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PROCEEDINGS OF THE
SECTION OF SCIENCES

VOLUME X

JOHANNES MÜLLER :—: AMSTERDAM

:===== : JULY 1908 :===== :

Translated from: Verslagen van de Gewone Vergaderingen der Wis- en Natuurkundige
Afdeling van 24 Mei 1907 tot 24 April 1908. Dl. XVI.)

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PROCEEDINGS OF THE SECTION OF SCIENCES

VOLUME X
(— 1ST PART —)

JOHANNES MÜLLER :—: AMSTERDAM
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KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Friday May 24, 1907.

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C O N T E N T S.

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J. W. VAN WIJHE: "On the existence of cartilaginous vertebrae in the development of the skull of birds", p. 14.

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J. J. VAN LAAR: "On the course of the plaitpoint line and of the spinodal lines, also for the case, that the mutual attraction of the molecules of one of the components of a binary mixture of normal substances is slight". (Communicated by Prof. H. A. LORENTZ), p. 34.

W. DE SITTER: "On periodic orbits of the type Hestia". (Communicated by Prof. J. C. KAPTEYN), p. 47.

J. D. VAN DER WAALS: "Contribution to the theory of binary mixtures", IV, p. 56.

Palaeontology. — “*On fossil Trichechids from Zealand and Belgium.*” By Mr. L. RUTTEN. Communicated by Prof. C. E. A. WICHMANN.

(Communicated in the meeting of March 30, 1907).

Last summer a fisherman found opposite the village of Breskens in the West Scheldt a large skull, which Dr. S. SCHOUTEN secured for the Geological Institute of Utrecht University.

The fragment belongs to an old *Trichechus*, but differs in some respects from the now living Walrus. On closer examination it appeared that the skull must be of the tertiary *Trichechus Huxleyi*, of which until now only tusks were known, found in the “Red Crag” of Suffolk. These were described by RAY LANKESTER. Of the skull the description will be given here.

The plan suggests itself to compare the fragment first with the walrus of recent times and then with the already known fossil *Trichechids*.

1. *Description of the skull and comparison with the walrus.*

The most conspicuous point about the skull is its remarkably good state of preservation. It has this in common with some remains of diluvial mammals, also found in the river Scheldt. The skull, to be sure, arrived here in several pieces, but the broken edges were absolutely fresh and all parts fit perfectly together. Probably the fossil only broke when it was being dredged. As to completeness the skull leaves nothing to be desired, since only parts of the nasal, maxillary and frontal bones, part of the vomer, the conchae and a few teeth are wanting. The skull was filled with a fine-grained grey clay; the outer wall of the cranium was overgrown with Balanids and Bryozoa. Although the fossil is very heavy, a chemical analysis of a little piece of bone revealed nothing particular. It still contains pretty much organic matter and consists for the rest especially of CaO and P_2O_5 , while a small quantity of ferric hydroxide colours the bone dark brown.

That our *Trichechus* is full-grown, is proved by the fact that all sutures are absent and by the strong development of all ridges.

As material for comparison we had at our disposal 24 recent skulls and 26 pairs of tusks. Very unpleasant was the great variability of the recent walruses. Certain characteristics vary so strongly in different individuals that with the limited material one always remains uncertain whether the analogous characteristic in the fossil lies within

the limit of variation of the recent animals. It will appear that in most respects the fragment approaches the old walruses, but that the tusks deviate considerably from those of the living animals.

Looking at the hind side of the fossil skull, one is at once struck by its relatively great height. This is caused by the strong development of the mastoid process and by a high ridge on the lambda suture (crista lambdoidea).

Dividing the height of the skull and that of the mastoid process by the breadth of the skull, we obtain two quotients, which for the fossil are greater than for the walrus. In determining these quotients the height of the mastoid process was measured by the vertical distance between the lower edge of the foramen magnum and the base of the mastoid process. The differences found are small, however; for this characteristic the fossil stands consequently at the end of the variation series of the walrus. The strong relief on the mastoid process and the extraordinary size of the crista occipitalis externa as well as of the crista lambdoidea are characteristics which the fossil skull has in common with some old walruses.

Small deviations are also found in the vicinity of the foramen magnum. The canalis hypoglossi always opens with the walrus into the inside of the condyli of the cranium with two openings at each side, whereas the fossil only shows a single small opening. But this characteristic has not much value, since the aperture of the canalis hypoglossi always varies strongly. The foramen magnum is much more flattened dorsoventrally with the fossil than with the walrus, but this too is a very variable characteristic. Comparable numbers are here obtained again when the breadth of the foramen magnum is divided by its height. The condyli occipitales are in the fossil less strong than normally and present a shuttle-like appearance, while in the walrus they project more and more when we proceed upwards; also they here project above the upper edge of the foramen magnum, while there they remain below its upper edge.

Of all these small differences the shape of the condyli and of the foramen magnum have the greatest importance, while the height-ratios and the aperture of the canalis hypoglossi are of less value.

The base of the skull shows no more differences with the walrus than the back part. The length of the two skulls compared with their breadth, agrees entirely. The first difference concerns the position of the foramen lacerum and of the canalis alisphenoideus. These namely lie close behind each other with the walrus, while with the fossil they are separated by a rather massive bony lamella, extending from the os petrosum towards the pterygoid process. Also

the cup for the joint of the lower jaw is very broad and the result of this is again that the pterygoid processes have approached each other closely. Further changes appear at the frontal side of the skull-base by the size of the alveoles of the tusks.

The alveolar outer wall shows a lateral projection as with the walrus, only slightly more massive; the distance between the two projections is consequently also somewhat greater than with the walrus. We divide this distance again by the breadth of the skull: the fossil lies at the end of the variation series of the walrus. But the size of the tusk-alveoles produces still another difference. It causes namely the rows of teeth to be squeezed together, so that the distance of the two incisors, compared with the breadth of the skull, is extremely small. But this also occurs with some old walruses. Moreover the lower side of the upper jaw differs in shape: in the walrus it is broad and hollowed, in the fossil narrow and flat.

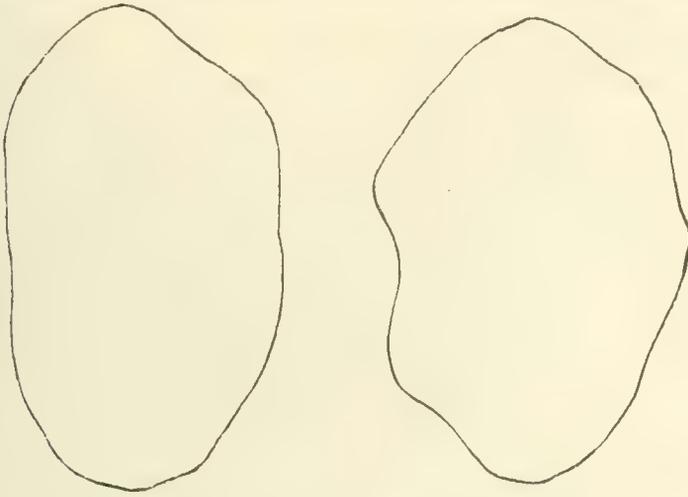
The lateral face of the skull presents in more than one respect a great difference with the walrus, namely in the shape of the tusks. In all the former properties the fossil approached the old walruses and if also the tusks had little deviated from theirs, the reasons for making a distinction would have been rather feeble. But this is not the case. The curvature of the tusks is with the fossil much stronger than with the walrus. If we determine the radius of curvature of all the tusks, we find it in the fossil to be 27 centimetres, for the walrus never under 38 cms. It must be noted that such a small radius of curvature only occurs with young, female walruses with relatively weak teeth. In older animals, however, with which we must compare our skull, the radius of curvature was never under 45 cms. and mostly over 50 cms. Whereas in the former properties the fossil resembles the old walruses, it deviates very strongly from them in regard to the shape of the tusks. Also the tusks are more elegant and the right tooth shows deep longitudinal grooves.

From the tusks we may also draw a conclusion as to the age of the animal. With young individuals the pulp cavity is very deep; with advancing age it gradually fills with osteodentine. With the fossil now the pulp cavity only had a depth of 3,5 cm. Also the thickness of the tusks is greater in the middle than at the base, which points to the period of strongest growth for the teeth being passed.

Again the considerable corrosion of the teeth point to an elderly individual.

If we now summarise the results of the comparison of the fossil with the walrus, we may state:

That the fossil skull belongs to a walrus-like animal, whose skull in general differs only little from recent animals as to strength of deve-



Cross-section of the base of the tusks of *Trichechus Huxleyi*.
Found near Breskens in the West Scheldt.

lopment, but deviates entirely from them by the shape of the tusks.

It is a pity that for comparative anatomy the value of the fragment is nil, for it presents many properties of *Trichechus*, such as the strong crista occipitalis externa, the big mastoid process and the massive bulla ossea still more typically than the recent animals.

2. Comparison of the skull with already known fossil *Trichechids*.

Fossil remains of *Trichechids* are known from North-America, England and Belgium. Also skulls of *Trichechus rosmarus* have been described from the subsoil of Paris (17), Hamburg (10) and Cologne (25), but it has been proved that they were carried there by man. The North-American finds seem to belong to the pleistocene and miocene (21). The tertiary skull deviates in the number of molars from *Trichechus rosmarus* 8); the pleistocene remains are identical with *Trichechus rosmarus*, although DE KAY has classed a skull fragment from Accomac County in Virginia under a fossil species *Trichechus virginianus*.

The English fossils were first found in the "Red Crag" of Suffolk, where, however, they are in a secondary "Lagerstätte"; probably they belong to the older pliocene. (19). Later they have also been found in the "Cromer Forest Beds" (32). They are only tusks, distinguished from the tusks of walruses by strong curvature,

smaller thickness and deeper longitudinal grooves. Also RAY LANKESTER is of opinion that in general they are bigger than those of the walrus, although the maximum size of the two is the same. So these tusks show a very good agreement with those of the Zealand fossil.

The Belgian Trichechids, occurring in the "Crag" of Antwerp, have been described by P. J. VAN BENEDEN in a work of splendid get up (26). The only pity is that the contents do not harmonise at all with the exterior, since to a dangerous imagination more scope has been given than to accurate description and careful criticism. When the fortifications round Antwerp were dug, fossil remains of Pinnipedia were found in very different places and at different times : they were treated by VAN BENEDEN in the following manner :

"Voici, comment nous avons procédé : Après avoir réuni tous les os de phoque, . . . nous avons réuni tous les os de même nature c'est à dire, les humérus, les fémurs etc. . . . Après cette première opération nous avons reparti les os longs après leur taille, ayant devant nous les mêmes os des espèces vivantes. Si l'on considère, que la plupart des pièces se répète plusieurs fois, il n'est pas difficile, . . . d'établir parmi eux des groupes génériques et spécifiques. Quand cette opération est faite pour les os comme les humérus et les fémurs, . . . on leur rapporte les autres os, en se guidant d'abord d'après leur dimension Nous avons alors étalé les humérus, les fémurs, les vertèbres etc. des diverses espèces européennes et nous nous sommes assurés, de quelles espèces vivantes nos phoques fossiles se rapprochent le plus. En répétant la même opération pour les autres os, nous sommes arrivés ainsi à composer nos espèces et à en établir un certain nombre avec une certitude entière."

Hence when VAN BENEDEN had established a new species by means of a single bone, he added to this bone what fitted best in size, a method which theoretically has some good points, but which in practice, with the very incomplete Antwerp material, presents so many difficulties, that the determinations of VAN BENEDEN must *a priori* be received with some misgivings. In any case, only the first piece of bone, on which a species was founded, may be regarded as having been definitely determined ; all the other bones, added to it, must be critically re-examined.

VAN BENEDEN describes three species of Trichechids, which he refers to three genera. Of these *Trichechus rosmarus* is supposed to be diluvial ; the two others, *Trichecodon Koninekii* and *Alachtherium Cretsi* are tertiary.

Of the common walrus only a scaphoid and an incomplete vertebra are described : "qui ont été quelque temps confondues . . . avec les

animaux quaternaires terrestres. On les avait placés à côté de Rhinocéros”!

The genus *Trichecodon* was based on a small fragment of a tusk which has now even been lost and of which a cast only is left at Brussels. This strongly rounded fragment, however, is not typified by a single characteristic, and so it will always be impossible to ascertain whether later remains really belong to it. But in this way the genus *Trichecodon* loses any right of existence and the bones, referred to it, must be regarded as undetermined.

Alachtherium is first mentioned in an oration of Viscount du Bus. Of this *Trichechid* only half a mandible was known then. Also in this case VAN BENEDEX has added to this lower jaw a whole series of bones from the neighbourhood of Antwerp. Among these also a fragment of a skull occurs, of which VAN BENEDEX gives the following description, accompanied by some large, but not very happy illustrations.

“En comparant la tête d’*Alachtherium* avec celle du Morse, nous voyons des différences fort grandes dans la disposition de certains os. Vu par devant, le crâne est beaucoup plus élevé et les parties latérales, formées par le temporal surtout, sont plus étendues en dehors et en dessous. Il en résulte, que par la partie supérieure, le crâne se rapproche plus de celui des Otaries et par les parties latérales de celui des Morses. Le crâne est brisé en avant de manière que la boîte est restée entière, et les os frontaux ne prennent qu’un faible part à la formation de la cavité cranienne. Le crâne, vu par la face postérieure, montre l’os occipital s’élevant verticalement très haut comme dans certaines Otaries adultes et les parties latérales et inférieures, formées par le temporal, sont très massives en même temps qu’elles descendent fort bas. Les deux condyles sont brisés. . . . Vu sur le côté, le crâne présente l’aspect d’un casque; il est beaucoup plus élevé que dans le Morse et la conformation de toutes les régions est complètement différente. . . . Tout le dessus du crâne est aplati et une bordure véritable sépare cette région supérieure en avant des os de la face, sur le côté des os des tempes. Les pariétaux sont fort bien indiqués au devant de l’occipital et sont disposés de manière à ressembler au premier abord, à des os nasaux”.

Not to mention a few inaccuracies, the differences with the walrus might have been indicated in less vague a manner, so that a new comparison does not seem superfluous:

Looking at the skull in front, we notice several differences with the walrus, which all have led to an alteration in the shape of the parietal. For with the walrus this bone is clearly convex at both sides while with *Alachtherium* a concavity is found which only for

a small part must be ascribed to decrease in size of the cranium. It is chiefly caused by the development of a gigantic crista lambdoidea and by the big mastoid process, as well as by the form of both. The crista lambdoidea namely by its size draws the parietal upwards and since in a median direction it extends far to the front, it exerts this influence over a great part of the circumference of the parietal. The strongly developed mastoid process, especially by its frontal position, draws the lower edge of the parietal and the upper edge of the squamosum outwards and therefore has the same effect at the lower side of the parietal as the crista lambdoidea at the upper and posterior side: the two together produce the concave shape of the parietal. While now the strong development of the crista lambdoidea and the increase in height of the mastoid process also cause an increase in height of the skull as compared with the walrus, it is at the same time broadened by the frontal position of the mastoid processes. For these are placed with the walrus in a slanting forward direction and are also smaller than with *Alachtherium*. So we cannot wonder that the absolute height and breadth of the skull exceed the corresponding dimensions of the walrus, but that their ratio lies within the limit of variation of recent animals.

Other differences with *Trichechus rosmarus* are found at the base of the skull. With the walrus a very large bulla ossea extends from the external edge of the basioccipital and basisphenoid as far as the mastoid process and as far forwards as the fossa glenoidea; a more or less distinct groove separates this bulla ossea, in a rostral and a caudal part. With *Alachtherium*, however, the bulla ossea is very small and in this respect it deviates distinctly from the *Trichechus* type. Corresponding to some extent to this circumstance the fossa glenoidea lies far backwards in *Alachtherium*: the space between the articulation and the mastoid process is very small. While now with the walrus the fossa glenoidea extends on the jugal process of the squamosum, so that above it the squamosum rises in a slanting upward direction, with *Alachtherium* it lies far less free and in front of it the squamosum rises steeply. We shall see later on that this is of importance when dealing with the mandibles to which the cranial fragment was said to correspond by VAN BENEDEX.

A further difference is found at the border between basioccipital and basisphenoid. Not to mention a frontally diverging crista and a roughness on the occipital on both sides behind it, these bones pass gradually into each other in the walrus. In *Alachtherium* the outside of the sphenoccipital suture is strongly thickened; a real knob has formed which only at the left side has been preserved. The back of

the basisphenoid now rises at the left to a great height against this knob and, instead of following the cranial base normally, it is situated here almost sagittally: the complete basisphenoid must consequently have shown a deep median groove. Analogous changes have occurred at the basioccipital, which towards the knob shows a deep concavity. By this the angle between basisphenoid and basioccipital has also become more acute than in the walrus.

In order to compare the back of the skull with that of the walrus we placed the fragment in such a position that the upper edge of the parietal has a slight forward inclination, as this is also the case in a walrus skull, placed on the table without tusks.

Then with both the basisphenoid rises slightly in a forward and the basioccipital in a backward direction, so that their positions may be considered as corresponding.

With *Alachtherium* the outline of the posterior part of the skull then shows one important change, caused by the strong crista lambdoidea and the broad mastoid processes. For this causes the back of the skull to consist of a narrow supra-occipital and a very broad temporal part, a phenomenon which also with old walruses is sometimes indicated to some extent, but never so strongly as with *Alachtherium*. Moreover *Alachtherium* has a very small cranium: all bones are uncommonly thick. Further in *Alachtherium* the crista lambdoidea runs in a median direction far to the front and even shows a tendency to pass into a sagittal ridge. As compared with the walrus this phenomenon becomes very striking by the complete absence of the crista occipitalis externa. Where consequently with old walruses the occipital superius is strongly convex by the massive crista occipitalis, it shows with *Alachtherium* a deep median fold. Compared with *Trichechus* the hind skull of *Alachtherium* thus shows three modifications: absence of the crista occipitalis externa, size of the mastoid processes, and shape of the crista lambdoidea. The differences at the side of the skull are not very great and we are certainly not justified in stating, as VAN BENEDEN does: that "la conformation de toutes les régions est complètement différente." Especially with the skulls of old male walruses *Alachtherium* shows many points of resemblance and it would almost appear as if VAN BENEDEN had only a small material for comparison at his disposal for his description.

The many changes which the skull of *Alachtherium* shows when compared with the skull of *Trichechus*, have not deprived it, though, of its *Trichechid* character. But there are three phenomena which bring it nearer the *Otaridae*: the smallness of the bulla ossea, the

absence of the crista occipitalis and the tendency of the crista lambdoidea to develop on the skull into a sagittal ridge¹⁾.

What reasons had v. BENEDEK to coordinate the cranial fragment to the mandible on which *Alachtherium* is based? He does not state them anywhere:

“On ne possédait d’abord de ce curieux Amphitérien d’autre os que le maxillaire Nous rapportons à ce même animal, le crâne que nous représentons, les vertèbres cervicales etc. etc. . . . La forme du maxillaire inférieur indique une conformation toute particulière de tous les os en face”

The question now is whether on the contrary it cannot be shown that cranial fragment and mandible do not belong to each other. This seems indeed to be the case:

The lower jaw deviates strongly from that of *Trichechus* and points in fact to a “conformation toute particulière de tous les os en face”. But as the hind skull of *Alachtherium* does entirely conform to the *Trichechus* type, it is unjustifiable to assume for the lacking part an entirely deviating shape, only in order to be able to fit the skull to the lower jaw.

The hind skull has much more massive and coarse bones than *Trichechus*, the mandible on the other hand is larger than that of the walrus, but of a much more elegant and fine build: also in their structure skull and lower jaw have consequently opposite characters. Also the lower jaw is too big for the cranial fragment. For the cranium belongs to an old animal and so the lower jaw should certainly not be much too big for the skull. If we now divide two dimensions of the walrus and of *Alachtherium*, we find:

	Walrus	<i>Alachtherium</i>	<i>Alach.</i> : Walrus
distance of the fossae glenoideae	13.5	14.5	107
length of the mandible	24.7	35.7	144

Hence we here obtain again such abnormally great differences of two dimensions between the walrus and *Alachtherium* that the otherwise considerable analogy of the skulls does not permit us to ascribe mandible and skull to one species. The principal argument, however, is found in the shape of cranium and mandible.

We saw that with *Alachtherium* the fossa glenoidea has a much less free situation than with *Trichechus*. With the latter the mandible has a short, vertical coronoid process, which consequently easily finds a place in front of the squamosum. The lower jaw of *Alach-*

¹⁾ J. A. ALLEN (30) states about *Trichechus obesus* that this also has a small bulla ossea.

therium, on the other hand, shows a long coronoid process, which has a backward and slightly inward direction, and which on account of the un-free situation of the articulation of the cranial fragment, must inevitably come in collision with the squamosum; hence the two bony pieces cannot possibly belong together.

So we may conclude that the skull described as *Alachtherium* does not belong to this species. No more can it belong to *Tricheodon*, since this genus must disappear from literature. So it must be regarded as undetermined.

Finally the "Musée d'Histoire naturelle" at Brussels possesses still another *Trichechid* skull, floated ashore near Heyst and considered to be diluvial. It is the cranium of a very old male: the sutures have all disappeared and the tusks are almost entirely used up. The preservation is exactly as that of the Zealand fossil: the bones have turned brown and the teeth entirely black; the skull is very heavy and perhaps has become partly siliceous. Besides its shortness and a strong development of the alveoles of the tusks, the fossil shows no differences with the walrus: these two characteristics, however, give it a very square appearance. But these small differences give us no right to regard the skull as a new species: it seems to be an ordinary *Trichechus rosmarus*. After having dealt with the known skulls, we must assign a place in the system to the Zealand fossil and to the Antwerp hind skull. They belong to different species. The resemblance of the tusks of *Trichechus Huxleyi* with those of the Zealand cranium was already pointed out. The curvatures of the tusks of *Trichechus Huxleyi*, drawn by RAY LANKESTER, are:

21, 27, 30, 38, > 50, > 50 cms.

Hence they agree much better with the Zealand tusks than with those of the walrus. Also the cross-sections of the tusks showed analogies and so we may safely class the Zealand skull under *Trichechus Huxleyi*.

If we ask what age must be attributed to the skull from the Scheldt, we must bear in mind that the good state of preservation precludes a long transport. Hence the skull must have been dislodged out of the bottom of the river. What sort of soil do we find there? Formerly already Dr. DE MAN has described remains of diluvial terrestrial mammals (35), which were also fished from the Scheldt and partly even very near the spot where also the *Trichechus* was found (36). Now it is very improbable that the *Trichechus* and the terrestrial mammals come from the same layer, since both are well preserved. In the year 1879 Dr. SEELHEIM published some profiles

of Zealand, based on borings and showing that in the West Scheldt occasionally tertiary layers occur. Since it is of course impossible to indicate the precise layer from which the fossil got free, yet it appears that the skull may belong to the tertiary pleiocene. Also in this respect it would correspond to *Trichechus Huxleyi*.

The cranial fragment from Antwerp does not belong to *Trichechus Huxleyi*, since it deviates considerably from the Zealand skull. Hence it must be a new species. It does not seem desirable to establish a new genus for a fragment, showing so much analogy with *Trichechus*. The name of this *Trichechid* may be, after the spot where it was found:

Trichechus Antverpiensis.

The recent walrus skulls from the "Rijks Museum voor Natuurlijke Historie" at Leyden and from the Zoological collections of the University at Utrecht and Amsterdam were placed at my disposal through the kindness of Dr. F. A. JENTINK and Profs. A. A. W. HUBRECHT and MAX WEBER. At Brussels I was enabled by the kindness of Dr. L. DOLLÒ to study the fossil and recent material of the "Musée royal d'Histoire naturelle de Belgique". Finally I have to thank Prof. WICHMANN who lent me the fossil for description and without whose assistance I should certainly not have succeeded in collecting all the literature.

FIGURES.

- Figs. 1. 3. 5. Hind view, base and side of the skull of *Trichechus Huxleyi*.
Find: opposite Breskens in the West Scheldt.
- Figs. 2. 4. 6. Hind view, base and side of the skull of *Trichechus Antverpiensis*.
Find: near Antwerp.

LITERATURE ON FOSSIL TRICHECHIDS.

- | | | |
|----|------------|--|
| 1. | 1823. | CUVIER. <i>Osséments fossiles</i> 2e éd. 1e partie p.264, 2e partie p.521. |
| 2. | 1828. | MITCHILL, SMITH and COOPER. Discovery of the fossil Walrus in Virginia. <i>Ann. Lyceum of Nat. Hist. of New-York</i> II p. 271. |
| 3. | 1834. | R. HARLAN. Critical notices of various organic Remains, hitherto discovered in North America. <i>Edinb. New. Phil. Journ.</i> XVII p. 360. |
| 4. | 1835. | R. HARLAN. <i>Physical and medical Researches</i> p. 255. |
| 5. | 1839. | H. DUCROTAY DE BLAINVILLE. <i>Ostéographie</i> , II, p. 45 et 49. |
| 6. | 1843. | CH. LYELL. <i>Edinb. New. Phil. Journ.</i> p. 187. |
| 7. | 1842—1843. | CH. LYELL. <i>Proc. geol. Soc.</i> IV p. 31. |
| 8. | 1844. | CH. LYELL. On the tertiary strata of the Island of Marthas Vineyard. <i>Amer. Journ. of Sc.</i> XLVI. p. 319. |

9. 1845. CH. LYELL. Travels in North America I p. 257—258.
10. 1845. K. G. ZIMMERMANN. Brief an den Geheimrath v. Leonhard. Neues Jahrb. f. Min. p. 73. (The here described walrus skull from the subsoil of Hamburg seems to have been carried there by man; Communications of Prof. C. GOTTSCHÉ).
11. 1847. GIEBEL. Fauna der Vorwelt. I. p. 232.
12. 1851. AGASSIZ. Proceed. Amer. Assoc. f. the Adv. of science. p. 252, 348.
13. 1853. EICHWALD. Lethaea rossica. 3. p. 890.
14. 1853—1856. BRONN. Lethaea geognostica. Thl. 6. p. 786.
15. 1853. PICTET. Paléontologie I. p. 233.
16. 1857. J. LEIDY. Notice on the Remains of the fossil walrus on the coast of the United States. Transact. Phil. Soc. Philadelphia New Series. vol. XI. p. 83.
17. 1858. GRATIOT. Bull. Soc. géol. de France. 2e série. V.
18. 1859. P. GERVAIS. Zoologie et Paléontologie française. II éd. p. 275.
19. 1865. E. RAY LANKESTER. On the Sources of the Mammalian Fauna in the Red Crag and on the Discovery in that Deposits of a new Mammal, allied to the Walrus. Proc. Geol. Soc. XXI. p. 233—232.
20. 1867. Vicomte DU BUS. Sur quelques mammifères du Crag d'Anvers. Bull. Acad. roy. Belgique. 2e série 24. p. 562—577.
21. 1869. J. LEIDY. The extinct Fauna of Dakota and Nebraska Journ. Acad. Nat. Science of Philadelphia VII. p. 416.
22. 1869. LE HON. L'Homme fossile. p. 304.
23. 1871. P. J. v. BENEDEN. Les Phoques de la mer Scaldisienne. Bull. Acad. roy. Belgique. 2e série 32. p. 5—18.
24. 1876. P. J. v. BENEDEN. Les Phoques du bassin d'Anvers. Bull. Acad. roy. Belgique. 2e série T. 41. p. 783—812.
25. 1876. SCHAAFHAUSEN. Trichechus rosmarus in Cöln. Sitz. Ber. Niederrh. Ges. f. Natur- und Heilk. S. 427.
26. 1877. P. J. v. BENEDEN. Description des ossements fossiles des environs d'Anvers I. Pinnipedia Ann. Musée d'Hist. Nat. de Belg. T. I.
27. 1878. W. DAVIS. On a collection of pleistocene Mammals, dredged off the Eastern Coast. Geol. Mag. (2). V. p. 97. 98.
28. 1880. E. T. NEWTON. Notes on the Vertebrata of the preglacial Forest Beds of the Coast. of England. Geol. Mag. (2). VII. p. 152—159.
29. 1880. E. RAY LANKESTER. On the Tusks of the fossil Walrus found in the Red Crag of Suffolk. Transact. Linn. Soc. (Zoology) (2). II. p. 213—221.
30. 1880. J. A. ALLEN. History of North American Pinnipeds U. S. Geol. Survey. Miscellaneous Publications 12.
31. 1880. E. RAY LANKESTER. Journ. Linn. Soc. (Zoology) XV London, p. 144—146.
32. 1881. E. T. NEWTON. The vertebrata of the pliocene Deposit of Britain. Mem. Geol. Surv. of the United Kingdom, p. 17—18.
33. 1891—1893. K. ZITTEL. Handb. d. Paläontologie IV p. 680—685.
34. 1898—1899. TROUESSART. Catalogus Mammalium p. 375—376.

LITERATURE ON THE FINDS OF TRICHECHUS HUXLEYI.

- 35 1875. J. C. DE MAN. Beenderen van den mammoth en van het uitgestorven rund, opgevischt in den omtrek van Zeeland. Arch. Zeeuwsch Genootsch. der Wetensch. III 2. p. 101—127.
- 36 1878. J. C. DE MAN. Een elandshoren, opgevischt in de Schelde. Mededeeling over eenige beenderen, in of nabij Zeeland gevonden. Arch. Zeeuwsch Gen. III 3. p. 1—22.
- 37 1879. F. SEELHEIM. Grondboringen in Zeeland. Verhand. Kon. Akad. der Wetenschappen Afd. Natuurk. Amsterdam.
- 38 1880. J. C. DE MAN. Derde Mededeeling over in de Schelde gevonden beenderen. Arch. Zeeuwsch Genootsch. V. 1. p. 161—170.

Anatomy. — “*On the existence of cartilaginous vertebrae in the development of the skull of birds*”. By Prof. J. W. VAN WIJHE.

(Communicated in the meeting of April 26, 1907).

It is a well-known fact that at a certain stage of development the notochord in all vertebrates extends forward as far as the hypophysis cerebri and backwards as far as the tip of the tail.

Over the whole length of the trunk and also in the occipital region of the head the dorsal part of the mesoderm is separated into segments or somites.

In the lower vertebrates: Selachians and Petromyzontes, the somites are not restricted to the occipital region, but extend forward as far as the hypophysis, i.e. equally far as the notochord.

The greater part of the voluntary muscular system is formed from the somites and in *Amphioxus* the segmentation of this muscular system is permanent and distinct from the anterior to the posterior end of the body.

The original function of the somitic muscles of the Chordates existed in my opinion¹⁾ in the to and fro movement of the notochord and so of the whole body during swimming.

In the Craniotes this muscular system is interrupted in the region of the auricular organ and in my opinion the presence of the auricular capsule is the cause of this. This capsule, which also encloses the organ for equilibrium, needed a firmer attachment than could be afforded by the connective tissue and found it in the parachordal cartilage, through the stiffness of which the muscular fibres in this region could no longer operate and consequently disappeared, partly even in their origin.

¹⁾ Cf. VAN WIJHE “Ueber die Homologisirung des Mundes und die primitive Leibegliederung der Wirbelthiere.” PETRUS CAMPER, Vol. IV. 1906.

The effect of this was also felt in the region in front of the auricular organ, but here part of the somitic muscles remained on account of a change of function. They became attached to the here developing eye-ball and now served for the movement of this latter and no longer for the movement of the whole body. This was accompanied by far-reaching shiftings, which can still be followed in the individual development.

The cartilaginous skeleton forms a system which appears only late in the development of the vertebrates and long after the appearance of the muscular system. As soon as the first cartilage may be observed, the muscular system in the head has undergone the changes here indicated. In the auricular region the somitic muscles have degenerated; partly they were not even indicated; in the region in front of the auricular organ they have entirely changed in place and shape and have entered into the service of the eye-ball. Only in the region behind the auricular organ — the occipital region — the myotomes — generally numbering three — still stand in the original order, like the myotomes of the trunk.

Head and trunk are separated in the ontogeny — although the border is later somewhat shifted in a caudal direction — already before the cartilaginous spinal chord appears, and I see no reason for assuming that this separation should not have taken place also in phylogeny before the appearance of the spinal chord.

The segmentation of the spinal chord depends on that of the muscular system. The body of a vertebra is not formed opposite the middle of a myotome, but opposite the border of two successive myotomes. BALFOUR has given the explanation of this at first sight curious phenomenon: the first muscular fibres occupy the whole length of a myotome and lie laterally of the tissue, surrounding the notochord. Now it is no more than natural that the solid points of attachment which in this tissue are formed for the muscular fibres, namely the origin of the vertebral bodies, are formed opposite the borders of two successive myotomes.

If we now ask where the appearance of vertebral bodies in the head must be expected, the answer must be that this cannot be in the auricular region, since here the myotomes have disappeared at the time of the appearance of the cartilage. No more can this be the case in the region in front of the auricular organ, for here the myotomes have entirely altered their place and have entered into the service of the eye-ball.

Only in the occipital region one would expect the appearance of two or three vertebrae. Yet until recently nobody has observed them

here, although this region has been investigated not only by the method of sectional series, but also by the methylene blue method, by which the investigation is so much easier. By this method Dr. NOORDENBOS did not find them in the vertebrate skull any more than myself in the skull of Selachians.

Instead of vertebrae we found the well-known parachordal cartilage accompanying the notochord in the occipital and auricular regions.

Certain authors have indeed spoken of the origin of vertebrae in the occiput, but the parts observed by them, were not cartilaginous but only badly outlined cell-heaps, not deserving the name of vertebrae.

So I was greatly surprised when my former assistant, Mr. F. SONIES, discovered by the methylene blue method two cartilaginous vertebral bodies in the occiput of embryos of the chick of the sixth breeding day and of ducks in a corresponding stage.

It will be asked how it is possible that these vertebrae have not been long known, since the embryos of the chick form the classical material for investigation in all embryological laboratories. The answer is that they were not discovered because the stage, in which they appear, is of so very short duration. One has to hit the moment in which the cartilage appears in the first two vertebrae of the neck. Before the cartilage appears in the remaining vertebrae, the two occipital vertebrae have already coalesced with the parachordal cartilage.

It is impossible to indicate the hour of the breeding day, since the development of the different eggs varies too much. By taking a large quantity of material, however, it is always possible to obtain the desired stage. It would require an immense expenditure of time to work all this material by the sectional method. With the methylene blue method, however, one is ready in a few days.

So the parachordal cartilage of birds does not originally form a morphological unity. With SONIES we may distinguish two parts in it: an anterior praevertebral part, situated in a region where the myotomes are degenerate or abortive and a posterior or vertebral part, occurring in the shape of two vertebral bodies, which soon coalesce with the anterior part.

Corresponding to these two vertebral bodies later also two vertebral arches appear on each side, which soon coalesce, but the locality of which remains indicated by two openings for the two roots of the nervus hypoglossus.

For further particulars and for several new discoveries about the development of the cartilaginous skull and the spinal column of

birds, I refer to the academical thesis of Mr. SONIES, which is now going through the press and will soon be published, also in "Petrus Camper". I will only mention that the small polar cartilage, discovered by NOORDENBOS in mammals and which also appears in Selachians, was found by SONIES also in birds.

Microbiology. — "*On Lactic acid fermentation in milk*". By Professor Dr. M. W. BELJERINCK.

(Communicated in the meeting of April 26, 1907).

In milk left to itself, which in consequence of spontaneous infection, contains the more generally distributed germs, with certain regularity some special floras are observed, whose composition is chiefly controlled by two factors: temperature and oxygen pressure. If the latter is very slight, that is, if the microbes of the milk are reduced to more or less anaërobic conditions, the floras become simple of composition and produce certain fermentations. The three principal of these are the Aërobacter-, the Butyric acid- and the Lactic acid fermentations, of which the two first are always characterised by the evolution of hydrogen and carbonic acid, whilst in the lactic acid fermentations, which may occur under different forms, beside the lactic acid, no gas at all, or carbonic acid only is formed. Sometimes this fermentation is accompanied by a vigorous slime formation, which slime consists of the swollen cell walls of the inferred lactic acid ferments.

For domestic purposes the lactic acid fermentation should be considered as useful; both the others as noxious.

The fermentation experiment the dairy industry applies to judge of the purity of milk has for its object to determine the commonness or the rarity of the germs of Aërobacter and of the butyric acid ferment. To this end a high standing glass is filled with milk, placed in a water bath of 40° C. and it is observed whether any fermentation gas is evolved, and if so, after how much time. In good milk this production of gas does not occur because then the lactic acid ferments develop so quickly that the other microbes are expelled. Artificially the Aërobacter fermentation is easily obtained by infecting non-acidified milk with faeces, soil or canal water and cultivating at about 37° to 40° C. After 6 to 12 hours production of gas is observed originating from *Aërobacter coli* or more rarely from *A. aërogenes*. The nature of the thereby obtained varieties changes with the temperature.

At temperatures beneath 40° the Aërobacter fermentation, after

lasting some hours, is replaced by a butyric acid fermentation which again, after some time is succeeded by a lactic acid fermentation. Externally the *Aërobacter* and the butyric acid fermentations cannot be distinguished, but this can be done easily with the microscope.

If 3 to 5 % chalk is added to a culture in a stoppered bottle at 35° to 40° C., the butyric acid fermentation can go on longer, and by early transplanting, likewise in milk with chalk and with exclusion of air, check the development of lactic acid ferments, without, however, quite dispelling them.

Microscopically the butyric acid fermentation may be recognised by the long, thin, at neutral reaction highly motile rods, sometimes mixed with elongated or more rounded clostridia, colouring blue by iodium, all belonging to the species *Granulobacter saccharobutyricum*.

To accumulate from such a crude butyric acid fermentation in milk the lactic acid ferments, which hardly ever lack there, it will suffice to transplant some drops into milk without chalk, and, if necessary, to repeat this after the butyric acid fermentation, which always sets in at first, is finished. Whether this be done in open or closed bottles or tubes, at 37° to 40° C., lactic acid rods of the genus *Lactobacillus* will be seen to appear, which by repeated transplantations completely dispel the butyric acid ferments.

If in these experiments instead of using fresh, unheated infection material, the soil, water, or faeces are previously heated to 80° or 95°C., by which only spore-forming microbes can develop in the milk, the fermentations of *Aërobacter* and the lactic acid ferments do not arise, their germs producing no spores, but a butyric acid fermentation is obtained, from which the aërobic spore-formers may be dispelled by repeated transplantation at exclusion of air.

1. *Properties of the active lactic acid ferments.*

As many bacteria of the most different groups can produce lactic acid it seems not superfluous to indicate what are the characteristics of the lactic acid ferments proper.

The active forms of dairy industries, yeast manufactories, distilleries, tanneries, and breweries, although joined by transitions, may be practically classified into the physiological genera *Lactococcus*, *Lactobacillus* and *Lactosarcina*, of which the two first only occur in the dairy products¹⁾.

¹⁾ In the chief floras of milk- and dairy products occur, to my knowledge, no species of *Lactosarcina*. When EMMERLING asserts to have found a yellow *Sarcina* in Armenian mazum (Centralbl. f. Bacteriologie, 2^{te} Abt. Bd. 4, p. 418, 1898), this can only have been a common infection from without. Also in butter sarcine species may accidentally occur but they do not belong to the chief flora, which consists of lactic acid ferments and lipophili.

They are always immotile, no-spore forming bacteria, which bear drying very well and which, by heating to 65° or 75° C., in which they just remain alive, while these temperatures are deadly to most other non sporeproducers, may be separated from these ("lacticisation"). They require for nitrogenfood peptones, such as are found in milk, malt extract, or other juices of plant- or animal origin, and for carbon food certain sugars, which may differ for different species. They do not peptonise proteids and, thus, do not liquefy gelatine; the secreted lactic acid can dissolve a certain quantity of caseine, but chemically this substance remains unchanged. These circumstances regulate their distribution in nature, where they are by no means general, but may rapidly multiply, especially under the influence of man. They are, however, found in the soil and can, by methods mentioned below, be accumulated and cultivated in a condition of pureness.

They are always more or less distinctly microaërophilous, some species or varieties can, however, grow very well at the air; other forms cannot and behave as real anaërobics. Access or absence of air is commonly of no consequence to the acid formation, but in the yeast industry a species is used, which at full atmospheric pressure produces no acid, and in the dairy industry are also forms which display the same property.

Always, even on good nutrient media, to which belong in particular maltextract agar, and milk- or whey-agar, the growth of the colonies remains limited, especially if the air and the produced acid can act simultaneously. If the acid is neutralised by chalk the growth of the colonies at the air may also become important. Yet, in most cases, the recognition of these ferments may repose on the smallness of their colonies compared with those of other bacteria.

Catalase is constantly absent, and hereupon an excellent diagnosis can be based, for which it is only necessary that a culture plate, on which all kinds of bacteria may occur, be flowed with strongly diluted hydrogensuperoxyd which is by all microbie species, except the lactic acid ferments, indifferently whether they belong to *Lactococcus*, *Lactobacillus* or *Lactosarcina*, changed into a seum of little oxygen bubbles.

Even the lately described¹⁾ large celled *Sarcina*, which in consequence of continued research I now consider as identic with the stomach sarcine (*Sarcina ventriculi*), and whose acid producing power is very slight, — i. e. 3 c.c. of normal acid per 100 c.c. of

¹⁾ These Proceedings 25 Februari 1905. Archives Néerlandaises T. 1 and 2. T. 11, p. 200, 1906.

maltextract or glucose broth, - does not at all decompose hydrogen superoxyd.

If we consider how generally catalase is met with in the animal and vegetable kingdom, as also in the microbes, its very absence in the lactic ferments appears in a peculiar light.

All active lactic acid ferments from milk invert sugar (invertase reaction) and can more or less easily decompose esculine and indican (emulsine reaction). The reaction on esculine is demonstrated by introducing, for example, 0.1% of this substance and a few drops of ferrictrate solution into whey agar or whey gelatin. Streaks drawn on it of species which decompose esculine produce intensely brown or black diffusion fields of esculetiniron, brown at more alcaline, black at more acid reaction, so that the lactic acid ferments become recognisable by the black fields in the midst of which their colonies are placed¹⁾. So long as esculine is present it is recognised by the magnificent blue fluorescence of the whole plate at feeble alcaline reaction. Indican may be used in a corresponding way but then no iron salt is wanted as the indoxyl produced from the glucosid oxidises of itself at the air to indigo blue. The lactic acid ferments decompose these two glucosides, slowly indeed, yet these reactions are very characteristic and useful. Amygdaline is not decomposed by the lactic acid ferments.²⁾

To the most remarkable properties of the lactic acid ferments belongs their power of reducing levulose to mannite,³⁾ which latter substance may even in concentrated nutrient solutions be recognised by its ready cristallisation at evaporation. A single drop dried on the object glass, commonly gives at microscopical investigation full certainty as to the existence of this reaction.

The lactic acid ferments thereby strongly contrast with the so nearly allied vinegar bacteria, in as much as the latter do just the reverse, i. e. they change by oxidation mannite into levulose.

Like so many other bacteria the lactic acid ferments possess, also with regard to various pigments, a strongly reducing power,

1) The knowledge of this extremely sensitive reaction, which has been applied for years in my laboratory, I owe to my colleague Mr. H. TER MEULEN.

2) Amygdalin is decomposed with much more difficulty by the action of microbes in general than the other glucosides named in the text. Moulds mostly decompose it into amygdalinate of amonium; beer yeast into amygdalonitril glucosid and glucose. Splitting under production of bitter almond oil, hydrocyanic acid and glucose I detected hitherto only with *Saccharomyces apiculatus* and with the anaërobic ferment of butyric acid fermentation, *Granulobacter saccharobutyricum*.

3) Ferments lactiques de l'industrie. Archives Néerlandaises 1901. KAYSER, Fermentation lactique. Annales de l'Institut agronomique 1904.

as is easily shown by inoculation into deep test-tubes of boiled milk coloured with litmus. The red litmus is first in the depth, later till near the surface quite discoloured, to turn red again by shaking with air. The thickness of the red layer in the curdled milk admits an accurate measure of the intensity of the growth and of the reduction process. The thinner the red layer the more intensive both functions must be.

2. *Factors of variability.*

Many, perhaps all lactic acid ferments display a high degree of variability as well in physiological as in morphological properties. Nevertheless this variability in different stocks, coming from different isolations of the same species, is not always equal by far, which may give rise to trouble in the study of the specific properties. The circumstances causing the variability are but partly known; decidedly belongs to them an oxygen pressure, too far above or too far beneath the optimum for the vital functions, which may, especially for the bacterium of the long whey (*Lactococcus hollandiae*), be demonstrated with exceeding clearness.

This remarkable species is characterised by a vigorous slime formation when cultivated in milk or whey, but loses this power at temperatures above 20° C., as well at the ordinary pressure of the atmospheric oxygen, as at complete exclusion of it, if the changed influence is allowed to act during some time on the growing microbes. This is shown by cultivating the whey in a closed bottle; the upper layer, just beneath the stopper, where a little air can find access, becomes quite liquid and contains a hereditarily constant, common *Lactococcus*, forming little acid and no slime. Also by cultivating the long whey microbe in tubes of boiled milk with access of air, after one or two re-inoculations, a *Lactococcus* is produced, which forms no slime at all. If the material for the reinoculation is secured from the depth of the cultures grown in closed flasks, at places where the access of air is impossible, and the inoculation is repeated once or more in the same way, a *Lactococcus* is likewise obtained which displays no trace of slime production.

At some depth beneath the surface, however, is a zone in which unchanged, slime forming, hereditarily constant material is found.

What in this case can be very easily ascertained, proves, at accurate investigation, also to be true for the other species of lactic acid ferments, namely, that they only then continue to display constant specific characters, when they are continuously cultivated at

a certain pressure of the oxygen, else, these characters are seen to disappear, whilst in fact, or apparently, new ones originate. Hence, in some cases it may be proved, in others the probability is shown, that each species must occur in three varieties, joined by intermediate forms, i. e. the normal form, a "high pressure variant", and a "low pressure variant".

As in wholly different groups of bacteria corresponding facts may be observed, there is cause to assign a fundamental signification to them.

A decisive factor which may cause the production of variants is furthermore the temperature, for experience proves that a prolonged cultivation above the optimum temperature of growth, gives rise to the appearance of forms distinctly different from the original stock.

In other cases the cause of the variability is unknown; not seldom for example, we find at the very first culture of a species taken from nature, strongly varying colonies, which prove to belong to the same species only because many colonies by sector-variation display the genetic alliance of the variants to the wild stock.

But then, too, there is reason to admit that the new vital conditions, to which the microbes are subjected just by the change of oxygen pressure and temperature, are the chief factors of the variation process which is, as it were, seen in action. This observation is of so general a nature and is so closely related to the essence of life, that it must be considered as probable, that also in higher plants and animals, local changes in the access or exclusion of oxygen, in connexion with temperature, play an important part in the morphogenesis.

As the examination of other species of microbes shows that the absence of certain nutrient substances in the culture medium, at free aëration and during growth, may cause hereditary variation, for example in *Schizosaccharomyces octosporus*, which in old cultures changes into the spore-free variant, totally differing from the chief form, there is reason also to believe, that also the said factor must be considered to explain the great variability of the lactic acid ferments; but the observations there about are not yet fit for definite conclusions.

3. *Elective culture of the microbes of the slimy lactic acid fermentation.*

There is reason to assume that the slime producing lactic acid ferments are the normal forms and the non-slime formers, species or variants derived from them. Hence, the former deserve to be considered in the first place.

To the typical slime producing species belongs the microbe of the long whey (*Lactococcus hollandiae*), which particularly before the introduction of pure cultures in the dairy industries, played an important part in the fight against cheese defects in North-Holland, and is still here and there practically used to that end.

Further I have found that the popular food known in Norway as "tjaette molken", a sample of which I owe to the kindness of Mr. PENNIX of Rotterdam, consists of milk, in which the long whey microbe, or at least a nearly allied form, secretes acid and slime.

Other materials in which these and allied microbes occur, were till now unknown, evidently because of the uncertainty about culture conditions and the lack of a good accumulation method. Taking the idea "species" in the broad sense, I think there is no objection as to bringing the group of forms, found in the manner described below, to the species just mentioned.

Starting from the following properties, the most characteristic for the microbes of the slimy lactic acid fermentation:

1st. The optimum temperature for their growth is at 20° or lower,

2nd. they can only compete in anaërobic cultures with the other microbes, and

3rd. the medium must consist of substances containing peptones as nitrogen and carbonhydrates as carbon source, I succeeded in finding a method giving rise to their accumulation.

It is true that I only examined a single material in this way, the common baker's yeast, but the investigation of the soil of fermenting or fermented substances, in short of materials of most varying description may be done in a corresponding way.

The experiment is arranged as follows.

Into a 30 c.c. closed bottle, filled with maltextract, to which is added $\frac{1}{2}\%$ of peptone siccum and which contains c.a. 10% extract, a little pressed yeast is introduced, for instance $\frac{1}{2}$ gram. Placed at a temperature of 18 to 20° C. a quiet fermentation sets in, which is allowed to continue 24 to 72 hours, whereby, because of the absence of air the yeast hardly grows, but the various lactic acid ferments reproduce quickly. Other microbes do not develop. Not seldom in this first culture have the contents of the flask already become somewhat slimy.

Whether this be the case or not, a not too small quantity from it is transplanted into a bottle quite filled with boiled, air-free milk, for instance $\frac{1}{2}$ c.c. into 30 c.c. of milk. At the same low or a somewhat higher temperature only a flora of lactic acid ferments can develop, and if the slime-forming species is present, it is the most

vigorous. We then see that after 2 or 3 days the milk becomes slimy and by inoculation into milk whey, a culture will start which sometimes differs so little from the ordinary long whey, that we may conclude to an identity of species.

Of course, I cannot foretell that such microbes occur in any yeast sample taken at random, hence I must add that for my experiments I used pressed yeast from the Yeast and Alcohol manufactory at Delft.

Such a culture in milk differs it is true in some respects from what is obtained by growing long whey from North Holland in milk, as in the former case short rods or oblong cocci are observed, and in the latter, shorter forms more reminding of the common micrococci.

I expect that by repeating this experiment various deviating varieties will be found, and by application of the method to other infection material perhaps new species of slime lactic acid ferments may be discovered.

4. *Elective culture of the lactococci of cream souring.*

As the lactococci and lactobacilli, which both occur in spontaneously or otherwise soured milk, in cheese, and various other dairy products, seem to grow nowhere better than in milk,¹⁾ the culture experiments here considered, should be taken with milk.

In order out of the innumerable microbes of the crude milk practically to come to a pure culture of *Lactococcus*, the management is as follows.

The optimum of growth is at 30° C. or lower, and as all species of *Lactococcus* (like those of *Lactobacillus*) are strongly microaërophilous, sometimes even anaërobic (i. e. cannot grow at all at full atmospheric pressure on plates), it is best to cultivate in absence of air.

A stoppered bottle is quite filled with commercial milk and placed at 30 C. After 24 hours or somewhat later a *Lactococcus*-flora begins to replace the other microbes, while not seldom a feeble fermentation of *B. coli* or *B. aerogenes* has preceded.

After one or two re-inoculations under the same conditions, but

¹⁾ It is not impossible that there are "peptones" which, together with glucose or lactose, are still better food for the lactic acid ferments than milk itself. How very differently peptones of dissimilar origin act on microbes is easily observed in yeast species which in general grow better on "plant peptones" than on "animal peptones". The introduction of the word "bios" to denote those nitrogen compounds which are best fit as yeast food, is an attempt to circumscribe the peptoneproblem has been given. The relation between "peptones" and the lactic acid ferments is still closer than between these substances and the different yeasts; but it is here not the place to insist on this point.

into well boiled milk, which is done by transferring a trace of the first culture to the second bottle, quite filled with boiled, air-free milk, and so on, the lactococci free themselves completely from all foreign microbes and a material is obtained, which displays a high degree of purity and of practical usefulness. If the acidifying power of the microbes obtained by the experiment is lower than wished for, for example 5, whilst 8 to 10 c.c. of normal acid on 100 c.c. milk is desired, this must be attributed to the accidentally present stock. It is necessary then to begin a new experiment, following the same way as described, or it can be advisable to perform the first inoculation with some good butter-milk.

As buttermilk, however, very often contains lactose yeast, in the latter case a vigorous alcohol fermentation may at first be expected in the bottles. But it soon disappears by inoculation into milk rendered free from oxygen by boiling.

If in this way, thus in absence of air, the culture has been prolonged, a fairly constant acid amount is obtained at each renewed inoculation, which does not, however, rise above 10—12 c.c. normal in 100 c.c. of milk. On whey agar or whey gelatin plates the growth at the air of the thus obtained lactococci is different, as sometimes a great many aërobic colonies arise, which cause the same acidification as the cultures in the bottles, while in other cases nothing is seen to grow.

The first group corresponds with the usual commercial forms destined for the souring of cream, which commonly consist of cultures of the microbes dried on milk sugar or starch; moreover there are commercial aërobic pure cultures in milk or whey, which are sold in bottles.

The second group, that is the cultures non-growing at the air, may still better be used for the cream souring than the aërobic stocks, as the anaërobic forms of *Lactococcus* show more aptness to secrete the flavour desired in butter, than the more aërophilous bacteria. ¹⁾

As well for this reason as for the great purity of the cultures made after this "bottle method", there is reason to prefer them in dairy work to the commercial so-called pure cultures, which for the greater part are by no means pure, but mostly contain, besides lactococci, numerous contamination germs of the milk. In consequence of frequent investigations I can therefore advise interested persons to use the here described method. Best would be if these cultures were

¹⁾ Of late I have also met with such like anaërobic lactic acid bacteria in commercial preparations.

prepared in the creameries themselves, but also the sellers of pure cultures, by following the above prescriptions, will obtain a better product than by the more usual way of selection of aërobic colonies. Besides, the management is simpler and more scientific.

To my opinion there is no satisfying ground to class the aërobic and anaërobic forms of *Lactococcus*, which can be produced after the said method, in separate species. They are but variants of one and the same species, whose oxygen requirements are different, which also appears from the fact that in the course of time one and the same stock shows considerable differences with regard to the said relation. Moreover, by several isolations all transitions between the more or less aërobic stocks may be obtained.

Finally it should be borne in mind, that by applying the "bottle method" at low temperature, in rare cases instead of a culture of real *Lactococcus* a *Lactobacillus* is obtained, which may likewise be had by colony selection from cheese. Using this *Lactobacillus* I did not observe at all the pleasant flavour of the anaërobic lactococci, so that I do not recommend these bacilli for cream souring.

5. *Elective culture of the lactic acid bacilli.*

If milk, soured spontaneously by *Lactococcus lactis*, or still better, buttermilk, is placed at exclusion of air in a thermostat of ca. 40° C., the original acid amount of 8 to 12 c.c. will in most cases rise after some days to about 18 or 20 c.c. per 100 c.c. of milk. For this experiment it is best to use a stoppered bottle of 250 to 300 c.c. capacity quite filled with milk. If for the first experiment a smaller quantity is used the result becomes uncertain, either by the disturbing influence of the air, or by the scarcity of the inferred bacteria.

The first change commonly observed in the sour milk is a moderately vigorous alcoholic fermentation, caused by the hardly ever lacking lactose yeast, and at the same time a complete separation of the caseine, which is driven to the surface of the liquid by the carbonic acid.

Microscopically we find that the lactococci present at first, are succeeded by more lengthened forms, truncated at the ends and united in chains, whereby the acid titer may considerably diminish, for instance in 12 hours from 8 c.c. to 6 c.c., which should be ascribed to the lactose yeast, for which the free lactic acid can serve as carbon food. By transference, at exclusion of air, the lactose yeast, as in the elective culture of lactococci, is rapidly dispelled by the then stronger lactic acid ferments.

Real lactobacilli mostly appear after 2 or 3 days and then the acid rises rapidly parallel to their multiplication to 20, even to 25 c.c. normal per 100 c.c. of milk. When this degree of souring is reached, there is usually no further increase observed, not even after several days, and whenever this does take place, there should be thought of aëration, by which the growth of vinegar bacteria and acetic acid formation from alcohol, have become possible.

The pure culture of lactobacilli is sometimes easy, in other cases, with more anaërobic stocks, it is more difficult. Always, however, it is troublesome with these pure cultures to obtain a considerable souring in milk and there is most chance of success (but even then the success is not quite certain) by souring lactobacilli together with *Lactococcus* which serves for the first souring to 8 c.c. If this amount of acid is reached, and the pressure of the oxygen sufficiently diminished, which in a stoppered bottle is likewise brought about by the presence of the lactococci, the lactobacilli can develop and cause further souring.

From the observation that by the described experiment more or less perfectly anaërobic lactobacilli are obtained, follows that here as in the case of *Lactococcus* different varieties may be expected. At a continued research the differences prove to extend over other characteristics also and may become so great, as well from a morphologic as from a physiologic point of view, that it seems necessary to create new species.

Especially the dimensions of the rods, the more or less branched state of the colonies on agar plates, the slime formation, the either or not originating of carbonic acid as fermentation gas beside the lactic acid, and the action or non action on different sugars give rise to this consideration. The deeper however we enter into these distinctions, the more troublesome it becomes to devise such descriptions as are wanted to present to other investigators an image of the results of our own researches; so numerous become the forms which nature, or better perhaps, which culture produces, and so slight are the differences by which these forms are distinguished, if we do not confine ourselves to the extremes of the groups.¹

If the latter is done, two distinct forms call attention, which on a former occasion I named²) *Lactobacillus caucasicus* and *L. longus*.

1) For further information see W. HENNEBERG, Zur Kenntniss der Milchsäurebakterien. Sonderabdruck aus Zeitschrift für Spiritusindustrie. No. 22--31, 1903. PAREY, Berlin.

2) Sur les ferments lactiques de l'industrie. Archives Néerlandaises. Sér. 2, T. 6, p. 212, 1901.

Without attributing a special value to this classification I yet wish to keep to it as I think that the facts to be mentioned are fairly well comprised thereby.

The longusgroup is characterised by its not acting on maltose, so that in maltextract no, or very little acid it formed, but it does decompose milksugar. In milk the forms of this group, if grown after a previous culture of *Lactococcus* which has produced 5 to 8 c.c. of lactic acid per 100 c.c. of milk, will once more produce a certain, even a like quantity of acid so that ca. 16 c.c. may be titrated, the latter amount being however an exception. Generally no evolution of carbonic acid is observed but sometimes it is, and then so much gas can arise that a milk beverage is acquired foaming like champagne.

By a series of transitions, the longus forms obtained at 40° C., are joined with lactobacilli which at a lower temperature find their optimal vital conditions, but which are rarer in milk.

The caucasicus group comprises those lactobacilli, which are able, independently of lactococci to produce in milk a very high acid formation. At 37 to 40° C. it is possible after three days of their action to titrate 20 to 25 c.c. of normal acid per 100 c.c. of milk. When that amount is reached further acid formation stops. In this case, too, there is a parallel form which, beside much lactic acid, also evolves carbonic acid. What by-product is then formed from the lactose molecule beside the carbonic acid is not yet clear; probably it is aethylalkohol. G. BERTRAND has proved that these ferments can produce succinic acid. They greatly owe their notoriety to their presence in kephir, which subject I have touched before¹⁾. Later however I have come to the conclusion²⁾ that their distribution is by no means restricted to kephir only, but that they also occur in our climate, sometimes in buttermilk, in cheese and even in common baker's yeast.

6. *Yoghurt and maya.*

The use of soured milk as drink and food is so familiar to many Eastern countries, and dates from so remote an antiquity that there can be no doubt as to its favourable effect on health, and the establishment of various societies which try to popularise new preparations of that nature, seems to prove that the attention of the Western nations begins to be drawn towards it.

Both in the preparations of the Eastern nations and in those of

¹⁾ Sur le Kefyr. Archives Néerlandaises. T. 23, p. 428, 1891.

²⁾ Ferments lactiques de l'industrie l. c.

industry are always found lactic acid ferments of the genus *Lactobacillus*, mostly of *Lactococcus* too. These lactic acid ferments alone determine the character of the "leben raib" of Egypt,¹⁾ of the "yoghurt" of Bulgaria,²⁾ and probably also that of the "prostokwacha" and the "véranetz" of Russia, which METCHNIKOFF mentions. In the "kephir" of the Caucasus, the "koumys" of Central Asia,³⁾ and the "mazun" of Armenia,⁴⁾ occurs moreover, lactose yeast, which may, however, under certain circumstances be wanting, without the character of these beverages being lost. All other microbes, which are mentioned in literature as occurring in the said beverages or their ferments, such as *Oidium*, *Mucor*, other moulds, torula, red yeast, vinegar bacteria, butyric acid ferment, proteolytic bacteria, are only present by deficient preparation, so that it may be said that in all examined cases a pure lactic acid fermentation proves to be the wanted process, whilst eventually also an alcoholic fermentation is wished for or suffered⁵⁾.

Hence, in the commercial preparations which start from yoghurt, only lactic acid ferments are cultivated. I have in particular investigated the products of "Le Ferment", mentioned beneath, as also a substance, sold as "maya" or Bulgarian ferment,⁶⁾ to which my attention was drawn by Dr. DE LINT at Scheveningen. Here I will shortly describe the latter preparation.

It consists in a yellowish strongly acid reacting powder, composed, after chemical, microscopical and bacteriological examination, of caseine lactic acid, lactose, fat and lactic acid bacteria; it is evidently nothing

1) Annales de l'Institut Pasteur. T. 16. p. 65, 1902.

2) MASSOL et GRIGOROFF, Revue médicale de la Suisse romande 1905 p. 716. BERTRAND et WEISWELLER, Action du ferment Bulgare sur le lait. Ann. de l'Institut Pasteur, T. 20, p. 977, 1906.

3) For Kephir and Koumys see WEIGMANN in LAFAR Technische Mykologie. Bd. 2. p. 128. 1905.

4) Centralblatt für Bacteriologie, 2te Abt. Bd. 15, p. 577, 1906.

5) The study of literature leads at first view to a quite other result, as many microbiological descriptions are made by beginners, not sufficiently acquainted with the properties of lactic acid ferments, and who have attributed an exaggerated weight to the different kinds of infections named above.

6) On the bottle stands: Maya bulgare, Société de la maya bulgare, GARNIER & Co., Paris, 16 Rue Popincourt. The Société de Pury, Montreux, brings into commerce a ferment of the same nature under the name of "maya bacilline", and the Société HENNEBERG, Geneva, a liquid preparation as "lacticose". Besides there are to be had in Paris Lactobacilline de METCHNIKOFF in "Le Ferment", Fournisseur de l'Assistance publique, 77 Rue Denfert-Rochereau, who sells also, the "Biolactyle" of FOURNIER and the "Bacilline paralactique" of TISSIER (the preparations of this firm make a very good impression).

else but yoghurt evaporated at low temperature, perhaps in the vacuum. As to the preparation of the "yoghurt" itself by means of this ferment, it is done as follows and gives good results.

Milk is evaporated to half its volume, cooled to a (not nearer indicated) temperature, for which I took 40° , as 45° proved too high and 37° too low, and on a quantity of 250 c.c., so much ferment is strewn as can be put in a little spoon distributed with the flacon containing the maya. After 6 hours already the curdling of the milk becomes perceptible, after 24 hours I titrated 12 c.c. and after 3×24 hours 20 to 23 c.c. of normal lactic acid per 100 c.c. of the evaporated milk, which by that time is changed into yoghurt.

As a titer of 10 c.c. corresponds to 0.9 % of lactic acid, the titer 20 corresponds to somewhat less than 2 % of the vanished milk sugar. Supposing that the evaporated milk contains about 9.6 % of milksugar it follows that 7 % of milksugar has remained undecomposed. The caseine is of course curdled and the whole has changed into a solid but soft, sweet tasting mass.

The evaporation of the milk is not necessary, but when prepared from ordinary milk, the yoghurt remains more liquid, and as the acid formation is equally strong as in evaporated material, there remains about 2.5 % of the original 4.8 % milksugar, so that in this case the taste is much less sweet.

If in the said way yoghurt has been prepared in the presence of air and is re-inoculated into a new quantity of milk, then the result is yoghurt of the same acidity as the first time. But after 3 or 4 transferrings difficulties arise and only with great quantities of infection material further souring can be obtained. The experiment succeeded much better when the yoghurt was prepared in a quite filled stoppered bottle; the transferring can then be longer continued, but I do not know whether this will do in the long run. Evidently the difficulty here, too, is the right choice of oxygen pressure, whereby the inferred lactic acid bacteria preserve their properties unchanged; and this difficulty is still increased by the presence of two different forms, with unequal optima as to temperature, and probably as to oxygen pressure also.

One of these forms is again a *Lactococcus*, the other a *Lactobacillus*.

The former deviates somewhat from the common *Lactococcus*, in as much as it is more extended, reminding of short rods, and furthermore by possessing a higher optimum as to the temperature whereby the growth is quickest, which optimum proves nearer to 37° than to 30° C. Hence, this form is as it were a transition to a *Lactobacillus*. Isolation on milk agarplates was very easy, even at 30° C.

As to the second species, the *Lactobacillus* proper of yoghurt, it was troublesome to grow its colonies on milk agar plates, but on malt extract agar it was more easily obtained. In literature it has been named *Bacillus Massol* by GRIGOROFF, but I think that name superfluous as the characters correspond fairly well with those of the kephir bacilli which also occur in our country; for instance, as has been observed before, in yeast and buttermilk. Sown in slightly soured milk this *Lactobacillus* can produce the strong acid mentioned above, without the help of other bacteria. Evolution of carbonic acid does not take place and the product has a very pure taste, although a beginning of fat cleavage seems inevitable at such a high amount of acid.

METCHNIKOFF ascribes a very favourable influence to the use of yoghurt, as it diminishes the phenomena of autointoxication starting from the intestinal canal, and he explains this effect by accepting that the *Lactobacillus*, after passing the stomach, continues active in the intestine, and checks ¹⁾ the formation of the obnoxious products which derive from other bacteria species. I do not doubt but this may be brought about by the lactic acid, but I think it highly improbable that the presence of the lactic acid bacteria from the yoghurt themselves should be required in the intestine. I think this conclusion is necessary, first because, without the use of yoghurt or other soured milk preparations, there occur in the intestine lactic acid ferments of different species, and second, because the conditions for lactic acid formation by the active ferments are wanting or must at least be very unfavourable there.

As to the first point I refer to the following experiments.

If sterile milk is infected with faeces of different origin (man, cattle) and treated as described for the elective culture of *Lactococcus*, without access of air and repeatedly reinoculated at a temperature between 23° to 26° C., the said genus of microbes is indeed obtained by which as good cream souring can be obtained as with the pure cultures prepared in the before described way.

If sterile milk is infected in a corresponding way and exposed to the conditions wanted for *Lactobacillus*, that is, if cultivated in absence of air at 40 to 45° C., a fermentation of coli will first arise and later or simultaneously a butyric acid and no lactic acid ferment-

¹⁾ Quelques remarques sur le lait aigri. Rémy, Paris, 1907. In this paper METCHNIKOFF gives many assertions but no decisive experiments. Besides, his bacteriological elucidation, p. 26, is not clear. The elaborate and interesting work of Dr. A. COMBE, L'autointoxication intestinale, Paris 1907, is neither quite convincing from a micro-biological point of view.

tation, which latter would inevitably arise if the lactic acid ferments were present in a rather considerable number. Only by repeated transferences *Lactobacillus* is produced, which after some inoculations forms 10 to 13 c.c. of normal acid.

Hence, there is no doubt as to the presence of *Lactobacillus* and *Lactococcus* in normal faeces. They are, however rare, and belong by no means to the intestinal flora proper, like *coli*, but to the accidental flora, which consists of all that is introduced and is able to pass the stomach and intestines alive, without multiplying. There seems to be no cause to attribute any important influence to this fact.

As to the second point, why in the intestinal canal the conditions for the growth of the active lactic acid ferments are wanting, it is that in the contents of the intestines an alcalic reaction exists, and that the sugars which are formed or introduced there, in as much as they are not absorbed by the intestinal wall, will surely be attacked by *coli*, which in these circumstances is the stronger and dispels all competitors.

Why *coli* (and *aërogenes*) so completely defeat the lactic acid ferments, should, to my opinion, be explained by the important fact, not sufficiently considered in literature, that the first mentioned species can quite well live on peptone only, and multiply at its expense, while the active lactic ferments completely lack this faculty and, beside peptone, require a carbohydrate for food.

If, moreover, it is borne in mind that *coli* in the presence of a carbohydrate can also feed on other sources of nitrogen than peptone, for example on amines and ammonium salts, whereas the active lactic acid ferments cannot, and decidedly want peptones for nitrogen food, it is clear that for the different forms of *coli* practically every where in the intestinal contents a good feeding material is present, and that in the few localities where it would also be sufficient for the lactic acid ferments, it will be seized upon by *coli*. Where only peptones occur, *coli* will moreover increase the already alcalic reaction of the contents and thus, not for itself but for the lactic acid ferments, render the conditions of life more unfavourable.

Hence it seems evident why in the intestinal canal a coliflora can exist but no lactic acid flora.

The yellow coloured faeces of babies during the lactation period may be alleged to support this view. They consist microscopically almost solely of bacteria, for far the greater part of common colibacteria¹⁾, among which there occur real lactic acid ferments, but

¹⁾ For different children not always the same varieties; sometimes, for instance non-fermenting forms reminding of *Lactobacillus*, for which I before indeed took such bacteria.

as in the case described before in quite an inferior number. This fact acquires a special significance when we consider that ESCHERICH, the discoverer of the colibacillus, has proved that this condition exists directly behind the baby's stomach, where *coli* and *aërogenes* are predominant which, in reference to the preceding, necessitates the conclusion that even at those portions of the intestines where a lactic acid flora should first be looked for, it is evidently unable to sustain itself.

There is no doubt but here too, the strongly disinfecting action of the stomachal hydrochloric acid plays a part, as this acid, at a much lower titer than the lactic acid checks the growth of the lactic acid ferments, but hence can be neutralised by much less alkali, which is not indifferent to *coli*, which produces alkali.

In so far as the theory of METCHNIKOFF and COMBE is right, after which yoghurt or other sour milk preparations counteract the auto-intoxication from the intestinal canal, it seems certain that here should more be thought of the influence of a milk diet and the free acid taken up with the milk, than of a specific intestinal flora. But in how far the apparently proved decrease of indol and phenol, whose quantity is considered as determining the degree of auto-intoxication, deviates, at a nutrition with soured milk preparations instead of meat, from this decrease when non-soured milk is used, — to my opinion the real core of the question, — has not been considered by the said authors.

Admitting that the soured preparations really deserve to be preferred, I think that especially in Holland, it must be possible with good buttermilk in as simple a way to reach the wished for end, as with the various exotic ferments, whose descriptions give the impression that the preparators are but imperfectly acquainted with the general phenomena of the lactic acid fermentation in milk.

Although I see no fundamental difference between the use of buttermilk and yoghurt, it is certain that the latter may be prepared in a very simple way under medical control, and hence, to my meaning, deserves to be recommended in certain cases.

Summarising the preceding I come to the following conclusion.

In milk three chief forms of lactic acid fermentation, determined by temperature, are to be distinguished, namely at very low temperature, the slimy lactic acid fermentation; at a middle temperature the common lactic acid fermentation caused by *Lactococcus*; and at higher temperature the lactic acid fermentation by *Lactobacillus*.

The elective culture of the microbes of the slimy fermentation, succeeds by cultivating baker's yeast in absence of air between 15°

and 18° C. in malt extract and transferring to boiled milk or whey at a somewhat higher temperature. The acidity obtained remains low and amounts to 3 to 5 c.c. of normal acid per 100 c.c. of milk.

The elective culture of *Lactococcus* takes place by allowing milk to sour in a stoppered bottle at 20° to 25° C. and transfer it repeatedly to boiled milk at that temperature. The thereby obtained stocks of *Lactococcus lactis* are mostly anaërobic but specifically not to be distinguished from the more aërobic forms which may be produced by the same experiment. The acid mostly remains at about 8 c.c. of normal acid per 100 c.c. of milk, but may become 10 to 12 c.c.

The elective culture of *Lactobacillus* succeeds best by cultivating buttermilk in absence of air at 37° to 40° C. and inoculating it into boiled milk, at 30° C. and higher, the acidity can rise from 18 to 25 c.c. of normal acid per 100 c.c. of milk.

The active lactic acid ferments are very variable; as factors of hereditary constant variation are recognised cultivation at too high or too low oxygen pressure, and cultivation at a temperature above the optimum of growth.

Lactic acid ferments do not lack in the intestinal flora, but play there an inferior part.

A considerable difference between Eastern and Western lactic acid ferments does not exist.

Yoghurt and other such like sour milk preparations deserve the attention of hygienists.

Chemistry. — “*On the course of the plaitpoint line and of the spinodal lines, also for the case, that the mutual attraction of the molecules of one of the components of a binary mixture of normal substances is slight*”, by Mr. J. J. VAN LAAR. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of April 26, 1907).

1. In the latest volume of These Proceedings¹⁾ Dr. KEESOM (also in conjunction with Prof. KAMERLINGH ONNES) stated some important results, inter alia concerning his investigation on the special case that one, e.g. α_1 , of the two quantities α_1 and α_2 is very small; which is

¹⁾ KAMERLINGH ONNES and KEESOM, These Proc., Dec. 29, 1906, p. 501—508 [On the gas phase sinking in the liquid phase etc. (Comm. 96b)]; KEESOM, Ibid. p. 508—511 [On the conditions for the sinking etc. (Comm. 96c)]; KEESOM, Ibid. March 28, 1907, p. 660—664 (Comm. 96c continued); KAMERLINGH ONNES and KEESOM, Ibid. of April 25, 1907, p. 786—798 [The case that one component is a gas without cohesion etc. (Suppl. N^o. 15)].

realised, among others, for mixtures of He (α_1) and H₂ (α_2). In these papers, particularly in the last, a particular kind of plaitpoint line has been repeatedly mentioned, viz. one passing from the critical temperature T_0 , called "third" by me (KEESOM'S T_{kn}), to the *highest* of the two critical temperatures T_2 (KEESOM'S T_{k_1}).

Now the theoretical possibility of such a course of the plaitpoint line, i.e. *of one of its two branches*, has been first brought to light by me in a series of Discussions on this subject¹). Not only for the special case $b_1 = b_2$, for which among others, fig. 1 of June 21, 1905 holds, but for all possible cases (see specially TEYLER I and II). We found that such a course will always be found, when the ratio of the two critical temperatures $\theta = \frac{T_2}{T_1}$ is *larger* than the value of this ratio, for which the plaitpoint line has a *double point*. This type was called type I by me. (see also fig. 1 of Oct. 25, 1906).

The case that a plait *starts* from C_0 to C_2 , or also at the same time from C_2 to C_0 (when there is a minimum temperature in the plaitpoint line) is not new (see K. O. and KEESOM, p. 788 below), but has been before described and calculated by me in all particulars.

The *double point* in the plaitpoint line, discovered by me in 1905 (June 21), did not only give the key to the possibility of such a course, which had already been ascertained for mixtures of water and ether, of ethane and methylalcohol²); but also the *connection*

¹) These Proc. May 25, 1905, p. 646—657; Ibid. June 21, 1905, p. 33—48; Ibid. Aug 17, 1905, p. 144—152 (Cf. also Arch. Néerl. 1905, p. 373—413); Ibid. Jan. 25, 1906, p. 578—590 (Also Arch. Néerl. 1906, p. 224—238); Ibid. Oct. 25, 1906, p. 226—235. Further Arch. TEYLER (2) X, Première partie, p. 1—26 (1905); Ibid. Deuxième partie, p. 1—54 (1906). Henceforth I shall refer to papers in these Proceedings by mentioning the *date*, to papers in the Arch. TEYLER by putting TEYLER I or II.

²) I do not quite understand why in cases as for He + H₂ the plait considered is particularly called a "gasplait". With exactly the same right the two coexisting phases might be called *liquid* phases, especially at the higher pressures in the neighbourhood of the point C_0 . With reference to water-ether, etc. we speak of a gas phase and a liquid phase before the three phase equilibrium is reached, i.e. at higher temperatures; and when at lower temperatures the equilibrium mentioned has established itself, of two liquid phases. The "gas phase" is then determined by the branch plait of the original transverse plait (which latter has now the peculiar shape directed towards C_0 in the neighbourhood of the axis $x=0$). But I acknowledge that this is perfectly arbitrary, it being difficult to indicate where the pressure is high enough on such a plait to justify us in speaking of liquid phases. Would it not be better to follow here VAN DER WAALS' terminology, and speak of *fluid* phases, and to call the two phases *liquid* phases at temperatures where the three phase equilibrium is found? Otherwise in this latter case — keeping to K. O. and KEESOM'S terminology — we should have to speak of *three* coexisting *gas* phases, a rarefied one and two very dense ones, which latter, however, we should never refer to as gas phases in the *perfectly identical* case of water + ether.

of the different series of *hidden* plaitpoints, etc. etc., as has, inter alia, been indicated in Jan. 25, 1906 (cf. also TEYLER II). Dr. KEESOM does not mention that in his figure I (loc. cit. p. 794) besides the plaitpoint line from K_m to K_1 drawn there, there *always* exists also a *second* branch, which runs along the v -axis in the neighbourhood of $x=1$ from the point where $v=b$ to K_2 — and which gives rise to a *three phase equilibrium* at lower temperatures, as this has been explained by me. (also in Jan. 25, 1906 and TEYLER II).

The fact whether a plait extends in the way mentioned, depends therefore, as we said before, *in the first place* on the fact whether the values of $\frac{b_2}{b_1}$ and $\frac{a_2}{a_1}$ (so of $\theta = \frac{T_2}{T_1}$ and $\pi = \frac{p_2}{p_1}$) are such that θ is larger than that value of θ for which the plaitpoint line has a *double point* with given value of π . The knowledge of this double point, being therefore of so great importance for the distinction of the different types, I have carried out in TEYLER I the lengthy calculations required for this, and drawn up the results obtained in tables. [See also TEYLER II, where fig. 22 (p. 30) represents the results *graphically*].

Hence *not* the fact that $T_{km} > T_{k_1}$ [with perfect justice KEESOM says in a footnote (loc. cit. p. 794) that T_{km} may also be $< T_{k_1}$], but only the fact that θ lies *above the double point value*, determines the considered course of the plaitpoint line. (See also Oct. 25, 1906, where I summed up most of the results obtained by me).¹⁾

It is true that KEESOM mentions in a note (loc. cit. p. 786) that I have examined the plaitpoint line for the case $a_1 = 0$, but this statement is not quite complete, for I have not only examined such a plaitpoint line for this particular case $a_1 = 0$, which I cursorily mentioned in a note (June 21, 1905, p. 39), but for all cases. Qualitatively the plaitpoint line $C_0 C_1$ for the case $a_1 = 0$ is not distinguished in anything from that for the case $a_1 > 0$ (provided it remain in the case of type I), hence there was no call for a special investigation of the form of the spinodal line and of the plait for $a_1 = 0$, this having already been done for the general case. Moreover KEESOM himself considers later on the case a_1 *small*, and no longer $a_1 = 0$, which of course does not occur in practice.

Also the equation of the spinodal line (for molecular quantities):

$$RTv^3 = 2(1-x)(v\sqrt{a_1} - b_1\sqrt{a})^2 + 2x(v\sqrt{a_2} - b_2\sqrt{a})^2,$$

¹⁾ Prof. VAN DER WAALS says (These Proc., March 28, 1907, p. 621), "that as yet no one has succeeded in giving a satisfactory explanation of the different forms (of plaits)." I think I have done so to a certain degree in my papers of 1905—1906.

given by KEESOM, had already been drawn up by me (May 25, 1905, p. 652) in the identical form:

$$RTv^3 = 2 \left[x(1-x)(av - \beta\sqrt{a})^2 + a(v-b)^2 \right],$$

where $a = \sqrt{a_2} - \sqrt{a_1}$ and $\beta = b_2 - b_1$.

2. The answer to the question whether the plait extends from C_0 to C_2 with or without double point in the *spinodal* curve, i. e. with or without *minimum* plaitpoint temperature, in other words the answer to the question whether the plait passes from C_0 to C_2 undivided, or whether two plaits extend on the Ψ -surface, one starting from C'_0 , the other from C'_2 , which meet at the minimum temperature — depends on the value of $\theta = \frac{T_2}{T_1}$ (on which also $\frac{T_2}{T_0}$ depends)

for given value of $\pi = \frac{p_2}{p_1}$. The condition for this I derived in Aug. 17, 1905, p. 150, and Jan. 25, 1906, p. 581. In the summer of 1906 I calculated the place of the minimum itself (Cf. Oct. 25, 1906, 234, line 18—16 from the bottom), but seeing that the paper, which at that time had already been completed and sent to the editor of the Arch. Teyler, has not yet been published (it may be even some time before it is), I think it desirable to publish already now the calculation in question.

Like the calculations of KEESOM, VERSCHAFFELT and others, it starts from the supposition that a and b do not depend on v or T , and that these quantities may be represented by

$$a_x = [(1-x)\sqrt{a_1} + x\sqrt{a_2}]^2 \quad ; \quad b_x = (1-x)b_1 + xb_2.$$

So in conformity with BERTHELOT and others we assume that $a_{12} = \sqrt{a_1 a_2}$. Some time ago Prof. VAN DER WAALS raised his voice against this supposition¹⁾, and it seems to me that there is really much to be said in favour of a_{12} being *in general* not $= \sqrt{a_1 a_2}$. But as a first approximation the equation put may be accepted, the more so as also the variability of b with v and T is neglected. That in consequence of the assumption $a_{12} = \sqrt{a_1 a_2}$ the left region, mentioned by VAN DER WAALS, would be compressed to an exceedingly small region, can hardly be adduced as an argument against this supposition; rather the fact that the attractions are *specific* quantities, and that therefore ε_{12} need not be $= \sqrt{\varepsilon_1 \varepsilon_2}$.

For the calculation of the minimum we start from the equation of the spinodal curve, derived by us (loc. cit.):

¹⁾ These Proc., March 28, 1907, p. 630—631.

$$RT = \frac{2}{v^3} \left[x(1-x)(av - \beta \sqrt{a})^2 + a(v-b)^2 \right], \dots \quad (1)$$

or

$$RT = \frac{2a^2}{v} \left[x(1-x) \left(1 - \frac{\beta \sqrt{a}}{v} \right)^2 + \frac{a}{a^2} \left(1 - \frac{b}{v} \right)^2 \right],$$

which with $\frac{b_1}{v} = \omega$, $\frac{\beta}{v} = n\omega$, $\frac{\sqrt{a_1}}{a} = \varphi$ passes into

$$RT = \frac{2a^2}{\beta} n\omega \left[x(1-x) \left(1 - n\omega(\varphi+x) \right)^2 + (\varphi+x)^2 \left(1 - (1+n\omega)\omega \right)^2 \right]. \quad (1a)$$

For $\frac{\sqrt{a}}{a} = \frac{\sqrt{a_1+x\alpha}}{a} = \varphi+x$ and $\frac{b}{v} = \frac{b_1+x\beta}{v} = \omega + n\omega = (1+n\omega)\omega$.

Now the spinodal curve must show a double point, in other words:

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial \omega} = 0,$$

when f represents the second member of (1a). The first equation gives:

$$(1-2x)(1-z)^2 - 2x(1-x)(1-z)n\omega + 2(\varphi+x)(1-y)^2 - 2(\varphi+x)^2(1-y)n\omega = 0,$$

when for the sake of brevity $n\omega(\varphi+x) = z$ and $(1+n\omega)\omega = y$ is put.

Bearing in mind that $n\omega = \frac{z}{\varphi+x}$, we get for the last equation:

$$(1-2x)(1-z)^2 - \frac{2x(1-x)}{\varphi+x} z(1-z) + 2(\varphi+x)(1-y)^2 - 2(\varphi+x)(1-y)z = 0. \quad (a)$$

The second equation yields, when in (1a) the factor ω is brought within []:

$$x(1-x) \left[(1-z)^2 - 2\omega(1-z)n(\varphi+x) \right] + (\varphi+x)^2 \left[(1-y)^2 - 2\omega(1-y)(1+n\omega) \right] = 0,$$

or

$$x(1-x) \left[(1-z)^2 - 2z(1-z) \right] + (\varphi+x)^2 \left[(1-y)^2 - 2y(1-y) \right] = 0,$$

i. e.

$$x(1-x)(1-z)(1-3z) + (\varphi+x)^2(1-y)(1-3y) = 0. \dots \quad (b)$$

From (b) we solve:

$$x(1-x) = -(\varphi+x)^2 \frac{(1-y)(1-3y)}{(1-z)(1-3z)}. \dots \quad (\beta)$$

Also from (a):

$$1-2x = \left[-2(\varphi+x)(1-y)^2 + 2(\varphi+x)(1-y)z + \frac{2x(1-x)}{\varphi+x} z(1-z) \right] : (1-z)^2,$$

or

$$1-2x = \left[-2(\varphi+x)(1-y)^2 + 2(\varphi+x)(1-y)z - \frac{2(\varphi+x)(1-y)(1-3y)z}{1-3z} \right] : (1-z)^2,$$

when for $x(1-x)$ the value from β is substituted. Further reduction yields:

$$1-2x = \left[-2(\varphi+x)(1-y)^2 + 2(\varphi+x)(1-y)z \left\{ 1 - \frac{1-3y}{1-3z} \right\} \right] : (1-z)^2,$$

or

$$1-2x = -\frac{2(\varphi+x)(1-y)}{(1-z)^2} \left[(1-y-3z) \frac{y-z}{1-3z} \right],$$

or

$$1-2x = -\frac{2(\varphi+x)(1-y)}{(1-z)^2} \cdot \frac{(1-y)-3z(1-z)}{1-3z} \quad \dots \quad (\alpha)$$

From (α) and (β) follows, as $(1-2x)^2 = 1-4x(1-x)$:

$$1+4(\varphi+x)^2 \frac{(1-y)(1-3y)}{(1-z)(1-3z)} = \frac{4(\varphi+x)^2(1-y)^2}{(1-z)^4} \cdot \frac{[(1-y)-3z(1-z)]^2}{(1-3z)^2},$$

i. e.

$$1 = \frac{4(\varphi+x)^2(1-y)}{(1-z)^4(1-3z)^2} \left[(1-y) \left\{ (1-y)-3z(1-z) \right\}^2 - (1-3y)(1-z)^3(1-3z) \right].$$

Arrangement according to the powers of z yields for []:

$$(3y^2-y^3) - 6z(y+y^2) + 3z^2(1+5y+2y^2) + z^3(-8-12y) + 6z^4,$$

or

$$y^2(3-y) - 6yz(1+y) + 3z^3(1+5y+2y^2) - 4z^3(2+3y) + 6z^4,$$

which may be reduced to

$$(y-z)^2(6z^2-8z+3-y),$$

so that we find:

$$1 = \frac{4(\varphi+x)^2(1-y)(y-z)^2(6z^2-8z+3-y)}{(1-z)^4(1-3z)^2},$$

from which may be solved:

$$(\varphi+x)_m^2 = \frac{(1-z)^4(1-3z)^2}{4(1-y)(y-z)^2(6z^2-8z+3-y)}, \quad \dots \quad (2)$$

through which $\varphi+x$ is expressed in the two parameters y and z .

In consequence of this (β) passes into

$$x_m(1-x_m) = -\frac{(1-z)^3(1-3z)(1-3y)}{4(y-z)^2(6z^2-8z+3-y)}, \quad \dots \quad (3)$$

from which x_m may be calculated with given values of y and z

Then φ_m is also known through (2), i.e. expressed in y and z .

Further we now find for RT'_m according to (1a):

$$RT_m = \frac{2\alpha^2}{\beta} n\omega \left[\frac{(1-z)^2(1-3z)(1-3y)}{4(y-z)^2(6z^2-8z+3-y)} + \frac{(1-z)^4(1-3z)^2(1-y)}{4(y-z)^2(6z^2-8z+3-y)} \right],$$

as $n\omega(\varphi+x) = z$ and $(1+nx)\omega = y$. Reduction yields:

$$RT_m = \frac{2\alpha^2}{\beta} n\omega \frac{(1-z)^4(1-3z)}{4(y-z)^2(6z^2-8z+3-y)} \left[1-3z)(1-y) - (1-z)(1-3y) \right].$$

The expression between [] is $= 2(y-z)$, hence, $\frac{n}{\beta}$ being $\frac{1}{b_1}$, we get:

$$RT_m = \frac{\alpha^2\omega}{b_1} \frac{(1-z)^4(1-3z)}{(y-z)(6z^2-8z+3-y)}.$$

Let us express this in T_1 , the critical temperature of one component. ($T_1 < T_2$). We find:

$$T_1 = \frac{8}{27} \frac{a_1}{b_1} = \frac{8}{27} \frac{\alpha^2\varphi^2}{b_1},$$

as $\frac{\sqrt{a_1}}{\alpha} = \varphi$ was put. At last we get:

$$\frac{T_m}{T_1} = \frac{27}{8} \frac{\omega}{\varphi^2} \frac{(1-z)^4(1-3z)}{(y-z)(6z^2-8z+3-y)} \dots \dots \dots (4)$$

Now

$$z = n\omega(\varphi+x) \quad ; \quad y = (1+nx)\omega,$$

from which we solve:

$$n\omega = \frac{z}{\varphi+x} \quad ; \quad y = \omega + \frac{xz}{\varphi+x},$$

hence:

$$\omega = y - \frac{xz}{\varphi+x} \quad ; \quad \frac{1}{n} = \frac{y}{z}(\varphi+x) - x \dots \dots \dots (5)$$

Now ω and n have been expressed in y and z , as $(\varphi+x)_m$ and x_m had already been expressed in y and z by (2) and (3).

As further:

$$1+n = 1 + \frac{\beta}{b_1} = \frac{b_2}{b_1} = \frac{\theta}{\pi},$$

and

$$1 + \frac{1}{\varphi} = 1 + \frac{\alpha}{\sqrt{a_1}} = \frac{\sqrt{a_2}}{\sqrt{a_1}} = \frac{\theta}{\sqrt{\pi}},$$

when $\theta = T_2 : T_1 = \frac{a_2}{b_2} : \frac{a_1}{b_1}$ and $\pi = p_2 : p_1 = \frac{a_2}{b_2^2} : \frac{a_1}{b_1^2}$, we have also:

$$\theta = \frac{\left(1 + \frac{1}{\varphi}\right)^2}{1+n} \quad ; \quad \pi = \frac{\left(1 + \frac{1}{\varphi}\right)^2}{(1+n)^2}, \dots \dots \dots (6)$$

so that also θ and π can be expressed in y and z .

Reversely we may now also think the corresponding values of ω , x and T_m to be solved for any given pair of values of π and θ , though explicitly this is impossible, so that we shall have to be satisfied with the set of equations from (2) to (6).

The further discussion of these equations, particularly with regard to the branch C_0A of the plaitpoint line, in connection with the longitudinal plait, will be found in the paper, which will shortly appear in the Arch. Teyler. There the course of the pressure is also examined, which we no further discuss here. It is only desirable to calculate the data for the "third" critical temperature C_0 , viz. x_0 and T_0 — not because these data are indispensable for the following considerations, but because KEESOM includes them in his considerations, and it is profitable in any case to know something concerning the relation $\frac{T_0}{T_1}$ or $\frac{T_0}{T_2}$.

As $v = b$ for the point C_0 , so $y = \frac{b}{v} = 1$, and the equation of the c, x -projection of the plaitpoint line (Aug. 17, 1905, p. 146; Teyler I and II), viz.

$$(1-z)^3(1-2x-3x(1-x)n\omega) + 3(\varphi+x)(1-y)^2(1-z)(1-2z) + \frac{(\varphi+x)^3(1-y)^2(1-3y)}{x(1-x)} = 0,$$

is reduced to

$$1 - 2x_0 - 3x_0(1-x_0)n\omega_0 = 0,$$

or as $y = (1 + nx)\omega$, and hence $\omega_0 = \frac{1}{1 + nx_0}$, to

$$(1 - 2x_0)(1 + nx_0) - 3x_0(1 - x_0)n = 0,$$

from which follows:

$$x_0 = \frac{(n+1) - \sqrt{n^2 + n + 1}}{n} \dots \dots \dots (7)$$

From this is seen that the situation of C_0 depends only on the value of n or $1 + n = \frac{b_2}{b_1}$.

The corresponding value of T_0 is found from (1a). For $y = 1$ we find:

$$RT_0 = \frac{2a^2\omega_0}{b_1} x_0(1-x_0)(1-z_0)^2,$$

in which $\omega_0 = \frac{1}{1 + nx_0}$ and $z_0 = nx_0(\varphi + x_0)$.

As $T_1 = \frac{8}{27} \frac{a^2\varphi^2}{b_1}$ (see above), we have:

$$\frac{T_0}{T_1} = \frac{27 \omega_0}{4 \varphi^2} x_0 (1 - x_0) (1 - z_0)^2 \dots \dots \dots (8)$$

Hence we can immediately calculate x_0 and T_0 from (7) and (8) for any given set of values of θ and π , or φ and n .

3. For our case (a_1 small) it is now important to know, when a minimum occurs in the plaitpoint line $C_0 C_2$, when not. For this purpose we shall derive the condition that the minimum is to appear exactly in the point C_2 . Evidently this condition will then indicate the limit between the two cases that there occurs a minimum in the neighbourhood of C_2 or not — in other words whether the line of the plaitpoint temperatures in C_2 descends first and rises later on to T_0 in C_0 ; or whether there is an immediate rise from T_2 to T_0 . (We call to mind that with us T_2 is always the *highest* of the two critical temperatures T_1 and T_2).

Now $y = \frac{b}{v} = \frac{1}{3}$ in the point C_2 , while $x = 1$. Hence equation (2) passes into

$$(\varphi + 1)^2 = \frac{(1-z)^4 (1-3z)^2}{4 \times 2^{2/3} (1/3 - z)^2 (6z^2 - 8z + 2^{2/3})} = \frac{81 (1-z)^4}{16 (2-3z)^2},$$

from which follows:

$$\varphi + 1 = \frac{9 (1-z)^2}{4 (2-3z)},$$

hence

$$\frac{\sqrt{a_2}}{\sqrt{a_1}} = 1 + \frac{1}{\varphi} = 9 \frac{(1-z)^2}{(1-3z)^2} \dots \dots \dots (a)$$

From

$$z = n\omega (\varphi + x)$$

follows further, as $\omega = \frac{b_1}{v} = \frac{b_2}{v} \times \frac{b_1}{b_2} = \frac{1}{3} \frac{1}{1+n}$ and $x = 1$:

$$z = \frac{1}{3} \frac{n}{1+n} (\varphi + 1) = \frac{3}{4} \frac{n}{1+n} \frac{(1-z)^2}{2-3z}.$$

This yields:

$$\frac{1+n}{n} = \frac{3}{4} \frac{(1-z)^2}{z(2-3z)},$$

or

$$\frac{b_2}{b_1} = 1 + n = \frac{3 (1-z)^2}{(1-3z)(3-5z)} \dots \dots \dots (b)$$

When we put:

$$\sqrt{\frac{a_2}{a_1}} = \kappa \quad ; \quad \frac{b_2}{b_1} = \lambda,$$

the simple relation

$$\lambda \overset{=}{>} \frac{\kappa}{3 + 2\sqrt{\kappa}}, \dots \dots \dots (c)$$

follows from (a) and (b) after some reduction, in which the sign > refers to the existence of a minimum in the neighbourhood of C₂.

The condition (c) found by us is quite identical with that, which we derived before from the formula for $\frac{1}{T_1} \left(\frac{dT_x}{dx} \right)_0$ found by us (Aug. 17, 1905, p. 150 and Jan. 25, 1906, p. 580). This condition was:

$$\theta < \frac{4\pi\sqrt{\pi}}{(3\sqrt{\pi}-1)^2}.$$

With this difference, however, that we then considered $\frac{1}{T_1} \left(\frac{dT_x}{dx} \right)_0$ at C₁, whereas we have now examined the branch of the plaitpoint line which starts from C₂, so that we have to calculate $\frac{1}{T_2} \left(\frac{dT_x}{dx} \right)_1$, and to derive the condition of the minimum from this. But it is immediately seen that it is obtained by substituting $\frac{1}{\theta}$ for θ and $\frac{1}{\pi}$ for π in the above condition.

So we find:

$$\theta < \frac{4}{\left(\frac{3}{\sqrt{\pi}} - 1 \right)^2} \frac{\pi\sqrt{\pi}}$$

or

$$\theta > \frac{\sqrt{\pi}(3-\sqrt{\pi})^2}{4} \dots \dots \dots (c')$$

And it appears immediately that (c) is identical with (c'), when we substitute $\frac{\theta^2}{\pi}$ for κ^2 and $\frac{\theta}{\pi}$ for λ in (c).

This furnishes a good test, both of the accuracy of the above derived formula (c), and of the condition (c'), derived by us before.

Let us now examine what values of λ and κ correspond according to the condition (c), so that the minimum still appears exactly in C₂. The corresponding values of z required for the calculation of T'_0 , may be found from (a), giving:

$$z = \frac{1}{3} \frac{\sqrt{\kappa}-3}{\sqrt{\kappa}-1}.$$

The subjoined table combines the calculated values. We call attention to the fact that the minimum in the neighbourhood of C_2 can only belong to the branch C_2C_0 for type I ($\theta >$ the double point value), and never to the branch C_2C_1 for type II or III ($\theta <$ the double point value). For $T_2 > T_1$ being put, the minimum on C_2C_1 cannot possibly lie at C_2 , but it can lie in the neighbourhood of C_1 .

	$\kappa = \sqrt{\frac{a_2}{a_1}}$	$\lambda = \frac{b_2}{b_1}$	$\theta = \frac{\kappa^2}{\lambda}$	$\pi = \frac{\kappa^2}{\lambda^2}$	x_1	$\frac{T_0}{T_1}$	$\frac{T_0}{T_2} = \frac{T_0}{T_1} \times \frac{1}{y}$
$\kappa = 1/3$	∞	∞	∞	∞	0	∞	1 (Case $a_1 = 0$)
$1/6$	25	$11^{2/13}$	325	169	0,279	364	1,12
$1/9$	16	$1^{5/11}$	176	121	0,365	209	1,19
0	9	1	81	81	0,500	108	$1^{1/3}$
$-1/3$	4	$4/7$	28	49	0,694	—	—
-1	$2^{1/4}$	$3/8$	$13^{1/2}$	36	0,800	$30^{3/8}$	$2^{3/4}$
$\mp \infty$	1	$1/5$	5	25	0,896	—	—
$5/3$	$1/4$	$1/16$	1	16	0,968	9,30	9,30

That is to say: for a gas *without cohesion* as one of the components of the mixture ($a_1 = 0, \kappa = \infty$) λ would have to be larger than the limiting value ∞ , for a minimum to appear in the line C_2C_0 in the neighbourhood of C_2 . (Then $T_0/T_2 < 1$ would be at the same time). For finite values of λ this cannot be satisfied, and the line C_0C_2 proceeds with $T_0 > T_2$ without a minimum.

For a gas with *feeble cohesion*, where e.g. $\kappa = \sqrt{\frac{a_2}{a_1}} = 16$, $\lambda = \frac{b_2}{b_1}$ must be $> 1^{5/11}$, for a minimum to appear. T_0/T_2 is then $< 1,19$.

For He—H₂ $\frac{a_2}{a_1}$ is about 175, hence $\kappa = 13,2$ according to an estimation of KEESOM (These Proc., March 28, 1907, p. 661; Ibid. April 25, 1907 p. 794). To this corresponds according to formula (c) the limiting value $\lambda = 1,29$. Now KEESOM estimated (loc. cit.) this value at about 2 for He—H₂, and 2 being $> 1,29$, there is a minimum in the plaitpoint line in the case of He—H₂. This minimum can be fully calculated by the aid of the formulae (2) to (8). The value of T_0/T_2 is then smaller than about 1,25.

For $z = 2^{1/4}$, λ must be $> 3/8$, and then $T_0/T_2 < 2^{1/4}$. Etc., etc.

The larger therefore the value of a_1 — the smaller in other words the value of z — the smaller also the limiting value of λ , above which a minimum is to be expected, the sooner this will therefore appear, and at comparatively large corresponding values of T_0/T_2 .

But as we already observed in § 1, all this refers only to the existence or non-existence of a *minimum* in the line $C_0 C_2$. That this line has the shape in question, depends on quite different circumstances — viz., as I already showed in June 21, 1905, p. 33—48 for $b_1 = b_2$, and further extended to the general case in later papers (particularly TEYLER I), it depends only on this, whether for the given value of τ the value of θ is found above that at which the *plaitpoint line* has a *double point* or not. And the criterion for this is fig. 1 of Oct. 25, 1906 (see also TEYLER II). If we are *above* the limiting line $DBPAC'$, we are in the region of type I, where one of the branches of the plaitpoint line runs from C_0 to C_2 (the other from A to C_1 — see e.g. fig. 1 of June 21, 1905 and fig. 1 of Jan. 25, 1906). And *below* the limiting line we are in the region of type II (or III), where the branches of the plaitpoint line are $C_1 C_2$ and AC_0 . But for all this consult the papers cited.

April 1907.

APPENDIX. After I had written the above considerations, the Continuation of the last cited paper by K. ONNES and KEESOM appeared in These Proceedings, April 25, 1907, p. 795—798. There a condition is derived for the appearance of a minimum plaitpoint temperature, which is identical with that which I published Jan. 25, 1906 (formula (3), p. 581), at which result also VERSCHAFFELT (These Proc., April 24, 1906, p. 751) arrived a month later.

For on p. 796 K. O. and KEESOM give the condition — see formula 2 :

$$\sqrt[4]{\frac{a_2}{a_1}} = \frac{1}{3} \left[-1 + \sqrt{1 + 3 b_2/b_1} \right].$$

Now in my notation $a_2/a_1 = 1/2$ (see above; I denote viz. the component with the smallest value of a by the index 1; KEESOM does the reverse). Further $b_2/b_1 = 1/2$, so that the above formula passes into

$$\sqrt[4]{\frac{1}{z}} = \frac{1}{3} \left[-1 + \sqrt{1 + 3/2} \right],$$

from which follows:

$$\lambda = \frac{x}{3+2\sqrt{x}},$$

being my above formula (c). And concerning this we have just proved that it is identical with my relation and that of VERSCHAFFELT (Jan. and April 1906), viz.

$$\theta = \frac{4\pi\sqrt{\pi}}{(3\sqrt{\pi}-1)^2},$$

which is of general application, irrespective whether the branch of the plaitpoint line starts from C_2 towards C_1 or towards C_0 . As we already observed, this expression holds on the side of the component I, when $\theta = T_2/T_1$ and $\pi = \nu_2/\nu_1$, so for the branch starting from what is point C_1 with me. For C_2 (KEESOM's K_1) θ and π must simply be replaced by $1/\theta$ and $1/\pi$ (see above in §3).

So in my opinion the footnote on p. 795 in the paper by K. O. and K. of April 25, 1907 is not accurate, for according to the above the conclusion of VERSCHAFFELT (and mine) does not require any qualification, because the formula¹⁾ given by us holds for any course of the plaitpoint line, irrespective of the fact whether the considered branch runs from C_2 to C_1 or to C_0 . For the transition of the two types takes place gradually through the double point of the plaitpoint line, and hence the two types are analytically included in the same formula, so that only one expression exists for $\frac{dT_x}{dx}$,

which holds equally for the two cases. And if any doubt should remain, this must be removed, when from the above the identity is seen between the relation derived last by K. O. and K., and the general one of VERSCHAFFELT and me.

It will be superfluous to observe that the so-called (homogeneous) "double *plaitpoint*" in the branch of the plaitpoint line C_0C_2 , of which K. O. and K. speak, is identical with the fully discussed minimum and with the double point in the spinodal line, and should not be confounded with the "double *point*", found by me in the (whole) plaitpoint line, where the two branches of this line intersect, and which separates the two types I and II (or III), the data for which double point can be calculated for the general case only with great difficulty. (see Teyler I).

¹⁾ In the footnote on p. 795 it says maximum temperature; this must of course be minimum temperature.

Astronomy. — “*On periodic orbits of the type Hestia.* By Dr. W. DE SITTER. (Communicated by Prof. J. C. KAPTEYN).

The problem, of which some particular solutions will be treated here, is the following. Two material points S and J , having the masses 1 and μ , move with uniform angular velocity $n' = 1$ in circles in one and the same plane round their centre of gravity. The constant distance SJ is adopted as unit of length. Another material point P , with an infinitely small mass, moves in the same plane under the influence of the Newtonian attractions of S and J . This is the problem which has (for $\mu = 0.1$) been so exhaustively treated by DARWIN in Vol. XXI of the *Acta Mathematica*. The particular solutions which are treated below are those in which the orbit of P is periodic and its limit for $\text{Lim. } \mu = 0$ is an ellipse with a small excentricity, described round S as a focus with a mean motion not differing much from 3. If this limiting orbit (i.e. the undisturbed orbit) is a circle, then the solution is, in POINCARÉ's phraseology, of the first sort (*sorte*), and its period is $T = \frac{2\pi}{n-1}$. If the excentricity of the

undisturbed orbit differs from zero, the solution is of the second sort, and the limiting value of the period for $\text{Lim. } \mu = 0$ is $\text{Lim. } T = 2\pi$. These solutions of the second sort are at the same time of the second genus (*genre*) relatively to those of the first sort.

The solutions of the first sort are the orbits of DARWIN's “Planet A”. This family of orbits undergoes within the range here considered a transition from stability to instability, which has been discussed by POINCARÉ in an investigation contained in the articles 383 and 384 of his “*Méthodes Nouvelles*” (Vol. III, p. 355—361). The results there reached will be derived here by a different (and, as it seems to me, simpler) reasoning.

DARWIN's work also presents an example of an orbit of the second sort, viz. the orbit figured by him on page 281 and designated as $x_0 = - .337$. Although POINCARÉ proves the existence of solutions of this kind, he seems to have overlooked the fact that DARWIN had actually computed one of them.

These solutions and their stability I wish to consider from the point of view of the general theory developed by POINCARÉ in the first and third volumes of the “*Méthodes Nouvelles*”. The following is a summary of those general theorems, proved by POINCARÉ, which will be used here. They are true for every problem capable of being reduced to two degrees of freedom, containing one variable parameter, and admitting for each value of this parameter a finite

number of periodic solutions. It need hardly be mentioned that their valency is restricted to a certain domain of the several variable quantities of the problem, of which it will however not be necessary to transgress the limits.

A periodic solution is completely determined by the values of the parameter and of one constant of integration or "element". The periodic solutions occur in families, the members of which are classified according to increasing or decreasing values of the parameter. These families may be graphically represented by curves $\Phi(\alpha, \beta) = 0$, where α is the parameter of the problem and β the determining element.

The stability or instability is determined by a certain quantity α , which is by POINCARÉ called the characteristic exponent. If the period is T , then values of α differing by a multiple of $\frac{2\pi i}{T}$, must be considered as identical. The following three cases are possible:

- αT purely imaginary the solution is *stable*
 αT real the solution is *evenly unstable*
 αT complex, with imaginary part $=\pi i$: the solution is *unevenly unstable*¹⁾.

A solution having the period T can as well be conceived to have the period $T' = 2T$. If it is unevenly unstable with reference to the period T , it is evenly unstable with reference to the period T' .

Within each family the exponent α and the period T vary continuously with the parameter α . The product αT and the differential coefficient $\frac{d\Phi}{d\beta}$ become equal to zero for the same values of α . The curve $\Phi = 0$ then either has a multiple point, or is tangent to a line $\alpha = \text{const}$. The family splits into two branches, or, which comes to the same thing, two families have one member in common. If (α_0, β_0) is the point representing this common member, then we have the following rules.

The number of branches of the curve $\Phi = 0$ (i. e. the number of families of periodic solutions) for $\alpha > \alpha_0$ differs by an even number from the number of branches for $\alpha < \alpha_0$.

The branches which part from the point (α_0, β_0) towards the direction of increasing α are alternately stable and evenly unstable²⁾. The

¹⁾ The names *even* and *uneven instability* have been introduced by DARWIN. POINCARÉ distinguishes them as instability of the first and second "classe". The relation of DARWIN'S quantity c to the exponent α is given by the formula $\alpha T = i\pi c$.

²⁾ To avoid circumlocution I speak of "stable and unstable branches", meaning branches whose points represent stable and unstable solutions respectively.

same thing is true of the branches on which z decreases. The two branches between which lies the part of the line $z = z_0$ on which $\beta < \beta_0$, are either both stable, or both unstable, and similarly the two branches enclosing the other half of the line $z = z_0$. If T is the period of one of the branches and T' of another, and if T_0 and T'_0 are the values of these periods in the point (z_0, β_0) , then T_0 and T'_0 are mutually commensurable. If T''_0 is their least common multiple, then $\alpha_0 T''_0 = 0$. If e. g. $T'_0 = 2 T_0$, then the instability is even with reference to the period T' .

As an illustration of these general rules I may be allowed to mention a few of the simplest cases.

1. The curve $\phi = 0$ is tangent to the line $z = z_0$. There are two families, springing from a common member, which come into existence at this value of the parameter. One of them is stable, and the other is evenly unstable. An example of this is presented by DARWIN'S families B and C of satellites.

2. The curve has a double point. Two families are "crossing" each other, at the same time exchanging their stability.

3. The curve consists of one branch tangent to the line $z = z_0$ and another branch intersecting the first in the point of contact. The two families which come into existence at this value of the parameter are both stable or both unstable. The third family, which exists both for $z > z_0$ and for $z < z_0$, becomes stable if it was unstable and unstable if it was stable.

The cases 2 and 3 are the only ones occurring in the present investigation.

The proof of the above supposes that the problem can be reduced to the second order, so that there are only two characteristic exponents ($+a$ and $-a$). The choice of the parameter is determined by the way in which this reduction is effected, or is conceived to be effected. DARWIN uses the integral of JACOBI for this reduction. Consequently his parameter is the constant C to which this integral is equal. This constant C is a function of the two elements a and e . The first of these can be replaced by the mean motion n , or by the period $T = \frac{2\pi}{n-1}$. In consequence of the reduction of the problem by means of the integral of JACOBI one of these elements, say T , is eliminated. This therefore appears no longer as an arbitrary constant of integration, but is entirely determined by C and e . On the other hand C is entirely determined by T and e . Now DARWIN'S calculations show that T continually increases if C decreases. It is therefore irrelevant for our purpose whether we consider C or T as the

parameter of the problem. The parameter which I will use here is $T'' = 2 T'$. This change from C to T'' can also be conceived as no more than a simplification of language. Instead of saying: "the solution corresponding to the value of C for which the period of the solution of the first sort is $\frac{1}{2} T''$ ", I say: "the solution corresponding to the value T'' ".

In DARWIN'S work μ has the constant value 0.1. If now we choose a convenient element ξ , we can conceive the curves $\phi(T'', \xi)$ to be drawn. Next imagine the same thing to be done for other values of μ , and take μ , T'' and ξ as rectangular coordinates. The curves $\phi(T'', \xi)$ belonging to the various values of μ then produce a surface, every point of which represents a periodic solution.

If, on the other hand, we take for T'' a fixed value T''_1 , considering μ as the variable parameter, then we have another problem, also admitting families of periodic solutions, which can be represented by curves $\psi(\mu, \xi) = 0$. If T''_1 varies these curves describe again the same surface. The form of this surface will now be investigated. Its section by the plane $\mu = 0.1$ then gives all periodic solutions of DARWIN'S problem.

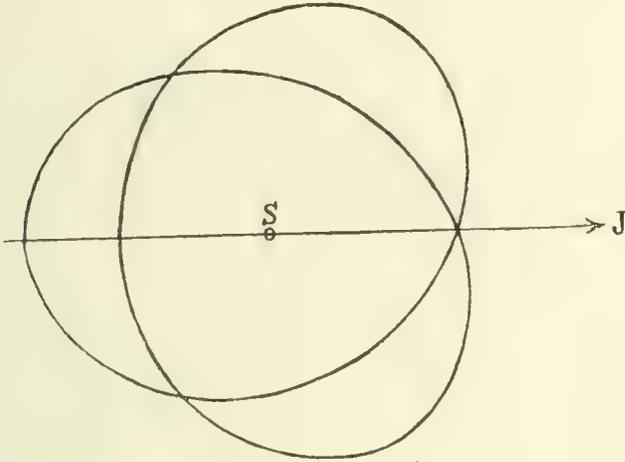
The element which I will use is $\xi = e_0 \cos \tilde{\omega}_0$, where e_0 is the excentricity and $\tilde{\omega}_0$ the longitude of the perihelion of the undisturbed orbit, which is the limit of the orbit of P for $\lim. \mu = 0$. The longitude $\tilde{\omega}_0$ is counted from a *fixed* axis which at the beginning of the period co-incides with SJ . The orbit of P is not periodic unless $\tilde{\omega}_0$ has one of the two values 0 or π . Moreover at the beginning of the period P must be on the line SJ , i.e. there must be either opposition or conjunction.

Solutions of the first sort are characterised by $\xi = 0$. These solutions can have any period, therefore the whole plane $\xi = 0$ is a part of our surface. The line $\xi = 0$, $\mu = 0.1$ represents DARWIN'S family A . For a value of $T'' = 2T'$, which lies between 330° and 354° , i.e. between 1.83π and 1.97π , this family loses its stability and becomes unevenly unstable. So there must be another family which at this point has a member in common with the family A . This new family must have the period T'' , and is therefore of the second sort. If for the sake of argument we assume the change of stability to take place at the value $T'' = 1.9\pi$, then we know of the branch of the curve $\phi = 0$, which represents this family, that for $T'' < 1.9\pi$ it is evenly unstable and for $T'' > 1.9\pi$ it is stable.

Now there are only four possible periodic solutions of the second sort, distinguished by the following positions of P at the beginning of the period :

B	:	P	in opposition in aphelion	($\tilde{\omega}_0 = 0, \tilde{\xi} = +e_0$)
B'	:	„	„ „ „ perihelion	($\tilde{\omega}_0 = \pi, \tilde{\xi} = -e_0$)
C	:	„	„ conjunction „ perihelion	($\tilde{\omega}_0 = 0, \tilde{\xi} = +e_0$)
C'	:	„	„ „ „ aphelion	($\tilde{\omega}_0 = \pi, \tilde{\xi} = -e_0$)

With reference to rotating axes, of which the axis of x co-incides with SJ , the orbits B and B' are identical, and similarly C



Orbit of family B or B'

Fig. 1.

and C' . The orbits B and B' are of the form represented in fig. 1. The orbits C and C' are of the same form, rotated through 180° , i.e. with the double point away from J .

The families B and B' are stable C and C' are unstable. This is easily found by considering the equation which determines the exponent α . This equation is (see POINCARÉ, Acta Math. XIII, p. 134):

$$n_1^2 \alpha^2 = \frac{d^2 \Psi}{d\tilde{\omega}_2^2} (n_1^2 C_{22} - 2 n_1 n_2 C_{12} + n_2^2 C_{11})$$

Now using the variables employed by POINCARÉ l. c. pages 128 and 171, we find easily,

$$n_1 = -1 \quad n_2 = 3 \quad C_{11} = C_{12} = 0 \quad C_{22} = -3\sigma_2^{-1}.$$

If further in Ψ (i. e. the average value of the perturbing function over one period) we neglect the terms which contain a higher power of e than the second, we find

$$\Psi = \mu K e^2 \cos \varepsilon \quad \varepsilon = \tilde{\omega}_2 + 3 \tilde{\omega}_1$$

where ε is the mean longitude of P at the beginning of the period, and K is a positive constant.

We find thus

$$a^2 = 3 \mu K e^2 x_2^{-4} \cos \varepsilon.$$

Thus, for positive values of μ , a^2 is negative, and therefore the orbit is stable, when there is opposition at the beginning of the period.

For positive values of μ therefore BB' is stable and CC' is unstable, for negative values ¹⁾ of μ BB' is unstable and CC' is stable. It is evident that, for $\xi = 0$, B and B' co-incide, and similarly C and C' . The branch of $\varphi = 0$ which intersects $\xi = 0$ in the point $T' = 1.9 \pi$ therefore represents either the family BB' or the family CC' . In the first case it is stable, and therefore it must on both sides of the point of intersection bend round towards the right. In the other case it is unstable and encloses the stable part of the line $\xi = 0$.

Now DARWIN has, for $C = 39.0$, i. e. $T' = 1.97 \pi$, actually computed and drawn an orbit, which shows the form of fig. 1, viz.: the orbit $x_0 = -.337$ which has already been quoted. This orbit thus belongs to the family B , but it also belongs to B' . It belongs to B if P is in aphelion at the beginning of the period and in perihelion in the middle of the period (being at both times in opposition to J), and to B' in the opposite case. The branch of the curve $\varphi = 0$ which passes through the point $T' = 1.9 \pi$ therefore represents the family BB' , and not CC' . Consequently it is stable, and that part of the section of our surface by the plane $\mu = 0.1$, which lies to the left of the line $T' = 2 \pi$, is thereby completely determined. This section is represented in Fig. 2. Stable families are there, and in the following figures, represented by heavy full lines, unevenly unstable families by broken lines, and evenly unstable ones by dotted lines.

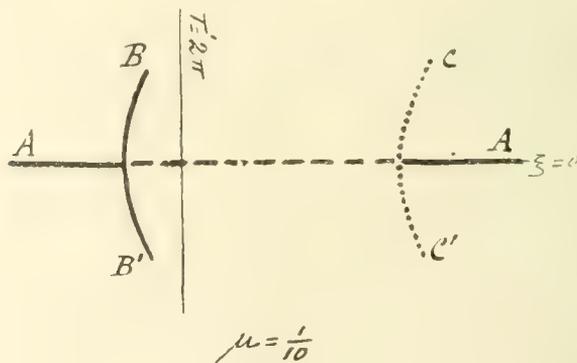


Fig. 2.

We next consider the section of our surface by the plane $\mu = 0$.

¹⁾ The meaning of a negative value of μ is that the force emanating from J is repulsive, the force from S remaining attractive.

We know then that there are stable periodic solutions of the first sort with an arbitrary period, and of the second sort with the period $T' = 2\pi$ and an arbitrary excentricity. The section therefore consists of the line $\xi = 0$ and the part of the line $T' = 2\pi$ between the points $\xi = +1$ and $\xi = -1$. I wish, however, to confine myself to *small* values of ξ . This section is represented in Fig. 3.

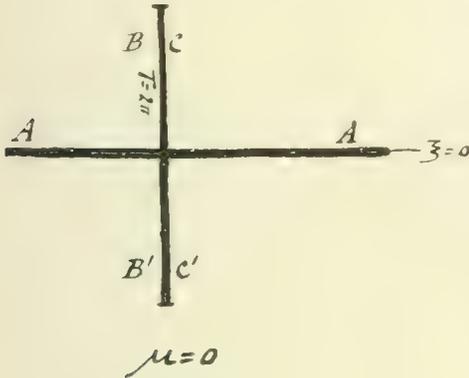


Fig. 3.

BB' . This family being stable, that branch must on both sides of the point of intersection bend upwards, as is represented in fig. 4a.

Consider now the section of our surface by a plane parallel to, and at a very small distance from, $\xi = 0$. The orbits represented by the curves $\chi(\mu, T')$ in this plane are all of the second sort. We can imagine these orbits to arise by a variation of μ from the undisturbed periodic orbit of the second sort. They then appear as

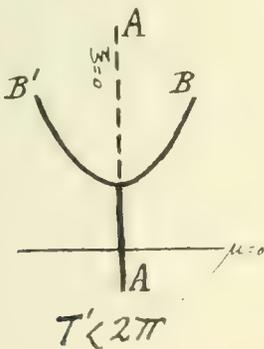


Fig. 4a.

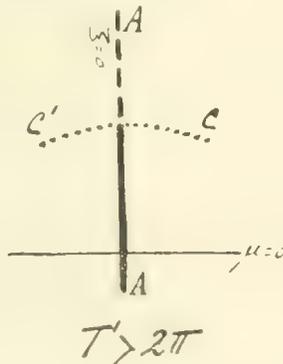


Fig. 4b.

solutions of a problem, in which the parameter is μ , ξ being kept constant, and thus T' (or C) now is our element. These solutions have been studied by SCHWARZSCHILD (Astr. Nachr. 3506). For $\mu = 0$ the period is 2π . For small values of μ there are (for each value of ξ) two solutions, viz. B and C when ξ is positive, B' and C'

when it is negative. The curve $\chi = 0$ thus consists of two branches, both passing through the point $\mu = 0, T'' = 2\pi$, and there exchanging their stability. Since now it has already been shown that the stable branch B is, for positive values of μ , situated on the left side, the unstable branch C must be on the right side. The curves are represented in fig. 5.

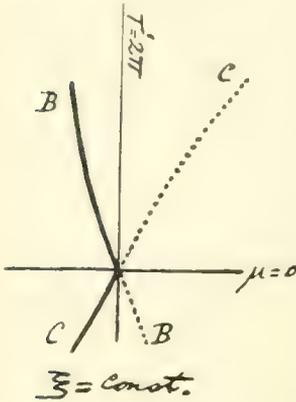


Fig. 5.

Our surface has thus been shown to consist of the plane $\xi = 0$ and of two sheets, which pass through the line $\mu = 0, T'' = 2\pi$, and then deviate to the left and to the right of the plane $T'' = 2\pi$. The points of the left-hand sheet represent the stable family BB' , those of the right-hand sheet the unstable family CC' . This latter sheet therefore intersects the plane $\mu = 0.1$ in a curve which on both sides of its point of intersection with the line $\xi = 0$ bends off towards the right. In this same point of intersection the family A regains its stability, the stable part of the line $\xi = 0$, which represents this family, being enclosed between the two unstable branches of the section just considered. This state of things is rendered in the right-hand part of fig. 2. Also the form of the section of the surface by a plane $T'' = T''_2 > 2\pi$, will need no further explanation. It is represented in fig. 4b. Whether this right hand sheet does reach up to the plane $\mu = 0.1$, so as to produce a real section, cannot be decided by this reasoning. If there is a point of intersection with the line $\mu = 0.1, \xi = 0$, this must correspond to a value of T'' exceeding $414.3 = 2.23\pi$, since for this value the family A is still unevenly unstable, as is shown by DARWIN'S work. That the left-hand sheet does actually intersect the plane $\mu = 0.1$ is shown by the existence of DARWIN'S orbit $x_0 = -.337$, belonging to the family BB' (and also by the change of stability of the family A).

Thus all results have been derived which have been found by POINCARÉ in the "Méthodes Nouvelles", already quoted. Naturally POINCARÉ also must leave the question, whether his results still hold for $\mu = 0.1$, unanswered.

It is not uninteresting to consider the solutions B and C from the point of view of the theory of perturbations. This can, of course, not teach us anything about their stability, but it will give information about the form of the curves $\chi(\mu, T'') = 0$ and $\psi(\mu, \xi) = 0$ for small values of μ and ξ . The period of the undisturbed solution is 2π .

By the perturbing influence of J this is changed to $T' = 2\pi + \tau$. The conditions that the perturbed orbit shall be periodic are:

$$\int_0^{T'} \frac{d\tilde{\omega}}{dt} dt = \tau \qquad \int_0^{T'} \frac{d\lambda}{dt} dt = 6\pi + \tau,$$

where λ is the mean longitude of $-P$. For the computation of the integrals we must use the mean motion affected by perturbations, i. e. $n = 3 + \sigma$. The left-hand members of these equations of condition are therefore functions of τ and σ , and these two unknowns can be determined from them.

If in these equations of condition we neglect the square and higher powers of e , they become

$$\left. \begin{aligned} \tau &= \frac{na}{4} (2\pi + \tau) \mu [B^{(1)} \pm \{21 A^{(3)} + 10 A_1^{(3)} + 2 A_2^{(3)}\}] \\ 6\pi + \tau &= (3 + \sigma) (2\pi + \tau) - na (2\pi + \tau) \mu A_1^{(0)} \end{aligned} \right\} . \quad (1)$$

The upper sign in these equations must be used for the family CC' , the lower sign for BB' . The sum within the $\{ \}$ being larger than $B^{(1)}$, we find that for the family BB' τ is negative, while for CC' it is positive, as has also been found above. Further the first equation shows that the numerical value of the differential coefficient $\frac{d\tau}{d\mu}$ for the first family (BB') decreases if μ increases, while for the other family it increases. Thus the left-hand branch of $\chi(\mu, T') = 0$ has its concave side towards the line $T' = 2\pi$, and the right-hand branch its convex side, as is shown in fig. 5.

In the numerical computation we must not forget that the formulas (1) can only be considered as approximatively true. The solution of the equations is easily effected by means of the tables of RUNKLE, the argument for the determination of the different functions $A_p^{(i)}$ being computed by

$$n = 3 + \sigma \qquad n^3 a^3 = f = \frac{10}{11}.$$

I find in this manner for the two families:

$$B: \quad \frac{\tau}{2\pi} = -0.085 \qquad T' = 1.83 \pi$$

$$C: \quad \frac{\tau}{2\pi} = +0.29 \qquad T' = 2.58 \pi$$

These are the periods of those orbits of the two families, which have $\xi = 0$, and which therefore co-incide with a member of the

family A , whose period is $T = \frac{1}{2} T''$. DARWIN'S computations show that the value of T'' for which the families A and B co-incide must lie between 1.836π and 1.97π . The point of co-incidence of A and C is outside the region explored by DARWIN, the corresponding value of T'' must therefore be larger than 2.23π .

If in the equations (1) we take account of the square of e , the right-hand member of the first must be multiplied by $\sqrt{1-e^2}$. In the second $A_1^{(0)}$ must be replaced by

$$A_1^{(0)} + \frac{1}{4} e^2 \left(B_1^{(1)} \pm \{31 A_1^{(3)} + 24 A_2^{(3)} + 6 A_3^{(3)}\} \right)$$

and $\frac{1}{2} e^2 \tau$ must be added to the second member. Now if we take $T' = \text{const.}$ then τ is constant and also σ can be taken to be constant. The second equation (1) then is of the form

$$\text{const.} = \mu (P + Qe^2) (2)$$

Now we have $\xi^2 = e^2$, therefore (2) is approximately the equation $\psi(\mu, \xi) = 0$. For the family BB' P and Q are of opposite signs, for CC' they have the same sign. Thus the form of these curves as drawn in the figures 4a and 4b is confirmed.¹⁾

Physics. — *“Contribution to the theory of binary mixtures. IV.”*

By Prof. J. D. VAN DER WAALS.

Continued, see p. 849 vol. IX.

THE BINODAL CURVE.

We might think that for the determination of the binodal curve we could follow the following course. It is required for coexistence that besides the temperature three other quantities are equal, i. e. p , q and $M_1 \mu_1$. If we now also trace the lines on which $M_1 \mu_1$ is equal, we should have to seek in order to find a point of a binodal curve, the points satisfying the condition that the p , q and $M_1 \mu_1$ lines passing through this point intersect in still another point of the field. This search, however, being exceedingly difficult would give moreover no clear survey of the results. We shall, therefore not follow this course. Still I shall make some prefatory remarks on the course of this third group of lines. For it is by no means devoid of interest to know in which phases of a binary system the

¹⁾ This last paragraph has been added in the English translation.

molecular potential of one of the two components has the same value. We shall call this third group of lines "potential lines".

THE POTENTIAL LINES.

The value of $M_1 \mu_1$ is equal to $\psi - v \frac{d\psi}{dv} - x \frac{d\psi}{dx}$; by differentiation we find:

$$d M_1 \mu_1 = -v d \frac{d\psi}{dv} - x d \frac{d\psi}{dx}$$

or

$$d M_1 \mu_1 = v dp - x dq.$$

If we want to know the shape of such a potential line, we must know $\frac{dv}{dx}$ for such a line, which quantity we shall represent by

$\left(\frac{dv}{dx}\right)_{Pot}$. For the value of this quantity we find then the expression:

$$\left(\frac{dv}{dx}\right)_{Pot} = \frac{v \frac{d^2\psi}{dx dv} + x \frac{d^2\psi}{dx^2}}{v \frac{d^2\psi}{dv^2} + x \frac{d^2\psi}{dx dv}}$$

which may also be written:

$$\left(\frac{dv}{dx}\right)_{Pot} = \frac{v \frac{dv}{dx} \frac{dx_q}{dx_p} + x \frac{dx_q}{dx_p}}{v \frac{dx_q}{dx_p} + x \frac{dx_p}{dx_p}}$$

So there is a locus in whose points $\left(\frac{dv}{dx}\right)_{Pot} = \infty$, and another in whose points $\left(\frac{dv}{dx}\right)_{Pot} = 0$. The former takes place when $\frac{v}{x} = \frac{dx}{dx_p}$ i.e. this locus is the series of points in which lines drawn from the origin touch the p -lines. On the other hand $\left(\frac{dv}{dx}\right)_{Pot} = 0$ if $\frac{v}{x} = \frac{dx}{dx_q}$; for the points of the spinodal curve in which $\frac{dx}{dx_p} = \frac{dx}{dx_q}$, also $\left(\frac{dv}{dx}\right)_{Pot}$ is equal to $\frac{dv}{dx_p}$.

The shape of the locus $v = x \left(\frac{dx}{dx_p}\right)_p$ is different, according as the p -lines have the course as in the left region of the general p -figure, or as is the case in the middle region or in the right region. The

course of the p -lines being modified by the temperature, the value of T will also influence this shape.

Let us first put a left region at a value of T below T'_{k_1} and also below T'_{k_2} . Then tangents may be drawn to all p -lines from the origin. The points of contact on the side of the small volumes then form a continuous series of points which begins in the point in which the liquid branch of the curve $\frac{dp}{dv} = 0$ intersects the 1st axis, and

moves further and further away from this curve as it approaches the 2nd axis, remaining all the time at smaller volumes than those of the curve mentioned. The points of contact on the side of the large volumes also form a continuous series of point, which starts in

the point in which the vapour branch of the curve $\frac{dp}{dv} = 0$ intersects the 1st axis, and also moves further and further away from this curve as it draws near the 2nd axis. This series of points has always

larger volume than the curve $\frac{dp}{dv} = 0$. So when a potential line

passes through such a series of points it is directed parallel to the V -axis. The locus of the points in which a potential line runs parallel to the X -axis, and which is found by drawing tangents from the origin to the q -lines, is a curve consisting of one single branch,

which at small volumes crosses the field from a certain point of the first axis to the point $v = b$ and $x = 1$. But the shape of this curve is very different, dependent on the more or less complicated shape of the q -lines. Without entering into further details we shall only observe, that when q -lines run as is the case in the absence of $\frac{d^2\psi}{dv^2} = 0$, this curve will have no point in common with the preceding

one; but if $\frac{d^2\psi}{dv^2} = 0$ exists, and intersects $\frac{d^2\psi}{dv^2} = 0$, the curve on

which $\left(\frac{dv}{dx}\right)_{Pot} = 0$, passes round $\frac{d^2\psi}{dv^2} = 0$, and twice intersects the line,

on which $\left(\frac{dv}{dx}\right)_{Pot} = \infty$. These two points of intersection are again of

importance for the shape of the potential lines. Then again a loop-potential line passes through one of these two points. In this case the double point is the point of intersection on the right, and the point of intersection lying on the left serves then again as isolated point, round which a series of potential lines run in closed figures. That in this case the point lying on the right is the double point, is in

connection with this that all potential lines terminate in the point $v=b$ and $x=1$. $M_1\mu_1$ is infinitely large on the line $v=b$, and on the second axis $M_1\mu_1$ is negative infinite. In the point $v=b$ and $x=1$ the value of the potential for the first component must therefore be indefinite. When arriving at this point all potential lines touch the line $v=b$. In fig. 15 the course of the potential lines has been schematically represented for this case of non-miscibility in the liquid state. The first axis is cut or touched by the potential

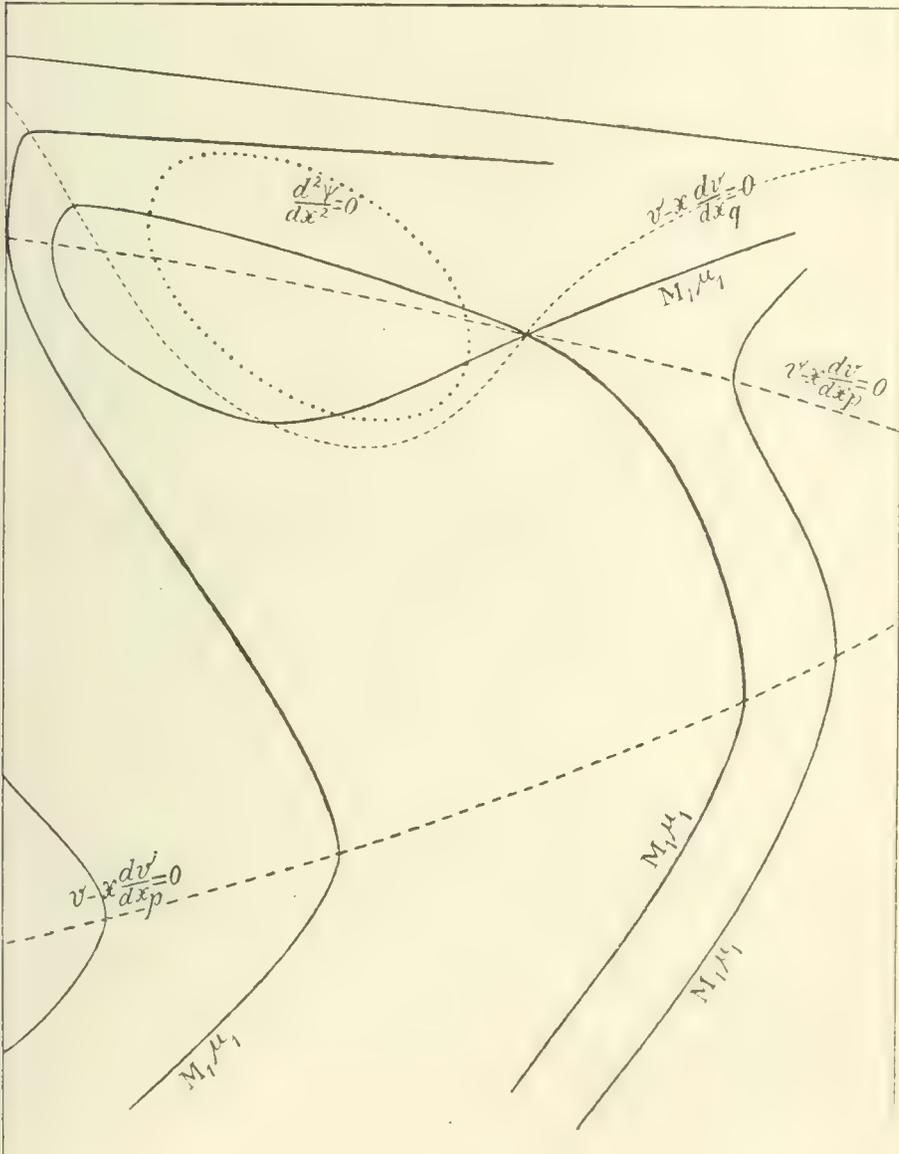


Fig. 15.

lines of every degree. $M_1\mu_1 = -\infty$ for $v = \infty$. If v decreases, $M_1\mu_1$ increases till the potential has reached a highest value in the point of maximum pressure $\left(\frac{dp}{dr} = 0\right)$. With further decrease of v the potential diminishes, till the final point of the unstable state is reached, where $\frac{dp}{dr}$ is again equal to 0. There $M_1\mu_1$ is minimum. If the point $v = b$ is reached, $M_1\mu_1 = \infty$. With very large volume $M_1\mu_1$ is approximately equal to $MRT \log \frac{1-x}{v}$, in which also a function of T is left out, which may generally be left out in the construction of the μ -surface for definite value of T ; from this shape for $M_1\mu_1$ it is seen that the portions of the potential lines which start from the 1st axis for large volume, may almost be considered as straight lines directed to the point $x = 1$ and $v = 0$. If the potential line starts from the volume v_1 , the equation of the initial portion is $v = v_1(1-x)$. If v_1 should be $= \infty$, and so $M_1\mu_1 = -\infty$, the value of $M_1\mu_1$ is negative infinite for every value of x for $v = \infty$, which it is also all along the second axis. The rule that for very large volumes the initial portions of the potential lines may be considered as straight lines already follows from the law of DALTON that each of the components in a mixture of gases behaves as if it alone was present in the volume. If $v = v_1(1-x)$, the density of the first component has the same value, and the quantities determined by the density, are the same; e. g. the pressure and the potential. If the circumstances are as assumed in fig. 15, there is of course also a locus where $\left(\frac{d^2v}{dx^2}\right)_{M_1v_1} = 0$, which is again a loop-line passing through the double point of the potential lines. If the locus $v - x \frac{dv}{dx} = 0$ does not intersect the other $v - x \frac{dv}{dx}$, all the potential lines have the simple shape which they have on the left side and on the right side in fig. 15.

If we suppose a left region at a value of T above T_{k_2} , the locus $v - x \frac{dv}{dx} = 0$ is subjected to a modification. Then the two branches of $\frac{dp}{dr} = 0$ have joined, and in the same way the two branches of this locus will join; but both lying outside $\frac{dp}{dr} = 0$ the point of

junction will lie at larger x than the point of junction of the branches of $\frac{dp}{dv} = 0$. This junction must then take place in a point of inflection of a p -line, as is immediately seen when in a p -figure the tangents are drawn from the origin in the circumstances mentioned, in which it also appears that the point of contact then lies on a p -line of maximum value. So the point of junction mentioned is a point in which the tangent of a p -line in its point of inflection passes through the origin. From the differential equation of $v - x \frac{dv}{dx_p} = 0$, if v is taken as function of x and p , follows for this locus:

$$\frac{dp}{dx} = \frac{\frac{d^2 v}{dx^2}}{\frac{dp_x}{dx} - x \frac{d^2 v}{dp dx}}$$

The potential lines of low degree have then lost the points in which they are directed vertically, and have then a very simple shape. With decreasing volume they no longer run back to smaller value of x .

In the second place let us choose a region in the middle, where the two points of intersection of $\frac{dp}{dv} = 0$ and $\frac{dp}{dx} = 0$ are found.

Even though the two branches of $\frac{dp}{dv} = 0$ remain entirely separated,

this is not necessarily the case with the two branches of $v - x \frac{dv}{dx_p} = 0$.

It is easy to see that the branch at the smaller volumes lies above $\frac{dp}{dv} = 0$ only from $x = 0$ to the double point of the p -lines. With

higher value of x it lies below it. In the same way the branch of $v - x \frac{dv}{dx_p} = 0$ at larger volumes lies below $\frac{dp}{dv} = 0$ only from $x = 0$

to x of the double point. This lower branch passes through the double point, and lies above $\frac{dp}{dv} = 0$ with greater value of x . The

two branches join as soon as there exists a p -line, for which the tangent in the point of inflection is directed to the origin. If at

minimum critical temperature the line $\frac{dp}{dv} = 0$ possesses a splitting

point, the curve $v - x \frac{dv}{dx_p} = 0$ is restricted to the left part, and is

closed for smaller value of x than that of the splitting point. If, however, the region extends far to the right, then also the right part of $\frac{dp}{dv} = 0$ can again contain a closed part of $v - x \frac{dv}{dx_p} = 0$, with a top at a certain value of x , and the open side at $x = 1$. Also for regions lying entirely on the right side it remains of force that $v - x \frac{dv}{dx_p} = 0$ lies within $\frac{dp}{dv} = 0$; so that if $\frac{dp}{dv} = 0$ no longer extends over the entire width, $v - x \frac{dv}{dx_p}$ cannot extend any longer over the entire width either.

If also in such middle region, and at the same time in a right region we examine the course of the locus $v - x \frac{dv}{dx_q}$, where the potential lines are directed horizontally, we see when consulting figs. 5 and 6 that the locus mentioned remains restricted to smaller volumes than those of the line $\frac{dp}{dx_v} = 0$ so long as the curve $\frac{d^2\psi}{dx^2} = 0$ does not exist, or if it does, for all points outside this curve.

If $\frac{dp}{dx_v} = 0$ cuts the curve $\frac{d^2\psi}{dx^2} = 0$, the locus mentioned passes through these points of intersection. Within $\frac{d^2\psi}{dx^2} = 0$ the line $v - x \frac{dv}{dx_q}$ lies at larger volumes than those of $\frac{dp}{dx_v} = 0$. But then no intersection of $v - x \frac{dv}{dx_q} = 0$ and $v - x \frac{dv}{dx_p} = 0$ inter se is to be expected. Hence there is no question of a loop-potential line. The result would have been perfectly different, if we had also examined the course of $M_2 \mu_2$. But this may be considered superfluous, now that we know the course of the q -lines, so of $M_2 \mu_2 - M_1 \mu_1$ and of $M_1 \mu_1$. This by no means exhausts the properties of the course of the potential lines, but as we are not going to avail ourselves of this third group of lines for the determination of the binodal line, I think that it will suffice to mention the above properties.

For the determination of the course of the binodal line we shall make use of the equation of p. 57, viz. :

$$dM_1 \mu_1 = v dp - x dq.$$

But first some preliminary remarks. Among all the lines to be discussed in a theory of mixtures the isobars and the binodal lines are to be

considered as the most important ones, because they can be the subject of experimental investigation. Though it is necessary for a clear insight that for a simple substance we know that below certain temperature the isotherm possesses unstable parts, and that we can indicate the limits of these unstable parts, yet the determination of the points of coexisting equilibrium is of the greatest importance for the experiment. In the same way it is, indeed necessary for a clear insight into a binary mixture that the existence of the unstable phases and their limits are known, so the spinodal curve; but the knowledge of the binodal line is of still more importance, and to determine the latter must be taken as the final end of all considerations, because it can constitute the subject of experimental investigation, and the results derived from our considerations can only be tested by experience in so far as they refer to the binodal line. If we are to admit an exception to this rule, this applies to the plaitpoints to whose existence could be concluded without an examination of the binodal curve being necessary. But moreover, it deserves attention that not even the whole of the binodal line can be realised by the experiment. The binodal line can possess portions lying in the unstable region, and others which are metastable. This has already been observed in the *Théorie Moléculaire* (Cont. p. 14), but appears in an ampler and more complete measure from the diagrams occurring in *These Proc.* March and June 1905. At the same time it appears there how very complicated the binodal line can be, when the spinodal curve hardly deviates from the usual shape. Hence if the more or less complexity of a plait is to be judged according to its spinodal curve or according to its binodal curve, a very different opinion will be arrived at.

Thus paying attention to the properties of the binodal curve I have been able to speak of a main plait and a branch plait in the last cited paper. In the same way, regarding only the binodal line and its nodal lines, we may speak of a transverse plait and a longitudinal plait, whereas, regarding only the spinodal curve, we shall have to consider these two as one single plait. However, to prevent confusion, it is desirable to follow one and the same terminology. At the moment it seems most desirable to me to consider particularly the spinodal curve when choosing the name, leaving that part out of account that may also sometimes exist, but which then encloses the concave-concave part of the ψ -surface. If no plaitpoint exists on the spinodal curve, or only one and then a realisable one, such a plait might be called a normal one. If besides there are a couple of heterogeneous plaitpoints found, we

might speak of an abnormal, or as I did in preceding pages of this communication, of a complex plait. If the spinodal curve has split up at certain value of T , which may take place in consequence of the curve $\frac{dp}{dr} = 0$ having split up, there are two plaits, one of which might be called the right plait, and the other the left plait. If it has split up in consequence of a separation between the curves $\frac{dp}{dr} = 0$ and $\frac{d^2\psi}{dr^2} = 0$, we might distinguish the two plaits by the names "transverse plait and longitudinal plait". Every time that the separation into two plaits takes place, two *homogeneous* plaitpoints make their appearance. With transition of a normal plait to a complex one a couple of *heterogeneous* plaitpoints appear. If then we wish to pay attention to properties of the binodal curve, other names might be thought desirable, but then it would be advisable to state distinctly that this is done to call attention to the special shape of the binodal line.

The equation $dM_1\mu_1 = vdp - v dq$ simplifies for a simple substance to $dM_1\mu_1 = v dp$, and may be considered in this form to lead to the construction for the point of coexistence. This construction can be carried out directly if as axes a p -axis and a $M_1\mu_1$ -axis is chosen, in which case we get a curve intersecting itself (Cont. II p. 4 fig. 1), or we can choose as axes a v -axis and a p -axis, and apply the law of MAXWELL. In the latter case we think $dM_1\mu_1 = v dp$ written in the form: $dM_1\mu_1 = d(pv) - p dv$, the integral of which is

$$(M_1\mu_1)_b - (M_1\mu_1)_a = (pv)_b - (pv)_a - \int_a^b p dv.$$

For coexistence we must have $(M_1\mu_1)_b = (M_1\mu_1)_a$, and $p_a = p_b = p_{\text{Coex}}$, so that we get:

$$p_c(v_b - v_a) = \int_{v_a}^{v_b} p dv.$$

For a binary mixture we get for the determination of coexistence, so for the determination of the points of the binodal curve, the same simple equation:

$$dM_1\mu_1 = v dp,$$

when following the series of points for which $dq = 0$ and so a q -line, in the execution of the construction.

Let us assume that we wish to apply MAXWELL's law. Then following a q -line, we draw the value of p at every value of v , and

seek how many times a straight line may be drawn parallel to the

v -axis, so that $p(v_b - v_a) = \int_a^b p dv$. If this can take place only once,

the extremities of this straight line indicate the value of v of the phases coexisting with each other, and the distance of this straight line above the v -axis the value of the pressure for this pair of coexisting phases, and the chosen q -line cuts then no other branches of the binodal line. This may take place several times, when the chosen q -line passes \pm times through the binodal curve, or when there are 6 points of the binodal curve on the chosen q -line. To ascertain whether this can take place 0 times, or 1, 2 or more times, we have to pay attention in the first and foremost place whether or not the chosen q -line intersects the spinodal curve, and if it does, how many times. For every time when a q -line cuts the spinodal curve, there is either maximum pressure or minimum pressure for the points of this q -line. In the points of the spinodal curve a p -line touches the chosen q -line, and one and the same p -line, having either larger or smaller value than the p -line which touches, will pass through two points lying on either side of the spinodal line. Thus in fig. 7 (p. 738) there is maximum pressure in point 4 of the q_1 -line, and minimum pressure in point 2, but for larger volume than that of point 4 the pressure is always smaller than in 4, and the smaller as v is larger, and in points of the same q -line in which v is smaller, the pressure is always larger than in 2, and the larger as we follow the q_1 -line to its initial point, where $p = \infty$. If we now construe p as function of v , the p -line has a shape similar to that of an ordinary isotherm. For $v = \infty$, $p = 0$, there is a maximum and a minimum pressure, and for $v = b$, $p = \infty$. MAXWELL'S rule may then be applied, but only once.

So this q_1 -line will possess two points of the binodal curve. In fig. 7 this will be the case for every q -line. For the line $q = \infty$, or for the first substance we find the coexisting phases of that substance, and for $q = -\infty$ or for the second substance, the coexisting phases of the second substance. If starting from a certain point of the v, x -diagram we draw both the p -curves as function of v , viz. the p -curve when we follow the q -line which passes through the chosen point, and the p -curve when we remain at constant value of x , then the 2nd curve has always greater value of p than the first for all values of v smaller than that of the point chosen. Thus in fig. 7 the pressure in a point lying more to the left to which the q -line moves is smaller than is the case for constant value of x at the same value of v .

Now let the point from which we start be the point of the binodal curve lying on the vapour side. Then, if we apply MAXWELL'S rule to the two curves, it follows from the circumstance that p is always larger for the curve at constant x , in the first place that MAXWELL'S line for this p -curve lies higher than that for the p -curve when we follow the q -line, and in the second place that on the vapour side the binodal curve for given x always lies at larger volumes than the vapour volumes would be when every mixture was to be considered as homogeneous. In the same way on the liquid side at smaller volumes. Just as the binodal line lies outside the line $\frac{dp}{dv} = 0$, the binodal line lies outside the phases which would coexist if every mixture should behave as a simple substance. Properties which also immediately follow from the ψ -surface.

In fig. 7*b* only q -lines of lower degree intersect the spinodal curve. The q -line of the highest degree which still has points in common with the spinodal curve, which points are coinciding points is that passing through the plaitpoint. When we follow this q -line maximum and minimum pressure will have coincided, and drawing p as function of v , we get a line which has an horizontal tangent in the plaitpoint, and at the same time a point of inflection, just as an ordinary isotherm in the critical point. This is a remark which always holds for a plaitpoint, also for a hidden plaitpoint; but then the special point in the p -line where $\left(\frac{dp}{dv}\right)_q$ and $\left(\frac{d^2p}{dv^2}\right)_q$ is equal to 0, lies on the unstable branch. There is a third possibility for the situation of this special point, viz. that it lies on what we might call the liquid branch of the p -line, as will presently appear.

Let us now consider the case of fig. 8, and let us choose there a q -line which intersects the spinodal curve 4 times, as is the case with one of the q -lines drawn. If starting at large volume we follow this q -line, we meet, still at large volume, the spinodal line in a point where p has a maximum value; in the second point where the q -line leaves the unstable region for the first time, there is maximum pressure. In the third point where this q -line enters the unstable region again, there is again maximum pressure, and in the fourth point when the unstable region is finally left there is again minimum pressure. Now to draw p properly as function of v , we must know the value of $\left(\frac{dp}{dv}\right)_q$. Now:

$$\left(\frac{dp}{dv}\right)_q = \frac{dp}{dx} \left(\frac{dx}{dv}\right)_q + \frac{dp}{dv_x}$$

which equation may be written in the following form:

$$\left(\frac{dp}{dv}\right)_q = -\frac{\frac{d^2\psi}{dv^2} \frac{d^2\psi}{dx^2} - \left(\frac{d^2\psi}{dx dv}\right)^2}{\frac{d^2\psi}{dx^2}}.$$

From this form we see that $\left(\frac{dp}{dv}\right)_q$ is positive in the unstable region only when $\frac{d^2\psi}{dx^2}$ is positive. If $\frac{d^2\psi}{dx^2}$ is negative, then $\left(\frac{dp}{dv}\right)_q$ is again negative in the unstable region, and when the q -line intersects the curve $\frac{d^2\psi}{dx^2} = 0$, $\left(\frac{dp}{dv}\right)_q = \infty$. In fig. 16 the course of p as function of v when this q -line is followed, has been schematically represented.



Fig. 16.

Now we have to examine how many points of the binodal line lie on this q -line. For this discussion I shall represent the branch right of point 1 by a ; the branch between 1 and 2 be then the b -branch etc. The number of times that MAXWELL'S rule can now be applied, is equal to the number of combinations in two of 4 quantities. Thus branch a could be combined, not with branch b , but with branches c , d and e . The branch b may be found combined with d and e . And finally branch c with e . We do not mean to say that the application in those 6 cases is always actually feasible. This will be discussed presently when we discuss other q -lines. But for the q -line chosen here, it is really possible to trace those 6 MAXWELL lines. And then this q -line must cut the binodal curve 12 times. These 12 points of intersection are to be found in fig. 17. In this figure the q -line has the shape of fig. 8. It intersects the

spinodal curve, which has also been drawn in this figure, four times. It has a maximum and minimum volume. Between the points of largest and smallest volume the locus $\frac{d^2\psi}{dv^2} = 0$ must be thought.

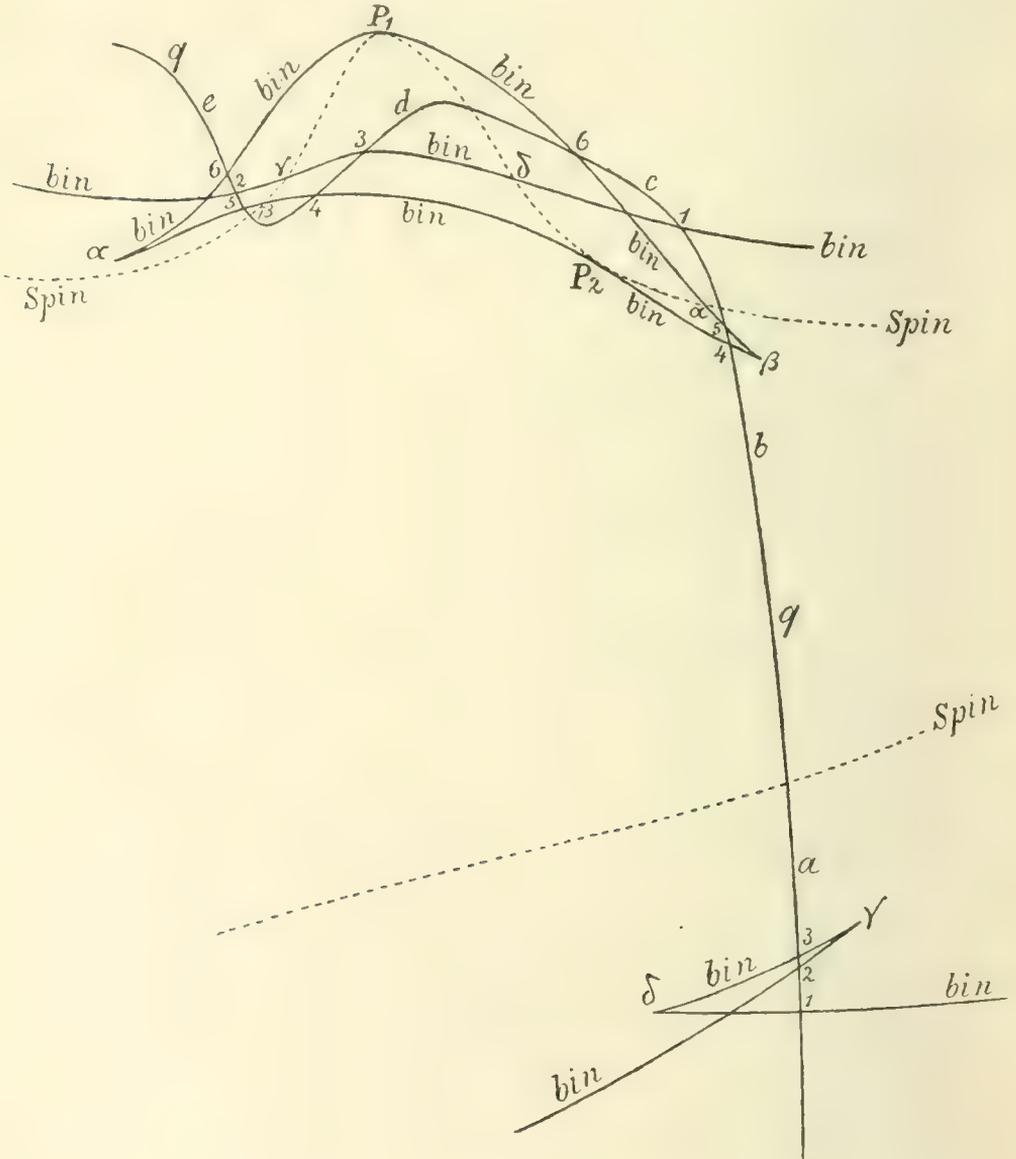


Fig. 17.

In this fig. 17 the binodal line has further been drawn, and on account of its intricate shape, it has been several times indicated by the sign *bin*. We may consider this binodal curve as consisting

of two separate parts. First that part that we might call vapour-liquid binodal curve. The liquid branch of this part has a regular course, but the vapour line branch has the well-known shape with two cusps. The nodal line belonging to the cusp γ , has its other extremity in the point γ , where the liquid branch of this binodal line passes through the spinodal curve. In the same way the two points indicated by σ belong together as extremities of one same nodal line. The remaining part of the binodal curve forms a curve closed in itself. For this part of the binodal curve the two heterogeneous plaitpoints P_1 and P_2 are in the first place of importance. The points on the right and on the left of P_1 lie in the stable region, the points on either side of P_2 in the unstable region. If we continue the branch on the right of P_1 , and pass through the spinodal curve in the point α , then to this point as an extremity of a nodal line belongs another point α as the other extremity of this nodal line, and there must again be a cusp for the binodal curve for this second point α . In this second point α the binodal curve returns again to higher value of x , and if it then meets the spinodal curve in the point indicated by β , another point β belongs to this, at which the right branch has a cusp. From this point the remaining part of the binodal curve has only points in the unstable region, and the points lying between the two points β are extremities of nodal lines which approach each other and coincide in P_2 .

To find the 12 points in which this q -line cuts the binodal curve, let us apply MAXWELL'S rule to that portion of the p -figure with the branches a , b and c , and determine the points denoted by 1. Let us also add the branch d , then the equality between the areas above and below the straight line would be disturbed, if the same straight line is retained, i. e. in this sense that the total amount of the areas above the straight line would be too large. From this follows that we must trace the straight line higher. For the points of the binodal curve which are determined by the combination of a with d , the pressure is, therefore, larger, while, as the figure shows, the volumes are both smaller than those of the corresponding points 1. The points determined by this combination have been indicated by 3. If we now also add the branch e , the pressure must again decrease. Then we determine the points denoted by 2. It will presently appear that the pressure in 2, though it is diminished, is still larger than in the points 1. By means of the combination of b with d , both branches in the unstable region, we determine the points 4; and after addition of the branch e the points 5, which must have lower pressure than the points 4. Finally the combination of c with e remains. Now

the situation of the q -line which we have chosen, is such, that the branch c remains on the right of the points of three-phase-pressure. From this ensues that if we have construed the p -line in fig. 16 correctly, the application of MAXWELL'S rule to the combination (c, e) must yield a larger pressure for the points 6 than for the points 1; but it also follows from this that the pressure for the points 2 (combination of a, e) lies between p_1 and p_6 — and so $p_2 > p_1$. But not all these 12 points are realisable. Every time an unstable branch occurs in the combination the nodes determined by this combination are not to be realised. So the points 3 (combination a, d), the points 4 (combination b, d), and the points 5 (combination b, e) are not to be realised under any circumstances. Thus already 6 of the 12 points are excluded as belonging to unstable coexisting equilibria. Of the remaining 6 points 2 more are excluded, if metastable states are set aside. So summarising we determine the following points by means of the combination put by the side of it:

points	combination	
1 . . .	a, c . . .	stable
2 . . .	a, e . . .	metastable
3 . . .	a, d . . .	unstable
4 . . .	b, d . . .	unstable
5 . . .	b, e . . .	unstable
6 . . .	c, e . . .	stable

To construe all the points of the binodal curve we should have to treat all the q -lines in a similar way. For the first component ($q = -\infty$) the p -line is the ordinary isotherm, in the same way for the second component ($q = +\infty$) the isotherm for this component. So with increase of the value of q such a gradual change of the q -line must take place that it passes from the first shape to the second. With very large volume these extreme shapes may be considered to coincide. This is also the case with all intermediate forms. The modification remains chiefly restricted to the smaller volumes, and in the case of $b_1 = b_2$ such a conclusion would be admissible also for the exceedingly small volumes. So long as the q -line (see fig. 4 and fig. 8) is still of so low a degree that it does not even pass through the lowest point of $\frac{d^2\psi}{dx^2} = 0$, the p -line has still the usual shape of an isotherm. Not before the q -line touches $\frac{d^2\psi}{dx^2} = 0$, does a special point make its appearance in the unstable branch. For this point of contact

$\left(\frac{dp}{dr}\right)_q = \infty$, but $\left(\frac{dp}{dr}\right)_i$ still remains positive on the left and on the right of that point. With somewhat higher degree of q , $\frac{d^2\psi}{dx^2} = 0$ is intersected twice, and two points may be pointed out in the p -line in the unstable branch where it is directed vertically. Between these 2 points $\left(\frac{dp}{dx}\right)_q$ is negative. But then too the p -line has but 3 branches, and so MAXWELL'S rule can only be applied once; then we find only two points of the binodal curve, viz. a point indicating a liquid volume lying in the left side of the figure, and a point indicating a vapour volume, lying much more to the right side, but still remaining sufficiently on the left of the double point of the vapour binodal curve. Then the q -line cuts the binodal curve in no other points on the vapour side. If the value of q rises higher, a third special point appears again on the unstable branch of the p -line, i. e. when the q -line begins to have 4 points of intersection in common with the spinodal curve. This will be the case when it passes through the hidden plaitpoint P_2 (see fig. 17). Then it touches the spinodal curve, but in such a way that $\left(\frac{d^2v}{dx^2}\right)_q$ has the reversed sign of $\left(\frac{d^2v}{dx^2}\right)_{spin}$. The rule that in a plaitpoint the p -line and the q -line envelop the plait is, accordingly, restricted to the realisable plaitpoints. It must run exactly the other way about for hidden plaitpoints. So there $\left(\frac{d^2v}{dx^2}\right)_p = \left(\frac{d^2v}{dx^2}\right)$ has the reversed sign of $\left(\frac{d^2v}{dx^2}\right)_{spin}$ and of $\left(\frac{d^2v}{dx^2}\right)_{bin}$. In this third special point of the unstable branch of the p -curve

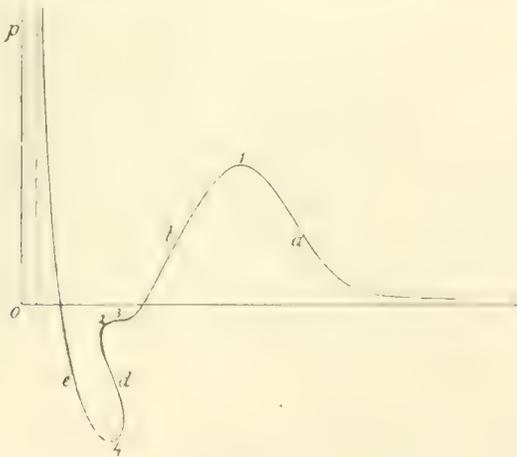


Fig. 18.

$\left(\frac{dp}{dr}\right)_q = 0$, and also $\left(\frac{d^2p}{dv^2}\right)_q = 0$, and then the p -curve has the shape of fig. 18.

For q above this value the spinodal curve is cut in 4 points. The two new points of intersection lie then on the left and on the right of P_2 , and at the beginning in the neighbourhood of this point. Then a portion lying in the stable region has been added to the q -line, from which we derive that p is smaller in the point of intersection lying on the right than that lying on the left. Not until now has the p -line the shape of fig. 16, but the branch c is still very small then, and the pressure of point 3 of this figure only little higher than of point 2. From this moment there could be question of the application of MAXWELL'S rule to the 5 branches a , b , c , d and e , and so of the determination of the 12 points of the binodal curve. But at the beginning not all these 12 points are real. The application for the combination of the first and the last branch is certainly feasible, and it yields a couple of realisable points for the binodal curve, and in contradiction with our result when we treated this combination for the q -line in fig. 17, the points defined in this way are not metastable but stable. No less is the application possible for the combination (b, d) , and the two points determined then lie in the unstable region, and can be represented by the points 4 of fig. 17, provided they are shifted nearer the point P_2 . The rule cannot be applied to the remaining 4 combinations. For the possibility of the application to the combinations (a, c) it is required that the length of branch c be such that the pressure of point 3 (fig. 19) be at least

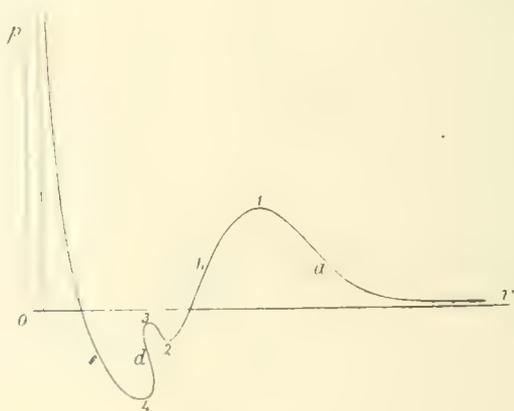


Fig. 19.

positive; and even this is not sufficient. If, namely, we have from point 3 a line // v -axis, and if then the area between the branches

b and c and this line parallel to the v -axis is smaller than the area between the branches a and b above this parallel line, MAXWELL's line would have to lie higher, and hence is not possible. A fortiori the combination (a, d) , which would require a still higher value of the pressure of MAXWELL's line, will be excluded. For similar reasons the combinations (b, e) and (c, e) must be rejected. From this follows that the q -line which is of somewhat higher degree than that passing through P_2 must remain on the left side before the point a of fig. 1, and on the right side of the ridge of the vapour branch of the binodal line. If we continue to raise the value of q , the possibility of the combinations (a, c) and (a, d) , begin simultaneously, i. e. when the pressure of the point 3, which may be considered as the top of c and d , has risen so high that the MAXWELL line for the combination (a, c) would just go through point 3. In the same way the possibility for the combinations (b, e) and (c, e) begins at the same time, i. e. when the pressure of point 2, which is the lowest point of the branches b and c , has descended so low, that the MAXWELL line for the branches c and e would just pass through point 2. If all these possibilities exist, the twelve points can be pointed out on the q -line.

Which of these two simultaneously beginning possibilities presents itself first on rise of the degree of the q -line, will probably not be bound to a general rule. If we now follow such a q -line, beginning at small volume on the left side of fig. 17, we first meet point 2 on the binodal curve, which proceeds regularly from left to right on the liquid side; then 6 and 5 follow before we pass through the spinodal curve. When the q -line rises again, we meet 4 and 3, which have then to lie more to the right than on the q -line, for which fig. 17 has been drawn. When the q -line again descends we first meet point 1, then 6, afterwards 5 and 4, and at last on the vapour side the points 3, 1 and 2 in this succession. But of all these points only the points 2 are stable. The points 1 and 6 are metastable. The others are unstable. And on further rise of q we reach that special q -line which is to be considered as the principal one for the phenomena of coexistence, and which, with three-phase-equilibrium, passes through the three coexisting phases. This coexistence of three phases is met with when (see fig. 16) the MAXWELL line for the combination (a, c) is the continuation of the line for the combination (c, e) . At the same time this line is also the MAXWELL line for the combination (a, e) . Then the points 1 and 2 or 2 and 1 coincide on the vapour side. On the liquid side on the left the points 2 and 6 or 6 and 2 coincide, on the right on the liquid side the points 1 and 6 or 6 and 1. The points 3, 4 and 5 have remained; of them

3 and 4 are unstable coexisting equilibrium, and 5 is metastable. In this case of three phase pressure the second component occurs in the vapour in a greater measure than in the two liquids, in connection with the circumstances which give rise to this figure, viz. that the second component has higher value of b and lower T'_i than the first. In fig. 3 Cont. II, p. 11 the course of the pressure is represented for the vapour-liquid binodal curve for this case.

With continued rise of the degree of q the p -curve, which entirely deviates from the shape of a simple isotherm for the last chosen values of q , must return to such a simple shape without abrupt changes. Thus the existence of 5 branches ceases when the q -line passes through P_1 . The branches c , d and e have then decreasing pressure with increasing volume. Only there is then a point where $\frac{dp}{dv_q}$ and $\frac{d^2p}{dv^2_q}$ is equal to 0 on this descending branch. But with still higher value of q also this particularity has vanished, and we approach to the usual shape of an isotherm. Already beforehand the q -line which above touches $\frac{d^2\psi}{dx^2} = 0$, was not found to run back to larger volumes in the unstable branch (d^1).

If we increase the temperature to T_{k_2} a new plaitpoint P_3 makes its appearance at $x=1$ and $v=(v_k)_2$. With further increase of the temperature the characters of the two realisable plaitpoints P_1 and P_3 begin to approach to each other. In fig. 17 the closed binodal curve belongs to P_1 . Above a certain temperature, which I called transformation temperature (These Proc. March 1905), this closed binodal curve passes to P_3 . At this transformation temperature the pairs of points β and γ have coincided on the spinodal curve in fig. 17, and two branches of the binodal curve touch, and $\frac{d^2v}{dv^2}$ is the same for these two branches. But for further particulars I refer to the already frequently cited communication. We must only bear in mind that in the case treated here $T_{k_2} < T_{k_1}$, whereas in the figure which I gave before for this transformation it was assumed that $T_{k_2} > T_{k_1}$. Regarding the properties of the binodal curve we may then speak of a principal plait and of a branch plait. At much higher T , P_3 and P_2 have coincided, and the binodal curve has become a normal simple line. (To be continued).

¹) Strictly speaking the change of the p -line with increasing value of q is not a moving away from and then a return to the shape of an isotherm. It must be regarded as a progressive development, which proceeds in the same sense. To the last q -line belongs then also the infinitely large pressure along the line $v=b$. This portion is, however, not necessary for the description of the binodal curve, at least when the plaitpoint P_1 exists.

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday June 29, 1907.

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Afdeling van Zaterdag 29 Juni 1907, Dl. XVI).

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Geology. — “*On ore veins in the province of Limburg*”. By Prof. A. WICHMANN.

(Communicated in the meeting of March 30, 1907.)

In the spring of 1856 the ex-colonel of the Dutch East-Indian Army P. VAN SWIETEN, at the Hague, founded a “Mining Society for the Netherlands”¹⁾, which obtained the concession for the mining field “Marie” in the southernmost part of the province of Limburg²⁾, in order to search for coal. After the first borings at Epen and Simpelveld had remained unsuccessful, the hamlet of Bommerig³⁾, community of Wittem, was taken, where on Oct. 11, 1856, a lode of ore was discovered of 0,80 meter thickness, at a depth of 56,20 M. and consisting chiefly of quartz and galena. Although it was suspected at once that this lode communicated with the one worked at Bleiberg in Belgium, situated to the SSE., yet the working of this lode was not taken in hand because of the great expense, involved in sinking a shaft. After this the Society continued its investigations in other parts of the mining field with insufficient results until it was dissolved after the available funds had been exhausted.

It has been known for a long time that the devonian and carboniferous strata in the environs of Aachen (Aix-la-Chapelle), extending mainly from the North-east to the South-west, are cleaved by faults directed almost perpendicularly to them and which appeared to be of great importance for the formation of ores⁴⁾. These masses of ores were found sparingly in the devonian system, mostly in those, belonging to the carboniferous limestone and only once — it was thought — in those of the coal-measures, namely at Bleiberg.

1) Nieuwe Rotterdamsche Courant, Thursday, May 22, 1856, N^o. 140. The foundation act dates from June 11, 1856 (Dutch State Gazette, Thursday, July 10, 1856 N^o. 162).

2) Situated a little over a kilometer to the North-east of Epen and 2 kilometers south of Mechelen.

3) Nieuwe Rotterdamsche Courant, Thursday, October 16, 1856; N^o. 286. P. VAN SWIETEN. Rapport sur les opérations de la Société de l'union minérale pour la Néerlande de 1856 à 1857. Annales des Travaux publ. de Belgique XVI. Bruxelles 1857—58 p. 266—267 Pl. V.

4) C. DANTZ. Der Kohlenkalk in der Umgebung von Aachen. Zeitschr. d. D. geolog. Gesellsch. XLV. 1893 p. 599—683, Taf. 26. W. SCHULZ. Führer des Berg- und Hütten-Ingenieurs durch die Umgegend von Aachen. Aachen 1886, p. 37—41 m. Karte. G. D. UYLENBROCK. Le sud-est du Limbourg néerlandais. Annales de la Soc. géolog. de Belgique XXXII. Liège 1904—05. M. pag. 151—104., Pl. V.

G. DEWALQUE. Essai de carte tectonique l. c. Pl. IV.

On the most important fault — called by UJLENBROEK the “Geul Valley fault” — lie the mines Fossey, near Hergenraed (Rhenish Prussia), Moresnet neutral territory and Bleiberg (Belgium). Excepting the contact seams, containing calamine, the ores are galena, being the oldest formation as usual, zinc-blende and pyrites. The vein, found at Bommerig more than 50 years ago in the lowest stratum of the productive carbon, shows that from the south-east to the north-west the ores seek more and more the younger strata¹⁾ and that the direction of the Geul valley fault begins to deviate more towards the north-north-west after Bleiberg.

For years numerous borings were made in a more northern part of Limburg, which led to the sinking and working of some coal-pits. In December 1905 Mr. L. RUTTEN at Utrecht found on the dump of the mine “Carl”²⁾ some pieces of ore which he presented to the Mineralogical-Geological Institute at Utrecht. Further investigations, undertaken by him, showed that these ores originated from a vein, met when sinking the shaft, at a depth of 278 metres, but of which the dip and direction had not been determined. He succeeded in securing a number of pieces, belonging to private people. The vein has only a thickness of 0.20 M. On the clay-containing salband pyrites has deposited, while the vein mass proper consists of calcite, developed in the cavities in the form of crystals, on which sometimes also crystals of pyrites are found. Beside this vein ores were also found, likewise on joints of the sandstone of the mine “Carl”, namely pyrites, but also zinc-blende, copper, pyrites, and galena. Moreover crystals of calcite are always found and generally dolomite.

In the mine Oranje-Nassau, near Heerlen, similar formations seem to occur, at any rate crystals of calcite, covered with pyrites, are found here on joints. Peculiar is here the regular coalescence, caused by the small cubes of pyrites accumulating at the poles, then continuing themselves on the obtuse edges of the scalenohedra and here gradually disappearing.

We finally point out that while in the Stolberg district the veins of galena, pyrites, zinc-blende, and calcite are still mostly bound to the carboniferous limestone these minerals occur in the more western Worm district on joints of the carboniferous sandstone, which is a more recent horizon, a phenomenon which repeats itself at Heerlen.

¹⁾ At Eupen they still occur in the devonian system.

²⁾ Situated at 1½ kilometers east of Heerlen.

Physiology. — “*Nerve influence on the action of the heart. First communication. Genesis of the alternating pulse.*” By Dr. L. J. J. MUSKENS. (Communicated by Prof. H. ZWAARDEMAKER).

(Communicated in the meeting of April 26, 1907).

Whereas in 1897¹⁾ the writer has shown, that in the “pulsus regulariter intermittens” of the frog we have to see a result of slowed conduction between sinus and auricle²⁾, or between auricle and ventricle, which was later accepted by WENCKEBACH³⁾ and recently proved by MACKENZIE⁴⁾ for man, the problem of the pulsus alternans did not profit by the application of physiology on the diseased heart. It is true, that already years ago Traube directed the attention on those types of P. A. in which the period between the weak beat, and the next stronger one is smaller than between the larger and the smaller contraction, where i. o. w. the greater contraction commences too early. But until now only OEHRWALL⁵⁾, W. STRAUB⁶⁾ and TRENDELENBURG⁷⁾ have gone into the analysis of allied heart-curves of the frog, following up the way which had led to the elucidation of the intermittent pulse.

It is clear, that for this analysis we must not join those observers, who think that the explanation of the P. A. by inotropic influence amounts to something more than a simple periphrase of the fact, that every other pulse is weaker. For this misconception brings with it the additional drawback, that it cuts off the way for all further analysis.

In the alternating contraction of the ventricle we have to do not with a simple pathological phenomenon but rather with a general physiological function that makes its appearance in many circumstances, which tends to appear as well in vertebrate as invertebrate animals under uncommon conditions. We have here to deal with a capacity of the cardiac muscle which enables the ventricle to go on with rhythmical contractions even under those abnormal conditions.

Digitalis dyalisata, injected subcutaneously in the frog, brings about after some time peculiar changes in the heart-beat, after a

¹⁾ Geneesk. Bladen. 1897. 4e Reeks. Bo. 4. p. 77.

²⁾ American Journal of Physiology. Vol. I. 1898 p. 509.

³⁾ WENCKEBACH. Nederl. Tijdschr. v. Geneesk. 1899. I. Blz. 666.

⁴⁾ MACKENZIE. British. Medical Journal. 24 October 1906.

⁵⁾ OEHRWALL. Skandinavisches Arch. f. Physiologie. Bd. 8. 1898.

⁶⁾ STRAUB. Arch. f. experimentelle Pathologie u. Pharmakologie. B. 45.

⁷⁾ TRENDELENBURG. Arch. f. Physiologie. (Anat. u. Physiol.) 1903. p. 284.

certain period of normal contractions. In the transition of the normal period into the period of slow contractions, regularly curves are recorded, which together with curves formerly published of the living frog-heart (loc. cit.) throw light, on one of the, at least four varieties of P. A. which in my opinion must be distinguished, at least in the physio-pathology of the frog-heart. As to the three other types of P. A. I think, that the way may become clear to explain them equally.

First form of P. A. with equal intervals.

The ventricle beats in regular rhythmus alternatingly stronger and weaker, the beginning of the weak contraction is separated by the same length of time from the preceding and following contraction (fig. 1). This variety was described by ENGELMANN¹⁾ and ascribed to momentarily diminished conductivity. F. B. HOFMANN²⁾ has shown, that this form of P. A. is often dependent on slight changes of the frequency of the heart-beat. STRAUB proved, that this P. A. under influence of antiarine easily gives way to "Puls-halbirung", which I often saw under influence of digitalis dyalisata.

Second form of P. A. with retarded small contraction.

In a former publication³⁾ I described an example of P. A. observed in the dying frog-heart, where the interval between the greater and smaller V_s was longer than between the smaller and larger contraction. There it appeared, that the auricle continued to beat regularly. By comparing the intervals $A-V$, preceding the greater and smaller contractions, we concluded then, that the contractionwave in the $A-V$ bundle, eventually in the ventricle itself, might be slowed, which was the cause that the V_s not only came too late, but was also weaker.

The supposition, that this P. A. in certain cases might depend on changes of conductivity within the ventricle, had right of discussion, although this could not be strictly proved, as is remarked by WENCKEBACH. For one can never with certainty conclude to a change of conductivity within a heart cavity (e. g. within V) if the interval between the contractions of two cavities (e. g. $A-V$) remained equal. But with the same certainty this meritorious observer is mistaken, when he, from the few curves (10a and 10b) of ENGELMANN generalises to the contrary i. e. to the exclusion of a similar relation in other

1) ENGELMANN, Arch. f. d. ges. Physiologie. Bd. 62. 1896. p. 556 seq.

2) HOFMANN. Arch. f. d. ges. Physiologie. Bd. 84. 1900. p. 165.

3) L. J. J. MUSKENS. Nederl. Tijdschr. v. Geneeskunde. 1902. No. II. Blz. 591.

cases. These curves cannot be looked upon as deciding in this point, because not there as in my cases, the auricle beats regularly, and therefore entirely different factors must be present, which has been overlooked by the writer. He appears to disregard, that in different lower animals, it was proved, that under influence of the vagus-nerve, at the same time the conductivity in one cavity can be improved, in others can be inhibited, which was confirmed by ENGELMANN, when he observed, how on the three bridges, veins-sinus, sinus-auricle, auricle-ventricle, independently of each other, conductivity might be changed; sufficient to show, that only very direct proofs could force us to admit, that under pathological circumstances this independence of conductivity in various parts should be lost. As well here, as also there, where WENCKEBACH explains the early smaller contraction by the quicker course of the weaker pulse-wave in the vessels, WENCKEBACH's conclusions appear to be much exposed to discussion. Also his conclusion, that there is no principal difference between P.A. with too early and retarded small contraction wave does not appear to be warranted by any well-known fact, certainly not by WENCKEBACH's suppositions. Sufficient facts can be adduced now, that here in different ways the same result can be arrived at.

To my former curves of P.A. brought about by poor nutrition, I now can add similar curves of P.A. brought about by injection of digitalis dyalisata (fig. 2).

In this case the interval $S_i^s - V_s$ can easily be determined. This amounts to 20.6; 22.3; 20.9. In every case this interval is lengthened where it precedes a smaller contraction; i.e. the contraction-wave, which culminates in a smaller V_s , found more resistance on its way from the sinus to the ventricle and there was an undoubtable slowing of the conduction.

Looking carefully at the curve, one finds that the sinus contraction preceding a weak V_s shows a flattened top. By measuring the intervals of S_i^s it becomes equally clear, that the sinus does not contract regularly and that it is the sinus contraction that comes too early, that is followed by a smaller V_s . Also the A_s preceding the weak V_s appears to be diminished in size.

Although it is not the place here, to go into detail about the new fact, that there exists a relation between the force of the sinus and auricular contraction and the force of the ultimate V_s , I will only remark, that in many similar experiments this relation was found. The question arises indeed, if in fact different parts of the three principal heart-cavities do maintain a special relationship in such a

way, that a completer sinus and auricular contraction tends to give rise to a completer ventricular contraction. If so, the next problem appears to be, whether this relation is kept up by special muscular arrangements or else whether nervous and ganglionic influence may play a rôle in it.

Third Form of P.A. with retarded smaller contraction.

Figure 3 is an example, where we find moreover reappearance of normal pulsation. Simple inspection of the pulsating heart made already the impression, that we had to deal with an antiperistaltic contraction: that the contraction-wave reaching the ventricle from the auricle, returned again to the auricle. BRANDENBURG ¹⁾, also PAN ²⁾, HERING ³⁾, VOLHARD ⁴⁾, and SCHMOLL ⁵⁾ have observed antiperistaltic contraction, after the writer had long ago shown ⁶⁾, that antiperistaltic contractions are a very constant phenomenon in the sinus of the turtle-heart.

In this case of fig. 3 we have to deal for the interpretation with 2 possibilities, 1. we may have to do with a real extra-contraction of auricle and ventricle, which results only in a very small elevation of the lever, because of its appearing in the beginning of the diastole, the ventricle being in the refractory period; or secondly we have to do with an antiperistaltic contraction wave, which on account of insufficient restoration of conduction in the $A-V$ bundle and the ventricular musculature, can only give rise to a weak V_s .

