisations. We assume that the values given in column C are the correct values of the pressure and apply a small correction to A to make the resulting differences as small as possible.

A correction to the volume of the following form [2]) appears to give the most satisfactory results.

If V is the volume as formerly calculated V' the new volume and R the reading in cms

$$v = V' - V = -0.34 + (100 - R)\,0.0109\ mm^3$$

P' is then calculated for each case with this new value of V'.

The differences between the new pressures thus calculated for column A and colume C are given in column E.

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[1]) The difference between these values and those in the Dutch text is due to an improved calculation of the calibration.

[2]) A systemic correction is here applied in place of the former empirical one.

TABLE XVII. Manometer.

A	B	C	D	E
20.741	*O*	20.746	— 0.005	— 0.018
21.501	*A III*	21.485	+ 0.016	+ 0.003
21.701	»	21.690	0.011	— 0.002
21.700	»	21.693	0.007	0.006
21.892	*O*	21.867	0.005	— 0.008
25.862	*A III*	25.847	0.015	+ 0.002
27.731	*A IV*	27.700	+ 0.031	+ 0.018
27.723	*O*	27.733	— 0.010	— 0.023
27.824	*A IV*	27.806	+ 0.018	+ 0.005
27.944	»	27.929	0.015	0.002
27.999	»	27.946	0.031	0.018
28.003	»	27.991	+ 0.012	+ 0.000
29.120	»	29.138	— 0.018	— 0.030
29.153	»	29.165	— 0.012	— 0.024
29.298	*A III*	29.263	+ 0.055	+ 0.043
30.261	*A IV*	30.253	0.008	— 0.004
31.729	*O*	31.710	0.019	+ 0.007
34.456	*A III*	34.395	0.061	0.051
35.405	*A IV*	35.394	0.011	+ 0.001
36.106	»	36.100	0.006	— 0.003
36.114	»	36.104	0.010	+ 0.001
36.134	»	36.130	+ 0.004	— 0.005
36.143	»	36.147	— 0.004	— 0.013
39.147	*O*	39.095	+ 0.052	+ 0.054
40.329	»	40.294	0.025	0.023
40.300	*A IV*	40.303	0.007	0.005
40.367	»	40.365	0.002	0.000
43.477	»	43.461	0.014	0.013
43.489	»	43.486	0.001	0.000
46.452	*O*	46.533	0.019	0.000
47.586	*A IV*	47.587	+ 0.001	+ 0.002
47.638	»	47.663	— 0.075	— 0.073
50.817	*O*	50.829	— 0.012	— 0.007
53.237	*A IV*	53.237	0.000	+ 0.009
53.268	»	53.280	0.012	— 0.003
61.274	»	61.334	0.060	0.038
61.374	»	61.412	0.038	0.016
62.003	»	62.033	— 0.026	— 0.004

(May 28, 1902.