~ Indeed, the first supposition could not be discarded if it were to be admitted that spontaneously under similar conditions in the frog such an extra- A_s , followed by a very weak V_s could occur; an extra-contraction, which moreover was followed by an uncomplete compensatory pause. This conception is however hardly acceptable, if we take note of the systematical mode, in which this P. A., so to say, is prepared by the two abnormal contractions, which precede the very small V_s . Because these changes in the two ventricular contractions occur regularly at least in so far, as in my curves I come across similar cases, I think, that the other interpretation gains considerably in probable correctness. This supposition is therefore as follows: Under influence of the drug the conducting power within the ventricle is

1) K. BRANDENBURG. Arch. f. Anat. u. Physiol. Abt. 1904. Supp. p. 216.

2) O. PAN. Deutsche Zeitschr. f. klin. Medizin. Bd. 78. 1903. p. 128.

3) HERING. PFLÜGER'S Arch. Bd. 82. p. 1.

4) VOLHARD. Zeitschr. f. klin. Medizin. 1904. Bd. 53. p. 574.

5) SCHMOLL. Arch. f. klin. Medizin. 1907. p. 507.

6) Ned. Tijdschr. v. Geneesk. 1898. Deel II. Blz. 568 and Americ. Jnl. of Physiology; Vol. I. 1898. p. 504.

sensibly reduced. In the first only slightly weakened V_s only a part of the ventricular musculature could contract as a result of this disturbance of conduction. In the following contraction of the ventricle the wave spreads, however more slowly than in the normal cases (hence the stretched form of V_s) over the entire ventricular musculature. As the conduction of the contraction wave can take place after the modern doctrine of GASKELL and ENGELMANN in all directions, the contraction wave in this case winds its way through this lengthened V_s to arrive antiperistaltically again at the auricle. After this only a part of the ventricular musculature has regained its conducting power sufficiently and a weakened V_s will join the antiperistaltic A_s . It is clear, that on account of the antiperistaltic contraction the wave from A to V , to A again; from A returning to another limited part of V ; then again to A , etc. will give rise to a pulsus alternans, in this case temporarily, whereby the interval between the commencement of the great contraction and that of the smaller one, is smaller than that between the small contraction and the greater contraction. One can among the dialysata-experiments recognise these cases there, where after a maximal toxic dose the frequency first became considerably slower, but finally quicker again. Whereas in the vena cava curve previously the pulsations of the sinus were easily visible, one does not find any indication of sinus contraction after the premortal pulse acceleration has set in.

We find therefore here a form of cardiac activity, which shows the same particularities as the P. A. formerly described by the writer for the poorly nourished frog-heart (Loc. cit. 1902) of which however the mode of origin was quite a different one.

The only objection, which can be adduced against this interpretation, is a theoretical one. Until now, it was looked upon as a dogma, that the unimpaired ventricular musculature under no circumstances shows the phenomena of dissociation. This dissociation between the different heart cavities and in every one separately, was explicitly described by the writer in several publications¹⁾, especially regarding the sinus; its significance for our understanding of nerve influence on the heart, was more than once urged. Although in these experiments and those of ENGELMANN the occurrence of similar dissociations also of the unimpaired ventricle had to be acknowledged, the direct proof of its existence as far as I know, has never been

¹⁾ Geneesk. bladen 1897. p. 75. Proceedings of the American Academy of Arts and Sciences 1898. Vol. XXXIII. No. 71. p. 188. Americ. Jnl. of Physiol. 1898. p. 503 seq. Ned. Tijdschr. v. Geneesk. 1898. Deel II. Blz. 572 and 1902 Deel II. Blz. 583.

proved. Where under influence of *digitalis dialisata* the tendency of the cardiac muscle to dissociation, as we saw above, is accentuated, there we could expect that if we combine this influence with the equally dissociating vagus influence, the evidence of dissociation also in the ventricle might come out. Indeed, during the influence of the vagus nerve on such an intoxicated heart, I found a curve which is apt to illustrate this dissociation. We have here to deal with the transition of an alternating pulse into a normal one, after a direct vagus stimulation and shortly after the inundation of the entire heart by a physiological salt solution (In the ventricular curve this is visible).

In my mind there is no doubt, that the small elevation after the reduced ventricular contraction cannot be interpreted, either as an auricular contraction (because nowhere in this or other tracings an A_s of this considerable height was observed) nor as an ordinary extra-systole of the ventricle. In the latter case it could not be explained not only why here an extra systole arose, nor why the preceding ventricular contraction coming at the right time, was so exceedingly diminished in size. We have here undoubtedly to do with a dissociation in time of two parts of the ventricular musculature (eventually also of the "Reizleitungssystem") and only when *after the pulsus alternans* a not completely synchronic contraction of these parts has taken place, and the entire musculature comes again at the same time in the refractory period, normal contractions can follow. According to this interpretation the difference between the great and the small contractions of the preceding P. A. is to be ascribed to the fact, that only in the great contractions a particular part of the muscular mass is reached by the contraction wave; whereas this part of the muscle is excluded from the contraction in the small V_s .

Fourth form of P. A. with retarded great contraction.

Of this type of P. A. I cannot adduce any curve met with in lower animals. The only specimen I have come across, is registered from a case of Basedow, who suffered from an exceedingly rapid and at the same time irregular heartbeat. It appears to me that here we have to deal with an automatically beating ventricle or better with a ventricle, in whom two divisions are beating independently, only every other V_s (the greater one) causing an antiperistaltic contraction reaching the auricle. Comparable curves have been published by MACKENSIE ¹⁾ and WENCKEBACH ²⁾, so that its occurrence in men can

¹⁾ MACKENSIE, British medical Journal, 1905, III. Comp. fig. 12 perhaps also fig. 5.

²⁾ WENCKEBACH, Arrhythmie. 1903. P. 107.

be doubtlessly stated. In the latter curves no registration of the jugular vein was added to the ventricular curve, so that I am not in the situation to suppose or deny for these curves the same origin as in my curve.

All in all it seems to me, that in these curves and their analysis we have important arguments, which tend to prove, that from the physiological side more special research is needed regarding the conduction within the individual divisions of the heart. For the first of the four described types of P. A. we have shown, that from physiological side the cause has to be sought in changed conductivity by which in the weak contractions, the contraction wave is limited to a part of V. For the second type of P. A. we thought we were able to bring direct proofs, while it became probable for the third type of P. A., that they are results of the conduction between the sinus and the ventricle becoming slower; and for the third form of P. A. it appeared probable that the P. A. is the result of the antiperistaltic contraction wave, so that we had here not a quantitative, but a qualitative change in the conductivity. Regarding the fact which TRENDELENBURG stated, that by stimulating the ventricle artificially the frequency of the ventricular rhythm may become much greater if slowly the stimulation is quickened, before "Halbierung" of the heart-beat makes appearance, then if within a short period a great frequency is attained, it is equally to be interpreted as follows: that by slowly increasing frequency the conductivity is enabled to adapt itself to the great demands; so that the moment is delayed where necessarily only partial contractions of the ventricle arise. Regarding the pathology, it appears to me that it is of importance for the knowledge of the pulsus trigeminus, discussed by WENCKEBACH, that from physiological side the importance of dissociation of the ventricle under certain circumstances as also the importance of the antiperistaltic contraction wave has been proved. With the statement that partial contractions do occur, it appears to me, that the necessity becomes evident, that the law of Bowditch has to be limited, in so far, that certainly every ventricular muscle fibre which contracts, does so with maximum force; but on the other hand, we have not to accept that necessarily in every ventricular contraction all muscular bundles contract equally.

Where we have to interpret curves like those of TSCHIRJEW¹⁾ (cited by WENCKEBACH) of O. PAN²⁾, R. FINKELBURG³⁾ and HAY

1) TSCHIRJEW. Archiv für Physiologie. 1877.

2) O. PAN. Deutsche Zeitschrift. f. klin. Medizin. Bd. 78. 1905. p. 128.

3) R. FINKELBURG. Cited by WENCKEBACH. 1905. Heft 1 and 2. p. 586.

and MOORE¹⁾ we should not neglect the value of these phenomena. For the absence of the compensatory pause (WENCKEBACH) finds in the above interpretation its complete explication. At the same time we can now regard HERING's opinion, that all pulsus bigemini should always depend on extra systoles as definitively rejected.

With FUNKE I agree finally in this that a further discussion about the question, whether apart from the pulsus alternans also the existence of a pulsus bigeminus must be acknowledged, is completely superfluous. On the other hand it might be desirable, if the experimental results of KNOLL and those of HERING in warmblooded animals about hemisystolia and heart-trigemini, should be taken up again, also in regard to the recent anatomical data.

Regarding these questions three recent researches must be regarded as important, firstly the observations of W. EINTHOVEN²⁾, whose accurate illustrations also of partial contractions, appear to promise a good deal for further analysis. Moreover the important researches of TAWARA³⁾ in ASCHHOF's Laboratory, which has shown, how far the division of functions in the ventricle of the warm- and perhaps also of the cold-blooded animals has gone.

Finally the observations of MACKENZIE⁴⁾ who has shown us the possibility, to get information also in man about the movement of the auricle under pathological circumstances, so that we may expect also this field of work becoming fertile for scientific analysis.

The next thing should be, to bring also the sinus of man under the scope of the graphical method. WENCKEBACH⁵⁾ thinks to have reason to believe, that dissociation of the sinus described by me in lower animals⁶⁾ might be observed equally in men.

To physiology the task to examine what influences are able to dissociate the two principal bundles of the "Reizleitung" system and to get information about nerve influence as well regarding that system itself as upon the muscular mass of the ventricle.

¹⁾ HAY and MOORE. *Lancet*. 1906. p. 1274.

²⁾ EINTHOVEN. *Tijdschr. v. Geneesk.* II. No. 22.

³⁾ TAWARA. *Das Herzleitungssystem*. 1906.

⁴⁾ MACKENZIE. *British medical Jnl.* 1902. Nov. p. 1411.

⁵⁾ WENCKEBACH. *Arch. f. Physiologie*. 1906. p. 361.

⁶⁾ *American Journal of Physiology*. Vol. I. 1898. No. IV. p. 503.

Zoology. — “*On the structure of the nerve-cells in the central nervous system of Branchiostoma lanceolatum.*” (First comm.)
By Dr. J. BOEKE. (Communicated by Prof. G. C. J. VOSMAER).

(Communicated in the meeting of April 26, 1907).

The methods of staining the elements of the nervous system, published in recent years by RAMON Y CAJAL, DONAGGIO, and especially by BIELSCHOWSKY, have enabled us to study the minute structure of the ganglion-cells not only of the lower animals but also of the vertebrates with more success than before. After having published in These Proceedings, some years ago¹⁾, the results of my former investigations on the structure of the nerve-cells of Branchiostoma, then studied by means of the goldmethod of APATHY, it seemed advisable to describe here too the results of my recent investigations on the same subject by means of the methods mentioned above, because they extend and complete my former results in several directions.

Contradictory to the results of EDINGER²⁾, the only author who studied the central nervous system of amphioxus by means of the method of BIELSCHOWSKY, viz. that the method gave only scanty results for the neurofibrillae in the cells, in my preparations, stained after the method of BIELSCHOWSKY—POLLACK, in a great number of nerve-cells of several specimens of Branchiostoma a very clear and distinct picture was obtained of the neurofibrillae, not only in the nerve-fibres, but also in the body of the nerve-cells.

Preparations of material preserved in a mixture of platinum chloride-osmic acid-acetic acid and corrosive sublimate³⁾, and stained in thin sections with iron-haematoxylin after HEIDENHAIN, were used as control and for the study of the protoplasmic structures between the neurofibrillae.

The different cell-forms of the central nervous system gave there, where they were satisfactorily stained, as a rule the same mode of arrangement of the neurofibrillae in the cell-body; therefore I will restrict myself to describe here only some cell-forms at length and only refer briefly to the structure of other cells. At another place I hope soon to give more and fuller details.

¹⁾ Proceedings Roy. Akad. of Sc. Amsterdam, of the meeting of Oct. 25, 1902.

²⁾ Anat. Anzeiger, Bd. 28, No. 17, 18, 24 April 1905.

³⁾ According to Dr. LEGROS the best method for the preservation of the nervous system of Branchiostoma. I can fully agree with him in this statement. This mixture gives better results than all the others I tried.

1. As is well known, the very large nerve-cells ("Kolossalzellen") lying at about equal distances from each other in the axis of the spinal cord, possess a thick axonic fibre, that after leaving the cell-body describes a characteristic curve and passes into one of the colossal nerve-fibres that run in a longitudinal direction through the spinal cord, and a number of dendrites, springing from the cell-body at different points.

Sections of these cells, stained after the method of BIELSCHOWSKY, give a very clear picture of the neurofibrillar structure. In a section in which only some of the dendrites are to be seen, and not the axonic fibre with its "cône d'entrance" (of which more later on) these cells show an arrangement of the neurofibrillae as shown in fig. 1.

The cell is surrounded by a glious capsule, composed of fine interwoven fibrillae. The preservation of the nervous system in formol, necessary for the BIELSCHOWSKY-reaction, causes the cells to shrink a little, so that the pericellular cavity is larger than it is in normal life and in well-preserved specimens. Within the cell-body the neurofibrillae form a very distinct and regular network. Everywhere they anastomose with each other, nowhere I could discover free running fibres. The meshes are regular, round or many-sided, and nearly all of about the same size. A subperipheral zone is formed, where the meshes are somewhat smaller and the composing neurofibrillae a little coarser. From this zone a few coarse neurofibrillae may be followed in the network radiating to the central zone around the nucleus. The nucleus itself is not coloured in these preparations, but is only to be seen as a clear round or oval spot in the midst of the darkly stained network of the neurofibrillae. There where a dendrite leaves the cell-body, the meshes of the network are elongated in the direction of the processus (fig. 1, 4*b*, 6). In the dendrites themselves, at least in the coarser ones, the anastomosing of the neurofibrillae is to be seen still at some distance from the cell-body. In fig. 1 is drawn a section of 7μ thick. In three of the following sections, passing through the dendrites, whose origin is shown in fig. 1, I could still see the anastomosing of the composing neurofibrillae. In fig. 2 is drawn one of the large dendrites of a similar colossal cell there where it branches into two. The network of the neurofibrillae, several coarser (and more darkly stained) fibrillae, and the continuity of the network in both branches is clearly to be seen. In the finer dendrites the neurofibrillae seem to become isolated sooner after having left the cell-body (fig. 1 at *b*). The same is to be seen in the smaller nerve-cells of the spinal cord (figs. 4*b*, 6).

The axon of the colossal nerve-cells has a somewhat different structure. As I described in my former paper¹⁾, the colossal nerve-fibres contain a great number of closely set exceedingly fine separate fibrillae, which in well-preserved preparations are distributed regularly through the whole extent of the fibre. There where the axon enters the cell, this bundle of neurofibrillae may be followed some way into the cell-body; we see the fibres describe a curve or vortex around the nucleus, and then the thin fibres melt into the somewhat coarser network of the neurofibrillae described above.

The smaller, mediumsized and smallest nerve-cells of *Branchiostoma*, such as those that are drawn in figg. 4, 5 and 6, at the same scale as the cell figured in fig. 1, show the same arrangement of the neurofibrillae as the colossal nerve-cells, viz. a regular network, the meshes elongated there where a dendrite or axon leaves the cell, more or less rounded in the centre of the cell-body. The subperipheral zone with finer meshes and coarser fibrillae I could not find here; the network seemed everywhere to be regular throughout the cell-body. In fig. 4*a* and 4*b* two sections through the same mediumsized nerve-cell are drawn. In fig. 4*a* the nucleus is to be seen, and on it a very regular network of neurofibrillae, with only one layer of meshes, and therefore giving a very clear idea of the regularity of the network. This section passes through the centre of the cell-body. Fig. 4*b* shows the peripheral part of the same cell. The meshes are here more elongated in the direction of the processus, and in the network some fibrillae are coarser and more darkly stained; all of these run in the direction of the dendrite and leave the cell-body there; inside the cell they form part of the general network; in the dendrite they run more or less parallel to each other and do not anastomose any more (see page 2). The same features are to be seen very clearly in fig. 6, showing the neurofibrillar structure of another mediumsized nerve-cell lying somewhat more cephalad in the spinal cord.

In fig. 5 is drawn a very small ganglion cell (magnified to the same scale as the foregoing figures). Here too the network of the neurofibrillae is easily to be seen and the meshes are of about the same size as in the mediumsized nerve-cells described above, though smaller than in the colossal ganglion cells.

Fusiform cells, in which the neurofibrillae simply pass through the cell-body from one processus to the other without interruption, as I described them in my former paper, I was not able to find in

¹⁾ These proceedings. Meeting of Oct. 25, 1902.

the preparations stained after BIELSCHOWSKY. Where the neurofibrillae were visible, they formed a network. In my preparations stained with chloride of gold after APATHY, which I looked over for these cells I however found them again. The uninterrupted course of the neurofibrillae was clearly to be seen. They are however only very rarely met with.

So we find in nearly all the cells a network of neurofibrillae with regular meshes. In full-grown animals the meshes in different cells are of about the same size. But when we examine the same kind of cells (for example the colossal ganglion cells) in very small animals, we find a neurofibrillar network of the same regularity but with much smaller meshes. So when we compare fig. 1, a colossal ganglion cell of a fullgrown Branchiostoma of 48 m.m. in length, with fig. 3, an analogue cell of an animal of 6 m.m. in length, we find a much smaller-meshed network. Those small animals have finished their metamorphosis already, and present nearly the same organisation as the adult animal. The nerve-cells therefore seem to have assumed already the definite arrangement of their neurofibrillar structure, but the meshes are much smaller. During the following growth of the nerve-cells the reticulum grows, but the structure remains the same. In different adult specimens the size of the meshes seemed always to be of the same order, and only to present the slight differences mentioned above.

When we compare this with the neurofibrillar structure, described for the ganglion cells of other animals, I will here especially call attention to the description of APATHY for Hirudineae and Vermes, of BOCHENEK for Helix, of DONAGGIO, CAJAL, MICHOTTE, LEGENDRE and the many authors, who have studied the ganglioncells of the higher vertebrates by means of the new elective histological methods. Among the descriptions by these authors of the neurofibrillar structure in the nerve-cells of the representatives of different classes of the animal kingdom, that of Branchiostoma takes just the place, we generally give to that animal in the animal series. In fig. 7 is drawn a sensory cell of a Pontobdella, with the neurofibrillar structure stained after APATHY. We see a very coarse network around the nucleus, with fibrillae radiating to the periphery and forming there a second network. The ganglion cells of Helix give according to BOCHENEK¹⁾ a much finer network. The meshes of this network are still much larger than those of the nerve-cells of Branchiostoma; these in their turn are larger and the fibrillae coarser than the neurofibrillar struc-

¹⁾ Le Nevraxe, Vol. III, Fasc. 1. 1901. page 85.

ture, as it presents itself in well-stained preparations of the nerve-cells of the higher vertebrates (as for example in the splendid figures of DONAGGIO). It seems that the higher is the organisation of the animal, and in consequence that of the nerve-cells, the finer and more regular is the network of the neurofibrillae in the nerve-cells. (cf. BOCHENEK).

The network of the neurofibrillae has no definite connection with the protoplasma-reticulum. In preparations, preserved in a mixture of HERMANN'S fluid and corrosive sublimate, and stained with iron-haematoxylin, the protoplasma has a very fine granular or fibrillar structure, and in the centre of many cells are shown curious diversely-shaped differentiations that remind us of the pseudochromosomes described by HEIDENHAIN, and of the rings, described in the ganglion cells of vertebrates (Teleostei, Rana). But it would take us too far, to describe these details here at some length.

2. An entirely different type of cells we find in the nerve-cells which form the large group of ganglion cells lying dorsally in the foremost part of the spinal cord just behind the brain ventricle, the so-called oblongata, extending from the niveau of the infundibular organ till beyond the first pigmented eye-cells. It is characteristic of the peculiar difficulties, with which the investigation of the histology of the nervous system of Branchiostoma is encumbered, that of the large number of authors, who have studied the subject, only JOSEPH¹⁾ two years ago gave a nearly accurate account of the structure of these cells. Even HEYMANS and VAN DER STRICHT in their very elaborate study of the histology of the nervous system of Branchiostoma, published in 1898, do not say a word about it, and only in one of the many beautiful drawings, with which their paper is illustrated, in two cells a slight indication of it is to be seen. JOSEPH says of these cells, that they present at the surface a finely striated border of minute rods, only at the side of the cell turned towards the surface of the animal, and underneath this striated border a coarsely granular darkly staining protoplasin. The same structure JOSEPH described in the cells lying close to the central canal in the spinal cord, covered by a pigment-cap, and being supposed to be light-perceiving cells. On these grounds JOSEPH put forward the suggestion, that the dorsal group of cells too consists of eye-cells, light-perceiving cells, differing only from the cells of HESSE by the absence of a pigmented cap-shaped cell.

This far-reaching suggestion is, I think, not proved, nor even made probable, by the facts. Even in the most carefully prepared sections

¹⁾ H. JOSEPH: Ueber einige Zellstructuren im Zentralnervensystem von Amphioxus Verh. d. Anatom. Gesellschaft. Jena 1904. p. 16—26.

in which the structure of both cells was very clearly to be seen, the two types still present some marked differences, both in the nuclei and in the structure of the protoplasm and the differentiations on the surface of the cells. According to JOSEPH the nuclei of the dorsal cells and of the ventral eye-cells possessed a similar granular structure, differing from that of the other nerve-cells. In some cases this is true, but in other cases the same structure is found in the nuclei of other cells, and, when we examine a number of preparations, the structure of the nuclei both of the dorsal cells, of the ventral light-perceiving cells and of the other nerve-cells presents so many differences and varieties, that there cannot be drawn any conclusion out of that. But the capital difference between the two cell-forms lies in the absence of a pigmented cap-cell in the dorsal cells and the totally different form and structure of the two types.

The light-perceiving cells of the spinal cord possess a border of short minute rods, lying close against the cap-shaped pigment cell. The processus of the dorsal cells are much longer, and not rods, but exactly shaped like hairs, or cilia. These hairs (fig. 8–11) are rather long, slender and thickly set and their course is often more or less wavy. On the same cell they seem to be all of about the same length; the hairs on different cells do not vary much in length.

The ventral light-perceiving cells are all of the same regular form. The dorsal cells however present the most different forms.¹⁾ Some are rather regular (fig. 8), some are long and slender (fig. 9), some are of a very irregular shape, but in most cases these cells, when we reconstruct them from the thin sections, appear to have a very typical cup-shape. In fig. 10 I have drawn the median section through one of these cups, in which the central hole in the cell is figured, in fig. 11 such a cup-shaped cell is cut vertically to the axis of the cup.

These cells are surrounded by a glious basket of closely interwoven fibres (in the figures this network is represented by a dark colour) and the cells seem to fill up the room left by this basket so that between the surface of the cell and the inside of the basket there remains an open space, in which the hair-like processes of the cell-surface are seen. In well-preserved sections this space has the same width on all sides of the cell, where the surface carries the hair-like structures. The hairs reach from the surface of the cell nearly to

¹⁾ Only such cells are described here, which seemed to be perfectly preserved. All those cells, of which the irregular form seemed to be caused by bad preservation, are left out of the discussion.

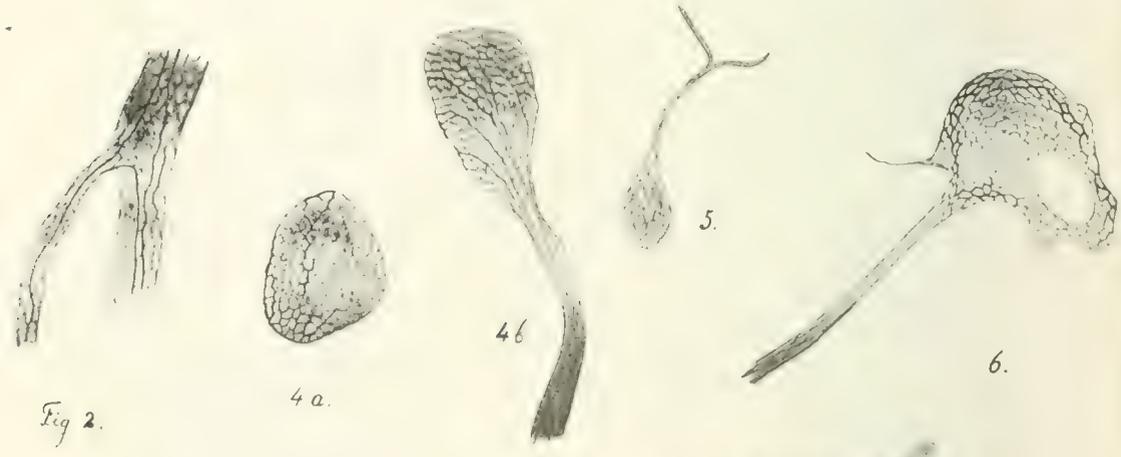
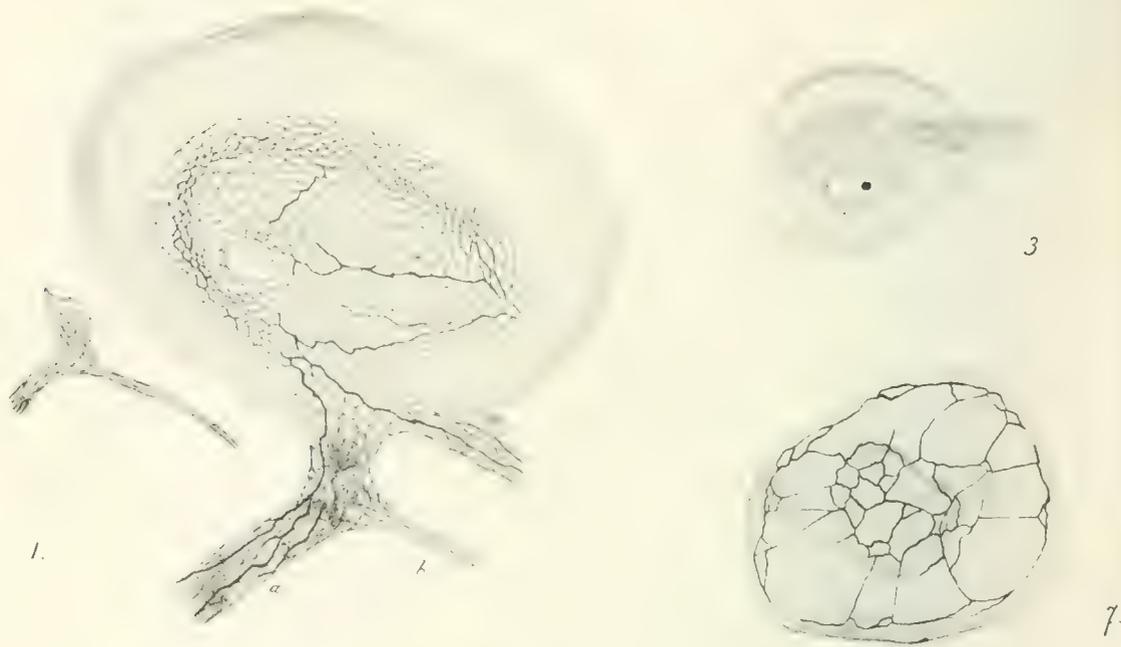
the inside of the basket, as may be seen in the figures. There where no hairs are developed, the glious fibres lie close against the surface of the cell-protoplasm (fig. 8, 10, 11).

Where JOSEPH considers the hair-like processes to be only present at that side of the cell which is turned towards the surface of the animal, I cannot agree with him. When we compare horizontal and transverse sections carefully with each other, we must draw the conclusion, that the hairs may be developed on all sides of the cell, except there where the cell-body sends a dendritical process through the glious basket. Even there where the cell is shaped like a cup or calix, at both sides of the cup the cilia may be present (figg. 10, 11). The cilia at the inside of the cup are separated from each other by an ingrowth of the fibres of the glious basket (fig. 10). When the cell is cut at right angles to the axis of the calix, this may lead to the appearance of a ring of protoplasm, at both sides covered with the hairs, and surrounding a mass of coiled-up fibres, the ingrowing fibres of the glious basket (fig. 11).

The protoplasm of these cells shows a regular network of neurofibrillae, which differs from the network of the other cells of the spinal cord by its much larger meshes (compare figg. 8—10 with figg. 1—6); only at the periphery of the cell, under the hair-like processes, a finer network of neurofibrillae is to be seen (fig. 8). The hairs themselves seem to be implanted on a layer of small darkly staining granules or small rods, of which the definite structure is difficult to be seen. In many cases it is only represented by a somewhat more coarsely granular layer of protoplasm there where the cell-body is covered with the hairs.

All these things seem to point to the conclusion that these cells do not possess a light-perceiving function, as suggested by JOSEPH. The shape of the cells and the peculiar structure at least are not favourable to the hypothesis. But it is sure, that this group of cells, all presenting the same peculiar structure, has a distinct and peculiar function. The structure of the cells reminds us in the first place of a static organ, and especially cells as drawn in fig. 9 and fig. 11, seem to suggest such a function. The peculiar baskets of fibres surrounding the cells remind us of the cells of PURKINJE of the brain of the craniotes, but bearing in mind the very little we know about these cells and about the cells just described, it is more advisable to stop at these general suggestions and not to try to go more into details. The suggestion of JOSEPH at all events seems to me to be untenable.

A third type of cells differing from those which I described here, is that of the cells of the so-called infundibular organ in the ventral



wall of the brain-vesicle. These cells I mean to describe in my second paper.

Leiden, 25 April '07.

DESCRIPTION OF FIGURES ON THE PLATE.

All the figures are magnified 1600 times, and are drawn with a camera lucida of ABBE directly after the preparations. Apochrom. oil-immersion lens of ZEISS and compens-ocular No. 8.

- Fig. 1. Colossal nerve-cell with neurofibrillar network, of a Branchiostoma of 4.8 cM. in length (BIELSCHOWSKY—POLLACK's method).
- 2. Dendrites of a similar cell of an animal of 5 cM. in length (same method).
 - 3. Neurofibrillar network of a colossal nerve-cell of a Branchiostoma of 6 mM. in length.
 - 4 *a* and *b*. Sections of a medium-sized nerve-cell of the same spinal cord as fig. 2.
 - 5. Section of a very small nerve-cell, with neurofibrillar network.
 - 6. The same as in fig. 4.
 - 7. Section of a sensory cell of Pontobdella, of 10 μ , treated after the gold-method of APATHY.
 - 8—11. Sections through different cells of the dorsal group of cells lying behind the brain-vesicle, taken from preparations of several adult specimens of Branchiostoma. In fig. 8 some of the adjoining cells are drawn, to demonstrate the similarity of structure of the nuclei in the two cell types.
- In fig. 10 and fig. 11 are drawn two typical sections through cup-shaped cells of the dorsal group of cells. The body contained in the centre of the cell of fig. 11 is the prolongation of the glious basket surrounding the cell. Compare fig. 10.

Physiology. — "*On a third heart sound*". By W. EINTHOVEN, in collaboration with Messrs. J. H. WIERINGA and E. P. SNIJDERS, assistants at the physiological laboratory at Leyden.

When continuing the investigation of the heart sounds by means of the string galvanometer¹⁾, we noticed that in some cardiophonograms, especially with the apex sounds of *Wi*, recorded in February last, shortly after the vibrations of the second sound still another vibration was present, which admitted of no other interpretation than by regarding it as a third heart sound.

We could not at once explain how this third sound was produced, and we put off the closer investigation of this phenomenon, however

¹⁾ See: Die Registrirung der menschlichen Herztöne mittels des Saitengalvanometers. PFLÜGER's Arch. f. d. gesammte Physiol. Vol. 117, p. 461, 1907.

interesting it seemed to us, since for the present our time was taken up by other work.

A couple of months afterwards Dr. A. G. GIBSON of Oxford — to whom our former publications on the recording of heart sounds were known, but who could not be acquainted with our later observations — asked whether in our collection of cardiophonograms of normal persons there were any in which an extra sound was visible in the diastolic phase. GIBSON occupied himself with an investigation of the venous pulse¹⁾ and had noticed that with some persons, without a morbid affection of the heart, a low pitched sound could be heard at the apex during the cardiac pause, something like a distant 2nd sound, but feebler and much lower in pitch. The sound is clear and nothing like a murmur. This particular sound is of varying intensity being louder during the interval between the end of an expiration and the beginning of the subsequent inspiration.

We hope elsewhere to publish in a more extensive paper the cardiophonograms we obtained; here we shall only deal briefly with them. When we try to predict from the shape and dimensions of the curves what impression the third heart sound must make on the ear of the observer, we cannot describe it otherwise than GIBSON did: a distant diastolic sound of low pitch and clear tone, varying in intensity, but always feeble.

There can be no doubt that the sound, heard by GIBSON at Oxford, is the same sound we recorded at Leyden.

The measurements made with some cardiophonograms, show that with *W* i the beginning of the third sound falls on the average 0.13 sec. (varying between 0.11 and 0.15 sec.) after the beginning of the second sound and on the average 0.32 sec. before the beginning of the following first sound. In the same curves the duration of the first sound is about 0.08 sec., of the second about 0.05 sec., of the third 0.02 to 0.03 sec. The first two sounds are murmurs, composed of tones of irregular pitch. The mutual distance of some tops in the curves shows that we have here tones of more than a hundred double vibrations per second, whereas the third sound seems to be built up of but one double vibration, the period of which amounts to about 0.02 sec.

The intensity of the third sound varies. While in some cardiac beats it is entirely absent, the amplitude of its vibrations reaches in other beats $\frac{1}{7}$ of that of the first and second sounds. Putting the

¹⁾ GIBSON'S investigation will shortly be published in "The Lancet" under the title: "The significance of a hitherto undescribed wave in the jugular pulse".

ratio of the amplitudes of the first or second sound to that of the third $a = 7$, and the ratio of the frequencies $b = 2$, the ratio of the intensities is $a^2 b^2 = 196$. Hence the third sound is at its maximum still about 200 times feebler than the first or second.

While the above given figures refer to the objective intensities, a comparison of the intensities of perception is still much less in favour of the third sound, since a tone of frequency 50 per second has objectively to be a little over a hundred times stronger¹⁾ than a tone of 100 vibrations a second, in order to produce an equally strong auditory impression. Consequently, if the third sound attains such an intensity that it is just audible still, the first and second sounds may be 20.000 times weakened, before also the auditory impression they produce, vanishes.

This explains the difficulty of the investigation by the method of auscultation. GIBSON²⁾ emphasises this particularly and says that in order to hear the sound, accidental sounds must be excluded as much as possible, while one has to strain one's attention during the interval in which the sound occurs. Although the cardiophonograms leave no doubt as to the existence of the third heart sound with W_i , we have been unable to hear it by means of a stethoscope.

Regarding the explanation of the third sound we refer to the above mentioned more extensive paper which will shortly be published elsewhere. Here we will only state our conclusion that the sound cannot be put on a line with a prae-systolic murmur of the mitral valve, nor with a duplication of the second sound by non-simultaneous action of the aortal and pulmonal valves, but that it is probably caused by a second vibration of the valvulae semilunares aortae and must be regarded as a phenomenon of pretty common occurrence.

Astronomy. — *“On some points in the theory of Jupiter's satellites.”*

By DR. W. DE SITTER. (Communicated by DR. E. F. VAN DE SANDE BAKHUYZEN).

The following pages contain a short account of some investigations, which will soon be published, together with other results, in N^o. 17 of the publications of the astronomical laboratory at Groningen.

A few words are necessary in explanation of the notations em-

¹⁾ Calculated according to MAX WIEN, PFLÜGER'S Arch. f. d. gesammte Physiol. Bd. 97. p. 1. 1903. H. ZWAARDEMAKER and F. H. QUIX give in ENGELMANN'S Arch. f. Physiol. p. 25. 1904, differences in the same sense, but of a different order of magnitude.

²⁾ l. c.

ployed. The notations used by different writers on the theory of the satellites are discordant in a most regrettable manner. The tables, both those of DAMOISEAU and of DELAMBRE, distinguish the four satellites by the numbers 1, 2, 3, 4. This example is followed by MARTH, and I have also in all my previous work on the satellites used this notation, as is also done by Mr. COOKSON in the discussion of his observations. The theoretical writers, on the other hand, LAPLACE, TISSERAND, SOULLART use the suffixes 0, 1, 2, 3 or a corresponding number of accents. Another fundamental difference is in the designation of the perijoves. The letter $\bar{\omega}$ in the writings of DAMOISEAU, MARTH, COOKSON and myself represents the "own" perijove; SOULLART and TISSERAND use it for the osculating perijove. There are many more differences of this kind, which need not be enumerated here. Though thoroughly convinced of the great importance of a consistent notation, I am, reluctantly, compelled in this communication to depart from the notations employed by me elsewhere. In the first article of the present communication, which treats of a theoretical point, I have, to avoid the writing out at length of many well known formulas and results, closely followed TISSERAND's very clear argument in the fourth volume of his *Traité de Mécanique Céleste*. Accordingly in this first article, I will adopt TISSERAND's notation, with one exception. In the further articles I will return to the notation employed in my previous work.

1. *Theory of the libration.* As has been explained, the notations employed are TISSERAND's excepting the mean longitudes, which I denote by l_1, l_2, l_3 instead of by l, l', l'' . In addition to the quantities F, F', G, G' defined by (19) page 11¹⁾ I wish to introduce

$$G_1 = \frac{4a}{a'} - 3a' A^{(1)} - a a' \frac{\partial A^{(1)}}{\partial a}$$

$$G_1' = \frac{4a'}{a''} - 3a'' A'^{(1)} - a' a'' \frac{\partial A'^{(1)}}{\partial a'}$$

TISSERAND assumes $G_1 = G$ and $G_1' = G'$, which is only approximately true. If it is not desired to introduce this approximation, then on page 11, formula (20) we must in R_1 replace G by G_1 and similarly in R_1', G' by G_1' .

The only further difference from TISSERAND's notation is in the definition of the libration. I put

¹⁾ The references of pages and formulas are to those of TISSERAND, volume IV.

$$\vartheta = l_1 - 3l_2 + 2l_3 + 180^\circ, \dots \dots \dots [1]$$

TISSERAND, however, has

$$\vartheta = l - 3l' + 2l''.$$

The angle ϑ , as defined by [1] is the angle to which the name *libration* was first applied by LAPLACE, and which is by him called ω . (*Mécanique Céleste*, Livre VIII, art. 15, *Oeuvres*, tome IV pages 75 and 79 of the edition of 1845).

The differential equation determining the libration is

$$\frac{d^2\vartheta}{dt^2} = -\beta^2 \sin \vartheta \dots \dots \dots [2]$$

This equation is derived by the combination of the three equations

$$\left. \begin{aligned} \frac{d^2l_1}{dt^2} &= -Q_1 \sin \vartheta \\ \frac{d^2l_2}{dt^2} &= -Q_2 \sin \vartheta \\ \frac{d^2l_3}{dt^2} &= -Q_3 \sin \vartheta \end{aligned} \right\} \dots \dots \dots [3]$$

We have thus

$$\beta^2 = Q_1 - 3Q_2 + 2Q_3 \dots \dots \dots [4]$$

From these equations the whole theory of the libration is derived in the well known manner, on which, however, I will not dwell, my sole object being at present the determination of the quantities Q_1 , Q_2 and Q_3 .

For that purpose we start from the formulas given by TISSERAND at the top of page 20, which must however be completed as follows:

$$\frac{d^2\varrho}{dt^2} = -\frac{3}{a^2} \left(\frac{\partial R_1}{\partial \varepsilon} + \frac{\partial R_4}{\partial \varepsilon} \right) \dots \dots \dots [5]$$

and two similar equations for ϱ' and ϱ'' .

Introducing the same auxiliary angles u and u' that are used by TISSERAND (formula (12) page 20), we get instead of TISSERAND's equations (B):

$$\begin{aligned} \frac{d^2 Q}{dt^2} = & \frac{3}{2} m' n^2 \left[F(k \sin u - h \cos u) + \frac{a}{a'} G_1(k' \sin u - h' \cos u) \right] \\ & - 3n \left[a_{0,1} \left(\{k^2 - h^2\} \sin 2u - 2kh \cos 2u \right) \right. \\ & \quad \left. + \frac{m' \sqrt{a'}}{m \sqrt{a}} a_{1,0} \left(\{k'^2 - h'^2\} \sin 2u - 2h'k' \cos 2u \right) \right. \\ & \quad \left. - 2b_{0,1} \left(\{kk' - hh'\} \sin 2u - \{kh' + hk'\} \cos 2u \right) \right]. \end{aligned}$$

$$\begin{aligned} \frac{d^2 Q'}{dt^2} = & - 3mn'^2 \left[G(k' \sin u - h' \cos u) + \frac{a'}{a} F(k \sin u - h \cos u) \right] \\ & + \frac{3}{2} m'' n'^2 \left[F'(k' \sin u' - h' \cos u') + \frac{a'}{a''} G'_1(k'' \sin u' - h'' \cos u') \right] \\ & + 6n' \left[a_{1,0} \left(\{k'^2 - h'^2\} \sin 2u - 2k'h' \cos 2u \right) \right. \\ & \quad \left. + \frac{m \sqrt{a}}{m' \sqrt{a'}} a_{0,1} \left(\{k^2 - h^2\} \sin 2u - 2kh \cos 2u \right) \right. \\ & \quad \left. - 2b_{1,0} \left(\{kk' - hh'\} \sin 2u - \{kh' + hk'\} \cos 2u \right) \right] \\ & - 3n' \left[a_{1,2} \left(\{k'^2 - h'^2\} \sin 2u' - 2k'h' \cos 2u' \right) \right. \\ & \quad \left. + \frac{m'' \sqrt{a''}}{m' \sqrt{a'}} a_{2,1} \left(\{k''^2 - h''^2\} \sin 2u' - 2k''h'' \cos 2u' \right) \right. \\ & \quad \left. - 2b_{1,2} \left(\{k'k'' - h'h''\} \sin 2u' - \{k'h'' + h'k''\} \cos 2u' \right) \right]. \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{d^2 Q''}{dt^2} = & - 3m'n''^2 \left[G''(k'' \sin u' - h'' \cos u') + \frac{a''}{a'} F'(k' \sin u' - h' \cos u') \right] \\ & + 6n'' \left[a_{2,1} \left(\{k''^2 - h''^2\} \sin 2u' - 2k''h'' \cos 2u' \right) \right. \\ & \quad \left. + \frac{m' \sqrt{a'}}{m'' \sqrt{a''}} a_{1,2} \left(\{k'^2 - h'^2\} \sin 2u' - 2k'h' \cos 2u' \right) \right. \\ & \quad \left. - 2b_{2,1} \left(\{k'k'' - h'h''\} \sin 2u' - \{k'h'' + h'k''\} \cos 2u' \right) \right]. \end{aligned}$$

To derive from these the formulas [3] we must for $h, k, h' \dots$ substitute the values

$$\begin{aligned} h &= B \sin u + B_1 \sin u' \\ k &= B \cos u + B_1 \cos u', \end{aligned} \quad \text{etc.} \tag{7}$$

which are given by TISSERAND at the bottom of page 21. In the

result we then reject all terms which do not contain the argument

$$u' - u = \vartheta + 180^\circ,$$

or its multiples. We thus find easily

$$\begin{aligned} \frac{d^2 Q}{dt^2} &= \frac{3}{2} m'n^2 \left[FB_1 + \frac{a}{a'} G_1 B_1' \right] \sin(u-u') \\ &\quad - 3n \left[a_{0,1} B_1^2 + \frac{m'\sqrt{a'}}{m\sqrt{a}} a_{1,0} B_1'^2 - 2b_{0,1} B_1 B_1' \right] \sin 2(u-u') \\ &\quad - 6n \left[a_{0,1} B B_1 + \frac{m'\sqrt{a'}}{m\sqrt{a}} a_{1,0} B' B_1' - b_{0,1} (B B_1' + B_1 B') \right] \sin(u-u') \\ \frac{d^2 Q'}{dt^2} &= -3mn'^2 \left[G B_1' + \frac{a'}{a} F B_1 \right] \sin(u-u') \\ &\quad + \frac{3}{2} m''n'^2 \left[F' B' + \frac{a'}{a''} G_1' B'' \right] \sin(u'-u) \\ &\quad + 6n' \left[a_{1,0} B_1'^2 + \frac{m\sqrt{a}}{m'\sqrt{a'}} a_{0,1} B_1^2 - 2b_{1,0} B_1 B_1' \right] \sin 2(u-u') \\ &\quad + 12n' \left[a_{1,0} B' B_1' + \frac{m\sqrt{a}}{m'\sqrt{a'}} a_{0,1} B B_1 - b_{1,0} (B B_1' + B_1 B') \right] \sin(u-u') \\ &\quad - 3n' \left[a_{1,2} B^2 + \frac{m''\sqrt{a''}}{m'\sqrt{a'}} a_{2,1} B''^2 - 2b_{1,2} B' B'' \right] \sin 2(u'-u) \\ &\quad - 6n' \left[a_{1,2} B' B_1'' + \frac{m''\sqrt{a''}}{m'\sqrt{a'}} a_{2,1} B'' B_1'' - b_{1,2} (B' B_1'' + B_1' B'') \right] \sin(u'-u) \\ \frac{d^2 Q''}{dt^2} &= -3m'n''^2 \left[G' B'' + \frac{a''}{a'} F' B' \right] \sin(u'-u) \\ &\quad + 6n'' \left[a_{2,1} B''^2 + \frac{m'\sqrt{a'}}{m''\sqrt{a''}} a_{1,2} B^2 - 2b_{2,1} B' B'' \right] \sin 2(u'-u) \\ &\quad + 12n'' \left[a_{2,1} B'' B_1'' + \frac{m'\sqrt{a'}}{m''\sqrt{a''}} a_{1,2} B' B_1' - b_{2,1} (B' B_1'' + B_1' B'') \right] \sin(u'-u) \end{aligned} \quad [8]$$

We now put

$$\begin{aligned} \sin(u-u') &= \sin \vartheta \\ \sin 2(u-u') &= -2 \sin \vartheta. \end{aligned}$$

Further we introduce the approximate values of B, B', \dots which TISSERAND gives in the middle of page 22, viz.:

$$B = m CG \quad B_1' = m'' C F'' \quad B_1 = B'' = 0, \dots [9]$$

where C is a constant, the value of which is indifferent to our argument, and can easily be derived by comparison with TISSERAND,

We then neglect the squares and products of $B, B' \dots$, and also the difference of G_1 and G , and we put

$$n^2 a^3 = n'^2 a'^3 = n''^2 a''^3 = f, \quad [10]$$

which also is only approximately true, and

$$- \frac{3}{2} \frac{f}{a'} C F' G = K,$$

Introducing all these simplifications we find the equations (22) of TISSERAND, viz.:

$$\frac{d^2 l_1}{dt^2} = - \frac{m' m''}{a^2} K \sin \vartheta$$

$$\frac{d^2 l_2}{dt^2} = 3 \frac{m m''}{a'^2} K \sin \vartheta$$

$$\frac{d^2 l_3}{dt^2} = - 2 \frac{m m'}{a''^2} K \sin \vartheta,$$

In comparing these with TISSERAND it must not be forgotten that our ϑ differs 180° from TISSERAND'S. We have thus, if all the above mentioned approximations are introduced

$$Q_1 = \frac{m' m''}{a^2} K, \quad Q_2 = - 3 \frac{m m''}{a'^2} K, \quad Q_3 = 2 \frac{m m'}{a''^2} K. \quad [11]$$

The values [9], however, are only approximately true; they contain only the perturbations of the first order in the masses. Nevertheless the deviations of the values of Q_i from the truth caused by the adoption of these approximate values, and similarly by [10] and by the neglect of difference of G and G_1 , are not of a serious nature. The neglect of the terms of the second degree in $B, B' \dots$ on the other hand, is very serious.

Now discarding all these simplifications, with the exception of $B_1 = B'' = 0$, which we continue to adopt, we find for the complete values of Q_1, Q_2, Q_3 :

$$\left. \begin{aligned} Q_1 &= - \frac{3}{2} m' n^2 \frac{a}{a'} G_1 B_1' - 6n \left[\frac{m' \sqrt{a'}}{m \sqrt{a}} a_{1,0} (B_1'^2 - B' B_1') + b_{0,1} B B_1' \right] \\ Q_2 &= + 3 m n'^2 G B_1' + \frac{3}{2} m'' n'^2 F' B' + \\ &\quad + 12n' [a_{1,0} (B_1'^2 - B' B_1') + b_{1,0} B B_1'] + \\ &\quad + 6n' [a_{1,2} (B'^2 - B' B_1') + b_{1,2} B' B_1''] \\ Q_3 &= - 3 m' n''^2 \frac{a''}{a'} F' B' - 12n'' \left[\frac{m' \sqrt{a'}}{m'' \sqrt{a''}} a_{1,2} (B'^2 - B' B_1') + b_{2,1} B' B_1'' \right]. \end{aligned} \right\} [12]$$

Using the numerical data adopted by SOUILLART, and putting

$$m_1 = 10000 m, \quad m_2 = 10000 m', \quad m_3 = 10000 m''.$$

we find from formula [11]

$$Q_1 = + 0.03201 m_2 m_3$$

$$Q_2 = - 0.03794 m_1 m_3$$

$$Q_3 = + 0.00994 m_1 m_2,$$

From the formulas [12], on the other hand, we have:

$$Q_1 = \{ + 0.03009 - 0.00460 m_1 - 0.01156 m_2 - 0.00958 m_3 \} m_2 m_3 = \\ = + 0.01815 m_2 m_3$$

$$Q_2 = \{ - 0.03436 + 0.00389 m_1 + 0.00933 m_2 + 0.00809 m_3 \} m_1 m_3 = \\ = - 0.02438 m_1 m_3$$

$$Q_3 = \{ + 0.00794 - 0.00020 m_1 - 0.00016 m_2 - 0.00042 m_3 \} m_1 m_2 = \\ = + 0.00751 m_1 m_2.$$

The numerical coefficients depend almost exclusively on the ratios of the major axes, i.e. on the mean motions, and they can be taken as correct to the last figure given.

The corresponding periods, computed by the formula

$$T = \frac{2\pi}{\beta},$$

are, expressed in years:

from formula [11] $T = 6.318$

from formula [12] $T = 7.985,$

The difference is considerable.

The question naturally arises: why have these important terms of the second degree been overlooked by LAPLACE and SOUILLART? For LAPLACE, the answer is very simple: he has neglected the part R_4 of the perturbing function throughout. For SOUILLART it is different. It is one of SOUILLART's great merits to have discovered the importance of this same part of the perturbing function, especially for the determination of the quantities $B, B' \dots$ The corrections which have been added by SOUILLART on this account to these coefficients, amount to a considerable part of the whole. Also SOUILLART evidently intended to find the expression for the period of the libration as completely as possible. On the pages 46 and 47 (Memoirs of the Royal Astronomical Society, Vol. XLV) he considers the different parts of the perturbing function, which can in the differential coefficients of the mean longitudes introduce the argument $l_1 - 3l_2 + 2l_3$. He, however, rejects them all, as giving negligible coefficients, and retains only the terms which had already been discovered by LAPLACE. Among the rejected terms are also the new terms treated above, which are discarded by SOUILLART on the ground that they are of the second degree in the eccentricities (page 47, bottom). He here overlooks

that in these terms, for the same reason as in those of the first degree, the excentricities must be replaced by their perturbations with the arguments u and u' , in order to find the terms determining the libration. These terms thus are of the second degree, not in the excentricities, but in the quantities $B, B' \dots$ and of *these* the squares are not negligible, as we have seen.

The question further arises: do not the terms of the third degree in the excentricities, i. e. those of the types

$$\begin{aligned} P e^3 \cos (2 l' - l - \tilde{\omega}), & \quad Q e^2 e' \cos (2 l' - l - 2 \tilde{\omega} + \tilde{\omega}'), \\ R e^3 \cos (6 l' - 3 l - 3 \tilde{\omega}), & \quad S e^3 \cos (4 l' - l - 3 \tilde{\omega}), \text{ etc.} \end{aligned}$$

also contribute appreciably towards the coefficients Q_i ? To find the answer to this question I have computed all the terms of this kind in Q_1 . These terms of the third degree, which are of the fourth order in the masses, are:

$$\begin{aligned} \delta Q_1 = \{ & + \cdot 00012 m_1^2 + \cdot 00079 m_2^2 + 00034 m_3^2 + \cdot 00061 m_1 m_2 + \\ & + \cdot 00050 m_1 m_3 + \cdot 00124 m_1 m_3 \} m_2 m_3 = + \cdot 00071 m_2 m_3. \end{aligned}$$

They are thus not wholly negligible. I have, however, not carried out the computation — which is rather complicated — for Q_2 and Q_3 , nor have I computed the terms of the fourth degree (i. e. of the fifth order in the masses). The development of the period T in powers of the masses evidently converges very slowly, and the period computed by the formulas [12] may very well be erroneous by a few tenths of a year.

2. *The equations of the centre.* The large inequalities, which in the integration by the method of variation of elements appear as perturbations of the excentricities and perijoves (formula [7] above), are in practice added to the longitudes and radii-vectores, and the excentricities and perijoves are conceived to be affected by their secular, but *not* by their periodic perturbations. I now return to the notations used in all my other work on the satellites, and I denote the excentricities and perijoves, defined in this way, by E_i and Ω_i . We have then ¹⁾

$$\left. \begin{aligned} h_i &= 2 E_i \sin \Omega_i = 2 \sum_j r_{ij} e_j \sin \tilde{\omega}_j \\ k_i &= 2 E_i \cos \Omega_i = 2 \sum_j r_{ij} e_j \cos \tilde{\omega}_j \end{aligned} \right\} \dots \dots [13]$$

The sums extend over the values of j from 1 to 4; e_i and $\tilde{\omega}_i$ are the "own" excentricities and perijoves of LAPLACE, the values of e_i are constant and $\tilde{\omega}_i$ are linear functions of the time. Further

¹⁾ These h_i and k_i are thus *not* the same quantities as those denoted by $h, k, h' \dots$ by TISSERAND.

$\tau_{ii} = 1$, the other ratios τ_{ij} , and the motions $\frac{d\tilde{\omega}_i}{dt}$ depending on the masses. Thus if certain values of the masses are adopted, the ratios τ_{ij} are thereby determined. If then h_i and k_i of the four satellites are known from the observations, then from the eight linear equations [13] (consisting of two sets of four each, with the same coefficients) we can determine the eight unknowns $e_i \sin \tilde{\omega}_i$ and $e_i \cos \tilde{\omega}_i$, and from these again e_i and $\tilde{\omega}_i$. The method is exactly the same as the one used by me for the determination of the inclinations and nodes (see these Proceedings, 1906 March, pages 767—780). The values of h_i and k_i have been determined from the heliometer-observations made at the Cape Observatory, in 1891 by Sir DAVID GILL, and in 1901 and 1902 by Mr. BRYAN COOKSON. The results from these observations have been treated by the method just delineated, in two different suppositions regarding the masses, i. e. regarding the ratios

Satellite	Epoch	e			$\tilde{\omega}$			$\tilde{\omega}_{1900^0}$	
		System I	System II	$p.e.$	System I	System II	$p.e.$	System I	System II
I	1891·75	0°036	0°036	±°009	158°	157°	± 15°	248°	235°
	1901·61	·055	·055	± 22	136	136	± 36	48	50
	1902·60	·022	·021	± 17	262	270	± 27	120	131
II	1891·75	0·018	0·020	±·006	169	166	± 16	300	274
	1901·61	·020	·019	± 14	318	315	± 37	292	294
	1902·60	·026	·026	± 9	302	301	± 24	261	267
	Mean	0·021	0·022					284	278
III	1891·75	0·086	0·086	±·003	179·7	179·6	± 2·0	201·4	200·4
	1901·61	·100	·101	± 9	198·2	198·1	± 5·6	193·9	193·8
	1902·60	·080	·080	± 6	219·0	218·8	± 4·0	212·2	212·3
	Mean	0·089	0·089					202·5	202·2
IV	1891·75	0·4284	0·4280	±·0015	142·28	142·29	± 0·20	148·19	147·83
	1901·61	·4228	·4216	± 30	148·92	149·05	± 40	147·76	147·96
	1902·60	·4261	·4262	± 25	149·06	149·03	± 34	147·20	147·28
	Mean	0·4258	0·4253					147·72	147·69

τ_{ij} and the motions $\frac{d\tilde{\omega}_i}{dt}$. The results are collected in the following table. The values of $\tilde{\omega}_i$ for 1900.0, given in the last two columns, have been derived from those for the individual epochs for each system separately by means of the motions $\frac{d\tilde{\omega}_i}{dt}$ corresponding to the assumed masses. The perijoves are counted from the assumed vernal equinox of Jupiter, whose longitude in 1900.0 is $135^\circ.45$.

The values of these elements, on which SOUILLART's theory is based, are:

(1900.0)	
$e_1 = 0.001$	$\tilde{\omega}_1 = 305^\circ$
$e_2 = 0.006$	$\tilde{\omega}_2 = 177$
$e_3 = 0.064$	$\tilde{\omega}_3 = 206.1$
$e_4 = 0.4160$	$\tilde{\omega}_4 = 152.69$

The results from the two systems are practically identical. The corrections to SOUILLART's values for the satellites II, III and IV, are considerable, and on the whole much larger than the deviations of the three epochs *inter se*. These corrections are thus undoubtedly real. The most remarkable of them is certainly the large own eccentricity of II. The value of this element, assumed by DELAMBRE and DAMOISEAU is zero. The value used by SOUILLART in his theory is a pure arithmetical result, and has no weight whatever as a determination of the element. DAMOISEAU, however, has suspected the existence of an eccentricity of practically the same amount as is found here. This is shown by the following quotation from his unpublished memoir, written in explanation of the construction of his tables, which I quote after SOUILLART¹). DAMOISEAU says there: "Nous avons des motifs de soupçonner dans l'orbite du second satellite une équation du centre propre de 32^s en temps synodique (ce qui correspondrait à une excentricité propre de 0.00032738), mais notre incertitude sur la position du périjove, dont le mouvement est encore à calculer par la théorie, nous a fait remettre cette recherche à un autre temps." This eccentricity, expressed in arc is $0^\circ.0188$, and it is therefore practically the same as the value found by me. The reason adduced by DAMOISEAU for not using it in his tables sounds somewhat strange: as a matter of fact the motion of the perijove had been determined long ago by LAPLACE.

With regard to Satellite I it is clear that the apparent equations

¹) *Mémoires des Savants étrangers*, tome XXX, page 28.

of the centre derived from the observations — which moreover are only little larger than their probable errors — do not represent a true excentricity. It is not impossible that they are produced by the existence of surface markings on the disc of the satellite, causing the centre of light, which is observed by the heliometer, to be displaced relatively to the centre of gravity, the displacement being different at different epochs. Any attempt to explain the observed h_i and k_i on this hypothesis would, however, necessarily involve so many undeterminate quantities, that its success would be no proof of its representing a true fact of nature.

3. *Determination of the libration from the observations.*

In a communication made by me in 1905 to the “Nederlandsch Natuur- en Geneeskundig Congres”,¹⁾ I have shown :

that the libration probably has an appreciable coefficient,

that the determination from the observations, not only of the phase and amplitude, but also of the period of the libration, is of the highest importance for the derivation of the masses, especially of the mass of Satellite I,

that this determination is possible from the observations made at the observatories at the Cape, Helsingfors and Pulkowa,

that most probably the period differs considerably from the value adopted by LAPLACE and SOULLIART, and

that this determination is intricately connected with an investigation of the long-periodic inequalities in the longitudes of the satellites, and that consequently the whole problem can only be solved by successive approximations.

In number 17 of the Publications of the Astronomical Laboratory at Groningen, which will soon be published, all these conclusions are confirmed and the successive approximations are carried out. In this communication I cannot dwell upon the details of this investigation, nor upon the difficulties which were encountered. I must confine myself to a brief statement of the results.

The observations used are the heliometer-observations of the Cape Observatory already quoted above, and further photographic plates taken at Helsingfors in the years 1892—93, 1893—94, 1894—95, 1895—96 and 1897, at Pulkowa in 1895—96, 1897 and 1898, and at the Cape in 1904. I thus had at my disposition ten oppositions

¹⁾ “*Over de libratie der drie binnenste grootte satellieten van Jupiter en eene nieuwe methode ter bepaling van de massa van Satelliet I.*” Handelingen van het 10de Congres, pages 125—128.

in all. For each of these corrections Δl_i to the assumed longitudes of the satellites were derived. These direct results from the observations can, however, not be used as they stand. There are, as has been mentioned above, in the longitude of each satellite four inequalities, whose periods are between 400 and 500 days, and whose coefficients are of the same order of magnitude as the libration. These inequalities therefore, during the few months over which each of the ten series of observations extends, are practically constant, and the correction Δl_i derived from the observations consequently contains, in addition to the correction $\Delta \varepsilon_i$ to the mean longitude, and the libration, also the correction to the assumed values of these inequalities.

Now the coefficients of these inequalities are proportional to the eccentricities and depend on the masses, and are therefore uncertain to the same extent as these, i.e. to a very large extent. The periods of the four inequalities are so nearly equal, that they cannot be separated from each other. Further the period of the most important of them — important both by its magnitude and by its uncertainty — differs just so much from the average interval of one opposition to the next that, when we consider only the values at the epochs of opposition, the inequality presents itself as one having approximately the period of the libration, and can therefore not be separated from the libration itself. For all these reasons it was impossible to determine the libration *and* the long-periodic inequalities from these observations alone.

For the determination of the masses, leaving for the moment the mass of IV out of consideration, we have the following data:

1. the large inequalities in the longitudes of the satellites I, II and III,
2. the motion of the perijove of satellite IV,
3. the period of the libration.

The motion of the perijove of IV also depends on the compression of the planet, which must thus also be investigated, and is determined by

4. the motion of the node of satellite II.

The data mentioned under 1, 2 and 4 are those used by LAPLACE, 3 has for the first time been pointed out by me in the communication to the "Nederlandsch Natuur- en Geneeskundig Congres", quoted above.

The method by which the approximations have been conducted is the following. Certain values of the masses, approximately verifying the conditions 1, 2, and 4, are assumed, and the corresponding

values of the long-periodic inequalities are computed. Let these be $\delta l_i'$, and let δl_i^0 be the values used in computing the tabular places which were compared with the observations. Then evidently the correction to the mean longitude corresponding to the assumed masses (and equations of the centre) is

$$\Delta l_i' = \Delta l_i - (\delta l_i' - \delta l_i^0).$$

From these $\Delta l_i'$ we then determine the amplitude, the phase and the period of the libration. If this period co-incides with the one computed from the assumed masses, then the approximation is sufficient, if not, then the whole process is repeated with different masses.

The communication of the different approximations and of the residuals remaining after the substitution of the finally adopted values, would exceed the limits set to this paper. The formula finally derived for the libration is

$$\vartheta = 0^\circ.158 \sin \frac{t - 1895.09}{7.0}.$$

The adopted masses are

$$m_1 = 0.0000 \ 256$$

$$m_2 = 0.0000 \ 231$$

$$m_3 = 0.0000 \ 820$$

and the corresponding ratio of the distribution of the libration over the longitudes of the three satellites is given by

$$\frac{\vartheta_1}{\vartheta} = + 0.175 \qquad \frac{\vartheta_2}{\vartheta} = - 0.260 \qquad \frac{\vartheta_3}{\vartheta} = + 0.0225$$

The mean longitudes (excluding libration) on 1900 January 0, Greenwich mean noon, are (counted from the point Aries)

$$l_1 = 142^\circ.604$$

$$l_2 = 99.534$$

$$l_3 = 167.999$$

$$l_4 = 234.372.$$

By a comparison of these with the values at the epoch 1750.0 the following sidereal mean daily motions¹⁾ were derived

$$n_1 = 203^\circ.4889 \ 5652$$

$$n_2 = 101.3747 \ 2411$$

$$n_3 = 50.3176 \ 0790$$

$$n_4 = 21.5710 \ 7132.$$

I have added no probable errors, which in the absence of the details of the observational material can only have a subjective value.

¹⁾ i.e. sidereal mean motions in a mean solar day.

Geology. — *“Considerations on the Starvingian “Zanddiluvium””*.

By P. TESCH. (Communicated by Prof. K. MARTIN).

The sandy areas form a great part of the Dutch land. When from the so-called diluvial half of our country the gravelous diluvium with the boulder-clay, the alluvial moors and river deposits and the regions where the wind has influenced the bottom, are subtracted, the districts are resting where the surface consists of sand without or with little gravel and which are gathered by STARING under the collective name of “zanddiluvium”. These districts have in the different parts of our country a different appearance and a different fertility. Therefore the neutral name of “zanddiluvium” has been chosen by STARING for a very good reason, in which name the origin rests undecided by this name. Yet he speaks on the pages 114 to 121 of the second volume of his “Bodem van Nederland” about the origin of these sand deposits as follows:

“Evidently it has been formed in the last part of the diluvial period or in the very first part of the alluvial period; for everywhere where it is found, it rests upon the gravelous diluvium and is covered by the alluvial beds.”

“This form (the horizontal position of the composing strata) connected with its position upon the gravelous diluvium and at the foot of the hills formed by this, permits to decide, with great probability, about the presumptive origin of the “zanddiluvium”. The sand with boulders and gravel being transported to the places where it is found now and having taken its present form, still a long time must have passed, before the surface was fastened by the vegetation and the currents were streaming in their present beds. During this period frost and rain will have had a stronger influence on these accumulations of sand and boulders than afterwards when their surface was protected by a thick crust of humus. The rocks which are capable of disintegration, many granites, mica slates, sand-stones and grits, have been converted into gravel, sand and clay, and the rain water has transported it for a great part to the valleys. These valleys were filled up and at the same time the hills became lower and took a more rounded form than was originally the case. That this sand represents the detritus of the gravelous diluvium and has been formed during the transition-time to the alluvial period, is also to be concluded from the reflection that such a formation must exist and that this formation cannot be pointed out in another than in this sand.”

STARING was however not ignorant of the fact that still other factors have contributed to the formation and the distribution of this sand.

This may be sure when we see, how on his geological map on some places along the great rivers alluvial sand-banks are placed on the "zanddiluvium", isolated from the present river-bed and how on the index of the colours "zanddiluvium" and "rivages diluviens" are marked with different character Z and Z' and yet this difference has not been sustained on the map. Here the same colour and the same character Z signifies "zanddiluvium" as well as "diluvial sand-banks" (map 19) and "river banks" (map 20). Apparently STARING would not decide, though he was convinced that those formations are not equivalent. He may not have said it clearly, yet the honour of having first recognized the problem is due to him.

So STARING's point of view in the matter was as follows:

The "zanddiluvium" includes all sanddeposits which have been formed after the glacial period and which are not surely alluvial. It has been washed away by rain water from the gravel-hills in the neighbourhood; yet on many places the possibility of another origin may be taken into consideration.

After STARING only three geologists, as far as is learned by the literature I can dispose of, have been engaged in the study of the postglacial "zanddiluvium".

At the 7th Physical and Medical Congress in 1899 (Transactions page 450) Dr. H. VAN CAPPELLE spoke on "de oorsprong van het heide- of hellingzand".

In the STARINGIAN "zanddiluvium" formations of a different age must be represented. The orator observed on many places (West-Drenthe, Gaasterland, Amersfoort etc.) between the sand which may be considered with great probability as the product of the washing from the gravel-hills, old surfaces which he connects with the interglacial period by the following reasoning:

"the younger diluvial currents which have formed the level sand of the valleys, have eroded this "heidezand". The sand of the valleys being formed in the period of the melting away of the first glacier, for the formation of the older "heidezand" only two subsections of the diluvial period rest: 1st. the second glacial period and 2nd. the period of the melting away of the first glacier."

"The first age is possible for the sand which covers the mentioned vegetable beds. So these vegetable beds must be interglacial. The sand which covers the boulder-clay directly, may also be a deposit of the first glacier."

Before continuing, I will add some remarks. The old surfaces, observed by Dr. VAN CAPPELLE, prove nothing else but the fact that in the formation of this sand periods of rest have existed, in which organic life could develop on the sand. But this fact does not yet prove that two different geological periods are necessary. Indeed Dr. VAN CAPPELLE himself says that the vegetable beds lie between the sand in the shape of wedge and so are very local. They have not any value for a further determination of the age. I agree perfectly well with Dr. VAN CAPPELLE where he says on page 451 :

“this author (Dr. J. MARTIN at Oldenburg) will not believe that the interglacial period has passed without leaving behind traces in our country, though a convincing proof is not to be given, because the ground moraine of the second glacier fails”, and indeed it is not to be thought otherwise that among our so-called postglacial formations, deposits exist which are equivalent to such of the second interglacial period and the third glacial period in Germany, but a decisive proof is not to be given. Old surfaces between the sand have not the least demonstrative power in this matter. To show the point of view of Dr. VAN CAPPELLE in 1899 in regard to the origin of the “heidezand”, I will cite the conclusion at the end of the mentioned communication :

The “heidezand” has been formed in part by the melting waters of the retiring first glacier, in part after the interglacial period by the brooks which were streaming from the gravelhills in the time of the approaching of the second glacier ; also the alluvial period has contributed to the increase of the “heidezand” by the washing of the hills.”

Dr. J. LORIÉ has laid stress upon the fact that the sandy plains which accompany our great rivers and many smaller ones ought to be considered as river-beds of the diluvial times. This sand has been washed away from the banks and removed down the current. Two examples will do.

At the 4th Physical and Medical Congress in 1893 (Transactions page 393) Dr. LORIÉ speaks about “the peat-moors of Brabant-Limburg.” The speaker demonstrates that these moors owe their origin to existing grooves in the surface which had no drainage. Those grooves represent old branches of the Meuse. This whole region ought to be considered as a diluvial delta of the Meuse.

“From the stadium of the “wild waters” when the Meuse was still streaming without a definitive bed, a compound network was born which decreased gradually, until only one current, the present Meuse, remained.”

In the communications on the geology of the Netherlands, collected by the geological committee, number 35, Dr. LORRÉ shows convincingly that the surface of the "Geldersche Vallei" represents a terrace of a faded branch of the Rhine.

"The peat is followed by a thick layer of sand which has not been washed away from the hills on either side, but has been supplied by a branch of the Rhine which thus built a terrace". (p. 95).

The fact, that the fluviatile sand reposes upon the marine fauna of the "Eemstelsel" with temperate character and is still separated from it by peat and clay, proves in my opinion that the terraces have been formed by the rivers in an old-alluvial period with our present climate. I hope to revert soon to this subject.

The third examiner of the Dutch sand districts was Dr. J. L. C. SCHROEDER VAN DER KOLK, and it may be superfluous to draw out his great merits another time, also in regard to this geological problem. To his accurate sand inquiry we owe this division¹⁾.

I. Quartz-amfibol-sands:

- a.* quantity less than 0,4. Southern Diluvium.
- b.* quantity more than 0,4. Northern Diluvium.

II. Quartz-garnet-sands:

- a.* small quantity. Increase. Alluvium.
- b.* great quantity. Decrease. Alluvium.

He applied his rules to this study of the environs of Deventer with much acuteness.

SCHROEDER VAN DER KOLK could not continue his successful researches because of his busy life in Delft and his feeble health, and his death has put an end to all further expectations.

The difficulties are far from being conquered now. In general the rules of SCHROEDER VAN DER KOLK are quite right. Yet it is not sufficient for the classification of each sand to determine the quantity of heavy minerals. The activity of the forces of nature being too much complicated, the effect cannot appear in a so simple form in all cases. In each case all circumstances must be taken into consideration or we shall often come to an erroneous conclusion. The following example shows this necessity.

The lower parts of the Rhine-diluvium in Limburg are formed by grey sands. In these sands grains of basalt are tolerably numerous. Even when the grains larger than 2 mM. and those smaller than 0,2 mm. are removed, the quantity of heavy minerals is still 0,5 to 0,6, the specific weight of basalt being 2,9 to 3,1. The grains of

¹⁾ In this form SCHROEDER VAN DER KOLK gave his division at the 6th Physical and Medical Congress in 1897 (Transactions page 409).

basalt not being considered as a *mixture* of minerals, it would be concluded that these sands belong to a glacial sheet which had reached our country before the gravel of the Rhine-diluvium had been deposited. Such a contradiction would give rise to many difficulties.

There is still more. In the gravelous diluvium in situ, the practical limit of 0,4% will probably give good results, but in the "zand-diluvium", the materials of which must have been derived from the gravelous diluvium, the quantity of heavy minerals must have been changed, according to the different manners of derivation. Here the greatest prudence is wanted.

When a characteristic can be pointed out which may be considered as the effect of these different manners of derivation, a further step has been made. I believe I have found a specific which may aid in some cases to take a decision, where the rules of SCHROEDER VAN DER KOLK do not help. I add immediately that this specific is not at all universal.

It will be necessary to tell in a very general manner how in my opinion the beginning of the diluvial period found our country and how our gravelous diluvium has been deposited.

As much as we know now, the tertiary base of our country is marine (excepted South-Limburg). In the western part of Zeeuwseh-Vlaanderen the base is formed by the "rupelleem", in the eastern part it is the marine deposit of the Diestien. Along our southern frontiers the marine deposits of the Poederlien, Diestien and Bolderien exist. The sands of the Moséén appertain in my opinion to the old fluvial diluvium. The miocene land- and fresh-water deposits seem to be restricted to South-Limburg. Along our eastern frontiers the base is formed by upper-oligocene sea-sands, to the north these strata are covered by miocene deposits. In Gelderland and Overysseel the same miocene clay forms the surface. In North-Brabant and North-Limburg this miocene is still covered by sands which probably appertain to the pliocene. To the north and west LORÉ has shown a pliocene sea-ground at several places (Grave, Arnhem, Goes, Gorkum, Bergen op Zoom, Utrecht and Amsterdam) which declines to the north-west. So at the beginning of the diluvial period the greater part of our country was covered by the pliocene sea and only in the east and south-east a coast existed.

Now the rivers Rhine and Meuse supplied the enormous quantity of sand and gravel which form everywhere in our country the base (excepted Zeeuwseh-Vlaanderen and South-Limburg). A delta was built which filled up the basin of Holland and the Southern North-sea as far as the chalk rocks of Norfolk and Suffolk.

Then a part of this delta was covered by the northern glacier which deposited its moraines on the surface of the delta. The glacial sheet being retired, a new period begins. The river water remained at the south of the morainal barrier Laren, Rheden, Nymegen, Cleef, Xanten, and the rivers came from the stadium of the "wild waters" into the stadium of a compound network. The gravelous diluvium has been deposited and we have to examine in what way the surface can have been changed now.

In the sandy areas which are very numerous in our gravelous diluvium, we may expect;

- a.* the sand which has been removed by the wind.
- b.* the sand washed away from the hills by rain water.
- c.* the sand deposited by a river in its bed.

How to distinguish these sands? The sand *a* must be deposited in a depression and does not show a stratification. The composition and the largeness of the grains do not offer a regularity and the surface must be hilly. This origin may be possible for sand districts which are surrounded by gravel-hills.

The sand *c* may be found along the rivers or in a groove which represents a faded river bed. In general the surface declines down the current and the sand is stratified in a horizontal position. The composition and the largeness of the grains give no difference in the direction of the stream, neither in vertical direction.

The sand *b* may be found in depressions as well as in grooves, has also to show a horizontal stratification, but in the composition and the largeness of the grains differences may be expected:

1. in the direction of the groove, when the bordering hills differ in composition, as in the case of the hills of Holtten, Markelo and Lochem.

2. in vertical direction. When grains of sand are washed down the hills by rain water, the grains become smaller, while the inclination decreases gradually. So the grains of the sand *b* has to become smaller in the higher parts of the layer.

The next may show whether this characteristic can practically be used.

In the last year the Dutch Government drilled at three places in the Peel. As appeared the peat reposes on sand and then the gravelous diluvium follows. The question is whether this sand is a sand *b* or a sand *c*. When the lower parts have indeed a greater quantity of large grains than the higher parts, it may be a sand *b*, else we have better consider it as a sand *c*.

The hill-slope from Meyel to Deurne is about 18 K.M. long and has a direction from N. N. W. to S. S. E. At the east of this ridge are situated the three places (at the station of Helenaveen, at the village of Helenaveen and at the north of the village of Helden) which I shall call 1, 2 and 3, from the north to the south.

The position of the sand bed is as follows:

	Meters + A.P.		thickness
	from	to	
at the first place	31.32	24.92	6.40
at the second place	32.01	25.76	6.25
at the third place	30.19	23.54	6.65

I accept four sorts of grains:

Sort A: grains larger than 2 mm.

Sort B: grains smaller than 2 mm. and larger than 1 mm.

Sort C: grains smaller than 1 mm. and larger than $\frac{1}{2}$ mm.

Sort D: grains smaller than $\frac{1}{2}$ mm.

The grains were separated by sieving and the quantity was weighed.

The results are found in the following tables:

FIRST PLACE.

Depth of the sample in meters + A. P.	Sort	Sort	Sort	Sort
	A.	B.	C.	D.
31.42	0.18	0.39	38.7	60.7
30.42	0.17	0.33	35.5	64.0
29.38	0.43	0.66	41.2	57.7
28.38	0.45	0.75	41.7	57.1
27.62	0.87	0.74	45.9	52.5
26.52	0.93	1.58	47.5	50.0
25.42	2.80	1.88	46.0	49.3

SECOND PLACE.

Depth of the sample, in meters + A. P.	Sort	Sort	Sort	Sort
	A.	B.	C.	D.
21.36	0.25	0.56	39.2	60.0
29.56	0.58	0.58	37.8	61.0
28.56	0.61	0.92	42.3	56.2
27.56	1.83	2.80	44.3	51.4
26.56	3.50	2.78	44.5	50.2

THIRD PLACE.

Depth of the sample, in meters + A. P.	Sort	Sort	Sort	Sort
	A.	B.	C.	D.
28.04	0.09	0.52	29.8	69.6
27.04	1.25	2.83	29.5	66.4
25.04	2.79	2.92	34.6	59.6

These lists show that indeed in the higher parts of the sand bed are more small grains and fewer large grains. So the question is decided in favour of the sand *b*.

It occurs to me that we may accept:

1st. that the groove originally was the bed of an old branch of the Meuse. However this river did not wash away sand from the banks.

2nd. that afterwards the groove has been filled up with sand *b*. As I remarked already, this specific is not at all universal, but can only give an indication in some cases.

Finally a remark. The glacial sheet being retired the glacial period ends for our country. Meanwhile we know that afterwards another approach of the glacier came, which did not reach our country. As Dr. VAN CAPPELLE remarked already, interglacial deposits must exist in our country. We do not know however deposits with an arctic fauna and so the points of comparison fail to divide our postglacial diluvium. Therefore we cannot give a further determination of the age for the deposits between our gravelous diluvium

and the modern formations. In my opinion it is not possible to make a map of our country which is *strictly* geological. But it is just for that reason that a geological survey would have to do good work, by finding means to conquer the difficulties.

For the moment we must be content with a temporary division which I propose as follows:

A. Glacial and fluvial (fluvio-glacial) diluvium. The expression "preglacial" can be applied only to the surface of the delta, where the deposits of the northern glacier repose on it. At the south of the glacial front the surface of the delta may be formed contemporaneously with the glacial diluvium.

B. Postglacial diluvium and old-alluvium.

Only in some cases it will be possible to draw the line.

C. Recent formations.

Within these geological limits only petrographical and genetical distinctions can be made.

Venlo, June 1907.

Physiology. — "*On the adsorption of the smell of muscon by surfaces of different material.*" By Prof. H. ZWAARDEMAKER.

(Communicated in the meeting of May 24, 1907).

In 1906 H. WALBAUM discovered the odorous principle of musc in a ketone of constitution $C_{16}H_{30}O$, to which the name muscon was given¹⁾. Through the kindness of the firm SCHIMMEL & Co. I was enabled to make some olfactological investigations with this preparation, which at my request was mixed with myristic acid for this purpose. With this fatty acid, melting at $54^{\circ}C.$, it forms a mixture, containing 0.627 % muscon which could easily be cast into an olfactometric cylinder of 8 millimetres lumen. Exposition of 0.15 cm. of such a cylinder to a passing air-current of 100 cubic centimetres per second gives a just recognisable impression of the smell of muscon, a soft perfume, not admitting of further definition and soon growing tiresome. With further dilution this perfume does not change its character. Hence the odorimetric coefficient of the mixture, used by us, was 6,7²⁾.

¹⁾ H. WALBAUM in SCHIMMEL & Co's Berichte, April 1906, p. 99.

²⁾ The odorimetric coefficient of a smelling substance, offered in a certain condition, is defined as the reciprocal value of the length in centimetres of the cylinder, corresponding to the so-called "threshold value" (for olfactometric cylinders of 8 mm.

The olfactometric measurements showed:

1. that the volatilised muscon adheres strongly to the glass walls along which it passes.

2. that rubbing such a glass wall with cotton wool gives rise, instead of an odour of muscon, to a smell, reminding of muse.

This smell of muse was also noticed with glass wool, cotton wool, feathers or paper, placed in the path, but not with asbestos wool and platinum sponge, the time of exposition in all these cases having been about $\frac{1}{4}$ minute.

This led to a closer investigation, which I undertook the more readily, since an investigation by J. AITKEN in 1905 showed that the odorous principle of muse must be regarded as a gas¹⁾. Thus the above-mentioned olfactometric cylinder, containing 0,627 % muscon in myristic acid and having a length of 10 cm. and a diameter of 0.8 cm., was connected by means of a short brass piece with equally long and wide tubes of all sorts of material, in such a way that these tubes could, if required, be kept at a pre-determined temperature by a water-jacket. The thus formed canal passed into an aerodromometer²⁾, i.e. a vertical glass tube in which an aluminium disc is suspended between two spiral springs, the displacement of which indicates the velocity of the air-current by means of an empirical scale. After the aerodromometer finally followed a large tin cone in which an electrically driven fan maintained a suction from the narrow towards the wide end. The connection between the different pieces could be removed and re-established in a moment.

The air, passing through this system, went successively through:

1. the olfactometric cylinder over its full length of 10 cm.
2. the tube of which the adsorption is to be examined.
3. the aerodromometer.

In the experiments here mentioned the velocity of the air-current was perfectly constant; 84 cm.³ passed per second. Each exposition lasted accurately 5 minutes. Between the experiments the olfactometric

lumen, i.e. 0.5 cM² cross-section). Cf. on this point *Physiol. des Geruchs*, Leipzig 1895, p. 185. The significance of this coefficient, which rises and falls with the smelling power of a substance, is at once seen, when one recognises the close relation between it and the quantity b in FECHNER's celebrated formula $\gamma = k \log \frac{I}{b}$ (*Psychophysik* II, p. 13). It deserves notice that the odorimetric coefficient of muscon in liquid paraffine is zero.

¹⁾ J. AITKEN. Evaporation of musk and other odorous substances. *Proc. Roy. Soc.* Vol. 25, p. 894, 1905.

²⁾ H. ZWAARDEMAKER. *Arch. f. Anat. u. Physiol.* (Physiol. Abth.), 1902, p. 417.

cylinder was always kept closed, while controlling experiments with an entirely similar cylinder of pure myristic acid showed that by this alone no lasting adsorption of odour is produced.

Adsorption of odour appeared to be entirely absent with some materials (porous porcelain, carbon, ebonite, steel, iron), with others to exist in a small degree (aluminium, silver, sulphur), with others again to exist in a more or less considerable degree (tin, copper, nickel, glass, lead alloyed with tin, lead). Judging by the impression received immediately after the experiment, the substances may be arranged in the following series of increasing condensation :

Porous porcelain *, are lamp carbon, ebonite *, steel, iron, aluminium*, silver*, sulphur*, tin, copper, nickel, glass, lead alloyed with tin, lead.

The substances marked with an asterisk are not entirely odourless at the temperature of the room.

The first members of this series have no adsorption odour whatever after the muscon-containing air has passed for five minutes, nor do they acquire it by heating. The final members, especially tin, copper, nickel, glass and lead have a distinct adsorption odour which during the first minutes or even hours has an unmistakable muscon character. At last, however, a change takes place, consisting in :

1. an alteration of the quality of the odour, so that finally it resembles musc. This holds for tin, copper, nickel, glass, glazed porcelain, lead alloyed with tin, lead.

2. an increase in the smelling power of the adsorption odour, so that for lead, at any rate, a maximum is obtained after about 3×24 hours.

3. a subsequent decrease in the smelling power, so that finally the tubes lose all odour.

Free from any acquired odour the tubes become :

For porous porcelain	in 0 days
„ carbon	„ 0 „
„ steel	„ 0 „
„ iron	„ a few minutes
„ sulphur	„ less than 24 hour
„ aluminium	„ „ „ 24 „
„ glass	„ „ „ 24 „
„ silver	„ „ „ 2 days
„ copper	„ „ „ 4 „
„ tin	„ „ „ 4 „
„ nickel	„ „ „ 4 to 9 days
„ lead alloyed with tin	„ „ „ 6 days
„ lead	„ „ „ 11 to 12 days

Meanwhile these figures have only an approximate value, since the temperature of the room varied considerably during the last spring. In the first place I investigated whether the adsorption of the muscon odour must be explained as an electrical phenomenon. The muscon gas, led over a sensitive electrometer, appeared to impart no charge to it, but it is not impossible that the method was not sensitive enough for this purpose. Therefore in the above described apparatus numbered nickel-plated copper tubes were placed, facing an insulated axially mounted steel rod of 3 millimetres thickness, so that an air condenser was formed with a distance of 2.5 millimetres between the cylindrical charged surfaces. The odd numbers are charged +, the even ones — from the 220 volt continuous current main. Each time, the exposition lasted a minute, the dielectric carrying the muscon passing in the ordinary way at a rate of 84 cm.³ of air per second. The cylinders appeared to have assumed the odour of muscon with about the same intensity, to acquire later an odour of muse in the same manner and to lose this in about the same time. The comparisons between the tubes were made by three observers, trained in these experiments and independently of each other.¹⁾

Next the influence of temperature was investigated, first on the adsorption and next on the change of the smell of muscon into that of muse. For this purpose tubes of an alloy of lead and tin were exposed for ten minutes at 0°, 13° and 100°.

		immediate impression	odourless in
exposition at	0°	strong smell of muscon	5 days
„	„ 13°	distinct smell of muscon	2 „
„	„ 100°	no smell of muscon	1 day

Then numbered, nickel-plated copper tubes were exposed during two minutes to the ordinary air-current, passing over the muscon-myristic acid. The odd numbers were placed in the ice-box, the even ones were left to themselves at the temperature of the room, each placed in a wide-necked glass stoppered bottle. After 24 hours there appears to be no statistical difference of any importance. All cylinders whether even or odd, appear to have assumed a smell of muse in a distinct though feeble degree. So the temperature coefficient of the surface-action, exerted by nickel on the phenomenon of the transition of muscon into muse, cannot be great.

¹⁾ One of these observers has an ordinary acuteness of smell for the odour of muscon, but cannot state with certainty the transition of muscon into muse. He also has in other respects strongly deviating peculiarities of his organ of smell which will soon be extensively communicated.

Finally I wish to state that capillary glass tubes of 1 mm.² cross-section, after air, carrying muscon, had been passed through them for five minutes, did not show a perceptible change of surface tension with water (the tension being measured by the height of the water column) and that repetition of the other experiments with tubes, heated beforehand and with air that had been dried by means of calcium chloride and cotton wool, gave no deviating results.

At present it is impossible to give a theory of these phenomena. As a preliminary working hypothesis one might suppose the adsorbed muscon to be dissolved in the layer of condensed water vapour and air which covers all objects and it might be further assumed that the change of muscon into musc only then takes place at a perceptible rate when the surface action of the metal, of the glass or of the glazed porcelain produces a particularly great density of the dissolved muscon in immediate contact with the surface in question. This hypothesis is in harmony with equilibrium experiments, made with dried air at 0°, 10°, etc.. These experiments are in progress but not completed yet.

Physiology. — “*On the adsorption of the smell of muscon by surfaces of different material*”. By Prof. H. ZWAARDEMAKER.
Continuation of a former communication.

When air, charged with muscon, is passed through tubes of an alloy of lead and tin, in the manner described in the communication of May 24, the inner surface of these tubes appears to adsorb muscon in quantities, the amount of which may to some extent be estimated from the time, during which the tubes preserve the odour of muscon. This assumption is based on the supposition that the adsorption takes place in one and the same dissolving substance, namely the condensed layer which is said to cover all objects.

The dilution at which the muscon is present in the air in these experiments, can be kept constant when the current velocity is controlled by means of an aerodromometer. Moreover, it may vary between certain limits, since experiments, made to this purpose, showed that it makes no difference in the results whether the muscon, volatilised per second from the smelling source, is contained in 42, 84 or 126 cc. of air. Tubes of lead, alloyed with tin, lose under similar conditions the adsorption odour in the same degree and in the same time, say 5 to 6 days.

Under the just stated conditions the adsorption equilibrium is reached at a temperature of the room of 19° centigrade in about ten minutes, as will appear from the following table:

	after 1 min. exposition	odourless	within 1 day
„ 2	„	„	after 1 „
„ 3	„	„	„ 2 days.
„ 4	„	„	„ 3 „
„ 5	„	„	„ 4 „
„ 10	„	„	„ 5 „
„ 20	„	„	„ 4 „

ADSORPTION ODOUR.

Preceding Exposition	at 0°	at 20°	at 40°	at 60°	at 100°
5 min.	8 days	3 à 5 days	2 days	1½ days	1 day
10 „	10 „	5 „	3 „	2 „	2 days
15 „	a little over 10 days	a little over 5 days	4 „	2 „	2 „

Nickel-plated copper tubes, treated in the same way, show saturation after an exposition of about 5 minutes, it making no difference whether this takes place at 0°, 20° or 40°. Complete loss of adsorption odour was found in these cases after respectively 4, 2 and 2 days.

From these experiments follows that a higher temperature during the exposition causes the state of saturation to be reached only little sooner, but that the degree of adsorption is much smaller at a higher temperature. This proves that with higher temperatures the equilibrium is shifted in the direction of minimum adsorption.

The facts, stated until now, agree very well with the hypothesis of a solution of muscon in the layer of condensed water-vapour, carbonic acid and air which covers all objects. Assuming this, we are led to believe that on nickel-plated copper this layer is thinner than on lead, alloyed with tin, and that consequently in the former case the equilibrium during exposition is reached sooner than in the latter, while temperature has the same influence on them both and in the same degree. The fact that tubes, heated beforehand and treated in dry air, give the same results, is not at variance with this since we may not expect that the condensed layer will by this treatment be completely removed. Also the transition of the smell of muscon into that of musc must take place in this layer, the only curious point being that temperature has so little influence on the rate of

this process of transition which yet must be of a chemical nature.

But a great difficulty to the theory arises from the fact that adsorption of odour on metal surfaces appears to be a general phenomenon. This appears from similar systematic experiments as with muscon for two other characteristic smelling substances. I chose ionon, a substance dissolving in water as well as in liquid air and scatol, a substance for which this has not yet been investigated.

Ionon, when diluted 1 to a million in an aqueous solution of 0,5 % antifebrin, and evaporating into a passing air-current which in the well-known manner passes through cylinders of different material, leaves an adsorption which disappears almost immediately with porous porcelain, arc-lamp carbon, glass, silver, sticks to tin for a very short time, to lead, containing tin, scarcely for a day to nickel and copper for about two days, to aluminium for $2\frac{1}{2}$ days and to iron and steel for about four days.

Scatol, when dissolved in proportion of 1 to 1000 in liquid paraffin and evaporating into passing air and passing in the well-known manner through cylinders of different material, leaves an adsorption which disappears almost at once with porous porcelain and arc-lamp carbon, in a few hours with glass, sticks to lead, containing tin, to lead, silver and tin for about a day, to copper 3 days, to iron 4 to 5 days, to steel 10 to 13 days, to aluminium over 10 days.

Hence ionon adheres most to the substance which does not take up muscon at all, i. e. to steel; scatol most to aluminium which shows a comparatively very small adsorptive power for muscon (aluminium does not keep muscon for 24 hours).

In order to explain these deviations one is forced either to assume a peculiar modification of the solubility, caused by the dissolution of the specific metallic particles in the condensation layer, or to assume an absorption in the metal itself. To me it would seem that the collected facts do not at present admit of a choice between these two possibilities, although the small influence of the muscon density in the air would point to an adsorption compound with the metal.

Physics. — “*Contribution to the theory of binary mixtures.*” V.

By Prof. J. D. VAN DER WAALS.

Continued, See p. 74.

Up to now we have assumed in the determination of the binodal line that the second component, for which the quantity b is larger than for the first component, has a lower critical temperature, so that we suppose $(T_k)_2 < (T_k)_1$. In the opposite case, so $(T_k)_2 > (T_k)_1$, we meet with some new complications, which we shall shortly discuss. So we choose now a region from the general p -figure, which lies more to the right, and in which the line $\left(\frac{dp}{dx}\right)_v = 0$ is found. Fig. 14 of These Proceedings April 26, 1907 may be serviceable for this discussion. In this figure the points 1, 2, 3, 4, 5 and 6 are points of the spinodal line. If we had inserted the spinodal line itself in the figure, this curve would have an ordinary shape on the vapour side, remaining all the time at larger volumes than those of the line $\left(\frac{dp}{dx}\right)_v = 0$. But on the liquid side the normal course of the spinodal line has been strongly modified by the presence of the line $\frac{d^2\psi}{dx^2} = 0$. On the left-hand side it begins in the point $\frac{dp}{dv} = 0$ of the first component, proceeds then to smaller volumes, till the presence of $\frac{d^2\psi}{dx^2} = 0$ forces it back to very small volumes, and is the cause that the distance between the spinodal line and the line $\frac{dp}{dv} = 0$ is abnormally enlarged. In the points where $\frac{d^2\psi}{dx^2} = 0$ and $\frac{d^2\psi}{dx dv} = 0$ intersect, the spinodal curve touches the curve $\frac{d^2\psi}{dx^2} = 0$. Two plaitpoints occur, viz. the realisable plaitpoint at very small volume, and the hidden plaitpoint in the neighbourhood of the points 2 and 3. This hidden plaitpoint lies in this case on the left-hand side in accordance with the shape of the q -lines. In fig. 17 this hidden plaitpoint lies on the right-hand side, and the shape of the q -lines in the region where $\frac{dp}{dx}$ is positive, is such that there is a q -line which may be drawn tangent to the spinodal curve, the hidden plaitpoint being the point of contact. In fig. 17 the q -lines

in this region turn their concave side to the axis of the 1st component. In the case to be discussed now they turn their convex side towards the 1st component, and hence the hidden plaitpoint must lie on the other side, as a point in which a q -line touches the spinodal line in the unstable region. The drawn q -line intersects the spinodal line in 6 points, and the p -line, thought as function of v , must have 3 maxima and 3 minima, when this q -line is followed; a maximum value in the points 1, 3 and 5, a minimum value in the points 2, 4 and 6. In fig. 20 this p -curve is represented and the different

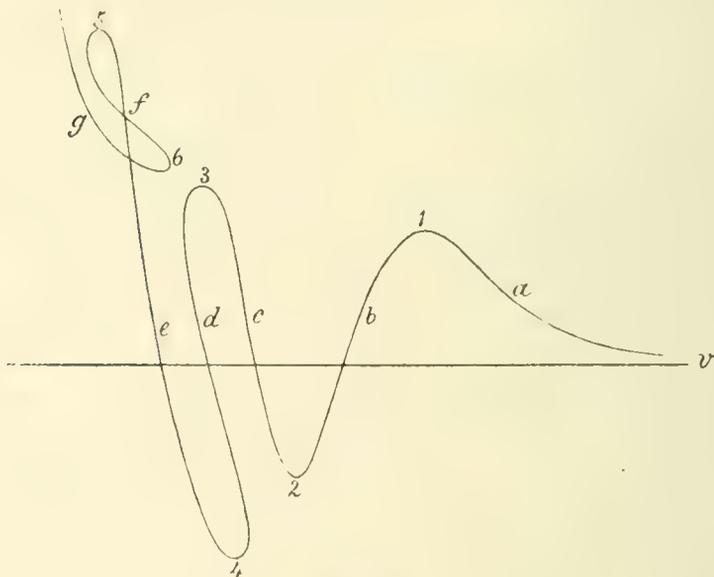


Fig. 20.

branches of this line are indicated by the letters $a \dots g$. The branches d and f traverse the region where $\frac{d^2\psi}{dx^2}$ is negative, and accordingly have two points each, where $\frac{dp}{dv} = \infty$. The complication which the p -line presents in this case compared with the p -line of fig. 16, consists only in this that the branch e , which before ran directly to infinity and continually to smaller volumes, has now got a maximum in the point 5, and as soon as the q -line passes into the region where $\frac{d^2\psi}{dx^2}$ is negative, runs back to larger volumes. In the point 6 the minimum value has been reached, which however must be larger than the maximum value of the pressure in the point 3. If the

value of q is lowered, the points 6 and 3 draw nearer to each other, and they coincide for the loop- q -line which passes through the point of intersection of $\frac{d^2\psi}{dx^2} = 0$ and $\frac{d^2\psi}{dx dv} = 0$. Then the branches c and d intersect at an acute angle, just as the branches f and g . When q is lowered further, and the q -line has split up into two separate portions, the p -line too divides into two separate parts; the branch g is then the continuation of c , and the branch f the continuation of d . Fig. 21 illustrates the course of p as function of v

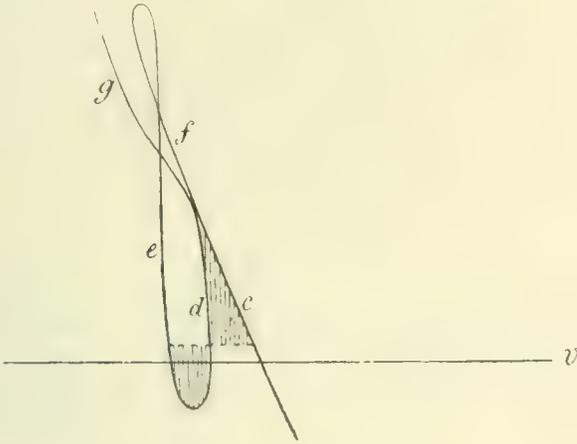


Fig. 21.

for such a q -line which has divided into two separate portions; then the branches c , g , which have united to one branch cut the united branch d , f , and the branch e .

When applying MAXWELL'S rule for the determination of the binodal line we are confronted with some difficulties, which I will now discuss. Already when the p -line runs as is represented by the branches e , f and g in fig. 20, so when the middle one of the 3 branches cuts one of the outside lines, we must pay proper attention to the sign of the areas when applying the rule for drawing MAXWELL'S line. If the straight line is drawn lower than the point of intersection of e and f , the area below this line, which according to the rule must be equal to the area above this line, must of course be all that is contained between the branches g and f below this line. But the area above the line, which consists of two parts, viz. the area of the loop, and the part that lies below the double point above the line, must not be considered as the sum of these two parts. On account of the branch f running back, the latter part must be taken

negative. This may be considered to be sufficiently self-evident and not to require an elaborate proof. But when the q -line has divided into two separate parts, and when the p -line runs as is represented in fig. 21, we meet with another difficulty, which indeed, calls for a somewhat closer discussion. The joined branches c and g form a curve which cuts the branches d , e and f , which have joined to a loop-like curve, in two points, but such a point of intersection must really be considered as two altogether different points. Such a point of intersection represents two perfectly different phases according as it is considered as point of e , g or of d , e , f . Hence when drawing the straight line we must bear in mind that the point of intersection of c and d and of e and g does not represent the same phase, and if the line is drawn as in fig. 21, where the two hatched areas are equal, the points at the extremities of this line are not points of the binodal line. To see how the straight line must be drawn in such cases we revert to the general equation:

$$dM_1\mu_1 = vdp - xdq.$$

Now to get from one point to the point with which it coexists, we can no longer follow one q -line, but we shall partly have to follow a way which joins the two separate branches of the q -line, and for this we choose the isobar of the point of intersection that the branches c , g and d , f have in common. We obtain then the equation:

$$(M_1\mu_1)_c - (M_1\mu_1)_e = \int_e^c vdp - \int_e^c xdq,$$

where in $\int vdp$ the value of v must be taken which corresponds to the chosen value of q , and in $\int xdq$ the value of x which corresponds to that p -line that passes through the point of intersection. Let us call the value of the volume of the point of intersection v_s and the values of x for the points where the isobar of the point of intersection cuts the two branches of the q -line, x_2 and x_1 . The above equation assumes then the following form:

$$(M_1\mu_1)_c - (M_1\mu_1)_e = \left[p(v_c - v_e) - \int_e^c p dv \right] - \left[q(x_2 - x_1) - \int_1^2 q dx \right].$$

Now if $(M_1\mu_1)_c$ must be $= (M_1\mu_1)_e$, then $p(v_c - v_e) - \int_e^c p dv$ is

not equal to 0, but to $q(x_2 - x_1) - \int_1^2 q dx$. For the loop- q -line the length of the isobar along which $\int x dq$ must be taken, is equal to 0, and x_2 and x_1 coincide. For a q -line of lower degree x_2 and x_1 differ. In the above equation it is supposed that branch c is taken as starting point, and that a course is followed necessary to reach branch c . The point from which we start, lies on the closed circle of the q -line and in the stable region. We now follow indifferently either the lowest branch of this circle or the highest, but dependent on the pair of coexisting phases that is to be determined. Let us suppose that we follow the lowest course, then we get to branch d , and meet the point of intersection of the isobar which we must follow to meet the other branch of the q -line in a point which has equal volume v_s . As this isobar must pass through the line $\left(\frac{dp}{dx}\right)_v = 0$, where maximum volume exists, the equality of the volumes v_s is possible¹⁾, but the values of x which we have called x_2 and x_1 , are different, viz. $x_2 < x_1$. For x_1 the value of q is the chosen one and for x_2 the value of q is again the same. Between x_1 and x_2 this value is variable. Now:

$$\left(\frac{dq}{dx}\right)_p = \frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dx dv} \frac{dv}{dx_p}$$

or

$$\left(\frac{dq}{dx}\right)_p = \frac{\frac{d^2\psi}{dx^2} \frac{d^2\psi}{dv^2} - \left(\frac{d^2\psi}{dx dv}\right)^2}{\frac{d^2\psi}{dv^2}}$$

$\frac{d^2\psi}{dx^2}$ (see fig. 14) being positive, $\left(\frac{dq}{dx}\right)_p$ is positive outside the spinodal line, and negative inside it. Along the p -line, starting from smaller value of x , the value of q is, therefore, increasing, maximum on the spinodal curve, then decreasing, minimum on the spinodal curve, after which it increases continually, as represented in fig. 22.

1) The same observation holds for all points which are points of intersection of different branches of the p -line in figs. 20 and 21. In such a point of intersection p and v are equal, and this could only occur when the phases denoted

by such a point of intersection lie on either side of the line $\frac{dp}{dx} = 0$.

Now :

$$p(v_c - v_e) - \int_e^c p \, dv = - \left\{ q(x_1 - x_2) - \int_2^1 q \, dx \right\}$$

or

$$\int_e^c p \, dv - p(v_c - v_e) = - \left\{ \int_2^1 q \, dx - q(x_1 - x_2) \right\}.$$

For the loop- q -line x_1 and x_2 coincide, and for a q -line but little lower $\int_2^1 q \, dx$ is larger than $q(x_1 - x_2)$. As x_1 always lies on the left

of the value of x for which q has minimum value, $\int_2^1 q \, dx > q(x_1 - x_2)$

always holds. From this follows that for the lowest pair of coexisting phases of fig. 21 the straight line must be drawn in such a way that the area of the hatched part above this line, to which the area of the hatched part of fig. 22 is added, is equal to the hatched part of fig. 21 which would lie below this line. So the pressure of the lowest pair of coexisting phases for this q -line is greater than would follow from the application of the rule if the point of intersection of c, g and d, f was an identical point, or rather represented one and the same phase. But we shall not pursue this course any further.

Now that we are obliged to include the quantity $\int x \, dq$ in our considerations, we can find the coexisting phases for the liquid volumes in a simpler way by the aid of this quantity. For such volumes lie on a p -line which can be followed without interruption when we proceed from one point of the pair of coexisting phases to the second point. And when we proceed along a p -line $dM_1 \mu_1 = -x \, dq$, and so $(M_1 \mu_1)_2 - (M_1 \mu_1)_1 = - \int_1^2 x \, dq$. Hence we need only choose two points on the chosen p -line, satisfying the requirement that $-\int_1^2 x \, dq = 0$,

or

$$q(x_2 - x_1) = \int_1^2 q \, dx.$$

Then we have to carry out the same construction on the q -line

as was carried out above on the p -line, and so for the p -line, for which fig. 22 would represent the course of the q -line, we have to

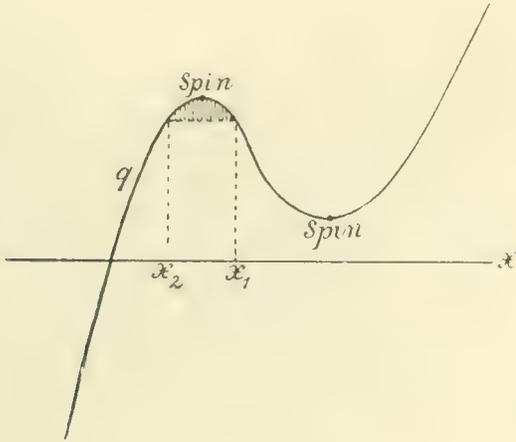


Fig. 22.

draw a straight line in such a way that its height indicates the middle value of the ordinates of the q -curve. That from the outset we have not followed this course for the determination of the coexisting phases in which the values of x_2 and x_1 for given value of p are determined, is due to the fact, that this way of determination is again possible without correction term only when the whole p -line is found between the two coexisting phases without interruption in the v, x -diagram; and as for the equilibrium between vapour and liquid phases as a rule this condition is not satisfied, and as it is only by way of exception that the q -line has split up into two branches, the determination of coexistence by the first mentioned method may as a rule be considered as possible. But nevertheless, in some cases the determination by means of the properties of the value of q , following a p -line, is to be preferred. If we do so in the case discussed for the determination of the coexistence of a liquid phase with a second liquid phase, we must choose every time other p -lines, and along each of these p -lines the course of q as function of x is as drawn in fig. 22; and with the simple shape of such a q -line there is only question of a single straight line along which

$$q_c(x_2 - x_1) = \int_1^2 q dx.$$

The binodal curve for the coexistence of liquid with liquid has therefore a simple shape and is restricted to the stable region.

Indeed, this was also to be derived from the p -figure (fig. 20) where the branches f and g must lie higher than the branches c and d , and therefore can never combine for the application of the rule for coexistence; but then only for those q -lines which are of higher degree than the loop- q -line; whereas the rule for finding the conditions for coexistence from the values of q when a p -line is followed, holds for all p -values without exception. Let us consider the case that this part of the plait has got quite detached from the transverse plait as a closed longitudinal plait, and has the two realisable plaitpoints, then a highest and a lowest p -line may be drawn, along which the maximum and the minimum in the q -line have coincided, and in the point where they coincide they yield the value of x for the two plaitpoints.

We had already repeatedly occasion to call attention to the reciprocity between $\frac{d^2\psi}{dx^2}$ and $\frac{d^2\psi}{dv^2}$, and between q and p or $\frac{d\psi}{dx}$ and $\frac{d\psi}{dv}$. Let us also do so in the case discussed. Here we have intersection in two points of $\frac{d^2\psi}{dx^2} = 0$ and $\frac{d^2\psi}{dxdv} = 0$, and it appeared that then separate portions of q -lines occur, so that it was not always possible to pass without a leap from one part of a q -line to another part of such a line. Then it is desirable for the determination of the coexisting phases not to follow such a q -line, but on the contrary to go along a p -line and to use the corresponding value of q . The reciprocal case is found in case of intersection of $\frac{d^2\psi}{dv^2} = 0$ and $\frac{d^2\psi}{dvdv} = 0$, in which case the course of the p -lines is as is indicated in the middle region of the general p -figure. Then there are p -lines, namely those of higher degree than the loop- p -line, which have divided into two separate parts; if we followed a p -line also then, in order to arrive at the coexisting phases by means of the values of q , we should be confronted by the same difficulties as we have met now when following the q -line. If for a p -line of lower degree than the loop- p -line we draw the value of q , then such a course for q follows from

$$\left(\frac{dq}{dx}\right)_p = \frac{\frac{d^2\psi}{dx^2} \frac{d^2\psi}{dv^2} - \left(\frac{d^2\psi}{dxdv}\right)^2}{\frac{d^2\psi}{dx^2}}, \text{ as has been drawn in fig. 23, where}$$

the 1st, 3rd and 5th branches lie in the stable region, and the 2nd and 4th branches lie in the unstable region, if we take into account that such a p -line passes 4 times through the spinodal line, in which

points $\left(\frac{dq}{dx}\right)_p = 0$, and also 4 times through the line $\frac{d^2\psi}{dv^2} = 0$, in which points $\left(\frac{dq}{dx}\right)_p = \infty$. Only for the loop- p -line the second minimum

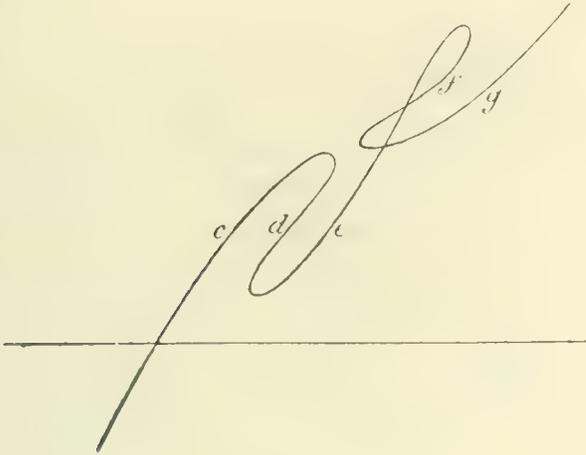


Fig. 23.

coincides with the first maximum, but for lower p -lines it lies higher, as in the drawing. We have exactly the same shape for q as function of x , as in fig. 20 for p as function of v . Only one figure must be turned over to cover the other, in accordance with the circumstance that $q = \frac{d\psi}{dx}$ and $p = -\frac{d\psi}{dv}$. The combination c, d and e now yields a pair of coexisting phases, and the combination e, f and g another pair. No other combinations are possible; and we should be justified in concluding that the binodal line has a simple course and remains limited to the stable region. But this conclusion would be perfectly valid only for all pressures not higher than those of the loop- p -line, though there are also coexisting phases with higher value of p . In this case it is certainly preferable to follow a q -line, and to construe p as function of v , which we have called preferable already above for other reasons. We know that then a highest pressure exists for the coexisting phases, viz. when $x_1 = x_2$; this is only possible if the chosen q -line passes through the line $\frac{dp}{dx} = 0$, for only then this is the case for values of x within certain limits. From this circumstance of the reciprocal case we conclude that in the case under consideration, in which $\frac{d^2\psi}{dx^2} = 0$ is cut by $\frac{dp}{dx} = 0$,

there is a minimum value of q for the coexisting phases, viz. if $v_2 = v_1$. Then the line joining these phases runs parallel to the x -axis, just as it runs parallel to the v -axis in the reciprocal case. This too can only occur, if the coexisting phases lie on either side of the line $\frac{dp}{dv} = 0$; for the isobar that passes through the two points of coexistence, can only have two equal values of v if between them maximum or minimum value for the volume occurs. The equality of v_1 and v_2 for minimum value of q , to which we have concluded from the principle of reciprocity, follows from the simple equation, which holds for two successive points of a binodal curve, viz.:

$$(v_2 - v_1) dp = (x_2 - x_1) dq.$$

For a pair of coexisting phases M_1u_1 is the same, and for a following pair of such phases dM_1u_1 is also the same; now the above equation follows from $dM_1u_1 = v_1 dp - x_1 dq = v_2 dp - x_2 dq$. If $x_2 - x_1 = 0$ and $v_2 - v_1$ is different from zero, then dp must be $= 0$; in the same way $dq = 0$ involves the equality of v_2 and v_1 , if dp is not equal to 0. We can also derive from this equation, how the nodal lines lie on either side of the special nodal line for which $x_2 = x_1$ or $v_2 = v_1$, i. e. to which direction they diverge in a fanlike way. Let us first consider the case $x_2 = x_1$, so maximum pressure on the vapour-liquid binodal curve. On the left of this nodal line the sign of $v_2 - v_1$ is positive on the vapour side, and the sign of dp , if we do not limit ourselves to an infinitely small value of dp , negative. Then also the sign of $(x_2 - x_1) dq$ must be negative, and the sign of dq being negative, $x_2 - x_1$ must be positive. On the right of this nodal line the sign of $v_2 - v_1$ and of dp must be what it was in the preceding case; but dq now being positive, $x_2 - x_1$ is negative. So the nodal lines converge towards the vapour side. It would be just the reverse if the pressure was minimum for $x_2 = x_1$, for then dp is positive. Let us now consider the case $v_2 = v_1$, so minimum value of q on the liquid-liquid binodal curve. Let us choose the right side, so where $x_2 > x_1$, and let us ascend, so put dp positive, then q being minimum, dq will be positive. The second member is positive, and so we find $v_2 - v_1$ positive, whereas for negative dp the value of $v_2 - v_1$ would be negative. So the nodal lines converge towards the right side, and we may consider the nodal line for which $v_2 = v_1$, as axis of such a converging pencil. This shows us at the same time how and where the plaitpoints must lie. As the tangent to the binodal curve in the plaitpoint is to be considered as the limiting direction of the nodal lines, both the p -line and the

q -line must have such a course in the upper plaitpoint that they descend towards the right, which moreover could be put à priori.

For every q -line when it still lies above the line $\frac{dp}{dx} = 0$, and does not pass through $\frac{d^2\psi}{dx^2} = 0$, descends when it proceeds towards the right. But in the lower plaitpoint, i. e. in the plaitpoint with the larger volume that lies below the line $\frac{dp}{dx} = 0$, the q -line which touches in that plaitpoint, must descend as it proceeds to smaller value of x , in accordance with the course of the nodal lines. We should also have found this course of the nodal lines confirmed, if we had paid attention to the course of the p -lines.

Everything we have discussed in this Vth communication rests on fig. 14; so we have thought that $\frac{dp}{dx} = 0$ and $\frac{d^2\psi}{dx^2} = 0$ intersect. There is, however, also a possibility, and it will even be the rule, that the two curves exist, but do not intersect. Then two cases are to be distinguished, viz. that $\frac{d^2\psi}{dx^2} = 0$ remains restricted to smaller volumes than those of $\frac{dp}{dx} = 0$, or to larger ones¹⁾. When tracing the two curves with respect to each other we must take care that the points in which tangents may be drawn to $\frac{d^2\psi}{dx^2} = 0$ parallel to the v -axis, lie on the line $\frac{d^2p}{dx^2} = 0$, and that also the point in which $\frac{dp}{dx} = 0$ has minimum volume, lies on this line too. Now the line $\frac{d^2p}{dx^2} = 0$ has a simple course. The value of $\frac{dx}{dx}$ for this line is equal to $\frac{db}{dx} \frac{3}{1+2\frac{b}{r}}$. From this follows that this line $\frac{d^2p}{dx^2} = 0$ consists of a single branch, which from a point of the 1st axis moves regularly to the right to points of continually larger volume. So if the line $\frac{dp}{dx} = 0$ cuts the line $\frac{d^2\psi}{dx^2} = 0$, the two points in which tangents parallel to

¹⁾ See These Proceedings April 26, 1907, p. 833 seq.

the v -axis can be drawn to $\frac{d^2\psi}{dx^2} = 0$, and the point where $\frac{dp}{dx} = 0$ has minimum volume must lie in such a way, that the last point lies between the two first mentioned. If the line $\frac{d^2\psi}{dx^2} = 0$ is restricted to smaller volumes than $\frac{dp}{dx} = 0$, then $\frac{d^2\psi}{dx^2} = 0$ must also lie at smaller x than the point where $\frac{dp}{dx} = 0$ has the smallest volume and the

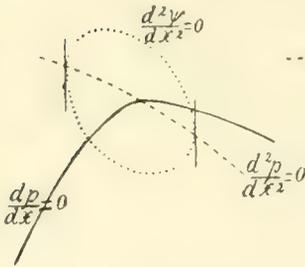


Fig. 24a

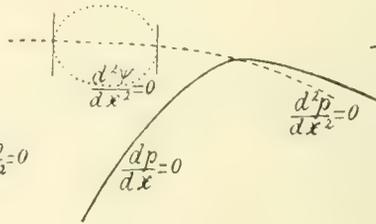


Fig. 24b

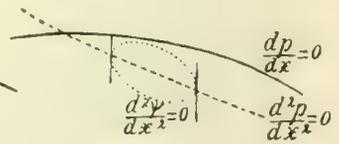


Fig. 24c

reverse: this has been represented in fig. 24a, fig. 24b and fig. 24c, but has not always been kept in view in preceding schematical figures, which were plotted for the representation of other particularities.

After these remarks we may examine more in details what happens when $\left(\frac{dp}{dx}\right) = 0$ and $\frac{d^2\psi}{dx^2} = 0$ intersect, and the temperature is raised.

With rise of T $\frac{d^2\psi}{dx^2} = 0$ contracts to the point in which this curve

must disappear. Also the curve $\frac{dp}{dx} = 0$ contracts. If the point in which

$\frac{d^2\psi}{dx^2} = 0$ must disappear, lies at smaller volume than $\frac{dp}{dx} = 0$, then

with contraction of $\frac{d^2\psi}{dx^2} = 0$ the right-hand point where the latter curve is

directed // v -axis, will have to pass through the minimum volume of

$\frac{dp}{dx} = 0$. Even then there is still intersection, but with further contraction

the two curves will touch, and get detached. Above the temperature at

which they touch, the complicated course of the q -lines has disappeared

in so far that no q -lines occur any more which have split up into

two separate branches; then we get a group of q -lines as drawn in

fig. 3, These Proc. March 30, 1907 with a maximum and a minimum volume, but moreover when they afterwards cut $\frac{dp}{dx} = 0$, with a maximum value of x . But when the point in which $\frac{d^2\psi}{dx^2} = 0$ must disappear lies at larger volume than $\frac{dp}{dx} = 0$, then with rise of T the left-hand point where $\frac{d^2\psi}{dx^2} = 0$ is directed parallel to the v -axis, must pass through the point where $\frac{dp}{dx} = 0$ has minimum volume. Then intersection still takes place, but with further rise of T the curves touch and get detached — and then the q -lines run as has been drawn in fig. 5. So contact between $\frac{d^2\psi}{dx^2} = 0$ and $\frac{d^2\psi}{dx dv} = 0$ may take place in two ways and we may already conclude to this from the condition for contact. From the equality of $\frac{dv}{dx}$ for the two curves follows namely:

$$-\frac{d^3\psi}{dx^3} \frac{d^2p}{dx dv} = \left(\frac{d^2p}{dx^2}\right)^2.$$

The value of $\frac{d^2p}{dx dv}$ being negative for the points of $\frac{dp}{dx} = 0$, $\frac{d^3\psi}{dx^3}$ must be positive in the point of contact. That is to say, that for the curve $\frac{d^2\psi}{dx^2} = 0$ the point of contact must lie to the right of the line which joins the minimum and the maximum volume. Only with the two kinds of contact which we have described, this condition can be fulfilled. If the first described contact takes place, the minimum volume of $\frac{dp}{dx} = 0$ must lie to the right of the point of contact. In the second case of contact this point must lie on the left, or even be entirely wanting in the figure, in which case $\frac{dv}{dx}$ is positive in all points of the line $\frac{dp}{dx} = 0$.

From all this follows that if the spinodal curve entirely envelops the curve $\frac{d^2\psi}{dx^2} = 0$ in a closed line, and the latter remains entirely restricted to smaller volume than the volumes of $\frac{dp}{dx} = 0$, there are

indeed still two realisable plaitpoints on this spinodal line, but that the course of the nodal lines in all this longitudinal plait is as was the case in the upper half of the above discussed longitudinal plait — so that in both plaitpoints the tangent p -line and the tangent q -line descend to the right. All over this longitudinal plait $v_2 > v_1$, if v_2 represents the right-hand point of coexistence. But if $\frac{d^2\psi}{dx^2} = 0$ remains restricted to volumes larger than those of $\frac{dp}{dx} = 0$, the course of the nodal lines is such that $v_2 < v_1$, and the plaitpoint has such a situation that $\frac{dv}{dx_p}$ and $\frac{dv}{dx_q}$ is negative for the p -line and the q -line which pass through the plaitpoint. I speak of the plaitpoint, because I think I can demonstrate that then there cannot be question of two realisable plaitpoints, nor of a detached closed longitudinal plait. For when a spinodal line splits up, not only $\frac{dv}{dx_p} = \frac{dv}{dx_q}$ in this splitting point, but also $\frac{d^2v}{dx_p^2} = \frac{d^2v}{dx_q^2} = 0$. Properly speaking I have already discussed this point (These Proc. April 26, 1907 p. 848), but on account of the great importance of the question further elucidation is perhaps not uncalled for. The following remarks may serve for this purpose.

Let us in the first place consider a mixture represented by a region of the general p -figure lying on the right side, and so much to the right that the point where $\frac{dp}{dx} = 0$ has minimum volume, no longer occurs, or lies at very small value of x . Then the point where $\frac{d^2\psi}{dx^2} = 0$ disappears at $T = T_g$, because it must lie on the line $\frac{d^2p}{dx^2} = 0$, lies at smaller volume than those of the line $\frac{dp}{dx} = 0$; and if this curve is still found at temperatures below T_g , the points in which this curve intersects the line $\frac{dp}{dx} = 0$, lie in the region where $\frac{dp}{dx}$ is negative. If we now suppose that the temperature rises, and the spinodal line might split up, this splitting point must lie between the larger volumes of $\frac{d^2\psi}{dx^2} = 0$ and the volumes of $\frac{dp}{dx} = 0$, so also in the region in which $\frac{dp}{dx}$ is negative. Now the question is if in this

region a point of inflection of the p -lines and of the q -lines can lie. It appears from what has been observed about the loci of these points of inflection (These Proc. March 30, 1907 p. 736) that this is possible for the q -lines. But from what has been observed on the course of the locus of the points of inflection of the p -lines (These Proc. Febr. 23, 1907 p. 628) appears that in the stable part of that region no point of inflection can occur for these lines.

Let us now take the other case, viz. that the point with minimum volume of $\frac{dp}{dx} = 0$ exists, and is not found at very small value of x . If the spinodal line has split up into two parts, then there is a part which we might consider as belonging to $\frac{dp}{dx} = 0$, and another part that surrounds $\frac{d^2\psi}{dx^2} = 0$. Now the splitting point lies again in the region where $\frac{dp}{dx}$ is negative, but in a part of that region where as well points of inflection of the p -lines as of the q -lines may occur, at least if $\frac{dp}{dx} = 0$ still intersects the curve $\frac{d^2\psi}{dx^2} = 0$. Two branches on which $\frac{d^2v}{dx^2q} = 0$, start from the point in which $\frac{dp}{dx} = 0$ cuts the curve $\frac{d^2\psi}{dx^2} = 0$. One of these branches passes through the region where $\frac{d^2\psi}{dx^2}$ is negative, and leaves this region only at the point where $\frac{d^2\psi}{dx^2} = 0$ has the maximum volume. The second branch runs right of the loop- q -line to larger volumes. But there is also a locus on which $\frac{d^2v}{dx^2p} = 0$, which runs right of $\frac{dp}{dx} = 0$, and passes through the two following points. 1st the point where $\frac{dp}{dx} = 0$ has minimum volume, and 2nd the point where $\frac{dp}{dx} = 0$ cuts the line $\frac{dp}{dx} = 0$. If the spinodal line splits up, this will have to take place in the point of intersection of the line on which $\frac{d^2v}{dx^2p} = 0$ with the second mentioned branch on which $\frac{d^2v}{dx^2q} = 0$. If this case of splitting occurs,

the detached closed longitudinal plait is cut by the line $\frac{dp}{dx} = 0$, and has the above discussed plaitpoints.

But though on the supposition of this way of splitting up we do not meet with a definite contradiction, yet there is one circumstance which makes me doubt whether it will occur frequently or universally. If we draw the point of intersection of the mentioned loci on which $\frac{d^2v}{dx^2_p} = 0$ and $\frac{d^2v}{dx^2_q} = 0$, we find a point which lies on the left side of $\frac{d^2\psi}{dx^2} = 0$, whereas after the detaching we should sooner expect the place of the plaitpoint with the largest volume, according to the course of the nodal lines, on the right side of $\frac{d^2\psi}{dx^2} = 0$. Indeed, another way of detaching is possible. The splitting may take place in a point on the left of $\frac{dp}{dx} = 0$. Then $\frac{d^2\psi}{dx^2} = 0$, which curve must disappear in a point of $\frac{d^2p}{dx^2} = 0$, must already have contracted so far that it lies entirely in the region where $\frac{dp}{dx}$ is positive. As we observed before, there runs a branch on which $\frac{d^2v}{dx^2_p} = 0$ also there and for the loop-line on which $\frac{d^2v}{dx^2} = 0$ (These Proc. March 30 1907 p. 736) there must be a closed figure, which has got detached from the branch right of $\frac{dp}{dx} = 0$, because the double point, the point in which $\frac{dp}{dx} = 0$ and $\frac{d^2\psi}{dx^2} = 0$, no longer exists. Then we have again a detached closed longitudinal plait, but one which is not intersected by $\frac{dp}{dx} = 0$, and which has two plaitpoints in which the p - and q -lines which touch have $\frac{dv}{dx_p} = \frac{dv}{dx_q} =$ positive, in accordance with the course of the nodal lines. In fig. 25 the circumstances after the splitting have been represented for this case. First of all the lines $\frac{dp}{dx} = 0$ and $\frac{dp}{dv} = 0$ occur in the figure; further $\frac{d^2p}{dx^2} = 0$, which passes through the point where $\frac{dp}{dx} = 0$ has minimum volume.

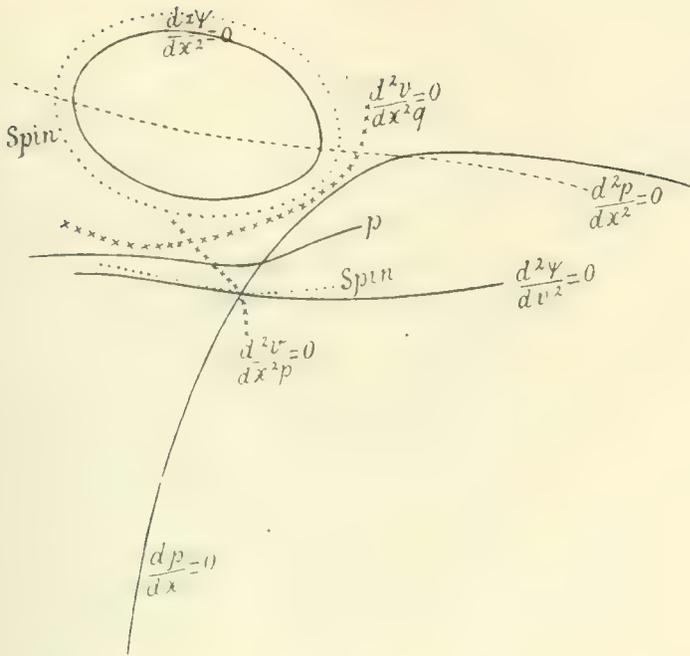


FIG. 25.

On the left of $\frac{dp}{dx}=0$ and at smaller volumes also $\frac{d^2\psi}{dx^2}=0$ has been drawn. Where this curve passes through $\frac{d^2p}{dx^2}=0$ it has tangents // v -axis. The spinodal line has split up and the two parts have been drawn far apart for the sake of lucidity. One part surrounds $\frac{d^2\psi}{dx^2}=0$, and the other part touches $\frac{d^2v}{dx^2}=0$ in the point in which this curve is intersected by $\frac{dp}{dx}=0$. Further a p -line has been drawn with two points of inflection. The right-hand point of inflection is of no importance for our case. And finally the detached branch of the locus of the points of inflection of the q -lines has been drawn. Now too the point in which $\frac{d^2v}{dx^2p}=0$ and $\frac{d^2v}{dx^2q}=0$ intersect, is to be expected on the left side of the spinodal curve, which has got detached. But for this case $v_2 > v_1$ for all nodal lines of the longitudinal plait, and the second plaitpoint is really to be expected on the left. I suspect these two ways of detaching to be connected with the two series of values of v_j , for which $\frac{d^2\psi}{dx^2}=0$ disappears in the

region where $\frac{dp}{dx}$ is positive. (These Proc., April 26, 1907, p. 833) — either for a very great difference in the size of the molecules of the components, or for a small difference. In the latter case the highest and the lowest points of $\frac{d^2\psi}{dx^2} = 0$ are to be found at almost the same value of x . But this is one of the many particularities which is to be left to a later investigation.

Particularly the last described way of splitting up of the spinodal curve takes place far to the left of the point where $\frac{dp}{dx} = 0$ has minimum volume, and so at a value of x_1 not very different from that for which $x_2 = x_1$ on the vapour binodal curve, and maximum pressure exists; and so this leads to the opinion that this detaching of a longitudinal plait is to be found for mixtures with minimum pressure and very different size of the molecules; but also this supposition must be further defined by a fuller investigation.

The following remarks may serve for a full characterization of the course of the spinodal line before and after the splitting. Before the splitting the curves $\frac{d^2\psi}{dx^2} = 0$ and $\frac{d^2\psi}{dv^2} = 0$ must be thought as intersecting, as in fig. 8 (These Proc. March 30, 1907), but the line $\frac{d^2\psi}{dx^2} = 0$ as having moved to smaller volumes. This figure holds indeed for a left-hand region of the p -figure, but this figure would change little in its essential features if we also insert the line $\frac{dp}{dx} = 0$ in it, but place it on the right so that $\frac{d^2\psi}{dx^2} = 0$ is no longer intersected by it. For a region of the left-hand side extended towards the right is the same as a region of the right-hand side extended towards the left. If $\frac{d^2\psi}{dx^2} = 0$ and $\frac{d^2\psi}{dv^2} = 0$ intersect there is a complicated plait, with the hidden plaitpoint on the right side. If now with rise of temperature the two curves get further apart, because they both contract, splitting up of the spinodal curve does not always immediately follow. For this to be brought about the curves must be pretty far apart, and intersection of $\frac{d^2v}{dx^2_p} = 0$ and $\frac{d^2v}{dx^2_q} = 0$ must take place between the two curves, and the temperature must be reached at which this point of inter-

section lies on the spinodal line. Then a point of the left-hand side of the spinodal line coincides with a point of the right-hand side of this line, but not in the hidden plaitpoint. Consult also fig. 17 (These Proc. May 24, 1907 p. 68). Then there are 4 plaitpoints, viz. P_1, P_2 and the double plaitpoint in the splitting point of the spinodal line. The course of the binodal curve on the liquid side has been represented in fig. 26. On the liquid side the binodal curve of

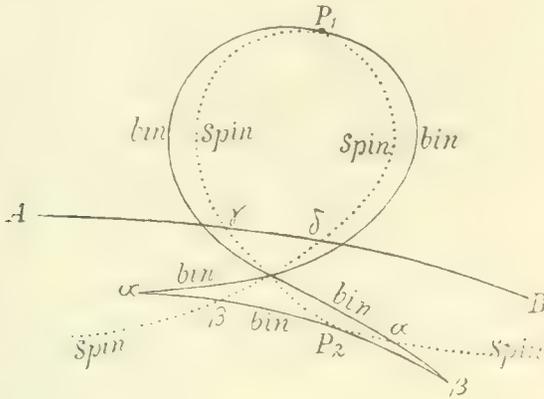


Fig. 26.

the vapour-liquid equilibria passes in two more points, γ and δ , through the spinodal curve. And so nothing appears yet of the detaching of the longitudinal plait for the experiment. Only at higher temperature the detached binodal curve, and then with its newly obtained plait-point, will pass through the binodal curve AB , and with still higher value of T the binodal line has split up into two quite separate branches.

(To be continued).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday September 28, 1907.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige

Afdeeling van Zaterdag 28 September 1907, Dl. XVI).

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Physiology. — “*Quantitative researches on phagocytosis. A contribution to the biology of phagocytes.*” By Prof. H. J. HAMBURGER and Dr. E. HEKMA.

(Communicated in the meeting of June 29, 1907.)

I. *Introduction and method of investigation.*

The investigations of which an abridged account is given in this paper ¹⁾ are a continuation of those begun several years ago by one of us ²⁾, with the object of ascertaining the influence exercised by solutions of various concentration on the red corpuscles and other cells. These researches had been for the greatest part confined to the study of chemical and volumetrical alterations experienced by the cells through the modification of their media and of their significance with regard to the functions of the body. But until now, *the influence of these agents on the life of the cell itself*, had not been the object of a systematic investigation, although the plan had existed for some time and the expediency of the method had been proved ³⁾. The importance of such an investigation will be readily admitted. In the first place, because it enhances the value of the chemical and volumetrical researches mentioned above, and secondly, because the phenomena produced by the agency of solutions undangerous to life, are in fact nothing else but the effects of reaction, which finally will help us to penetrate farther into the chemical structure of the living cell. The red corpuscles, which were mostly used for the chemical and volumetrical researches, however, are no suitable objects for the study of the influence of reagents upon life, for they do not afford sure tests of vitality, nor is it possible to measure the value of their life functions.

We therefore looked elsewhere for our material and our choice fell on the phagocytes, for the twofold reason that they are simple, isolated cells in which it is possible to follow the effect of the chemical exchange with their natural medium, and to rate their very life by quantity; besides the phagocytosis is an essential factor in the functions of life. In support of this latter contention, we refer to the important place assigned by METCHNIKOFF to these cells in the struggle of the body against disease; a theory which he has

¹⁾ For detailed account, see “*Biochemische Zeitschrift*”.

²⁾ HAMBURGER, Zittingsverslag der Koninkl. Akad. v. Wetensch. 29 December 1883.

³⁾ HAMBURGER, Het gedrag van witte bloedlichaampjes tegenover cyaankalium, Bijdrage tot de kennis der celpermeabiliteit. Feestbundel voor Rosenstein, 1902.

defended with such admirable acumen and unflagging energy. According to the same investigator, the part they play in the healthy body is no less important. The only thing therefore which remains to be done with regard to these cells, is to get a nearer insight into their conditions of life; as yet scarcely anything is known on this subject, a fact mentioned with regret by METCHNIKOFF, in the paper he read last year before the students of the University of Amsterdam, on: "Réactions phagocytaires" ¹⁾).

The method of investigation employed by us was the following: White corpuscles from the blood of a horse, after having been transported into various media, were brought into contact with carbon and afterwards it was ascertained what percentage of the leucocytes had taken up particles of carbon. *This percentage was the measure for the degree of phagocytosis and gave the value of the influence of various agents on that function of life.*

These calculations were based on the principle that the phagocytarian power of the phagocytes present in a suspension is of unequal extent; i. e. the more detrimental the action of the agent is, the smaller must be the number of phagocytes able to take up carbon.

Our selecting a neutral indifferent substance of bacteria, had its ground in the fear that otherwise our work would have become too complicated. We here refer to the recently established fact that most kinds of bacteria, before they can be taken up by the phagocytes, must undergo a certain amount of preparation ²⁾. Hence it follows that not only will the intensity of phagocytosis be influenced by the agent as such, but also by the degree of preparation it has undergone. Another fact which had to be borne in mind, is that the bacteria sometimes secrete poisons which have an injurious effect on the phagocytosis.

1) "Nous ne sommes qu'an début. Lorsqu'on connaitra mieux *la physiologie des phagocytes* (the italics are ours) on cherchera des méthodes pour augmenter l'activité de ces éléments dans la lutte contre les microbes et on cherchera d'autres pour préserver contre l'attaque des phagocytes les cellules nobles de notre corps. En poursuivant ce but, il faudra tenir compte de ce que les phagocytes sont non seulement les destructeurs des microbes, mais qu'ils sont capables aussi de s'incorporer des poisons solubles et de les rendre inoffensifs. Leur rôle n'en devient que plus important."

2) WRIGHT and DOUGLAS, *Proceed. of the Royal Society* 72, 1903, p. 357 and later studies prepared under WRIGHT. Further HEKTOEN and RÜDIGER, *Journ. of Infect. diseases* 2, 1905, p. 128 and other studies prepared under HEKTOEN.

The selection of the indifferent substance carbon, instead of the usual grains of carmine was based on the greater facility and more accurate certainty with which the taking up of carbon can be ascertained. It was also for this reason that carbon had been the substance selected in former investigations on the action of carbonic acid¹⁾ and the action of cyanate of potassium²⁾ on phagocytosis.

On the present, as well as on the former occasion, the leucocytes used in our investigations, were taken from the blood of a horse³⁾. They were obtained by shaking blood with pieces of glass in a closed bottle and straining the defibrinated blood through a piece of muslin. The red corpuscles sink to the bottom, and the serum which covers them holds all the leucocytes. When this turbid fluid has been poured off we have a suspension of leucocytes in serum; this suspension can be made richer in leucocytes, by centrifugalizing it, removing part of the clear serum and mixing the leucocytes which have fallen to the bottom, with the remaining serum. A detailed description of this method, the process of preparing the carbon, the mode of bringing it into contact with the leucocytes, and the method of determining the percentage of the cells which have taken up carbon, will be found in our article in the *Biochemische Zeitschrift*⁴⁾.

II. *The effects produced on phagocytosis by the addition of water.*

Our first experiments were directed to the solution of the question in how far phagocytosis is affected by the addition of water.

With this object in view, equal quantities of the suspension of leucocytes were mixed with serum, previously diluted with known quantities of water. The following table shows the results of one of the series of experiments. It will need no further explanation.

1) HAMBURGER. VIRCHOW'S Archiv, **156**, 1899. S. 329. Osmot. Druck u. Ionenlehre. **1**. S. 416.

2) HAMBURGER Het gedrag van witte bloedlichaampjes tegenover Cyaankalium in ROSENSTEIN'S Feestbundel. 1902.

3) At Groningen we experienced great difficulty in obtaining a regular supply of horses' blood. Mr. K. HOEFNAGEL, the Director of the abattoir at Utrecht, had the great kindness fully to meet our wants, for which we here beg to tender him our best thanks.

4) Compare also HAMBURGER, Osmot. Druck u. Ionenlehre, Bd. **1**. S. 401.

TABLE I.

Effect of lessening the concentration of the serum.

Serum diluted with	Number of examined white corpuscles	Number of white corpuscles which have taken up coal	Percentage of number of white corpuscles containing coal	Decrease of the phagocytarian power
0% water	886	331	37	
20 "	754	246	32	13.5%
50 "	732	154	21	43.2 "
100 "	636	81	12½	66.2 "
140 "	520	0	0	
200 "	546	0	0	

From this table it appears that in serum, to which no water has been added, of the 886 examined leucocytes, 331 had taken up carbon, i.e. 37 %.

We must here point out that in the circulating, and also in the coagulated blood, the percentage of phagocytes is actually much smaller.

By a certain proceeding, however, we contrived that in our experiments, the leucocytes used for examination should contain a great number of phagocytes. This process is based on the fact that among the leucocytes, the phagocytes are the cells which soonest sink to the bottom. After this explanation, it will cause no surprise to find that, in normal serum, the percentage of the phagocytes which have taken up carbon, continually varies in different series of our experiments.

An addition of 20 % water, already lessens the phagocytarian power with 13½ %. In calculating this loss on an addition of 5 % water, supposing the diminution to be proportionate, the decrease of the phagocytarian capacity would have amounted to:

$$\frac{5}{20} \times 13.5\% = 3.4\%.$$

In other words, *when the osmotic concentration of the blood plasma is lessened by 5%, a loss which in a healthy individual may be of daily occurrence*¹⁾, the phagocytarian power falls about 3.4 %.

By the side of this great sensitiveness of the phagocytes to the increase of their percentage of water, stands the fact, as shown from

¹⁾ Compare a o. KOEPPE, PFLÜGER's Archiv. 62, 1896, S. 567. In his experiments, KOEPPE noticed a decline below the mean osmotic pressure of over 10 %.

the table, that on the other hand, there are a great number of phagocytes which can stand a dilution of their serum with 100 % of water. Former experiments have proved that this dilution causes an increase in the bulk of the cells, of considerably over 30 %.¹⁾

We will now pass on to the following question: *Is this decrease in the phagocytosis of a permanent nature?*

In order to find an answer, we brought the white corpuscles which had been submitted to the action of diluted serum back into the normal, undiluted serum, and then tested again their power of taking up coal.

T A B L E II.

After exposure to the action of diluted serum, the phagocytes are brought back into normal serum.

Serum diluted with	Number of white corpuscles examined	Number of white corpuscles which have taking up carbon	Percentage of white corpuscles containing carbon, in normal serum
0% water	500	105	21
20 "	500	99	19.8
50 "	500	107	21.4
70 "	500	96	19.2
100 "	500	78	15.6
200 "	500	61	12.2

This table shows that phagocytes, which had some time remained in serum, diluted with 20 % or 50 % water, dilutions which as the former series of experiments indicates, caused a reduction of the phagocytarian capacity of 13.5 % and 43.2 % respectively, after having been brought back into normal serum, *entirely recovered their original phagocytarian power.*

The addition to the serum of 100 % water, has on part of the phagocytes a lasting deleterious effect; the addition of 200% water is even more detrimental. Still, it is interesting to observe that, although in serum which had been diluted with 200 % water, all the phagocytes

¹⁾ HAMBURGER. Archiv. f. (Anat. u.) Physiol. 1898. S. 317.

had lost the power to take up carbon, after being replaced in the normal serum, over 50% of the phagocytes recovered their original capacity.

So, *the greater part of the phagocytes can support a considerable volume of water without permanent loss of their phagocytarian capacity.*

Here may be asked: On what does it depend, whether a phagocyte will regain its phagocytarian power? It is not impossible, nay, it is even probable, that here as well as in the case of the red corpuscles some lose their contents in serum diluted with 70% of water ¹⁾. If the quantity of water added be raised to 100%, the number of destroyed erythrocytes will be found considerably larger. When the red corpuscles, which *have not lost their haemoglobin*, are removed from the serum diluted with 100% of water into undiluted serum, they entirely recover; they change from small globules into biconcave discs, which even arrange themselves like piles of coins.

However, this only applies to the cells which have not lost their colouring matter. These which have actually lost haemoglobin cannot recover. Now our microscopical investigations have revealed the fact, that in serum + 70% water, some of phagocytes lose a part of their contents: in that case we see a granular substance lying by their sides. In serum to which 100% water has been added, the effect is more apparent still. Then the number of leucocytes which have expelled granular matter is still larger. It is easy to understand that these cells, when again placed in normal serum, have lost the power of taking up carbon. The difficulty of ascertaining this with certainty however, is very great: in the transmission there is every chance of disturbing the granular substance by the side of the cells, and it is impossible to know whether one deals with a phagocyte which has lost part of its contents or not. Anyhow, taking into consideration the striking analogy existing between white and red corpuscles, both with regard to their permeability and to the osmotic pressure of their interior substance, and even to the percentage of the volume of their watery contents ²⁾, we seem justified in our conjecture that the same fluid, which causes a loss of colouring matter in the least resisting of red corpuscles, also brings about the irremediate destruction of the phagocytarian power of the least resisting phagocytes.

It is a fact worthy of notice that the resisting power of the phagocytes reaches a higher maximum than that of the erythrocytes. In serum diluted with 200% water, all the erythrocytes of the horse are destroyed, and not quite half of the phagocytes.

III. *Effect on phagocytosis by the reduction of water.*

A similar method as had been used for studying the effect on phagocytosis by dilution of the serum, was now applied to ascertain the

¹⁾ HAMBURGER. Transactions of the Royal Academy of Sciences, 25th March 1885.

²⁾ Osmot. Druck u. Jonenlehre I, S. 401—435.

influence of concentration. With this object in view, common salt was dissolved in the serum in quantities of 0.1, 0.2, 0.3, 0.4 % and more. The results of these experiments are shown in the following table :

T A B L E III.

Effects on phagocytosis by increased concentration of the serum.

Serum in which is dissolved	The liquids in the preceding column are corresponding to :	Percentage of leucocytes which have taken up carbon	Decrease of the phagocytarian power
0 % Sod. Chl.	Sod. Chl. 0.9%	$\frac{208}{832} \times 100 = 26\%$	
0.1 "	" 1	$\frac{184}{874} \times 100 = 21.5$	17.3%
0.2 "	" 1.1	$\frac{184}{1005} \times 100 = 18.3$	29.6
0.3 "	" 1.2	$\frac{76}{941} \times 100 = 8$	69.2
0.4 "	" 1.3	$\frac{43}{793} \times 100 = 5.4$	79.2

Here we see that the injurious effect is very great, much greater than is the case when the osmotic concentration has been diminished. Then we observed that by diluting the serum with 20 % water, phagocytarian capacity fell 13.5 %; here we find that by raising the osmotic concentration by 10 %, the phagocytarian power is lowered by 17.3 %. This effect must already be perceptible within the physiological boundaries in which the osmotic pressure of the blood plasma usually varies in the normal body. For it may happen every day that in a normal individual the osmotic pressure of the liquor sanguinis, a few hours after dinner, is still raised by that of 0.1 % common salt¹⁾.

Here again, as we did before when studying the decrease of osmotic pressure, we ask whether the loss of phagocytarian power

¹⁾ KÖEPPE, l. c.

D. SCHOUTE. Het fysisch-chemisch onderzoek van menschelijk bloed in de kliniek. Diss. Groningen 1903.

See also Osmot. Druck u. Ionenlehre B. I S. 540 ff.; B. II S. 279 and 310 ff.

can be restored, by replacing the white corpuscles in the normal serum. The answer will be found in the following table.

T A B L E IV.

After being exposed to the action of increased concentration, the leucocytes were replaced into normal serum.

Serum in which is dissolved	After replacement into normal serum, the phagocytarian power stands at
0% Sod. Chl.	$\frac{273}{700} \times 100 = 39\%$
0.2 "	$\frac{246}{646} \times 100 = 38.4$
0.7 "	$\frac{226}{685} \times 100 = 33$
1.2 "	$\frac{170}{567} \times 100 = 30$
1.5 "	$\frac{149}{713} \times 100 = 21$
2 "	$\frac{87}{625} \times 100 = 14$
3 "	$\frac{57}{590} \times 100 = 9$

From this table it may be seen that after exposure to the action of serum in which 0.2% common salt had been dissolved, a solution which had lowered the phagocytarian capacity by 29.6% (see table III), replacement in normal serum brings it back to its original value. The action of serum in which 0.7% of salt had been dissolved, however, causes a permanent loss of phagocytosis. Still, this loss is not so great considering that, in the serum with 0.7%, not a single cell has taken up carbon, — in other terms, the phagocytosis has been entirely paralysed.

Now the phagocytes had only been exposed for half an hour to the action of the concentrated media. This certainly may be considered long enough for the small cells to readjust themselves to their new medium. Still, it may be asked whether *after a more prolonged exposure the normal value of the phagocytarian power would be restored too*. This question is of great importance for the functions of normal life, in

which the increase of osmotic concentration often lasts longer than half an hour. For this reason, the experiments in which the leucocytes were exposed to an action of much longer duration, were made with serum containing only 0.1 and 0.2% NaCl; higher osmotic concentration does not occur in the body. The leucocytes were placed in the serum of increased osmotic concentration for 2, 24 and 48 hours, and then transferred into normal serum.

The experiment showed that after an exposure of 24 and 48 hours, the phagocytarian power had been diminished; but an *equal* decrease of vitality was also observed in phagocytes which had remained for 24 and 48 hours respectively in *normal* serum. This proved that the prolonged action of serum of increased osmotic concentration had had no *permanent* injurious effect on the phagocytarian capacity.

Thus we may conclude that, in the living body an increase in the osmotic concentration of the blood plasma, as well as a decrease of the same, has a deleterious effect on the phagocytarian power, but that the loss may be recovered; for as soon as the osmotic pressure has been restored to the normal, the phagocytes also entirely regain their inherent power.

If from these experiments we may conjecture, that what we have observed in the phagocytes, will also be applicable to other cells with semipermeable walls, it is reasonable to conclude from the results shown in tables II and III, *that the vital functions of the cell are in a large measure influenced by slight oscillations in the osmotic concentration of the environment and consequently of the cells themselves.*

IV. *Effect of simple solutions of Salt.*

1. Solutions of Sodium Chloride.

Now the question arises whether the loss of vitality described above, must be attributed to the variations of the quantity of water as such, or to the modification in the concentration of one or more of the substances.

In order to examine this question systematically, we might have alternately reduced the several elements in the diluted serum to their original concentration and then studied the extent of the improvement. But as in the mean time it had been clearly demonstrated to us that in a pure solution of Sodium Chloride of 0.9% the phagocytes take up carbon in equal or almost equal quantities as in normal serum, we decided to abandon this mode of investigation.

Here we must incidentally remark that, after all that has been said by

LOEB¹⁾ and others, of the injurious action of a pure solution of a simple salt on the life of young moving larvae and the vital processes of higher animals, such as the beating of the heart and the movements of the intestines, we were at first rather astonished at the almost perfect innocuousness of similar solutions in regard to the phagocytes. However, we can easily find an explanation for this seeming inconsistency. Whenever a cell is surrounded by a simple isotonic solution of salt, two things are likely to happen: an exchange of ions may take place, thus causing a modification in the chemical structure of the cell, which interferes with certain of its vital functions. This is the case with the larvae of fundulus with the muscle of the heart and that of the intestines. A supply of specified ions is then required to restore the chemical structure of the cell to its normal state. But — and this is the second possibility — if the permeability of the cell to ions is highly limited, a pure isotonic solution of salt will not cause any, or only a very slight alteration in the chemical structure of the cell. This is the case with the white corpuscles, the slight permeability of which to ions of salts has already been demonstrated in the most convincing manner.²⁾

Bearing this fact in mind, it can cause no surprise that, contrary to the results of the observations on eggs and muscles, a pure solution of Sodium Chloride leaves the phagocytarian power entirely or almost entirely intact.

Under these circumstances, for determining the influence of the water as such, it was indicated to take solutions of Sodium Chloride of various strength.

Table V shows the action of diluted solutions of Sodium Chloride on phagocytosis.

TABLE V.

Effect of hyper-isotonic solutions of Sodium Chloride on phagocytosis.

Solutions of Salt.	Percentage of leucocytes which have taken up carbon.
NaCl-sol. 0.9%	$\frac{235}{756} \times 100 = 31\%$
NaCl 0.75% = NaCl 0.9% + 20% water	$\frac{208}{741} \times 100 = 28$
„ 0.6 = „ 0.9 + 50 „	$\frac{221}{1012} \times 100 = 21.8$
„ 0.45 = „ 0.9 + 100 „	$\frac{83}{745} \times 100 = 11.1$

¹⁾ J. LOEB, American Journal of Physiol. 3 1900 p. 327 and 383; 5 1901 p. 362 Pflüger's Archiv 80 1900 S. 229.

LINGLE, Americ. of Journal of Physiol. 4 1900 p. 265.

MISS MOORE, Ibid. 1900 p. 386 etc.

²⁾ HAMBURGER. Zeitschr. f. Biol. 35. 1897 S. 252 and S. 280; Proceed. of the Royal Academy of Sciences 11 April 1897.

Archiv f. (Anat. u.) Physiol. 1898 S. 31 and S. 317.

VIRCHOW'S Archiv 156 1899 S. 329.

HAMBURGER and VAN DER SCHROEFF, Archiv f. (Anat. u.) Physiol. 1902. S. 251.

Here we see the marked effect of a diminution in the concentration of the salt solution.

TABLE VI.

Effect of hyper-isotonic solutions of salt.

Solutions.	Percentage of leucocytes containing carbon.	Decrease of the phagocytarian power.
NaCl 0.9%	$\frac{250}{722} \times 100 = 34.6\%$	
" 0.95	$\frac{293}{875} \times 100 = 33.5$	30%
" 1	$\frac{95}{802} \times 100 = 11.84$	60.6
" 1.1	$\frac{105}{981} \times 100 = 10.8$	69
" 1.2	$\frac{7}{990} \times 100 = 0.7$	98
" 1.3	$\frac{0}{200} \times 100 = 0$	
" 1.4	$\frac{0}{150} \times 100 = 0$	
" 1.5	$\frac{0}{150} \times 100 = 0$	

The surprisingly rapid decline of phagocytosis observed in serum of increased concentration (table III) is again clearly demonstrated in this instance.

Even the slight increase of 0.9 to 1% lowers the phagocytarian power already 60.6%. Another illustration of this rapid decline is afforded by the observation that in the 1% concentration of Sodium Chloride, the amount of carbon present in the coal containing phagocytes is far less than in those that have stayed in the solution of 0.9%.

Now, by comparing tables VI and III, we see at a glance that, when the experiments were made with a solution of Sodium Chloride of 0.9%, to which afterwards salt had been added, *the decline in the phagocytarian power is more marked* than when they are

made in serum supplemented with an equal quantity of salt. This proves that besides the osmotic pressure, which must principally be made accountable for the decline, there is still another factor at work, *and this factor can be no other than the modification — however slight — produced by a pure solution of NaCl in the chemical structure of the phagocytes.* Some time ago, one of us, in conjunction with Dr. VAN DER SCHROEFF¹⁾, already demonstrated that the leucocytes the same as the red corpuscles are in any case permeable to *anions*. It is therefore evident that, owing to their chemical structure being interfered with, the cells most lose some of their vitality (phagocytarian power) under the action of pure salt solutions, — or rather, that they should lose *more* than in an isosmotic serum.

We have submitted this hypothesis to further experiments, starting from the following reasoning: If it is a fact that in a hyper-isotonic solution of salt, the phagocytes undergo a chemical variation through exchange of ions, it must be possible to restore this loss of phagocytarian capacity resulting from their modification in their structure, by replacing them in normal serum, and that this recovery will not be complete by immersion in a 0.9% solution of salt. The following table proves that we were correct in our surmise.

TABLE VII.

Effect of solutions of salt on the chemical structure
of the phagocytes.

White corpuscles immersed for 2½ hours in the following solutions.	Phagocytarian power after being transferred into.	
	Normal Serum	Salt solutions of 0.9%
NaCl 0.9%	$\frac{319}{942} \times 100 = 33.9\%$	$\frac{284}{811} \times 100 = 35\%$
" 1	$\frac{258}{775} \times 100 = 33.3$	$\frac{251}{760} \times 100 = 33$
" 1.1	$\frac{233}{790} \times 100 = 29.5$	$\frac{209}{735} \times 100 = 28.6$
" 1.2	$\frac{202}{722} \times 100 = 28$	$\frac{175}{677} \times 100 = 26$

¹⁾ HAMBURGER and VAN DER SCHROEFF. l. c.

It is clearly demonstrated that the phagocytes, which have been exposed for two hours to the action of solutions of Sodium Chloride, exhibit a greater phagocytarian power when they are transferred into serum, than when they are placed into salt solution of 0.9 %.

No doubt the observation will strike the attentive reader as contradictory, that this is only the case with the phagocytes which had been exposed to the action of sodium chloride of 1 %, 1.1 % and 1.2 %, but not with those which for the same space of time had been immersed in a similar solution of 0.9 %; then the effect of this salt-solution and the serum is quite the reverse. This, however, is not actually the case; for in serum the phagocytes are likely to stick together and on this account do not offer as large a surface to the carbon as in the salt-solutions in which they remain more isolated. If then, as must be the case in an *isotonic* solution, the injurious effects of the Cl-ions of the pure salt solution are comparatively small, they may easily be exceeded by the unfavourable position of the cells caused by the serum.

When, however, by the use of *hyper-isotonic* solution of sodium chloride, the injurious action of the Cl-ions be increased, it may exceed the detractory influence of the agglomeration of the cells, and produce the results shown in the table.

With regard to these statements it may here be asked why, in *isotonic* solutions of sodium chloride, the injurious effect on the phagocytes cannot be determined, but is easily demonstrated when hyper-isotonic solutions are used, and the more readily in proportion as the concentration of the salt solutions are increased in strength. This question is very natural, because it concerns such a small increase in the considerable amount of ions of Cl or of Na already present. Here we are involuntarily reminded of the fact stated by HEDIN¹⁾ with regard to the red blood-corpuseles. The minute investigations of this scientist have brought to light the fact, that in isosmotic *isotonic* solutions of salt, the corpuseles possess an equal volume, but that in isosmotic anisotonic solutions their relative volume is no longer equal. HEDIN has not given an explanation of this important fact; but anyhow, it proves that simple solutions of salt, when anisotonic, exercise still another kind of action beyond that of their osmotic pressure. We propose to investigate this matter somewhat further: it is very probable that by a modification in the dissociation of the contents of the cell, an altered condition for the exchange of ions is produced.

¹⁾ HEDIN. Skandinavisches Archiv f. Physiol, 1895 S. 377.

2. Solutions of Chloric Potash.

In our description of the influence of sodium chloride, we attributed it to the ions of chlorine. This was based on the results of investigations in which we compared the action of sodium chloride and of potassium chloride, of which a few items here follow.

These investigations proved that isosmotic solutions of sodium chloride and potassium chloride have almost the same effects on phagocytosis.

T A B L E VIII.

Comparison of isosmotic quantities of sodium chloride and potassium chloride.

		Percentage of leucocytes containing carbon.
Serum		$\frac{253}{722} \times 100 = 35\%$
NaCl-sol. 0.9%		$\frac{300}{836} \times 100 = 36$
KCl-sol. 1.15% (isot. m. NaCl-sol. 0.9%)		$\frac{258}{277} \times 100 = 34$
Serum + 0.1 % NaCl		$\frac{183}{672} \times 100 = 27$
" + 0.127 " KCl		$\frac{181}{715} \times 100 = 25$
" + 0.3 " NaCl		$\frac{45}{630} \times 100 = 7$
" + 0.38 " KCl		$\frac{54}{683} \times 100 = 8$
" + 0.3 " NaCl	} afterwards placed into normal serum	$\frac{184}{600} \times 100 = 30$
" + 0.38 " KCl		$\frac{185}{621} \times 100 = 30$

Two other parallel-experiments in solutions of 0.9% of sodium chloride produced the following results:

$$\frac{198}{863} \times 100 = 23\% \text{ of leucocytes containing carbon}$$

$$\text{and } \frac{146}{677} \times 100 = 21.5\% \text{ ,, ,, ,, ,,}$$

in the isosmotic solution of KCl 1.15 %:

$$\frac{128}{615} \times 100 = 21 \% \text{ leucocytes containing carbon}$$

$$\text{and } \frac{165}{732} \times 100 = 22.5 \% \quad , , \quad , , \quad , ,$$

Hence we may conclude, that there is no difference between the action of chloride of potassium and chloride of sodium.

3. Effect of chloride of calcium.

The great importance which, according to the most recent investigations must be ascribed to the ions of calcium,¹⁾ in the constitution of the fluid-matter of the tissues, induced us to test also the effect of this medium on phagocytes.

With this object in view, we dissolved various quantities of chloride of calcium in the serum of the blood of a horse and mixed the suspension of leucocytes thus obtained with carbon.

T A B L E IX.

Effect of calcium chloride.

Serum +	Percentage of leucocytes containing carbon	Increase of the phagocytarian power
0% CaCl ₂ 6 aq.	$\frac{132}{612} \times 100 = 21.2\%$	
0.01%	$\frac{225}{861} \times 100 = 26$	22.6%
0.1	$\frac{180}{652} \times 100 = 27.6$	30.2
0.5	$\frac{162}{598} \times 100 = 27$	27.3
1	$\frac{0}{724} \times 100 = 0$	

An addition of 0.01 % of Ca Cl₂ 6 aq. to the serum already produces an increase of the phagocytarian capacity of 22.6 %; by the addition of 0.1 % Ca Cl₂ 6 aq., the effect is somewhat

¹⁾ See especially the investigations of LOEB. Publications of the University of California and of LANGENDORFF and HUECK. Pflüger's Archiv 96 1903 S. 473; for the complete bibliography on the subject until 1904, see Osmotischer Druck und Ionlehre B. III, S. 107 etc. Comp. also A. NETTER, Importance biologique du Calcium. Paris. Masson et Cie, 1907.

greater, and by the addition of 0.5 % Ca Cl_2 6 aq., it again decreases.

The result registered in the first instance, which is produced by the addition of 0.01 % Ca Cl_2 , must be considered the most valuable, for it denotes the nearest unalloyed effect of the calcium chloride. In the experiments where quantities of 0.1 %, 0.5 % and 1 % of Ca Cl_2 were added, the increase of phagocytosis is counteracted by the unfavourable influence of the raising of osmotic pressure.

This experience is in strict accordance with the observations made by LANGENDORFF, who found that the injection of very small quantities of calcium, causes the heart to beat with greater force. We ascribe this manifestation to the action of the ion of calcium on the contractile substance, and we may conclude that the muscular fibre and the phagocytes also, are permeable to this cation.

4. Effect of citras natrius.

The frequent use which, in consequence of the experiments of WRIGHT and DOUGLAS,¹⁾ is made of this medium at the present day by the bacteriologists, in order to prevent the coagulation of the blood, actuated us also to experiment with this substance for the sake of determining its action on the phagocytosis. The following table gives a survey of the results.

The customary solutions of 1 % and 2 % of citras natrius in 0.9 % solution of sod. chl. were used in these experiments.

T A B L E X.

Effect of citras natrius.

	Percentage of leucocytes containing carbon
(a) 1 cc. suspension of leucocytes + 2 cc. solution of 1% citras natr. in 0.9% Sod.Chl.	0
(b) 1 cc. suspension of leucocytes + 2 cc. solution of 2% citras natr. in 0.9% Sod.Chl.	0
(c) leucocytes from (a) transferred in Sod.Chl. 0.9 %	$\frac{260}{686} \times 100 = 38 \%$
(d) leucocytes from (b) transferred in Sod.Chl. 0.9 %	$\frac{255}{731} \times 100 = 35 \%$
(e) 1 cc. suspension of leucocytes + 2 cc. solution of 0.9 % Sod.Chl. (Control test)	$\frac{369}{725} \times 100 = 50 \%$

¹⁾ WRIGHT and DOUGLAS, Proceed. of the Roy. Soc. **72**, 1903, p. 357; **73**, 1904, p. 128.

From the above table it is shown: 1. That in 1—2% solutions of citras natrius in 0.9% of Sod. Chl. the phagocytarian power is nil.;

2. that the phagocytarian capacity again *partially reappears*, when the cells are transferred into 0.9% solutions of Sod. Chl. *The permanent decline of the phagocytarian power still amounts to 28%.*

5. Effect of Fluornatrium.

Fluornatrium being also much used for preventing the coagulation of the blood, it seemed important to us also to study the effect of this medium on the phagocytosis.

T A B L E XI.

Effect of Fluornatrium.

	Percentage of leucocytes containing carbon	
	Before being transferred into 0.9% Sod. Chl.	After being transferred into 0.9% Sod. Chl.
2cc suspension of leucocytes + 2cc NaFl 0.65% (isot. with NaCl 0.9%)	0%	$\frac{91}{677} \times 100 = 14\%$
" + 2cc NaFl 1%	0	$\frac{30}{514} \times 100 = 6$
+ 2cc NaFl 2%	0	0
+ 2cc NaCl 0.9%		$\frac{369}{725} \times 100 = 50$

Here we see that when the leucocytes have been exposed to a solution of Fluornatrium of 2%, 1% or 0.65% (isot. with 0.9% NaCl) the phagocytarian power is entirely paralysed, yea, that even after transferring of the phagocytes in a solution of 0.9% Sod. Chl. it shows to have been entirely destroyed for ever. *Hence we may conclude that NaFl is a powerful poison for the protoplasma of the phagocytes.*

V. Effect of acid and alkali.

1. Effect of acid.

The important part which the alkaline reaction of the blood-plasma seems to play, not only in connexion with the degree of

oxydation taking place in the body, but also in infectious diseases, induced us to study its effects on the increase or decrease of the phagocytarian power.

The results of one of the experiment are shown in the following table.

T A B L E XII.

Diminution of the alkaline reaction of the serum.

1 cc $\frac{1}{2}$ n. H_2SO_4 +	Amount of acid added	Percentage of white blood corpuscles containing carbon
9 cc serum	$\frac{1}{20}$ norm.	0
14 cc "	$\frac{1}{30}$ "	$\frac{43}{308} \times 100 = 4.3\%$
19 cc "	$\frac{1}{40}$ "	$\frac{35}{398} \times 100 = 9$
49 cc "	$\frac{1}{100}$ "	$\frac{165}{724} \times 100 = 21.4$
299 cc "	$\frac{1}{600}$ "	$\frac{255}{612} \times 100 = 41.7$
499 cc "	$\frac{1}{1000}$ "	$\frac{256}{530} \times 100 = 43.5$
normal serum		$\frac{227}{530} \times 100 = 43$

We observe that even the small addition of $\frac{1}{600}$ n-acid is injurious to the phagocytosis.

Now, we know that according to titration with lacmoïde, 100 c.c. horse serum in the mean is equivalent to 75.5 cc. $\frac{1}{25}$ n-acid¹⁾; consequently it is calculated that serum represents an alkaline fluid of $\frac{1}{30}$ normal.

The addition of $\frac{1}{600}$ n-acid, therefore lowers its alkaline reaction by 5%.

Consequently, a diminution of the alkaline reaction of the serum by 5% is already injurious to the phagocytes.

This result is in strict accordance with the injurious effect experienced

¹⁾ HAMBURGER, Verhandl. d. Koninkl. Akad. v. Wetensch. Second section, Vol. VI, N^o. 1, 1897.

by administering acid per os, and we are fully justified in ascribing the poisonous effects of the acid, to a diminution in the process of oxydation.

The results agree also with the observations recently published by J. LOEB, on the influence of the traces of NaOH (OH-ions) on the artificial fructification of the eggs of sea-urchins. The author has clearly demonstrated that the primary cause of this effect might be found in the acceleration of chemical reactions.¹⁾

T A B L E XIII.

Increase of the alkaline reaction of the serum.

1cc $\frac{1}{2}$ n. NaOH +	Amount of alkali added	Percentage of white corpuscles containing carbon.
29cc Serum	$\frac{1}{60}$	$\frac{25}{622} \times 100 = 4$
37cc "	$\frac{1}{78}$	$\frac{57}{840} \times 100 = 6.8$
49cc "	$\frac{1}{100}$	$\frac{114}{707} \times 100 = 16$
99cc "	$\frac{1}{100}$	$\frac{179}{716} \times 100 = 25$
199cc "	$\frac{1}{100}$	$\frac{143}{531} \times 100 = 27$
399cc "	$\frac{1}{800}$	$\frac{149}{580} \times 100 = 25.7$
normal serum		$\frac{177}{664} \times 100 = 26.5$

It is seen from this table that, within a large margin, the addition of OH-ions to the serum does not exercise a perceptible influence on the phagocyterian power; it remains unaltered until the value is increased by $\frac{1}{200}$ normal: i.e., with 15 % of the original alkaline reaction. An additional supply of alkali causes a lessening of the phagocyterian power.

More pronounced still is the effect of acid and alkali on the phagocytes, when these substances, instead of being added to serum, are

¹⁾ J. LOEB, PFLÜGER'S Archiv 118, 1907, H. 3/4, S. 181.

introduced in solutions of 0.9 % sod. chl. A more detailed account of the results of these investigations will follow later.

We also made a number of experiments to test the influence of other media on the phagocytarian power, e.g. with *ureum*, *chinine*, *argentum colloidalale*, *heterogenous serum*, etc., the results of which will appear in a subsequent paper.

Summary.

The following are the principal conclusions derived from the above described experiments.

1. The action exercised by various media on the phagocytarian power of white corpuseles, can be accurately determined by counting the percentage of cells which have taken up particles of carbon.

2. *The addition of water to the inherent medium of the phagocytes i.e. to their own serum, acts injuriously on the phagocytarian power.*

Even a decrease in the osmotic concentration as may daily occur in a normal individual, causes a perceptible decline in the phagocytarian power.

So, it was shown in one of the experiments that, whilst in normal undiluted serum 37 % of the leucocytes had taken up carbon, in serum which had been diluted with 20 % of water the amount of cells containing carbon was only 32 %: this corresponds to a decline in the phagocytosis of $\frac{37-32}{37} \times 100 = 13.5\%$.

By the addition of 50 % water, the percentage of phagocytes containing carbon fell to 21 %; thus in this case a decrease of phagocytosis of $\frac{37-25}{37} \times 100 = 43\%$.

By addition of 140 and of 200 % water, the percentage of the carbon-containing leucocytes was lowered to nil, — in other words the phagocytarian power had been suspended; but only *temporarily, for*

3. *by replacing the cells damaged by the addition of water, into their own serum, the phagocytarian power is entirely or partially restored.*

So the recovery was complete, when the serum had been diluted with 20 % or 50 % of water, and only partial when 70 tot 100 % water had been added. Even when it had been diluted with 200

a figure at which, it is shown under 2, the phagocytosis had been entirely suspended, — a recovery took place in the phagocytarian power to half of its original amount.

4. *The observations, here made with the phagocytes, correspond with those previously observed in the red corpuscles.*

1. The phagocytes, the same as the red corpuscles, can support a considerable quantity of water ($\pm 60\%$) without a single cell being destroyed;

2. The modifications produced in the phagocytes by the addition of water, unless they have led to their entire destruction, may, judging from the phagocytarian capacity, be entirely obviated by replacing them in normal serum.

5. *A heightening of the osmotic concentration of the serum, as well as a lowering of the same, (comp. sub 2) has a very injurious effect on the phagocytosis. It was obvious that an increase of the osmotic concentration had even a more pronounced deleterious action than the decrease at the same ratio.* Already an addition of 0.1% NaCl to the serum caused the phagocytarian power to decline 17.3% .

By the addition of 0.4% NaCl this decrease amounted to 79.2% and by the addition of 0.5% Sod. Chl., the phagocytarian power was reduced to nil; but this considerable loss was but temporary, for

6. *when the cells which had been damaged by an addition of sodium chloride to the serum, were replaced in their original blood-serum, their phagocytarian capacity was again entirely or partially restored; entirely when only from 0.1% — 0.2% of the substance had been added; partially when a greater amount had been used.*

7. If thus, as shown under 2 and 5, the phagocytarian power is specially impaired by modification of the normal osmotic concentration of the blood-serum, this capacity will be entirely restored as soon as the blood-plasma, principally owing to the activity of the kidneys, has recovered its normal osmotic concentration. The experiments have demonstrated that this recovery is still possible after the agency of the anisotonic serum for 24 hours and more.

8. *In solutions of 0.9% NaCl the phagocytarian power is almost equal to that of serum.* It considerably decreases under the action of weaker and stronger solutions of this salt, even more so than in serum which has been made isosmotic with these salt-solutions.

9. *This result leads to the conclusion that the decline of the phagocytarian capacity produced by anisotonic serum, has its cause principally in the alteration of the amount of water in the cells.*

10. *Besides the modification of the amount of water in the cells, another factor comes into play, namely the chemical change, which takes place consequently on the exchange of the contents of the cells with those of their environment and which, as a matter of course, is greater when the cell is surrounded by a simple solution of NaCl than when placed in an isosmotic serum. This accounts for the fact, that phagocytes which have been submitted to the action of hyperisotonic solutions of NaCl, when replaced into serum, exhibit a somewhat greater phagocytarian power, than when they are transferred to a 0.9% solution of NaCl. In the latter case they have not the opportunity, given them in the former, of regaining the ions which they have lost in the anisotonic solutions of salt.*

11. *It is very probable that the ions of Ca and of OH belong to this category.*

With regard to calcium, it has been proved that by the addition to the serum of the minute quantity of $0.01\% \text{ CaCl}_2$ 6 aq, i.e. about $0.005\% \text{ CaCl}_2$, the phagocytarian power was raised by about 22.6% . The inference is that ions of calcium must have penetrated into the phagocytes.

On the other hand it may be surmised that the phagocytes will lose ions of calcium when the amount of calcium in the medium is lower than that to which the phagocytes are accustomed. This loss of ions of calcium must cause a diminution of the phagocytarian power.

We observe a similar result in the case of the OH-ions; for our experiments have demonstrated that decrease of these ions causes a lowering of the phagocytarian power. *A 5% diminution of the alkaline reaction of the serum, which necessarily must lower the amount of alkali in the phagocytes, produces a noticeable decline in the phagocytarian capacity.*

12. LOEB and after him other investigators have pointed out, that a pure solution of NaCl must be considered injurious to the larvae of lower sea animals, the muscles of the heart, and those of the intestines. *This opinion does not hold for the phagocytes.* The proof of this assertion is found in the fact that in a solution of NaCl isotonic with serum, the phagocytosis is almost as powerful as in the serum itself.

This seeming contradiction may be met by the explanation that the exchange of substance between the leucocytes and the solution of NaCl, especially when the latter is isotonic with the serum, is very small; whilst in the case of other cells (ciliated cells, muscular fibre cells) the conditions of the exchange of ions are not so restricted, and consequently the chemical structure of these cells is more easily modified. And it is obvious that a modification of their chemical structure causes a disturbance in their inherent functions.

13. From the facts here recorded, it is evident that in studying the action of the phagocytes on bacteria *in vitro*, the degree of osmotic concentration and of the alkaline reaction of the medium, must be taken into account. This condition has been lost sight of in several of the experiments. They ought therefore to be repeated.

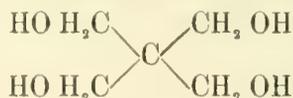
Groningen, June 1907.

Chemistry. — “*The decomposition of penta-erythritol tetraformate on heating.*” By Prof. P. VAN ROMBURGH.

(Communicated in the meeting of June 29, 1907).

As the heating of the diformate of *s.* divinylglycol had led in such a simple manner to hexatriene 1.3.5, investigations have been set on foot in my laboratory for studying the decomposition of formic esters of polyhydric alcohols, the results of which will be gradually communicated.

If for penta-erythritol we accept the formula:

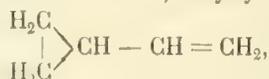


and if the reaction took place in a similar manner as with *s.* divinylglycol diformate, we might expect on heating the tetraformate¹⁾ the formation of a hydrocarbon of the formula:



in which occurs twice a 3-ring.¹⁾

¹⁾ GUSTAVSON C. R. 123 (1896) 242 obtained from the tetrabromide of penta-erythritol, by the action of zinc and alcohol, vinylcyclopropan:



instead of the above cited hydrocarbon.

The reaction, however, proceeds in quite a different sense, for instead of the hydrocarbon we only obtain carbon monoxide, while penta-erythritol is regenerated.

In order to prepare the tetraformate of penta-erythritol, this substance is heated with an excess (8 mols.) of concentrated formic acid in a flask connected with a condenser, in such a manner that the excess of acid distils over slowly with the water formed in the reaction. When the thermometer placed in the liquid shows 120° , the heating is stopped, and the distillation is repeated with a fresh quantity of formic acid. After this the heating is repeated twice with 100% acid. There then remains in the flask an oily liquid, which, when placed in a dish over sulphuric acid, abundantly deposits crystals after some time. These are collected at the pump, and then thoroughly pressed between filterpaper.

The solid substance thus obtained, after having been recrystallised from dry benzene, melts at 55° . After repeated recrystallisation from that solvent, the melting point rose to 57° and then remained constant.

This formate is sparingly soluble in ether; from a solution in benzene it is precipitated by ether: It has a slightly bitter taste.

On boiling with a titrated solution of potassium hydroxide the formic acid formed may be readily estimated.

Found 74.21 and 74.16% formic acid

Calculated = 74.18%.

The ultimate analysis gave the values expected for penta-erythritol tetraformate :

Found	Calculated for $C_5H_{12}O_8$
C 43.6 43.75	43.57%
H 5.26 5.16	4.88%

On heating this formate, a plainly visible evolution of gas commences at 220° , which is fairly strong at 230° . The gas evolved consists of pure carbon monoxide and when the evolution of gas has ceased, there remains in the flask pure penta-erythritol, which may be readily identified as such by its properties. The amount of gas evolved is that required by theory.

I wish to express my thanks to Mr. VAN ENDT, who has assisted me in these experiments with care and zeal.

Mr. RENIER, who is engaged in the study of the formates of glycols found that from 2.5 dimethyl-hexandiol 2.5, the well known tetramethyldihydrofurane, is formed by simply heating with formic acid.

On the other hand the formic ester of pentandiol 2.4 is very stable towards heat.

At about 400°, however, the ester is decomposed and a liquid is formed boiling at 42° which is most probably 1.2 dimethyl-cyclopropane.

Mr. VAN MAANEN, is engaged in the study of the decomposition of the formic esters of mannitol.

Utrecht, Org. Chem. Lab. University.

Botany. — “*On the influence of wound stimuli on the formation of adventitious buds in the leaves of Gnetum Gnemon L.*” By Mr. J. VAN BEUSEKOM. Communicated by Prof. F. A. F. C. WENT.

(Communicated in the meeting of June 29, 1907).

It had been observed for a long time already that adventitious sprouts were formed on the leaves of a specimen of *Gnetum Gnemon L.*, cultivated in the Botanic Garden at Utrecht.

In January 1906 my attention was drawn to this circumstance by Prof. WENT, who advised me to study the development of these adventitious sprouts, and to try to discover the origin of their formation.

The results of this investigation will be concisely communicated here.

The adventitious buds appear on the tips of the leaves, while these are still attached to the plant.

As far as I have been able to find out, the formation of these adventitious buds has never before been observed with *Gnetum Gnemon*, neither in its natural sites, nor in botanic gardens¹⁾ except at Utrecht. The Utrecht Garden possesses three specimens of *Gnetum Gnemon*. One of these has been continually cultivated in a hothouse where in winter the temperature is kept at about 25° C., and the air is very damp. The other two were, when I began my investigation, in an other hothouse where the temperature is lower (in winter on an average 15° C.), and the humidity less. Whereas of the former I have always obtained leaves in different stages of bud-formation, the other two showed the phenomenon only after they had been conveyed to the warmer and damper hothouse.

Although all three plants, apart from the formation of adventitious buds, are evidently healthy and do not make a morbid impression

1) On this point I gained information from the other botanic gardens in our country, from that at Munich and also from that at Buitenzorg.

at all, they flower very rarely. Personally I only observed it with one of the plants from the cooler hothouse. This latter plant produced one single ♂ inflorescence, which enabled me to check the accuracy of the determination.

The first external change, noticed with a leave which will form adventitious buds, is that on the tip extremely small yellow dots appear, which are seen best when light is falling through the leaf. They remind us in this respect of the oil dots in the leaves of the *Rutaceae* or *Hypericum*, but as a rule they are bigger and less densely spread than these.

With the bigger ones a hand-magnifier will show that where the dots are, the epiderm of the upper or lower side or of both together is slightly bulged, so that we have to do with small vesicles.

It will be shown presently that these vesicles are caused by the sting of a scale-insect *Aspidiotus* spec., and as such are not restricted at all to the tip of the leaf. Normally, however, it is only the tip which can form adventitious buds. The remaining part of the leafblade can only form adventitious buds when the organic relation with the tip has been disturbed in some way or other. But even then they arise apically in this part. Hence only the vesicles which have originated on the apical part of a leaf, form the introduction to the process of bud-formation.

For the sake of simplicity we shall in what follows, only mention the tip of the leaf, since the statements referring to the tip also apply to the other cases.

After some time also the region, surrounding the vesicles, becomes discoloured; as a rule the tip of the affected leaf soon becomes distinctly yellow, although in some cases it long keeps a more or less greenish tint.

At the same time with this discoloration the tip of the leaf becomes thicker. This thickening is at first not easy to observe macroscopically; gradually, however, it becomes stronger and at last generally advances so far that the tip becomes stiff and difficult to bend.

Of the yellow vesicles nothing can then be seen any longer.

The extent of this region of discoloration and thickening varies much in a basal direction; along the edge it generally extends farther basipetally than in the middle; always, however, the phenomenon is restricted to the apical part of the leaf. A new stage sets in, when the surface of the thickened leaf-tip which until now had remained smooth, on account of the swelling being even, becomes uneven: as well on the lower as on the upper surface this may as a rule be observed: on the upper surface it is generally more pronounced.

During the first weeks generally no striking changes are observed until after about a month a varying number of local elevations, yellow like the leaf-tip that produces them, becomes prominent and reveals the differentiation of special proliferating centres.

As a rule we see these grow to real knobs, especially in the direction perpendicular to the surface of the leaf.

While the knobs are still relatively small, brownish grey streaks begin to appear on their tops, which gradually extend, so that finally the whole knobbed surface becomes brown.

For some time such a knob shows nothing particular, except that it becomes larger and thicker. Next on a certain day a small opening is formed in its top, through which a small green point projects, which will grow out into a leafed sprout.

As well on the lower as on the upper surface of the leaf-tip knobs may form. Mostly they form on the upper surface, though. With some leaves I have observed knobs on both sides at the same time.

The observations on the time, needed by a leaf in order to form "ripe" knobs, after the yellow vesicles have appeared, have led to somewhat diverging results. The shortest period was observed with leaves on the upper branches or in the periphery of the crown, which consequently occupied the most favourable position with regard to light. On these good-sized, brown knobs had generally formed half a year after the appearance of the yellow vesicles.

Also for the question, how old and how large a knob must be in order to open and give the adventitious bud an opportunity for sprouting, no rule can be fixed. I saw one sprout five weeks after the knob had first been observed as a special elevation, while others were still closed after five to seven months.

About the size of the knobs we may state that some knobs, scarcely rising more than a millimetre above the surface of the leaf-tip, opened, while others of double and even treble the height remained obstinately closed. Yet these latter contain as well an adventitious bud and not seldom even more than one.

The microscopical investigation was for the greater part carried out on microtome preparations. For fixing the material I used the mixture: zincchloride-glacial acetic acid-alcohol, (2 grams of zincchloride and 2 ccm. of glacial acetic acid to 100 ccm. of alcohol of 45—50 pCt), recommended by JUEL¹⁾. The particular hardness of

¹⁾ H. O. JUEL, Ueber den Pollenschlauch von Cupressus. (Flora. Bd. 93. 1904. pag. 56—62).

the leaf tissue made it necessary to treat the material, before being embedded in paraffin, during 3 to 4 × 24 hours with a 40 pCt. aqueous solution of hydrofluoric acid. After this treatment it was then washed for 8 to 10 hours in streaming water, dehydrated by the usual method and after treatment with chloroform embedded in paraffin (melting point 62° C.).

For staining the sections I used at first Haematoxylin-Delafield and saffranin, according to the prescriptions given in CHAMBERLAIN'S "Methods in Plant Histology" ¹⁾; but this method proved unsatisfactory for differentiating the very thin-walled meristem cells. Therefore I afterwards always stained with methyl green and acid fuchsin ²⁾, by which very good results were obtained.

A consequence of the treatment with hydrofluoric acid was that the microtome preparations were not suitable for all observations. In these cases I used hand-cut preparations, if necessary stained with Haematoxylin-Delafield.

The anatomy of the normal leaf, on which something may be found in literature with BERTRAND ³⁾, DE BARY ⁴⁾, SCHEIT ⁵⁾ and HABERLANDT ⁶⁾, is as follows.

The epiderm of the lower and upper surface consists of relatively small, cubical cells, the outer wall of which is strongly thickened and provided with a strong cuticle and from which capriciously shaped and canaliculate outgrowths project into the lumen of the cell. (cf. BERTRAND, l. c. Pl. II fig. 6, 7, 8.). In the epiderm of the lower side numerous, irregularly placed stomata are found.

Under the upper epiderm lies the palissade parenchyma, formed by one continuous row of cells, slightly elongated in the direction perpendicular to the leaf surface. (dimensions 13–21 μ by 21–30 μ). Between the palissade cells and the lower epiderm lies the spongy parenchyma, consisting of tubular cells, the diameter of which is on the average 18 μ , as a rule is not more than 9 μ at a partition wall between two tubes and does not reach more than 28 μ . Between the cells of the spongy parenchyma remains a system of large inter-

¹⁾ G. J. CHAMBERLAIN, Methods in Plant Histology 2nd ed. Chicago. 1905. pag. 30, 38 and 54.

²⁾ CHAMBERLAIN. l. c. p. 40, 44 and 68.

³⁾ G. E. BERTRAND, Anatomie comparée des tiges et des feuilles chez les Gnétacées et les Conifères. (Annales d. Sc. nat. Botanique 5^{ème} série Tome XX. 1874).

⁴⁾ A. DE BARY, Vergl. Anatomie der Vegetationsorgane der Phanerogamen und Farne. (Handb. der Physiol. Botanik von W. HOFMEISTER. Leipzig. 1877.)

⁵⁾ M. SCHEIT, Die Tracheiden-Säume der Blattbündel der Coniferen etc. (Jenaische Zeitschr. f. Naturw. Bd. XVI. Neue Folge Bd. IX. 1883.)

⁶⁾ G. HABERLANDT, Physiologische Pflanzenanatomie 2te Aufl. Leipzig. 1886

cellular spaces. In the spongy parenchyma numerous thick-walled sclerenchyma fibres are found, which are generally ramified and often have an enormous length.

A section through a leaf-tip on which yellow vesicles are found, shows that these vesicles are caused by hypertrophy of cells of the spongy parenchyma which have there entirely lost their tubular shape and among which specimens are found, measuring 91 by 109 μ , 100 by 73 μ , 100 by 113 μ etc. Among the cells, constituting the vesicle, some are always found which in unstained preparations are conspicuous by their wall being more or less swollen and brown. In preparations, treated with acid fuchsin and methyl green, the wall of these cells is blue, those of the other cells red. Applying the usual reactions we find that these walls have become suberized. Also of the palissade parenchyma some cells may have become larger, but always in a small degree.

While in the vesicles themselves the process goes a little further still on account of partition walls forming in some of the enlarged cells, whose walls have not become suberized, also the region, surrounding the vesicles, evidently answering to a stimulus, proceeding from them, begins to undergo similar changes. Macroscopically we detect this by the more or less yellow tint, assumed by the vicinity of the vesicles. A microscopical examination of the section shows that now also outside the vesicles the cells of the spongy parenchyma are hypertrophical. As with the formation of the vesicles the chlorophyll is disorganised in the hypertrophical cells.

At this stage no function of importance may be ascribed yet to the cells belonging to the palissade parenchyma.

In most preparations now already the peculiar behaviour is noticed of those spongy parenchyma cells which border immediately on the palissade parenchyma. While the other cells of the spongy parenchyma swell as evenly as possible in all directions, those which lie immediately below the palissade parenchyma become enlarged especially in a radial direction, thus making the impression of a second layer of palissade cells. Since also in later stages they will repeatedly draw our attention, I shall in what follows call these cells subpalissade cells, instead of using the cumbrous longer definition.

This extension, especially in a radial direction, of the subpalissade cells, is illustrated by the following two tables.

A Subpaliss. cells of a normal leaf-tip	Height μ : 16	16	14.5	18	14.5	16	14	14.5	14.5	22
	Breadth μ : 31	18	16	25.5	27	27	18	25.5	31	31
B Subpaliss. cells of a yellow leaf-tip	Height μ : 33	26	38	44	47	42				
	Breadth μ : 31	29	18	31	33	33				

In a leaf-tip which macroscopically is distinguished, besides by the yellow colour, by a distinct thickening, the intercellular cavities of the spongy parenchyma are found to have entirely disappeared, excepting a small corner here and there. The mutual pressure which the cells consequently begin to exert on each other, causes them to assume a more polygonal shape. The cells, bordering on the subpalissade cells and often also the rows, turned towards the lower epiderm, show a tendency to stretch themselves in a direction perpendicular to the surface of the leaf. In many spongy parenchyma cells partition walls have formed.

The part, played by the subpalissade cells in the process of thickening, is generally a very important one. So I found in a leaf-tip in a place, where it was 332 μ thick (the section of a normal tip is on the average 170 μ), cell rows, formed of subpalissade cells, partitioned by two or three walls and measuring 90, 110, 115 and 127 μ in height. — The palissade rows were in these places not sensibly enlarged.

That the different tissues also in the same leaf-tip do not everywhere play the same part in the process of thickening, appears from the following figures, from measurements in two different places of the same leaf-tip;

Height of the palissade cells.	36 to 45 μ	} most cells with 2 or 3 partitions	27 μ	non-partitioned.
Height of the subpaliss. cells.	73 to 82 μ (mostly partitioned)		146 μ	} with numerous partition walls.
Height of the spongy parench.	273 μ .		273 μ .	

The special thickened outgrowths, mentioned on page 3, are caused by the same processes of proliferation which cause the thickening of the whole leaf-tip, and which in some places go on with particular activity, while the surrounding region seems to come to rest.

On the upper surface they are formed by locally strong proliferation of the subpalissade cells. Sometimes also the underlying cells, originating from the spongy parenchyma contribute to them and then it cannot be ascertained as a rule what part is derived from the subpalissade cells and what from the original spongy parenchyma. In most cases the contribution of the palissade parenchyma to the formation of the special elevations is rather unimportant.

The special thickenings on the lower surface of the leaf are entirely formed by cells which genetically belong to the spongy parenchyma but for the rest in exactly the same way as those on the upper surface. Since the cells from which they are built up, divide parallelly to the surface of the leaf and the so-formed division-

cells stretch themselves again, these special thickenings, which at first appear as small unevennesses, grow out into the knobs, already mentioned on p. 3.

How has the epiderm been able to follow the increase of surface, accompanying these thickening processes?

In a normal leaf-tip we find for the dimensions of the epiderm cells 9 to 18 μ height and 9 to 29 μ breadth, while in the epiderm, covering a special thickening, amidst cells of normal dimensions others are found which measured :

Height μ :	9	11	13	13	13	9	11	11	9	9
Breadth μ :	31	36	45.5	36	49	45.5	45.5	42	54.6	45.5

Hence some epiderm cells seem really to broaden; whether this is only an extension or active growth, I dare not decide.

Besides, the epiderm soon gives way and is rent. Like the part of the epiderm which gives way to the pressure, some cells of the tissue underneath die off, the cell-walls turning brown. In this way arise the brown streaks on the surface of the knobs which finally by extension in tangential direction of this suberizing process becomes entirely brown. A special suberizing meristem, a phellogen, is not formed.

The regular structure of these cell-hills is lost as soon as the differentiation of a meristem commences. Some cells, assembled in a small group, then enter a new stage of strong growth, which makes them conspicuous in the preparations by a more rounded form amidst the adjoining cubical cells. A number of the surrounding cells are compressed by the pressure which these primordial cells cause by their growth, and die.

Soon the primordial cells divide into a number of small filial cells with extremely thin walls and dense contents, after which the primordium has become meristem.

For answering the question in what place in a knob the meristem is formed and what is the descent of the initial cells, we have the following data. An otherwise 415 to 450 μ thick leaf-tip had by local swelling to about 840 μ , formed a knob, which by a small depression in the middle was, so to speak, divided into two halves, each of which contained a primordium of a meristem. The surface of the knob was entirely suberized to a fairly considerable depth. In one half the primordium lay 220 μ below the top of the knob and its cells in all probability descended from the subpalissade cells, in the other half the primordium lay 180 μ below the surface and was of the same origin as in the former case.

While in another case a primordium was noticed which genetically belonged to the original spongy parenchyma I found in a small knob which was still covered by an intact epiderm, and did not rise more than 85μ above its surroundings, and in this case had been formed by special proliferation of the palissade parenchym, a distinct young meristem immediately below the epiderm. *The epiderm cells themselves however, did not take part in the formation of the meristem.*

In a word, meristem formation may take place as well by cells, descending from the spongy parenchyma and the subpalissade cells, as from such as have been formed by hyperplasia of the palissade parenchyma, the epiderm, however, plays no part. In other words: *the adventitious buds on the leaves of Gnetum Gnemon are endogenic formations.*

In the beginning the young meristem increases in size by its own active growth as well as by new cells from the immediate vicinity becoming meristematic.

When the meristem has reached certain dimensions, it partly becomes loose from the surrounding tissue. This is brought about by some of the cells, forming the transition between the meristem and the surrounding tissue, being dissolved and resorbed.

This dissolution process proceeds along the whole upper side of the meristem, so that the growing point of the adventive bud comes to be placed in a slit-shaped space.

The greater the depth at which the meristem was originally formed inside the knob, the thicker is the layer of tissue which ultimately separates the bud from the outer world and the further the development within the enclosure proceeds. This explains how it is possible that knobs, no larger than 1 millimetre, open, while much larger ones remain persistently closed.

The appearance of two meristems within the same knob is a very common occurrence; once I found as many as four meristems in one knob.

The buds assume a green colour while they are still entirely enclosed within the knob and hence must have the power, like the germinating plants of Ephedra and the Coniferae, to form chlorophyl independent of light.

The sprouts growing out of the adventitious buds always remain short and tender. The biggest I observed reached a height of about 3,5 centimetres and consisted of a stem with 5 internodes (including the basal part) of which the upper one reached the greatest length (almost 2 centimetres), while the leaflets on the last node became

largest (about 3 cms. long). The position of the leaves is alternating, the innervation of the leaf the typical one for *Gnetum Gnemon*¹⁾. The leaflets on the first node as a rule remain scale-shaped; in some cases, however, they develop to leaflets, differentiated into stalk and blade.

Although in the axils of the leaflets axillary buds are certainly formed, I never saw the adventitious sprouts ramify themselves, except in a single case, when, as I surmise, of the basal piece of an adventitious sprout the terminal bud did not develop for some reason or other, and instead the buds in the axils of the scale-leaves sprouted.

On a differentiation of histogens at the vegetative cone I have not been able to form a definite opinion from my preparations of adventitious sprouts.

The numerous attempts which I made, in order to induce the adventitious sprouts to produce roots, have all failed. This agrees with the circumstance that in my preparations I have never been able to discover anything that resembled root-formation. The sprout-carrying leaves which had been planted in wet sand did not form roots either. As far as I know formation of adventitious roots does not occur at all with *Gnetum Gnemon*.

A connection between the vascular system of an adventitious sprout and the nerve system of the mother leaf is established by procambial bundles, formed by cells of the tissue, situated between the meristem and a leaf bundle.

As a rule we find as the first indication of this vascular bundle connection in the immediate vicinity of very young meristems even, some tracheids and cells, changing into them. The degree of development, reached at a certain moment by this vascular bundle connection, is not directly dependent on the degree of development of the adventitious bud in question, but seems to me to stand in close relation to the distance between meristem and leaf bundle and to the diameter of this latter. When a complete connection has been established we see the vascular bundles of the adventitious sprout within the knob in which the bud has formed, bend towards each other and unite with a more or less cylindrical group of locally formed vessels and tracheids, the ramifications of which are connected with the vascular bundles of the mother leaf.

When describing the changes, macroscopically observed with a

¹⁾ Viz. Nr. 3 of the leaf nervations, distinguished by KARSTEN for the species of *Gnetum* [G. KARSTEN, Untersuchungen über die Gattung *Gnetum*. I. (Ann. du Jardin Bot. de Buitenzorg Volume XI. 1893. p. 195—218)].

leaf in which adventitious buds are forming, it has already been briefly stated that the yellow vesicles, initiating the process of bud-formation, are caused by a scale-insect, *Aspidiotus* spec.

That suspicion fell on this *Aspidiotus* had a very simple reason.

Although not nearly all the leaves, showing yellow vesicles, carried scale-insects, yet the reverse was generally true and it soon became apparent that the leaves, carrying scale-insects generally also had some yellow vesicles.

But unexpected difficulties were experienced when it was attempted by means of microtome preparations to obtain certainty and a clearer insight in what had been rendered probable by macroscopical observation. A great difficulty was that the majority of the insects refused to stick to the bits of leaf from which the microtome preparations were going to be made. While a great part already loosed their hold during the treatment preceding the embedding itself, their example was followed by most others when they were put into the melted paraffin. It was supposed that perhaps the reason of this was that the scale-insects, when coming into the fixing solution, withdrew their suction organ from the tissue of the leaf, possibly on account of a pre-mortal reactional movement. After this unfavourable result the leaves carrying the insects were always treated before fixation with an anaesthetic, namely aether. This precaution, however, did not materially improve the results.

Among the microtome preparations which I obtained in spite of these difficulties, there is not a single one in which a scale-insect may be seen in a sucking position. But always in the places where an insect was on the leaf, in the tissue the changes were found which we described as characteristic for the yellow vesicles.

Here also hypertrophy, accompanied by disorganisation of the chlorophyll, of cells, belonging to the mesophyll; some of these cells have brown walls. Also in the epiderm on which the scale-insect is found, some cells are found, the walls of which are suberized and which besides are sometimes slightly swollen.

I was more fortunate with hand-made preparations, some of which show the suction apparatus of the scale-insect inside the leaf tissue. From these we see that in the yellow vesicles those cells, the wall of which has become suberized, have been in direct contact with the suction apparatus of the scale-insect and that the other cells, which become hypertrophical, only react to a stimulus, exercised by the wounded cells. On the character of this stimulant action we shall speak presently.

That here the enlargement of the cell should take place at the

expense of its own contents, as is stated for many similar hypertrophical processes, is not the impression I received. Although in many of the very strongly hypertrophical cells a large central vacuole may be observed, yet I saw nowhere reduction of the protoplasm to a very thin wall-lining. The nucleus does not show any deviation and the cell-wall does not become perceptibly thinner.

That the sting of the scale-insect not only causes the formation of the yellow vesicles but through them also all further changes, including the formation of the adventitious buds, has become clear to me by :

1. the microscopical examination of a very large number of preparations, relating to these stages ;

2. the continued observation of a number of leaves on the tree, showing that those leaves on which scale-insects or the yellow vesicles caused by them, were seen, underwent the above described changes, while the control leaves remained free from them. On Sept. 13, 1906, the top of that plant which forms adventitious buds most strongly and one of its branches were each surrounded by a muslin balloon, after they had first been carefully inspected and cleaned. These balloons were supported by skeletons of galvanised iron wire and closed below by pulling them on to a pad of cotton wool, placed round the sprout. At the top of the plant were then only young leaves, on the branch full-grown ones, all of them free from scale-insects and vesicles. The balloon, surrounding the top of the plant had repeatedly to be replaced by a bigger one as the top grew.

On January 22, 1907, the balloon was removed from the branch and the leaves were examined. Of two of these leaves the extreme part of the top had turned yellow. A microscopical examination of these leaf-tips showed, however, that here was no initial stage of bud-formation. Hypertrophical cells, such as we ought to have found in this case in the mesophyl, were not present. The yellow colour was caused by the dying of the tissue, the cell-contents then discolouring.

On May 10 the top of the plant was liberated. A number of full-grown leaves which at the beginning of the experiment were still young and young leaves at lateral sprouts which during the isolation had been formed by sprouting of the axillary buds, were now seen. All these leaves were perfectly normal, healthy and strong with a normal green colour ; on none of them anything could be detected of yellow vesicles or spots, of none the top showed any discoloration or thickening. The isolation by means of the muslin balloon

had not hindered these leaves at all in their normal development. Only scale-insects and other animals had been prevented from settling on the leaves with the formerly described result.

Similar tumours as the yellow, thickened leaf-tips of *Gnetum Gnemon* really are, have also been repeatedly observed with other plants and described under the name of "yellow specks" (Gelbfleckigkeit), oedemata or intumescences. The word "Intumescencia" was introduced into phytopathological nomenclature by SORAUER with the definition ¹⁾: "Intumescencia" sind "diejenigen Erscheinungen, die das gemeinsame Merkmal haben, als kleine knötchenförmige oder drüsige Auftreibungen der Blätter aufzutreten, die meist an diesen Stellen gelb verfärbt erscheinen und eine aussergewöhnliche Zellstreckung ohne wesentliche Zellvermehrung zeigen". That the thickened leaf-tips of *Gnetum Gnemon* are not indeed "kleine Auftreibungen" and do present "wesentliche Zellvermehrung" need not necessarily prevent us from counting them among the intumescences, since as well very large ²⁾ or mutually coalescent ³⁾ as typically hyperplastic ⁴⁾ intumescences have been described for other plants.

We cannot now deal with the very divergent views of different investigators about the cause of the formation of intumescences; we will only mention that in most cases it has been stated that a high temperature and great humidity of the air are essential factors.

Experiments enabled me to form an idea about the character of the stimulus exerted by the scale-insects on the tissue of the leaf of *Gnetum Gnemon*, on which the formation of the intumescences is the reaction. After I had tried artificially to produce intumescences in leaves by mechanically wounding them in all sorts of ways and treating them with poisons, without obtaining the desired result, I arrived at the conclusion that either my method of wounding, compared with that of the scale-insects, was too coarse or that the insect injected some stimulating substance into the leaf. In order to settle these points the following experiments were carried out: 1). In

¹⁾ P. SORAUER, Die symptomatische Bedeutung der Intumescenzen (Bot. Zeitg. 48 Jahrg. 1890. p. 241).

²⁾ H. v. SCHRENK, Intumescences formed as a result of chemical stimulation. (Missouri botan. garden. 16th ann. report. 1905. p. 125).

³⁾ Miss G. E. DOUGLAS, The formation of intumescences in potato-plants. (Bot. Gazette Vol XLIII. 1907. p. 233.)

⁴⁾ E. KÜSTER, Über experimentell erzeugte Intumescenzen. (Ber. deutsch. bot. Ges. Bd. XXI. 1903. pag. 452). P. SORAUER, Ueber Gelbfleckigkeit. (Forsch. a. d. Geb. d. Agrik. Phys. h. v. Dr. E. WOLLNY. Bd. IX. 1886. pag. 387). and Intumescenz bei *Solanum floribundum*. (Zeitschr. f. Pflanzenkrankh. Bd. VII. 1897. p. 122).

the leaf-tips extremely small wounds were made by means of the sterilised, very fine point of an injection syringe. 2). The same was done after the point had first been stuck into yellow vesicles, caused by the scale-insects. 3). A number of yellow and thickened leaf-tips were ground in a mortar and a very small portion of the so obtained pulp, mixed with some diluted glycerin, injected in several places in leaf-tips. 4). The same operation as in 3 was applied after the pulp had first been heated to 100° C.

The result was exactly the same in *all* cases.

After some ten days small, brown specks were visible in the wounded places, which afterwards could still increase somewhat in size. A month after the wounding the brown specks had become surrounded by a very thin, more or less transparent, yellow margin. The brown specks were formed by the cells which had died in consequence of the wounding, and the walls of which had turned brown. In the yellow margin a complex was found of relatively small cells, leaving no intercellular cavities. These cells had thick walls and their protoplast still contained remnants of the chlorophyll grains. The complex was formed by hyperplasia of the whole mesophyll. On the border between this complex and the normal tissue some cells of the spongy parenchyma had become greatly enlarged, their chloroplasts having become disorganised. After another month it was noticed that the leaf-tips in the neighbourhood of the wounded spots assumed a somewhat yellow colour, which gradually became more and more distinct: Microscopically it could be stated that where externally this yellow discoloration was visible, the tissue round the wounded spots had undergone precisely the same changes as take place round the yellow vesicles, caused by the scale-insects, namely a general hypertrophy of the cells of the spongy parenchyma, while here and there even a partition wall had already been formed in the enlarged cells.

As was stated above, this result was obtained in all cases, also in those in which small wounds had been made without anything else. From which we may conclude that the leaf of *Gnetum Gnemon* may be stimulated to the formation of intumescences and hence of adventitious buds by wounding, provided this is very light and that consequently the process must be regarded as a reaction on a *wound stimulus*.

In a disease of carnations which also consists in the formation of a sort of intumescences¹⁾ and for which it has been shown by

¹⁾ H. v. SCHRENK l. c. p. 39.

WOODS¹⁾ that it is caused by the sting of aphides, among others, WOODS thinks the growth of the yellow, thickened spots must be ascribed to the diffusion of an irritant, injected by the insect²⁾.

That a similar hypothesis is superfluous for the "stigmonose" of *Gnetum Gnemon*, appears clearly enough from our experiments. The specific point about the wound, made by the suction organ of the scale-insect is only that it is so trifling. Only a few cells, namely those which are distinguished in the yellow vesicles by brown walls, have undergone the direct consequences of it, while the whole subsequent formation of the intumescences takes place as a reaction on the stimulating action, proceeding from these few wounded cells.

KÜSTER³⁾ calls all cataplasms after vulneration, as far as they have a parenchymatical character, callus. According to this terminology also the tissue of which the thickened leaf-tips of *Gnetum Gnemon* consist, is a "callus" and the buds, formed in them, are callus-buds⁴⁾.

Why the two plants from the cooler hothouse did not form callus or buds on their leaves, is now also clear. The *Aspidiotus*, playing such an important part in this formation of callus, is also found in the cooler hothouse; but for callus-formation the chief condition is humidity. This condition was only to some extent fulfilled by the cooler, but completely by the hotter house, while also the higher temperature in this latter could not but favour the formation of callus with these tropical plants.

Why only the apical part of a leaf is capable of forming callus and buds, may be explained in the following manner. The small wound causes an afflux of nutrient matter in an apical direction. If now an accumulation of this matter, which is necessary for the hyperplastic formation of callus, shall be possible, the afflux must not be able to pass by, i.e. it must be stopped apically of the wound. And this condition is normally only fulfilled in the tip of the leaf, in another part of the blade only when the organic relation with the tip has been disturbed.

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1) A. F. WOODS, Stigmonose: a disease of carnations and other pinks. (Bull. no. 19. U. S. Dept. Agr. Div. Veg. Phys. and Path. 1900).

2) l. c. p. 24.

3) E. KÜSTER, Pathologische Pflanzenanatomie. Jena. 1903, p. 154.

4) See also: E. KÜSTER, Histologische und experimentelle Untersuchungen über Intumescenzen. (Flora oder allgem. bot. Zeitg. 96 Bd. 1906, p. 527—537).

Crystallography. *"The system of crystallization of the diamond"*
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 (Comm. by Mr. MOLENGRAAFF).

(Communicated in the meeting of June 29, 1907).

There still exists some doubt about the system of crystallization of the diamond. Although the tetrahedral hemihedrism of the diamond is pretty generally accepted, still the opinion that it belongs to the regular system, also finds support.

The existing uncertainty is caused by the lack of physical research with regard to this question. Such a research into the existence or non-existence of polarity of the trigonal axes has now been made. With that object in view the trigonal axes of the diamond were tested for pyro-electricity according to P. P. KOCH's method¹⁾. Tourmaline, boracite and quartz, which evinced strongly pronounced pyro-electrical characteristics, were used as testminerals. The result arrived at is absolutely negative. The diamond is not pyro-electric, and the trigonal axes do not possess polarity.

Besides, researches were made into the crystalline form of the diamond out of a collection of 367 uncut diamonds collected by Mr. MOLENGRAAFF. Practically all types of crystallization were represented here, to explain which tetrahedral hemihedrism for the diamond had been accepted in numerous writings of GROTH, SADEBECK, MARTIN and others. The result of this investigation, which is not yet at an end, is that a rational explanation of all irregular and apparently tetrahedral-hemihedral crystalline forms of the diamond can be found in the peculiar octahedral lamellar structure of the diamond.

On this ground it may be accepted that the diamond crystallizes in the holohedral division of the regular system.

¹⁾ P. P. KOCH. Ueber eine neue Methode zur Untersuchung auf Pyroelektricität. Inaug. Dissert. München, Mainz 1902.

Physics. — “*Contribution to the theory of binary mixtures. VI. The plaitpoint line.*” By Prof. J. D. VAN DER WAALS.

Continued. See p. 123.

By the plaitpoint line we understand the continuous series of points, in which the mixture is in the plaitpoint state. If we think the points of the surface of saturation determined by the coordinates T , p and x , then the plaitpoint line is a curve lying on this surface, and its projections on the planes of coordinates are expressed by: $p = f_1(T)$, $p = f_2(x)$ and $x = f_3(T)$. If the surface of saturation is given by the coordinates T , v and x , its projections have the form: $v = f_4(T)$, $v = f_5(x)$ and $x = f_6(T)$. The two surfaces of saturation mentioned may be derived from each other by the aid of the relation $p = \varphi(x, v, T)$. If we have the first mentioned surface, the substitution of p leads to the second. However, we might also have eliminated T , and obtained a surface of saturation of the form $F(p, v, x) = 0$, also one of the form $F_1(p, v, T) = 0$. A point of saturation being determined and known in all respects if the 4 quantities T , x , v , and p are known and the equation of state giving a relation between these 4 quantities, we may imagine as many surfaces of saturation as the number of combinations of 4 quantities three and three. The number of projections of the plaitpoint line is then the number of combinations two and two. For the direction of the projections $\frac{dT}{dx}$, $\frac{dp}{dx}$, $\frac{dv}{dx}$, $\frac{dp}{dT}$, $\frac{dp}{dv}$, and $\frac{dv}{dT}$ present themselves for consideration, which of course, are not independent of each other.

The best known shape of the plaitpoint line is that for which the initial point lies in the critical point of the first component, and the final point in the critical point of the second component.

In this case there is a point in which the plaitpoint line begins, and another in which it terminates; but such initial and final points lie necessarily in such places as are to be considered as natural boundary points. Thus initial and final points might also occur for boundary volumes ($v = b$) — but a plaitpoint line can never have an initial or final point for arbitrarily chosen value of v and x . Thus in the case that there is minimum or maximum T_k the mentioned well-known shape of the plaitpoint line will, it is true, make its appearance only in a certain point with gradual increase or decrease of the temperature for certain definite value of T — but such a point is then necessarily a double plaitpoint, and the plaitpoint line itself retains its character of continuous series of points; the double

plaitpoint mentioned is then an homogeneous double plaitpoint. If then in the v, x -projection the plaitpoint line is drawn, it proceeds again continuously from the left to the right side — and this continues to be the case also when the plaitpoint line mentioned has more intricate properties, e. g. when there are two heterogeneous double plaitpoints, as discussed in “Contribution etc.” and also treated in These Proc. March 25, 1905, p. 621 and These Proc. June 24, 1905, p. 184. However, besides this plaitpoint line, another is possible. The latter does not cross from the left side to the opposite side. So only the possibility is left either that it is a closed curve in the v, x -projection, or that it begins and terminates on the borders $v = b$.

We shall proceed to discuss some properties of the special points of this line, particularly of the double points of this line. KORTEWEG has demonstrated that these double points are of two kinds. Either it is a double point in which two homogeneous plaitpoints originate or coincide — or it is a double point in which this is the case for two heterogeneous plaitpoints. Though physically such plaitpoints bear such different characters — mathematically they satisfy the same criteria, and on the plaitpoint line such an heterogeneous double point is the transition point for a series of plaitpoints which might be realized, and for a series of unrealisable plaitpoints.

Minimum or maximum temperature for the plaitpoint line.

If we suppose a double plaitpoint to originate or to disappear on the ψ -surface at a certain value of T , two plaitpoints are found at somewhat higher or lower value of T . This holds both for the case that the double point is an homogeneous and an heterogeneous double point, as we shall briefly call them. For the plaitpoint line $\frac{dT}{dx}$ and $\frac{dT}{dv} = 0$ in this case. But for an homogenous double point $\frac{dT}{dp}$ is also $= 0$. This property follows from the shape of $\frac{dp}{dT}$, which has been derived in Verslag Kon. Ak. v. Wet. Deel IV p. 20 and p. 82 because $\frac{d^2v}{dx^2_p} = 0$ in an homogeneous double point (Contribution etc. These Proc. March 30, 1907, p. 745). For an heterogeneous double point $\frac{dT}{dp}$ is not equal to 0, as also appears from the value given for $\frac{dp}{dT}$, as for such a double point $\frac{d^2v}{dx^2_p} = 0$, but $\frac{d^2v}{dx^2_p} = \frac{d^2v}{dx^2_q}$. That in an heterogeneous

double point $\frac{dT}{dp}$ is not equal to 0, is also immediately seen when we consider that for such a point also $dp = 0$, and $\frac{dT}{dp}$ appears therefore in an indefinite form, the value of which we shall presently determine. Hence $\frac{dv}{dx} = \left(\frac{dv}{dx} \right)_p$ for such a double point, which is not the case for an homogeneous double point.

Of the 6 differential quotients which come in for discussion, three are equal to 0 for an homogeneous double point, and three are left the value of which is still to be determined, viz. $\frac{dv}{dx}$, $\frac{dv}{dp}$ and $\frac{dp}{dx}$.

If we write:

$$\frac{dv}{dx} = \frac{\frac{dT}{dx}}{\frac{dT}{dv}}$$

$$\frac{dv}{dp} = \frac{\frac{dT}{dp}}{\frac{dT}{dv}}$$

and

$$\frac{dp}{dx} = \frac{\frac{dT}{dx}}{\frac{dT}{dp}},$$

in all these three expressions both numerator and denominator is equal to 0. If in the first we differentiate numerator and denominator with respect to x , in the second with respect to v , and in the third with respect to p , we find:

$$\frac{dv}{dx} = \frac{\frac{d^2T}{dx^2}}{\frac{d^2T}{dv^2} \frac{dv}{dx}}$$

$$\frac{dv}{dp} = \frac{\frac{d^2T}{dp^2} \frac{dp}{dv}}{\frac{d^2T}{dv^2}}$$

and

$$\frac{d^2 T}{dx^2} = \frac{d^2 T}{dv^2} \frac{dv}{dx} \frac{dp}{dx}$$

or

$$\left(\frac{dv}{dx}\right)^2 = \frac{\frac{d^2 T}{dx^2}}{\frac{d^2 T}{dv^2}}, \quad \left(\frac{dp}{dx}\right)^2 = \frac{\frac{d^2 T}{dx^2}}{\frac{d^2 T}{dp^2}} \text{ and } \left(\frac{dp}{dv}\right)^2 = \frac{\frac{d^2 T}{dp^2}}{\frac{d^2 T}{dv^2}}.$$

We may verify these properties by writing for the immediate neighbourhood of the minimum or maximum plaitpoint temperature :

$$T = T_1 \pm \alpha (x - x_1)^2 = T_1 \pm \beta (v - v_1)^2 = T_1 \pm \gamma (p - p_1)^2,$$

in which the sign $+$ holds for minimum value of T , and reversely the sign $-$ for maximum value.

From this follows :

$$\alpha (x - x_1)^2 = \beta (v - v_1)^2 = \gamma (p - p_1)^2,$$

or

$$\pm (x - x_1) \sqrt{\alpha} = \pm (v - v_1) \sqrt{\beta} = \pm (p - p_1) \sqrt{\gamma},$$

and

$$\frac{dv}{dx} = \pm \sqrt{\frac{\alpha}{\beta}}, \quad \frac{dp}{dv} = \pm \sqrt{\frac{\beta}{\gamma}} \text{ and } \frac{dx}{dp} = \pm \sqrt{\frac{\gamma}{\alpha}}.$$

As $\frac{dv}{dx} \times \frac{dp}{dv} \times \frac{dx}{dp} = +1$, we have to take all the signs positive, or one positive and two negative. Thus in the case that there is minimum or maximum plaitpoint temperature, and we choose the direction of x such that $\frac{db}{dx}$ is positive, $\frac{dv}{dx} > 0$ and $\frac{dp}{dx} < 0$, and so also $\frac{dp}{dv}$ negative. It is, however, not always the case that $\frac{dp}{dv}$ is negative.

Thus for a plaitpoint line with maximum value of p , $\left(\frac{dp}{dT} = 0\right)$ also $\frac{dp}{dx} = 0$ and $\frac{dp}{dv} = 0$. So in such a case reversal of the sign of $\frac{dp}{dv}$ must take place.

If we examine the criteria for an heterogeneous double point, we have in the first place $\frac{dT}{dx} = 0$ and $\frac{dT}{dv} = 0$. Then $\frac{dT}{dp}$ is not equal to 0. But in its stead there are two other differential quotients which

are equal to 0. From $\frac{dp}{dT} = \frac{dp}{dx} \frac{dx}{dT}$ follows, if we take into consideration that $\frac{dp}{dT}$ has a finite value, and $\frac{dx}{dT} = \infty$, that $\frac{dp}{dx}$ must be $= 0$, and also $\frac{dp}{dv} = 0$. That p has maximum or minimum value in the case of an heterogeneous double point has already been represented by us in a drawing. (These Proc. March 25, 1905, p. 621, and June 24, 1905, p. 184). So of the 6 differential quotients for the projections of the plaitpointline 4 are zero. Two are left whose value is to be determined, viz. $\frac{dp}{dT}$ and $\frac{dv}{dx}$.

If we write: $\frac{dp}{dT} = \frac{\frac{dp}{dx}}{\frac{dT}{dx}}$, we find by differentiation of numerator and denominator:

$$\frac{dp}{dT} = \frac{\frac{d^2p}{dx^2}}{\frac{d^2T}{dx^2}}.$$

If we write: $\frac{dv}{dx} = \frac{\frac{dT}{dx}}{\frac{dT}{dv}}$, we find:

$$\left(\frac{dv}{dx}\right)^2 = \frac{\frac{d^2T}{dx^2}}{dv^2}.$$

What follows may serve as a verification. Let us again write as holding in the immediate neighbourhood of the double point:

$$T = T_0 \pm \alpha (x - x_0)^2 = T_0 \pm \beta (v - v_0)^2$$

and

$$p = p_0 \pm \gamma (x - x_0)^2 = p_0 \pm \delta (v - v_0)^2.$$

For minimum value of T and p the positive sign must be chosen, and reversely. So we have the following relations:

$$\alpha (x - x_0)^2 = \beta (v - v_0)^2,$$

and

$$\gamma (x-x_0)^2 = \delta (v-v_0)^2$$

from which

$$\frac{\alpha}{\gamma} = \frac{\beta}{\delta}.$$

Further

$$\frac{T-T_0}{\alpha} = \pm \frac{p-p_0}{\gamma}.$$

In this last equation the sign $+$ must be chosen, if, as is the case, T and p have at the same time either maximum or minimum value. We find then :

$$\frac{dp}{dT} = \frac{\gamma}{\alpha}$$

and

$$\frac{dv}{dx} = \pm \sqrt{\frac{\alpha}{\beta}} = \pm \sqrt{\frac{\gamma}{\delta}}.$$

That this value of $\frac{dv}{dx} = \left(\frac{dv}{dx}\right)_{p,T}$ follows from the derivation.

So we find a definite value for $\frac{dp}{dT}$, and as no lower value of T exists for minimum value of T , and no higher value for maximum value, the p, T -projection of the plaitpoint curve must possess "cusps". That this value of $\frac{dp}{dT}$ is positive, and so p and T are at the same time maximum or minimum, follows inter alia from the equation :

$$dp = \left(\frac{dp}{dv}\right)_{xT} dv + \left(\frac{dp}{dx}\right)_{vT} dx + \left(\frac{dp}{dT}\right)_{vx} dT.$$

For, as $\frac{dv}{dx} = \frac{dv}{dx_{pT}} = -\frac{\left(\frac{dp}{dx}\right)_{vT}}{\left(\frac{dp}{dv}\right)_{xT}}$, this equation reduces to :

$$\frac{dp}{dT} = \left(\frac{dp}{dT}\right)_{vx}.$$

Other special points of the plaitpoint curve.

It appears from the form for $\frac{dp}{dT}$ (Versl. Kon. Ak. Deel IV p. 20) that also the case that $\frac{dp}{dT} = 0$ is possible, and for some mixtures a maximum value of p in the p, T -projection has been experimentally

shown. As $\frac{dp}{dT} = \frac{dp}{dx} \frac{dx}{dT} = \frac{dp}{dv} \frac{dv}{dT}$, we derive that in this case also

$\frac{dp}{dx} = 0$ and $\frac{dp}{dv} = 0$, save in exceptional cases. Then p is the highest,

or the lowest pressure that can occur on the plaitpoint curve. Of

the 6 differential quotients 3 are again equal to zero, and 3 others

are again to be determined, viz: $\frac{dx}{dT}$, $\frac{dv}{dT}$ and $\frac{dv}{dx}$. We find then from :

$$\frac{\frac{dx}{dT}}{\frac{dx}{dT}} = \frac{\frac{dp}{dT}}{\frac{dp}{dT}}$$

$$\frac{dx}{dT} = \frac{dp}{dp}$$

$$\frac{d^2x}{dT^2} = \frac{d^2p}{d^2p}$$

$$\left(\frac{dx}{dT}\right)^2 = \frac{d^2p}{d^2p}$$

From

$$\frac{dv}{dT} = \frac{dp}{dp}$$

$$\frac{dv}{dT} = \frac{dp}{dp}$$

we find :

$$\left(\frac{dv}{dT}\right)^2 = \frac{d^2p}{d^2p}$$

And from :

$$\frac{dv}{dx} = \frac{dp}{dp}$$

$$\frac{dv}{dx} = \frac{dp}{dp}$$

we find :

$$\left(\frac{dv}{dx}\right)^2 = \frac{d^2p}{d^2p}$$

which may be again verified from the equations :

$$p = p_1 \pm \alpha(T - T_1)^2 = p_1 \pm \beta(x - x_1)^2 = p_1 \pm \gamma(v - v_1)^2$$

For plaitpoint lines which do not run from $x = 0$ to $x = 1$, and which therefore either form a closed figure, or run from a point of

the line $v = b$ to another point of this line, the value of x may be maximum or minimum. Then $\frac{dx}{dT}$, $\frac{dx}{dv}$ and $\frac{dx}{dp} = 0$, and the three remaining differential quotients must be determined — and lastly also v might be maximum or minimum; then $\frac{dv}{dT}$, $\frac{dv}{dp}$ and $\frac{dv}{dx}$ would be equal to zero, and $\frac{dT}{dx}$, $\frac{dp}{dx}$ and $\frac{dp}{dT}$ would have to be determined.

Three phase pressure and final point of the three phase pressure on the plaitpoint line.

If at a certain temperature three phase pressure exists, there must be a hidden plaitpoint on the ψ -surface, as appears from the foregoing remarks. If the spinodal curve is closed on the side of the small volumes there is moreover a realisable plaitpoint, and there can even be another realisable plaitpoint if the temperature is above the T_k of one of the components. Let us call x_1 and v_1 , x_2 and v_2 , x_3 and v_3 the compositions and volumes of the three phases, assuming the first two to be liquid phases and the third to be a gas-phase, and let us put $x_2 > x_1$. Now three cases may occur, viz.: $x_3 > x_2 > x_1$; $x_2 > x_1 > x_3$, and $x_2 > x_3 > x_1$. The first case occurs when the gas phase contains more of the second component than each of the liquid phases, and so when $\left(\frac{dp}{dx}\right)_{rT}$ is always positive; the second case when the gas phase contains less of the second component, and so when $\left(\frac{dp}{dx}\right)_{vT}$ is negative, and the third case requires that the line $\left(\frac{dp}{dx}\right)_{vT} = 0$ runs between the two liquid phases. Of the first case an example may be found in the mixture water in SO_2 , mixtures of ethane and some alcohols (above methylalcohol) constitute an example of the second case, and of the third case the mixture water and phenol is an example.

As we have an equilibrium which is independent of the size of the volume, when for a mixture of 2 substances there exists equilibrium of 3 phases, the formula of CLAPEYRON may serve for the computation of the value of $\frac{dp}{dT}$, and we may put:

$$T \frac{dp}{dT} = \frac{W}{v}$$

if W represents the heat which is released with decrease of volume when part of the middle phase is converted into the state of the

two other phases, and u the degree of this decrease of volume. We arrive at the same result if we follow the course (Verslag Kon. Akad. v. Wetensch. Deel V, p. 482) indicated there, viz.:

$$\frac{dp}{dT} = \frac{\begin{vmatrix} 1 & x_1 & \eta_1 \\ 1 & x_2 & \eta_2 \\ 1 & x_3 & \eta_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & v_1 \\ 1 & x_2 & v_2 \\ 1 & x_3 & v_3 \end{vmatrix}} = \frac{(x_1 - x_3)(\eta_2 - \eta_3) - (x_2 - x_3)(\eta_1 - \eta_3)}{(x_1 - x_3)(v_2 - v_3) - (x_2 - x_3)(v_1 - v_3)}.$$

We find the same equation when we have three phase equilibrium for a binary system of solid, liquid, and gaseous. And the course of the line $p = f(T)$ is then known. It is a line, consisting of two branches lying above each other, which smoothly pass into each other at a certain maximum temperature, and the upper branch of which possesses maximum pressure. In this case, however, the course is simpler. For the equilibrium of solid, liquid, and gaseous two branches occur; on one branch the liquid is richer in one of the components than the solid body, and on the other branch the reverse. Where these branches meet, the value of x has the same amount for the solid body and for the liquid, and in that point the line $p = f(T)$ has an element in common with the melting-point line.

This is seen from the value of $\frac{dp}{dT}$, if e. g. $x_2 = x_3$ is put in it, in

which case $\frac{dp}{dT} = \frac{\eta_2 - \eta_3}{v_2 - v_3}$. And it has, therefore, often been stated

as a fixed rule, that when two phases have the same concentration, the variation of equilibrium with the temperature depends only on these two phases, and is independent of the third. Also for equilibrium of 2 liquid phases and one gas phase, however, equality of concentration may occur between two phases. Thus one of the liquid phases may get the same concentration as the gas phase, or the two liquid phases may get the same value of x . Then the above mentioned rule does not hold. When a solid body has the same concentration as a liquid, and e. g. $x_2 = x_3$, then η_2 is not equal to η_3 , and v_2 not equal to v_3 . Then there are, indeed, two phases of the same concentration, but not two *identical* phases. But when a liquid phase has the same concentration as a gas phase this expression means that in the three phase triangle one of the sides has been reduced to zero, and these two phases have become identical. Then we find

after division of numerator and denominator by $x_2 - x_3$:

$$\frac{dp}{dT} = \frac{\eta_1 - \eta_3 - (x_1 - x_3) \left(\frac{d\eta_3}{dx_3} \right)_{p,T}}{v_1 - v_3 - (x_1 - x_3) \left(\frac{dv_3}{dx_3} \right)_{p,T}}$$

or

$$T \frac{dp}{dT} = \frac{W_{13}}{v_{13}}$$

It appears from the form for $\frac{dp}{dT}$, that this value is equal to $\left(\frac{dp}{dT} \right)_{x_2}$, if a section is made through the surface of saturation for $x = x_3$.

In other words: The three phase triangle in its extreme position touches the section mentioned — and to this we might also at once have concluded. It will also be immediately seen, that the coinciding of the points x_2 and x_3 of the three phase triangle takes place in a plaitpoint, and that therefore the final point of the line $p = f(T)$ lies on the plaitpoint line. Then we have a plaitpoint in the point where x_2 and x_3 coincide, and the p, T -projection of the plaitpoint line being the envelope of the p, T -projection of the sections of the surface of saturation for constant values of x , the plaitpoint line and the p, T -projection of the sections touch, and so also the final point of the p, T -projection of the three phase pressure, as in that final point the last element of this pressure coincides with the section mentioned. This contact has not yet been taken into consideration in former diagrams. If there are two final points of the three phase pressure, then there are two separate portions of the realisable portion of the plaitpoint line, which are joined by the three phase pressure, the meeting-points being again *cusps*, just as is the case with the hidden portion of the plaitpoint line. Now, however, rises the following question. We know from the shape of the section of the surface of saturation at given value of x , that in the simplest case it consists of two branches, and that on the upper branch the value of $\frac{dp}{dT}$ may also be negative. Can now also $\frac{dp}{dT}$ be negative for the three phase pressure? As far as I know this has never been observed; but the observations on the rise of the three phase pressure with the temperature, and the other circumstances, viz. the values of x and v , have been only little examined

as yet. If it should be possible that $\frac{dp}{dT}$ becomes negative, and for the present I do not see any reason to consider this impossible, this can only take place in the case of a plaitpoint line descending with the temperature. Accordingly the final point of the three phase pressure, so the plaitpoint, lies on that part of the section of the surface of saturation which lies between minimum pressure and critical point of contact, and it is known, that then also the plaitpoint line must descend in its p, T -projection, because it is the envelope of the sections of the surface of saturation. If $\frac{dp}{dT}$ is negative at the final point, this value must have passed through 0; this will then require that no heat is released with conversion of the middle phase into the two others, and so that if heat is released with conversion into one of the two extreme phases, the conversion into the other extreme phase is attended by heat-absorption¹⁾. And without further investigation this cannot be pronounced as impossible.

Finally we point out that $\frac{dp}{dT}$ cannot become infinite. For this it would be required that the denominator is equal to zero without this being the case with the numerator. Then the area of the three phase triangle must be equal to zero or the 3 points must lie on a straight line. This is the case when two points coincide, but then the numerator is also equal to zero. Now a p -line — for the three points always lie on the same isobar — can indeed be intersected by a straight line in 3 points, but in this case this would have to occur in the same three points with a q -line; this observation will most likely suffice to put down this case as one that does not occur.

So we have in the p, T -projection of the threephase pressure a curve which, at least as a rule, ascends with the temperature; under every point of this line is a point of the plaitpoint line (hidden point) and above every point is a second point of this line (realisable point). This second point is wanting if the plait should not be closed at the bounding volume.

Shapes of plaitpoint lines (p, T -projection).

According to the above considerations I shall describe a possible shape of plaitpoint line for the case of two components, for which

¹⁾ The diagrams p. 126 Cont. II, in which the value of v_{21} and w_{21} for coexisting phases has been represented, must be supplemented, when also incomplete miscibility is assumed.

the ratio $\frac{T_{k_2}}{T_{k_1}}$ is a high value, and for which the temperature, at which $\frac{d^2\psi}{dv^2} = 0$ has contracted to a point, is much higher than T_{k_1} .

As an example take the mixture helium and hydrogen investigated by KAMERLINGH ONNES and KEESOM partly experimentally and further theoretically, or the mixture helium and water. As, however, there are two shapes possible, I shall describe them both, not stating as yet, which of these shapes is the correct one in these cases.

As b for hydrogen will be higher than b for helium, helium is the first component. In the first place we observe that there must be a complex plait for $T < T_{k_1}$, which extends over the whole width.

$\frac{d^2\psi}{dv^2} = 0$ has closed on the helium side for $T > T_{k_1}$; but $\frac{d^2\psi}{dx^2} = 0$

is a closed curve, which extends outside $\frac{d^2\psi}{dv^2} = 0$ on the helium

side, and so there is intersection of $\frac{d^2\psi}{dv^2} = 0$ and $\frac{d^2\psi}{dx^2} = 0$. The

spinodal curve, which remains near $\frac{d^2\psi}{dv^2} = 0$ on the side of H_2 ,

moves further away from this line as we approach the helium side,

and remains also outside $\frac{d^2\psi}{dx^2} = 0$. I shall continue to assume that the

spinodal line remains closed on the side of the small volumes. The changes which must be made if this should not be the case, will be easily applied in the result at which we arrive. Then there are three plaitpoints for this $T > T_{k_1}$. With very small difference of T and T_{k_1} there is first the ordinary plaitpoint on the helium side; and further there are two heterogeneous plaitpoints, viz. a realisable one at the very small volumes, and a hidden one (see inter alia figs. 12 and 13 of the preceding communications).

If now the first mentioned plaitpoint should coincide with the hidden one, as is assumed in the discussion of these figures, only one single plaitpoint would remain; but another, more intricate case is possible.

If $\frac{d^2\psi}{dv^2} = 0$ and $\frac{d^2\psi}{dx^2} = 0$ are quite detached, as will happen with

increasing temperature, the spinodal line may viz. either continue to run round the two curves, as I have repeatedly drawn, or it may split up between the two curves. For splitting it will be required that they are so far apart that a point is found between them, in

which not only $\frac{d^2\psi}{dx^2}$ and $\frac{d^2\psi}{dv^2}$ are positive, but the product is equal to

$$\left(\frac{d^2\psi}{dx dv}\right)^2 \text{ — which may the sooner take place when the line } \left(\frac{dp}{dx}\right)_{vT} = 0$$

is in the neighbourhood. In this case there originate two new realisable plaitpoints. Then there are 5 plaitpoints at somewhat higher temperature, because 2 new ones have been added to the three above mentioned ones. And now, as I demonstrated when I discussed such a splitting up, at somewhat higher temperature the hidden plaitpoint will coincide with one of the newly formed realisable ones, and vanish as a couple of heterogeneous plaitpoints. So there are 3 realisable plaitpoints left viz. one that is the plaitpoint of the half of the plait (transverse plait) on the hydrogen side. And the two others, which are the upper and lower plaitpoint of the half of the plait which has got detached (longitudinal plait). In other words: one half is a plait that surrounds the curve $\frac{d^2\psi}{dv^2} = 0$, and the other half runs round

$$\frac{d^2\psi}{dx^2} = 0. \text{ In this half } \frac{d^2\psi}{dx^2} = 0 \text{ performs in many respects the function}$$

which $\frac{d^2\psi}{dv^2} = 0$ performs as a rule. The splitting up of the spinodal line, so that a closed longitudinal plait detached itself, can, therefore, take place in such a way that this longitudinal plait is found at temperatures at which $\frac{d^2\psi}{dv^2} = 0$ still exists for the same value of x , but is then restricted to very small volumes (mixtures of water and phenol); or it may take place in such a way that $\frac{d^2\psi}{dv^2} = 0$ no longer exists for the same value of x , but then the volumes need not be so very small. We might say: the detaching might take place in such a way that the two parts of the plait exist above or by the side of each other.

Moreover the case may occur that $\frac{d^2\psi}{dv^2}$ has quite disappeared, and

$\frac{d^2\psi}{dx^2}$ alone exists. Then only a longitudinal plait is found. This can only take place for temperatures above T_{k_1} and T_{k_2} , and if what I have called T_g , is larger than T_{k_1} and T_{k_2} .

In fig. 27 I have drawn the p, T -projection of the plaitpoint line, which in the v, x -plane is again a line which proceeds continuously from the left to the right side. At T_k for H_1 there are three points

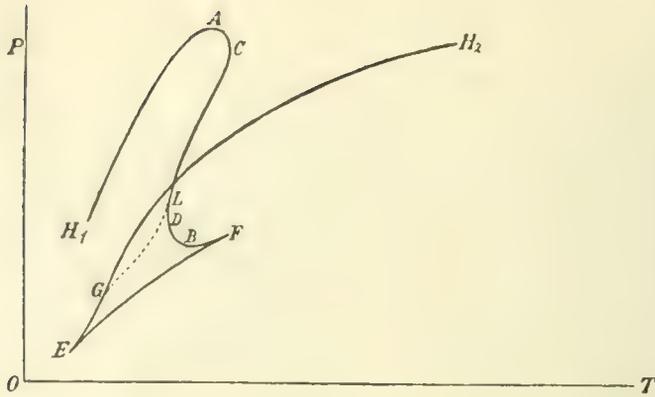


Fig. 27.

of this line; the downmost is the hidden plaitpoint lying between E and F . The place of the points E and F has been arbitrarily chosen, so that it may also be possible that E must lie more to the right than T_{k_1} . At the value of T_D , the splitting temperature, 2 new points appear. At T_F a couple of heterogeneous plaitpoints unite. At T_C the detached longitudinal plait would disappear. Between T_C and T_D there are two plaitpoints for the detached plait. The three phase pressure runs between G and L , the extremity L having been chosen such that the point D (splitting point) lies below the three phase triangle, and can, therefore, not be observed. Hence only the following 3 parts can be realized by means of the experiment: 1. H_2G , 2. H_1 , ACL , 3. GL . And now, if the hypothesis that the plait is closed on the side of the limiting volumes, should be incorrect, we have only to open the upper part at A and C , and to make the open branches run asymptotically towards infinitely high. So this plaitpoint line is essentially the same as that with the double point which I have drawn. Only one of the branches, i.e. the left branch, has in addition got a maximum and a minimum pressure, and a maximum and a minimum temperature. If we drew the T, x -projection, there would be 2 maxima and two minima — also in the p, x -projection. But the v, x -projection remains simple. If the plait is closed at the limiting volumes, there is a minimum volume, in

1) In this figure the shape of the plaitpoint line has been drawn when really the spinodal line could run between $\frac{d^2\psi}{dv^2}=0$ and $\frac{d^2\psi}{dx^2}=0$. Further investigation will have to decide whether or no this complication can occur. If it occurs, the righthand side of the plait (transverse plait) will be much narrower, then when this complication is not met with. In the latter case the righthand part of the plait is a complex plait.

the opposite case two points for which $v = b$, come in its stead. For neither with an homogeneous double point, nor for an heterogeneous double point, $\left(\frac{dv}{dx}\right)_{pl} = 0$. And in the point in which $\left(\frac{dv}{dx}\right)_{pl} = 0$ or $\left(\frac{dx}{dv}\right) = 0$, (see p. 185) $\frac{dp}{dT}$ has not a value which presents any particularity.

The second form will differ from the one described here in so far that the temperature at which the detaching of the longitudinal plait takes place, is assumed to be equal to T_{k_2} (critical temperature of the second component).

This may take place if the temperature at which $\frac{d^2\psi}{dx^2} = 0$ disappears, is not only higher than T_{k_1} , but also higher than T_{k_2} (a case to which I alluded already before in these Contributions).

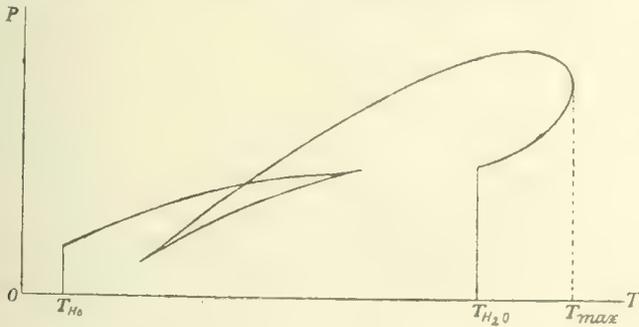


Fig. 28.

Then the p, T -projection is given by fig. 28. The highest temperature which then occurs, is that at which $\frac{d^2\psi}{dx^2} = 0$ disappears.

That for the mixture helium and hydrogen the second shape of the plaitpoint line can occur, and that therefore T_g can be $> T_{k_2}$, follows immediately from the formula for the value of T_g (These Proc. May 24, 1907) viz.:

$$MRT_g = 2 \frac{a_1 + a_2 - 2a_{12}}{b} x(1-x) \frac{1-y}{(1+y)^2}$$

For the case, namely, that a_1 and a_{12} may be neglected, and $b_1 < b_2$ and $x > \frac{1}{3}$, we find really $T_g > T_{k_2}$. Thus we find for $x = 0,4$, which belongs to $\frac{b_1}{b_2 - b_1} = 0,3704$, $MRT_g > \frac{8}{27} \frac{a_2}{b_2}$. For still higher value of x , this value of T_g would be found still higher, but

if $a_1 + a_2 - 2a_{12}$ becomes appreciably smaller than a_2 , this may, of course, be different. In this case the plait remains a complex plait as far as $T = T_{k_2}$. At this value of T has $\frac{d^2\psi}{dv^2} = 0$ disappeared, and $\frac{d^2\psi}{dx^2} = 0$ still exists. So above T_{k_2} the complex plait is to be considered as a longitudinal plait.

If in the case described above we have a plaitpoint line that proceeds continuously in the v, x -plane, starting on the lefthand side in the critical point of the first component, and terminating in the critical point of the second component, though a maximum value of x , and then also a minimum value may be possible, still another case is possible, and most likely this case is met with in the mixture water and phenol. Of course the first mentioned line, which starts and terminates in the critical point of the components, must continue to exist, if we continue to assume that the plait remains closed on the side of the limiting volumes. Else it splits up into 2 parts, which I, however, consider as two parts of one and the same branch of the plaitpoint curve. If another branch is possible, it must be a separate closed curve — which, however, if the plait is supposed open on the side of the limiting volumes, may be considered as starting in a point of the line $v = b$, and terminating in another point of this line. We meet with this case when the longitudinal plait detaches itself at a temperature which is lower than T_{k_1} and T_{k_2} . As has been described above, the longitudinal plait will have quite retreated to volumes smaller than those of the liquid branch of the binodal line of the transverse plait at a certain value of T higher than the temperature of detaching. Then the three phase pressure no longer exists, and the first mentioned branch of the plaitpoint line, which joins the critical points of the components, has its simplest shape. In fig. 29

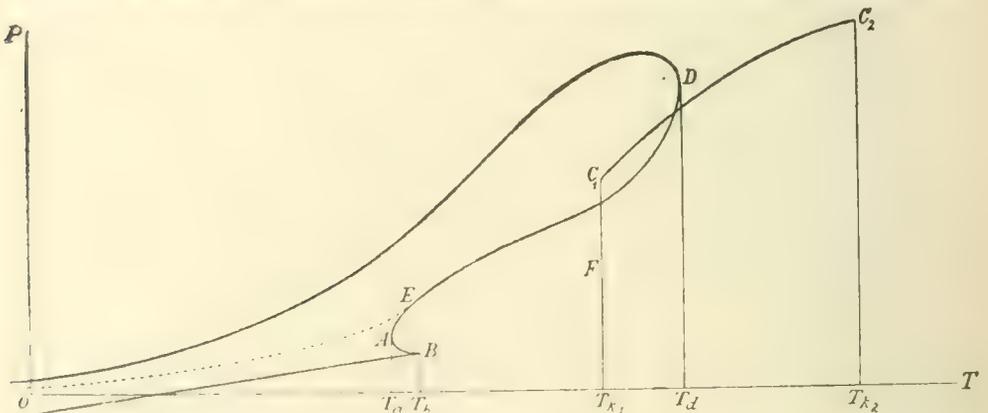


Fig. 29.

the p, T -projection has been drawn. At $T = T_d$, which is lower than T_{k_1} and T_{k_2} , the detaching takes place, and there is an homogeneous double point. At $T = T'_b$ there is an heterogeneous double point, and at T_d again an homogeneous double point. If we suppose the longitudinal plait to be open towards $v = b$, p_D must be thought infinitely large, and the upper part of this second branch disappears. Without doubt the three phase pressure line, which terminates in E , will have its other extremity, i. e. its initial point, at $T = 0$.

We should have a very simple and remarkable case of a closed curve for the second branch of the plaitpoint line if the lowest temperature at which an heterogeneous double point is formed, lies little below the temperature at which this double point vanished again — and this temperature lies below T_{k_1} and T_{k_2} . Then also the temperature at which again an heterogeneous double point exists,

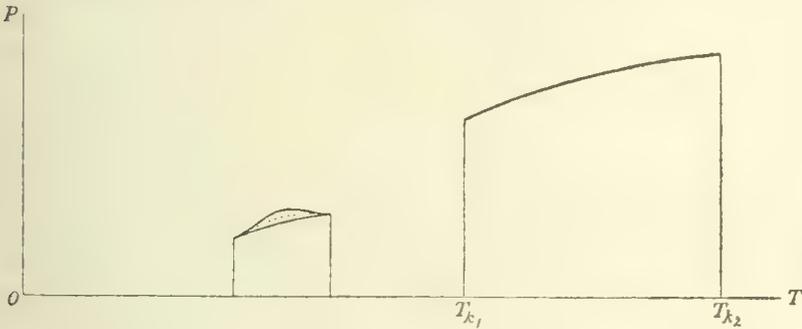


Fig. 30.

will lie only little higher than the first. Fig. 30 gives then again the p, T -projection for such a case. There can then be a three phase pressure indicated by a dotted line. Then the liquid begins to split up into two phases at a temperature lying much below T'_{x_1} and T'_{x_2} , becoming homogeneous again at somewhat higher temperature — at least if the value of x has been chosen between that belonging to the extremities of the three phase pressure. In the v, v -projection we have then a small closed figure with maximum and minimum volume.

So many different shapes of plaitpoint lines, however, may be deemed possible, that they would require a special study. If they are found by the experiment, I expect that the rules given in these contributions, will prove sufficient to render them intelligible.

However, I intend shortly to indicate the circumstances in which the forms discussed are met with, more fully by means of some mathematical developments.

Physics. — “*On the measurement of very low temperatures. XV. Calibration of some platinum-resistance thermometers.*” By Prof. H. KAMERLINGH ONNES and J. CLAY. Communication N^o. 99^b from the Physical Laboratory at Leiden.

(Communicated in the meeting of June 29, 1907).

§ 1. *Introduction.* The investigation on the variation of the resistance of metals (pure ones and those with known admixtures) set on foot many years ago (see Comm. N^o. 77 § 1 These Proc. Febr. 1902) at Leiden, comprises besides the determination of the galvanic resistance of conductors made of the different metals, also the determination of the expansion for each of these metals. We have only little advanced as yet with the latter part of this investigation, the expansion has only been investigated for platinum, which was chosen as standard metal, and then only down to -182° .¹⁾ We hope shortly to publish a Communication on the expansion down to -252° C. For the present, however, the knowledge of this expansion is not yet of much importance for the investigation of the variation of the specific resistance with the temperature. When in this investigation we descend to very low temperatures, the correction for the expansion becomes so small compared with the disturbance in consequence of other influences which are still further to be investigated, that we may disregard it for the moment.²⁾

The investigation consists then in the calibration of different resistance thermometers. The wires treated in this Communication being chiefly of importance to us as resistance thermometers, we have inserted their calibration in this series.

¹⁾ In Comm. N^o. 85 (These Proc. April 1905) it was observed for the first time that in order to represent the expansion of glass from -180° to 0° a formula of the second degree with other constants was required than for the range from 0° to $+100^{\circ}$. We found this confirmed in Comm. N^o. 95^b (These Proc. Sept. '06), and also applicable to platinum, for which a formula of the third degree, as we gave one for glass, proved necessary between -180° and $+100^{\circ}$. Afterwards (Dec. '07) SCHEEL, who was at first (Zeitschr. f. Instr.k. April '06) of opinion that a formula of the second degree could be found for platinum between -190° and $+100^{\circ}$, come to the same opinion as we, and gave the three constants for platinum. Our formula of the second degree for platinum between 0° and -180° quoted by SCHEEL was used by us to prove, that for platinum between -180° and $+100^{\circ}$ a formula of the second degree is *not sufficient*, but that a formula of the third degree is required. In order to show this with given values at $+100^{\circ}$, 0° and -190° observations at a temperature about halfway between 0° and -190° , as our -87° , are more suitable than observations at a temperature between 0° and $+100^{\circ}$, as those by SCHEEL at $+56^{\circ}$.

²⁾ Here it is left entirely undecided whether the variation of the resistance with the temperature is not in close connection with the expansion.

§ 2. *Particulars on the comparison and on the investigated wires.* In these calibrations we have taken the platinum wire which was compared with the hydrogen thermometer in Comm. N° 95^c (These Proc. Sept. '06) and which we shall call Pt_I , as standard. We determined the variation of the resistance of the other wires by bringing them together with Pt_I at the desired temperature, and by then comparing their resistance with that of Pt_I . The two platinum wires Pt_{III} and Pt_V were brought in the same cryostat (see § 4 Comm. 95^c) together with Pt_I , and whereas the temperature was kept constant with one resistance according to the indication of the WHEATSTONE-bridge, the ratio of the resistance of the other to Pt_I was determined by means of the differential galvanometer. Pt_V was also measured separately with the WHEATSTONE-bridge. The difference of the results by the two methods amounted only to 0,02 % at the lowest temperatures.

Just as Pt_I , Pt_{III} and Pt_V were supplied by HERAEUS; they were delivered at the same time, but later than Pt_I . The diameter of all three was 0.1 m.m. After having been treated and wound round the glass (see Comm. N°. 95^c § 3, in the same way, they were heated for a long time in an annealing furnace for glass. Pt_{III} and Pt_V differed only in this respect that after being heated Pt_V was partly unwound, and then wound again, and was not heated in the annealing furnace again.

To obtain also a resistance thermometer of very small dimensions a platinum wire of 0,05 m.m. diameter was wound round a tube of 1 c.m. diameter and about 8 c.m. long. The thin platinum wire was welded to thick platinum wires which were fused in the glass. Consequently the thermometer could be cleaned by means of acids if necessary. The thin wire Pt_A used for this thermometer, was also furnished by HERAEUS.

A fourth wire was investigated to get an idea of the

§ 3. *Invariability of the resistance thermometers for low temperatures with the time, viz.* the resistance thermometer with which the observations were made by MEELINK in 1902, and which we shall call Pt_M . The zero point appeared to have remained unchanged to one 300000th). This was also the case with Pt_I , after measurements had been made at very low temperatures with the resistance thermometer for two years.

Repetition of the calibration at low temperatures of 1902 did not give an equally good harmony. We found:

1) The thermometer had got defect in consequence of the bursting of the glass cylinders. However carefully it was repaired; yet this gave rise to a diminution of length of the wire of 3 mm. or 0,039 %, for which a correction was applied.

TABLE IV.

Hydrogen th.	1902	1907	Deviation in Ω	Deviation in $^{\circ}$ C.
0°	110.045	110.048	- 0.003	
-182.63	28.692	28.605	+ 0.087	+ 0.25°
-197.08	21.877	21.999	- 0.022	- 0.05
-209.93	16.025	15.934	+ 0.089	+ 0.25

We think that we cannot draw another conclusion from it than that the reliability of the measurements with the hydrogen thermometer in 1902 was not yet so great as it has become now as appears from Comm. N^o. 95^d.

§ 4. *Results.* The measurements have yielded for the resistance of each of the wires expressed in that at 0° as unity:

TABLE V. Comparison of different platinum resistance thermometers

Temperature	Pt_I	Pt_{III}	Pt_V	Pt_M	Pt_d
0°	1.	1.	1.	1.	1.
- 30.53	0.87892	0.87846	0.87799		
- 58.58	0.76685	0.76632	0.76643		
- 87.55	0.64991	0.64918	0.65039		
-103.83	0.58345			0.58720	
-109.09	0.56204	0.56025	0.56126		
-140.19	0.43311	0.43195	0.43182		
-159.11	0.35368	0.35240	0.35214	0.35979	
-182.75	0.25283	0.25141	0.25009	0.26022	0.27374
-195.10	0.20045	0.19894	0.19858	0.20812	0.22298
-204.68	0.15974	0.15816	0.15880		0.18355
-212.20	0.12816	0.12653	0.12625	0.13622	0.15285
-216.63	0.11024	0.10853	0.10824		
-252.82	0.01421				0.040637
-255.18	0.01244				0.03766
-259.10	0.01053				0.03645

It appears that wires delivered at the same time show the same course with only small deviations. A considerable difference in deformation of the wire has had only a slight influence for Pt_V . The great difference with wires delivered at different times points to the fact that the originally used material and the treatment in drawing the wires out decide on the change of the resistance. How great the influence of the treatment in drawing is appears from the comparison of Pt_{III} and Pt_d . They were supplied by HERAEUS about the same time and are therefore probably made of platinum of the same degree of purity. Yet the thinner wire Pt_d decreases much less in resistance than the thicker one. At the temperature of liquid hydrogen the differences become very large. In view of the results obtained for gold, which have been inserted in the following Communication (N^o. 99^c), the most plausible explanation is this that the admixtures in the platinum of the wires sent by HERAEUS, either due to their being less pure by nature or to the way of drawing, were less with the platinum sent later than with that sent earlier. We come back to this in Comm. N^o. 99^c. Here we may still mention that DEWAR's wire gave 0,30521 to ours 0,25344 at -182° , and that only the thickest (0,2 m.M.) of HOLBORN's wires gave a smaller value than ours, viz. 0,21253 to ours 0,21786 at -191° .

§ 5. *Calibration formulae for the new wires.* Just as for Pt_I we have also calculated the constants for each of the wires Pt_{III} and Pt_V in a calibration formula which is adjusted down to -217° and does not give to great deviations at the hydrogen temperatures. To be adjusted to the hydrogen temperatures too formulae of another form are required. The above mentioned formulae of the form (A) :

$$\frac{W_t}{W_0} = 1 + a.t.10^{-2} + b.t.10^{-4} + c.t.10^{-6} + d\left(\frac{10^2}{T} - \frac{10^2}{(273.09)^2}\right)$$

give for the adjustment which we distinguish by A_I :

A_I	a	b	c	d
Pt_V	+0.401819	+0.0007403	+0.0052641	+0.020666
Pt_{III}	+0.398291	-0.0026645	+0.0039442	+0.016843

The mean error proved to be greater, for Pt_V even considerably greater than in the calibration in Comm. 95^c, which can be ascribed only partly to the indirect method of the determination of the resistance.

Physics. — “*Isotherms of diatomic gases and their binary mixtures.*
VI. Isotherms of hydrogen between — 104° C. and — 217° C.”
 (Continued). By Prof. H. KAMERLINGH ONNES and C. BRAAK.
 Communication N°. 99^a from the Physical Laboratory at Leiden.

(Communicated in the Meeting of June 29, 1907).

§ 14. *Survey of the determinations.*

The determinations mentioned in this Communication constitute one whole with those of Communication N°. 97^a. They may partly serve to control the earlier determinations at — 104° and — 136°, which, being the oldest observations, are not quite so reliable as the others. For the determinations of isotherms at lower temperatures they are a valuable supplement for the smaller densities from 70 to 100 times the normal one. With the exception of the isotherm of — 217° the determinations communicated now may also be considered as a whole in themselves. To complete this set of determinations a part of the isotherm mentioned for the density at about 170 times the normal one is still wanting. We hope soon to publish the additional determinations referring to this. To the standard-temperatures at which we determined the isotherms, we have still added — 164° C. From the data mentioned in Comm. N°. 97^a may be derived (see § 13 of the communication mentioned) that the point where the inclination of the *p*-curve for exceedingly small densities becomes zero, lies at about this temperature. The purpose of the determinations at — 164° is to determine this point, which we shall call the *Boyle-point*, more definitely.

The determinations, with the exception of that at — 140°, were made at temperatures which differed little from the standard-temperatures of Comm. N°. 97^a. They may be reduced to these standard-temperatures by a simple correction (See Comm. 97^a § 6). This reduction has not yet been carried out for the isotherms mentioned below. In Table XIX the temperatures are given at which the measurements were made. They were determined and calculated in exactly the same way as those of Comm. N°. 97^a; just as to these latter temperatures the correction of Table XVIII Comm. N°. 97^b is still to be applied to them.

We may still remark about the measurement of the pressure (cf. § 3), that for the lowest pressures a direct connection with the open manometer was required, because the closed auxiliary manometer cannot be used below 20 atms.

§ 15. *Remarks about the manometers and the piezometers.*

When the determinations were finished, the auxiliary manometer

was once more compared with the open standard-manometer. It proved that the normal volume had again undergone a slight diminution, i.e. of 0.00026 of the original value. This comparison was made at about 22, 28 and 55 atmospheres. If the calculations of the pressure are carried out with the corrected normal volume, the remaining differences between the indications of the open and the closed manometer are smaller than $\frac{1}{10000}$ of the total pressure.

The steeltube f'_4 with hexagonal portion f'_5 on the stem of the piezometer b_5 (cf. fig. 2 Pl. II Comm. N^o. 69, for the details at the top of the tube compare fig. 4 ibid f_1, f_2, f_4) was soldered to the glass stem b_5 in the way described in Comm. N^o. 94^b. Now the packing could be pressed down more tightly (cf. § 4 Comm. N^o. 97^a) without danger of the block sliding from the stem.

The dimensions of the different parts of the piezometer were about the same as in the determinations of series I of Comm. N^o. 97^a. The glass stem had a greater length and a volume of about 12 cm³., which enabled us to determine a greater part of the isotherms than was possible in series I. The reservoir had the somewhat smaller volume of 5.1583 cm³.

§ 16. *Second group of values of pv_A .*

In table XIX the results of the determinations have been given in the same way as in Table XII of Comm. N^o. 97^a.

In conclusion we express our hearty thanks to Mr. J. CLAY for his valuable assistance in this investigation.

TABLE XIX. H_2 . Series IV. Values of pv_A .

N ^o .	t	p	pv_A	d_A
1	-103. ^o 71	28.423	0.63208	44.967
2		38.154	0.63648	59.944
3		48.682	0.64143	75.897
4		58.217	0.646.8	90.222
5	-139. ^o 88	25.432	0.49452	51.428
6		33.774	0.49697	67.960
7		41.273	0.49867	82.600
8		48.558	0.50232	96.667
9		25.380	0.49466	51.308
10	-164. ^o 14	22.818	0.40065	56.952
11		28.688	0.40164	71.427
12		34.387	0.40253	85.427
13		39.947	0.40376	98.936
14	-183. ^o 18	20.409	0.32562	62.677
15		24.705	0.32550	75.898
16		28.374	0.32521	87.248
17		32.416	0.32522	99.673
18		20.400	0.32557	62.663
19	-195. ^o 17	18.554	0.27867	66.581
20		23.337	0.27765	84.655
21		27.879	0.27622	100.933
22	-204. ^o 69	16.752	0.24040	69.684
23		20.456	0.23880	85.658
24		24.019	0.23695	101.367
25	-212. ^o 82	15.416	0.20644	74.679
26		18.038	0.20430	88.296
27		20.643	0.20228	102.051
28	-217. ^o 40	14.638	0.18742	78.403
29		16.787	0.18495	90.766
30		18.857	0.18293	103.080

Physics. — “*On the change of the resistance of the metals at very low temperatures and the influence exerted on it by small amounts of admixtures*” I. By Prof. H. KAMERLINGH ONNES and J. CLAY. Communication N^o. 99^c from the Physical Laboratory at Leiden.

(Communicated in the meeting of June 29, 1907).

§ 1. *Introduction.* In Comm. N^o. 99^b we called attention to the very large differences in the change of the galvanic resistance with the temperature, which different platinum wires show when we descend to the low temperatures which are to be reached with liquid hydrogen. Such differences were still more pronounced for different gold wires which we investigated. With this metal (see Comm. N^o. 95^d/ These Proc. Sept. 1906) we had taken in hand the investigation of the influence of small amounts of admixture announced in Comm. N^o. 77, because the influence of admixing silver would probably be important and the percentage of silver could be determined very accurately, the possibility of drawing out wires of the different kinds of gold and its high melting point moreover rendering this metal preferable to the for the rest very suitable mercury.

Besides, the inquiry into the influence of small amounts of admixture on the change of the resistance of gold with the temperature proved at once useful as we thought that the gold resistance thermometer would be preferable to the platinum resistance thermometer. Dr. C. HOITSEMA, who already obliged us before (see Comm. N^o. 95^d) by putting pure gold at our disposal, has had the kindness of supplying us again with different samples of gold of high purity, further of preparing for us different alloys with accurately known small percentages of admixture, and of determining the impurity which was finally left in the wires after they had been melted down. For all this valuable help and for the information which Mr. HOITSEMA was enabled to give us by his wide experience we express our hearty thanks.

The investigation of the different gold wires with very small amounts of admixture may of course also be considered as the calibration of different gold resistance thermometers. (Comp. Comm. N^o. 99^b § 1). We prefer, however, to consider it as a part of our more general investigation (Comm. N^o. 77) on the change of the resistance with the temperature for pure metals, and on the influence which small amounts of admixture exert on it.

As to the change of the specific resistance for the pure metals reduced to the most normal state, attention is drawn to the temperature of the

point of inflection $\frac{d^2r}{dT^2} = 0$ the temperature of the point of proportionality $\frac{dr}{dT} = \frac{r}{T}$ and the temperature of the minimum $\frac{dr}{dT} = 0$. It is clear that the situation of these points, and the correspondence¹⁾ and difference of their situation and of the coefficient of variability of the resistance with the temperature in general for different metals of different classes must furnish important data for the theory of electrons.

The investigation of these points is only possible by the aid of liquid hydrogen. Only in some cases — and even then the purity of the metal is open to doubt — the point of inflection, whose existence was indicated by DEWAR, was found high enough to be ascertained without measurements for hydrogen temperatures. For metals in the purest and normal state the point of proportionality lies probably still below the temperatures which are to be reached with liquid hydrogen. It is true that DEWAR derived from his measurements at two hydrogen temperatures that it was surpassed for some of his metal wires. Our measurements, however, point to this that as a metal is brought to a purer and more normal state, the point of proportionality is found to be lower. The metal wire which came nearest to this *ideal* state, was one of our gold wires. Even at the lowest temperature the point of proportionality was not yet reached for that wire. Probably DEWAR's wires were further removed from this ideal state.

With the low situation which we find for the point of proportionality measurements at two hydrogen temperatures do not suffice, but we have made determinations at at least three hydrogen temperatures, because they were necessary to determine the probable situation of the point of proportionality by the aid of $\frac{d^2r}{dt^2}$. Until we have reached the proportional point, we need not discuss the question of the minimum point.

In the inquiry into the properties of the metals in the ideal state we must know first of all in how far the metal is in this state, and else how we can derive what would be found in this state. The influence of small deviations in the nature of the metals on the change of the resistance with the temperature, becomes so exceedingly great for the hydrogen temperatures, that a special investigation is necessary for them. Here two things have to be paid attention to: to small amounts of admixture, and to differences in

¹⁾ In this respect something is to be derived from the formulæ given by us in this and in the preceding communications.

hardness etc. For the present we have left the latter out of account; in the investigation of the influence of the admixtures, however, the influence of the hardness was as much as possible eliminated by our treating the different samples of metals exactly in the same way when comparing them, and by reducing them to the same state of softness.

The most natural explanation of the whole of the results obtained as yet (in this and the preceding Communication) is to ascribe the deviations for the different wires of one and the same metal to impurities in the metal, which may also come in during the drawing if efficient precautions do not prevent it, and which even in very small quantities exert a very great influence on the changes of the resistance with the temperature.

The influence of the drawing is altogether lost for mercury, in which it is also easy to ensure uniform distribution of small quantities of admixture. This enhances the importance of the study of this metal for the investigation of the influence of admixture. In the first place we have measured its resistance at hydrogen temperatures which had not yet been determined; it is given in § 4. It proved that for pure mercury ¹⁾ the inflection point falls in the region of liquid hydrogen temperatures. This is a drawback for the inquiry into the change of the resistance with the temperature for pure metals.

Just as the gold wire *AuV* (see § 2), also the silver wire *AgI* and the platinum wires of the preceding Communication are probably purer than DEWAR's wires of the same metals. For bismuth, on the other hand, DEWAR has most likely had a purer sample than we. The change of the resistance at hydrogen temperatures for this metal, which had not yet been measured by him, has been given in § 5. The observations for lead for those temperatures, which were still wanting up to now, have been given in § 6.

The high degree of purity for some of the metals which were at our disposal, and the lower temperatures to which we descended (solid hydrogen evaporating at 2,5 m.m. pressure) render the decrease of the resistance in some cases many times larger than was observed by DEWAR. To this it is also owing that we have observed the great influence, which very small changes in the nature of the metal obtain on the change of the resistance at hydrogen temperatures. We may account for this by paying attention to the difference of the resistance of a wire of pure metal at the temperature T , r_{1T} , with

¹⁾ Perhaps in connection with the low melting point. Possibly the point of proportionality is first reached for osmium.

that of a wire of the same metal with a proportion of admixture x at the same temperature, r_{xT} . According to a theorem of MATTHIESSEN¹⁾ derived from observations between 0° and 100°, this difference (the theorem refers to a difference that is about the same as that considered here) is constant for different temperatures. FLEMING²⁾ found this theorem about confirmed down to — 200°. As we have found, this theorem no longer holds for hydrogen temperatures. But the deviation is not of such a nature as to affect our conclusions. So if to form an idea of the influence of the admixtures, we put $r_T = r_{iT} + px$, further p constant and large, then it is clear that — when r_{iT} becomes as small as is the case (see Table I *AuV*) for pure metals and hydrogen temperatures — the resistance of a metal for the case that x gets an appreciable amount, will be owing almost exclusively to the admixture. The small amounts of admixture obtain a remarkable influence³⁾.

Analogues are easily found in the important influence of small amounts of admixture on the density in the neighbourhood of the critical temperature of a substance, in a space becoming opaque by a cloud depositing on a minimum quantity of dust. But for a further discussion the systematic investigation of the influence of small amounts of admixture should be more advanced. At all events the changes of the resistance with temperature at hydrogen temperatures proves to be a highly sensitive criterion to decide about the nature of a metal.

§ 2. *Gold.* The different samples of pure gold were all supplied by Dr. C. HOITSEMA. With the exception of two the wires were all treated in exactly the same way, drawn out by HERAEUS to 0,1 mm. diameter, and treated at every pull with diluted sulphuric acid and nitric acid. The gold wire *AuVI* was drawn in a different way and made strongly impure. The exact amount has not yet been ascertained.

1) Pogg. Ann. Bd. CXXII.

2) Proc. Royal Institution June 1896 p. 9.

3) To a less degree of purity of the examined metal wires it is perhaps to be ascribed that NICCOLAI, Att. Linc. 16, 1st sem. p. 906 finds a smaller decrease of resistance at — 189° than we do, as appears by comparison with Tables I, III, V of this communication and V of Comm. N^o. 96^b. Indeed NICCOLAI finds:

	silver	gold	lead	platinum
0°	1	1	1	1
— 189°	0.2784	0.3068	0.3357	0.3198

[Added in the translation].

Au_0 is the wire which was calibrated in Comm. N°. 95^d. After melting it down Dr. HOITSEMA found 0,03% impurities. As to the wires Au_{III} , Au_{IV} and Au_V , Dr. HOITSEMA found about 0.015% admixture for Au_{III} , about 0.005% admixture for Au_{IV} and Au_V . Au_{AgI} was made of an alloy which after the wire had been remelted, contained 0.4% admixture (probably chiefly silver). Wound round the same glass cylinder all wires except Au_0 and Au_{AgI} were heated at the same time in an annealing furnace for a long time, and slowly cooled, as had also been done with Au_0 and Au_{AgI} , so that they were perfectly malleable.

In table I the resistances have been given expressed in that at 0° as unity. These were about 9Ω.

TABLE I.

Change of the resistance of different gold wires with the temperature.

Temperature	Au_{III}	Au_{IV}	Au_V	Au_{VI}	Au_0	Au_{AgI}
0°	1	1	1	1	1	1
−103.83	0.56601	0.59389	0.59306	0.64827	0.60545	0.64549
−183.00	0.27653	0.27177	0.27096	0.37053	0.30070	0.37099
−197.87	0.21456	0.20963	0.20871	0.31659	0.23908	
−205.01			0.17897		0.20992	
−215.34	0.14058	0.13407	0.13337	0.16822	0.16681	
−252.93	0.01602	0.008743	0.008103	0.13669	0.04554	0.13942
−255.13			0.005691			
−258.81	0.01095	0.004265	0.003601	0.13241	0.03982	0.13288
[−261]			0.002713			
[−262]		0.003257	0.002526			

To facilitate comparison we may observe that DEWAR found 0.03290, whereas we found 0.008103 at −252°.93 for Au_V .

§ 3. *Mercury*. It was doubly distilled and brought into a glass spiral. The latter was protected by a bath of pentane, which was slowly cooled from the bottom upwards before it was immersed into the bath of the cryostat. We found:

Temp.	Resistance
0°	97.126
— 183.00	7.2650
— 197.87	6.0103
— 205.01	5.3900
— 215.34	4.5057
— 252.93	1.2613
— 258.81	0.7534

§ 4. *Silver*. This was also supplied by Dr. C. HOITSEMA and drawn to a wire of 0.1 m.m. by HERAEUS, during which operation it was treated in exactly the same way as the gold and the platinum. After the resistance had been determined, the composition was controlled; the silver contained then 0.18% impurity. The zero point

Temperature	Resistance Ag_1	$O-C_D$	$O-C_{AIV}$
99°.76	1.41089	0	0
0 before	1	0	0
0 after	1.09037		
— 103.81	0.58087	— 0.00042	— 0.00042
— 139.87	0.43282	+ 42	+ 42
— 183.57	0.24679	— 17	0
— 195.17	0.19703	+ 29	— 2
— 204.67	0.15528	— 31	
— 252.92	0.008913	0	
— 259.22	0.00642	0	

of the silver wire (resistance at 0°) changes slightly by being drawn out in consequence of the difference in expansion of silver and glass. The preceding table gives the resistance expressed in that at 0° as unity. (The resistance at 0° was 21.519Ω).

Column $O-C_D$ contains the deviations from a formula adjusted from $+100^\circ$ to -259° of the new form:

$$\frac{W_t}{W_0} = 1 + a \cdot 10^{-2} \cdot t + b 10^{-4} t^2 + c 10^{-6} t^3 + d \left(\frac{10^6}{T^3} - \frac{10^6}{(273.09)^3} \right) + e \left(\frac{10^{10}}{T^5} - \frac{10^{10}}{(273.09)^5} \right) \quad (D)$$

which agreed best with the values:

D	a	b	c	d	e
Ag_I	+0.402746	+0.004355	+0.004806	+0.00955	-0.000013

For a comparison with platinum and gold the column $O-C_{A_{IV}}$ gives the deviations from a formula of the form A (see Comm. N^o. 95^c and 95^d):

$$\frac{W_t}{W_0} = 1 + a.t.10^{-2} + b.t^2.10^{-4} + c.t^3.10^{-6} + d \left(\frac{10^2}{T} - \frac{10^2}{273.09} \right)$$

with values which we distinguish by A_{IV} on account of the other way of adjusting the coefficients.

A_{IV}	a	b	c	d
Ag_I	+0.40355	+0.03968	+0.005232	+0.008662

§ 5. *Bismuth*. The measured resistance, which we shall also investigate in the magnetic field, was that of a bismuth spiral of HARTMAN and BRAUN N^o. 301. The resistance expressed in that at 0° as unity (the resistance at 0° was 17.3138Ω) was:

Temperature	Resistance Bi_I	$O-C_{A_{III}}$
12°64	1.05148	0
12°70	1.05165	
0	1	
- 103.71	0.63649	+ 0.00952
- 139.88	0.52865	- 127
- 164.05	0.46246	- 144
- 182.73	0.41435	- 69
- 195.17	0.38478	+ 144
- 204.68	0.36064	+ 127
- 216.01	0.33014	- 69
- 253.01	0.22329	- 92
- 255.34	0.21388	- 2
- 258.86	0.19574	0

This column $O-C_{A_{III}}$ gives the differences with a formula of the form A (see § 4), with values which we shall indicate by A_{III} on account of the adjustment at two hydrogen temperatures and over the region to 0° :

A_{III}	a	b	c	d
Bi_I	+0.390037	+0.051928	+0.0038155	-0.0079700

§ 6. *Lead.* The knowledge of the resistance of lead is of particular importance on account of the fact that this metal does not show the THOMSON-effect. Probably the lead used by us, contained no more than 0,015 % admixture.

The resistance of a narrow strip cut out from the flatted lead and protected against chemical action by paraffine expressed in the resistance at 0° (3.18114 Ω) as unity was found to be:

TABLE V.

Change of the resistance of lead with the temperature.

Temperature	Resistance
+ 46.33	1.0652
0°	1
-103.63	0.59548
-183.65	0.29439
-195.15	0.25257
-204.52	0.21742
-216.61	0.17129
-252.78	0.03032
-255.07	0.02314
-258.70	0.01311

Physics. — *Repetition of DE HEEN'S and TEICHER'S experiments on the critical state*", by Prof. H. KAMERLINGH ONNES and G. H. FABIUS. Communication N°. 98 from the Physical Laboratory at Leiden.

(Communicated in the meeting of April 26, 1907).

§ 1. *Introduction.* Experiments have been repeatedly made from which the conclusion was drawn, that a substance can assume different densities above its critical temperature with the same pressure and the same temperature, which densities it can retain for hours according to some investigators¹⁾. That in reality this is not the case, and that the

¹⁾ TRAUBE *Ztschr. f. phys. Chem.* 58 p. 477. 1907, cf. also MATHIAS, *Le point critique des corps purs* p. 250.

When with change of density dissociations or variations of volume of the molecules themselves should made their appearance which clearly require more time than the establishment of temperature equilibrium through conduction of heat and convection, we should when a phase was kept at constant volume after having suffered variation of density, have to find an increase of the pressure both for liquid and vapour phases and for phases above the critical temperature; thermodynamically it follows from this that the density of liquid in equilibrium with vapour would then have to be a function of the time.

Gf. TRAVERS and USHER on variations of density in consequence of false equilibria.

differences found are to be ascribed to admixtures or to differences in the pressure or the temperature of the phases compared appeared already when KUENEN (Comm. n^o. 11, Verslag Kon. Ak. v. Wet. May and June 1894) repeated GALITZINE's experiments (Wied. Ann. 50, 1893), and found but trifling differences remaining. Afterwards when DE HEEN (Bull. Ac. Belg. 3e S. t. XXXI '96) had again found the differences of density in question by another way, it was shown by repetition of his experiments at Leiden (Comm. N^o. 68, These Proc. April 1901 p. 628 and p. 691), that also these differences of density vanish almost entirely for pure CO₂ when attention is paid to the differences of temperature. In the last few years, however, it has been particularly TEICHNER's¹⁾ experiments that have given new support to the opinion that after all these differences of density really exist (Drudes Ann. 13, 1904).

In the first place we have repeated DE HEEN's experiment in different ways. Already with the earlier repetition (1901) thermo-elements had been introduced for the determination of the difference in temperature of the two metal reservoirs of the apparatus, which were separated by a cock, the apparatus for the rest resembling that of DE HEEN as closely as possible. One of the thermo-elements, however, was damaged during the experiments. Though it could be ascertained that the differences in density even without correction for the temperature were considerably smaller than those found by DE HEEN, probably on account of the greater purity of the CO₂, the exact amount of the difference remaining after temperature correction could not be determined. To replace these measurements by better ones a new, improved apparatus with thermo-elements was built, resembling for the rest DE HEEN's apparatus as closely as possible, and with this apparatus we made the observations communicated in § 3. They confirm that the differences in density derived by DE HEEN from his experiments do not exist for a pure substance when temperature and pressure are uniform²⁾.

¹⁾ In the TEICHNER's tube the same differences of density which GALITZINE and WILIP (Congr. Intern. de Physique I 668, 1900) had found were shown by GILBERT-FARADAY's density-bulbs. What holds for TEICHNER's experiments applies therefore also to those of GALITZINE and WILIP.

²⁾ So if there exist processes as meant in p. 215. note 1, they pass so quickly that it is not possible to demonstrate them by methods which require that the equilibrium of pressure and temperature has first been established. As yet nothing has been found that points to the fact that the establishment of the temperature equilibrium is retarded an appreciable time on account of changes of energy which increase in course of time to a definite limiting value, the volume remaining constant.

It was demonstrated in Comm. N^o. 68 that admixtures and differences of tem-

Further we have repeated TEICHNER's experiment¹⁾ with more precautions than had been taken by this observer. Especially a thermoelement (platinum-platinum-iridium not to sacrifice the security which the glass apparatus offers for the preservation of the purity of the substance) was adjusted in the upper and in the lower end of the TEICHNER tube, just as in DE HEEN's modified apparatus, to enable us to follow the differences of temperature in the tube.²⁾ By using CO₂ for the experiment, a high degree of purity could be reached, and we came into a region where the temperature could be kept constant up to a very small amount.

If we wish to prevent diffusion between the higher phases and the lower ones, the modified TEICHNER tube (at least when not a capillary constriction has been made in it³⁾) is inferior to the modified apparatus of DE HEEN. Moreover when we wish to reach the equilibrium of temperature quickly, the bad conductivity of heat of the glass is a drawback, but it has the great advantage, that the changes of density can be observed at the same time with the other phenomena for the critical state. With regard to these phenomena TRAVERS and USHER (*Ztschr. f. phys. Chem.* 57, p. 365, 1906) and YOUNG (*ibid* p. 262) published important papers, after we had made the experiments mentioned in § 7. In the main points our observations agree with the descriptions given by TRAVERS and

perature lead to systematic disturbances as in DE HEEN's experiments. Both give rise to disturbances of the same character. In the discussion of the influence of the differences of temperature the valuable paper by VILLARD *Ann. d. Ch. et d. Phys* (7) 10. 1897 has been overlooked there. That TEICHNER's results might be ascribed to small admixtures has appeared in details from the calculations by VERSCHAFFELT (*Comm. Suppl. n^o. 10, (Dec. 1904).*

To the influence of admixtures on phenomena in the neighbourhood of the critical point attention has also been drawn by YOUNG *Journ. de Chim. Phys.* 4 (1906) p. 475. To this may be added that KEESOM, *Comm. No. 88, These Proc.* Jan. 1904 p. 593 did not only consider the increase of the pressure during condensation with constant temperature as a proof of the presence of admixtures, but that it served him further to arrive at an opinion on the quantity of the admixture.

¹⁾ This was already mentioned *Comm. Suppl. No. 10 These Proc.* Dec. 1904. Lately TRAUBE strongly urged the advisability of a repetition.

²⁾ In a CAGNIARD-LATOUR tube thermometers were fused by VILLARD. Our tube may just as well be called a VILLARD tube with density-bulbs as a TEICHNER tube with thermo-elements.

³⁾ Such an apparatus, if necessary provided with a valve which is worked magnetically might be serviceable in the investigation of the variation of density with temperature. [After this was printed we noticed that the device of a capillary constriction was used by RAMSAY, *Proc. Roy. Soc.* 30 (1880) p. 327. Note added in the translation].

USHER and by YOUNG, and supplement them by giving the variations of the densities.

After what the repetitions of DE HEEN's experiments had taught us again about the asserted differences of density at the same pressure and the same temperature above the critical temperature, our repetition of TEICHNER's experiments has become rather a first contribution to the study of the variations of density with temperature and pressure by this way, than a refutation of the conclusions derived from TEICHNER's experiments. We have, however, been able to show sufficiently by our experiments that these conclusions are erroneous.

§ 2. *Repetition of one of DE HEEN's experiments.* As we can refer to Comm. n°. 68 with regard to the choice of the experiment which is to be repeated (on account of the systematic character of the deviations only one need be repeated), and as on another occasion a full description of the apparatus used and the different operations will be given, we think that the following remarks on the arrangement of the experiments will suffice here.

1. The pure carbonic acid was prepared by distillation. The admixtures are to be estimated at no more than 0,00027 (cf. KEESOM Comm. N°. 88 II, § 2 and V § 10 These Proc. Jan. 1904). In the apparatus it comes into contact only with metal, glass, and cork packings of this gave a perfect closure after having been repeatedly tightened during a week.).

2. The apparatus, the conduits, and the further auxiliary arrangements, among which also two metal bottles with the purified CO₂, are all in connection with a mercury airpump. One of the bottles with pure CO₂ serves for rinsing. From the second the desired quantity is conveyed into the apparatus by distillation.

3. The density in every reservoir is determined by making the carbonic acid flow from it into a large reservoir with mercury manometer kept at constant temperature. In the volumetric calculations the corrections are applied according to the empiric equation of state V s. 1 of Comm. n°. 74 (Arch. Néerl. (2) 6, 1901). Errors in the density caused by leakages in the reservoirs at high pressure are excluded. It was ascertained by separate control experiments that the total amount of CO₂ in the apparatus remained unchanged during the experiments.

4. The apparatus was kept at uniform constant temperature by means of flowing water, a xylene thermoregulator (see Comm. N°. 70 III § 3 These Proc. May 1901) and a valve stirrer (see Comm.

N^o. 83 III § 4 These Proc. Febr. 1903). Sufficient precautions were taken to prevent conduction of heat from outside to parts of the apparatus.

§ 3. *Variations of density found in the repetition of the experiment of DE HEEN after correction of the difference in temperature of the reservoirs.*

DE HEEN brings the temperature of the two reservoirs (see § 2 beginning) from 28° to about 35° C, and opens the cock between them 6 times four seconds during the heating; then when the temperature has become constant at about 35°, he opens the cock once more 6 times 4 seconds. Then he assumes that temperature and pressure are the same in the two reservoirs.

When repeating the experiment (being very careful to prevent drops from being scattered from one reservoir into the other) we found confirmed by reading the thermo-elements (nickel-iron), what was observed in Comm. N^o. 68 viz. that every time when the cock between is opened for adjustment of pressure, a difference of temperature arises between the two reservoirs, and that when closing the cock at the end of the experiment a difference of temperature remains, which must be taken into account.

In order to find out in how far the equilibrium of temperature and pressure has been reached, we have in the first place made three determinations, in which the cock was opened respectively 2, 4, and 6 seconds every time (probably our cock allowed comparatively less substance to pass than that of DE HEEN). The results have been given in the following table: v denotes the upper, l the lower reservoir, so q_l is the density in the lower reservoir; the numbers of times the cock was opened during the heating (distributed over 15 minutes) and then at constant temperature (distributed over half an hour) have been separately given; q'_l is q_l corrected for $t_v - t_l$.

SERIES I.

Establishment of the equilibrium by opening the cock	ρ_v	ρ_l	ρ_l/ρ_v	t_v	$t_v - t_l$	Corrected for difference of temperature	
						ρ'_l	ρ'_l/ρ_v
6 times 2 + 6 times 2 sec.	0.418	0.566	1.21	34.25	0.27	0.456	1.09
6 times 4 + 6 times 4 sec.	0.424	0.495	1.17	34.40	0.22	0.454	1.08
6 times 6 + 6 times 6 sec.	0.427	0.489	1.15	34.20	0.16	0.456	1.07
6 times 6 + 6 times 6 sec.	0.437	0.501	1.15	34.30	0.20	0.466	1.07

The temperature corrections have been borrowed from the graphical representation derived in Comm. N^o. 68 from AMAGAT's determinations. The uncertainty which still prevails with respect to the correct course of the isotherms in the neighbourhood of the critical state is, of course, also found in these corrections.

The rather rapid process of the heating from 28° to 35° prevents further that the whole apparatus has already assumed the temperature of the waterbath, so that also the observed differences of temperature themselves are not quite certain.

In the following series of determinations we proceeded in the same way till the 12th opening (the 6th at constant temperature). It was put off for 3 hours. When the cock was opened the course of the deviations of the galvanometer appeared to be the same as in the preceding series; the remaining temperature corrections, however, were somewhat smaller than in the 1st series, which we ascribe to this that all parts of the apparatus have had time to assume the temperature of the waterbath. An increase of pressure in the lower reservoir, which should have been found when e. g. in this lower reservoir molecules were dissociated during these three hours (see § 1, p. 216 footnote 2), and which should have given rise to a greater galvanometer deviation when the cock was opened for the last time, could not be traced. We found:

SERIES II.

Establishment of the equilibrium by opening the cock	p_v	p_l	p_l/p_v	t_v	$t_v - t_l$	Corrected for difference of temperature	
						p'_l	p'_l/p_v
6 + 5 + 1 time 2 sec.	0.430	0.497	1.16	34.55	0.20	0.466	1.08
6 + 5 + 1 „ 4 sec.	0.440	0.489	1.11	34.85	0.16	0.456	1.04
6 + 5 + 1 „ 6 sec.	0.439	0.485	1.10	34.40	0.15	0.452	1.03

In virtue of what the preceding determinations had taught us as to the reaching of equilibrium of temperature and pressure the cock was opened 12 times 12 seconds in a following determination, and finally two more determinations were made in which the cock was opened 12 times 4 seconds and at last once five minutes. We found:

SERIES III.

Establishment of the equilibrium by opening the cock	ρ_v	ρ_l	ρ_l/ρ_v	t_v	$t_v - t_l$	Corrected for differences of the temperature	
						ρ'_l	ρ'_l/ρ_v
6 + 6 times 12 seconds	0.446	0.488	1.09	34.70	0.20	0.456	1.02
6 + 6 times 4 sec. } at last 5 minutes }	0.427	0.445	1.04	34.05	0.06	0.422	1.01
	0.462	0.478	1.03	34.00	0.06	0.467	1.01

From this follows that as the equilibrium of pressure and temperature is obtained better, the densities of the phases become more and more equal, and that at last after the application of the temperature correction only very small differences remain.

The much more considerable deviations found by DE HEEN ($\rho_l/\rho_v = 1.19$, see Comm. No. 68) must, therefore, be attributed to admixtures and differences of temperature. ^{1) 2)}

§ 4: *Example of differences of density as found by DE HEEN, caused by a slight impurity of the CO₂.*

The great influence of small quantities of admixture is very convincingly shown by the following results.

In a group of determinations ending with a repetition of experiment 2 Series I we found:

Establishment of the equilibrium by opening the cock	ρ_v	ρ_l	ρ_l/ρ_v
6 + 6 times 4 sec.	0.392	0.531	1.35

It appeared that in consequence of carelessness in the cleaning after it had been repaired, in the metal bottle in which the pure

¹⁾ That during the opening of the cock no important exchange of liquidogeneous and gasogeneous molecules between the upper and the lower reservoir can have taken place (to use the terminology of DE HEEN), appears from this, that when the upper reservoir was filled with air and the lower reservoir with air with 31 pCt. CO₂, and the equilibrium was established without the cock being opened, after the cock had been opened for 5 minutes, only 0,33 percent of CO₂ had passed into the upper reservoir.

²⁾ They give both systematic errors of the same character (cf. § 1 p. 2 note 1 above).

carbonic acid was kept, a trace of oil had been left which had diffused in the carbonic acid. When we thought that the whole apparatus had been sufficiently cleaned by blowing pure carbonic acid through it, we found:

Establishment of the equilibrium by opening the cock	ρ_v	ρ_l	ρ_l/ρ_v
Only 6 times 4 sec. during the rise of the temperature.	0.385	0.555	
6 + 3 times 4 sec.	0.399	0.521	1.30
6 + 6 times 4 sec.	0.417	0.505	1.21

which by the side of Series I and II gives at the same time an idea with what degree of approximation equilibrium of pressure is reached by the repeated opening of the cock. Repeating the last determination after continued blowing we found:

$$6 + 6 \text{ times } 4 \text{ sec.} \quad | \quad 0.426 \quad | \quad 0.496 \quad | \quad 1.19$$

The last observation harmonizes pretty well with Series 1 N°. 2, of which it is a repetition, the same deflections of the galvanometer being moreover found. It leads to the same ρ'_l/ρ_v .

It appears how misleading even the presence of slight impurities may be, and that in Comm. N°. 68 the leather packings have been justly called a fundamental defect of DE HEEN's apparatus (cf. 1 § 2 above).

§ 5 *Correction for gravity and for slight admixtures in the experiment of DE HEEN. Result.*

In order to be able to calculate the correction which is to be applied to ρ'_l/ρ_v for gravity, we must know the density as function of the level in a column of pure CO_2 . The accurate shape of the isotherms in the critical state being very uncertain ¹⁾, this function is not accurately known. GOUY has taken SARRAU's equation of state as basis for his calculation. We have started from the equation of state V s. 1 of Comm. N°. 74. Only 0,0002 was found for the correction at 34° and 8 c.M. difference of level, so that it may be neglected. As to the correction for the admixtures for the CO_2 , when we put their amount as given in § 2, at 0,00027, it becomes

¹⁾ It is just to the knowledge of this shape that observations as made in § 3 and § 7 may contribute.

1 ‰ à 1,5 ‰ at 34°.5 according to the calculations of VERSCHAFFELT, Comm. Suppl. N°. 10. If the uncertainty of this correction is taken into account, we must come to the conclusion, the limit of accuracy of our experiments not being higher than 1 ‰ either, that the differences of density derived by DE HEEN from his experiments do not exist for pure CO₂ when sufficient care is taken to ensure equilibrium of pressure and temperature.

§ 6. *Repetition of TEICHNER's experiment.* An elaborate description of the apparatus and the operations will be given on another occasion. Here the following remarks may suffice:

1. Repetition with CO₂ was considered to be desirable also by TEICHNER because it gives more warrants for purity.

2. A platinum-platinum-iridium thermo-element (used with a magnetically protected galvanometer of DUBOIS) was successfully fused into TEICHNER's tube at the top and at the lower end, so that the tube remained proof against a pressure of 150 atmospheres. However we did not succeed in making the thermo-elements free from disturbing electromotive forces, nor did they give with certainty the accuracy of 0°.01 we wished. The places of contact were found at $\frac{1}{4}$ and $\frac{3}{4}$ of the height of the tube. A third thermo-element to compare the temperature in the tube with that in the bath would be desirable.

3. The critical density of CO₂, 0.469 ¹⁾, being smaller than that of CCl₄, with which substance TEICHNER worked, it was much more difficult to obtain the required density-bulbs (small glass bulbs) of 0.365, 0.390 and 0.405. We owe them as well as the fusion of the thermo-elements to the skill of Mr. O. KESSELRING, chief of the glass-blowing department of the laboratory. By means of CLEBSCH's formula it was found that the decrease of volume of the bulbs at the highest pressures can only amount to from $\frac{1}{600}$ to $\frac{1}{400}$.

4. Still greater care was devoted to the purification of the CO₂, than in the repetition of DE HEEN's experiment. From a metal bottle of CO₂ as used for the latter, $\frac{1}{3}$ is once more blown off, and then $\frac{1}{3}$ distilled over into a second bottle from which under weighing, so much is suffered to escape that a fixed quantity remains. This second bottle is connected by glass tubes with the experimental tube, a mercury manometer, a mercury airpump and an auxiliary bottle (also of metal) with pure CO₂ for rinsing the conduits, after which the fixed quantity which it contains, is quite distilled over into the

¹⁾ Derived by KEESOM Comm. N°. 88 These Proc. Jan. 1904, p. 574, from his observations by means of the rule of the rectilinear diameter.

experimental tube, the latter being immersed in liquid air; at last the experimental tube, still immersed in liquid air, is connected with the mercury airpump, and fused off. By weighing it is ascertained that the desired quantity has been transferred to the experimental tube.

5. A mode of heating which does not give rise to convection currents inside the tube, is considered of great importance also by TEICHNER and TRAUBE, to prevent mixing of what is at the upper end and what is at the lower end of the tube. TEICHNER does not accept YOUNG's refutation of his experiments (see TEICHNER, loc. cit.) because in YOUNG's apparatus convection currents are not so well prevented as in that of himself. The thermostat used by us, however, satisfies much higher demands than that of TEICHNER. The tube is immersed in a liquid bath in a double walled non-silvered vacuum-glass which is hermetically closed with a badly conducting lid. The glass is provided with a valve-stirrer and an arrangement to heat the bath electrically from above, and is itself again immersed in a bath with double glass walls, which is likewise provided with a valve-stirrer. With the exception of two windows, the space between these two walls is filled with cotton wool, by which the outer wall too is surrounded. Like the bath in which the improved apparatus of DE HEEN (§ 2) was placed, the outer bath was kept at constant temperature (up to $0^{\circ},02$) by flowing water with the aid of a xylene thermoregulator (§ 2). In this way the temperature of the bath in which the experimental tube is, can be kept constant up to $0^{\circ},002$. The heating takes place according to the indication of thermometers divided into 50^{th} of a degree, which are placed in the inner and in the outer bath, and is regulated in such a way that everything that might give rise to convection currents is as much as possible avoided.

6. When manipulated the tube was always efficiently shielded for the protection of the observer.

§ 7. *Observations.* The bulbs correspond with the densities 0.365, 0.390, 0.405, 0.421, 0.443, 0.450, 0.466, 0.483, and 0.510.

The position of the bulbs and that of the meniscus (indicated in what follows by the read number of the mark of division between []) was read on a millimeter scale etched on the tube; in the middle of the tube the mark 30 is found, the zero point is 20 mm. above the bottom.

In the *first experiment* the heating took place very slowly; at first also the inner bath was (electrically) heated, t refers to the outer bath, t_i to the inner one. After three hours small gas bubbles were seen to rise from the downmost thermo-element, probably

caused by conduction of heat along the threads, in consequence of t_c having increased too much. We observed:

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time	t_c	t_i	men.
11 ^h	27°. ⁸	27°. ¹⁵	[31]
11 30		27. ⁹²	
12 20		30. ⁰⁰	
12 25	30.4	30.39	the rising of the gas bubbles has finished, the bulbs begin to show a tendency to divide
12 30		30.69	0.365 rises
12 33		30.83	0.365 at 33
12 36		30.88	[33] 0.510 begins to descend slowly to 25
		30.94	0.510 at 5
		30.99	0.510 on the bottom
			the separate (electrical) heating of the inner bath is stopped,
12 40	31.00	31.00	0.365 at 36 0.483 begins to descend
12 44		31.00	[34] 0.365 between 38 and 39 0.483 „ 21 and 22
			t_i rises very slowly whereas t_c was all the time kept somewhat higher to 2 ^h .50
12 48			0.483 between 10 and 11 0.365 rises to the top with accelerated motion 0.483 continues to descend very slowly
12 52			[34.5] 0.483 has arrived at the bottom all except 0.510, 0.483, and 0.365 in meniscus
1			[35] 0.390 begins to rise
1 3			0.390 between 37 and 38 with 0.405 by its side somewhat lower
1 10			0.390 between 38 and 39 with 0.421 below it in a slanting direction
1 15			[35] 0.421 begins to rise, for the rest like 1 ^h 10'.

Looking from the top downward in a slanting direction through the tube we observe a slight grey mist which is denser in the part below the meniscus. When we look straight through the tube, with a light behind the tube, the whole shows a light brown colour, which is somewhat darker under the meniscus.

The bulbs continue to move slowly apart, the rising ones

moving faster than the descending ones, and the velocity of the rising bulbs increasing as they get higher, the same thing taking place though in a smaller degree with the descending ones.

2 ^h 15	31.032	the meniscus has gradually got fainter and is hardly to be distinguished, only a slight constriction of the light band is to be perceived at 35.
2 20	31.036	0.421 between 42 and 43 0.443 „ 36 and 37 0.450 „ 34 and 35 0.466 „ 32 and 33
2 50	31.22	the outer bath is further kept at this temperature
2 50	31.050	the rise of the temperature of the inner bath is now exceedingly regular
4 30	31.096	0.443 in the middle of 53 0.450 between 32 and 33 0.466 „ 24 and 25
		The mist in the tube is now equally dense everywhere, and becomes gradually less, the moving apart of the three still descending bulbs continues slowly and regularly
8 7	31.210	0.466 lies just on the bottom 0.450 between 36 and 37; has risen 4 mm in 2 ^h 15' and so shows a tendency to move to the top.
8 10		The cooling takes place by reducing the outer bath to a lower temperature
8 30	31.133	0.466 begins to rise from the bottom 0.450 at the same place
8 35		Throughout the tube a bluish mist appears
8 40		This mist gets denser
8 50		At [15] a thick milky white mist is formed, which spreads rapidly upwards and downwards.

30.984 At [10] the meniscus appears. From the upper place of contact of the thermo-elements drops fall down, from the lower place of contact smaller gas bubbles rise upwards. (The cooling proceeds too rapidly). The meniscus rises, 3 bulbs fall quickly into it from above, and 1 rises towards it from the bottom.

The differences of temperature within the tube were found to be between 0°.02 and 0°.03, but in consequence of disturbances they were often not to be observed. In the observation of 8^h7 they were no more than 0°.01.

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In the *second experiment* the heating took place somewhat more rapidly. In 1^h10' the inner bath was brought at

9^h 10' 29°.99 The separation of the bulbs has, of course, not advanced so far as in the first observation, and the deflections of the galvanometer were larger. The tube remained now farther behind the temperature of the waterbath.

9 40 31.60 [36] The meniscus is still dimly to be distinguished here, and a thin light mist is visible through the whole tube. Under the meniscus a somewhat darker band of a light brown colour is seen.

The position of the bulbs is now

0.365 at the top

0.390 between 43 and 44

0.405 ,, 42 and 43

0.421 in the middle 41

0.443 ,, ,, 40

0.450 ,, ,, 39

0.466 between 37 and 38

0.483 and 0.510 on the bottom.

31.7 0.390 and 0.405 now go very rapidly to the top followed by 0.421.

At the place of the meniscus now only a slight constriction and a light brown mist band are to be perceived.

10 45 31.900 The temperature is then kept constant till 11^h37 with no greater deviation than 0°.004 and then from

11 37 31.888 till 2^h15 with no more deviation than 0°.002.

2 15 The mist has now entirely vanished and the bulbs are all apart.

0.510, 0.483, 0.466 on the bottom

0.450 and further ones in the top, none of the bulbs remain suspended in the body of the tube; the bulb 0.450 was the last to go to the top, whereas 0.466 had already been on the bottom for some time.

A difference of temperature is no longer to be perceived.

At first the cooling took place more slowly.

3 31.550 A slight mist is perceived

3 45 31.040 the mist has become distinctly denser.

5 10 30.985 [5]. In this position the meniscus originates in a milky cloud in the lower part of the tube. Further the lowering of the temperature proceeds too rapidly.

5 25 30.950 [14]. Three bulbs float in the meniscus.

A disturbance in the regulation of the temperature of the outer bath put a stop to the experiment.

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A *third experiment* did not give any difference with the first with slow heating. The temperature was kept still closer to the critical, and the heating went still more slowly. Now the meniscus descended, and the brown colour under the meniscus was particularly pronounced, it was darkest just below the meniscus, and gradually faded downward. After a slow heating which extended over $10\frac{1}{2}$ hours the meniscus disappeared at

6^h 30°.986. The position of the bulbs was now
 0.443 between 34 and 35
 0.450 in the middle 27
 0.466 between 25 and 26.

In consequence of a slight disturbance in the regulation of the outer bath t_i descended to

6 45 30.984 [25] the meniscus appears, the position of the highest bulb does not undergo any change, the two others 0.450 and 0.466 float in the meniscus.

9 15 31.010 the meniscus disappeared somewhat lower, the brown colour in the lower half is much more intense, at last it contracted to a dark brown band of ± 1 mM. width just below the meniscus. Gradually it grew lighter and shortly after the meniscus it also disappeared;

 0.443 between 41 and 42
 0.450 23 and 24
 0.466 in the middle 22.

The temperature at the top of the tube was $0^{\circ}.01$ higher than at the bottom.

§ 8. *Conclusions and remarks.* An elaborate discussion of the observations in connection with other peculiarities of the net of isotherms in the neighbourhood of the critical state (cf. Comm. n°. 74 and Comm. n°. 88 IV § 5, Jan. 1904 and Comm. Suppl. n°. 10), and as to those on the mist in connection with the observations of Gouy,

TRAVERS and USHER and YOUNG ¹⁾, must be deferred to a later occasion. However, some conclusions are obvious.

As the temperature rises more slowly and the equilibrium in the tube is better reached, we can get nearer to equality of density of the vapour and liquid phases. We think we found a smaller difference in density in our measurements than any of the observers before us ²⁾.

We did so in the third experiment. The critical temperature was then fixed between $30^{\circ}.984$ and $30^{\circ}.986$ ³⁾. That at $30^{\circ}.984$ only a small difference in density existed between liquid and vapour appears as follows: When the meniscus appeared we found 0.443 for the density of the vapour at the height 35. So we estimate the density at 0.452 at the height of the meniscus (25) according to the correction of § 5 (doubtful); bulb 0.450, however, floats on the meniscus. The density on the bottom is < 0.483 (height (see § 7 beginning) 5 cm. under 30), so we estimate the density of the liquid at 0.468 at the height of the meniscus; bulb 0.466 floats. Vapour and liquid differ, therefore, certainly less than $\frac{1}{10}$, and probably no more than $\frac{1}{20}$ in density. In the first experiment we found $q_v > 0.421$, $q_l < 0.483$ from which $q_v > 0.430$, $q_l < 0.468$ follow with the estimated correction for gravity; so under these less favourable circumstances a difference of less than $\frac{1}{10}$ is most likely realized. These results concerning the closer and closer approach of the density of liquid and vapour; which quite agree with the views of ANDREWS-VAN DER WAALS, deprive the much larger differences of phases at the same temperature and pressure above the critical temperature, which TEICHNER derives from his experiments, of all importance ⁴⁾.

¹⁾ This includes the discussion of the mist stage of VON WESENDONCK, which would constitute the transition stage in the neighbourhood of the critical state, and which in any case can only extend over a small part of the region of density and temperature where a mist can be seen.

²⁾ YOUNG, Journ. Chem. Soc. 71 (1897) p. 455 stated at $0^{\circ}.05$ below the critical temperature a difference of 14% between the liquid and vapour densities. [Note added in the translation].

³⁾ The readings of the temperature have been reduced to those on a thermometer which had been controlled with an air thermometer accurate up to $0^{\circ}.01$ by the Phys. Tech. Reichsanstalt. Our result agrees with that of KEESOM, $30^{\circ}.98$ (Comm. N^o. 88 see above), made with the same thermometer. Moreover, besides and after the determinations of the critical temperature of CO_2 cited in LANDOLT-BÖRNSTEIN-MEYERHOFFER'S Phys. Chem. Tables are to be mentioned: VERSCHAFFELT Zitt. Versl. Juni '96 ($31^{\circ}.0$), VON WESENDONCK Verh. d. Deutsch. Phys. Ges. 5 p. 238 ($30^{\circ}.95$), BRINKMAN Diss. Amsterdam 1904 ($31^{\circ}.12$).

⁴⁾ It is true we might assume that the equilibrium of liquid and vapour with so slight a difference of density as we observed, is only reached after so long a time as was allowed in this experiment to obtain equilibrium of temperature and that at first states with greater difference of density of liquid and vapour (cf. note 1 p. 1 § 1) appear at the same temperature, which gradually pass into the final

It is in harmony with this that a tolerably sharply defined critical density can be assigned. We derive 0.460 from the density of vapour and liquid in the third experiment for it, which agrees with the mean which would follow from experiment 1 at $2^{h}15$, i.e. 0.450, and 0.470, derived from experiment 3 on account of the appearance of the meniscus at 25 (cf. further § 6 p. 223 footnote 1).

Differences of density as TEICHNER finds, were also found by us; it was in the second experiment at the moment that the meniscus disappears with comparatively rapidly rising temperature. However, after the tube has been kept at the same temperature above the critical temperature for 3 hours, and the temperature in the tube has become uniform probably up to less than $0^{\circ},01$ $0^{\circ},9$ above the critical temperature, they have been reduced to less than 0,466 -- 0,450, or less than 3,5 %, i. e. after correction for gravity $< 3,3$ % over 10 cm.

The remaining difference in the first experiment $0^{\circ},23$ above the critical temperature after 6 hours' heating above the critical temperature can be derived from the fact that bulb 0.450 floats 5.6 cm. above 0,466. This difference is no more than 3,5 %, and corrected for gravity 2,9 %.

From VERSCHAFFELT's calculations follows that at $0^{\circ},23$ above the critical temperature 0,0001 molecule of admixture may cause about 12 % difference of density. Differences of temperature and admixtures which may account for remaining differences such as those just treated are scarcely to be avoided even with the precautions taken by us.

Nothing has been observed of an "Entmischung" by cooling when the critical temperature is approached, as TRAUBE i.e. p. 477 mentions.

TEICHNER observed that, after the disappearance of the meniscus, at the place where it was found last a transition zone exists towards which the differences of density concentrate, whereas outside it the changes are only insignificant. In our observations the contrary appeared, and the changes in density continue regularly with increase of temperature, only the motion of the bulbs was slightly accelerated as they approach the top of the tube, and they cover the last 1 or 2 cm. very rapidly. In a less degree but in the same way this takes place with the bulbs which descended. We consider these phenomena to be connected with heating and cooling of phases by compression and expansion¹⁾.

state with simultaneous change of the pressure of coexistence (cf. note 1 p. 1 § 1). But the absence of subsequent rise of pressure in the repetition of DE HEEN's experiment (see § 3) has taught us, that already after a very short time after-changes of the density no longer occur.

¹⁾ They just as the "Entmischung" assign that the temperature differences within the tube in the experiments of TEICHNER were probably greater than in ours. [Note added in the English translation].

Physics. — “Contributions to the knowledge of the Ψ -surface of VAN DER WAALS. XV. The case that one component is a gas without cohesion with molecules which have extension. Limited miscibility of two gases.” (Continuation). By Prof. H. KAMERLINGH ONNES and Dr. W. H. KEESOM. Supplement N^o. 15 to the Communications from the Physical Laboratory at Leiden.

(Communicated in the meeting of May 24, 1907).

§ 8. *On the temperatures and the pressures on the gas-gasplait.* In order to form a provisional opinion as to the experimental conditions which must be satisfied that limited miscibility in the gas state may be observed, and to be able to derive what pairs of substances must be considered suitable for this purpose, it is desirable to examine for some cases what temperatures and pressures occur on the gas-gasplait¹⁾. In the case (§ 2, March '07) of a component without cohesion²⁾ or almost without cohesion ($m < m_1$, see § 7), the gas-gasplait will occur for all temperatures between the critical temperature of the less volatile component and the critical temperature of complete miscibility³⁾. Its pressures will then be larger than

¹⁾ In our opinion Mr. VAN LAAR (These Proc. May '07, p. 35, note 2) is wrong in thinking that in the case of a three-phase-equilibrium, as e.g. in the system water-ether, our terminology with regard to the distinction between gas and liquid does not agree with that used by VAN DER WAALS. According to what has been said about this distinction in § 4, for a three phase equilibrium of a system of the type water-ether the denser phase which is rich in water, *must* be called liquid as belonging to the liquid branch of the connode, just as the less dense phase which is rich in ether, *must* be called gas phase; whether the denser phase which is rich in ether is to be called liquid or gas, is not determined by the principle of the continuity of the phase along the connode; if the reduced temperatures at which this phase appears, are taken into account, also the last mentioned phase will be called liquid for the system ether-water in accordance with what has been said in § 4.

²⁾ The second branch of the plaitpoint curve in Fig. 1 § 2 (These Proc. March '07, p. 787), about which VAN LAAR speaks in These Proc. May '07, p. 36, has there ($a_{22} = 0$) contracted to a point $x = 1$, $v = b_{22}$. It is true that in the case of our § 7 (Plate II, These Proc. March '07) a second branch of the plaitpoint curve occurs, but it has been explicitly stated there (p. 795 at the bottom and p. 797) that we did not discuss the spinodal curves at the lower temperatures at which this branch of the plaitpoint curve makes its influence felt, referring inter alia with a view to these temperatures to VAN LAAR's papers.

³⁾ As mentioned in § 1 this idea was introduced by VAN DER WAALS, who also gave the formulae for the calculation of this temperature (VAN LAAR calls it “third critical temperature”).

the critical pressure of the least volatile component. If the suppositions mentioned in § 2 might be applied for this, and the values of a_M and b_M for He might be borrowed from Comm. N^o. 96^c, Febr. '07, p. 660 footnote 2, so that $a_{MH_e} = 1/175 a_{MH_2} = 0.0000024$ and $b_{MH_e} = 1/2 b_{MH_2} = 0.00044$, this case would be realized for mixtures of He and water¹⁾. Then we should find $T'_{km} = 1.056 T'_{k_1}$, so that the gas-gasplait would occur over a range of temperature of 36° above 365° C., and at pressures above 195 à 200 atms.

In the case that the molecules of the least volatile component act on each other feebly, but still exert such an attraction that a double plaitpoint²⁾ occurs in the net of the spinodal curves, the pressure in this plaitpoint and its temperature in connection with the critical temperature of complete miscibility give important indications as to the pressures and temperatures of the gas-gasplait.

In table II these data, calculated for the case that the suppositions

T A B L E II.

Pair of substances	T_{km}/T_{k_1}	T_{dpl}/T_{k_1}	T_{dpl}/T_{kx}	P_{dpl}/P_{k_1}
Hydrogen-helium	0.933	0.915	4.44	41.6
Oxygen-helium	0.962	0.957	8.64	61.3
Argon-helium	0.970	0.962	7.90	64.5
Neon-helium	1.007	0.961	3.72	48.8
NO-helium	1.031	0.931	3.76	13.1
NH ₃ -helium	1.009	0.969	6.20	25.2
H ₂ S-helium	0.972	0.970	13.79	171
CO ₂ -helium	0.9540	0.9536	15.89	1045

¹⁾ The a_M and b_M for water have been borrowed from LANDOLT-BÖRNSTEIN-MEYERHOFFER's Physik. Chem. Tabellen.

²⁾ The appearance of a double plaitpoint near K_m was already observed by VAN LAAR (These Proc. May 1905 p. 42). The conditions for its appearance, however, were not correctly defined by him (cf. § 9 II). In view of this latter fact we thought that we drew sufficient attention to this result of VAN LAAR by referring the reader to VAN LAAR's papers. (see p. 797 footnote 1). The detaching of a longitudinal plait at high temperatures, which leaves the \downarrow -surface with its open side turned to $v = b$, follows immediately (see These Proc. April '07 p. 848 footnote) from the general considerations and calculations of VAN DER WAALS Cont. II § 19 sqq. and VAN DER WAALS' diagram in Zittingsverslag Kon. Akad. Nov. 1894 p. 133, when the case *a*) cf. § 9. I, a case which VAN LAAR has not included in his considerations, does not occur.

mentioned in § 2 might be applied, have been given for some helium mixtures. T_{km} has here been calculated according to VAN DER WAALS Cont. II, p. 43 (cf. § 6), T_{dpl} according to the formula mentioned in § 7, p_{dpl} from the equation of state with the just mentioned T_{dpl} and the v_{dpl} also represented in formula in § 7¹⁾.

The reduced temperature of the double plaitpoint T_{dpl}/T_{kz} , mentioned in this table, gives an idea in how far the phases in its neighbourhood behave as compressed gas-phases.

The values of a and b of the different components have been borrowed from KOHNSTAMM, LANDOLT-BÖRNSTEIN-MEYERHOFFER's Physik. Chem. Tabellen; for those of helium see above; for neon we have made use of the ratio of its refractive power²⁾ to that of helium according to the determinations of RAMSAY and TRAVERS³⁾, and of the estimation concerning the critical temperature by TRAVERS, SENTER and JAQUEROD⁴⁾.

It appears from table II that when the gas-gasplait can make its appearance, the range of temperature within which this is the case (between T_{km} and T_{dpl}), on the mentioned suppositions is small for most of these pairs of substances, for some even exceedingly narrow.

For the pressures on the gasplait higher values than p_{dpl} will have

¹⁾ Though originally we did not consider the developments which led us to the explicit expressions for the double plaitpoint mentioned These Proc. March '07 pp. 796 and 798 of sufficient importance, now that VAN LAAR (see These Proc. May '07 p. 41) thinks the derivation of such like expressions impossible there is a reason for communicating them on a following occasion.

²⁾ HAPPEL, Habilitationsschrift Tübingen 1906, p. 30, found that the refractive power for argon, krypton and xenon would yield values for b which greatly deviate from the b 's derived from the critical data. When according to the principle of the corresponding states (cf. HAPPEL loc. cit. p. 31, note 1) we compare the ratios of the refractive powers for these gases with those of their critical volumes (derived from p_k and T_k) the deviations are far less considerable. So with regard to this property, these one-atomic gases form a group, just as is the case with the bi-atomic and with a great many more-atomic substances (GUYE, Journ. de phys. (2) 9 (1890) p. 312).

³⁾ RAMSAY and TRAVERS, Phil. Trans. A197 (1900) p. 81. Yet we must remark that when comparing this ratio for helium and argon according to RAMSAY and TRAVERS with the ratio of b_{He} according to our estimation and b_A derived from p_k and T_k , we should find an important deviation (cf. note 1). Also in view of this the data concerning mixtures of helium and neon are very uncertain.

⁴⁾ TRAVERS, SENTER and JAQUEROD, Phil. Trans. A 200 (1902) p. 177. Their views, however, on a connection between atomic weight and critical temperature lead to an unlikely result for the critical temperature of helium.

The determinations of isotherms of neon by RAMSAY and TRAVERS, loc. cit. have been of as little use to us as those of helium for the determination of a and b (different particulars in the course of the isotherms of the one-atomic gases given by these scientists in plate 2 loc. cit. do not seem very probable to us).

to be expected as a rule. Thus it appears from table II that these pressures become very high, if the circumstances are not very favourable.

It would have a very favourable influence on the circumstances of temperature and pressure at which limited miscibility in the gas state might be observed, if it should prove that for mixtures of helium with another gas, a_{12M} is smaller than is expressed by $\sqrt{a_{11M} a_{22M}^2}$.

§ 9. *Mr. VAN LAAR's remarks*, (These Proc. May '07 p. 34—46) which imply that we have set forth some of our results as new, whereas they had been already derived and published by him before, compel us to the following explanations in order to show the incorrectness of these assertions.

I. As to part of these observations, they are best refuted by shortly repeating the train of thought followed by us.

When we applied the equations laid down by VAN DER WAALS with regard to the spinodal curve²⁾ in Cont. II, § 19 sqq., transferred to the ψ -surface for the unity of weight, to the case that one of the components is a gas without cohesion³⁾ with molecules which have extension, we arrived on the suppositions⁴⁾ mentioned in § 2 at a *plait which starts from the side of the small volumes, comes*

¹⁾ These Proc. March '07 p. 796 note 1, and VAN DER WAALS These Proc. April '07 p. 831.

²⁾ The equation for the spinodal curve of the molecular ψ -surface (cf. VAN LAAR These Proc. May '07 p. 37 at the top) was given by VAN DER WAALS in Cont. II, p. 45,

equation (1) in a form which after substitution of $\frac{da}{dx}$, $\frac{db}{dx}$ and $\frac{d^2a}{dx^2}$ passes imme-

diately into that used by VAN LAAR. (See VAN LAAR, These Proc. May '05 p. 33 at the bottom). The equation given by us p. 788 referred to and was derived from the equation for the ψ -surface for the unity of weight (These Proc. Dec. '06 p. 510). For the rest we differ from the opinion repeatedly expressed by VAN LAAR (inter alia These Proc. May '05 p. 34), that it would be more difficult and more elaborate to derive the equation of the spinodal curve and also that of the plaitpoint curve from the ψ -function than to do the same from the ζ -function.

³⁾ This investigation was announced in Comm. No. 96b, Dec. '06 p. 502.

⁴⁾ When we were not allowed to put $\frac{\partial^2 b}{\partial x^2} = 0$ for and in the immediate neighbourhood of $v = b$, as we did (cf. VAN DER WAALS Cont. II p. 42), the spinodal curve will always be closed towards the side $v = b$ as VAN DER WAALS observes l.c. and These Proc. April '07 p. 848. It is then to be expected, at least for small $\frac{\partial^2 b}{\partial x^2}$, that the plait in question makes its appearance for the first time at a maximum plaitpoint temperature, and for the rest extends to the large v 's in the same way as the plait described here.

into contact with the line $x = 0$ at a certain temperature, and crosses in a slanting direction from $v = b$ to the side $x = 0$ at lower temperature (§ 2 These Proc. March '07 p. 787). Comparison of this result with VAN LAAR'S papers induced us then to cite (p. 786 footnote 1) that the latter already treated the projection of the plaitpoint curve on the v, x -plane for the case of a gas without cohesion, but without further investigating the shape of the spinodal curve and of the plait for this case. Now that VAN LAAR (These Proc. May '07 p. 35) says: "The case that a plait starts from C_0 ¹⁾ to C_2 ²⁾, or also at the same time from C_2 to C_0 (when there is a minimum temperature in the plaitpoint line) is not new see KAMERLINGH ONNES and KEESOM, p. 788 below), but has been before described and calculated by me in all particulars", we have once more looked through his papers.

It would have been good if Mr. VAN LAAR had indicated the place where we had to look for this description of the plait treated in § 2 and indicated by VAN LAAR in the italicized words (the italics are ours); we have not been able to find this description in his preceding papers even on this renewed careful perusal³⁾.

That the shape of the plait described by us occurs for temperatures above the critical temperature of the least volatile component led us to the considerations on limited miscibility in the gas state mentioned in § 3 sqq.

Always availing ourselves of the above mentioned equations of VAN DER WAALS, we examined then if also with $a_{22} > 0$ such a plait may occur for values as they are to be expected for mixtures with helium. We saw in § 7 (These Proc. March '07 p. 795) that for the case of the plaitpoint curve running from K_1 to K_m (called type I by VAN LAAR) 3 cases are to be distinguished: *a*) that with falling temperature the plaitpoint gets from K_m on the ψ -surface, and proceeds regularly towards K_1 ; *b*) that with falling temperature a plaitpoint coming from K_m and one coming from K_1 unite to a double plaitpoint; *c*) that the plaitpoint gets from K_1 on the ψ -surface and proceeds regularly towards K_m (without double plaitpoint with minimum

1) Our K_m .

2) Our K_1 .

3) On the contrary he says in his paper These Proc. Sept. 1906 p. 231 (cf. VAN LAAR, These Proc. May 1905, p. 42 at the bottom): "In former papers it has been demonstrated that in the neighbourhood of C_0 a minimum plaitpoint temperature makes its appearance both with type I in the line C_0C_2 and with type II in the line C_0A , and that therefore with decrease of temperature a *separate plait* begins to detach itself starting from C_0 at a definite temperature T_0 (the plaitpoint temperature in C_0), which plait will merge into the main plait (or its branch plait) later on in an homogeneous double point.

plaitpoint temperature). The conditions for the occurrence of these cases were defined by us by means of the equations (2) and (3) there. From this appeared that with very feeble attraction the case *a*) occurs, with greater attraction the case *b*), whereas with still greater attraction case *c*) occurs (supposing the system to belong to type I).

We have found neither the case *a*) as we already observed above, nor the case *c*) in VAN LAAR. We did find the case *b*), chiefly with regard to the treatment of what takes place at lower temperatures, when three-phase-equilibria occur. For this treatment we referred to VAN LAAR (cf. These Proc. March '07 p. 797).

From the fact that VAN LAAR has declared this shape *b*) to hold universally for type I (cf. p. 235 footnote 3; see also VAN LAAR p. 36) it appears in our opinion, that VAN LAAR has not only left the cases *a*) and *c*) unmentioned, but has decidedly overlooked them.¹⁾

II. One more remark remains to be discussed. In § 7 we put as the *two* criteria of the case *b*), the course of the plaitpoint curve being from K_1 to K_m (see above), in which case a minimum plaitpoint temperature occurs (supposing $b_{22M} < b_{11M}$):

$$\sqrt[3]{a_{22M}/a_{11M}} > \frac{1}{3} \left\{ -1 + \sqrt{1 + 3 b_{22M}/b_{11M}} \right\}$$

and

$$\sqrt[3]{a_{22M}/a_{11M}} < -(1 - b_{22M}/b_{11M}) + \sqrt{1 - b_{22M}/b_{11M} + (b_{22M}/b_{11M})^2}.$$

Mr. VAN LAAR points out (These Proc. May '07, p. 45, appendix), that the first-mentioned condition corresponds with a condition for the occurrence of a minimum plaitpoint temperature, derived by him These Proc. Dec. '05, p. 581 (and VERSCHAFFELT These Proc. March. '06 p. 751). In our opinion, however, Mr. VAN LAAR is mistaken when he thinks that the *one* condition stated by him is sufficient in all cases to decide as to the occurrence of a minimum plaitpoint tem-

¹⁾ We might consider the course of the spinodal curves in case *b*), if this is also extended to values of $x > 1$ and < 0 , and of $v < b$, as a more general case, from which the cases *a*) and *c*) might be obtained, at least qualitatively and when we restrict ourselves to the region of the ψ -surface ($1 > x > 0$ and $v < b$) that is of importance for the treatment of mixtures. This might be done by cutting out a region bounded by $x = 0$ and $x = 1$, and a suitable line $v = b$ in the same way as VAN DER WAALS These Proc. Feb. '07, p. 621 sqq. treats the course of the isobars (cf. § 7 p. 796 of this Communication). We have not found a single indication that VAN LAAR's description of case *b*) is to be interpreted in this way; from the phrase, quoted p. 235 footnote 3 e.g. we should much sooner conclude to the contrary.

At any rate the distinctions which are of physical importance, have not been made.

perature. Nor can his considerations of Dec. 1905 give an indication to conclude to the occurrence of a minimum plaitpoint temperature in the branch $K_1 K_m$ of the plaitpoint curve. For there VAN LAAR starts from the value of dT_{xpl}/dx at the critical temperature of the most volatile component called T_{k_2} by us. The condition that at T_{k_2} , the *lower* of the critical temperatures of the components, $dT_{xpl}/dx < 0$, implies that *if the plaitpoint curve crosses from K_2 to K_1 , a minimum temperature must occur in it.*¹⁾ VAN LAAR (These Proc. May '07 p. 43), considers now the value of dT_{xpl}/dx for the least volatile component (for K_1). It is clear in our opinion, that at the *highest* critical temperature the condition $dT_{xpl}/dx < 0$, which coincides with the first of the inequalities mentioned (cf. p. 797 last sentence of the alinea at the top), does not give any indication to conclude to the occurrence of a minimum plaitpoint temperature. That if for K_1 $dT_{xpl}/dx < 0$, really *one* of the conditions for the occurrence of a minimum plaitpoint temperature in the branch $K_1 K_m$ of the plaitpoint curve has been fulfilled, VAN LAAR has, in our opinion, only shown by his considerations on the situation of the double plaitpoint, not given until April (translated in These Proc. of May) at the same time with his remarks, which considerations agree with those which had led us a month before to the statement of the two conditions mentioned.

¹⁾ Only if the plaitpoint curve crosses from K_1 to K_2 , this condition is sufficient to conclude to the occurrence of the minimum plaitpoint temperature (cf. VAN LAAR p. 46); for if the branch of the plaitpoint curve starting from K_2 bends round to VAN LAAR's point A ($x=1$, $v=b_{22}$ in our notation), we cannot speak of a minimum plaitpoint temperature in the sense which is generally attached to this word.

A similar consideration gave rise to our remark in note 2 p. 795, which remark we are obliged to maintain in spite of Mr. VAN LAAR's contradiction, p. 46. (Wrongly Mr. VAN LAAR thinks there, note 1, that in our note instead of "maximum-temp." "minimum-temp." should be read, which might also immediately appear by comparison with the cited text of VERSCHAFFELT).

Physics. — “Some remarks on the last observations of Prof. H. KAMERLINGH ONNES and Dr. W. H. KEESOM.” By Mr. J. J. VAN LAAR. (Communicated by Prof. H. A. LORENTZ).

In the Proceedings of May 24 (1907) Prof. KAMERLINGH ONNES and Dr. KEESOM answered some remarks which I published in the Proceedings of April 26 (1907) (in reference to their last papers).

I may be allowed to revert briefly to the chief points of this answer.

a. P. 235, line 14—18: “It would have been good”, etc.

In fact the course of the *plait* in the case referred to by Prof. KAMERLINGH ONNES and Dr. KEESOM has not been given by me. I confined myself to investigating the course of the *plaitpoint line* in general. The results which I obtained on this point harmonize fully with what was found by KAMERLINGH ONNES and KEESOM.¹⁾

b. P. 236, line 7—15.

In my papers published in These Proceedings only the case *b* has been mentioned (in which a minimum temperature occurs on the branch C_0C_2 of the *plaitpoint line*). The reason of this is, that I then (7 June 1905 and 10 January 1906) only worked out the case $b_1 = b_2$, in which such a minimum always occurs.²⁾ The formulae which I developed afterwards for the general case $b_1 \begin{matrix} < \\ > \end{matrix} b_2$ (see TEYLER I, II and III), [which last publication (viz.

III) was delayed by circumstances independent of my will], contained the *possibility* of the three cases *a*, *b* and *c*. As a matter of course Prof. KAMERLINGH ONNES and Dr. KEESOM could not take this yet unpublished investigation into account, and I only mention the fact to prove that the *possibility* of the cases *a* and *c* had not escaped my notice.

c. P. 236. “In our opinion, however, Mr. VAN LAAR is mistaken when he thinks that the one condition stated by him is sufficient in all cases to decide as to the occurrence of a minimum *plaitpoint* temperature.”

¹⁾ At the same time I am glad to declare that I completely acknowledge the priority of Prof. KAMERLINGH ONNES and Dr. KEESOM in bringing to light the possibility of *plaits* which proceed without minimum from C_1 to C_2 and inversely (cases *a*. and *c*. of KAMERLINGH ONNES and KEESOM), the knowledge of which is necessary for understanding the behaviour of binary mixtures, containing as one component a substance with weak attraction.

(Note added in the English translation).

²⁾ Afterwards I have seen that also for $b_1 = b_2$ the case *a*. does occur, when it is not supposed that $T_0 < T_2$. (Note added in the English translation).

I may remark about this that I have meant nothing else — which it seems, I ought to have set forth more clearly — but to conclude to the existence of the minimum temperature, when the plaitpoint temperature at the extremity of the branch of the plaitpoint line where the condition in question is fulfilled, is *lower* than at the other end of that branch, in which case this one condition is really sufficient.¹⁾

E R R A T A.

In the Proceedings of the meeting of February 1907:

p. 660 l. 5 from the bottom: for “least” read: “most”

p. 662 table II in the heading read $M_2/M_1 = 2$

In the Proceedings of the meeting of March 1907:

p. 765 l. 13 from the top: read:

$$\left[\frac{(pv_A)^{t''} - (pv_A)^{t'}}{t'' - t'} \right]_{v_A} = \left(\frac{d(pv_A)}{dt} \right)^{t''}$$

table VII, for: α_p read: $\frac{d(pv_A)}{dt}$

p. 778 l. 13 and 14 from the top: for α_p read: α_c

p. 785 l. 17 from the bottom: Insert:

For the temperatures in liquid hydrogen first the correction is to be applied which will prove to be necessary according to Comm. N^o. 95^a § 3*b* and § 8.

p. 796 the value for $\frac{x_M}{1-x_M}$ given in equation (1) is to be multiplied

by $\sqrt{a_{11M}/a_{22M}}$.

¹⁾ But I readily concede that Prof. KAMERLINGH ONNES and Dr. KEESOM have been the first to deduce both conditions for the minimum and to take in consideration the cases *a* and *c*. (Note added in the English translation).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING
of Saturday October 26, 1907.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige
Afdeling van Zaterdag 26 October 1907, Dl. XVI).

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Physiology. — “*About Odour-affinities*”. By Prof. H. ZWAARDEMAKER.

(Communicated in the meeting of September 28, 1907).

The great number of odours occurring in nature and in technics may, by virtue of the current opinions in literature, be divided into 9 classes, which may be indicated by historical names¹⁾, chiefly borrowed from Linnaeus. These classes are: ethereal odours, aromatic odours, odores fragrantes, moschus odours, allyl odours, empyreumatic odours, capryl odours, narcotic odours, odores nauseosi. I have selected from each class a representative which, chemically well characterized, can in a very simple way be made fit for olfactometric investigation (by dissolving the chemically pure odoriferous material in paraffinum liquidum²⁾), with one exception, viz. muscon, which is odourless in by itself odourless paraffine, and which therefore has been used mixed with myristine acid. They are:

isoamylacetate	0.5%
nitrobenzol	5%
terpineol	2.5%
muscon	0.627%
aethylbisulfide	1‰
guajacol	1‰
valerian acid	1‰
pyridine	1%
scatol	1‰

If in the double olfactometer two of these materials are joined together, there occur among the 36 combinations thus obtained no real mixed odours, but because they are counterbalancing each other, either an odourless mixture or rivalry is obtained. Odourlessness, respectively indefiniteness of odour, with change into odourlessness by rarefaction of the airmixture, is met with, when the stimuli are weak; rivalry, i. e. the alternate preponderance of one or the other of the two mixed odours is met with, when the stimuli are strong. However, also in the last case the impression, made by the mixture, is considerably weaker than the impression that every odour by itself brings about.

As unit of smelling-power I take the olfaction, i. e. the smallest quantity of odour of a definite quality, which can be recognized

¹⁾ H. ZWAARDEMAKER, Physiologie des Geruchs. Leipzig 1895 p. 207.

²⁾ Not water, but paraffinum liquidum has been chosen as a solvent because it is desirable to work for months at a stretch with the same cylinders without alteration of the solution.

by a normal organ of smelling ("Erkennungsschwelle", "recognizing-limit").

If the above mentioned paraffinic solutions are evaporated at a temperature of 15° C. and give off smelling material to a current of air brushing past with a velocity of 75 cub. cm. per sec., one olfaction will be communicated to the latter per 50 à 100 cub. cm. of air by shoving out the cylindric evaporating surface to a definite cylinderlength. This cylinderlength amounted on an average :

for the isoamylacetate-solution to	0.2	cm.
„ „ nitrobenzol-solution	„ 0.03	„
„ „ terpineol-solution	„ 1.1	„
„ „ muscon-mixture	„ 0.15	„
„ „ aethylbisulfide-solution	„ 0.015	„
„ „ guajacol-solution	„ 0.3	„
„ „ valerian acid-solution	„ 0.03	„
„ „ pyridine-solution	„ 0.02	„
„ „ scatol-solution	„ 0.003	„

In judging of this table, it should be taken into consideration that the numbers have sometimes been obtained by taking the cylinderlengths during the experiment actually 10 or 12 times longer and by rarefying the air in the reservoir that is smelled at, 10 or 100 times. Thus the objection was obviated, on the one hand that small cylinderlengths should not be capable of being read with precision, on the other hand that the evaporation should not regularly take place from very narrow strips.

The mixing of the odours, which are to be joined together two by two, took place in a common reservoir, which, provided with two stop-cocks had been placed downstream the double olfactometer. The symmetry of the latter, in view of the resistance offered to the current of air, was previously examined by means of a bridge of Wheatstone applied to air-streams, while the purification of odours adhering by absorption between each number of two experiments has taken place by a permanent stream of air and electric warming of the wall of the reservoir. The reservoir in which the mixture takes place was smelled at by means of a separate tube, the downstream cock being opened.

The 36 combinations procured indefiniteness of odour or rivalry, if p olfactions of one odour and 9 olfactions of the other showed a proportion of p/q , of the following amount :

T A B L E.

Mixture	$p \cdot q$ ¹⁾	Degree of completeness of the compensation ²⁾
isoamylacetate and nitrobenzol	0.44	indefiniteness*
" " terpineol	1.32	moderately acid additional odour
" " muscon	0.0625	indefiniteness
" " aethylbisulfide	0.244	complete compensation*
" " guajacol	3	indefiniteness
" " valerian acid	0.01	rivalry
" " pyridine	3.6	indefiniteness*
" " scatol	0.0037	indefiniteness
nitrobenzol	1.375	indefiniteness*
" " terpineol	0.434	"
" " muscon	0.111	"
" " aethylbisulfide	0.65	"
" " guajacol	0.03	"
" " valerian acid	3.	"
" " pyridine	0.012	indefiniteness*
" " scatol	0.125	"
terpineol	0.067	somewhat aromatic remainder
" " muscon	1.	rivalry
" " aethylbisulfide	0.05	almost complete compensation
" " guajacol	0.53	" " "
" " valerian acid	0.12	" " "
" " pyridine	1	indefiniteness
" " scatol	0.03	tolerably satisfactory
muscon	2.7	almost complete compensation
" " guajacol	1.2	rivalry
" " valerian acid	0.2	indefiniteness*
" " pyridine	0.056	rivalry*
" " scatol	1.2	indefiniteness
aethylbisulfide	3.2	very indefinite empyreum. odour
" " guajacol	0.0075	indefiniteness
" " valerian acid	0.03	"
" " pyridine	0.016	indefiniteness*
" " scatol	0.0007	"
valerian acid	4	"
" " guajacol	1.2	"
" " pyridine	0.42	indefiniteness
" " scatol		

The details of these experiments and more particularly the absolute quantities which were made to be the foundation of each combination of odours, will soon be communicated elsewhere in a more extensive treatise.

The proportional number given in the table holds good, besides for a definite average combination, also for an adjacent group of stronger, respectively weaker stimuli. For this zone holds good the rule that if p olfactions of an odour are compensated by q olfactions of another odour, this must likewise be the case for np and nq olfactions. FECHNER called the intensities of stimulus and sensation,

¹⁾ $p < q$.

²⁾ Marked with an asterisk are those cases in which by an intentional experiment it has been proved that the proportional number is applicable to a certain zone of stimuli.

mutually increasing and decreasing in due proportion, the cardinal values of stimulus and sensation. By analogy we might speak here of cardinal values of the stimuli counterbalancing each other, leaving altogether out of the question whether this proportion will prove as easily explainable as that which FECHNER has in view. The zone for which the proportional number of the table holds good, may therefore be called the zone of cardinal proportions.

From the fact that at a simultaneous impression two odours can neutralize each other, it follows that the action of these stimuli on the organ may be represented by two vectors, standing as it were for two forces, which in general act more or less in opposite directions, the direction of the vector of the strongest odour (answering to q in the table) being chosen in such a way that the co-sine of the angle that it forms with the continuation of the vector of the weakest odour (answering to p in the table) is exactly equal to the proportion found for p/q in the combination concerned. For in this case the vector of the strongest odour may be thought to be replaced by the sum of two other vectors: one in a direction opposite to the vector of the weakest odour, and one at right angles to it (in the plane of the original vectors). If, moreover, the two original vectors are given equal length, each with such a unit of length as the proportional number implies, i. e. for the vector q and its components of q/p times more weight than for the vector p , the neutralization of actions that has to be symbolized by the original vectors, will have been accurately expressed. For the vector p and one of the components of vector q will represent equal, but opposite forces. We shall only have to consider the direction of the other component of vector q as direction of odourlessness, in order to have duly accounted for the complete lack of sensation.

A second set of vectors can be placed in the same system, provided the two sets have one vector in common. Starting from of a new proportional number $p'q'$ the new third odourvector that has been introduced, may then be given a definite direction with regard to the first odourvector; also the second and third vectors may be given their relative directions by means of a third proportional number $p''q''$. The latter, it is true, can be done in two ways, according as the third vector is reached by a right- or a left-handed rotation starting from the vector answering to p , but of these two one may be chosen. To the combination $p'q'$ belongs a vector of odourlessness at right angles to the vector of weakest odour and to the combination $p''q''$ a vector of odourlessness at right angles to the vector of weakest odour. The units of length of these vectors will in general

never be the same and also differ according to the combination that one has in view. As, however, we have never mutually combined three, but always two odours at a time, it will never be necessary to occupy ourselves with the units of length of the three at the same time, nor does this change of units, depending on the case considered, raise any objection. Even in our further demonstration this does not give rise to any difficulty, as we are never going to mutually compare vectors but when they have the same direction with regard to an independent vector that is at the same time considered, in other words possess with respect to the latter about the same units of length¹⁾).

A third set of vectors can, speaking generally, not be placed in the same system, even though it has one vector in common with the two preceding systems, for the fourth vector will in general have to be given different directions, according as it is considered in connection with the first and second, with the first and third or with the second and third. But what is in general impossible, may in special cases prove quite practicable. Let us consider this.

If we number our nine standard-odours with the figures 1 to 9 and likewise the corresponding vectors, each time two of these vectors can be fixed and the rest arranged with regard to these two vectors, which are definite in their situation. The question we put just now, comes to this: Is the mutual relation between the odours perhaps so as to make some of these last seven vectors coincide? In consequence of mistakes in the experiment a complete coincidence will no doubt be out of the question, but let us consider whether it happens within a margin of error of at most 1% of 2π difference of direction ($= 3.6^\circ$). For this purpose we have first combined 1 and 2, considering all the others with regard to these two; then 1 and 3 are fixed, the rest arranged according to this, etc. till all combinations, 36, have occurred. In each of the combinations seven vectors were met with, whose situation with regard to the two vectors previously chosen had to be traced in order to see whether they coincided or not. For each set of two previously determined vectors this gives rise to 42 judgments, so that in all 1512 judgments have

1) The proportional numbers as they have been empirically composed and taken together in our table, form 252 possible constellations of three vectors. Among them there is only one which, also as to the units, is completely satisfactory for all three proportions at the same time. It is the constellation in which terpineol, scatol and valerian acid are combined. The length of the vectors measured by means of a joint unit of length amounts in this case to 1 for the terpineol-vector, 8 for the scatol-vector and 20 units for the valerian acid vector.

been necessary for 36 combinations. However, as each case is repeated once, there are in reality 756 separate judgments.

Just now we said it was necessary that never other vectors than those about equally directed, which accordingly possess about the same unit of length, should be mutually compared. If we keep this in view, the following constellations are identical within a limit of 1 % :

isoamylacetate and nitrobenzol	}	each four times
valerian acid and scatol		
isoamylacetate and guajacol	}	each three times
nitrobenzol and guajacol		
muscon and scatol		
aethylbisulfide and valerian acid		
isoamylacetate and terpineol	}	each once
isoamylacetate and aethylbisulfide		
nitrobenzol and terpineol		
terpineol and guajacol		
muscon and aethylbisulfide		
muscon and valerian acid		
aethylbisulfide and scatol		

together 27 constellations¹⁾.

Some of these 27 constellations give rise to reciprocity, in such a way, that the vectors which are identical to two previously determined vectors, make the latter identical, when they themselves are previously determined in their mutual situation. This occurs in :

isoamylacetate	}	←	{	muscon	}	→	{	isoamylacetate
and		with regard to	and	and		with regard to	and	
terpineol			←	{		scatol		←

Though the coincidence of two odour-vectors, if considered with respect to two other odour-vectors, is already very remarkable, a coincidence of 3 vectors is still more interesting. This has been realized in the following cases, if we extend the limit of error to 2° :

¹⁾ This figure rises to 42, if besides complete coincidences also coincidences with reciprocal values are taken into consideration.

isoamylacetaat	}	with regard to muscon and scatol
terpineol		
guajacol		
nitrobenzol	}	with regard to aethylbisulfide and scatol
terpineol		
guajacol		
muscon	}	with regard to isoamylacetate and terpeneol
valerian acid		
scatol		
aethylbisulfide	}	with regard to nitrobenzol and terpeneol
valerian acid		
scatol		

Even a coincidence of 4 vectors, has been found once, and that for :

isoamylacetate	}	with regard to valerian acid and scatol
nitrobenzol		
terpineol		
guajacol		

The planes in which the vectors of odourlessness, belonging to these odours, meet, have the form of a cone, respectively with the vector of valerian acid and that of scatol for their axis.

If we cast a glance at the sum of the results arrived at, it is especially the coincidence of several vectors at the same time which draws our attention: 4 vectors with regard to valerian acid and scatol, 4 sets of 3 vectors each time with regard to two others. Evidently there exists between the coinciding vectors agreement in action on our consciousness for those definite cases. But some of the coinciding vectors are repeatedly found together. Their mutual agreement must therefore be of a more intimate nature, otherwise it could not reveal itself so frequently and in so many different circumstances. This closer connection exists e.g. between isoamylacetate and nitrobenzol, which in no less than 4 cases become reciprocally identical to 1° of the circumference of the circle; a connection only a little less close between nitrobenzol and terpeneol, nitrobenzol and guajacol and isoamylacetate and guajacol, which do the same in 3 cases; a connection not quite wanting between isoamylacetate and terpeneol and terpeneol and guajacol, where also these appear to coincide to 1° .

But also the vectors with regard to which the coincidence of

several vectors comes about, are closely connected with each other. Valerian acid and scatol, with regard to which no less than 4 vectors become identical, have 4 mutual coincidences, to 1%, and also the vectors with regard to which the coincidence of three vectors takes place, show in 3 cases out of 4, a plurality of identity. Considered from certain definite points of view, therefore, they must have something in common in their action upon our consciousness.

If we ask ourselves what physical importance the relations found might have, it must be this that odour-mixtures formed from the coinciding vectors possess fixants in common and these fixants will have to be found in the odour-classes to which the vectors with regard to which they are placed, belong. In perfumery are known a number of such fixants rendering useful services with regard to certain definite perfume-mixtures, which otherwise would not be durable. In my "Physiologie des Geruchs" I have given a series of such examples. Here we quote one borrowed from G. COHN'S "die Riechstoffe". Artificial jasmin is obtained by joining together benzylacetate, linalylacetate, linalool and benzylalkohol, mixed with some indol, which serves "als Fixiermittel und zur Auffrischung des Geruchs", and which may be replaced by methylketol, scatol, propyldimethylindol, propylaethylindol, allylmethylindol, etc. With the aid of our table it must be possible to devise mixtures that will furnish available bouquets with fixants to be specified beforehand. Their practical fitness for the perfume-industry will depend, besides on the pure compensation-proportions, also on the velocity of evaporation and diffusion of the materials used. In a practically available perfume the latter should not differ too much.

The multidimensional character of the organ of smelling prevents, alas, projecting a clear representation of all proportions of the different qualities in their action upon consciousness. This is only partially possible, for separate vectors, isolated from the whole. Yet it appears that in general there exists a contrast between

isoamylacetate	}	with regard to	}	aethylbisulfide
nitrobenzol				valerian acid
terpineol				scatol
guajacol				

From a phylogenetic point of view the first group might be called the food-odours, the second the putrid odours, if not in many a case also muscon should be added to the last category, for which reason it is perhaps safer to refrain from any denomination. The arrangement in each group is governed by the above mentioned reciprocities.

Physiology. — “*A method to extract enzymes and pro-enzymes from the mucous membrane of the digestive tube and to establish the topic distribution of them.*” By Prof. H. J, HAMBURGER.

I. INTRODUCTION ; PRINCIPLE OF THE METHOD.

The method applied as yet to extract enzymes and pro-enzymes from the mucous membranes of the stomach and the intestines consists in preparing the mucous membrane and extracting it in a fine state of division, with or without the addition of antiseptics; by repeated precipitation and dissolution the body to be examined is finally obtained in a more or less pure state. If we wish to be informed as to the distribution of the enzyme over the various parts of the mucous membrane, in other words to establish the topic distribution of it, extracts are made of equal weights or of equal surfaces, and of these the specific action is determined quantitatively.

It need hardly be said that these methods are rather complicated and lengthy as well. A great drawback more especially is, that in extracting, the enzyme is polluted with so many other substances of the mucous membrane.

Now, we have occupied ourselves for some time with the question by what force enzymes (pro-enzymes) are brought to the surface of the mucous membrane, and more especially tried to determine whether we have to do with kataphoresis, in other words whether in normal life enzymes (pro-enzymes) are carried along by the electric current arising when the secretory nerve fibres are stimulated in the natural way.¹⁾ We will not dwell on the results of these investigations now. Let us only observe here that the method consisted in laying on the mucous membrane a small column of solidified agar-agar, into which a platinum electrode had been melted; then it was investigated whether under the influence of a weak electric current, moving from the muscular side of the mucous membrane, to the free surface of it, enzyme or pro-enzyme passed from the epithelium cells into the agar agar.

That, if the enzyme or pro-enzyme was indeed moved by kataphoresis it would also pass into the agar-agar, we had a reason to expect after the investigations of GRAHAM²⁾, VOIGTLÄNDER³⁾ and others

¹⁾ HAMBURGER, Osmotischer Druck u. Ionenlehre. Bd. II. S. 433 ff.

²⁾ GRAHAM, Liebig's Annalen 121, 1862 S. 1.

³⁾ VOIGTLÄNDER, Zeitschr. f. physik. Chemie. 3, 1889 S. 316.

For the literature on this subject compare, COHEN, Vorträge für Aerzte über Physikalische Chemie 2e Aufl. 1907 S. 128.

according to which the velocity of diffusion in colloids is as great as in the water in which the colloid is dissolved.

If it should be objected that these experiments were made only with crystalloids, the investigations of C. EYKMAN⁴⁾ have shown conclusively that colloids can diffuse into other colloids (gelatine into agar-agar).

But before trying to establish the influence of a constant electric current on the transition of enzyme into the agar-agar, we wished to know to what extent the ferment would diffuse into the agar-agar without the introduction of the electric current.

Evidently this transition took place. This fact seemed to suggest a means of extracting in an easy manner enzymes and perhaps pro-enzymes as well, from the mucous membrane in not too impure a state. Perfect extraction would, it is true, be unattainable in this way, but there was a likelihood that the method might be employed to determine in a simple manner the relative amounts of enzyme in the various parts of a mucous membrane.

II. EXPERIMENTAL METHOD.

Parts of a glass tube having in our experiments an internal diameter of 22 mm. and a height of 30 mm., were ground flat at one end by means of emery and placed with that side on a glass plate, plate glass being the best for this purpose.

By means of a pipette 3 cc. of liquid agar-agar were put into each little cylinder. I shall not discuss the way to prepare this liquid: it is to be found in all handbooks on the technics of bacteriology. It must be observed, however, that it is advisable to let the agar-agar solution cool down to $\pm 45^{\circ}$ before measuring it in the pipette; otherwise there is a danger of its flowing partly away from underneath the glass cylinder.

After some time the agar columns have become solid and are placed, still surrounded by the glass tube, on the spread out parts of the membrane which, if necessary, has been previously cleaned. For this cleaning which also may serve to remove the mucus, we take NaCl 0,9%. Investigations especially made for this purpose on the gastric mucous membrane have shown, however, that for this organ at least washing with water gives satisfactory results.

On the mucous membrane, which if necessary has been cleaned, the agar-agar columns remain for 8 hours or more, in order to enable enzymes and pro-enzymes to diffuse into the agar-agar.

⁴⁾ C. EYKMAN, Centralbl. f. Bakteriol. 29, 1901, S. 841.

If the experiment bears upon the *pepsin incl. the pepsinogen of the gastric mucous membrane*, the agar columns which have been on it, are cut fine and mixed with 3 cc. HCl of 0.4%. For this we use cylindrical bottles with close fitting glass stoppers: they have a diameter of 24 mm. and a height of 48 mm. Into these bottles we put albumen columns prepared according to METT's method. When these have been in contact with the agar-suspension for 10 hours or more at 37.5 C., we determine by measurement how much has been digested: then the albumen columns are placed in it again and the measurements are repeated a few hours later. In each bottle we generally had two albumen tubes. Perhaps it will be objected that the presence of solid particles of agar-agar must impede the action of the pepsin on the albumen. This proves not to be the case: in the first place we observe that on all 4 sides of the 2 albumen columns always about the same column of albumen has been digested, which most likely would not be the case if now and then an agar-particle prevented the entrance of the digesting fluid. And secondly we noticed that when the experiment is made with a liquid, from which the agar particles have been removed by filtration, the rate of digestion is the same as when the agar particles were still in the fluid.

If the experiment bears only upon the *pepsinogen* of the gastric mucous membrane, we place alkalic instead of neutral agar-agar on it, viz. a quantity of agar of 2% in Na_2CO_3 of 3 p. mille. The investigations of LANGLEY¹⁾ have shown that in this concentration pepsin is decomposed by Na_2CO_3 , pepsinogen on the other hand not.

It stands to reason that besides pepsin and pepsinogen, *chymosin* and *prochymosin* will also be absorbed by the neutral agar-agar. It was found indeed that the agar-mass had obtained the faculty of coagulating milk.

In a similar way as the gastric mucous membrane the intestinal mucous membrane may be experimented upon. We found that the neutral agar absorbs both enterokinase and crepsin. The quantity of *enterokinase* present in the agar is determined by cutting fine the agar, mixing it with water, *filtrating*, and bringing the extract thus obtained, into contact with inactive juice of a fresh pancreas gland and two albumen tubes.

The attentive reader will notice that here no agar particles are present at the digestion of the albumen as in the case of the gastric

¹⁾ LANGLEY Journal of Physiology 3 1882 p. 253.

LANGLEY and EDKINS Ibid 7 1886 p. 371.

juice. They were removed before the action of the fluid on the albumen tubes, because it was observed that the conversion of albumen by *trypsin* was greatly retarded by the presence of the agar-agar.

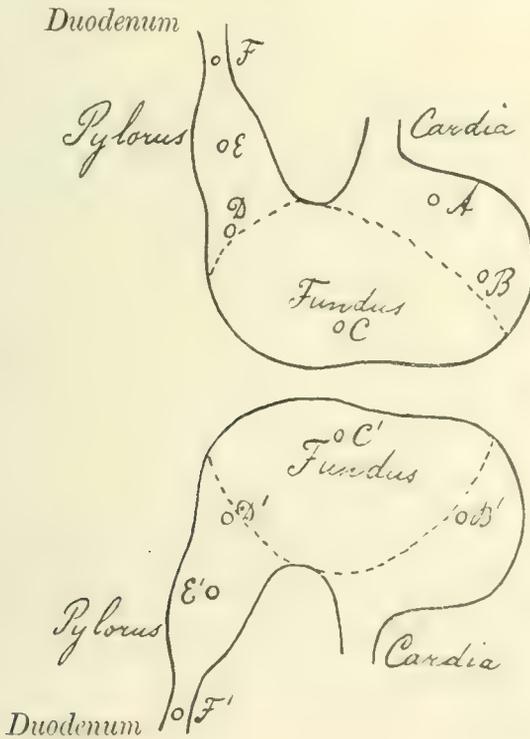
To determine the quantity of *crepsin*, drawn into the agar, we made the *clear* extract act upon the peptone.

We here append the results of some experiments carried out in accordance with the method described above. More explicit statements will be published elsewhere. Some further particulars concerning the method of investigation are mentioned below.

III. SOME EXPERIMENTS.

1. *Distribution of pepsin (incl. pepsinogen) over the gastric mucous membrane.*

A pig's stomach was cut into two symmetrical halves along the great and small curvature and washed with NaCl 0.9 ‰. Then both halves were spread out flat and on the spots marked below A, B, C, etc. columns of neutral agar-agar of 2 ‰ were placed.



As will be seen *A* is situated in the cardia region.

B in the border region between cardia and fundus part

C in the fundus region

D in the border region between fundus and pylorus part

E in the pylorus region

F on the duodenum near to pylorus.

With respect to this figure we must point out that for *all* our experiments with gastric mucous membrane, the letters have the same meaning.

In the experiment of which table I gives the results, the agar columns of 3 cc. remained during $14\frac{1}{2}$ hours on the mucous membrane. Then the agar was cut fine and mixed with 3 cc 0.4 % HCl and each of the mixtures thus obtained was made to exert its digestive influence on two tubes of albumen.

The four numbers which in the following table are connected by + represent the lengths of the four albumen columns, digested at the 4 sides of the two tubes.

TABLE I.

Digested after $12\frac{1}{2}$ hrs.

<i>A</i> $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2$ m.M.	<i>A'</i> $\frac{1}{2} + \frac{1}{4} + \frac{1}{2} + \frac{1}{2} = 1\frac{3}{4}$ m.M.
<i>B</i> $\frac{3}{4} + \frac{3}{4} + \frac{1}{2} + \frac{3}{4} = 2\frac{3}{4}$ "	<i>B'</i> $\frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} = 3$ "
<i>C</i> $2 + 2 + 2 + 2\frac{1}{4} = 8\frac{1}{4}$ "	<i>C'</i> $2 + 2 + 2 + 2 = 8$ "
<i>D</i> $1\frac{1}{2} + 1\frac{1}{4} + 1\frac{1}{2} + 1\frac{1}{2} = 5\frac{3}{4}$ "	<i>D'</i> $1\frac{1}{2} + 1\frac{1}{2} + 1\frac{1}{2} + 1\frac{1}{2} = 6$ "
<i>E</i> $1\frac{1}{4} + 1\frac{1}{4} + 1\frac{1}{2} + 1\frac{1}{4} = 5\frac{1}{4}$ "	<i>E'</i> $1\frac{1}{4} + 1\frac{1}{4} + 1\frac{1}{4} + 1\frac{1}{2} = 5\frac{1}{4}$ "
<i>F</i> $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1\frac{1}{3}$ "	<i>F'</i> $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1\frac{1}{3}$ "

From this table is seen:

1. That in the cardia region (*A* and *A'*) the amount of pepsin is small, increases towards the fundus (*B* and *B'*), reaches its climax there (*C* and *C'*) and decreases towards the pylorus (*D* and *D'*). In the duodenum we also meet with pepsin, but its quantity is small.

2. The table shows that in both halves of the stomach the quantity of pepsin was equal in corresponding parts.

If we speak here of pepsin we mean pepsin and pepsinogen. As has been said before and will further be demonstrated pepsinogen

too enters the agar. It is by mixing with HCl converted into pepsin and so determined quantitatively with the pepsin.

2. *Distribution of pepsinogen.*

As has been said, the investigations of LANGLEY¹⁾ have shown that, contrary to pepsin, pepsinogen is not destroyed by a solution of Na_2CO_3 0.3%. We have made use of this fact to try if we could withdraw pepsinogen from the mucous membrane.

For this purpose agar columns were placed upon the mucous membrane containing 2% of agar in a Na_2CO_3 solution of 0.3%. The column again had a diameter of 22 mM., the contents being 3 cc.

It must be casually observed that separate experiments had shown that in such an alkalie agar mass, pepsin at once loses irretrievably its digestive power.

To the method of experimenting we have not much to add. Let us only mention that the alkalie agar, after having been in contact with the mucous membrane was cut fine, neutralized with diluted hydrochloric acid, then mixed with 3 cc. HCl of 0.4%. The purpose of this was, to liberate the pepsin from the pepsinogen. The digesting-experiments with albumen-tubes gave the results tabulated below. Here the lengths of the $\frac{1}{2}$ digested albumen columns have each time been added together.

TABLE III.

The agar columns were kept on the mucous membrane for 20 hours.

Digestion of albumen after 8 hours.	Digestion of albumen after 18 hours.
<i>A</i> . . . 0 m.M.; <i>A'</i> . . . 0 m.M.	<i>A</i> . . . 0 m.M.; <i>A'</i> . . . 0 m.M.
<i>B</i> . . . 0 " ; <i>B'</i> . . . 0 "	<i>B</i> . . . 1½ " ; <i>B'</i> . . . 1½ "
<i>C</i> . . . 8 " ; <i>C'</i> . . . 8 "	<i>C</i> . . . 10 " ; <i>C'</i> . . . 9½ "
<i>D</i> . . . 4.4 " ; <i>D'</i> . . . 4.2 "	<i>D</i> . . . 6.2 " ; <i>D'</i> . . . 6.9 "
<i>E</i> . . . 4 " ; <i>E'</i> . . . 4 "	<i>E</i> . . . 5 " ; <i>E'</i> . . . 5.2 "
<i>F</i> . . . 0 " ; <i>F'</i> . . . 0 "	<i>F</i> . . . 1 " ; <i>F'</i> . . . 1 "

It will be seen that from the *cardea* part (*A* and *A'*) no pepsinogen was extracted. This need not surprise us; for in several experiments with neutral agar, no pepsin could be extracted from it either.

In the border region between *cardea* and *fundus*, pepsinogen was found, but in a small quantity. It was considerable in the *fundus* (*C*), gradually growing less towards the *pylorus* (*D* and *E*).

¹⁾ LANGLEY, l. c.

3. *To what extent does the length of time during which the agar columns are on the mucous membrane, influence the quantity of enzyme and pro-enzyme absorbed?*

For these experiments the stomach was divided into two symmetrical halves. On one half two *neutral* agar-columns were placed at *A*, *B*, *C* etc., and on the corresponding places of the other half *A'*, *B'*, *C'* etc. two columns of *alkalic* agar. One of the two columns near *A*, near *B*, near *C* etc. was taken away and treated, after having been on the mucous membrane for 18 hours; the same was done with those near *A'*, *B'*, *C'* etc. The other series *A*, *B*, *C* etc. *A'*, *B'*, *C'* etc. was left on the membrane for 36 hours. In both series of experiments the thus activated agar was made to act for 20 hours on the albumen columns. The tables will be plain now without further explanation.

TABLE IV.

Determination of the quantity of pepsin and pepsinogen which had passed into the neutral agar, after the latter had been on the mucous membrane for

	18 hours	36 hours
<i>A</i>	0 m.M. digested	2½ m.M. digested
<i>B</i>	1 " "	3½ " "
<i>C</i>	4.8 " "	16 " "
<i>D</i>	3.4 " "	7 " "
<i>E</i>	3 " "	5.4 " "
<i>F</i>	0 " "	2 " "

TABLE V.

Determination of the quantity of pepsinogen which had passed into the alkalic agar, the latter having been on the mucous membrane during

	18 hours	36 hours
<i>A'</i>	0 m.M. albumen digested	1½ m.M. albumen digested
<i>B'</i>	0 " " "	2¼ " " "
<i>C'</i>	3.8 " " "	10 " " "
<i>D'</i>	2.4 " " "	5½ " " "
<i>E'</i>	1.2 " " "	3 " " "
<i>F'</i>	0 " " "	1 " " "

Both tables show that after 36 hours more pepsin and also more pepsinogen had passed into the agar than after 18 hours.

Further a comparison of tables IV and V makes it evident that the digestion of albumen in experiments made with neutral agar, is more considerable than where alkalie agar has been employed. This result may also tend to confirm the reliability of the method; for into the neutral agar pepsin and pepsinogen may enter, the latter of which under the influence of hydrochloric acid produces pepsin, whilst in the alkalie agar only pepsinogen is found. And as we have seen invariably in all our experiments, the quantity of enzyme and pro-enzyme at identical spots of the two symmetrical parts of the stomach turns out to be the same.

I wish to observe here that the digestion of serum-albumen takes place much more quickly than that of the egg-albumen used. GLÄSSNER was the first to point out the advantage of coagulated serum, and I may confirm it from my own experience. Serum albumen has moreover the advantage that without preparation such as cutting up and filtrating, it can be used after simply being coagulated in glass tubes.

Owing to accidental circumstances no serum-albumen has been used for the experiments described in this paper.

4. *Distribution of rennet-ferment.*

To demonstrate the presence of rennet-ferment and to know its distribution in the gastric mucous membrane, about the same method was applied as that used for the investigations relating to pepsin and pepsinogen. Only the agar-columns had a greater diameter than in the pepsin experiments, viz. 35 instead of 22 mM. The contents accordingly were 5 cc. instead of 3 cc. Moreover it was self-evident that the quantitative determination of the rennet-ferment had to be effected in another way. The columns having been on the mucous membrane for some hours, the agar was cut fine and mixed in a test-tube with $\frac{1}{2}$ cc. HCl 0.4% and afterwards with 10 cc. of milk. Then the test tube was plunged into a bath of water at 37.5° , after which it was noted down every half minute where coagulation had taken place.

The presence of some hydro-chloric acid did not impair the experiment. Previous tests had shown that in a mixture of 5 cc. neutral agar $\frac{1}{2}$ cc. HCl of 0.4% and 10 cc. of milk coagulation did not set in till more than an hour after. As the following series of experiments demonstrates the addition of only $\frac{1}{2}$ cc. of HCl can hardly

have had any influence of the coagulative action of the rennet-ferment.

TABLE VI.

Coagulation is visible:					
by <i>A</i> after 3 minutes;			by <i>A'</i> after 3 minutes.		
" <i>B</i> "	2	"	" <i>B'</i> "	2	"
" <i>C</i> "	1	"	" <i>C'</i> "	1	"
" <i>D</i> "	1½	"	" <i>D'</i> "	1½	"
" <i>E</i> "	2½	"	" <i>E'</i> "	2½	"

This experiment shows that the distribution of chymosin (including pro-chymosin) runs parallel with that of pepsin (including pro-pepsin), a result agreeing with the researches of others.¹⁾

Secondly the experiment shows the action of the rennet in corresponding parts of the two symmetrical halves of the stomach to be equal.

5. *Distribution of enterokinase in the intestinal mucous membrane.*

On the mucous membrane of the duodenum, jejunum, ileum, caecum and colon, cut open length-wise, agar-columns were placed, their contents being 3 cc. These having been left on it for 24 hours, the agar-agar was cut fine and extracted with 3 cc. of a Na Fl-solution of 2%. Then 6 cc. of diluted pancreatic juice were added to 2 cc. of the filtrate. The former had been obtained by pressing the pancreas gland of a newly killed pig, mixing the thick juice thus obtained with a Na Fl-solution of 2%, and filtrating the mixture.

In the mixture of 6 cc. diluted pancreatic juice thus obtained and 2 cc. agar-filtrate two albumen tubes were placed.

By the side of this, controlling-experiments were made with 2 cc. agar-filtrate and 6 cc. of Na Fl solution, instead of 6 cc. of the diluted pancreatic juice. The digestion of the albumen was noted down after 19 and 44 hours.

¹⁾ NENCKI u. SIEBER. Zeitschr. f. physiol. Chemie **32** 1901 S. 291; PEKELHARING. Ibid. **35** 1902 S. 8; PAWLOW u. PARASTSCHUK. Ibid **42** 1904 S. 415; SAWJALOW. Ibid **46** 1905 S. 307.

The following table gives the results of one of the series of experiments made.

TABLE VII.

	2 cc duodenum-agar extract		2 cc jejunum-agar extract		2 cc ileum-agar extract	
	+ 6 cc pancreatic juice	+ 6 cc NaFl. solution	+ 6 cc pancreatic juice	+ 6 cc NaFl. solution	+ 6 cc pancreatic juice	+ 6 cc NaFl. solution
Albumen digested after 19 hours	7.2 m.M.	0 m.M.	6.4 m.M.	0 m.M.	5.6 m.M.	0 m.M.
Albumen digested after 44 hours	12.4 "	0 "	11.2 "	0 "	10 "	0 "

This series of experiments shows *the quantity of enterokinase to decrease gradually downward, a result agreeing with that obtained by CHEPOWALNIKOW, DELEZENNE, FROUIN and FALLOISE.*

I shall not discuss the experiments here, which show that in the digestion of albumen by trypsin the presence of agar has a retarding influence, nor the influence which the time during which the agar has been in contact with the intestinal mucous membrane, has on the transition of the enterokinase. These questions will be further discussed in a more explicit account.

One experiment remains to be mentioned, showing how the enterokinase diffused into the agar, distributes itself over agar and water, after the agar has been cut up and mixed with water.

5 cc. of liquid agar are mixed with 2 cc. of watery extract of the intestinal mucous membrane. Of this mixture 2×3 cc. are taken and poured into the above mentioned cylindrical tubes. When the agar has become solid it is cut fine and mixed each time with 2 cc. of water. The mixture remains for an hour exposed to the temperature of the body in order to enable the agar to give up enterokinase.

After cooling down it is filtrated, twice 1 cc. is taken and mixed with 2 cc. of inactive pancreatic juice. In both mixtures I and II albumen tubes are placed.

Besides this experiment another one identical with it, is made; only instead of 5 cc. of agar, 5 cc. of water are taken of course. Cutting up is out of the question here. The quantities, however, remain the same.

TABLE VIII.

Liquid	Digestion after	Experiment I	Experiment II
5cc agar + 2cc intest. extr.; of this 2×3cc cut up; each 3 cc mixed with 2 cc water; of this 2×1 cc mixed each with 2cc pancreatic juice	4 hours	$1 + 1 + 1 + 1 = 4$ m.m.	$1 + 1 + 1 + \frac{3}{4} = 3\frac{3}{4}$ m.m.
	16 hours	$3\frac{1}{2} + 4\frac{1}{2} + 3\frac{1}{2} + 4 = 15\frac{1}{2}$ m.m.	$3\frac{1}{2} + 3\frac{1}{2} + 3\frac{1}{2} + 4 = 14\frac{1}{2}$ m.m.
	25 hours	$6 + 5 + 6 + 5 = 22$ m.m.	$5 + 5 + 6 + 5 = 21$ m.m.
5 cc water + 2cc intest. extr. of this 2×3cc; each 3 cc mixed with 2 cc of water; of this 2×1 cc mixed each with 2 cc of pancreatic juice.	4 hours	$1\frac{1}{4} + 1\frac{1}{4} + 1 + 1 = 4\frac{1}{2}$ m.m.	$1\frac{1}{4} + 1 + 1\frac{1}{2} + 1\frac{1}{4} = 5$ m.m.
	16 hours	$4 + 4\frac{1}{2} + 4 + 4 = 16\frac{1}{2}$ m.m.	$4 + 3\frac{1}{2} + 4 + 4\frac{1}{2} = 17$ m.m.
	25 hours	$6 + 5 + 5 + 6 = 22$ m.m.	$5 + 5 + 5 + 6 = 21$ m.m.

This table leaves no doubt, but the method of extracting the agar with water, gives reliable results. *They turn out to be the same as if the agar itself were water. The enterokinase must distribute itself equally over agar and water.*

We observed the same with pepsin.

Finally we shall describe a series of experiments, showing that erepsin too enters the agar, and that this supplies us with a means of determining its distribution over the intestinal mucous membrane.

6. Distribution of Erepsin.

The 2% agar used was not dissolved in water, but in NaCl of 2% because the amount of erepsin, passing from the intestine into the agar during the time taken up by the former experiments, was not great enough.

Therefore it was expedient to leave the agar for at least 24 hours on the mucous membrane, taking care to prevent putrefaction as much as possible.

The action of erepsin consists as we know in its power to change hemialbumose and peptone into products not giving the biuret-reaction.

VERNON¹⁾ has based on this a colorimetric method, to determine the degree of conversion brought about by erepsin and FALLOISE²⁾ among others, has successfully used it. We too have applied this method, in a somewhat modified form, however. It chiefly consisted in a solution of CuSO_4 being mixed with a NaOH solution. The fluid thus obtained imparts a violet-red colour to peptone. The more the peptone solution from which we started is converted by erepsin, the fainter the violet-red colour will be. It was now investigated, with how much water the standard liquid had to be diluted to produce the violet-red colour observed.

One of our experiments gave the following result.

In the peptone-solution (WITTE) on which the duodenal-extract has acted, are **46.2%** of the original quantity of peptone left. In the peptone-solution acted upon during the same time by the *jejunum*-agar-extract are still **16%** of the original quality of peptone left, and lastly where the *ileum*-agar-extract acted during the same time, **14%** of the original amount.

It follows from this that in jejunum and ileum there was more erepsin present than in duodenum, which corroborates so far FALLOISE's results inasmuch as we too found much more erepsin in jejunum than in duodenum. In the ileum, however, the amount of erepsin is much greater than in the duodenum; FALLOISE, indeed, notes a difference in the same direction, but it is only slight. It must be kept in mind though, that our experiments relate to the pig, FALLOISE's to the dog.

I may add that in PEYER's plâques hardly any erepsin or enterokinase was found.

A number of experiments, made for the researches described above, have been carried out by Mr. R. A. B. OOSTERHUIS, Med. cand., assistant at the physiological laboratory.

CONCLUSION.

The above researches have shown:

1. That when agar columns are placed upon the mucous membrane of stomach and intestines, enzymes and pro-enzymes are

¹⁾ VERNON. Journal of Physiology. 30, 1903, p. 330.

²⁾ FALLOISE. Archives internat. de Physiol. 2, 1903/4, p. 299.

absorbed from them and enter the agar. As such were examined pepsin, including pepsinogen, chymosin and prochymosin, enterokinase and erepsin.

2. The above-mentioned ferments can be extracted, partly at least, by water from the agar-agar. Quantitative investigations have shown even that pepsin + pepsinogen and enterokinase as well, distribute themselves equally over the agar and water.

3. The facts mentioned sub 1 and 2 suggest a simple means of extracting the above-named ferments from the mucous membrane, and of determining quantitatively the distribution of them.

We have only to leave solid agar-agar columns of equal dimensions on various parts of the mucous membrane for some time and make subsequently a comparative quantitative determination of the specific action of the watery agar extract.

4. The results obtained with this new method with respect to the distribution of the above-named ferments in the digestive tube of the pig confirm those obtained by most investigators with the usual extraction methods on the dog.

5. The advantages of the method over the usual one consist, besides in its greater simplicity, also in the fact that the enzyme under investigation is much less polluted by decomposition products of the mucous membrane.

Especially for the investigation of the distribution of enzymes in individuals who, when alive suffered from diseases of the stomach or intestinal canal (ulcers in the stomach, the intestines, etc.) the method seems to me likely to be of use.

Moreover it is to be expected that besides the ferments examined till now, others will also pass into the agar-agar, which will enable us to make quantitative determinations of them in a similar way.

Finally the method seems to me to deserve recommendation as it can be applied in experiments at a lecture; at the same time, by adding congo-red or a similar indicator to the agar, the amount of acid or alkali can be demonstrated ocularly.

Groningen, September 1907.

Mathematics. — “*The extension of the Configuration of KUMMER to spaces of $(2^{\nu}-1)$ dimensions.*” By MR. J. A. BARRAU. (Communicated by Prof. D. J. KORTEWEG.)

(Communicated in the meeting of September 28, 1907).

§ 1. If we represent by S_1 the system $\begin{matrix} a & b \\ b & a \end{matrix}$, built up out of two letters and by S_2 the same system in new letters c and d ; if likewise we represent by T the system of signs $\begin{matrix} + & + \\ + & - \end{matrix}$ and by $-T$ the opposite $\begin{matrix} - & - \\ - & + \end{matrix}$, we obtain by connecting these

$$\begin{array}{cc} S_1 & S_2 \\ S_2 & S_1 \end{array} \quad \text{and} \quad \begin{array}{cc} T & T \\ T & -T \end{array}$$

the two systems

$$\begin{array}{cccc} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{array} \quad \begin{array}{cccc} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{array}$$

By giving *each* row of four letters in turn the signs of *each* row of the system of signs sixteen quadruplets of algebraic quantities appear which, as is known ¹⁾, represent the elements of the $C_f(16_6)$ of KUMMER whether they are considered as homogeneous coordinates of points or as coefficients of planes in Sp_3 . For, to each element are incident the elements of another kind, represented by the three permuted letter quadruplets and for each of them with half of the sign combinations.

§ 2. If now we call S_1 and T the letter- and the sign-system of 4 resp. and if we repeat the combination described above such-like systems of 8 are formed of which that one of the letters furnishes the permutations of a *regular* G_3 of order 8 ²⁾, consisting exclusively of binary substitutions, whilst that of the signs is *anallagmatic* ³⁾, i. e. every two rows show as many sign variations as

¹⁾ See a.o. JESSOP *Line-Complex* p. 23 or HUDSON *Kummer's Surface* p. 5.

²⁾ Compare MILLER *Quart. Journ.* 28 p. 255, group 8 No. 4.

³⁾ LUCAS *Récréations Mathématiques* II p. 113; *Nieuw Archief voor Wiskunde* 7 p. 256.

sign-permanencies. The systems become (that of the signs somewhat differently arranged):

I	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	1	+	+	+	+	+	+	+	+
II	<i>b</i>	<i>a</i>	<i>d</i>	<i>c</i>	<i>f</i>	<i>e</i>	<i>h</i>	<i>g</i>	2	+	+	+	+	-	-	-	-
III	<i>c</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>g</i>	<i>h</i>	<i>e</i>	<i>f</i>	3	+	+	-	-	+	+	-	-
IV	<i>d</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>h</i>	<i>g</i>	<i>f</i>	<i>e</i>	4	+	-	+	-	+	-	+	-
V	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	5	+	-	-	+	-	+	+	-
VI	<i>f</i>	<i>e</i>	<i>h</i>	<i>g</i>	<i>b</i>	<i>a</i>	<i>d</i>	<i>c</i>	6	+	-	-	+	+	-	-	+
VII	<i>g</i>	<i>h</i>	<i>e</i>	<i>f</i>	<i>c</i>	<i>d</i>	<i>a</i>	<i>b</i>	7	+	-	+	-	-	+	-	+
VIII	<i>h</i>	<i>g</i>	<i>f</i>	<i>e</i>	<i>d</i>	<i>c</i>	<i>b</i>	<i>a</i>	8	+	+	-	-	-	-	+	+

By providing each of the rows of letters with each of the sign-combinations there appear sixty-four octuples of algebraic numbers to which we assign the notations I 1, I 2, . . . VIII 8. Whether we consider these numbers as homogeneous coordinates of points or as coefficients of equations of Sp_6 in a Sp_7 , each element is incident to $7 \times 4 = 28$ of another sort, namely to half of the sign combinations of each letter permutation; so a Cf . ($6\mathbb{1}_{28}$) appears, to be designated by K^{VII} .

As with K^{III} it is possible to combine the Cf -elements to simplexes A, B, C, D, E, F, G, H in various ways. Such an arrangement is i. a.:

	1	2	3	4	5	6	7	8
<i>A</i>	11	114	1115	1143	117	1168	1116	1112
<i>B</i>	12	117	1116	1148	114	113	1115	1111
<i>C</i>	13	116	1117	1141	115	112	1114	1118
<i>D</i>	14	111	1118	1146	112	115	1113	1117
<i>E</i>	15	118	1111	1147	113	114	1112	1116
<i>F</i>	16	113	1112	1144	118	117	1111	1115
<i>G</i>	17	112	1113	1145	111	116	1118	1114
<i>H</i>	18	115	1114	1142	116	111	1117	1113

The table indicates that eight vertices of e.g. the simplex A are resp. the points 11, 114 etc, according to the former notation, whilst at the same time the eight opposite side- Sp_6 of the simplex are represented by those same notations.

The connection of Cf -elements can now be represented by a diagram (pl. I) the rows of which indicate the Sp_6 , the columns the points, whilst incidence of a Sp_6 with a point is indicated by hatching the square common to the respective row and column.

We see that the diagram can be brought to a more condensed shape :

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>
<i>A</i>	<i>S</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>
<i>B</i>	<i>a</i>	<i>S</i>	<i>g</i>	<i>f</i>	<i>e</i>	<i>d</i>	<i>c</i>	<i>b</i>
<i>C</i>	<i>b</i>	<i>g</i>	<i>S</i>	<i>e</i>	<i>f</i>	<i>c</i>	<i>d</i>	<i>a</i>
<i>D</i>	<i>c</i>	<i>f</i>	<i>e</i>	<i>S</i>	<i>g</i>	<i>b</i>	<i>a</i>	<i>d</i>
<i>E</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>S</i>	<i>a</i>	<i>b</i>	<i>c</i>
<i>F</i>	<i>e</i>	<i>d</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>S</i>	<i>g</i>	<i>f</i>
<i>G</i>	<i>f</i>	<i>c</i>	<i>d</i>	<i>a</i>	<i>b</i>	<i>g</i>	<i>S</i>	<i>e</i>
<i>H</i>	<i>g</i>	<i>b</i>	<i>a</i>	<i>d</i>	<i>c</i>	<i>f</i>	<i>e</i>	<i>S</i>

Here *S* indicates a simplex-filling; each of the other letters a system (8_3) denoting the incidence connection between the elements of two simplexes. These systems (8_3) have all degenerated into two (4_3), each pair of our simplexes is thus connected in an equal way and forms a $Cf(16_{1,0})$ of the same type.

§ 3. Analogous to the well-known decomposition of K^{III} into four tetrahedra lying in pairs in a MÖBIUS-position, it is obvious to call the position of two of the simplexes, e.g. *A* and *B*, by that name. Each side- Sp_6 of one *S* contains three points, so a face, of the other; each vertex of one lies in three side- Sp_6 , so in a side- Sp_4 of the other; the correspondence is such that opposite elements of *A*, e.g. vertex A_1 and side-space A_1 also furnish opposite elements of *B*, namely resp. the side- Sp_4 : $B_1B_3B_6B_7B_8$ and the face $B_2B_3B_4$, just as this is the case with the tetrahedra in MÖBIUS-position.

There exists already however, provided with the same property, an extension of this notion, that of BERZOLARI¹⁾ where each side- Sp_6 of one *S* contains one vertex of the other, and is generated by operation with a focal system on an arbitrary simplex; let us call this position *MI*, then it is evident that the discussed more specialized *MII* is to be regarded as a threefold *MI*.

¹⁾ *Rendiconti del Circolo Matem. di Palermo* 22.

The elements of two simplexes A and B in $MIII$ can be arranged only in one other way to two suchlike simplexes, namely as

$$\begin{array}{l} \text{first simplex } P: A_1, A_2, A_3, A_4, B_5, B_6, B_7, B_8, \\ \text{second } \quad \quad Q: B_1, B_2, B_3, B_4, A_5, A_6, A_7, A_8. \end{array}$$

If we regard such a new simplex in connection with $C, D, \dots H$, it then shows with each of these a new sort of position; for all however of the same type, showing analogy to the pairs of tetrahedra in STEINER-position which can be separated in the same way from K^{III} . We find for the $cf'(16_{10})$ of two such simplexes a diagram of the shape:

$$\begin{array}{c} S x \\ x S, \end{array}$$

where x again represents a system (8₃) which however does not degenerate now, but is identical to the cyclic system which is obtained out of the initial row: 1 2 . . 5 . . .

Opposite elements of one simplex furnish, as in Sp_3 , no opposite ones of the other.

§ 4. The 28 operations determining in each cf -space the cf -points incident to them and reciprocally, are *focal-correlations*; thus e.g. the $Sp_6: A_1$

$$(+ a, + b, + c, + d, + e, + f, + g, + h)$$

is transformed into the point A_2 situated in it

$$(+ b, - a, + d, - c, + f, - e, + h, - g)$$

by operating with the skew-symmetrical determinant of transformation:

$$\begin{array}{cccccccc} 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}$$

These focalsystems are mutually in involution as the group of the letter substitutions as well as that of the sign variations are ABEL groups.

The 36 remaining reciprocities are polarities with respect to some 36 quadratic Sp_6 , which serve for K^{VII} as the 10 fundamental-surfaces of order two for K^{III} .

1) MARTINETTI, *Rendic. Palermo* 16 p. 196.

Their equations are of two types; namely *eight* of the form

$$\pm x_1^2 \pm x_2^2 \pm x_3^2 \pm x_4^2 \pm x_5^2 \pm x_6^2 \pm x_7^2 \pm x_8^2 = 0,$$

where the combinations of signs must be derived from the sign system; and twenty-eight of the form:

$$\pm x_1 x_2 \pm x_3 x_4 \pm x_5 x_6 \pm x_7 x_8 = 0,$$

where the connection of the indices is given by the seven binary substitutions of the regular G_8 , whilst the signs must be selected:

+	+	+	+
+	+	-	-
+	-	+	-
+	-	-	+

The sixty-three operations which transform an element into another of the same sort are *collineations*; so we obtain, analogous to the KLEIN G_{32} in Sp_3 , a geometrical ABEL group G_{128} , consisting of the identity and sixty-three collineations; twenty-eight focal systems in involution and thirty-six polarities.

§ 5. The twenty-eight points in each Sp_6 of $KVII$ lie on a quadratic Q_6 and reciprocally.

To prove this we regard the determinant of the terms of order two, formed of seven of the eight homogeneous coordinates; so this is of order $7 + \binom{2}{7} = 28$. The omission of a coordinate is geometrically the projecting out of a vertex of the fundamental simplex on the opposite Sp_6 ; if the projections of 28 points lie in it quadratically, then the points themselves do so in their Sp_6 .

Let us first restrict ourselves to $Sp_6 : A_1$.

The twenty-eight points are to be divided into seven quadruplets of the same order of letters; the purely quadratic terms within such a quadruplet are in each column alike, the mixed ones may differ in sign. Let us call the four terms in a column p, q, r, s , then the substitution

$$\begin{aligned} P &= p + q + r + s \\ Q &= p + q - r - s \\ R &= p - q + r - s \\ S &= p - q - r + s \end{aligned} \quad , \text{ the } \Delta \equiv \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix} \text{ of which is } \neq 0,$$

causes three of the four quadratic terms to disappear, the $\Delta_{3,3}$ breaks up into the product of a Δ_7 of quadratic and a $\Delta_{2,1}$ of mixed terms. Here

$$\Delta_7 = \begin{vmatrix} b^2 & a^2 & d^2 & c^2 & f^2 & e^2 & h^2 \\ c^2 & a^2 & a^2 & b^2 & g^2 & h^2 & e^2 \\ d^2 & c^2 & b^2 & a^2 & h^2 & g^2 & f^2 \\ e^2 & f^2 & g^2 & h^2 & a^2 & b^2 & c^2 \\ f^2 & e^2 & h^2 & g^2 & b^2 & a^2 & d^2 \\ g^2 & h^2 & e^2 & f^2 & c^2 & d^2 & a^2 \\ h^2 & g^2 & f^2 & e^2 & d^2 & c^2 & b^2 \end{vmatrix}$$

That in general $\Delta_7 \neq 0$ is evident i. a. from

$$h = 1, a = b = c = d = e = f = g = 0.$$

The Δ_{21} gets after change of signs of some rows the form:

0	0	0	0	0	bh	0	0	0	0	-ah	0	df	-de	0	-cf	ce	0	0	0	0
0	0	0	bf	-be	0	0	0	-af	ae	0	0	0	0	dh	0	0	-ch	0	0	0
0	bd	-bc	0	0	0	-ad	ac	0	0	0	0	0	0	0	0	0	0	0	fh	-eh
0	0	0	0	ch	0	0	0	dg	0	-de	0	0	-ah	0	-bg	0	be	0	0	0
0	0	0	cg	0	-ce	0	0	0	dh	0	0	-ag	0	ae	0	-bh	0	0	0	0
-cd	0	bc	0	0	0	ad	0	0	0	0	-ab	0	0	0	0	0	0	-gh	0	eh
0	0	0	dh	0	0	0	0	0	cg	-cf	0	0	-bg	bf	-ah	0	0	0	0	0
0	0	0	0	dg	-df	0	0	ch	0	0	0	-bh	0	0	0	-ag	af	0	0	0
cd	-bd	0	0	0	0	0	-ac	0	0	0	ab	0	0	0	0	0	0	gh	-fh	0
0	0	-eh	0	0	0	-fg	0	0	0	cf	0	0	bg	0	ah	0	0	0	0	-bc
0	-eg	0	0	0	ce	0	-fh	0	0	0	0	ag	0	0	0	bh	0	0	-ac	0
-ef	0	0	0	be	0	0	0	af	0	0	-gh	0	0	0	0	0	ch	-ab	0	0
0	-fh	0	0	0	0	0	-eg	0	0	de	0	0	ah	0	bg	0	0	0	0	-bd
0	0	-fg	0	0	df	-eh	0	0	0	0	0	bh	0	0	0	ag	0	0	0	-ad
ef	0	0	-bf	0	0	0	0	0	-ae	0	gh	0	0	-dh	0	0	0	ab	0	0
-gh	0	0	0	0	0	0	0	0	0	ah	-ef	0	de	0	cf	0	0	-cd	0	0
0	0	fg	0	-dg	0	eh	0	-ch	0	0	0	0	0	0	0	0	-af	0	0	ad
0	eg	0	-cg	0	0	0	fh	0	-dh	0	0	0	0	-ae	0	0	0	0	ac	0
gh	0	0	0	0	-bh	0	0	0	0	0	ef	-df	0	0	0	-ce	0	cd	0	0
0	fh	0	0	-ch	0	0	eg	-dg	0	0	0	0	0	0	0	0	0	-be	0	bd
0	0	eh	-dh	0	0	fg	0	0	-cg	0	0	0	0	-bf	0	0	0	0	0	bc

The sum of the numbers in each column amounts to zero; so

$$\Delta_{31} = 0.$$

As each element with the 28 incident to it can be transformed into any other by means of a direct or reciprocal projectivity, the quadratic position of every 28 is now proved.

§ 6. Each couple of Sp_6 of the cf has twelve points in common lying thus in a Sp_5 . No other Sp_6 containing these twelve, all these Sp_5 differ and their number is $\binom{2}{64} = 2016$. The cf -points form with them a cf ($64_{378}, 2016_{12}$).

There are triplets of Sp_6 which have six points in common, lying thus in a Sp_4 , each cf - Sp_6 has namely in still 32 Sp_6 six of its points. Such a sextuple can be deduced from three groups of twelve, their number is thus $\frac{2016 \times 32}{3} = 21504$; they form with the cf - Sp_5 a cf ($21504_3, 2016_{32}$).

There are quadruplets of Sp_6 having four points in common which therefore determine a Sp_3 ; each cf - Sp_6 has namely four of its six points in fifteen other cf - Sp_6 . Every Sp_3 can be derived from four Sp_6 , their number is thus $\frac{21504 \times 15}{4} = 80640$. They form with the cf - Sp_4 a cf ($80640_4, 21504_{15}$).

There are sextuplets of Sp_6 having three points of the cf in common, which therefore determine a Sp_2 ; each cf - Sp_6 has namely three of its four points in eight other cf - Sp_6 more, these eight Sp_6 furnish two by two however the same triplet; as furthermore each Sp_2 can be deduced from $\binom{2}{6} = 15$ Sp_6 , their number is $\frac{80640 \times 4}{15} = 21504$.

This could be expected as the whole consideration starting from the cf -points might have been put reciprocally, and would then have led on account of the self-reciprocity of the system to the same elements; so still 2016 Sp_1 are obtained, the right lines of connection of the pairs of points.

The further amounts of incidences of the kinds of elements mutually can now be easily deduced; the notation of K^{VII} becomes finally:

	Sp_0	Sp_1	Sp_2	Sp_3	Sp_4	Sp_5	Sp_6
	64	2016	21504	80640	21504	2016	64
incident to:							
Sp_0	—	2	3	4	6	12	28
Sp_1	63	—	3	6	15	66	378
Sp_2	1008	32	—	4	21	160	2016
Sp_3	5040	240	15	—	15	240	5040
Sp_4	2016	160	21	4	—	32	1008
Sp_5	378	66	15	6	3	—	63
Sp_6	28	12	6	4	3	2	—

By the method of intersecting and projecting triplets and doublets of consecutive kinds of elements are to be transformed into elements of Sp_3 or Sp_2 ; thus are formed e.g. a cf ($21504_{2,1}$) of points and planes, with 80640 cf -lines; and a plane cf ($2016_{3,2}$, 21504_3) of points and lines, or reciprocally.

§ 7. If we represent the system of letters and that of signs of 8 resp. by S_1 and T and if we repeat the combination

$$\begin{array}{ccc} S_1 & S_2 & T \quad T \\ & \text{and} & \\ S_2 & S_1 & T \quad -T, \end{array}$$

we obtain systems for 16 belonging to each other, etc., the operation allowing of indefinite continuation; one always arrives at a regular ABEL substitution group G_{2^p} and a suitable anallagmatical system for the signs.

These always furnish in R_{2^p-1} a cf , analogous to that of KUMMER with the notation:

$$Cf \left(2_{(2^p-1), 2^p-1}^{2^p} \right),$$

arising from an arbitrary starting element by an operation with a geometrical ABEL group:

the *identity* and $2^p - 1$ *collineations* on one hand
and $(2^p - 1) \cdot 2^{p-1}$ *focal systems* mutually in involution with
 $(2^{p+1} - 2^p + 1) \cdot 2^{p-1}$ *polarities* on the other hand.

The quadratic situation of the elements incident to one element can always be proved by reduction of the determinant according to the example of § 5¹⁾.

1) A more extensive treatment also for spaces of other numbers of dimensions will follow in the dissertation to be published: J. A. BARRAU, *Bijdragen tot de theorie der cff.* (Amsterdam 1907).

Mathematics. — “The theorem of GRASSMANN in a space of n dimensions.” By LUCIEN GODEAUX, at Morlanwelz (Hainault). (Communicated by Prof. P. H. SCHOUTE).

We shall designate by the letter S a linear space and the number of dimensions of this space shall be the index.

The notation V_j^i represents a variety, the locus of ∞^i elements and of order j .

The order of a variety, locus of spaces S_k occurring in an $(n-k)(k+1)-1$ times infinite number in a space S_n , is the number of S_k of an S_{k+1} through an S_{k-1} of this S_{k+1} and belonging to the variety.

1. In an S_2 the theorem of GRASSMANN can be read thus:

The locus of S_0 for which the S_1 which unite it to three fixed S_0 meet three fixed S_1 in three S_0 of the same S_1 is a variety V_1^3 .

In an S_3 it has been given in the two following forms:

The locus of an S_0 for which the S_1 which unite it to four fixed S_1 meet four fixed S_1 in four S_0 of a same S_1 is a V_2^4 . (LE PAIGE, *Sur la génération de certaines surfaces par des faisceaux quadrilatéraux*, *Bul. de Belgique*, 1884, 3^e série, tome VIII).

The locus of an S_0 for which the S_1 which unite it to four fixed S_0 meet four fixed S_2 in four S_0 of a same S_2 is a V_2^4 .

2. Let there be in an S_n k S_{r_i} which we shall designate by A_i and k S_{s_i} which we shall designate by B_i , ($i = 1, \dots, k$).

Let p be a number satisfying the $2k$ inequalities

$$r_i + p + 1 \leq n - 1 \dots \dots \dots (1)$$

$$r_i + s_i + p + 1 > n, \quad (i = 1, \dots, k) \dots (2)$$

A space S_p determines with the k spaces A_i k spaces S_{r_i+p+1} . These spaces meet the corresponding spaces B_i in k spaces $S_{r_i+s_i+p-n+1}$.

If these k spaces belong to an $S_{i=n}$, $\sum_{i=1}^k (r_i + s_i) + k(p - n + 2) - 1$,

the space S_p describes a variety $V_{n-p}(p+1)-1$ the order of which is to be found.

Let us suppose we have

$$\sum (r_i + s_i) + k(p - n + 2) = n + 1. \dots \dots (3)$$

Let C be an S_{p+1} and D an S_{p-1} of C .

Let us designate by Δ an S_p passing through D and situated in C .

Let us take $k-1$ spaces Δ and let us number them $1, \dots, j-1, j+1, \dots, k$.

These $k-1$ spaces Δ determine with $k-1$ spaces A_i suitably chosen $k-1$ spaces S_{r_i+p+1} . These spaces meet the corresponding spaces B_i in $k-1$ spaces $S_{r_i+s_i+p-n+1}$, ($i=1, \dots, j-1, j+1, \dots, k$).

These spaces determine an $S_{\sum_{i=1}^{i=j-1} (r_i+s_i) + \sum_{i=j+1}^{i=n} (r_i+s_i) + (k-1)(p-n+2) - 1}$

This space has in common with B_j a space

$$S_{\sum_{i=1}^{i=j-1} r_i + \sum_{i=j+1}^{i=n} r_i + \sum_{i=1}^{i=n} s_i + (k-1)(p+2) - kn - 1}$$

In its turn this space determines with A_i a space

$$S_{\sum_{i=1}^{i=j+1} (r_i + s_i) + (k-1)(p+2) - kn}$$

On account of the equality (3) the latter meets C in a single point, which determines with D a space Δ_j .

When j varies from 1 to k , one obtains k series of spaces Δ between which exists a $(1, 1, \dots, 1)$ correspondence. There are k coincidences.

The variety described by the space S_p is $V_{(n-p)(p+1)-1}^k$.

The locus of a space S_p for which the S_{r_i+p+1} which unite it to k fixed spaces S_{r_i} meet $k S_{s_i}$ in $k S_{r_i+s_i+p-n+1}$ of a same $S_{\sum (r_i+s_i) + k(p-n+2) - 1}$, ($i=1, \dots, k$), is a variety $V_{(n-p)(p+1)-1}^k$.

The spaces A_i are evidently principal spaces of the locus of S_p , principal space having the same meaning as principal point or plane of a complex of rays.

In S_0 we find the following theorem:

The locus of an S_1 for which the S_2 which join it to four S_0 meet four S_1 in four S_0 of a same space S_2 is a variety V_3^4 (complex of order four).

3. If we regard the ordinary space as if generated by right lines we have a geometry of four dimensions. We shall now show two generalizations of the theory of GRASSMANN in this geometry.

Let us imagine k linear congruences G_1, \dots, G_k , and k plane pencils $(P_1, \alpha_1), \dots, (P_k, \alpha_k)$. Let us imagine moreover to be given a linear system C of linear complexes to the amount in number of ∞^{6-k} .

An arbitrary right line g determines k linear complexes with the

k congruences G . These have in common with the k corresponding plane pencils k lines p_1, \dots, p_k .

Let us now find the locus of the line g when the k lines p belong to a same complex of the system C .

Let (A, α) be any plane pencil. Let us take $k-1$ lines of this pencil and let us number them $1, \dots, i-1, i+1, \dots, k$.

Each of these lines determines with the corresponding congruence G a linear complex, which has in common with the corresponding plane pencil (P, π) a line p . The $k-1$ lines p found in this way determine a complex of the system C . This complex has a line p_i in common with the plane pencil (P_i, π_i) . This line determines with G_i a complex having a line a_i in common with (A, α) . When i varies from 1 to k we have k series of lines a between which exists a $(1, 1, \dots, 1)$ correspondence. There are k coincidences.

The locus of a right line for which the linear complexes that it determines with k fixed linear congruences meet k fixed plane pencils in k lines of a linear complex of a system of $6-k$ terms is a complex of degree k (order and class) to which belong the given k linear congruences.

If $k=6$, we have a theorem of GRASSMANN.

4. Let us suppose five groups of three lines H_1, \dots, H_5 and five nets of lines R_1, \dots, R_5 .

An arbitrary line g determines with H_1, \dots, H_5 five linear congruences which meet the five corresponding nets in five lines. If these five lines belong to a selfsame linear congruence the line g describes a congruence.

Let π be a plane. Let us consider in this plane five series of lines p_1, \dots, p_5 .

Between the lines of these series it is easy to see that there is such a correspondence that to four right lines corresponds a fifth.

Let us suppose that three right lines are fixed, whilst the fourth describes a pencil. It is then easy to verify that the fifth also describes a pencil. According to an extension of the principle of ZEUTHEN there are fifteen coincidences.

The locus of a right line taken in such a way that the linear congruences which it determines with five systems of three lines have in common with five nets five lines of a same linear congruence is a congruence of the fifteenth class.

In the same way we can verify that this congruence is also of order fifteen and that it contains the generatrices of the same kind as the given lines of the five quadratic surfaces determined by these lines.

Physics. — “Contributions to the knowledge of the ψ -surface of VAN DER WAALS. XVI. On the gas phase sinking in the liquid phase for binary mixtures in the case that the molecules of one component exert only a feeble attraction.” By Prof. H. KAMERLINGH ONNES and Dr. W. H. KEESOM. Supplement N^o. 16 to the Communications from the Physical Laboratory at Leiden.

§ 1. *Introduction.* In Comm. N^o. 96^b, These Proc. Dec. 1906, p. 501 a gas phase sinking in a liquid phase¹⁾, the barotropic phenomenon, was treated for binary mixtures for a gas liquid plait, which crosses the ψ -surface as a transverse plait at lower temperature. Then the treatment for temperatures, at which the appearance of a longitudinal plait brings about a disturbance, was deferred to a later communication. Moreover, more special cases, as the appearance of minimum or maximum critical temperature or minimum or maximum pressure of coexistence, were left out of consideration, and the discussion was restricted to the case that retrograde condensation of the first kind occurs.

When for binary mixtures the conditions for the sinking of a gas phase in a liquid phase were treated in Comm. N^o. 96^c, These Proc. Dec. '06, p. 508 and Febr. '07 p. 660, it appeared in the first place that at least if the hypotheses mentioned there are valid, and pairs of substances are found with proper a_{22M}/a_{11M} , b_{22M}/b_{11M} and M_2/M_1 the theory of VAN DER WAALS' ψ -surface leads us to expect that barotropic plaitpoints²⁾ will occur.³⁾ Further that for mixtures with

1) Considerations which are not in accord either with the limited compressibility of a gas at high pressures, first stated by NATTERER in 1844, or with our present views on the mixing of two substances, induced JAMIN, C.R. 96 (1883) p. 1448, Journ. de phys. (2) 2 (1883) p. 389 to raise the question whether it should be possible that with compression of a mixture of CO₂ with air or with hydrogen, a liquid phase would collect above the gas phase. CAILLETET (JAMIN l.c.) did not succeed in realizing this.

2) On the peculiar phenomena which are met with in case of a barotropic plaitpoint, we hope shortly to make a communication.

3) That the barotropic plaitpoint found in Comm. N^o. 96^c, Dec. '06 § 5 belongs to the gas liquid plait (cf. Comm. N^o. 96^c, Febr. '07, p. 660 footnote 1) was derived from the shape of the spinodal curve for this case, in connection with the course of the plaitpoint curve. The same thing may appear as follows: By applying the criterion (3) in Suppl. N^o. 15, March '07, p. 796, we find that mixtures of a pair of substances of ratios indicated in the mentioned § belong to case (c), (cf. p. 276) while we may derive from VAN LAAR'S fig. 22, Arch. TEYLER (2) 10 (1907) p. 138, These Proc. Sept. '06, p. 226, fig. 1, that the plaitpoint curve crosses from the side $x=0$ to the side $v=b$ (VAN LAAR'S type I).

certain ratio v_{k2}/v_{k1} for not too large T_{k2}/T_{k1} (from 0 up to a certain value, see Table I loc. cit. p. 662) only one barotropic plaitpoint occurs, which in connection with Comm. N°. 96^b p. 503 and 504 pointed to the fact that for the knowledge of the course of the barotropic phenomena at lower temperature considerations in which only the transverse plait is taken into account, are not sufficient for these mixtures ¹⁾ (see Comm. N°. 96^c p. 663).

In Comm. N°. 96^c p. 660 footnote 2 an estimation ²⁾ was derived about the critical temperature of helium from the observation of the barotropic phenomenon for a mixture of helium and hydrogen described in Comm. N°. 96^a These Proc. Nov. '06, p. 459. In this estimation the supposition already mentioned in Comm. N°. 96^a, p. 460, that the molecules of helium exert only an exceedingly slight mutual attraction, was found confirmed.

This suggested the investigation already announced in Comm. N°. 96^b, p. 502 on binary mixtures one of whose components is a gas the molecules of which exert no or only feeble attraction (Suppl. N°. 15, These Proc. March '07, p 786). Here a plait was described for the first time which at descending temperature appears on the

¹⁾ For mixtures of pairs of substances as meant in table I p. 662, for which $0.2'9 > T_{k2}/T_{k1} > 0.196$, three barotropic plaitpoints will occur one of which, however, does not belong to the absolutely stable region. At least for the larger ones of the mentioned ratios T_{k2}/T_{k1} , the two others belong to a plait which enters the \downarrow -surface from K_1 , and crosses the \downarrow -surface as a transverse plait at lower temperature. For this the considerations of Comm. N°. 96^b will hold at least in so far as solid phases do not cause a disturbance. For the smaller ones of these ratios one of these two barotropic plaitpoints will also fall in the not absolutely stable fluid region, and so also for these we shall have to take the occurrence of three phase equilibria into account.

²⁾ For the calculations in note 2, p. 660 of Comm. N°. 96^c we availed ourselves for a and b of hydrogen of the values calculated for this by KOHNSTAMM (LANDOLT-BÖRNSTEIN-MEYERHOFFER's Physik. Chem. Tables 1905), which values had $T_k = 38.6$, $p_k = 20$ according to OLSZEWSKI, Wied. Ann. Bd. 56, p. 133, 1895 as starting point. If we derive the a and b for H_2 from $T_k = 29$ à 32, $p_k = 15$ according to DEWAR (B. A. Report 1902), the estimation for T_{kHe} yields about 1° , OLSZEWSKI's newer data, Ann. d. Phys. 17 (1905) p. 986: $T_k = 32.3$, $p_k = 14.2$, give it a value of more than 1° (the calculation according to note 2 l. c. yields $a_{22}M/a_{11}M = 1/50$, $T_{kHe} = 1.3$).

This would bring about these modifications in the classification of the helium mixtures mentioned in Suppl. N°. 15, Sept. '07, § 8, that mixtures of He with H_2O , O_2 , A, Ne, NO, NH_3 would belong to case (b), those with H_2 , H_2S , CO_2 to case (c). For the modification which another assumption about $a_{12}M$ (cf. p. 280) would cause in the circumstances under which the plait starting from $v = b$ occurs see Suppl. N°. 15 l. c. p. 234. A smaller $a_{12}M$ might even again bring about a shifting in the classification in the direction from (c) towards (a).

ψ -surface from the side of the small volumes, reaches the side $x=0$ at $T=T'_k$, and then passes into a plait crossing in a slanting direction from $v=b$ to $x=0$. This description was accompanied by remarks about limited miscibility in the gas state.

In Suppl. N^o. 15 § 7 These Proc. March '07, p. 795 three cases were distinguished for mixtures in which one component is a gas with a feeble attraction. They are indicated as cases (a), (b) and (c) in § 9, These Proc. Sept. '07, p. 235. Case (a) corresponds with the above mentioned one; in case (b) a plait coming from $v=b$ and one coming from $x=0$ join to a single plait in a double plaitpoint¹⁾; in case (c) a plait starts from $x=0$, comes in contact with $v=b$

1) On the suppositions mentioned in Comm. N^o. 96c p. 509 and p. 510 the data for the two double points in the net of spinodal curves, of which this double plaitpoint is one (a node) may be found in the following way (cf. Comm. Suppl. N^o. 15, p. 233, note 1):

The equation for the δ, x -projection of the spinodal curve on the molecular ψ -surface:

$$RTv_M^3 = 2(1-x_M)(v_M\sqrt{a_{11M}}-b_{11M}\sqrt{a_M})^2 + 2x_M(v_M\sqrt{a_{22M}}-b_{22M}\sqrt{a_M})^2 \quad (1)$$

(cf. Suppl. N^o. 15, March '07, p. 788) gives as conditions for the appearance of a double point after some obvious reductions:

$$(v_M\sqrt{a_{11M}}-b_{11M}\sqrt{a_M})^2 = 2b_{11M}\sqrt{a_{11M}} \cdot (1-x_M)(v_M\sqrt{a_{11M}}-b_{11M}\sqrt{a_M}) + \\ + 2b_{22M}\sqrt{a_{11M}} \cdot x_M(v_M\sqrt{a_{22M}}-b_{22M}\sqrt{a_M}) \dots \quad (2)$$

and

$$(v_M\sqrt{a_{22M}}-b_{22M}\sqrt{a_M})^2 = 2b_{11M}\sqrt{a_{22M}} \cdot (1-x_M)(v_M\sqrt{a_{11M}}-b_{11M}\sqrt{a_M}) + \\ + 2b_{22M}\sqrt{a_{22M}} \cdot x_M(v_M\sqrt{a_{22M}}-b_{22M}\sqrt{a_M}) \dots \quad (3)$$

From (2) and (3) follows:

$$\frac{(v_M\sqrt{a_{11M}}-b_{11M}\sqrt{a_M})^2}{\sqrt{a_{11M}}} = \frac{(v_M\sqrt{a_{22M}}-b_{22M}\sqrt{a_M})^2}{\sqrt{a_{22M}}} \dots \quad (4)$$

Extracting the root from this equation, we may (2) and (3) reduce to:

$$\frac{v_M\sqrt{a_{11M}}-b_{11M}\sqrt{a_M}}{\sqrt{a_{11M}}} = 2b_{11M}(1-x_M) \pm 2b_{22M}x_M\sqrt{a_{22M}/a_{11M}} \dots \quad (5)$$

and

$$\frac{v_M\sqrt{a_{22M}}-b_{22M}\sqrt{a_M}}{\sqrt{a_{22M}}} = 2b_{22M}x_M \pm 2b_{11M}(1-x_M)\sqrt{a_{11M}/a_{22M}} \dots \quad (6)$$

By eliminating v_M from (5) and (6) we obtain for x_M the equation (1) of Suppl. N^o. 15, March '07, p. 796 (cf. errata Proc. Sept. '07, p. 239).

The further derivation of v_M and T (see Suppl. N^o. 15, March '07, p. 798) may be left to the reader (compare with these developments VAN LAAR, These Proc. May '07, p. 38 sqq. and Arch. TEYLER (2) 11 (1907) 1^{re} partie § 5).

at lower temperature, and passes then into a plait crossing in a slanting direction¹⁾.

For this distinction a_{22M}/a_{11M} was always supposed so small in connection with the value of b_{22M}/b_{11M} , that the plaitpoint curve crosses from K_1 to the line $v=b$ and that three phase equilibria not yet occur at the temperatures under consideration²⁾.

Now that the estimations concerning the a and b of helium justify the supposition that the plaitpoint curve³⁾ crosses from $x=0$ to the

1) KUNDT, Berl. Sitzb. Oct. 1880, S. 812—824 was of opinion that it would always be possible to convert a liquid to the gas state by pressing in a gas. This view was maintained in VAN ELDIK's thesis (Leiden 1898, p. 7, cf. Comm. Phys. Lab. Leiden, Suppl. N^o. 3, p. 45), where it says that the operation mentioned, if it is realized, would be the determination of the plaitpoint pressure corresponding to the temperature of observation of the pair of substances which is subjected to the experiment. There it was tacitly assumed that with sufficiently high pressure the plaitpoint state could be reached for every temperature between the critical temperatures of the components as e.g. for mixtures of methyl chloride and carbonic acid, even though it would have to be found above 750 atmospheres for hydrogen and ether, as VAN ELDIK derived taking into account the diminution which with increasing pressure is found in the decrease of the surface tension caused by one and the same increase of pressure (KUNDT loc. cit. p. 818, VAN ELDIK Thesis, p. 5, cf. Suppl. N^o. 3, p. 52). If we pay attention to the possibility now foreseen by the theory, that this diminution continues outside the region of observation, it seems probable in the light of the observations mentioned, that it would not be possible — here we treat as infinite, pressures which exert forces on the molecules greater than those joining the parts of them — to reduce the surface tension to 0 for the pair of substances mentioned (and the same remark applies to hydrogen and ethylalcohol) at the temperature of observation (KUNDT 21°, VAN ELDIK 9°.5), so that already at that temperature a plait crossing obliquely from $x=0$ to $v=b$ would exist on the \downarrow -surface.

In fact we should also derive from VAN LAAR's figure cited p. 274 footnote 3 that ether-hydrogen (and also alcohol-hydrogen) belong to VAN LAAR's type I, while according to the criteria laid down by us, they should belong to case (c) of this type. VAN DER WAALS' equations Contin. II p. 43, however, would point out a critical temperature of complete miscibility of about -200°C . in the supposition of $a_{12M} = \sqrt{a_{11M}a_{22M}}$, so that according to these suppositions an obliquely crossing plait would only make its appearance below this temperature. If the existence of an obliquely crossing plait at the temperature of the above mentioned experiments should be confirmed, this might, among other things, point to the fact that a_{12M} would be considerably smaller than $\sqrt{a_{11M}a_{22M}}$ for the pair of substances mentioned (cf. p. 280).

2) According to this restriction case (c) cannot occur e.g. for b_{22M}/b_{11M} larger than a certain value (cf. Suppl. N^o. 15 These Proc. March '07, p. 797).

3) KUENEN, These Proc. Febr. '03, p. 473 was the first to find experimentally a plaitpoint curve starting from $x=0$, and directed to the side $v=b$ for mixtures of ethane and methylalcohol.

line $v = b$ for mixtures of helium and hydrogen¹⁾, it is desirable to subject the barotropic phenomena for the cases mentioned to a closer examination. In this discussion for mixtures one component of which is a gas with feeble attraction, we shall again restrict ourselves and suppose that in the considered cases at the considered temperatures the second branch of the plaitpoint curve (VAN LAAR, These Proc. May '05 p. 37. starting from K_2 , does not make its influence felt, so that three phase equilibria do not yet make their appearance.

§ 2. *The course of the barotropic phenomena for binary mixtures, one component of which is a gas whose molecules exert only a feeble attraction.*

In this discussion we shall have to distinguish the cases a , b , and c mentioned in § 1.

a . In this case a plait starting from $v = b$ and closed towards the side of the large v 's, appears for $T_{km} > T > T_{k_1}$, which plait we have called gas-gasplait in Suppl. N^o. 15, March '07, p. 793. If

$b_{22} < b_{11}$, then θ_{pl} will be $> \frac{\pi}{2}$ for $T > T_{bpls}$ (see Comm. N^o. 96^b

p. 504 : at $T = T_{bpls}$ a barotropic plaitpoint occurs (cf. Suppl. N^o. 15 March '07 Pl. I, fig. 1); at $T < T_{bpls}$ we find a barotropic nodal line on the gas-gasplait (cf. Suppl. N^o. 15, Pl. I, fig. 2). At $T = T_{k_1}$ the gas-gasplait passes into an obliquely crossing gas liquid plait. A barotropic nodal line will exist on it (see fig. 1) till it disappears under the three phase triangle, and so passes into the not absolutely stable region. As mentioned in § 1 we shall not give the description of what happens when three phase equilibria have appeared. In the same way we shall for the present disregard more complicated cases, as the appearance of two barotropic nodal lines on the gas-gasplait, three on the obliquely crossing plait etc., till further investigation may teach that these cases are possible.

If $b_{22} > b_{11}$, the plait coming from $v = b$ may reach the side $x = 0$, and pass into an oblique plait without it being necessary that a barotropic tangent chord occurs.

b . For $b_{22} < b_{11}$ a barotropic plaitpoint will occur at $T_{bpls} > T_{dpl}$ (cf. Suppl. N^o. 15, March '07 p. 798). This barotropic plaitpoint, and also at $T_{bpls} > T > T_{dul}$ the barotropic tangent chord, may

1) This follows also from VAN LAAR's fig. 22, Arch. TEYLER (2) 10 (1907) p. 38 with the mentioned estimations on the critical temperature and pressure of helium (cf. p. 275 note 2) and on the suppositions made (cf. These Proc. Dec. '06 p. 509 and 510).

occur both on the plait starting from $v = b$ and on that starting from $x = 0$ (figs. 2 and 3). If in the homogeneous double plaitpoint the isobar should run parallel to the x -axis, T_{bpl} would coincide with T_{dpl} . For $T < T_{dpl}$ a barotropic chord exists on the obliquely crossing plait, just as in case (a)¹.

For $b_{22} > b_{11}$, as in this case for a , the existence of barotropic tangent chords is not required.

c. If $b_{22} < b_{11}$, a barotropic plaitpoint will make its appearance for $T_{bpl} < T_{k_1}$ and $> T_{k_m}$; at lower temperatures a barotropic tangent chord is found on the plait starting from $x = 0$ and closed on the side of the small v 's, and at $T < T_{k_m}$ on the obliquely crossing plait (fig. 4). For $b_{22} > b_{11}$ as for a and b .

In fig. 5 the course of the spinodal curves (continuous) and of the connodal curves (lines consisting of dashes) on the ψ -surface for the unity of weight has been more fully represented for a case c . The figure has been construed with a view to mixtures of helium and hydrogen. In this we adopted the hypotheses mentioned in Comm. N^o. 96, Dec. '06, p. 509 and 510, and put for hydrogen $T_{l_1} = 32.3$, $p_{k_1} = 14.2$, for helium $T_{k_2} = 1.3$, $b_{\text{HHe}} = \frac{1}{2} b_{\text{H}_2}$ (p. 275 note 2)². The volume v is expressed in the theoretical normal volume of a molecular quantity as unity. The point K_m has been calculated according to VAN DER WAALS Cont. II, p. 43. The spinodal curves have been constructed as in Suppl. N^o. 15, March '07, p. 788. P_b is the barotropic plaitpoint, calculated in the way indicated in Comm. N^o. 96^c, Dec. '06, p. 510. Further the plaitpoint curve $K_1 K_m$ calculated according to the equation given by VAN LAAR,

¹) In the light of our present knowledge of the behaviour of mixtures and divested of the considerations which are incompatible with it (cf. p. 274 footnote 1) the phenomenon deemed possible by JAMIN, C. R. 96 (1883) p. 1451, Journ. phys. (2) 2 (1883) p. 383, would be described as follows: On compression of a gas above a suitable quantity of liquid (see p. 281 note 2), this liquid is made to dissolve at first under plaitpoint circumstances after which on further pressing in of the gas into the thus formed homogeneous phase a phase richer in the least volatile component (called by JAMIN liquid, by us in certain cases, cf. Suppl. No. 15, March '07, § 4, second gas phase) may separate *above* the phase which is richer in the most volatile component. If this phenomenon could be realized, we should have to deal with a case b for a temperature $T > T_{dpl}$, and in which the line RQ (see fig. 6) intersects the plait starting from $v = b$ in such a way that for the intersected connodal tangent chords $b > \frac{\pi}{2}$.

²) However, on account of the uncertainty which still prevails about T_{KHe} and p_{KH_2} , and in view of the probability that $a_{12M} < \sqrt{a_{11M} a_{22M}}$ (see p. 280) it is still to be considered as quite possible that He—H₂ belongs to case (b), as was supposed in Suppl. N^o. 15.

These Proc. April '05 p. 652 has been included in the diagram. The second branch of the plaitpoint curve is not to be distinguished from the straight line HK_2 on the scale on which the diagram has been drawn.

For the connodal curves the points of intersection with the line $x = 0$ representing the points of saturation for pure hydrogen have been calculated. For this purpose the constants of saturation have been used, which have been calculated by DALTON¹⁾ for a substance that follows the equation of state of VAN DER WAALS with constant a and b . For the rest the course of the connodal curves for which for $T > T_{km}$ also the plaitpoints are known, has been represented schematically. This applies particularly to the points of intersection of the connodal curve for $T = 20$ with the line $v = b$, so that also the course of the connodal curve, particularly of the gas branch, is uncertain in the neighbourhood of the line $v = b$. The line CD represents the experimentally determined barotropic tangent chord for $T = 20$ (see Comms. N°. 96^a and N°. 96^c Febr. '07 p. 660 footnote 2).

The situation of the line CD with respect to the connodal curves might point to T_{km} being higher than was calculated by us, which may be due either to the critical temperature of helium being lower than was assumed by us here, or to a_{12M} being $< \sqrt{a_{11M}a_{22M}}$ for mixtures of He—H₂²⁾.

The course of the barotropic plaitpoints and barotropic tangent chords in case (c), and also in case (b), if they occur on the plait starting from $x = 0$, corresponds for the higher temperatures with that for the case that the branch of the plaitpoint curve starting from K_1 crosses the ψ -surface from $x = 0$ to $x = 1$, for which case the course was described in Comm. N°. 96^b. For the lower temperatures we meet with this difference that in the cases considered in this comm. the barotropic tangent chord continues to exist on the plait, till it disappears under the three phase triangle, whereas in the cases considered in Comm. N°. 96^b the barotropic tangent chord may also vanish from the plait through a barotropic plaitpoint (lower barotropic plaitpoint temperature, see Comm. N°. 96^b p. 504). The latter must even be the case if for $T \geq T_{k_2}$ no three phase equilibria appear as yet (cf. p. 275 note 1).

1) J. P. DALTON, Phil. Mag. April 1907, p. 520.

2) The same remark concerning a_{12M} for mixtures of H₂ with other substances might be derived as follows: for CO₂—H₂ (the same holds for CO₂—O₂) from a comparison of the experimentally determined portion of the plaitpoint curve with that calculated in the same way as above for He—H₂; for H₂—ether and H₂—alcohol see § 1 p. 277 note 1).

VAN DER WAALS, These Proc. Jan. '07 p. 528 calls attention to the influence of $b_{22}b_{11}$ on the occurrence of barotropic phenomena by stating this rule: "When the most volatile substance has the greatest limiting density, the gas phase can be specifically heavier than the liquid phase." In connection with what was discussed above we may now supplement this rule as follows: If of a binary mixture the more volatile component has the greater limiting density, the gas phase will be made to sink in the liquid phase by compression with suitable concentration and temperature, provided the more volatile component has so feeble an attraction that pressing in of this latter component cannot make the liquid phase of the less volatile component dissolve in the gas phase at definite ¹⁾ temperatures even at the highest (comp. p. 277 note 1) pressures. ²⁾ It is implied in the terms of this rule that it has been supposed that no two liquid phases occur.

It is not excluded that also in other cases sinking of the gas phase in the liquid phase might occur. ³⁾

If we apply this rule to pairs of substances of which data are available for a_{22M} , a_{11M} and b_{22M} , b_{11M} , it appears that only for He—H₂ it may be expected on reasonable grounds ⁴⁾ that barotropic phenomena occur at not too high pressures⁵⁾. Further investigations will have to reveal whether for mixtures of pairs of substances as nitrogen

¹⁾ Also at higher temperatures than these barotropic phenomena may then occur.

²⁾ In the case of compression of a gas above a liquid, starting from the pure substance in the way as was done in KUNDT's experiments we describe on the ζ -surface a curve the v, x -projection of which is a straight line joining a point of the line $x = 0$ with the point $v = 0, x = 1$. For the liquid phase to disappear at a definite, suitable temperature just under plaitpoint circumstances, we must start from a definite quantity of liquid so that the volume is represented by SQ (see fig. 6). If the quantity of liquid from which we start, is smaller, the liquid phase will evaporate (be dissolved in the gas phase), if it is larger the gas is dissolved in the liquid phase (cf. VAN DER WAALS, Cont. II, p. 136). Only if the diffusion is not rapid enough to ensure equilibrium all through the tube, solution of the liquid under plaitpoint phenomena may be observed also with other quantities of liquid as corresponding with v_0 , as has been set forth by KUENEN's experiments on the influence of phenomena of retardation.

³⁾ See e.g. § 1, p. 275, note 1.

⁴⁾ Though for mixtures of e.g. helium and acetonitril the available data with application of the special hypotheses assumed in this § (concerning the equation of state etc.) would point to the fact that at high pressures barotropic phenomena might still just occur, it is impossible to express a definite expectation with regard to this on account of the influence of the uncertainties, both in the data and in the validity of the mentioned suppositions.

⁵⁾ This was mentioned in Comm. No: 96b Dec. '06 p. 504.

and a light oil with high critical temperature ¹⁾, nitrogen-lithium, argon-kalium, mercury-iron etc. sinking of the gas phase in the liquid phase could be realized.

§ 3. *On the conditions for the occurrence of barotropic phenomena.*

It appeared in § 2 that with a suitable ratio of the limiting densities the occurrence of the barotropic phenomena depends to a great extent on the ratio of the attractions of the molecules of the two components, hence on the ratio of the critical temperatures. The same thing may also be derived in the following way, more independent of the particular hypotheses which have led to the consideration of obliquely crossing plaits.

To bring about the phenomenon of the gas phase sinking in the liquid phase, the gas phase will have to be much more compressible than the liquid phase, and even on compression the gas phase must not dissolve in the liquid phase. For this the temperature will have to be pretty far below the critical temperature of the least volatile component (T_{k_1}), but still far above that of the second component (T_{k_2}). This points to a large difference between the critical temperatures of the components.

If for the pair of substances considered retrograde condensation of the first kind occurs, the coexisting phases indicated by the points L and G on the ψ -surface for the molecular quantity (see fig. 7) ²⁾, can only have the same density if $M_2 > M_1$.

Only when on the plait on the molecular ψ -surface connodal tangent chords appear for which the angle with the axis $x = 0$: $\theta > \frac{\pi}{2}$, the coexisting phases can have equal density for $M_2 < M_1$.

As the difference between x_g and x_l is larger, and so the connodal tangent chords deflect more rapidly from the side $x = 0$, a smaller difference between M_2 and M_1 will suffice to establish equal densities in G and L .

This will be the more the case the more the plait extends towards the side $v = b$.

The latter is particularly furthered by a small ratio a_{22M}/a_{11M} (cf. Comm. Suppl. N^o. 15 Pl. I fig. 1 and Pl. II), so by a small ratio of the critical temperatures, b_{22M} smaller than b_{11M} also tending in this direction.

1) Mr. F. M. GILLEY of Boston drew our attention to mixtures of air and oil.

2) The dotted lines indicate that the considerations of this § hold both for the case that at lower temperature the plait crosses the ψ -surface as a transverse plait, and for the case that it extends towards $v = b$.

Hence we get as conditions for the possibility of the occurrence of barotropic phenomena:

The second component must have: T_{k_2} small compared with T_{k_1} , and by preference also: $M_2 > M_1$ and $b_{22M} < b_{11M}$.

This becomes still clearer by the application of equations (2) and (4) of Comm. N^o. 79, April '02, p. 659:

$$x_l = x_g v \frac{\varphi'_{g'} - \varphi'_{l'}}{MRT}$$

$$\frac{\varphi'_{g'} - \varphi'_{l'}}{MRT} = \left\{ \alpha \frac{T}{p_m} \frac{dp_m}{dT} - \beta \right\} \frac{p_m (v_g - v_l)}{MRT}$$

which determine the ratio of the concentrations of gas and liquid phases of a binary mixture in which the quantity of one component is small, if the law of the corresponding states may be applied. The connodal tangent chord will rapidly deflect from the side $x = 0$, if the exponent of e assumes a considerable negative value. The greatest influence on this exerts $\alpha = \frac{1}{T_{k_1}} \left(\frac{dT_{xk}}{dx} \right)_{x=0}$, on account of

the value of the coefficient $\frac{T}{p_m} \frac{dp_m}{dT} (> 7)$; so T_{k_2} will have to be small with respect to T_{k_1} . The influence of $\beta = \alpha - \gamma$, if $\gamma = \frac{1}{v_{k_1}} \frac{dv_{xk}}{dx}$ (cf. Comm. N^o. 81, Oct. '02 p. 325) is only of secondary importance. To tend at least in the right direction, γ would have to be negative, so $b_{22M} < b_{11M}$ ¹⁾.

Physiology. — “*An investigation of* Mr. J. W. A. GEWIN, *on the relation of pepsin to chymosin.*” By Prof. C. A. PEKELHARING.

That gastric juice possesses the power, on the one hand to digest proteins under acid reaction, on the other hand, to curdle milk under neutral or scarcely acid reaction, is generally attributed to the presence of two different enzymes in the gastric juice, viz. pepsin and chymosin. This opinion is chiefly based upon an observation of HAMMARSTEN, who was the first to throw light on the changes that take place in milk when it is coagulated by means of rennet. HAMMARSTEN found that an extract from the mucous membrane of the stomach, which, when prepared fresh, could digest proteins as well

¹⁾ The more elaborate mathematical treatment of the conditions for the occurrence of barotropic phenomena, as sequel to Comm. N^o. 96c, will be postponed till further experiments call for a further discussion.

as curdle milk, after having been digested for a few days with hydrochloric acid at a temperature of 37° C., no longer showed the action of rennet, but had preserved its peptic action. From this it could not but follow that each of these actions depended upon a separate agent.

Meantime doubts have gradually arisen as to the correctness of this opinion. That there must at any rate be a very close connection between the proteolytic action of pepsin and the enzyme of rennet, was made probable by the experience that all enzym-solutions with a proteolytic action, no matter whether they are of animal or of vegetable origin, can also act like rennet. And, as I communicated some years ago in this Academy, and as was afterwards corroborated by NENCKI and SIEBER, it could also be proved that all kinds of preparations of pepsin, also when a long digestion with hydrochloric acid and a purification as careful as possible had preceded, are able to act like rennet.

In 1904 there appeared an investigation by PAWLOW and PARASTSCHUK ¹⁾, in which they demonstrated that pepsin and chymosin must be considered as the same substance. These investigators found that in different liquids containing enzyme not only the proteolytic and the curdling power are always found side by side, but that also a proportionately greater curdling power corresponds to a greater proteolytic action. That this is not found in some enzym-solutions of commerce appeared to be owing to the presence of other substances; as soon as their effect was destroyed, the proportionality came to light. A solution of rennet, according to HAMMARSTEN prepared by means of carbonate of magnesia from gastric juice, which, in his opinion, no longer contained any pepsin at all, appeared to be a very good digester of albumen, if only the noxious influence of magnesia-salts was taken away. No more was it proved by HAMMARSTEN, — as PAWLOW explained — that a pepsin-solution can be freed from rennet by digestion with hydrochloric acid, as the proteolytic action had been examined, while the liquid still had an acid reaction; the curdling action, on the other hand, after neutralization, by which the enzyme might be easily destroyed.

Against PAWLOW's explanation objections have been raised. Especially two Swedish investigators, BANG ²⁾ and SCHMIDT-NIELSEN ³⁾, have defended HAMMARSTEN's point of view. The investigation of Mr. GEWIN

¹⁾ Zeitschr. f. Physiol. Chemie, Bd. XLII, S. 415.

²⁾ Zeitschr. f. Physiol. Chemie, Bd. XLIII, S. 358.

³⁾ Ibid. Bd. XLVIII, S. 92.

⁴⁾ PFLÜGER's Archiv. Bd. LXXIX, S. 425.

refers principally to the grounds alleged by these two authors for the duality-hypothesis.

In the first place he has occupied himself with an inquiry into the correctness of the conclusion previously drawn by BANG⁴⁾ from a number of experiments, that there is not only a difference between pepsin and rennet, but also that even the enzym of rennet does not possess the same qualities in different kinds of animals. BANG continued to apply the old name, chymosin, to the enzym of rennet, as it is found in the calf. From this he distinguished by the name of parachymosin the enzym that can be got from the mucous membrane of the pig-stomach. The difference showed itself in the fact that parachymosin, when diluted, became sooner inactive than chymosin, that it showed a greater activity by the addition of chlorcalcium, was more proof against heating to 70° C. and less so against the action of alkali.

With reference to extracts from the mucous membrane of the stomach of calf or pig GEWIN could corroborate these differences; only he did not find the difference in the promotion of the activity by adding chlorcalcium as important as BANG. However, it was a different thing, if not the extracts themselves were examined but the enzym extracted therefrom by dialysis, and purified as much as possible in the way formerly communicated by me. The better the purification had taken place, the smaller the difference became. The extract from the mucous membrane of the calf-stomach loses its power to curdle milk when, neutralized, it is heated for 10 minutes to 70° C.; however, its power is but little reduced, if it is mixed with caustic soda to 0.01 °, and neutralized again after half an hour. With the extract of the mucous membrane of a pig-stomach it is just the reverse. With the enzym of the calf, purified as much as possible, the resistance against heating appeared to have become great, against alkali small. From this it must therefore be deduced that the difference does not lie in the enzym itself, but that it is caused by other substances occurring in the extract. Indeed, it could be proved experimentally that the extract from the membrane of the calf-stomach contains substances which protect the enzym against the action of alkali, but make it the more sensitive to heat. Of a solution of purified pig-enzym (which possesses the qualities of BANG's parachymosin) one half was diluted with water, the other with an extract from the mucous membrane of a calf-stomach, which extract had been deprived of all enzym by heating it for one hour to 80° C. and then neutralized. Of both solutions 2 cc. mixed with 8 cc. of milk caused curdling in 30 sec. A part of each was heated for

10 minutes to 70° C., another part for half an hour left in contact with 0.01 % Na HO and then neutralized again. Now the result was :

2 cc. enzyme with water	+ 8 cc. milk	After heating curdling in 2½ min.	After action of alkali no curdling
2 " " " extract	+ 8 " "		

GEWIN also examined two rennet-preparations of commerce, one Dutch of VAN HASSELT and one Danish of HANSEN. Both showed the qualities of BANG's chymosin. But when, by dialysis and precipitation with acetic acid, the enzyme had been isolated and at least for the greater part been freed from impurities, they had become much more susceptible to alkali and much less so to heating.

That the enzyme is destroyed not only by alkaline, but gradually also by neutral reaction has been made clear by PAWLOW and corroborated by GEWIN in numerous experiments. From this GEWIN explains the difference found by BANG between chymosin and parachymosin, the solution being diluted. What BANG calls chymosin is the enzyme mixed with substances protecting it from alkali. When those substances are present, it may be assumed that the enzyme is better proof against the dilution with water, by which the number of hydroxyl- and metal-ions increases. A solution of purified enzyme (parachymosin), possessing the same curdling power as a not purified solution of calf-enzyme (chymosin), shows, when diluted, sooner a decrease in action, and consequently must, also sooner, show the promoting influence of the addition of chloreacium.

So there is no reason for assuming different rennet-enzymes in different kinds of animals. The difference does not lie in the enzyme but in other substances originating from the mucous membrane of the stomach. If it is necessary to give a separate name to the enzyme of the gastric juice that can curdle milk, it is sufficient to use the word chymosin for it.

But is even this necessary? Should it be assumed that chymosin is different from pepsin?

To the solution of this question GEWIN has devoted the second part of his investigation. It was tried in vain to divide the enzyme into a proteolytic and a curdling part. It is a well-known fact that proteins, undissolved and at a temperature of 15° C. put into a pepsin-solution, take up and keep back this enzyme, so that it is not to be separated from it by washing it out. Coagulated and minced hen's albumen was put in a solution of purified pepsin in 0.2 % HCl. Now, if chymosin were a different matter from pepsin, only the last mentioned would perhaps be extracted from the solution by the albumen. It appeared, however, that the liquid filtered from the

albumen after some hours, had lost not only the peptic but also the curdling power. Indeed, the same negative result had already been arrived at by JACOBY, who for these experiments did not use hen's albumen but caseine ¹⁾.

In the second place it was examined whether a separation into two enzymes could be brought about by dialysis. When pepsin, dissolved in hydrochloric acid, is dialyzed against distilled water, it is partly precipitated, most completely at a low temperature, as soon as the quantity of acid has gone down to about 0.02% HCl. Always, however, a considerable part remains dissolved, which, with the aid of ammonium-sulfate — if the solution does not contain much albumose through 50% saturation with this salt — can be precipitated. If pepsin and chymosin are different matters, it cannot be deemed improbable that they also differ in solubility, that therefore the precipitate in the dialyser should contain more of either one or the other matter than the liquid filtered from it. Also in this way, however, a separation into two enzymes, did not succeed.

SCHMIDT—NIELSEN, however, has communicated experiments from which it appears that, though not a complete, still a partial separation of pepsin and chymosin is possible. A strongly active extract from the mucous membrane of the calf-stomach, prepared with hydrochloric acid, was divided into two parts. One was preserved at a low temperature, the other at 37° C. After some days the heated part had for the greater part lost its power to curdle milk, under a neutral reaction; at acid reaction, however, protein was still strongly digested. Now both liquids were neutralized and the one not heated so much diluted that the curdling power had become as weak as that of the one heated. After that the two liquids were rendered equally acid with hydrochloric acid and digested with fibrine. The fibrine was much quicker dissolved by the heated liquid than by the diluted one, not heated. During the process of heating, therefore, the chymosin had been chiefly lost, the pepsin however not.

This experiment would certainly be convincing, if the neutralization had the same effect on the heated liquid as on the one not heated. This, however, is not the case. The extract contains substances protecting the enzym from the action of alkali; also when no more of this is added than what is necessary to attain a neutral reaction. If the extract is preserved at a low temperature, these substances remain for a long time undisturbed, but if the acid extract is heated to 37° C., they are destroyed. GEWIN has proved this by ample experiments, of which a detailed account is given elsewhere. At the

¹⁾ Biochem. Zeitschr. Bd. I, S. 66.

outset the enzym in the extract of rennet is quite proof against neutralizing, but after having been digested for a few days at 37° C. this power of resistance becomes smaller, and then diminishes quickly. Now, if the neutralized liquid, in order to determine the curdling power, is mixed with milk, the reaction remains neutral and no curdling arises. However, if it is rendered acid again, soon after the neutralisation, a sufficient quantity of enzym is left to digest protein. Also if the not heated solution of the enzym, before being neutralized, is sufficiently diluted with 0.2% HCl, neutralisation herein causes a rapid decrease and at last an annihilating of the curdling power. Thus the curdling time of such a solution was, 8 times diluted, 10 seconds, directly after the neutralisation 2½ to 4 minutes, whilst the milk, mixed in the same proportion with the solution half an hour after the neutralisation, was not yet curdled after 20 minutes.

In all experiments, on the other hand, at which the noxious action of alkali was avoided, the curdling and proteolytic power of the enzym solutions appeared to keep pace with each other.

Summing up the result is therefore that not a single reason is left to assume a difference between pepsin and chymosin.

No more is there any reason to stick to the opinion of NENCKI and SIEBER, which I formerly shared, according to which pepsin should be considered as a molecule, which, through different groups of atoms, on the one hand should have a proteolytic, on the other hand a curdling action. The basis for such an idea, the opinion that the activity in one direction could be preserved, whilst that in the other direction was lost, I must now consider as having lost its foundation. The opinion defended by SAWJALOFF, is far more acceptable, who considers the alteration of caseine, of which the formation of cheese is the consequence, as the beginning of digestion, of proteolysis¹⁾. This opinion has, I believe, become more probable by the experiments made of late about the alterations caseine undergoes under the influence of rennet, particularly by the investigation made some time ago in my laboratory by Miss VAN HERWERDEN²⁾. From these it has appeared that caseine, at a very weak acid or neutral reaction and at a temperature not much lower than 37° C. in a solution of rennet — either a preparation of commerce or pepsin purified as well as possible — soon falls asunder into paracaseine, which when it does not directly become insoluble as a lime-compound, cheese, continues to change, and other substances, among which

¹⁾ Zeitschr. f. Physiol. Chem. Bd. XLVI, S. 307.

²⁾ Ibid, Bd. LII, S. 184.

spar are projected in the plane of the slit of the spectroscope. The two images could be examined separately.

“Bei richtiger Stellung des Kalkspaths bestand das eine Bild aus Licht, dessen elektrische Schwingungen in der Lichtquelle parallel den Kraftlinien vor sich gehen, das andere Bild aus Licht, dessen elektrische Schwingungen in der Lichtquelle auf den Kraftlinien senkrecht stehen. Dass die Ebene der Schwingungen nach dem Durchsetzen des Kalkspaths durch die Quarzlinse gedreht wird, thut nichts zur Sache”.

Bij means of this arrangement the components with vertical vibrations are undoubtedly separated from those with horizontal ones. The main object of RUNGE and PASCHEN's investigation being the connexion between series and magnetic separation there is no objection to be made. The case is changed however as soon as the relative intensities of the components in the emitted light are under investigation, for these under certain circumstances could be essentially altered. If vertical and horizontal vibrations are reflected differently by the grating, the rotation of the direction of vibration in the beams passing through the quartz lens of course will be apparent in the observed intensity.

Polarizing effects of gratings are well known and generally the direction of vibrations relatively to the grooves must be of importance.

I had not anticipated that this circumstance would give rise to such striking effects as were observed by me in some experiments with a large ROWLAND grating. I have only made some observations with the yellow mercury lines, observing in the spectrum of the first order. The incident rays made an angle of about 19° with the normal to the grating, and in this latter direction observations were made or photographs taken. A vacuum tube charged with some mercury ¹⁾ was placed in the magnetic field and by means of a glass lens an image was projected on the slit of the spectroscope. The light emitted at right angles to the horizontal magnetic lines of force was investigated.

In figure 1. a reproduction is given of the triplet in which the line 5769.4 is resolved. The distribution of intensities is in absolute contradiction with equation (1).

Observations with a calcspar and a sodium flame, the light of which was incident on the grating at about the same angle as above specified, the direction of observation being normal to the grating, showed at once that the light reflected from the grating

¹⁾ F. PASCHEN, Physik. Zeitschr. Jahrg. 1 p. 478. 1900.

was strongly polarized. The vertical vibrations were strongly preponderating.

The influence of a rotation of the plane of polarisation of the yellow mercury light on the distribution of intensities in the triplet was then examined. The plane of polarization was rotated by means of quartz plates with faces perpendicular to the axis placed in front of the slit. I had at my disposal two small plates of 2.15 resp. 4.17 mm. thick. According to GÜMLICH ¹⁾ the rotation for mercury light of wavelength 5769 in a quartz plate 1 mm. thick, is at $t = 20^\circ$ $22^\circ.718$ and hence the rotation in my plates amounted to

$$22.72 \times 2.15 = 48^\circ.90 \text{ en } 22.72 \times 4.17 = 94^\circ.7.$$

The change in the distribution of light is at once apparent. In figure 3 the outer components are hardly visible. The negative reproduced corresponds to the case in which the plate rotating the plane of polarization $94^\circ.7$ is in front of the slit.

Figure 2 corresponds to the case in which the incident vibrations are inclined at about 45° to the slit. It may be remarked that in this case the real distribution of intensities between the components, as existing in the emitted light, is observed.

Vertical and horizontal vibrations now being equally present in each of the components, and hence the circumstances as to vibrations being the same for the three components, the polarization by the grating is eliminated.

The distribution of light in figure 2 is certainly not in contradiction with equation (1) and eye observation seems to confirm it numerically also. Of course a photographic reproduction is not sufficient for a comparison of intensities and a numerical test must be reserved for a future paper.

For an estimation of the real ratio of intensities of the components of a divided spectral line henceforth care must be taken that for the region of the spectrum under review the vibrations in the incident light are inclined at an angle of 45° C. to the slit.

If in the case of a complicated division of a spectral line some components are weak, it will sometimes be possible to strengthen these components by placing a quartz plate of suitable thickness in front of the slit. This will be feasible in all cases in which the incident vibrations are not those most favoured by the grating.

Of course also with other spectroscopes this device does apply

¹⁾ GÜMLICH. Wied. Ann. Bd. 64 p. 333. 1898.

e. g. in the case of a MICHELSON echelon spectroscope, if the incident light has been previously analyzed by means of an auxiliary spectroscope. Reflection and refraction in the glass prisms of course weakens to different amounts vertical and horizontal vibrations.

Cases in which relation (1) fails are to be observed in some spectra with many lines (e. g. iron). Among adjacent magnetic triplets some are to be detected in which the distribution of intensity in one resembles figure 1, in the other figure 2. Without further analysis one may conclude that for the one or for the other relation (1) fails.

EXPLANATION OF THE PLATE.

The figures are thirtyfold enlargements of negatives concerning the mercury line 5769, 4.

In all cases the image of the source was projected on the slit with a glass lens.

Fig. 1. No quartz plate in front of slit.

Fig. 2. Quartz plate, rotating plane of polarization 45° in front of slit. The distribution of light corresponds to that in source.

Fig. 3. Quartz plate, rotating plane of polarization 90° in front of slit. Though the time of exposition was thrice that used with the other figures, there appear only traces of the outer components in the original negatives.

Chemistry. — “*On lupeol.*” By Prof. P. VAN ROMBURGH.

In the Comptes rendus of June 24, 1907, JUNGLEISCH and LEROUX state that lupeol cinnamate occurs in the gutta percha of *Palaquium Treubii* BRCK.

I have demonstrated previously that cinnamic acid may be obtained from this species of gutta, whereas lupeol cinnamate proved to be a constituent of different commercial varieties of gutta percha¹⁾.

JUNGLEISCH and LEROUX have now studied the lupeol obtained by them and state that this substance on heating suddenly on the “bloc Maquenne” melts at 190° — 192° , then immediately solidifies and melts again at 212° . They explain this phenomenon by assuming that lupeol loses water and is converted into a hydrocarbon, melting at 212° , to which they give the name of lupeylene. At 130° lupeol would lose water slowly, very rapidly so at 150° — 160° and suddenly at 190° .

¹⁾ B.B. 37 (1904) 3442.

²⁾ Diss. Utrecht 1906.

P. ZEEMAN. The intensities of the components of spectral lines
divided by magnetism.



1. No quartz plate in front of slit.



2. Quartz plate, rotating plane of polarization 45° in front of slit.
Intensities as in source.



3. Quartz plate, rotating plane of polarization 90° in front of slit.

On treating lupeol with acetic anhydride and sodium acetate at 170° , they did not obtain lupeol acetate, but lupeylene, and they argue that the acetate cannot be obtained by the ordinary methods owing to the lupeol losing water so readily.

But some time ago (*loc. cit.*) I obtained with Dr. v. D. LINDEN an acetate, by acetylating lupeol; whilst Dr. COHEN also prepared this ester and studied several of its reactions.

It, therefore, did not seem to me superfluous to repeat the experiments of these French chemists, and to again prepare and analyse the lupeol acetate, so as to make sure that this substance really exists; and that Dr. COHEN, who did not analyse it, because the properties coincided with those of my preparation, and because a mixture of his acetate with lupeol exhibited a considerable lowering of the melting point, was really in possession of the substance.

In order to observe readily an eventual separation of water, and to see and weigh the same, I, first of all, heated lupeol for many hours in one of the limbs of a reverted U vacuum tube placed in an oilbath at 190° , whilst the other limb was cooled in a Weinhold's glass containing liquefied ammonia. In the limb containing the lupeol a sublimate of beautiful crystals had deposited above the oil surface whilst a slight deposit had also formed in the cooled limb.

The weight of the lupeol was 0.5403 gram.

The deposit in the cooled limb weighed 0.0065 gram.

On heating the same at 100° there remained 0.0046 gram.

Therefore only traces of water could have been present in the cooled limb.

The sublimate in the heated limb melted at 212° — 213° .

In another experiment, 1.0806 gram of lupeol was weighed in a glass boat and placed in a horizontal tube which could be heated in an airbath. The tube was connected to a reservoir with sulphuric acid and the whole apparatus was evacuated by means of a water-airpump. First I heated the apparatus for ten hours at 140° — 160° ; the loss of weight amounted only to 0.0066 gram, but it must be observed, however, that a sublimate had formed in the tube just above the boat. Then the substance was heated in the same manner for six hours at 190° — 200° . The total loss in weight then amounted to 0.041 gram, but as the sublimate weighed 0.039 gram it was in reality only 0.002 gram. A separation of water, which would have amounted to 0.040 gram, was therefore again out of the question.

The residue of the two experiments, after being recrystallised from

acetone, was combusted with lead chromate and the result showed that the lupeol had remained unaltered.

0.1991 gram yielded 0.2147 gram H_2O and 0.6172 gram CO_2

	H.	C.
Found :	12.08	84.54
Calculated :	11.49	84.85 (for $C_{31}H_{50}O$) ¹⁾ .

On boiling with acetic anhydride (10 parts) and sodium acetate (1 part), the residue could be converted readily into an acetate melting at 213° as shown by the analysis :

0.2191 gram yielded 0.2251 gram H_2O and 0.6588 gram CO_2

	H.	C.
Found :	11.51	82.04
Calculated :	10.93	82.41 (for $C_{33}H_{52}O_2$).

For the purpose of comparison the acetate of non-heated lupeol was prepared and analysed in the same manner. It melted at 212° .

0.2113 gram yielded 0.2169 gram H_2O and 0.6362 gram CO_2

	H.	C.
Found :	11.5	82.11
Calculated :	10.93	82.41

A mixture of the two acetates analysed also melted at 212° , whilst the acetate mixed with lupeol gave a strong depression of the melting point.

In another experiment, lupeol was heated at 200° for $2\frac{1}{2}$ hours in a current of dry carbon dioxide. A small calcium chloride-tube attached showed a slight increase of weight, but it appeared that a little solid matter had again volatilised with the current. The heating was then continued for six hours and the residue heated finally until the mass began to melt. On treating the same with benzoyl chloride and pyridine it was easy to obtain the lupeol benzoate, (m.p. 264°) proving that the lupeol has not passed into the hydrocarbon.

Finally I have also heated lupeol with acetic anhydride and sodium acetate in a sealed tube for three hours at 170° . The reaction product after being treated with water was recrystallised from a mixture of acetone and alcohol. The melting point of the product obtained was

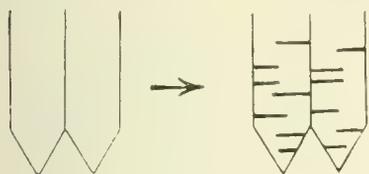
¹⁾ I will not go into the question whether it would be better to assign to lupeol the formula $C_{30}H_{50}O$.

212°. An addition of lupeol acetate from a previous preparation did not affect the melting point. If, however, it was mixed with lupeol or with lupeol which had been heated for some time at 190° a serious depression (about 20°) of the melting point could be observed.

The experiments described, therefore, prove convincingly that lupeol (obtained from bresk) is not converted into lupeylene under the circumstances mentioned by JUNGLEISCH and LEROUX.

As lupeol might perhaps exist in two modifications, Dr. F. M. JAEGER, to whom I wish to convey my best thanks, was kind enough to study the behaviour of lupeol on melting. Dr. JAEGER communicates to the following particulars me:

“If lupeol is melted to a singly-refracting liquid *L*, which takes place very sharply, the mass, on cooling, solidifies partly to an



aggregate of broad pointed needles *A*, glittering in high interference colours, partly to a horny singly-refracting mass *A'*, which frequently exhibits globular spherulites, resembling liquid droplets,

which are very feebly doubly-refracting. The needles *A* at once show a tremendous number of transversal clefts, whilst the splendour of the colours diminishes strongly; *A* passes here into a second modification *B*, the common form of lupeol. Meanwhile the horny mass has also burst and exhibits here and there a strained double refraction besides an increase in the number of droplets, that is to say, crystallisation nuclei in an embryonal condition. If heated carefully for a moment, it *crystallises* into the needles *A*, which pass immediately into *B* (by bursting etc.); the horny mass *A'* is *identical* with the needles *A*: it is *A* in a *supercooled condition*.

The crystallisation velocity is here nearly = 0, and by heating it is increased to such an extent (owing to the diminution of the internal friction etc.) that the mass begins to crystallise. This is a phenomenon well known to me; beautiful instances of crystallisation on heating are usnic acid and many cholesterol esters of fatty acids.

On remelting *B*, *A* is formed some times for a few short moments, afterwards *L*. The two modifications therefore appear to be related by enantiotropy.”

Utrecht, Org. Chem. Lab. Univers.

Statistics. — *“Relations between mortality of infants and high temperatures.”* By Dr. E. VAN EVERDINGEN. (Communicated by Prof. C. H. WIND.)

In the “Statistische mededeelingen uitgegeven door het bureau van statistiek der gemeente Amsterdam” the other day a treatise appeared as N^o 19: “Kindersterfte in Nederland (in de jaren 1881—1905)” by Prof. Dr. R. H. SALTET and Mr. PH. FALKENBURG.¹⁾ The authors point to the fact of the existence of a distinct maximum in the mortality of children under one year of age in the summer-months, and try to find among others a relation between the amplitude of this maximum in different places and in different periods, and the monthly means of temperature at neighbouring places in the same periods. The result is rather negative; hence they write as follows:

“Also from the chronological comparison of the mortality of infants and temperatures, as given by us here for Zealand and the town of Groningen, we are only able to draw a negative conclusion. If in a single case we may speak of parallelism, in the majority of the cases no direct relation between temperatures and mortality of infants can be traced. In so saying we do not imply that we have proved the statement, that the mortality of infants should be independent of the condition of the air. On the contrary, the diagrams we gave furnish the most evident proofs of a relation between the condition of the air in the summer and the mortality of infants. But it is not the height of the temperature which regulates this mortality. As we remarked before, in our opinion the probability remains, that the temperature-fluctuations of the summer — diurn or interdiurn — are the causes which exert an obnoxious influence. The data concerning these fluctuations fail however and cannot in our opinion be replaced by data about temperature-frequencies, which the Meteorological Institute would be able to furnish. We cannot but with a single word refer here to the theory which connects the summer mortality of babies with the presence in this season of a larger number of insects, bearers of disease-germs. Positive facts enabling to further investigate this matter are lacking at present.

Hence there is reserved here a vast field of research for the zealous investigator of the future.

¹⁾ A German translation of this treatise appeared under the title: “Statistische Mitteilungen des Statistischen Amtes der Stadt Amsterdam N^o. 19. Kindersterblichkeit besonders in den Niederlanden, bearbeitet von Prof. Dr. R. H. SALTET and Dr. PH. FALKENBURG.

2. We think we can show that the writers have expressed themselves here in too definite terms, and that a distinctly positive result is obtained if another method of research is followed.

Already in furnishing the mean monthly temperatures for periods of 5 years, which the writers used for their research, the present writer expressed his doubt as to whether these data were fitted for the purpose aimed at. If the fluctuations of the mortality of infants were merely directly proportional to the fluctuations of the mean monthly temperature, the relation sought for ought indeed to appear also in this way. As soon however as these data are otherwise connected — e.g. the increased mortality occurs only after the temperature exceeding a certain limit — it is no longer allowed to use mean values without further inquiry.

The research of Prof. SALTET and Mr. FALKENBURG did indeed not, as we saw before, reveal any relation between mean monthly temperatures and mortality of infants. We will demonstrate this here clearly once more by giving for a single town, Groningen, the deviations from the mean for 25 years of the 5-annual means of the mortality of infants in the summer months May-September, and by their side, in italics, the same deviations for the temperature. The former are given in hundredths per diem, the latter in tenths of a degree Celsius.

GRONINGEN 1881—1905.

Period	May	June	July	Aug.	Sept.
1881—1885	10 —5	12 —9	32 1	0 —8	—9 —4
1886—1890	9 3	17 1	24 —11	—4 —6	9 —2
1891—1895	3 5	—10 —10	—8 —1	—13 4	—2 4
1896—1900	—14 —5	—5 15	—20 6	9 11	4 3
1901—1905	—7 2	—17 4	—16 6	6 —2	—4 —2

If high temperature-means for 5-year-periods were accompanied by high mortalities of infants, at least the signs of the two kinds of deviations ought to agree generally. We find however 12 concordances, 12 discrepancies and 1 undecided case; hence in this way no relation is apparent.

3. It might be supposed that this very unfavourable result is caused partly by adding up the numbers for five years. In this connection we might mention that Dr. OLLAND¹⁾ formerly traced a

¹⁾ A. G. OLLAND, Invloed van het weer op het sterftecijfer. Thesis, Utrecht, 1896.

difference in the general mortality by comparing very hot and very cold summermonths. The principal causes however are different.

We now take as an example the mean mortality of infants at Groningen over the whole period 1891—1905, expressed for every month in percentages of one twelfth of the year-mortality; then we find in round percentages (i.e. p. 74, Table XLIII B).

April	May	June	July	Aug.	Sept.	Oct.
103	93	100	110	133	103	83

We will compare these numbers with the mean monthly temperatures, and moreover with the number of days with excessively high temperatures, which occurred during a similar period. In this respect we think in the first place of the days with maximum temperatures above 25°C ., for which the normals occur in the "Maand-overzicht der weersgesteldheid in Nederland"¹⁾.

We thus find for the total number during the period 1894—1906.

April	May	June	July	Aug.	Sept.	Oct.
2	20	64	100	51	22	1

whereas the mean monthly temperatures in the period 1891—1905 are

8.6	13.0	16.6	18.4	17.6	14.7	9.6
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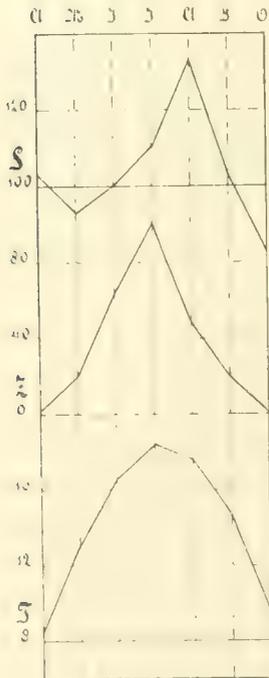


Fig. 1.

There is much more resemblance between the form of the maxima in the first two series of numbers than between those of the first and third series, as is clearly shown by fig. 1. (*S* mortality, *Z* number of days with max. temp. $> 25^{\circ}$, *T* mean monthly temperature); but also for the second series the epoch of the maximum does not coincide with that of the first: it seems as if there is a retardation of the mortality as compared with the high temperatures.

In itself this does not look improbable. Though the writer enters here a field where he is scarcely entitled to a judgment, he thinks he may risk the supposition, that perhaps the high temperatures favour the development of diseases which only after a certain lapse of time cause death. If this be the case, then a better agreement between the fluctuations of temperature and mor-

¹⁾ (Monthly weather review in the Netherlands) Publ. N^o. 94 Kon. Ned. Meteor. Inst.

tality must be obtained if the latter are compared with mean temperatures or numbers of days with high temperatures for a period which begins and ends a little earlier.

We have tested this conclusion by calculating the mean temperatures as well as the numbers of days with temperature above 25° — which, in accordance with a terminology used in meteorology, we will call henceforth “summer days” — once for the calendar months, once for periods from the 16th of one month till the 15th of the next month.

Both the choice of the “shift” of 15 days and that of the temperature-limit 25° are somewhat arbitrary. For a preliminary research like ours there is however no objection to this. From the table below it appears that at least for some parts of our country the shift has about the size which serves the purpose; by *S* are indicated the mean values of the mortality of infants for calendar months during the period mentioned in the first column, given in percentages of one twelfth of the yearly death-rate, as we will continue doing in the following pages; by *T* the mean numbers of summerdays in 5 years, for periods of one month shifted over 15 days.

Period	Place of observation	May-June	June-July	July-Aug.	Aug.-Sept.
1891—1905	<i>S</i> Groningen	100	110	133	103
	<i>T</i> Groningen	19	21	32	16
1881—1905	<i>S</i> Utrecht	102	130	133	97
	<i>T</i> Utrecht—de Bilt	18	25	22	15
1881—1905	<i>S</i> Zeeland	72	101	159	159
	<i>T</i> Flushing	4	12	12	7

For Groningen and Utrecht the maxima now coincide wholly or almost so. For the province of Zeeland, only partially well represented as to its climate by Flushing, a difference of a month in the epoch of the maximum remains.

4. We first give here the results of the investigation for Groningen, in which we compare successively with the deviation of the mortality in 5-year periods the deviations of:

- a.* the common mean monthly temperatures;
b. the means for the periods May 16th—June 15th etc.;
c. the numbers of summerdays for calendarmonths;
d. the number of summerdays for the periods May 16th—June 15th etc.

GRONINGEN 1881—1905.

Period	June	July	Aug.	Sept.
--------	------	------	------	-------

a. Mean temperatures for calendarmonths

1891—1895	-3	-13	-5	-5	-28	0	-7	2
1896—1900	7	12	-6	2	10	7	5	1
1901—1905	-4	1	11	2	19	-6	2	-1

b. Mean temperatures May 16th—June 15th etc.

1891—1895	-3	-7	-5	0	-28	-7	-7	1
1896—1900	7	3	-6	-2	10	8	5	5
1901—1905	-5	3	11	2	19	-2	2	-8

c. Summerdays calendarmonths.

1891—1895	-3	-11	-5	-12	-28	1	-7	-2
1896—1900	7	14	-6	6	10	2	5	3
1901—1905	-4	-2	11	6	19	-3	2	-1

d. Summerdays May 16th—June 15th etc.

1891—1895	-3	-7	-5	-1	-28	-16	-7	5
1896—1900	7	8	-6	-2	10	9	5	2
1901—1905	-4	-2	11	3	19	7	2	-8

First regarding only the signs, we find:

	concordances	discrepancies	Undecided
<i>a</i>	6	5	1
<i>b</i>	7	4	1
<i>c</i>	8	4	—
<i>d</i>	10	2	—

The differences between *a* and *b*, and between *c* and *d* give the effect of the substitution of number of summerdays for mean-temperatures, the differences between *a* and *b*, and between *c* and *d* the effect of a shift of 15 days. It appears that the improvement in the concordance of the signs is largest for the transition from *b* to *d*; in the transition from *c* to *d* in the first place the improvement also in the quantitative agreement of the deviations of mortality and number of summerdays is remarkable. Finally the agreement in case *d* may be called so satisfactory, that but little doubt remains whether the high temperatures must be considered as the cause of the increased mortality of infants.

Hence the negative result arrived at by Prof. SALTET and Mr. FALKENBURG was due partly to the use of mean temperatures instead of temperature frequencies, partly to their non considering a retardation of the mortality with respect to the cause of death.

We will now test the agreement in case *d* also for Utrecht and Zealand.

UTRECHT (de Bilt).

Period	June		July		Aug.		Sept.	
1881—1885	16	— 7	17	10	—29	— 3	—13	— 4
1886—1890	15	— 4	—22	—11	—11	—14	— 2	1
1891—1895	—10	— 2	—10	— 7	— 6	—10	— 2	1
1896—1900	— 7	8	7	4	8	21	15	8
1901—1905	—21	5	7	6	46	8	3	— 6

ZEALAND (Flushing)

Period	June		July		Aug.		Sept.	
1881—1885	— 3	1	11	9	21	3	—12	— 3
1886—1890	4	2	3	— 4	—30	— 3	7	0
1891—1895	9	0	4	— 1	—12	—10	— 8	1
1896—1900	—13	— 1	—11	— 5	21	7	38	7
1901—1905	4	0	— 7	0	1	4	—24	— 5

Hence so far as the signs are concerned we find :

	Concordance	Discrepancy	Undecided
Utrecht :	13	7	—
Zealand :	12	4	4

The number of concordances in these cases is large enough to show that the fluctuations in the number of summerdays play an important part in the matter of the mortality of infants, especially if we consider that at Utrecht as well as in Zealand most concordances occur in the months of large mortality of infants: July and August for Utrecht, August and September for Zealand. Doubtless there are besides them other important circumstances, not connected with temperature, the influence of which even in means for 5 years is not yet eliminated. This need not surprise us — it rather may be called remarkable that in the case of Groningen not *more* is shown of such circumstances. Whether a better agreement might be obtained by assuming a higher temperature limit for Utrecht, for Zealand a lower temperature limit and another value for the shift we have not tried so far.

The mean temperatures for the periods May 16th—June 15th etc. were not calculated for Utrecht—de Bilt and Flushing; a comparison of the methods *a*, *c* and *d* however favours the view, that no improvement would ensue. For, the result for Utrecht was :

	Concordance	Discrepancy	Undecided
<i>a</i>	13	7	—
<i>c</i>	13	6	1
<i>d</i>	13	7	—

for Flushing

<i>a</i>	7	7	6
<i>c</i>	12	7	1
<i>d</i>	12	4	4

It would seem as if the result for Utrecht is practically the same in the three cases. If however also the quantitative agreement is considered, this appears to be wholly lacking in the case *c*, but to be much better in case *d* than in case *a*; hence the results obtained for Groningen are also here, though not so generally, valid.

5. A drawback of the use of means or sums for periods of 5

years is this, that the total number of data for comparison is rather small, and that there remains an uncertainty as to whether really the increased mortality, appearing from these means, occurred in the *same* years in which also the number of summerdays was largest. For this reason we have compiled from the "Ned. Staatscourant" the numbers expressing the death-rate of children less than one year of age for the period 1881—1905, both for Utrecht and Groningen; we have expressed the monthly values in percentages of one twelfth of the yearly death-rate, and compared these data after the method of the preceding §§, by comparing the deviations from the means for the period 1881—1905 respectively 1891—1905, with the deviations of the number of summerdays between May 16th and June 15th etc.

The results are given in the following tables.

GRONINGEN 1891—1905.

Year	June		July		Aug.		Sept.	
1891	-4	-4	-32	-1	-47	-5	0	-2
92	35	2	-26	-5	-9	-4	2	2
93	-26	-2	12	5	-22	-2	-37	4
94	-39	-2	2	-1	61	-1	-7	-
95	25	-2	11	0	4	-2	13	5
96	41	3	48	2	-15	2	-44	-3
97	-2	4	-12	0	24	1	7	-3
98	-7	-1	-67	-6	42	-2	-2	5
99	-3	-2	8	3	-33	5	-8	2
1900	12	3	-11	-2	36	5	61	2
01	-5	-3	10	3	2	13	-30	0
02	48	2	-21	1	-30	-5	-9	-2
03	-11	0	-15	-3	1	-4	33	-1
04	-21	0	13	-1	134	5	18	-1
05	-38	-2	75	5	-21	0	6	-3

UTRECHT 1881-1905.

Year	June	July	Aug.	Sept.
1881	9 — 3	-14 2	-12 1	- 9 — 3
82	-17 — 3	3 — 3	-49 — 2	10 — 3
83	35 — 2	0 3	-61 — 3	- 9 — 1
84	25 — 4	35 7	7 6	-40 5
85	39 3	42 1	-41 — 3	-15 — 2
86	26 — 1	-35 — 3	-15 — 2	43 10
87	23 — 4	-17 — 1	- 9 0	-25 — 1
88	- 9 — 1	-21 — 1	-29 — 2	-27 — 3
89	27 4	16 — 1	- 8 — 4	10 — 2
90	21 — 4	-45 — 5	- 3 — 2	-15 — 2
91	-21 — 4	-28 — 3	-15 — 2	- 8 — 2
92	3 4	-22 — 4	-42 — 3	16 2
93	1 0	18 4	40 0	-21 2
94	-12 — 2	10 — 1	0 — 1	-17 — 3
95	- 4 — 2	- 6 — 2	-14 — 2	21 2
96	3 0	0 4	-13 1	8 — 3
97	17 6	19 4	0 6	- 6 — 2
98	3 2	-33 — 5	15 — 1	57 8
99	-46 — 1	27 2	14 8	43 3
1900	- 8 3	41 — 1	18 9	- 5 2
01	-17 0	20 1	21 9	-12 0
02	- 9 1	-40 1	-48 — 3	11 — 1
03	-25 3	-18 1	18 — 3	27 0
04	-45 — 1	35 — 2	174 7	6 — 2
05	- 8 0	44 5	36 0	-17 — 3

For comparison we give the same table, with the mean temperatures at Groningen for the periods May 16th—June 15th etc. (case *b* § 4). The deviations for these temperatures are given in tenths of degrees.

GRONINGEN 1891—1905 (*b*).

Year	June		July		Aug.		Sept.	
1891	-4	-23	-32	2	-47	-9	-27	3
92	35	9	-26	-16	-9	-17	4	10
93	-26	6	12	13	-22	-4	-35	7
94	-39	-18	2	3	-61	-1	-5	-19
95	25	-6	11	-4	4	-3	15	20
96	41	11	48	6	-15	1	-42	-1
97	-2	27	-12	2	24	12	9	-8
98	-7	-9	-67	-22	42	-5	0	25
99	-3	-13	8	9	-33	14	-6	5
1900	12	3	-11	-4	36	15	63	3
1	-5	-4	10	2	2	21	-23	-8
2	48	-2	-21	2	-30	-29	-7	-6
3	-11	0	-15	-14	1	-15	35	-13
4	-21	8	23	-4	134	13	20	-4
5	-38	10	75	24	-21	1	8	-8

It was to be expected that in these tables more cases would be found where the signs do not agree, or the magnitude of the deviation in one series of numbers is not proportional to that in the other series; indeed, all disturbing causes exert their full influence here. If however we leave out of consideration those cases where one of the deviations is zero, and those where the deviation of the number of summerdays is but one, or the deviation of the mean temperature not more than 0.2 (Groningen *b*) then we obtain the following résumé :

Signs equal.			Signs opposite.		
Number of cases	<i>T</i>	Sums <i>S</i>	Number of cases	<i>T</i>	Sums <i>S</i>
31	114	927	13	42	191
Groningen			Utrecht		
50	186	1275	19	54	308
Groningen (<i>b</i>)			Groningen (<i>b</i>)		
31	38,4	836	17	14,9	366

Hence in the two first tables the number of concordances is much larger than that of the discrepancies, and, moreover, in the favourable cases the mean deviations of number of summerdays and mortality are greater, so that in our opinion no doubt remains concerning the relation between the two phenomena.

The résumé for the third table clearly shows that the parallelism with the deviations of the mean monthly temperatures is not so good as with the number of summerdays.

We may now try to find the factor, which can express this relation approximately. For this purpose we take the sum of all positive deviations of the number of summerdays, and compute the algebraic sum of the corresponding deviations of the mortality of infants. Likewise for the negative deviations of the summerdays. We thus find:

	Sum pos. dev. T'	Sum S	Sum neg. dev. T'	Sum S
Groningen	+ 81	+ 463	— 83	— 352
Utrecht (town)	+ 120	+ 502	— 120	— 465

Without giving all numbers in detail we may add here the result for the province of Utrecht, where 54 concordances occur against 17 discrepancies:

Utrecht (prov.)	+ 133	+ 543	— 118	— 433
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The difference between sum S for positive and negative deviations of T' , rather large for Groningen, is probably an indication of a non linear relation. If we overlook this and combine the two kinds of deviations, we find for the factor sought for Groningen about 5.0, for Utrecht (town) about 4.0, Utrecht (prov.) about 3.9.

6. What precedes still leaves open to doubt whether the high temperatures themselves cause the increased mortality, or, as Prof. SALTET and Mr. FALKENBURG supposed, temperature-fluctuations connected with them. It might indeed be imagined that, as a rule, numerous high temperatures would be accompanied by numerous large temperature fluctuations. Though, from a meteorological point of view, this supposition did not look very probable, we have tested it likewise, for which purpose the annals of the Royal Netherlands Meteor. Institute are quite sufficient. For the same periods from the middle of one month to the middle of the next for which the summerdays had been counted, we computed for Groningen the sum of the differences of the maximum temperature from one day to another, for simplicity neglecting the tenths of degrees. The results were dealt with in the same way as before: hence the deviations of the

temperature fluctuation sums are expressed in entire degrees. The result follows here:

GRONINGEN 1891—1905.

Period	June		July		Aug.		Sept.	
1891—1895	— 3	—19	— 5	22	—28	—16	— 7	20
1896—1900	7	31	— 6	9	10	45	5	— 1
1900—1905	— 4	—11	11	—31	19	—30	2	—18

Among 12 cases there is concordance in 5, discrepancy in 7 cases, so that in this way no relation between the temperature-fluctuations from day to day and the fluctuations of the mortality of infants is shown. Also the annual range of the mean amplitude of the temperature-fluctuations from day to day has another character than that of the frequency of high temperatures: the maximum of the temperature-fluctuations falls in June, not in August. Therefore we thought it useless to separately investigate the *large* or the diurn temperature-fluctuations.

7. In concluding our investigation we are well aware that it is far from complete and leaves room for various questions. If we succeeded in convincing the reader that one of the prominent causes of increased mortality of infants is an increase of the number of very hot days, we express a hope that others, may it be competent in medical matters, will feel inclined to trace the more direct relations, to explain the different character of the phenomenon in various parts of the country, perhaps with the aid of other temperature limits and other shifts. The data the Meteorological Institute is able to furnish for this purpose, also concerning other elements than temperature, will readily be put at the disposal of the future investigator.

The results obtained so far may be summed up as follows:

1st. The fluctuations of the mortality of infants bear hardly any relation to those of the mean monthly temperatures in the same months, neither with those of the mean amplitude of the temperature-fluctuations from day to day.

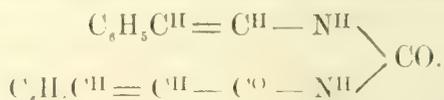
2nd. The number of days with * maximum temperature above

25° C. ("summerdays") counted for periods from May 16th to June 15th, June 16th—July 15th etc shows fluctuations, which in a large majority of cases agree in sign with those of the mortality of infants in June, July etc.

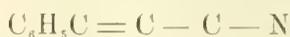
3rd. If a simple proportionality is assumed between the deviation from the normal of the number of "summerdays" and that of the mortality in a period beginning and ending 15 days later, then for each summerday above or below the normal number the mortality of infants is increased or diminished at Groningen with 5, at Utrecht with 4% of the mean monthly death-rate.

Chemistry. — "*Action of potassium hypochlorite on cinnamide*", (2nd communication). By Dr. R. A. WEERMAN. (Communicated by Prof. S. HOOGEWERFF).

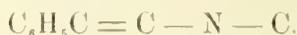
It has been stated in a previous communication ¹⁾ that from cinnamide and potassium hypochlorite was obtained cinnamoylstyrylurea.



This proved that in the action of potassium hypochlorite on cinnamide an intramolecular rearrangement of atoms takes place, and that it therefore becomes possible to arrive from a compound with the atomic grouping:



at one with the atomic grouping:



A compound of this structure may be very readily obtained from cinnamide by treating this in methylalcoholic solution with an alkaline solution of potassium hypochlorite. In this way a yield of about 70% of the urethane is obtained:



styrylaminoformic methyl ester.

B. p. 181°—182° at 14 mm. M. p. 122°—123° (corr.)

0,1674 gr. yielded 0,0914 gr. H₂O and 0,4141 gr. CO₂

0,1733 „ „ 12 cc. N₂ at 14° and 761 mm.

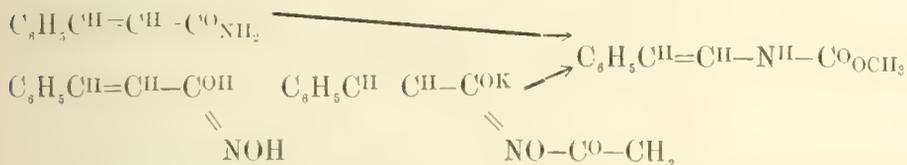
Found: 67,45% C; 6,11% H and 8,12% N.

Calculated for C₁₀H₁₁O₂N: 67,76% C; 6,27% H and 7,91% N.

¹⁾ Proc. 1906, 303.

A substance of the same structure has been described by THIELE and PICKARD¹⁾, who prepared it from the potassium salt of the acetylated cinnamo-hydroxamic acid. As they give the melting point as 115°, and as, in another respect, their observations do not quite agree with mine, the urethane was prepared by Mr. W. OCHTMAN, in the manner indicated by THIELE and PICKARD for the purpose of comparison.

The two substances appeared to be quite identical; the melting point was found to be 122°—123° and a mixture of the two melted at the same temperature.



This ready formation of urethane in an aqueous-alcoholic alkaline medium is remarkable.

I ascertained that this reaction also takes place with a derivative of cinnamic acid. From o-nitro-cinnamide is formed the o-nitrostyryl-aminofornicmethyl ester:



This crystallises in bright yellow needles mp. 149°—150°.

0,2009 gr. yielded 0,3956 gr. CO₂ and 0,0781 gr. H₂O.

0,1457 „ „ 15,8 cc. N₂ at 15° and 758 mM.

Found: 53,70 % C; 4,36 % H and 12,60 % N.

Calculated for C₁₀H₁₀O₄N₂: 54,03 % C; 4,55 % H and 12,61 % N.

A fuller communication will follow in the Recueil.

Delft, July 1907.

*Chemical Laboratory of the
Technical High School.*

Meteorology. — “*The analysis of frequency-curves of the air-temperature.*” By Dr. J. P. VAN DER STOK.

1. The question in what way the characteristic details of frequency-curves of different kinds may be pointed out in a striking way in a plegant, analytical form has again been treated extensively in a recent work²⁾.

The aim of this communication is to fix the attention on the

¹⁾ Ann. 309, 197.

²⁾ H. BRUNS, Wahrscheinlichkeitsrechnung und Kollektivmasslehre, Leipzig und Berlin, TEUBNER, 1906.

method of treatment suggested in it and to give some applications of it to frequency-numbers concerning air-temperature, deduced from observations made on board the lightship "Schouwenbank".

2. The method suggested by BRUNS deserves the more the consideration of all who are occupied with the treatment of frequencies as it is based on the classical works of BESSEL and FECHNER and can be regarded as a logical outcome of the principles indicated by these investigators.

As a basis is taken the well-known function

$$\Phi_0(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt (1)$$

for which in various works tables are given; the first derivative of this function:

$$\Phi_1(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} (2)$$

assumes after substitution of hx for x and multiplication by h the form of the specific probability of a deviation x according to the law of errors of GAUSS in its simplest form. The derivatives of higher order can be written thus:

$$\left. \begin{aligned}
\Phi_2(x) &= \frac{2^2}{\sqrt{\pi}} e^{-x^2} \cdot 1! \left[-\frac{x}{0!1!} \right] \\
\Phi_3(x) &= \frac{2^3}{\sqrt{\pi}} e^{-x^2} \cdot 2! \left[\frac{x^2}{0!2!} - \frac{1}{1!0!2^2} \right] \\
\Phi_4(x) &= \frac{2^4}{\sqrt{\pi}} e^{-x^2} \cdot 3! \left[-\frac{x^3}{0!3!} + \frac{x}{1!1!2^2} \right] \\
\Phi_5(x) &= \frac{2^5}{\sqrt{\pi}} e^{-x^2} \cdot 4! \left[\frac{x^4}{0!4!} - \frac{x^2}{1!2!2^2} + \frac{1}{2!0!2^4} \right] \\
\Phi_6(x) &= \frac{2^6}{\sqrt{\pi}} e^{-x^2} \cdot 5! \left[-\frac{x^5}{0!5!} + \frac{x^3}{1!3!2^2} - \frac{x}{2!1!2^4} \right]
\end{aligned} \right\} . (3)$$

etc.

Now BRUNS' suggestion is as follows: the specific probability (sum of the numbers = 1) of a deviation from a groundvalue assumed arbitrarily be represented by the series:

$$y = h [D_0 \Phi_1(hx) + D_1 \Phi_2(hx) + D_2 \Phi_3(hx) + \dots] . . . (4)$$

from which ensues, that the integral of this equation, called the curve of the sums, is expressed by the form

$$D_0\Phi_0 + D_1\Phi_1 + D_2\Phi_2 + \text{etc.} \dots \dots \dots (5)$$

where Φ means $\Phi(hv)$, as is the case in what follows.

From (4) it is immediately evident when regarded in connection with (3) that the suggested analysis of the curve (called by BRUNS not frequency-curve but curve of distribution) shows a resemblance to the development of a function in terms of a FOURIER series.

In the different Φ_p terms appear polynomial functions of order $p-1$; the Φ_p curve shows p maxima and minima and intersects the x axis in $p-1$ points and alternately will be found for $x=0$ either an extreme value (order uneven) or a point of intersection (order even).

The constants D are determined in the well-known way by evaluating the moments of various orders with respect to the y -axis through the origin of coordinates; if we take for this origin the value of x corresponding to the arithmetical mean and if we put:

$$\int_{-\infty}^{\infty} x^n y dx = \mu_n,$$

we find evidently $D_0 = \frac{1}{2}$ on account of μ_0 being equal to 1; furthermore μ_1 must be $= 0$ on account of the choice of the origin, so D_1 must be put equal to 0, whilst, if one defines the value of the constant h in such a way that

$$2 h^2 \mu_2 = 1,$$

it is then easy to deduce that also D_2 must be 0.

The expressions (4) and (5) can thus be simplified and they become

$$y = h [\frac{1}{2} \Phi_1 + D_3 \Phi_3 + D_4 \Phi_5 \dots] \dots \dots \dots (4^a)$$

and

$$\frac{1}{2} \Phi_0 + D_3 \Phi_3 + D_4 \Phi_4 \dots \dots \dots (5^a)$$

The constants D_3, D_4 , etc. can easily be calculated by means of the formula (3) where, with a view to this, the above mentioned form is given.

To calculate $2D_n$ we have namely, to consider the form appearing between square brackets in the expression for Φ_{n+1} and to substitute in it $h^n \mu_n$ for x^n .

Our finding in this way $2D_n$ instead of D_n is due to the same reason why D_0 must be put equal to $\frac{1}{2}$; namely to the form of (1) in which the number 2 stands as coefficient.

TABLE I.

Air temperature. Schouwenbank. Frequencies of Daily-means.

Celsius	January	February	March	April	May	June	July	August	September	October	November	December
-8.9 to -8.0	1	—	—	—	—	—	—	—	—	—	—	1
-7.9	—	—	—	—	—	—	—	—	—	—	—	—
-6.9	—	1	—	—	—	—	—	—	—	—	—	1
-5.9	2	2	—	—	—	—	—	—	—	—	1	—
-4.9	2	3	—	—	—	—	—	—	—	—	—	3
-3.9	5	5	—	—	—	—	—	—	—	—	1	7
-2.9	18	14	3	—	—	—	—	—	—	—	—	10
-1.9	32	23	6	—	—	—	—	—	—	—	3	27
-0.9	53	37	15	—	—	—	—	—	—	—	2	21
0.1	56	48	25	—	—	—	—	—	—	—	—	27
1.1	77	89	50	—	—	—	—	—	—	—	13	39
2.1	112	123	69	8	—	—	—	—	—	—	23	85
3.1	134	134	111	13	—	—	—	—	—	—	30	77
4.1	150	176	150	36	—	—	—	—	—	—	45	101
5.1	147	142	202	96	—	—	—	—	—	1	55	163
6.1	113	111	157	167	41	—	—	—	—	6	81	163
7.1	70	82	95	206	36	4	—	—	—	28	131	122
8.1	24	40	69	186	112	3	—	—	—	57	158	91
9.1	4	—	25	142	161	3	—	—	—	73	158	48
10.1	—	—	15	97	215	49	—	—	4	105	167	14
11.1	—	—	2	45	166	71	—	—	6	194	86	—
12.1	—	—	6	3	130	165	8	—	51	189	41	—
13.1	—	—	—	1	87	180	27	4	67	182	4	—
14.1	—	—	—	—	46	218	85	49	149	87	1	—
15.1	—	—	—	—	32	160	170	152	222	53	—	—
16.1	—	—	—	—	4	94	275	254	230	18	—	—
17.1	—	—	—	—	—	57	245	245	152	3	—	—
18.1	—	—	—	—	—	49	130	137	71	4	—	—
19.1	—	—	—	—	—	7	64	106	29	—	—	—
20.1	—	—	—	—	—	—	24	32	15	—	—	—
21.1	—	—	—	—	—	—	1	15	4	—	—	—
22.1	—	—	—	—	—	—	1	3	—	—	—	—
23.1	—	—	—	—	—	—	—	3	—	—	—	—

For the values of $\Phi_1, \Phi_2 \dots \Phi_n$ BRUNS has given tables, so there are no farther difficulties about the calculations which can easily be done after some practice.

We must suffice with this short and for that reason incomplete survey of the method; for further details we must refer to the above-mentioned work, where all questions which may arise are discussed extensively.

3. When applying this method to observations of air-temperature it has been assumed that the series need not be continued farther than to the third term, so that only asymmetrical (D_3) deviations and symmetrical ones (D_4) of order one of the simple law are regarded, which, with these kinds of curves not differing much from the bell-shape, proves to be sufficient. When introducing terms of higher order the disadvantage moreover appears that with the evaluation of the higher moments the single extreme deviations, therefore inaccurately determined, play an unduly important part. As first example have been selected the daily-means of the air-temperature, because with these frequency-curves their obliquity changes sign along with the season and can therefore be regarded as a climatological factor. The daily-means are calculated from observations on temperature taken six times a day during the years 1882—1904.

In Table I the frequencies are given from degree to degree, calculated at a total of 1000; the number of data amounts of course for every month to about:

$$23 \times 30 = 690 \text{ or } 23 \times 31 = 713.$$

The obliquity is immediately evident; in winter we find extreme temperatures or negative deviations which are not compensated by equally large positive deviations; in summer we find on the contrary important positive deviations not contrasted by negative ones. The constants of the curve, namely, the mean temperature M indicating the origin of coordinates, the factor of consistency h and the coefficients D_3 and D_4 by which the deviations of the curve from the regular bell-shape are determined are found in survey in Table II; the last two quantities having reference to $\mu_0 = 1$, so that they must still be multiplied by 1000 for the calculation of the numbers comparable to the frequencies of Table I.

TABLE II.
 Constants of the frequency-curves.
 Daily-means of air-temperature.

	M	h	D_3	D_4
January	3.762 C.°	0.2587	+ 0.01657	- 0.00043
February	3.842	0.2773	+ 0.01959	+ 0.00048
March	5.321	0.2908	+ 0.00214	+ 0.00213
April	7.953	0.3738	+ 0.00313	- 0.00115
May	11.068	0.3540	- 0.00847	- 0.00166
June	14.320	0.3708	- 0.00015	+ 0.00114
July	16.871	0.4442	- 0.00216	+ 0.00007
August	17.367	0.4380	- 0.01170	- 0.00026
September	16.032	0.3893	- 0.00634	+ 0.00106
October	12.142	0.3335	+ 0.00470	- 0.00002
November	8.399	0.2662	+ 0.02657	+ 0.00659
December	5.174	0.2377	+ 0.02463	+ 0.00390

From these results of the calculation it is evident that in contrast to most frequency-curves, the D_4 deviations from the simple exponential law for the frequency-curve of the daily-means of the air-temperature are slight and may be regarded as being within the limits of the errors of observation.

In order to investigate how far the calculation agrees with the observation, the numbers of Table I for the months of January and December are taken together as being the most asymmetrical. The numbers obtained in this way are indicated in Table III by O (observed).

The constants of these curve are:

$$M = 4.469 \text{ C}^\circ$$

$$h = 0.2403$$

$$D_3 = + 0.01764$$

$$D_4 = + 0.00130$$

TABLE III.

Analysis of the frequencies Jan. + Dec. of Table I.

O	C_0	$O-C_0$	C_3	$O-C_0-C_3$	C_4	$O-\Sigma C$
1	0	1.0	0.1	0.9	—	1
0	0.1	— 0.1	0.2	— 0.3	0.1	0
0 ^s	0.2	0.3	0.3	0.0	0.2	0
1	0.4	0.6	1.3	— 0.7	0.3	— 1
2 ^s	1.4	1.1	2.5	— 1.4	0.4	— 2
6	3.4	2.6	4.0	— 1.4	0.4	— 2
14	8.4	5.6	5.0	0.6	0.1	1
29 ^s	17.5	12.0	3.7	8.3	— 0.9	9
37	32.9	4.1	— 0.3	4.4	— 2.0	6
41 ^s	54.9	— 13.4	— 8.7	— 4.7	— 2.5	— 2
58	81.5	— 23.5	— 16.0	— 7.5	— 1.7	— 6
98 ^s	108.0	— 9.5	— 25.2	15.7	0.6	15
105 ^s	127.9	— 22.4	— 11.9	— 10.5	3.1	— 13
125 ^s	134.9	— 9.4	0.4	— 9.8	4.1	— 14
155	127.0	28.0	12.6	15.4	2.9	13
138	106.6	31.4	18.2	13.2	0.5	13
96	79.7	16.3	15.7	0.6	— 1.8	2
57 ^s	53.3	4.2	8.2	— 4.0	— 2.6	— 1
26	31.7	— 5.7	0.6	— 6.3	— 1.9	— 4
7	16.9	— 9.9	— 3.9	— 6.0	— 0.8	— 5
—	8.4	— 8.4	— 5.0	— 3.4	0.1	— 4
—	3.0	— 3.0	— 3.9	0.9	0.4	1
—	1.2	— 1.2	— 2.4	1.2	0.5	1
—	0.3	— 0.3	— 1.2	0.9	0.3	1
—	0.3	— 0.3	— 0.5	0.2	0.1	0
—	0.1	— 0.1	— 0.2	0.1	0.1	0
$\Sigma = 1000$		$\Sigma = 214.6$		$\Sigma = 118.4$		$\Sigma = 117$

In the second column is given under f'_n (calculated) the distribution derived according to the simple exponential law; in the fourth column we find the values of the second term in the series having D_3 as factor; from the third and fourth columns is evident that the sum of the differences is lessened by this term from 215 to 118 a thousand. As has been noticed before, the influence of the third term with D_4 is slight.

The sum of the differences remains 12%, also after introducing this term, which can be called satisfactory considering that the total number of observations is not more than :

$$2 \cdot 31 \cdot 23 = 1426$$

and that the most unfavourable months have been taken as an example. In fact, from the regular course of the differences it is evident that there might be a possibility of making the differences smaller still by addition of a fourth or a fifth term with D_5 and D_6 .

For D_5 we find the value -0.00036 , from which ensues that of by far the greater part the differences are due to incompleteness of the material of observation, so that extension of the series would avail but little.

4. As fitting material for a second application of the method to meteorological quantities all the observations of temperature have been chosen, taken six times a day in the month of July on the same lightship during the years 1882—1906. The number of observations is now six times greater than for the daily-means and amounts to 4516.

On account of this greater number the frequency-curve will have a more regular shape and the obliquity which was easily discernible for the daily-means also for the summer months, will now come more clearly to the front.

The observations are arranged according to the different quarters of the wind, so that we obtain (Table IV) frequencies of the so-called thermic windrose. On board the lightships the direction of the wind is determined in accordance with the indications of the compass; for the period 1882—1906 we can assume that these observed directions of the wind can be reduced to the proper direction by applying as correction the mean deviation, -15° .

TABLE IV. Frequencies of air-temperatures for different quarters of the wind (Magnetic), Schouwbank, July, 1882—1906.

	N	NNE	NE	ENE	E	ESE	SE	SSE	S	SSW	SW	WSW	W	WNW	NW	NNW	C	Total
10.6—11.5 C	1	—	—	—	—	—	—	—	4	4	—	—	—	1	—	—	—	4
11.6—12.5	4	4	4	—	—	—	—	—	0	0	—	—	1	0	3	13	—	23
12.6—13.5	29	17	5	1	2	2	—	—	1	1	2	2	0	2	13	25	—	102
13.6—14.5	44	26	12	2	14	4	5	4	3	4	15	11	9	20	43	51	3	267
14.6—15.5	71	49	35	21	18	9	13	4	13	9	30	65	69	72	77	49	18	624
15.6—16.5	67	66	80	44	27	14	18	18	49	27	70	164	152	73	50	42	24	952
16.6—17.5	56	38	79	39	35	18	28	45	37	31	94	182	147	51	44	37	37	968
17.6—18.5	34	29	56	38	32	23	25	45	22	22	62	140	119	31	41	26	27	742
18.6—19.5	26	13	48	16	27	13	16	7	14	14	35	67	52	13	21	40	27	419
19.6—20.5	9	10	29	16	16	6	13	4	5	10	19	17	24	6	9	6	18	217
20.6—21.5	4	4	18	10	10	1	6	6	5	4	2	5	6	1	6	3	7	98
21.6—22.5	2	2	7	3	2	4	4	3	3	4	4	2	5	4	2	4	15	61
22.6—23.5	2	2	3	3	1	0	2	1	4	0	1	—	1	0	2	1	4	24
23.6—24.5	—	1	1	0	2	1	—	—	—	1	3	—	—	0	—	—	1	10
24.6—25.5	—	—	4	1	1	—	—	—	—	—	—	—	—	4	—	—	4	5
25.6—26.5	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	4
Sum	349	258	375	191	185	95	430	77	424	425	337	656	585	272	311	264	182	4516

More clearly than in the numbers of Table IV does the influence of the direction of the wind on the temperature show itself in the mean temperature M and the factor of consistency h , arranged in Table V.

TABLE V.

Direction of wind Magn.	Number of observ.	Mean Temp. M .	h
<i>N</i>	349	15.10	0.3485
<i>NNE</i>	258	16.33	0.3478
<i>NE</i>	375	17.50	0.3477
<i>ENE</i>	191	17.61	0.3561
<i>E</i>	185	17.68	0.3155
<i>ESE</i>	95	17.53	0.3491
<i>SE</i>	130	17.77	0.3527
<i>SSE</i>	77	17.67	0.3436
<i>S</i>	124	17.39	0.3713
<i>SSW</i>	125	17.38	0.3754
<i>SW</i>	337	17.23	0.4037
<i>WSW</i>	656	17.68	0.5124
<i>W</i>	585	17.10	0.4685
<i>WNW</i>	272	16.32	0.4391
<i>NW</i>	311	16.27	0.3512
<i>NNW</i>	264	15.65	0.3418
Calm	182	18.56	0.3295

From this table is evident that in this summermonth by far the highest temperatures are observed when there are calms; for the rest we have the lowest temperatures with the northerly seawind, the highest with a landwind; the transition from NNE (W $7^{\circ}.5$ E prop. dir.) to NE (N 30° E prop. dir.) is sharp, much sharper than that from SW (N 210° E prop. dir. landwind) to WNW (N $277^{\circ}.5$ E proper dir. seawind).

This sharp difference we do not find for the factor of consistency, which shows for WSW wind a distinct maximum and for calm a minimum.

The numbers of observations being rather slight for many directions

of the wind, the numbers of Table IV have been arranged in Table VI to five groups where as much as possible comparable series have been added together.

TABLE VI.
Frequencies deduced from Table IV.

	\overline{WNW} \overline{NNE}	\overline{NE} \overline{E}	\overline{ESE} \overline{SW}	\overline{WSW} \overline{W}	C	Total
10.6—11.5 C°.	2	—	2	—	—	4
11.6—12.5	21	4	0	4	—	23
12.6—13.5	86	8	6	2	—	102
13.6—14.5	184	25	35	20	3	267
14.6—15.5	318	74	78	135	18	623
15.6—16.5	298	148	166	316	24	952
16.6—17.5	226	153	223	329	37	968
17.6—18.5	161	126	169	259	27	742
18.6—19.5	83	91	99	119	27	419
19.6—20.5	40	61	57	41	18	217
20.6—21.5	18	38	24	11	7	98
21.6—22.5	8	12	19	7	15	61
22.6—23.5	7	7	5	4	4	24
23.6—24.5	1	3	5	—	1	10
24.6—25.5	1	3	—	—	1	5
25.6—26.5	—	1	—	—	—	1
Sum	1454	751	888	1241	182	4516

The comparatively low temperatures for WNW—NNE winds with a small factor of consistency (great distribution) in contrast to the high temperatures and small distribution for WSW—W winds is very clear from this table.

At the same time is evident from these data how the combination of series with different mean values decreases the obliquity in the total, so that we can expect that the obliquity factor D_s will be considerably smaller for the total series than D_s , calculated for the various series, which is confirmed by the following table.

TABLE VII.

Constants of the frequency-numbers of Table VI.

	M	h	D_3	D_4
<i>WNW-NNE</i>	16.135	0.3576	- 0.01778	0.00385
<i>NE-E</i>	17.589	0.3428	- 0.01657	0.00343
<i>ESE-SW</i>	17.430	0.3710	- 0.01540	0.00477
<i>WSW W</i>	17.089	0.4869	- 0.01300	0.00297
<i>C</i>	18.555	0.3295	- 0.01153	0.00108
Total	16.971	0.3603	- 0.01116	0.00331

The (negative) obliquity D_3 is therefore strongest for the northerly seawinds with low temperature and decreases further regularly with the azimuth counted from North through East. The symmetrical deviation D_4 is greatest for southerly winds and smallest for calms.

TABLE VIII.

Analysis of the frequencies of Table VI.

O	C_0	$O-C_0$	C_3	$O-C_0-C_3$	C_4	$O-\Sigma C$
1	2.6	- 1.6	- 3.5	1.9	2.9	- 1
5	8.7	- 3.7	- 4.4	0.7	1.0	0
23	27.2	- 4.2	- 3.3	- 0.9	- 3.6	3
59	65.5	- 6.5	4.5	- 11.0	- 8.8	- 2
138	122.7	15.1	15.1	0	- 6.1	6
211	178.4	32.6	14.7	17.9	7.3	11
214	201.2	12.8	- 0.5	13.3	15.4	- 2
164	175.8	- 11.8	- 15.3	3.5	6.4	- 3
93	119.2	- 26.2	- 14.5	- 11.7	- 6.5	- 5
18	62.7	- 14.7	- 4.0	- 10.7	- 8.9	- 2
22	25.5	- 3.5	3.6	- 7.1	- 3.0	- 4
14	8.1	5.9	4.3	1.6	1.1	1
5	2.0	3.0	2.3	0.7	1.7	- 1
2	0.4	1.6	0.8	0.8	0.8	0
4	0.0	1.0	0.2	0.8	0.3	1
$\Sigma = 1000$		$\Sigma = 144.2$		$\Sigma = 82.6$		$\Sigma = 42$

Let us remark here that a negative D_3 refers to the ascending slope of the curves on the left being steeper than the descending slope on the right and that a positive sign of D_4 means that small deviations appear in greater number than would be the case in accordance with the simple exponential law.

In order to show clearly the part played by the various terms of the series in the composition of the curve of distribution a comparison has been given in table VIII, as in table VI, of the observed and calculated frequency-numbers of the last series of table IV; the number of observations 4516 has here been reduced in the first column under O to 1000.

From this table is evident that if only a great number of observations is at hand, the frequency-curve of the air-temperature can be very satisfactorily determined by the three constants of the series of BRUNS, the total of the differences between observation and calculation amounting in round numbers to 4 %.

Anthropology. — “*Is red hair a nuance or a variety?*” By Prof. L. BOLK.

Concerning the anthropological importance of red hair the literature relating to it contains up till now little more than opinions based upon general impressions or suppositions, founded on statistical data, which when looked at more closely are open to more or less unfavourable criticism. There is in those opinions and suppositions a definite main current according to which it is generally assumed that a closer affinity of redhairiness exists to what, for the sake of brevity, I shall indicate as the blonde race, characterized as to the pigmentation by blonde hair and blue eyes.

The nature of the relation between blonde and red-haired people is expressed by TOBINARD¹⁾ as follows: the red-haired type has arisen from the blonde type “par une action des milieux”. Also BEDDOE and RIPLEY, to mention the principal English and the best known American anthropologist, assume a closer connection between blonde and red hair. VIRCHOW looks upon the subject from a somewhat different standpoint, when he says that redhairiness probably arises in two manners, viz. by a decrease of pigment in brown hair or an increase in blonde hair²⁾. This opinion of VIRCHOW is based upon

1) *Éléments d'Anthropologie générale*. Paris 1885 p. 334

2) Das jedoch scheint mir nicht unwahrscheinlich zu sein, dass es eine doppelte Art von Rothhaarigkeit giebt, von denen die eine als eine Steigerung des Pigments bei den Blonden, die andere als eine Verminderung desselben bei den Braunen anzusehen ist. *Archiv für Anthropol.* XVI Bnd. p. 338.

his statistics of the extension of redhaired people in Germany. Now it would not be difficult to prove that VIRCHOW was in no way entitled to such a conclusion on the ground of his statistics; his data were very incomplete and the relations found by him he himself calls "ganz unzutreffend". I intend to revert to this in another place, but would like to examine another side of VIRCHOW's conclusion somewhat more closely. For where he says that redhairiness arises, either by an increase or by a decrease of the hair-pigment, this implies that in VIRCHOW's opinion redhairiness is the consequence of quantitative difference, and that, in other words, this quality is consequently only a question of gradation. Moreover I do not wish to enter into the question whether VIRCHOW has a right to place blonde or brown over against each other as primary or pure hair-colours. Let it suffice for the present to state that VIRCHOW sees no contrast between red and blonde hair, but that the former is only a nuance, either of blonde or of brown. I know only one anthropologist who, in contradistinction to the great majority, raises his voice against the existence of a closer relationship between red and blonde hair, viz. AMMON, who in his Anthropology of the Baden population hazards the suggestion that the difference between blonde and red hair is not founded on a quantitative difference of the pigment, but on a qualitative distinction. So AMMON is more inclined to the opinion that in redhairiness not a nuance, but a variety renders itself manifest¹⁾.

In working up my anthropological material concerning the population of Holland I have naturally come to the question about the importance of redhairiness, and the conclusion at which I have arrived deviates from the general opinion. The extension of redhairiness in our country causes me to deny every closer relationship with the blonde race.

Let me begin by pointing out that the composition of our population is very favourable for an answer to this question. A few years ago I had the pleasure in this meeting to throw light upon the main features of the composition of our population from the so-called blonde and brown-haired race. And I could then establish how the composition of our population differs, if the northern part of our country is compared with the southern. The blonde type decreases regularly in a southern direction, going hand in hand with an increase of mixed types, and though of course in a smaller proportion, an increase of the pure brown type. The differences

¹⁾ Die von manchen Anthropologen beliebte Vereinigung der roten Haare mit den blonden, halten wir für unzulässig, denn die roten stehen in vielen Fällen den braunen näher und sind jedenfalls stärker pigmentiert, haben vielleicht ein Pigment von anderer Beschaffenheit. Zur Anthropologie der Badener. blz. 129.

between the northern and southernmost parts of our country are in the end rather considerable, and it is for this very reason that our population is so extremely fit to answer a question like this. If it should after all be true that redhairiness is more closely related to blondness, then the variation in the number of blondes cannot but cause a similar change in the number of redhaired people.

The materials for the following illustration have again been borrowed from my inquiry made at the time into the distribution of the colour of hair and eyes among the population of Dutch schools. On the schedules that were distributed for that purpose I distinguished four colours of hair: blonde, brown, red and black, and four colours of eyes: blue, grey, brown and brownish-green. The total number of children examined, amounted, with the exception of the Israelites, to 478,976. The total number of redhaired individuals among them is 11772, so that there are on an average 2.45% redhaired children. The figures from which this proportion has been borrowed, are high enough to consider this as the exact average.

The first question we shall answer is: in what proportion do red-haired persons occur in the different provinces of our country. This appears from Table I. In the first column is found the total number of the children examined in each province, in the last the number of red-haired ones among them, also in the proportion expressed by the percentage. What appears from this last column? Suppose that in round numbers the general average is 25 red-haired individuals in 1000 inhabitants, then we see that in four provinces: Friesland, Gelderland, N. Holland and Utrecht the same proportional number appears, that there occurs in Z. Holland only one in 1000, in Groningen 2 in 1000 and in Overijssel and Limburg 3 in 1000 less — in N. Brabant 1 and in Drenthe 2 more in 1000 inhabitants. These figures differ so little, also from the general average, that we are in my opinion fully entitled to conclude that in the provinces mentioned the extension of red-haired persons is much the same everywhere.

This slight difference in the percentage of redhaired persons in our country is corroborated by Table II in which the absolute numbers and the proportions are mentioned of all the places in our country in which the number of the children examined was more than 2500. It was to be expected that where the absolute numbers are sometimes relatively low here, the variation of the percentage would be greater. But yet nowhere does the proportion fall below 2% and only once a percentage of 2.9 is reached as the most favourable proportion. Where the absolute figures are high, as in Amsterdam and Rotter-

TABLE I.

Province	Total	Blondhaired- blue-eyed	Blackhaired brown-eyed	Redhaired
Friesland	33,053	44,282=43.2%	566=1.7%	857=2.5%
Groningen	32,223	43,401=41.3	446=1.4	755=2.3
Drenthe	45,169	5,959=39.4	205=1.3	422=2.7
Overijssel	41,389	14,713=35.5	689=1.6	919=2.2
Gelderland	46,786	16,078=34.4	1340=2.8	1198=2.5
Zuid Holland	108,073	33,943=31.4	2,712=2.5	2640=2.4
Noord Holland	97,050	30,291=31.2	1791=1.8	2472=2.5
Utrecht	22,017	6,626=30.1	528=2.4	545=2.5
Zeeland	20,986	5,707=28.4	834=4.1	395=1.8
Noord Brabant	41,455	9,173=22.3	1661=4.—	1072=2.6
Limburg	21,902	4,790=21.8	1013=4.7	497=2.2

TABLE II.

Municipality	Total	Redhaired	Percent.
the Hague	43,184	276	2.01
Enschede	3,667	77	2.1
Maastricht	3,812	86	2.2
Utrecht	8,668	205	2.3
Haarlem	9,908	229	2.3
Hengelo	2,876	68	2.3
Rotterdam	25,828	647	2.5
Amsterdam	44,418	1164	2.6
Dordrecht	4,586	123	2.6
Zwolle	3,618	101	2.7
Deventer	3,754	105	2.7
Leeuwarden	3,562	102	2.8
Leiden	5,648	161	2.8
Gouda	3,640	102	2.8
Groningen	5,039	142	2.8
Arnhem	6,269	179	2.9

dam, the general figure of proportion, which has been found, reappears again.

Thus far I have left one province out of consideration, Zeeland. There is no denying that this province takes up a place somewhat separated from the others, since here the number of redhaired persons falls suddenly to 1.8%. This contrast with the other provinces is too great not to see here the influence of a definite cause. Yet this decrease in redhairiness in Zeeland, as will be proved higher up, cannot be attributed to a rise in the number of brunettes, which really occurs here, for N. Brabant, which is no less brunette than Zeeland, does not show this decrease. I will not enter into the cause of this decrease, I only wish to point out that already repeatedly both by Belgian and Dutch investigators attention has been drawn to the fact that from an anthropological point of view our Zeeland population takes up quite a peculiar place among the inhabitants of our country. It seems to me that this opinion is corroborated by the proportion found for the redhaired persons.

This much concerning the extension of redhaired persons in our country in general. The general conclusion to which we come, may be expressed thus, that with the exception of Zeeland this extension is a very regular one all through the country. This fact was really contrary to my expectation, as I myself, when beginning to work up my data, held the general opinion that there was a closer relationship between redhairiness and blondness. So I expected that, where in our country the blonde type varies so strongly, the influence of this would also come forth in the variation of redhairiness. Let us now, in order to prove the independence of the two phenomena, pay some more attention to Table I. For through this table we also get an insight into the decrease of the blonde and the increase of the brunette race, in a direction from North to South. As I said before, I distinguished on the schedules sent round four colours of hair and four of the eyes, making together 16 combinations. Of these combinations there are two which are really characteristic for the race, namely the combination blonde hair and blue eyes for the blonde race and the combination black hair and brown eyes for the brunette race. The other 14 combinations may be considered as mixed forms between the two races. Now, in order to keep the foundation of my reasoning as pure as possible, I have inserted in Table I only these two combinations to mutually compare them.

In the third column we find the number and percentage of blond-haired blue-eyed individuals in the different provinces. Now it appears that the number of pure blondes decreases very regularly from North

to South. The number is greatest in Friesland, viz. 43.8%, smallest in Limburg 21.8%, so reduced to half of the number for Friesland. The fourth column affords a survey of the increase of pure brunettes. Herein Drente shows the smallest number, 1.3%, Limburg the greatest 4.6%. The figures in this column point to a distinct increase in a southern direction. From this table something else appears that is important for the characterisation of redhairiness. If namely the percentages of the "pure" types are added up, so the blondes and the brunettes, this gives for Friesland a total of 44.9%, for Limburg only 26.4%, while between these two numbers those of the other provinces are regularly grouped. So the number of mixed types is in the south of our country nearly 20% higher than in the north. As a general result we may state a decrease of the pure blondes, an increase of the pure brunettes and the mixed types in a southern direction.

And notwithstanding in Friesland twice as many pure blondes are found as in Limburg, a change in the number of redhaired individuals is not perceptible. Therefore I think I am entitled to deny the existence of any relation between the two phenomena on the ground of the figures found. But my table also induces me to reject the opinion which is sometimes given, that redhairiness should be a consequence of a crossing between a blonde and a brunette individual. If this were the case, an increase might be expected of the number of redhaired persons in a southern direction in connection with the increase of mixed types.

Have I therefore to deny relationship between the blonde and the redhaired type on the ground of the data mentioned, a still stronger proof for this is afforded by another fact, which I had expected as little as the others which have been explained. It is namely the extension of redhairiness among the Jewish school-population. The total number of Jewish children examined at Amsterdam, the Hague and Rotterdam amounted to 9155. Of these 228 were redhaired, i.e. 2.47%, whilst for the not Jewish population a proportion of 2.45% had been found. The agreement between the two figures is surprising and the importance of the fact for the question put by us, shows itself clearly, when I point out the fact that pure blondes i.e. blondhaired blue-eyed Jews occur only in a proportion of 8.2% pure brunettes, i.e. black-haired brown-eyed in 18.1%. From this it proceeds that in our country among the Jewish schoolpopulation with 8.2% pure blondes, there occur as many redhaired persons as in Friesland with 43.2% pure blondes. A stronger proof that there is no direct relation between redhairiness and blondness cannot be desired.

Thus far we examined redhairiness with regard to the increase or decrease of the number of blondes among our population, and we came to the conclusion that the two phenomena are independent of each other. We can now look upon the phenomena from another point of view. If it were true that redhairiness showed a preference for the blond race, the consequence of this must needs be that among the children who, as to pigmentation, belong to the blonde race, accordingly such as have blue or grey eyes, there are more redhaired individuals than among those with brown or brownishgreen eyes.

How far this is true is shown by Table III.

TABLE III.

Colour of eyes.	Total	Blondhaired	Brown	Black	Red
Blue . . .	186.033	83.34%	11.81%	2.38%	2.47%
Grey . . .	152.072	79.67	14.66	3.06	2.63
Brownish-green	58.531	60.68	28.64	8.—	2.55
Brown . .	82.338	45.05	38.61	14.28	2.03

The first column mentions the total number of children with one of the four different iris-colours, and in the four following columns we find consecutively the percentage of the combination of the iris-colour with one of the four haircolours. Phenomena make themselves manifest therein, which were to be expected beforehand. Of the blue-eyed individuals for example, 83 % have blonde hair, of the brown-eyed only 45 %; on the other hand the number of brown-haired persons with the last is more than three times as large as with the blue-eyed, and a relatively still stronger rise is found with the blackhaired. Generally speaking, it appears that with an increase of the pigmentation of the iris also the pigmentation of the hair increases. This holds good for blonde, brown and black hair. But in contradistinction to this there appears to be no relation between the degree of pigmentation of the iris and the hair with redhaired individuals. For of the blue-eyed 2.47 % are redhaired, of the grey-eyed 2.63 %, of the brownish-green-eyed 2.55 % and of the brown-eyed 2.03 %. It is true, this last figure is the lowest, but it seems to me that the difference is not so great that therein the proof may be seen that redhairiness shows less relationship to the brunette race. Moreover, this opinion could directly be refuted by the fact that I

find a somewhat lower percentage of redhaired individuals among the blue-eyed children than with the grey or brownish-greeneyed.

Red hair is therefore a quality altogether independent of the degree of pigmentation of the iris. No matter from what side we look upon the redhairiness with regard to the other phenomena of pigmentation of hair or eyes, there is — at least on the score of my researches --- not a single proportion to be alleged on behalf of the opinion that redhairiness should by preference occur in the blonde race. So I cannot but reject as incorrect the opinion of those who, reducing redhairiness to a quantitative difference of pigment, see in it nothing but a nuance. And these results of our investigation naturally lead to considering red hair as a variety, in which the pigment is qualitatively different from that in blonde and black hair. Between these two last there is properly speaking no real difference; gradually, through numerous shades, flax-blonde hair passes into jet-black, by an increase of the quantity of pigment; also in red hair a great number of shades can be distinguished; in proportion to the quantity of pigment, the colour varies between gold-blonde and fiery red.

How is the appearance and the regular extension of redhaired individuals among our population to be accounted for? It need hardly be said that, on the ground of the proportions found, I am not entitled to give any explanation. For this purpose anthropological researches of another nature would be necessary. Only for the sake of completeness I mention the opinion of TOPINARD, who explains redhairiness from a former mixing with an originally redhaired race, which in pre-historic times is said to have inhabited the plains of Russia, Siberia and Turkistan and of which one of the groups of the Finnish population (the Letts and the Esths) are said to be the purest descendants ¹⁾.

¹⁾ L'histoire de cette race est à faire. Jusqu'à nouvel ordre j'admets qu'elle a occupé le sol de la Russie aux époques préhistoriques, antérieurement à l'invasion des Asiatiques, répondant à l'un des deux types finnois actuels, je n'ose dire de celui qui a apporté la langue du Kalevala. Elle est représentée dans la plupart des kourgans anciens de la Russie. Le type en est signalé dans les annales des Han antérieurement à l'ère chrétienne en Sibérie et dans le Turkestan Chinois. — *Éléments d'Anthrop. générale.* p. 334.

E R R A T A.

Proceedings of the meeting of September 1907.

p. 229 l. 1 and p. 230 l. 7 from the bottom: for p. 1 read p. 215

Proceedings of the meeting of March 1907.

p. 788 l. 6 from the bottom: for $1.299 < \tau < 1.040$
read $1.299 > \tau > 1.040$

.. .. l. 5 for $1.040 < \tau < 1$
read $1.040 > \tau > 1$

.. .. l. 19 for "becomes" read "comes"

.. .. l. 18 for "united" read "unite"



KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday November 30, 1907.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige
Afdeling van Zaterdag 30 November 1907, Dl. XVI).

C O N T E N T S.

G. VAN RIJNBERK: "On the segmental skin-innervation by the sympathetic nervous system in vertebrates, based on experimental researches about the innervation of the pigment-cells in flat fishes and of the pilo-motor muscles in cats". (Communicated by Prof. C. WINKLER), p. 332.

H. KAMERLINGH ONNES and J. CLAY: "Some remarks on the expansion of platinum at low temperatures", p. 342.

J. W. LANGELAAN: "On the development of the Corpus callosum in the human brain". (Communicated by Prof. T. PLACE), p. 344. (With one plate)

W. KAPTEYN: "On an infinite product, represented by a definite integral", p. 347.

P. ZEEMAN: "Magnetic revolution of spectral lines and magnetic force", (2nd part), p. 351. (With 2 plates).

Mr. and Mrs. DOCTERS VAN LEEUWEN-REYNVAAN: "On a double reduction of the number of chromosomes during the formation of the sexual cells and on a subsequent double fertilisation in some species of *Polytrichum*". (Communicated by Prof. F. A. F. C. WENT), p. 359.

Erratum, p. 366.

Physiology. — *“On the segmental skin-innervation by the sympathetic nervous system in vertebrates, based on experimental researches about the innervation of the pigment-cells in flat fishes and of the pilo-motor muscles in cats.”* By Dr. G. VAN REINBERK. (Communicated by Prof. C. WINKLER.)

(Communicated in the meeting of October 26, 1907.)

We possess numerous, though dispersed, data, obtained either by means of experiments on animals or founded on clinical observations, tending all to confirm the opinion, that in vertebrates and in man, the efferent nervefibres, intended for the skin, which are conducted by the grey connecting branches from the lateral column of the N. sympathicus towards the mixed spinal nerves, are distributed within the area of the skin that is supplied with afferent fibres by the spinal nerve. As moreover, in general, save slight deviations, the efferent sympathetic fibres of the grey connecting branches have their origin in the ganglia of the column in which these branches apparently originate, we may assume that the zones of the skin, innervated by the ganglia of the sympathetic column are nearly identical in their distribution with the zones of the skin, supplied by the different corresponding spinal ganglia. Less numerous are the data about the relative extent of the sympathetic and spinal zones of the skin: but indirect indications apparently support the view that the zones of the skin innervated by the spinal ganglia are more extensive than the zones of the different corresponding sympathetic ganglia. With this reservation however we apparently may hold it very probable, that the innervation of the skin both by the sympathetic and by the spinal ganglia is taking place according to the self-same morphological scheme. Hitherto nevertheless no direct proofs have been given by demonstrating on the self-same object the relative distribution and extent of these innervation-areas. It has been my purpose to do this now by means of a few simple experiments.

A. *The sympathetic innervation of the pigment-cells and the spinal innervation for sensibility of the skin in flat fishes.*

Since the elaborate researches of G. POUCHET¹⁾ we know that in several species of fishes the phenomenon of the variability of colouring

¹⁾ G. POUCHET. Des changements de coloration sous l'influence des nerfs. — Journal de l'anatomie et de la physiologie. Tome 12 p. 1—90, and p. 113—165, Paris 1876.

in the skin is directly influenced by the sympathetic nervous system. If in a turbot the connecting branches of some spinal nerves or these nerves themselves, in that upward turned half of the body containing the eyes, are cut through, there appears on the skin a more or less sharply defined dark zone. POUCHET considered this phenomenon to be caused by a paralysis of the pigment-cells in consequence of the section of the nerves, and he called the dark zones appearing after section, "paralytic" zones. He made however no further researches as to the significance of these zones, when considered as innervation-areas of sympathetic ganglia, and since, to my knowledge, nobody has taken up again these yet so extremely interesting researches. I have done so at the present time, and added unto this a comparative investigation about the sensible innervation of the skin.

For objects I got numerous specimens of *Solea* (*impar, vulgaris, monochir*) and *Rhomboidichthys* (*mancus seu podas*). This latter species in particular, and likewise *Solea impar*, have furnished me with excellent results, and the more detailed demonstration is principally based on experiments made on these animals. The operative part of these experiments was very simple. By a longitudinal incision, cleaving skin and muscles, and passing along the lateral line of the organ of sense in the ventral portion of the skin of the caudal part of the pigmented half of the body bearing the eyes, the origins of a few haemal vertebral spinous processes were laid bare and the

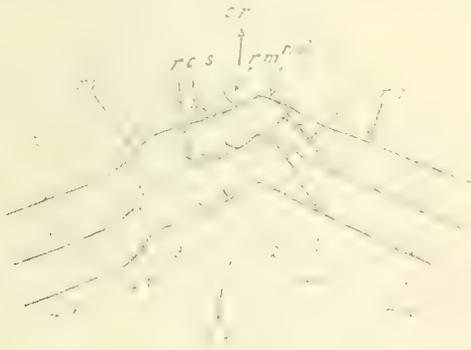


Fig. 1.

Scheme of course and distribution of the main trunks of the spinal nerves in the caudal portion of the Pleuronectidi (taken from a preparation of *Rhombus faevis*), 1, body of vertebra, 2, neutral spinous process, 3, haemal spinous process. $4'd, 4''d, 4'v, 4''v$, first and second longitudinal septum of the dorsal and ventral muscles, — cr, ca , cranial and caudal boundaries of the preparation. — $r.d., r.m., r.v.$, ramus dorsalis, medius and ventralis of the spinal nerves — $r.c.d., r.c.s.$, ramus communicans and ramus spinosus of the dorsal nerve-trunks. — $r.c.s.$, left sympathetic connecting branch.

ventral branches of the spinal nerves were sought. In most cases these were caught up and torn off together with the connecting branches of the N. sympathicus.

In all cases the visible consequence of these operations consisted constantly in the appearance on the skin of a more or less sharply defined dark field, i. e. darker than the surrounding skin. Distribution and extent of these dark fields were dependent on the place in the segmental arrangement of the sectioned nerves and on their number. The shape of these fields was always identical, being that of a band, beginning in the dorsal marginal fin, going somewhat obliquely cranialward to the lateral line of the organ of sense, and thence somewhat obliquely caudalward towards the ventral marginal fin, wherein it terminated. Thus much for the shape and the general distribution of these zones. As regards their extent, the following may be stated. After destroying the connecting branch of one single spinal nerve I never observed any plainly visible change in the colouring of the skin. After destroying the connecting branches of two consecutive nerves, usually a narrow, not very dark zone was observed, that might be not easily defined. Only when three consecutive branches were destroyed, there appeared a plainly visible, sharply defined dark zone.



Fig. 2. 1)

Rhomboidichthys mancus, dark zone appearing after cutting through three spinal nerves and the sympathetic connecting branches.

If more than three branches were destroyed, there was found a dark zone, identical as to shape and position, only broader. If after a first section of viz. three branches, still another couple of branches,

1) This figure and fig. 3, 4, 6, 7, 8, 9 are reproductions of photographs counterdrawn in outline.

lying next to these first ones either cranially or caudally, were destroyed, the originally observed dark zone was afterwards constantly found uniformly broadened, either the cranial boundary being removed cranialward or the caudal boundary being removed caudalward, according to the case. By these means a series of indications was furnished, tending to prove that the skin-areas supplied with pigmento-motor fibres by each connecting branch or by the ganglia of the sympathetic column, are themselves likewise uninterrupted, zone-shaped fields. Still further data on this subject were obtained in the following manner.

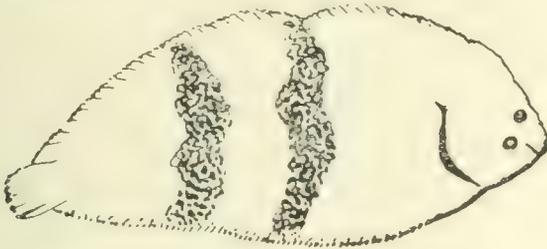


Fig. 3.

Solea impar. Isolation of four spinal nerves between four nerves cut through cranially and four other ones cut through caudally of them.

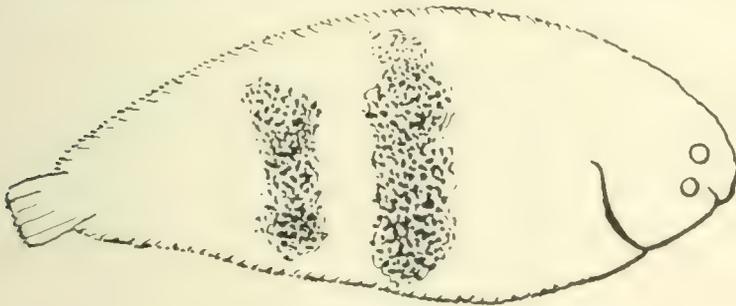


Fig. 4.

Another solea, on which a similar operation had been made.

If a few, viz. four connecting branches were destroyed, and again also four other ones cranially or caudally from these, leaving intact e. g. four branches between the two, two dark zones appeared of about equal breadth, enclosing between them a somewhat broader zone of lighter colouring, corresponding to the uninjured branches. (Fig. 3 and 4). By means of similar experiments the supposition that the ganglia of the sympathetic column innervate zone-shaped skin-areas becomes nearly a certainty. Some results too were obtained as to the extent of these areas. Comparative calculations, as shown before, starting from measurements of the darker and lighter zones, made with as much accuracy as was possible, have shown that the cranial-

caudalward breadth of a skin-area innervated by a ganglion of the sympathetic column, having an average length of 20 cm. may be approximated at 7 mm., and that the areas overlap one another somewhat more than half.

The foregoing having been duly stated, a comparison between the scheme of the spinal and that of the sympathetic innervation of the skin lay very near indeed. Once the ventral (and dorsal) branches of a couple of spinal nerves having been cut through together with the sympathetic connecting branches, it is easy enough to define the extent and the distribution of the insensible zone of the skin resulting from this operation, and to establish a comparison between these and those of the dark zone. In order to facilitate this definition, I augmented the irritability for reflex actions in the animals by intoxicating them with a small quantity of a solution of the sulphuric salt of strychnia in sea-water (1 : 10,000). After this a slight scratching of the skin by means of a pin's point was sufficient to produce a plainly visible general reaction, making it possible to define the boundaries between the sensible and insensible areas with great precision. I found the results of a series of experiments to be nearly invariable, so that I may communicate them here with sufficient certainty.

Generally then the anaesthetic areas and the dark zones, observed after the section of spinal nerves and their corresponding sympathetic connecting branches are found to accord completely as regards their extent, distribution and arrangement. Consequently the pigmento-motor sympathetic fibres, originating in a certain ganglion of the N. sympathicus and its connecting branch, are distributed precisely within that area of the skin that is supplied with sensory fibres by the corresponding spinal ganglion. Both schemes therefore cover one another completely, and the above given particulars about the innervation of the pigment-cells, holds good likewise for the sensory innervation of the skin. Thus the central innervation of the skin in Pleuronectidae is divided into a series of segmental areas, which considered in their functional significance, may be distinguished in sensory and pigmento-motor skin segments, but according completely as regards their distribution and extent.

B. The sympathetic innervation of the pilo-motor muscles and the spinal sensory innervation of the skin in cats.

The well-known researches of LANGLEY (1893)¹⁾ have shown that

¹⁾ J. N. LANGLEY. — Preliminary account of the arrangement of the sympathetic nervous system, based chiefly on observations upon pilomotor nerves. Proceedings

the sympathetic nerve-fibres, intended for the pilo-motor muscles of the skin of the trunk in cats, originate in the series of ganglia of the column of the N. sympathicus, that they are conducted along the grey connecting branches towards the relative corresponding spinal nerves, thence following the primary dorsal nerve-trunks and the (dorsal) skin branches of these, to terminate in the pilo-motor muscles of the dorsal portion of the skin. Besides he has demonstrated that by far the greater number of the nerve-fibres, originating in the sympathetic ganglion, the "pilo-motor" nerves as he called them were conducted along the selfsame grey connecting branch towards the one spinal nerve segmentally corresponding with it, and that along the dorsal skin-branch or branches of this nerve, they jointly reach the skin, where they are distributed within one uninterrupted area, that may be sharply defined. He found further more, that these skin-areas, supplied with pilo-motor nerve-fibres by the series of sympathetic ganglia form a regular series, arranged on both sides of the mid-dorsal line of the body. As regards the relation between the innervation of the skin by fibres for the pilo-motor muscles from the sympathetic ganglia, and the innervation by sensory fibres from the spinal ganglia, he confined himself to comparing the arrangement of the pilo-motor skin-areas innervated by the sympathetic ganglia with the results of the researches made by TÜRK and SHERRINGTON about the spinal innervation in the dog and the monkey. Direct comparisons between the sensory and the pilo-motor innervation of the skin were not made by him. These have been made recently by me.

The way in which to do this was clearly indicated. At present, especially after the anatomical studies of BOLK on man, we may take it for granted, that there does not occur an interchange of nerve-fibres destined for the skin between the spinal nerves in the trunk-area in mammalia. Consequently the serially arranged skin-branches of the dorsal portion of the body represent separately the different spinal and sympathetic nerve-fibres intended for the dorsal portion of the skin of the trunk from the spinal nerves and sympathetic connecting branches in which they originate. In order therefore to obtain a knowledge of the innervation of the dorsal skin-portion relatively by the spinal and by the sympathetic ganglia, it is sufficient to define separately and then to compare the different areas of dis-

of the R. Society of London, vol. 52, n°. 320, p. 547—556 Februari 1893. London.
 J. N. LANGLEY. The arrangement of the sympathetic nervous system based chiefly upon pilomotor nerves. *Journal of Physiology* (Foster) vol. 15 n°. 3 p. 176—244, 1893. Cambridge.

tribution of the pilo-motor and of the sensory fibres having their course in the dorsal nerve-branches of the skin. This may be done in a very simple way.

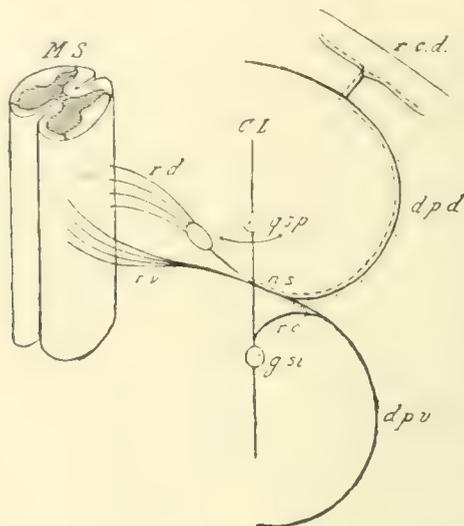


Fig. 5.

Scheme of the course of the (post ganglionic) pilomotor- and of the sensory nerve-fibres toward the skin of the trunk-area in cats.

M.S. = medulla spinalis. — *r.d. r.v.* = dorsal and ventral root. — *N.S.* = mixed spinal nerve. — *d.p.d.-d.p.v.* = dorsal and ventral trunk of the spinal nerve. — *r.e.d.* = dorsal ramus cutaneus. — *C.L.* = lateral column of the N. Sympathicus. — *g.sp.-g.si.* = ganglion spinale, sympathetic ganglion. — *r.c.* = grey connecting branch — — — — define the course of the spinal sensible fibres . . . that of the pilomotor fibres.

I obtained this scheme by defining first by means of SHERRINGTON'S method of isolation the surface of the sensible area innervated by a certain skin-branch and next by the stimulus of an induced current applied to the same branch, causing the surface of the skin-area, innervated by the pilomotor fibres from this branch, to become visible. In order to do so, the hair on the trunk of the cats I made use of, were first cut uniformly by means of a so-called tondeuse to a length of about half a c.M. Afterwards, under narcotics and with aseptic precautions (as far as possible, the skin not being shorn) a longitudinal incision was made in the skin along the mid-dorsal line, and the skin was folded back to both sides. The connective tissue having been prepared the series of dorsal skin-nerves was in most cases pretty plainly distinguishable, and it was very easy to choose a definite branch for isolating and to section the three branches lying next to this one both cranialward and caudalward, either after

having loosened them from the adjacent blood-vessels that mostly follow the same course, or else together with the bloodvessels between a double ligature. The skin was then sketched and the animal was allowed a few days quiet. After this the sensible and the insensible areas to be found in the skin were defined and their boundaries carefully indicated by means of coloured demarcation-lines. Finally the animal was again brought under narcosis, the incision in the skin was reopened and the isolated nerves were laid bare and stimulated. The area-field, on which the hair was rising, was demarcated by another colour.

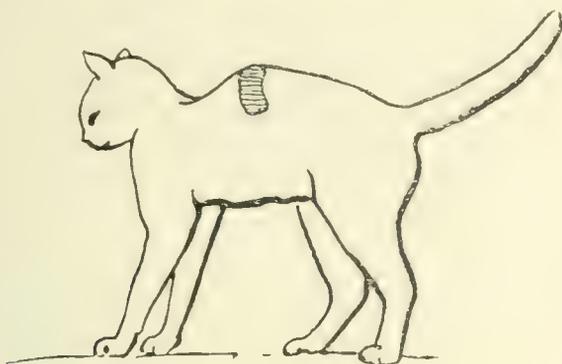


Fig. 6.

Pilomotor area of the 7th thoracic nerve.

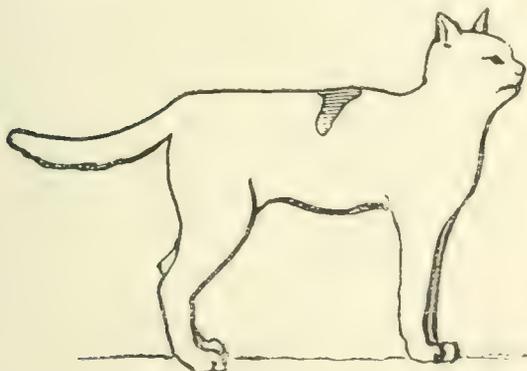


Fig. 7.

Pilomotor area of the 8th thoracic nerve in the same cat.

As regards the pilomotor nerves I can be short, as I have hardly anything to add to the very accurate communications of LANGLEY on this subject. Like him I found in my experiments that the areas in which during the irritation of different skin-branches and on different animals the hair rose, showed rather important

differences, both as to their extent, shape and boundaries and as to the intensity of the phenomenon itself. Usually the field, in which the hair rose, was nearly rectangular, and save for a slight deviation caudal-ward, it was lying vertically on the mid-dorsal line. In the most successful experiments the pilo-motor areas extended nearly unto the dorsal axilla-inguinal line over a dorso-ventral surface of almost 60 m.M. The cranio-caudal breadth amounted on the average to 26 m.M. For an instance of the proportions of the pilomotor areas in an exceptionally favourable case I refer to the photographs represented in fig. 6 and 7.

The isolated sensible areas usually presented a shape not greatly different from that of the pilomotor areas described above. Like these they were generally nearly rectangular, lying almost vertically on the mid-dorsal line, and they showed likewise a slight deviation caudal-ward, perhaps even somewhat more marked. Cranially and caudally they were bounded by the insensible areas; ventrally they passed without any distinct boundaries into the lateral part of the body, where sensibility was retained wholly intact. The cranio-caudal breadth of the sensible areas was on the average 30 m.M., their dorso-ventral extent of course was not to be defined; that of the insensible areas was on the average 60 m.M.

We may now pass on to a comparison between the sensible and the pilomotor skin-areas. On account of what I remarked before about the variability both as to shape and extent of these latter ones, it may be inferred already that the results of this comparison presented likewise great differences. On one important point however the results of all my experiments are in accord: the pilomotor skin-area was always to be found within the sensible area of the isolated nerve-branch. In this respect the principal problem I had put before me in all my experiments, may be considered to have been solved, at least for that portion of the skin of the trunk on which I made my experiments. As regards further the relative extent of the sensible and of the pilomotor skin-areas, and the exact situation of the latter within the former, I found, as remarked before, great differences. Sometimes the pilomotor field area had an extent nearly equal to that of the sensible field, both fields being consequently almost identical. In the majority of cases however the pilomotor skin-area was less extensive in all directions than the sensible area. The place, occupied by the pilomotor field within the sensible field differed greatly in different cases. Generally it was lying almost in the midst of it, as is shown in the cats, photographs of which are represented in fig. 8 and 9.

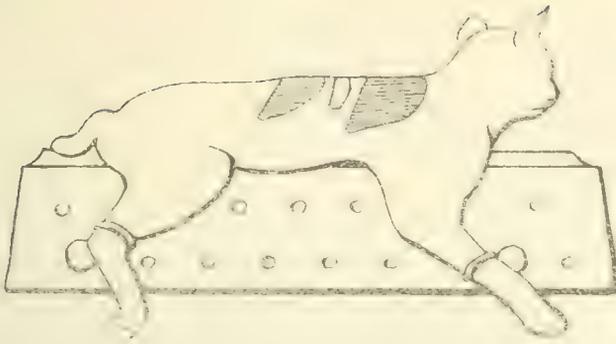


Fig. 8

Situation of the pilomotor skin-area (white) within the sensible area nearly isolated by insensible areas (hatched transversally).

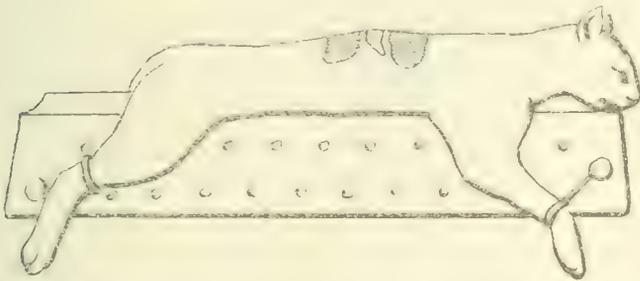


Fig. 9.

The same in another cat.

In another case however it lay nearer to the cranial or caudal boundary of the sensible area, I have not been able to state a definite rule in this respect.

Returning now to the principal problem aimed at by my researches, we find that from the above statements it has become evident that the pilomotor nerve-fibres and the sensory fibres having their course in the dorsal skin-branches of the skin of the trunk in cats, are distributed within areas of the skin that are in accordance as to situation and arrangement but not as regards their extent. Thence it follows that the sympathetic ganglia and the spinal ganglia innervate the skin after the same scheme, and although the relations in cats are less simple than those found in flat fishes, still I believe that here likewise the scheme of the pilomotor innervation of the skin by means of the marginal column of the sympathetic nervous system may be called a "segmental" scheme.

Physics. — “*Some remarks on the expansion of platinum at low temperatures*”. By Prof. H. KAMERLINGH ONNES and J. CLAY. Supplement N^o. 17 to the Communications from the Physical Laboratory at Leiden.

(Communicated in the meeting of September 28, 1907).

The communication from the “Physikalisch-technische Reichsanstalt” by K. SCHEEL in the meeting of Jan. 11, 1907 of the “Deutsche physikalische Gesellschaft” led us to make a remark already in the Meeting of June 29, 1907 (These Proc. Sept. 1907 p. 200). In Communication N^o. 95^b (These Proc. Sept. '06 p. 199) we had given a quadratic formula for the expansion of platinum *below* 0°, from which followed that, as was remarked in the Introduction of that Communication, a formula of the third degree is required if we wish to represent the expansion of platinum from -180° to $+100^{\circ}$ by one polynomial with increasing powers of t , and if we have to deal with observations which if repeated a sufficient number of times, allow us to reach an accuracy (comp. § 1 of Comm. N^o. 85, June '03, These Proc. April '05) of $\frac{1}{200}$ in the expansion. We found this confirmed by the measurements of SCHEEL, who arrived at the same result by determining a quadratic formula for the expansion of platinum *above* 0°, and by measuring the length at -190° .

We now consider the striking difference of the expansion at low temperatures according to the formula given by us, and that according to SCHEEL's formula, viz.: 43μ for the expansion of a bar of 1 meter between -183° and $+16^{\circ}$, (cf. SCHEEL loc. cit. p. 19, note 1), a difference much greater than could be accounted for by the inaccuracy of the observations.

For an explanation of this discrepancy we call attention to the difference of the observations of Dec. 16 1904 and Febr. 3, 1905 in Table II of Comm. N^o. 95^b, which give as length of the platinum bar provided with the two glass extremities, at $16^{\circ 1)$ before it had ever been reduced to low temperature, 1027.460 m.m., and a long time after it had been reduced to low temperature for the last time, 1027.457 m.m., mean 1027.458 m.m., with that of Dec. 19, 21 and 23 in the same table which yield the mean value 1027.441 mm. (from 1027.441, 1027.442 and 1027.440) for the length at 16° , which was observed on return to the ordinary temperature a day after

1) In Table II of this communication under $L_{16^{\circ}}$ for the ordinary temperatures the length of the bar at 16° reduced on the measuring rod at 16° has been given and not the length at \mathcal{S} as in the tables of Comm. N^o. 85.

the cooling. Indeed this former mean value is 17μ larger than the latter.

Now this difference of 17μ , which refers to a bar of platinum of 840 mm. (for a bar of 1 M. it would be 20μ) exceeds the errors which may be ascribed to the inaccuracy of the observation by about half the difference which exists between SCHEEL's formula and our formula of June 1906.

As basis for the calculation of our formula the mean¹, of the two lengths has been taken. We arrive at values for the expansion nearer to those of SCHEEL when for the length at the ordinary temperature we take that which was found immediately after cooling, instead of the mean of this length and the length which was found long before and after the cooling, as was done in the calculation of our formula of June 1907. If we now make use of the first-mentioned length, that which was found immediately after cooling, in order to find the coefficients now distinguished by (a) and (b) from the former α and β in the formula:

$$l_t = l_0 \left(1 + \left\{ (a) \left(\frac{t}{100} \right) + (b) \left(\frac{t}{100} \right)^2 \right\} 10^{-6} \right)$$

we find:

Platinum (—183° to +16°)	}	(a) 877.7	KAMERLINGH ONNES
		(b) 35.7	and CLAY (1905)
		whereas	
		(a) 861.5	SCHEEL (1906)
		(b) 37.0	

It is true that the now remaining difference of 34μ per M. with an expansion of —183° to +16° remains considerably larger than the accuracy of the observations would lead us to expect, but it is considerably smaller than that found originally, and taking into consideration the different sources of uncertainty whether we observe really what we think we observe, the small number of measurements, and the difference of the methods applied at low temperatures for the first time, it is not great.

We had hoped to obtain further information on the difference in length of our bar at ordinary temperature immediately after the cooling and long after it, but have not yet been able to do so.

Differences as the one discussed now have more occurred in our measurements. We have pointed this out in Comm. N^o. 95^b and

¹) In the calculations for the glass the values of the length immediately after the cooling, Dec. 23 in Table I, and April 15 and 16 in Table III, have been left out of account in connection with the further observations.

for glass we have expressly investigated the possibility of thermal hysteresis on cooling to the lowest temperatures. In connection with what has been said in Comm. N^o. 95^b we fear that for the above treated difference an irregularity in the behaviour of the place of fusion of the glass points to the platinum bar has played a part, to prevent which further experiments ought to be made with still greater care. If what we now think probable, is verified, observations in which a difference as the one considered just now, manifests itself, should be rejected.

Besides the formula of the second degree for temperatures below 0°, we have also calculated a formula of the third degree

$$l_t = l_0 \left[1 + \left\{ (a') \frac{t}{100} + (b') \left(\frac{t}{100} \right)^2 + (c') \left(\frac{t}{100} \right)^3 \right\} 10^{-6} \right]$$

for the expansion of platinum between -183° and $+80^\circ$ by the aid of BENOIT'S observations from 0° to $+80^\circ$, in which formula (a'), (b'), (c') refer to the length at the ordinary temperature immediately after the cooling.

The agreement of

Platinum	}	$+ 80^\circ$	(a') 875.3	}	BENOIT and KAMERLINGH ONNES and CLAY (1905)	
		$- 183^\circ$	(b') 31.6			(c') - 1.49
Platinum	}	$+ 100^\circ$	(a') 874.9	}	SCHEEL (1906)	
		$- 190^\circ$	(b') 31.41			(c') - 6.94

is pretty satisfactory. Substitution of SCHEEL'S values for those of BENOIT would bring about only a slight change in the first group of coefficients.

Anatomy. — *“On the Development of the Corpus callosum in the human Brain.”* By Prof. J. W. LANGELAAN. (Communicated by Prof. T. PLACE).

The points that at this moment seem of interest in the history of the development of the corpus callosum have been clearly formulated by RERZIUS¹⁾ in the form of questions. Two of these are: 1. Where does the corpus callosum originate? 2. Of what

¹⁾ Rerzius. Das Menschenhirn. Stockholm 1896. p. 6.

elements is it composed at its first appearance? The third question of RETZIUS has been amplified by ZUCKERKANDL¹⁾ and may be formulated as follows: what are the changes occurring in the mesial wall of the pallium in consequence of the development of the corpus callosum?

For the answering of the first question a human embryo of the beginning of the fourth month was at my disposal. The fronto-occipital diameter of the corpus callosum amounted to but 0.5 m.m. Figure I shows a frontal section through the more posterior part of the lamina terminalis.

The plane of section deviating a little from the frontal plane, that which is shown in the right part of the drawing is more frontally placed than that which is shown in the left part.

As appears from the drawing the corpus callosum lies in the lamina terminalis; especially on the left this is clearly evident, where the underborder of the pallium goes over into a taenia (*T*) which is bent in and passes over into the lamina terminalis (*L.t.*). The fact that the ependyma of the lamina terminalis, which is continued into the ependyma of the taenia, also spreads underneath the corpus callosum, obviates all doubt as to the existence of this relation. If now the sections are examined more frontally, it will be seen, that the more frontal part of the corpus callosum no longer lies in the lamina terminalis. This part of the corpus callosum exceeds the limits of the lamina and is situated in the zone of union of the mesial walls of the pallium. This zone is built up of glia-tissue and in immediate continuity with the glia-layer covering the fore-side of the lamina terminalis.

On the ground of this observation I believe that the corpus callosum originates in the lamina terminalis, very soon, however, in consequence of the enlargement of the commissure, preponderantly in a frontal direction, it encroaches on the lamina and lies partially in the zone of union of the pallia.

Another embryo, of the middle of the fourth month, exhibits a corpus callosum with a maximum diameter of 2.5 m.m. Here the commissure is still entirely situated in front of the foramen Monroi. Figure II shows a frontal section through the more posterior part of the corpus callosum. In this section the corpus callosum (*C.c.*) lies most dorsally, laterally going over into the mesial wall of the pallium. In this wall, aside from the callosum, we find the fornix

¹⁾ ZUCKERKANDL. Sitzb. K. Acad. der W. Math. Naturw. cl. Bd. CX. h. VIII. Wien 1901. p. 234.

(*F.*) which, in the mesial wall of the pallium, is not clearly distinguishable from the corpus callosum. Downwards, the fornix may be followed as far as the anterior commissure (*Ca.*). In the angle, where callosum and fornix meet, lies a bundle of fibres (*Ps.*) ventrally from the corpus callosum and coming from behind. This bundle crosses in the middle-line another bundle of the same kind coming from the opposite direction. This crossing-system is the fornix-commissure. More frontally this commissure is wanting and only the callosum and the fornix are present in the relation I just now described.

From the topographical relation of the corpus callosum to the fornix-commissure the deduction may be made that the more posterior part of the callosum is equivalent to the splenium. In the same way it follows from the relation of the callosum to the fornix-bundle that the more anterior part of the callosum corresponds with the genu of that structure. The origin of the corpus callosum therefore comprises the whole commissure, and consequently the growth of the corpus callosum does not take place by means of the apposition of new systems of fibres, but by an equable enlargement in correspondence with the growth of the pallium.

The most preponderant change in the structure of the mesial wall of the pallium at the place of origin of the callosum consists in the cortex-layer bending a little inward and ending with a sharp edge. The middle-layer of the wall of the pallium gets richer in nuclei; these nuclei surround the callosum and the fornix like a cap. Along the lower edge of the cortex-layer they penetrate into the marginal-zone of the wall of the pallium. By this process the marginal-zone disappears as a separate layer.

In the zone of union of the mesial walls of the pallium the changes in the structure of this wall are more considerable; the observation, that the most mesial bundles of the fornix pass through the glia-tissue of this zone of union, seems of importance here, as from this fact may be derived, that the re-constructed mesial wall of the pallium — the later septum lucidum — comprises more than the original mesial wall.

Fig. I.

Frontal section of the more posterior part of the lamina terminalis. Section 20 μ stained with haematoxylin and eosin. Enl 16.5 diam.

Ca. Anterior commissure.

C.c. Corpus callosum.

C.ch. Corpus chorioideum.

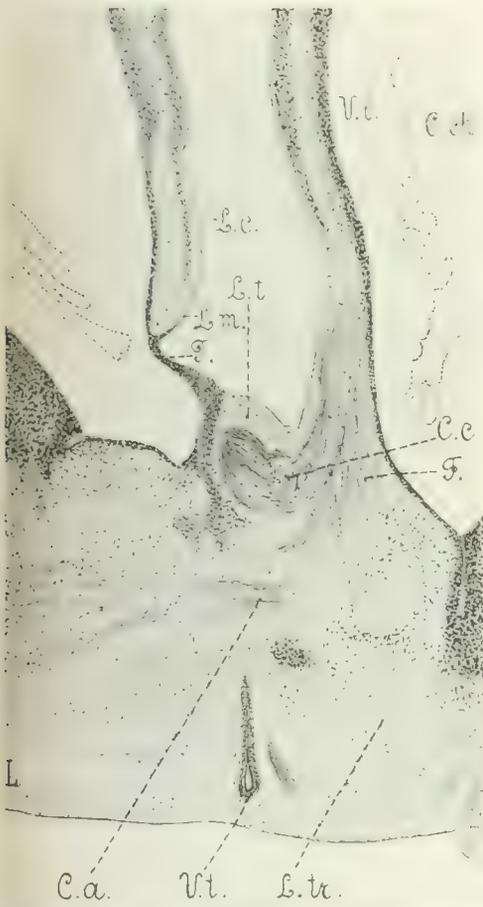
L.t. Lamina terminalis.

L.tr. Lamina trapezoidea.

T. Taenia.

Fig. II.

Fig. I.



F. Fornix.*V.l.* Lateral ventricle.*L.c.* Limbus corticalis.*V.t.* Third ventricle.*L.m.* Limbus medullaris.

Fig. II.

Frontal section through the more posterior part of the corpus callosum. Section 15 μ stained with haematoxylin and eosin. Enl. 13 diam.

C. Zone of union of the pallia.*L.c.* Limbus corticalis.*C.a.* Anterior commissure.*L.m.* Limbus medullaris.*C.c.* Corpus callosum.*L.tr.* Lamina trapezoidea.*C.ch.* Corpus chorioideum.*Ps.* Fornix commissure.*C.str.* Corpus striatum.*V.l.* Lateral ventricle.*F.* Fornix.*V.t.* Third ventricle.

Mathematics. — “On an infinite product, represented by a definite integral.” By Prof. W. KAPTEYN.

The object of this paper is to write the infinite product

$$\prod_{s=0}^{\infty} \left(1 + \frac{v^2}{(u+s)^2} \right)$$

in the form of a definite integral.

This product is connected with *mod.* $\Gamma(u + iv)$, for

$$\text{mod. } \Gamma(u + iv) = \Gamma(u) \cdot e^{-P(u,v)} \quad (u > 0)$$

where

$$P(u,v) = \frac{1}{2} \sum_{s=0}^{\infty} \lg \left(1 + \frac{v^2}{(u+s)^2} \right) \quad ^1)$$

thus

$$\text{mod.}^2 \quad \Gamma(u + iv) = \frac{\Gamma^2(u)}{\prod_{s=0}^{\infty} \left(1 + \frac{v^2}{(u+s)^2} \right)}$$

and

$$\prod_{s=0}^{\infty} \left(1 + \frac{v^2}{(u+s)^2} \right) = \frac{\Gamma^2(u)}{\text{mod.}^2 \Gamma(u + iv)}$$

To write the second member of this equation in the form of a definite integral, we start from WEIERSTRASS' definition

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_W e^t t^{-z} dt$$

where the integral is taken along a curve W commencing at negative

¹⁾ Nielsen. Handbuch der Theorie der Gammafunctionen p. 23.

infinity, circulating around the origin in the positive direction, and returning to negative infinity again; thus

$$\frac{2\pi i}{\Gamma(z)} = e^{\pi iz} \int_0^{\infty} e^{-t} t^{-z} dt - e^{-\pi iz} \int_0^{\infty} e^{-t} t^{-z} dt$$

and if $z = u + iv$

$$\begin{aligned} \frac{2\pi i}{\Gamma(u+iv)} &= e^{-\pi v} \left(\cos(\pi u) + i \sin(\pi u) \right) \int_0^{\infty} e^{-t} t^{-u} \left(\cos(vlg t) - i \sin(vlg t) \right) dt \\ &\quad - e^{\pi v} \left(\cos(\pi u) - i \sin(\pi u) \right) \int_0^{\infty} e^{-t} t^{-u} \left(\cos(vlg t) - i \sin(vlg t) \right) dt. \end{aligned}$$

Writing

$$\begin{aligned} \int_0^{\infty} e^{-t} t^{-u} \cos(vlg t) dt &= M \\ \int_0^{\infty} e^{-t} t^{-u} \sin(vlg t) dt &= N \\ \frac{2\pi}{\Gamma(u+iv)} &= \alpha + i\beta \end{aligned}$$

we obtain

$$\alpha = (e^{\pi v} + e^{-\pi v}) \sin(\pi u) M + (e^{\pi v} - e^{-\pi v}) \cos(\pi u) N$$

$$\beta = (e^{\pi v} - e^{-\pi v}) \cos(\pi u) M - (e^{\pi v} + e^{-\pi v}) \sin(\pi u) N$$

and

$$\alpha^2 + \beta^2 = (e^{2\pi v} - 2 \cos 2\pi u + e^{-2\pi v}) (M^2 + N^2).$$

Now we have

$$M^2 = \int_0^{\infty} e^{-x} x^{-u} \cos(v'g x) dx \cdot \int_0^{\infty} e^{-y} y^{-u} \cos(vlg y) dy$$

$$N^2 = \int_0^{\infty} e^{-x} x^{-u} \sin(vlg x) dx \cdot \int_0^{\infty} e^{-y} y^{-u} \sin(vlg y) dy$$

so

$$M^2 + N^2 = \int_0^{\infty} \int_0^{\infty} e^{-(x+y)} (xy)^{-u} \cos\left(vlg \frac{y}{x}\right) dx dy$$

or in polar coordinates, putting

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$M^2 + N^2 = \int_0^2 \int_0^{\infty} e^{-r(\cos \theta + \sin \theta)} (r^2 \sin \theta \cos \theta)^{-u} \cos(vlg \tan \theta) r dr d\theta.$$

This double integral may be reduced to a single one, for

$$\int_0^{\infty} e^{-r(\cos \theta + \sin \theta)} r^{-2u+1} dr = \frac{\Gamma(2-2u)}{(\cos \theta + \sin \theta)^{2-2u}} \quad (u < 1)$$

therefore

$$M^2 + N^2 = \Gamma(2-2u) \int_0^{\frac{\pi}{2}} \cos(vlg \, tg \, \theta) \frac{(\sin \theta \cos \theta)^{-u}}{(\cos \theta + \sin \theta)^{2-2u}} d\theta$$

or

$$M^2 + N^2 = 2\Gamma(2-2u) \int_0^{\frac{\pi}{4}} \cos(vlg \, tg \, \theta) \frac{(\sin \theta \cos \theta)^{-u}}{(\cos \theta + \sin \theta)^{2-2u}} d\theta.$$

If in this integral, we change the variable by the substitution

$$tg \, \theta = e^{-2t}$$

it takes the form:

$$M^2 + N^2 = 4 \Gamma(2-2u) \int_0^{\infty} \frac{\cos(2vt) dt}{(e^t + e^{-t})^{2-2u}}$$

With this value we find

$$\frac{4\pi}{\text{mod}^2 \Gamma(u+iv)} = 4 \Gamma(2-2u) (e^{2\pi v} - 2 \cos 2\pi u + e^{-2\pi v}) \int_0^{\infty} \frac{\cos(2vt) dt}{(e^t + e^{-t})^{2-2u}}$$

and finally

$$\prod_{s=0}^{\infty} \left(1 + \frac{v^2}{(u+s)^2}\right) = \frac{\Gamma^2(u) \Gamma(2-2u)}{\pi^2} (e^{2\pi v} - 2 \cos 2\pi u + e^{-2\pi v}) \int_0^{\infty} \frac{\cos(2vt) dt}{(e^t + e^{-t})^{2-2u}}$$

which holds for all values of v , and for values of u between 0 and 1.

If for instance we put $v = \frac{z}{2\pi}$, $u = \frac{1}{4}$ and $\frac{3}{4}$ we obtain

$$\begin{aligned} \left(1 + \frac{4z^2}{\pi^2}\right) \left(1 + \frac{4z^2}{25\pi^2}\right) \left(1 + \frac{4z^2}{81\pi^2}\right) \dots &= \\ &= \frac{\Gamma^2\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{2}\right)}{\pi^2} (e^z + e^{-z}) \int_0^{\infty} \frac{\cos\left(\frac{zt}{\pi}\right) dt}{(e^t + e^{-t})^{\frac{3}{2}}} \end{aligned}$$

and

$$\begin{aligned} \left(1 + \frac{4z^2}{9\pi^2}\right) \left(1 + \frac{4z^2}{49\pi^2}\right) \left(1 + \frac{4z^2}{121\pi^2}\right) \dots &= \\ &= \frac{\Gamma^2\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{2}\right)}{\pi^2} (e^z + e^{-z}) \int_0^{\infty} \frac{\cos\left(\frac{zt}{\pi}\right) dt}{(e^t + e^{-t})^{\frac{5}{2}}} \end{aligned}$$

Writing $u = 1 - u'$ we may also conclude from the preceding that

$$\frac{1}{\text{mod}^2 \Gamma(1 - u' + iv)} = \frac{\Gamma(2u')}{\pi^2} (e^{2\pi v} - 2 \cos 2\pi u' + e^{-2\pi v}) \int_0^\infty \frac{\cos(2vt) dt}{(e^t + e^{-t})^{2u'}}$$

or, because

$$\Gamma(u' + iv) \Gamma(1 - u' + iv) = \frac{\pi}{\sin \pi (u' + iv)}$$

$$\text{mod}^2 \Gamma(u' + iv) = 4 \Gamma(2u') \int_0^\infty \frac{\cos(2vt) dt}{(e^t + e^{-t})^{2u'}}$$

which formula holds not only for $0 < u' < 1$, but also for $u' > 1$.

Introducing in this equation, the infinite product, we have

$$\int_0^\infty \frac{\cos(2vt) dt}{(e^t + e^{-t})^{2u'}} = \frac{\Gamma^2(u')}{4 \Gamma(2u')} \frac{1}{\prod_{s=0}^\infty \left(1 + \frac{v^2}{(u' + s)^2}\right)}$$

a formula which enables us to write the integral in a finite form in two cases viz. $u' = n$ and $u' = n - \frac{1}{2}$. If $u' = n =$ positive number

$$\prod_{s=0}^\infty \left(1 + \frac{v^2}{(n + s)^2}\right) = \prod_{s=n}^\infty \left(1 + \frac{v^2}{s^2}\right);$$

with

$$\frac{e^{\pi v} - e^{-\pi v}}{2\pi v} = \prod_{s=1}^\infty \left(1 + \frac{v^2}{s^2}\right)$$

this gives

$$\int_0^\infty \frac{\cos(2vt) dt}{(e^t + e^{-t})^{2u}} = \frac{\pi v \Gamma^2(n)}{2 \Gamma(2n)} \frac{\prod_{s=1}^{n-1} \left(1 + \frac{v^2}{s^2}\right)}{e^{\pi v} - e^{-\pi v}}.$$

If $u' = n - \frac{1}{2}$, we have

$$\prod_{s=0}^\infty \left(1 + \frac{v^2}{(n - \frac{1}{2} + s)^2}\right) = \prod_{s=n-1}^\infty \left(1 + \frac{v^2}{(\frac{1}{2} + s)^2}\right)$$

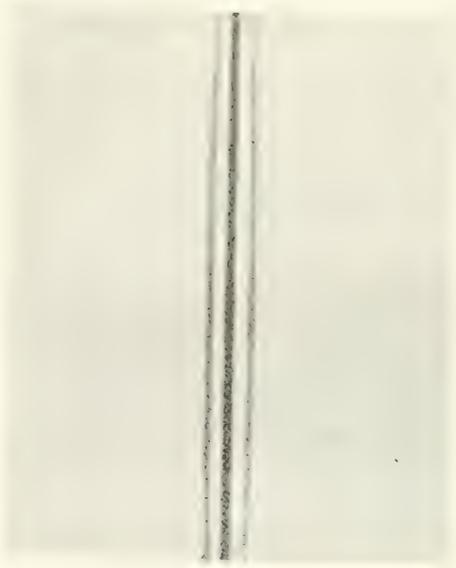
which gives with

$$\frac{e^{\pi v} + e^{-\pi v}}{2} = \prod_{s=0}^\infty \left(1 + \frac{v^2}{(\frac{1}{2} + s)^2}\right)$$

this result

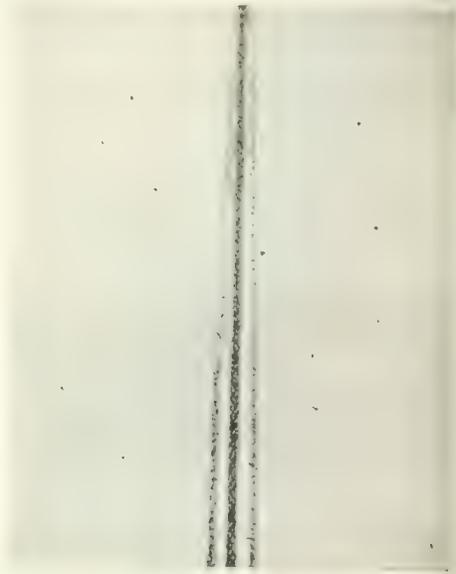
$$\int_0^\infty \frac{\cos(2vt) dt}{(e^t + e^{-t})^{2n-1}} = \frac{\Gamma^2(n - \frac{1}{2})}{2 \Gamma(2n - 1)} \frac{\prod_{s=0}^{n-2} \left(1 + \frac{v^2}{(\frac{1}{2} + s)^2}\right)}{e^{\pi v} + e^{-\pi v}}.$$

Fig. 2.



5791
Enlargement of middle part of
5791. Fig. 1.

Fig. 3.



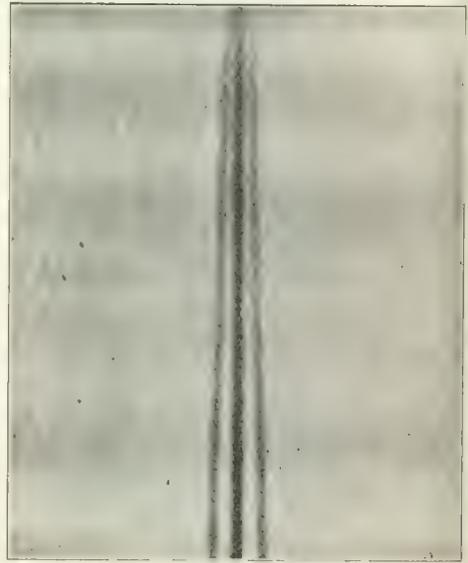
5791
Enlargement of point of
5791. Fig. 1.

Fig. 4.



5770
Enlargement of middle part of
5770. Fig. 1.

Fig. 5.



5770
Enlargement of point of
5770. Fig. 1.

Physics. — “*Magnetic resolution of spectral lines and magnetic force.*” By Prof. P. ZEEMAN. (Second part). ¹⁾

Asymmetry in strong fields.

2. By means of *the method of the non-uniform field*, described in the first part of this communication, it is possible to survey at one glance a phenomenon dependent upon the intensity of the magnetic field for a series of different intensities, all other circumstances surely being the same.

I there proposed to use this method for a more minute study concerning an asymmetry of the resolution of spectral lines first predicted from theory by VOIGT ²⁾ and lately considered by LORENTZ ³⁾ from another point of view.

The theoretical result of VOIGT, applying to the case of resolution into a *triplet*, may be given in his own words: “*dass das normal zu den Kraftlinien wahrnehmbare Duplet der parallel zu R [magnetic force] polarisirten Componenten bei kleineren Feldstärken in der Weise unsymmetrisch ist, dass die nach Rot liegende Componente die grössere Intensität, die nach Violett hin liegende aber den grösseren Abstand von der ursprünglichen Absorptionslinie besitzt.*” VOIGT here mentions an absorption line because he considers the so called inverse effect, by reason however of the parallelism of the phenomena of emission and absorption, the emission lines show analogous phenomena.

The amount of the asymmetry of the distances, i. e. the difference of the distances of the outer components from the middle line, ought to be on VOIGT's theory independent of the strength of the magnetic field. Moreover it is to be inferred that the described asymmetry must be scarcely observable.

On a former occasion ⁴⁾ I have given some examples of asymmetrical resolution and measurements since published by other physicists undoubtedly go far towards confirming these results.

A more minute investigation of the course of the magnetic separation, when the scale of field intensities from large to small values is traversed, is I think still of great theoretical interest. The most interesting parts of the scale are of course the very strong and the weak fields.

¹⁾ Continued from Proceedings of April 1906.

²⁾ VOIGT. Ann. d. Phys 1. p. 376. 1900.

³⁾ LORENTZ. These Proceedings November, December 1905.

⁴⁾ ZEEMAN. *ibid.* December 1899. Archiv. Néerl. (2) T. 5. 237—242. 1900.

The most striking example of asymmetrical resolution that I know of, occurs in the case of one of the yellow mercury lines (5791). The structure of a line like this one cannot be made out by means of MICHELSON'S interferometer. Indeed the assumption of symmetry, which is, as has been proved by Lord RAYLEIGH¹⁾, necessary to deduce the structure from the visibility curve in this case certainly is unjustified.

3. Following the method described in the first part of this paper I have made some experiments concerning the mentioned spectral line in strong fields. For the ROWLAND grating used in my observations I am indebted to the dutch Society of Sciences at Haarlem. Presently I hope to give an account of results obtained in weak fields by means of an interference method.

The grating has 10.000 lines to the inch and a radius of curvature of 6.5 M., the divided part being of 14 cm. width. In the use of my method the grating necessarily should be mounted in such a manner that to every point of the slit corresponds only one point of the spectral image. ROWLAND'S concave grating can be mounted in a non-astigmatic manner as has been remarked by RUNGE and PASCHEN²⁾ and this arrangement was made use of in former investigations by myself,³⁾ HALLO and GEEST.⁴⁾

All observations recorded in the present paper were made with the spectrum of the first order.

4. Whereas the mercury line 5791 is resolved asymmetrically, the neighbouring line 5770 is resolved by the magnetic field into a perfectly symmetrical triplet, or at least very approximately so. I have used this circumstance for applying the optical method of measurement of field (see § 1), the mentioned yellow lines being easily photographed simultaneously.

Fig. 1 represents a ninefold enlargement of one of the negatives. According to measurements of FABRY and PEROT the difference of wavelength of the yellow mercury lines is $5790.66 - 5769.60 = 21.06$ A.U., hence 1 m.m. in Fig. 1 corresponds to 0.551 A.U. Inspection of Fig. 1 clearly shows that line 5791 is asymmetrically resolved. Perhaps this is still more evident in the enlargements Figures 2 and 3 of parts of Fig. 1.

1) RAYLEIGH. Phil. Mag. November 1892.

2) RUNGE and PASCHEN. Wied. Ann. Bd. 61. p. 641. 1897.

3) ZEEMAN. Archiv. Néerl. (2) T. 5. 237. 1900, T. 7. 465. 1902. These Proc. May 1902, May 1903, Dec. 1904.

4) HALLO. Archiv. Néerl. (2) T. 10. p. 148., GEEST. (2) T. 10 p. 291. 1905.

Our object of investigation is the relation between asymmetry and strength of field.

The measurements were made in the following way. The negative was placed on the comparator, in such a manner, that the middle line of one of the triplets was contained between the two parallel wires in the reading microscope. The parallel wires had been placed previously at right angles to the direction of motion of the negative. It appeared that if with one of the triplets the desired coincidence had been obtained, this was also the case with the other. An extra system of cross wires, crossing under an angle of about 50° , was used in the measurements and made it possible to determine the resolution in the selected point of the lines.

The resolution of one line having been measured for a definite value of the magnetic force, the corresponding resolution in the corresponding point of the second line was determined immediately afterwards.

The line 5770 appeared to be divided almost exactly symmetrically, so that the resolution could be taken as a measure of the magnetic force.

On the obtained negatives 34 series of measurements were made. They relate to different points of 10 negatives made at different times.

The vacuum tubes used were intentionally made somewhat dissimilar.

In order to control the results the negatives were taken with different maximum intensity of field.

Finally the negatives obtained can be distributed into two groups, differing by the position of the grating. After taking 7 negatives I resolved to rotate the grating in its own plane through 180° in order to see whether this had some influence on the asymmetry.

This appeared to be not the case, but the apparent distribution of intensities changed in a remarkable manner. Whereas in one position of the grating figures 1—5 were obtained, the middle line being strong and the outer components rather weak, the distribution of light after rotation became reversed. In this position of the grating Fig. 1 of my last paper was taken. (See these Proceedings October 1907). The middle line is very weak and the outer components predominate.

5. The results of the measurements are dealt with in the following way. The amounts of separation of line 5791 towards the red and towards the violet are supposed to be functions of the separation of line 5770, which may be treated as proportional to the magnetic force. The separations of line 5770 may be taken as abscissae, the two other separations as ordinates.

Groups of four or five single proximate results simply were combined by assigning to each mean abscissa the mean ordinate.

The 2×7 principal values thus obtained are given in the first three columns of the following table.

Mean separation 5770	Separation 5791		Asymmetry	Intensity of field in Gauss
	towards red	towards violet		
270	234	259	25	14800
328	283	312	29	18020
362	313	345	32	19860
399	353	388	35	21910
440	394	431	37	24140
453	404	442	38	24880
532	475	523	48	29220

All these differences of wavelength are given in thousandths of an ÅNGSTRÖM unit.

The fourth column in like manner gives the amount of the asymmetry.

6. The last column contains the field intensity in Gauss. In calculating it I have assumed proportionality between separation and magnetic force.

Increasing accuracy of the measurements has furnished continually increasing arguments for this proportionality and the investigations of FÄRBER¹, WEISS and COTTON²), PASCHEN³) and STETTENHEIMER⁴) have given a high degree of certainty to this simple law.

The numbers in the fifth column are deduced from those in the first by means of the separations of line 5770 $+ 0.414$ and $- 0.415$ given by RUNGE and PASCHEN for the field used in their investigation.

The measurements of RUNGE and PASCHEN concerning the mercury lines refer, as Prof. PASCHEN has kindly communicated to me, to a field of 22750 Gauss according to measurements made in his laboratory by Frl. STETTENHEIMER, and of 22780 Gauss according

¹) FÄRBER. Diss. Tübingen, 1902; Ann. d. Phys. **9**, 886, 1902.

²) WEISS and COTTON. Journal de Physique. Juin 1907.

³) PASCHEN. Physik. Zeitschr. 8 Jahrgang N^o. 16. 522, 1907.

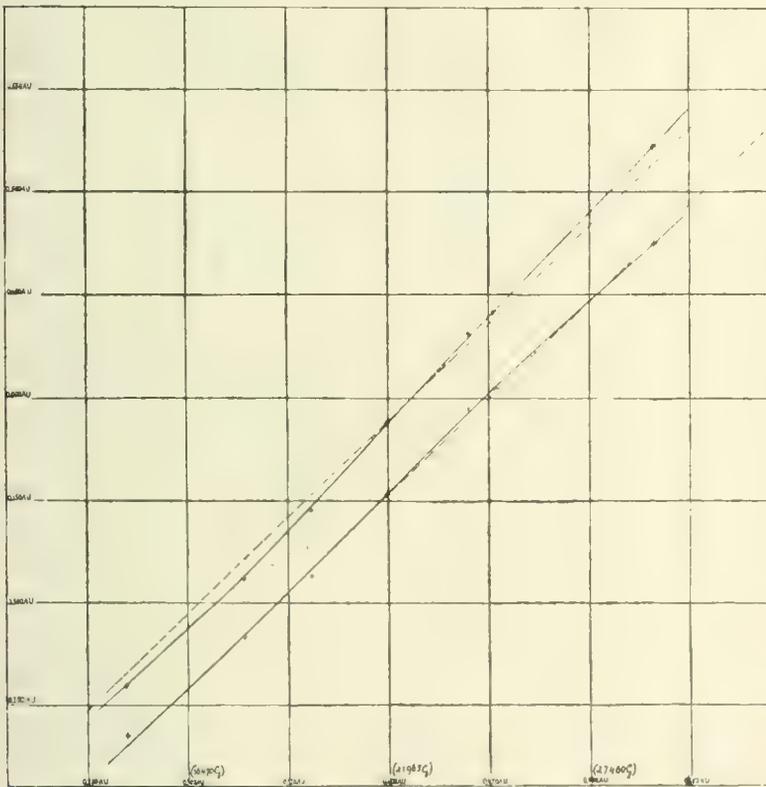
⁴) STETTENHEIMER. Diss. Tübingen, 1907; Ann. d. Phys. **24**, 384 1907.

to measurements, not hitherto published, by GMELIN. In reducing my observations I have therefore taken 22765 as the value of the magnetic force belonging to the mean of the two numbers mentioned for the separation.

7. The results are graphically represented in the following diagram. The abscissae are the separations of line 5770 in A.U., and the corresponding field intensities in Gauss, the ordinates the corresponding separations of 5791. The small crosses represent the observations of the table in § 5.

The full freehand lines give the best mean result of the observations. (See § 9).

The signification of the upper dotted straight line is the following.



The mean of the 34 single values of the asymmetry is 0.036 A.U. The straight dotted line has been traced parallel to the lower line at a distance of 36 thousandths of an ANGSTROM unit, measured along the ordinate.

8. We may infer from our observations that with the fields used, from 15000 to 30000 Gauss, an asymmetry undoubtedly exists which

to say the least bears a very striking resemblance to the one deduced from theory by VOIGT.

In both cases, the theoretical one and that following from our experiments, there is a difference of the distances between the central line and the two outer components, in this sense that the component towards the red is nearer to the middle line than that towards the violet, just as predicted by theory.

There exists also an asymmetry of *the intensities* of the outer components in the sense indicated by theory.

An inspection e. g. of the original negative of which Fig. 1 is a nine-fold reproduction, or of the reproduction Fig. 1, or better of reprints on photographic paper of the 29-fold enlargement given in Fig. 2 or even of that figure reveals the existence of a very small asymmetry of intensity. This is perhaps most clearly seen by looking at the figure from a not too small distance, covering the central line with a small strip of paper. No trace of asymmetry can be seen in the triplet of line 5770, see also the enlargements Fig. 4 and Fig. 5 of the middle and outer parts of the right of Fig. 1.

On the other hand there seems to be a difference between theory and observation in this respect, that the amount of asymmetry appears to be not constant. The table of § 5 and the graphical representation clearly indicate that when the magnetic force decreases from 30000 to 15000 Gauss the asymmetry also is nearly halved.¹⁾

An error of an amount sufficient to bring a single point of the upper line on the dotted one is not absolutely excluded (see § 9). [For the right part of the diagram the error ought to be three times the probable error of *one single* of the principal values (see § 5) and would happen therefore on the average in one out of every twenty three cases].

We have however reasonable security against a combination of errors which would move all the points of the full line to the dotted one.

Of course we cannot deduce from the now determined part of the upper line whether or not it will approach asymptotically to a finite distance of the lower one.

9. We may now consider the question as to the best fitting *straight* lines to our two systems of points.

Measuring the divergencies at right angles to the line the best fit will be obtained if we make the sum of the squares of the perpen-

¹⁾ An excellent series of measurements made after the writing of this article gives a somewhat lower rate of decrease, the mean value of the asymmetry being the same.

diculars from the system of points to the line a minimum. The line thus determined will be the principal axis of inertia of the system of points²⁾).

Performing this calculation we find that the best fitting lower line passes through the point with the coordinates 398, 351, at a slope determined by $\theta_1 = 43^\circ 6'$. For the upper line these numbers become 398, 386, whereas $\theta_2 = 45^\circ 35'$.

In order to judge of the accuracy obtained in the representation of the observations by these straight lines the following table may serve. The third and fourth column, resp. the sixth and seventh column contain the errors of the abscissae and ordinates of the two point systems to be assumed, if the straight lines are supposed correct.

Mean separation 5770	Separation 5791 towards red	Δa_1	Δo_1	Separation 5791 towards violet	Δa_2	Δo_2
270	234	+ 1.6	- 1.8	259	+ 1.4	- 1.3
328	283	- 1.2	+ 1.4	312	- 1.6	+ 1.6
362	313	- 2.1	+ 2.2	345	- 2.0	+ 2.0
399	353	+ 0.1	- 0.1	388	0	0
440	394	+ 1.3	- 1.5	431	+ 1.0	- 0.9
453	404	+ 0.2	- 0.2	442	0	0
532	475	- 1.0	+ 1.2	523	0	0

It appears from this table that the lines completely represent the observations, if we admit mean uncertainty of 0.0013 A. U. in the observations concerning line 5770 and of 0.0014 A. U. resp. of 0.0011 A. U. in the determination of the components towards the red resp. towards the violet in the case of line 5791; we must admit these as appears from the distribution of deviations.

10. The position of one point of each line may still be checked by the observations of RUNGE and PASCHEN. They give for the separation in the case of line 5770 towards the red resp. towards the violet $+ 414 \pm 1.7$ resp. $- 415 \pm 1.7$, whereas for the same magnetic

²⁾ KARL PEARSON. On Lines and Planes of closest Fit to Systems of Points in space. Phil. Mag. p. 559. Vol. 2. 1901. Here we read: "The best fitting straight line for a system of points in a space of any order goes through the centroid of the system" cf. KEESOM. These Proceedings 31 May, 1902.

force these numbers for line 5791 become $+366 \pm 6.7$ resp. 399 ± 6.7 , the values preceded by \pm indicating the mean error. According to our observations to the abscissa 415 correspond the ordinates 368 and 403, hence a very good agreement.

11. From the extremely small amount of the asymmetry viz. 0.036 A. U. one might infer after comparison with the width of the spectral lines in our figures that the asymmetry is only a small part of the real width of the line. Such a conclusion would however be too rash.

It is true that from our figures and from their originals follows an apparent width of the outer components of about 0.190 A. U. The negative of Fig. 1 was not taken however with extremely narrow slit; but with a width of slit of 0.08 m.m. Other photographs taken with a width of slit of 0.02 m.m. gave a somewhat smaller apparent width of the spectral line as the first result.

To be sure however of the real width of the line, which is of some importance here, I made an independent determination by means of an echelon spectroscope of high resolving power, the mercury tube being under the same circumstances, as in the experiments under review. The width of the spectral line appeared to be the $\frac{1}{11}$ th part of the distance of two successive orders of the echelon. In the vicinity of the yellow mercury lines this distance is 0.694 A. U.

hence the width of these lines is about $\frac{0.694}{11} = 0.063$ A. U.

We may compare this result with a value we may deduce from results obtained by MICHELSON. MICHELSON'S analysis ¹⁾ by means of the interferometer shows that in a field of 10000 Gauss the *whole* separation of the yellow mercury lines is 0.36 A. U. From his diagram on pag. 354 l. c. we infer that the width of the spectral line was under the circumstances of the case one fourth part of the separation or 0.09 A. U.

Hence taking a mean value for the width of 0.07 A. U. we conclude that the asymmetry amounts to about one half of the width of the line or at any rate that width and asymmetry are of the same order of magnitude.

¹⁾ MICHELSON. Phil. mag. Vol. 45, p. 348. 1898.

EXPLANATION OF PLATES II AND III.

Plate II. Fig. 1. The figure is an enlargement (about nine-fold) of the original negative. The yellow mercury lines 5791 and 5770 in a non-uniform field. 1 mm. in figure is 0.551 A. U.

Plate III. Fig. 2—5 enlarged 29 times after the original.

Fig. 2. Middle part of line 5791 in Fig. 1. } asymmetrical

Fig. 3. Point of line 5791 in Fig. 1. } separation.

Fig. 4. Middle part of line 5770 in Fig. 1. } symmetrical

Fig. 5. Point of line 5770 in Fig. 1. } separation.

The letters *r* and *v* indicate the parts towards the red and towards the violet ends of the spectrum.

Botany. — “*On a double reduction of the number of chromosomes during the formation of the sexual cells and on a subsequent double fertilisation in some species of Polytrichum.*” By Dr. W. DOCTERS VAN LEEUWEN and Mrs. J. DOCTERS VAN LEEUWEN-REYNVAAN. (Communicated by Prof. F. A. F. C. WENT).

In 1904 there appeared an investigation by IKENO¹⁾ on spermatogenesis in *Marchantia polymorpha*. Since then quite a number of researches on this subject have been carried out with liverworts. Here and there an occasional reference to the true Mosses has been made, but, as far as we are aware, nothing has been published on their spermatogenesis since the appearance of IKENO's paper.

The older publications, e. g. those of GUIGNARD²⁾ and of STRASBURGER³⁾, treat exclusively of the final changes of the spermatids to spermatozoids. For this reason we began the present investigation in 1904 soon after the publication of IKENO's memoir; we obtained results, differing so widely from the ordinary conceptions, that we investigated, not only the spermatogenesis, but also the development and the fertilisation of the ovum.

The material was fixed at a suitable time, mostly in the field, by a sublimate mixture, and was afterwards stained with iron-haematoxylin according to HEIDENHAIN. We used *Polytrichum piliferum*, *P. juniperinum* and *P. formosum*. It is our intention to give a more detailed account of the work and of the methods which we have employed, in the *Recueil des Travaux Botaniques Néerlandais*.

IKENO made the remarkable discovery that in the antheridial cells, immediately before division, a small round body passed out of the

1) IKENO *Beihfte zum Botan. Centralblatt*. Bd. 16, 1903.

2) GUIGNARD. *Revue gén. de botanique* I, 1889.

3) STRASBURGER. *Hist. Beitr.* Heft IV, 1892.

nucleus into the cytoplasm and divided itself into two parts, which during mitosis wandered to the poles of the spindle, like true centrosomes. Since then these centrosomes have been found again in many other liverworts, but according to some observers, they seem to be absent from the divisions in the antheridia of *Pellia*.

At the diaster stage the centrosome disappeared, only to emerge again from the nucleus at a subsequent division. At the last division only of the antheridial cells it remained in the cytoplasm and was transformed into the blepharoplast. For this reason IKENO considers the blepharoplast of the liverworts to be homologous with the centrosome.

Many arguments, both for and against this view, have afterwards been advanced, which we will not discuss further. Without a detailed review of the literature such a discussion would scarcely be possible.

Our results with *Polytrichum* agree in part with those published by IKENO and others, but also at the same time differ from them in certain respects.

I. *On the growth of the antheridial cells and on spermatogenesis.*

In the antheridia the cells are closely packed. The nuclei are spherical and contain at their centre a substance, which is deeply stained by iron-haematoxylin. We do not propose to discuss whether this is, or is not, a nucleolus. There is no agreement in the literature on this point and different investigators designate by nucleolus the most widely different structures. It is, however, usual in botanical literature to call such a body a nucleolus, though it also takes up the other chromatin stains very readily. However this may be, the dark mass lies in the middle of the nucleus, and slightly more towards the periphery there is, in addition, another fairly large, black corpuscle.

If nuclei are examined in various stages of rest and mitosis, those, which are furthest removed from their next division only show the central black mass. A little later the corpuscle also appears, at first connected to the central mass by a thin black thread. Soon this connexion disappears and the corpuscle approaches more and more the nuclear membrane. After some time it emerges from the nucleus and remains imbedded in the cytoplasm, in contact with the nuclear membrane. The corpuscle which was at first round, now becomes rod-shaped and afterwards undergoes constriction in the middle, thus assuming a dumb-bell shape. It subsequently divides into small spheres, which move along the nuclear membrane and which become more

and more widely separated. At this stage each little corpuscle is surrounded by a light border, which becomes especially noticeable, when the corpuscles separate from the nucleus.

During mitosis the two corpuscles are found at the tops of the spindle and hence may well be called centrosomes. There are, especially during division, many black granules in the cytoplasm. This always renders the investigation more difficult, but in the first place the two centrosomes are larger than the other granules and secondly they are surrounded by a lightborder. In the case of animal cells their centrosome nature would not be doubted, but with vegetable cells a certain amount of reserve is still very desirable. We do not hesitate, however, to call these bodies centrosomes. Of course they do not stain well in all mitoses. Any one who has searched for centrosomes in animal tissues, knows, that the staining of these corpuscles is difficult, even in objects which are famous for them. The centrosomes of *Polytrichum* accordingly have a chromatic origin. They originate in the nucleus and divide into two in the cytoplasm.

IKENO describes these corpuscles as disappearing in the diaster stage. This is not the case in *Polytrichum*. They do not remain in their places, but may be found in various cells moving more and more to the other side of the chromosomes, so that at last they lie opposite each other among the spindle threads, which unite the two chromosome masses. When the daughter nuclei have only just been formed and the chromosomes are therefore still more or less clearly visible, the corpuscle lies between them; afterwards everything becomes a black mass. At the last division of the antheridial cells the centrosome is also taken up in the nucleus and there is here no deviation from what is found in liverworts.

After this, such changes begin, as finally lead to the formation of the spermatozoids.

We did not succeed in finding young sporogonia with many young spore mother-cells undergoing division.

We found, however, numerous dividing nuclei in the vegetative cells of young sporogonia and hence it was not difficult to make out the actual number of the chromosomes. The chromosomes are small, but they are sharply differentiated and may be especially well recognised in the equatorial plane. We found that the cells of the sporogonium have 12 chromosomes.

Judging from analogy with what is known of liverworts and vascular cryptogams, it was safe to assume that in the formation of the spores a reduction of the chromosomes would take place, and

that their number in the gametophyte would hence be six. We did in fact, always find six chromosomes in the cells of the antheridium and in those of the female plants.

When the antheridia have arrived at the end of their development, the chromosomes assume a different appearance. At first they are, relatively to their length, fairly thick rodlets. At the last but one division, however, they have the same length, but become much thinner and are no longer so smooth. At this stage six chromosomes can always still be clearly observed. At the last division, and therefore immediately before the actual formation of the spermatozoids, three of the six chromosomes go to one pole and three to the other pole.

The nuclei of the spermatozoids therefore do not contain six, but three chromosomes, i. e. a quarter of the number contained in the nuclei of the vegetative generation.

The cells, in which the reducing division has taken place, and which therefore are about to develop into spermatozoids, may be recognized by their almost invisible cell wall and by their beginning to round themselves off. The nucleus has again a central mass of chromatin, which is, however, appreciably smaller than that in the younger cells.

This mass again extrudes a chromatin granule in the usual manner which travels to the periphery and then emerges from the nucleus. The corpuscle arises therefore in the same way as the centrosome in the cells undergoing division. It does not, however, divide but goes at once to the periphery of the cell. Meanwhile a piece is again separated from the mass of chromatin in the nucleus, and this time the part separated off is so large, that it is often almost equal to the remainder. At first the two portions remain connected but afterwards they become completely separated and finally the part of the chromatin which has been split off, wanders out of the nucleus.

IKENO also describes in the changed spermatids of *Marchantia* the occurrence of a chromatin body by the side of the nucleus, when the centrosome has already quite reached the periphery.

Where it comes from, he does not know, nor what subsequently happens to it; he only says that the organ disappears again later and calls it "chromatoide Nebenkörper", which name we may retain.

Having arrived outside the nucleus, it changes its shape in *Polytrichum* and extends itself to a bent rodlet. This rodlet grows further, till at last it becomes a closed circular body. Afterwards it again becomes indistinct and in subsequent stages it can only be seen as a dotted ring, which finally disappears completely. We have not been able to discover anything about the significance of this body.

Meanwhile the centrosome has also changed its form. It has become somewhat longer and more or less cudgel-shaped. At the obtuse end a thin band then becomes visible, which goes along the periphery of the cell. The progressive differentiation of this band starts from the centrosome and proceeds in the direction of the nucleus. The latter has also travelled to the periphery, at that side, which is opposite the centrosome.

A similar band, which extends from the blepharoplast to the nucleus, has also been described by IKENO. According to him it originates in the cytoplasm and is stained in the same way as the latter, but more intensely.

In our preparations which were stained with iron-haematoxylin, it is very clearly visible and sharply marked out in black, but a difference from the staining of the chromatin may nevertheless be observed.

While this band slowly grows out and the "chromatoïde Nebenkörper" disappears, a quantity of chromatin is separated off for the third time from the chromatin mass of the nucleus. This time however, only a very small body is formed, which also emerges from the nucleus, but mostly remains very close to the nuclear membrane; the latter can only be seen very indistinctly.

In a somewhat later stage the band extends along half the circumference of the cell and has therefore nearly reached the nucleus. The third chromatin body is found at the end of the band and in contact with the nucleus, so that in the spermatozoid it lies between the band and the modified nucleus.

The changes which the nucleus itself undergoes in the formation of the spermatozoid have already been described in detail by STRASBURGER and others; it seems to us therefore unnecessary to investigate this matter further.

II. *Development of the Ovum and Fertilisation.*

In the young archegonia the mother cell of the ovum is especially large. During the further development of the archegonium this cell divides into two and thus gives rise to an ovum and a ventral canal-cell. A point of difference from many other mosses is, that in the species of *Polytrichum*, which we have examined, the two cells are of exactly the same size. These cells now round themselves off and then lie loose in the venter of the archegonium. The venter increases in size and the cells which have been rounded off, separate from each other, till one lies at the base of the venter, and the

other close to the first of the neck canal-cells. Meanwhile these latter degenerate, i.e. their walls disappear and they become somewhat rounded, so that they lie detached in the neck.

The top of the neck opens and through the opening the neck canal-cells pass out. This could be seen in living specimens with mature archegonia; when they were placed in water, the neck soon opened and the cells appeared one by one.

At the stage when the neck-cells have become separated and the neck itself is about to open, a large number of the neck-cells may be found, in fixed preparations, in the venter of the archegonium. They lie loose round the ovum and the ventral canal-cell.

The ventral canal-cell now approaches the ovum and applies itself to the latter. No demarcation between the cytoplasm of the two cells can then be observed. The two nuclei lie side by side and gradually fuse. This was observed by us several times and in all the successive stages. The rest of the ventral canal-cell shrivels up and is extruded like the neck canal-cells.

Finally the ovum lies by itself in the venter with a large, normal, round nucleus.

A transformation of the neck canal-cells into mucilage, as described by GAYET¹⁾ and others, does not occur. Mucilage may indeed be found later in the neck, and may serve to attract spermatozoids, but it is probably secreted by the neck-cells themselves.

It now became of great importance to know the number of chromosomes in the nucleus of the ovum. Unfortunately, as has already been remarked, only a very limited number of nuclear divisions can be found in the tissues of Mosses, (except in the antheridia) and hence most of the ova were either in a stage before, or in a stage after that of nuclear division. In the other dividing cells of the archegonium there were always six chromosomes. At the stage immediately preceding mitosis, the nucleus of a young cell, which after division would form an ovum, showed six pieces of chromatin. Happily we found one very good mitotic stage. Here there was a large spindle, parallel to the axis of the archegonium, which proved that we had lighted on the division of the egg mother-cell. There were six chromosomes, and though they were still in contact with each other in pairs by one end, the other end was already directed to the top of the spindle. It was highly probable, that of the six chromosomes three were going to one and three to the other pole. The discovery of a nuclear fusion also leads to the supposition, that

¹⁾ L. GAYET. Ann. des Sc. nat. Bot. Série 8, T. III, 1897.

the number of chromosomes was again doubled in the ovum, which now awaited fertilisation in the venter; the latter already communicated with the exterior.

These two discoveries seem to us to justify the conclusion that before fertilisation the ovum contains six chromosomes.

We next had to attempt to find how there could be again twelve chromosomes after fertilisation. There were six chromosomes in the ovum and three in the spermatozoid. If fecundation were to take place in the ordinary manner, there would still only be nine chromosomes, whereas the sporogonia have twelve.

For this purpose we fixed and cut several hundred female plants of *Polytrichum*. There was of course only a small chance, that a given plant would contain a fertilized archegonium, fixed at the right moment. In a number of cases we found, however, the desired stages and so now possess a fine series of preparations, illustrating in regular succession, the fertilisation process from the penetration and modification of the spermatozoid onwards.

The number of spermatozoids which enter the venter of the archegonium is sometimes very great, but after some have penetrated into the ovum, the others no longer closely surround the ovum, but lie more in the direction of the neck. Hence it would appear that here also the fertilized egg exerts a repulsive action.

The youngest stage, which we now possess, and which has been observed several times, shows near the periphery, but without a doubt imbedded in the cytoplasm of the ovum, two spermatozoids; their length, their shape, everything agrees with that view.

In a later stage both are in contact with the nucleus; they have become thicker and shorter. This thickening and shortening proceeds until there are two oblong corpuseles, containing a few dark granules in their interior, and lying against the nucleus.

We also found a few examples of the next stage, namely an ovum which clearly showed three nuclei, each with a thick mass of chromatin, and another ovum, in which the demarcation between the three nuclei was no longer so obvious; the circumference of the nucleus in the latter case was, however, still indented and inside the nucleus there were three dark masses of chromatin. We find therefore, that the cells of the sporophyte contain twelve chromosomes, that those of the gametophyte have six, and that the spermatozoids have three. The ovum has again six chromosomes after fusion with the ventral canal-cell, and after fecundation by two spermatozoids there are once more twelve chromosomes.

E R R A T U M.

In the Proceedings of the meeting of September 1907.
p. 201 l. 8 from the bottom: for 300.000 read 30.000.



(December 24, 1907).

KONINKLIJKE AKADEMIE
VAN WETENSCHAPPEN
-:- TE AMSTERDAM -:-

PROCEEDINGS OF THE
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(Translated from: Verslagen van de Gewone Vergaderingen der Wis- en Natuurkundige
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KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday December 28, 1907.

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Afdeling van Zaterdag 28 December 1907, Dl. XVI).

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- J. SCHMUTZER: "About the oblique extinction of rhombic crystals". (Communicated by Prof. A. WICHMANN), p. 368.
- S. J. DE LANGE: "On ascending degeneration after partial section of the spinal cord". (Communicated by Prof. C. WINKLER), p. 386. (With 2 plates).
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Crystallography. — "*On the oblique extinction of rhombic crystals.*"

By J. SCHMUTZER. (Communicated by Prof. A. WICHMANN).

(Communicated in the meeting of November 30, 1907).

As late as the year 1901 ALFRED HARKER in a brief communication¹⁾ pointed out the fact that with rhombic crystals the oblique extinction on planes making a small angle with the *c*-axis, is to be neglected only when the angle of the optic axes has no great value. That this was not superfluous is perhaps partly owing to the fact that, in the application of the theoretically deduced results concerning the extinction of crystal-sections mineralogists have confined themselves to monoclinic (and triclinic minerals, preferably to the feldspars.²⁾) It seems, therefore, that the circumstance that rhombic minerals as a rule show an oblique extinction and only exceptionally a straight one, is not laid sufficient stress on, though the fact is of course well-known.³⁾ That is why, even in the younger petrographical literature, it is often alleged, in verification of the rhombic nature of a mineral, that all its sections show a straight extinction, whilst at partly straight, partly oblique extinction of the crystal-sections the monoclinic nature of the mineral is considered to have been proved.⁴⁾ A separation of rhombic and monoclinic pyroxenes, olivine and diopside, zoisite and klinzoisite however on the ground of the character of the extinction is not to be insisted upon; only in case of small axis-angles this characteristic has some value as a criterion. What gave rise to the calculation of the angles of extinction for olivine was that considerable extinctions were found with respect to a particularly well developed pinacoidal cleavage of this mineral, whilst, to compare them with the results obtained from this, I have also made the same calculations for talc.

1) Mineralogical Magazine, XIII, 1903, p. 66—68.

2) MICHEL LÉVY, Ann. d. Mines, (7), XII, 1877, p. 392—471, Abstract Zeitschr. f. Kryst. III, 1879, 217—231; Minéraux des Roches, 1888, p. 9 seq.; FOUQUÉ et MICHEL LÉVY, Minéralogie Micrographique, Paris 1879; A. HARKER, Min. Mag. X, 1893, p. 239—240; G. CÉSARO, Mém. cour. Acad. Roy. Belg. LIV, 1895; DALY, Proc. Americ. Acad. Arts a. Sc. XXXIV, 1899, p. 311—328; A. A. FERRO, Riv. di Min., Padua XX, 1898; Atti Soc. Lig. di Sc. nat. Genova, IX, 1898, Abstract Zeitschr. f. Kryst. XXXII, 1900, 532; VICENTE DE SOUSA BRANDÃO, Comunicações da direcção d. serviq. geol. de Portug. IV, 1901, 13—126.

3) Cf. FOUQUÉ et MICHEL LÉVY, Minéralogie Micrographique, p. 55—57.

4) Cf. LAEROIX, about FOUQUÉ in Contributions à l'étude des gneiss à pyroxène et des roches à wernérite, Bull. Soc. franç. de Minéralogie XII, Paris 1889, p. 328.

Olivine and Talc.

If O be the intersection of the acute bisectrix with the globe of projection $g=1$, A and B the projections of the optic axes, ZO the axis of a zone, from which ZQb represents an arbitrary plane with its pole N , then, according to FRESNEL, the extinction on the plane ZQb , with respect to the zone-axis, is represented by the curve Zc , when the plane cN divides the angle BNA into two equal parts. Suppose we call $\sphericalangle OQ$, the inclination of the plane (N) with regard to the acute bisectrix, x , and the angle of extinction with respect to the zone-axis, $\sphericalangle Zc = y$, then, according to MICHEL LÉVY¹⁾ the value of y can be calculated from the equation:

$$\cot 2y = \cot (aZ + bZ).$$

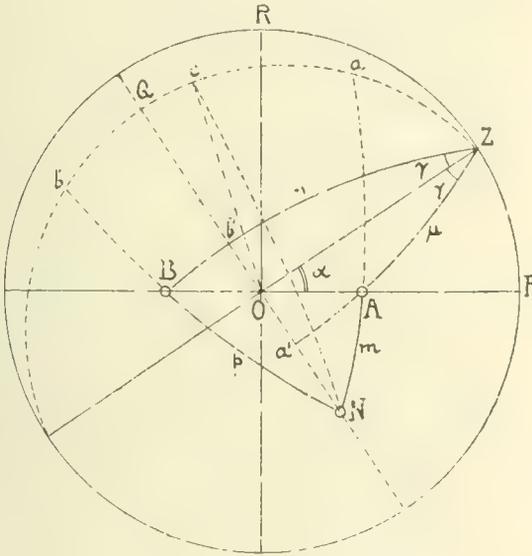


Fig. 1.

$$\sphericalangle aZ = \frac{\pi}{2} - \sphericalangle ANa'.$$

Now in the right-angled $\triangle ANa'$

$$\operatorname{tg} ANa' = \frac{\operatorname{tg} Aa'}{\sin Na'} = \frac{\cot \mu}{\cos (x + \gamma)}$$

so that:

$$\operatorname{tg} aZ = \cot ANa' = \operatorname{tg} \mu \cos (x + \gamma).$$

In the same way we find:

¹⁾ Les Minéraux des Roches, p. 9.

$$tg bZ = tg r \cos (x - \gamma).$$

Now

$$\cot 2y = \cot (aZ + bZ) = \frac{1 - tg aZ tg bZ}{tg aZ + tg bZ} = \frac{1 - tg \mu tg r \cos (x + \gamma) \cos (x - \gamma)}{tg \mu \cos (x + \gamma) + tg r \cos (x - \gamma)}. \quad (1)$$

As is indicated in the figure, here the particular case is considered that the zone-axis lies in the plane forming right angles with the acute bisectrix, so that $\mu + r = \pi$. If we let the zone-axis successively make different angles a with OP , varying from 0 to π , and if, at the same time, we let this plane perform a revolution about this axis, then N passes through the whole surface of the globe and consequently the extinction with regard to OZ can be calculated as a function of a and x for each arbitrary section through the crystal.

As $\mu + r = \pi$, the formula (1) can be simplified as follows:

$$\cot 2y = \frac{1 + tg^2 \mu \cos (x + \gamma) \cos (x - \gamma)}{tg \mu [\cos (x + \gamma) - \cos (x - \gamma)]}$$

from which we derive:

$$\begin{aligned} \cot 2y &= \frac{-(\cos^2 \mu + \sin^2 \mu \cos^2 \gamma) + \sin^2 \mu \sin^2 x}{\sin 2\mu \sin \gamma \sin x} = \\ &= -\frac{\cos^2 \mu + \sin^2 \mu \cos^2 \gamma}{\sin 2\mu \sin \gamma} \cdot \frac{1}{\sin x} + \frac{\sin^2 \mu}{\sin 2\mu \sin \gamma} \cdot \sin x \quad \dots \quad (2) \end{aligned}$$

$$= \frac{A}{\sin x} + B \sin x \quad \dots \quad (3)$$

Now in $\triangle ZOA$ $\cos \mu = \sin OA \cos a = \sin V \cos a$

and in $\triangle ZPA$ $\cos \angle AZP = \frac{tg PZ}{tg AZ}$

or

$$\cos \left(\frac{\pi}{2} - \gamma \right) = \sin \gamma = \frac{tg a}{tg \mu}.$$

If in (2) we substitute the values in a and V for μ and γ , we get:

$$\begin{aligned} \cot 2y &= -\frac{1 - \sin^2 \mu \sin^2 \gamma}{\sin 2\mu \sin \gamma} \cdot \frac{1}{\sin x} + \frac{\sin^2 \mu}{\sin 2\mu \sin \gamma} \cdot \sin x = \\ &= -\frac{1 - tg^2 a \cos^2 \mu}{2 \cos^2 \mu tg a} \cdot \frac{1}{\sin x} + \frac{1 - \cos^2 \mu}{2 \cos^2 \mu tg a} \cdot \sin x = \\ &= \frac{1 - \sin^2 V \sin^2 a}{\sin 2a \cdot \sin^2 V} \cdot \frac{1}{\sin x} + \frac{1 - \sin^2 V \cos^2 a}{\sin 2a \sin^2 V} \cdot \sin x \quad \dots \quad (4) \end{aligned}$$

From this form we can deduce what follows. For $x = 0$, y

becomes $= \frac{\pi}{2}$ for all values of α , and the same thing takes place with $\alpha = 0$ for all values of x . Consequently on all planes parallel to the acute or to the obtuse bisectrix the extinction with respect to these bisectrices is straight. If the direction of the mean index of refraction (OR) becomes zone-axis, with $\alpha = \frac{\pi}{2}$, a certain particularity shows itself. For this value of α (4) assumes the following form :

$$\begin{aligned} \cot 2y &= \frac{1}{\sin 2\alpha} \left(-\frac{1 - \sin^2 V}{\sin^2 V} \cdot \frac{1}{\sin x} + \frac{\sin x}{\sin^2 V} \right) = \\ &= \frac{1}{\sin 2\alpha} \left(-\frac{\cos^2 V}{\sin^2 V} \cdot \frac{1}{\sin x} + \frac{\sin x}{\sin^2 V} \right) \dots \dots (5) \end{aligned}$$

For $x = 0$ becomes $y = \frac{\pi}{2}$.

For $x = \frac{\pi}{2} - V$ (5) changes into :

$$\cot 2y = \frac{1}{\sin 2\alpha} \left(-\frac{\cos V}{\sin^2 V} + \frac{\cos V}{\sin^2 V} \right) = 0.$$

y becomes indefinite; the pole N of the plane at this moment coincides with an optical axis.

Finally for $x = \frac{\pi}{2}$ y becomes $= 0^\circ$. So the extinction is $\frac{\pi}{2}$ for a value of x between 0° and $\frac{\pi}{2} - V$, next becomes indefinite and remains 0° for $x = \frac{\pi}{2} - V$ to $\frac{\pi}{2}$, as the sign for $\cot 2y$ shows.

As to the values of y in general, the following may be observed.

In (4), if $\frac{\pi}{2} > V > 0$ is assumed, is always

$$\begin{aligned} 1 &> 1 - \sin^2 V \sin^2 \alpha > 0 \\ 1 &> 1 - \sin^2 V \cos^2 \alpha > 0 \end{aligned}$$

For a given value of α $\cot 2y$ keeps the same sign, if x varies between 0 and π ; it gets, however, negative values for x between 0 and $-\pi$. If we confine ourselves to a variation of x between the limits 0 and $\frac{\pi}{2}$, then the sign of $\cot 2y$ becomes negative for the

values of α , lying between 0 and $\frac{\pi}{2}$; it becomes positive, however for $\frac{\pi}{2} < \alpha < \pi$, whilst the absolute values of y are equal for two poles, lying symmetrically with regard to the plane RO . The same thing holds good for the extinction on planes, lying symmetrically with regard to the plane OP , so that the isogyres drawn upon the globe will lie symmetrically with respect to the planes RO , OP and also RP . Just as the symmetry with regard to RO and OP is accompanied by a change of sign, so also for the plane RP .

The extinction with regard to the variable zone-axis OZ is easy to reduce to that with respect to the acute bisectrix, as the latter is yielded by $\angle ONc = \sphericalangle Qc = \frac{\pi}{2} - y = y'$.

$$y = \frac{\pi}{2} - y'$$

$$\cot 2y = \cot (\pi - 2y') = - \cot 2y'.$$

from which follows according to (3)

$$\cot 2y' = - \frac{A}{\sin x} - B \sin x \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which:

$$A = - \frac{\cos^2 \mu + \sin^2 \mu \cos^2 \gamma}{\sin 2\mu \sin \gamma}$$

$$B = \frac{\sin^2 \mu}{\sin 2\mu \sin \gamma}.$$

For the determination of the greatest extinction with regard to the acute bisectrix with $x = \text{constant}$ and a variable angle α , we may set about as follows¹⁾. If we call $\angle ANO = \psi$, $\angle BNO = \psi'$, we find from the triangles ANO and BNO

$$\begin{aligned} \text{tg } \psi &= \frac{\sin \angle AON}{\sin ON \cot V - \cos ON \cos \angle AON} = \\ &= \frac{\sin V \cos \alpha}{\cos x \cos V - \sin x \sin V \sin \alpha} \end{aligned}$$

and

$$\text{tg } \psi' = \frac{\sin V \cos \alpha}{\cos V \cos x + \sin V \sin x \sin \alpha}.$$

$$\text{Now } 2y' = \psi - \psi'$$

$$\text{tg } 2y' = \text{tg } (\psi - \psi') = \frac{-2 \sin^2 V \sin \alpha \cos \alpha \sin x}{\sin^2 V \cos^2 \alpha + \cos^2 V \cos^2 x - \sin^2 V \sin^2 \alpha \sin^2 x},$$

which gives:

¹⁾ A. HARKER, Min. Mag. XIII, 1903, p. 66-67.

$$\cot^2 \alpha (\sin^2 V + \cos^2 V \cos^2 x) - 2 \cot \alpha \sin^2 V \sin x \cot 2\gamma' + (\cos^2 x - \sin^2 V) = 0$$

$$\cot \alpha = \frac{\sin^2 V \sin x \cot 2\gamma'}{\sin^2 V + \cos^2 V \cos^2 x} \pm \sqrt{\left(\frac{\sin^2 V \sin x \cot 2\gamma'}{\sin^2 V + \cos^2 V \cos^2 x} \right)^2 - \left(\frac{\cos^2 x - \sin^2 V}{\sin^2 V + \cos^2 V \cos^2 x} \right)}$$

As long as the second term remains smaller than the first, the condition for which being $\cos x > \sin V$, or $x < \frac{\pi}{2} - V$, this equation will yield two positive roots, and accordingly two values of α between 0 and $\frac{\pi}{2}$ will satisfy it at a given value of 2γ smaller than the maximum. The extinction will have reached the maximum, when the two roots are equal, so if

$$(\sin^2 V \sin x \cot 2\gamma')^2 = (\sin^2 V + \cos^2 V \cos^2 x) (\cos^2 x - \sin^2 V)$$

or:

$$\sin 2\gamma'_{max} = \frac{\sin V \operatorname{tg} V}{\cos x \cot x} \dots \dots \dots (7)$$

whilst the corresponding value of $\alpha^1)$ is found from

$$\cot \alpha_{max} = \frac{\sin^2 V \sin x \cot 2\gamma'_{max}}{\sin^2 V + \cos^2 V \cos^2 x} \dots \dots \dots (8)$$

The plane ON , in which lies the corresponding pole, then makes with the plane OP an angle $\left(\frac{\pi}{2} - \alpha \right)$.

If we take for olivine the value $2V = 87^\circ 2)$, this gives according to the above formulas the following figures:

TABLE I.

γ	α	γ	A	B
15°	48°19'35"	43°48'20"	-4.0852	2.3539
30	53 24 25	25 23 40	-2.1480	1.5708
45	60 52 23	33 51 47	-1.6103	1.6103
60	69 52 7	39 25 0	-1.5708	2.1482
75	79 44 14	42 30 39	-2.3548	4.0868

1) $\alpha(\gamma'=max)$ is denoted by α_{max} , wherever it could not give rise to ambiguity
 2) Min. d. Roches, p. 248.

from which the following extinctions with respect to the acute bisectrix are calculated

TABLE II.

x	Values of y' at $z =$				
	$z = 45^\circ$	$z = 30^\circ$	$z = 45^\circ$	$z = 60^\circ$	$z = 75^\circ$
1	1°53' 6"	3°36' 38"	4°53' 14"	5° 8' 26"	3°32' 41"
30	4 4 28	7 56 59	11 14 42	12 54 21	10 16 52
45	6 49 59	13 42 48	20 38 46	27 27 29	33 6 56
60	10 14 10	20 52 53	32 32 7	46 20 3	64 40 45
75	13 32 29	27 22 46	41 47 36	57 4 59	73 14 21
90	15 (+18")	30 (+11")	45	60 (-4")	75 (-1")

The values for y' found by calculation with $x = 90^\circ$, which accordingly represent the limit of extinction with regard to the acute bisectrix on the plane making right angles with the latter, give a measure for the exactitude of the values found. The errors successively amount to $+18''$, $+11''$, $0''$, $-4''$, $-1''$.

The greatest extinction for different values of x with the corresponding angle $\left(\frac{\pi}{2} - \alpha\right)$ are now to be calculated from the formulas (7) and (8); we come to the following result:

x	y'_{max}	$\frac{\pi}{2} - \alpha$
15	5°13' 12"	34°36' 4"
30	12 54 29	29 25 6
45	33 44 35	10 40 21
$2 - \frac{1}{2} \sqrt{16^2 - 30^2}$	45	0

To calculate from

$$\cot \alpha_{max} = \frac{\sin^2 V \sin \alpha + \cos^2 V \cos \alpha}{\sin^2 V + \cos^2 V \cos^2 \alpha}$$

the value of α_{max} when $\sin \alpha = 0$; we eliminate y' .

As

$$\sin 2\psi'_{max} = \frac{\sin V \operatorname{tg} V}{\cos x \cot x}$$

we get

$$\begin{aligned} \cot \alpha_{max} &= \frac{\sin^2 V \sin x \sqrt{1 - \left(\frac{\sin V \operatorname{tg} V}{\cos x \cot x}\right)^2}}{\sin V \operatorname{tg} V} \\ &= \frac{\sin^2 V + \cos^2 V \cos^2 x}{\sin^2 V + \cos^2 V \cos^2 x} = \\ &= \frac{\sin^2 V \sin x \sqrt{\cos^2 x \cot^2 x - \sin^2 V \operatorname{tg}^2 V}}{\sin V \operatorname{tg} V (\sin^2 V + \cos^2 V \cos^2 x)} = \\ &= \frac{\sin x \cos V \sqrt{\frac{\cos^4 x}{\sin^2 x} - \frac{\sin^4 V}{\cos^2 V}}}{\sin^2 V + \cos^2 V \cos^2 x} = \\ &= \frac{\sqrt{\cos^4 x \cos^2 V - \sin^2 x \sin^4 V}}{\sin^2 V + \cos^2 V \cos^2 x} \dots \dots \dots (9) \end{aligned}$$

When $x = 0$, is

$$\cot \alpha_{max} = \pm \cos V.$$

From which for olivine follows the value:

$$\begin{aligned} \left(\frac{\alpha}{2} - \alpha\right) &= b\eta \operatorname{tg} (\pm) \cos V = \\ &= b\eta \operatorname{tg} (\pm) \cos 43^\circ 30' \\ &= (\pm) 35^\circ 57' 22''. \end{aligned}$$

In the following figure these results are graphically represented. The black lines connect the poles of planes with equal positive, the lines in black and white those of planes with equal negative extinction. Herein the angles have been considered positive from the acute bisectrix in the direction of the hands of the clock; negative in the opposite direction.

The curves MM' and NN' , going through the optic axes, connect the poles of the planes with the greatest (positive and negative) extinction and with the same inclination with regard to the acute bisectrix. The point in which the curves mentioned intersect an isogyre, has on that isogyre the greatest angular distance from O . For the rest very little need be added to what is to be read from the figure. It shows clearly that an extinction with regard to the acute bisectrix, which deviates little from 0° , is confined to the immediate neighbourhood of the principal planes of symmetry.

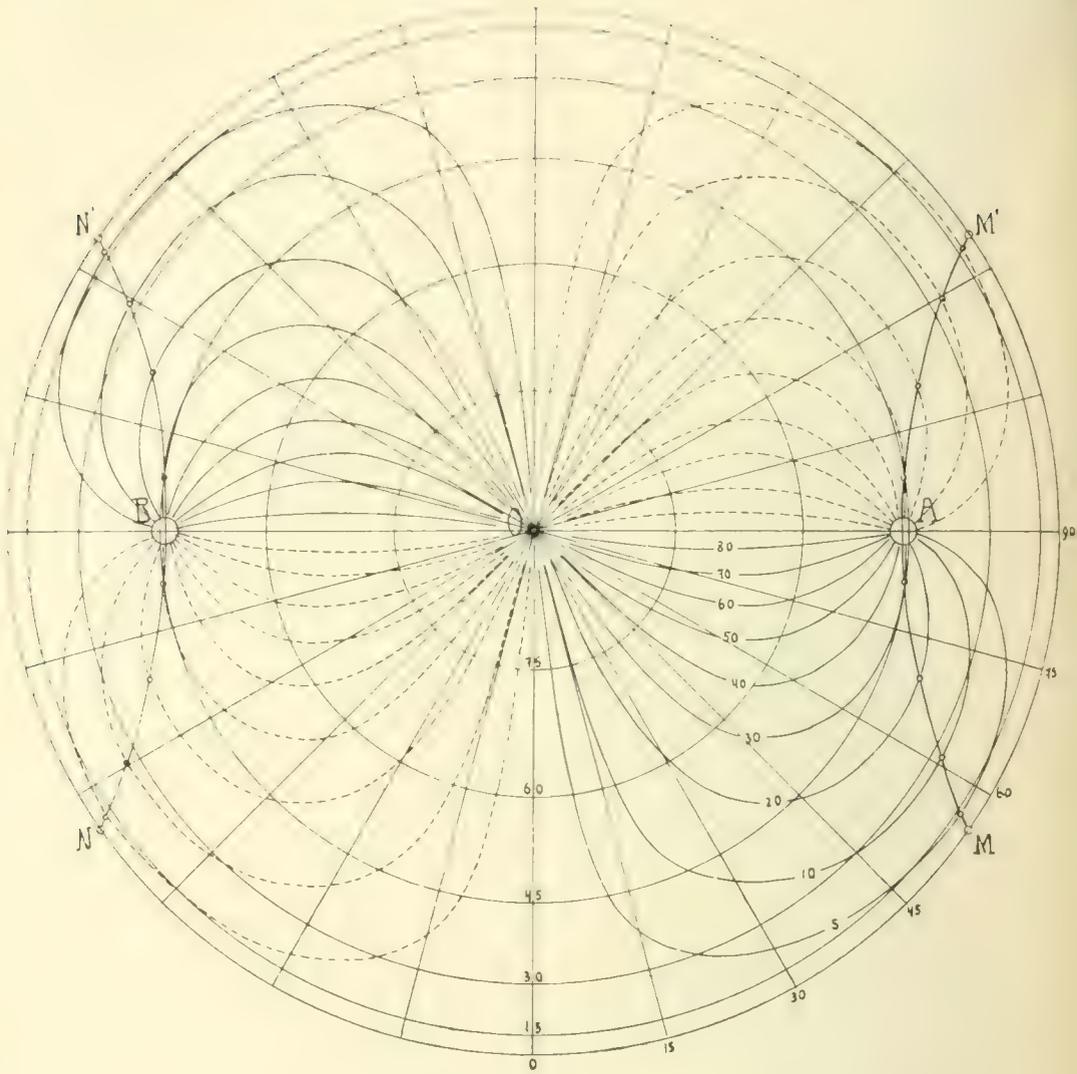


Fig. 2

For tale ($2I^r = 13^\circ$) the following values are calculated in the same way:

1) HINTZE, Handbuch der Mineralogie, II, 1897, p. 815, cf. BAUER, Pogg. Ann. 1869, CXXXVIII, 368.

x	α	δ	A	B
15	83° 43' 21"	1° 42' ($\pm 1'$)	— 156.05	154.32
30	84 22 26	3 15	— 89.846	89.268
35	84 40 45	3 44	— 82.815	82.45
40	85 1 25	4 11	— 78.895	78.718
45	85 24 31	4 36	— 77.628	77.628
50	85 49 38	4 59	— 78.717	78.891
55	86 16 38	5 20	— 82.31	82.672
60	86 45 19	5 38	— 89.268	89.846
65	87 15 5	5 53 30"	— 100.62	101.46
70	87 46 48	6 6	— 120.16	121.38
75	88 19 16	6 46 30	— 154.32	156.05
80	88 52 25	6 24	— 225.49	228.14
85	89 26 4	6 28 30	— 443.61	449.28

to which correspond the extinctions:

x	$\gamma = 15$	$\gamma = 30$	$\gamma = 35$	$\gamma = 40$	$\gamma = 45$	$\gamma = 50$	$\gamma = 55$
15	0° 3' 4"	0° 4' 54"	—	—	0° 6' 9"	—	—
30	0 7 9	0 12 44	—	—	0 14 46	—	—
45	0 8 12	0 26 53	—	—	0 31 19	—	—
60	0 36 53	1 4 59	—	—	1 16 39	—	—
70	1 21 38	2 26 12	2° 40' 54"	2° 51' 34"	2 57 13	2° 57' 48"	2° 52' 14"
75	2 17 13	4 11 16	4 39 53	5 1 35	5 15 26	5 21 1	5 17 6
80	4 23 11	8 24 35	9 31 30	10 33 35	11 23 43	12 1 43	12 24 4
85	9 27 9	19 10 28	22 33 26	26 2 27	29 36 9	33 23 25	37 31 53
90	15° ($\pm 53''$)	30 ($-50''$)	—	—	45°	—	—

x	00	05	50	75	80	85	90
15	0 5 21	—	—	0 3 14	—	—	0°
30	0 12 52	—	—	0 7 28	—	—	0
45	0 27 21	—	—	0 15 56	—	—	0
60	1 7 57	—	—	0 30 55	—	—	0
70	2 41 57	2 20' 46"	2 4' 20"	1 7 40	1° 7' 27"	0 34' 26"	0
75	5 1 13	4 35 53	3 58 51	3 9 35	2 12 10	1 7 58	0
80	12 49 51	11 53 44	10 58 49	9 9 38	6 42 42	3 33 47	0
85	41 34 43	46 43 1	53 20 59	59 24 49	67 38 33	78 3 57	90
90	60 (+51)	—	—	75 (- 8")	—	—	90°

For the greatest extinction and corresponding angle we find:

x	\mathcal{V}'_{max}	$\frac{\tau}{2}$
0	0	44° 48' 56"
15	0 6' 9"	44 48 31
30	0 14 47	44 41 24
45	0 31 21	44 27 2
60	1 16 50	43 38 28
75	5 21 33	39 38 9
$\frac{\tau}{2}$	$\mathcal{V} = 45$	0
	83° 30'	

Figure 3 affords a general view of the results. Suppose the part of the globe-surface, falling outside the parallel-circle of 60° but within the isogyre of 1° , to be equal to the part falling within the same circle outside the isogyre, then it appears that at about $\frac{2}{3}$ of the sphere an extinction of less than 1° is observed, so practically a straight extinction. Now the sections, yielding greater extinctions, lie so much in the neighbourhood of the planes, making right angles with the optic axes, that they are for the greater part impracticable for the determination of the direction of extinction. A comparison of figures 2 and 3 shows the result that with rhombic

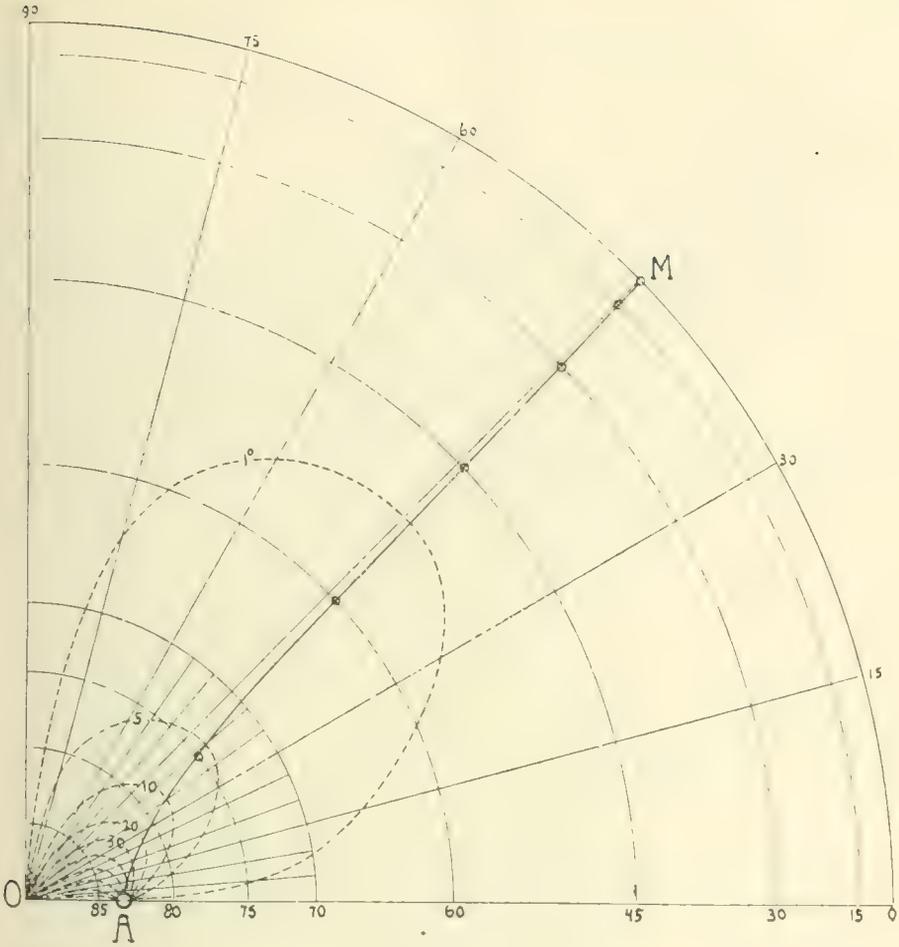


Fig. 3.

crystals with a great axis-angle the oblique, with those with a small axis-angle the straight extinction will predominate, when we have to do with arbitrary sections, as in a rock slice. However in the two cases we as rarely find an absolutely straight extinction. With hexagonal and tetragonal crystals, however, exclusively straight extinction with regard to the optic axis occurs, as for $V=0$ the equation (4)

$$\cot 2\eta = \frac{1}{\sin^2 V} \left(-\frac{V - \sin^2 V \sin^2 \alpha}{\sin 2\alpha} \cdot \frac{1}{\sin x} + \frac{1 - \sin^2 V \cdot \cos^2 \alpha}{\sin 2\alpha} \sin x \right)$$

always becomes ∞ .

In fig. 4 the maximum extinction as a function of x is represented for one globeoctant; MA_1 refers to tale, MA_2 to olivine, MA_3 to a mineral with an axis-angle $2V=160^\circ$. The values $O'A_1$, $O'A_2$

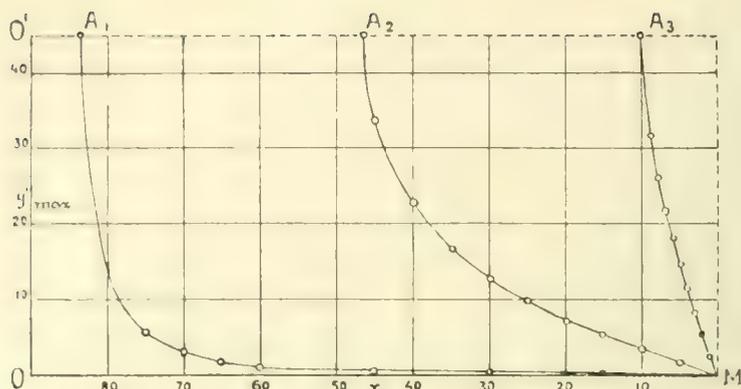


Fig. 4.

and $O'A_3$ give the size of V . The general equation of the curves MA is:

$$\sin 2y'_{max} = \frac{\sin V \operatorname{tg} V}{\cos x \cot x},$$

$$y'_{max} = \frac{1}{2} \operatorname{tg} V \sin \left(\frac{\sin V \operatorname{tg} V}{\cos x \cot x} \right)$$

from which

$$\begin{aligned} \frac{dy'_{max}}{dx} &= \frac{\sin V \operatorname{tg} V}{2 \sqrt{1 - \left(\frac{\sin V \operatorname{tg} V}{\cos x \cot x} \right)^2}} \cdot \left(\frac{1 + 2 \operatorname{tg}^2 x}{\cos x} \right) = \\ &= \frac{\sin V \operatorname{tg} V (1 + 2 \operatorname{tg}^2 x) \cot x}{2 \sqrt{\cos^2 x \cot^2 x - \sin^2 V \operatorname{tg}^2 V}} = \\ &= \frac{\sin^2 V}{2} \cdot \frac{1 + \sin^2 x}{\cos x \sqrt{\cos^4 x \cos^2 V - \sin^2 x \sin^4 V}} \dots \dots \dots (10) \end{aligned}$$

So for $x = 0$ the direction of the tangent is given by :

$$\frac{dy'_{max}}{dx} = \frac{+ \sin^2 V}{(-) 2 \cos V}$$

For $x = \frac{\pi}{2} - V$ by

$$\frac{dy'_{max}}{dx} = \infty$$

as the term under the root-mark becomes $= 0$. So the tangents in A on the curves form right angles with the direction MO .

It further follows from the formula (10) that the rise of the curve for the same value of x grows smaller as the value of V diminishes, as is also shown by fig. 4. If V becomes $= 0$, as in the hexagonal and tetragonal system, then we also have

$$\frac{dy'_{max}}{dx} = 0,$$

so that the curve MA coincides with the abscissa-axis MO . Finally with regard to the form of the curve which represents the angle $\left(\frac{\pi}{2} - \alpha_{max}\right)$ as a function of x , it appears already from a comparison of figures 2 and 3, that this curve MA , the axis-angle becoming smaller, gradually approaches the straight line that divides into two equal parts the angle between OA and the normal to it in O .

Indeed (9) yields

$$\begin{aligned} \cot \alpha_{max} &= \frac{\sqrt{\cos^4 x \cos^2 V - \sin^2 x \sin^4 V}}{\sin^2 V + \cos^2 V \cos^2 x} \\ &= \operatorname{tg} \left(\frac{\pi}{2} - \alpha_{max} \right) = \frac{\sqrt{\cos^4 x - (\cos^4 x + \sin^2 x \sin^2 V) \sin^2 V}}{\cos^2 x + \sin^2 x \sin^2 V}. \end{aligned}$$

If V becomes smaller, $\operatorname{tg} \left(\frac{\pi}{2} - \alpha_{max} \right)$ increases, and with $V = 0$ reaches the greatest value $\frac{+}{-} 1$, so that then α_{max} becomes $= 45^\circ$. The curves $M'AM$ and NBN' then pass into two straight lines which intersect in O , thus forming right angles, whilst they have shifted 45° with regard to direction AB .

Of great practical importance is the solution of the problem, how

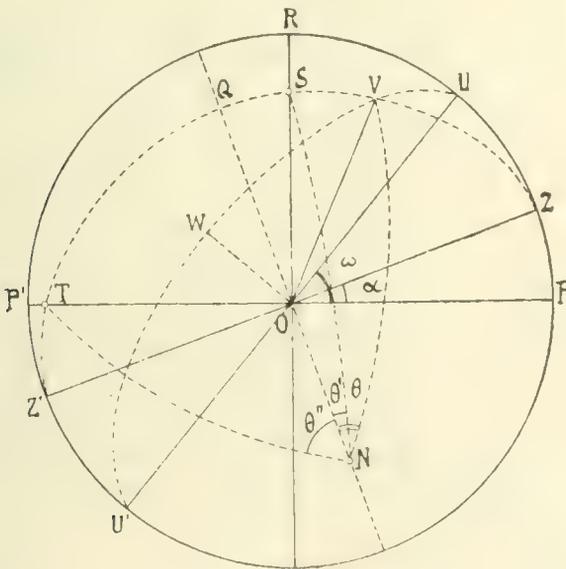


Fig. 3.

great the extinction is with regard to the trace of a cleavage plane. For this, if the angle of extinction with regard to the acute bisectrix be known, is only necessary the value of the apparent angle between the trace mentioned and the same bisectrix. One value need only be subtracted from the other.

If ZZ' be the axis of a zone, in which ZQZ_1 represents an arbitrary plane, N the corresponding pole, and be the plane determined again by α and $OQ = x$; if UWU' be an arbitrary cleavage plane, determined by ω and $WO = y$, then VO is the line of intersection of both planes, $VQ = \angle QNV = \theta$, the apparent angle between the acute bisectrix (O) and VO .

Now $VQ = \frac{\pi}{2} - VZ$, and in ΔVUZ is

$$\angle Z = \frac{\pi}{2} - x$$

$$\angle U = \frac{\pi}{2} + y$$

$$UZ = \omega - \alpha$$

So

$$\begin{aligned} \cot VZ = \operatorname{tg} \theta &= \frac{\sin\left(\frac{\pi}{2} - x\right) \cot\left(\frac{\pi}{2} + y\right) : \cos\left(\frac{\pi}{2} - x\right) \cos(\omega - \alpha)}{\sin(\omega - \alpha)} \\ &= \frac{-\cos x \operatorname{tg} y + \sin x \cos(\omega - \alpha)}{\sin(\omega - \alpha)} \dots \dots \dots (11) \end{aligned}$$

If we apply this formula to the cleavage planes $h^1(100)$ and $g^1(010)$ of olivine, then with

$$h^1(100) \dots \omega = 0, y = 0$$

$$g^1(010) \dots \omega = \frac{\pi}{2}, y = 0$$

and (11) passes into:

$$\operatorname{tg} \theta' = \sin x \operatorname{tg} \alpha$$

$$\operatorname{tg} \theta'' = -\sin x \cot \alpha$$

in which θ' and θ'' are successively the apparent angles between the traces of $g^1(010)$ and $h^1(100)$ on the plane (N).

Now if we think both x and α to vary between 0 and $\frac{\pi}{2}$, we find the following values for θ' and θ'' :

at (-)	x = 0	x = 15	x = 30	x = 45	x = 60	x = 75	x = 90	at (+)
0	indet.	90°	100°	90°	90°	90°	90°	90°
1	0	86 8 30	88 0 2	88 35 56	88 50 43	88 57 53	89	89
2	0	82 18 57	86 0 17	87 10 22	87 41 27	87 55 46	88	88
3	0	78 33 11	84 0 59	85 45 49	86 32 13	86 53 39	87	87
4	0	74 52 5	82 2 49	84 21 9	85 23 2	85 51 34	86	86
5	0	71 19 25	80 4 30	82 56 43	84 13 33	84 49 28	85	85
10	0	55 44 4	70 34 28	75 59 33	78 29 29	79 39 17	80	80
15	0	44 0 26	61 48 38	69 14 38	72 48 29	74 29 45	75	75
30	0	24 8 46	40 33 34	50 46 7	56 18 35	59 7 56	60	60
45	0	14 30 39	26 33 55	35 45 53	40 53 36	44 0 24	45	45
60	0	8 29 56	16 6 8	22 12 28	26 33 54	29 8 50	30	30
75	0	3 58 2	7 37 51	10 43 43	13 3 52	14 39 38	15	15
80	0	2 36 57	5 2 18	7 6 25	8 40 56	9 39 57	10	10
85	0	1 17 49	2 30 17	3 32 24	4 19 58	4 49 49	5	5
86	0	1 2 13	2 0 9	2 49 59	3 27 56	3 51 50	4	4
87	0	0 46 38	1 30 4	2 7 20	2 35 55	2 33 52	3	3
88	0	0 31 4	1 0 1	1 24 52	1 43 56	1 55 55	2	2
89	0	0 15 32	0 30 0	0 42 26	0 51 53	0 57 57	1	1
90	0	0°	0°	0°	0°	0°	0	0

As is also shown in the table

$$\theta' = b g t g \sin x t g a$$

becomes:

for $x = 0$, $a < \frac{\pi}{2}$ equal to 0,

for $x = 0$, $a = \frac{\pi}{2}$ indefinite, and

for $a = 0$ also every time 0, whilst

$$-\theta'' = b g t g \sin x \cot a$$

for $x = 0$, $a > 0$ equal to 0,

for $x = 0$, $a = 0$ indefinite, and lastly

for $a = \frac{\pi}{2}$ always becomes 0.

In order to know the extinction of the plane (N) with respect to the trace of g' (010) or h' (100), we combine this table with table II.

For $\frac{\pi}{2} \geq \alpha \geq 0$, $\frac{\pi}{2} \geq x \geq 0$ the extinctions successively become $(-\theta' + y')$ and $(-\theta'' + y'')$; we find the following values:

$$q' = y' - \theta'.$$

x	$x = 0$	$x = 15$	$x = 30$	$x = 45$	$x = 60$	$x = 75$	$x = 90$
0	0°	0°	0	0	0	0°	0°
15	0	- 2° 4' 56"	- 3 33' 23	- 3 43' 44"	- 2 49' 42	- 0°58' 9"	0
30	0	- 4 53' 18	- 8 9 9	- 8 20' 40	- 5 44' 1	1 46' 4	0
45	0	- 9 37' 25	-15 19' 13	-14 37' 7	- 8 21' 29	2 12' 48	0
60	0	-19 0 20	-27 59' 13	-23 18' 38	9 58' 32	- 2 2 57	0
75	0	-40 27' 45	-51 31' 46	-36 7 52	8 7 44	-1 15' 24	0
90	indif.	-90°	-90°	-90	0	0	0

When $\alpha = 90^\circ$ and $x = \left(\frac{\pi}{2} - V\right) = 46^\circ 30'$ the extinction becomes indefinite; it is here that the transition from 90° to 0° takes place. In the same way we find for

$$q'' = y'' - \theta''.$$

x	$x = 0$	$x = 15$	$x = 30$	$x = 45$	$x = 60$	$x = 75$	$x = 90$
0	indefinite	90°	90°	90°	90°	90°	90°
15	0	45°53' 32"	65°53' 6"	76° 4' 47"	83° 2' 39"	88° 2' 14"	90
30	0	27 45' 24	48 50' 34	64 28' 55	77 41' 28	86 30' 42	90
45	0	19 23' 53	37 48' 37	55 54' 49	73 25' 43	85 48' 0	90
60	0	13 48' 22	29 0 29	49 39' 57	72 53' 57	86 13' 49	90
75	0	7 39' 43	17 54' 43	43 50' 39	77 44' 37	87 44' 59	90
90	0	0	0	0	90°	90°	90

Also here the extinction for $\alpha = 90^\circ$ and $x = \left(\frac{\pi}{2} - V\right) = 46^\circ 30'$ becomes indefinite.

The shape of the q -isogyres is represented in fig. 6. The black lines refer to φ'' , those in black and white to φ' ; φ'' gives the value of the positive extinction with regard to the trace of h' (100), φ'

that of the negative extinction with reference to the trace of g' (010) on the plane (ΔV).

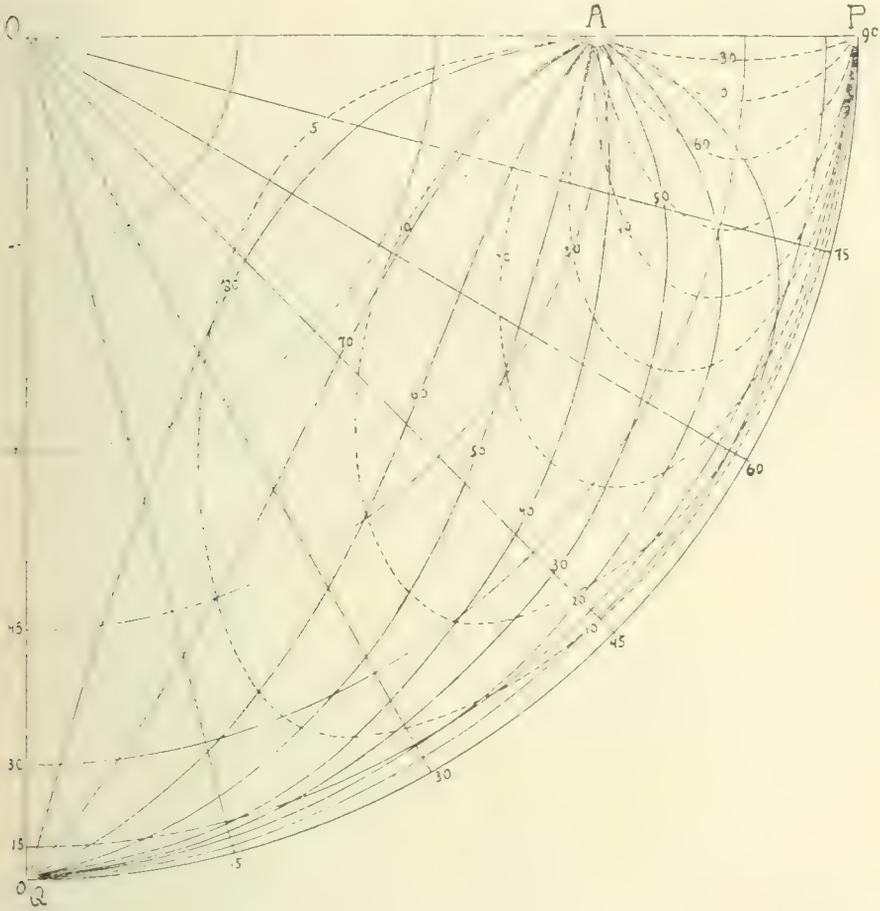


Fig. 6.

The figure is with regard to the axes OP and OQ quite symmetrical again, in the same way as fig. 2 shows for the isogyres with respect to the acute bisectrix, because the points here correspond to the value:

$$\varphi' = \frac{1}{2} b\eta \cot \left(\frac{1 - \sin^2 V \sin^2 \alpha}{\sin 2\alpha \cdot \sin^2 V} \cdot \frac{1}{\sin x} + \frac{1 - \sin^2 V \cos^2 \alpha}{\sin 2\alpha \sin^2 V} \cdot \sin x \right) - b\eta \tan(\sin x \tan \alpha) = \eta' - \theta'.$$

$$\varphi'' = \frac{1}{2} b\eta \cot \left(\frac{1 - \sin^2 V \sin^2 \alpha}{\sin 2\alpha \cdot \sin^2 V} \cdot \frac{1}{\sin x} + \frac{1 - \sin^2 V \cos^2 \alpha}{\sin 2\alpha \sin^2 V} \cdot \sin x \right) + b\eta \tan(\sin x \cot \alpha) = \eta'' - \theta''.$$

Now the signs of y' , θ' and θ'' remain unchanged for $0 < x < \pi$ and for α varying between 0 and $\frac{\pi}{2}$, so that also here the sign of the extinction in adjoining globe octants will be alternately positive and negative. The points in which equivalent isogyres in the systems φ' and φ'' intersect, indicate the places of the poles of the sections, in which symmetrical extinction with respect to the cleavage planes h' (100) and g' (010) visible in the slice is observed. So there

$$\begin{aligned}\varphi' + \varphi'' &= 2y' - (\theta' + \theta'') = 0. \\ 2y' &= \theta' + \theta''\end{aligned}$$

The curve $OA = V$. If the axis-angle diminishes, A gradually approaches O ; the isogyres φ' and φ'' approach a symmetrical direction with regard to the axes OP and OQ , so that the curve, connecting their points of intersection, draws nearer and nearer to the straight line, and at last, when V has become $= 0$, and A coincides with O , passes into the straight line which divides the angle POQ into two equal parts. For $V = 0$ y' becomes $= 0$, so:

$$\begin{aligned}\theta' + \theta'' &= 0 \\ \sin x \operatorname{tg} \alpha - \sin x \cot \alpha &= 0 \\ \operatorname{tg} \alpha &= \cot \alpha\end{aligned}$$

so that the geometrical place of the points of intersection of the isogyres φ' and φ'' , i.e. that of the points of symmetrical extinction, is represented by the line

$$\alpha = 45^\circ.$$

Anatomy. — “*On ascending degeneration after partial section of the spinal cord.*” By Dr. S. J. DE LANGE. (Communicated by Prof. C. WINKLER).

(Communicated in the meeting of November 30, 1907).

The following researches have been made with the purpose of investigating whether there exist any connections between the spinal cord and the ascending fasciculus longitudinalis dorsalis, and on the other hand to ascertain once more the course of the ascending anterolateral fascicle of GOWERS and its relation to the dorso-lateral fasciculi of the cerebellum.

From the extensive literature on this subject I will but rarely quote something, whenever the results obtained by others do not accord with my observations. The drawings are taken after four

animals used in the experiments: a full-grown rabbit, in which a hemisection was made at the origin of the medulla oblongata (fig. 1) and three young cats, on which lesions were produced in different portions of the spinal cord (fig. 2, 3 and 4), indicated by C⁷ C⁴ and D⁴.

The central nervous system of these animals was treated by the MARCHI-method, and afterwards cut into serial sections 25 μ thick.

In all these animals a plainly visible degeneration was found after the operation in the tractus spino-cerebellaris anterior or the fascicle of GOWERS. In the dorso-lateral fasciculus there is likewise found a certain degree of degeneration, but far less intense, excepting only in the first animal on which the hemisection was made near the origin of the medulla oblongata. For in this latter case there is found a very compact degeneration of the corpus restiforme, being caused however not only by spino-cerebellar fibres, but likewise and for a great part by bulbo-cerebellar fibres, that were injured when the lesion was produced ¹⁾.

Figures 5-11 represent the degeneration as observed in my preparations. In the first sections we see the dorsal and the ventral spino-cerebellar system still undivided and taking a longitudinal course. Gradually the fibres of the dorso-lateral portion are deviating from the longitudinal direction, whilst the ventral portion still continues to follow it. In a slow spiral-line the dorsal fibres run towards the corpus restiforme; consequently by and by several important portions of the medulla oblongata are found to be encompassed between the two parts of the lateral fasciculi ad cerebellum. This fact becomes most plainly evident in fig. XV, taken from the first cat, on which an almost complete section had been made at C⁷. There being no degeneration of the olivo-cerebellar tract in this case, the demarcation between corpus restiforme and antero-lateral fascicle is much more distinct.

I have represented from both these animals the radiation of the corpus restiforme into the cerebellum (dorsal portion of the lateral fasciculi to the cerebellum), and that of the fascicle of GOWERS (antero-lateral portion of the lateral fasciculi to the cerebellum): fig. 10 and 11, 17 and 18. The first radiation is produced through the pedunculus ad cerebellum inferior, that, of the fascicle of GOWERS through the pedunculus ad cerebellum superior.

In order to reach this latter the fibres of the tractus spino-cerebel-

¹⁾ To understand this rightly, the exact place of lesion must be more accurately circumscribed. The lesion was made just underneath the most caudal root of the hypoglossus, and directed somewhat obliquely upward. Haemorrhage was still to be traced in those sections where the olivary body shows the largest profile.

laris anterior, that continued for a long while their longitudinal course, begin to assume an oblique dorsal direction in the region of the oliva superior and at the first appearance of fibres belonging to the corpus trapezoides. They cross the fibres of the corpus trapezoides in dorso-lateral direction, in consequence of which they present in the sections a peculiar distribution by layers (fig. 9). Until the place of exit of the trigeminus their situation remains nearly unaltered, the fibres lying only more closely together, till they take a sudden turn in dorsal direction, somewhat frontal from the trigeminus, thus joining the course of the lateral ribbon (*laqueus lateralis*) and staying for the greater part at its outside. Through the pedunculus ad cerebellum the greater part of the fibres now reach the cerebellum, where they take a retrograde direction, extending in the shape of a fan into the vermis inferior. Only a few fibres follow the course of the remaining portion of the lateral laqueus, and reach the corpus quadrigeminum posticum of the same side. Some authors assert that fibres may be traced likewise unto the corpus quadrigeminum anticum (THOMAS, WALLENBERG, BIANCHI, COLLIER and others), but VAN GEHUCHTEN, EDINGER and MOTT agree on this point with LOEWENTHAL, who has been the first to give an accurate description of the fascicle of GOWERS.

As I remarked before, the image of the lesions of the spinal cord presents differences only as regards the dorsal portion of the lateral fasciculi to the cerebellum, the degeneration found in this portion being far less intense than it was in the case, where the section was made near the origin of the medulla oblongata. This result, however, must be ascribed not only to the absence of degenerated bulbo-cerebellar fibres, but partly also to the fact that a great number of fibres from the dorsal fascicle find a provisory termination at the passage of the spinal cord into the myelencephalon, as is shown by the preparations. Whilst therefore in the ventral portion we find almost without exception only long fibres, in the dorsal part of the lateral fasciculi of the cerebellum long and short fibres are intermixed, and very probably the connection is composed for the greater part of two neurones.

In all my preparations, some degeneration is observed likewise in the fasciculus longitudinalis posterior. This receptacle of numerous fibres, ascending and descending ones, originating in very different portions of the nerve-trunk, shows degeneration over the whole of its length, and it seems as if from this fascicle all nuclei of motor nerve-fibres are provided with afferent fibres, the MARCHI-granulae being found even in the motor roots. In this paper I will not digress longer on the possible significance of this MARCHI-granulation, though

FIG. X



FIG. XII

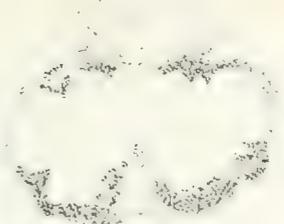


FIG. XIV



FIG. XVI

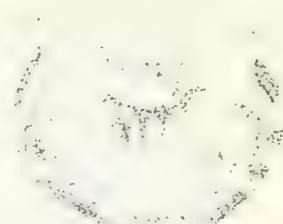


FIG. XVI CORPUS TRAPEZOIDES.



FIG. XVII



FIG. XVIII



FIG. XIX



FIG. XX



FIG. XXI



FIG. XXII



FIG. XXIII



FIG. XXIV



FIG. XXV

FASC. LONG. DORS.



FIG. XXVI



..... CORP. QUADR. POST.

..... FASC. LONG. DORS.

perhaps it may be of very great importance. Other methods of investigation will be more efficient in bringing this problem nearer to its solution. In fig. 19 and 20 this granulation is represented, as it appeared in a cat after a central lesion of the 4th dorsal segment (fig. 3). Yet I think I am fully justified in concluding from my preparations to the existence of long ascending fibres, that have their origin probably somewhere near the central canal. In the spinal cord these fibres then continue their course in the anterior fascicle, part of them also in the anterolateral fascicle (fig. 12). The former ones turn more centralward at the origin of the medulla oblongata, and issue in the fasciculus longitudinalis dorsalis. The second ones take a bent near the most caudal root of the hypoglossus, and proceed along the course of that root centralward to the fasciculus longitudinalis (fig. 13, 14 and 25). These ascending fibres are still found even in the nucleus of the N. oculomotorius, as is shown in fig. 26.

RAMON Y CAJAL also assumes the existence of these fibres, v. GEHUCHTEN however has found ascending degeneration only after a lesion of the nucleus of DEITER, but he does not mention whether he has made observations on sections obtained after central lesions of the spinal cord. In the laboratory of VAN GEHUCHTEN similar experiments were made by LUBOUSCHINE, by the injection of a drop of water. But it was his purpose to destroy part of the posterior horn in order to trace the origin of the anterolateral fascicle. He found this fascicle to be degenerated, but there did not exist any degeneration of the fasciculus longitudinalis, because the central portion of the spinal cord remained uninjured.

After the hemisection near the origin of the medulla oblongata, there appeared also degeneration of the tractus bulbo-cerebellaris, which takes its course from the olivary body towards the periphery of the corpus restiforme as *fibrae arcuatae internae* and *externae*. The greater part of these fibres are crossed, still a few of them originate on the same side.

Besides these fibres there are found in the basal portion of the *formatio reticularis* a great number of degenerated fibres, ascending longitudinally and probably belonging to the secondary sensible tract (fig. 7—11). Part of these fibres transgress the median line and seek the median ribbon (*laqueus medialis*) of the non-injured side, the rest of them are united near the *oliva superior* and continue their course in the *laqueus lateralis* more centralward than the fibres of the fascicle of GOWERS. They may be traced into the *corpus quadrigeminum posterius*.

The significance of the granulation in the *corpus trapezoides*, which

was to be observed in all cases, did not become clear to me, still I will not pass in silence the fact of the constant appearance of this granulation.

As the summary of my results, I find that after onesided lesion of the spinal cord an ascending degeneration is observed in the following systems:

1. The homo-lateral posterior columns, where it may be traced as far as into the nuclei of GOLL and BURDACH.

2. The lateral fasciculi to the cerebellum,

a. the dorsal portion almost without exception only on the operated side,

b. the antero-lateral portion on both sides, but still principally on the operated side.

3. The fasciculus longitudinalis dorsalis on both sides.

4. The corpus trapezoides on both sides (?)

The descending degeneration is represented in figures 21, 22, 23 and 24. It includes:

1. The anterior columns, principally on the operated side, probably centrifugal fibres from the fasc. long. dors.

2. The pyramidal lateral fasciculus on the operated side.

3. The tractus rubro-spinalis, in the lateral columns (VAN GENUCHTEN).

4. De tractus vestibulo-spinalis, frontal of the anterior horn (EDINGER).

5. Fibres in the posterior columns, being situated partly along the sulcus longitudinalis posterior, and partly along the entering posterior roots, to all probability presenting a homologon to the oval area and the comma of SCHULTZE.

Physics. — *“Motion of molecule-systems on which no external forces act.”* By Dr. O. POSTMA. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the Meeting of November 30, 1907).

§ 1. Up till now two ways have been mainly followed to show that a gas mass left to itself, on which no external forces act, in consequence of the collisions of the molecules will finally pass into a state, in which the molecules are probably about uniformly distributed over the vessel and possess MAXWELL'S distribution of velocities.

The first is the method of BOLTZMANN, who assuming that the density all through the vessel is already the same, and further starting from the assumption that there is no regular arrangement of the molecules as regards the velocity, demonstrates that a certain

quantity $H = \int f \cdot lf' \cdot d\omega$ constantly decreases by the collisions till the stationary state is reached, with which, as appears, MAXWELL'S distribution of velocities exists.

The second is the method half followed by BOLZMANN, and entirely by JEANS, by which it is demonstrated that on certain hypotheses the state with uniform density and that with Maxwell's distribution of velocities are the most probable. These hypotheses are, as regards the distribution of place, that every time there would be an equal chance to any place in the vessel for every molecule; with regard to the distribution of velocities that there would every time be an equal chance that the point of velocity of a molecule would get into any arbitrarily chosen volume-element, in which we should finally have to reckon with the fact that the total energy has a certain definite value.

I have tried to show¹⁾ that there is something contradictory in this, which might be avoided by assuming that the gasmass is arbitrarily chosen from a microcanonical ensemble, of which all the systems possess the energy which the gasmass must have. For in this all the combinations of place and all the combinations of velocities with the same energy are equally numerous, and so we have the same chance to hit upon them for the system chosen.

So another proof for the above mentioned result is furnished, when we show that an arbitrary ensemble of systems with the same energy, left to itself, passes into a microcanonical ensemble. GIBBS endeavours to demonstrate this in the XIIth Chapter of his "Statistical Mechanics"; the reasoning is made clearer by LORENTZ²⁾, though the latter goes no further than calling the assumption that we should finally get a microcanonical distribution, very plausible.

However in a recent paper³⁾ POINCARÉ called attention to a property in the light of which, in my opinion, the above reasoning is no longer tenable. There POINCARÉ shows, namely, that the quantity

$S = \int P \log P dx_1 \dots dx_n$ (in which $x_1 \dots x_n$ represent the variables

which determine every system of a certain ensemble, and $P = \frac{D}{N}$

the coefficient of probability, the integration being extended over the

¹⁾ These Proc. Febr. 21, 1906 and Jan. 24, 1907.

²⁾ "Über den zweiten Hauptsatz der Thermodynamik"; Abhandlungen über Theoretische Physik, Leipzig 1906, p. 289.

³⁾ "Réflexions sur la Théorie cinétique des gaz"; Journal de Physique, 1906, p. 369.

whole region occupied by the ensemble) is constant if the external circumstances are unchanged and the relation $\sum \frac{\partial X_i}{\partial x_i}$ exists (for which $X_i = \frac{dx_i}{dt}$). In this POINCARÉ takes as variables the coordinates and velocity-components of the molecules; so the quantity referred to above differs from the quantity η introduced by GIBBS only by a constant factor.

GIBBS shows that in a canonical ensemble — η has the properties of the entropy, POINCARÉ calls the quantity S itself entropy, also for an arbitrary ensemble. Hence this quantity will have to decrease, where — η increases.

The property in question may be derived as follows: P has the properties of a density, so:

$$\frac{\partial P}{\partial t} = - \sum \frac{\partial P X_i}{\partial x_i} = - \sum X_i \frac{\partial P}{\partial x_i} \left(\text{as } \sum \frac{\partial X_i}{\partial x_i} = 0 \right).$$

Now, an arbitrary function of P will also behave as a density. Namely:

$$\begin{aligned} \frac{\partial f(P)}{\partial t} &= f'(P) \frac{\partial P}{\partial t} = - f'(P) \sum X_i \frac{\partial P}{\partial x_i} = - f'(P) \sum X_i \frac{\partial P}{\partial x_i} - f(P) \sum \frac{\partial X_i}{\partial x_i} = \\ &= - \sum X_i f'(P) \frac{\partial P}{\partial x_i} - \sum f(P) \frac{\partial X_i}{\partial x_i} = - \sum \frac{\partial [X_i f'(P)]}{\partial x_i}. \end{aligned}$$

So the $f'(P)$ also satisfies an equation of the same form as the P itself, which equation represents the extension of the wellknown equation from hydrodynamics:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

to a space of n dimensions.

Now $S = \int P \log P d\tau$, (in which $d\tau = dx_1 \dots dx_{n_1}$) $= \int f(P) d\tau$, is the integral of such an $f'(P)$, integrated over the whole extension occupied by the ensemble. To ascertain the change of S with the motion of the ensemble, we must every time integrate over the variable space (though constant in size), over which the phases extend or where P and also $f'(P)$ have values. So we can perfectly compare $\frac{dS}{dt}$ with the increase in time of a quantity of liquid, taken over all the places where it is. This increase, however, is equal to zero.

However, when $S = \int P \log P dx_1 \dots dx_n$ is constant, the ensemble cannot move in the direction to the microcanonical distribution. For then P would become constant all over the phase-extension in course of time, and so the integral would get a minimum value.

Now, however, the question suggests itself: when the function S or η is constant, is there not another quantity characterising the ensemble, which by its variation in a certain direction indicates the motion of the ensemble in the direction of the uniform distribution of place and MAXWELL'S distribution of velocities? For this motion is hardly open to doubt, and in a special case such a function has been found for one system by BOLTZMANN in the quantity H .

POINCARÉ supposes he has found such a function in his "entropie grossière", a quantity of the same form as S , but in which the elements of the area over which the summation is made, are not taken infinitely small, but so small that practically we cannot distinguish between systems lying within the same element. This quantity may, therefore, be represented by $\sum H \log H \cdot \delta$, in which δ represents the element and H the mean density in it. In contradistinction with this entropy S is called the "entropie fine", and it may easily be shown that the "entropie grossière" is always smaller than the "entropie fine". It is less easy to see that the "entropie grossière" gradually decreases, nor does POINCARÉ prove this. For it is not easy to see that the quantity $S = \iint P \log P dl d\omega$, of which he tries to prove in some cases that it has decreased, represents an "entropie grossière", while the proof too rests on an assumption which is unjustified in my opinion. It is true that we shall demonstrate further on, that there are quantities of this form which decrease, but for them the name of "entropie grossière" is not very appropriate, as the elements of the extension over which the summation is made are just as well infinitely small, though of lower order of magnitude than the original elements.

§ 2. A very suitable introduction in the theory of gases is supplied by the *problem of the small planets*¹⁾ repeatedly treated by POINCARÉ. There the problem is discussed what in course of time the distribution along the ecliptic will become of a number of small planets, which at some time were placed in their orbit in such a way that chance has decided at least the distribution of the velocities. POINCARÉ shows

¹⁾ Cf. l.c. and also: "Calcul des Probabilités", Paris 1896 and "La Science et l'Hypothèse" Paris 1904.

that if the number is large, and the planets do not interfere with each other, in the long run the planets will most likely get about uniformly scattered over the ecliptic.

If we should wish to treat the problem in exactly the same way as Gibbs, we should have to consider an ensemble of systems each consisting of n planets. As, however, the planets do not interfere with each other, we may also take an ensemble of systems of only one planet, in which case the ensemble represents all possibilities which may occur in the placing of a planet. When now such an ensemble, satisfying certain simple conditions, gradually spreads uniformly over the ecliptic, there is for every planet chosen at random from this ensemble, finally an equal chance to any place of the ecliptic, so that, if we have to choose a planet from such an ensemble n times, they will most probably be distributed about uniformly over the ecliptic, if n is large.

It is assumed that the orbits are circular, and lie in the plane of the ecliptic, so that every planet is determined by the variables l (length) and ω (angular velocity), in which ω is constant, and $l = l_0 + \omega t$, if also larger angles than 2π are admitted. The function S (POINCARÉ'S entropic fine) is, accordingly, here $\iint P \log P dl d\omega$ integrated over all the phases.

As X_1 corresponding to l is equal to ω , and X_2 corresponding to ω is equal to 0, here $\sum \frac{\partial X_i}{\partial x_i} = \frac{\partial \omega}{\partial l} = 0$, so the function S remains constant.

Yet the ensemble approaches uniform distribution over the ecliptic, which, however, is an altogether different thing from the density P becoming constant. Then S would, of course, decrease (it must be observed that $\iint P dl d\omega$ and $\iint dl d\omega$ remain also constant). This approach to uniform distribution is perhaps most readily seen, when we consider only that part of the ensemble that had originally a length between l_0 and $l_0 + dl_0$, the angular velocities lying between ω_1 and ω_2 . This part of the ensemble, being originally found in one point of the ecliptic, will get disintegrated by the different velocities, and gradually spread over the ecliptic, till finally the ecliptic is taken up a very large number of times. At a definite point of the ecliptic there are now parts, which were originally spread over a large number of elements of the extension, always at equal distances from each other, and it is easy to see, that if the function representing the original density, and its derivative are finite and continuous, the

density along the ecliptic will finally be the same everywhere ¹⁾. The adjoined fig. 1 represents the motion of the ensemble. Every point of the originally horizontal elementary area moves upwards in a vertical direction with constant velocity, so that the extension always occupies a slanting area with an inclination determined by $\tan \alpha = t$.

The horizontal areas at distances 2π from each other indicate the parts of the extension, which are in the same point of the ecliptic. Originally these parts have been in parts of the original area at equal distances from each other. These distances become smaller and smaller, the surface elements becoming more numerous and at the same time smaller.

Instead of the constant quantity $S = \iint P \log P dl d\omega$ (l thought to be continuous) we get a variable, when we immediately take together the

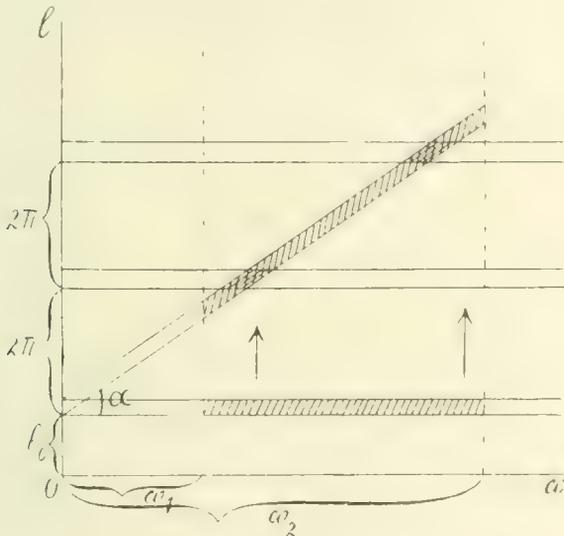


Fig. 1.

surface elements which come to the same thing with respect to the place in the ecliptic. So we get the quantity $S_p = \int_0^{2\pi} P' \log P' dl$, in

which P' represents the quantity found per unit of length in the conjoint areas, of a width dl and at distances 2π apart, which give the same placing in the ecliptic. So in the case referred to, that originally a horizontal area whose width is dl_0 was occupied,

$P' dl = \sum P_i dl_0 dl \cot \alpha = \sum P_i \frac{dl_0 dl}{t}$ so $P' = \frac{dl_0}{t} \sum P_i$, in which P_i re-

¹⁾ Cf. POINCARÉ l.c.

presents the density in each of the points of the original area which now come to the same thing.

Now this P' becomes, as we saw, in course of time a constant and so S_p minimum. This quantity might be called *entropy of place*; "entropic grossière" seems less appropriate, because the elements of extension are infinitely small. This final approach to a minimum must not be taken as a continual decrease. It is easy to see that the original function of density might be chosen in such a way that there are also times, at least in the beginning, at which the densities P' rather diverge than draw nearer to each other. Then the quantity S_p would increase; so $\frac{dS_p}{dt}$ is not like BOLTZMANN'S $\frac{dH}{dt}$ negative all through.

If, however, we compare times, in which first an angle 2π , then 4π etc. is occupied, we may say with a pretty high degree of certainty, that S_p has always diminished. If now instead of a horizontal area an arbitrarily chosen ensemble is considered, the above reasoning may be applied for every horizontal elementary area from it. So now too the ensemble is finally uniformly spread over the ecliptic, and the quantity $S_p = \int_0^{2\pi} P' \log P' dl$ becomes minimum. Now, however,

$P' dl = \Sigma \int P dl d\omega$ or $P' = \Sigma \int P d\omega$, every time integrated over all ω 's which fall within the horizontal area determined by dl . This P' becomes finally constant. The motion of the ensemble in this more general case is expressed by fig. 2.

The above mentioned inaccuracy in POINCARÉ'S reasoning is this: he considers the quantity $S = \iint P \log P dl d\omega$, in which he integrates with regard to l from 0 to 2π . So the P from this formula has evidently arisen by summation for the different values of l which come to the same thing, but not by integrating with respect to ω . Hence this P is the sum of the densities of the elements obtained by taking a definite ω , and then successively $l, l + 2\pi$ etc. In fig. 2 these elements are cross-hatched for one value of ω . However, POINCARÉ assumes further that finally for $t = \infty$ this P , or rather this ΣP no longer depends on l , but only on ω ¹⁾. This seems to me incorrect. For then for every vertical elementary area in fig. 2 the sum of the densities in the elements, which are every

¹⁾ Réflexions sur la Théorie cinétique des gaz; p. 381, p. 385 etc.

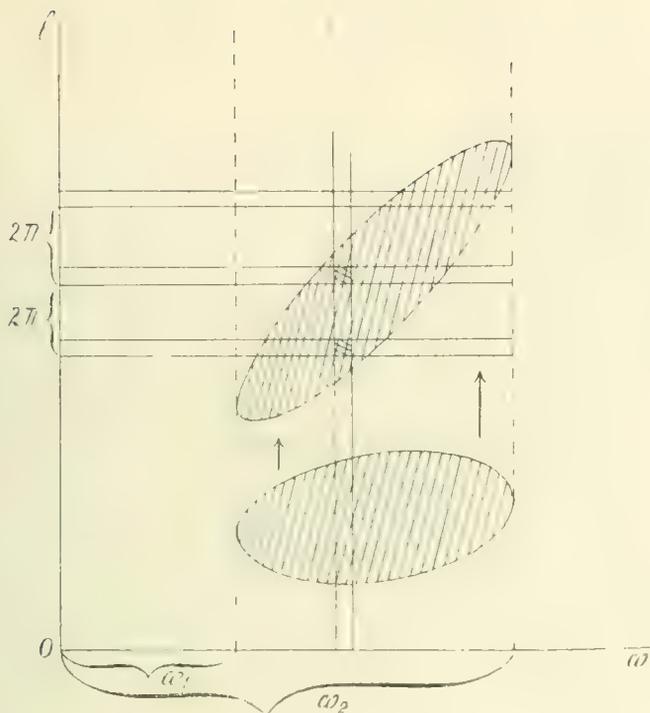


Fig. 2.

time at a distance 2π from each other, would finally have to become constant. This, however is only possible if the number of terms of this sum becomes at last infinitely large, which is by no means the case. Every vertical distance within the extension remains, namely, of the same length, so that the number of elements within each vertical area which are to be taken together, remains always finite. Only when the occurring ω 's extend over a finite distance, the number of terms of the sum for $t = \infty$ can also become infinitely large.

A second partial entropy, that *with regard to the velocities* is obtained by taking these elements of the extension together which give the same velocity. So this is here $S_s = \int_{\omega_1}^{\omega_2} P'' \log P'' d\omega$, in which

$P'' = \int P dl$ is integrated along the vertical areas. This entropy, is indeed, also smaller than the "entropic fine", but the difference remains constant, and so also S constant.

§ 3. A transition case from that of the planets to that of the gas molecules is furnished by the case of a gas of one dimension. By this POINCARÉ understands a gas, all the molecules of which move

velocities, and to ascertain whether the P integrated and summed over these spaces becomes constant in course of time.

It stands to reason that here the problem of the distribution of velocities will be the simplest, because it is modified directly by the impact, and the distribution of place only indirectly.

In agreement with § 2 where we considered an area from the original extension lying between l_0 and $l_0 + dl_0$ in order to examine the placing, we shall take here a part from the extension determined by limits of velocity lying infinitely near each other, but covering a finite part of the $3n$ -dimensional space of coordinates. In connection with the condition that $\Sigma v^2 = C$, we take from the $3n$ -dimensional space of velocities an element of a spherical shell, whose radius is $\sqrt{\Sigma v^2}$. To this corresponds a prismatic or cylindrical part of the extension, the base of which is represented by the element in question. With regard to the distribution of velocities the points from these and similar prismatic or cylindrical tubes come to the same thing. The elements of the spherical shell represent the projection of the tubes on the space of velocities. Now it remains to investigate whether the quantity of substance, which originally is found above the element mentioned in a given tube, will not finally have spread uniformly over all the tubes, so that the same quantity will be found above every element of the same size. If so, $S_s = \int P' \log P' d\tau$ will again become minimum, if $d\tau$ represents the size of such an element, and $P' d\tau$ the quantity which is projected in $d\tau$.

We may also call the points from the element of the spherical shell the points of velocity of the systems, and the vector, which joins the origin with such a point of velocity, represents all the velocities of the system both with regard to magnitude and to direction; the projections of the vector on the $3n$ axes of the space are the components of the molecular velocities.

The best way of setting forth the gradual uniform dispersion of these points over the mentioned hypersphere is perhaps by availing ourselves of BOREL'S mode of representation, and by partially following his method.¹⁾

BOREL imagines that in the same $3n$ -dimensional space in which the coordinates of the molecules are laid out (so that we get in this way a point representing the total arrangement for every system) also the components of velocity are projected starting from the representing point mentioned. The vector then obtained represents the velocity

¹⁾ "Sur les Principes de la Théorie cinétique des gaz" par EMILE BOREL. Annales de l'Ecole Normale Supérieure IIIe Série, 1906, N^o. 1, p. 9.

of the representing point, and we may now examine what takes place with this vector during the motion. It is now obvious that the representing point moves in a space inclosed by surfaces and spaces as $x_1 = r$, $x_1 = a - r$ etc. with which it collides when one of the molecules of the system strikes against a wall of the vessel. When two molecules collide, the representing point strikes against a surface, the equation of which has the form :

$$(x_m - x_n)^2 + (y_m - y_n)^2 + (z_m - z_n)^2 = 4r^2.$$

Now BOREL shows that for this impact the same rules hold as for ordinary collisions, so that the lines along which the point moves before and after impact, lie in the same plane as the normal, the normal dividing the angle of the first two lines into two equal parts. The velocity, too, remains the same. According to the above we must now imagine a finite space filled with such representing points, an infinitely small pencil of vectors of velocity starting from each point, mutually equal, and also equal for all the points. Now a number of these representing points strikes against one of the above mentioned surfaces, e.g. with equation

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = 4r^2;$$

this implies then that for these systems the first and the second molecule collide. This surface is that of a cylindre of revolution of the $3n-3^{\text{th}}$ degree, against the outside of which the points strike.¹⁾ The base of the cylindre is a sphere, the descriptive lines have here become descriptive spaces, namely plane spaces of $3n-3$ dimensions.

In the collision referred to, the extension from which the points come being large compared to the dimensions of the section of the cylindre (or at least of the same order) the infinitesimal pencil will extend in the directions of the perpendicular section of the cylindre, so here in 3 dimensions, to a pencil of finite width. If from this we take again an infinitesimal part, it comes from a definite point of each section of the sphere, and so from the points of a descriptive space of the cylindre.²⁾ Part of this strikes again against another cylindre (which e.g. involves collision of the 1st and the 3rd molecule), and the infinitely narrow pencil extends again to finite width; etc.

The representing points which have not taken part in these collisions strike again against another surface, and the pencil extends every time to one an infinite number of times wider, but every time in other

¹⁾ If the new coordinates $\xi, \xi', \nu, \nu', \zeta$ and ζ' determined by $\xi = \frac{1}{2}(x_1 - x_2) \div 2$, $\xi' = \frac{1}{2}(x_1 + x_2) \div 2$ etc. are introduced, the equation of the surface becomes $\xi^2 + \nu^2 + \zeta^2 = 2r^2$.

²⁾ Properly speaking a narrow region in the direction of this descriptive space,

spaces. However, when in a system the same molecule has had a great number of collisions; the extension has had a projection on the space determined by the axes of the coordinates of the 1st molecule for each of these collisions. So when every molecule has had a great number of collisions the above mentioned vector of velocity has passed round the sphere on which the points of velocity lie, a great many times in every direction. The points of velocity which originally covered an element of the spherical shell, will now occupy the whole spherical surface many times. As, however, the points of the sphere, where a point of velocity is after one, two etc. revolutions in a certain direction, come to the same thing with respect to the distribution of the velocities of the molecules, as in all the preceding cases we may again assume that finally the density is the same all over the spherical surface.

For the rest of the elements of the same spherical shell originally occupied the same reasoning holds. If there are also systems in the ensemble with another kinetic energy, the points disperse also here homogeneously in spherical layers; as, however, one kind cannot pass into another, the density may be different for the layers. It is the same as in the distribution of place when the gas masses are in different vessels. ¹⁾

Now the problem of the placing of the molecules. For this purpose we consider a part of the phase-extension, originally determined with regard to place by limits lying indefinitely near each other, but occupying a finite part of the space of velocities. Now we have to demonstrate

¹⁾ Also without BOREL's way of representation the above mentioned dispersion may be imagined to a certain extent. In each of a number of systems the molecules have the same velocities, but different places. Now it will entirely depend on the mutual situation e.g. of the molecules 1 and 2 in connection with their velocities, what the direction of the normal becomes in the collision, and so up to a certain extent, what the final velocities will be. In any case we get an infinitely large number from a single pair of velocities. Whereas we had before infinitely narrow limits between which the components of velocity had to lie, now we get a finite region. If we have chosen a definite one from the pairs of final velocities so also a definite velocity of the 1st molecule, this molecule may have all kinds of positions with regard to the 3rd molecule, against which it will presently strike, so also the normal of collision can have all kinds of directions, and so the limits thought infinitely narrow are again removed to a finite distance, etc. If we now take as variables the angles of the general vector of velocity with the axes of velocity instead of the components of velocity, the moving apart of the limits will yield a larger amount of occupied angles, so that finally there is occupied an amount of a large number of times 2π . If we now take into account, that increase of such an angle by an amount 2π has no effect, we arrive at considerations and results of quite the same nature as in the problems treated above.

that finally this ensemble will be uniformly scattered over all the combinations of place. In this case there will again be a partial entropy S_p which has become minimum.

We may try to make use of BOREL'S way of representation also here, and shall take as introductory case an ensemble of systems consisting of 2 molecules moving along the same line perpendicular to two parallel walls between these walls.

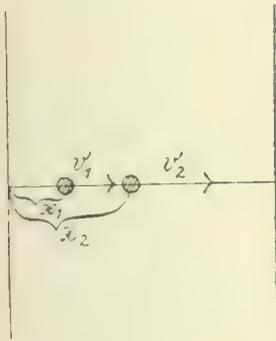


Fig. 3a

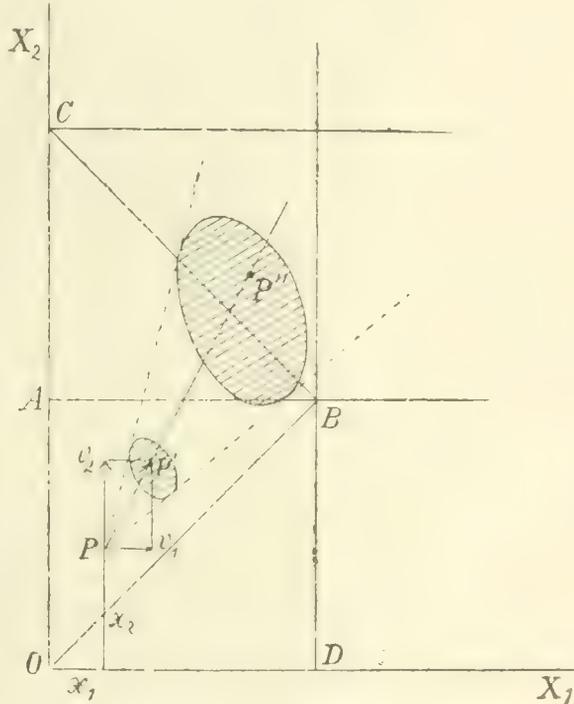


Fig. 3b

Fig. 3^a represents one system from the ensemble, fig. 3^b the motion of the whole ensemble, when the dimensions of the molecules are disregarded. Originally all the systems are in an element placed at $P(x_1, x_2)$, the points of velocity lying within an arbitrary figure. Here the representing points will be found after 1 sec., whereas they would occupy a continually extending figure if there was no collision. Collision of the molecules with the walls is here represented by collision of the representing points with the lines OA and AB , collision of the molecules inter se by collision with the line OB .

Now, however, we may also think the motion after the collision e.g. with AB continued past AB , if the triangle ABO which

would then be passed through for the second time with the velocity after impact is thought to be turned over along AB . Now the point pursues its course uninterrupted. A following collision with OB now becomes collision with BC ; now we may proceed again in the same way. The pencil then goes on without any disturbance, but we must take into account that the elements of surface, which in this way proceed from each other, represent the same placing of the two molecules. The continually extending figure of the representing points will finally contain a very large number of elements of surface of every kind coming to the same thing, or a very large number of turned over triangles, so that finally the points will be uniformly distributed over the sums of the elements of surface. So every situation of the two molecules represented by a point in ABO , occurs equally frequently. A point of BDO , however, is never reached; so every situation in which the 2nd molecule is on the right of the 1st is equally probable, but the 2nd cannot get to the left side of the 1st.

Now we should have to extend this reasoning to the case of more than two dimensions. The reflection against the walls does not affect our reasoning. The striking of the molecules against each other, however, is now represented by striking against a cylindrical surface. Though this obstructs the way, it no longer shuts off a part of the space. The case may be compared with that of fig. 3_b if the line OB is replaced by a circle. I have not succeeded in solving the problem for this general case. However, it seems very plausible that the finite number of cylindres will not be able to prevent that the uniform distribution of the representing points over the sums of the elements of volume which come to the same thing, which distribution would finally come about as we saw in § 3 if there was no collision, will be established also now. So all the combinations of place of the molecules would then occur equally frequently.

§ 5. Finally it may still be shown that when all the combinations of place and all the combinations of velocity occur with equal frequency, it follows from this that for the great majority of the systems the molecules are distributed about uniformly over the vessel, and have MAXWELL'S distribution of velocities. So far we have always distinguished between the individual molecules, now we shall have to take into consideration, that exchange of the molecules does not affect the *distribution* of place or velocity, so far as we can know it. So all the combinations which arise from each other by exchange of molecules, now come to the same thing. Hence if of s molecules s_1 are in the first element of volume, s_2 in the 2nd etc., there are

$\frac{s!}{s_1! s_2! \dots s_n!}$ combinations yielding the same distribution of place. As BOLTZMANN has shown, the denominator may be represented by $e^{\int f(xy z) \log f(xy z) dx dy dz}$ by approximation, in which $f(xy z)$ represents the function of distribution of the molecules over the vessel. The integral is minimum if $f(xy z) = C$, so the number of combinations is then maximum, or the uniform distribution is the most frequently occurring one. To show that the deviation from this distribution is not large as a rule, we may examine how many combinations yield a distribution, in which instead of $\frac{s}{n}$ molecules, $\frac{s}{n} + x_1, \frac{s}{n} + x_2$ etc. molecules occur in the elements. This number is:

$$\frac{s!}{\left(\frac{s}{n} + x_1\right)! \left(\frac{s}{n} + x_2\right)! \dots \left(\frac{s}{n} + x_n\right)!}$$

By putting $s! = s^{s+\frac{1}{2}} e^{-s} \sqrt{2\pi}$ etc., we get for this, taking into account that $x_1 + x_2 + \dots + x_n = 0$:

$$\frac{n^{s+\frac{n}{2}}}{(\sqrt{2\pi s})^{n-1} \left(1 + \frac{nx_1}{s}\right)^{\frac{s}{n} + x_1 + \frac{1}{2}} \dots \left(1 + \frac{nx_n}{s}\right)^{\frac{s}{n} + x_n + \frac{1}{2}}}$$

Now by approximation

$$\log \left(1 + \frac{nx_1}{s}\right)^{\frac{s}{n} + x_1 + \frac{1}{2}} = \left(\frac{s}{n} + x_1 + \frac{1}{2}\right) \left(\frac{nx_1}{s} - \frac{n^2 x_1^2}{2s^2} + \dots\right) = x_1 + \frac{nx_1}{2s} - \frac{nx_1^2}{2s} \dots$$

So the \log of the denominator (with the exception of the first factor) is by approximation:

$$\sum x + \frac{n \sum x}{2s} + \frac{n}{2s} \sum x^2 = \frac{n}{2s} \sum x^2.$$

The number of combinations becomes now:

$$\frac{n^{s+\frac{n}{2}}}{(\sqrt{2\pi s})^{n-1}} e^{-\frac{n}{2s} \sum x^2};$$

if we put $\sum x^2 = nu^2$, the chance, that the mean deviation is smaller than u , may be represented by:

$$e^{-\frac{n}{(\sqrt{2\pi s})^{n-1}} \int_0^u e^{-\frac{t^2}{2s}} dt} = \frac{1}{\sqrt{2\pi}} \int_0^u e^{-t^2} dt.$$

Very soon, however, this value is very large, when u is only a few times $\frac{1}{n} \sqrt{2s}$ as yet; then u or the mean x , however, is still small compared to $\frac{s}{u}$, the mean number of molecules per volume element.

In a similar way the problem of the distribution of velocities may be treated. Here the denominator of the expression $\frac{s!}{s_1! \cdot s_2! \cdot \dots \cdot s_N!}$

may be reduced to $C e^{\int f(\vec{s}) \log f(\vec{s}) d\vec{s}}$, in which $f(\vec{s})$ represents the function of distribution of the points of velocity. The integral is minimum, taking into consideration that $\sum v^2$ is constant, when $f(\vec{s}) = a e^{-b(v^2 + v'^2)}$. Now it remains to investigate what is the chance to a given deviation from this distribution. We may define this deviation by the figures $x_1, x_1', \dots, x_2, x_2', \dots$ etc. indicating the relative surplus of points of velocity in the elements of volume, respectively with the velocities v_1, v_2 etc. In the first element is then the quantity $s_1 = a e^{-b v_1^2} (1 + x_1)$, in the second $s_2 = a e^{-b v_2^2} (1 + x_2)$ etc., so that the number of combinations to be taken together:

$$\frac{s!}{[a e^{-b v_1^2} (1 + x_1)]! [a e^{-b v_1'^2} (1 + x_1')]! \dots [a e^{-b v_2^2} (1 + x_2)]! \dots}$$

The first factor of the denominator is equal to (by approximation):

$$(a e^{-b v_1^2})^{a e^{-b v_1^2} (1 + x_1) + \frac{1}{2}} \times (1 + x_1)^{a e^{-b v_1^2} (1 + x_1) + \frac{1}{2}} \times e^{-a e^{-b v_1^2} (1 + x_1)} \times \sqrt{2\pi} = \\ = C_1 (a e^{-b v_1^2})^{a e^{-b v_1^2} x_1} \times (1 + x_1)^{a e^{-b v_1^2} (1 + x_1) + \frac{1}{2}} \times e^{-a e^{-b v_1^2} x_1}.$$

If we multiply by the other factors, the latter part vanishes, as $\sum a e^{-b v_1^2} x_1 = 0$. So we keep:

$$C (a e^{-b v_1^2})^{a e^{-b v_1^2} x_1} (1 + x_1)^{a e^{-b v_1^2} (1 + x_1) + \frac{1}{2}} \times (a e^{-b v_1'^2})^{a e^{-b v_1'^2} x_1} \\ (1 + x_1')^{a e^{-b v_1'^2} (1 + x_1') + \frac{1}{2}} \times \text{etc.}$$

If we take the Nap. log. the former part vanishes, as also $\sum a e^{-b v_1^2} x_1 = 0$, so that log. denominator:

$$= \sum [a e^{-b v_1^2} (1 + x_1) + \frac{1}{2}] \log (1 + x_1) + C.$$

1) This a is equal to the above one multiplied by $d_2 d_3 d_4$.

This is by approximation:

$$ae^{-bv_1^2} \left\{ x_1 + \frac{x_1^2}{1.2} + \dots \right\} + ae^{-bv_2^2} \left\{ x_2 + \frac{x_2^2}{1.2} + \dots \right\} + \text{etc.} + C$$

or

$$ae^{-bv_1^2} \frac{\sum x_1^2}{1.2} + ae^{-bv_2^2} \frac{\sum x_2^2}{1.2} + \dots + C,$$

so that the whole form may be represented by

$$C e^{- \left[a_1 \frac{\sum x_1^2}{1.2} + a_2 \frac{\sum x_2^2}{1.2} + \dots \right]}$$

or if we denote the most probable quantities per element by a_1, a_2 etc.:

$$C e^{- \left[a_1 \frac{\sum x_1^2}{1.2} + a_2 \frac{\sum x_2^2}{1.2} + \dots \right]}$$

This exponent agrees perfectly with that obtained for the preceding problem. We may reduce the latter to the form $-\Sigma \left(\frac{ax}{\sqrt{2a}} \right)^2$,

and the former to the form $-\Sigma \left(\frac{x}{\sqrt{\frac{2s}{n}}} \right)^2$; now they represent

the negative sum of the squares of the absolute deviations divided by the root from twice the normal number. Just as in the preceding problem the chance to a combination of deviations for which the root from this latter sum does not amount to more than a few times unity is now very large. If we now take as measure

for the deviation the mean relative deviation or $\sqrt{\frac{\sum x^2}{N}}$, we see

that this value is very small compared to $\sqrt{\frac{\sum ax^2}{2}}$, so that this mean deviation will be very slight¹⁾.

To conclude we may still remark that in order to get in the end both uniform distribution of the molecules over the vessel and MAXWELL'S distribution of velocities, originally both a finite part of the space of velocities and of the space of coordinates must have been occupied. Or there must be such an uncertainty as to the original situation and velocities of the molecules that we must consider possible a finite whole of combinations with regard to both. This finite whole of possible combinations constitutes the ensemble, which we follow in its course instead of the system unknown within certain limits.

1) It having been assumed in the calculation that a considerable number of points of velocity still occur in every element, we must not think of the whole of the space of velocities when estimating the number of elements N .

Crystallography. — “*On the permissible orders of the axes of symmetry in crystallography.*” By Prof. W. VOIGT at Göttingen. (Communicated by H. A. LORENTZ).

(Communicated in the meeting of November 30, 1907.)

In one of the articles of the second part of his collected papers, H. A. LORENTZ took up the question — which is equally important in both crystallography and crystalphysics — of the permissible order of an axis of symmetry of the first or the second kind. In this investigation he proceeds from the principle of the rational duplicate ratio, from which he first proves that it is consistent with itself, and therefore a suitable basis for crystallographical deductions.

The study of this interesting treatise led me to the thought, that for the purpose at hand another fundamental principle of crystallography — viz. that of the rational indices — might well form a more convenient starting point. The continuation of this thought led me to the following development, which, I believe, attains the end in view in a remarkably simple and short manner. I will prove for this useful fundamental principle, as LORENTZ did for the principle of the rational duplicate ratio, that it does not contradict itself and then derive from it the permissible orders of the axes of symmetry.

1. The principle of the rational indices, as is well known, is as follows.

If we select three arbitrary boundary surfaces of a crystal polyhedron and draw through any point O parallels to their lines of intersection to form a system of axes; if we choose further two other arbitrary positions through this system of axes, and then the intercepts of the planes upon the axes are

$$\begin{aligned} u = OA, \quad v = OB, \quad w = OC \quad \text{on the one hand,} \\ u' = OA', \quad v' = OB', \quad w' = OC' \quad \text{on the other,} \end{aligned}$$

the principle of the rational indices maintains then, that,

$$\frac{u'}{u} : \frac{v'}{v} : \frac{w'}{w} = z_1 : z_2 : z_3 \dots \dots \dots (1)$$

forms at all times a ratio of whole numbers.

In order that this principle should lead to no contradiction, it is necessary that if one proceeds from three *other* boundary surfaces of the polyhedron and uses *their* lines of intersection as the fundamental system of axes, then the polyhedral surfaces have *also on these axes* intercepts with the above mentioned relation, if the principle held with reference to the first system of axes.

For the former the proof follows by repeated application of the law of sines, which gives according to the figure

$$\frac{u}{\sin \varphi''} = \frac{r}{\sin \psi'} = \frac{t}{\sin \varphi'}, \quad \frac{u'}{\sin \psi''} = \frac{r'}{\sin \varphi'}, \quad \frac{v''}{\sin \chi} = \frac{t''}{\sin \psi''},$$

while

$$\chi = \psi' + \varphi' + \psi'' + \varphi', \quad \pi = \varphi' + \psi'' + \varphi'' + \varphi' + \psi''.$$

From these we get

$$r = \frac{\sin(\varphi' + \varphi'') \sin(\varphi' + \varphi'')}{\sin \varphi' \sin \varphi''}, \quad \varrho = \frac{\sin(\varphi' + \varphi'') \sin(\varphi' + \varphi'')}{\sin(\varphi' + \varphi' + \varphi'') \sin \varphi'}.$$

The relation between r and ϱ , is most easily obtained by determining φ' from the first formula and substituting this value in the second. We thus obtain

$$r / (r - 1) = \varrho.$$

This shows that a rational r leads to a rational ϱ , which was to be proved.

The last part of the proof can still be simplified, according to a suggestion by LORENTZ, if we assume the MENELAUS' Theorem as known.

The desired proof is also given, when from

$$\frac{r'}{r} = r' \text{ and } \frac{u}{u} = r''$$

$$\frac{r''}{r} = \varrho' \text{ and } \frac{t''}{t} = \varrho''.$$

follow.

The former of these we have considered above; relative to the latter, MENELAUS' Theorem gives according to the figure

$$\frac{BD}{AD} = \frac{OA'}{AA'} \cdot \frac{BB'}{OB'},$$

i. e.

$$\frac{t''}{t} = \frac{u'}{u' - u} \cdot \frac{v - v'}{v'}.$$

Hence the rationality of u'/u and v'/v gives directly the rationality of t''/t .

2. The determination of the permissible order n of an axis of symmetry follows from any one of the fundamental principles of Crystallography, but only for the case when $n \geq 5$, because each of these principles places five similar crystallographic elements in relation. We usually so proceed, that the general property which the principle gives for the cases $n > 5$ is also demanded for the cases $n < 5$. We can however for the latter *limited* number of cases rely upon

experience, and apply the principle only for the former *unlimited* number of cases.

Since the principle of the rational indices permits surfaces of the crystal polyhedron to be translated parallel to themselves, therefore for its application axes of symmetry of the first and second kind have exactly the same value. A difference lies only in that for axes of the second kind n must necessarily be an even number.

We start with a construction upon a sphere of unit radius, through the center of which we lay all the directions that come into consideration. (Fig. 2). Let A be trace of the n -fold axis, P_1, P_2, \dots, P_3 the traces of the normals of 5 related surfaces (1), (2), \dots (5) of the polyhedron, such that $\varphi = 2\pi/n$. The P_h 's are then designated as the *poles* of these surfaces.

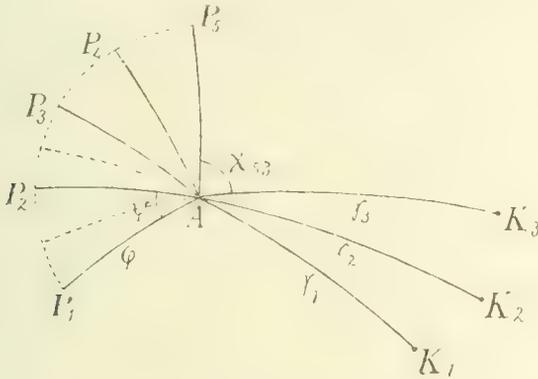


Fig. 2-

Let $K_1, K_2, K_3,$ be the traces of the lines of intersection of the surfaces (2, 3), (3, 1), (1, 2), so that

$$K_1P_2 = K_1P_3 = \frac{1}{2}\pi, \quad K_2P_3 = K_2P_1 = \frac{1}{2}\pi, \quad K_3P_1 = K_3P_2 = \frac{1}{2}\pi.$$

Let K_1, K_2, K_3 form the system of axes, and (4) and (5) the pair of surfaces for the application of the principle of rational indices. It is now a question of determining the intercepts which the surfaces (4) and (5) make upon the edges K_i .

If we give the surfaces such positions that they are tangent to the sphere in their poles, then the sections σ_{hi} are identical with the reciprocals of the $\cos(K_i P_h)$ where $i=1, 2, 3, h=4, 5$. Consequently the values of these cosines are to be determined.

If we write γ_i instead of $\angle AK_i$, g for $\angle AP_h$ and χ_{hi} for the $\angle P_h AK_i$ then from the $\triangle K_i AP_h$ in the figure we get

$$\cos(K_i P_h) = \cos g \cos \gamma_i + \sin g \sin \gamma_i \cos \chi_{hi}; \quad \dots \quad (4)$$

further, from the $\triangle P_3AK_1$, P_3AK_2 , P_3AK_3 we get

$$tg \gamma_1 = tg \gamma_3 = \frac{ctg \varphi}{\cos \frac{1}{2} \psi}, \quad tg \gamma_2 = \frac{ctg \varphi}{\cos \psi}, \dots \dots \dots (5)$$

while direct from the figure we get

$$\left. \begin{aligned} \chi_{41} &= \pi - \frac{3}{2} \psi, & \chi_{42} &= \pi - 2\psi, & \chi_{43} &= \pi - \frac{5}{2} \psi, \\ \chi_{51} &= \pi - \frac{5}{2} \psi, & \chi_{52} &= \pi - 3\psi, & \chi_{53} &= \pi - \frac{7}{2} \psi. \end{aligned} \right\} \dots \dots \dots (6)$$

If we write the principle of rational indices

$$\frac{\delta_{41}}{\delta_{51}} : \frac{\delta_{42}}{\delta_{52}} : \frac{\delta_{43}}{\delta_{53}} = z_1 : z_2 : z_3, \dots \dots \dots (7)$$

and observe that in the quotients of each two δ_{hi} 's, since

$$1/\delta_{hi} = \cos(K_i P_h) = \cos \gamma_i \cos \varphi (1 + tg \varphi tg \gamma_i \cos \chi_{hi})$$

the factor standing before the brackets always cancel, then we can easily introduce the values (5) and (6) and obtain from (7)

$$\frac{\cos \frac{1}{2} \psi - \cos \frac{5}{2} \psi}{\cos \frac{1}{2} \psi - \cos \frac{3}{2} \psi} : \frac{\cos \psi - \cos 3\psi}{\cos \psi - \cos 2\psi} : \frac{\cos \frac{1}{2} \psi - \cos \frac{7}{2} \psi}{\cos \frac{1}{2} \psi - \cos \frac{5}{2} \psi} = z_1 : z_2 : z_3.$$

This gives directly

$$\frac{\sin \frac{3}{2} \psi}{\sin \frac{1}{2} \psi} : \frac{\sin 2\psi \sin \psi}{\sin \frac{3}{2} \psi \sin \frac{1}{2} \psi} : \frac{\sin 2\psi}{\sin \psi} = z_1 : z_2 : z_3 \dots \dots \dots (8)$$

If we now take the first and last members of this double proportion we have

$$\frac{\sin \frac{3}{2} \psi}{\sin \frac{1}{2} \psi} : \frac{\sin 2\psi}{\sin \psi} = r$$

i. e. equal to a rational fraction, or also

$$\frac{1 + 2 \cos \psi}{2 \cos \psi} = r \quad \text{i. e.} \quad \cos \psi = \frac{1}{2(r-1)} = r',$$

where r' is also rational.

This requirement, when $\psi = 2\pi/n$ and $n > 5$ is fulfilled only for $n = 6$.

If we introduce this value of ψ in equation (8) we get

$$2 : \frac{3}{2} : 1 = \varepsilon_1 : \varepsilon_2 : \varepsilon_3$$

which is entirely consistent with the double proportion.

Consequently $n = 6$ is the only value ≥ 5 that is consistent with the principle of the rational indices. If we extend the requirement that $\cos \psi$ must be rational to the case $n < 5$ then the values $n = 2, 3, 4$, are also permissible for axes of the first kind, and the values $n = 2, 4$ for axes of the second kind.

Göttingen, November 1907.

Physics. — *“Isotherms of diatomic gases and their binary mixtures.*

VI. *Isotherms of hydrogen between -104°C. and -217°C. ”*

(Continued). By Prof. H. KAMERLINGH ONNES and C. BRAAK.

Communication N^o. 100^a from the Physical Laboratory at Leiden.

(Communicated in the meeting of November 30, 1907).

§ 17. *Survey of the determinations. Remark on the apparatus.*

The measurements mentioned in this Communication comprise in the first place the supplementary determinations to which we already alluded in § 14 of Comm. N^o. 99^a (Sept. 1907). These are three determinations at -217° at a density of about 170 times the normal one.

The obvious thing to do further was to repeat the other determinations of series II with the same piezometer arranged for the determinations mentioned above, this piezometer being one of about the same dimensions as that of series II of Comm. N^o. 97^a (March 1907). As a matter of fact a comparison of the values of pv_A obtained in this series with those yielded by the series III and IV teaches that the former lie somewhat, though only slightly, lower than the latter. This may be due to a systematical error as the filling in the later series was accomplished with more precautions (compare § 5 of Comm. N^o. 97^a). In the series now given, just as in series IV, distilled hydrogen was used.

Both the steel tubes on the stem of the piezometer and those on the stem of the piezometer reservoir were soldered to the glass (cf. § 15 of Comm. N^o. 99^a). This ensures a gas-proof connection with the steel capillary. With sealing wax it is difficult to make the connection gas-proof, because sometimes the nut begins to slide off when the flange is tightly screwed on.

§ 18. *Values of pv_{Λ} of series V.*

In table XX the results of the determinations have been represented in the same way as in table XII of Comm. N^o. 97^a. The temperatures at which the measurements were made were: —182°·74, —195°·16, —204°·62, —212°·91 and —215°·94. In table XX the reduction to the standard temperatures of table XII has been carried out. It was effected by interpolation by means of virial coefficients, which were derived in § 12 of Comm. N^o. 97^a, which enabled us to abandon the elaborate method of § 8. The computation of the temperatures took place in the same way as in Comm. N^o. 95^e (Nov. 1906). They may be reduced to the normal hydrogen scale by means of table XVIII of Comm. N^o. 97^b (March 1907).

TABLE XX, H₂. Series V. Values of pv_{Λ} .

N ^o .	t_s	p	pv_{Λ}	d_{Λ}
1	—182°·81	48·431	0·32746	147·90
2		55·499	0·32857	168·91
3		62·889	0·33028	190·44
4	—195°·27	42·304	0·27362	154·61
5		47·782	0·27351	174·70
6		52·808	0·27360	193·01
7	—204°·70	36·999	0·23165	159·72
8		41·258	0·23061	178·91
9		44·631	0·23001	194·04
10	—212°·82	32·035	0·19414	165·01
11		34·611	0·19270	179·61
12		37·275	0·19149	194·66
13	—217°·41	28·955	0·17318	167·63
14		31·191	0·17152	181·85
15		33·480	0·17005	195·12

§ 19. *Values of pv_{Λ} of series IV.*

If in the same way as in the preceding § the results of table XIX of Comm. N^o. 99^a are reduced to the standard temperatures the values of the adjoined table are obtained. For —139°·88 this

TABLE XXI, H₂. Series IV. Values of $p v_A$.

N ^o .	t_S	p	$p v_A$	d_A
1	— 103° 57	28.447	0.63261	44.967
2		38.186	0.63702	59.944
3		48.724	0.64198	75.897
4		58.368	0.64694	90.222
5	— 139° 88	25.406	0.49459	51.368
6		33.774	0.49697	67.960
7		41.273	0.49967	82.600
8		48.558	0.50232	96.667
9	— 164° 14 ¹⁾	22.818	0.40065	56.952
10		28.688	0.40164	71.427
11		34.387	0.40253	85.427
12		39.947	0.40376	98.936
13	— 182° 81	20.496	0.32704	62.670
14		24.818	0.32699	75.898
15		28.506	0.32672	87.248
16		32.568	0.32675	99.673
17	— 195° 27	18.527	0.27827	66.581
18		23.303	0.27724	84.055
19		27.837	0.27580	100.933
20	— 204° 70	16.749	0.24036	69.684
21		20.453	0.23876	85.658
22		24.015	0.23691	101.367
23	— 212° 82	15.416	0.20644	74.679
24		18.038	0.20430	88.296
25		20.643	0.20228	102.051
26	— 217° 41	14.635	0.18738	78.103
27		16.784	0.18491	90.766
28		18.853	0.18289	103.080

¹⁾ This temperature has been derived from the comparison of the resistance thermometer with the hydrogen thermometer of July 3, '07 (see table I, Comm. N^o. 101^a).

reduction has not been carried out, as it is better to take this temperature as standard temperature instead of $-135^{\circ}.71$. Here the reduction would have to be made for a comparatively large difference of temperature, and would become inaccurate. It is, therefore, preferable to leave the values of table XIX for this temperature intact, and if necessary apply the reduction to those of series I, which are much less reliable. The temperature $-164^{\circ}.14$ has been adopted as new standard temperature.

The determinations 5 and 9 as well as 14 and 18 of table XIX have been united to a mean.

§ 20. *Comparison of the series I and II with the control-determinations.*

For reasons mentioned in § 17 the points of series I and II have been doubly determined in a mutually perfectly independent way. They can be easily compared with the control determinations of IV and V by reducing them to the same density and temperature by means of virial coefficients. If in this way for $-103^{\circ}.57$ Nos. 2, 3 and 4 of series I (see table XII) are compared with 1, 2 and 3 of series IV (see table XXI), we find for the differences of pv_{Λ} for IV—I respectively:

$$+ 0.00001 \quad , \quad + 0.00007 \quad , \quad - 0.00019$$

and for $-139^{\circ}.88$ for IV (5,6) — I (2,3):

$$- 0.00085 \quad , \quad - 0.00036.$$

Dealing in the same way with the series II and V (see tables XII and XX), we find respectively for the temperatures $-182^{\circ}.81$, $195^{\circ}.27$, $-204^{\circ}.70$, and $212^{\circ}.82$.

$$V(1, 2) \quad - \quad II(2, 3) \quad = \quad + 0.00007, + 0.00010$$

$$V(4, 5, 6) \quad - \quad II(2, 3, 4) = + 0.00012, + 0.00026, + 0.00017$$

$$V(7, 8, 9) \quad - \quad II(2, 3, 4) = + 0.00020, + 0.00019, + 0.00034$$

$$V(10, 11, 12) - II(2, 3, 4) = + 0.00013, + 0.00008, + 0.00021$$

The differences are to be ascribed chiefly to condensation of impurities, as they diminish with increase of the temperature. This was considered as sufficient ground to reject the results of the series I and II for the further calculations as less reliable. This was also done for -104° , though the series I and IV harmonize very well for this temperature. When we disregard the influence of the probable condensation the very regular course in the situation of the points is an indication about the accuracy of the measurements themselves also for the other isotherms.

So there remain the determinations of the series III, IV and V, which, reduced to the standard temperatures, occur in the tables XII, XX and XXI. With these data the further calculations have been carried out. Plate I gives a survey of the situation of the points in the diagram of isotherms; on this plate $\frac{pv_A}{T}$ has been given as function of the density. (T absolute temperature). By I and II the isotherms of $100^{\circ}.20$, and 0° , which will be treated in the following communication, are indicated, by the other Roman figures ascending with decrease of temperature, those to $-217^{\circ}.41$.

§ 21. *Individual virial coefficients.*

In the same way as has been explained in § 12 of Comm. N^o. 97^a the first three virial coefficients of the development into series considered there were calculated for every isotherm, by means of the earlier and the new data. They have been put together in the subjoined table.

TABLE XXII. H₂ Individual virial coefficients.

t_s	A_A	$10^3 B_A$	$10^6 C_A$	$10^{12} D_A$	$10^{18} E_A$
$-103^{\circ}.57$	0.62048	+0.24409	+0.5300	+0.9113	-0.648
$-139^{\circ}.88$	0.48765	+0.11175	+0.4034	+0.6753	-0.378
$-164^{\circ}.14$	0.39891	+0.00732	+0.4148	+0.4970	-0.208
$-182^{\circ}.81$	0.33063	-0.07947	+0.3908	+0.3809	-0.088
$-195^{\circ}.27$	0.28508	-0.12309	+0.3165	+0.2892	-0.016
$-204^{\circ}.70$	0.25074	-0.17328	+0.3398	+0.2166	+0.031
$-212^{\circ}.82$	0.22103	-0.22271	+0.3599	+0.1514	+0.066
$-217^{\circ}.41$	0.20424	-0.24539	+0.3558	+0.1122	+0.082

In the same way as in Comm. N^o. 97^a the differences between the observed values of pv_A and those calculated by means of the found virial coefficients are put together in a table, which we subjoin.

The second column contains the differences for the points of the hydrogen thermometer (see table XII of Comm. N^o. 97^a), the following columns refer to the series IV, V and III in the order given here, the values being arranged according to the ascending densities for each series.

TABLE XXIII H₂. Deviations from the formula.

t_s	$10^5(O_1 - C_1)$									
-103.57	-1	+8	-1	-11	+6					
-139.88	-2	+13	-15	0	+4					
164.44	0	3	+7	6	+2					
-182° .81	-1	-15	+13	+3	+12	-15	-9	+12		
-195° .27	-4	-2	+25	-11	-16	0	+8			
-204° .70	-18	+4	+36	+22	-22	-24	-21	+23		
-212° .82	-14	+3	+12	+22	-5	-10	-4	-22	+18	
-217° .44	-14	+13	0	+15	-2	+1	-3	-22	-6	+18

It appears that on the whole series IV gives higher values than series V. The calculated curves may serve for the adjustment of the series mutually. Undoubtedly their points will be more reliable than those of the separate determinations. In future we shall, therefore, start from the virial coefficients of XXII.

§ 22. *Minima of pv_A .*

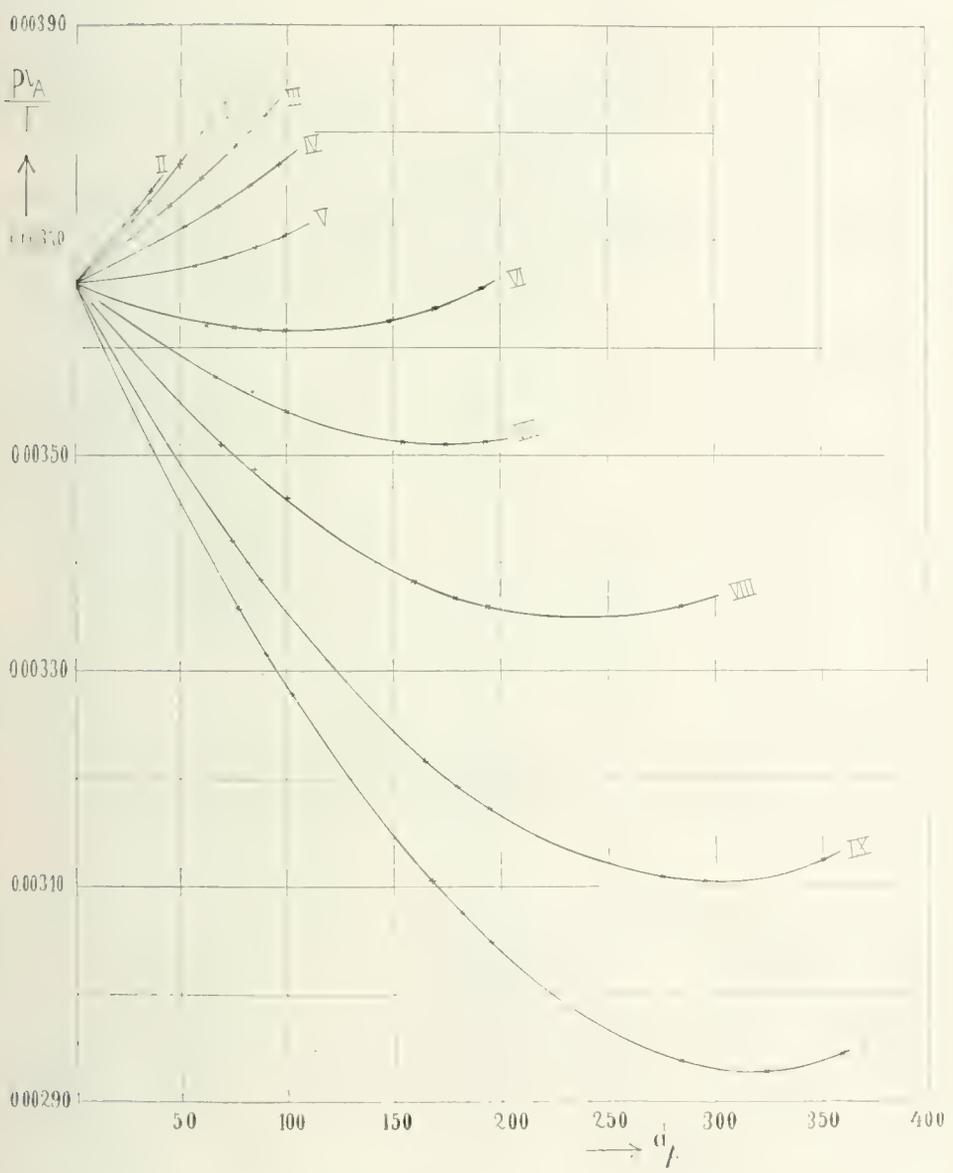
With the now available data the minima of the pv_A curves were now again determined for the lowest five temperatures, and as before the coefficients P_0 , P_1 and P_2 of a parabola calculated. The columns of table XXIV have the same meaning as those of table XV of Comm. N°. 97^a.

TABLE XXIV. H₂. Minima of pv_A .

t_s	pv_A	d_A	p	$O-C$
-182° .81	0.32663	99.70	32.57	-0.29
-195° .27	0.27348	183.40	50.07	+1.36
-204° .70	0.22945	238.27	54.67	-0.21
-212° .82	0.18782	287.99	54.09	-0.84
-217° .44	0.16342	321.51	42.54	+0.19
	0.39292	0	0	-0.23

H. KAMERLINGH ONNES and C. BRAAK. "Isotherms of diatomic gases and their binary mixtures. VI. Isotherms of hydrogen between -104° C. and -217° C." (Continued).

Pl. I.



For the calculation of the coefficients of the parabola a sixth point was used which has been inserted at the bottom of the table, and was obtained by means of the isotherm of $-164^{\circ}.14$. For this temperature the value of B_A is very slight, and by means of interpolation the BOYLE-point can be easily determined. For this is found, measured on the absolute scale:

$$\theta = -165^{\circ}.72$$

to which corresponds a value of $pv_A = 0.39292$.

For the coefficients of the parabola we find:

$$P_0 = -14.8370$$

$$P_1 = +676.563$$

$$P_2 = -1624.31$$

The differences of the last column are slight, except for $-195^{\circ}.27$. For this temperature C_A appears also to be too small (see table XXII).

Both deviations must be owing to the not quite accurate position of one or more of the points of the isotherms. From the diagram of Plate I it is already to be seen that the middle point of series IV probably lies too high.

The parabola cuts the ordinate $p = 0$ in two points where pv_A is respectively $= 0.39330$ and 0.02323 , with which agree the absolute temperatures:

$$T_1 = 6.3 \quad T_2 = 107^{\circ}.5.$$

For the top of the parabola $pv_A = 0.20826$, with which corresponds a pressure of 55.61 atmospheres. From this follows for the absolute temperature of the isotherm which passes through the top:

$$T = 64^{\circ}.2.$$

Physics. — *“Isotherms of diatomic gases and their binary mixtures. VII. Isotherms of hydrogen between $0^{\circ} C.$ and $100^{\circ} C.$ ”* By Prof. H. KAMERLINGH ONNES and C. BRAAK. Communication N^o. 100^b from the Physical Laboratory at Leiden.

(Communicated in the meeting of November 30, 1907).

§ 1. *Survey of the determinations.*

The reservoir of 5 cm.^3 of the piezometer of series IV (Comm. N^o. 99^a Sept. 1907) was replaced by one of 10 cm.^3 With this apparatus two isotherms were determined, in ice and in vapour of boiling water. To obtain constant temperatures the same instruments were used as in Comm. N^o. 60 (Sept. 1900). The water manometer (c.f. § 8 of Comm. N^o. 27 (June 1896)) was read, but the difference

of pressure amounting to no more than 0.5 mM., the corresponding correction for the temperature might be neglected. For the determination of the temperature of the steel capillary 3 thermometers were suspended along the capillary. For the determination of 100° a paper screen had been applied to turn off the rising current of heated air: the spiral with cold water, however, above the wool-packing of the boiling apparatus, had been omitted. The difference of temperature between the thermometers amounting to no more than 2°, this was permissible.

The corrected indication of the aneroid barometer amounted to 765.4 mM., from which for the temperature of the boiling-point follows 100°.20.

§ 2. *Values of pv_A .*

In the subjoined table the results of the determinations have been represented. The columns have the same meaning as in table XIX of Comm. N°. 99^a.

TABLE I. H₂. Values of pv_A .

N ^o .	t	p	pv_A	d_A
1	0°	27.333	1.01511	26.926
2		35.602	1.02002	34.903
3		43.413	1.02505	42.352
4		50.583	1.02964	49.127
5	100°.20	30.970	1.38619	22.342
6		39.796	1.39143	28.601
7		50.254	1.39788	35.951

§ 3. *Individual virial-coefficients.*

As has been done in § 12 of Comm. No. 97^a we may avail ourselves of the data of table I to derive the first two virial coefficients for every isotherm. On account of the small densities which occur in these measurements, in formula (1) of § 12 of Comm. No. 97^a, D_A , E_A and F_A have been put = 0, so that the formula reduces to:

$$pv_A = A_A + \frac{B_A}{v_A} + \frac{C_A}{v_A^2} \dots \dots \dots (1)$$

Only a small number of points having been given, and the densities being small, as was observed before, C_A cannot be determined with sufficient accuracy. We borrowed the values of this coefficient from Comm. No. 71 § 3, where $C_{A_0} = 0.0_6670$ and $C_{A_{100}} = 0.0_6606$.

In order to determine the course of the pv_A curves more accurately the value of pv_A was chosen for a density corresponding to that in the hydrogen thermometer of Comm. No. 60 with which (comp. Comm. No. 97^b XV § 1) 0.0036629 was found for the pressure coefficient for hydrogen at 1090 mm. zero point pressure. By successive approximations this value of pv_A is to be derived by means of these determinations of isotherms. We find for 0°:

$$pv_{A\ 0^\circ, 1100\ mm.} = 1.000256$$

and for 100°.20 with the pressure coefficient 0.0036629:

$$pv_{A\ 100^\circ, 2} = 1.367373.$$

For the density $d_A = 1.44$ may be put in both cases.

Now we obtain five values of pv_A and d_A for 0°, and four for 100°.20, from which by the method of least squares the coefficients A_A and B_A of equation (1) may be determined. These values are:

For 0°:

$$A_A = 0.99924$$

$$B_A = 0.5800 \times 10^{-3}.$$

For 100°.20:

$$A_A = 1.36626$$

$$B_A = 0.8632 \times 10^{-3}.$$

For 100°.00 we may calculate from this:

$$A_A = 1.36553$$

$$B_A = 0.8626 \times 10^{-3}.$$

The differences which remain between the values of pv_A of table I and those calculated according to formula (1) with the coefficients found here are respectively:

for 0°:

$$+ 0.00018, \quad - 0.00023, \quad - 0.00028, \quad + 0.00004, \quad + 0.00029$$

for 100°.20:

$$- 0.00013, \quad + 0.00034, \quad - 0.00001, \quad - 0.00019.$$

The first value always refers to the point calculated for the hydrogen thermometer. The differences are slight, and do not or

only very slightly exceed $\frac{1}{4000}$ of pv_A .

Physics. - - "On the measurement of very low temperatures. XVII. Determinations for testing purposes with the hydrogen thermometer and the resistance thermometer. Communication N°. 101^a from the Physical laboratory at Leiden by Prof. H. KAMERLINGH ONNES, C. BRAAK and J. CLAY.

(Communicated in the meeting of November 30, 1907).

§ 1. *Introduction.*

In communication N°. 95^e (Nov. 1906) the results of a number of measurements are recorded which show the possibility of measuring temperatures down to -217° with the hydrogen thermometer accurately to $\frac{1}{50}$ deg. The results obtained with several fillings showed

that with our measurements to -217° this accuracy has been reached indeed. It was our plan to make also the following measurements:

1st. more testing determinations between 0° and -217° in order to establish still better the limit of the accuracy of the temperature measurements with the hydrogen thermometer and the accuracy of the definition ¹⁾ once for all of special temperatures by definite resistances of a resistance thermometer;

2nd. the extension of the testing determinations to measurements in liquid hydrogen;

3rd. the determination of definite standard temperatures by means of the boiling points and melting points of hydrogen, oxygen and other substances that can be easily purified. ²⁾

4th (comp. Comm. N°. 95^a § 1, Sept. 1906) temperature measurements with the helium thermometer, *a.* for a direct or an indirect comparison with the hydrogen thermometer, *b.* in order to get a firm basis for the determination of the lowest temperatures, especially with a view to the reduction to the absolute scale.

The investigation mentioned sub 3 and 4 has advanced a good

¹⁾ In investigations the reading of temperatures with a resistance thermometer will as a rule for simplicity be preferred to reading them with the hydrogen thermometer.

²⁾ When we possess the fixed points meant here, the hydrogen thermometer can for calibrations be replaced by boiling point apparatus filled with pure gas and placed in the same bath as the apparatus to be calibrated. This is a great simplification in cases where only these *few special* temperatures are required. Moreover in these fixed points we have the means for a comparison between gas thermometers filled with different gases (for instance H₂ and He) or between thermometers in different laboratories.

deal. We now intend to communicate some measurements relating to 1st and 2nd.

§ 2. *Survey of the determinations.*

With regard to the controls meant sub 2nd two independent determinations have been made with entirely the same apparatus for comparison. The measurements meant sub 1st did not entirely succeed owing to a small reparation which the resistance thermometer required.¹⁾ These measurements, however, have thereby acquired a signification in another respect, namely as a new calibration from -104° to -259° of the resistance thermometer used in Comm. N^o. 95^c, they allow us to judge in how far after similar reparations, which in the long run will be inevitable, the same temperature coefficients will remain valid for the resistance thermometer.

The communicated measurements determine also with a greater accuracy a couple of temperatures (comp. however note 2 in § 3, 2^o) which hitherto had not been determined with the desired reliability (comp. §§ 3 and 5)²⁾.

The results have been combined in a table following below. The second and third columns contain the readings of the hydrogen thermometer and of the resistance thermometer. Those of the hydrogen thermometer are calculated in the way of N^o. 95^e (designated by t) and therefore require the corrections mentioned in Comm. N^o. 97^b (March 1907). They have not been applied here because this had not been done in any of the preceding communications and mutual comparison is thereby facilitated. The next column shows the resistances of column 3 recalculated with the factor 1.01806, which is the ratio between the resistances at 0^o C. before and after the breaking of the wire. These values have been compared with formula A_I of § 6 of Comm. N^o. 95^c. The fifth column contains the deviations from this formula. The sixth column contains the differences which were to be expected according to Comm. N^o. 95^c. The seventh column contains the differences between the two resistance thermometers in Ω .

¹⁾ When the resistance broke only $\frac{1}{60}$ of the wire was lost, yet on account of this one might allege that if the latter is not perfectly homogeneous variations in the coefficients of temperature are not entirely excluded. These are especially to be feared as a result of the new winding of the wire.

²⁾ These measurements are used in table V, Comm. N^o. 99^b (Sept. 1907) at $-252^{\circ}.82$ and $-255^{\circ}.18$ deviating from table I, Comm. N^o. 95^c (Sept. 1906).

TABLE I. Comparison of the platinum thermometer Pt_I with the hydrogen thermometer.

Date 1907	Temperature according to the hydrogen thermometer	Resistance in Ω	Resistance recalculated	$O_{PtI}-C_{Al}$	$O_{PtH}-C_{Al}$	$O_{Pt}-C_{Pt}$
3 July	— 103°.671	79.431	80.551	— 0.000	— 0.023	+ 0.023
3 July	— 119°.730	70.190	71.450	+ 0.003		
3 July	— 164°.113	45.054	45.863	+ 0.047		
25 March	— 182°.352	34.492	35.111	— 0.008	— 0.029	+ 0.021
29 June	— 216°.610	14.936	15.204	+ 0.016	+ 0.028	— 0.012
19 March	— 252°.822	1.9208	1.9553	+ 2.4131	+ 2.432	— 0.019
1 July	— 252°.839	1.9243	1.9588	+ 2.4180	+ 2.432	— 0.014
19 March	— 255°.177	1.6852	1.7154	+ 2.0518		
1 July	— 258°.864	1.4522	1.4783	+ 0.4855	+ 0.499	+ 0.287

§ 3. Results.

In order to be able to make conclusions from table I we remark that the later determinations have been made with the same thermometer filled with distilled hydrogen¹⁾.

From the results of table I and some earlier determinations at the same temperatures made with the resistance thermometer before it was broken, we may derive.

1. A comparison between the indication of the resistance thermometer before and after the breaking of the wire, abstracting from the reading errors of the hydrogen and the resistance thermometer, by means of the determinations of March 25, June 29, and July 3. They give only small differences for the observations on the last two dates. Indeed if we compare the differences $O-C_{Al}$ of table II of Comm. N°. 95^c

¹⁾ With the earlier determinations this was the case for only some among them. We may derive from former measurements (comp. Comm. 95^c), that this will give no difference till — 217°, but for measurements in hydrogen this has not been proved experimentally. The method of filling by means of electrolytic hydrogen, provided it be done carefully, may be considered as perfectly satisfactory also for these temperatures. As this method, however, involves a more complicate system of auxiliary apparatus the other must be preferred with regard to the reliability.

with those of table I of this Communication, the differences for -103° , -183° and -217° ¹⁾ respectively are:

$$+ 0.023 + 0.021 \text{ and } - 0.012$$

corresponding to

$$0^\circ.040, 0^\circ.036 \text{ and } 0^\circ.022.$$

From this we derive that down to -217° the variations in the temperature coefficients owing to the new winding of the wire, though not imperceptible are extremely small.

2. A comparison between two resistance calibrations, for which the same hydrogen and resistance thermometers were used, in the neighbourhood of the boiling point of hydrogen by means of the determinations of March 19 and July 1. The difference is $0.0049 \Omega = 0^\circ.046$ and exceeds $\frac{1}{50}$ deg. which has been derived as the limit of accuracy for measurements to -217° . This must probably be ascribed to the fact that the measurements of the resistance are less accurate because they are made with the WHEATSTONE bridge and not with the differential galvanometer ²⁾.

3. A comparison between the indications of the thermometer filled with distilled hydrogen and the one used before and filled with electrolytical hydrogen, by means of the determination of July 1, abstracting from the errors of observation of the hydrogen

¹⁾ For -217° this difference just reaches the limit of accuracy derived in Comm. N^o. 95^e and for the two other temperatures the difference only little exceeds this limit.

For -183° another reason may be given for this difference. To a certain extent it must probably be ascribed to the circumstance that the earlier determinations (of June 30 and July 6 '06) like those at -217° of June 30 '06 must be considered as less reliable. It appeared namely during an investigation started in Dec. 1906 that the steel capillary was no longer absolutely tight and this may also have been the case when the measurements under discussion were made. The latter becomes probable when we direct our attention to the great variations of the zero during these determinations, viz. 0.33 mm., to which we alluded in § 11 of Comm. N^o. 95^e without being able to explain it then.

The fault may have arisen because at the end of May '06 the thermometer has partly been taken to pieces and the capillary was bent too much. The observations made before June '06 were not influenced by this fault.

²⁾ The accuracy of the WHEATSTONE bridge is perfectly sufficient for higher temperatures below 0° C. (comp. Comm. N^o. 99^b § 2 for temperatures till -217°), but owing to the disadvantageous influence of the connecting resistances, it falls short for measurements in hydrogen where the variation of the resistance becomes so small. Therefore, simultaneously with the measurements of table I in hydrogen made with the WHEATSTONE bridge, we have calibrated another thermometer *P₁* with the differential galvanometer in order to fix the temperatures below -217°

and the resistance thermometer and variations in the temperature coefficients of the resistance¹⁾.

The difference appears to be larger than we should expect after the experience made with the higher temperatures. It may be that in the measurement of May 5 '06, the first measurement made in liquid hydrogen, in the measurement of the resistance or the reading of the hydrogen thermometer a systematic error has crept in which escaped our attention. At any rate it will be necessary to repeat the calibration at these lowest temperatures.

The differences treated in this section, in so far as they go beyond the expected limit of accuracy, point partly to abnormal sources of error, partly to errors which in future may be prevented (as for instance by always measuring small resistances by means of the differential galvanometer) and it is probable that when we avail ourselves of the experience made we shall reach also for temperatures below -217° an accuracy to $0^{\circ}.02$.

§ 4. In the same way as Comm. N^o. 95^c § 7 the following observations, where two resistance thermometers were simultaneously immersed in the same bath, allow us to judge of the accuracy with which a temperature is fixed by a given resistance.

With Pt_I we have made an adjustment to a definite temperature at which the resistance of Pt_{III} was determined, then the temperature was changed a little and again read on Pt_I and then the resistance of Pt_{III} was determined and reduced to the first temperature.

	temp. on Pt_I	— $87^{\circ}.54$,	resistance Pt_{III}	103.950 Ω ²⁾	
new	„ „ „	reduced — $87^{\circ}.54$,	„ „	103.959	difference 0.009 Ω
					or $0^{\circ}.014$
	temp. on Pt_I	— $216^{\circ}.65$		17.379	
new	„ „ „	reduced — $216^{\circ}.65$		17.385	difference 0.006 Ω
					or $0^{\circ}.009$

¹⁾ Although it is not excluded that here the variation of the temperature coefficients is larger than to -217° , this can by no means explain the large deviation because the wire had previously been carefully annealed. Moreover it is difficult to assume that impurities of the gas in the thermometer would be the cause, for then we must accept that about 0.7% of air has been present in the gas, which is rather impossible on account of the great carefulness observed when the thermometer is filled.

²⁾ For small differences in the calibrations of Pt_{III} and Pt_I we refer to Comm. N^o. 99^b where also the zero's are given. The differences result from a more accurate determination of the ratio between the arms of the WHEATSTONE bridge and of the resistance of the conducting wires. In observations which were made from 1905—1907 it appeared that the zeros had remained unchanged to less than $\frac{1}{20000}$ (comp. also Comm. N^o. 99^b).

The probable error of an adjustment to the resistance thermometer appears to be equal to that of a reading on the hydrogen thermometer (comp. Comm. N^o. 95^e § 7)¹⁾.

The following observations related to the defining of a temperature over a longer recording period.

Pt_{III} and Pt_V were calibrated immediately after each other with Pt_I , thence we derived the temperature reading according to Pt_I on Pt_{III} and Pt_V . Afterwards and adjusted to the same temperature we have compared the gold thermometer Au_0 with Pt_{III} and Pt_V immediately after each other. The readings were

temp. Pt_I on Pt_{III} ,	temp. Pt_I on Pt_V ,	Au_0 resistance ²⁾
— 58°.56	— 58°.56	40.324
— 87°.43	[— 87°.50]	34.638
— 159°.07	— 159°.08	20.393
— 216°.27	— 216°.29	8.459

If we disregard the large deviation in brackets which indicate that it must be ascribed to an irregularity, it appears that the definition of a temperature by means of a single determination on a resistance thermometer has about the same probable error as a single determination with the hydrogen thermometer and is generally reliable to an amount which remains below 0°.02.

For the present the accuracy of the determination of a temperature which is kept constant during a long time by means of the hydrogen thermometer may be considered equal to the accuracy of the definition of a temperature by means of the resistance thermometer, where however the necessary readings, even when they are repeated require less time.

§ 5. According to § 3 the observations of June 30 and July 6, '06 must be rejected. They have been used for the calibration of the thermo-element and the resistance thermometer of Comms. Nos. 95^a, 95^c and 95^f (Nov. 1906) and hence the values — 217°.411 and — 182°.75 of the tables VIII of Comm. N^o. 95^a, I, II of Comm. N^o. 95^c and IX of Comm. N^o. 94^f must be modified.

The determination of June 30, '06 at — 183° shows a perceptible difference with that of March 25, '07, the only one of the required

¹⁾ In Comm. N^o. 95^c § 7 we for the time being took as a starting point that the error in the adjustment of the hydrogen thermometer was negligible.

²⁾ These values deviate a little from the values given in table III of Comm. N^o. 95^f owing to corrections which were afterwards determined.

reliability. In observation 11 of table VIII we must therefore replace $-182^{\circ}.75$ by $-182^{\circ}.79$ and an analogous modification is required in the tables I, II and IX.

Instead of the values which relate to $-182^{\circ}.75$ of table II we now get:

TABLE II. Values for -183° instead of those of table II of Comm. N^o. 97c.

t	Number of observations	O.-C. <i>AI</i>	O.-C. <i>AII</i>	O.-C. <i>B</i>	O.-C. <i>C</i>
-182.79	2	-0.008	$+0.048$	$+0.104$	-0.014

The agreement is thereby greatly improved, as O.-C._{AI} is changed from -0.029 to -0.008 .

In the tables VIII of Comm. N^o. 95^a and IX of Comm. N^o. 95^f the three temperatures -183° from which a mean has been derived, are all diminished by $0^{\circ}.04$. O.-C. thereby becomes less by 0.0008 and the agreement improves in the same rate.

As to the determination of July 6, '06 the modification is very small.

In the tables VIII of Comm. N^o. 95^a and IX of Comm. N^o. 95^f no change occurs if we omit this last determination in the mean, in tables I and II it is not used.

With regard to this and to what has been said in note 1 of § 13 of Comm. N^o. 95^a, no new calculation has been made.

As a supplement to columns 2 and 3 of table I towards a complete calibration of Pt_I' we can use Pt_I in table V of Comm. N^o. 99^b with the zero 135.438Ω (comp. note 1 of § 4), where we have to add the corrections of table XVIII of Comm. N^o. 97^b.

Physics. — “On the measurement of very low temperatures. XVIII. The determination of the absolute zero according to the hydrogen thermometer of constant volume and the reduction of the readings on the normal hydrogen thermometer to the absolute scale.” Communication N^o. 101^b from the Physical Laboratory at Leiden by Prof. H. KAMERLINGH ONNES and C. BRAAK.

(Communicated in the meeting of November 30, 1907).

§ 1. *The determination of the absolute zero.*

D. BERTHELOT¹⁾ has used the observations of CHAPPUIS on the pressure coefficients between 0° C. and 100° C. and the slopes of the *pv*-lines at 0° C. and 100° C. to derive the mean relative pressure coefficient from 0° C. to 100° C. which the investigated gas would possess for densities in the state of AVOGADRO, (so we call for shortness the state in which the deviations from the law of BOYLE-GAY-LUSSAC-AVOGADRO may be neglected). In the same way we may use for this purpose the data of Comm. N^o. 100^b and Comm. N^o. 60 (Sept. 1900).

If for the pressure coefficient of the hydrogen thermometer for a zero pressure of 1090 mm. and a density of 1.44 found in Comm. N^o. 60 we derive the value 0.0036629, we may derive the pressure coefficient for the state of AVOGADRO (represented by α_{AV}) from the data of Comm. N^o. 100^b for B_{Λ} for 0° C. and for 100.°20 C. by means of the formula:

$$100 \times 0.0036629 = \frac{A_{\Lambda_0} \times 100 \alpha_{AV} + (B_{\Lambda_{100}} - B_{\Lambda_0}) \frac{1090}{760}}{A_{\Lambda_0} + B_{\Lambda_0} \frac{1090}{760}} \quad (1)$$

where we must replace the value of A_{Λ_0} found in § 3 of Comm. N^o. 100^b by a more accurate value:

$$A_{\Lambda_0} = 1 - B_{\Lambda_0} - C_{\Lambda_0} = 0.999419$$

and where $B_{\Lambda_{100}}$ has been derived from B_{Λ_0} and $B_{\Lambda_{100.20}}$ by interpolation²⁾. Hence follows for the desired pressure coefficient

$$\alpha_{AV} = 0.0036619.$$

The reverse gives the temperature of the freezing point measured on the absolute scale. Hence:

¹⁾ Sur les thermomètres à gaz. Travaux et Mémoires du Bureau international des Poids et des Mesures, T. XIII.

²⁾ In formula (1) the curvature of the *pv*-lines is left out of account, which is permissible.

$$T_0^{\circ} C. = 273^{\circ}.08 K$$

where K (KELVIN) stands for degrees on the absolute scale of which 100 occur between the freezing point and the boiling point of water.

The data are not sufficiently accurate to allow us to determine the last decimal to less than unity ¹⁾ ²⁾. The value found here agrees very well with that which may be derived from the most reliable data of other observers ³⁾.

The method followed here of deriving the pressure coefficient for infinitely small densities by means of determinations of isotherms at pressures between 25 and 50 atmospheres is preferable to using either the data of CHAPPUIS or those of AMAGAT. It is true that in the former case the coefficient C may be neglected without error arising, but the small difference of pressure has a bad influence on the determination of B . On the contrary with higher pressures, such as with AMAGAT's determinations, the coefficients C and the higher ones have too much influence to allow an accurate derivation of the value of B . In our determinations C is of so small account that an error in the estimation of C may be neglected for the determination of B ⁴⁾.

While therefore the influence of errors in C_A may be neglected we find on the other hand that the pressures are so large that an error in the pressure coefficient passes diminished to about

1) In discussing the isotherms we intend to refer to a small systematic difference between the isotherms of hydrogen at 20° C. according to Comm. N°. 70 (June 1901) and those at 0° C. and 100° C. of Comm. N°. 100^b. It rather points at $T_0^{\circ} C. = 273^{\circ}.07 K$.

2) We intend to determine this value still more accurately with nitrogen and helium by means of determinations of isotherms at 0° C. and 100° C. and of pressure coefficients between 0° C. and 100° C. where we proceed according to Comm. N°. 60 (the determination of H₂ is also repeated), but as a higher degree of accuracy is wanted (designated by that now reached with the determinations of the isotherms) we now take a reservoir of 300 c.c.

3) Comp. for this the note of § 2. XIV Comm' N°. 97^b.

4) If for instance in the adoption of C_A an error of 15% has been made, which with a view to the data of table XXII of Comm. N°. 100^a probably includes the higher limit for the error for lower temperatures, this becomes only 0.0000001 which, considering that the greatest density which occurs in the determinations of Comm. N°. 100^b amounts to about 50 times the normal one, would cause for 0° C. an error in B_A which remains below 0.000005. As such a systematic error would change the value of $C_{A,100^{\circ}}$ in nearly the same way, the error in this difference will be much smaller and, for instance, the error arising thence in the difference between $B_{A,0^{\circ}}$ and $B_{A,100^{\circ}}$ may be estimated at 0.000001, i. e. $\frac{1}{5}$ of the error in the absolute value of B_A . The error in the absolute zero arising from the two influences combined remains below 0².01 C.

$\frac{1}{25}$ into the value of B_A . Hence in this way we have obtained data at a rather large difference of pressure, from which B_A may be derived unambiguously.

§ 2. *Reduction of the readings on the normal hydrogen thermometer to the absolute scale.*

By means of formula (3) of § 2 of Comm. N^o. 97^b and with the new values of B'_T (comp. formula (2) of § 2 Comm. N^o. 97^b) which may be derived from the data of table XXII, we have determined anew the corrections of the readings of the hydrogen thermometer of constant volume to the absolute scale. For this we have started from the individual virial coefficients and not as in Comm. N^o. 97^b from a general temperature formula, because the course of the separate isotherms has now been ascertained sufficiently to render a similar previous equalization superfluous.

For B'_0 and B'_{100} we have adopted other values than in Comm. N^o. 97^b. We have namely used the results obtained from direct determinations of isotherms at 0° C. and 100° C., of which the results are laid down in Comm. N^o. 100^b. These values are:

$$B'_0 = 0.0005807 \qquad B'_{100} = 0.0006321.$$

for the absolute zero we have adopted

$$t = -273^{\circ}.08 \text{ C.}$$

TABLE XXV. H_2 . Corrections to the absolute scale.

t_s	θ	$B'_T \cdot 10^3$	Δt_s	Δt	(O. C.) $\times 10^3$
- 103° .56	- 103° .54	+ 0.394	+ 0.017	+ 0.016	0
- 139° .87	- 139° .84	+ 0.229	0.029	0.026	- 1
- 164° .13	- 164° .09	+ 0.018	0.043	0.039	+ 1
- 182° .80	- 182° .75	- 0.241	0.056	0.051	+ 3
- 195° .26	- 195° .20	- 0.432	0.059	0.054	- 3
- 204° .63	- 204° .62	- 0.632	0.069	0.063	- 2
- 212° .81	- 212° .73	- 1.003	0.079	0.072	+ 1
- 217° .40	- 217° .32	- 1.293	+ 0.083	+ 0.076	+ 1

derived in the previous § from the same determinations of isotherms. The results have been combined in the preceding table in the manner of table XVI of Comm. N^o. 97^b.

The differences with the earlier values remain, notwithstanding we have used entirely different data, far within the limits of the accuracy mentioned in § 3 (*loc. cit.*). The coefficients of the formula given there become:

$$a = 0.007117$$

$$b = + 0.005962$$

$$c = 0.000185$$

$$d = + 0.001330$$

With this formula the temperatures of the second column have been determined. The differences between the data of the last column but one and the formula are given in the last column.

For -273° the new formula gives the same value as the one before of Comm. N^o. 97^b i.e. $\Delta t = +0.14$, for 0° C. and $+100^{\circ}$ C. it gives $\Delta t = 0^{\circ}$. For the temperatures between 0° C. and 100° C. the formula yields much larger negative values than those which BERTHELOT has derived with his equation of state (*loc. cit.* IX). For 20° , 50° and 80° are found: according to BERTHELOT:

$$\Delta t = -0^{\circ}.00046 \quad -0^{\circ}.00053 \quad \text{and} \quad -0^{\circ}.00033$$

according to our formula:

$$\Delta t = -0^{\circ}.0012 \quad -0^{\circ}.0020 \quad \text{and} \quad -0^{\circ}.0014$$

According to the general equation of state of hydrogen derived previously (comp. § 1 Comm. N^o. 97^b) these values would be respectively:

$$-0^{\circ}.0026 \quad , \quad -0^{\circ}.0047 \quad , \quad -0^{\circ}.0036.$$

Geology. — “*On the terms Schalie, Lei and Schist.*” By Mr. J. SCHMUTZER. (Communicated by Prof. C. E. A. WICHMANN).

East-Indian mining engineers, by whom almost exclusively the existing descriptions of rocks in Dutch have been written, under the influence of the German literature relating to this subject, have used the terms *schiefer* and *lei* side by side, on the whole with hardly any difference. The cause of this is the want of a fixed geological terminology in our language; a want which of late has also been felt by the “Ned. Mijnbouwkundige Vereeniging”, as appears from its attempt to create such a terminology under the supervision of Prof. G. A. F. MOLENGRAAFF. The purport of this communication is to aid in solving the problem, *what* terms are best fitted to denote in Dutch those rocks which in German bear the name of *Schiefer* in the amplest sense of the word.

The starting-point is formed by the consideration, that, for the sake of clearness, different notions should be rendered by different words, and that, in the choice of terms, as long as there is a choice everything that might give rise to ambiguity, should be avoided.

The distinct difference which in nature exists between exclusively sedimentary slate, metamorphosed in a relatively slight measure and the finer or coarser crystalline, more or less distinctly foliated metamorphosed rocks, partly of sedimentary, however also partly of eruptive origin, is greatly discounted by the application of the term *lei* also to the latter, although the term "crystalline" be added. This explains partly, how the word *schiefer* could introduce itself into the Dutch terminology, though in a meaning that is far from being a fixed one; the want was felt of another presentive word. A bad choice, however, was made with the word *schiefer*¹⁾, as the latter, by taking up a place by the side of the word *lei*, must necessarily assume a more limited meaning in our language, since in the language from which it was borrowed, it occurs in different widely-diverging combinations²⁾. The consequence of this is an absurd state of things, which is not to be improved by reintroducing the disused *scheversteen*³⁾, which has been tried in the shorter form of *schever*⁴⁾. In order to prevent a possible confusion of ideas, which lies concealed in the analogy with the German cognate word, this word seems to be less fitted. The corresponding English *shiver*, which has maintained itself among miners in the meaning of "flake of stone, shale, slaty debris"⁵⁾, does not occur as a scientific term⁶⁾ and could not as such be put side by side with *shale* and *slate*.

The solution of the problem is considerably simplified by the circumstance that the English *shale* is etymologically identical with the Dutch *schalie*, a word not used in the north of Holland. Whilst the English *shale* as a secondary form of *scale* and *shell* may be directly reduced to

1) Already used in most Dutch dictionaries.

2) As is well known also the Danish-Norwegian *skifer*, the Swedish *skiffer* is used to denote the English terms *shale* as well as *slate* and *schist*.

3) PLANTIN, cf. E. VERWIJS and J. VERDAM, *Middelned. Woordb.*, IV, 336—337; VII, 224; J. v. W. GRIMM, *Deutsch. Wörtl.* IX, 1, sq. KILIAEN gives the word in the sense of 1^o. *schalie*, 2^o. *sicamb.*, i. e. the word used in Cleves for Latin *silex* (comm. of Prof. J. W. MULLER).

4) See also KLUGE, *Etym. Wörtl. d. deutsch. Spr.* 337; GRIMM, l. c., E. MÜLLER, *Etym. Wörtl. d. engl. Spr.* II, 378.

5) WRIGHT, *Eng. Dial. Dict.* V, 392.

6) Cf. ARCH. GEIKIE, *Text Book of Geology*, JAS. GEIKIE, *Struct. a. Field Geology*; TEALL, *British Petrography*.

the old-English *scate*, *schale*¹⁾, this *schalie* is formed from the old-French *escaille* (Fr. *écaille*, Ital. *scaglia*), which, in its turn however, has been borrowed from Germanic²⁾. Now, where these two words are closely connected both in origin and in meaning, it cannot be subject to serious objection to identify the Eng. *shale*: "applied to all argillaceous strata . . . which split up more or less perfectly in their line of bedding,"³⁾ with the Dutch term *schalie*.

Prof. J. W. MÜLLER had the kindness to communicate to me as follows⁴⁾: As far as I can go into the question, the state of things with these two words is the following: Time out of mind *lei* in Holland, Utrecht and the (north and south) eastern provinces (cf. also Lorelei, Erpeler Lei on the Rhine, etc.), *schalie* on the other hand in the southwest: Flanders, Zeeland and South-Holland isles, have been the only word for both kinds of rocks, now more closely distinguished by geologists. Now you wish to confine the north-eastern word to the one, the south-western to the other kind. Of course this is something arbitrary, but in a scientific terminology, such unnatural and artificial distinctions are necessary, and there is nothing against it; the Flemish people will continue calling everything *schalie*, the Dutch *lei*, and there is no objection to this either". Prof. G. A. F. MOLENGRAAFF was so kind as to write to me that already at his lectures he availed himself of the term *schalie*, as an equivalent of the English *shale*⁵⁾; a happy circumstance, which not only pleads for the weight of the grounds alleged, but will at the same time contribute much to make this term enter into general use⁶⁾.

The English *slate*⁷⁾ refers to argillaceous rocks, which by a metamorphosis, in which the pressure has predominated but a chemical

1) MÜLLER, II. 365.

2) Cf. Anglo-saxon *scēalu*, putamen, gluma; Gotic *skalja*, "Ziegel, eigentlich wohl Schindel, Schuppenartiges", KLUGE, 331; FRANCK, Etym. Woordb. d. ned. Taal, 827; MÜLLER, II, 365; KLUGE, 294, 351; VERWIJS and VERDAM, VII, 224; GRIMM, VIII, 2060—2064.

3) NICHOLSON cf. WRIGHT, V, 348. ARCH. GEIKIE'S definition runs (op. cit. 2d edit 1885, 164), *shale*, (synon. Fr. *schiste*, G. *Schiefërthon*), "clay that has assumed a thinly stratified or fissile structure," see further JAS. GEIKIE, 62. By the side of the French *argile schisteuse* we find the Ital. *scisto argilloso* (argilloscisto) and *argilla scagliosa*, lit. *kleischalie*.

4) Letter of 24 Dec. '07.

5) Dec. 16th '07: for argillaceous on other rocks "with indistinct, more or less shelly stratification."

6) The term *schalie* was used by the late Prof. J. L. C. SCHROEDER v. D. KOLK for the Eng. *crystalline schist*, but could not maintain itself in this meaning

7) Old-Eng. *slat*, *sclat*, *sklat*, old-Fr. *esclat*, Fr. *éclat*; COTGRAVE says: "*esclat*, a shiver, splinter, also a thin lath or shingle" cf. MÜLLER, II, 400—401.

change is not excluded, have got a thinly fissile structure, always parallel to the axis of the synclinal or anticlinal folds of system and consequently can make widely diverging angles with the original line of bedding) ("false bedding"). It is to be recommended, in accordance with the original ¹⁾, now generally current, meaning of the word, to confine the term *lei* to these rocks, of which roof-slate is the best known representative. Therefore on the ground of the definition given, the above affix, apart from the well-known Rhenish devonian slates, ought to be given to the greater part of the so-called "Fleckschiefer", "Knotenthonschiefer", andalusite-, staurolite-, chiasolite slates, etc.

The English term *schist* ²⁾ is used for such heterogeneous rocks that the existence of this word by the side of *lei*, *slate* is fully justified. Not only does *schist* denote the more or less foliated peripheral facies of purely eruptive rocks, which often without any distinct boundary-line pass into adjoining strata of sedimentary rocks, — whether these eruptive rocks occur in smaller laccolithes and dykes or in extensive intrusions or effusions (as many "trapps"); — also sedimentary rocks can by contact change into *schists*, metamorphosing agencies in strongly disturbed regions can without any distinction on a large scale change sedimentary and eruptive rocks into crystalline rocks, more or less distinctly, sometimes excellently foliated, such as gneiss, eyegabbro, amphibolite, etc. Though the term *schist* in its various shades of meaning ³⁾ and especially the adjective derived from it ⁴⁾, has got a much wider meaning in Romance languages, yet it finds also there, by the simultaneous use of the equivalents of *lei* and *slate*, a more limited application than in German. In connection with this it is perhaps recommendable, more particularly after English

¹⁾ JAS. GEIKIE, op. cit. 76, 220, sq.; WRIGHT, 504; ARCH. GEIKIE, 125—126: "In England the term *slate* or *clay-slate* is given to argillaceous, not obviously crystalline rocks possessing this cleavage structure", (syn. *argillaceous schist*, Fr. *phyllite*, *phyllade*; *schiste ardoise*; G. *Thonschiefer*, *Thonglimmerschiefer*).

²⁾ PLANTJN, "schalie, leye oft scheversteen", une ardoise, ardozia, scandula, cf. VERWIJS and VERDAM, IV, 336-337; VII, 224, by the side of which stood the meaning "rots, leisteen", (cf. also "een leye der scandaliseringhe"); old-Saxon. *leia*, *rots*, cf. FRANCK, 558; KLUGE, 243. Mdl. Dutch had the word in the meaning of "Slate used for roofs", VERWIJS and VERDAM, IV, l. c.

³⁾ From $\sigma\chi\acute{\iota}\zeta\alpha$, to split, to sever, to cleave, to divide; adj. verb. $\sigma\chi\iota\sigma\tau\acute{\iota}\varsigma$; ARCH. GEIKIE, 124: "a rock possessing this crystalline arrangement into separate folia is in England termed a *schist*. As Prof. MOLENGRAAFF communicated to me, this term was already used by him.

⁴⁾ Sp. also *pizarra cristalina*.

⁵⁾ which coincides with Germ. *schiefrig*, Dan. Norw. *skifrig*, Swed. *skiffrig*.

usage, to denote also in Dutch the *krystalline Schiefer* by means of the term *schist* (pl. *schisten*) borrowed from the Greek, and to confine one's self in the application of the derived adjective *schistens* and the substantive *schistositeit* exclusively to this class of metamorphosed rocks.

While, in conclusion, I express my special gratitude to Prof. J. W. MÜLLER for his kindly furnished contributions and for some valuable hints, I cannot omit tendering my best thanks to Prof. A. WICHMANS and Prof G. A. F. MOLENGRAFF for their lively interest and the support with which they obliged me.

Chemistry. — “On the question as to the miscibility and the form-analogy in aromatic Nitro- and Nitroso-compounds”. By Dr. F. M. JAEGER. (Communicated by Prof. A. P. N. FRANCHIMONT).

1. The following communication contains a further contribution to the knowledge of the mutual behaviour of the aromatic nitro- and nitroso-derivatives, of which several particulars were given previously. The entire miscibility in the solid condition, and the form-analogy bordering on isomorphism, were established in the case of *p*-Nitro- and *p*-Nitrosodiethylamine¹⁾, while afterwards, in a more extended paper of a more general nature²⁾, the mutual comparison of *p*-Nitro- and *p*-Nitrosophenol and of *o*-Nitroso- and *o*-Dinitrobenzene was discussed. It then appeared that no *general* rule could be laid down as to the connection of the two classes of compounds.

A new pair of similar comparable substances which are interesting from more than *one* point of view, namely *o*-Nitro- and *o*-Nitrosoacetanilide were now studied. The result was different from that obtained with *p*-Nitro- and *p*-Nitrosodiethylamine although analogy is present in *one* of the axial relations, and a solid solution of the two components appears to be possible to some slight extent.

§ 2. Ortho-Nitro-Aceto-Anilide.

$C_8H_7(NO_2) \cdot N H \cdot CO CH_3$; meltingpoint: $87\frac{1}{2}^{\circ} C$.

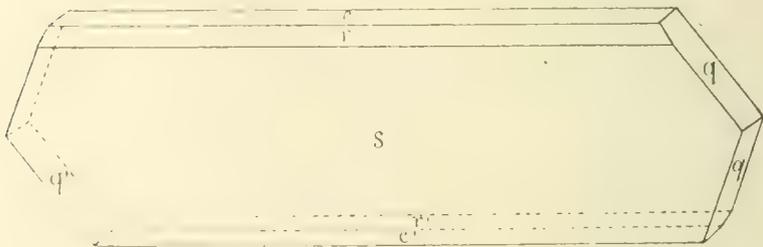


Fig. 1.

¹⁾ F. M. JAEGER, These Proc. VII, p. 660,

²⁾ idem, Ueber Mischbarkeit von festen Phasen, Z. f. Kryst. **42**, 236—276 (1906).

On account of its great solubility in most solvents it is very difficult to obtain this compound in a properly crystallised form. The greatest success is met with by very slow evaporation of its solution in dilute alcohol when it crystallises in pale yellow, very thin laminae with extended hexagonal periphery; the crystals are very transparent and give sharp signal-images.

The symmetry is *monoclinic-prismatic*. In the choice of the plane-symbols adopted here the axial relation is calculated on:

$$a : b : c = 0.8935 : 1 : 1.9198$$

$$\beta = 83^{\circ}51'$$

Forms observed: $s = \{101\}$, strongly predominant and yielding sharp reflexes; $r = \{101\}$ much narrower, but gives a good reflection; $c = \{001\}$, narrower than r ; $q = \{011\}$, also narrow, but reflects well. The habit is flattened along $\{101\}$ with considerable elongation along the b -axis.

The following angular values were determined:

	<i>Measured:</i>	<i>Calculated:</i>
$c : q = (001) : (011) =$	* $62^{\circ}21'$	—
$q : s = (011) : (10\bar{1}) =$	* $80^{\circ}57'$	—
$c : r = (001) : (101) =$	* $60^{\circ} 4'$	—
$r : s = (101) : (10\bar{1}) =$	$49^{\circ}45'$	$49^{\circ}45'$
$s : c = (10\bar{1}) : (00\bar{1}) =$	$70^{\circ}11'$	$70^{\circ}11'$

A distinct plane of cleavage was not found.

In the orthodiagonal zone the direction of extinction is orientated everywhere perpendicular to the direction of the b -axis. On $\{101\}$ no perceptible dichroism is observable.

The optical properties of the substance in convergent polarised light are very remarkable.

For the red, yellow and most of the green rays of the spectrum the axial plane = $\{010\}$; extraordinarily strong, inclined dispersion; the axial angle for the red is much larger than that for the green. The character of the double refraction is positive.

On the other hand, the axial angle for the blue and violet rays is situated perpendicular on $\{010\}$ with a horizontal dispersion. The axial angle for all rays is but small.

The curious colourphenomena in white light exhibited by this substance, which thus possesses at the same time inclined and horizontal dispersion, lend themselves particularly well to the demonstration of anomalous dispersion in biaxial crystals.

The sp.gr. of the crystals is 1.419 at 15° ; the equivalent volume is therefore 126.85.

Topic parameters: $\chi : \psi : \omega = 3,7578 : 4,2058 : 8,0744$.

§ 3. Orthonitroso-Aceto-Anilide.

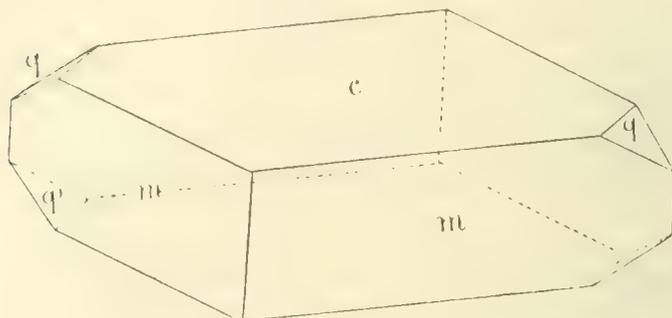
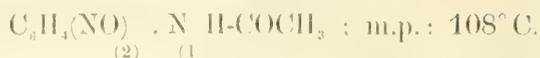


Fig. 2.

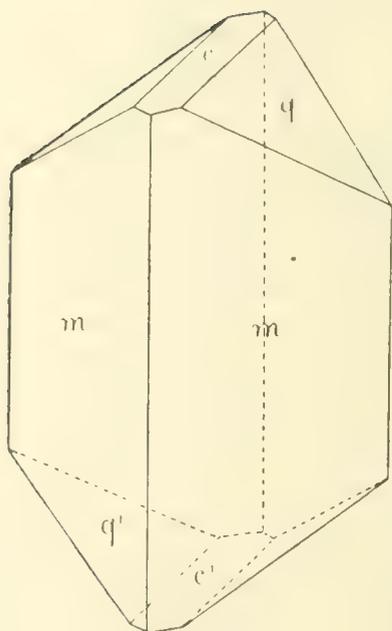


Fig. 3.

Through the kindness of Dr. F. LEUCUS of *München*, who was the first to prepare this substance (Ber. 1907 40. 1083), I received a small quantity of the crystallised compound obtained by cooling the hot alcoholic solution. The crystals exhibited the habit of fig. 2; from a mixture of ether and benzene I obtained the thick prismatic crystals of fig. 3. The prism-planes, which often act as resting planes of the crystals in the mother liquor, were in consequence mostly curved and unsuitable for accurate measurement, whilst the forms *c* and *q* always gave ideal reflexes. The crystals have a brilliant emerald green colour and are quite transparent.

The symmetry is *monoclinic-prismatic*; the axial relation was calculated on:

$$a : b : c = 0,8940 : 1 : 0,7295$$

$$\beta = 82^\circ 6'$$

Forms observed: $c = \{001\}$, predominant in the crystals obtained from alcohol and always very shining; $m = \{110\}$, well developed but often with curved plane; $q = \{011\}$, giving ideal reflexes and mostly exhibiting rather large planes; $a = \{100\}$, exceedingly narrow and dim. The habit is flattened along c or long prismatic parallel

the *c*-axis, with flattening along two parallel planes of *m*; perfectly cleavable towards {001}.

The following angles were measured:

	<i>Measured:</i>	<i>Calculated:</i>
<i>c</i> : <i>m</i> = (001) : (110) =	* 84° 51 $\frac{1}{4}$ '	—
<i>c</i> : <i>q</i> = (001) : (011) =	* 35° 51'	—
<i>m</i> : <i>m</i> = (110) : (110) =	* 83° 3'	—
<i>m</i> : <i>q</i> = (110) : (011) =	61° 55'	61° 51 $\frac{1}{2}$ '
<i>m</i> : <i>q</i> = (110) : (011) =	72° 18'	72° 15'
<i>m</i> : <i>a</i> = (110) : (100) =	41° 34'	41° 31 $\frac{1}{2}$ '

On {001} not observable but on {110} very distinctly dichroïc; for vibrations parallel to the *c*-axis, grass-green; and yellowish-green for vibrations perpendicular thereon. The angle of extinction is very difficult to determine; it amounts to about 12° in regard to the *c*-axis on the planes of {110}.

The optical axial plane is {010}; on {001} one axis is visible at a small angle with the normal to that plane; the inclined dispersion is extraordinarily strong: *q* < *c*.

The sp. gr. of the crystals is 1.351 at 15°, the equivalent volume is 121.39.

Topic parameters: $\chi : \psi : \omega = 5.1206 : 5.7277 : 4.1784$.

4. A small addition of the nitroso-compound to the nitro-compound causes a perceptible depression of the melting point of the latter substance. As, therefore, no certainty is obtained as to the formation of mixed crystals of these closely related derivatives — for a melting curve with an absolute minimum might be also present — a few preliminary quantitative experiments were carried out, which showed that we have here indeed an ordinary binary meltingpointline with eutecticum. It was further shown by more detailed microscopic tests that from mixed fusions of the two compounds is always deposited a *mixture* of the *yellow* nitroaceto-anilide crystals and the *green* nitrosoaceto-anilide crystals with a totally different aspect. At the side of the nitroso-compound a perceptible quantity of the nitro-derivative is carried down by the deposited crystals as solid solution; at the side of the nitro-derivative a formation of solid solutions could not be ascertained in this way. In any case, if there should be a slight mixing, it is limited on the side of the nitroso-compound to a few percent of the nitro-compound; the hiatus is therefore enormously extended.

Finally be it observed here that both substances are very volatile; the vapour of the nitroso-compound is *yellowish green*.

A few experiments were tried to sublime mixtures of the two compounds. The deposit on the concave coveringglass then consists of a network of dendritic, strongly pleochroïc (colourless — yellowish-green) little crystals, between which are found the square-shaped crystals of the nitroso-compound, besides the yellow individuals of the nitro-compound, united in clusters.

The first mentioned crystals contain chiefly the nitroso-compound and probably to a very small amount also the nitro-compound, so this may be a new case of the formation of solid solutions by sublimation. Probably, this pair of compounds lends itself to the measuring of the vapour tensions of these solid solutions.

Zaandam, December 1907.

Physics. — *“Observation of the magnetic resolution of spectral lines by means of the method of FABRY and PEROT.”* By Prof. P. ZEEMAN.

The interference method of the parallel semi-silvered plates, worked out with so much ingenuity by FABRY and PEROT¹⁾ excels above all other spectroscopic modes of procedure by the accuracy with which its theoretical foundations may be practically realized.

The principal task of the experimenter in applying this method has become to effect the perfect parallelism of the reflecting silvered plates.

In order to test by an independent method some recent results obtained in an investigation of the magnetic resolution of spectral lines²⁾ the method of FABRY and PEROT seemed most appropriate. Especially it appeared possible to extend at the same time the investigation to the behaviour of the lines in weak fields. The present paper is preliminary to a discussion of numerical results. I think it beforehand very improbable that errors of ruling of the ROWLAND grating will turn out to be the reason of the asymmetrical resolution of some lines, which I have described.

The method of FABRY and PEROT is applied in the present paper for the first time to an investigation of the magnetic separation of spectral lines. In some places in the literature of the subject the opinion is expressed that the method of interference fringes of silvered layers cannot be used for the subject under review. The

¹⁾ FABRY et PEROT, *Ann. de Chimie et de Physique* 1899—1904.

²⁾ ZEEMAN, *These Proceedings*, November 1907.

main objection is derived from the great loss of intensity in the apparatus of FABRY and PEROT. The present paper proves that this objection is not insuperable.

2. Of the two forms in which the method of the parallel plates is employed, the simplest, also requiring the least costly apparatus, has been used for the actual measurement of wave-lengths by FABRY and PEROT¹⁾, lord RAYLEIGH²⁾ and EVERSHEIM³⁾. This form of instrument called *Stalon*, I also have preferred. The distance of the silvered surfaces is here constant. The glass plates are held up to rounded distance-pieces (made of steel), by adjustable springs, which permit to regulate the pressure. By variation of the pressure the steel and the glass can be deformed in an extremely small degree and the accurate parallelism of the glass plates effected; the parallelism being secured already very approximately beforehand by the accuracy of finish of the distance-pieces.

3. The theory of the comparison of wave-lengths by means of this apparatus is simple, and has been given by FABRY and PEROT. We will apply it to the magnetic resolution of spectral lines and especially to the most simple case, the division into a triplet.

Let λ_0 be the wave-length of the original line (afterwards therefore the middle line of the triplet). To this corresponds a system of rings; let P_0 be the ordinal number of the first from the centre. The ordinal number at the centre ρ_0 is then the integer number P_0 , augmented with a fraction ε_0 , hence $\rho_0 = P_0 + \varepsilon_0$. Ordinarily $0 < \varepsilon_0 < 1$.

The diameter of a ring increases with ε . Let e be the thickness of the plate of air, the order of interference at the centre is $\rho_0 = \frac{2e}{\lambda_0}$. At an angle i with the normal to the plate the order of interference is $\rho_0 \cos i$.

If x_0 be the angular diameter of the ring P_0 , then we have, observing in the focal plane of a lens, $\rho_0 \cos \frac{x_0}{2} = P_0$. Developing the cosinus

¹⁾ FABRY et PEROT, Ann. de Chim. et de Phys. T. 25, Janvier 1902. C.R. 27 Mars 1904. FABRY et BUISSON, C. R. 16 Juillet 1906. BARNES Astrophysical Journal. Vol. 19. p. 190. 1904.

²⁾ Lord RAYLEIGH, Phil. Mag. Vol. 11, p. 685, 1906.

³⁾ EVERSHEIM, Zeitschr. f. wissenschaftl. Photographie, Band 5 p. 152, 1907.

$$P = P_0 \left(1 + \frac{x_0^2}{8} \right)$$

or

$$\epsilon_0 = P_0 \frac{x_0^2}{8} \dots \dots \dots (1)$$

Let λ_r be the wavelength of the outer component of the triplet towards the red then, if P_r , ϵ_r and x_r have a significance corresponding to that of P_0 , ϵ_0 and x_0 ,

$$\epsilon_r = P_r \frac{x_r^2}{8}$$

We have however $\lambda_0 (P_0 + \epsilon_0) = \lambda_r (P_r + \epsilon_r)$, whence

$$\lambda_r = \lambda_0 \frac{P_0}{P_r} \left(1 + \frac{x_0^2}{8} - \frac{x_r^2}{8} \right) \dots \dots \dots (2)$$

In like manner, λ_v , P_v , x_v determining the component of the triplet towards the violet, we have

$$\lambda_v = \lambda_0 \frac{P_0}{P_v} \left(1 + \frac{x_0^2}{8} - \frac{x_v^2}{8} \right) \dots \dots \dots (3)$$

In the case of radiation in a magnetic field this expression may often still be simplified. In many cases we may choose

$$P_0 - P_r = P_r \dots \dots \dots (4)$$

Looking at the system of rings corresponding to λ_0 , while the magnetic force is slowly but gradually increased one sees at the same time rings which proceed from the system λ_0 and are moving outwards and others which are moving inwards. The rings corresponding to λ_r are contracting, those corresponding to λ_v are expanding.

It depends upon the value of ρ of the étalon and upon the intensity of the magnetic field how great the expansion and contraction of the rings, in comparison with the distance of the rings λ_0 , will be.

The value of ρ and the maximum magnetic force will determine whether in the centre new rings will appear or respectively will disappear.

In the case one does not select for measurement the smallest rings but if the rings λ_r and λ_v , which originate from the same ring λ_0 , are suitable to be measured, ϵ can become larger than unity.

When we select the rings thus specified the equality (4) applies and then we may determine λ_r and λ_v from the angular diameters of the rings and the value of λ_0 , regarded as known; the result is then

independent of the accurate value of the thickness of the plate of air.

Of course the position of the new rings between the rings λ_0 will, with a given value of the magnetic force, be determined by the thickness of the plate of air and what might be called "the sensibility" of the system of rings to magnetic forces will increase with the thickness of the plate of air. A limit of this sensibility is (often too soon) attained by the effective width of the spectral lines under consideration.

In some cases it will be desirable to select for measurement rings different from the three specified ones. There are no difficulties about the significance of P ; it always means the ordinal number of the measured ring.

However if P_0 differs from P_r or P_v , their values must be known for the calculation according to (2) and (3).

4. Besides the simplification resulting from equation (4) there is still another one to be considered in the investigation of the radiation in a magnetic field.

I mean that the quantity $e = \rho \frac{\lambda}{2}$, the optical thickness of the plate of air may be treated as an absolute constant.

Ordinarily this thickness depends upon λ . In consequence of the change of phase by reflection upon the silver, which varies with wavelength, the comparison of different coloured systems of rings finally requires a knowledge of the optical thickness for each separate colour.

It is clear that in the application to the subject now under review only systems of rings corresponding to rays differing extremely little in wavelength are considered, hence the variation of thickness with wavelength needs not to be taken into account.

5. Figures 1 and 2 may give an idea of the aspect of the magnetic resolution of the spectral lines observed by means of the method of FABRY and PEROT. They are about sixfold enlargements of negatives taken with an étalon with an interval of nearly 5 m.m. between the plates. The source of light in the magnetic field was a small vacuum tube charged with mercury. The order of interference at the centre for the mercury line 5791 is at 16° about 17265.7.

The system of rings was formed in the focal plane of a small achromatic lens of 18 m.m. aperture and of 12 c.m. focus. Its focal plane coincides exactly with the plane of the slit of a small

spectroscope. When the slit is opened widely each spectral line is seen as a rectangle with bright rings or parts of rings as the case may be. The part of the spectrum in the figures refers to the two yellow and the green mercury lines. In fig. 1 the two rectangles corresponding to the two yellow mercury lines are superposed. The green mercury line is largely overexposed. I have reproduced it also in order to give an idea of the dispersion used. The intensity of the magnetic field in figures 1 and 2 was about 5000 Gauss.

It is a very beautiful sight to watch the moving system of rings, while the magnetic force is slowly increased. The rings λ_r and λ_v are first seen approaching, then coinciding, separating, coinciding for a value of the field of about 15000 Gauss with the next ring λ_o , passing over this ring, etc.

For measurements it is necessary to reduce the width of the slit, as in Fig. 2. Owing to rise of temperature the rings have somewhat expanded.

6. For measurements, which I hope to communicate in a future paper, I have used not only the method of diameters resumed above (§ 3) but also *the method of the coincidences*¹⁾ for the distinct values of the magnetic force, which bring to coincidence λ_r and λ_v , or λ_r and λ_v with λ_o .

Concerning the difficulties attending the use of the method of coincidences FABRY and PEROT¹⁾ remark:

“Même avec ce perfectionnement, la méthode présentait des inconvénients assez graves :

1. La nécessité d'éclairer simultanément l'appareil par les deux sources entraîne des pertes de lumière assez importantes ;

2. Les coïncidences ne sont bien observables que lorsque les deux systèmes d'anneaux ont des éclats comparables, et cette condition n'est pas toujours facile à réaliser ;

3. La recherche de la coïncidence entraîne toujours des tâtonnements et l'on n'est jamais sur (lorsque la période est courte) d'en rencontrer une qui soit exacte.”

The drawbacks to the method, mentioned sub 1. and 2. are eliminated in the application to radiation in a magnetic field, now under consideration. By variation of the current in the electromagnet the coincidence can be attained with the desired degree of accuracy and hence also the third objection is obviated.

¹⁾ FABRY et PEROT, Ann. de Chim. et de Phys. p. 12, T. 25, Janvier 1902

P. ZEEMAN. "Observation of the magnetic resolution of the spectral lines
by means of the method of Fabry and Perot."



1. The yellow mercury lines in magnetic field. Very wide slit. Green mercury line overexposed.



2. The same lines. Narrow slit for measurement of the yellow mercury lines.

7. Some further details concerning the apparatus may finally be given.

The mounting and the plates of the 5 m.m. étalon are by JOBIX. The inner surfaces of the plates are accurately flat. The outer surfaces need only ordinary flatness, they are inclined at an angle of $1'$ to the inner ones. The plates of the étalon are vertical, and the whole apparatus is capable of the necessary adjustments in azimuth, while also a horizontal sliding motion parallel to the plates of the étalon was provided for.

An image of the vacuum tube was focussed upon the étalon by means of an achromatic lens of 12 cm. focus, the enlargement being four times. All optical pieces were mounted upon double T -pieces and therefore rigidly connected.

The figures clearly indicate that for the investigation of the magnetic separation of the yellow mercury lines, it would be of no value to use an étalon of greater optical thickness of the plate of air. On the contrary the effective width of the yellow mercury lines when under magnetic influence is rather large, so that the limits of the method in this case are being rapidly approached.

Physics. — *“Isotherms of monatomic gases and their binary mixtures.*

I. Isotherms of helium between $+100^{\circ}$ C. and -217° C.”

Communication N^o. 102^a from the Physical Laboratory at Leiden. By Prof. H. KAMERLINGH ONNES.

§ 1. On account of the important rôle, which VAN DER WAALS' theory plays in many chapters of thermodynamics, experimental data concerning the equation of state of a substance are of the greater value as the interaction of the molecules of this substance conforms the better to the hypotheses from which VAN DER WAALS started. The knowledge of the equation of state of the monatomic gases, whose molecules we must consider as the simplest for the present, is of the greatest importance from this point of view.

In Comm. N^o. 69 (April 1901) on the isotherms of diatomic gases and their binary mixtures it was already observed that the investigation of the net of isotherms of argon and of helium promised still more important results than the completion of the net of isotherms of the gases formerly called permanent, particularly of hydrogen, at low temperatures, on which subject my attention had been chiefly fixed since the establishment of the cryogen laboratory (cf. Comm. N^o. 14, Dec. '94). But the difficulty of obtaining argon and helium in so pure a state and in such quantities as are required for

the determination of isotherms, retarded the determination of the equation of state of helium and argon for a long time after Comm. N^o. 69.

The investigations on the isotherms of hydrogen are in progress, and yielded already results laid down in communications N^o. 78, 97^a, 99^r and 100^r, which, I hope, will soon be followed by others. In the meantime, however, also the difficulty of obtaining pure helium has been quite, that of obtaining pure argon nearly overcome. The successful preparation of pure helium was chiefly due to the hydrogen circulation (Comm. N^o. 94^f) yielding the required liquid hydrogen. So the first measurements from the series which will refer to the monatomic gases and their binary mixtures, can already be communicated.

They concern the isotherms of helium, which have now taken the place occupied by the isotherms of hydrogen before the hydrogen was liquefied. Among others the isotherms must lead to the calculation of the critical quantities for helium. From the now communicated determinations of the compressibility along different isotherms at densities which are comparatively small and differ only slightly, the critical temperature can already be calculated by approximation.

§ 2. *Survey of the determinations.*

This investigation comprises some six determinations of isotherms. The temperatures at which they were made, were kept constant and determined in the same way as in the determinations of isotherms for hydrogen published in preceding Communications N^o. 97^a (April 1907), N^o. 99^r (Sept. 1907), N^o. 100^r (Jan. 1908). The readings of the hydrogen thermometer were reduced to the absolute scale by means of formula (4) of Comm. N^o. 97^b (Sept. 1907) with the new coefficients of § 2 of Comm. N^o. 101^b. The six temperatures thus reduced to which the isotherms refer, are :

$$+ 100^{\circ}.35, 20^{\circ}.00, 0^{\circ}, - 103^{\circ}.57, - 182^{\circ}.75 \text{ and } - 216^{\circ}.56.$$

Besides the measurements at the two standard temperatures 0^o C. and 100^o C.¹⁾ and those at low temperatures a determination was made at 20^o C. to obtain data for the calculation of the quantity of gas in the stem of the piezometer and in the other parts, which remain at the ordinary temperature during the measurements.

For all these isotherms the densities, at which the pressure was observed, lie about between the same limits which were set by the

1) The results at 0^o C. and 100^o C. are incompatible with those of RAMSAY and TRAVERS, which, indeed, show strange deviations.

dimensions of the piezometer and by the manometer. The utmost limits of the density are 25 and 54 times the normal one. The piezometer and further auxiliary apparatus were perfectly the same as have served for the determinations of C. BRAAK and me (see Comm. N^o. 100^b Jan. 1908) with hydrogen at 0° C. and 100° C. The satisfactory results obtained then, enhance at the same time the reliability of the measurements considered now.

§ 3. Results for pv_A .

The subjoined table contains the results of the determinations. The first column gives the number of observation, the second the temperature measured above 0° C. on the absolute scale, the third the pressure in atmospheres, the two following ones the product pv_A , and the density d_A , in which the volume of the gas v_A is expressed in the normal volume (that at 0° C. and 1 atmosphere) and the density d_A in the normal density (that at 0° C. and 1 atmosphere). (Compare the corresponding tables of the above mentioned determinations of isotherms of hydrogen).

The calculation of these results was made as follows :

First the points of the isotherm of 20° C. were calculated, (cf. also § 8 of Comm. N^o. 79 (April 1902)) and the coefficients A_A and B_A of the curve

$$pv_A = A_A + \frac{B_A}{v_A} + \frac{C_A}{v_A^2} \dots \dots \dots (1)$$

were determined by the 3 points by the aid of the method of least squares. For C_A a definite value was assumed, the densities being too small for this coefficient to be determined with sufficient certainty. If we write VAN DER WAALS' equation with the second correction for the size of the molecules in the form :

$$pv = RT + \frac{RTb-a}{v} + \frac{5}{8} \frac{RTb^2}{v^2}$$

where v is the volume of the gas under the pressure p at the absolute temperature T , expressed in the theoretical normal volume (see Comm. N^o. 71 § 3) and if we put the value of A_A at 0°, $A_{A_0} = 1$, which approximation is allowed for our purpose, we find (cf. Comm. N^o. 71 § 3) for the value C_{A_T} of C_A at T

$$C_{A_T} = \frac{5}{8} RT b^2,$$

where $R = 0.0036619$. The value of b was first estimated at 0.0005 (cf. the note to § 6 of Comm. N^o. 96^c Jan. 1907), afterwards at 0.000432, see § 4. With the coefficients A_A and B_A obtained for 20°

TABLE I. He. Values of pv_A .

N ^o .	θ	p	pv_A	d_A
1	+ 100° 35	42.574	4.38725	30.689
2		54.459	1.39314	39.091
3		66.590	1.39929	47.589
4	+ 20° 00	27.539	1.08664	25.343
5		36.303	1.09028	33.297
6		53.708	1.09918	48.862
7	0°	26.634	1.01392	26.268
8		38.565	1.01851	37.864
9		50.240	1.02521	49.004
10	-- 103° 57	20.580	0.63135	32.597
11		24.100	0.63296	38.075
12		29.185	0.63597	45.891
13		33.383	0.63845	52.288
14	-- 182° 75	13.751	0.33787	40.699
15		16.019	0.33898	47.257
16		18.189	0.34025	53.457
17	-- 216° 56	9.564	0.21132	45.259
18		10.502	0.21171	49.606
19		11.448	0.21219	53.951

in this way, the reductions to 0° for the gas which is outside the reservoir at a temperature of 20° and for a small part at the temperature of the room, were carried out in first approximation. With the three points which were thus found on the isotherm of 0°, the virial coefficients A_A and B_A were then calculated also for this temperature.

From this follows A_{A_0} , the value of pv_A for $d=0$ by means of the formula:

$$A_{A_0} = 1 - B_{A_0} - C_{A_0}.$$

With the pressure coefficient from 0° C. for the state of AVOGADRO, 0.0036619 (cf. § 1 of Comm. N^o. 101^b) follows for 20°:

$$A_{A_{20}^{\circ}} = A_{A_0} (1 + 0.0036619 \times 20),$$

so that on the isotherm of 20° a fourth point is acquired, which renders the slope of the pv_A line a great deal more certain (cf. the conclusion of § 1 of Comm. N^o. 101^b). Then the calculation of A_A and B_A was repeated, and by the aid of these corrected coefficients the isotherm of 0° C. was again calculated, and this calculation by approximation was continued till it caused no longer any change. In this way we found for 20° C. (for $b=0.000432$):

$$pv_{A_{20}} = 1.07273 + 0.0005337 d_{20} + 0.000000125 d_{20}^2 \dots \quad (3)$$

With this formula the corrections have been calculated for the determinations of isotherms. For the rest the latter were treated as in the preceding communications.

§ 4. *Individual virial coefficients.*

We may avail ourselves of the data of table I in order to derive the coefficients A_A and B_A by the aid of the method of least squares. C_A was assumed according to formula (2) of the preceding §. For every isotherm $pv_{A,d=0}$ was calculated, and this value was added to the others as if it concerned a new observed point. This calculation was effected by the aid of the value $A_A = 0.99949$, which may be derived from the value for the coefficients B_A and C_A for 0° finally obtained in the calculation by approximation from the conclusion of the preceding §. Table II contains the virial coefficients and at the same time the differences between the given pv_A 's and the calculated ones. These differences are arranged according to the ascending densities. So the first column of differences refers to $pv_{A,d=0}$, the others to the data of table I in the above succession.

The calculation of the B_A 's is still uncertain, because for C_A estimated values have been assumed. Determinations of pv_A at greater densities, which will render an independent determination of C_A possible, are in preparation.

That the estimations of C_A are not too inaccurate, may be made probable as follows. For 100° follows from table II $B_{A(100)^\circ} = 0.000673$. On the suppositions on which VAN DER WAALS' equation rests, the value of b may be derived from the value for two temperatures of $B = RTb - a$ with $B = B_A (A_{A_0})$ and then $b = 0.000432$ is found, which does not differ much from the value 0.0005, which was first assumed by way of estimation on other grounds. Though the calculation followed here is very uncertain, yet the found value was preferred to the first estimated one, and for this reason the calculations which were first made with 0.0005, have been repeated with this new estimation. The differences of the results lie within the limit of errors of observation.

TABLE II. He. Individual virial coefficients.
Deviations of the $p v_A$ from the calculated ones.

	A_A	$10^3 B_A$	$10^6 C_A$	$10^5 (O-C)$			
+100° 35	1.36667	+0.673	+0.16	+10	-21	-10	+22
20° 00	1.07273	+0.534	+0.13	-3	+31	-36	+8
0°	0.99970	+0.512	+0.12	-20	+80	-75	+15
103° 57	0.62026	+0.337	+0.07	+14	-7	-32	0 +27
182° 75	0.33066	+0.176	+0.04	1	-4	-8	+8
216° 56	0.20693	+0.096	+0.02	0	+1	-3	+3

§ 5. *Determination of the critical temperature of helium.*

From the data of table II we may arrive at a first estimation concerning the critical temperature of helium, which will be found from determinations of isotherms within the now accessible region of temperatures.

Extrapolation proves that the BOYLE-point will lie in the neighbourhood of -250° C. For hydrogen -166° was found for this. (cf. Comm. N°. 100ⁿ). If we assume 30° K. for the critical temperature of hydrogen, then follows from this for helium

$$T_{kHe} = 6^\circ \text{ K.}$$

If this value of T_k is adopted, the region of temperature -217° to -183° for helium corresponds with that of 0° to $+200^\circ$ for hydrogen. By applying the law of the corresponding states to the slopes of the $p v_A$ -lines for the two substances in the neighbourhood of these equivalent limits of temperature, we arrive at a slightly lower value of the critical temperature viz.

$$T_{kHe} = 5^\circ.3 \text{ K.}$$

This value too I think I may still consider as a highest limit for the critical temperature of He, as it seems probable to me, that He with respect to H_2 will deviate from the law of the corresponding states in this sense that the critical temperature will be found lower than would follow from the application of this law to corresponding states for values of the reduced temperatures many times larger than 1.

Now there can only be question of a first estimation based on determinations of isotherms. The determination of the isotherms of -253° and -259° , which is in progress, will, I hope, soon lead to a more reliable estimation.

In conclusion I gladly express my thanks to Mr. C. BRAAK for his assistance in this investigation.

(January 23, 1908).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday January 25, 1908.



(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige

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Zoology: — “*Physiological regeneration of neurofibrillar endnets (tactile discs) in the organ of Eimer in the mole.* By Dr. J. BOEKE and Dr. G. J. DE GROOT. (Communicated by Prof. G. C. J. VOSMAER).

(Communicated in the Meeting of November 30, 1907).

In recent years several authors (RANVIER, VON LENNIOSEK inter alia) have called attention to the fact, that there where intraepithelial nerves in mucous membranes or in the epidermis are found penetrating even between the superficial layers of epithelium cells covering the sensory surface (so for example in the peribulbar nerve-endings between the taste-buds in the papillae of the tongue, etc.), we have to draw the conclusion, that at the same time as the superficial cells degenerate and are cast off, the sensory nerves with their knob-like end-swellings or end-loops of the neurofibrillae must undergo a perpetual change and growth. But then these are always the fine ramifications and endings of the nerves, which branch between the deeper layers of epithelium cells. Real neurofibrillar endnets like those which are formed round the base of the tactile cells of MERKEL, are always found in the deeper layers of the epidermis, where they lie protected by the other epithelium cells. These tactile cells nowhere degenerate so quickly as it is the case with the superficial cells of the upper layer of the epithelium, and need not be replaced by other cells coming from the deeper strata. There is no need of a quick regeneration of the neurofibrillar endnets (and the tactile cells).

But suppose we had a tissue, where in the uppermost strata of a stratified epithelium, in which the superficial cells quickly degenerate and are cast off, we find real tactile cells with distinct neurofibrillar endnets, which therefore must degenerate at the same time as the surrounding cells, how would the process of regeneration of the neurofibrillae take place there?

In the course of investigations carried on in the histological laboratory at Leiden we found a favourable object to study this question in the sensory organs in the snout of the earth-mole (*Talpa europaea*).

Here we find an extremely sensitive tissue (the organ of EIMER) the elements of which are only protected by a very thin horny layer, and which by reason of its lying at the tip of the snout must, on account of the well-known habits of the animal, continuously form new horny cells for the protecting horny layer above, because otherwise the functional cells would very soon come to lie at the surface and be liable to be injured.

The structure of the peculiar organ first described by EIMER 1870 and the innervation of it, have been studied in the course of this year (1907) by two authors¹⁾ by means of the recent improved methods of staining the neurofibrillae. Both give about the same description but arrive at different conclusions.

As is well known, the organ of EIMER consists of thickenings of the epidermis, formed by columns of epithelial cells in the shape of an hourglass, which form small round prominences on the surface of the snout, and which, because the columns of cells are longer than the thickness of the epidermis at the place where they are found, project with their base into the corium, and form here a bulging out of the epithelium, generally described as "buffershaped". Each of the columns is made up of several strata of more or less flattened epithelial cells, which at the base of the column do not reach from one side to the other, but are wedgeshaped and overlapping each other with the thinned-out ends. Nearer the surface the cells gradually become flattened and larger, until only two cells lying at the same niveau, fill out the entire cross-section of the sensory column (fig. 1, 5). There the column ends as it reaches the horny layer. The cells of the column are, according to BOTEZAT, true spiny cells like the other cells of the epidermis (fig. 3).

In the axis of the column a thick nerve fibre, the axial fibre, runs through the whole length of it, penetrating into the epithelium at its base. Sometimes there are two or three axial fibres. Around the column of cells a set of 18 or 19 thin, unbranched nerve fibres, closely set, somewhat zigzag, run upwards between the outer ends of the cells of the column and the adjoining epidermis-cells, until they reach the horny layer. These are called rand-fibres to distinguish them from the axial fibre. At the base of the column between the epidermis-cells a small number of tactile cells of MERKEL are found, and underneath the epidermis in the corium one or two small Pacinian corpuscles.

EIMER already described small varicosities or knoblike swellings of the nerve-fibres in the upper part of the columns. The nerve-fibres run more or less zigzag between the cells. EIMER himself and after him HUSS (1898) thought that these knoblike varicosities were lying intracellular, the nerve-fibres running between the cells. The varicosities are therefore attached laterally to the nerve-fibres.

¹⁾ EUGEN BOTEZAT. *Anat. Anzeiger*, 30 Bd. 1907.

M. BIELSCHOWSKY. *Anat. Anzeiger*, 31 Bd. 1907.

In this year (1907) BIELSCHOWSKY¹⁾ has investigated the nerves of the organ of EIMER by means of his method of staining the neurofibrillae, and although he does not give much that is new, as he says himself in his paper, his study is interesting because with that by BOTEZAT it is the only one, in which the new staining methods for the nervous system are used for this organ. We may quote here what he says about the course and the peculiarities of the nerve-fibres, because this makes clear his opinion better than a long description. The course of both the axial fibre and the randfibres he assumes to be entirely intercellular: "irgend ein näherer Konnex der Fasern zu den Epithelzellen findet nicht statt; ihr Verlauf ist ein rein intercellulärer. Im Bereiche der äusseren Schicht weisen sie in scheinbar regelmässigen Abständen die bekannten punktförmigen Varikositäten auf. Die Varikositäten sind offenbar nur auf Zerfallsvorgänge zurückzuführen. Dafür spricht der Umstand, dass sie immer erst in der Verhornungszöone des Epithels deutlich hervortreten. Aehnliche Beobachtungen kann man auch am Schweinerüssel and anderen rüsselförmigen Säugerschnauzen machen". (i.e. p. 189).

In his last paper, published some months before the paper by BIELSCHOWSKY appeared (April 1907), BOTEZAT²⁾ who in his paper of 1903 pronounced the same opinion as EIMER and HUSS, viz. that the knoblike thickenings of the nerve fibres penetrate into the cells of the column of EIMER, adopts the view that they are epicellular, after a study of the nerves coloured with methylene blue and after the method of RAMON Y CAJAL. "Der Beweis hierfür lässt sich am besten dadurch erbringen, dass man die Terminalknöpfchen fast genau zwischen den Zellen des Organs liegen sieht." BOTEZAT states that the varicosities possess a netlike structure. Because they are excessively small, the extreme sensibility of the snout must be due to the very large number of the terminal knobs ("tactile discs") and not to their great perceptibility. A column of EIMER consists of about 15 layers of cells, and in each layer about 20 of these tactile knobs are to be found. The total number therefore is in each organ of EIMER 300, and for the entire snout more than 100000³⁾. According

1) M. BIELSCHOWSKY, Ueber sensible Nervenendigungen in der Haut zweier Insectivoren (*Talpa europaea* und *Centetes ecaudatus*). Anat. Anzeiger. Bd. 31, p. 187—194, 1907.

2) EUGEN BOTEZAT. Ueber die epidermoidalen Tastapparate in der Schnauze des Maulwurfs etc. Archiv für Mikroskopische Anatomie. Bd. 61. p. 730—764. 1903.

3) EUGEN BOTEZAT. Die fibrilläre Struktur von Nervenendapparaten in Hautgebilden. Anat. Anzeiger. Bd. 30. p. 321—344. 1907.

to BIELSCHOWSKY the total number is $\pm 150,000$, together with more than 5000 end-bulbs and numerous cells of MERKEL.

Although he does not ascribe to the varicosities a high degree of perceptibility, BOTEZAT assumes all of them (both of the axial fibre and of the rand-fibres) to be tactile discs, in accordance with most authors. A difference in structure between the different tactile discs or knobs he mentions without paying much attention to it.

Now the facts seem to us to point to a different conclusion.

The opinion of BIELSCHOWSKY, that the varicosities of the nerve-fibres are due to "Zerfallsvorgänge", seems to us to be erroneous. In the first place these varicosities do not appear first in the horny zone. On the contrary, as soon as the cells are transformed into horny cells, the fibres and their varicosities degenerate, and the first varicosities appear seven to eight layers of cells lower down. In the second place the varicosities are much too regular and are distributed with a far too great regularity to be the mark of degeneration, and are always present in nearly the same number. In the third place their structure does not point at all to "Zerfallsvorgänge."

But in his description BOTEZAT too does not seem to have hit the point. He does not give an explanation of the difference in structure of the varicosities and of their mode of attachment to the nerve-fibres, and of the fact that they are only to be found in the peripheral part of the nerve-fibres and not in the basal half.

When we treat a small piece of the snout of the mole, after fixation in formaline, according to the method of BIELSCHOWSKY—POLLACK, and study a correctly differentiated preparation in thin (6μ) longitudinal sections (that is a longitudinal section of the nerve-fibres and of the column of cells, the section being made at right angles to the surface of the epidermis of the snout), the following details will be seen: the structure and form of the varicosities ("Terminalknöpfchen, Seitenknöpfchen") are not the same in the course of the nerve-fibres. When we follow a rand-fibre from the base of a column of EIMER to the top, the first swellings appear at a distance of 10 to 12 cell-layers from the top (fig. 1, 5). The swellings are here only loosely built small nets, lying in the course of the nerve-fibres, nothing but a local slackening of the bundle of neurofibrillae in the fibres, the fibrillae probably forming a few anastomoses. From this point upwards we see these networks appearing with great regularity in the course of the nerve-fibres where the fibre passes another cell of the column, and each time the reticular structure becomes finer and more distinct (fig. 1, 3).

In the upper four to five rows of cells a change in the form and

arrangement of the networks becomes visible. The small swellings of the nerve-fibre no more lie in the course of the nerve-fibres, but more and more pass to the side of it (fig. 1, 3) and at last they lie entirely beside the nerve-fibre, being connected with it by means of a very small and short stalk (fig. 1, 2, 3). The swellings of the rand-fibres always pass to that side of the fibre lying close against the cells of the column of EIMER, and so project centripetally (fig. 1, 6). So when we look at a rand-fibre from the outside of the column, as in fig. 5, we see nothing or only very little of this change of place of the varicosities, and only when we play up and down with the micrometer-screw of the microscope, we are able to make out that the peripheral rows of varicosities lie in reality underneath the fibres.

So in the first place we see a very regularly occurring change of place of the varicosities, as the fibre approaches the surface of the epithelium. When we only take the place of the fibre we are examining in the section into account, this change is always found to take place with perfect regularity.

In the second place the following change may be seen: the nerve-fibres of the organ of EIMER (both the rand-fibres and the axial fibres) run between the cells of the epithelium. The first swellings or varicosities, the small loose nets lying in the course of the fibres, of course also appear between the cells. But as soon as these varicosities get larger and change their places, so that they come to lie besides the fibres, they push their way *into* the substance of the cells of the sensory column and not between these cells. They become *intracellular*. In the preparations stained after the method of BIELSCHOWSKY the cells and their margins and nuclei are so clear and distinct, that when we only take care to examine thin sections (5—6 μ), this fact may be stated with perfect clearness. Fig. 1, 2 and 3 give a good idea of it; when we examine longitudinal sections of the rand-fibres, the section passing through the axis of the sensory column, we see as it were the varicosities or knobs push their way into the protoplasm of the cells. In cross-sections now and then we come across places, one of which is figured in fig. 4. The tactile knob growing into the flat epithelial cell, pushes its way into the protoplasm apparently with some force so that the flattened nucleus is curved in by it. Similar drawings are given by HUSS.

Another question is, whether these varicosities or tactile knobs lie in the protoplasm of the cell, become an integrating part of it. The facts seem to point to the contrary. On observing

the fibres and their tactile knobs closely under the highest power, we get the impression that even there where the knobs lie intracellularly, the neurofibrillae are still surrounded by a very thin layer of perifibrillar substance, taking a different stain from the protoplasm of the cell itself. But of course this layer of perifibrillar substance must be continuous with the surrounding protoplasm. The neurofibrillar network remains entirely independent, but a trophic connection of the perifibrillar substance and the protoplasm surely must be present. This seems to us to be beyond doubt, and we may venture to suggest, that only now the varicosities reach their full development, are real tactile discs; as long as they lie between the cells, the varicosities are only parts of the nerve-fibres where the neurofibrillae are getting looser and growing out, but only when they pass to the side of the fibres and grow into the cells, they become real tactile neurofibrillar end-nets. The rows of varicosities are merely stages of development of the tactile discs.

The end-knobs or terminal discs in the upper row of cells of the sensory column, which are already on the point of passing into the horny layer, are for the greater part already lying loose in the cells, the nerve-fibres themselves and the connecting stalks atrophying. So in fig. 5 the four knobs, represented by black spots in the upper row of cells, are entirely separated from the nerve-fibres below, and the same fact is to be seen in the fig. 1 and 3, where a part of the nerve-fibre (the stalk of the end-knob) was still stained. The argument, that this independence of the terminal knobs is due to the connecting stalks not being cut in the section examined, is annihilated by a close study of many sections. Thus we can state with perfect accuracy, that the connecting fibre really does not exist any more (at least, is not stained as the functional fibres are).

The axial fibre shows the same peculiarities as the rand-fibres, but the tactile nets are larger and more rounded; the axial fibre too runs between the cells until its end; even there where, in the upper part of the column, the entire cross-section is composed of two cells, the line between these cells runs just through the middle of the transverse plane (cf. HUSS) and leaves a small room just in the axis of the column, occupied by the axial fibre (fig. 6). The tactile nets grow out from the fibre now at one side and then at the other, and grow into the cells of the sensory column just as it was described for the rand-fibres.

So we find the same peculiarities of structure in all the nerve-fibres and their tactile neurofibrillar networks. The same cause seems to us to underlie all these differentiations, which we may describe

as a getting more and more differentiated and independent of the tactile discs (or varicosities) as we draw nearer to the surface of the sensory column.

When we see now, that the horny layer above the columns of the organ of EIMER is always thinner than in the adjacent parts of the epidermis (so for example in fig. 1, st. c.), as it was the case in all the preparations examined, and when we bear in mind, that these prominences on the surface of the snout of the mole are continually exposed to all sorts of mechanical insults, the question, put at the beginning of this paper, may be answered in the following manner:

The horny layer above the cells of the column of EIMER being very thin and composed of a few layers of cells, and the horny scales being lost very soon by desquamation, there must be a continual moving upwards of the cells of the deeper layers of the column of EIMER, to take the place of the thrown off cells. With these cells the nerve-fibres must grow upwards at the same rate. About in the middle of their course these nerve-fibres begin to form tactile discs. These corpuscles first appear as places in the course of the fibres where the neurofibrillar structure is looser; these first varicosities little by little pass out of the course of the fibre and grow into the cells of the column of EIMER, and so become real tactile discs. These tactile discs lying at the side of the nerve-fibres and remaining attached to them by a short stalk, are a direct argument for the growing upwards of the nerve-fibres together with the cells of the column. Otherwise the cells would take with them the tactile corpuscles and sever them from the fibres they belong to or draw out the stalks in an oblique direction. Of this no trace is to be found anywhere. It is very probable, that only when the varicosities grow out to small tactile discs and come to lie intracellularly, they acquire a heightened perceptive faculty. As they are continually travelling upwards to the surface, new varicosities are formed underneath in the course of the same nerve-fibre. As soon as the cells undergo the transformation into horny scales, the tactile discs and their connecting stalks and the nerve-fibres atrophy, the former remaining visible longer than the latter. Only the upper rows of tactile discs, of the form of the networks of fig. 2 and fig. 3, seem to be fully developed.

They are continually replaced by others, coming from below. The nerve-fibres of the column of EIMER chiefly grow at the base of the column and atrophy at its top.

Perhaps these views may be extended to other intraepithelial



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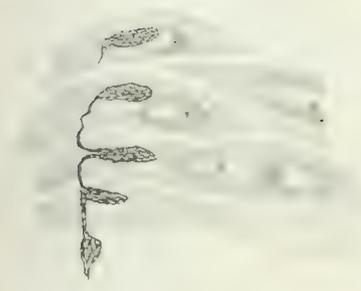


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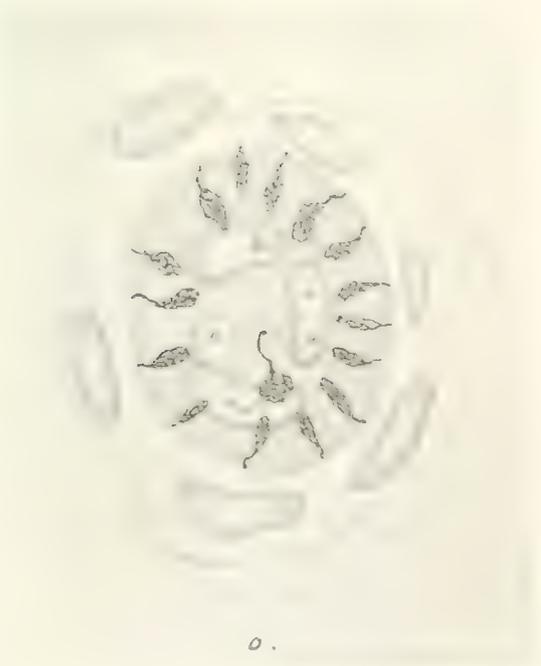
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6.

J. Boeke a. n. del.

nerve-endings. It will be difficult to find an object of study as favourable as the organ of EIMER.

Leiden, Anatomical Cabinet.

DESCRIPTION OF THE FIGURES ON THE PLATE.

All the figures are drawn from life from preparations made after the method of BIELSCHOWSKY-POLLACK, with a camera lucida of ABBE. Fig. 1 and 5 are enlarged 1200 times, the others 1600 times. Apochromate-oil-immersion. Sections 5 and 6 μ .

Fig. 1. Longitudinal section of the upper part of a column of EIMER of the earth-mole. A rand fibre (*rf*) and a part of an axial fibre (*mf*) are seen. The horny layer (*stc*) above the column of EIMER is distinctly thinner than at both sides of it.

Fig. 2. Longitudinal section of a flat cell of the upper part of a column of EIMER, with two tactile discs, growing into the same cell. The netlike structure and the curious drawing in of the connecting fibre, is clearly shown.

Fig. 3. Longitudinal section of the upper part of a column of EIMER, to show the developing of the tactile discs, and the final atrophy of the nerve-fibre.

Fig. 4. From a cross-section through the upper part of a column of EIMER. A nucleus curved in by a tactile disc.

Fig. 5. Longitudinal section through the peripheral part of a column of EIMER. Three rand-fibres are shown. The tactile discs lie behind the nerve-fibres. The intracellular position of the tactile discs is clearly to be seen. The upper cell, in which lie four tactile discs, is being transformed into a horny cell. The nerve-fibres degenerate.

Fig. 6. Cross-section through the upper cells of a column of EIMER. In the section of 6 μ four cells were to be seen, lying two and two in the same niveau.

The tactile discs of the rand-fibres all grow centripetally into the cells, the axial fibre runs between the cells.

Astronomy. — " *β Lyrae as a double star.*" By J. STEIN, S. J. at Rome. (Communicated by Prof. H. G. VAN DE SANDE BAKHUYZEN).

1. As far as I know, Professor E. C. PICKERING was the first who, led by his spectroscopic investigations, suggested that β Lyrae might be a close double, the components of which describe circular orbits in a light-period¹).

This surmise was confirmed by BELOPOLSKY²) in 1892. He measured the displacement of the luminous *F*-line on some fourteen spectographs. They were found to show a minimum (in absolute value) at the time of the minima and a maximum at the time of the maxima of

¹) Spectrum of β Lyrae. By Prof EDWARD C. PICKERING. A. N. 3051 (1891).

²) Les changements dans le spectre de β Lyrae. A. BÉLOPOLSKY. *Memorie della Società degli Spettroscopisti Italiani*. Vol. XXII, 1893.

the star's light, in such a way that they correspond to an approach before the principal minimum and to a recession after that time. From these observations he derived a circular orbit for that component which eclipses the other at the time of this minimum. The investigation of the Potsdam spectographs equally led Prof. VOGEL¹⁾ to the conclusion that the displacement of the lines can hardly be explained otherwise than as a consequence of the motion of different bodies having unequal spectra. He does not succeed however in determining the position of the lines with sufficient accuracy. He thinks that the photometric data would lead to the assumption of two bodies of unequal luminosity moving either in a fairly circular orbit or in an ellipse having its major axis in the visual line. On the other hand the spectroscopic investigations would lead to the assumption of two bodies, one showing a spectrum with luminous, the other a spectrum with absorption-lines, which would describe very excentric orbits the major axes of which would make a considerable angle with the visual line. It would be impossible, in his opinion, to satisfy the two phenomena at the same time. In 1896 Dr. MYERS²⁾ subjected ARGELANDER's lightcurve (Verh. pro 1850) to an elaborate theoretical investigation. His result is that the whole curve of the lightvariation is represented satisfactorily by assuming two elongated revolution ellipsoids the major axes of which are in each other's prolongation, circulating around each other in nearly circular orbits.

The next year BELOPOLSKY found the duplicity confirmed³⁾. This time it was the displacements of the dark *Mg*-line ($\lambda = 448.2 \mu\mu$), which enabled him to derive a slightly excentric orbit for the second component viz. of that component which is eclipsed during the principal minimum. Father W. SIDGREAVES, in his latest spectrographic investigation of β Lyrae⁴⁾ arrives at the same result as Prof. VOGEL: rather considerably excentric orbit, the major axis of which makes a great angle with the visual line.

In conformity with what had already been suggested before by

1) Ueber das Spectrum von β Lyrae. Von H. C. VOGEL. Sitzungsberichte der K. Preussischen Ak. der Wiss. zu Berlin. 8 Februar 1894.

2) Untersuchungen über den Lichtwechsel des Sternes β Lyrae. Inauguraldissertation... von G. W. MYERS, München 1896. — The system of β Lyrae. id. The Astroph. Journ. Vol. VII N^o. 1.

3) Recherches nouvelles du spectre de β Lyrae, par A. BÉLOPOLSKY. Memorie della Società degli Spettrosc. It. vol. XXVI, 1897. — New Investigations of the Spectrum of β Lyrae, id. Astroph. J. Vol. VI N^o. 4.

4) A spectrographic Study of β Lyrae. By Rev. WALTER SIDGREAVES S. J. Monthly Notices of R. A. S., Jan. 1904.

Dr. MYERS, Prof. CH. ANDRÉ¹⁾, basing himself on different numerical data, thinks himself justified in assuming, that the excentricity of the orbit has increased since the time of ARGELANDER, and also that the major axis has been displaced. On this supposition ANDRÉ tries to find an explanation of the terms of a higher order in the formula of ARGELANDER as corrected by Dr. PANNEKOEK²⁾. Finally Dr. L. TERKÁN has brought forward some short considerations in A. N. n° 4067³⁾. Afterwards a more elaborate investigation has appeared in the Memoirs of the Hungarian Academy of Sciences⁴⁾.

We think that this enumeration covers the principal literature about what has been put forward in *explanation* of the light-variation.

2. The original plan of the author of the present paper was a treatment by the method of MYERS of the light curve derived by Dr. PANNEKOEK, in order to ascertain whether any important change of the elements of the orbits since the time of ARGELANDER, might be established.

The first part of MYERS' thesis in which, as a first approximation, a circular orbit is derived, is generally fairly correct. But the second part in which this orbit is changed to a slightly excentric one, by the aid of differential formulae, appeared to call urgently for a fresh treatment. Erroneous normal equations have been derived from incorrect differential formulae. The former have been wrongly solved and finally the close adjustment of the theoretical curve to that of ARGELANDER, chiefly in the vicinity of the principal minimum, seems to have been obtained by a happy coincidence of numerical errors. It is of no use to enter into further particulars on the subject. As an instance we give in the 2nd column of the following table the light-intensities (I_B), as derived by MYERS from the observed grades (Stufen) of ARGELANDER during the period of from 30 hours before to 30 hours after the principal minimum. In the next column are contained the light-intensities (I_C) given by MYERS as resulting from the definitive elements of his orbit⁵⁾, the 4th col. shows these same quantities freed from numerical errors. In the three last columns the

1) *Traité d'Astronomie Stellaire* par CH. ANDRÉ, 2me p. NN. 460—1.

2) *Untersuchungen über den Lichtwechsel von β Lyrae*. Dr. A. PANNEKOEK. *Verhandelingen der Kon. Ak. van Wetensch. te Amsterdam*, Vol. 5, N°. 7. id. A. N. N°. 3456.

3) *Beitrag zur Berechnung der Bahnelemente von β Lyrae*. Dr L. TERKÁN.

4) β Lyrae pályaelemeinek kiszámítása spektroskopikai és photometriai adatokból. TERKÁN Lajostól. — *Mathematikai és Természettudományi Ertesítő*, XXIV kötet 3 füzetéből Budapest 1906.

5) *Inaugural-dissertation*, p. 48; A. J. l.c. p. 16.

same quantities have been given reduced to light-grades (σ). I do not find mentioned what is the value of a light-grade of ARGELANDER according to MYERS. From the light-intensities in the two minima I find 0.130 magnitudes, a value to which I have adhered. The intensity of the maximum has been taken for unit. The light-grades of A which, from 3.35 in the principal minimum, rise to the value 12.35 at a maximum, have been reduced to the interval of 3.00 to 12.00 for the sake of convenience.

t	I_B	I_{C_1}	I_{C_2}	σ_B	σ_{C_1}	σ_{C_2}
-30^h	0.7296	0.7525	0.7586	9.27	9.61	9.67
-24	.5836	.6019	.6674	7.40	7.73	8.60
-18	.4336	.4993	.5627	4.83	6.15	7.16
-12	.3661	.4275	.4506	3.55	4.85	5.29
- 6	.3484	.3487	.3500	3.10	3.13	3.16
0	.3433	.3433	.3433	3.00	3.00	3.00
+ 6	.3499	.3488	.3477	2.15	3.12	3.44
+12	.3988	.4275	.4462	4.30	4.85	5.21
+18	.5306	.5591	.5586	6.67	7.41	7.40
+24	.6572	.6624	.6635	8.46	8.54	8.55
+30	.7644	.7528	.7553	9.70	9.61	9.64

In what follows we have tried, first of all, to give correct formulae for the derivation of a slightly excentric orbit from the variation of the light. These have then been used for the curve of ARGELANDER and for that of Dr. PANNEKOEK. Afterwards the spectroscopic data of BELOPOLSKY have also been freshly reduced, because there is some uncertainty about the resulting orbit ¹⁾. This is perhaps to be attributed to the method of LEHMAN—FILHÉS ²⁾. This method is excellent for a satisfactory determination of the excentricity if it is large; but it is less suitable for a very small excentricity. For the drawing of the graphical velocity-curve remains always slightly arbitrary and this fact exerts too strong an influence in the case that e is small.

3. We thus start from the following hypothesis:

¹⁾ In the "Recherches nouvelles" (Memorie etc.) BELOPOLSKY gives $e = 0.04$; in his "New investigations" (A. J.) $e = 0.07$, as the result of the same observations.

²⁾ A. N. n. 3242.

Two similar, elongated revolution-ellipsoids move about their common centre of gravity in elliptic orbits. We assume that the major axes of the ellipsoids are continually in each other's prolongation while their centres move about their common centre of gravity in obedience to the laws of KEPLER. Required the intensity of the light as it appears to our eye, if we assume that the ellipsoids may be exchanged for their uniformly illuminated projections on the sphere.

As unit of length we take the semi major axis of the larger ellipsoid (E_1); as unit of brightness the maximum of β -Lyrae.

Further let be:

α the semi major axis of the smaller body:

q the proportion of the major axis to the diameter of the equator;

f the proportion of the major axis of the ellipse, which is the projection of one of the ellipsoids on the sphere, to the major axis of that same ellipsoid;

a the semi major axis of the relative orbit of the smaller body (E_2), e the excentricity, v the true anomaly, r the radius vector in the true relative orbit.

β the angle formed by the radius vector in the true orbit with the projection of the visual line on the plane of the orbit (on the further side of the sphere); this angle increases with the motion in the orbit;

ω the longitude of the periastron, counted in the same way as β ;

i the angle between the plane of the orbit and a plane tangent to the sphere;

ρ the projection of r on the sphere;

M the common part of two circles the radii of which are resp. $= 1$ and $= \alpha$, having their centres at a distance of $\rho' = \frac{\rho}{f}$;

λ the proportion of the brightness (per unit of surface) of the larger of the elliptic projections to the smaller one;

J the apparent total light-intensity at the time t , (as seen from the earth).

As long as E_1 and E_2 do not cover each other, we have:

$$J = f.$$

When E_2 is covered by E_1

$$J = f \left(1 - \frac{M}{\pi (\lambda + \alpha^2)} \right).$$

When E_1 is covered by E_2

$$J = f \left(1 - \frac{\lambda M}{\pi (\lambda + \alpha^2)} \right).$$

Let $2\varphi'$ and 2φ be the angles formed by the common chord of the circles, which define M , as seen from their respective centres, then

$$M = \frac{1}{2} \{ (2\varphi' - \sin 2\varphi') + \kappa^2 (2\varphi - \sin 2\varphi) \}$$

$$\sin \varphi' = \kappa \sin \varphi; \quad \cos \varphi = \frac{\varrho'^2 + \kappa^2 - 1}{2\kappa \varrho'}$$

φ' is always $< \frac{\pi}{2}$; φ may become $= \pi$, in the case that the smaller disc is seen projected wholly within the larger one.

Furthermore:

$$\varrho^2 = r^2 (1 - \cos^2 \beta \sin^2 i); \quad \beta = \omega + \nu.$$

These formulae agree with those of Dr. MYERS.

Computation of f .

The equation of the cylinder, enveloping the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ the axis of which makes the angles φ, χ, ψ with the X, Y - and Z -axis, is:

$$\left(\frac{\cos^2 \varphi}{a^2} + \frac{\cos^2 \chi}{b^2} + \frac{\cos^2 \psi}{c^2} \right) \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) = \left(\frac{x \cos \varphi}{a^2} + \frac{y \cos \chi}{b^2} + \frac{z \cos \psi}{c^2} \right)^2.$$

The surface of an orthogonal section of this cylinder is:

$$\Omega = \pi \sqrt{a^2 b^2 \cos^2 \psi + b^2 c^2 \cos^2 \varphi + c^2 a^2 \cos^2 \chi}.$$

or, putting

$$a = b = \frac{1}{q}, c = 1$$

$$\Omega = \frac{\pi}{q} \sqrt{1 - \varepsilon^2 \cos^2 \psi}, \quad \varepsilon^2 = \frac{q^2 - 1}{q^2}.$$

The semi minor axis of the section is $\frac{1}{q}$, therefore the semi major axis

$$f = \sqrt{1 - \varepsilon^2 \cos^2 \psi}; \quad \cos^2 \psi = \cos^2 \beta \sin^2 i;$$

because ψ is the angle between the major axis of the ellipsoid and the visual line.

In the computation of f Dr. MYERS, instead of taking the instantaneous projection of the ellipsoid on the sphere, takes the intersection of the ellipsoid with a plane through the centre, at right angles to the visual line. In his opinion this is allowable "wenn die Abplattung

nicht ungeheuer gross ist". He therefore puts

$$i = \frac{1}{\sqrt{1 + (q^2 - 1) \cos^2 \psi}}.$$

As the rigorous formula is at least equally simple there is no reason for the substitution. If we put the expression of MYERS = f' , we have

$$\frac{f}{f'} = \sqrt{1 + \left(\frac{q^2 - 1}{2q}\right)^2 \sin^2 2\psi}.$$

Greatest values, for $\psi = \frac{\pi}{4}$

q	1.2	1.3	1.4	1.5
$\frac{f}{f'}$	1.02	1.03	1.06	1.08

There thus is introduced a systematic error, which, already for small elongations, cannot be neglected.

4. As soon as, with the aid of provisional elements, a light-curve has been calculated, we try to vary these elements in such a way that the differences between observation and computation are diminished. We have to investigate, therefore, in what way the light-intensity varies with the elements.

We have already: $J = F(f, M, \lambda, \alpha^2)$. We will now, first of all, express df and dM in function of $d(\alpha^2)$, $d\beta$, dQ .

In the first place we have to consider that λ is fully determined by α^2 , in the case, which as we shall presently see must be admitted, that during the minima E_2 is projected wholly on E_1 . For, if $f_m = \sqrt{1 - \varepsilon^2 \sin^2 i}$ (= value of f in both the minima), then, on the same supposition:

$$f_m \cdot \frac{\lambda}{\lambda + \alpha^2} = \text{const.}_1 \quad (= \text{intensity at the principal minimum})$$

$$f_m \left(1 - \frac{\alpha^2 \lambda}{\lambda + \alpha^2}\right) = \text{const.}_2 \quad (= \text{,, ,, ,, secondary ,, })$$

From these, after division:

$$d\lambda = \frac{\lambda(1-\lambda)}{\alpha^2} d(\alpha^2) \quad \text{and} \quad \frac{df_m}{f_m} = \frac{\lambda}{\lambda + \alpha^2} d\alpha^2 = -\frac{d(\varepsilon^2 \sin^2 i)}{2f_m^2}.$$

With the aid of the latter formula we get without difficulty

$$df = \frac{\varepsilon^2 \sin^2 i}{2f} \sin 2\beta d\beta + \frac{\cos^2 \beta}{f} (1 - \varepsilon^2 \sin^2 i) \frac{\lambda}{\lambda + \alpha^2} d(\alpha^2);$$

which is independent of the variation of i .

The computation of dM directly from the formulae is rather lengthy¹⁾; by considering the geometrical meaning of M , dM is found at once. Evidently M is purely a function of x and ϱ' . If x increases by the amount Δx , the increment of M is a strip $2x\varrho\Delta x$; if ϱ' increases by $\Delta\varrho'$, the increment of M is *negative* and equal to a strip (crescent) $2 \sin \varphi' \cdot \Delta\varrho' = 2x \sin \varphi' \cdot \Delta\varrho'$. Therefore

$$\begin{aligned} dM &= \varrho d(x^2) - 2 \sin \varphi' \cdot d\varrho' \\ &= \varrho d(x^2) - \frac{2 \sin \varphi'}{f} d\varrho + \frac{2\varrho \sin \varphi'}{f^2} \cdot df \end{aligned}$$

If in this expression we substitute the value, given above, of df , we get dM expressed as a function of $d(x^2)$, $d\beta$, $d\varrho$.

5. *Calculation of $d\beta$ and $d\varrho$ in function of the variations of the elements of the orbit and of the epoch.*

If $\sin \varphi = e$, then (*vide* BAUSCHINGER, die Bahnbestimmung der Himmelskörper, n^o. 197):

$$\begin{aligned} dv &= \left(\frac{a}{r}\right)^2 \cos \varphi \{(t-T) d\mu - \mu dT\} + \frac{a}{r} \cos \varphi \sin v \left(1 + \frac{r}{p}\right) d\varphi. \\ \frac{d\varrho}{\varrho} &= \frac{da}{a} - \sin(P-\Omega) \cos(P-\Omega) \sin i \operatorname{tg} i \cdot d\omega - \sin^2(P-\Omega) \operatorname{tg} i \cdot di \\ &+ \left(\frac{a}{r}\right)^2 \left\{ e \sin E - \sin(P-\Omega) \cos(P-\Omega) \sin i \operatorname{tg} i \cos \varphi \{(t-T) d\mu - \mu dT\} \right. \\ &\left. - \left(\frac{a}{r}\right)^2 \left\{ (\cos E - e) \cos \varphi + \sin(P-\Omega) \cos(P-\Omega) \sin i \operatorname{tg} i \sin E \left(\frac{r}{a} + \cos^2 \varphi\right) \right\} d\varphi. \right. \end{aligned}$$

According to the definition of ω adopted above, we have to put:

$$\varrho \sin(P-\Omega) = r \cos i \cos(\omega + v) = r \cos i \cos \beta$$

$$\varrho \cos(P-\Omega) = -r \sin(\omega + v) = -r \sin \beta$$

We now pass to the following particular case:

a. the original orbit is circular;

b. if $i = 90^\circ - i'$, then i' is so small that 3rd and higher powers may be neglected; the same is true for $\sin \varphi$.

Furthermore let $d\mu = 0$; $d\omega = 0$; $2e \cos \omega = x$, $2e \sin \omega = y$;
 $n = \frac{2\pi}{T}$. $T =$ period $= 12.91$ days). $t_1 =$ time counted from "superior conjunction"; $dM_0 = -\mu dT$.

Then with sufficient approximation

¹⁾ See: Untersuchungen über den Lichtwechsel des Sternes β Persei, von J. HARTING. (München 1889) p. 41.

$$d\beta = dM_0 + x \sin nt_1 - y \cos nt_1 \dots \dots \dots (a)$$

$$\frac{dQ}{Q} = \frac{da}{a} + \frac{a^2}{2Q^2} \cos^2 nt_1 i'^2 + \frac{a^2}{2Q^2} \sin 2nt_1 \cdot dM_0 -$$

$$- \frac{1}{2} x \left(\cos nt_1 - \frac{a^2}{Q^2} \sin 2nt_1 \sin nt_1 \right) - \frac{1}{2} y \left(\sin nt_1 + \frac{a^2}{Q^2} \sin 2nt_1 \sin nt_1 \right) \dots (b)$$

As these differential expressions have led several astronomers¹⁾ into error, we will derive them in still another way.

From :

$$\beta = v + \omega$$

we get :

$$d\beta = dv + d\omega.$$

In the circular orbit $v = M$; in the elliptic orbit this becomes :

$$v = M + 2e \sin M + \dots + dM_0.$$

If we substitute $M = nt_1 - \omega$, and put $d\omega = 0$, we get, neglecting higher powers of e :

$$d\beta = dM_0 + x \sin nt_1 - y \cos nt_1$$

If in :

$$Q^2 = r^2 \sin^2 \beta + r^2 \cos^2 i \cos^2 \beta,$$

$$i = 90^\circ \dots i',$$

then, neglecting higher powers of i' :

$$\frac{dQ}{Q} = \frac{dr}{r} + \frac{r^2}{2Q^2} \sin 2\beta d\beta + \frac{r^2}{2Q^2} \cos^2 \beta \cdot i'^2 \dots \dots \dots (c)$$

In the elliptic orbit we have :

$$r = \frac{a(1-e^2)}{1+e \cos v} = a_0 + da - ae \cos(\beta - \omega) + \dots =$$

$$= a_0 + da - \frac{1}{2} a x \cos nt_1 - \frac{1}{2} ay \sin nt_1 \dots$$

Therefore :

$$dr = da - \frac{1}{2} ax \cos nt_1 - \frac{1}{2} ay \sin nt_1$$

and, substituting this in (c), we get the expression already given of

$$\frac{dQ}{Q}$$

¹⁾ Dr. MYERS puts $d\beta = 0$ for $t_1 = 0$ and at the same time $dM_0 = 0$; this is incompatible with (a). Prof. HARTWIG, in his paper: "Der veränderliche Stern vom Algoltypus Z Herculis" (Bamberg 1900) p. 39, puts $\sin(P-\Omega) \cos(P-\Omega) \sin i \operatorname{tg} i = 0$ for $i = 90^\circ$, whereas, according to our formulas, it becomes $-\frac{1}{2Q^2} \sin 2\beta$.

(See also A.N. 3644).

Dr. PANNEKOEK quotes another instance in his Thesis on Algol (p. 22-3).

6. We thus have consecutively expressed dM and df in function of $d(x^2)$, $d\beta$ and $d\varrho$; and afterwards $d\beta$ and $d\varrho$ in function of da , i'^2 , dM_0 , x and y . If now we differentiate the expression

$$J_1 = f \left(1 - \frac{M}{\pi(\lambda + x^2)} \right),$$

valid in the vicinity of the first minimum, we find, by consecutive substitution, the following expression for dJ_1 :

$$\pi(\lambda + x^2) dJ_1 = K_1 d(x^2) + A_1 da + I_1 i'^2 + X_1 x + Y_1 y + \Delta_1 (dM_0 - y),$$

in which :

$$K_1 = \frac{\pi\lambda}{q^2 f^2} J_1 \cos^2 nt_1 + \pi \left(1 + \frac{\lambda - \lambda^2}{x^2} \right) (f - J_1) - f\varrho - \frac{2\lambda}{q(\lambda + x^2)} \cdot \frac{\varrho}{f^2} \cdot \cos^2 nt_1 \sin \varphi';$$

$$A_1 = \frac{2\varrho \sin \varphi'}{r}; \quad I_1 = \frac{r^2 \cos^2 nt_1 \sin \varphi'}{\varrho};$$

$$X_1 = \Delta_1 \sin nt_1 - \varrho \sin \varphi' \cos nt_1; \quad Y_1 = \Delta_1 (1 - \cos nt_1) - \varrho \sin \varphi' \sin nt_1;$$

$$\Delta_1 = \frac{r^2 \sin 2nt_1 \sin \varphi'}{\varrho} + \frac{\varepsilon^2}{2f^2} \sin 2nt_1 \{ \pi(\lambda + x^2) J_1 - 2\varrho \sin \varphi' \}.$$

If we treat in the same way the expression:

$$J_2 = f \left(1 - \frac{\lambda M}{\pi(\lambda + x^2)} \right),$$

valid in the vicinity of the second minimum, we find, putting

$$t_2 = t_1 - \frac{U}{2};$$

$$\pi \frac{\lambda + x^2}{\lambda} dJ_2 = K_2 d(x^2) + A_2 da + I_2 i'^2 + X_2 x + Y_2 y + \Delta_2 (dM_0 + y)$$

in which :

$$K_2 = \frac{\pi}{q^2 f^2} J_2 \cos^2 nt_2 + \pi (f - J_2) - f\varrho - \frac{2\lambda}{q(\lambda + x^2)} \cdot \frac{\varrho}{f^2} \cdot \cos^2 nt_2 \sin \varphi';$$

$$A_2 = \frac{2\varrho \sin \varphi'}{r}; \quad I_2 = \frac{r^2 \cos^2 nt_2 \sin \varphi'}{\varrho};$$

$$X_2 = -\Delta_2 \sin nt_2 + \varrho \sin \varphi' \cos nt_2; \quad Y_2 = -\Delta_2 (1 - \cos nt_2) + \varrho \sin \varphi' \sin nt_2,$$

$$\Delta_2 = \frac{r^2 \sin 2nt_2 \sin \varphi'}{\varrho} + \frac{\varepsilon^2}{2f^2} \sin 2nt_2 \left(\pi \frac{\lambda + x^2}{\lambda} J_2 - 2\varrho \sin \varphi' \right).$$

7. If the observations do not give the light-intensity, but the brightness expressed in magnitudes or in grades, then we have still to express the variation of the number indicating the magnitude or the grade, in the variation of the light-intensity.

Let J_0 represent the intensity at the maximum, G , the corresponding magnitude, J and G the same quantities at the time t , then, by the formula of POGSON :

$$G - G_0 = 2.512 (\log J_0 - \log J).$$

Consequently :

$$dG = - 2.512 m \cdot \frac{dJ}{J}, (m = \text{modulus of Brigg's log.})$$

$$dG = - 1.092 \frac{dJ}{J}.$$

Now, if $-\frac{1}{v}$ is the equivalent in magnitudes of a grade, then, σ_0 and σ being the number of grades:

$$\sigma_0 - \sigma = v(G - G_0) = 2.512v (\log J_0 - \log J)$$

Therefore:

$$d\sigma = 1.092v \frac{dJ}{J}$$

Putting the value of ARGELANDER'S grade for the light-curve of β Lyrae at 0.130 magnitudes, then:

$$d\sigma = 8.413 \frac{dJ}{J}.$$

8. In the hypothesis which we adopted, the main phases (min., max., min., max.) take place for the values $\beta=0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ of β_1 . Let v_1, v_2, v_3, v_4 represent the true anomalies for these values; M_1, M_2, M_3, M_4 the corresponding mean anomalies. If, as is the case with β Lyrae, the intervals are nearly equal, e must be small and we may put approximately:

$$v_1 = M_1 - y; v_2 = M_2 + x; v_3 = M_3 + y; v_4 = M_4 - x.$$

$$(x = 2e \cos \omega; y = 2e \sin \omega)$$

or :

$$v_2 - v_1 = \frac{\pi}{2} = (M_2 - M_1) + x + y$$

$$v_3 - v_2 = \frac{\pi}{2} = (M_3 - M_2) - x + y$$

$$v_4 - v_3 = \frac{\pi}{2} = (M_4 - M_3) - x - y$$

If the differences $M_2 - M_1 \dots$ ¹⁾ are known with equal and

¹⁾ The time-equation for the reduction to the common centre of gravity, computed from the spectroscopic orbit, is found to reach a value of somewhat over ± 100 seconds and may consequently be neglected.

sufficient precision, we find from these formulae the most probable values of x and y as follows:

$$\left. \begin{aligned} 4x &= -\pi + (M_4 - M_3) + 2(M_3 - M_2) - (M_2 - M_1) \\ 4y &= \pi + (M_4 - M_3) - 2(M_3 - M_2) - (M_2 - M_1) \end{aligned} \right\} \dots I$$

If we combine only similar phases, we get

$$\left. \begin{aligned} 2x &= -\pi + (M_4 - M_2) \\ 2y &= \pi - (M_3 - M_1) \end{aligned} \right\} \dots II$$

The two solutions are identical, if

$$(M_2 - M_1) + (M_4 - M_3) = \pi.$$

In A. N. n^o. 3456 Dr. PANNEKOEK summarises the intervals, counted from the principal minimum, for different observers between 1842 to 1895.

Dividing this period in two, he finds on an average: ($U = 12^d.91$)

	max ₁ —min ₁	min ₂ —min ₁	max ₂ —min ₁
1842—1870	3 ^d .12	6 ^d .40	9 ^d .54
1870—1895	3 ^d .32	6 ^d .48	9 ^d .73.

From these values we find, for the first period:

according to form. (I):

according to form. (II):

$$\left\{ \begin{aligned} e \sin \omega &= -0.0052; \quad e = 0.009 \\ e \cos \omega &= +0.0076; \quad \omega = 326^\circ \end{aligned} \right\} \left\{ \begin{aligned} e \sin \omega &= +0.0067; \quad e = 0.008 \\ e \cos \omega &= -0.0043; \quad \omega = 123^\circ. \end{aligned} \right.$$

Similarly for the second period:

$$\left\{ \begin{aligned} e \sin \omega &= -0.0040; \quad e = 0.013 \\ e \cos \omega &= -0.0125; \quad \omega = 162^\circ \end{aligned} \right\} \left\{ \begin{aligned} e \sin \omega &= -0.0030; \quad e = 0.006 \\ e \cos \omega &= -0.0055; \quad \omega = 209^\circ \end{aligned} \right.$$

The only conclusion to be derived from these results is that e was very minute in both periods, and hardly exceeding 0.01.

9. A single glance at the numbers communicated by Dr. PANNEKOEK shows that a trial to derive something more definite from the results of the *separate* observers would be quite hopeless. In particular we may allege the considerable difference between the results obtained by LINDEMANN and PANNEKOEK, in their reduction of the observations of PLASSMANN. It thus seems to be out of place, from these observations alone, to draw the conclusion that the excentricity has increased.

Dr. L. TERKAN has proposed the following method of deriving the inclination of the orbit.¹⁾

1) A. N. nr 4067.

The minimum or maximum of light takes place when ϱ takes "extreme" values, consequently when

$$\frac{d\varrho}{dv} = \cos v \sin v \sin^2 i + e \sin v = 0 \quad . \quad . \quad . \quad (2)$$

In formula (2) $\sin v = 0$ for the principal minimum, $\cos v = -\frac{e}{\sin^2 i}$ for the "secondary maximum". This is in the assumption that the time of the principal minimum coincides with the time of periastron. Therefore if, at the moment of the maximum, we know e and v , then we know also i . TERKAN adopts the value 0.07, derived for e by BELOPOLSKY from his spectroscopic observations¹⁾. He determines the mean anomaly at the maximum from the interval found by PLASSMANN²⁾:

$$II \text{ min.} - II \text{ max.} = 3,05 \text{ days,}$$

and then expands this anomaly in a series³⁾ by the aid of $\cos v = -\frac{e}{\sin^2 i}$. This series has an argument α , which contains $\sin^2 i$. He thus finds

$$i = 51^\circ.3.$$

Afterwards, in his Hungarian paper⁴⁾, he takes $e = 0.06$. From his own observations he derives: I min. — I max. = 3,48 days and then finds, using the usual equations of KEPLER, by the aid of $\cos v = -\frac{e}{\sin^2 i}$:

$$i = 30'.$$

Even if we disregard the very doubtful value of the numerical data, the hypothesis seems unfounded that the maximum of the light occurs at the moment that ϱ is a maximum. If, moreover, we assume with Dr. TERKAN, both the celestial bodies to be spherical, then the light must be constant as long as the two spheres do not cover each other as seen by the observer. This is not confirmed by observation. Besides there can be no question of a clearly defined epoch of maximum in such a case. The way in which Dr. TERKAN meets this objection by saying: "that our eye or the telescope is unable to separate the system and that the rays of light which in space come

¹⁾ See p. 462. 1st footnote.

²⁾ A. N. nr 3242.

³⁾ In this series e has been erroneously substituted for $\sin \varphi \cos \varphi$ and $\frac{\sqrt{4-e^2}-e}{\sqrt{4-e^2+e}}$ (instead of $\sqrt{\frac{1-e}{1+e}}$) for $tg(45^\circ - \frac{1}{2}\varphi)$.

⁴⁾ β Lyrae palyaclemeinek etc. p. 412.

from the same distance, but from a larger field, are united to a larger disc" ¹⁾, seems little satisfactory.

10. *Determination of the elements of the orbit etc. by means of the light-curve of ARGELANDER* ²⁾.

As a first approximation we put $i = 90^\circ$, $e = 0$.

An approximate value of q is furnished by the general course of the curve in the vicinity of the maxima. As long as it is symmetrical in regard to the ordinate of the maximum, we may assume that the eclipse has not yet begun, so that

$$J = \sqrt{1 - \varepsilon^2 \sin^2 nt_M};$$

t_M being the time counted from the maximum.

From the light-curve we take the decrease of $(\sigma_0 - \sigma)$ in grades, for equal intervals of time before and after the two maxima.

t_M	$(\sigma_0 - \sigma)_I$	$(\sigma_0 - \sigma)_{II}$	t_M	$(\sigma_0 - \sigma)_{mean}$	t_M	$(O - C)_I$	$(O - C)_{II}$
-30 ^h	0.76	0.50					
-24	0.47	0.31	$\pm 6^h$	0.025	-18 ^h	-0.03	+0.04
-18	0.25	0.18	± 12	0.093	-12	0.00	+0.03
-12	0.10	0.07	± 18	0.220	-6	0.00	0.00
-6	0.02	0.02			+6	-0.01	-0.01
+6	0.03	0.03			+12	+0.01	-0.01
+12	0.09	0.11			+18	+0.01	-0.01
+18	0.21	0.24					
+24	0.38	0.43					
+30	0.59	0.67					

An increasing dissymmetry begins to show itself for both the maxima at about 24 hours distance from these epochs.

With the mean values $(\sigma_0 - \sigma)_{mean}$ the light-intensities were now computed by the formula:

$$\sigma_0 - \sigma = - \frac{2.512}{0.13} \log J$$

and the relation

$$\varepsilon^2 \sin^2 nt_M = 1 - J^2$$

¹⁾ β Lyrae palyaelemeinek etc. p. 417.

²⁾ De Stella β Lyrae variabili commentatio altera. Scripsit FREDERICUS ARGELANDER Bonnae a. 1859. — Curva "vera" pro 1850.

furnished the data :

$$0.015 \varepsilon^2 = 0.006$$

$$0.058 \varepsilon^2 = 0.022$$

$$0.127 \varepsilon^2 = 0.051$$

leading to the most probable values $\varepsilon^2 = 0.397$; $q = 1.288$.

The deviations Obs.-Comp. have been given in the two last columns.

Having found q , we get \varkappa and λ from the light-intensities at the two minima. The values of these being 0.3433 and 0.6365, we obtain, for $i = 90^\circ$, the two relations

$$\frac{1}{q} \cdot \frac{\lambda}{\lambda + \varkappa^2} = 0.3433; \quad \frac{1}{q} \left(1 - \frac{\varkappa^2 \lambda}{\lambda + \varkappa^2} \right) = 0.6365,$$

whence :

$$\varkappa = 0.6387; \quad \lambda = 0.3233.$$

Finally, at the moment at which the eclipse begins :

$$\frac{q}{f} = \varrho' = 1 + \varkappa.$$

The consideration of the asymmetry, shows that this must be the case shortly after 18^h ($nt_M = \pm 20^\circ 55'$). We therefore put :

$$\frac{q}{f} = \frac{a \cos 21^\circ}{\sqrt{1 - \varepsilon^2 \sin^2 21^\circ}} = 1.6387 = 1 + \varkappa$$

from which :

$$a = 1.710.$$

We thus have, as a first approximation, the following elements : $\varkappa = 0.6387$; $\lambda = 0.3233$; $q = 1.288$; $a = 1.710$; $e = 0$; $i = 90^\circ$ and, as for the "epoch", we assume, that the central eclipse of E_2 by E_1 , coincides with the principal minimum of ARGELANDER'S curve.

11. In the following table the 2nd column, headed O_1 , shows the light-grades of ARGELANDER'S curve for equal intervals before and after the principal minimum: the 7th column, headed O_2 , similarly shows the same element before and after the half period = 6^d.455 (*not* therefore before and after the secondary minimum, which ARGELANDER places at 6^d.375 from the principal minimum). The columns C_{s_1} and C_{s_2} contain the light-grades, computed by the aid of the elements given just now.

12. As will be remarked, the deviations $O - C_s$ are in the main negative before, positive after the two minima. We conclude that, by shifting the theoretical light-curve in a negative direction with regard to the time, we may obtain improved agreement. The excen-

t	O	C_{S_1}	C_{M_1}	$O_1 - C_{S_1}$	$O_1 - C_{M_1}$	O_2	C_{S_2}	C_{M_2}	$O_2 - C_{S_2}$	$O_2 - C_{M_2}$
-72	11.95	11.98	11.98	-0.03	-0.03	11.87	11.98	11.98	-0.11	-0.11
-66	11.84	11.91	11.93	-0.07	-0.09	11.79	11.91	11.93	-0.12	-0.14
-60	11.69	11.80	11.84	-0.11	-0.15	11.66	11.80	11.84	-0.14	-0.16
-54	11.48	11.58	11.72	-0.10	-0.24	11.48	11.62	11.72	-0.14	-0.24
-48	11.20	11.34	11.45	-0.14	-0.25	11.25	11.37	11.52	-0.12	-0.27
-42	10.82	10.75	11.04	+0.07	-0.22	10.99	11.06	11.26	-0.07	-0.27
-36	10.29	10.07	10.44	+0.22	-0.15	10.68	10.68	10.92	0.00	-0.24
-30	9.27	9.14	9.64	+0.13	-0.37	10.30	10.22	10.50	+0.08	-0.20
-24	7.40	7.91	8.55	-0.51	-1.15	9.78	9.70	10.00	+0.08	-0.22
-18	4.83	6.23	7.13	-1.45	-2.30	9.04	9.13	9.44	-0.09	-0.40
-12	3.55	4.22	5.26	-0.67	-1.71	8.42	8.56	8.82	-0.14	-0.40
-6	3.10	3.05	3.15	+0.05	-0.05	8.21	8.23	8.25	-0.02	-0.04
0	3.00	3.00	3.00	0.00	0.00	8.20	8.19	8.19	+0.01	0.00
+6	3.15	3.05	3.15	+0.10	0.00	8.38	8.23	8.25	+0.15	+0.13
+12	4.30	4.22	5.26	+0.08	-0.96	8.91	8.56	8.82	+0.35	+0.09
+18	6.67	6.28	7.13	+0.39	-0.46	9.64	9.13	9.44	+0.51	+0.20
+24	8.46	7.91	8.55	+0.55	-0.09	10.25	9.70	10.00	+0.55	+0.25
+30	9.70	9.14	9.64	+0.56	+0.06	10.75	10.22	10.50	+0.53	+0.25
+36	10.50	10.07	10.44	+0.43	+0.06	11.14	10.68	10.92	+0.46	+0.22
+42	10.97	10.75	11.04	+0.20	-0.07	11.43	11.06	11.26	+0.37	+0.17
+48	11.21	11.34	11.45	-0.03	-0.14	11.64	11.37	11.52	+0.27	+0.12
+54	11.57	11.58	11.72	-0.01	-0.15	11.79	11.62	11.72	+0.17	+0.07
+60	11.75	11.80	11.84	-0.05	-0.09	11.91	11.80	11.84	+0.11	+0.07
+66	11.88	11.91	11.93	-0.03	-0.05	11.98	11.91	11.93	+0.07	+0.05
+72	11.91	11.98	11.98	-0.07	-0.07	12.01	11.98	11.98	+0.03	+0.03

tricity, besides displacing the maxima and minima, also causes a slight dissymmetry in regard to the minima. In order to separate the influence of the excentricity on the asymmetry from that of the epoch, we may divide the equations of condition into two groups. For the coefficients K , A , I and X are even functions; Y and Δ uneven functions of nt_1 , resp. nt_2 . If, therefore, we take the sum and the difference of two equations of condition, corresponding to

the times $+t_1$ and $-t_1$, the former of the resulting equations will only contain the quantities $d(z^2)$, da , t^2 and x , the latter only y and dM .

In this way the following equations have been derived for successive intervals of six hours counted from the 1th, resp. 2nd minimum.

They do not rest however on the above elements but on those

P R I N C I P A L M I N I M U M.

t_1	$\pm 72^h$	0.03	$d(z^2)$	0.00	x	$=$	0.05	0.02	y	0.70	$(dM_0 - y)$	
	06	.13	+	.00	4.41	—	.07	—	1.12	1.45	—	0.02
	60	.30	+	.00	2.02	—	.42	—	1.11	2.15	—	0.03
	54	.04	+	.02	2.59	—	.49	—	4.55	3.20	—	0.04
	48	.40	+	.06	3.16	—	.19	—	0.94	4.76	—	0.05
	42	.50	+	.14	3.55	—	.44	—	7.76	6.37	—	0.07
	36	.40	+	.28	3.80	—	.04	—	6.0	8.48	—	0.10
	30	.06	+	.50	3.89	—	.15	—	.44	10.32	—	0.51
	24	.60	+	.91	3.78	—	.62	—	.31	12.92	—	0.53
	18	1.75	+	1.70	3.43	—	4.38	—	.18	16.08	—	0.92
	12	3.42	+	3.38	2.69	—	4.33	—	.07	19.57	—	0.37
	6	2.97	+	4.93	1.02	—	0.02	—	.01	14.41	—	0.02

S E C O N D A R Y M I N I M U M.

t_2	$\pm 72^h$	0.03	$d(z^2)$	0.00	x	$=$	0.04	0.02	y	0.05	$(dM_0 + y)$	
	06	.12	+	.00	.10	—	.04	—	.07	10	—	0.02
	60	.30	+	.00	.44	—	.05	—	.09	15	—	0.09
	54	.33	+	.00	.26	—	.08	—	.01	.60	—	0.15
	48	.30	+	.02	.44	—	.07	—	.07	.90	—	0.19
	42	.38	+	.05	.59	—	.05	—	.06	1.62	—	0.22
	36	.52	+	.10	.71	—	.01	—	.05	2.01	—	0.23
	30	.70	+	.18	.79	+	.02	—	.04	2.65	—	0.23
	24	.90	+	.30	.80	+	.01	—	.03	3.35	—	0.23
	18	1.11	+	.52	.73	—	.40	—	.01	4.02	—	0.30
	12	1.22	+	.89	.55	—	.15	—	.00	4.48	—	0.24
	6	0.62	+	1.09	.47	+	.04	—	.00	2.71	—	0.08

which have been derived by repeated approximations from ARGELANDER'S curve by Dr. MYERS. In his opinion these are the best possible circular elements:

$$a = 1.8955; \quad z = 0.7580; \quad q = 1.1993; \quad \lambda = 0.4023; \quad i = 0$$

By their aid I computed the light-grades C_{M_1} and C_{M_2} of the preceding table. As will be remarked, the deviations $O - C_{M_1}$ are rather considerable in the vicinity of the principal minimum.

In deriving the following normal equations, the equations of condition for $t_1 = \pm 6^h$ and $t_2 = \pm 6^h$ have been neglected.

Normal-equations :

$$18.54 d(x^2) + 15.17 da + 16.98 (10t'^2) + 15.31 x = - 7.670$$

$$15.17 \quad \text{..} \quad + 41.68 \quad \text{,,} \quad + 18.14 \quad \text{,,} \quad + 53.17 \quad \text{,,} = - 9.673$$

$$16.98 \quad \text{,,} \quad + 18.14 \quad \text{,,} \quad + 20.29 \quad \text{,,} \quad + 20.75 \quad \text{,,} = - 7.720$$

$$15.31 \quad \text{,,} \quad + 53.17 \quad \text{,,} \quad + 20.75 \quad \text{,,} \quad + 101.96 \quad \text{,,} = - 13.227$$

$$255.69 y + 160.37 (dM_0 - y) = 10.242$$

$$160.37 y + 1125.52 (dM_0 - y) = 37.759$$

Solution of the first four equations :

$$x = - 0.026; \quad 10t'^2 = - 0.044; \quad da = - 0.0741; \quad d(x^2) = - 0.2915.$$

As t' becomes imaginary, we put $t' = 0$ in the equations of condition, and then find :

$$x = - 0.026; \quad da = - 0.0799; \quad d(x^2) = - 0.3268.$$

$$y = + 0.021; \quad dM_0 - y = + 0.031,$$

which lead to the improved elements :

$$a = 1.8156; \quad z = 0.4978; \quad \lambda = 0.2249; \quad q = 1.3859; \quad e = 0.017; \quad \omega = 141^\circ.3.$$

The correction for x^2 is particularly large, more than half its original value (0.5746). As in such a case dJ cannot any longer be considered to be proportional to $d(x^2)$, we should have to compute a new light-curve by the aid of the new elements; we should then have to calculate the differential coefficients in order — if necessary — to find a new approximation.

In the following table the columns C_1 and C_2 show the light-grades calculated by means of the improved elements.

In fig. I has been given a graphical representation of these numbers. The agreement in the vicinity of the principal minimum is considerably improved. It is true that there remains a deviation exceeding a light-grade, at 18 hours before the minimum. It might perhaps be further diminished by a repetition of the whole process. If, however, we take into account the uncertainty mostly existing when we draw the curve for the vicinity of the minimum, then it seems hardly worth while to repeat the elaborate calculation. At all

t	C_1	O_1	$O_1 - C_1$	O_2	C_2^*	$O_2 - C_2$
-72h	11.95	11.95	0.00	11.87	11.97	-0.10
-66	11.84	11.84	.00	11.79	11.88	-.03
-60	11.69	11.68	+ .01	11.66	11.73	-.07
-54	11.48	11.47	+ .01	11.48	11.52	-.04
-48	11.20	11.22	-.02	11.25	11.26	-.01
-42	10.82	10.93	-.11	10.99	10.96	+ .03
-36	10.29	10.42	-.13	10.68	10.55	+ .13
-30	9.27	9.47	-.20	10.30	10.05	+ .25
-24	7.40	8.06	-.66	9.78	9.47	+ .31
-18	4.83	6.01	-1.18	9.04	8.84	+ .20
-12	3.55	3.46	+ .09	8.42	8.35	+ .07
-6	3.10	3.05	+ .05	8.21	8.22	-.01
0	3.00	3.00	.00	8.20	8.21	-.01
+6	3.15	3.06	+ .09	8.38	8.30	+ .08
+12	4.30	3.88	+ .42	8.91	8.73	+ .18
+18	6.67	6.39	+ .28	9.64	9.35	+ .29
+24	8.46	8.32	+ .14	10.25	9.94	+ .31
+30	9.70	9.66	+ .04	10.75	10.45	+ .30
+36	10.50	10.54	-.04	11.14	10.87	+ .27
+42	10.97	11.01	-.04	11.43	11.18	+ .25
+48	11.31	11.30	+ .01	11.64	11.44	+ .20
+54	11.57	11.55	+ .02	11.79	11.66	+ .13
+60	11.75	11.75	.00	11.91	11.83	+ .08
+66	11.88	11.89	-.01	11.98	11.94	+ .04
+72	11.91	11.98	-.07	12.01	12.00	+ .01

events the small coefficients of y in the equations of condition, show clearly that it is impossible to explain any appreciable asymmetry by an eccentricity of a few hundredths. The improved agreement near the principal minimum is obtained in the main by shifting the theoretical principal minimum by $\frac{dM_0 - y}{2\pi} = U = 0.063$ in the direction of the negative time-axis, while at the same time the secondary minimum occurs 0.069 days before the minimum of ARGELANDER'S curve.

13. A second set of elements was derived by making the plausible assumption, that the first minimum in ARGELANDER'S curve occurs 0.08 days earlier. We thus get rid of the greater part of the dissymmetry. The second theoretical minimum is assumed to coincide with the observed minimum. The interval between the two minima thus becomes just half the period ($6^d.375 + 0^d.08 = 6^d.455$). As a consequence the orbit must be either circular or elliptic with

t	O_1	C_1	C'_1	O_1-C_1	$O_1-C'_1$	O_2	C_2	C'_2	O_2-C_2	$O_2-C'_2$
- 72 ^h	11.97	11.97	11.97	0.00	0.00	11.89	11.97	11.97	-.08	-.08
- 66	11.88	11.89	11.83	-.01	+.05	11.83	11.89	11.95	-.06	-.12
- 60	11.74	11.75	11.66	-.01	+.08	11.71	11.75	11.84	-.04	-.13
- 54	11.54	11.56	11.44	-.02	+.10	11.54	11.56	11.67	-.02	-.13
- 48	11.29	11.31	11.18	-.02	+.11	11.33	11.31	11.44	+.02	-.11
- 42	10.95	10.98	10.86	-.03	+.09	11.08	11.01	11.15	+.07	-.07
- 36	10.48	10.35	10.16	+.13	+.32	10.78	10.62	10.71	+.16	+.07
- 30	9.65	9.42	9.20	+.23	+.45	10.42	10.26	10.40	+.16	+.02
- 24	8.06	8.00	7.85	-.03	+.21	9.95	9.63	9.75	+.32	+.20
- 18	5.48	6.29	6.05	-.81	-.57	9.28	9.06	9.15	+.22	+.13
- 12	3.80	4.07	3.90	-.27	-.10	8.56	8.52	8.57	+.04	-.01
6	3.18	3.05	3.05	+.13	+.13	8.25	8.25	8.26	.00	-.01
0	3.00	3.00	3.00	.00	.00	8.19	8.20	8.20	-.01	-.01
+ 6	3.07	3.05	3.05	+.02	+.02	8.30	8.25	8.26	+.05	+.04
+ 12	3.76	4.07	3.90	-.31	-.14	8.71	8.52	8.57	+.19	+.14
+ 18	6.01	6.29	6.05	-.28	-.04	9.40	9.06	9.15	+.34	+.25
+ 24	7.98	8.09	7.85	-.11	+.13	10.08	9.63	9.75	+.45	+.33
+ 30	9.37	9.42	9.20	-.05	+.17	10.61	10.26	10.40	+.35	+.21
+ 36	10.11	10.35	10.16	-.04	+.15	11.04	10.62	10.71	+.42	+.33
+ 42	10.84	10.98	10.86	-.14	-.02	11.35	11.01	11.15	+.34	+.20
+ 48	11.22	11.31	11.18	-.09	+.04	11.58	11.31	11.44	+.27	+.14
+ 54	11.50	11.56	11.44	-.06	+.06	11.75	11.56	11.67	+.19	+.08
+ 60	11.71	11.75	11.66	-.04	+.05	11.87	11.75	11.84	+.12	+.03
+ 66	11.84	11.89	11.83	-.05	+.01	11.96	11.89	11.95	+.07	+.01
+ 72	11.91	11.97	11.97	-.06	-.06	12.00	11.97	11.97	+.03	+.03

the major axis at right angles to the line of the nodes. In this way we find:

$$a=1.7209; \alpha=0.5015; \lambda=0.2276; q=1.3944; i'=7^{\circ}, 25; e=0.04; \omega=180^{\circ}.$$

In the next tables the columns C_1 and C_2 show the light-grades computed by means of the first five elements, neglecting the eccentricity. The columns C_1 and C_3 contain the same quantities taking into account the eccentricity.

The mean deviation of the values in the columns O_1-C_1' and O_2-C_2' is ± 0.17 light-grades, whereas ARGELANDER assigns the value ± 0.16 to the mean error of the ordinates of his light-curve (prob. error 0.1095). It would be quite illusory therefore to endeavour to obtain an improved agreement. Against the elliptic orbit there is however the grave objection that it gives the first maximum 0.18 days after —, the second maximum 0.10 days before the corresponding maxima of ARGELANDER'S light-curve. In the circular orbit the first maximum lies only 0.02 days, the second 0.06 days later, whereas the agreement is still very satisfactory.

14. Finally we communicate a set of circular elements obtained by a repeated approximation from the light-curve of Dr. PANNEKOEK :

$$a = 1.5378; \alpha = 0.5378; \lambda = 0.2900; q = 1.4609.$$

In deriving them we assumed that a cannot fall short of $1 + \alpha$. We further assumed the theoretical principal minimum to coincide with the observed minimum.

In the following table t is the number of hours before and after the *theoretical* principal and secondary minimum; O_1 and O_2 are the light-grades at the same moments, as read off from Dr. P.'s light-curve; C_1 and C_2 the light-grades of the theoretical curve.

The results have been graphically represented in Fig. II. The remaining deviations are mainly positive before the first minimum; after that they are negative. At the secondary minimum the signs are reversed. The deviations might be rather considerably diminished if, with a small eccentricity ($e \sin \omega = 0.016$), we place the principal minimum in Dr. PANNEKOEK'S light-curve 0.063 days later, the secondary minimum 0.069 earlier. In this way, however, the interval in time min. I — min. II is diminished more considerably than seems admissible.

For the rest it need not be said, that in the present case, where two gaseous bodies seem to be in contact, the Keplerian equations of motion must give only a rough approximation, while the action of the tides must contribute its part to mask the influence of the

t	O_1	C_1	$O_1 - C_1$	O_2	C_2	$O_2 - C_2$
-72h	11.96	11.97	-0.01	11.96	11.97	-0.01
-66	11.88	11.87	+ .01	11.87	11.88	- .01
-60	11.72	11.67	+ .05	11.62	11.71	- .09
-54	11.53	11.37	+ .16	11.33	11.46	- .13
-48	11.26	10.95	+ .31	10.89	11.15	- .26
-42	10.87	10.37	+ .50	10.42	10.75	- .33
-36	10.23	9.61	+ .62	9.92	10.27	- .35
-30	9.06	8.60	+ .46	9.38	9.72	- .34
-24	7.17	7.27	- .10	8.86	9.10	- .24
-18	5.45	5.53	- .08	8.34	8.43	- .09
-12	3.80	3.53	+ .27	7.92	7.82	+ .10
-6	3.20	3.09	+ .11	7.61	7.57	+ .04
0	3.00	3.00	.00	7.50	7.50	.00
+6	3.16	3.09	+ .07	7.75	7.57	+ .18
+12	3.77	3.53	+ .24	8.11	7.82	+ .29
+18	5.07	5.53	- .46	8.68	8.43	+ .25
+24	6.75	7.27	- .52	9.26	9.10	+ .16
+30	8.36	8.60	- .24	9.80	9.72	+ .08
+36	9.40	9.61	- .21	10.30	10.27	+ .03
+42	10.07	10.37	- .30	10.75	10.75	.00
+48	10.62	10.95	- .33	11.08	11.15	- .07
+54	11.13	11.37	- .24	11.42	11.46	- .04
+60	11.52	11.67	- .15	11.61	11.71	- .10
+66	11.78	11.87	- .09	11.87	11.88	- .01
+72	11.93	11.97	- .04	11.94	11.97	- .03

excentricity on the course of the light-curve. We conclude that, from the light-curves we can only infer that the orbit is nearly circular.

At all events there is no reason to assume an increase or decrease of the, certainly very small, excentricity. A comparison of the elements a and q might lead us to conjecture that the distance of the two celestial bodies has diminished since the time of ARGELANDER. The

increase in q is in agreement with such a supposition, but the continual lengthening of the period seems to clash with it.

15. *Computation of the orbit from the spectrographs of* BELOPOLSKY.

In the computation of the orbit from the velocities in the line of sight as derived by B. from the displacement of the bright F -line in the spectrographs of 1892, the method of WILSING¹⁾ has been adopted. For very small excentricities it is to be preferred to that of LEHMAN-FILHÉS.

The first column contains the mean time of observation at PULKOWA; the 2nd gives the phase in the light-period of 12.91 days. We have assumed, in accordance with the formula of ARGELANDER, as corrected by PANNEKOEK, that the principal minimum occurred on 1892 Sept 25. 781 M. T. Greenwich (= 25^d.865 M. T. PULKOWA).

The 3rd column contains the velocities, expressed in geographical miles, reduced to the sun. They have been taken, with slight modifications, from the *Memorie della Soc. d. Spettr. It.*, vol. XXII. For BELOPOLSKY has applied a constant correction — 2.1 G.M. for the velocity of the earth, whereas in reality this velocity varies between — 1.6 and — 2.3 G.M.

T	Phase	Veloc. in G.M.	V_r	$O-C$
	d			
Sept. 23.3	10.34	—11.2	—11.25	+0.05
24.4	11.44	—11.6	—10.09	—1.51
25.4	12.44	— 4.4	— 2.58	+1.82
27.3	1.44	+ 4.8	+ 5.28	—0.48
30.3	4.44	+10.7	+10.50	+0.20
Oct. 2.3	6.44	+ 1.7	+ 2.09	—0.39
3.3	7.44	— 3.6	— 2.71	—0.89
7.3	11.44	— 9.5	—10.09	+0.59
11.3	2.53	+10.1	+10.29	—0.19
19.3	10.53	—12.4	—11.29	—1.11
20.3	11.53	—10.3	— 9.84	—0.46
26.3	4.62	+10.6	+ 9.98	+0.62
Nov. 25.2	8.70	— 6.7	— 7.85	+1.15
26.2	9.70	—10.2	—10.62	+0.42

¹⁾ Dr. J. WILSING. Ueber die Bestimmung von Bahnelementen enger Doppelsterne aus spectrokopischen Messungen der Geschwindigkeitcomponenten. A. N. no. 3198.

With the notations of WILSING these observations lead to the following normal equations:

$$\begin{aligned}
 + 14 g_0 - 3.17 a_1 + 1.56 b_1 - 3.41 a_2 - 1.28 b_2 &= - 42.00 \\
 - 3.17 g_0 + 7.64 a_1 - 1.68 b_1 + 1.81 a_2 + 1.02 b_2 &= + 90.78 \\
 + 1.56 g_0 - 1.68 a_1 + 6.36 b_1 - 2.71 a_2 - 0.27 b_2 &= - 37.81 \\
 - 3.41 g_0 + 1.81 a_1 - 2.17 b_1 + 8.52 a_2 + 0.32 b_2 &= + 30.74 \\
 - 1.28 g_0 + 1.02 a_1 - 0.27 b_1 + 0.32 a_2 + 5.48 b_2 &= + 8.57
 \end{aligned}$$

Solution :

$g_0 = - 0.097$ G.M. = constant velocity towards the sun.

$a_1 = - an \sin i \sin(\omega' + M_0) = + 11.196$; $b_1 = an \sin i \cos(\omega' + M_0) = - 2.953$

$a_2 = - ean \sin i \sin(\omega' + 2M_0) = + 0.498$; $b_2 = ean \sin i \cos(\omega' + 2M_0) = - 0.708$

ω' is the longitude of the periastron, counted from Ω_0 ; M_0 the mean anomaly at the beginning of the light-period, consequently :

$$an \sin i = 11.579; \omega' = 115^\circ 20'; M_0 = 139^\circ 54'; e = 0.075.$$

As $\omega' + M_0 = 180^\circ + 75^\circ 12'$, the elements belong to the body which, during the principal minimum, eclipses the other. Conjunction takes place, when $\omega' + v = 270^\circ$, i.e. 0^d.39 days after the time of the principal minimum, as computed from the empirical formula.

In the following table the 4th column shows the computed velocities, the 6th the outstanding deviations.

16. Spectrographs of 1897.

The velocities (in G. M), derived by *B.* from the displacements of the dark *Mg*-line $\lambda = 448.16 \mu \mu$, have been taken unchanged from the *Memorie della Soc. degli Spettr. It.* vol XXVI. The empirical formula leads to the epoch 1897 June 22 16th. 24 M. T. PULKOWA for the principal minimum.

In a first approximation I determined a circular orbit and found:

$$g_0 = - 2.094; an \sin i = 24.210; \omega' + M_0 = 89^\circ.30'.1.$$

Afterwards corrections were derived by the aid of the formula :

$$\begin{aligned}
 d \frac{dz}{dt} = dg_0 + KndT \sin(\omega' + v) + dK \cdot \cos(\omega' + v) + \\
 + Ke \cos \omega' \cos 2(\omega' + v) + Ke \sin \omega' \sin 2(\omega' + v).
 \end{aligned}$$

in which $K = an \sin i$, $T =$ time of periastron-passage.

This formula is obtained from the general differential-formula¹⁾, by putting $du = 0$, $d\omega = 0$ and by further neglecting 2nd and higher powers of e . We thus find the following normal equations :

¹⁾ Vide: BAUSCHINGER, Die Bahnbestimmung der Himmelskörper, N^o. 199.

$$\begin{aligned}
& + 26dg_0 - 1.35K\mu dt - 2.01dK - 2.22K\epsilon\cos\omega' - 2.03K\epsilon\sin\omega' = -0.05 \\
& -1.35 \text{ ,, } +13.18 \text{ ,, } - 1.02 \text{ ,, } + 1.31 \text{ ,, } + 0.16 \text{ ,, } = +6.17 \\
& -2.01 \text{ ,, } - 1.02 \text{ ,, } +12.82 \text{ ,, } - 0.28 \text{ ,, } + 0.26 \text{ ,, } = 0.00 \\
& -2.22 \text{ ,, } + 1.31 \text{ ,, } - 0.28 \text{ ,, } +13.88 \text{ ,, } - 0.31 \text{ ,, } = +4.75 \\
& -2.03 \text{ ,, } + 0.16 \text{ ,, } + 0.26 \text{ ,, } - 0.31 \text{ ,, } +12.12 \text{ ,, } = -5.98
\end{aligned}$$

	<i>T</i>	Phase	Veloc. in G.M.	<i>V_r</i>	<i>Q-C</i>	
June	20	^h 11.5	^d 10 ^h 17.0	+18.27	+19.29	-1.02
	22	12.0	12 17.6	- 2.60	+ 0.28	-2.88
	23	12.4	0 20.2	-10.62	-10.98	+0.36
	24	12.1	1 19.9	-20.40*)	-19.92	-0.48
	28	11.6	5 19.4	-11.14	-10.77	-0.37
	30	11.1	7 18.9	+14.16	+12.41	+1.75
July	2	11.9	9 19.7	+21.38	+22.42	-1.04
	8	12.3	2 22.3	-24.97*)	-25.56	+0.59
	9	11.4	3 21.4	-25.68	-25.32	-0.36
	10	11.1	4 21.1	-21.27	-19.87	-1.40
	11	11.0	5 21.0	- 8.83	-10.00	+1.17
	12	11.5	6 21.5	+ 3.24	+ 2.34	+0.90
	13	11.4	7 21.4	+13.15	+13.43	-0.28
	15	11.4	9 21.4	+24.15	+22.34	+1.81
	17	11.2	11 21.2	+10.34	+ 9.35	+0.99
	21	11.2	2 23.4	-27.52	-25.65	-1.87
	22	11.2	3 23.4	-23.48	-25.07	+1.59
	24	10.3	5 22.3	- 9.28	- 9.37	+0.09
	25	10.2	6 22.2	+ 0.53	+ 2.70	-2.17
	26	10.0	7 22.0	+12.77	+13.68	-0.91
	27	10.2	8 22.0	+21.03	+20.86	+0.17
30	10.1	11 22.1	+10.11	+ 9.16	+0.95	
31	10.2	0 0.6	- 1.03	- 2.01	+0.98	
Aug. 2	9.7	2 0.1	-20.36	-21.64	+1.28	

*) Mean of two observations.

Solution :

$$dq_0 = + 0.0124 \quad ; \quad K\mu dT = + 0.450 \quad ; \quad dK = + 0.054 \quad ;$$

$$Ke \cos \omega' = + 0.292 \quad ; \quad Ke \sin \omega' = - 0.489.$$

from which we get the elements :

$$g_0 = - 2.082 \text{ GM.} \quad ; \quad an \sin i = 24.264 \quad ; \quad e = 0.0235 \quad ;$$

$$\omega' = 300^\circ 51' \quad ; \quad \omega' \mp M_0 = 88^\circ 26' \quad ;$$

whereas the conjunction coincides perfectly with the principal minimum, the difference in time amounting to less than 0.01 days. Evidently this is the principal minimum. This is in accordance with the fact that the difference of the longitudes found for the two periastra deviates but slightly from 180° ($300^\circ 51' - 115^\circ 20'$). This may be partly due to a fortunate coincidence.

Meanwhile the excentricity of the 2nd orbit is more than three times smaller than that of the 1st, while the velocity in the direction towards the sun found for the whole system is 2 Geogr. miles greater in the 2nd case.

If the latter difference is real, this acceleration would have caused a shortening of the period between 1892 and 1897. As, however the measures of 1892, according to the judgment of Prof. H. C. VOGEL "nicht als ganz einwurfsfrei angesehen werden können" ¹⁾, we suspend our judgment to the time that Prof. BELOPOLSKY will again have taken up his beautiful investigations on the spectrum of β Lyrae, particularly about the *F*-line. Already in 1897 he communicated his intention to do so.

If we put $i = 90^\circ$, the semiaxes major are :

$$a_1 = 2056000 \text{ G. M.} \quad ; \quad a_2 = 4307000 \text{ G. M.}$$

From KEPLER's third law we derive therefrom, roughly

$$m_1 = 17.1 \text{ sun's masses} \quad ; \quad m_2 = 8.1 \text{ sun's masses.}$$

17. In our opinion the preceding considerations justify the conclusion that the data about β Lyrae do not furnish a sufficient basis for a decision about any change in the elements, in particular in the excentricity. For the rest, owing to our ignorance on the circumstances in such a close system, the adopted explanation of the light variation can only claim to give a rough approximation — a rude imitation of a very complicated mechanism.

¹⁾ Ueber das Spectrum von β Lyrae. Sitzungsab. Ak. Berlin. 1894 VI.

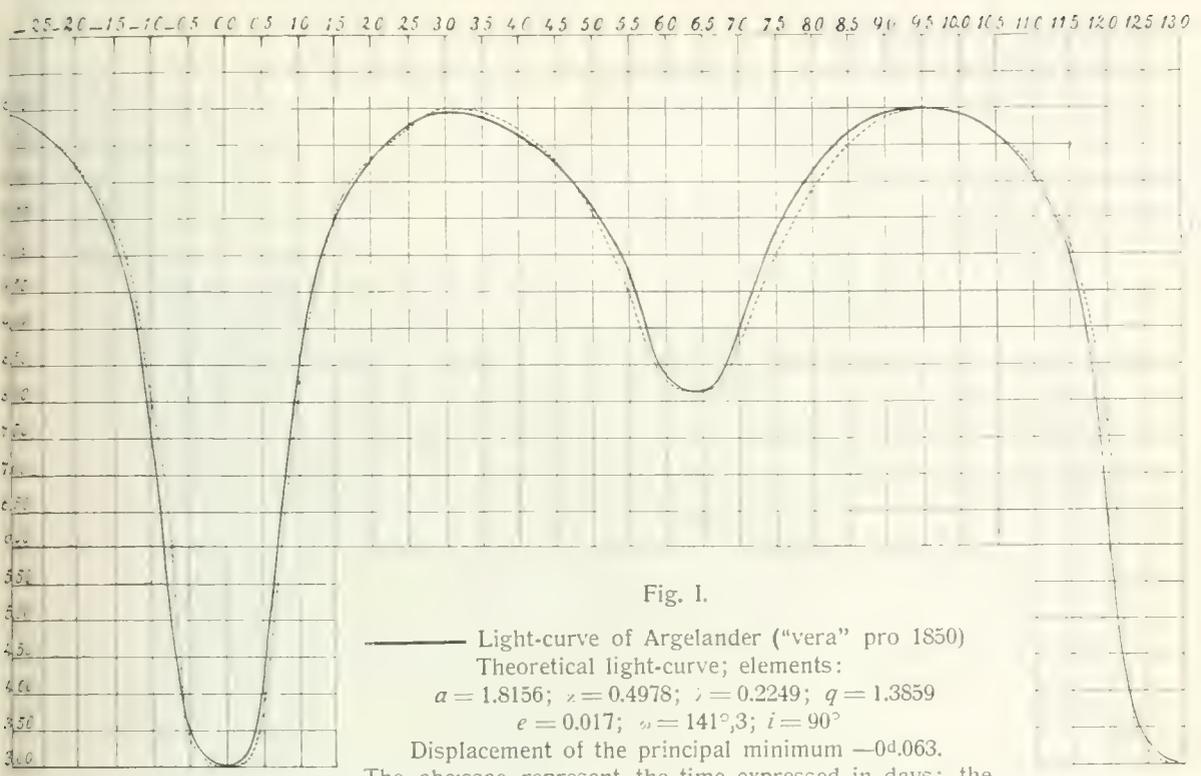


Fig. I.

— Light-curve of Argelander ("vera" pro 1850)
 Theoretical light-curve; elements:
 $a = 1.8156$; $\nu = 0.4978$; $\lambda = 0.2249$; $q = 1.3859$
 $e = 0.017$; $\omega = 141^\circ.3$; $i = 90^\circ$
 Displacement of the principal minimum $-0^d.063$.
 The abscissae represent the time expressed in days; the
 ordinates the light-intensity in grades.

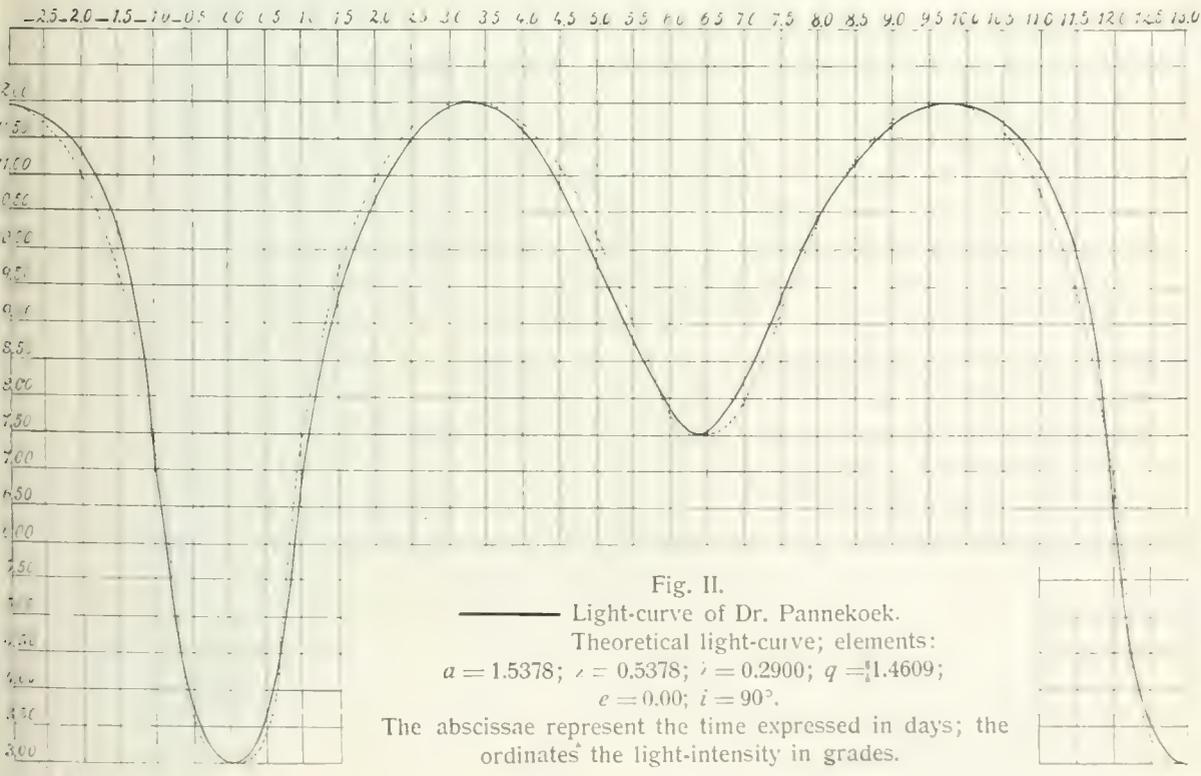


Fig. II.

— Light-curve of Dr. Pannekoek.
 Theoretical light-curve; elements:
 $a = 1.5378$; $\nu = 0.5378$; $\lambda = 0.2900$; $q = 1.4609$;
 $e = 0.00$; $i = 90^\circ$.
 The abscissae represent the time expressed in days; the
 ordinates the light-intensity in grades.

Mathematics. — “*The section of the measure-polytope M_n of space Sp_n with a central space Sp_{n-1} perpendicular to a diagonal.*”
By Prof. P. H. SCHOOTE.

(Communicated in the meeting of December 28, 1907).

We determine the indicated section in three different ways:

1. by means of the projection of M_n on the diagonal,
2. with the aid of the projection of M_n on a plane through two opposite edges intersecting the diagonal,
3. by regarding regular simplexes.

I. *The projection of M_n on a diagonal.*

1. We can easily prove both analytically and synthetically the following theorem:

“The vertices of the measure-polytope M_n project themselves on a “diagonal in $n + 1$ points, namely in the ends of the diagonal and “in the $n - 1$ points, which divide the latter into n equal parts; in “these $n + 1$ points are projected successively

$$1, n, \frac{1}{2}n(n-1), \dots, \frac{1}{2}n(n-1), n, 1$$

“points, where these numbers are the coefficients of the terms “of $(a + b)^n$ ”.

From this general theorem ensue the results for $n = 4, 5, 6, 7, 8$ given in the diagrams added here (see the expanding plate). An explanation of the sketch belonging to $n = 4$ will sufficiently explain the others.

The horizontal lines of this figure always represent the same diagonal on which the projection takes place; on these ten lines are successively indicated the projections of vertices, of edges, of faces and of bounding bodies. In order to find space for the figures indicating the numbers, the thick projection-lines have been broken off, where such was necessary.

If we designate the five points on the diagonal by a, b, c, d, e , — see the bottom line of the ten horizontal ones — then in these places — see the topmost of the ten lines — 1, 4, 6, 4, 1 vertices are projected there — bear in mind $(1 + 1)^4$.

On the four equal segments ab, bc, cd, de are projected successively 4, 12, 12, 4 edges — think of $4(1 + 1)^3$.

In like manner the three equal segments ac, bd, ce are successively the projections of 6, 12, 6 faces — think of $6(1 + 1)^2$.

Finally on the two equal segments ad, be are projected successively 4, 4 bounding bodies — think of $4(1 + 1)$.

It is easy to deduce from this the results given in the other diagrams for $n = 5, 6, 7, 8$, if we keep in mind, that the coefficients by which $(1 + 1)^4, (1 + 1)^3, (1 + 1)^2, (1 + 1)$ are multiplied are 1, 4, 6, 4 and so by addition of unity at the end pass into a repetition of $(1 + 1)^4$.

2. More generally holds the following theorem, comprising the preceding:

“The vertices of each bounding M_p of M_n ($p \leq n$) are projected on “the diagonal of M_n in $p + 1$ successive points of division of that “diagonal; here again the projections are distributed according to “the coefficients 1, $p, \frac{1}{2} p(p - 1), \dots$ of $(a + b)^p$ over these $p + 1$ “successive points.”

The vertices of a bounding square are projected in three of the $n + 1$ points, which naturally demands the division 1, 2, 1. The vertices of a bounding cube are projected in four of the $n + 1$ points, which of necessity must lead to the division 1, 3, 3, 1 as by the preceding the division 2, 2, 2, 2 is excepted.

From this ensues then directly the following theorem:

“The section of a space $S_{p_{n-1}}$ perpendicular to the diagonal of M_n “forming the axis of projection, with the space S_{p_p} bearing a bounding “ M_p of M_n is an $S_{p_{p-1}}$ in S_{p_p} perpendicular to the diagonal of “ M_p connecting the two vertices of M_p projecting themselves in the “ends of the projection of M_p .”¹⁾

But there is more. If p' ($M_{p'}$) represents the section of a measure-polytope M_p with a space $S_{p_{p'-1}}$ of its space S_{p_p} perpendicular to one of its diagonals in a point of which the distance to the centre of the diagonal in the diagonal as unity amounts to $\frac{1}{2} - p'$, from

which is evident that $p' \leq \frac{1}{2}$, the two theorems hold:

“For even n a bounding measure-polytope M_p of M_n is intersected “by the central space $S_{p_{n-1}}$ perpendicular to the diagonal of M_n

1) The indicated diagonal d_p of M_p is the projection of the axis of projection d on the space S_{p_p} of M_p ; so we can obtain the projections of the vertices of M_p on d by projecting these vertices first in S_{p_p} on d_p and projecting afterwards on d the points found on d_p by the preceding means.

As d_p and d in the edge of M_n as unity are represented by \sqrt{p} and \sqrt{n} and d_p is projected on d as $\frac{p}{n}$ of d , the cosine of the angle between d and S_{p_p} is

equal to $\frac{1}{n} \sqrt{np}$.

“according to an $\frac{a}{p}(M_p)$, where a according to circumstances can

“assume for even p one of the $\frac{p}{2}$ values $1, 2, \dots, \frac{p}{2}$, for odd p one

“of the $\frac{p-1}{2}$ values $1, 2, \dots, \frac{p-1}{2}$.”

“For odd n the measure-polytope M_p is intersected under the same
“circumstances according to a $\frac{2a-1}{p}(M_p)$ where a can assume for

“even p one of the $\frac{p}{2}$ values $1, 2, \dots, \frac{p}{2}$, for odd p one of the $\frac{p+1}{2}$

“values $1, 2, \dots, \frac{p+1}{2}$.”

We shall now, instead of losing ourselves in further generalities, give the full results of the diagrams for the cases $n = 4, 5, 6, 7, 8$ to make clear the above. In order to be able to indicate easily ratios of measure we shall suppose the edge of M_n to be unity of length.

3. Case $n = 4$. The space — see first diagram — perpendicular in the centre c of diagonal ae to this diagonal contains the six vertices of M_4 , projecting themselves in c and cuts — see lines 3 and 4 — no edge; so the section has six vertices. This same space cuts twelve faces — see line 7 — according to $\frac{1}{2}(M_2)$ and eight bounding bodies — see lines 9 and 10 — according to $\frac{1}{3}(M_3)$; so the section has twelve edges with a length $\sqrt{2}$ and eight equilateral triangles as faces. So the section is a (6, 12, 8) and, indeed, the regular octahedron with edges $\sqrt{2}$.

Case $n = 5$. We find — see second diagram — thirty vertices generated by intersection of edges, sixty edges, forty faces and ten bounding bodies, so a (30, 60, 40, 10). The vertices are of the same kind, the edges have as $\frac{1}{4}(M_2)$ the length $\frac{1}{2}\sqrt{2}$. The forty faces consist of twenty $\frac{1}{2}(M_3)$ and two times ten $\frac{1}{6}(M_3)$, i. e. of twenty hexagons and twenty triangles, both regular¹⁾ with sides $\frac{1}{2}\sqrt{2}$.

¹⁾ Where the regularity is obvious — as e. g. with the triangles by the equal length of all edges, etc. — the additional “equilateral” or “regular” will in future be left out.

Each of the ten bounding bodies is as $\frac{3}{8}(M_4)$ — compare in the first diagram the section with a space perpendicular to ae in the point in the middle between c and d — a (12, 18, 8) bounded by four $\frac{1}{2}(M_3)$ and four $\frac{1}{6}(M_3)$, i. e. by four of the hexagons and four of the triangles, and therefore a tetrahedron truncated regularly at the vertices, i. e. the first of the equiangular semi-regular (Archimedean) bodies.

Case $n=6$. Out of the third of the diagrams we read that the section is a (20, 90, 120, 60, 12). All the edges have a length $\sqrt{2}$, all the faces are triangles. The bounding bodies are for one half (30) as $\frac{1}{2}(M_4)$ octahedra, for the other half (15 + 15) as $\frac{1}{1}(M_4)$ tetrahedra. The twelve bounding polytopes are as $\frac{2}{5}(M_5)$ — compare now again the second diagram — polytopes (10, 30, 30, 10) bounded by five of the octahedra and five of the tetrahedra, which can be regarded as regular five-cells, regularly truncated at the vertices as far as half of the edges, so as to lose all the original edges by this truncation.

Case $n=7$. We arrive at a (140, 420, 490, 280, 84, 14). The length of the edges is $\frac{1}{2}\sqrt{2}$. The 490 faces consist of 210 hexagons and 280 triangles, the 280 bounding bodies of 210 truncated tetrahedra and 70 tetrahedra, the 84 four-dimensional bounding polytopes of 42 polytopes $\frac{1}{2}(M_5) = (30, 60, 40, 10)$ found already under $n=5$ and 42 polytopes $\frac{3}{10}(M_5) = (20, 40, 30, 10)$ bounded by five truncated tetrahedra and five tetrahedra — regular five-cells truncated at the vertices as far as a third of the edges. The 14 five-dimensional bounding polytopes are as $\frac{5}{12}(M_6)$ polytopes (60, 150, 140, 60, 12) bounded by six (30, 60, 40, 10) and six (20, 40, 30, 10).

Case $n=8$. Here a (70, 560, 1120, 980, 448, 112, 16) is the result. The length of the edges is $\sqrt{2}$, all faces are triangles. The

980 bounding bodies consist of 420 octahedra and 560 tetrahedra the 448 four-dimensional bounding polytopes of 336 polytopes $\frac{2}{5}(M_5)$ and 112 polytopes $\frac{1}{5}(M_5)$, i. e. of 336 five-cells truncated as far as half of the edges, found under $n = 6$, and 112 five-cells. The 112 five-dimensional bounding polytopes are as far as one half is concerned $\frac{1}{2}(M_6) = (20, 90, 120, 60, 12)$ already found above,

as far as the other half is concerned $\frac{1}{3}(M_6) = (15, 60, 80, 45, 12)$ bounded by six five-cells truncated as far as half the length of the edges and six five-cells. Finally the sixteen six-dimensional bounding polytopes are as $\frac{3}{7}(M_7)$ polytopes $(35, 210, 350, 245, 54, 84)$ bounded by seven $(20, 90, 120, 60, 12)$ and seven $(15, 60, 80, 45, 12)^1$.

From this all we easily deduce the following general laws :

“The vertices of the section are vertices of M_n for even n , for odd n they are centres of edges of M_n ; they are always of the same kind ²⁾.”

“The common length of the edges is $\sqrt{2}$ for even n and $\frac{1}{2}\sqrt{2}$ for odd n ; they are always of the same kind³⁾.”

“The faces are triangles for even n , hexagons and (smaller) triangles ⁴⁾ for odd n .”

“The bounding bodies are octahedra and tetrahedra for even n , truncated tetrahedra and (smaller) tetrahedra for odd n ”.

“The four-dimensional bounding polyhedra are five-cells truncated as far as halfway the edges and five-cells for even n , five-cells

¹⁾ If we had set to work, when enumerating the results, in that sense inversely that with each new value of n of the bounding polytopes with the greatest number of dimensions we had descended to the vertices, we should have furnished a geometrical variation of the well known nursery-book : “the house that Jack built”. However with two differences. When descending from every one round higher of the ladder we pass *every other time* again the same stadia and the ladder is a Jacob's ladder with an infinite number of rounds.

²⁾ That is, in each vertex as many edges meet in the same way, etc.

³⁾ The cases $n = \text{odd}$ seem to be an exception to this, as there are for the truncated tetrahedra two kinds of edges, namely : sections of two hexagonal faces and sections of an hexagonal and a triangular face. However, this is only apparently. For, for each edge we find that in the section itself always again the number of faces passing through it of each of the two sorts is steadfast, thus for $n = 5$ two hexagonal faces and one triangular one.

⁴⁾ We do not mention here, that for $n = 3$ only an hexagon appears. Neither that of the bounding bodies the tetrahedra do not appear for $n = 4$, etc.

truncated as far as a third of the edges and (smaller) five-cells for odd n ."

Etc., etc.¹⁾.

The above results are for the greater part given in the general theorems mentioned above.

II. The projection of M_n on a plane through two opposite edges cutting the diagonal.

4. For each value of n the indicated projection — see fig. 1 for $n=8$ and $n=9$ — is a rectangle $PQ'Q'P'$ with the sides 1 and

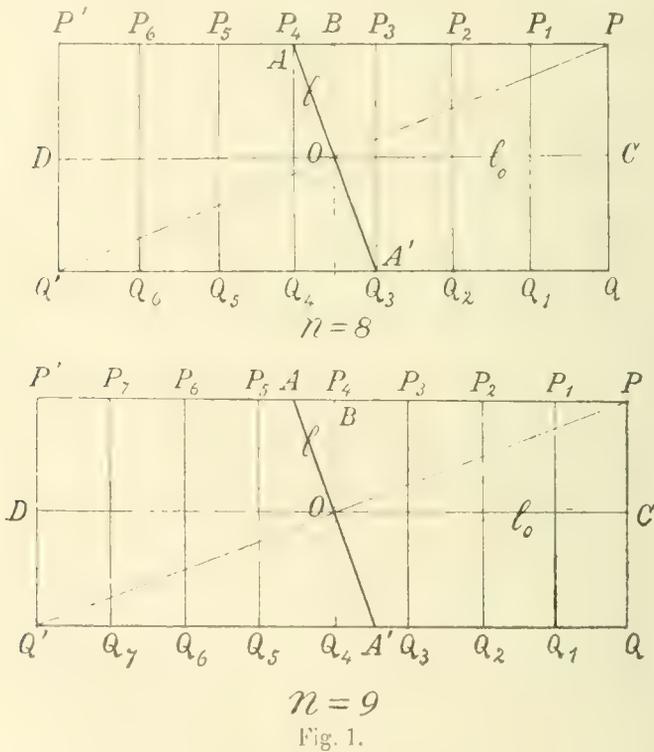


Fig. 1.

$\sqrt{n-1}$, which is divided by $n-2$ lines $P_1Q_1, P_2Q_2, \dots, P_{n-2}Q_{n-2}$ parallel to the shorter sides $PQ, P'Q'$ into $n-1$ equal rectangles.²⁾

1) We break off here because not until the third division do we indicate that everything making its appearance in the section can be regarded as simplex or truncated simplex.

The symbol which indicates the numbers of vertices, edges, faces, etc. for arbitrary n is purposely omitted as its form is rather complicated.

2) To compare the treatise "On the sections of a block of eightcells, etc." (Verhandelingen der K. A. v. W., vol IX, N^o. 7).

The diagonal on which the intersecting space Sp_{n-1} is at right angles is one of the diagonals of the rectangle, e.g, PQ' . If the normal erected in the centre O of PQ' on this line, representing the projection of the intersecting space Sp_{n-1} , cuts the side PP' in A , this point A always lies at a distance $\frac{1}{2\sqrt{n-1}}$ from the centre B of PP' . For in the right-angled triangle AOP we find that B is the foot of the normal let down out of O on AB and from this ensues $AB \cdot BP = OB^2$ and therefore $AB = \frac{1}{4} : \frac{1}{2} \sqrt{n-1}$. So A coincides for even n with the point of division $\frac{P_n}{2}$ and this point lies for odd n in the middle between $\frac{P_{n-1}}{2}$ and $\frac{P_{n+1}}{2}$. From this it is again evident that the vertices of the section are vertices of M_n for even n and centres of edges of M_n for odd n .

In the paper quoted above which restricts itself to the case $n=4$ we find in a note how we can regard the section under observation as a "rhombotope" truncated at both sides; the course of thoughts is as follows. Let us imagine in the direction of the edges $PQ, P'Q'$ on either side an infinite number of measure-polytopes M_n piled on each other and let us then remove the measure-polytopes M_{n-1} , projecting themselves on PP', QQ' and lines parallel to these, with which the successive polytopes M_n bound each other; then a prism is formed with M_{n-1} as right section. If this prism is intersected by a space Sp_{n-1} which projects itself along the perpendicular l_0 let down out of O on PQ , the section is thus an M_{n-1} . What variation does this section M_{n-1} of the prism undergo when we substitute for the intersecting space projecting itself along l_0 an other one which projects itself along a line l_φ through O , enclosing with l_0 an angle φ ? As is easy to see from the figure this variation consists of a regular enlargement of the perpendiculars let down out of the boundary of M_{n-1} on the space Sp_{n-2} , projecting itself in O , which enlargement means a multiplication of those perpendiculars by $\sec \varphi$ and can be regarded as a *stretching* in the direction of the diagonal CD . As for $n=4$, where M_{n-1} becomes a cube, such a stretching makes a rhombohedron of a cube, out of M_{n-1} is formed in general what we call a rhombotope.

Just as the rhombohedron regarded as a whole passes into itself when it is revolved 120° about the axis, or — in other words — just as the axis of the rhombohedron has a period three, the axis of the rhombotope under consideration has a period $n-1$. Let us

now imagine this rhombotope, for the special case that the projection of the intersecting space Sp_{n-1} — so also the projection of the rhombotope itself — falls along OA and let us truncate it by the two spaces Sp_{n-2} standing normal to the plane of projection in the ends A, A' of the segment AA' of that projection lying inside the rectangle and cutting the axis of the rhombotope therefore at right angles; we then find the required section, to be indicated according to the number of its dimensions by D_{n-1} . We directly determine the length of the axis of the untruncated rhombotope and of D_{n-1} , but before this we shall deduce some general theorems easy to find.

5. The edges of M_n project themselves on the assumed plane *either* along one of the n lines $PQ, P_1Q_1, P_2Q_2, \dots, P_{n-2}Q_{n-2}, P'Q'$, or as parts of PP' or QQ' . Because the vertices of D_{n-1} must be vertices of M_n or points of intersection with edges of M_n , these points project themselves — compare fig. 1 for $n=8$ and for $n=9$ — for even n exclusively in the ends A, A' , for odd n exclusively in those ends and in the centre O .

From this ensues for $n=2n'$ the general theorem:

“The section $D_{2n'-1}$ of $M_{2n'}$ is a $2n' - 1$ -dimensional prismoid with respect to each pair of opposite bounding spaces $Sp_{2n'-2}$ and so in $2n'$ ways”.

Here follow two theorems holding for arbitrary n :

“Each line through the centre O normal to two opposite bounding spaces Sp_{n-2} is axis of D_{n-1} with the period $n-1$.”

“Each space Sp_{n-2} through O parallel to a bounding space Sp_{n-2} divides D_{n-1} into two congruent $n-1$ -dimensional prismoids.”

In the demonstration of these three theorems the entire equivalence of a pair of opposite bounding spaces Sp_{n-2} with any other pair has the chief part; moreover the third causes us to inquire how the space Sp_{n-2} through the centre parallel to a bounding space intersects D_{n-1} . We prove as follows that this section is a D_{n-2} .

If the projection l of the intersecting space Sp_{n-1} revolves round O , the $Sp_{n-2}^{(0)}$ normal to the plane of projection in O remains in its place and Sp_{n-1} thus describes a pencil with this $Sp_{n-2}^{(0)}$ as axial space. Therefore then the varying section keeps going through the section of $Sp_{n-2}^{(0)}$ with M_n . We can easily know the nature of this section of $n-2$ dimensions by regarding the case in which l coincides with l_0 . Then our D_{n-1} is an M_{n-1} and this measure-polytope projecting itself along l_0 is intersected according to a D_{n-2} by the space $Sp_{n-2}^{(0)}$, which is in O normal to the plane of projection and

which therefore bisects the diagonal CD of this M_{n-1} . This D_{n-2} is the section of D_{n-1} with the space $Sp_{n-2}^{(0)}$, through O parallel to the spaces $Sp_{n-2}^{(0)}$, which are in A_1 and A' normal to the axis and which truncate the rhombotope. So we find:

“Each space $Sp_{n-2}^{(0)}$ through the centre O parallel to a bounding space Sp_{n-2} intersects D_{n-1} according to a D_{n-2} of which O is again the centre.”

From this follows again more generally:

“Each space $Sp_p^{(0)}$ ($0 < p < n - 1$) through the centre O parallel to a bounding space Sp_p , intersects D_n according to a D_{p-1} , of which O is again the centre”.

Thus we find ascending from below:

“Each chord of D_{n-1} through O parallel to an edge has a length $\sqrt{2}$, each plane through O parallel to a face intersects D_{n-1} according to a regular hexagon with sides $\frac{1}{2}\sqrt{2}$, each space through O parallel to a bounding body intersects D_{n-1} according to a regular octahedron with edges $\sqrt{2}$, etc.”

6. We retrace our steps and determine of the above mentioned rhombotope the length of the axis before and after the truncation. Out of the similitude of the triangles AOB and POC follows in connection

with the length $\frac{1}{2}\sqrt{n-1}$, $\frac{1}{2}\sqrt{n}$, $\frac{1}{2}$ of OC , OP , OB for OA the value $\frac{1}{2(n-1)}\sqrt{n(n-1)}$ and so for half of the unmutilated axis which

is $n-1$ times as large $\frac{1}{2}\sqrt{n(n-1)}$. If we represent by $Rh_p [q, r]$

a rhombotope with p dimensions of which q is the length of the axis, r are the parts of the axis removed by the truncation, the section D_{n-1}

has to be represented by the symbol $Rh_{n-1} \left[\sqrt{n(n-1)}, \frac{n-2}{2(n-1)} \right]$

So the theorem holds:

“We obtain the section D_{n-1} , if we allow the measure-polytope M_{n-1} to pass in the indicated way by stretching in the direction of a diagonal as far as \sqrt{n} times the original length into a rhombotope with a length of axis $\sqrt{n(n-1)}$ and if we truncate this rhombotope by two spaces Sp_{n-2} normal to the axis to a

$$Rh_{n-1} \left| \sqrt[n]{n-1}, \frac{n-2}{2(n-1)} \right|^{n-1}$$

III. *Explanation in details of the connection of D_{n-1} with regular and regularly truncated simplexes.*

7. We consider in the space $S\rho_n$ a rectangular system of coordinates with an arbitrary point O as origin and OX_1, OX_2, \dots, OX_n as axes, and we now call the 2^n th part of that space which is the locus of the point with only positive coordinates the " n -edge $O(X_1 X_2 \dots X_n)$ ".

If A, A' are two opposite vertices of a measure-polytope M_n of $S\rho_n$ and if AA_1, AA_2, \dots, AA_n are the edges passing through A and $A'A'_1, A'A'_2, \dots, A'A'_n$ the edges parallel to these but directed oppositely, then M_n can be regarded as the part of the space $S\rho_n$ common to the two n -edges $A(A_1 A_2 \dots A_n)$ and $A'(A'_1 A'_2 \dots A'_n)$.

If we intersect this figure of the two oppositely orientated n -edges and the measure-polytope M_n common to both by an arbitrary space $S\rho_{n-1}$, the two n -edges are intersected along two oppositely orientated simplexes and the section of M_n with that space $S\rho_{n-1}$ appears as the part of that space that is enclosed at the same time by both simplexes situated in that space. If the selected space is normal to the diagonal AA' , connecting the vertices of the n -edges, the simplexes are regular and they have the point of intersection P of the intersecting space $S\rho_{n-1}$ with AA' as common centre of gravity. So the general theorem holds:

"The section of M_n with a space $S\rho_{n-1}$ normal to a diagonal can always be regarded as a part of that space $S\rho_{n-1}$ enclosed by two definite concentric, oppositely orientated, regular simplexes of that space".

If we wish to make use of this theorem we must determine in a more detailed way the length of the edges of those oppositely orientated regular simplexes with common centre of gravity.

8. If we think the intersecting space $S\rho_{n-1}$ to be normal to the

¹⁾ This theorem shows distinctly why the sections of an octahedron parallel to two faces must be identical to those of a cube by planes normal to a diagonal in points of the middle third part of that line. The same in other words: If we truncate a cube with the unity of edge at two opposite vertices by planes normal to the connecting line in the points dividing this diagonal into three equal parts and if we compress an octahedron with edges $\sqrt{2}$ in the direction of the normal on two parallel faces as far as half the thickness, then we cause the same solid to be generated in two different ways.

diagonal AA' in the first point of division A_1 , at a distance $\frac{1}{n} \sqrt{n}$ from A , the section is a simplex with edge $\sqrt{2}$. So the two simplexes, generated when an arbitrary point P of AA' is substituted for point A_1 , have edges of a length of $AP\sqrt{2n}$ and $A'P\sqrt{2n}$, wherefore we indicate them, also with reference to the number of vertices, by $S_n(AP\sqrt{2n})$ and $S'_n(A'P\sqrt{2n})$. So the theorem holds:

“If we shove an M_n , of which the diagonal AA' is normal to a given space $S_{p_{n-1}}$, in the direction of that diagonal through that space $S_{p_{n-1}}$, so that the spaces $S_{p_{n-1}}$ of the bounding polytopes M_{n-1} move parallel to themselves, the section of $S_{p_{n-1}}$ with the moving polytope M_n is at every moment the part of that space $S_{p_{n-1}}$ that is enclosed within two concentric, yet oppositely orientated, regular simplexes $S_n(p\sqrt{2n})$ and $S'_n(p'\sqrt{2n})$ where p and p' are connected in such a way that the sum $p + p'$ is equal to \sqrt{n} . During that movement of M_n the common centre of gravity of the two simplexes remains in its place and the spaces $S_{p_{n-2}}$ of the bounding simplexes S_{n-1} and S'_{n-1} move parallel to themselves; whilst simplex S_n expands itself from this common centre of gravity to a simplex $S_n(n\sqrt{2})$, simplex S'_n inversely contracts from a simplex $S'_n(n\sqrt{2})$ to this point”.

At the moment when this process has got halfway and the two simplexes are of the same size we find:

“The section D_{n-1} is the part of the intersecting space $S_{p_{n-1}}$ enclosed by two definite equal concentric yet oppositely orientated regular simplexes $S_n\left(\frac{1}{2}n\sqrt{2}\right)$ and $S'_n\left(\frac{1}{2}n\sqrt{2}\right)$.”

Thus for $n = 3$ the regular hexagon with sides $\frac{1}{2}\sqrt{2}$ is the figure enclosed by two triangles with sides $\frac{3}{2}\sqrt{2}$ — think of the well-known trademark —, thus for $n = 4$ the regular octahedron with edges $\sqrt{2}$ is the figure enclosed by two tetrahedra with edges $2\sqrt{2}$ — think of the two tetrahedra described in a cube and the octahedron common to both. So in general the problem in the space of n dimensions is reduced to another problem in space of $n - 1$ dimensions and moreover the connection of the result with regular simplexes is explained.

If we think the simplex S_n to be white and the simplex S'_n to be black, the n bounding spaces $S_{p_{n-2}}$ of D_{n-1} originating from S_n will be white, those originating from S'_n will be black. From this ensues that it must be possible to colour the $2n$ bounding spaces

$S_{p_{n-2}}$ of D_{n-1} in such a way in turns white and black, that two opposite bounding spaces $S_{p_{n-2}}$ have a different colour. The octahedron is really the only one of the regular bodies that allows this operation. ¹⁾

9. If the simplex S_n expands from a point to an $S_n(n\sqrt{2})$ and at the same time S'_n contracts from an $S'_n(n\sqrt{2})$ to a point, then S_n lies at the beginning of the process within S'_n and at the end inversely S'_n lies within S_n . Gradually first the vertices, then the edges, then the faces, etc. of S_n have passed outward. We shall now investigate when that takes place.

From the diagrams of the expanding plate given in the first part it is evident, that the section of M_n with a space $S_{p_{n-1}}$ changes its nature when the point of intersection P of that space $S_{p_{n-1}}$ with the diagonal AA' passes one of the $n - 1$ points of division A_1, A_2, \dots . As the nature of the section of course also changes when bounding elements of S'_n lying inside S_n pass outward, the latter must take place at those moments when those points of division of the diagonal AA' of the moving M_n pass through the fixed space $S_{p_{n-1}}$. This theorem then really holds:

“In the translation of M_n in the direction of AA' through the space $S_{p_{n-1}}$ in succession the vertices, the edges, the faces, bounding bodies, etc. of S_n come entirely outside S'_n at those moments that the point of intersection P of the diagonal AA' with the space $S_{p_{n-1}}$ coincides successively with the points of division $A_1, A_2, A_3, A_4, \dots$ etc.”

We regard — in order to prove this theorem — the arbitrary stadium of the simplexes $S_n(AP\sqrt{2n})$ and $S'_n(A'P\sqrt{2n})$, divide the n vertices of S_n in an arbitrary way into two groups β and γ of p and $n-p$ points, and indicate by β' and γ' the groups of the p and $n-p$ corresponding vertices of S'_n , by B, C, B', C' (fig. 2) the centres of gravity of the point-groups $\beta, \gamma, \beta', \gamma'$ — i.e. the

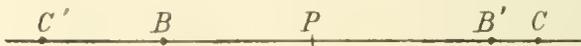


Fig. 2.

¹⁾ In contradiction to this seems that for $n = 5$ through each edge *three* faces pass and thus *three* bounding bodies (12, 18, 8) lie around it. This contradiction however is only apparent; it is annulled by the remark that two bounding bodies (12, 18, 8) having a face in common agree or differ in colour according to the face being triangular or hexagonal. Of the three faces one is triangular, two are hexagonal; the bounding bodies to which the two hexagonal faces belong, differ in colour from the two others, these agreeing in colour.

centres of the bounding simplexes $S_p, S_{n-p}, S'_p, S'_{n-p}$ with these points as vertices. Then the five points B, C, B', C', P lie in such a way upon the same right line, that B and C' lie on one side of P and B' and C on the other side, and we have

$$\left. \begin{aligned} p \cdot BP &= (n-p) \cdot PC \\ (n-p) \cdot C'P &= p \cdot PB' \end{aligned} \right\} \frac{AP}{PA'} = \frac{BP}{PB'} = \frac{CP}{PC'}$$

We can now assert that the bounding simplex S_p of the vertices β of S_n lies entirely or partly inside S'_n when B is between C' and P , whilst S'_p lies entirely outside S_n when C' lies between B and P . In other words: as AP increases, the bounding simplex S_p of S_n comes entirely outside S_n when B coincides with C' and the spaces $S_{p_{p-1}}$ and $S_{p_{n-p-1}}$ of S_p and S'_{n-p} , crossing each other in general entirely perpendicularly, become incident because they get the point $B = C'$, then common centre of gravity, as point of intersection.

Under the condition $BP = C'P$ follows from the equations

$$\frac{BP}{PC} = \frac{n-p}{p}, \quad \frac{PC}{C'P} = \frac{AP}{PA'}$$

the relation

$$(n-p) \cdot AP = p \cdot PA',$$

which shows that P must coincide with the p^{th} dividing point A_p of AA' .

10. If P coincides with A_p the spaces $S_{p_{p-1}}$ and $S_{p_{n-p-1}}$ of S_p and S'_{n-p} have, as we saw above, the common centre of S_p and S'_{n-p} in common. As this point of intersection of S_p and S'_{n-p} becomes vertex of the section, — if we call this again $\frac{p}{n}(M_n)$ in connection with preceding investigations — the theorem holds:

“The centres of the $\binom{n}{p}$ bounding simplexes S_p of a regular simplex $S_n(p\sqrt{2})$ form the vertices of a polytope congruent to $\frac{p}{n}(M_n)$ for $p = 1, 2, \dots, n-1$.”

For even $n = 2n'$ we have specially:

“The centres of the $\binom{2n'}{n'}$ bounding simplexes $S_{n'}$ of a regular simplex $S_{2n'}(n'\sqrt{2})$ form the vertices of a $D_{2n'-1}$.”

11. If P lies between A_p and A_{p+1} the vertices of the section of the two simplexes S_n and S'_n are furnished by the points of intersection of each bounding simplex S_{p+1} of S_n with the $p+1$

bounding simplexes S'_{n-p} of S'_n which have the property of counting among their $n-p$ vertices only one vertex corresponding to a vertex of this S_{p+1} ; in each bounding simplex S_{p+1} these $p+1$ points of intersection form the vertices of a new regular simplex S_{p+1} which is concentric to the assumed one but oppositely orientated. We determine the length of the edges of this new simplex, for the definite case that P lies just in the middle between A_p and A_{p+1} , with the aid of reflections in quite close connection with the preceding.

If B, C, B', C' (fig. 3) are successively the centres of gravity of

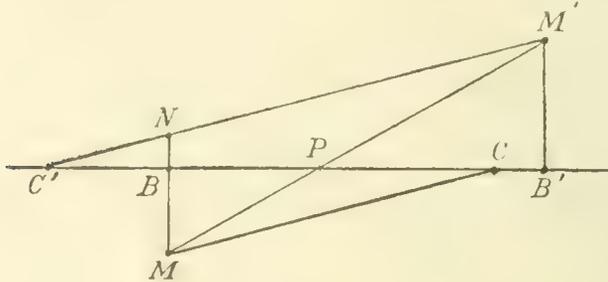


Fig. 3.

the bounding simplex S_{p+1} , of the bounding simplex S_{n-p-1} of the remaining vertices of S_n and of the bounding simplexes S'_{p+1} , and S'_{n-p-1} of the groups of vertices of S'_n corresponding with the vertices of S_{p+1} and S'_{n-p-1} , these points lie on a same right line through P again, viz.: B and C' on one side and C and B' on the other side of P . If furthermore M and M' are corresponding vertices of S_{p+1} and S'_{p+1} these points lie in parallel normals erected in B and B' on BB' and the line connecting M and M' passes through P . The point of intersection N of BM and $C'M'$ is the vertex of \bar{S}_{p+1} corresponding to the vertex M of S_{p+1} . From CM and $C'M'$ being parallel follows

$$\frac{BN}{MB} = \frac{C'B}{BC} = \frac{C'P - BP}{BP + PC},$$

whilst the relations

$$\frac{AP}{PA'} = \frac{BP}{PB'} = \frac{CP}{PC'} = \frac{2p+1}{2n-2p-1}$$

and

$$\frac{BP}{PC} = \frac{B'P}{PC'} = \frac{n-p-1}{p+1}$$

enable us to express $C'P$ and BP in PC . Substitution gives the result

$$\frac{BN}{MB} = \frac{1}{2^{\rho+1}}$$

So the theorem holds:

“If we describe in the spaces $S_{\rho\rho}$ bearing the bounding simplexes $S_{\rho+1}\left(\frac{2\rho+1}{2}\sqrt{2}\right)$ of a regular simplex $S_n\left(\frac{2\rho+1}{2}\sqrt{2}\right)$ simplexes $S_{\rho+1}\left(\frac{1}{2}\sqrt{2}\right)$ concentric and oppositely orientated to the original ones we find the $(\rho+1)\binom{n}{\rho+1}$ vertices of a $\frac{2\rho+1}{2n}(M_n)$.”

For odd $n = 2n' + 1$ we have in particular:

“If we describe in the spaces $S_{\rho n'}$, bearing the bounding simplexes $S_{n'+1}\left(\frac{2n'+1}{2}\sqrt{2}\right)$ of a regular simplex $S_{2n'+1}\left(\frac{2n'+1}{2}\sqrt{2}\right)$ simplexes $S_{n'+1}\left(\frac{1}{2}\sqrt{2}\right)$ concentric and oppositely orientated to the original ones we find the $(n'+1)\binom{2n'+1}{n'+1}$ vertices of a $D_{2n'}$.”

In connection with the results found above the length $\frac{1}{2}\sqrt{2}$ appearing here for the edges of the new simplexes contains a confirmation.

Mathematics. — “On five pairs of four-dimensional cells derived from one and the same source.” By MRS. A. BOOLE STOTT and Prof. P. H. SCHOUTE.

(Communicated in the meeting of December 28, 1907).

Introduction.

As this paper must be regarded as a short completion of the handbook of the “Mehrdimensionale Geometrie” included in the Sammlung SCHUBERT we keep the notation used there.

We regard in succession each of the six regular cells C_5 , C_6 , C_{16} , C_{24} , C_{120} , C_{600} of the space Sp_4 and derive from these two new four-dimensional cells. The first, which has the centres K_0 of the edges of the regular cell as vertices is formed by a regular truncation at the vertices as far as the centres of the edges; the second is the reciprocal polar of the first with respect to the spherical space of the points K_0 .

Because the regular $C_{1,6}$ leads us to find the regular $C_{3,4}$, the number of pairs of new cells is not six but five.

I. General observations.

1. If we understand for the regular cells by e, k, f, r successively the number of the vertices, edges, faces, bounding bodies, by p, q the number of bounding bodies through an edge, through a point, by e', k', f' the number of vertices, edges, faces of the bounding bodies, then besides the relations

$$e + f' = k + r \quad , \quad e' + f'' = k' + 2$$

of EULER the equations hold

$$qe = re' \quad , \quad pk = rk' \quad , \quad 2f' = rf'' \quad ,$$

out of which number of five we can easily deduce the relation

$$(q - 2)e = (p - 2)k. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The following table furnishes these quantities for the six regular cells of Sp_4 .

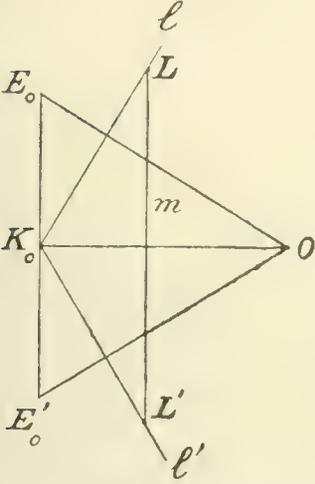
e	k	f	r	p	q	e'	k'	f'
5	10	10	5	3	4	4	6	4
8	24	32	16	4	8	4	6	4
120	720	1200	600	5	20	4	6	4
16	32	24	8	3	4	8	12	6
24	96	96	24	3	6	6	12	8
600	1200	720	120	3	4	20	30	12

2. We shall now endeavour to express the characteristic numbers E, K, F, R of the first of the two new cells — and what is also possible for these P, Q — in the characteristic numbers e, k, f, r, p, q of the regular cell.

“The number of vertices of the new cell is k , i. e. $E = k$.”

If we project the regular cell (see the diagram) on the plane through one of the edges E_0E_0' and the centre O , the two new bounding spaces passing through the centre K_0 project themselves according to the perpendiculars l, l' let down out of K_0 on the axes OE_0, OE_0' . The section of the regular cell with a plane normal to the plane of projection in a point lying close to E_0E_0' being an equilateral triangle,

a square or a regular pentagon, with the assumed point always as centre, according to p having the value 3, 4 or 5, the section of the system of the $p + 2$ bounding spaces of the new cell passing through K_0 with a space normal to OK_0 — e.g. with a space which according to the normal m to OK_0 is perpendicular to the plane of projection — is a right p -lateral prism, of which the segment LL' of m enclosed between l, l' is the axis and the perpendicular endplanes project themselves in L and L' . From this ensues:



“Through a vertex pass $p + 2$ bounding spaces, i.e. $Q = p + 2$ ”.

“Through an edge pass three bounding spaces, i.e. $P = 3$ ”.

“Through a vertex pass $2p$ edges, so pk is the number of edges, i.e. $K = pk$.”

“The system of the bounding spaces consists of two groups, namely of e regular polyhedra with q faces, and r semi-regular polyhedra (e', k', f') with equivalent vertices truncated at the vertices as far as the centres of the edges, i.e. $R = e + r$ ”.

“As the polyhedra of the second group have $e' + f' = k' + 2$ faces and a face is common to two bounding spaces, the number of faces is half the sum of qe and $r(k' + 2)$ or qe and $pk + 2r$, i.e. $2F = qe + pk + 2r$ ”.

Thus the result is :

“The first of the two cells, (E, K, F, R, P, Q), deduced out of the regular cell (e, k, f, r, p, q) has the characteristic numbers

$$E = k, \quad K = pk, \quad F = \frac{1}{2}(qe + pk) + r, \quad R = e + r, \\ P = 3, \quad Q = p + 2.”$$

Here the law of EULER $E + F = K + R$ may serve as verification. In reality the difference of the two members of this equation

$$E + F - (K + R) = k + \frac{1}{2}(qe + pk) + r - (pk + e + r) \\ = k - e + \frac{1}{2}(qe - pk) \\ = \frac{1}{2}\{(q - 2)e - (p - 2)k\}$$

is equal to zero in consequence of the relation (1).

3. The second of the new cells deduced out of the regular cell is enclosed by the polar spaces of the centres K_0 of the edges with respect to the spherical space through those points, i.e. by the

tangential spaces to that spherical space in those points, i. e. by the spaces in the points K_0 normal to the axes OK_0). By polar inversion of the result found above we arrive with respect to this second new cell at the following results :

“The number of bounding spaces of the new cell is k , i. e. $R' = k$.”

“The bounding bodies have $p + 2$ vertices and are double pyramids with a regular polygon with p -sides as base lying in a plane bisecting the connecting line of the vertices at right angles.”

“The faces are isosceles triangles.”

“In a bounding space lie $2p$ faces, so pk is the number of faces, i. e. $F' = pk$.”

“The system of the vertices consists of two groups, namely of e regular vertices and r semi-regular vertices, i. e. $E' = e + r$.”

“The number of edges K' is $\frac{1}{2}(qe + pk) + r$.”

So the result is :

“The second of the two cells, (E', K', F', R') , deduced out of the regular cell (e, k, f, r, p, q) is bounded by double pyramids with a regular polygon with p -sides as base and has the characteristic numbers

$$E' = e + r, \quad K' = \frac{1}{2}(qe + pk) + r, \quad F' = pk, \quad R' = k.”$$

It might appear as if it were possible to deduce more pairs of new cells out of the regular cells by doing for the ends F_0 of the axes OF_0 the same as has been done above for the points K_0 . This is, however, not the case, because for each regular cell the centres F_0 of the faces form the centres K_0 of the edges of another regular cell which is for the cells C_6, C_{24} dualistically related to themselves a cell of the same kind, for the cells related in pairs to one another (C_8, C_{16}) (C_{120}, C_{600}) a cell dualistically related. And as is immediately evident, the pointgroups E_0 and R_0 can neither lead to new results.

We conclude these general observations with the remark that the two cells deduced from the regular cell (e, k, f, r) show much regularity ; of the former the vertices and the edges, of the latter the faces and the bounding bodies are mutually equivalent groups of elements, whilst the faces and the bounding bodies of the former and the vertices and edges of the latter form groups of elements consisting of two subgroups. Do these new cells furnish the maximum amount of regularity for polytopes not entirely regular? We do not intend to go into further details here, as the Mathematical Society at Amsterdam is proposing a prizequestion about what is to be understood by “semi-regular polytopes”.

1) The handbook quoted above contains in Vol II pages 256—261 some communications about the corresponding polytopes in the space S_{p_n} .

The following table shows the results which are obtained by substituting the values of e, k, f, r, p, q for the five different cases. For the sake of completeness those quantities are also included which indicate how many vertices are situated in face and bounding space. We must here notice that, the first cell having two kinds of faces and bounding bodies, we are obliged to take four new quantities, namely the numbers of vertices S_1 and T_1 in face and bounding body of one, the numbers of vertices S_2 and T_2 in face and bounding body of the second kind. Here S_2 and T_2 will relate to the truncating bodies with faces of the same kind and S_1 and T_1 refer to the truncated bodies, where we must then consider as far as S_1 is concerned those faces which the truncated bodies keep in common. We must likewise for the second cell, with two kinds of vertices and edges, introduce the four new quantities P_1', P_2', Q_1', Q_2' .

As is evident $T_1 = Q_1' = k'$, whilst $T_2 = Q_2'$ is the number of vertices of the regular polyhedron with q faces.

	e	E, R'	K, F'	F, K'	R, E'	P, S'	Q, T'	S_1, P_1'	S_2, P_2'	T_1, Q_1'	T_2, Q_2'
C_5	5	10	30	30	10	3	5	3	3	6	4
C_8	16	32	96	88	24	3	5	4	3	12	4
C_{24}	24	96	288	240	48	3	5	4	4	12	8
C_{60}	120	720	3600	3600	720	3	7	3	3	6	12
C_{120}	600	1200	3600	3120	720	3	5	5	3	30	4

In a second part we shall submit each of these five pairs of new cells to a separate investigation.

Mathematics. — “*The analogon of the Cf. of KUMMER in seven-dimensional space*”. By Dr. J. A. BARRAU. (Communicated by Prof. D. J. KORTEWEG).

(Communicated in the meeting of December 28, 1907).

§ 1. In a preceding communication a method was indicated to generate the $Cf'f'$ in spaces of (2^p-1) dimensions, which can be regarded as analoga of the $Cf.$ of KUMMER¹⁾.

¹⁾ A quotation in HUDSON'S *Kummer's Quartic Surface* (p. 187) drew my attention to the fact, that these $Cff.$ have already been obtained by an altogether different method by W. WIRTINGER (*Göttinger Nachrichten*, 1889, page 474; *Monatshefte für Mathematik und Physik* I, page 113; *Mathem. Annalen* 40, page 74). In these papers the varieties are also investigated, for which the elements of such $Cff.$ are singular in the same way as those of the $Cf.$ (16₆) for KUMMER'S quartic surface.

In the following the K^{VII} is under closer investigation especially with a view to the $Cff.$ obtainable out of it by omission of certain elements. To this end it is necessary to construct an other diagram than that of eight simplexes, which can only clearly show the $Cff.$ $(56_{2,5})$; $(48_{2,2})$; $(40_{1,9})$; $(32_{1,8})$; $(24_{1,2})$ and $(16_{1,6})$, formed by omission of the elements of 1, 2, 3, 4, 5, 6 simplexes, all (except the first) in different types, likewise of $Cff.$ (24_9) and $(32_{1,2})$, constructed exclusively of fillings (8_3) .

§ 2. If we isolate in the $Cf.$ of KUMMER a point and a plane not incident to it, the remaining fifteen elements of each kind are divided into a sextuple incident to one of the isolated elements and a nonuple. Each one of the two sextuples forms with the 15 elements of the other kind a free $Cf.$ $(6_3, 15_2)$ which means nothing else but that each of the fifteen right lines connecting the $Cf.$ -points in one plane bears another $Cf.$ -plane and reciprocally.

The two nonuples of elements, however, of both kinds together form one $Cf.$ (9_4) , the structure of which is identical to that of $Cf.$ (9_4) III of the classification of MARTINETTI ²⁾.

This arrangement can be done in $16 \times 10 = 160$ ways.

We can likewise isolate out of K^{VII} in $64 \times 36 = 2304$ ways a point and an Sp_6 not incident to it, by which the sixty-three other elements of each kind are divided into a group of twenty-eight incident to the isolated element of the other kind and a remaining group of thirty-five. The two groups of twenty-eight form together a scheme $(28_{1,2})$; each group of twenty-eight with that of thirty-five of another kind a scheme $(28_{1,5}, 35_{1,2})$; addition of $(28_{1,2})$ and $(28_{1,5}, 35_{1,2})$ furnishes a scheme $(28_{2,7}, 63_{1,2})$; the two groups of thirty-five form together a scheme $(35_{1,8})$.

This arrangement made for the two elements $A1$ is shown in the plate, where the same notations are assigned to the elements as in the diagram of simplexes of which it is a transformation.

We have but to explain how the regular composition indicated by the thicker lines is obtained.

§ 3. Let us first take into consideration the scheme $(28_{2,7}, 63_{1,2})$ of points (columns) and Sp_6 (rows). Every number of twelve points on a row lying in two different $Cf.$ - Sp_6 lies in an Sp_6 ; so we can take the $Cf.$ to consist of twenty-eight points and sixty-three Sp_6 in Sp_6 ; each of the sixty-four $Cf.$ - Sp_6 of the K^{VII} contains such a $Cf.$ (and reciprocally).

²⁾ Atti della R. Accademia Peloritana XV.

We find for the complete notation of such a $Cf.$ in a way analogous to that of § 6 of the former paper :

	Sp_0	Sp_1	Sp_2	Sp_3	Sp_4	Sp_5
	28	378	2016	5040	1008	63
incid. to:						
Sp_0	—	2	3	4	6	12
Sp_1	27	—	3	6	15	66
Sp_2	216	16	—	4	20	160
Sp_3	720	80	10	—	15	240
Sp_4	216	40	10	3	—	32
Sp_5	27	11	5	3	2	—

By projection and intersection we find from this in Sp_5 a $Cf.$ $(378_{10}, 2016_3)$; a $(2016_{10}, 5040_4)$ and a $(5040_3, 1008_{15})$ of points and right lines, in Sp_3 a $(378_{80}, 5040_6)$ and a $(2016_{10}, 1008_{30})$ of points and planes.

Although the number of $Cf.$ -points is $28 = 4 \times 7$ one cannot succeed in forming four simplexes S_4 out of the $Cf.$ -elements; after isolation of such a simplex (which is possible in several ways) we can form out of the remaining elements (also in several ways) at most a scheme S_5 , and then an S_4 , S_3 , S_2 and S_1 after which of the (28_{14}) an element of each kind is left, mutually not-incident, which we join to an " S_0 ". In the figure $S_5 + S_4 + S_3 + S_2 + S_1 + S_0$ are taken together to a scheme (10_6) which we indicate by T .

The arrangement of the thirty-five remaining elements follows now by our regularly putting down the combinations 3 by 3 of the seven points chosen for S_6 ; it is evident that the entire diagram contains along the chief diagonal only schemes S or T , whilst outside a couple of new fillings appear amongst which we notice a (10_4) , complementary to T .

It is by addition of these partial schemes that we can obtain a great number of $Cf.$ included in the total figure; we restrict ourselves to the *forced* $Cf.$ which are those of which each element shows more incidences than are sufficient to determine it and of which for this reason the existence is remarkable from a geometric point of view.

Of the $Cf.$ $(28_{37}, 63_{12})$ in Sp_6 the twenty-eight points form evidently with twenty-eight Sp_6 a dual $Cf.$ (28_{12}) , the same points with the thirty-five other Sp_6 a $Cf.$ $(28_{16}, 35_{12})$.

By omission out of (28_{12}) of a S_6 remains a (21_{10}) ; by omitting S_6 and S_5 a (15_8) remains the scheme of which is *anallagmatic*: each couple of its $Cf.-Sp_6$ has four $Cf.$ -points in common. The same number of 15 points forms with 15 other Sp_6 (namely $H1$ as far as $B1$ included out of the number of thirty-five) a $Cf.$ (15_7) of which the scheme is complementary to the *anallagmatic* (15_8) .

Out of the $Cf.$ (35_{16}) is formed by omission of S_4 a $Cf.$ (30_{14}) ; by omission of T a $Cf.$ (25_{12}) ; of S_4 and T a $Cf.$ (20_{10}) ; of two different T a $Cf.$ (15_8) , identical to the already mentioned one, its points lie in an Sp_6 , the (35_{16}) has in each of the twenty-eight other $Cf.-R_6$ such a (15_8) .

If we add to the $Cf.$ (30_{14}) a system T out of (28_{12}) a $Cf.$ (40_{18}) is formed.

The $Cf.$ (35_{16}) is also obtainable out of the diagram of simplexes of the K^{VII} , the simplex A then falls away entirely, of each of the seven other ones three elements of each kind disappear. The diagram (35_{16}) consists of seven systems S_4 in the chief diagonal, mutually connected by fillings (5_2) , which all degenerate into (3_2) and (2_2) .

By omitting 1, 2, 3, 4 from this S_4 we obtain $Cf.$ (30_{14}) , (25_{12}) , (20_{10}) and (15_8) . The (30_{14}) is identical to the already mentioned one, the (15_8) however is of another type: not *anallagmatic*, neither do its points lie in *one* Sp_6 .

§ 5. In each of the Sp_6 formed by intersection of two $Cf.-Sp_6$ of K^{VII} lie 12 $Cf.$ -points, of which thirty-two sextuples are also common to a third $Cf.-Sp_6$; such a sextuple lies thus in an Sp_4 , the twelve points form with the thirty-two Sp_4 a $Cf.$ $(12_{16}, 32_6)$.

We can give to the diagram of such a $Cf.$ the following form: (see table p. 507).

If e.g. we take the $Cf.-Sp_6$, formed by the intersection of the $Cf.-Sp_6$: $A1$ and $A2$, the twelve points become respectively:

$$\begin{array}{ll} A3 = P1 & B4 = Q1 \\ A4 = P2 & B3 = Q2 \\ A5 = P3 & C6 = Q3 \\ A6 = P4 & C5 = Q4 \\ A7 = P5 & H8 = Q5 \\ A8 = P6 & H7 = Q6 \end{array}$$

The entire $Cf.$ consists evidently of two simplexes S_6 : P and Q in MÖBIUS-position forming together the part (12_6) whilst moreover every triplet of vertices of one simplex with the three non-conjugate ones of the other lie in *one* Sp_4 , i.o.w. each face of one intersects the non-conjugate one of the other.

This connection is for the first time possible in Sp_3 , the analogon in Sp_3 would be: two tetrahedra in MÖBIUS-position, where each edge of one intersects the non-conjugate one of the other; of this $Cf.$ $(8_7, 14_4)$, although it is possible to design the diagram, the execution is evidently impossible.

We find for the complete notation of $Cf.$ $(12_{16}, 32_6)$:

	Sp_0	Sp_1	Sp_2	Sp_3	Sp_4
	12	60	160	240	32
incid. to					
Sp_0	—	2	3	4	6
Sp_1	10	—	3	6	15
Sp_2	40	8	—	4	20
Sp_3	80	24	6	—	15
Sp_4	16	8	4	2	—

By projecting and intersecting are formed out of these e.g. in Sp_2 a $Cf.$ $(60_8, 160_3)$ and a $Cf.$ $(160_6, 240_4)$ of points and right lines; in Sp_3 a $Cf.$ $(60_{24}, 240_6)$ and a $Cf.$ $(160_4, 32_{20})$ of points and planes.

§ 6. The points of Sp_5 can be conjugated one to one to the linear complexes of the usual three-dimensional space, the Sp_4 become linear systems of α^4 of these complexes, the $Cf.$ $(12_{16}, 32_6)$ can be represented in our space.

It is however possible to take the twelve complexes simultaneously special and to regard them as right lines, the thirty-two Sp_4 then become linear complexes which each contain a sextuple of the right lines; the $Cf.$ $(12_{16}, 32_6)$ is then realized in right lines and linear complexes of our space.

We can easily give line-coordinates for such a number of twelve right lines by omitting from the point-coordinates of the twelve points a couple, e. g. X_7 and X_8 , and by letting the six remaining ones satisfy the fundamental relation

$$X_1 X_6 + X_2 X_5 + X_3 X_4 = 0$$

So we obtain e. g. the right lines

$$\begin{array}{ll}
 P1 = (c, -d, -a, 0, -g, h) & Q1 = (d, c, 0, -a, -h, -g) \\
 P2 = (d, c, 0, -a, h, g) & Q2 = (c, -d, -a, 0, g, -h) \\
 P3 = (0, -f, g, -h, -a, 0) & Q3 = (f, 0, h, g, 0, -a) \\
 P4 = (f, 0, -h, -g, 0, -a) & Q4 = (0, -f, -g, h, -a, 0) \\
 P5 = (g, -h, 0, f, c, -d) & Q5 = (h, g, -f, 0, d, c) \\
 P6 = (h, g, f, 0, -d, -c) & Q6 = (g, -h, 0, -f, -c, d),
 \end{array}$$

if besides is satisfied

$$ch + dg = af - gh = 0.$$

The peculiarity appearing with this example taken for simplicity's sake, that the right lines show mutually some incidences, is lost by submitting the coordinates in Sp_5 first to a linear transformation.

In the same way, indeed, we can formulate for all $Cff.$ indicated in spaces of a lower number of dimensions an analytical definition by deducing the coordinates of their elements from those of the elements of K^{VII} .

Chemistry. — “On the constitution of VAN GEUNS's oxymethyldinitrobenzonitrile”. By Dr. J. J. BLANKSMA. (Communicated by Prof. A. F. HOLLEMAN).

By the action of potassiumcyanide on meta-dinitrobenzene in methylalcoholic or ethylalcoholic solution, LOBRY DE BRUYN¹⁾ obtained in 1882 the oxymethyl- or oxyethylnitrobenzonitrile $C_6H_3(OCH_3)CNNO_2$, 1. 2. 3.

The investigation of these substances was continued afterwards by VAN GEUNS²⁾ who succeeded in saponifying these nitriles to acid-amines and in preparing the corresponding acids thereof. At the same time VAN GEUNS showed that in both substances a further nitro-group can be introduced by nitration with nitric and sulphuric acids thus yielding the compounds $C_6H_2(OCH_3)CN(NO_2)_2$, m.p. 113° and $C_6H_2(OC_2H_5)CN(NO_2)_2$, m.p. 63°. These two compounds contain a movable nitro-group which may be readily replaced by OH, OCH_3 , NH_2 , $NHCH_3$, NHC_6H_5 etc.

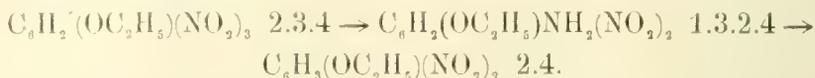
As, however, the place where the nitro-group had been introduced had remained unknown, the constitution of these derivatives was consequently also unknown.

When VAN GEUNS, owing to his departure for India was obliged

¹⁾ Recueil 2, 205.

²⁾ Dissertation Amsterdam 1903.

to discontinue this research I tried at the request of the late Prof. LOBRY DE BRUYN to determine this constitution. After a few trials which led to no result the method was followed which had proved successful in the determination of the constitution of 2.3.4 trinitrophenetol¹⁾. The constitution of that substance was shown to be :



Oxymethyldinitrobenzotrile was now treated in an analogous manner; by the action of alcoholic ammonia one NO_2 -group was replaced by NH_2 and this was then in turn removed by diazotation and boiling with alcohol. In this manner was obtained an oxymethylnitrobenzotrile (m.p. 126°) $\text{C}_6\text{H}_2(\text{OCH}_3)\text{CN}(\text{NO}_2)_2 \rightarrow \text{C}_6\text{H}_2(\text{OCH}_3)\text{CN.NH}_2\text{NO}_2 \rightarrow \text{C}_6\text{H}_3(\text{OCH}_3)\text{CN.NO}_2$.

This shows that the NO_2 -group at 3 is replaced by NH_2 as otherwise the original oxymethylnitrobenzotrile $\text{C}_6\text{H}_3(\text{OCH}_3)\text{CN.NO}_2$ 1.2.3 m.p. 171° would have been reobtained. Now it remained only to determine the constitution of this substance. On treatment with nitric and sulphuric acids an oxymethyldinitrobenzotrile was obtained which melts at 71° and which possesses the following constitution: $\text{C}_6\text{H}_2(\text{OCH}_3)\text{CN}(\text{NO}_2)_2$ 1.2.4.6²⁾.

The constitution of this substance was determined in the following manner. If this compound is treated in alcoholic solution with ammonia or methylamine the OCH_3 group is readily substituted by NH_2 or NHCH_3 and dinitrocyano-aminobenzene m.p. 219° or dinitrocyano-methylaminobenzene m.p. 161° is formed which substances were prepared previously from the corresponding oxyethyl compound³⁾.

The oxymethylnitrobenzotrile m.p. 126° was then heated at 150° with hydrochloric acid for 5 hours. On opening the tube a gas escaped which burnt with a green-bordered flame (CH_3Cl) whilst in the tube there were present crystals which after recrystallisation from water melted at 228° and proved to be 5-nitrosalicylic acid ($\text{C}_6\text{H}_3\text{COOH, OH, NO}_2$ 1. 2. 5.) In the motherliquor the presence of NH_3 was detected, formed by saponification of the cyano-group. For the purpose of identifying the substance obtained a little of the preparation was mixed with an equal quantity of 5-nitrosalicylic acid (m.p. 228° prepared by nitration of salicylic acid⁴⁾. The melting point

¹⁾ Recueil **27**, 49.

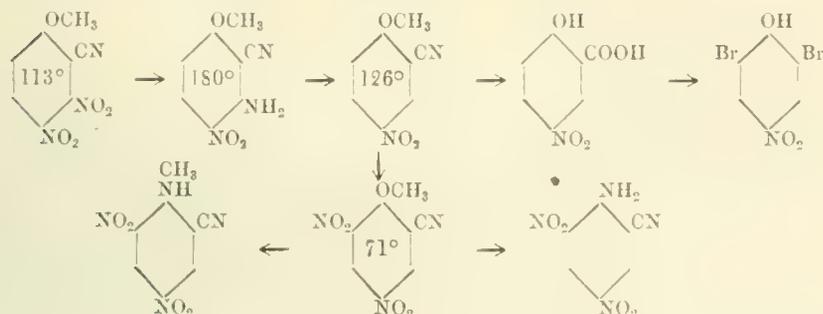
²⁾ This shows that in oxymethylnitrobenzotrile m.p. 126° the nitro-group is placed on 4 or 6.

³⁾ BLANKSMA. Rec. **20**, 413. **21**, 274.

⁴⁾ HÜBNER. Ann. **195**, 31.

was not altered thereby. Both preparations could also be converted readily into 2,6-dibromo-4-nitrophenol m.p. 141° by treatment with bromine water¹⁾.

The following reactions were applied:

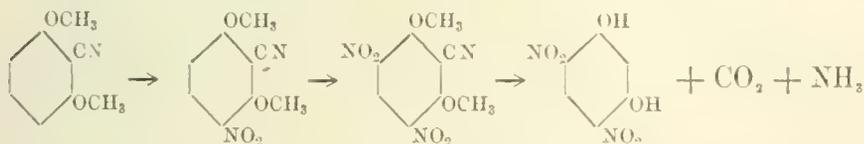


This proved that the constitution of the oxymethyl dinitrobenzocyanide prepared by VAN GEUNS is $C_6H_2(OCH_3)_2CN, (NO_2)_2$, 1, 2, 3, 4.

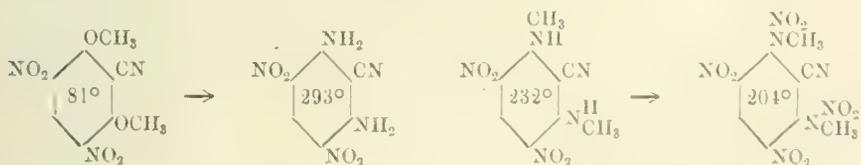
At the same time it was shown that the movable NO_2 -group in this substance is placed at 3; consequently we now know the constitution of the compounds obtained from it by substitution of the NO_2 -group by OH, OCH_3 , etc.

Finally, the constitution was determined of the dinitrodimethoxybenzocyanide obtained by the nitration of $C_6H_3(OCH_3)_2CN$ 1, 3, 2, or of the nitrodimethoxybenzocyanide $C_6H_2(OCH_3)_2CN, NO_2$ 1, 3, 2, 4.²⁾

This compound was converted into 4,6-dinitro-resorcine m.p. 215° by being heated for 5 hours at 150°–160° with hydrochloric acid (30% HCl); from this follows that its constitution is $C_6H(OCH_3)_2CN, (NO_2)_2$ 1, 3, 2, 4, 6.



4,6 dinitro-2-cyano 1,3 dimethoxybenzene on treatment with alcoholic ammonia or methylamine readily yields compounds which perfectly resemble the compounds which have been obtained in a similar manner from 2,4,6 trinitroresorcinoldimethylether.



¹⁾ LELLMANN and GROTHMANN. Ber. 17, 2731.

²⁾ Dissertation VAN GEUNS. p. 69.

Geophysics. — “*Registration of earth-currents at Batavia for the investigation of the connection between earth-current and force of earth-magnetism.*” By Dr. W. VAN BEMMELEN at Batavia. (Communicated by Mr. J. P. VAN DER STOK).

(Communicated in the meeting of December 28, 1907.)

Notwithstanding the great progress in our knowledge of the phenomena of earth-magnetism, the desired improvement has not yet been noticed in the explanation of these phenomena.

That the different variations to which the magnetic needle is liable are the consequence of the changes of electric currents has become highly probable and the place, too, where in that case the currents are to be found is no longer entirely unknown to us.

So SCHUSTER¹⁾ has proved that the daily variation is in general caused by extra-terrestrial currents, whilst I myself have indicated²⁾ that this is likewise the case for that part of the magnetic disturbances which shows a regular daily variation.

Suchlike electric currents have, however, not been shown experimentally and their indication in those unapproachable regions is for the present not to be expected.

Only *one* part of the earth is accessible to us, viz. the outer crust and numerous are the investigations on the electric currents circuiting in that crust.

However, all these investigations have but poorly advanced our knowledge about the connection between those currents and the magnetic variations.

The reason is not only to be found in the great experimental obstacles and the lack of cooperation in the various investigations, but especially in the complicated relations of the system of currents in those zones of the earth where those investigations have been made, viz. between 40° and 70° latitude.

My supposition that in the equatorial zone, just as for other geophysical phenomena, simpler relations must exist, has been proved to be right by the investigations which I wish to communicate here.

The annotations of the earth-current executed by me these last three years at the observatory of Batavia can be divided into two series.

During the first period, March—November 1905, I registered the

1) Phil. Trans. Vol. 180, p. 667.

2) Natuurk. Tijdschrift voor N. I. Dl. 63, p. 227.

earth-current between Cheribon and Batavia, with the aid of the intercommunal telephone-line. The variations were written down photographically, the velocity of the registering strip amounted to the usual 15 mm. an hour.

The important results obtained by this method incited me to go on and in the period now come to a close I could at night make use, by the kind cooperation of the officials of the Telegraph Service, of different telegraph-lines to Anjer, Buitenzorg, Soekaboemi, Billiton, Poerwakarta, Cheribon, Samarang, Soerabaja and Makassar): greater velocity of registration was applied too (60 and 240 mm. an hour).

Besides continuing the different registrations of the earth-current during longer or shorter time to obtain statistic results, I also made experiments. When the registration pointed to a new connection between earth-current and magnetic variation, other registrations of the earth-current were organised to get a closer investigation of that connection. When questions on the influence of wire or groundplate cropped up, it was tried to answer them by experimentation.

The instrumental arrangement was contrived in such a way, that on a strip 20 cm. wide beside the variations of two earth-current circuits those of the corresponding magnetic components were noted down. Corresponding means here: the component normal to the direction of the earth-current circuit.

The sensibility was chosen in such a way, that the corresponding variations of earth-current and magnetic force did not differ too much in magnitude. To this end great sensibility of the magnetic variation instruments (up to 0.1 γ an mm.) was necessary; as however only the relative position during one night was considered it was easy to arrange those variation instruments quite simply.

I intend to give a more extensive communication about this at some other time.

The daily variation.

The obstacles met with in investigating the daily variation were very great. For the most important part of that variation takes place about noon, but during the day time the electric field of Batavia is disturbed by the electric tram and moreover I was allowed the use of the lines only at night.

If still I succeeded in coming to useful results, this is due to the kindness of the Superintendent of the Intercommunal Telephone Company, Mr. S. W. BAINTS, who allowed hourly readings of the amount of the earth-current to be done at the office of the Company at Batavia. The hours were 8^h45^m A. M., 9^h45^m A. M. etc. until

4^h45^m P. M.; an ordinary galvanometer with pointer was used.

I have chosen from these readings those falling on magnetically very calm days and evincing moreover not to have suffered from disturbances on the line or from other irregularities. For those same days I have used the observatory-notation during the hours of the night (6 P. M.—5 A. M.).

Two unknown quantities remained, viz. the ratio of the values of the scale division and the difference of the central position.

The former I determined one evening during a magnetic storm at the office of the Company by alternately reading the galvanometer and allowing the Observatory to register. The reduction to a same central position I got to a plausible result by using the Sunday notations. For, on those days I could use the line already after 12 at noon and from a score of magnetically fairly calm Sundays I deduced the difference between the hours 4³/₄ and 6 p.m.

Graphically I then interpolated the 24 values of the hours of the day.

For the employed magnetically calm days finally was calculated the daily variation of the magnetic component normal to the direction Cheribon-Batavia from the Buitenzorg magnetograms.

Daily variation of the earth-current Cheribon-Batavia and of the magnetic horizontal component normal to that direction.

Earth-current in Volt per Kilometer $\times 10^{-5}$ (Direction Ch.-Bat. = +)	Magnetic Component in 10^{-5} C. G. S. (NE = +)	Earth-current in Volt per Kilometer $\times 10^{-5}$ (Direction Ch.-Bat. = +)	Magnetic Component in 10^{-5} C. G. S. (NE = +)
1 a. m. — 38	—11.1	1 p. m. + 33	22.2
2 — 36	— 9.6	2 — 6	13.9
3 — 40	— 9.3	3 — 33	3.3
4 — 33	— 7.9	4 — 40	— 4.5
5 — 32	— 6.5	5 — 32	— 9.4
6 — 22	— 3.0	6 — 23	—10.2
7 11	3.4	7 — 51	—11.0
8 81	8.8	8 — 60	—13.1
9 153	15.9	9 — 51	—13.6
10 154	24.2	10 — 47	—14.6
11 117	29.5	11 — 46	—13.7
12 84	28.5	12 — 39	—12.9

Out of the curves, indicating according to the above numbers the daily vibration of earth-current and magnetic component, is evident:

that this vibration for them corresponds;

that the direction of the earth-current is such that it can be regarded as causing the variations of the magnetic component;

further: that the magnetic component is retarded with respect to the earth-current and finally ;

that the ratio of the amplitudes of corresponding vibrations decreases with the duration of that vibration, so that those of the earth-current are relatively larger with a shorter duration.

The chief maximum in the afternoon is reached by the earth-current about an hour and a half earlier, the chief maximum at night about two hours earlier.

The secondary vibration in the evening-hours is for the earth-current much stronger.

It is an indication to apply here the harmonic analysis and to employ for it the formula

$$A = A_n \sin n (t + c_n).$$

The results of the harmonic analysis confirm in full what the mere observation taught us.

Especially the increase of the earth-current as the duration of the corresponding variation of the earth-magnetism becomes shorter is very distinctly expressed.

This dependence can be pretty accurately expressed by the following formula.

Let the amplitude of the magnetic component be M ; the duration expressed in days T , and the amplitude of the earth-current A , then

$$A = 0.8 \sqrt[4]{\frac{1}{TM}}.$$

The values in the above column "calc" (on the next page) have been computed with the aid of this formula.

Yet not much value must be attached to that correspondence, as the higher terms of the harmonic analysis are very untrustworthy on account of the inaccuracy of the hour-values employed.

The difference of phase increases regularly as far as the 5th term, and then drops again to the value it had for the 3rd term; but the phase differences found for the higher terms deserve little confidence.

I have been successful in obtaining a confirmation of a part of these results with the aid of the cable Batavia—Billiton. The four months March—June 1906 gave for the nightly course proper results.

	AMPLITUDE				PHASE.				
	Earth-current Volt. p. K.M. $\times 10^{-5}$	Magn. Comp. C.G.S. $\times 10^{-5}$	Earth-current Magn. Comp.		Δ C-0	Earth- current	Magn. Comp.	Earth- current Magn. Comp.	
			Observ.	Calc.					
A ₁	75	18.6	4.0	3.9	-0.1	C ₁	299°	286°	13°
A ₂	44	7.6	5.7	5.9	0.2	C ₂	73	53	20
A ₃	28	3.0	9.4	8.0	-1.4	C ₃	122	98	24
A ₄	9	0.9	10.0	10.8	0.8	C ₄	53	15	38
A ₅	7	0.5	13.1	13.6	0.5	C ₅	103	56	47
A ₆	6	0.4	15.7	15.9	0.2	C ₆	57	34	23

The variation of the corresponding magnetic component changed its nature pretty much (as was to be expected) during these months.

The earth-current really followed this variation whilst the maxima and the minima kept preceding those of the magnetic component.

Out of the average for the 4 months this is obvious.

	7	8	9	10	11	Midnight	1	2	3	4	5
Earth-current Volt. p. K.M. $\times 10^{-5}$	0	-27	-25	-29	-17	12	20	28	26	0	-14

Magnet. comp. 0 -2.1 -3.7 -5.3 -5.5 -4.9 -4.1 -3.3 -3.4 -4.5 -5.4
 10^{-5} C. G. S.

For the earth-current the maximum comes one hour, the minimum about half an hour earlier. Let us suppose this minimum to belong to the preceding vibration of $3^h 30^m$ duration, then the difference in phase is 26° .

The ratio of the amplitude is $26^\circ.0$, whilst I deduced roughly out of the Sunday notation for the great vibration 16.0 when the cable was at my disposal from 0^h till 4^h p.m.

So we meet here too with decrease of the ratio between earth-current and corresponding magnetic component together with increase of the duration of the vibration, but by the side of it a much stronger earth-current than for the line Cheribon—Batavia.

Annual inequality in the daily vibration.

At Batavia the amplitude of the daily vibration of the magnetic force is liable to a single- and a double-yearly inequality, where the maxima are attained in March and September, the minima in January and in June. The two maxima and the two minima are of the same magnitude.

From the continued measurements at the office of the Intercommunal Telephone Company I could deduce that the variations of the earth-current show the same annual inequality.

This series of measurements shows two breaks.

First in January '06 the lines were permanently disturbed and secondly in August '06 errors seem to have slipped into the observations, on account of which repeatedly improbably large values were read. After my having pointed this out, the readings in December next were again serviceable.

Out of each month I have taken those days which were in the first place magnetically calm and for which in the second place as much as possible complete and useful readings of earth-currents were at hand.

Of the mean hourvalues for each month was then taken the difference of the smallest and the greatest value.

The maximum generally fell in with the $8^{\frac{3}{4}}$ a.m. or $9^{\frac{3}{4}}$ a.m. observation, the minimum with that of $3^{\frac{3}{4}}$ p.m. or $4^{\frac{3}{4}}$ p.m.

These differences expressed in Volt pro kilometer $\cdot 10^{-5}$ follow here.

	J.	F.	M.	A.	M.	J.	J.	A.	S.	O.	N.	D.
1905		266	194	208	190	129	127	127	170	173	167	131
1906		171	177	127	125	109	135	233 (?)				122
1907	81	118	113	90	88.							

Notwithstanding the imperfection in these measurements, the double annual period and its correspondence with that of the magnetic component is so distinctly expressed that doubt is not possible.

Variations of short duration.

The second period of registration, November 1905—October 1907, was chiefly devoted to the study of the connection of the vibrations of short duration in earth-current and magnetic component.

The usual velocity of registration was here 1 mm. a minute, which with sharp photographic lines allows the measurement of variations with the duration of half a vibration of 0.2 to 0.3 minute, but in numerous nights the velocity was enlarged to 4 mm. a minute, when it was possible to measure accurately differences of time of 0.1 minute.

By the continued registration of the earth-current along different lines, each one accompanied by that of the corresponding magnetic component, an extensive material of curves was collected, from which in general the following could be gathered.

For the earth-current along about east-west lines ¹⁾ to each vibration answers a similar one of the magnetic component. For that of the nearly north-south lines that correspondence seems also to exist in part, but it is greatly disturbed by the circumstance that the earth-current keeps following more or less the vibrations of the east-west line.

So also near the equator we find complicated phenomena, but only in part, for as far as the east-west current is concerned we meet with such a striking correspondence that it is possible to deduce simple laws; the two same laws, which were found for the daily variation:

1. *the vibration in the earth-current precedes that of the magnetic component with a certain difference in phase;*

2. *the ratio of amplitude of earth-current and magnetic component decreases when the duration of the vibration increases;*

1) The east-west lines were:				The north-south lines were:			
		Direction	Distance			Direction	Distance
Bat. — Anjer	W	6° N	106 K.m.	Bat. — Billiton	N	13° E	392 K.m.
, — Poerwakarta	E	40 S	78 ,	, — Buitenzorg	S	5 E	46 ,
, — Cheribon	E	18 S	200 ,	, — Soekaboemi	S	9 E	84 ,
, — Semarang	E	12 S	406 ,				
, — Soerabaja	E	10 S	665 ,				
, — Makassar	E	5 N	1486 ,				

appear distinctly from this material of registering curves with its thousands and thousands of shorter and slower vibrations.

To deduce the real amplitudes and phases of those variations we should have to execute for each separate case an immensely extensive harmonic analysis and, this being quite impossible, corresponding variations had to be chosen and measured discriminately. Therefore all the measurements have been done by me personally.

The precedence of the earth-current.

This precedence was rule; it was quite exceptional if it was not met with. When choosing cases for measurement I always avoided those where by a superposed oscillation of greater length and amplitude the time of the turning point was made to appear much earlier or later.

The difference of phase proved to vary from case to case but to be already constant in the mean of a small number.

For the lines Batavia—Soerabaja and Batavia—Poerwakarta the difference in phase was determined with respect to that of Batavia—Cheribon and not with respect to the magnetic component.

For the lines Batavia—Buitenzorg and Batavia—Billiton the deduction was accompanied by great difficulties, as perfectly corresponding cases between earth-current and the magnetic component seldom made their appearance on account of the interference of the east-west component.

For Batavia—Soekaboemi I have therefore desisted from making a calculation and the difference in phase for the two first lines must be mentioned with reserve.

Difference in phase.

Batavia—Poerwakarta	22°	Batavia—Buitenzorg	23° (?)
„ —Anjer	14	„ —Billiton	28 (?)
„ —Cheribon	22		
„ —Semarang	36		
„ —Soerabaja	32		

The difference in phase found here for Cheribon and Billiton shows a striking resemblance to that found for the daily variation.

Cheribon.

Variations of short duration. Average of 6 terms of the daily variation.

22°

27°

According to this table the increase of the amplitude when the duration of the vibration diminishes seems to reach a maximum value at 0.5 min. and moreover the ratio of amplitudes seems to be dependent on the amplitude itself and in such a way that with equal duration it increases with the diminishing of the amplitude.

A complete confirmation of these results was found in 312 cases for the Anjer-line.

Batavia-Anjer.

Amplitude Earth-current in Volt p. K.m.
Amplitude Magnetic Component in C. G. S.

Duration of half vibration.	Magnetic Component in 10^{-5} C. G. S.				
	0.19—3.0	0.53—0.68	1.36—1.83	2.30—3.89	5.20—5.48
0.4 min.	96	95	97		
0.6	97	104	95		
0.9	86	90	89		
1.8	92	84			
2.3			64		
5.3		86	74		
6.3				68	
7.7	83				
9.2			68		
9.7					60
10.8		68			
12.3			73		
13.6		58			
19.9				75	
22.0					63

Here too with a short duration a maximum is found, viz. with 0.6 min.

Beside this correspondence we find the unexpected result, *that the Batavia—Anjer current is about four times stronger than the Batavia—Cheribon current.*

That ratio is constant for different duration of vibration, as was proved by the arrangement according to groups of mean equal duration.

Duration of half a vibration	Batavia-Anjer
	Batavia-Cheribon
0.33 min.	4.28
0.54	3.94
0.66	4.36
0.76	4.21
0.96	4.21
1.67	3.79
5.6	4.25
7.6	3.84
15.1	5.21
	Mean 4.23

When we wish to compare the values for the ratio of amplitudes of magnetic force and earth-current of vibrations of shorter duration with those of the six terms of the daily variation, it is best to inscribe all values in one diagram with abscis $\sqrt[4]{M}$ and ordinate $\sqrt[4]{1/T}$.

When now the formula $A = 0.8 \sqrt[4]{\frac{1}{TM}}$ found above for the 6 terms of the daily variation will still hold, then the values which lie on the same radius-vector through the origin must be the same.

It is evident that this is only the case for the middle part of the diagram, namely for the radius-vector where $0.8 \sqrt[4]{\frac{1}{TM}} = 150$.

The ratio $\sqrt[4]{1/T}$ and $\sqrt[4]{M}$ is therefore = 2.

If the amplitude of the magnetic component is relatively larger, then the radii with equal values are bent gradually to the axis of abscissae, and if $\frac{1}{T}$ is relatively larger, then they are bent to the axis of ordinates.

For a great $\frac{1}{T}$; i. e. for a duration of vibration of about one minute, they turn again to the axis of ordinates and a maximum seems to be formed.

It will be possible to force these curves in a formula, but we

must not expect that that formula will give the real formula, as it must be very complicated on account of the nature of the phenomenon.

What according to me is clear from the diagram is that the ratio of amplitudes for the vibrations of shorter duration will gradually pass into those of longer duration (the six terms of the daily variation) and so that between these two phenomena there is also a gradual transition.

The results for the earth-current Buitenzorg-Batavia and Billiton-Batavia are again uncertain on account of the lack of agreement with the magnetic component. I found :

Buitenzorg-Batavia.

Duration of half a vibration	Amplitude Magn. Comp.	Amplit. earth-current in V. p Km.		Number of cases
		Amplit. magn. Comp. in C. G. S.		
1.1 min.	0.38 J.	75		20
3.7	0.96	79		20
7.6	1.20	71		28
				68

Billiton—Batavia.

0.7	0.39	63		14
1.3	0.29	74		14
3.5	0.49	66		14
7.0	0.50	58		23
22.4	1.44	41		5
				70

If indeed these figures are trustworthy then the ratio of amplitudes of the earth-current with respect to the magnetic component decreases here too when the duration increases.

The increase at the outset with very small duration is also found in the above figures, even in much greater ratio than for the Anjer- and Cheribon-current.

The numbers for Buitenzorg are a little smaller and for Billiton smaller than for Anjer, but still larger than for the Cheribon-current.

The line to Poerwakarta I had but for two nights at my disposal.

I have then allowed the Poerwakarta-current to be registered at the same time as the Cheribon-current and I have found a complete correspondence between them for vibrations of a duration from 0.8 to 15.5 minutes.

Not before the last months of the registering-period have I extended the investigations to the lines to Semarang and Soerabaja and to my surprise I found a pretty great deviation from the circumstances which appear on the lines Cheribon and Anjer.

Semarang--Anjer.

Duration of half a vibration	Amplitude Anjer-current 10^{-6} V. p Km.	Amplitude Anjer		Number of cases
		Amplitude Semarang		
0.33 min.	38	1.70		20
0.69	74	2.06		20
1.04	61	2.46		20
1.71	57	2.32		20
7.43	160	3.96		20
20.81	290	4.61		16
				<u>116</u>

Soerabaja—Anjer.

0.37	43	2.17		20
0.93	92	2.87		20
2.65	140	3.27		20
11.85	326	4.71		20
34.28	739	7.46		7
				<u>87</u>

So the influence of the duration on the amplitude of the earth-current is here much greater for the Semarang- and Soerabaja-current than for the Anjer-current.

It is remarkable that here too the increase of the influence with the duration takes place about according to $\sqrt{1/T}$.

With respect to the first value for $t = 0.33$ min., respect. 0.37 minutes duration we get:

	$\frac{A_1}{t_1}$	$\frac{A_2}{t_2}$		$\frac{A_3}{t_3}$	$\frac{A_4}{t_4}$
Semarang	1.20	1.21	Soerabaja	1.26	1.32
	1.33	1.27		1.61	1.51
	1.51	1.36		2.38	2.17
	2.18	2.33		3.10	3.30
	2.82	2.71			

Direct comparison of the Batavia-Semarang current to the magnetic component furnished: (June 18—21, '07):

Duration of half a vibration	Amplit. of the Magnet. Comp.	Amplit. earth-current in Volt p. K.M.	Number of cases
	in 10×10	Amplit. earth-current in C.G.S.	
0.6 min.	1.3	43	14
1.0	0.9	35	15
1.4	0.9	31	13
5.0	0.7	18	10
11.5	2.2	13	9

The last two values fit in very well with the scheme of the values found for the Cheribon-current, but the first three show a much quicker increase when the duration of the vibration is shorter.

This peculiar increase immediately strikes one when regarding the registered lines. In order to investigate whether that increase of the influence of the duration was connected with the increase of the distance of the two stations between which the earth-current was measured, I asked for and obtained direct connection with Makassar. The loss by defective isolation on the line however was so great, that the real distance had not obtained any lengthening of importance.

On the trustworthiness of the results.

A certain doubt has always been left when observing earth-currents whether the results arrived at do give an idea of the real existing earth-current.

According to SCHUSTER (Terr. Magn. III, p. 130) the intensity of the current is really to be determined by switching on in the circuit a cell of known E. M. F. I have therefore always used this

simple means and connected to it often, by introducing a resistance of 100 or 1000 Ohm, a measurement, though a rough one, of the total resistance of the whole circuit.

The results generally showed a mutual correspondence, only for longer lines a distinct loss by defective isolation was often discernible.

For the earth-current this loss by defective isolation is of not much consequence: for, if two points lying at a distance L from each other with potentials P and $P + L\rho$, are connected by a wire the potential will vary along that wire proportional to the distance of P to $P + L\rho$ and will be in a point between the two, say at $1/a$ of the distance, $P + \frac{L\rho}{a}$. But there the potential of the earth will also be

$P + \frac{L\rho}{a}$ if that earth potential likewise varies proportionally to the distance (which we shall suppose to be true at first computation). There will thus be no difference of potential between line and point of contact with the ground, neither loss of current.¹⁾

However there is loss of current, when I switch on a cell, thus when I generate a drop of potential along the wire, that does not at all run parallel to the earth potential.

This explains that the image of the earth-current rose and fell so regularly with the magnetic component, whilst so often a great loss by defective isolation took place on the line, so that the determination of the values of the scale division by means of inserting a cell gave abnormal values.

When, however, an investigation must be made of the regular or non-regular increase of the earth-potential with the distance, then this loss by defective isolation is disturbing. That is why the registering with the continuous connection Batavia—Makassar shed no light upon the subject.

Influence of the material of the line.

I could experiment accurately on the possibility of the influence of the material of the conductive wire on amplitude and phase of the earth-current by registering simultaneously the currents between Cheribon and Batavia, resp. through the copper telephone wire and the iron telegraph one.

¹⁾ If we suppose the earth-current to form a closed circuit passing round the earth and our wire to have contact with it in three points viz. at the two end stations and the point of contact, there is a distribution of current according to WHEATSTONE and the contact is the bridge of WHEATSTONE.

The resistance of the former circuit was generally about 10 times smaller than that of the other.

At Batavia the two circuits were on the same groundplate, at Cheribon the two groundplates hung in the same well. Moreover the wires ran for the greater part on the same telegraphpoles.

An all but perfect correspondence was now found, so that all influence of the material (especially with respect to magnetic induction) may be regarded as non-existing.

Influence of the current of polarisation.

I was more anxious about a disturbing influence of the polarisation of the ground-plates to which repeatedly from various sides attention has been drawn. For, the polarisation might be able to explain the difference in phase and the change in the ratio of amplitudes.

Let us suppose that the earth-current together with the magnetic component increases, then the resisting current of polarisation also grows. If now the increase of the earth-current and of the magnetic component passes into a decrease, the current of polarisation will for the moment keep increasing and consequently the observed current (i.e. earth-current minus current of polarisation) will sooner reach its turning-point than the magnetic component. If the vibration increases in duration the current of polarisation will also increase first faster, then slower, and therefore the observed current will always lose with respect to the magnetic component and the ratio of amplitudes — as was really found — will decrease first faster, then slower.

Though the influence to be expected of the polarisation had therefore to agree with what was found, yet we could not believe that it could be the cause of those phenomena, as for the observations made at the office of the Telephone Company the connection with the earth was made every hour only for a few moments and so there could be no question about a continual increase of the polarisation.

To find out the influence of the polarisation I have taken the following experiment. To begin with I measured the current of polarisation directly. To that end I buried a second ground plate a few meters from the old one and made a new connection: old earthplate—galvanometer—new ground-plate.

The old ground-plate I polarised strongly by switching on a cell into the line (Cheribon—Batavia). After breaking the con-

nection with Cheribon I immediately closed the new one, in consequence of which the depolarisation-current passed through the second galvanometer.

I actually found the polarisation with its characteristic qualities, but its intensity was hardly more than a few percents of the chief current and thus really too small to serve as cause of difference in phase and change in ratio of amplitudes.

After this investigation I have placed a set of non-polarising ground-plates (amalgamated zinc plates immersed in a solution of $Zn SO_4$ in porous pots)¹⁾ on the garden of the Observatory, and repaired to Cheribon to place a corresponding set there. The repetition of the experiment described above showed really the non-appearance of polarisation.

After this I connected one of the two telephonewires between Batavia and Cheribon with the old polarising ground-plates, the other with the new non-polarising ones, and allowed the two earth-currents to register simultaneously on the same strip with the same sensibility and a velocity of registration of 24 cm. an hour together with the magnetic component. The experiment could hardly have been taken more accurately.

As I expected the result for the difference in phase was a very slight influence in the sense mentioned above; for the ratio of amplitude I found for *one* night also a very small influence in the expected direction, but during two other nights a somewhat greater difference in opposite sense. I think I must attribute those last influences to the unavoidable inaccuracy of the determination of the values of the scale division (by switching on a cell of known E. M. F.).

At any rate I had proved sufficiently that the current of polarisation was not the cause of the found phenomena, so I can take those phenomena to be real.

Connection between earth-current and magnetic force.

If we wish to investigate more closely the connection between the variation of earth-current and magnetic component it is necessary to regard the variations of the latter quite by themselves.

The general rule holds at Batavia that the two horizontal components change simultaneously, i. e. that generally between the turning-points of X and Y only a small difference in time exists and that on the other hand Z generally has a difference of phase of 90° with X and Y .

AD. SCHMIDT (Met. Zeitschrift 1899) has pointed out, that the

¹⁾ C. A. BRANDER. Inaugural Dissertation, Helsingfors, 1888.

variations of the magnetic component might be explained by the passage of electric current vortices.

Following this explanation we should have to conclude for Batavia to the passing of vortices whose centre remains far from Batavia, so that only the outside slightly bent pieces of current pass by.

So as a first approximation we may assume that the extra-terrestrial current is almost rectilinear at Batavia and when varying in intensity has but slight oscillations in direction.

The average direction must be *WSW — ENE* for almost without exception an increase of the *N*-component is accompanied by a weakening of the *E*-component and so $\Delta X > \Delta Y$.

That current we really find back in the diagram of the equipotential lines of the daily variation according to SCHUSTER-VON BEZOLD,¹ which equipotential lines follow at first approximation the current-lines.

Also for the explanation of the phenomenon found by me of the *earthmagnetic after-disturbance*, it was a matter of fact to take a current encircling the earth and this current too had to have a suchlike direction as was mentioned above, but the angle with the equator was at Batavia much smaller than is found now.

Each varying extra-terrestrial current will induct an intra-terrestrial one and the magnetic variation observed at the surface is the sum of the influence of the two. LAMB (see the paper quoted above of SCHUSTER on the daily variation) proves that the ratio of the potential of the primary and the secondary field is complex and that therefore difference in phase exists. The horizontal component caused by the extra-terrestrial current is in advance compared to the one generated by the inducted currents; so the resulting component will be in retardation compared to the extra terrestrial current.

By SCHUSTER however no difference of phase is found for the vertical force and LAMB has pointed to the fact that this can be the consequence of increasing electric conductivity of the earthstrata towards the depth. The results of the new seismological observations point to an iron nucleus of the earth and therefore to a very great increase.

We may therefore probably assume that the difference in phase is very little, at any rate that difference in phase is slight for variations of short duration.

The magnetic force observed consisted of a primary and of a secondary part, which have the same sign as far as the horizontal component is concerned,

¹) Sitz. Ber. der Berliner Akademie für 1895.

The ratio c' between the secondary and primary part increases with the frequency of the vibrations of the current.

Let us call the magnetic force X , then

$$X = \text{primary} + \text{secondary}$$

$$X = \text{primary} (1 + c'),$$

so the primary part of $X_0 = \frac{X}{1 + c'}$.

The extra-terrestrial current which we can put $S = s \sin 2\pi \frac{t}{T}$ will induct in the upper earthstrata a current S' :

$$S' = \varrho f(ST) \frac{2\pi}{T} \cos 2\pi \frac{t}{T}.$$

The induction will depend on the distance and the latter possibly on intensity and duration of the vibration of the current, moreover on the conductivity ϱ of the upper earthstrata.

The primary magnetic force will at first approximation (the distance being about the same) depend in the same manner on the extra-terrestrial current. So:

$$X_0 = f(sT) \sin 2\pi \frac{t}{T}$$

and

$$X = (1 + c') f(sT) \sin 2\pi \frac{t}{T}.$$

The existence of a vertical conducting current having been proved we must also take for granted that part of the extra-terrestrial current is closed by the earth and that a current is generated equally directed as the current of induction.

Already the properties of the conductivity of the atmosphere point to a dependence of magnitude and duration of the extra-terrestrial current, also on the conductivity of the upper earthstrata.

So we put for the current

$$\varrho \psi (sT) \sin 2\pi \frac{t}{T}.$$

And for the total current we find:

$$A = \varrho \left[\frac{2\pi}{T} f(sT) \cos 2\pi \frac{t}{T} + \psi (sT) \sin 2\pi \frac{t}{T} \right]$$

or

$$A = \varrho \left[\psi^2 (sT) + \frac{4\pi^2}{T^2} f^2 (sT) \right]^{1/2} \sin \frac{2\pi}{T} \left(k + \frac{T}{2\pi} Bg \operatorname{tg} \frac{f(sT) 2\pi}{\psi(sT) T} \right),$$

whilst we found above

$$X\bar{t} = (1 + c') f(sT) \sin 2\pi \frac{t}{T}.$$

For the difference in phase we found a constant part of T or

$$\frac{T}{2\pi} Bg \operatorname{tg} \frac{f(sT) 2\pi}{\psi(sT) T} = \frac{T}{K}.$$

For angles of $\pm 23^\circ$, found for Cheribon for a shorter duration of the vibration, we may write here approximately

$$\frac{f(sT)}{\psi(sT)} = \frac{T}{K}.$$

So the ratio of amplitudes becomes

$$\frac{A}{X} = \frac{\rho}{1 + c'} \frac{\psi(sT)}{f(sT)} \left[1 + \frac{4\pi^2}{K^2} \right]^{1/2},$$

$$\frac{A}{X} = \frac{\rho}{1 + c'} \frac{1}{T \sqrt{K^2 + 4\pi^2}} = C \frac{\rho}{1 + c'} \frac{1}{T}.$$

For $1 + c'$ we find according to LAMB-SCHUSTER for that part of the potential which is to be expanded in terms of a spherical function of order 2 (for that part which is to be expanded in terms of spherical functions of a higher order, the increase is quicker).

$\sigma = \frac{\text{constant}}{T}$	$1 + c'$
10	1.172
20	1.278
30	1.337
40	1.374
50	1.399
100	1.466
900	1.605
6400	1.643

So for higher frequency (for duration of half a vibration = 1 minute σ is 7200) $\frac{1}{1 + c'}$ is about constant.

So we get $\frac{A}{X}$ proportional to $\frac{1}{T}$.

The observations, however, give for Cheribon and Anjer proportionality with $\sqrt[8]{\frac{1}{T}}$ (for still smaller T , even inversion, and for

Semarang and Soerabaja proportionality to $\sqrt[4]{\frac{1}{T}}$.

AD. SCHMIDT Met. Zeitsch. 1902 p. 94) brings the supposition forward that the current can be generated in the wire by induction only, thereby supposing the wire to be closed by the earth.

Then putting the case very simply we arrive by application of the rule of AMPÈRE at :

$$A_{E-W} = -\gamma \frac{dX}{dt} + \sigma \frac{d^2 Z}{d\varphi dt}.$$

This gives the difference with respect to the above that the variation Z makes its appearance.

The Z , however, changes but little in equatorial regions, so it cannot make the theory correspond to the observations.

The slow increase of the earth-current when the frequency increases does *not* point to induction, but rather to direct connection with the primary current.

The quantity $\frac{1}{1+c'}$ has indeed, compared to $\frac{1}{T}$, rather a slow course.

That difference in phase is according to SCHUSTER-LAMB rather decreasing for quicker vibration whilst for the earth-current it proves to be constant.

But whence the difference in phase?

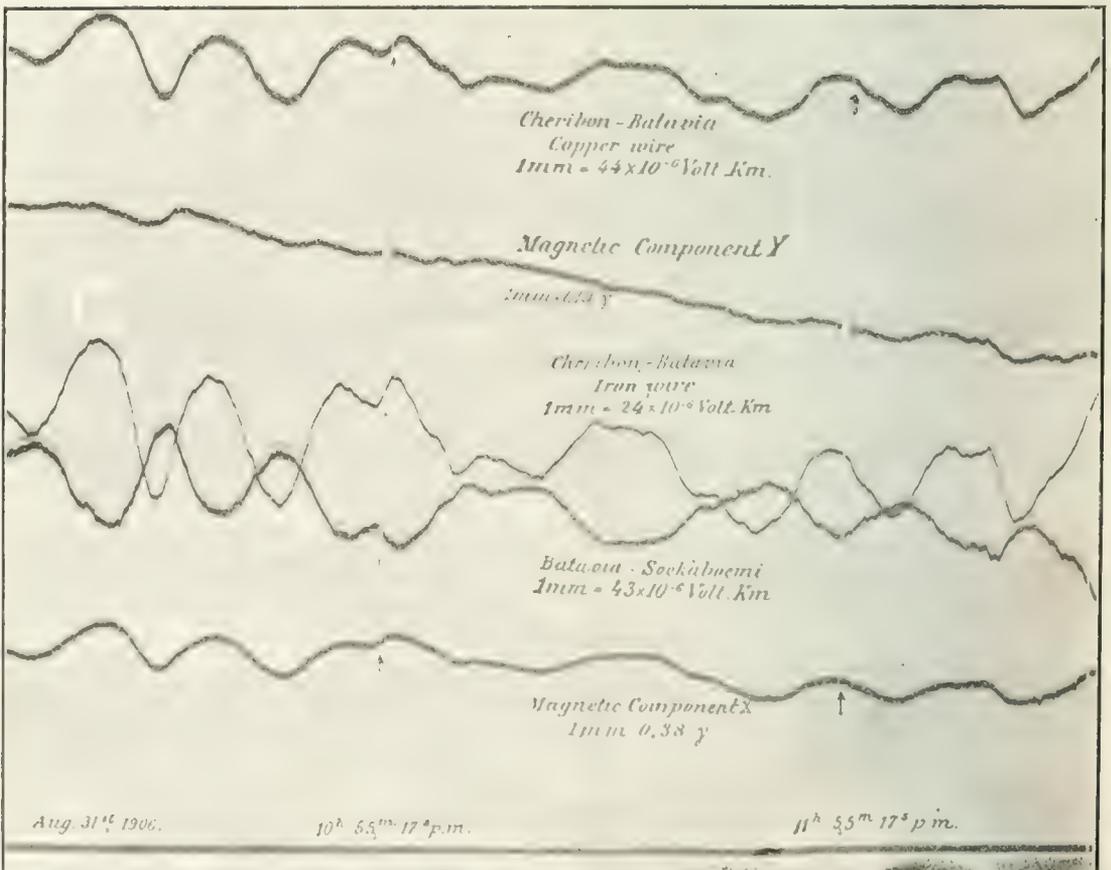
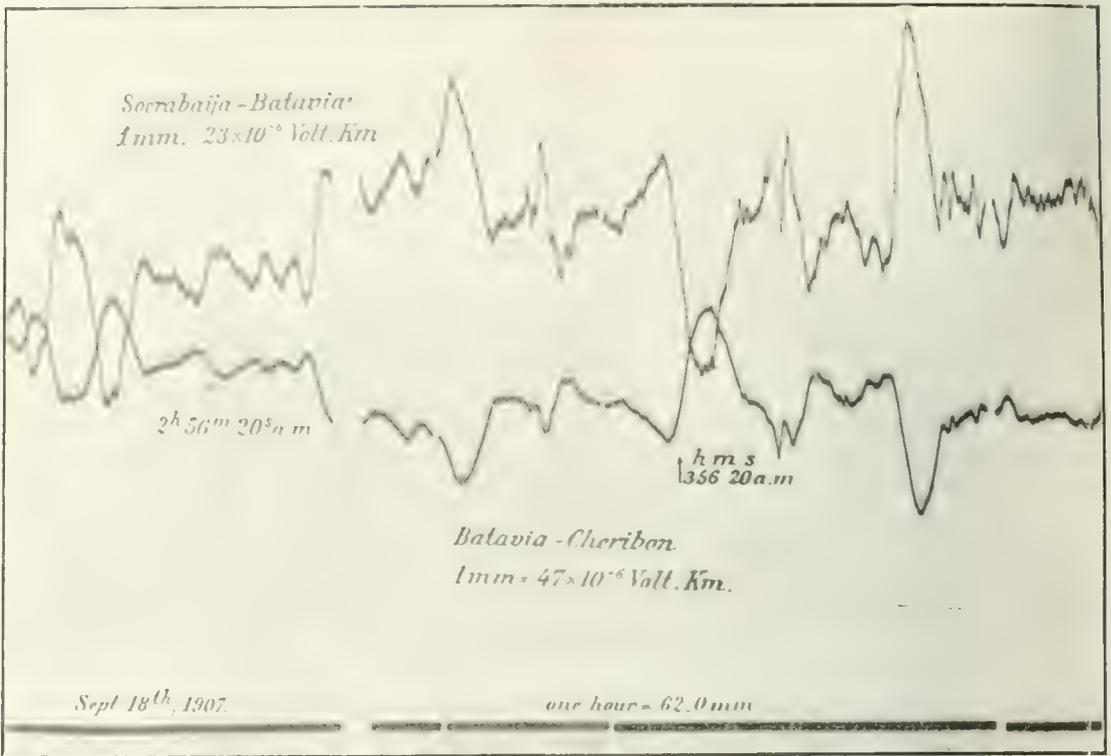
The differences in the intensity and the difference in phase of the earth-current for the lines to Anjer, Cheribon, Billiton and Buitenzorg can be explained also by the difference in conductivity of the ground between those places and Batavia.

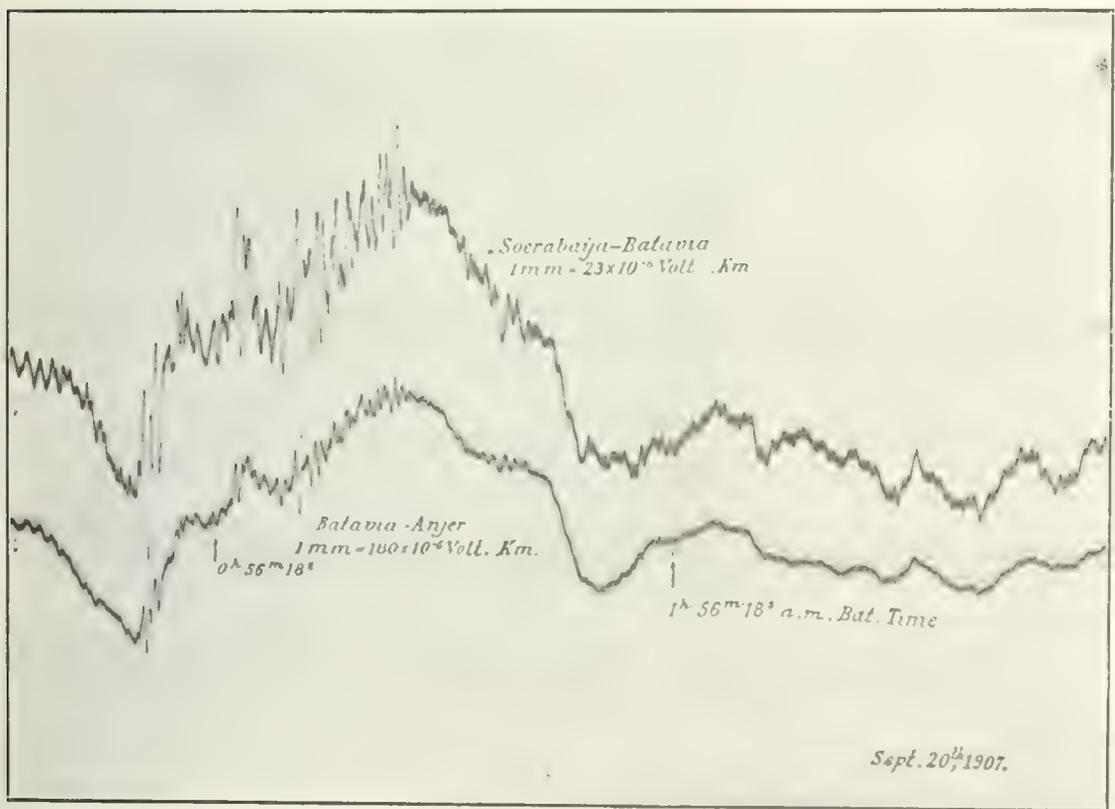
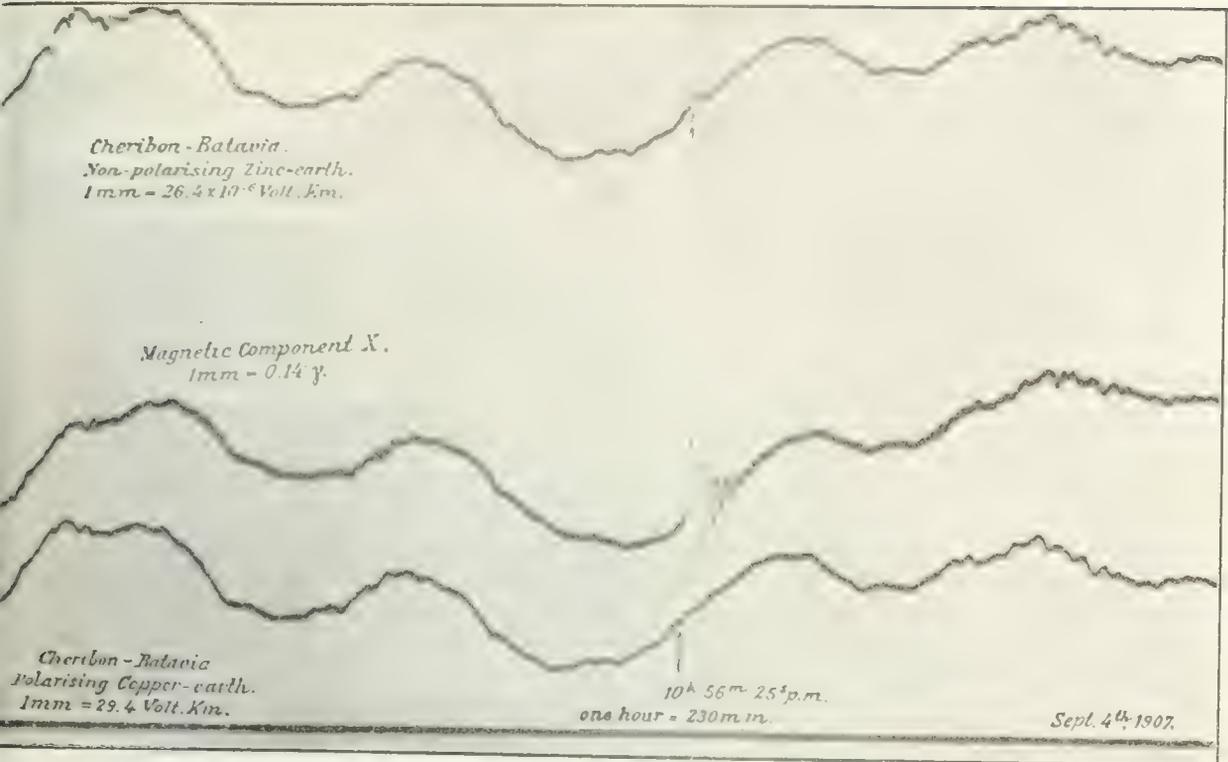
For instance between Anjer and Batavia lies the volcano Karang and therefore the conductivity is probably greater than between Batavia and Cheribon, and the fact that the earth-current is four times as strong can be attributed to it.

The great intensity of the earth current for Buitenzorg—Batavia may be partly attributed to the same reason and moreover to the difference in height (280 M); that between Batavia and Billiton to the well conducting seawater.

For the lines Semarang—Batavia and Soerabaja—Batavia we find however for the ratio of amplitude a distinct difference for vibrations of short duration. Each attempt at explanation of the connection between earth-current and magnetic variation will be in vain as long as this has not been confirmed and expounded.

To explain it out of the loss by isolation is impossible, as the difference would have to appear less for the lines Anjer-Batavia (106 K. M.) and Cheribon-Batavia (200 K. M.) which is not the case.





Neither can it be explained by mutual induction of the two lines, passing partly along the same telegraphpoles, as that influence would just work inversely.

There is a circumstance which causes the lines to Anjer and Cheribon to differ greatly from those to Semarang and Soerabaja; that is the greatest depth below the surface of the earth which the chord reaches between those places and Batavia.

It is for	Batavia—Anjer	1 K. M.
	Batavia—Cheribon	3 ..
	Batavia—Semarang	14 ..
	Batavia—Soerabaja	37 ..

When thus the variations of short duration cause a current chiefly at a greater depth, where the conductivity is very different from than at the surface, a distinct difference might appear. The opposite however is more to be expected.

To conform that difference it will be necessary to register at Semarang the current between Cheribon-Semarang and Soerabaja-Semarang.

If we find for that the same as for Batavia-Cheribon and Batavia-Anjer, then indeed we must attribute the greater increase of amplitude with short duration for the lines Batavia-Semarang and Batavia-Soerabaja to the greater distance.

ERRATUM.

In the Proceedings of the Meeting of March 30, 1907:

p. 770 l. 3 from the bottom: for 46.419 read 46.491.

p. 779 l. 10 from the top: for VII H₂ 1 read VII H₂ 2.

(February 20, 1908).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING
of Saturday February 29, 1908.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige
Afdeling van Zaterdag 29 Februari 1908, Dl. XVI).

C O N T E N T S.

- P. H. SCHOUTE: "Fourdimensional nets and their sections by spaces", 1st Part, p. 536.
Z. P. BOUMAN: "Contribution to the knowledge of the surfaces with constant mean curvature".
(Communicated by Prof. JAN DE VRIES), p. 545.
W. KAPTEYN: "On the multiplication of trigonometrical series", p. 558.
D. H. BRAUNS: "On a crystallised *d.* fructose tetracetate". (Communicated by Prof. A. P. N.
FRANCHIMONT), p. 563.
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the pressure coefficient of helium for the international helium thermometer and the reduction
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the rare earths at the temperatures obtainable with liquid hydrogen, and their change by the
magnetic field, p. 592. (With 5 plates).
H. KAMERLINGH ONNES and W. H. KEESOM: "On the equation of state of a substance in the
neighbourhood of the critical point liquid-gas. I. The disturbance function in the neighbourhood
of the critical state, p. 603. (With one plate) II. Spectrophotometrical investigation of the
opalescence of a substance in the neighbourhood of the critical state", p. 611. (With one plate).
F. M. JAEGER: "On the form-analogy of Halogene-derivatives of Hydrocarbones with open
chains". (Communicated by Prof. A. P. N. FRANCHIMONT), p. 623.
Errata, p. 623.
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Mathematics. — “*Fourdimensional nets and their sections by spaces*”. (First part). By Prof. P. H. SCHOUTE.

(Communicated in the meeting of January 25, 1908).

Out of the table

$$C_5 \dots 75^\circ 31' 21'', \quad C_{16} \dots 120^\circ, \quad C_{120} \dots 144^\circ \\ C_8 \dots 90^\circ, \quad C_{24} \dots 120^\circ, \quad C_{600} \dots 164^\circ 28' 39''$$

of the angles formed by two bounding bodies meeting in a face of the regular cells of space Sp_4 it is immediately evident that only for the cells C_8, C_{16}, C_{24} can there be any question about each respectively filling that space. It is well known, that this is really the case. In the handbook included in the Sammlung SCHUBERT “*Mehrdimensionale Geometrie*” (vol. II, page 241) is indicated how the two nets of the cells C_{16} and C_{24} can be deduced by transformation from the net of cells C_8 , the existence of which is clear in itself. We repeat this here in a somewhat different form to add new considerations to it.

1. The points with the coordinates $(\pm 1, \pm 1, \pm 1, \pm 1)$ are the vertices of an eightcell $C_8^{(2)}$ with double the unit of length as length of edge, the origin of the coordinates as centre and the directions of the axes as directions of the edges. These vertices can be easily arranged in two groups of eight points, one group of which contains the points with a positive product of coordinates, the other group the points with a negative one. Each of these groups has the property that no two of the eight points are united by an edge of $C_8^{(2)}$; therefore we call them groups of non-adjacent vertices. Let us join for each of these groups the two points lying in the same face of $C_8^{(2)}$ by a diagonal, then the systems of edges of two cells $C_{16}^{(2V2)}$ are generated; as each of the bounding cubes of $C_8^{(2)}$ is circumscribed about one of the 16 bounding tetrahedra of each of the two $C_{16}^{(2V2)}$, we call these last inscribed in $C_8^{(2)}$, where one may be called positive, the other negative.

Let us now suppose the net of the C_8 to be composed of alternate white and black eightcells, so that two C_8 with a common bounding body differ in colour — from which it follows, that two C_8 in contact of edges do this too, whilst on the other hand two C_8 in face or in vertex contact bear the same colour —, and let us assume that in each white C_8 is inscribed a positive C_{16} and in each black C_8 a negative one; then it is clear that both groups of C_{16} do not

yet fill the whole space Sp_4 . For to make of a C_8 the inscribed C_{16} , we must truncate from this measure polytope at each of the eight vanishing vertices a fivecell rectangular at this point, of which the four edges passing through this point have a length 2. Because a vertex which vanishes for one of the sixteen cells C_8 , to which it belongs, does this for all, there will remain round this point sixteen alternate white and black rectangular five-cells and these will form together a $C_{16}^{(2V^2)}$ of which this point is the centre. Thus a space-filling for Sp_4 is formed by three equally numerous groups of cells $C_{16}^{(2V^2)}$ with the property that all cells C_{16} of the same group can be made to cover one another by translation.

To show how striking the regularity of the net of the C_{16} is we must suppose three cells $C_{16}^{(2V^2)}$ of which no two belong to the same group, to be removed parallel to themselves to a common centre, the origin of coordinates. We then see immediately that the vertices of the three $C_{16}^{(2V^2)}$ together form the vertices of a $C_{24}^{(2)}$. For the two inscribed cells $C_{16}^{(2V^2)}$ together again furnish the vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$ of the original eightcell $C_8^{(2)}$ and the coordinates of the vertices of the third cell $C_{16}^{(2V^2)}$ are

$$(\pm 2, 0, 0, 0), (0, \pm 2, 0, 0), (0, 0, \pm 2, 0), (0, 0, 0, \pm 2),$$

from which is evident what was assumed (compare "*Mehrdimensionale Geometrie*", vol II, p. 205).

We shall presently use this observation to trace the connection between the four groups of axes of the three systems of cells C_{16} with the groups of axes of C_8 .

2. To transform the net of the cells C_8 into a net of cells C_{16} , we must again suppose the cells of the former alternately coloured white and black in order to break up each of the black cells into eight congruent pyramids with the centre of the eightcell as common vertex and the eight bounding cubes as bases. By adding to each white eightcell the eight black pyramids having a bounding cube in common with it, the net of the cells $C_{24}^{(2)}$ is generated; in reality to the sixteen vertices of the eightcell supposed to be white with the origin of coordinates as centre, viz. to the points $(\pm 1, \pm 1, \pm 1, \pm 1)$ the eight vertices mentioned above

$$(\pm 2, 0, 0, 0), (0, \pm 2, 0, 0), (0, 0, \pm 2, 0), (0, 0, 0, \pm 2)$$

are added.

The transformation of the net of the $C_8^{(2)}$ into that of C_{16} can also take place in the following simple way. Divide each of the cells $C_8^{(2)}$

into 16 equal and similarly placed cells $C_8^{(1)}$ by means of four spaces through the centre O parallel to the pairs of bounding spaces. Then divide each of the sixteen parts $C_8^{(1)}$ (fig. 1) by the space in the midpoint of the diagonal concurring in the centre O of $C_8^{(2)}$ normal to this line into two equal halves; here the section as is known is a regular octahedron $A_{12} A_{13} \dots A_{34}$. We now direct our attention first to the half cells $C_8^{(1)}$ surrounding the point O ; they form a $C_{24}^{(V2)}$. Of the 24 bounding octahedra sixteen are furnished by the sections $A_{12} A_{13} \dots A_{34}$, whilst the eight remaining ones are obtained by joining

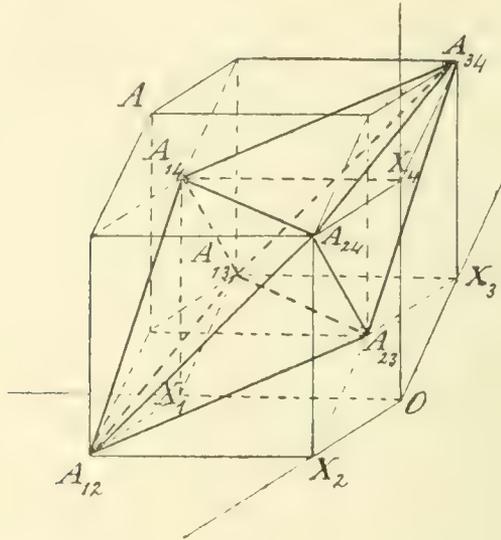


Fig. 1.

in each of the eight ends of the chords along the four axes OX_1 , OX_2 , OX_3 , OX_4 through O , e. g. in X_1 , the eight rectangular tetrahedra $X_1(A_{12} A_{13} A_{14})$, where it is clear that in X_1 eight of those tetrahedra really meet, because we can reverse the direction of each of the segments $X_1 A_{12}$, $X_1 A_{13}$, $X_1 A_{14}$. Furthermore we observe that around an arbitrary vertex A of the original cell also 16 half cells $C_8^{(1)}$ are lying and that these form in exactly the same way a $C_{24}^{(V2)}$. By this the net of the $C_8^{(2)}$ has been transformed into a net of cells $C_{24}^{(V2)}$, where the centres and the vertices of the cells $C_8^{(2)}$ form the centres of the cells $C_{24}^{(V2)}$ placed in the same way.

If we add to the considered sixteenth part $C_8^{(1)}$ (fig. 1) the three parts generated by reversing the sign of one of the two axes OX_1 and OX_2 or of both, it is immediately evident that A_{34} is the centre of a face of the original cell $C_8^{(2)}$. From this is evident to the eye

the truth of the wellknown theorem, that the centres of the faces of a $C_8^{(2)}$ — and therefore also the centres of the edges of each of the two inscribed cells $C_{16}^{(2V^2)}$ — are the vertices of a $C_{24}^{(V^2)}$.

3. Before examining more closely the nets of the cells C_8, C_{16}, C_{24} — or, as we shall express ourselves, the nets $(C_8), (C_{16}), (C_{24})$ — in their mutual connection we put to ourselves the question whether it is possible to fill Sp_4 entirely with *different* regular cells. Here the table given above points to two possibilities. We can either complete the sum of the angles $75^\circ 31' 21''$ and $164^\circ 28' 39''$ with 120° to 360° or by combination of one of the two cells C_{16}, C_{24} with twice the other arrive at 360° . The latter is however already excluded by the fact that C_{16} and C_{24} differ in bounding bodies, which obstacle does not occur when one tries to arrange the three cells C_8, C_{16}, C_{600} with the same length of edges around a face. Yet, though this is possible, neither in this way does one arrive at the object in view. If the indicated space-filling had taken place then two bounding tetrahedra of C_8 , having always a face in common, would have to differ from each other in this, that one would at the same time have to belong to a C_{16} and the other to a C_{600} and this is impossible. For one cannot colour the bounding tetrahedra of a C_8 alternately white and black for the mere reason, that the number five of those tetrahedra is odd. So there is no space-filling of Sp_4 where *different* regular cells appear.

4. We shall now consider more closely the systems of points formed by the centres of the regular cells of the nets $(C_8), (C_{16}), (C_{24})$ which we shall indicate by the symbols $(P_8), (P_{16}), (P_{24})$.

Of the systems of points $(P_8), (P_{16}), (P_{24})$, which we might call fourdimensional "assemblages of BRAVAIS", (P_8) is the simplest. If the axes of coordinates are assumed through the centre of a definite cell $C_8^{(2)}$ parallel to the edges of this cell, then (P_8) is the system of the points $(2a_1, 2a_2, 2a_3, 2a_4)$ with only even integer coordinates which we indicate by means of abbreviated symbols by the equation $(P_8) = (2a_i)$.

Of the two other systems of points, (P_{24}) can be most simply expressed in P_8 . Out of the second mode of transformation of the cells $C_8^{(2)}$ into the cells $C_{24}^{(V^2)}$ it was clear to us that (P_{24}) is found by joining the system (P_8) to the system of the vertices of the cells $C_8^{(2)}$. Now this system of the vertices can be deduced out of (P_8) by a translation indicated in direction and magnitude by the line-segment connecting the centre of the eightcell, which served to determine the

system of coordinates, with one of the vertices; thus this system of vertices is indicated in the same symbols by $(2a_i + 1)$ and we find $(P_{24}) = (2a_i) + (2a_i + 1)$, i. e. (P_{24}) is the system of the points with integer coordinates which are either all even or all odd.

Finally (P_{16}) is derived from (P_{24}) by adding to (P_8) not the whole system of the vertices of the cells $C_8^{(2)}$, but only that half which is not occupied by the vertices of the inscribed $C_{16}^{(2V2)}$. We express this by means of the equation $P_{16} = (2a_i) + \frac{1}{2}(2a_i + 1)$.

Here we have to understand by $\frac{1}{2}(2a_i + 1)$ that system of points of which the coordinates are only odd integer numbers under the condition that half the sum is either always even or always odd. If in the cell $C_8^{(2)}$ which furnished us above with the system of coordinates a positive $C_{16}^{(2V2)}$ is inscribed, which for the future we shall always suppose, then the point $(1, 1, 1, 1)$ is occupied by a vertex of the inscribed $C_{16}^{(2V2)}$ and so for the non-occupied vertices $\frac{1}{2}(2a_i + 1)$ half the sum of the four quantities a_i is odd.

If we make the connection between the systems of points (P_8) , (P_{16}) , (P_{24}) in the indicated way, then the number of points of (P_{24}) is twice, and the number of points (P_{16}) is one and a half times as large as that of (P_8) and so the fourdimensional volumes of $C_8^{(2)}$, $C_{16}^{(2V2)}$, $C_{24}^{(42)}$ have to be in the same ratio as the numbers $1, \frac{2}{3}, \frac{1}{2}$. This can be easily verified. To make a $C_{16}^{(2V2)}$ of $C_8^{(2)}$ we have truncated at eight vertices a rectangular fivecell, which is $\frac{1}{24}$ of $C_8^{(2)}$; so $\frac{2}{3}$ of $C_8^{(2)}$ remains. And to make of $C_8^{(2)}$ the cell $C_{24}^{(42)}$ contained in the former we have halved each of the sixteen parts $C_8^{(1)}$.

5. By the "transformation-view" of each of the nets (C_8) , (C_{16}) and (C_{24}) with respect to a space Sp_3 of the bearing space Sp_4 as screen we understand the intersection varying every moment, of this non-moving space with the fourdimensional net moving along in the direction normal to this space. If for this movement we interchange the relative and the absolute, we can also take this transformation-view to be generated by the intersection of the non-moving fourdimensional net with a space Sp_3 , moving along in a perpendicular direction and remaining parallel to itself; there we can again assume that this view is observed by one who shares the movement of the space

Sp_3 . The chief aim of this communication is to indicate how we can connect the transformation-views of the nets (C_{16}) , (C_{24}) with that of the net (C_8) , which is by far the simplest. Because the three views furnish at every moment a filling of the intersecting space, this investigation can lead to new threedimensional space-fillings, even though they be not entirely regular.

To be able to design a transformation-view of the net (C_{16}) we must know for each of the component cells C_{16} the place *of* the centre and the position *about* the centre; as the coordinates of the centres of the cells are given above, we have only to occupy ourselves further with the position about the centre. We designate that position by means of the four diagonals of each C_{16} and we then notice that these four lines for each of the two kinds of inscribed cells C_{16} are also diagonals — groups of non-adjacent diagonals — of the circumscribed cells C_8 , whilst for the cells C_{16} of the third group they are parallel to the axes of coordinates.

If we suppose the centre of a cell $C_{16}^{(2V^2)}$ of the third group to be at the same time the centre of a cell $C_8^{(4)}$, the edges of which are parallel to the axes of coordinates, the $C_{16}^{(2V^2)}$ is inscribed in this new eightcell in such a sense, that the vertices of $C_{16}^{(2V^2)}$ are the centres of the eight bounding cubes of $C_8^{(4)}$. For an obvious reason we call this $C_{16}^{(2V^2)}$ *polarly* inscribed in $C_8^{(4)}$ — and now to distinguish, we call the cells of the two other groups *bodily* inscribed in the cells $C_8^{(2)}$. For, as was observed above, in each of the eight bounding cubes of $C_8^{(2)}$ a bounding tetrahedron of $C_{16}^{(2V^2)}$ is inscribed, whilst each of the remaining eight bounding tetrahedra of $C_{16}^{(2V^2)}$ has with respect to each of the four pairs of opposite bounding cubes of $C_8^{(2)}$ three vertices of one and one vertex of the other cube as vertices.

In this way each of the cells $C_{16}^{(2V^2)}$ of the net (C_{16}) is packed up in a C_8 as small as possible, of which the edges are parallel to the axes of coordinates; here the fourdimensional *cases* of the “erect” cells C_{16} of the third group are cells $C_8^{(4)}$, those of the “inclining” cells C_{16} of the first and the second group are cells $C_8^{(2)}$. Whilst the cases $C_8^{(2)}$ of the inclining cells C_{16} fill the space Sp_3 , the cases $C_8^{(4)}$ of the erect cells C_{16} do so eight times, because $C_{16}^{(2V^2)}$ is the $\frac{1}{24}$ part of $C_8^{(4)}$, — as is immediately evident when one divides the erect $C_{16}^{(2V^2)}$ and its case $C_8^{(4)}$ by spaces through the common centre parallel to the pairs of bounding spaces of $C_8^{(4)}$ into sixteen equal parts

and when one compares the rectangular fivecell of $C_{16}^{(2V^2)}$ to the $C_8^{(2)}$ of $C_8^{(4)}$ —, and the erect C_{16} together fill a third of Sp_4 .

In the second mode of transformation of the cells $C_8^{(2)}$ of the net (C_8) into the cells $C_{24}^{(V^2)}$ of a net (C_{24}) the vertices of the $C_{24}^{(V^2)}$ concentric to $C_8^{(2)}$ are the centres of the faces of these $C_8^{(2)}$, from which it follows that the six centres of the faces of each of the eight bounding cubes of $C_8^{(2)}$ are vertices of a bounding octahedron of $C_{24}^{(V^2)}$ and so this cell may again be called inscribed — and *bodily* inscribed too — in $C_8^{(2)}$. Also the remaining bounding octahedra can be directly indicated with respect to these circumscribed $C_8^{(2)}$; through each of the sixteen vertices of $C_8^{(2)}$ pass six faces of this cell, of which the centres form the vertices of a bounding octahedron of $C_{24}^{(V^2)}$.¹⁾

From the preceding it follows, that the fourdimensional cases, inclosing the cells $C_{24}^{(V^2)}$ and having edges parallel to the axes of coordinates, consist of two nets (C_8) of cells $C_8^{(2)}$, which by exchange of centres and vertices pass into each other.

6. We conclude this first part by indicating the connection existing between the systems of axes of the five different cells with the origin of coordinates as common centre, which can be obtained by parallel translation of one of the cells $C_8^{(2)}$, one of each of the three groups of cells $C_{16}^{(2V^2)}$ and one of the cells $C_{24}^{(V^2)}$. We indicate these cells for brevity by C_8 , C_{16} , C'_{16} , C''_{16} , C_{24} where C_{16} represents the polarly inscribed sixteencell and C'_{16} and C''_{16} successively the positive and the negative *bodily* inscribed one. Further here too — according to the notation of the handbook mentioned above — E , K , F , R will denote a vertex, midpoint of edge, centre of face, centre of bounding body and therefore OE , OK , OF , OR will have to denote the axes converging in these points. Thus OE_8 is an axis OE of C_8 , OK_{16} an axis OK of C_{16} , OF'_{16} an axis OF' of C'_{16} , etc.

The numbers of axes OE , OK , OF , OR of each of the three different cells are always the halves of the numbers of the elements E , K , F , R ; they are contained in the following table.

Here C_{16} of course represents the three cells C_{16} , C'_{16} , C''_{16} .

We now indicate the connection of the systems of axes of the

¹⁾ By doubling the radii vectores of the six centres of the faces from the chosen vertex of these $C_8^{(2)}$ we find the central section normal to the diagonal of this point.

	<i>OE</i>	<i>OK</i>	<i>OF</i>	<i>OR</i>
C_8	8	16	12	4
C_{16}	4	12	16	8
C_{24}	2	48	48	12

five cells C_8 , C_{16} , C'_{16} , C''_{16} , C_{24} by giving the coordinates of the points E , K , F , R belonging to these concentric cells with respect to two systems of axes of coordinates with the common centre of the cells as origin, the systems (OX_i) of the four axes OR_8 and the system (OY_i) of the four axes OE'_{16} (fig. 2) between which the relations

$$\left. \begin{aligned} 2y_1 &= x_1 + x_2 + x_3 + x_4 \\ 2y_2 &= x_1 - x_2 - x_3 + x_4 \\ 2y_3 &= -x_1 + x_2 - x_3 + x_4 \\ 2y_4 &= -x_1 - x_2 + x_3 + x_4 \end{aligned} \right\}$$

exist. ¹⁾

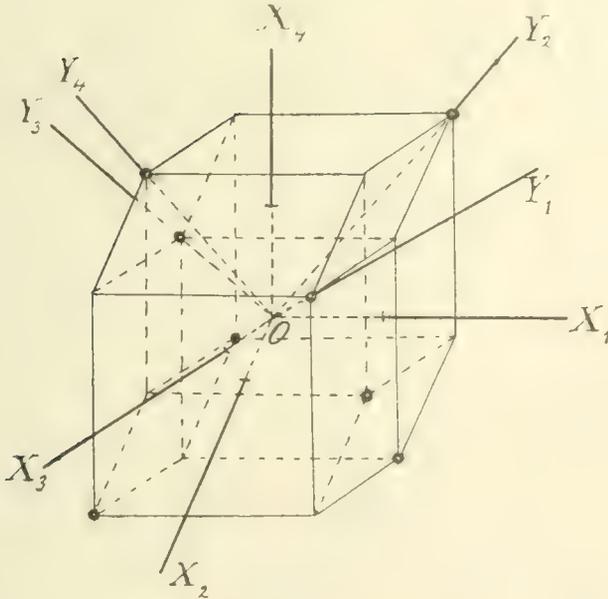


Fig. 2.

¹⁾ We selected this transformation T , because it causes the octuples of vertices of C_{16} and C'_{16} to pass into each other and those of C_{16} into itself. It satisfies the condition $T^4 = -1$, so that first T^8 gives unity. We find that T^2 is a rectangular double-rotation round O by which (x_1, x_4) passes into $(-x_4, x_1)$ and (x_2, x_3) into $(-x_3, x_2)$.

We shall now give in both systems of coordinates the coordinates of the vertices of the five concentric cells and we divide in doing so — see the following table — the sixteen vertices of $C_8^{(2)}$ into the eight vertices of C'_{16} and the eight vertices of C''_{16} ; to that end it is necessary for distinction to indicate whether the product of the coordinates is positive or negative.

Cells	Number of vertices	Coordinates (OXi)	Product	Coordinates (OYi)	Product
C_8 and C'_{16}	8	$(\pm 1, \pm 1, \pm 1, \pm 1)$	+	$(\pm 2, 0, 0, 0)$	
C_8 and C''_{16}	8	$(\pm 1, \pm 1, \pm 1, \pm 1)$	-	$(\pm 1, \pm 1, \pm 1, \pm 1)$	-
C_{16}	8	$(\pm 2, 0, 0, 0)$		$(\pm 1, \pm 1, \pm 1, \pm 1)$	+
C_{24}	24	$(\pm 1, \pm 1, 0, 0)$		$(\pm 1, \pm 1, 0, 0)$	

With the aid of this it is easy to find both quadruples of coordinates of the systems of the points K, F, R of the five cells. They are given in the following table, which after all the preceding is clear in itself.

Cells					Number of axes	Coordinates (OXi)	Product	Coordinates (OYi)	Product
C_8	C_{16}	C'_{16}	C''_{16}	C_{24}					
E	$2R$	E	$\frac{4}{3}R$	$2R$	4	$(\pm 1, \pm 1, \pm 1, \pm 1)$	+	$(2, 0, 0, 0)$	
E	$2R$	$\frac{4}{3}R$	E	$2R$	4	$(\pm 1, \pm 1, \pm 1, \pm 1)$	-	$(\pm 1, \pm 1, \pm 1, \pm 1)$	-
K	$\frac{3}{2}F$	-	-	$\frac{3}{2}F$	16	$(\pm 1, \pm 1, \pm 1, 0)$		$(\pm \frac{3}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$	-
F	K	K	K	E	12	$(\pm 1, \pm 1, 0, 0)$		$(\pm 1, \pm 1, 0, 0)$	
R	$\frac{1}{2}E$	R	R	R	4	$(\pm 1, 0, 0, 0)$		$(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$	+
-	-	F	-	F	16	$(\pm 1, \pm \frac{1}{3}, \pm \frac{1}{3}, \pm \frac{1}{3})$	-	$(\pm \frac{2}{3}, \pm \frac{2}{3}, \pm \frac{2}{3}, 0)$	
-	-	-	F	F	16	$(\pm 1, \pm \frac{1}{3}, \pm \frac{1}{3}, \pm \frac{1}{3})$	+	$(\pm 1, \pm \frac{1}{3}, \pm \frac{1}{3}, \pm \frac{1}{3})$	+
-	-	-	-	K	48	$(\pm 1, \pm \frac{1}{2}, \pm \frac{1}{2}, 0)$		$(\pm 1, \pm \frac{1}{2}, \pm \frac{1}{2}, 0)$	

Of course the axes, of which the number is given each time, agree in nature with the points connected by them with O . So the

four axes given in the first row are axes OE for C_8 and C'_{16} , axes OR for C_{16} , C''_{16} and C_{24} ; moreover the coefficients $2, \frac{4}{3}, 2$ of $2R, \frac{4}{3}R, 2R$ indicate that the quadruples of coordinates appearing in this row relate to the point which is obtained by multiplying the observed axis OR of C_{16}, C''_{16}, C_{24} as far as the length from O goes by $2, \frac{4}{3}, 2$.

With the preceding we have pointed out the position of each axis of one of the cells of the three nets (C_8), (C_{16}), (C_{24}) with reference to each of the two systems of coordinates and so we have furnished in connection with the preceding the material by which it is possible to deduce easily all the spacial sections of these three regular nets connected in a simple way with these axes. To give an example here already we observe that a space normal to one of the twelve axes OF_8 is normal to an axis OK for all the cells of the net (C_{16}); if it now proves possible to determine such a space in such a way that it is equally distant from the centres of all the cells C_{16} which are intersected, then in the intersecting space a more or less regular space-filling is generated by a selfsame body in three different positions.

In a future part we hope to commence with the determination of the remarkable spacial sections of the nets (C_8), (C_{16}), (C_{24}).

Mathematics. — “*Contribution to the knowledge of the surfaces with constant mean curvature*”. By Dr. Z. P. BOUMAN. (Communicated by Prof. JAN DE VRIES).

(Communicated in the meeting of January 25, 1908).

§ 1. As is known the great difficulty connected with the study of the surfaces with constant mean curvature is the integration of the differential equation

$$\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} = - \sinh \theta \cdot \cosh \theta.$$

The course followed here leads to two simultaneous partial differential equations of order one and of degree two.

In Gauss' symbols the value of the mean curvature H of a surface is indicated by

$$H = \frac{2FD' - ED'' - GD}{EG - F^2}.$$

As independent coordinates on the surface we choose those which are invariable along the lines with length zero and we represent them by ξ and η . So we find,

$$H = -2 \frac{D'}{F}, \text{ whilst } E = G = 0.$$

Let us multiply both members of the first equation by X (cosine of the angle of the normal with the X -axis); we then find:

$$FHX = -2D'X.$$

But

$$D'X = \frac{d^2x^2}{\partial \xi \partial \eta}$$

and moreover²⁾:

$$FX = \frac{1}{i} \frac{\frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \xi}}{\left| \begin{matrix} \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{matrix} \right|} = \frac{1}{i} \begin{pmatrix} y & z \\ \xi & \eta \end{pmatrix},$$

where x, y, z represent the Cartesian coordinates of the surface with respect to a rectangular system of axes.

So we find

$$H \begin{pmatrix} y & z \\ \xi & \eta \end{pmatrix} = -2 \frac{\partial^2 x}{\partial \xi \partial \eta},$$

or:

$$\frac{\partial^2 x}{\partial \xi \partial \eta} = -2i \begin{pmatrix} y & z \\ \xi & \eta \end{pmatrix}$$

and likewise:

$$\frac{\partial^2 y}{\partial \xi \partial \eta} = -2i \begin{pmatrix} z & x \\ \xi & \eta \end{pmatrix} \dots \dots \dots (I)$$

$$\frac{\partial^2 z}{\partial \xi \partial \eta} = -2i \begin{pmatrix} x & y \\ \xi & \eta \end{pmatrix}$$

Moreover x, y and z must satisfy

$$E = G = 0,$$

therefore

$$\begin{aligned} \Sigma \left(\frac{\partial x}{\partial \xi} \right)^2 &= 0 \\ \Sigma \left(\frac{\partial x}{\partial \eta} \right)^2 &= 0 \end{aligned} \dots \dots \dots (II)$$

¹⁾ BIANCHI, Vorlesungen über Differential-Geometrie, translation into German by MAX LUKAT, page 89.

²⁾ l. c. page 86.

The equations (I) and (II) give back for $H=0$ the problem of the minimal surfaces.

For $-\frac{H}{2i}$ we shall introduce for brevity the symbol Q .

§ 2. To satisfy beforehand (II) we put

$$\left. \begin{aligned} \frac{\partial x}{\partial \xi} + i \frac{\partial y}{\partial \xi} &= u \frac{\partial z}{\partial \xi}, & \frac{\partial x}{\partial \eta} + i \frac{\partial y}{\partial \eta} &= v \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \xi} - i \frac{\partial y}{\partial \xi} &= -\frac{1}{u} \frac{\partial z}{\partial \xi}, & \frac{\partial x}{\partial \eta} - i \frac{\partial y}{\partial \eta} &= -\frac{1}{v} \frac{\partial z}{\partial \eta} \end{aligned} \right\} \dots (III)$$

where u and v are functions to be determined of ξ and η .

When we substitute the equations (III) into (I), we find the equations which u and v must satisfy, whilst moreover $\frac{\partial x}{\partial \xi}$, $\frac{\partial x}{\partial \eta}$, $\frac{\partial y}{\partial \xi}$ and $\frac{\partial y}{\partial \eta}$, derived from (III) must obey the conditions of integrability.

The latter furnish

$$\frac{\partial u \frac{\partial z}{\partial \xi}}{\partial \eta} = \frac{\partial v \frac{\partial z}{\partial \eta}}{\partial \xi},$$

and

$$\frac{\partial \frac{1}{u} \frac{\partial z}{\partial \xi}}{\partial \eta} = \frac{\partial \frac{1}{v} \frac{\partial z}{\partial \eta}}{\partial \xi},$$

which is clear.

Writing out we find

$$\left. \begin{aligned} (a). \quad \frac{\partial u}{\partial \eta} \cdot \frac{\partial z}{\partial \xi} + u \frac{\partial^2 z}{\partial \xi \cdot \partial \eta} &= \frac{\partial v}{\partial \xi} \cdot \frac{\partial z}{\partial \eta} + v \frac{\partial^2 z}{\partial \xi \cdot \partial \eta} \\ (b). \quad \frac{1}{u^2} \frac{\partial u}{\partial \eta} \cdot \frac{\partial z}{\partial \xi} - \frac{1}{u} \frac{\partial^2 z}{\partial \xi \cdot \partial \eta} &= \frac{1}{v^2} \frac{\partial v}{\partial \xi} \cdot \frac{\partial z}{\partial \eta} - \frac{1}{v} \frac{\partial^2 z}{\partial \xi \cdot \partial \eta} \end{aligned} \right\} \dots (IV)$$

If we now also substitute the values of $\frac{\partial x}{\partial \xi}$, $\frac{\partial x}{\partial \eta}$, $\frac{\partial y}{\partial \xi}$ and $\frac{\partial y}{\partial \eta}$ into the equations (I) whilst we put $Q = -\frac{H}{2i}$ we find:

$$\begin{aligned} \left(\frac{\partial u}{\partial \eta} + \frac{1}{u^2} \frac{\partial u}{\partial \eta} \right) \frac{\partial z}{\partial \xi} + \left(u - \frac{1}{u} \right) \frac{\partial^2 z}{\partial \xi \partial \eta} &= Q \left(u + \frac{1}{u} - v - \frac{1}{v} \right) \frac{\partial z}{\partial \xi} \cdot \frac{\partial z}{\partial \eta}, \\ \frac{1}{i} \left(\frac{\partial u}{\partial \eta} - \frac{1}{u^2} \frac{\partial u}{\partial \eta} \right) \frac{\partial z}{\partial \xi} + \frac{1}{i} \left(u + \frac{1}{u} \right) \frac{\partial^2 z}{\partial \xi \partial \eta} &= Q \left(v - \frac{1}{v} - u + \frac{1}{u} \right) \frac{\partial z}{\partial \xi} \cdot \frac{\partial z}{\partial \eta}, \\ \frac{\partial^2 z}{\partial \xi \partial \eta} &= \frac{Q}{2i} \left(\frac{u}{v} - \frac{v}{u} \right) \frac{\partial z}{\partial \xi} \cdot \frac{\partial z}{\partial \eta}. \end{aligned}$$

From these three last equations we derive directly with the aid of (IV):

$$\begin{aligned}
 (a). \quad & \frac{\partial u}{\partial \eta} \cdot \frac{\partial z}{\partial \xi} + u \frac{\partial^2 z}{\partial \xi \partial \eta} = Qi (v - u) \frac{\partial z}{\partial \xi} \cdot \frac{\partial z}{\partial \eta} \\
 (b). \quad & \frac{1}{u^2} \frac{\partial u}{\partial \eta} \cdot \frac{\partial z}{\partial \xi} - \frac{1}{u} \frac{\partial^2 z}{\partial \xi \partial \eta} = Qi \left(\frac{1}{v} - \frac{1}{u} \right) \frac{\partial z}{\partial \xi} \cdot \frac{\partial z}{\partial \eta} \\
 \text{whilst} \quad & \frac{\partial^2 z}{\partial \xi \partial \eta} = \frac{Q}{2i} \left(\frac{u}{v} - \frac{v}{u} \right) \frac{\partial z}{\partial \xi} \cdot \frac{\partial z}{\partial \eta}
 \end{aligned} \quad (V)$$

We can easily show that one of the equations (V) is dependent on the two others, as is clear.

If we divide both members of (V,a) by u^2 and if we add (V,b), we find:

$$\frac{\partial z}{\partial \eta} = \frac{2v}{Qi(v-u)^2} \cdot \frac{\partial u}{\partial \eta}.$$

From (IV,a) follows:

$$\frac{\partial u}{\partial \eta} \cdot \frac{\partial z}{\partial \xi} - \frac{\partial v}{\partial \xi} \cdot \frac{\partial z}{\partial \eta} = (v-u) \frac{\partial^2 z}{\partial \xi \partial \eta} = (v-u) \left(\frac{u^2 - v^2}{uv} \right) \frac{Q}{2i} \frac{\partial z}{\partial \xi} \cdot \frac{\partial z}{\partial \eta}.$$

By substituting here $\frac{\partial z}{\partial \eta}$ we find:

$$\frac{\partial z}{\partial \xi} = - \frac{2u}{Qi(v-u)^2} \cdot \frac{\partial v}{\partial \xi}.$$

We can now write down out of (III) the following set of equations:

$$\begin{aligned}
 \frac{\partial x}{\partial \xi} &= \frac{1}{2} \left(u - \frac{1}{u} \right) \cdot \frac{\partial z}{\partial \xi} = \frac{-(u^2 - 1)}{Qi(v-u)^2} \cdot \frac{\partial v}{\partial \xi} \\
 \frac{\partial x}{\partial \eta} &= \frac{1}{2} \left(v - \frac{1}{v} \right) \cdot \frac{\partial z}{\partial \eta} = \frac{v^2 - 1}{Qi(v-u)^2} \cdot \frac{\partial u}{\partial \eta} \\
 \frac{\partial y}{\partial \xi} &= \frac{1}{2i} \left(u + \frac{1}{u} \right) \cdot \frac{\partial z}{\partial \xi} = \frac{u^2 + 1}{Q(v-u)^2} \cdot \frac{\partial v}{\partial \xi} \\
 \frac{\partial y}{\partial \eta} &= \frac{1}{2i} \left(v + \frac{1}{v} \right) \cdot \frac{\partial z}{\partial \eta} = \frac{-(v^2 + 1)}{Q(v-u)^2} \cdot \frac{\partial u}{\partial \eta} \\
 \frac{\partial z}{\partial \xi} &= \frac{-2u}{Qi(v-u)^2} \cdot \frac{\partial v}{\partial \xi} \\
 \frac{\partial z}{\partial \eta} &= \frac{2v}{Qi(v-u)^2} \cdot \frac{\partial u}{\partial \eta}
 \end{aligned} \quad \dots \quad (VI)$$

So, as soon as u and v are known, the problem will be solved.

§ 3. In order now to write down the equations which u and v must satisfy, we can make use of (IV) and (VI) , or we can use the conditions of integrability.

(IV, a) gives :

$$\frac{\partial u}{\partial \eta} \cdot \frac{\partial z}{\partial \xi} = -\frac{2u}{iQ(v-u)^2} \cdot \frac{\partial v}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} + \frac{2v}{iQ} \left(\frac{2}{(v-u)^2} \cdot \frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} + \frac{1}{v-u} \cdot \frac{\partial^2 u}{\partial \xi \partial \eta} \right).$$

(IV, b) gives :

$$\frac{\partial u}{\partial \eta} \cdot \frac{\partial z}{\partial \xi} = -\frac{2u}{iQ(v-u)^2} \cdot \frac{\partial v}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} + \frac{2u}{iQ} \left(\frac{2}{(v-u)^2} \cdot \frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} + \frac{1}{v-u} \cdot \frac{\partial^2 u}{\partial \xi \partial \eta} \right).$$

Out of (VI) we find :

$$\frac{\partial}{\partial \xi} \left(\frac{\partial z}{\partial \eta} \right) = -\frac{2}{iQ} \frac{v+u}{(v-u)^2} \cdot \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi} - \frac{2v}{iQ(v-u)} \left(\frac{2}{(v-u)^2} \cdot \frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} + \frac{1}{v-u} \cdot \frac{\partial^2 u}{\partial \xi \partial \eta} \right),$$

$$\frac{\partial}{\partial \eta} \left(\frac{\partial z}{\partial \xi} \right) = -\frac{2}{iQ} \frac{v+u}{(v-u)^2} \cdot \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi} + \frac{2u}{iQ(v-u)} \left(\frac{2}{(v-u)^2} \cdot \frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta} - \frac{1}{v-u} \cdot \frac{\partial^2 v}{\partial \xi \partial \eta} \right),$$

and

$$\frac{\partial^2 z}{\partial \xi \partial \eta} = \frac{Q}{2i} \left(\frac{u}{v} - \frac{v}{u} \right) \cdot \frac{\partial z}{\partial \xi} \cdot \frac{\partial z}{\partial \eta} \text{ gives } \frac{\partial^2 z}{\partial \xi \partial \eta} = -\frac{2}{iQ} \frac{v+u}{(v-u)^3} \cdot \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi}.$$

The equations given above show that all the conditions of the problem can be satisfied in the only way by putting :

$$\frac{2}{(v-u)^2} \cdot \frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} - \frac{1}{(v-u)} \cdot \frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \text{ and } -\frac{2}{(v-u)^2} \cdot \frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta} + \frac{1}{(v-u)} \cdot \frac{\partial^2 v}{\partial \xi \partial \eta} = 0,$$

which equations we write in the form :

$$\left. \begin{aligned} 2 \frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} + (v-u) \frac{\partial^2 u}{\partial \xi \partial \eta} &= 0 \\ 2 \frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta} - (v-u) \frac{\partial^2 v}{\partial \xi \partial \eta} &= 0 \end{aligned} \right\} \dots \dots (VII)$$

So the problem is entirely reduced to the integration of these two simultaneous differential equations which are of order two and non-linear.

It is easy to deduce from (VII) , that the conditions

$$\frac{\partial}{\partial \eta} \left(\frac{\partial x}{\partial \xi} \right) = \frac{\partial}{\partial \xi} \left(\frac{\partial x}{\partial \eta} \right) \text{ and } \frac{\partial}{\partial \eta} \left(\frac{\partial y}{\partial \xi} \right) = \frac{\partial}{\partial \xi} \left(\frac{\partial y}{\partial \eta} \right),$$

are satisfied.

We find namely always :

$$\frac{\partial^2 x}{\partial \xi \partial \eta} = -\frac{2(uv-1)}{Q(v-u)^3} \cdot \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi}, \quad \frac{\partial^2 y}{\partial \xi \partial \eta} = \frac{2(uv+1)}{Q(v-u)^3} \cdot \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi},$$

whilst

$$\frac{\partial^2 z}{\partial \xi \partial \eta} = - \frac{2(v+u)}{iQ(v-u)^2} \cdot \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi}$$

After substitution we get :

$$D = -\frac{H}{2} F \text{ and } X^2 + Y^2 + Z^2 = 1,$$

so that really all the conditions of the problem prove to be satisfied by the equations (VII). Thus only the solution of (VII) is left to be found.

§ 4. We already know, that for the coordinates ξ and η

$$D = -\frac{H}{2} \cdot F$$

must be satisfied.

But moreover follows from the equations of CODAZZI¹⁾:

$$\frac{\partial D}{\partial \eta} = 0 \quad \text{and} \quad \frac{\partial D'}{\partial \xi} = 0$$

So

$$D = f_1(\xi) \text{ and } D' = f_2(\eta), \quad . \quad . \quad . \quad (VIII)$$

where f_1 and f_2 are respectively functions of ξ and η only.

The case that either D or D' is equal to zero offers no difficulties, but nothing remarkable either.

The case that D and D' are both equal to zero, leads, as is immediately clear, to the sphere as the simplest form of a surface with constant mean curvature. We can namely write down the condition for umbilical points, which is as follows with the omission of infinitesimals of higher order:²⁾

$$\frac{E}{D} = \frac{F}{D'} = \frac{G}{D''}$$

When for each point of the surface $E = G = 0$ then each point is an umbilical point, as soon as always $D = D' = 0$; and these surfaces are (in as far as it concerns the real solution) spheres only.

§ 5. We shall now take the matter a little more generally.

Let us regard the total curvature of a surface as a simultaneous differential-invariant of both groundforms, we then find³⁾:

¹⁾ BIANCHI, l. c. p. 91. In using the coordinates ξ and η the CHRISTOFFEL symbols are all zero, except $\begin{Bmatrix} 1 & 1 \\ & 1 \end{Bmatrix}$ and $\begin{Bmatrix} 2 & 2 \\ & 2 \end{Bmatrix}$. By making use of $D = -\frac{H}{2} F$, we prove what was said in the text.

²⁾ See e. g. V. and K. KOMMERELL, Allgemeine Theorie der Raumkurven und Flächen, II, p. 21.

³⁾ BIANCHI, l. c. p. 68.

$$\begin{aligned} \text{Total curvature} &= \frac{DD'' - D'^2}{EG - F^2} = \frac{H^2}{4} - \frac{f_1(\xi)f_2(\eta)}{F^2} = \\ &= \frac{1}{2iF} \frac{\partial}{\partial \eta} \left(\frac{2}{iF} \frac{\partial F}{\partial \xi} \right) = -\frac{1}{F} \frac{\partial^2(LF)}{\partial \xi \partial \eta} \quad (IX) \end{aligned}$$

(We notice moreover that, as is directly to be seen,

$$\frac{2}{F} = \frac{1}{r_1} - \frac{1}{r_2},$$

where r_1 and r_2 are the principal radii of curvature).

Let us now deduce from (VI) the value of F , we then find:

$$F = -\frac{2}{Q^2(v-u)^2} \cdot \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi},$$

or:

$$F = \frac{8}{H^2(v-u)^2} \cdot \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi}.$$

We substitute this value of F into (IX) by means of the following calculations. Out of (VII) follows

$$\begin{aligned} \frac{1}{F} \frac{\partial F}{\partial \eta} &= \frac{\frac{\partial^2 u}{\partial \eta^2}}{\frac{\partial u}{\partial \eta}} + \frac{2}{v-u} \cdot \frac{\partial u}{\partial \eta}, \\ \frac{\partial}{\partial \xi} \left(\frac{1}{F} \frac{\partial F}{\partial \eta} \right) &= \frac{\frac{\partial^2 u}{\partial \eta^2}}{\frac{\partial u}{\partial \eta}} \cdot \frac{\partial u}{\partial \xi} - 2 \frac{\frac{\partial u}{\partial \eta} \cdot \frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi}}{(v-u)^2}. \end{aligned}$$

This must be equal to

$$-\frac{H^2}{4} F + \frac{f_1(\xi) \cdot f_2(\eta)}{F} = -\frac{2}{(v-u)^2} \cdot \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi} + \frac{H^2(v-u)^2 f_1(\xi) f_2(\eta)}{8 \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi}},$$

and so we find:

$$\frac{H^2 \cdot f_1(\xi) \cdot f_2(\eta) \cdot (v-u)^2}{8 \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi}} = \frac{\frac{\partial^2 u}{\partial \eta^2}}{\frac{\partial u}{\partial \eta}} + \frac{\frac{\partial^2 u}{\partial \xi^2} \cdot \frac{\partial \eta}{\partial \eta}}{v-u}.$$

The second member can be once more reduced by means of (VII), and we find:

$$\frac{H^2 f_1(\xi) f_2(\eta) (v-u)^2}{8 \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi}} = \frac{2}{(v-u)^2} \cdot \frac{\partial v}{\partial \eta} \cdot \frac{\partial u}{\partial \xi}.$$

So

$$H^2 f_1(\xi) f_2(\eta) = \frac{16}{(v-u)^4} \frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta} \quad \dots \quad (\text{X})$$

§ 6. Let us now return to equation (VII). We see immediately that a solution, which does not cause $F' = \frac{8}{H^2 (v-u)^2} \cdot \frac{\partial v}{\partial \xi} \cdot \frac{\partial u}{\partial \eta}$ to vanish, is given by

$$u = \varphi(\eta) \quad , \quad v = \psi(\xi),$$

where φ and ψ are respectively functions of η and ξ only.

It is clear that equation (IX) is satisfied, when $f_1(\xi) = f_2(\eta) = 0$, so when $D = D' = 0$ (§ 4).

It is worth noticing, that when $u = \varphi(\eta)$ and $v = \psi(\xi)$ are substituted into the equation for F , this form becomes a solution of

$$-\frac{H^2}{4} F = \frac{\partial}{\partial \xi} \left(\frac{1}{F} \cdot \frac{\partial F}{\partial \eta} \right)$$

and so this tallies perfectly, because we have here the differential equation of LIOUVILLE. Indeed, the problem of the surfaces with constant mean curvature always leads to an extended equation of LIOUVILLE, as (IX) does, in whatever way we treat it.

That we really find a sphere here must follow from (VI). These equations give for $u = \varphi(\eta)$ and $v = \psi(\xi)$,

$$z = \frac{1}{Q} \frac{v + u}{v - u},$$

$$x = \frac{1}{Q} \frac{uv - 1}{v - u},$$

$$y = -\frac{1}{Q} \frac{uv + 1}{v - u},$$

the wellknown formulæ for the sphere in minimal coordinates.

We find:

$$x^2 + y^2 + z^2 = -\frac{1}{Q^2} = \frac{4}{H^2},$$

i.e. a sphere with radius $\frac{2}{H}$, as is necessary.

Now that we have regarded the special case $f_1(\xi) = f_2(\eta) = 0$, we can put both functions equal to 1 by introducing new functions

$$f_1(\xi) = \xi_1 \quad \text{and} \quad f_2(\eta) = \eta_1,$$

which we shall again indicate by ξ and η . This is of high importance, if eventually the solution of equation (VII) were to be found.

§ 7. We can now put the question whether the equations (VII) can be solved by putting u equal to $f(v)$, where for the present f is arbitrary.

From (VII) can be deduced

$$\frac{\frac{\partial^2 v}{\partial \xi \partial \eta}}{\frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta}} + \frac{\frac{\partial^2 u}{\partial \xi \partial \eta}}{\frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \eta}} = 0.$$

For $u = f(v)$ this leads to

$$\begin{aligned} \frac{\partial u}{\partial \xi} &= f'(v) \cdot \frac{\partial v}{\partial \xi}, \\ \frac{\partial^2 u}{\partial \xi \partial \eta} &= f''(v) \cdot \frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta} + f'(v) \cdot \frac{\partial^2 v}{\partial \xi \partial \eta}. \end{aligned}$$

So:

$$\frac{\frac{\partial^2 v}{\partial \xi \partial \eta}}{\frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta}} + \frac{f''(v) \cdot \frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta} + f'(v) \cdot \frac{\partial^2 v}{\partial \xi \partial \eta}}{f'(v) \frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta}} = 0,$$

or

$$\{f''(v) + f'(v)\} \frac{\partial^2 v}{\partial \xi \partial \eta} + f''(v) \cdot \frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta} = 0.$$

Then, according to (VII), $\frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta} = \frac{v-u}{2} \cdot \frac{\partial^2 v}{\partial \xi \partial \eta}$.

So:

$$f''(v) + f'(v) + \frac{v-f(v)}{2} f''(v) = 0.$$

One integral of this is sufficient to recognize the nature of the surfaces found. We find that satisfies

$$f(v) = -v^2.$$

¹⁾ Prof. W. KAPTEYN was so kind as to draw my attention to the following general solution of the differential equation.

Put

$$f(v) = y,$$

then

$$\frac{v-y}{2} \frac{d^2 y}{dv^2} + \left(\frac{dy}{dv}\right)^2 + \frac{dy}{dv} = 0.$$

Now put

$$y = v + w,$$

so

$$\frac{dy}{dv} = 1 + \frac{dw}{dv}, \quad \frac{d^2 y}{dv^2} = \frac{d^2 w}{dv^2},$$

so that

The equations (VII) become

$$\frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} - u \frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \quad \text{and} \quad \frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta} - v \frac{\partial^2 v}{\partial \xi \partial \eta} = 0,$$

which are satisfied by a function and its opposite. From this we deduce:

$$\frac{\partial}{\partial \xi} \left(\frac{\partial (lu)}{\partial \eta} \right) = 0.$$

Therefore e. g.

$$u = e^{\psi(\eta) + \varphi(\xi)}, \quad v = -e^{\psi(\eta) + \varphi(\xi)}.$$

By quadratures we find out of (VI),

$$2Qiz = -\Psi(\eta) + \varphi(\xi),$$

$$4Qix = e^{\psi(\eta) + \varphi(\xi)} + e^{-\psi(\eta) - \varphi(\xi)},$$

$$4Qiy = -e^{\psi(\eta) + \varphi(\xi)} + e^{-\psi(\eta) - \varphi(\xi)}.$$

The surface is a cylinder of revolution. Its section with the plane XOY is a circle, as we find

$$y^2 + x^2 = -\frac{1}{4Q^2} = \frac{1}{H^2}.$$

The radius of the circle is therefore $\frac{1}{H}$, as it has to be.

We can furthermore easily show that our solution agrees with the differential equations (IX), when we put

$$f_1(\xi) = f_2(\eta) = 1.$$

We find namely that the second member becomes zero, so that

$$F = \frac{H}{2} = \frac{1}{2} \left(\frac{1}{r_1} + \frac{1}{r_2} \right).$$

As moreover $\frac{1}{F} = \frac{1}{2} \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$, as we saw before, r_2 is therefore $= \infty$.

§ 8. We can now investigate what in the equations (VII) the significance would be of a solution $u = \chi(\xi)$, if it were possible.

$$-\frac{w}{2} \frac{d^2 w}{dv^2} + \left(\frac{dw}{dr} \right)^2 + 3 \left(\frac{dw}{dr} \right) + 2 = 0.$$

Let $\frac{dw}{dr} = p$, so $\frac{d^2 w}{dv^2} = p \frac{dp}{dr}$,

then:

$$-\frac{w}{2} p \frac{dp}{dw} + (p+1)(p+2) = 0,$$

from which ensues, $\frac{(p+2)^2}{p+1} = kw^2$ ($k = \text{const.}$)

For $k=0$ this solution gives the one used in the text.

The equation

$$\frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} + (v - u) \cdot 2 \frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

is satisfied by $u = \chi(\xi)$.

So there remains to be integrated

$$2 \frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta} - (v - u) \frac{\partial^2 v}{\partial \xi \partial \eta} = 0,$$

when $u = \chi(\xi)$.

We find:

$$\frac{\partial v}{\partial \xi} = \frac{1}{2} (v - \chi(\xi))^2 \cdot f(\xi),$$

with $f(\xi)$ as arbitrary function of ξ .

The solution $u = \chi(\xi)$ furnishes (see (V^T)) the value zero for $\frac{\partial x}{\partial \eta}$, $\frac{\partial y}{\partial \eta}$ and $\frac{\partial z}{\partial \eta}$; whilst for $\frac{\partial x}{\partial \xi}$, $\frac{\partial y}{\partial \xi}$ and $\frac{\partial z}{\partial \xi}$ the wellknown formulæ are found back for the minimal curves.

Entirely the same (with exchange of u and v , ξ and η) is found by putting $v = \chi_1(\eta)$.

This solution therefore shows what relations there are between the minimal surfaces and those under consideration. For the former we have but to join the two solutions found to get the complete solution with two arbitrary functions. So we see that the minimal surfaces are translation surfaces, generated by moving a minimal curve out of a set along the various points of a curve out of the second set; i. o. w. we have found back the integration of the minimal surfaces and in the usual form too.

Because of H tending to zero there is in this case no fear of F becoming 0.

§ 9. Now that the special cases of sphere (plane), cylinder and minimal surfaces are excluded, the integration of the equations (VII) would remain. I have not been able to attain more than the lowering of the order of the two differential equations, which is perhaps a step onward to a complete solution or to solutions for definite series of surfaces.

To this end we put:

$$\frac{\partial v}{\partial \xi} \cdot \frac{1}{(v-u)^2} = \frac{w_1}{2}, \quad \frac{\partial u}{\partial \eta} \frac{1}{(v-u)^2} = -\frac{w_2}{2},$$

where w_1 and w_2 are functions of ξ and η .

From these we derive, by differentiation with respect to ξ and η respectively

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = (v-u) \cdot w_1 \cdot \left(\frac{\partial v}{\partial \eta} - \frac{\partial u}{\partial \eta} \right) + \frac{1}{2} (v-u)^2 \cdot \frac{\partial w_1}{\partial \eta},$$

and

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = -(v-u) \cdot w_2 \cdot \left(\frac{\partial v}{\partial \xi} - \frac{\partial u}{\partial \xi} \right) - \frac{1}{2} (v-u)^2 \cdot \frac{\partial w_2}{\partial \xi}.$$

By means of the two non-differentiated equations and by equation (VII), we deduce from our last equation:

$$w_1 \cdot \frac{\partial v}{\partial \eta} = w_1 \left(\frac{\partial v}{\partial \eta} - \frac{\partial u}{\partial \eta} \right) + \frac{1}{2} (v-u) \cdot \frac{\partial w_1}{\partial \eta}$$

$$\text{and } -w_2 \cdot \frac{\partial u}{\partial \xi} = w_2 \left(\frac{\partial v}{\partial \xi} - \frac{\partial u}{\partial \xi} \right) + \frac{1}{2} (v-u) \cdot \frac{\partial w_2}{\partial \xi},$$

or:

$$w_1 \frac{\partial u}{\partial \eta} = \frac{1}{2} (v-u) \cdot \frac{\partial w_1}{\partial \eta} \quad \text{and} \quad w_2 \frac{\partial v}{\partial \xi} = -\frac{1}{2} (v-u) \cdot \frac{\partial w_2}{\partial \xi},$$

from which ensues:

$$-w_1 w_2 (v-u) = \frac{\partial w_1}{\partial \eta} \quad \text{and} \quad -w_1 w_2 (v-u) = \frac{\partial w_2}{\partial \xi}.$$

So we may put:

$$w_1 = \frac{\partial f}{\partial \xi} \quad \text{and} \quad w_2 = \frac{\partial f}{\partial \eta},$$

where f is a function of ξ and η which has however to satisfy a new differential equation.

So we have:

$$\frac{\partial v}{\partial \xi} \cdot \frac{1}{(v-u)^2} = \frac{1}{2} \frac{\partial f}{\partial \xi} \quad \text{and} \quad \frac{\partial u}{\partial \eta} \cdot \frac{1}{(v-u)^2} = -\frac{1}{2} \frac{\partial f}{\partial \eta},$$

whilst moreover:

$$v - u = -\frac{\frac{\partial^2 f}{\partial \xi \partial \eta}}{\frac{\partial f}{\partial \xi} \cdot \frac{\partial f}{\partial \eta}}.$$

Out of (VII) follows:

$$v - u = 2 \frac{\frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta}}{\frac{\partial^2 v}{\partial \xi \partial \eta}} \quad \text{and} \quad v - u = -2 \frac{\frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \eta}}{\frac{\partial^2 u}{\partial \xi \partial \eta}}.$$

By substitution of $v - u$, $\frac{\partial v}{\partial \xi}$ and $\frac{\partial v}{\partial \eta}$ we thus find :

$$\begin{aligned}
 1 &= \frac{(v-u) \frac{\partial f}{\partial \xi} \cdot \frac{\partial v}{\partial \eta}}{\frac{\partial^2 v}{\partial \xi \partial \eta}}, & 1 &= \frac{(v-u) \frac{\partial f}{\partial \eta} \cdot \frac{\partial u}{\partial \xi}}{\frac{\partial^2 u}{\partial \xi \partial \eta}} \\
 1 &= - \frac{\frac{\partial^2 f}{\partial \xi \partial \eta} \cdot \frac{\partial v}{\partial \eta}}{\frac{\partial^2 v}{\partial \xi \partial \eta} \cdot \frac{\partial f}{\partial \eta}}, & 1 &= - \frac{\frac{\partial^2 f}{\partial \xi \partial \eta} \cdot \frac{\partial u}{\partial \xi}}{\frac{\partial^2 u}{\partial \xi \partial \eta} \cdot \frac{\partial f}{\partial \xi}} \\
 \frac{\frac{\partial^2 v}{\partial \xi \partial \eta}}{\frac{\partial v}{\partial \eta}} &= - \frac{\frac{\partial^2 f}{\partial \xi \partial \eta}}{\frac{\partial f}{\partial \eta}}, & \frac{\frac{\partial^2 u}{\partial \xi \partial \eta}}{\frac{\partial u}{\partial \xi}} &= - \frac{\frac{\partial^2 f}{\partial \xi \partial \eta}}{\frac{\partial f}{\partial \xi}}.
 \end{aligned}$$

After integration we find :

$$\frac{\partial v}{\partial \eta} \cdot \frac{\partial f}{\partial \eta} = F_2(\eta) \quad \text{and} \quad \frac{\partial u}{\partial \xi} \cdot \frac{\partial f}{\partial \xi} = F_1(\xi).$$

Joining these equations to the values of $\frac{\partial v}{\partial \xi}$ and $\frac{\partial u}{\partial \eta}$, we find :

$$\frac{\partial v}{\partial \eta} \cdot \frac{\partial u}{\partial \eta} = - \frac{1}{2} (v-u)^2 F_2(\eta) \quad \text{and} \quad \frac{\partial u}{\partial \xi} \cdot \frac{\partial v}{\partial \xi} = \frac{1}{2} (v-u)^2 F_1(\xi).$$

These equations must be regarded as the intermediate integrals; they contain the arbitrary functions F_2 and F_1 , and it is easy to prove that by differentiation they lead back to the two equations (VII) of order two.

It goes almost without saying that F_2 and F_1 appearing here are closely connected to f_1 and f_2 appearing in (VIII).

From the equations just found follows :

$$\frac{\partial v}{\partial \eta} \cdot \frac{\partial v}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} \cdot \frac{\partial u}{\partial \xi} = - \frac{1}{4} (v-u)^4 \cdot F_2(\eta) \cdot F_1(\xi),$$

or

$$F_2(\eta) \cdot F_1(\xi) = - \frac{\frac{\partial v}{\partial \eta} \cdot \frac{\partial v}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} \cdot \frac{\partial u}{\partial \xi}}{(v-u)^4}.$$

If we compare this to (X), then :

$$- 4 F_2(\eta) \cdot F_1(\xi) = H^2 f_1(\xi) f_2(\eta).$$

The first integrals found satisfy therefore all the conditions entirely. We have transformed our original coordinates in such a way that

$f_1'(\xi)$ and $f_2'(\eta)$ both became 1 and so now we can take in accordance with it:

$$F_2(\eta) = \frac{H}{2i} \text{ and } F_1(\xi) = \frac{H}{2i},$$

so that the first integrals become:

$$\frac{\partial v}{\partial \eta} \cdot \frac{\partial u}{\partial \eta} = -\frac{H}{4i}(v-u)^2 \text{ and } \frac{\partial u}{\partial \xi} \cdot \frac{\partial v}{\partial \xi} = \frac{H}{4i}(v-u)^2,$$

or

$$\frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \eta} = \frac{Q}{2}(v-u)^2 \text{ and } \frac{\partial u}{\partial \xi} \cdot \frac{\partial v}{\partial \xi} = -\frac{Q}{2}(v-u)^2 \quad \dots \quad (A)$$

By replacing moreover $v-u$ by s_1 and $v+u$ by s_2 the final equations become:

$$\left(\frac{\partial s_1}{\partial \eta}\right)^2 = \left(\frac{\partial s_2}{\partial \eta}\right)^2 - 2Qs_1^2 \text{ and } \left(\frac{\partial s_1}{\partial \xi}\right)^2 = \left(\frac{\partial s_2}{\partial \xi}\right)^2 + 2Qs_1^2 \quad \dots \quad (B)$$

These are still to be solved.

Mathematics. — “*On the multiplication of trigonometrical series.*”

By Prof. W. KAPTEYN,

1. If $f(x)$ and $\varphi(x)$ are two functions which are finite and continuous in the interval from $x=0$ to $x=\pi$, we have

$$f(x) = \frac{1}{2} a_0 + a_1 \cos x + a_2 \cos 2x + \dots$$

$$f(x) = b_1 \sin x + b_2 \sin 2x + \dots$$

$$\varphi(x) = \frac{1}{2} a'_0 + a'_1 \cos x + a'_2 \cos 2x + \dots$$

$$\varphi(x) = b'_1 \sin x + b'_2 \sin 2x + \dots$$

where

$$a_n = \frac{2}{\pi} \int_0^\pi f(\omega) \cos n\omega \, d\omega \quad b_n = \frac{2}{\pi} \int_0^\pi f(\omega) \sin n\omega \, d\omega$$

$$a'_n = \frac{2}{\pi} \int_0^\pi \varphi(\omega) \cos n\omega \, d\omega \quad b'_n = \frac{2}{\pi} \int_0^\pi \varphi(\omega) \sin n\omega \, d\omega.$$

In the same way the product $f(x) \cdot \varphi(x)$ may be developed, this product being finite and continuous in the same interval; therefore

$$f(x) \cdot \varphi(x) = \frac{1}{2} A_0 + A_1 \cos x + A_2 \cos 2x + \dots$$

$$f(x) \cdot \varphi(x) = B_1 \sin x + B_2 \sin 2x + \dots$$

where

$$A_n = \frac{2}{\pi} \int_0^\pi f(\omega) \varphi(\omega) \cos n\omega \, d\omega, \quad B_n = \frac{2}{\pi} \int_0^\pi f(\omega) \varphi(\omega) \sin n\omega \, d\omega.$$

We shall now investigate the relations which exist between the integrals A_n , B_n and the coefficients a_n , b_n , a'_n , b'_n .

Substituting in A_n for $\varphi(\omega)$ the series of cosines, we obtain

$$\begin{aligned} A_n &= \frac{2}{\pi} \int_0^\pi f(\omega) \cos n\omega \left[\frac{1}{2} a'_0 + a'_1 \cos \omega + a'_2 \cos 2\omega + \dots \right] d\omega \\ &= \frac{1}{2} a'_0 a_n + \sum_1^\infty \frac{a'_m}{2} \cdot \frac{2}{\pi} \int_0^\pi f(\omega) [\cos(m+n)\omega + \cos(m-n)\omega] d\omega \\ &= \frac{1}{2} a'_0 a_n + \sum_1^\infty \frac{a'_m}{2} (a_{m+n} + a_{m-n}). \end{aligned}$$

This equation may be written in another form; for, because $a'_{-p} = a'_p$,

$$\sum_1^\infty \frac{a'_m}{2} a_{m-n} = \sum_1^\infty \frac{a'_m}{2} a_{n-m} + \frac{1}{2} \sum_{n+1}^\infty a'_m a_{m-n}$$

or, putting $m+n$ instead of m in the summation from $n+1$ to ∞

$$\sum_1^\infty \frac{a'_m}{2} a_{m-n} = \frac{1}{2} \sum_1^n a'_m a_{n-m} + \frac{1}{2} \sum_1^\infty a_m a'_{m+n}.$$

Hence

$$A_n = \frac{1}{2} \sum_0^n a'_m a_{n-m} + \frac{1}{2} \sum_1^\infty (a'_m a_{m+n} + a_m a'_{m+n}) \dots \quad (I)$$

If now we substitute in A_n for $\varphi(\omega)$ the series of sines, we have

$$\begin{aligned} A_n &= \frac{2}{\pi} \int_0^\pi f(\omega) \cos n\omega [b'_1 \sin \omega + b'_2 \sin 2\omega + \dots] d\omega \\ &= \sum_1^\infty \frac{b'_m}{2} \cdot \frac{2}{\pi} \int_0^\pi f(\omega) [\sin(m+n)\omega + \sin(m-n)\omega] d\omega \\ &= \sum_1^\infty \frac{b'_m}{2} (b_{m+n} + b_{m-n}) \end{aligned}$$

or, as $b'_{-p} = -b_p$

$$A_n = -\frac{1}{2} \sum_1^n b'_m b_{n-m} + \frac{1}{2} \sum_1^\infty (b'_m b_{m+n} + b_m b'_{m+n}) \dots \quad (II)$$

In the same way we find

$$\begin{aligned} B_n &= \frac{2}{\pi} \int_0^\pi f(\omega) \sin n\omega \left[\frac{1}{2} a'_0 + a'_1 \cos \omega + a'_2 \cos 2\omega + \dots \right] d\omega \\ &= \frac{1}{2} a'_0 b_n + \sum_1^\infty \frac{a'_m}{2} (b_{m+n} - b_{m-n}) \end{aligned}$$

or, after a slight reduction

$$B_n = \frac{1}{2} \sum_0^n a'_m b_{n-m} + \frac{1}{2} \sum_1^\infty (a'_m b_{m+n} - b_m a'_{m+n}) \dots \dots \dots (III)$$

and

$$\begin{aligned} B_n &= \frac{2}{\pi} \int_0^\pi f(\omega) \sin n\omega [b'_1 \sin \omega + b'_2 \sin 2\omega + \dots] d\omega \\ &= \sum_1^\infty \frac{b'_m}{2} (a_{m-n} - a_{m+n}) \end{aligned}$$

or

$$B_n = \frac{1}{2} \sum_1^n b'_m a_{n-m} + \frac{1}{2} \sum_1^\infty (a_m b'_{m+n} - b'_m a_{m+n}) \dots \dots \dots (IV)$$

2. If we suppose

$$\begin{aligned} f^2(x) &= \frac{1}{2} \mathfrak{U}_0 + \mathfrak{U}_1 \cos x + \mathfrak{U}_2 \cos 2x + \dots \\ &= \mathfrak{C}_1 \sin x + \mathfrak{C}_2 \sin 2x + \dots \end{aligned}$$

the four preceding equations give immediately, by putting $\varphi(x) = f(x)$

$$\mathfrak{U}_n = \frac{1}{2} \sum_0^n a_m a_{n-m} + \sum_1^\infty a_m a_{m+n} \dots \dots \dots (1)$$

$$\mathfrak{U}_n = -\frac{1}{2} \sum_1^n b_m b_{n-m} + \sum_1^\infty b_m b_{m+n} \dots \dots \dots (2)$$

$$\mathfrak{C}_n = \frac{1}{2} \sum_0^n a_m b_{n-m} + \frac{1}{2} \sum_1^\infty (a_m b_{m+n} - b_m a_{m+n}) \dots \dots \dots (3)$$

3. From the four equations of Art. 1, the beautiful theorem of PARSEVAL may be easily deduced. For, supposing that for all the values on the circumference of the circle $mod z = 1$, we have

$$\begin{aligned} \frac{1}{2} a_0 + a_1 z + a_2 z^2 + \dots &= \varphi(z) \\ \frac{1}{2} a'_0 + \frac{a'_1}{z} + \frac{a'_2}{z^2} + \dots &= \psi(z), \end{aligned}$$

it is evident, if we assume in succession $z = e^{i\omega}$ and $z = e^{-i\omega}$, that

$$\begin{aligned} F_1(\omega) + i F_2(\omega) &= \varphi(e^{i\omega}) & G_1(\omega) - i G_2(\omega) &= \psi(e^{i\omega}) \\ F_1(\omega) - i F_2(\omega) &= \varphi(e^{-i\omega}) & G_2(\omega) + i G_1(\omega) &= \psi(e^{-i\omega}). \end{aligned}$$

Multiplying these equations and adding the results we obtain

$$2 [F_1(\omega) G_1(\omega) + F_2(\omega) G_2(\omega)] = \varphi(e^{i\omega}) \psi(e^{i\omega}) + \varphi(e^{-i\omega}) \psi(e^{-i\omega})$$

where

$$\begin{aligned} F_1(\omega) = F_1 &= \frac{1}{2} a_0 + a_1 \cos \omega + a_2 \cos 2\omega + \dots \\ G_1(\omega) = G_1 &= \frac{1}{2} a'_0 + a'_1 \cos \omega + a'_2 \cos 2\omega + \dots \\ F_2(\omega) = F_2 &= a_1 \sin \omega + a_2 \sin 2\omega + \dots \\ G_2(\omega) = G_2 &= a'_1 \sin \omega + a'_2 \sin 2\omega + \dots \end{aligned}$$

If now we put $n = 0$ in the equations (I) and (II) we find

$$\frac{2}{\pi} \int_0^\pi F_1 G_1 d\omega = \frac{1}{2} a_0 a'_0 + a_1 a'_1 + a_2 a'_2 + \dots$$

$$\frac{2}{\pi} \int_0^\pi F_2 G_2 d\omega = a_1 a'_1 + a_2 a'_2 + \dots$$

thus

$$\frac{1}{2\pi} \int_0^\pi \{ \varphi(e^{i\omega}) \psi(e^{i\omega}) + \varphi(e^{-i\omega}) \psi(e^{-i\omega}) \} d\omega = \frac{1}{4} a_0 a'_0 + a_1 a'_1 + a_2 a'_2 + \dots$$

which is the theorem in question.

4. From the preceding formulae we may also deduce the values of several interesting series. For, if the series for $f(x)$ and $\varphi(x)$ are given, and the integrals

$$\int_0^\pi f(\omega) \varphi(\omega) \cos n\omega d\omega \quad \text{and} \quad \int_0^\pi f(\omega) \varphi(\omega) \sin n\omega d\omega$$

are to be found, the values of the series in the second members of the given equations may be determined. To show this, we shall make the following application of the formulae (1), (2) and (3).

Suppose $f(x) = x$, then

$$x = \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{\cos x}{1^2} - \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} - \dots \right)$$

$$x = 2 \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right)$$

and

$$A_n = \frac{2}{\pi} \int_0^\pi \omega^2 \cos n\omega d\omega = 4 \frac{\cos n\pi}{n^2}$$

$$C_n = \frac{2}{\pi} \int_0^\pi \omega^2 \sin n\omega d\omega = -\frac{2\pi \cos n\pi}{n} - \frac{4(1 - \cos n\pi)}{\pi n^3}$$

Now the formula (1) gives, because

$$a_2 = a_4 = a_6 = \dots = 0$$

$$A_0 = \frac{1}{2} a_0^2 + a_1^2 + a_3^2 + a_5^2 + \dots$$

$$A_2 = \frac{1}{2} a_1^2 + a_1 a_3 + a_3 a_5 + a_5 a_7 + \dots$$

$$A_4 = a_1 a_3 + a_1 a_5 + a_3 a_7 + a_5 a_9 + \dots$$

$$A_6 = a_1 a_5 + \frac{a_3^2}{2} + a_1 a_7 + a_3 a_9 + a_5 a_{11} + \dots$$

.

therefore

$$\begin{aligned} \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots &= \frac{\pi^4}{96} \\ \frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 7^2} + \frac{1}{5^2 \cdot 9^2} + \dots &= \frac{\pi^2}{16} - \frac{1}{2} \\ \frac{1}{1^2 \cdot 5^2} + \frac{1}{3^2 \cdot 7^2} + \frac{1}{5^2 \cdot 9^2} + \dots &= \frac{\pi^2}{64} - \frac{1}{9} \\ \frac{1}{1^2 \cdot 7^2} + \frac{1}{3^2 \cdot 9^2} + \frac{1}{5^2 \cdot 11^2} + \dots &= \frac{\pi^2}{144} - \frac{137}{4050} \end{aligned}$$

According to formula (2) we have

$$\begin{aligned} \mathfrak{A}_0 &= \sum_1^{\infty} b_m^2 \\ \mathfrak{A}_1 &= \sum_1^{\infty} b_m b_{m+1} \\ \mathfrak{A}_2 &= -\frac{1}{2} b_1^2 + \sum_1^{\infty} b_m b_{m+2} \\ \mathfrak{A}_3 &= -b_1 b_3 + \sum_1^{\infty} b_m b_{m+3} \\ \mathfrak{A}_4 &= -\frac{1}{2} (2b_1 b_3 + b_2^2) + \sum_1^{\infty} b_m b_{m+4} \\ \mathfrak{A}_5 &= -(b_1 b_4 + b_2 b_3) + \sum_1^{\infty} b_m b_{m+5} \\ \mathfrak{A}_6 &= -\frac{1}{2} (2b_1 b_5 + 2b_2 b_4 + b_3^2) + \sum_1^{\infty} b_m b_{m+6} \end{aligned}$$

From which may be deduced

$$\begin{aligned} \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots &= \frac{\pi^2}{6} \\ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots &= 1 \\ \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \dots &= \frac{3}{4} \\ \frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 6} + \dots &= \frac{11}{18} \\ \frac{1}{1 \cdot 5} + \frac{1}{2 \cdot 6} + \frac{1}{3 \cdot 7} + \dots &= \frac{25}{48} \\ \frac{1}{1 \cdot 6} + \frac{1}{2 \cdot 7} + \frac{1}{3 \cdot 8} + \dots &= \frac{137}{300} \\ \frac{1}{1 \cdot 7} + \frac{1}{2 \cdot 8} + \frac{1}{3 \cdot 9} + \dots &= \frac{49}{120} \end{aligned}$$

• • • • •

In the same way the formula (3) gives

$$\begin{aligned} \mathfrak{C}_1 &= \frac{1}{2} a_0 b_1 + \frac{1}{2} (a_1 b_2 - b_1 a_2 + a_2 b_4 - b_4 a_5 + \dots) \\ \mathfrak{C}_2 &= \frac{1}{2} (a_0 b_2 + a_1 b_1) + \frac{1}{2} (a_1 b_3 - b_1 a_3 + a_3 b_5 - b_3 a_5 + a_5 b_7 - b_5 a_7 + \dots) \\ \mathfrak{C}_3 &= \frac{1}{2} (a_0 b_3 + a_1 b_2) + \frac{1}{2} (a_1 b_4 - b_1 a_5 + a_3 b_6 - b_4 a_7 + a_5 b_8 - b_6 a_9 + \dots) \\ \mathfrak{C}_4 &= \frac{1}{2} (a_0 b_4 + a_1 b_3 + a_2 b_1) + \frac{1}{2} (a_1 b_5 - b_1 a_6 + a_3 b_7 - b_3 a_7 + a_5 b_9 - b_5 a_9 + \dots) \\ \mathfrak{C}_5 &= \frac{1}{2} (a_0 b_5 + a_1 b_4 + a_2 b_2) + \frac{1}{2} (a_1 b_6 - b_1 a_7 + a_3 b_8 - b_4 a_9 + a_5 b_{10} - b_6 a_{11} + \dots) \\ \mathfrak{C}_6 &= \frac{1}{2} (a_0 b_6 + a_1 b_5 + a_2 b_3 + a_3 b_1) + \\ &\quad + \frac{1}{2} (a_1 b_7 - b_1 a_7 + a_3 b_9 - b_3 a_9 + a_5 b_{11} - b_6 a_{11} + \dots) \\ &\dots \dots \dots \end{aligned}$$

from which the following relations may be obtained

$$\begin{aligned} \frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \dots &= \frac{\pi^2}{16} - \frac{1}{2} \\ \frac{1}{4 \cdot 1^2} - \frac{1}{2 \cdot 5^2} + \frac{1}{6 \cdot 3^2} - \frac{1}{4 \cdot 7^2} + \dots &= \frac{\pi^2}{12} - \frac{31}{54} \\ \frac{1}{5 \cdot 1^2} - \frac{1}{1 \cdot 5^2} + \frac{1}{7 \cdot 3^2} - \frac{1}{3 \cdot 7^2} + \dots &= \frac{\pi^2}{16} - \frac{4}{9} \\ \frac{1}{6 \cdot 1^2} - \frac{1}{2 \cdot 7^2} + \frac{1}{8 \cdot 3^2} - \frac{1}{4 \cdot 9^2} + \dots &= \frac{7\pi^2}{60} - \frac{347}{900} \\ \frac{1}{7 \cdot 1^2} - \frac{1}{1 \cdot 7^2} + \frac{1}{9 \cdot 3^2} - \frac{1}{3 \cdot 9^2} + \dots &= \frac{\pi^2}{24} - \frac{187}{675} \\ &\dots \dots \dots \end{aligned}$$

Chemistry. — “*On a crystallised d. fructose tetracetate*”, by Dr. D. H. BRAUNS. (Communicated by Prof. A. P. N. FRANCHIMONT).

Very few crystallised derivatives of *d.* fructose have as yet been obtained. A pentacetate was described by ERWIGS and KOENIGS as a gummy substance. A number of researches have shown, however, that the high temperature at which the reactions generally took place causes a conversion or decomposition of the fructose. As no satisfactory results were obtained with acetic anhydride and acetyl chloride acetyl bromide was employed which reacts at a comparatively low temperature. The greatest possible precautions were taken to exclude moisture and to let the reaction take place at a low temperature. The details will be published in full later on.

Refrigerated *d.* fructose in fine powder was mixed with a little more than 5 mols. of acetyl bromide at -15° and after starting the reaction by touching one spot with a tube having the ordinary

temperature. I waited until most of the hydrogen bromide had been evolved and the reaction was consequently nearly over. The excess of acetyl bromide was then distilled in a high vacuum and the *product*, consisting of a tenaceous, yellow mass, treated with iced water, then dissolved in alcohol and placed in a desiccator containing caustic potash and kept at a low temperature. A crystallised mass was obtained which after being submitted to pressure was recrystallised at a low temperature when beautiful crystals, free from bromine, were deposited.

These crystals are colourless, odourless, taste bitter and melt at 131° — 132° . In a high vacuum they may be sublimed even at 95° more rapidly at 105° ; the sublimate has the same melting point.

The ultimate analyses gave a mean result of C 48.26%, H 5.86%.

The molecular weight determination by the lowering of the freezing point of benzene gave a mean of 355.

The acetyl determination was carried out by saponification with $n/_{10}$ sodium hydroxide at a low temperature. Blank experiments made under similar conditions showed that fructose is not altered or converted into acids. The saponification was nearly complete after two hours and quite so in 18 hours; after 28 hours no sensible decomposition of the fructose had set in and about the same figures were obtained as those in 18 hours. The average amount of acetic acid found was 69.42%.

It is, therefore, a fructose tetracetate $C_{14}H_{20}O_{10}$ for which theory requires C. 48.25% H. 5.86%, molecular weight 348 acetic acid 68.96%.

This compound is but little soluble in water, ether, benzene and ligroin, readily so in alcohol and chloroform.

The chloroform solution was used to determine the rotatory power. It polarises to the left and the specific rotation of d. fructose tetracetate at 20° was found $[\alpha]_D^{20} = -91^{\circ}.38$.

Dr. F. M. JAEGER was kind enough to investigate the crystals and reported as follows:

d. Fructose tetracetate (BRAUNS).

$C_{14}H_{20}O_{10}$; Melting point 132° C.

Sp. Gr. of the crystals at $15^{\circ} = 1.388$; Mol. Vol. = 250.72.

From ethyl alcohol + ether, it is obtained on slow evaporation, in beautiful, colourless, shining little crystals which may be readily measured and which possess a pure geometrical structure.

The compound is *hemimorphous*; its symmetry is that of the *monoclinic-sphenoidal* class. It, therefore, does not possess a single

symmetry-plane or a symmetry centre; but only one single unipolar, twin axis. All the crystals which I investigated represented the same variety of the two possible enantiomorphous forms.

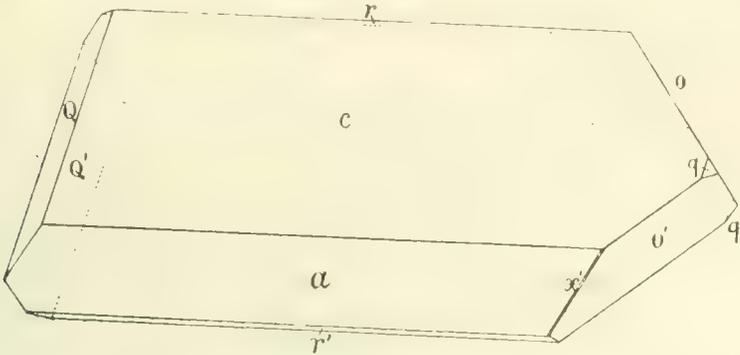


Fig. 1. d. Fructose tetracetate (BRAUNS).

The symmetry assigned here to the crystals is not only proved by their habit, but also proved beyond all doubt by the investigation of the etched figures obtained by means of 95% alcohol; these were very distinct particularly on {100} and {001}.

Parameters: $a : b : c = 1, 3463 : 1 : 1, 5733$

$\beta = 52^\circ.12'$

Forms observed: $c = \{001\}$, broad and very shining; $a = \{100\}$, somewhat narrower; $o = \{\bar{1}11\}$, large and yielding sharp reflexes; $q = \{011\}$, small but reflecting well; $Q = \{0\bar{1}1\}$, large and shining; $r = \{\bar{1}02\}$, very narrow and dull; $x = \{\bar{9}11\}$ exceedingly narrow and measurable only with difficulty. Once or twice one plane of $\{\bar{1}\bar{1}1\}$ was observed, rudimentary and striped parallel to the plane {001}.

<i>Angular values:</i>	<i>Measured:</i>	<i>Calculated:</i>
$c : a = (001) : (100) =$	$52^\circ.12'$	—
$o : o = (\bar{1}11) : (11\bar{1}) =$	75.41	—
$c : o = (001) : (\bar{1}11) =$	79.37	—
$a : q = (100) : (011) =$	$67.21\frac{1}{2}$,	$67^\circ.24\frac{1}{2}'$
$q : o = (011) : (\bar{1}11) =$	$43.10\frac{1}{2}$,	$43.17\frac{3}{4}$
$x : o = (91\bar{1}) : (11\bar{1}) =$	60.44 (about)	$60.53\frac{1}{2}$,
$x : a = (91\bar{1}) : (100) =$	8.36 (about)	$8.27\frac{1}{4}$,
$a : o = (100) : (11\bar{1}) =$	$69.29\frac{1}{2}$,	$69.20\frac{3}{4}$,
$q : q = (011) : (01\bar{1}) =$	77.39	$77.37\frac{1}{2}$,
$c : q = (001) : (011) =$	$51.10\frac{1}{2}$,	$51.11\frac{1}{4}$,
$c : r = (001) : (\bar{1}02) =$	35.44	$35.43\frac{1}{2}$,
$r : a = (\bar{1}02) : (\bar{1}00) =$	92.4	$92.4\frac{1}{2}$,

Readily cleavable parallel to a and c .

The optical axial plane is $\{010\}$. Very faint, inclined dispersion: $\rho > \nu$; double refraction negative. On c one optical axis emerges at a small angle with the normal.

Topic axial relation: $\chi : \psi : \omega = 7.1503 : 5.3109 : 8.3556$.

Physics. — “*New observations concerning asymmetrical triplets*”. By Prof. P. ZEEMAN.

Asymmetry investigated by means of FABRY and PEROT's method.

1. In the second part of the paper “Magnetic resolution of spectral lines and magnetic force” I¹⁾ investigated, by means of a method, which I called that of the non-uniform field, the asymmetry predicted from theory by VOIGT²⁾ in the case the original line is resolved into a triplet.

A glance at Plate II of my paper immediately shows that observation seems to confirm strikingly VOIGT's theoretical result that the component of the triplet towards the red is at a somewhat smaller distance from the middle line than the one towards the violet.

In order to exclude however all doubt as to the reality of this experimental result I thought it desirable to continue my work in a direction independent of ROWLAND's method.

I have shown³⁾ that the resolution of spectral lines by magnetic forces can be investigated by means of the semi-silvered parallel plates of FABRY and PEROT.

Using the special form of instrument in which the distance of the silvered surfaces is constant, the étalon, we may yet choose between two ways of comparison of the wavelengths of the centre line and of the components, originating by the action of the magnetic field.

Firstly we may measure, the intensity of the field being arbitrarily chosen, the diameters of the interference rings. By combining only measurements of rings originating from the same ring the calculation becomes very simple; for as shown in my last paper even a knowledge of the ordinal number of the rings then is unnecessary.

2. We may use however also the method of coincidences, regulating

1) ZEEMAN. These Proceedings 30 November 1907.

2) VOIGT. Ann. d. Phys. 1. p. 376. 1900, see also the last paper by VOIGT. Physik. Zeitschrift 9. p. 122. 1908.

3) ZEEMAN. These Proceedings 28 December 1907.

the magnetic force in such a manner that a ring which expands by increasing magnetic intensity coincides with a contracting ring.

The rings corresponding to components towards the red then coincide with rings due to components towards the violet side of the spectrum. The intensity of the coinciding rings is then only slightly inferior to that of the unmodified one, a circumstance favourable to the accuracy of the measurements.

Let λ_0 be the wavelength of the middle component of the triplet, λ_r that of the component towards the red, λ_v that of the component towards the violet then we may perform the calculation, ignoring the value of the ordinal numbers of the rings, by the following procedure.

Let P_0, P_r, P_v be the ordinal numbers of rings with angular diameters x_0, x_r, x_v then we have in general :

$$\lambda_r = \lambda_0 \frac{P_0}{P_r} \left(1 + \frac{x_0^2}{8} - \frac{x_r^2}{8} \right)$$

$$\lambda_v = \lambda_0 \frac{P_0}{P_v} \left(1 + \frac{x_0^2}{8} - \frac{x_v^2}{8} \right).$$

If the magnetic force is increasing a contracting ring corresponds to λ_r , an expanding one to λ_v . As I remarked on a former occasion we can put in the case of radiation in a magnetic field $P_0 = P_r$ or $P_0 = P_v$ if only rings λ_r and λ_v are considered, which originate from the same ring λ_0 .

Hence in applying the method of coincidences the simplest procedure is to consider the ring formed by superposition of two other rings, once as a ring λ_v derived from a smaller ring λ_0 , and again as a ring λ_r derived from a larger one λ_0 .

By measuring three rings viz. the one due to the coincidence of the rings λ_r and λ_v (diameter $x_c = x_r = x_v$), further the larger ring with diameter x_0 and finally the smaller one with diameter x'_0 , the result may be found by the simple formulae :

$$\lambda_r = \lambda_0 \left(1 + \frac{x_0^2}{8} - \frac{x_c^2}{8} \right)$$

and

$$\lambda_v = \lambda_0 \left(1 + \frac{x_0'^2}{8} - \frac{x_c^2}{8} \right).$$

3. Using an étalon, with an interval of nearly 5 mm. between the plates, I have made by means of the method of coincidences some measurements of the magnetic resolution of the yellow mercury lines 5791 and 5770.

The system of rings was formed in the focal plane of a small achromatic lens of 18 m.m. aperture and of 12 cm. focus. This focal plane coincided exactly with the plane of the slit of a one-prism spectroscope. The width of the slit was so far reduced that the rings of the two yellow mercury lines could be observed separately. Reproductions of negatives (somewhat enlarged) are given on the Plate, the first with field off; the second showing the first coincidence (superposition of rings λ_r and λ_o); the third gives the second coincidence, the rings λ_r and λ_o being now in coincidence with λ_o . The plate refers to coincidences for 5770; negatives showing the coincidences for 5791 however scarcely present any difference with those now given.

By measurements on half a dozen of negatives concerning the first coincidence, the result was obtained that a separation equal to 0.166 Ångström units for line 5770, corresponds to a separation of line 5791 towards the red of 0.160 A. U., towards the violet of 0.177 A. U.

Now a separation of 0.166 A. U. corresponds, according to the data given in § 6 of my paper cited in § 1 above sub ¹), to a strength of field of 9130 Gauss.

Considering as the object of the investigation the determination of the numerical value of the asymmetry we infer from the given data that it is equal to 0.017 A. U. A discussion of the systematic errors of observations to be feared, shows that the values 0.015 A. U. and 0.019 A. U. are yet possible, that however the values 0.011 A. U. and 0.023 A. U. are very improbable.

Some measurements made by means of the method of diameters tend to show that the accuracy of results obtained by that method is somewhat superior to that now found.

The accuracy obtained is however in excellent accordance which what might be expected from data given by FABRY and PEROT ¹) if applied to our case

By our experiments with the method of silvered plates two points are clearly shown viz. first that the positive results concerning asymmetrical resolution in strong fields obtained on a former occasion by ROWLAND's method have a real significance, secondly that also in lower fields the asymmetry remains and has an amount such as to be expected, if strength of field and asymmetry are nearly proportional.

¹) FABRY et PEROT. Ann. de Chim. et de Phys. Janvier 1902.

Determination of the total charge of the electrons.

4. Taking for granted the existence and also the nature of the asymmetrical resolution as being in accordance with VOIGT's theory, it certainly is extremely interesting to interpret the result in the language of electronic theory.

LORENTZ¹⁾ has deduced VOIGT's equations from the theory of electrons or more accurately expressed he gives a system of equations which come to the same thing as those of VOIGT.

Let H be the intensity of the magnetic field, λ the wavelength, $\delta\lambda_1$ and $\delta\lambda_2$ the differences of wavelengths between the middle component and those towards violet and red, V the velocity of light in the aether, and $\frac{e}{m}$ the well-known ratio of charge and mass of the electron, then according to LORENTZ :

$$\frac{e}{m} = \frac{4\pi V}{H\lambda^2} \sqrt{\delta\lambda_1 \cdot \delta\lambda_2} \quad (1)$$

For $\delta\lambda_1 = \delta\lambda_2$ this formula changes into the equation, which first enabled us to determine $\frac{e}{m}$. This ratio is found in electromagnetic units.

If N denote the number of molecules per unit volume, one electron vibrating in each molecule, we have also according to LORENTZ

$$Ne = \frac{H}{2\lambda V} \frac{\delta\lambda_1 - \delta\lambda_2}{\sqrt{\delta\lambda_1 \cdot \delta\lambda_2}} \quad (2)$$

These formulae were already communicated by GEHRCKE and VON BAEYER²⁾.

My own observations concerning asymmetry (§ 5 of my paper cited³⁾ and § 3 above) seem at first sight to be in contradiction with this formula. One of my results being that the asymmetry varies with strength of field, according to (2) Ne must vary also, because H and $\sqrt{\delta\lambda_1 \cdot \delta\lambda_2}$ change in nearly the same ratio. Now an increase of Ne , or of the number of radiating particles per unit volume, must manifest itself in the radiating power of the vacuum tube. An inspection of Plate II (paper cited sub³⁾) shows that in my experiments the intensity of the light of the components really has been a maximum in the strongest part of the field. We must therefore conclude that the circumstances of the luminous mercury vapour in the Geissler tube were slightly different in the various parts of the non-uniform magnetic field.

1) LORENTZ. Rapports présentés au congrès international de physique 1900.

2) GEHRCKE u. v. BAEYER. Verhandl. deutsch. physik. Gesellsch. 7. p. 401. 1906.

3) ZEEMAN. These Proceedings 30 November 1907.

It therefore seems probable to accept with Prof. VOIGT¹⁾ that the change of value of the asymmetry is due to differences in the circumstances of the radiating vapour.

5. The following table embodies the result of the calculations according to (1) and (2) of my observations concerning line 5791.

MERCURY LINE 5791.

$\frac{e}{m}$	N_e	Mean resolution 5770	H
1.92×10^7	8.40×10^{-4}	0.532 A. E.	29220
1.92	6.24	0.440 "	24140
1.90	5.97	0.399 "	21910
1.87	5.03	0.328 "	18020
1.87	4.33	0.270 "	14800
(2.07	4.58	0.166 "	9130)

The last line in this table refers to the observations recorded in § 1 of this paper.

Dividing the numbers of the second column by those of the first one we infer that the vibrating mass (probably wholly electromagnetic) only amounts from $4 \cdot 10^{-11}$ to $2 \cdot 10^{-11}$ gram per cm^3 .

Accepting J. J. THOMSON'S value of e viz. $1,1 \cdot 10^{-20}$ electromagnetic units, we may find the number N . The number of electrons per unit volume causing the radiation of the mercury line 5791 in a vacuum tube, appears then in the circumstances of our experiments and according to the magnetic force to lay between $8 \cdot 10^{16}$ and $4 \cdot 10^{16}$ per cm^3 .

In these experiments the temperature of the vacuum tube may be taken as somewhere between 100° and 120° . According to HERTZ the pressures of mercury vapour corresponding to these temperatures are 0.29 resp. 0.78 m.m. From these facts in connection with other well known data we may conclude that the number of electrons participating in the emission of line 5791 is of the same order of magnitude as the number of atoms present.

There seems to be no obstacle in accepting this result and the hypothesis that all atoms participate simultaneously in the emission of light might even seem the most natural. It is however of some

¹⁾ Voigt. Physik. Zeitschr. 9. S. 120. 1908.

interest to compare with this result some consequences issuing from work done in the Amsterdam laboratory by HALLO on the magnetic rotation of the plane of polarisation in sodium vapour¹⁾, and by GEEST on magnetic double refraction in the same substance²⁾, and from one of the results of JEAN BECQUEREL³⁾ in his remarkable experiments concerning the behaviour of tysonite and other crystals at low temperature and in a magnetic field.

These physicists come to the conclusion that in the substances they have experimented on, only a small part of the atoms are participating simultaneously in the emission or absorption phenomena.

Of course there is not the least improbability in accepting that in a Geissler tube the circumstances are quite different, and to admit that in a vacuum tube the number of atoms vibrating at a given instant is very large.

Asymmetries of Wolframium and Molybdenum lines.
Observations of Mr. JACK.

6. Not only the lines of mercury and iron, which I investigated, but also those of other substances give in the magnetic field asymmetrical triplets. Some examples of very pronounced asymmetries, have been met with by Mr. JACK in the physical laboratory at Göttingen, and I am indebted to the kindness of Prof. VOIGT in being able to communicate these here. In the annexed table the wavelengths are given in ÅNGSTRÖM units, the separations however in m.m. as measured on the plates. For a knowledge of the relative asymmetry this is sufficient.

With some lines the asymmetry is reversed, the component towards the red being at a larger distance. According to the remarks of Mr. JACK it is not excluded however that in these cases the structure of the lines is not quite simple.

The intensities given can only have a relative value according to the results of my paper in these Proceedings of October 1907.

Observation parallel to the lines of force.

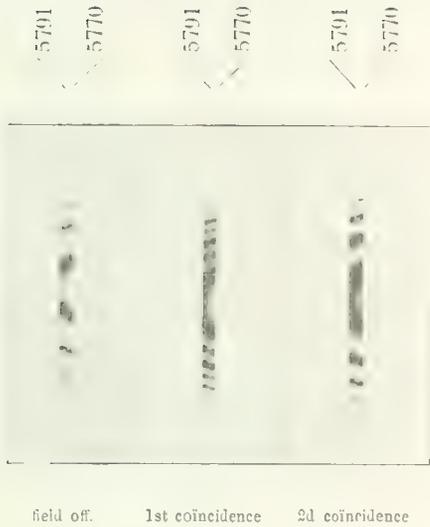
7. In a direction parallel to the magnetic force the two components of the doublet must be placed, according to the elementary theory, symmetrically relatively to the unmodified line. It seemed rather superfluous to test this point. However at the very outset

¹⁾ HALLO. Thesis, Amsterdam 1902. Arch. Néerl. (2) T. 10 p. 148. 1905.

²⁾ GEEST. Thesis, Amsterdam 1904 Arch. Néerl. (2). T. 10, p. 291, 1905.

³⁾ See especially JEAN BECQUEREL. Influence des variations de Température sur la dispersion. Le Radium. 1907.

P. ZEEMAN. New observations concerning asymmetrical triplets.



Looking at the doublets of the lines 5791 and 5770, which were very brilliant, I observed a narrow and extremely weak line between the components of the two lines. This weak line seemed with 5770 precisely midway between the components, with 5791 however it seemed to be displaced somewhat towards the red.

These weak lines evidently are due to reflection of light, radiating nearly at right angles to the direction of the magnetic force, from the inner surface of the capillary of the Geissler tube. LOHMANN¹⁾ investigating the neon lines, observed a similar, but in his case entirely symmetrical perturbation. I found the weak [line] to be linearly polarized, as was to be expected.

The whole image, apart then from the ratio of intensities and the character of the polarization, strikingly resembles the type of effect observed at right angles to the magnetic force. No good photographs showing the extremely weak line at the same time with the two components of the doublet were obtained.

I therefore tried to bring into the field of view the unmodified line at the same time with the doublet. It is well known that the use of a spectrum of comparison in measurements where a high degree of precision is wanted, is not without serious objections. KAYSER²⁾ therefore recommends as the most suitable method to produce the lines necessary for comparison in the source itself. In our case this is naturally out of question.

The sidelong displacement, which the luminous line in the vacuum tube undergoes by the action of the field, makes it already impossible, even if the position of the vacuum tube remains unchanged, accurately to compare a negative taken with the field off, with one taken when the field is on.

The best manner of procedure in the given circumstances therefore seemed to reflect into the spectroscope by means of a semi-silvered mirror the light of a separate vacuum tube placed sideways and to analyse this light simultaneously with that of the tube between the poles. However also this comparison succeeded only incompletely in view of the extreme accuracy wanted. In some comparisons the line of the unmodified source seemed to be in a symmetrical position for line 5770 as well as for 5791. I hesitate however to attach even a very moderate value to this result. The experiments however forcibly suggested the question :

1) LOHMANN, Beiträge zur Kenntniss des ZEEMAN-Phänomens. Dissertation. Halle a. d. S. S. 62. 1907. Zeitschr. f. Wissensch. Photographie. Band 6. Heft 1 u. 2. 1908.

2) KAYSER, Handbuch der Spectroscopie. Band I. p. 732.

Has the middle line of a triplet the same wavelength as the unmodified line?

9. The change of wavelength here contemplated undoubtedly must be extremely small, for no one of the physicists occupied with the radiation phenomena in a magnetic field has, to my knowledge, come across phenomena which decide the question put above this paragraph.

Some observations made with an echelon spectroscope have given me evidence, that different spectral lines and among these the mercury lines undergo in very strong fields displacements of the order of 6 or 10 thousandth parts of an Angström unit, in most cases towards the violet. The matter seems of sufficient interest to be treated in a separate paper, which I hope to give rather soon.

Physics. — *“Change of wavelength of the middle line of triplets.”*
(First Part). By Prof. P. ZEEMAN.

1. In dealing with radiation in a magnetic field it has been tacitly assumed by all experimentalists I know of, that the middle line of triplets or of other symmetrical separation figures occupies the same position in the spectrum as the unmodified line. During a rather detailed investigation of the asymmetrical separation shown by some lines (see the paper immediately preceding) experiments on the light emitted in the direction of the magnetic force showed that symmetry was not always present where it was expected.

The interest attaching to the encountered anomaly suggested the question whether the original line is displaced during magnetisation. The following paper gives sufficient evidence to assert that such is the case. The asymmetrical position of the very weak line observed between the components of the doublet of line 5791 (see § 8 of the paper immediately preceding) is not explained however by this displacement. The contrary is the case. The theoretical interest of the subject is probably intimately connected with the existence of couplings between vibrations parallel and perpendicular to the field¹⁾.

2. For the further discussion I will recapitulate here very briefly the formulæ²⁾ giving for MICHELSON'S echelon grating the angular

¹⁾ Cf. however VOIGT. Annalen d. Phys. Bd. 24, p. 195, 1907.

²⁾ MICHELSON. Journal de Physique, (3), Vol. 8, p. 305, 1899.

Fürst B. GALITZIN. Zur Theorie des Stufenspectroscops. Bull. de l'Acad. Imp. des Sciences St. Pétersbourg 1905 (5) T. 23. N^o. 1 et 2.

dispersion and the distance of adjacent orders of the same line.

Let λ denote the wavelength of the light considered, μ being the index of refraction, θ the angle of diffraction, t the thickness of the glassplates, s the width of the steps of the echelon.

Then in the case of normal transmission:

$$\frac{d\theta}{d\lambda} = \frac{t}{s \cdot \lambda} \left((\mu - 1) - \lambda \cdot \frac{d\mu}{d\lambda} \right) \dots \dots \dots (1)$$

A being the distance of adjacent orders, we have further:

$$A = \frac{\lambda^2}{t \left((\mu - 1) - \lambda \cdot \frac{d\mu}{d\lambda} \right)} \dots \dots \dots (2)$$

3. The most simple hypothesis that could be made was, that of the two lines under consideration only 5791, which exhibits asymmetrical separation and not line 5770, would show a displacement of the middle line.

In order to test this hypothesis I used the echelon spectroscope in such a manner that different lines could come *simultaneously* under comparison. The ordinary manner of using the echelon only permits the examination of one line at the same time or at least only of lines which differ by a small fraction of an ÅNGSTRÖM unit.

We may however place the steps of the echelon in a horizontal position, the slit of the echelon collimator being also horizontal, thus rotating these parts through 90° from the position commonly used. The slit of the auxiliary spectroscope may remain vertical. This arrangement, which is principally that of NEWTON'S crossed prisms or that of GEHRCKE'S "Interferenzpunkte", has the advantage of showing simultaneously the behaviour of different lines. To every spectral line correspond small horizontal lines, the length of which is determined by the width of the slit of the auxiliary spectroscope. It depends

a ————— upon the position of the echelon whether two or one of the orders of a line will be visible. Fig. 1 represents that part of the field of view, which relates to the yellow mercury lines. The lines a and b represent the successive orders of line 5770, a' the only visible order of line 5791, supposing that the small vacuum tube, charged with mercury vapour, is out of the field.

----- a''
 ————— a'

b —————

During the establishment of the magnetic field the well known components are seen moving upwards and downwards. Moreover

any change in wavelength of line 5791, that of the other line remaining constant, must manifest itself in a relative displacement, determined by equations (1) and (2).

Taking two negatives with the smallest possible interval of time any change of position of line a' can be made out by measurement. A small displacement of line a' to a position a'' , was noticed.

4. The annexed table gives in detail the results of measurements on negatives, taken according to the method of § 3 on different days and under somewhat different circumstances.

The echelon used, was described on a former occasion¹⁾; it has 30 plates 7,8 m.m. thick, the depth of the steps being 1 m.m.

The distance of line a to line b is measured in m.m. and indicated as distance $a-b$ and so on. H denotes the strength of field in Gauss.

Plate Nr.	Field on			H in Gauss	Field off		
	Distance				Distance		
	$a-b$	$a-a'$	$b-a'$		$a-b$	$a-a'$	$b-a'$
135	1.215	0.896	0.319	7830	1.219	0.898	0.321
139	1.200	0.891	0.309	10920	1.208	0.898	0.310
140	1.214	0.882	0.332	8580	1.221	0.905	0.316
141	1.147	0.861	0.286	7700	1.150	0.867	0.283
142	1.140	0.849	0.291	7180	1.147	0.862	0.285
144	1.140	0.855	0.285	15120	1.145	0.872	0.273
146	1.136	0.819	0.307	20340	1.143	0.861	0.282
150	1.093	0.746	0.347	23470	1.116	0.818	0.298

It appears from this table that the position of line a' relatively to a and b is changed by magnetization and that the displacement increases with the field intensity.

It is not less clear however that the observed displacement is due not only to a change of wavelength of line 5791, but to a superposition of change of frequency of the two lines considered. Indeed the distance $a-b$, i. e. the distance of the adjacent orders of line 5771, is always smaller in the first section of the table than in the second one. Hence we must conclude to a change of wavelength of

¹⁾ ZEEMAN. These Proceedings 30 November 1901.

line 5771. The amount can easily be given in A. U. The change of $a-b$ amounts to 0,023 m.m. in the strongest field of 23470 Gauss. *Half* this amount determines the change of wavelength. It becomes 0,007 A. U., the distance of two orders of the echelon = 1.116 m.m. corresponding to 0,689 A. U.

5. The simplicity of the results obtained by means of the method of § 4 is considerably diminished by the fact, that line 5770 undergoes a change of wavelength as well as line 5791. The sensibility of the method for discovering *relative* changes of wavelength is very clearly seen by a comparison of the two columns under $a-a'$.

In order however to be sure of a simple interpretation of results and also on account of gain in the intensity of the light I returned, the reality of a change of wave-length being now rather evident, to the arrangement as it is most commonly used. The slit of the auxiliary spectroscope is then parallel to that of the echelon.

The results obtained for the yellow mercury lines are given in the table.

Plate Nr.		Distance of orders, field off in m.m.	Distance of orders, field on in m.m.	H	δ_r in A. U.
160a	5770	1.081	1.067	20100	0.004
160b	5791	1.061	1.044	20100	0.006
160c	5791	1.058	1.050	8900	0.003
161a	5770	1.110	1.089	23800	0.007
161b	5770	1.110	1.106	9000	0.001
164	5770	0.855	0.834	24400	0.009
165a	5791	0.856	0.847	13750	0.004
165b	5770	0.859	0.843	16450	0.007

The observations recorded in the last three columns have been taken with other orders of the echelon. δ_r gives in A. U. the change of wavelength by magnetization. The largest change observed is one of 0.009 A. U. recorded on plate 164 for a field of 24400 Gauss.

The evidence from these experiments tends to confirm those obtained in § 4.

The separate numbers show some discrepancies which needs a discussion, which will be given later on. Before proceeding further,

I think it appropriate however to call attention to the fact that the change of wavelength of the middle line of a triplet seems not to be confined to the light emitted by a Geissler tube.

During the writing of this paper my attention was arrested by a passage in a Thesis of W. HARTMANN¹⁾:

“Es mag schon an dieser Stelle erwähnt werden, dass der Abstand der Ordnungen beim Einschalten des Magnetfeldes sich mehrfach änderte, und zwar im allgemeinen mit wachsender Feldstärke kleiner wurde.

Dieser Änderung würde rein äusserlich betrachtet eine Verkürzung der Wellenlänge entsprechen, doch konnte eine wirkliche Gesetzmässigkeit nicht constatirt werden.”

The observations of HARTMANN were made by means of an echelon spectroscope, the source of light being the self-induction spark in vacuum after MICHELSON'S arrangement. HARTMANN'S negatives concerning copper, iron, gold and chromium were made with fields ranging from 8000 to 12000 Gauss. Perhaps the author would have expressed his opinion with less reserve, if he had operated with stronger fields, in which case the phenomenon is more definite. In the light however of our own observations there seems to be sufficient evidence to conclude, that also the middle lines of the triplets of other metals undergo the kind of change existing in the case of mercury.

Physics. — “*The influence of temperature and magnetisation on selective absorption spectra*”. By Prof. H. E. J. G. DU BOIS and G. J. ELIAS. (Communication from the Boscha-Laboratory.)

§ 1. As soon as the unequalled paramagnetic properties of the compounds of so-called rare earth-metals had been demonstrated²⁾, attention was drawn to the fact that most likely also the magneto-optic phenomena would show important peculiarities; this was done in the following words: “La polarisation rotatoire magnétique “a le signe positif ou négatif pour les composés des différents “métaux de cette série, comme d'ailleurs pour ceux de la série “du fer. Je n'ai pas pu constater jusqu'ici un effet particulier de “l'aimantation sur le spectre d'absorption très caractéristique d'une

1) WALTHER HARTMANN. Das ZEEMAN-Phaenomen im sichtbaren Spectrum von Kupfer, Eisen, Gold und Chrom. Dissertation, Halle a. d. S. 1907. p. 10.

2) H. DU BOIS & O. LIEBKNECHT, Ann. d. Physik (4) 1 p. 196, 1900; St. MEIJER, Ann. d. Physik (4) 1 p. 664, 1900.

“solution d’erbium fortement paramagnétique ; d’ailleurs M. ZEEMAN “lui-même l’avait déjà cherché en vain pour le spectre d’émission “de l’erbine chauffée. Des expériences sont en préparation pour déter- “miner la rotation dans les raies d’absorption mêmes et aux alentours “immédiats”.¹⁾

After results had been published by SCHMAUSS, BATES and WOOD, the agreement of which left much to be desired, the experiments in question were taken up again in this laboratory in 1906, when one of us actually obtained a very peculiar dispersion curve of the magnetic rotation within and near a narrow region of absorption.²⁾ At the same time such determinations proved to be subject to many difficulties, which could only be surmounted by means of specially adapted apparatus ; moreover, simultaneous measurements of other optical properties of the absorbing substances are desirable for completeness’ sake ; this more extensive investigation is now being continued with such improved apparatus.

In the first negative experiments referred to a direct influence of magnetisation in the form of a displacement of the dimly defined absorption bands of an aqueous erbium-solution, in other words a ZEEMAN-effect in the usual sense, could not be observed, the grating used, however, being the same as that used in the present experiments. Naturally the observation of the last-mentioned effect is much simpler and more easily feasible than an adequate and trustworthy measurement of the rotation. However, the relation between those two modes of looking at one and the same phenomenon is so close that either remains undetermined without a rather complete knowledge of the other.

Mr. JEAN BECQUEREL Jr.³⁾ resumed such an investigation on the narrower and more sharply defined absorption bands of some exceedingly rare and small crystal fragments, which we had not at our disposal: xenotime, tysonite, parisite and others, the spectra of which had been formerly determined by HENRI BECQUEREL Sr.⁴⁾ Besides, the influence, already more or less known, of the temperature on such spectra was more fully investigated. The important results obtained may be assumed to be known.

¹⁾ H. DU BOIS, Rapp. Congr. de Phys, 2 p. 499, Paris 1900 ; Ann. d. Phys. (4) 7 p. 944, 1902.

²⁾ G. J. ELIAS, Physik. Zeitschr. 7 p. 931, 1906 (chloride of erbium).

³⁾ J. BECQUEREL, Compt. Rend. 142 pp. 775, 874, 1144, 1906. 143 pp. 769, 890, 962, 1133, 1906. 144 pp. 132, 420, 592, 682, 1032, 1336, 1907. 145 pp. 413, 795, 916, 1150, 1412. Also Physik. Zeitschr. 8 pp. 632, 929, 1907.

⁴⁾ H. BECQUEREL, Ann. Chim. & phys. (6) 14 p. 170, 1888.

§ 2. In a former attempt to classify all elements according to their magnetic properties and those of their compounds, the following remarks were made :

“Les éléments nouvellement reconnus : hélium, argon¹⁾, néon, krypton, xénon, n’ont pas encore été déterminés ; il n’est guère probable qu’ils soient autres que diamagnétiques. On peut classer 63 autres éléments, dont 37 diamagnétiques, 22 paramagnétiques, 4 ferromagnétiques à la température ambiante ; tandis qu’en 7 cas (Be, Mg, Sc, Nb, La, Ta, Th) la classification nous paraît encore plus ou moins douteuse. Dans le système naturel à masses atomiques croissantes, on peut distinguer 7 séries d’éléments paramagnétiques consécutifs, qui les comprennent tous, le signe de l’élément ouvrant chaque série étant seul encore incertain ; les séries d’ordre pair sont moins prononcées au point de vue paramagnétique que celles à numéro impair”²⁾.

These last-mentioned uneven series are :

1) O.

3) Sc (?), Ti, V, Cr, Mn, Fe, Co, Ni, Cu³⁾

5) La (?), Ce, Pr, Nd, Sa, Eu, Gd, Tb, Dp, Er, Yb.

7) Ra (?), Th (?), U.

Now the anorganic compounds⁴⁾ which chiefly absorb light selectively, evidently belong to these paramagnetic series ; this connection is so remarkable that it can hardly be an accidental one.

From this profuse supply of material only a few samples could be chosen ; we have thought that we ought to extend the investigation in the first place to matter which was comparatively easily to be had in larger pieces ; among others to some coloured gems, which are to be considered as dilute solid solutions, to certain microcrystalline salts, but also to amorphous solidified molten matter and to glassy solid solutions in an amorphous substratum, e. g. borax or glass. The crystalline structure gives rise to complications which render the phenomena very intricate, though they are most interesting in themselves. All this yields a rather extensive material of observation, possibly of importance in connection with molecular theories on solid and liquid substances.

With the available cryomagnetic arrangement we could expose matter in liquid air to a strong magnetic field ; in many cases we

¹⁾ Recently confirmed by P. TÄNZLER, Ann. d. Physik. (4) 24 p. 931, 1907.

²⁾ Rapp. Congr. d. Phys. 2 p. 487, Paris 1900.

³⁾ O. LIEBENECHT & A. P. WILLS, Ann. d. Physik (4) 1 p. 186, 1900.

⁴⁾ See H. KAYSER, Handb. d. Spectroscopie 3 ; Leipzig 1905.

could thus study the *simultaneous* influence of the two factors, temperature and magnetisation.

§ 3. For the observation or measurement of the absorption spectra we used, besides a hand-spectroscope, according to circumstances:

1. A RAPS¹⁾ spectrometer with a prism of heavy flint glass; dispersion C—F about 7.5° .

2. An autocollimator made according to our directions by the firm of C. ZEISS, a description of which will soon appear; dispersion C—F, about 25° .

3. The concave grating presented by ROWLAND to Berlin university, and kindly placed at our disposal by Prof. RUBENS; the radius is about 4 m., the number of lines 5684 per cm. The arrangement was as usual in a right-angled triangle with movable constant hypotenuse; a unit of the scale in the spectrum of the first order (half mm), corresponded to 0.23μ .

The calibration of these instruments in wave-lengths was made by means of lines of hydrogen, helium, potassium, and those of a mercury arc lamp between the limits 434 and 770 μ .

The sources of light were according to the required strength of illumination 1. a "Nernst lamp", 2. a "Lilliput" arc lamp (2 Amp.), 3. an arc lamp with horizontally directed and slowly rotating positive-carbon (25 Amp.), 4. sunlight.

The polar pieces of the large ring-electromagnet had slitted or rectangular openings, to which attention was paid in the dioptric determination of the path of the rays along the field's axis.

The cryomagnetic arrangement was that used before²⁾; the level of the liquid air was kept at the lower edge of the openings; the sample could be enclosed in a thick copper frame in order to let it have a temperature as uniform as possible.

Outside the field a small vacuum vessel was used with unsilvered strips for observation; the variations of temperature had to take place as gradually as possible to prevent the samples from bursting and cracking.

The use of higher temperatures up to about 200° does not give rise to any difficulty with this apparatus which may also be arranged pyromagnetically; we hope to revert to this question later on.

Results.

§ 4. We reserve the detailed description of the measured absorption

¹⁾ A. RAPS, Zeitschr. f. Instr. Kunde, 7 p. 269, 1887.

²⁾ H. DU BOIS & A. P. WILLS, Verh. D. Phys. Ges. 1 p. 169, 1899. — F. C. BLAKE, *ibid.* 9 p. 295, 1907.

spectra for a future occasion, and here confine ourselves to the principal characteristics. The wave-lengths have been expressed in $\mu\mu$ with an uncertainty of no more than 0,3 $\mu\mu$.

First series. The well-known rather narrow absorption bands of oxygen (among others 476,7—477,6) only played a secondary and inconvenient part in this preliminary investigation; for when the samples under observation are immersed in liquid air their absorption spectra become impure, which should be duly taken into account. We have not yet succeeded in observing the absorption spectrum of the strongly paramagnetic liquid oxygen in a field of sufficient intensity. O_3 , NO_2 and NO_3 are also of importance¹⁾.

§ 5. Third series. Here chromium is of special importance. It derives its name from its coloured compounds, which mostly show dichroism and the well-known transmutation of colour with change of temperature. We examined:

Chromium alum [$Cr K (SO_4)_2$]; diluted green aqueous solution. At 18° light band 662,7—672,5; fainter band 688,1—726,4. A plate of alum 2 m.m. thick exhibited pretty narrow bands in liquid air, some of which were slightly affected by magnetism.

Chromium-potassium oxalate [$Cr_2 K_2 (C_2 O_4)_6 + 6 H_2 O$]; strongly dichroitic (red-blue) small monoclinic crystals, which were cemented on a covering glass and ground to a thickness of about $\frac{1}{2}$ m.m.

Plane of polarisation \parallel long sides: at 18° a bright band 698,1—703,7; at -193° it lay 696,4—701,4.

Plane of polarisation \perp long sides: at 18° bright band 697,5—703,5; at -193° it lay 696,4—701,4.

An aqueous solution exhibited at 18° a broad band 693,2—702,3, the maximum of which lay at 695,4—699,3; moreover a very faint band 708,4—711,0.

A solution in glycerin had this broad band at 18° from 694,9—699,4. At a temperature higher than that of liquid air (roughly estimated at -130°): faint band 659,3—664,9 (possibly not simple), faint band 669,0—671,2, stronger band 674,7—676,8; halfshade limit at 681,8. Very strong band 694,8—698,1, shade to 700,3; beginning of region of absorption 706,0²⁾.

"*Chromium borax*" obtained by melting together 5—15 % chromium fluorite with anhydrous borax, in the way of the borax-pearls used in analytical chemistry; ground, polished and varnished in order to

¹⁾ E. WARBURG & G. LEITHÄUSER. Ann. d. Physik (4) 23, p. 209, 1907.

E. WIEDEMANN, Wied. Ann. 5 p. 515, 1878. W. LAPRAIK, Journ. f. prakt. Chemie (2) 47 p. 307, 1893. G. B. RIZZO, Nuov. Cim. (3) 35 p. 132, 1894.

prevent decomposition on exposure to air: smaragdine amorphous plates about 3 m.m. thick.

At 18° light band 673,7—681,4; vague indistinct band 695,3—736,9. At —193° light band 672,6—680,8; dimly defined band 692,3—747,6.

Natural emerald. [$\text{Be}_3\text{Al}_2(\text{SiO}_3)_6$]; hexagonal, coloured by a few permilles Cr_2O_3 ; sensibly dichroitic (grassgreen-seagreen). Worthless light green specimen not quite transparent, provided at the laboratory with parallel facets, thickness 6 mm.

In the *ordinary* spectrum at 18° rather strong band 679,0—680,7; somewhat stronger band 682,4—685,0; at —193° strong band 678,2—679,5, and still stronger band 681,8—683,4.

In the *extraordinary* spectrum the bands were much paler but at the same places, their relative widths being interchanged.

§ 6. *Ruby.* [Al_2O_3]; rhombohedral, solid solution of a little Cr_2O_3 ; dichroitic (purplered-brickred). By the kindness of Mr. M. A. WOLFF—DE BEER, manager of the Amsterdam diamond factory, several natural and several artificial rubies were placed at our disposal. The last-mentioned rough material is imported from Paris in the form of cones ¹⁾; by cleaving them we got pieces of about uniform colour and crystallographic orientation, as is to be observed by means of the dichroscope. With carborundum a square plate ($7 \times 7 \times 3$ mm.) was ground and polished at the laboratory, a side of which contained the direction of the axis; most of the experiments were made with it. There is no reason to suppose that natural ruby would show other properties than the artificial material.

This stone absorbs green and yellow light. In the above investigations of J. BECQUEREL we only found it briefly stated that “a group lying between 657 and 676 disappears in liquid air, that the band at 697 becomes thinner and the band at 705 broader and intenser than at ordinary temperature;” no mention is made of any magnetic influence ²⁾.

Further ruby was more closely investigated by MIETHE, who found the two principal absorption bands at 694 and 696, with a breadth of about $0.4 \mu\mu$, besides 6 bands of less importance. Moreover he described the remarkable fluorescence spectrum; the latter we have no further examined, as Prof. MIETHE intended to proceed with his experiments on this subject ³⁾.

¹⁾ M. DUBOIN, Compt. Rend. 134 p. 840, 1902; A. VERNEUIL, ibid. 135 p. 791, 1902.

²⁾ J. BECQUEREL, Physik. Zeitschr. 8 p. 932, 1907 (Sept.).

³⁾ A. MIETHE, Verh. D. physik. Ges. 9 p. 715, 1907 (Nov.).

§ 7. With the spectrometer we found in the *ordinary* spectrum of ruby: At 18° strong band 692,4—692,6; very strong band 693,9—694,2; at -193° strong line 691,7; stronger line 693,2. So with decrease of temperature a displacement of $0.7 \mu\mu$ takes place towards violet, the mutual distance of the double line retaining the value $1.5 \mu\mu$, however, as it is also found between the centre-lines of the two bands. In the extraordinary spectrum, at 18° band 692,5, fainter band 694,1; at -193° faint line 691,7, fainter line 693,2. In both spectra about 8 more bands of less importance, which cannot all be discussed here.

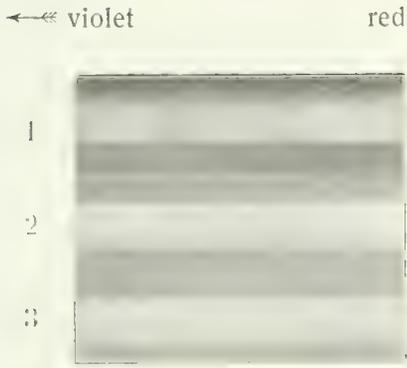
In the extraordinary grating spectrum the bands have faded away so much that an exact observation is impossible; hence only the *ordinary* spectrum was observed here. Though the bands are not sharply defined, they were estimated at about the same breadth by the two observers, viz. at 18° : band 692,5 at 0,23, band 694,1 at $0.33 \mu\mu$. In a longitudinal magnetic field, estimated at about 30 kilogauss the breadths amounted to 0,37 resp. $0.49 \mu\mu$ in a situation slightly shifted towards violet; the increase in breadth was therefore 0,14 and $0.16 \mu\mu$ respectively.

With unpolarised light the phenomena were pretty much the same as the photo shows; it was obtained on a "LUMIÈRE B-plate" sensitized for red light with alizarin blue and nigrosin with an exposition of 20 minutes: 1. indicates the position of the ruby bands in the solar spectrum; 2. those in the arc-lamp spectrum outside the field; 3. the same in the field. The apparent broadening is evidently caused by a doublet of vaguely outlined bands, as sufficiently appears from the connection with what follows.

If the ruby was cooled down to about -193° , band 692,5 appeared to contract to a breadth of about 0,06, band 694,1 to $0,08 \mu\mu$. In fact they have then become *absorption lines*, which though thicker than FRAUNHOFER lines, are yet no broader than e.g. those of a dense sodium vapour; by this gradual transition any distinction between absorption bands and lines vanishes.

In the field an ordinary ZEEMAN-effect was now observed as follows: line 692,5 resolves into two components at a mean distance of $0.25 \mu\mu$, the space between somewhat less bright than the spectral background; the components of line 694,1 had a distance of $0,35 \mu\mu$ with a somewhat darker space between; possibly due to the intermediate lines of a quadruplet; for both a slight displacement towards violet was to be perceived. So the separation at -193° seems to be greater than the above broadening at 18° . These numerical determinations cannot yet be considered as definite or accurate.

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1. Ruby bands in the solar spectrum.
2. Ruby bands outside the field at 18°.
3. Ruby bands in the field at 18°.

The phenomenon was very clearly visible, and the amount of the separation is evidently pretty large. Chiefly in consequence of rime only a short observation was possible, which frustrated photographing with long exposition. Our $\frac{3}{4}$ plate proving unfit, we could not yet determine the circular polarisation; nor did our arrangements allow the observation of a transversal effect for equatorial direction of rays; we hope to be able to remove these difficulties in course of time.

§ 8. Of the other metals of the third series there exist among others also the following compounds with characteristic absorption spectra ¹⁾: Sapphire, CoCl_3 , KMnO_4 , FeCl_3 , the last mentioned with strongly negative rotation, and some compound rhodanides. Of these we examined:

Cobalt-ammonium rhodanide $[(\text{NH}_4)_2\text{Co}(\text{CNS})_4]$ in rather diluted alcoholic solution: at 18° dimly outlined band 594—663; limit of absorption at 696. At -193° the blue solution became solid, but remained transparent, though of a paler colour; the bands become narrower as the temperature falls; at last vague bands are seen 580—586, 597—605, and 618—620; shade up to 645.

§ 9. Fifth series. This now includes, arranged according to increasing atomic weights from 140 to 175: Cerium, *Praseodymium*, *Neodymium*, *Samarium*, [*Europium*], Gadolinium, [*Terbium*, *Dysprosium*], *Erbium*, Ytterbium ²⁾; the three placed between brackets were not yet obtainable in 1899; the compounds of the others then proved strongly paramagnetic, with a maximum for erbium. The visible absorption spectra of the metal compounds printed in italics exhibit the well-known highly selective properties; possibly an accurate investigation would yield a similar result also for the "white" compounds of Ce, Gd and Yb, perhaps only in the ultra-violet or the infra-red spectrum. We have confined ourselves for the present to compounds of neodymium and erbium, which had been used for the magnetic measurements in 1899. We wish to express our indebtedness to Prof. ROSENHEIM and Dr. R. J. MEIJER for their kind assistance in this special chemical department.

¹⁾ Cf. J. M. HIEBENDAAL, thesis for the doctorate, Utrecht 1873.

²⁾ Cf. R. J. MEIJER, Handb. der anorg. Chemie 3 p. 129—338, Leipzig 1906. Scandium Yttrium and Lanthanum have lower atomic weights and diamagnetic compounds. Holmium and Thulium have not yet been sufficiently chemically determined. Further the following investigations have lately been published on the absorption spectra: W. RECH Zeitschr. wiss. Photogr. 3 p. 411, 1906; — HELEN SCHAEFFER, Physik. Zeitschr. 7 p. 822, 1906; B. SCHAEFFERS, Dissert. Bonn 1907.

Neodymium nitrate $[\text{Nd}(\text{NO}_3)_3 \cdot 6 \text{H}_2\text{O}]$. The water of crystallisation is expelled during moderate heating, and on cooling a fine rosy-red amorphous transparent mass is retained, which remains unchanged for some time. In course of time a kind of crystalline light foam deposits on the surface and the walls of the test-tube attended by absorption of water, which foam could be collected and be pressed into thin films by the aid of zapon-varnish.

The amorphous nitrate exhibited numerous bands, among which a rather well defined one at about 625 (with companion 626). At -193° it had a breadth of 0,3, in a field of about 25 kilogauss one of 0,4 $\mu\mu$. By chance we noticed that the crystalline nitrate foam at 193° exhibits much narrower bands than an amorphous layer of the same thickness; among others in the neighbourhood of D : 577.0, 578.7, 579.7, 582.0, 583.0; the three last of these have a breadth of about 0,15 $\mu\mu$; in the field a very slight, hardly measurable broadening seems to take place. The 5 sharply defined bands in the green between 500 and 525 behave in an analogous manner.

Neodymium magnesium nitrate $[2\text{Nd}(\text{NO}_3)_3 \cdot 3 \text{Mg}(\text{NO}_3)_2 \cdot 24 \text{H}_2\text{O}]$. Yields analogous products when treated as before.

It has a very complex absorption spectrum with a great number of narrow bands, which are very pronounced particularly at the temperature of liquid air. Particularly worth mentioning is the following, observed with a thickness of about 10 m.m. of the amorphous mass:

At 18° a region of absorption extending from 499.2—537.7, consisting of 499.2—513.9 very strong band, very badly defined towards violet; 516.9—528.2 very strong band, then shade; and 534.8 narrow faint line; 535.6—537.1 band with maxima 535.6—536.4 and 536.7—537.1.

At -193° this region extends from 498.1—526.6 and consists of 498.1—512.6 very strong band, then bands at 513.7—514.1, 514.7 515.3, 516.1—516.7, then shade, 517.6—518.5 band, then again shade, and 519.2—526.0 very strong band.

Another remarkable region extends at 18° from 616.9—629.3, which consists of 616.9—618.9 faint band; 621.0—622.9 ditto; 624.8—626.9 pretty strong band; 628.3—629.3 very faint band. At

193° we see what follows: 616.7—617.6 faint band; 618.9 thin, faint line; 621.1—623.0 faint band; 624.7—625.4 band with strong fine centre; 625.9—626.4 ditto; 627.1 thin faint line; 628.6—629.5 faint band.

In thinner layers these regions diverge even more; e. g. a rather narrow absorption region branches off from the region 560.1—587.5

on the red side, which with decrease of temperature first becomes more distinct, but then vanishes at still lower temperature. Also for this double nitrate the crystalline foam exhibited more sharply defined bands at -193° ; the 5 bands in the green from 500—525 particularly the first and the fourth were broadened in the field.

Neodymium borax, obtained by melting together about 5—10% neodymium oxide with anhydrous borax; pink amorphous mass.

With decrease of temperature the spectrum is also subjected to important modifications, but displays in general wider bands than the preceding one. Analogous in this respect are:

Neodymium glass, prepared for us by the firm of SCHOTT & Co. at Jena (V.S. 5255 and 5256), with 15% and 20% cerite; it contains so much neodymium that it has a pink colour, and exhibits strong selective absorption.

§ 10. *Erbium nitrate* [$\text{Er}(\text{NO}_3)_3 \cdot 6\text{H}_2\text{O}$]. By evaporation of the solution this, too, may be obtained as an amorphous transparent mass, which has an absorption spectrum rich in narrow bands, which, however, has not yet been further examined.

Erbium magnesium nitrate [$2\text{Er}(\text{NO}_3)_3 \cdot 3\text{Mg}(\text{NO}_3)_2 \cdot 24\text{H}_2\text{O}$].

Treated just as the neodymium salt; yellow transparent mass; readily absorbs water of crystallisation and becomes crystalline and opaque.

Like the preceding salt it shows a very complex spectrum, of which the following groups, measured for a sample of about 10 m.m. thickness, are particularly noteworthy:

At 18° : 514.5—527.3 group of bands, consisting of three intense bands, at 516.9—517.2; 517.7—519.5; 520.5—521.7. At -193° this becomes as follows: 513.7—521.8 group of bands, consisting of: 513.8—514.0 bands; 514.8—514.9 intense band; 515.1—515.7 band; 516.3—517.2 intense band; 517.7—519.3 intense band; 520.1 faint band; 521.0—521.8 intense band.

At -193° a number of double lines in the red are of importance, the principal of which lie at 641.4 and 642.6; 643.7 and 645.6; 647.7 and 649.4. At 18° these lines have entirely vanished.

The whole visible spectrum of this salt contains about 40 bands and lines, partly very faint, which vanish for a thinner layer.

Erbium borax obtained by melting together 15—20% erbium oxide with anhydrous borax; yellow, amorphous, transparent.

Here decrease of temperature has not so important an influence.

The most remarkable group of bands, extending at 18° from

516.7—525.6 contains three bands at 516.9—517.2; 518.0—519.1; 520.6—522.2.

At -193° this group extends from 516.2—523.9 and then exhibits three bands too, at 516.6—517.0; 517.8—519.0 (these two pretty sharply defined); 520.4—522.4 (less sharply defined, perhaps double).

Erbium glass was prepared for us by the firm SCHOTT & Co. at Jena (V.S. 5257, about the same as V.S. 3524).

It shows some narrow, not very sharply defined bands.

The well-known group of bands in the green extends at 18° from 516.8—523.0, and consists of: 517.0—517.3 band; 517.8 line; 518.4—518.9 band; 519.1 indistinct line; 520.0—520.2 band; 521.1—521.8 band.

At -193° it extends from 516.5—522.6 and shows what follows: 516.5—517.2 band; 517.9 line; 518.2—518.6 band; 519.0 line; 519.8—520.0 band; 521.0—521.2 band; 521.8—522.3 band.

Further the following lines are found at -193° :

648.9; 651.6; 655.5; 657.6, all faint. At 18° faint bands are seen at 650.0—654.0 and 656.4—661.1.

§ 11. Seventh series. We have until now investigated:

Uranyle nitrate $[\text{UO}_2(\text{NO}_3)_2 + 6\text{H}_2\text{O}]$, monoclinic crystals, not sensibly dichroitic; ground plate 2 m.m. thick. In the blue the well-known bands, of which we mention: at 18° two faint undefined bands 467.5—471.6 and 484.9—488.0; at -193° strong band 467.9—469.7 (possibly not simple), shade to 470.3; strong sharply defined band 484.5—484.9, then shade; idem 485.3—485.7, shade to 486.3

Berlin, Febr. 27, 1908.

Physics. — “On the measurement of very low temperatures. XIX. Derivation of the pressure coefficient of helium for the international helium thermometer and the reduction of the readings on the helium thermometer to the absolute scale”. Communication N^o. 102^b from the Physical Laboratory at Leiden. By Prof. H. KAMERLINGH ONNES.

§ 1. *Pressure coefficients of helium.* As the absolute zero is known with sufficient accuracy — from the Leiden observations on hydrogen may be derived $T_{0^{\circ}\text{C.}} = 273^{\circ}.10 \text{ K.}$ ¹⁾ a value which, because it agrees with other determinations, is probably not far from the true one — we may by means of the virial coefficients B_A for helium at 0° C. and 100° C. , determined in the preceding communication (102^a), derive the pressure coefficients of helium at different densities for this range of temperature. For the pressure coefficient of the *international helium thermometer* ²⁾ i. e. the mean relative pressure coefficient from 0° C. to 100° C. for helium with the density belonging to the zero pressure of 1000 m.m. $\left[\begin{matrix} 0^{\circ} \text{ C.} - 100^{\circ} \text{ C.} \\ \alpha_r \end{matrix} \right]_i$ or for shortness ${}_i\alpha_r$, the formula

$$100 \cdot {}_i\alpha_r = \frac{A_{A_0} \times 0.36617 + (B_{A,100^{\circ}\text{C.}} - B_{A,0^{\circ}\text{C.}}) \frac{100}{76}}{A_{A_0} + B_{A,0^{\circ}\text{C.}} \frac{100}{76}} \dots (1)$$

yields

$$\text{He } {}_i\alpha_r = 0.0036613.$$

If one considers that according to table II of Comm. N^o. 102^a the isotherm of 0° gives rather large values for Obs.—Comp., then it seems that the isotherm of 20° C. , where the Obs.—Comp. are only small, are more reliable for the derivation given above.

¹⁾ In Comm. N^o. 101^b the value $273^{\circ}.08$ is found, but as will be explained in Comm. N^o. 102^b, an erratum to Comm. N^o. 97^b XV, the pressure coefficient 0.0036627 for hydrogen at 1090 m.M. must be restored instead of 0.0036629 which was derived in the above mentioned communication and used for a certain time. It is to be noted that the difference introduced by this recalculation is not greater than the other observational errors. The small differences between some numbers of this communication with the Dutch text are the consequence of this correction.

²⁾ The scale of the hydrogen thermometer of constant volume at 1000 m.m. zero pressure is generally called the scale of the normal hydrogen thermometer (this was also done in Comm. N^o. 97^b). As 0° C. and 760 m.m. are accepted as the normal state for gases, it seems to me preferable to call the scale just mentioned the scale of the *international* hydrogen thermometer. In the same way we must speak of the international helium thermometer.

Therefore I have calculated $\left[\begin{smallmatrix} 0^{\circ} & 100^{\circ} \\ \alpha_p & \end{smallmatrix} \right]_{1000 \text{ mm.}}$ by means of the data for 20° C. and $100^{\circ} .35 \text{ C.}$ With neglect of the deviations from the absolute scale for the hydrogen thermometer at 20° C. , $B_{A, 0^{\circ} \text{ C.}}$ was determined by means of rectilinear extrapolation. This gave

$$B_{A, 0^{\circ} \text{ C.}} = 0.0,499,$$

whence

$$A_{A_0} = 0.99950.$$

With these new data we derive from formula (1) of this section

$$\text{He} \alpha_p = 0.0036616.$$

From the data for B_A of table II of the preceding Comm. and $T_{0^{\circ} \text{ C.}} = 273^{\circ} .10 \text{ K}$ we may determine in the way mentioned in § 2 of Comm. N^o. 97^b the corrections of the readings of the helium thermometer of constant volume with a given zero pressure to the absolute scale. They have been calculated for a zero pressure of 1000 m.m. and are combined in table I where the remaining columns have the same signification as the corresponding ones of table XVI of Comm. N^o. 97^b.

TABLE I. Correction of the international helium thermometer to the absolute scale.

θ	$10^3 \cdot B' T$	$(\Delta t)_a$	$(\Delta t)_b$
$100^{\circ} .00$	+ 0.492		
$0^{\circ} (a)$	+ 0.513		
$0^{\circ} (b)$	+ 0.500		
— $103^{\circ} .57$	+ 0.544	+ 0.0034	— 0.006
— $182^{\circ} .75$	+ 0.532	+ 0.0158	+ 0.002
— $216^{\circ} .56$	+ 0.463	+ 0.0252	+ 0.010

The corrections indicated with (a) are derived by means of the values of $B_{A, 0^{\circ} \text{ C.}}$ from the direct determination, for (b) we have used the value which is recalculated with $B_{A, 120^{\circ}}$ (comp. the preceding Comm.). It is probable that on account of what has been said in the preceding section those of column (b) are the most reliable.

§ 3. Determinations of other observers.

For a comparison with the results of the two preceding sections

we can only use the determinations of TRAVERS, SENTER and JACQUEROD.¹⁾ They have found:

1st. for the pressure coefficient of the helium thermometer at 700 m.m. zero pressure $\left[\alpha_v^{0^{\circ}\text{C}-100^{\circ}\text{C}} \right]_{700} = 0.00366255$ which agrees with 0.0036628 for $\left[\alpha_v^{0^{\circ}\text{C}-100^{\circ}\text{C}} \right]_i$ and

2nd. for the difference between the indications of the helium thermometer t_{He} and the hydrogen thermometer t_{H_2} (each of about 1000 m.m. zero pressure) at the boiling point of oxygen ($t_{\text{H}_2} - t_{\text{He}})_{-180^{\circ}\text{C.}} = 0^{\circ}.10$, and at that of hydrogen ($t_{\text{H}_2} - t_{\text{He}})_{-252^{\circ}\text{C.}} = 0^{\circ}.20$, which differences are so considerable that CALLENDAR²⁾ concludes thence that the corrections of the helium thermometer to the absolute scale are negative.

The two results which strongly deviate from mine may be readily explained if one adopts that the determination of the coefficient of pressure variation of helium by TRAVERS, SENTER and JACQUEROD has not yielded the true value. For if the differences in indication found by them between their helium- and their hydrogen thermometer are reduced by means of the corrections of each of these thermometers to the absolute scale which are given in Comm. 100^r and in Table I of this Comm., to the difference in readings on the absolute scale, which are found at the same temperature by means of the hydrogen thermometer which gives θ_{H_2} and by means of the helium thermometer which gives θ_{He} , there remains at -182° a difference

$$(\theta_{\text{H}_2} - \theta_{\text{He}})_{-182^{\circ}} = 0^{\circ}.10 - 0^{\circ}.049 - 0^{\circ}.002 = 0^{\circ}.05$$

while by extrapolation of the corrections found to -217° for -252° one would find

$$(\theta_{\text{H}_2} - \theta_{\text{He}})_{-252^{\circ}} = 0^{\circ}.20 - 0^{\circ}.12 - 0^{\circ}.02 = 0^{\circ}.06$$

When calculating the temperatures t_{H_2} and t_{He} the investigators mentioned have taken the pressure coefficient of the helium thermometer $\left(\alpha_v^{\text{He}} \right)_i$ Travers to be equal to that of the hydrogen thermometer at the same zero pressure (for 1000 m.m. therefore 0.0036626 according to our value of Comm. N^o. 60). If the corrections applied by me are right that pressure coefficient must therefore, at -182° in order that $\theta_{\text{H}_2} - \theta_{\text{He}} = 0$ be diminished by 0.0000010 so that

$$\alpha_v^{\text{He}} = 0.0036616$$

¹⁾ Phil. Trans. Ser. A. Vol. 200 p. 105-180. KUENEN and RANDALL (Proc. Roy Soc. Vol. 59) have made a determination, which, being only intended to show whether the helium behaved normally, is not made to the high degree of accuracy which is required for a comparison with isothermal determinations.

²⁾ Phil. Mag. [6] 5. 1903.

and in order $\theta_{\text{H}_2} - \theta_{\text{He}}$ at -252° by 0,0000013 to that
 $a_r^{\text{He}} = 0,0036614.$

The first value which has been derived without extrapolation and which is therefore the most reliable, appears to agree perfectly with the one derived by me from the isothermals in § 1.

With regard to the method of derivation followed here we may remark that it allows of a fairly large accuracy. Though the certainty of the determinations of temperature on which it is based may be doubted to the absolute value, yet the only difference which comes into account here is known with sufficient certainty. The calculation mentioned above therefore not only gives an explanation of the too large differences found by TRAVERS, SENTER and JACQUEROD, but is also a welcome control for the coefficient of pressure variation of helium found in section 1.

Physics. — “*The absorption spectra of the compounds of the rare earths at the temperatures obtainable with liquid hydrogen, and their change by the magnetic field*”, by JEAN BECQUEREL and H. KAMERLINGH ONNES. Communication N^o. 103 from the Physical Laboratory at Leiden.

§ 1. *Introduction.* The investigations of one of us (J. B.)¹⁾ proved that the absorption spectra of the compounds of the rare earths, cooled down to the temperature of liquid air, may serve to acquire new data for the nature, the number, and the motion of the electrons which play a part in the formation of these spectra. So it seemed to us of great importance to continue these investigations at the temperatures obtainable with liquid hydrogen, which are so many times lower and seem particularly adapted²⁾ to reveal the forces which the ponderable substance exerts on the electrons. For this purpose the apparatus used at Paris for the observation of the spectra were conveyed to the cryogenic laboratory at Leiden, so that we were enabled to obtain some three hundred of spectrograms which represent the observed phenomena. The study of these photographs will take a long time; we shall therefore confine ourselves on this occasion to the communication of some facts which immediately draw the attention.

¹⁾ JEAN BECQUEREL, Radium IV. 9, p. 328 and IV, 11, p. 385 (1907).

²⁾ H. KAMERLINGH ONNES, The importance of accurate measurements at very low temperatures. Comm. of the phys. lab. of Leiden Suppl. no. 9, p. 25 sqq. (1904).

§ 2. *Apparatus.* In the first place a few words about the arrangement of the experiments. It was the same for the experiments without and with the magnetic field. The crystals, fixed with wax on a small piece of platinum foil a , (a_2 , a_3 fig. 3^a, Pl. I), which was carried by a rod a_4 , were immersed in liquid hydrogen in a double-walled tube (b fig. 2, fig. 3^a), which is the continuation of a non-silvered vacuum glass b_2 , which contained liquid hydrogen and which is surrounded by another double-walled (c_{10} , c_{20}) tube c , also the continuation of a non-silvered vacuum glass with liquid air, on which it rests on pieces of cork b_3 . A clearance of $\frac{1}{2}$ mm. between the two glasses (fig 3^a) proved sufficient to allow the liquid air to circulate along the hydrogen tube. This protects the hydrogen so effectively from access of heat that the evaporation is insignificant, even when the two tubes are placed between the hot coils of the magnet and the crystal is exposed to strongly concentrated electric light.

The walls of the narrow part of the tubes are very thin, and because the radiation of heat is independent of the distance of the walls they have been brought to an exceedingly small distance from each other (0.5 mm.), but without being anywhere in contact. Owing to the skill of Mr. KESSELING, glassblower of the laboratory, who succeeded in doing this, we had at our disposal a tube of 4 mm. inner diameter filled with liquid hydrogen, protected by a tube of liquid air, the outer diameter of which is no more than 8 mm., which allows us to bring the poles of the magnet so near together that very strong fields are obtained even with hollow poles.¹⁾

The hydrogen tube must be closed hermetically. For this purpose it is fastened in a cap, d , which may be adjusted by means of a levelling board, f , with screws and sliding groove. The tube is brought from below into the cap, where it rests against a wooden cylinder, within d_0 (fig. 2), and it is fastened with a thin rubber ring e_1 , which lies round d_0 doubled over and is turned down when the tube is put in. To ensure tightness a rubber solution is put between ring and glass, and the rubber is pressed tight against the glass and the cap with copper wire. The cap is provided with: 1. the tube d_{30} , to which at d_{31} a head with packing cap d_{32} is screwed, in which the rod a_6 can turn (by means of a_7), and move up and down (by means of the nut d_{33}). 2. a tube d_1 to siphon over liquid hydrogen as

1) Instead of the usual poles of the WEISS magnet we have used auxiliary pieces, p_{30} (see figs. 2 and 3), which prolong the cone to a section of 6 mm. diameter, with conic perforations, which have a diameter of 3 mm. on the side of the crystal.

indicated in Comm. N^o. 94 from the supply bottle into the apparatus, which tube is closed in other cases with a rubber tube with cock. 3. an outflow tube d_2 (fig. 2), which leads along cock k (fig. 1 and fig. 4) to the gasholder with pure hydrogen, to a safety tube l (fig. 1), along k_2 to an airpump, and past k_4 to the vacuum bottle r , from which the liquid hydrogen is siphoned over (the operation is elucidated by the diagrammatic fig. 4, which does not call for a further description).

We first have convinced ourselves that when the air has been exhausted from the hydrogen tube surrounded by e , this tube exactly occupies its place between the poles, without being strained by the supports g and i , when these have a suitable position, we then fill it along k_1 with hydrogen from the gasholder, exhausting it repeatedly, then we pour liquid air through a funnel with filter into b_1 , which is covered with some cotton wool. The apparatus is then filled with liquid hydrogen through d_1 . In order to pass to the melting point of hydrogen, k_2 is opened till crystals appear on the surface of the liquid hydrogen, through which the gas bubbles which rise from the heated crystal, are seen to make their way. If the apparatus has been filled in the way described before, observations with the crystals may be made uninterruptedly for several hours. The precautions taken to prevent mixing of hydrogen and air are indispensable. Air entering the apparatus, would sink down, and be sucked up in front of the crystal as soon as the magnetic field is applied, and intercept the light.

For every filling of the apparatus $\frac{1}{4}$ liter of liquid hydrogen from the supply is generally used, and it was sufficient to do this twice a day to be able to observe all the day in case of ordinary as well as of low pressure: twice a week a quantity of 5 liters was prepared for these experiments, which was just sufficient to fill the apparatus also the second day after the preparation. As it was impossible to entirely prevent the hydrogen which evaporated at lowered pressure from being contaminated with air, it was not admitted again into the cycle. The hydrogen cycle proved its reliability by never failing us a single time in all these weeks.

I. PHENOMENA WHICH DEPEND SOLELY ON THE TEMPERATURE.

§ 3 *Simplification of the spectra.* On cooling to the temperature of liquid air ($T = 85^\circ$) one of us had found¹⁾ that almost all bands become narrower and divide, some new ones also appearing. In

¹⁾ JEAN BECQUEREL, l. c.

general their intensity increases. The bands which decrease in intensity or which vanish altogether, are exceedingly few in number. The measurements on anomalous dispersion in the neighbourhood of some bands of tysonite had proved that this increase of intensity is not only the consequence of the bands becoming narrower, but also of a modification which, according to the theory of electrons on the supposition of quasi-elastic forces, indicates the increase of the dielectric constant in every band, and implies that the number of electrons which determine such a band, has increased.

Passing to the temperature of liquid hydrogen ($T = 20^\circ$), we saw some bands continue to increase in intensity, but also others which showed an increasing absorption with fall of temperature down to that of liquid air, decrease both in intensity and in breadth. There are even bands having appeared in liquid air, which become almost invisible in liquid hydrogen. An example of such a change with the temperature is furnished by the bands 523.5 and 479.1 of tysonite.

The measurements of the anomalous dispersion in the neighbourhood of these bands had shown that the electrons belonging to these bands are about twice or three times as numerous at the temperature of liquid air as at the ordinary temperature. In liquid hydrogen the number has already become very small, and at the temperature of solid hydrogen (14°) hardly any electrons of this kind take part in the motion. Fig. 1, Pl. II, which represents the compensator fringes ¹⁾ in the neighbourhood of band 523.5 of tysonite at different temperatures and with different thickness, allows us to measure the disturbance in the fringe with regard to height and breadth. Figs. 2 and 3, which we treat in § 8 and ²⁾, and which represent the magneto-optic phenomena, may elucidate this.

§ 4. *Maximum of intensity of every band for a definite temperature.* It follows from the foregoing that several bands pass through a maximum of intensity with decrease of the temperature. In general the place of this maximum is different for different bands. When in the experiment with tysonite described in § 3 we wait till the last traces of hydrogen evaporate from the crystal, immediately after when the temperature of the crystal rises, the band 523.5 is seen to greatly increase in intensity. Without doubt the maximum for this band lies at a temperature not far above the boiling point of hydrogen. All the crystals of xenotime, tysonite, parisite, apatite, monazite, didymium sulphate, praseodymium sulphate, neodymium sulphate, exhibit

¹⁾ JEAN BECQUEREL, Radium IV no. 9 p. 328.

similar phenomena. The green line 523,5 of neodymium which is exceedingly fine and sharp at $T = 20^\circ$, has almost vanished at $T = 14^\circ$.

We have further examined the influence of the fall from $T = 91^\circ$ to $T = 58^\circ$ by immersing the crystals in liquid oxygen boiling at the airpump. The change in this region is only slight. This confirms the conclusion drawn from what was observed in heating from $T = 20^\circ$ upwards that the maximum must lie near this latter temperature and at all events far below $T = 58^\circ$.

Naturally the question obtrudes itself whether those few bands, whose intensity diminishes between the ordinary temperature and that of liquid air, do not also pass through a maximum either between $T = 290^\circ$ and $T = 95^\circ$, or at a temperature above $T = 290^\circ$. It will be difficult to decide the question, because in consequence of the broadening and overlapping of the bands the change of each of these bands in itself escapes observation.

§ 5. *Change in width.* In the previous experiments¹⁾ it had been found generally valid for all bands measured down to the temperature of liquid air, that the width of the bands was proportional to the square root of the absolute temperature. This is the law which for the case of a gas may be deduced from the formulæ formerly developed by LORENTZ²⁾

When we pass to the temperature of liquid hydrogen this law appears to be no longer valid for some bands, whereas for others the order of magnitude of the change seems to remain the same. In the figures 1, Pl. II obtained by the method of the compensator fringes, it is very clearly to be seen, that 523,5 of tysonite is not half as broad at $T = 20^\circ$ as at $T = 85^\circ$, as the law of the \sqrt{T} would require. And it was this very band which had served to show experimentally, that this law held down to $T = 85^\circ$ with a high degree of approximation.

The question whether there is a *minimum of width*, could not be solved yet. At first sight some bands do not seem to contract any further between $T = 20^\circ$ and $T = 14^\circ$, two of xenotime seem even to get wider.

With regard to the totality of the phenomena of change of width in liquid and solid hydrogen we may further observe that in these even more than in liquid air³⁾ the spectra manifest a pronounced

1) JEAN BECQUEREL, Radium IV no. 9 p. 328.

2) H. A. LORENTZ. Kon. Akad. v. W. VI p. 506 and p. 555 (1898).

3) That tysonite and xenotime have this tendency has been observed by JEAN BECQUEREL, Radium I. c.

tendency to assume the character of gas spectra when the temperature decreases. Some absorption lines of praseodymium and neodymium sulphate, cleared of broad bands that covered them, are even finer than the *D*-lines.

§ 6. *The approach to a limit of the double refraction of crystals in the non absorbed parts of the spectrum.* If we watch the bands, by the aid of which the double refraction is investigated, with change of temperature, we observe the following. If the crystal is heated above the ordinary temperature, they are greatly displaced. When the temperature is lowered to that of liquid air they move in the opposite direction. For a crystal of tysonite we have also examined them with further cooling with liquid hydrogen. In spite of the great difference of temperature the displacement is then hardly perceptible. This may point to the fact that the difference of the expansion of the crystals in the different directions approaches a limit at very low temperatures.

§ 7. *Connection of the change of the absorption bands occurring at very low temperatures with the electronic theory.* Already in § 3 we pointed out the connection of the change of the bands with that of the number of the electrons which are concerned with a certain band according to the electronic theory coupled with the assumption of quasi-elastic forces. The experimental problems raised by § 3 and § 4 may be defined as follows in the language of this theory¹⁾: to determine as functions of *T* on one side the number and on the other side the damping coefficient (proportional to the width of the band) of the electrons which belong to a certain band. We might make use of the position of the maxima to find mutually related bands, in the first place in the different spectra of one crystal. An investigation into the connection between what we already know about these functions and what the change of the electrical resistance of the metals leads us to expect about the action of forces exercised by the ponderable substance on the electrons naturally suggests itself.²⁾ At very low temperatures we shall no longer be justified in considering the electrons as a perfect gas, but we shall rather have to compare them to a vapour which precipitates on parts of the atoms (dynamides (LEXARD)), and solidifies at still lower temperature³⁾. When we approach these centres the paths of the electrons are subjected to changes which modify the free

¹⁾ Cf JEAN BECQUEREL. Radium I. c

²⁾ H. KAMERLINGH ONNES. Loc. cit.

³⁾ A metal would become transparent at very low temperature.

length of path in the same way as VAN DER WAALS' quantity b is subjected to a change by the forces exerted by the molecules on each other.¹⁾

The three states of aggregation which we used just now as an illustration of the behaviour of the electrons, might perhaps be considered as referring to the stability of different paths of the electrons, and the quasi-elastic force might be connected with the conditions for the electrons moving in these paths.

If we further note that it is the ratio of the absolute temperatures on which the degree of change of the spectra depends (compare the transition from $T=20$ to $T=14$ with that from $T=290$ to $T=95$), we may accept for the present as a heuristic image the idea that we may speak of corresponding states according to different units of temperature caused by mechanic similarity of the motion of the electrons round the centres.

II. PHENOMENA DEPENDING ON THE TEMPERATURE AND ON THE STRENGTH OF THE MAGNETIC FIELD.

§ 8. *Constancy of the change of the frequency of vibrations under the influence of the magnetic field at all temperatures.*

According to the experiments made by one of us previously (J. B.), when a uniaxial crystal is placed with its axis in the direction of the lines of force and of the ray of light, some absorption bands are resolved into two components, which belong to the absorption of two circularly polarized rays of opposite sense. The difference of frequency of vibration of the two components had then proved to be independent of the temperature. It follows now in a still more convincing way from the comparison of the divergence of the two bands at the temperature of liquid hydrogen with the divergence at the temperature of liquid air, that within the limits of errors of observation, the difference of frequency of vibration is entirely independent of the temperature. According to the theory of LORENTZ this constancy of the divergence of the bands, which is observed both for those which behave in the sense of the ZEEMAN-effect as for those which behave in opposite sense, must be considered as proceeding from the invariability of the relation e/m . Accordingly the observations in liquid hydrogen seem to furnish a strong support to the argument in favour of the existence of positive electrons derived from the constancy of this quotient.²⁾

¹⁾ Calculated by REINGANUM according to the theory of BOLTZMANN.

²⁾ *Le Radium* tom V. p. 17 1908.

§ 9. *Partial polarisation of the components of some bands.* In a foregoing communication (CR. 19 Aout 1907) one of us (J. B.) has demonstrated, that the band 624,97 of tysonite becomes double in each of the two spectra of left-handed and right-handed circularly polarized light, which are obtained by means of a plate of a quarter wavelength and a rhombohedron. Therefore in both components of the magnetic doublet of the band the polarisation is not perfectly circular. The band behaves as if it were owing both to positive and to negative electrons with the same period of vibration, and the same ratio e/m , in which the number of positive electrons is to be put as the largest, because the strongest component belongs to it.

At the temperature of liquid hydrogen the same phenomenon is observed with some bands which become at the same time fine and bright (fig. 2 Pl. I band 522. 1). In general the same thing is found on reexamining the spectra at the temperature of liquid air and at the ordinary temperature, though it is more difficult to see. Some time ago DUFOUR again found the same phenomenon in emission bands of fluorcalcium put into the flame.

§ 10. *Asymmetry of the right- and left-handed components.* The experiments at the temperature of liquid air had proved ¹⁾ that when the rays of light run parallel to the lines of force the right- and left-handed components very often differ in strength. No regularity had been found in these differences, the asymmetry was now in one, then in the other sense.

If we pass to the temperature of liquid, or better still, to that of solid hydrogen, the asymmetries, which sometimes change their sign, become exceedingly great; one component increases in intensity at the expense of the other, even to such a degree, that some components vanish almost entirely on the side of the greater wave lengths. An example is furnished by fig. 3, Pl. III referring to 654,2 and 643,4 of xenotime, one component of which is very intense, the other very faint. Apatite shows the same thing.

In solid hydrogen almost all the components which diverge towards the small wave lengths, become very sensibly intenser than those of opposite sign.

§ 11. *Variation of the magnetic rotation of the plane of polarisation in the neighbourhood of the absorption bands.*

a. Simple bands. The experiments of MACALUSO ²⁾, H. BECQUEREL ³⁾,

¹⁾ JEAN BECQUEREL *Le Radium* V. No. 1. p. 9. 1908.

²⁾ CR. CXXVII p. 548, 1898.

³⁾ CR. CXXV p. 679. 1897 CXXVII p. 899. 1891.

ZEEMAN¹⁾ have proved that in the neighbourhood of the bands which exhibit the ZEEMAN-phenomenon, the rotation of the plane of polarisation on both sides of the band is positive, and in the inside of the magnetic doublet negative. The experiments made with uniaxial crystals²⁾ with the axis placed parallel to the lines of force and to the beam of light either at the ordinary temperature, or at the temperature of liquid air, have proved that the regular change of the magnetic rotatory power with the wavelength of the light is subjected to a disturbance of the same kind on both sides of the band, and to an opposite disturbance at the middle of the band. This disturbance is positive outside the band for the bands belonging to negative electrons, and negative for the bands of positive electrons.

At the temperatures of liquid and solid hydrogen the same phenomena are observed, at least when the asymmetry of the left- and right-handed components is not too large. In the neighbourhood of some bands whose components are very unequal, opposite disturbances are observed on both sides of the band — as is easily explained by means of the usual figures of the anomalous dispersion. These phenomena are clearly visible on the figures. 4 Pl. III and 5 Pl. IV.

These figures have been obtained by a method which was already used in former experiments³⁾. Against the slit of the spectroscope a BABINET compensator was fixed between two crossed Nicols in such a way that the fringes were perpendicular to the slit. Before the compensator a plate of a quarter of a wave-length is placed in such a way that the two opposite circularly polarized vibrations are changed into two rays rectilinearly polarized parallel and normal to the principal direction of the compensator. The deviations of the fringe in the spectrum in the neighbourhood of the bands are proportional to the difference of phase of the circularly polarized rays in the crystal plate.

In the figures we find for band 522.15 fig. 4 the symmetrical case, for band 523.7 fig. 4, and 642.3 fig. 5 the dissymmetrical case with disturbance in the same direction, for band 537 fig. 4, and 654.2 fig. 5 the opposite disturbance on both sides of the band.

b. Compound bands. The phenomena of absorption at lower temperatures have shown that several bands may be resolved into two or more. These components behave differently with respect to the magnetic field, because some belong to positive, others to negative

1) Arch Neerl. VII p. 465. 1902.

2) JEAN BECQUEREL, Radium IV No. 2 p. 49. 1907, V No. 1 p. 5. 1908.

3) JEAN BECQUEREL, C.R. May 21 1906.

electrons. Therefore we meet with disturbances in the magnetic rotation which are different for the different bands, and whose effects are superposed. Thus two bands placed side by side, one of positive and the other of negative electrons, may give rise to disturbances in opposite direction in the dispersion of magnetic rotation. It is perhaps to this that we must look for the explanation of what is observed in band 577 of tysonite, which is clearly double in liquid hydrogen.

In general we may say that with regard to the theory of the magnetic rotation for absorption bands, the conclusions drawn from experiments at the ordinary temperature do not lead to a definite result. For at the ordinary temperature it is uncertain whether we have really to deal with a simple band. On the other hand at the low temperatures, at which the bands become narrow, and their change in the magnetic field may be closely followed, it is easy to find the true explanation of the different types of disturbances in the magnetic dispersion of rotation for the bands in the different cases.

§ 12. *Magnetic rotatory power of the paramagnetic crystals.* One of us (J. B.)¹⁾ had previously shown that the negative magnetic rotatory power of the crystals of tysonite and parisite increases considerably with decrease of temperature. The rotatory power is about inversely proportional to the absolute temperature. If this is brought into connection with the law of CURIE that the paramagnetic susceptibility is inversely proportional to T , it appears that the negative rotation of these crystals is probably a consequence of the increase of the paramagnetic polarisation of the crystal.

If these crystals are placed in liquid hydrogen we find that the increase continues in the same way with decrease of temperature, and the rotatory power rises to exceedingly high values. The exact numbers will be given later, but in round numbers the rotation of the plane of polarisation of the blue light amounts to 150° for a plate of tysonite of 1 mm. in a field of 10000 Gauss at the boiling point of hydrogen. Xenotime, which gives a very slight rotation at the ordinary temperature, shows a considerable rotatory power in liquid hydrogen.

§ 13. *Connection between the phenomena of the asymmetry of the left- and right-handed polarized components by the magnetic field at very low temperatures, and the electronic theory.*

In connection with § 4 the phenomena taken together give rise

¹⁾ JEAN BECQUEREL, Radium. Tom. V, N^o. 1, p. 5, 1908.

to the supposition that for the paths of the electrons there exist conditions (fields) of stability, which are determined by the temperature. The action of the magnetic force and the change in the rate of vibration would then bring about that some electrons enter these fields of stability or leave them, both changes occurring either in the direction of greater union with or further separation from the centres which determine the paths, and the increase of this action at low temperature would be in connection with the small velocity. The influence on the stability of the paths, which is here considered, would be the same as manifests itself in the change by temperature of the number of electrons (see § 7) which satisfy the conditions of the motions which may be ascribed to quasi-elastic forces.

In this connection the question suggests itself if the greater stability of vibrations in a certain direction will not give rise to paramagnetic properties.

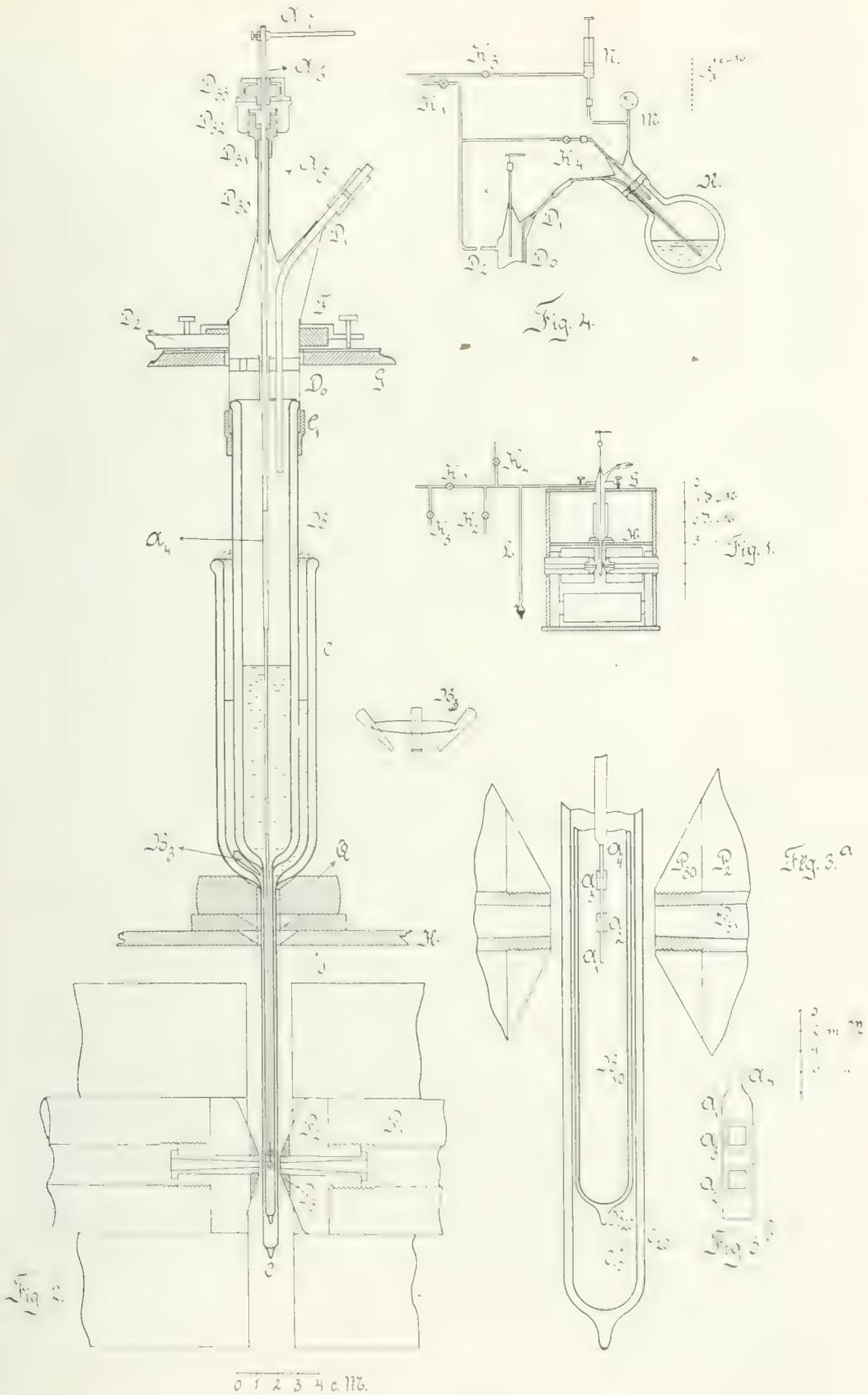
§ 14. *Variability of the mass of the electrons with the direction of the movement.* The theory of the magneto-optic phenomena in crystals (VOIGT ¹), JEAN BECQUEREL ²) leads to the following results.

The magnetic field gives rise to certain connections between the motions of the electrons in the different principal directions of the crystal. Let us consider the simple case which is repeatedly met with, viz. that the corresponding bands in the different spectra occupy the same place. In that case according to the theory the magnetic doublets will have to be symmetrical, and when the bands are sufficiently narrow to allow us to neglect the breadth, the deviations will be proportional to the square root of the product of the two magnetic constants which belong to the corresponding bands of the two spectra. If the beam of light and one of the principal directions 1, 2, 3 of the crystal are made to coincide with the direction of the magnetic field, those two of the three spectra of the crystal are observed which correspond with the vibrations normal to the lines of force.

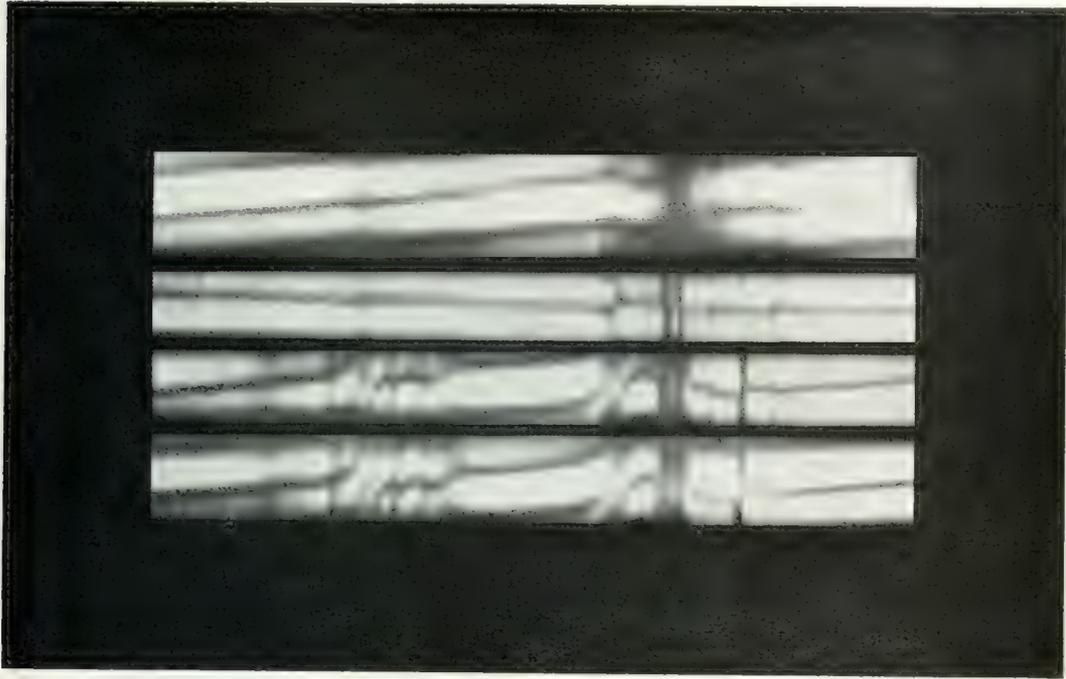
Observation shows that both for the uniaxial crystals of xenotime and tysonite and for the biaxial crystals of didymium sulphate, neodymium sulphate, and praseodymium sulphaté (which last exhibits some lines in liquid hydrogen as sharp as vapour lines) the doublets of the common band have the same divergence. A phenomenon of great importance is observed, when the spectra of vibrations normal to the lines of force are combined in different ways. If the directions 1, 2, 3 successively are placed in the direction of the

¹) Nachr. Kön. Ges. d. Wiss. Göttingen Juli 1906.

²) C.R. 19 Nov. 3. 10. 24 Dec. 1906. Radium IV n°. 3 Mars 1907.



517.6 523.5



$T = 293^{\circ}$ abs.

$T = 85^{\circ}$..

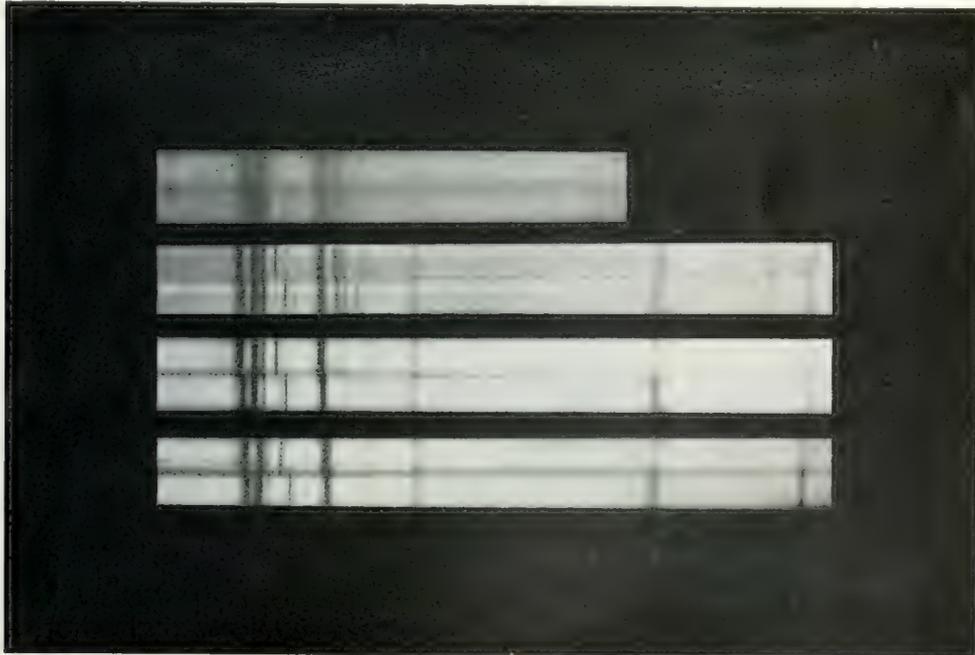
$T = 20^{\circ}$..

$T = 14^{\circ}$..

Fig. 1. *Anomalous birefringence*, tysonite, group in the green 2^d spectrum (ROWLAND grating), thickness of plates 0.71 mm. in 1, 2, 3, 4 and 0.41mm. in 2 (in 2 the ordinary and extraordinary ray are interchanged).

520.6 523.7

537



$T = 293^{\circ}$ abs.

$T = 85^{\circ}$..

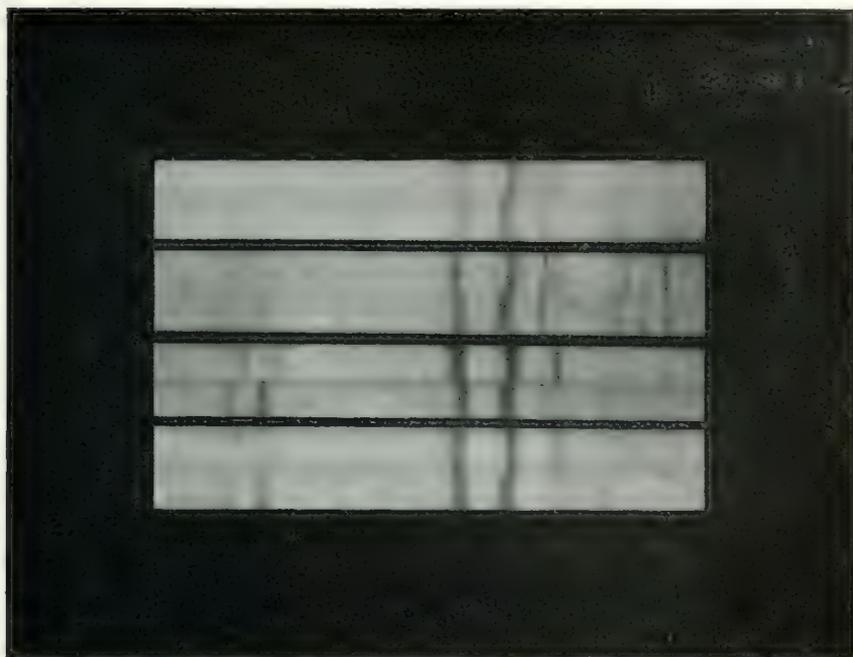
$T = 20^{\circ}$..

$T = 14^{\circ}$..

522.1

Fig. 2. *Left- and righthanded vibrations* in a field of 18000 Gauss nearly Xenotime, group in the green, 2^d spectrum (ROWLAND grating).

642.27 643.45 650.56 654.25 658.10



$T = 290^\circ$ abs

$T = 85$ „

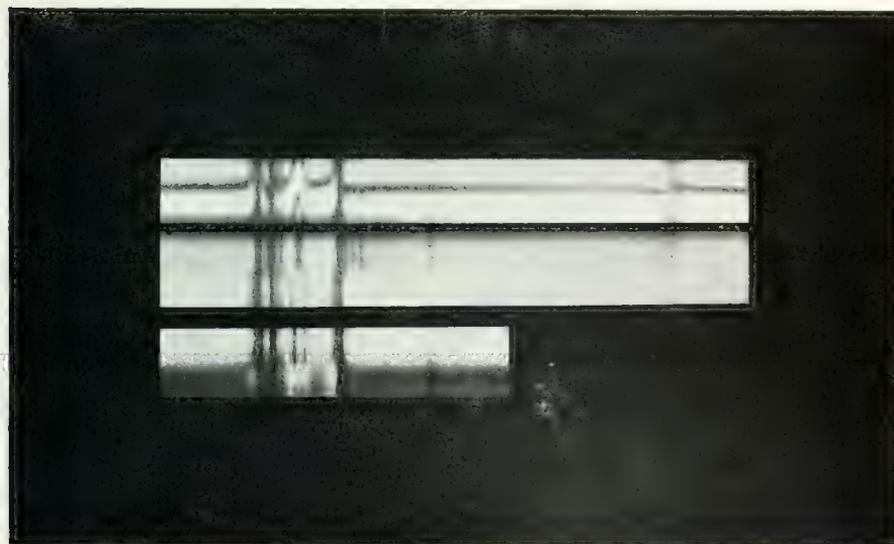
$T = 20$ „

$T = 14$ „

Fig. 3. *Left and righthanded vibration* in 3 a field of 18000 Gauss nearly, Xenotime, group in the red, 2_d spectrum (ROWLAND grating); panchromatic plates of WRATTEN and WAINWRIGHT.

520.6 522.15

537



$T = 20^\circ$ abs

$T = 20^\circ$ abs

$T = 14^\circ$ abs

5237

Fig. 4. *Xenotime*, group in the green, 2_d spectrum (ROWLAND grating)

1. magnetic circular birefringency, plate thick 0.80 mm., field 15000 Gauss
2. images by rhombohedron before slit, the incident light polarized to give equal intensities to the regions in the transparent part in the middle of the group. Field 15000 Gauss.
3. images given by rhombohedron before slit, incident light polarized under 45° with the horizon; field 18000 Gauss.

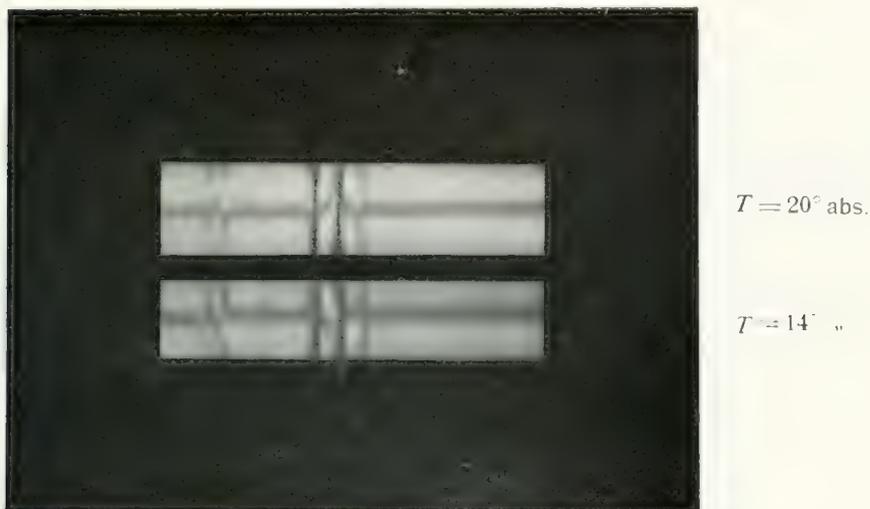


Fig. 5. *Magnetic rotation of the plane of polarization.* Xenotime $2d$ spectrum (ROWLAND grating); thickness 0.80 mM., field 18000 Gauss; (quarter of wavelength plate turned 90° in the one in respect to the other).

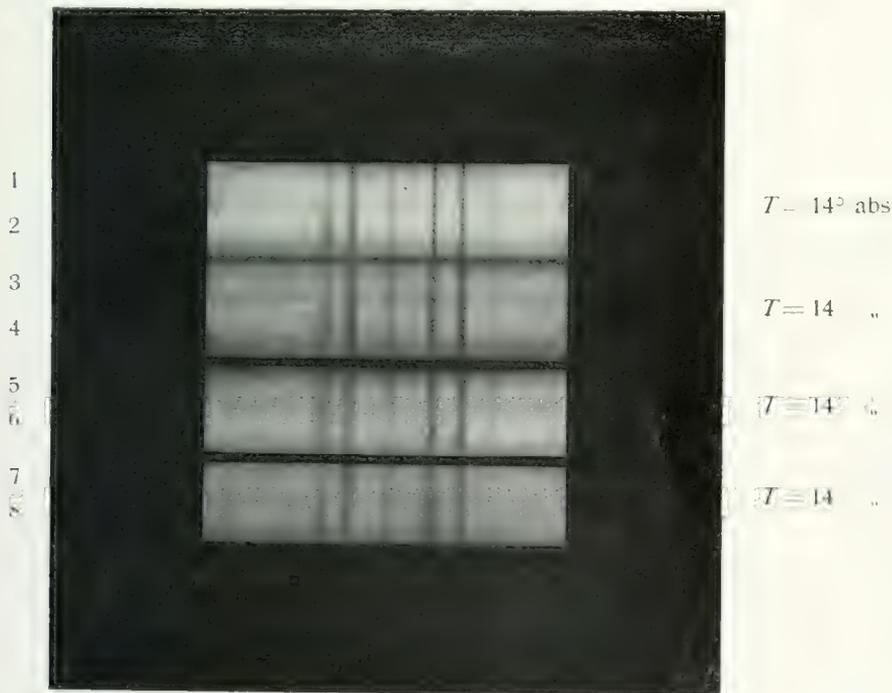


Fig. 6. *Sulphate of Neodyme group in the orange, $2d$ spectrum (ROWLAND grating) spectra of the vibrations:*

- 1 in the principal direction a , field = 0.
- 2 " " " " b , field = 0.
- 3 " " " " a , a and b normal to the field (18000 Gauss).
- 4 " " " " b , a and b " " " " "
- 5 " " " " a , a and c " " " " "
- 6 " " " " c , a and c " " " " "
- 7 " " " " b , b and c " " " " "
- 8 " " " " c , b and c " " " " "

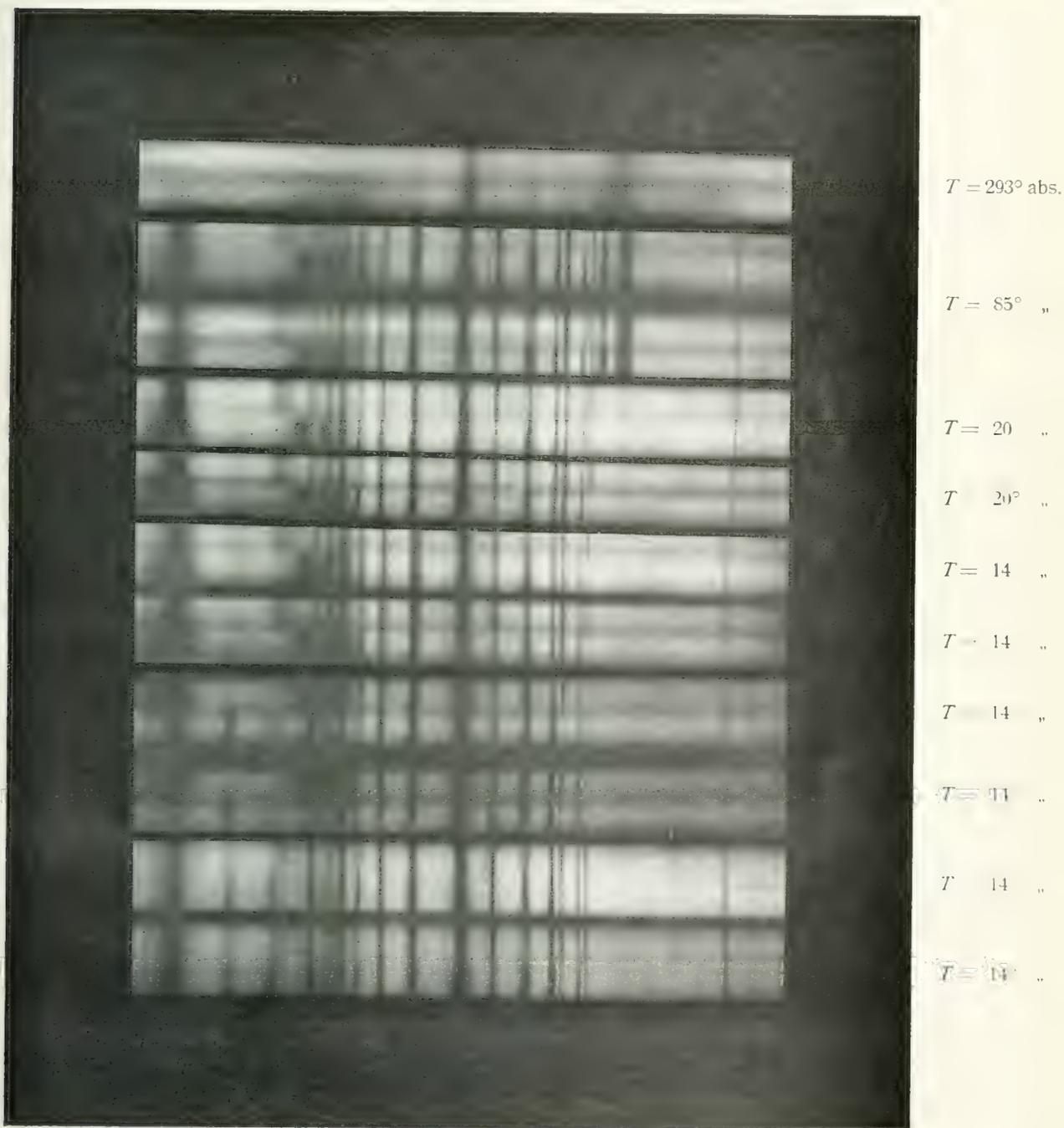


Fig. 7. Sulphate of neodymium; spectra of vibrations in the directions α, β, γ group in the ν_{10} spectrum (ROWLAND grating)

1, 3, 7, 11, 19	α without field
2, 4, 8, 12, 15	β " "
16, 20	γ " "
5 and 6, 9 and 10, 13 and 14	α and β normal to field (18000 Gauss)
17 and 18,	β and γ " " "
21 and 22	α and γ " " "

field, we get the combinations 2.3, 1.3, 1.2 for the vibrations normal to the field. Experiment shows that the divergences of the pairs of doublets in these three cases are very different. Thus for a band of spectrum 1, the vibration being normal to the field, the magnetic doublet is different according as the direction normal to the field has the principal direction 2 or 3. The phenomenon is clearly seen in the figure which represents the group of bands in the orange for neodymium sulphate at -259° . Fig. 7 Pl. V gives a survey of the phenomena of the changes with the temperature and the magnetic field in the blue of neodymium sulphate. According to theory it follows from this that each of the three different directions has a different magnetic constant, and that therefore the vibrating system presents three different masses for the three kinds of vibrations.

As the corresponding bands in the two spectra occupy the same or only slightly different places, it follows that in first approximation the constant of the quasi-elastic force in each of the three directions must be proportional to the mass in that direction.

Physics. — *“On the equation of state of a substance in the neighbourhood of the critical point liquid-gas. I. The disturbance function in the neighbourhood of the critical state.”* By Prof. KAMERLINGH ONNES and Dr. W. H. KEESOM. Communication N^o. 104^a from the Physical Laboratory at Leiden.

§ 1. The great compressibility of a substance in the neighbourhood of the critical point liquid-gas and the properties connected with this, (such as the small variation of the thermodynamical potential at isothermal compression etc.) — which are derived from VAN DER WAALS' original equation of state and better still from his latest considerations about the compressibility of a molecule¹⁾ — render it necessary that in deriving conclusions from observations in the neighbourhood of that condition we must take into account various circumstances, otherwise unnecessary for the experimental investigation of the equation of state of a homogeneous substance consisting of one component, which investigation includes that of the quantities of saturation etc.

It is well-known that owing to the great compressibility the thermodynamic equilibrium is difficult to attain, in fact it has often

¹⁾ Comp. VAN DER WAALS, Proceedings June '03,

happened that phenomena¹⁾ at the critical point have been described as abnormal and as being at variance with the views of ANDREWS-VAN DER WAALS, in cases where the thermodynamic equilibrium had not yet been attained either because small differences in composition had remained owing to the slow diffusion of very small quantities of admixture (KUENEN Comm. N°. 8, Oct. '93, N°. 11, May and June '94), or because differences of temperature resulting from variations of volume in different portions of the substance during the passage from one condition of temperature and pressure to the next had not yet been equalized (KAMERLINGH ONNES, Comm. N°. 68, March and April '01 and KAMERLINGH ONNES and FABIUS, N°. 98, May '07).

When the thermodynamical equilibrium is obtained either by keeping the substance in the neighbourhood of the critical point during a long time at a constant temperature or by repeated reversals of the sealed tube containing the substance (GOUY), or by stirring it electromagnetically (KUENEN) we must pay regard to the gravitation which on account of the great compressibility of the substance in that condition becomes of great influence²⁾ and also to small quantities of admixture which may occur and of which the nature and the quantity are known³⁾.

The consideration of these influences and those of capillarity and absorption phenomena near the walls of the vessel⁴⁾, things which in other cases are hardly to be considered, is indispensable at the critical point liquid-gas for the determination of the *experimental equation of state of a substance*, i. e. the relation between p , v and T for a substance consisting of one component in thermodynamic equilibrium subject to no other external forces than the pressure on the walls of the vessel.

§ 2. In this communication we intend to bring into connection some peculiarities in the experimental equation of state in the neighbourhood of the critical state with the great compressibility

1) For a survey of these phenomena comp. GRAETZ, WINKELMANN's Handbuch, III, 2te Aufl. p. 837.

2) GOUY. C. R. 115 (1892) p. 720 and 116, p. 1289. J. P. KUENEN, Comm. N°. 17 May '95.

3) Cf. Comm. N°. 75, Nov. '01, N°. 79, March '02, N°. 88 Nov. '03 (KEESOM), N°. 81, June and Sept. '02, Suppl. N°. 6, Febr. and May '03, N. 18, Dec '04, N°. 12, Jan. '07 (VERSCHAFFELT). On the influence of gravity a small quantity of admixture being present, cf. KUENEN, Comm. N°. 17, May '95 and KEESOM, Comm. N°. 88 VI, Nov. '03.

4) Comp. VAN DER WAALS, l.c. p. 106 and 107.

in this area. Therefore we compare the experimental equation of state of a substance near the critical point liquid-gas with an equation of state which we shall call *the special undisturbed equation of state for that substance* and which is derived by adjusting interpolation formulae to observations in areas where no disturbances occur such as in the neighbourhood of the critical point.

For we presume to be able to derive from the results of data at our disposal that the experimental equation of state differs from the special undisturbed one by the presence of terms which for the accuracy reached in the observations meant only deserve notice in the neighbourhood of the critical point, and which are intimately connected with the great compressibility in this area. We shall call the compound of these terms *the disturbance function in the equation of state in the neighbourhood of the critical point*.

In order to be able to derive from the special undisturbed equation of state and the disturbance function at the critical state the conditions of coexistence, vapour pressures, liquid and vapour densities, we must have investigated whether in that condition MAXWELL's criterium for a substance consisting of one component may be applied unmodified or not.

For the present we must include in this disturbance function the disturbances caused by admixtures which chemically may have an existence of their own, but which it was not possible to remove and which always occur in definite quantities, as long as the nature and the quantity of these admixtures are unknown. The investigation of substances with small quantities of admixture¹⁾ may help us towards a better judgment of the question whether this disturbance function may be entirely ascribed to admixtures which may exist separately. As long as this has not yet been decided it will be indispensable to pay regard also to admixtures which can have no existence of their own but which may always occur as electrically charged particles, or as portions of the substance of greater density which may give rise to differences of density distributed as nebulous drops and which in this area might be kept up by capillary force. It will also be necessary to take into account differences of density depending on the statistic equilibrium.

In order to arrive at some knowledge of such a disturbance

¹⁾ Comp. p. 604 note 3. For the influence of small quantities of admixture of substances of small volatility the following investigations are important: M. GENTNERSZWER, ZS. physik. Chemie 46 (1903) OSTWALD Jubelb. p. 427, 61 (1907) p. 356; M. GENTNERSZWER and A. PAKALNEET, *ibid.* 55 (1906) p. 303, M. GENTNERSZWER and A. KALNIN, *ibid.* 60 (1907) p. 441.

function, observations of greater accuracy are required over an area which comprises the critical state and also approaches it sufficiently. These observations must be accurate to within $\frac{1}{5000}$, as is usual in the Leiden laboratory in the investigations of bi- and monatomic substances and their binary mixtures, while the nature and the quantity of the separable admixtures ought to be known to $\frac{1}{10000}$ of the whole mass¹⁾.

§ 3. Our conclusion about the existence of a disturbance function in the equation of state in the neighbourhood of the critical point liquid-gas is based on the following data which may be arranged into three groups.

a. In Comm. N^o. 74 (Arch. Néerl. (2) 6 (1901) livre jub. Bosscha p. 874) has been pointed out that AMAGAT's observations of the isothermals of carbon dioxide near the critical point show systematic deviations from the values derived from the special undisturbed equation of state. This equation of state was derived from the empiric equation of state introduced in Comm. N^o. 71, June '01, by choosing the virial coefficients so (Comm. N^o. 74 § 4) that the agreement with the observations over the whole area of observations is as good as possible while the agreement with the general reduced equation of state at a reduced temperature lying far outside the area of observation was retained.

We get a similar series of observations if we compare the observations of carbon dioxide in the neighbourhood of the critical point — described in Comm. N^o. 88 (Jan. '04) — with the special undisturbed equation of state, while using the reduced virial coefficients *V* s. 1 (Comm. N^o. 74, p. 884) and the critical temperature and pressure found in Comm. N^o. 88.

It really appeared in Suppl. N^o. 14, Jan. '07 (KAMERLINGH ONNES and Miss JOLLES) that the critical quantities, derived according to $\frac{\partial p}{\partial v} = 0$, $\frac{\partial^2 p}{\partial v^2} = 0$ from the special undisturbed equation of state *V* s. 1, show great deviations from those derived experimentally.

A similar difference was found by AMAGAT (Journ. de phys. (3) 8 (1899) p. 353) when he derived the densities of saturated liquid and vapour from the equation of state (containing 10 constants) formed by him for carbon dioxide. The curve which represents the densities calculated thus as a function of the temperature, at lower tempera-

1) For such an investigation carbon dioxide would be fittest owing to the comparably small difficulties in preparing it perfectly pure and keeping its temperature sufficiently constant, and also because much is already known about its equation of state over a large area.

tures coincides almost with the curve given by the observations: according as we approach the critical temperature the calculated curve shows a displacement towards the small densities with regard to the observed curve. That this displacement is much larger than follows from the calculation of Suppl. N^o. 14 mentioned above must be ascribed at least partly to the circumstance that AMAGAT probably did not derive the liquid and vapour densities from his equation of state by means of the criterium of MAXWELL, but for shortness' sake calculated by means of his equation of state the densities for which p has the value for the saturation pressure furnished by experiment.

b. In Comm. N^o. 75 Nov. '01 attention was directed to the difference between the $C_s = \left[\frac{T}{p} \left(\frac{\partial p}{\partial T} \right)_v \right]_k$ derived from AMAGAT'S net of isothermals and the $C_a = \left[\frac{T}{p} \frac{dp_{\text{sat.}}}{dT} \right]_k$ resulting from his determinations of the saturated vapour pressure, which values must be equal for the undisturbed equation of state¹⁾. One of the reasons to undertake the observations about carbon dioxide of Comm. N^o. 88, Jan. '04, was the wish to obtain more certainty about these peculiarities in the behaviour of the substance in the neighbourhood of the critical point (comp. l.c. p. 566). The same difference viz. $C_s = 7.12$, $C_a = 6.71$ followed from these determinations. BRINKMAN (Thesis for the doctorate, Amsterdam 1904, p. 43) confirmed this difference not only for carbon dioxide, but he also found it for methyl chloride, while MILLS (Journ. phys. Chem. 8 (1904) p. 594, 635: comp. also 9 (1905) p. 402) for ethyl oxide (RAMSAY and YOUNG), isopentane and normal pentane (YOUNG) finds differences of 10 percent between C_a derived by means of the formula of BIOT for the saturated vapour pressures and C_s which with regard paid to the regular variation of b with temperature²⁾, follows from the data collected by RAMSAY and YOUNG (ethyl oxyde), YOUNG (isopentane) and ROSE-INNES and YOUNG (normal pentane) in order to judge of the equation of isochors $p = bT - a$.

c. In Comm. N^o. 88, Jan. '04, p. 575 table XXII the saturated vapour pressures of carbon dioxide between 25°.55 C. and the critical temperature (30°.98 C.) are compared with the formula $\log \frac{p}{p_k} = f \frac{(T - T_k) T_k}{T^2}$ which was obtained by keeping in the develop-

¹⁾ M. PLANCK, Wied. Ann. 15 (1882) p. 457; comp. also Comm. N^o. 75 § 3. The quantities C_s and C_a are both obtained by an extrapolation, C_s at $v = v_k$ of a higher temperature to T_k , C_a along the vapour pressure curve of lower T to T_k .

²⁾ S. YOUNG, Proc. Phys. Soc. London 1894/95, p. 602; comp. also Comm. Phys. Lab. Leiden No. 88 p. 54 note 1, KEESOM Thesis p. 86.

ment of $\log p$ in ascending powers of T^{-1} the second power¹). While for the other temperatures in the table mentioned the deviations did not exceed 0.01 atm., a deviation of Obs.—Comp. = 0.05 atm. was found for 30°.82 C.²). Although it was then held probable that this deviation was to be ascribed to an accidental error of observation, we have afterwards found that a deviation in the same sense and of about the same size also occurs in the results of other observers about saturated vapour pressures of carbon dioxide near the critical point.

A comparison of the results of BRINKMAN's observations (Thesis Amsterdam 1904 pp. 41 and 42) of saturated vapour pressures of methyl chloride and carbon dioxide with the pressures derived by him according to a formula of the same form as the one mentioned above, yields the following differences :

for methyl chloride ($t_k = 143.^\circ 12$) :

$$\begin{array}{l} \text{at } t = 137.^\circ 54, \quad 138.^\circ 92, \quad 140.^\circ 26, \quad 141.^\circ 66, \quad 142.^\circ 02 \\ O-U = + 0.02, \quad - 0.01, \quad - 0.02, \quad + 0.03, \quad + 0.08 ; \end{array}$$

for carbon dioxide ($t_k = 31.^\circ 12$) :

$$\begin{array}{l} \text{at } t = 24.^\circ 24, \quad 26.^\circ 08, \quad 28.^\circ 46, \quad 29.^\circ 86, \quad 30.^\circ 40 \\ O-U = + 0.02, \quad - 0.02, \quad + 0.03, \quad + 0.08, \quad + 0.07. \end{array}$$

In both substances investigated one finds below the critical temperature an obvious deviation resembling that found in Comm. N°. 88.

The observations of AMAGAT, Journ. de phys. (2) 1 (1892) p. 288, of the saturated vapour pressure of carbon dioxide fail to give any definite indication about the question treated here because AMAGAT has rounded off the pressures to 0.1 atm. In connection with the preceding statements, however, it deserves attention that TSURUTA, Journ. de phys. (3) 2 (1893) p. 272, when comparing these data with the formula $p = 34.3 + 0.8739t + 0.01135t^2$, also there found an obvious difference $O-U$ at 31.°25 which exceeds by 0.06 atm. that at 31.°35 (crit. temp. according to AMAGAT).

From the data mentioned here one might draw the conclusion that for carbon dioxide and methyl chloride the curve of the saturated vapour pressures, continued to near the critical point, with extrapolation to this point would lead us to expect a p_k somewhat larger than the critical pressure found experimentally. From the very careful

¹) In Physik. ZS. 8 (1907) p. 944, BOSE went still farther and kept the third power in this development which had been given by RANKINE, Misc. Scientif. Papers pp. 1 and 410,

²) As it appears from the columns Obs. and Comp. all the numbers in the column $O-U$ have wrong signs.

observations of YOUNG of isopentane Proc. Phys. Soc. London 1894/95, p. 613, however, a deviation as found above for carbon dioxide cannot be derived.

It may be that some connection exists between the above mentioned disturbance in the saturation pressure in the immediate neighbourhood of the critical point of carbon dioxide and a disturbance in the observations of Comm. N^o. 88 of the densities of saturated liquid and vapour of carbon dioxide. Plate I represents these densities

d_{liq} and d_{vap} , expressed in the theoretical normal density. $\frac{1}{2}(d_{liq} + d_{vap})$

is also represented. The straight line is the line which was drawn for the determination of the critical volume after the method of the rectilinear diameter of CAILLETET and MATHIAS in Comm. N^o. 88 (comp. Comm. N^o. 88 p. 574). The middle of the chord belonging to 30.8 lies clearly below this line. If for the determination of the rectilinear diameter only the three points at lower temperature are used, the difference is much larger. If this deviation cannot be ascribed to an error of observation, it would follow hence that the diameter of CAILLETET and MATHIAS for carbon dioxide shows a curvature in the immediate neighbourhood of the critical point¹⁾. K_{98} indicates the critical density which in Comm. N^o. 98 (KAMERLINGH ONNES and FABIUS) was derived from determinations less than 0.002 deg. below the critical temperature. If we might assume that the carbon dioxide of Comm. N^o. 98 and that of N^o. 88 possessed the same degree of purity, an assumption to which the agreement between the critical temperatures entitles, and also that the difference in the methods of density determination has not given rise to a systematic difference, then the situation of the point K_{98} would confirm the curvature of the diameter in the neighbourhood of the critical point.

A similar disturbance as we remarked above for the saturation volumes of carbon dioxide in the immediate neighbourhood of the critical point, cannot be derived either from YOUNG's observations of isopentane (comp. Proc. Phys. Soc. London 1894/95 p. 636) or from those of normal pentane (Trans. Chem. Soc. 71 (1897) p. 455),

¹⁾ This curvature is in another sense than the curvature found by KUENEN and ROBSON, (Phil. Mag. (6) 3 p. (1902) p. 624) at lower temperatures in the diameter for carbon dioxide and which agrees with the general rule given by YOUNG (Phil. Mag. (5) 50 (1900) p. 291) about this curvature at lower temperatures in connection with the value of $\frac{RT_k}{\rho_k v}$ and the slope of the diameter as compared with

the temperature axis.

which are continued down to $0^{\circ}.05$ below the critical temperature ¹⁾. It would be very desirable to investigate more closely in how far the disturbances mentioned sub *c* are connected with a disturbance in the equation of state, or must be ascribed to special circumstances of those experiments themselves (such as the difficulty to determine the moment at which begin condensation occurs).

§ 4. The disturbances mentioned in § 3 apparently point to the fact that the substance in the neighbourhood of the critical point occupies a smaller volume than would be expected according to the special undisturbed equation of state. In Comm. N^o. 88 p. 555 the possibility is mentioned that these disturbances are connected with differences of density which occur in the substance near the critical state, as it is indicated by the mist (the blue opalescence) in the neighbourhood of that state. The question was left aside whether those differences of density are to be interpreted either as condensations round condensation centres with an existence of their own (dust according to KONOWALOW ²⁾), electrically charged particles ³⁾ a third phase separated in small drops and for the greater part consisting of an admixture), or simply as spontaneously formed differences of density, either as accidental aggregations caused by molecular motion and governed by the statistic equilibrium (SMOLUCHOWSKI ⁴⁾), or because small drops still have a positive surface tension at temperatures at which larger drops are no longer stable (DONNAN ⁵⁾).

Whatever may be the cause of the blue mist, in all cases we may expect a close relation between the compressibility and the occurrence of it. In order to form a better judgment about this matter it was considered to be desirable to start an investigation of the conditions of existence of this mist in a substance consisting of one component in the neighbourhood of the critical point liquid-gas. For an optical research of these conditions of existence we refer to the next communication.

¹⁾ Nor can a similar disturbance be derived with certainty from BRINKMAN's observations of carbon dioxide and methyl chloride, which observations, however, are not continued so near to the critical point as those of comm. N^o. 88.

²⁾ D. KONOWALOW. Ann. d. Phys. (4) 10 (1903) p. 360.

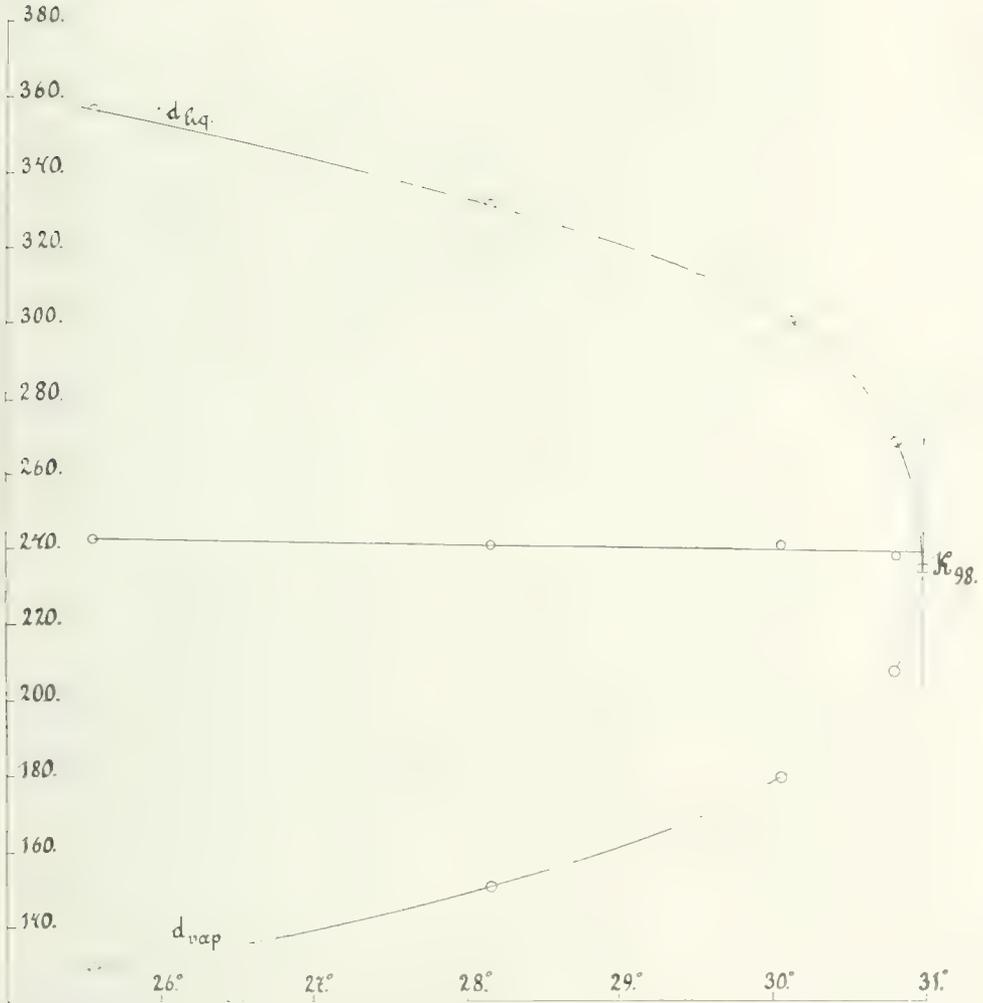
³⁾ Owing to the highly penetrating radiation from the radio-active portions of the crust of the earth (EVE, Phil. Mag. (6) 12 (1906) p. 189) in the atmosphere (STRONG, Physik. ZS. 9 (1908) p. 170), or in the surroundings of the building where the experiments are made, these particles would always be present to almost the same amount. In the mean time it follows from the experiment of FRIEDLÄNDER ZS. physik. Chem. 38 (1901) p. 385, on the stability of the mist in an electric field, that the particles which cause the opalescence are not charged.

⁴⁾ M. v. SMOLACHOWSKI, Ann. d. Phys. (4) 25 (1908) p. 205.

⁵⁾ F. G. DONNAN. Chem. News 90 (1904) p. 139.

Prof. H. KAMERLINGH ONNES and Dr. W. H. KEESOM. On the equation of state of a substance in the neighbourhood of the critical point liquid-gas. I. The disturbance function in the neighbourhood of the critical state.

Plate I.



Physics. — “*On the equation of state of a substance in the neighbourhood of the critical point liquid-gas. II. Spectrophotometrical investigation of the opalescence of a substance in the neighbourhood of the critical state*”, by Prof. H. KAMERLINGH ONNES and Dr. W. H. KEESOM. Communication N^o. 104*b* from the Physical Laboratory at Leiden.

§ 1. *Introduction.* The spectrophotometrical investigation¹⁾ of the opalescence will have to give an answer to the question how the intensity of the light of a certain wavelength scattered in a certain direction with respect to the incident light and included in a certain angle of vision, in connection with the polarisation state depends on the temperature and the density of the single substance in the neighbourhood of the critical point liquid-gas (cf. Comm. N^o. 104*a* § 4). A first quantitative contribution to this investigation is given in this communication²⁾.

We have confined ourselves in this first investigation to the determination of:

1. for different temperatures the ratio in which the rays of light of different wave lengths at the same temperature are scattered in a certain direction;

2. the way in which the intensity of the light of definite wavelength scattered in a certain direction and included in a certain angle of vision changes with the temperature.

On the supposition *a*, that the light emitted by the blue mist is owing to the scattering of the incident light in consequence of part of the substance condensing to particles of the same size (e.g. spheres) round centres which are uniformly spread through the space, the results of the investigation mentioned under 1 will enable us to form an opinion on the size of these particles³⁾, that under 2 on the way in which the total quantity of substance which has condensed, varies with the temperature.

¹⁾ This investigation was carried out before the interesting article of SMOLUCHOWSKI *Ann. d. Phys.* (4) 25 (Febr. 7, 1908) p. 205, appeared. We still had an opportunity, however, to compare it with some parts of the text (c.f. also the preceding Communication). [Added in the English translation].

²⁾ The colorimetric determinations of FRIEDLÄNDER, *ZS. physik. Chem.* 38 (1901) p. 385 constitute such a contribution for a mixture of two liquids in the neighbourhood of the critical point of separation.

³⁾ Already FRIEDLÄNDER *loc. cit.* p. 438 called attention to the importance of such an investigation for the knowledge of the internal structure of the critically turbid media

On the more general supposition *b*, that the opalescence is the consequence of differences of density e.g. governed by the statistic equilibrium, which extend over parts of the volume of irregular shape and size, a distance can be pointed out which is connected with the average size of these parts of the volume and so with the substance being more or less coarse-grained in that state, and which determines the optical phenomenon in a similar way as the size of the particles on supposition *a*. The investigation mentioned under 1 will then enable us to judge about this distance. When in future we speak of the size of the light-scattering particles, we shall refer to this distance. In this case the investigation mentioned under 2 will teach us something about the mean deviation of the density in these parts of the volume. This too will be implied in future in "the quantity of condensed substance".

The measurements made by us can, however, be only considered as preliminary ones. As, however, we have to put off the continuation of these measurements for some time, we think that we must not postpone the communication of these provisional results any longer.

§ 2. *The arrangement of the experiments* is represented in Pl. II fig 1. After having passed through a layer of water the light emitted by the luminous body of the NERNST-lamp *Ner* (70 HK) is concentrated by the lenses L_1 and L_2 (to an image of ± 1 c.M. height) in the tube *Et* filled with ethylene¹⁾.

The light scattered upwards in the direction of the axis of the tube²⁾ by the cloud is concentrated by means of the system of

¹⁾ This was obtained by distilling over so much from the ethylene circulation of the cryogenic laboratory into the glass tube with cock fused to it, which had been cooled in liquid air, and rinsed with ethylene, that after the gas phase left above the solid ethylene had been drawn off, and the tube was heated to the melting point, $\frac{1}{3}$ of it was filled. Then with the cock closed the tube was removed from the ethylene circulation, and fused together at a previously narrowed place while still partially placed in liquid air. When the temperature rose to that of the room, it appeared, when at 0° the rime cleared away, that in the gas space a thin white deposit was visible on the wall, which evaporated some degrees below the critical temperature of the ethylene. This deposit points to the presence of an admixture which is slightly less volatile than ethylene (cf. VILLARD, *Ann. de chim. et de phys.* (7) 10 (1897) p. 389). That it was not visible in the liquid space, probably points to a small difference in refrangibility with liquid ethylene.

When we stirred, and then slowly cooled the tube to below the critical temperature, the meniscus appeared in the top of the tube.

²⁾ The top of the tube is surrounded by a black cylindre in order to prevent rays of light received by this part from being reflected upwards, and being thrown into the spectroscope.

lenses L_3, L_4 and L_5, L_6 and the totally-reflecting prism Pr to an image of the beam of light crossing the ethylene tube on the slit of the spectroscope Sp (a direct-vision spectroscope of HILGER-CHRISTIE giving a spectrum of great intensity¹), in which an eye-piece slit has been made in order to confine a certain portion of the spectrum; by means of the screw Scr_2 different portions of the spectrum may be brought into the field). A beam of the light emitted by the NERNST-lamp is thrown on the slit of the spectroscope by means of plane mirrors through the polarizing prisms Nic_1, Nic_2, Nic_3 (after having been first made parallel by lens L_7) and then through lens L_8 and a totally-reflecting prism. The prisms Nic_1 and Nic_3 are rigid, which ensures that the light thrown into the spectroscope by the reflections on the mirrors with different positions of Nic_2 is reduced in the same proportion; the prism Nic_2 can turn round, and is provided with a graduated circle, which could be read up to 3'. The plane of polarisation of Nic_3 is horizontal so that the condition of polarisation in the two beams thrown into the spectroscope agrees in the main²). The plane of polarisation of Nic_1 has been put parallel to that of Nic_3 . After the tube of ethylene had been brought to the required temperature, and the temperature of the room had been regulated in such a way that the temperature of the tube of ethylene (read to 0.01) could be kept sufficiently constant (up to some hundredths of a degree) by the addition, when necessary, of some cold or hot water into the vacuum glass, the prism Nic_2 was adjusted by rotation so as to obtain equal intensity of the considered portions

¹ See ZEEMAN Comm. N^o. 5, June '92, more detailed Arch. Néerl. 27 (1893) p. 259 and Pl. V. The "halfprism" was used in our experiments with a view to the intensity in the magnifying position (CHRISTIE, Proc. Roy. Soc. 26 (1877) p. 8). Moreover the dispersion is greater in this position, whereas the loss of purity in the spectrum is of no importance here.

² Not too near the critical state the light emitted by the blue mist in a direction normal to the incident light, is polarized in the plane of incidence (RAMSAY, ZS. physik. Chem. 14 (1894) p. 486). It is to be expected that on approach to the critical state the light emitted in the direction mentioned becomes more and more partially polarized (cf. TYNDALL, Phil. Trans. 160 (1870) p. 348). It would be interesting to examine if then TYNDALL's residual blue (l. c.) could be observed (on the connection of this with the difference in refractivity of the scattering particles and the surrounding medium see RAYLEIGH, Phil. Mag. (4) 41 (1871) p. 454). Measurements on the condition of polarisation might also lead to an opinion on the size of the particles, see BOCK, Wied. Ann. 68 (1899) p. 674 (spectrophotometrical investigation of the light scattered by a jet of steam, measurement of the condition of polarisation, and determination of the size of the particles by means of diffraction rings) and PERNTER, Denkschr. Kais. Ak. d. Wiss. Wien 73 (1901) p. 301.

of the two spectra. With a view to this adjustment care had been taken that the two spectra were as close above each other as possible ¹⁾ and had about the same height. The adjustment and reading were made in the four different positions of *Nic*₂, which gave equality of intensity.

§ 3. *Observations.* Only observations above the critical temperature have been communicated here; in order to get unambiguous data for the dependence of the intensity of the opalescence on the temperature and the density below the critical temperature, a stirring-apparatus, or an arrangement to keep the temperature constant till the thermo-dynamic equilibrium should have been reached, would have been required. The observations were made after the tube of ethylene had been kept at higher temperature for 15 hours or longer, and had then been slowly cooled down to the temperature of observation. The measurements have been made for two wavelengths, corresponding to *D* and *F* in the solar spectrum ²⁾. In order to give an idea of the degree of accuracy of the adjustments, we have communicated the data of an observation at a mean intensity of the scattered light in table I.

TABLE I.

Series VI, No. 3, 13 Nov. 1907						
Wave length	Temperature	Adjustments of <i>Nic</i> ₂				?
<i>D</i>	11°.69	63°48'	36°36'	154°36'	125° 9'	14°10'
	11. 66	64 24	36 15	153 30	126 18	13 50
	11. 70	63 18	36 54	154 9	125	13 53
	11. 69					—
		temp. mean 11°.68				mean: 13°58'
<i>F</i>	11. 66	155°45'	124°48'	65°18'	34°33'	15°15'.5
	11. 68	156 15	124 54	66 9	33 54	15 54
	11. 67	157 18	122 57	67 30	34 6	16 56
	11 61					—
		temp. mean 11°.66				mean: 16°5'

¹⁾ The use of a HÜFNER'S prism would render more accurate adjustments possible.

²⁾ When the experiments are repeated under circumstances which admit of a more accurate spectrophotometric adjustment, an extension of the measurements to more wavelengths will be desirable.

The last column contains the angle of the plane of polarisation of Nic_2 derived from the other columns, for the adjustment at equal intensity, with this plane of polarisation when Nic_2 crosses Nic_1 and Nic_1 . In general the adjustment for the wavelength F was less accurate than for D on account of the slighter intensity in the spectrum for the former wavelength. The greater deviation which the latter angle φ shows for the wavelength F in table I from the preceding ones, may be explained from the difference in temperature.

The results thus obtained have been joined in table II.

T A B L E II.

Wavelength D		Wavelength F	
Temperature	φ	Temperature	φ
Series V, 12 Nov. 1907			
13° .53	8°27'	13° .59	10°11'
12 .55	9 45.5		
Series VI, 13 Nov. 1907			
12° .54	10°36'	12° .54	12°39'
11 .86 ³	12 37	11 .83	14 58.5
11 .68	13 58	11 .66	16 5
11 .42	17 52	11 .43	18 24
11 .24	22 18		

The observations of series VI ceased after the adjustments for the wavelength D at 11.°24, because after this the temperature fell below the critical temperature, which was determined at 11.°18¹ (cf. § 3 beginning).

The difference between the angles φ for Series V 12.°55 and Series VI 12.°54, wavelength D , is owing to this that between these observations a slight modification in the position of the lenses L_7 , L_8 has taken place. The observations mentioned here may serve

¹) Comparison of this value of the critical temperature with that of other investigators indicates that the critical temperature of the admixture (cf. § 2, p. 612, note 1) does not lie much higher than that of ethylene.

to bring connection between the series V and VI. The results of other series of observations are not communicated here, because for them all the precautions mentioned had not yet been taken.

From the data of table II the course of the intensity of the scattered light with the temperature (§ 2 2nd) will be derived in the first place. Let us call $H_{D,t}$ the intensity in the spectrum of the light scattered by the cloud at the temperature t of the wavelength D for a certain arrangement of the apparatus, which is further thought to be unmodified, $H_{D,comp}$ the intensity in the comparison spectrum when Nic_2 is parallel with Nic_1 and Nic_3 , then $i_{D,t} = H_{D,t}/H_{D,11.068} = \sin^4 \varphi_{D,t}/\sin^4 \varphi_{D,11.068}$. An investigation of the absolute intensity of the light scattered by the mist compared with that of the incident light (cf. § 6b) will have to reveal how to derive a quantity from $i_{D,t}$ which determines the intensity of the scattered light, independent of the particular circumstances of the arrangement. For an examination of the way in which the intensity of the scattered light depends on the temperature, the quantity $i_{D,t}$ is very suitable.

Table III contains the results obtained on this from table II:

TABLE III.

t	$i_{D,t}$	t	$i_{D,t}$
13°.53	0.490	11.068	1
12.54	0.337	11.42	2.61
11.86 ^s	0.671	11.24	6.11

These results have been represented in Pl. II fig. 2, where also a curve has been traced through the points of observation (see further p. 620).

The ratio $r_{F:D,t} = \frac{H_{F,t}/H_{F,comp.}}{H_{D,t}/H_{D,comp.}} = \frac{\sin^4 \varphi_{F,t}}{\sin^4 \varphi_{D,t}}$ yields data for the inquiry mentioned in § 2 1st. into the ratio in which the light of different wavelengths is scattered. Table IV contains the results.

TABLE IV.

t	$r_{F:D,t}$	t	$r_{F:D,t}$
13°.59	2.00	11°.68	1.66
12.54	2.01	11.43	1.18
11.56 ^s	1.85		

To this purpose the angles φ for D and F have been reduced to the same temperature by interpolation.

Above $12^{\circ}.54$ the ratio of the intensities of D and F seems to be constant. The fact already observed by several earlier observers that on approach of the critical temperature the mist changes from blue to almost white, is clearly set forth in the table. Measurements on this change of colour, however, have been communicated here for the first time.

§ 4. *On the size of the light-scattering particles*¹⁾. To be able to derive from $r_{F,D}$ the ratio of the intensities F and D of the light scattered in a certain direction by the mist compared with the ratio of the intensities F and D of the incident light on the mist, we must bear in mind: 1st that the two beams of light which are compared with each other in the spectroscope are subjected to different reflections and absorptions outside the spectroscope, which might bring about a change in the ratio of the intensities D and F , 2nd that the optical apparatus for observation of the scattered light not being perfectly achromatic might cause a similar change in the ratios of intensities, 3rd that if the condition of polarisation of the two beams is not exactly the same on their arrival in the spectroscope, the reflections in the spectroscope may also give rise to such a change²⁾.

The influences mentioned under 1 and 2 may be determined and eliminated by measurements of the scattered light when the substance in the neighbourhood of the critical state has been replaced by a suspension for which the ratios of intensities of the scattered light are known³⁾. Then it will have to appear in how far the deviation of the values 2.00 found in table IV at the higher temperatures from that which according to RAYLEIGH (Phil. Mag. (4) 41 (1871) p. 107) would be found if the scattering were brought about by non-conducting particles the dimensions of which are small with

1) Cf. § 1 p. 612.

2) Cf. CHRISTIE loc. cit.

3) Suspensions for which the intensity of the transmitted light is: according to

RAYLEIGH $I = I_0 e^{-k\lambda^{-4}x}$: mastic, Ag Cl, Cu_2S in water, emulsion of lemon-essence in water: ABNEY and FESTING, Proc. Roy. Soc. 40 (1886) p. 378, LAMPA, Wien. Sitz. ber. [2a] 100 (1891) p. 730, HURION, C.R. 112 (1891) p. 1431, COMPAN,

C.R. 128 (1899) p. 1226; according to CLAUSIUS $I = I_0 e^{-k\lambda^{-2}x}$: Ba SO_4 in a mixture of glycerin and water, etc.: COMPAN loc. cit. To ensure that in this experiment the light is subjected to the same reflections as in the experiments with the mist we should have to take a suspension in ethylene of the critical density.

respect to the wavelength: $\lambda^4_D/\lambda^4_F = 2.129$, is to be explained in this way ¹⁾.

About the influence of what was mentioned under 3 we have made a separate measurement. See for this § 5.

After the corrections indicated in this § have been applied, the data of table IV may serve to give an idea of the size of the particles by the aid of developments such as are given by LORENZ ²⁾. From the change of $r_{F:D}$ in table IV on approach to the critical temperature may already be deduced that the light-scattering particles must no longer be considered as small with regard to the wavelength at and below $11^\circ.86^s$ (i.e. $0^\circ.5$ above T'_k).

§ 5. *On the quantity of substance which is condensed in the light-scattering particles at different temperatures* ³⁾. To get to know the intensity of the scattered light at different temperatures, only a correction has to be applied to table III on account of the circumstance mentioned p. 617 under 3. Therefor the condition of polarisation of the scattered light at different temperatures must first be known (cf. p. 613 note 2). An upper limit for this correction may already be given as follows.

In the measurement mentioned in § 4 it appeared that light polarized normal to the slit was weakened to a greater degree in the spectroscope than light polarized parallel to the slit, in such a way that the ratio of the intensities in the spectrum is ⁴⁾:

$$H_{D\perp}/H_{D\parallel} = 0.82, \quad H_{F\perp}/H_{F\parallel} = 0.70.$$

If we now suppose that at $13^\circ.53$ all the light of wavelength D scattered in a direction normal to the incident light is polarized in the plane of incidence, and that at $11^\circ.24$ this light would be totally unpolarized, it follows from this measurement, that at $11^\circ.24$ the weakening of the D -light in the spectroscope would be 1.10 times the weakening of the D -light at $13^\circ.53$.

To be able to derive from the intensity of the scattered light at different temperatures how the quantity of condensed substance depends on the temperature, we should have to get a somewhat complete insight into the way in which the light is scattered by such

¹⁾ Also the fact that the light scattered by the mist must pass through a layer of a certain thickness (± 2 cM.) in the direction of propagation, may cause a deviation in the same direction.

²⁾ L. LORENZ. Vidensk. Selsk. Skr. Copenhagen 6 (1890). Oeuvres Scientifiques 1 p. 405.

³⁾ Cf. § 1 p. 612.

⁴⁾ Cf. with this the calculations of CHRISTIE Proc. Roy. Soc. 26 (1877) p. 24.

particles, and hence be acquainted with the structure of the particles (cf. § 1), in which also the origin (cf. Comm.) N^o. 104^a, § 4) would come in for discussion. However, it is to be expected that when the particles are small compared with the wavelength of the light, the intensity of the scattered light will increase proportional to the square of the quantity of condensed substance, whereas when the particles are no longer so small, the increase will take place more slowly.

To whatever cause we may attribute the occurrence of the differences in density, the great compressibility of the substance in the neighbourhood of the critical state will have a preponderating influence on it. Thus e.g. the mean deviation in density governed by the statistic equilibrium (SMOLUCHOWSKI)¹⁾ will be proportional to $\sqrt{\partial p / \partial \rho}$ (ρ = density). If we assume that the substance condenses round centres of attraction which exert forces on the surrounding particles of the substance which per unit of mass are only dependent on the distance, the quantity which is condensed round every centre of attraction is proportional to²⁾ $\partial p / \partial \rho$.

In order to examine what information the data in table III give on a connection between the intensity of the scattered light and the compressibility, we notice that in the neighbourhood of the critical point $\partial p / \partial \rho = q_{,1}(T - T_k)$, if the average density of the substance differs so little from ρ_k that the following term $3q_{,30}(\rho - \rho_k)^2$ may be neglected (so $T - T_k$ not too small).

TABLE V.

t	i_{DtO}	i_{DtC}	O—C in % of O
13.°53	0.190	0.213	— 12
12. 54	0.337	0.368	— 9
11. 865	0.671	0.730	— 9
11. 68	1	1	
11. 42	2.61	2.08	+ 20
11. 24	6.41	8.33	— 36

1) M. v. SMOLUCHOWSKI, Ann. d. Phys. (4) 25 1908 p. 205.

2) In this it is supposed that the condensation is so insignificant that ρ in a condensed part remains sufficiently near ρ_k .

In table V the data of table III have been compared with the formula: $i_{D,t} = \frac{0.5}{T - T_k}$ ($t_k = 11^\circ.18$, see p. 615).

The ———— curve in Pl. II fig. 2 represents $i_{D,t}$.

The differences $O - C$ are of two kinds:

1. The deviation at $11^\circ.24$: this was to be expected in the immediate neighbourhood of the critical temperature, as the formula for T_k would give an infinite intensity; here the influence makes itself felt of following terms in the development of $\partial\rho/\partial\theta$, or of the intensity of the scattered light as function of the quantity of substance (see p. 619);

2. also at temperatures further from the critical temperature there is a systematic deviation: the observed curve of intensity ascends here more rapidly than the calculated one. This might among others be in connection with the observation of TRAVERS and USHER ¹⁾, who found that the maximum of the intensity of the opalescence should not lie at T_k , but for SO_2 , $0^\circ.05$ above T_k .

Leaving these deviations out of account we may conclude that on the main the observations conform to the mentioned equation.

The deviations from a formula $i_{D,t} = \frac{0.25}{(T - T_k)^2}$ would have been much larger. The correction mentioned in the beginning of this § will not affect this conclusion.

On the supposition that at least when the dimensions of the volume elements in which appreciable condensations or rarefactions are found, are small with respect to the wavelength, the intensity of the scattered light is proportional to the square of the quantity of substance which has condensed round every centrum, or to the square of the mean deviation in density governed by the statistic equilibrium, it follows that our observations rather support the hypothesis of the condensations and the rarefactions caused by the molecular movement and governed by the statistic equilibrium, than the hypothesis of centres of attraction whose number remains constant with varying temperature.

If it appears from further investigations that the absolute value of the intensity of the light scattered by the mist is in harmony with what is to be expected according to the distribution law of BOLTZMANN (cf. SMOLUCHOWSKI, loc. cit.) a connection may be formed between the observations of the intensity of the light scattered by the mist and the disturbance function in the equation of state in

¹⁾ M. W. TRAVERS and F. L. USHER. Proc. Roy. Soc. A. 78 (1906), p. 247.

the neighbourhood of the critical point through considerations on the increase of the virial of attraction in consequence of the differences of density ¹).

§ 6. *Remarks on further experiments on the mist in the neighbourhood of the critical state.*

a. When through measurements as treated in § 3 the way in which the intensity of the light scattered by the mist depends on temperature and density, will have been sufficiently brought to light, the determination of this intensity at different heights in a CAGNIARD-LATOUR tube may be substituted for the method of the floating bulbs for the determination of the density at different heights in the tube (See Comm. N^o. 98, Sept. '07). If the establishing of the thermodynamic equilibrium is effected by keeping the temperature for a long time sufficiently constant, the determination of the intensity of the scattered light as function of the height in the tube would supply a method for the accurate determination of the experimental equation of state in the immediate neighbourhood of the critical state (cf. Comm. N^o. 98 § 1 p. 218).

b. Besides the before-mentioned measurements on the condition of polarisation (§ 2) and the measurements for the sake of the corrections mentioned in § 4, measurements on the ratio between the intensity of the scattered light and that of the incident light would also be desirable. (Cf. § 3 p. 616 and § 5 p. 620). For this purpose measurements might serve in which the ethylene is replaced by a silver mirror forming an angle of 45° with the axis of the tube ²).

¹) Cf. M. v. SMOLUCHOWSKI, *BOLTZMANN Festschrift* 1904, p. 626

²) We have in the meantime made a preliminary measurement of the absolute intensity of the scattered light by comparing it with the light reflected from a silver mirror (reflection constants for light polarized perpendicular and parallel to the plane of incidence calculated according to QUINCKE, *Pogg. Ann.* 128 (1866) p. 541 from determinations of the principal angle of incidence and the principal azimuth by JAMIN, *Ann. chim. phys.* (3) 22 (1848) p. 311). For this measurement the comparison spectrum had to be intensified by replacing the systems of lenses L_7 and L_8 by stronger combinations. From the angles $\varphi_{A\eta} = 31^\circ 33'$ and $\varphi_{Et} = 5^\circ 4.5'$ we derive that at $t = 11^\circ.93$ the intensity of the light of wavelength D scattered by 1 cM.³ of ethylene perpendicular to the direction of incidence per unit angle of vision is $s_D = 0,0007$, if the intensity of the incident (unpolarized) light = 1.

If we calculate according to RAYLEIGH, *Phil. Mag.* (5) 12 (1881) p. 86—88, LORENZ, *Oeuvres Scientif.* I p. 496, $s = \frac{2\pi^2}{N\lambda^4} \frac{(\Delta\mu)^2}{\mu_0^2}$ (N = number of light-scattering

c. It would be of interest to investigate whether for a single substance in the neighbourhood of the critical point an increase of viscosity is found as has been noted by OSTWALD¹⁾ for a liquid mixture in the neighbourhood of the critical point of separation from measurements of STEBUTT, and has been further determined by FRIEDLÄNDER (see p. 611). Perhaps the increase of the viscosity and the size of the light-scattering particles (§ 1) might be brought into relation, and so also the colour of the scattered light.

d. We could not ascertain an influence of RÖNTGEN-rays on the blue mist in ethylene. An investigation might be made as to whether the α -rays or the emanation of radium exert an influence on the mist.

e. FÜCHTBAUER²⁾ investigated a mixture of iso-butyric acid and water in the neighbourhood of the critical point of separation ultramicroscopically; he did not succeed in dissolving the cone of light. Nor could we ascertain³⁾ the presence of separate light-scattering particles in the mist for a mixture of amylene-aniline with the objective Homog. Imm. $\frac{1}{12}$, eye-piece 4, condenser AA (ZEISS) and as source of light an electric arc lamp (30 Ampère) or solar light (10 Dec. '07). We consider a repetition of this experiment with more intense solar light and with more precautions taken to keep the temperature of the mixture that is ultramicroscopically examined,

particles per cm^3 , $\Delta\mu$ deviation from the average refractive index (μ_0), and if we express $\Delta\mu$ in terms of the deviation in density according to LORENTZ-LORENZ, and

if according to SMOLUCHOWSKI we write $\bar{\delta}^2 = - \frac{RT_0}{rv_0^2 \left(\frac{\partial \rho}{\partial v}\right)_0}$ (v = number of

molecules in the light-scattering particle), in which $\left(\frac{\partial \rho}{\partial v}\right)_0$ for $v_0 = v_k$ can be

developed as $\frac{P_0}{r_n T_0} \nu_{11} (T - T_k)$ (Suppl. N^o. 6, May '03), we find at $T - T_k = 0.75$ for ethylene $s_D = 0.00075$.

Although our measurement is but preliminary, it leads us to conclude that, at least as far as the order of magnitude is concerned, the intensity of the light scattered by the blue mist in a single substance in the neighbourhood of the critical state agrees with the hypothesis of SMOLUCHOWSKI, that light is due to differences in density caused by molecular motion and governed by statistical equilibrium. [Note added in the English translation].

1) W. OSTWALD. Lehrbuch der allgemeinen Chemie II 2 (2te Aufl. p. 684).

2) CHR. FÜCHTBAUER, Zeitschr. physik. Chem. 48 (1904) p. 552.

3) We express our hearty thanks to Prof. M. DE HAAS of Delft for his kindness to lend us his ultramicroscopic apparatus.

at a constant temperature near the critical temperature of separation, desirable, and also such an investigation for a single substance in the neighbourhood of the critical point gas-liquid ¹).

This investigation in connection with what follows (see § 4) from measurements as mentioned in § 3 on the size of the light-scattering particles might give us an idea of the velocity of motion of the light-scattering particles or of the mean time of existence of definite aggregations governed by the statistic equilibrium.

Chemistry. — “*On the form-analogy of Halogene-derivatives of Hydro-carbones with open chains*”. By Dr. F. M. JAEGER. (Communicated by Prof. A. P. N. FRANCHIMONT).

(This paper will not be published in these Proceedings).

E R R A T A.

In the Proceedings of the meeting of March 30, 1907:

Plate II belonging to the Communication of Prof. H. KAMERLINGH ONNES and Dr. W. H. KEESOM: for $\tau = 1.18$ read $\tau = 1.08$; for $\tau = 1.05$ read $\tau = 1.035$.

p. 798 l. 4 from the bottom: for 0.966 read 0.996.

In the Proceedings of the meeting of September 28, 1907:

p. 211 l. 12 from the bottom: for 0.16822 read 0.25234.

In the Proceedings of the meeting of December 28, 1907:

p. 414 l. 7 from the bottom: for 28.955 read 29.030.

¹) The possibility is namely not excluded that then the light-scattering particles have larger dimensions and a greater mutual distance than at the critical point of separation of two liquids. To form an opinion on this point a spectrophotometric investigation for a liquid mixture, in the same way as we have made for a single substance (§ 3) would be useful.

(March 27, 1908).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday March 28, 1908.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige
Afdeling van Zaterdag 28 Maart 1908, Dl. XVI).

C O N T E N T S.

- J. C. KAPTEYN: "On the mean star-density at different distances from the solar system", p. 626.
P. H. SCHOUTE: "On fourdimensional nets and their sections by spaces", (2nd Part), p. 636.
(With 2 plates).
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Astronomy. — “*On the mean star-density at different distances from the solar system.*” By J. C. KAPTEYN.

(Communicated in the meeting of February 29, 1908).

In the meeting of April 20, 1901, I derived not only the so-called *luminosity-curve* but also the law according to which the star-density, i. e. the number of stars per unit of volume, diminishes with increasing distance from the solar system¹⁾. I assumed, and the assumption will again be made in the present paper, that there is no absorption of light in space. I then pointed out that the luminosity-curve is not very appreciably modified if we change, within admissible limits the data from which it was derived. On the other hand it was expressly stated that the determination of the change of density was only quite *provisional*. Its discussion was deferred to a subsequent communication, in which, along with the data then used, other elements might be taken into account. (l.c. p. 731).

These other elements are mainly: the total numbers of stars of different magnitude and their mean parallaxes. As to the first, the numbers: a short time ago I treated all the materials accessible to me (see Publications of the Astr. Lab. at Groningen N°. 18) and I think that I obtained very reliable results for the stars brighter than 11.5, fairly reliable ones down to the 15th magnitude. Now, though the mean parallaxes are still wanting, we are already able, by the numbers alone, to arrive at a considerable improvement in the distribution of the densities, at least for the larger distances. Such a derivation will be given in what follows.

As formerly a separate treatment of the regions of different galactic latitude was not yet attempted, because I think that it will be desirable that in carrying out such a separate treatment we investigate at the same time whether it be admissible or not to assume the same luminosity curve (mixture-law) for the different parts of the system.

Already I have collected fairly extensive materials for this purpose, but still some time will have to elapse before the investigation can be carried out with advantage.

In the main my purpose, in making the following determination, was simply to get first notions about the star-density at still greater distances than could be reached in our former investigation. The determination embraces also the smaller distances but it remains to be seen whether for these the correction found is or is not an improvement.

In the *Astron. Journ.* N°. 566 I derived analytical expressions,

¹⁾ See also: Publications of the Astr. Lab. at Groningen No. 11.

which fairly represent the numbers found in the communication just mentioned.

Let

$$\delta = 2\ 512 \dots = \frac{\text{Apparent brightness of a star of mag } m}{\text{'' '' '' '' '' '' '' '' } m + 1} \quad (\log. \delta = 0.4);$$

ϱ = distance from the solar system ($\varrho = 1$ for parallax = 0"1);

N_m = number of stars in the whole of the sky between the apparent magnitudes $m - \frac{1}{2}$ and $m + \frac{1}{2}$.

$\Delta(\varrho)$ = star-density = number of stars per unit of volume at distance ϱ (unit of volume = cube, each side of which = unit of distance).

h_m = apparent brightness of a star of the magnitude m ($h_{5.5} = 1$).

L = Luminosity = total quantity of light emitted ($L = 1$ for Sun).

$\varphi(L) dL$ = probability that the luminosity of a star, chosen at random, is contained between L and $L + dL$.

$$\psi(L) = \int_{\frac{L}{\sqrt{\delta}}}^{L\sqrt{\delta}} \varphi(z) dz = \text{probability that the luminosity is contained}$$

between the limits $L \pm \frac{1}{2}$ mag.

Now, if we assume that $\varphi(L)$ is not dependent on ϱ , we shall have

$$N_m = 4\ \pi \int_0^\infty \varrho^4 \Delta(\varrho) d\varrho \int_{\frac{h_m}{\sqrt{\delta}}}^{h_m\sqrt{\delta}} \varphi(h\varrho^2) d\varrho = 4\ \pi \int_0^\infty \varrho^2 \Delta(\varrho) \psi(h_m\varrho^2) d\varrho \quad (1)$$

The expressions derived in *Astron. Journ.* N^o. 566 are :

$$\psi(L) = \frac{\alpha^2 \cdot \text{mod.}}{\sqrt{\pi} L} e^{-\alpha^2 \log L - T^2} \dots \dots \dots (2)$$

$$\frac{\Delta(\varrho)}{\Delta(0)} = e^{-\beta\varrho} + \beta\varrho e^{-\gamma\varrho^2} \dots \dots \dots (3)$$

in which

$$\left. \begin{aligned} T &= 1.400 \\ \alpha^2 &= 0.385 \end{aligned} \right\} \dots \dots \dots (4)$$

$$\Delta(0) = 111.0 \dots \dots \dots (5)$$

$$\left. \begin{aligned} \beta &= 0.0220 \\ \gamma &= 0.0052 \end{aligned} \right\} \dots \dots \dots (6)$$

In a subsequent part of the same paper, the *numbers* of the stars, as given by PICKERING led to a new value of $\Delta(0)$ viz.

$$\Delta(0) = 136.9 \dots \dots \dots (7)$$

the difference of this value and the value (5) is wholly explained by the constant difference of the photometric scale of Potsdam, which was

used for the determination (5) and that of Harvard which served for the derivation of (7).

In what follows the magnitudes have also be reduced to the Harvard scale. I have adopted the luminosity-curve (2), in which the constants have the values (4), without any change. For the density curve (3), however a new determination was obtained by the aid of the total numbers of the stars of different apparent magnitude. In other words: by the aid of formula (1) I derived Δ as a function of ϱ from the *given* values N_m ($m = 2$ to 15) and the *given* form of ψ .

The introduction of the analytical functions (2) and (3) has the advantage of greatly facilitating the computations. Of course we have not to forget, however, that they can be relied on only just to the same extent as that for which we possess observational data. For the luminosity-curve, with the exception only of the stars belonging to the classes of the very greatest apparent brightness, the unlimited use of the formula will not easily give rise to appreciable errors, because extrapolation is only necessary for a very small fraction of the total. On the contrary, the density-curve (3) (which, as we already remarked, is not very accurately determined) furnishes values, which, for ϱ exceeding 60, are to be considered as wholly obtained by *extrapolation*. It will appear from what follows that up to $\varrho = 60$ the values derived from the new materials do not differ from those formerly obtained more than seems in accordance with their uncertainty. That on the other hand, the values for $\varrho > 60$, which we may extrapolate by means of formula (3), are *far* too small; to such an extent that for these greater distances the formula is evidently quite unsatisfactory.

To begin with, I ascertained how the formula (3), in which the constants have the values (6) and (7), represents the N_m of publication 18. A table of the integrals entering in the formula (1) has been given in *Astronomical Journal* N^o. 566 for values of m between 0 and 11. (table III)¹).

¹) In the calculation of the values of T_1 and T_3 a mistake has been discovered:

	T_1	T_3
For $m = 3.0$, instead of 9.12 read 9.13	9.12	9.13
4.0 " " 12.69 " 12.71	12.69	12.71
8.0 " " 17.04 " 17.10	17.04	17.10
6.0 " " 21.86 " 21.99	21.86	21.99
7.0 " " 26.63 " 26.88	26.63	26.88
8.0 " " 30.54 " 30.96	30.54	30.96
9.0 " " 32.80 " 33.42	32.80	33.42
10.0 " " 32.84 " 33.64	32.84	33.64
11.0 " " 30.54 " 31.51	30.54	31.51
Instead of 1.71 read 1.72		1.71
" " 1.37 " 1.38		1.37
" " 1.04 " 1.05		1.04

For $m = 14$ the values were now expressly computed. The result is as follows:

		Total number of stars.		
		Obs. (Publ. 18)	Comp. I. (by Form. (3))	Comp. II.
m				
4.5 tot	5.5	1 848	1 897	1 788
6.5	,, 7.5	17 940	18 420	18 650
8.5	,, 9.5	159 200	140 200	169 500
10.5	,, 11.5	1 275 000	808 200	1 335 000
13.5	,, 14.5	23 680 000	6 500 000	20 800 000

The deviation increases strongly with diminishing brightness and is excessive for magnitude 14. We conclude at once, that for the greater distances the formula (3) furnishes a value of the star-density which is much too small. Calculation shows that some approximate agreement is already obtained if we take the stars between $\varrho = 140$ and $\varrho = \infty$ to be 21 times more numerous.

As it thus appears that formula (3) is useless for considerable values of ϱ , I began by retaining that formula exclusively for the values of ϱ below 70 whereas for the values exceeding 70 I assumed that the density diminishes regularly (linearly) from 0.214 to zero.

It was easily ascertained that, if we choose the decrease of the density in such a way that it vanishes for $\varrho = 557$, we get considerably nearer to the truth, especially if we take:

$$\Delta_0 = 125.$$

The values obtained in this way were put down in the above table under the head Comp. II.

As a further approximation I also derived corrections for the star-density at distances below 70. It appeared that the results become more satisfactory if the linear decrease of the densities is assumed to begin for distances somewhat smaller than 70.

Having obtained this result I have no further continued these approximations, but I have given up the formula (3) altogether and have tried to determine the luminosity-curve directly in the assumption that, for the intervals between $\varrho = 0$ and $\varrho = 10$; $\varrho = 10$ and $\varrho = 30$; $\varrho = 30$ and 50 ; $\varrho = 50$ and $\varrho = g$ the density changes linearly in such a way that it vanishes for $\varrho = g$.

In this way the problem is reduced to the derivation of the 5 unknown quantities:

$$\Delta(0) ; \Delta(10) ; \Delta(30) ; \Delta(50) ; g.$$

For reasons given in the paper quoted above we have to assume

that $\frac{\partial \Delta}{\partial \varrho}$ vanishes for $\varrho = 0$. As a consequence $\Delta(10)$ will certainly be little different from $\Delta(0)$. Therefore, in order to reduce the number of unknown quantities as much as possible, I took, in agreement with what was formerly found :

$$\Delta(10) = 0.97 \Delta(0) (9)$$

The number of unknown quantities is thus lowered to 4.

Putting

$$\frac{\Delta(\varrho)}{\Delta(0)} = D_\varrho (10)$$

we have

$$D_\varrho = A\varrho + B (11)$$

in which, for the several intervals, A and B have the following values :

	A	B	
$\varrho = 0 \text{ to } 10$	$\left \begin{array}{c} \frac{1}{10} D_{10} - \frac{1}{10} \end{array} \right $	$\left \begin{array}{c} 1. - \end{array} \right $	
10 ,, 30	$\left \begin{array}{c} \frac{1}{20} D_{30} - \frac{1}{20} D_{10} \end{array} \right $	$\left \begin{array}{c} \frac{3}{2} D_{10} - \frac{1}{2} D_{30} \end{array} \right $	
30 ,, 50	$\left \begin{array}{c} \frac{1}{20} D_{50} - \frac{1}{20} D_{30} \end{array} \right $	$\left \begin{array}{c} \frac{5}{2} D_{30} - \frac{3}{2} D_{50} \end{array} \right $	
50 ,, g	$\left \begin{array}{c} -\frac{1}{g-50} D_{50} \end{array} \right $	$\left \begin{array}{c} \frac{g}{g-50} D_{50} \end{array} \right $. . . (12)

The practical advantage of the present form is that the expression (1) for N_m can now be reduced to the well known integral

$$\Theta(z) = \frac{1}{\sqrt{\pi}} \int_{-z}^z e^{-x^2} dx (11)$$

Numerical integration is thus avoided and we obtain relations which are linear in respect to the unknown quantities D_{30} , D_{50} and $\frac{1}{\Delta(0)}$.

If we denote by $(N_m)_0^z$ the number of stars between apparent magnitude $m - \frac{1}{2}$ and $m + \frac{1}{2}$ existing between the distance 0 and ϱ , we get, substituting (2) (10) (11) in (1) :

$$(N_m)_0^z = [G A + H B] \Delta(0) (12)$$

in which $\mu = mod.$ of the Nep-Log.

$$G = \frac{2\pi\alpha}{h_m} e^{-\frac{0.4m - 2.20 + T}{\mu} + \frac{1}{4z^2\mu^2}} \Theta \left[\alpha(2.20 - T - 0.4m + 2\log\varphi) - \frac{1}{2\alpha\mu} \right] \quad (13)$$

$$H = \frac{2\pi\alpha}{h_m} e^{-\frac{0.4m - 2.20 + T}{2\mu} + \frac{1}{16z^2\mu^2}} \Theta \left[\alpha(2.20 - T - 0.4m + 2\log\varphi) - \frac{1}{4\alpha\mu} \right] \quad (14)$$

and

$$\log h_m = 2.20 - 0.4m \dots \dots \dots (15)$$

As soon as the $(N_m)_0^{\epsilon}$ have become known we find the $(N_m)_{\rho_1}^{\rho_2}$ by simple subtraction.

I have carried through the solution for the values 400, 600, 800 and 1000 for g . It appeared that only when we come to the last value we get satisfactory results.

It seems superfluous to give all my calculations in full. I will only communicate some of the values obtained with the constant

$$g = 1000 \dots \dots \dots (16)$$

which was finally adopted. The value of the G and H were found to be as follows. (See table I p. 632).

Now, if for the stars of magnitude 2, 3, 4, 5, we take the numbers found by PICKERING for the whole of the sky, viz. resp. 58, 172, 577, 1848¹⁾ and for the remaining magnitudes, the numbers which we derive from table 2 of the Groningen Publication N^o. 18, by simply multiplying with 41 253 (the number of square degrees for the whole of the sky), we find equations of condition for the derivation of the unknown quantities $\Delta(0)$, D_{30} , D_{50} , such as this :

$$58 \frac{1}{\Delta(0)} = 0.2962 + 0.0411 D_{30} + 0.0244 D_{50}$$

etc. They get a more convenient form if we put $\frac{140}{\Delta} = Z$ and if we then divide all the equations by the coefficient of Z . In this way the equations of condition become as follows :

¹⁾ In Publ. 18. p. 8 I found, by countings made on the materials of PICKERING :

$$N_{1.495}^{2.495} = 58; N_{2.495}^{3.495} = 171; N_{3.495}^{4.495} = 574; N_{4.495}^{5.495} = 1837.$$

With the aid of the computed values communicated in the same publication it is easy to pass from these to the numbers $N_{1.5}^{2.5}$ etc. The results thus found are those of the text.

TABLE I.

	G.				H.			
	$p=10$	30	50	1000	$p=10$	30	50	1000
$m=2$	0.9341	2.365	3.105	4.834	0.2463	0.3302	0.3496	0.3670
3	3.406	10.75	15.39	30.29	0.7842	1.203	1.324	1.461
4	11.22	44.74	70.58	188.1	2.314	4.179	4.877	5.813
5	33.29	169.5	296.8	1151.	6.275	13.66	16.94	23.12
7	210.9	1795.	3937.	39430.	34.79	116.5	171.0	361.9
9	850.9	12380.	34570.	1407000.	128.0	696.0	1254.	5425.
11	2463.	54580.	195700.	23250000.	304.6	2778.	6288.	72510.
13	3436.	151700.	703600.	340600000.	460.9	7203.	20780.	784150.
15	3387.	264000.	1589000.	3329000000.	438.6	11900.	44120.	6275000.

$(m = 2)$	$0.099 D_{30}$	$+ 0.059 D_{50}$	$- Z = 0.715$	} . . . (17)
$(m = 3)$	0.186	$+ 0.146$	$- Z = 0.836$	
$(m = 4)$	0.272	$+ 0.287$	$- Z = 0.817$	
$(m = 5)$	0.375	$+ 0.534$	$- Z = 0.781$	
$(m = 7)$	0.527	$+ 1.474$	$- Z = 0.595$	
$(m = 9)$	0.508	$+ 3.108$	$- Z = 0.345$	
$(m = 11)$	0.341	$+ 5.187$	$- Z = 0.149$	
$(m = 13)$	0.159	$+ 6.926$	$- Z = 0.047$	
$(m = 15)$	0.050	$+ 7.148$	$- Z = 0.010$	

In solving these equations I have neglected those corresponding to the magnitudes 2 and 3. The reason is that for these the influence begins to be sensible of stars of so great a luminosity that *extrapolation* beyond the directly determined part of the luminosity-curve becomes necessary. These stars might therefore rather be used for a correction of this curve at its brighter extremity.

The remaining equations have been condensed into three by combining those for $m = 4$ and 5, those for 7, 9, 11 and those for 13 and 15. The solutions of these three equations is:

$$\begin{array}{l}
 Z = 1.002 \quad \text{therefore } \Delta(0) = 139.7 \\
 D_{30} = 0.460 \\
 D_{50} = 0.1315
 \end{array}
 \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} (18)$$

whereas we already assumed

$$D_{10} = 0.970 \quad (19)$$

If with these values we compute the numbers N_m and if further we interpolate those for $m = 6, 8, 10, 12, 14$ we get the following comparison between theory and observation:

TABLE II. TOTAL NUMBER OF STARS.

m	Obs.	Comp.	$O-C$ in fraction of whole obs. numb.
2	58	44.5	+ 0.233
3	172	161.5	+ 0.061
4	577	564.3	+ 0.022
5	1 848	1 889	- 0.022
6	5 816	6 025	- 0.036
7	17 940	18 450	- 0.028
8	54 040	54 580	- 0.010
9	159 200	157 200	+ 0.013
10	457 900	448 000	+ 0.022
11	1 275 000	1 256 000	+ 0.015
12	3 453 000	3 490 000	- 0.011
13	9 157 000	9 419 000	- 0.028 ^s
14	23 680 000	24 100 000	- 0.018
15	60 225 000	58 500 000	+ 0.029

If, in accordance with what has been said, we except the very brightest magnitudes, the deviations are doubtlessly below the uncertainties in the determination of the observed numbers of the stars. The somewhat irregular course of the numbers is probably due to the discontinuities in the density-curve as definitively adopted.

The following table may serve to get at least some insight in the distribution of the stars of a determined apparent magnitude over the different distances.

TABLE III. NUMBER OF STARS (N_m) ^{r^2}

r	$m=3$	5	7	9	11	13	15
0 to 10	108.	863	4770	17 500	42 000	63 000	60 000
10 > 30	45.5	779	8310	56 100	236 000	625 000	1 030 000
30 > 50	5.5	145	2350	3 400	144 000	543 000	1 250 000
50 > 1000	2.5	103	3000	59 900	835 000	8 188 000	56 450 000

or expressed in fractions of the totals

TABLE IV.

ρ	3	5	7	9	11	13	15
0 to 10	0.669	0.457	0.259	0.142	0.033	0.007	0.001
10 to 30	.282	.412	.452	.357	.188	.066	.018
30 to 50	.034	.077	.127	.150	.115	.058	.021
50 to 1000	.015	.054	.162	.382	.665	.870	.960

Summing up we find: that the total numbers of stars of different magnitude (Harvard scale) as derived from observation in *Publications of the Groningen Laboratory No. 18*, are well represented by adopting the luminosity-curve (2), with the values (4) of the constants and the following values of the star-density:

TABLE V. STAR-DENSITY.

ρ	Δ	ρ	Δ
0	1.000 $\Delta(0)$	100	0.125 $\Delta(0)$
10	0.970	200	0.111
20	0.715	300	0.097
30	0.460	400	0.083
40	0.296	500	0.069
50	0.131 ⁶	600	0.055
60	0.130	700	0.042
70	0.129	800	0.028
80	0.127	900	0.014
90	0.126	1000	0.000

in which $\Delta(0) = 139.7$.

It would be interesting to investigate what are the changes that can be made in these values without their ceasing to represent the observed numbers satisfactorily. I have deferred this investigation for the present because it will be desirable in such a discussion to include also the data furnished by the parallaxes.

Mathematics. — “*On fourdimensional nets and their sections by spaces.*” (Second part). By Prof. P. H. SCHOOTE.

(Communicated in the meeting of February 29, 1908).

The net (C_n).

1. The problem to determine the section of the net (C_n) with a given space can be naturally divided into two parts. The first part occupies itself with the question, how a series of spaces parallel to the given one intersects an eightcell; in the second is indicated, how the section of each of the eightcells intersected by the given space can be deduced from that section which determines this space in the eightcell assumed in the first part. Of course the four series of parallel spaces normal to an axis of the eightcell come here to the fore and then in the first part of the problem are investigated in the first place the so-called “transition forms” where the intersecting space contains one or more vertices of the eightcell, whilst between each pair of transition forms adjacent to each other a single intermediary form is introduced, namely that one by the space which bisects the distance between the two spaces bearing those transition forms. Generally this is sufficient for our end; moreover it is not difficult to interpolate where necessary other intermediary forms.

In the preceding communication of the same title we have packed up each of the cells C_{16} of the net (C_{16}) and each of the cells C_{24} of the net (C_{24}) in the smallest possible eightcell with edges parallel to the axes of coordinates, with the intention to connect the spacial sections of the nets (C_{16}) and (C_{24}) with those of the net (C_8) by cutting with each C_{16} and each C_{24} also the case C_8 enclosing these cells. With a view to this application we add to the above indicated four series of parallel intersecting spaces two others, viz. those normal to one of the two lines connecting the origin of coordinates with the point (3, 1, 1, 1) and the point (2, 1, 1, 0); indeed, these lines are — see the last table of the preceding communication — axes of one or more of the cells C_{16} and C_{24} enclosed in a cell C_8 . Also for these two new series we restrict ourselves to the forms of transition and the intermediate forms lying in the middle between two adjacent forms of transition.

In order to simplify the survey of the sections appearing in the six series of parallel spaces we give the results to which the first part — the determination of the section with one C_8 — leads in two different ways. In the first place we project all vertices, edges, faces, bounding bodies of the cell C_8 on the axis normal to each of the

six series of spaces to deduce the sections from this tabularly; in the second place we indicate the sections themselves in parallel perspective in the eightcell. To each of those two closely allied modes of transacting an extending plate is given.

To promote the uniformity we indicate the axes OE , OK , OF , OR by their ends $(1, 1, 1, 1)$, $(1, 1, 1, 0)$, $(1, 1, 0, 0)$, $(1, 0, 0, 0)$. Then we have to deal successively with the six series

$(1, 1, 1, 1)$, $(1, 1, 1, 0)$, $(1, 1, 0, 0)$, $(1, 0, 0, 0)$, $(3, 1, 1, 1)$, $(2, 1, 1, 0)$ and we have now to investigate for each of those six cases the two parts into which the problem was above divided.

2. *Case* $(1, 1, 1, 1)$. — This case was, as far as the first part of the problem is concerned, completely solved in a foregoing study (*Proceedings*, Jan. 1908, page 485). Hence the first part of the first plate with the superscription $(1, 1, 1, 1) OE_8$ is an extension of the first diagram $n = 4$ of the plate given then. In order to be able to indicate together with the projections of all bounding elements the projections of the vertices of these elements, which considerably promotes the insight into the spacial figure, the numbers of edges, faces, bounding bodies are denoted here outside the scheme on the righthand side. Moreover the sections of the eightcell with the spaces of transition and the intermediate spaces perpendicular to the diagonal of projection are mentioned tabularly; here use has been made of a method formerly (*Verhandelingen*, volume IX, n^o. 4) developed in all details which acquaints us not only with the characteristic numbers (e, k, f) of each section, but also with the nature of the faces. Thus the central section is a $(6, 12, 8)$, because it contains 6 vertices and does not cut an edge, intersects 12 faces and contains no edges, intersects 8 bounding cubes and contains no faces; this section is a regular octahedron in connection with which each cube of the two quadruples of bounding bodies is cut according to an equilateral triangle of the same size. In this way the adjacent intermediary section is a $(12, 18, 8)$, because 12 edges, 18 faces and 8 bounding cubes are intersected, viz. a tetrahedron regularly truncated at the vertices, i. e. the first of the semi-regular Archimedian polyhedra (*Proceedings*, page 488) because four of the bounding cubes are intersected according to regular hexagons, the four remaining ones according to equilateral triangles. Here the number of edges is found back as half of the total number of sides of the faces, thus 12 as half the product of eight and three, 18 as half the sum of four times six and four times three. Moreover, when indicating the polygons lying in the faces, we have underlined the figure of each group of regular polygons.

The second plate indicates the obtained sections in parallel perspective. The first diagram on the top leftside, represents an eightcell which indicates besides the diameters normal to the different series of parallel intersecting spaces a few other lines appearing in the solution; for our case (1, 1, 1, 1) to which the four following diagrams refer the axis EE' is this diameter. To characterize this case the mark (1, 1, 1, 1) is noted down to the right at the bottom in the rectangle; moreover the fractions $\frac{4}{8}, \frac{3}{8}, \frac{2}{8}, \frac{1}{8}$ placed to the left at the top of each diagram indicate the part of the axis EE' lying with E on the same side of the intersecting space. It is easy to follow in these diagrams the changes in form which each face of the regular octahedron forming the central section undergoes, when the point of intersection of the intersecting space with the axis OE moves from O to E . Thus the face lying in the upper cube of the eightcell, which is at the same time the visible upper plane of the octahedron regarded by itself, transforms itself first into a regular hexagon, then into an equilateral triangle of opposite orientation, etc.; if the eightcell is a $C_8^{(2)}$, then the sides of the triangles of the first and third diagrams are $2\sqrt{2}$, those of the hexagons and the triangles of the second and fourth diagrams are $\sqrt{2}$, whilst the series closes with the transition form consisting of the single vertex E to which the fraction $\frac{0}{8}$ answers.

We now arrive at the question how the remaining eightcells that are likewise cut by the intersecting space are intersected in each of the considered cases. To this end we suppose the above intersected eightcell to be the central one of the net and so we assume the centre of this cell to be the origin of the system of coordinates with respect to which we have determined in the first communication the coordinates of the centres of the remaining cells in the symbolic form $(2a_i)$. The equation of the central space perpendicular to the axis OE_8 towards the point (1, 1, 1, 1) is $x_1 + x_2 + x_3 + x_4 = 0$; the length of the normal let down out of the centre $(2a_i)$ on to this space is therefore Σa_i . So the eightcell with the centre $(2a_i)$ is cut by the central space $\Sigma x_i = 0$, when $-2 \leq \Sigma a_i \leq 2$, and here the five cases occur where Σa_i has one of the values $-2, -1, 0, 1, 2$. In other words:

if with the central cell the central section $\frac{4}{8}$ makes its appearance, then with the remaining cells the sections $\frac{0}{8}, \frac{2}{8}, \frac{4}{8}, \frac{6}{8}, \frac{8}{8}$ occur and

no others. The sections $\frac{0}{8}$ and $\frac{8}{8}$ being points and therefore not under consideration, we find as section of the net (C_8) a three-dimensional space-filling consisting of two groundforms, octahedron and tetrahedron, where the tetrahedron occurs in two positions of opposite orientation. From a close consideration of this result follows now that the fractional symbols of the intersected cells furnish *in general* differences of multiples of quarters with that of the central cell and are thus represented by $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ when the symbol of the central cell is $\frac{3}{8}$ or $\frac{1}{8}$. We find then again a three-dimensional space-filling consisting of two groundforms each of which appearing in two oppositely orientated positions, the first semi-regular Archimedian body and the tetrahedron. As we arrive again at eightcell and tetrahedron when starting from the section $\frac{2}{8}$ of the central cell, the above-mentioned two cases are for this series the only ones where the three-dimensional space-filling consists of two groundforms. In every other case — as e. g. the one answering to the fractions $\frac{1}{16}, \frac{5}{16}, \frac{9}{16}, \frac{13}{16}$ — we find four different groundforms and never more; we recommend the designing of the just mentioned quadruplet of sections as a good practice.

If we exchange the infinite system of cells $C_8^{(2)}$ by a finite block of k^4 cells $C_8^{(2)}$ forming together a $C_8^{(2k)}$, if we divide a diagonal of this block into eight equal parts and if we suppose the block to be intersected by a space standing in one of the points of division perpendicular to the diagonal, we then find according to circumstances either a finite system of octahedra $O^{(2\sqrt{2})}$ and tetrahedra $T^{(2\sqrt{2})}$ with edges $2\sqrt{2}$, or a finite system of Archimedian bodies $A^{(\sqrt{2})}$ and tetrahedra $T^{(\sqrt{2})}$ with edges $\sqrt{2}$, enclosed in an octahedron, a tetrahedron or an Archimedian body of greater size, viz., in the section of the block $C_8^{(2k)}$ with the intersecting space. In connection with the notes joined to the pages 15, 16 and 24 of the study "On the sections of a block of eightcells, etc." (*Verhandelingen*, volume IX, n^o. 7) we here indicate how large in each of those cases the number of the component parts ($O^{(2\sqrt{2})}, A^{(\sqrt{2})}, T^{(2\sqrt{2})}, T^{(\sqrt{2})}$) is. We restrict ourselves here to mentioning the results and we only remind the readers that the deduction of these are based on the actual connection

$\frac{1}{4} C_8^{(2k)} = T^{(2k) \vee 2}$	$T_p^{(2 \vee 2)} \dots (k+2)_3$ $T_n^{(2 \vee 2)} \dots (k)_3$	$O^{(2 \vee 2)} \dots (k+1)_1$
$\text{Sum } \frac{1}{2} k(k^2+1)$		
$\frac{3}{8} C_8^{(4k)} = A^{(2k) \vee 2}$	$T_p^{(2 \vee 2)} \dots \frac{1}{6} k(23k^2+6k-2)$ $T_n^{(2 \vee 2)} \dots \frac{1}{6} k(23k^2-6k-2)$	$O^{(2 \vee 2)} \dots \frac{1}{6} k(23k^2-1)$
$\text{Sum } \frac{1}{6} k(39k^2-5)$		
$\frac{1}{2} C_8^{(2k)} = O^{(2k) \vee 2}$	$T_p^{(2 \vee 2)} \dots 4(k+1)_3$ $T_n^{(2 \vee 2)} \dots 4(k+1)_3$	$O^{(2 \vee 2)} \dots \frac{1}{3} k(2k^2+1)$
$\text{Sum } k(2k^2-1)$		
$\frac{1}{8} C_8^{(4k+2)} = T^{(2k+1) \vee 2}$	$T_p^{(1 \ 2)} \dots (k+3)_3$ $T_n^{(1 \ 2)} \dots (k)_3$	$A_p^{(1 \ 2)} \dots (k+2)_1$ $A_n^{(1 \ 2)} \dots (k+1)_3$
$\text{Sum } \frac{1}{3} (2k^3+3k^2+7k+3)$		
$\frac{3}{8} C_8^{(4k+2)} = A^{(2k+1) \vee 2}$	$T_p^{(1 \ 2)} \dots \frac{1}{6} k(k+1)(2^3k+34)$ $T_n^{(1 \ 2)} \dots \frac{1}{6} k(23k^2+12k-11)$	$A_p^{(\vee 2)} \dots \frac{1}{6} (k+1)(23k^2+1^1k+6)$ $A_n^{(1 \ 2)} \dots \frac{1}{6} k(23^2+27k+10)$
$\text{Sum } \frac{1}{3} (46k^3+69k^2+29k+3)$		

between the coefficients of the different powers of x in the development of $(1 + x + x^2 + \dots + x^{k-1})^4$ and the numbers of cells $C_8^{(2)}$ of the block $C_8^{(2k)}$ which agree with each other in projection on a diagonal.

In the following table of results we have separated from one another the three cases leading to sections $\frac{1}{4} C_8^{(2)} = T^{(2\sqrt{2})}$, $\frac{1}{2} C_8^{(2)} = O^{(2\sqrt{2})}$ and the two cases leading to sections $\frac{1}{8} C_8^{(2)} = T^{(\sqrt{2})}$, $\frac{3}{8} C_8^{(2)} = A^{(\sqrt{2})}$.

Moreover, the two positions of opposite orientation appearing for T and A are distinguished from each other as T_p , T_n and A_p , A_n , and then those parts $T^{(\sqrt{2})}$ and $A^{(\sqrt{2})}$ get the same foot-index which answer not only as regards volume but also as regards position of juncture to the relation

$$A^{(\sqrt{2})} + 4 T^{(\sqrt{2})} = T^{(3\sqrt{2})},$$

whilst this index is a p (positive) for $T^{(2\sqrt{2})}$ when this tetrahedron agrees in position to $T^{(2k\sqrt{2})}$ and $A^{(2k\sqrt{2})}$, and can be taken arbitrarily in the third case $O^{(2k\sqrt{2})}$, where the two amounts are indeed equal.

In this table the symbols $(k+2)_3$, etc. represent binomial coefficients. The coming to the fore of the numerical factor 23 is connected with the relation holding only for the volume

$$A^{(1/2)} = 23 T^{(1/2)},$$

which ensues immediately from the one given above. It forms part of

$$\frac{O^{(2\sqrt{2})}}{32} = \frac{A^{(1/2)}}{23} = \frac{T^{(2\sqrt{2})}}{8} = \frac{T^{(1/2)}}{1},$$

of which we have availed ourselves when arranging the preceding table, either as an aid in the calculation or as control.

The cases (1, 1, 1, 0), (1, 1, 0, 0), (1, 0, 0, 0). — These three cases are so much simpler than the preceding one, that we can treat them collectively, now that the application of the results appearing here to the nets (C_{1a}) and (C_{2d}) make a short treatment necessary. The projection of the bounding elements on the corresponding axes OK , OF , OR are immediately found; in order to take into account the duality, appearing on one hand between OE and OR and on the other hand between OK and OF , the projections on OR are placed on the first plate next to those on OE , whilst the projections on OK and OF find a place there side by side. A single glance given to these diagrams already arouses the conviction that the sections in the direction of DE over OK and OF to OR must keep on becoming simpler. That this is really the case — and for what reason — is

clearly evident from the second plate, giving the sections for the cases OK and OF . As is shown in the three diagrams with the fractional symbols $\frac{3}{6}, \frac{2}{6}, \frac{1}{6}$ belonging to OK here one of the dimensions of the section, viz. the dimension in the direction of the edge with K as centre, is of constant length, by which the sections become prisms with a height 2, namely an hexagonal prism $H^{(\sqrt{2})}$, a triangular prism $P^{(2\sqrt{2})}$ and a triangular prism $P^{(\sqrt{2})}$; with these symbols H and P the indices $\sqrt{2}$ and $2\sqrt{2}$ indicate the length of the sides of the bases. As a matter of fact we can now assert that with these prisms of which the endplanes are the determining variable elements, the problem of the intersection has lost a dimension; for, in order to determine the prism we have only to ask how the ground-cube is intersected by a plane perpendicular to a diagonal of this bounding body of the eightcell, i. o. w. the problem has become threedimensional. In the same way we find in case OF rectangular prisms of which two dimensions remain constant, which has been indicated for the section of transition $\frac{2}{4}$ and the intermediary section $\frac{1}{4}$, whilst the section in case OR is an invariable cube, which is of course not designed.

It is almost superfluous to stop for the two space-fillings of case OK , that by $H^{(\sqrt{2})}$ and $P^{(\sqrt{2})}$ together and that by $P^{(2\sqrt{2})}$ alone, as they appear indeed as well-known plane-fillings. We suffice by giving the following relations:

$$P^{2k+1\sqrt{2}} = (k+2)_2 P_p^{(\sqrt{2})} + (k+1)_2 H^{(\sqrt{2})} + (k)_2 P_n^{(\sqrt{2})}$$

$$H^{(2k+1)\sqrt{2}} = 6(k+1)_2 P_p^{(\sqrt{2})} + (3k^2+3k+1) H^{(\sqrt{2})} + 6(k+1)_2 P_n^{(\sqrt{2})}$$

$$P^{(k\sqrt{2})} = (k+1)_2 P_p^{(\sqrt{2})} + (k)_2 P_n^{(\sqrt{2})}$$

$$H^{(k\sqrt{2})} = 3k^2 P_p^{(\sqrt{2})} + 3k^2 P_n^{(\sqrt{2})}$$

4. Case (3, 1, 1, 1). — If the vertex A of the eightcell $U_8^{(2)}$ — see first diagram of second plate — is point (1, 1, 1, 1) then the point $\left(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is obtained by dividing the inner diagonal AB of the cube lying in the space $x_4 = 1$ into three equal parts and then to take the first point of division (C^1). The line OC is for this case

1) By mistake in the diagram for AC has been taken $\frac{1}{3} AB$ instead of $\frac{1}{3} AB$.

the axis upon which we must project to determine the projection of the bounding elements. Now it is clear that the projection of the cube with AB as a diagonal is obtained by projecting first this bounding body on the projection AB of the axis OC on its space $x_1 = 1$ which furnishes with regard to the vertices the stratification 1, 3, 3, 1 and by determining then the projection on OC of these new points lying on AB . Now, angle BOC is a right one, for out of the coordinates $(1, -1, -1, -1)$ and $(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ of B and C follows immediately $OB^2 + OC^2 = BC^2$. So B projects itself on OC in O and so this of course is also the case with the vertex $(-1, 1, 1, 1)$ of the eightcell lying opposite B . So we find — see the first plate under head (3, 1, 1, 1) C_8 — the stratification of the 16 vertices by causing the group of points 1, 3, 3, 1 laid upon the axis of projection at equal intervals to be followed by a second group of the same structure in such a way that the first 1 of this second group coincides with the last 1 of the first group. It is from this that this projection has its type, as is indicated at the foot. One really finds without any difficulty all that is given on the scheme by representing to oneself the two bounding cubes indicated in the typical image — here lying in the spaces $x_1 = \pm 1$ — and to suppose that their corresponding vertices, edges, faces are united by edges, faces and bounding bodies.

If again we do not take the isolated point A into consideration, then we have to deal here with six different forms of the section, three intermediary forms and three forms of transition ; these are given with the addition of the corresponding fractional symbols $\frac{6}{12}, \frac{5}{12}, \dots, \frac{1}{12}$ on the second plate. We shall indicate somewhat in details how these diagrams are deduced by drawing, independently of the results of the first plate, and to this end we immediately notice that the space through A perpendicular to OC is represented by $3x_1 + x_2 + x_3 + x_4 = 6$ and that this space after a slight parallel displacement to O truncates from the edges of the eightcell passing through A segments which are in the ratio to each other of 1 : 3 : 3 : 3. If now the edge AB' drawn horizontally is parallel to OX_1 , we begin to set off, in order to obtain the first intermediary form, on the other edges through A — see the last of the six diagrams — segments AP_2, AP_3, AP_4 to the length of half the edge, i. e. of the unit, on the edge AB' a segment AP_1 with a length of a third of the unit, which causes the tetrahedron $P_1P_2P_3P_4$ corresponding to the symbol $\frac{1}{12}$ to be gene-

rated. The space $x_1 = 1$ contains of this tetrahedron the equilateral triangle $P_2P_3P_4$ with the side $\sqrt{2}$; the other faces $P_1P_3P_4$, $P_1P_2P_4$, $P_1P_2P_3$ lying in the spaces $x_2 = 1$, $x_3 = 1$, $x_4 = 1$ are isosceles triangles with basis $\sqrt{2}$ and sides $\frac{1}{3}\sqrt{10}$. So this section is not a

regular *tetrahedron* but a regular *triangular pyramid*, of which the perpendicular let down out of the vertex P_1 on to the groundplane $P_2P_3P_4$ is an axis with the period three; because the foot of this perpendicular lies on the diagonal AB of the right cube at a distance from A forming a sixth part of AB and as AP_1 is likewise a sixth of AB' this axis is parallel to the diagonal BB' of the eight-cell. It is now easy to deduce the changes of the section following from the displacement of the intersecting space by investigating either the parallel displacement of the edges of the section over the faces of the eightcell or the parallel displacement of the faces of the section through the bounding cubes of the eightcell. If the intersecting space has removed itself as far as double the distance from A , then

as is evident from both considerations — the tetrahedron of intersection has simply been multiplied by two from A . Passing on from this section $\frac{2}{12}$ it seems preferable to watch more closely the

edges. If the edges P_3P_2 and P_3P_1 of the section $\frac{2}{12}$ have arrived

in the positions P_3P_2 and $P_3'P_1$ of the section $\frac{3}{12}$ when the inter-

secting space has come at the threefold distance from the starting point A , it is sufficiently evident that the connection of the points P_3P_3' must furnish a new edge. So we see gradually how the entire

rhombohedron forming the section $\frac{6}{12}$ develops itself. We yet point

to the fact that the section in each position of the intersecting space during its parallel motion has an axis with period three, parallel to the diagonal BB' and at last passing into this line. Indeed, the diagonal AB of the bounding cube lying in space $x_1 = 1$ being an axis of revolution with the period three for that cube, so the plane through AB and AB' is a "plane of revolution" with the period three for the eightcell. As now the moving intersecting space is and remains normal to the line OC lying in this plane — see the first of the 20 diagrams — the line of intersection of this plane with the intersecting space, which line is of course normal to OC , must be an axis with the period three for the section. As was found

already above the line OB is really normal to OC and so the obtained axis is parallel to OB . Because the plane through AB and B' contains the perpendiculars OC and OR out of O on to the intersecting space and the space $x_1 = 1$ of the righthand cube, each line of it and therefore also OB must be normal to the plane determined by the intersecting space in the space $x_1 = 1$; so if we move the intersecting space in an opposite sense and return from $\frac{6}{12}$ by $\frac{5}{12}$, etc. to $\frac{1}{12}$ the

rhombohedron forming the central section, and then moving in the direction of the edge $B'A$ through the eightcell, is truncated normal to the axis by the plane determined in the space of that right cube. In fact, in the above mentioned paper (*Verhandelingen*, vol. IX, n^o. 7) has been found that the section is always a rhombohedron or a truncated rhombohedron when the intersecting space is normal to a plane through two opposite edges, which is here the case, as the plane through AB and B' contains the edge AB' and the opposite edge.

We now indicate the body corresponding to the fractional symbol

$\frac{n}{12}$ by D_n , where n can take one of the values 1, 2, ..., 11, 12 and

D_n and D_{12-n} represent the two oppositely orientated positions of a selfsame body, with a view to then investigating which of those parts make their appearance when the net (C_8) is cut by the central space $3x_1 + x_2 + x_3 + x_4 = 0$. From the distances of the points with the coordinates ($2a_i$), forming the system of centres of the net, follows immediately that the parts $D_2, D_4, D_6, D_8, D_{10}$ appear together and that thus the corresponding three-dimensional space-filling consists of three — and if we notice the orientation even of five — different groundforms. Now, as we know, the form D_6 alone already is able to fill the space and so this is also the case with the forms D_2 and D_8 and the forms D_4 and D_{10} together. What is more, from the condition that in the obtained space-filling with the three or five different groundforms the face of one of those forms must continue itself in faces of the surrounding forms, follows immediately that beside each D_2 must lie a completing D_8 , beside each D_4 a completing D_{10} and that recomposition of those parts completing each other to a D_6 must lead to a net of rhombohedra D_6 . We really cause this net of rhombohedra to be generated in a simpler way if, before cutting the net (C_8) by the assumed space, we suppose the series of the spaces $x_1 = 2a_1 + 1$ to have disappeared, a thing to which the use of the plane of projection through the two edges, here AB' and the opposite one, has led us involuntarily in the paper quoted last. By this the net (C_8) transforms itself into a threefold infinite net of an

infinite series of rectangular prisms which have a cube with the edge two as basis, and the section of this net of prisms is exactly the net of rhombohedra. That the sections which, when the intersecting space has an arbitrary position, are quite irregular parallelopipeda, here become rhombohedra is the result of the fact that the intersecting space forms with each of the three spaces $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ equal angles, angles with a cosine of the value $\frac{1}{6}\sqrt{3}$. Out of the

diagram with the symbol $\frac{6}{12}$ it is furthermore evident that the ends BB' of the axis of this rhombohedron lie in two consecutive spaces $x_1 = 2\alpha_1 + 1$ and that the distance of the parallel spaces of intersection of the intersecting space with these spaces, which spaces cut the net of rhombohedra in the intersecting space into pieces, must amount to 4. This tallies; for the angle between the spaces $3x_1 + x_2 + x_3 + x_4 = 0$ and $x_1 = 0$ has $\frac{1}{2}\sqrt{3}$ as cosine and therefore $\frac{1}{2}$ as sine, so that the distance of the planes must be $2 : \frac{1}{2}$.

From the preceding follows now likewise that the section with the space $3x_1 + x_2 + x_3 + x_4 = 1$ furnishes a space-filling consisting of the parts $D_1, D_3, D_5, D_7, D_9, D_{11}$; of course also this space-filling consisting of three groundforms each of which appearing in two opposite positions can be obtained by cutting up a net of rhombohedra. It is also clear that by taking an intermediary position of the space of intersection we are led to six quite different groundforms, which can be indicated by $D_{\frac{1}{2}}, D_{\frac{3}{2}}, \dots, D_{\frac{10}{2}}$, or in opposite orientation by $D_{\frac{1}{2}}, D_{\frac{3}{2}}, \dots, D_{\frac{11}{2}}$.

By cutting a block of k^4 cells C_n instead of a fourfold infinite net (C_n) we can also deduce how one of the forms $D_n^{(k)}$ of k -times greater linear size can be built up out of the above mentioned segments D_n . We avoid this not to become too longwinded.

5. *Case (2, 1, 1, 0)*. — When treating the case (1, 1, 1, 0) we have seen that the appearance of nought in the symbol causes prisms to be found with the constant height 2, by which the fourdimensional problem is reduced to a three-dimensional one. Thus we are placed before the consideration of the section (2, 1, 1) of the net of cubes which in various respects for the three-dimensional space forms the analogon of that of the section (3, 1, 1, 1) in Sp_4 .

If we suppose that the space, in which the section (2, 1, 1) is to be taken, contains the upper cube of the eightcell and the vertex P — see the first of the 20 diagrams — is taken as origin of a rectangular

system of coordinates with the edges passing through this point as axes, the edge PQ as axis corresponding to the figure 2 of (2, 1, 1), then the centre F' of the upper plane of that cube is the point (2, 1, 1) and PF' is therefore the axis normal to the series of intersecting planes¹⁾. Now it follows from the rectangle $APQE$ with the sides $AE=2$, $AP=2\sqrt{2}$, that AQ is normal to PF' and that the points A and Q project themselves on PF' in the same point. Thus we find the projection of the eight vertices of the cube under consideration on PF' by placing the projections (1, 2, 1) of the faces with PA and QE as diagonals so side by side that the last 1 of the first coincides with the first 1 of the last, by which the stratification 1, 2, 2, 2, 1 is arrived at, which, with a view to upper and lower cube, passes by doubling into 2, 4, 4, 4, 2. From this ensue then the results given on the first plate. If we now — returning to the second plate — set off on the three edges of the cube passing through P , in the assumed supposition that PQ agrees with the 2 of (2, 1, 1), from P segments $\frac{1}{2}$, 1, 1 then — see the last diagram — the triangle $P_1P_2P_3$ appears forming the upper plane of the triangular prism corresponding to the fraction $\frac{1}{8}$ and out of this the sections $\frac{2}{8}$, $\frac{3}{8}$, $\frac{4}{8}$ are developed in the same way as was pointed out above. Of triangle $P_1P_2P_3$ the line connecting P_1 with the middle of P_2P_3 is an axis with the period two, or to express it more simply a line of symmetry, and this line is parallel to the diagonal AQ of the first diagram. In each position of the intersecting plane the section has the line of intersection of this plane with the plane $APQE$ as line of symmetry: in connection with this the lozenge, unmutilated for the case $\frac{4}{8}$, which when following the reverse way to the case $\frac{1}{8}$ moves parallel to itself through the cube in such a way that the vertex Q describes the edge QP , is cut by the groundplane of the cube according to a perpendicular on the line of symmetry. If we imagine in the chosen space of the upper cube of the eightcell the threefold net of cubes and if we remove before passing to the intersection by the series of parallel planes the partitions parallel to the endplanes, we obtain in the intersecting plane a net of lozenges which are cut by the removed partitions into segments of the found form, etc.

In the ensuing parts we shall pass on to the intersection of the nets (C_{16}) and (C_{24}).

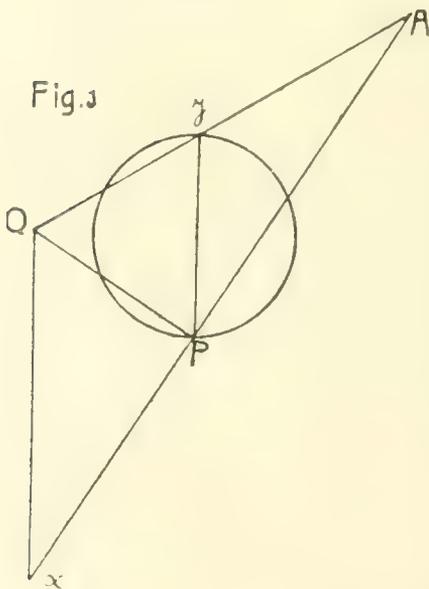
¹⁾ It is really inaccurate to speak of an upper plane of the upper cube; of course the plane is meant, which appears in the diagram as upper plane to the eye.

Mathematics. — “Some constructions deduced from the motion of a plane system.” By Prof. J. CARDINAAL.

(Communicated in the meeting of February 26, 1908.)

1. We precede two wellknown principles of motion.

a. Let the motion be generated by rolling a curve C_u (body centre) with which the system S is connected over a curve C_u' (space centre). If A, B, C, \dots are moving points of S and $\alpha, \beta, \gamma, \dots$ the centres of curvature of the orbits which they describe at a certain moment there exists between the system S and the system $\alpha, \beta, \gamma, \dots$ (Σ) a quadratic correspondence and such that if $\alpha, \beta, \gamma, \dots$ were moving points A, B, C, \dots would be the centres of curvature of their orbits. The conics of S corresponding to the right lines of Σ touch the tangent of body centre and space centre in the pole and osculate each other. The inflectional circle belongs to it. The reverse theorem is easy to deduce.



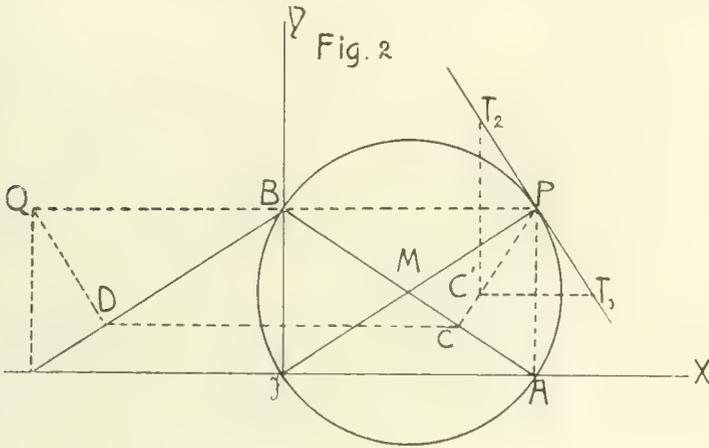
b. Let P be the pole (fig. 1), I the inflectional pole (common point of the tangents in the inflectional points); so the inflectional circle is known. Let A be a moving point, then α is determined as follows: Draw AI and AP ; determine the point of intersection Q of AI with the normal through P on AP . Draw out of Q the parallel to IP , which cuts P in α .

2. Application to the elliptic motion (fig. 2). Let AB (l) be the right line gliding with its points A and B along the rectangular axes IX and IY and let the demand be to construct the conic λ^2 corresponding to l .

The circumscribed circle (M) of $\triangle ABI$ is the inflectional circle; P is also directly known; the centre of curvature belonging to a point of AB can be constructed according to (1*b*) and so each point of λ^2 can be determined. However, some points of λ^2 are immediately known. The centre M of AB describes a circle having I as centre, the centres of curvature α and β belonging to A and B lie at infinite distance in the directions IB and IA ; λ^2 furthermore

touches circle (M) in P ; so λ^2 is an equilateral hyperbola passing through I with directions of asymptotes IB and IA and touching circle (M) in P .

The centre C of λ^2 can be determined in the following manner.



We suppose λ^2 to be constructed by means of the projective pencils formed by rays parallel to IX and IY . If the two points united in the point of contact P were separated then the two pairs of parallel rays through these points would determine two points A_1 and A_2 on IX and two points B_1 and B_2 on IY and the centre would be the point of intersection of A_1B_1 and A_2B_2 . It is true A_1 and A_2 coincide in A , and B_1 and B_2 in B ; but from the preceding follows that the centre C lies on AB . If in P we draw the tangent to λ^2 perpendicular to the normal PI , then a point of each asymptote lies at equal distance from P . So we measure $PT_1 = PT_2$ and we draw $T_1C' // IX$, $T_2C' // IY$; C' would be the centre of λ^2 if C' were situated on AB . However, out of the figure is evident that C' lies on a right line symmetric to T_1T_2 with respect to PA , and therefore perpendicular to AB . So the centre C of λ^2 is the footpoint of the normal let down out of P on AB .

If we consider different positions of AB and if we construct the successive positions of the point C , then the locus is an astroid on the axes IA and IB . The hyperbola λ^2 keeps touching the invariable circle with IP as radius; so the astroid is the locus of the centres of the equilateral hyperbolae with asymptote directions IA and IB passing through I and touching the last-mentioned circle.

The two diameters IA and IB of circle I form with the right line at infinity a polar triangle of the circle; so the points C have the signification of poles of one of the sides of that polar triangle.

It is to be proved that the locus of the poles of the two other sides with respect to λ^2 is likewise the astroid just found. To that end we construct the pole of IX .

If we take as centres of the projective pencils of rays generating λ^2 point I and the point of IX at infinity, then on account of the former reasoning the pole of IX lies on the parallel through B to IP ; at the same time this pole lies on a parallel drawn through C to IX ; so the point of intersection D of the latter right lines is the demanded pole. As D is symmetric to C with respect to IY , it also belongs to the astroid. In the same way we can prove that the pole of IY is likewise a point of the astroid.

By projective transformation the above problem can be put as follows :

Given a conic and a polar triangle of it. To determine the locus of the poles of the sides of that triangle with respect to the system of conics passing through the vertices and touching the original conic.

If we regard this as a problem by itself we arrive at the following algebraic solution:

Take the polar triangle as triangle of coordinates; then for the equation of the given conic can be written :

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0, \dots \dots \dots (1)$$

and for that of the conic described about that polar triangle:

$$p_1x_2x_3 + p_2x_3x_1 + p_3x_1x_2 = 0 \dots \dots \dots (2)$$

If we introduce the condition that (2) touches (1) then the coefficients of the latter satisfy the relation:

$$(a_1p_1^3 + a_2p_2^3 + a_3p_3^3)^3 = 27a_1a_2a_3p_1^2p_2^2p_3^2.$$

The pole of one of the fundamental sides, e.g. of $x_3=0$, is found by substitution of

$$p_1 = -p_2 \frac{x_1}{x_3}; p_3 = -p_3 \frac{x_2}{x_3}.$$

By this the equation of the locus of these poles becomes :

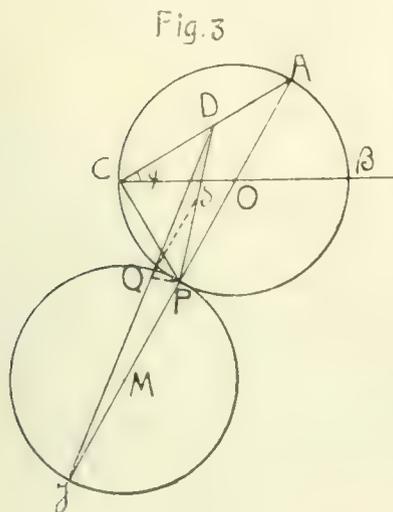
$$(a_1x_1^3 + a_2x_2^3 + a_3x_3^3)^3 = 27a_1a_2a_3x_1^2x_2^2x_3^2,$$

which can also be written in the form:

$$a_1^{\frac{1}{3}}x_1^{\frac{2}{3}} + a_2^{\frac{1}{3}}x_2^{\frac{2}{3}} + a_3^{\frac{1}{3}}x_3^{\frac{2}{3}} = 0.$$

We recognize in this the form of the astroid on oblique coordinates; the curve itself is a projective transformation of the common astroid. The locus of the poles of the other sides gives the same result.

3. Application to the cardioid motion. Let AC be the right line l , which passing through the fixed point C of circle (O) glides with one of its points A along the circumference; let now too the demand be to construct the conic λ^2 corresponding to l (fig. 3).



Circle (O) is the cuspidal circle; the pole P lies diametrically opposite to A and the inflectional circle (M) is symmetric to (O) with respect to the tangent in P . Now λ^2 , too, is to be constructed according to the preceding point by point; this takes place in the following way:

Let D be a point of l , draw DP and DI ; the normal in P on DP intersects DI in Q , the parallel to PI out of Q intersects DP in σ .

Just as with the elliptic motion we can again construct some particular points. If we apply the general construction to point C , it is evident that γ lies halfway CP ; O is evidently a point of λ^2 and β is the centre of curvature of the point at infinity on l ; so the conic λ^2 passes through γ , O , β , and osculates circle (O) in P .

Whilst thus the construction of λ^2 offers no difficulties, the generated system of conics is more intricate than the preceding.

Some properties are to be found geometrically; thus it is soon evident that the system contains two parabolae.

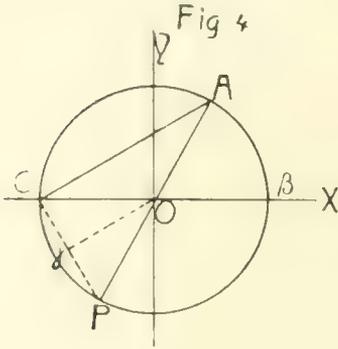
For a parabola is necessary that AC be a tangent to the inflectional circle (M) . Let us imagine the two touching circles (O) and (M) and if we draw from the endpoint A of the common diameter the tangents to circle (M) , we see that we can give two positions to (M) so that one of the tangents passes through C , so there are two parabolae belonging to the system. From the figure is evident:

$$\sin \varphi = \frac{1}{3} .$$

From this ensues: For all values of $\angle ACO$, which are lying between the values $\varphi = \sin^{-1} \frac{1}{3}$ on one side as well as on the other of CO , λ^2 becomes a hyperbola, for all values outside those limits λ^2 becomes an ellipse, the transition between the ellipses and the hyperbolae is formed by two parabolae.

The locus of the centres of this system of conics does not become

a simple curve. Simpler are the loci of the poles of the diameter OA , and the normal directed on to it out of O which we can take as axes for the calculation.



Let therefore (fig. 4) OA be the X -axis, the normal OY on it the Y -axis, then we find the equation of λ^2 as follows:

Let $OA = a$, $\angle AC\beta = \varphi$, thus $AO\beta = 2\varphi$. As λ^2 passes through O , β and touches circle (O) in P , its equation can be written:

$$y(x \cos 2\varphi + y \sin 2\varphi + a) + m(y - x \tan 2\varphi)(y - x \tan \varphi + a \tan \varphi) = 0;$$

the coefficient m is determined by the condition that the point $\gamma(-a \cos^2 \varphi, -a \cos \varphi \sin \varphi)$ lies on λ^2 . By substitution of the coordinates of γ for x and y and after reduction we get:

$$m = \cos \varphi \cos 2\varphi \sin \varphi$$

and the equation of λ^2 becomes:

$$\sin^2 \varphi \sin 2\varphi \cdot x^2 + (\cos 2\varphi - \cos \varphi \sin \varphi \sin 2\varphi - \sin^2 \varphi \cos 2\varphi) xy + (\sin 2\varphi + \cos \varphi \sin \varphi \cos 2\varphi) y^2 + a(1 + \sin^2 \varphi \cos 2\varphi) y - a \sin^3 \varphi \sin 2\varphi \cdot x = 0$$

or shorter:

$$2 \sin^3 \varphi \cdot x^2 + \cos \varphi (4 \cos^2 \varphi - 3) xy + \sin \varphi (3 - 2 \sin^2 \varphi) y^2 - 2a \sin^3 \varphi \cdot x + a \cos \varphi (3 - 2 \cos^2 \varphi) y = 0$$

The three derivatives become:

$$4 \sin^3 \varphi \cdot x + \cos \varphi (4 \cos^2 \varphi - 3) y - 2a \sin^3 \varphi = 0. \quad (1)$$

$$\cos \varphi (4 \cos^2 \varphi - 3) x + 2 \sin \varphi (3 - 2 \sin^2 \varphi) y + a \cos \varphi (3 - 2 \cos^2 \varphi) = 0. \quad (2)$$

$$2 \sin^3 \varphi \cdot x - \cos \varphi (3 - 2 \cos^2 \varphi) y = 0. \quad (3)$$

If we eliminate out of these equations two by two the value φ , we get the three loci.

Finally we shall deduce the simplest of these loci, namely the locus of the poles of the axis OX which is obtained by eliminating φ out of (1) and (3).

From (1) and (3) we deduce by subtraction the following two simpler equations:

$$3 y \cos \varphi = 2a \sin^3 \varphi, \quad (4)$$

$$2 \sin^3 \varphi \cdot x - \cos \varphi (3 - 2 \cos^2 \varphi) y = 0; \quad (5)$$

after substitution of the value $\sin^3 \varphi$ out of (4) into (5) we find:

$$\cos^2 \varphi = \frac{3(a-x)}{2a}$$

and from this finally for the locus

$$27 y^2 (a-x) = (3x-a)^2;$$

so this is a cissoid, whose cusp lies at a distance $x = \frac{1}{3}a$ from point O and whose asymptote passes through β .

OBSERVATION. The systems of conics treated in these two cases are simply infinite systems, where more than one conic pass through a point and more than one conic touches a right line; so they are distinguished from the ordinary pencils and the tangential ones.

Thus for the first mentioned system six conics pass through a point and twelve conics touch a right line.

Astronomy. — “*On the masses and elements of Jupiter’s Satellites, and the mass of the system*”, by DR. W. DE SITTER. (Communicated by Prof. J. C. KAPTEYN).

(Communicated in the meeting of February 29, 1908).

The determination of the elements and masses of the satellites of Jupiter and of the mass and the dynamical compression of the planet, which is communicated in the following pages, is based almost exclusively on heliometric and photographic observations made at the observatories at the Cape of Good Hope, Pulkowa and Helsingfors, in the years 1891 to 1904.

In addition to these I have also included in the discussion the observations made by BESSEL with the heliometer at Königsberg in 1832 to 1839, and the values of the node of the second and the perijove of the fourth satellite in 1750.0 (for which DELAMBRE’S values were adopted). These latter have however, as will appear later on, only a very slight effect on the final results. The determination of all masses and elements is thus practically independent of observations of eclipses.

Previous to 1891 no series of observations of the satellites except of the eclipses had been made with the purpose of determining the elements of the orbits. Such series of observations as had been executed in the first half of the nineteenth century by BESSEL, AIRY and others, were avowedly intended only to determine the mass of Jupiter. Accordingly the satellites were by these astronomers, so far as possible,

observed only near elongation. The series of observations made by GILL at the Cape in 1891, and the series of photographs taken in the same year by DONNER (on the suggestion of BACKLUND) and continued by him and by KOSTINSKY to 1898, are the first series of observations of the satellites in every point of their orbits. The discussion of the Cape observations of 1891 by me then suggested the desirability of further observations, which were executed by COOKSON with the Cape heliometer in 1901 and 1902, while photographic observations were made at the Cape in 1902, 1903 and 1904.

It will be conducive to a good understanding of what follows if I collect here at once all the notations used.

The theory, which was compared with the observations, is SOUILLART's¹⁾. This theory gives the longitudes and latitudes of the satellites, referred to the plane of Jupiter's orbit of 1850.0. As I have explained in *Cape XII. 3*²⁾ page 96, the orbit of 1900.0 has been used in its place.

The radii-vectores and the longitudes of the satellites in their orbits are given by the formulas:

$$\begin{aligned} r_i &= a_i Q_i \\ n_i &= l_i + \mathfrak{D}_i + \text{inequalities} \\ l_i &= n_i t + \varepsilon_i. \end{aligned}$$

We have the rigorous equations:

$$\begin{aligned} \varepsilon_1 - 3 \varepsilon_2 + 2 \varepsilon_3 &= 180^\circ \\ n_1 - 3 n_2 + 2 n_3 &= 0. \end{aligned}$$

1) *Théorie analytique des mouvements des satellites de Jupiter*, par M. SOUILLART, Mémoires R. A. S. XLV, 1880.

Théorie analytique des mouvements des satellites de Jupiter, seconde partie, par M. SOUILLART, Mémoires des savants étrangers, XXX, 1887.

2) *Annals of the Royal Observatory at the Cape of Good Hope*, under the direction of Sir DAVID GILL, K. C. B. Volume XII:

Part I. (Not published).

Part II. *Determination of the mass of Jupiter and orbits of the satellites*, by BRYAN COOKSON M. A. (1906).

Part III. *A determination of the inclinations and nodes of the orbits of Jupiter's satellites*, by Dr. W. DE SITTER. (1906).

Part IV. *Determination of the elements of the orbits of Jupiter's satellites*, by BRYAN COOKSON. (1907).

The titles of these papers, which I shall often have to quote, are referred to by the abbreviations used in the text above. I shall also often quote:

Publications of the Astronomical Laboratory at Groningen, N^o. 17. *On the Libration of the three inner satellites of Jupiter*, by W. DE SITTER, Sc. D. (1907), which is referred to as: *Gron. Publ. 17*.

The quantities ϑ_i are the libration, which is determined by the formulas:

$$\vartheta = l_1 - 3l_2 + 2l_3 - 180^\circ = k \sin \frac{t - t_0}{T} = k \sin \beta (t - t_0)$$

$$\vartheta_i = \frac{Q_i}{\beta^2} \vartheta, \quad \beta^2 = Q_1 - 3Q_2 + 2Q_3.$$

The quantities Q_i (and therefore β^2 and T) depend on the masses, and have been given in *Gron. Publ.* 17, Art. 18, up to terms of the third order.

The inequalities can be divided into three groups, according to their periods, of which the first group may be divided into three subdivisions. These are:

Ia. Equations of the centre. The osculating excentricities and perijoves — excluding their periodic perturbations (which are taken into account separately as inequalities of the longitudes and radii-vectores) — are determined by the formulas:

$$h_i = 2E_i \sin \Omega_i = 2 \sum_j \tau_{ij} e_j \sin \tilde{\omega}_j$$

$$k_i = 2E_i \cos \Omega_i = 2 \sum_j \tau_{ij} e_j \cos \tilde{\omega}_j.$$

Here e_i and $\tilde{\omega}_i$ are the *own* excentricities and perijoves of the four satellites. The angles $\tilde{\omega}_i$ vary proportionally to the time, and the coefficients τ_{ij} depend on the masses, τ_{ii} being unity. We have then

$$dv_i = -\cos l_i h_i + \sin l_i k_i$$

$$dQ_i = -\frac{1}{2} \sin 1^\circ (\sin l_i h_i + \cos l_i k_i).$$

The squares of E are negligible, except for the fourth satellite. The corresponding term is mentioned under *Ic*.

Ib. The great inequalities. These arise (as perturbations in h_i and k_i) through the commensurability of the mean motions of the three inner satellites. They are:

$$dv_1 = x_1 \sin 2(l_1 - l_2) \quad dQ_1 = -\frac{1}{2} \sin 1^\circ x_1 \cos 2(l_1 - l_2)$$

$$dv_2 = -x_2 \sin (l_1 - l_2) \quad dQ_2 = \frac{1}{2} \sin 1^\circ x_2 \cos (l_1 - l_2)$$

$$dv_3 = -x_3 \sin (l_2 - l_3) \quad dQ_3 = \frac{1}{2} \sin 1^\circ x_3 \cos (l_2 - l_3)$$

Ic. Minor inequalities of short periods. The periods of all the inequalities of group *I* are short (not exceeding 17 days).

II. Inequalities arising through the commensurability of the mean motions, and having periods between 400 and 500 days. These only exist for the satellites *I*, *II* and *III*. In the radii-vectores they are negligible. Their expressions are:

$$\begin{aligned} \delta v_i &= \sum_j x_{ij} \sin \varphi_j \\ \varphi_i &= l_i - 2l_s + \tilde{\omega}_i. \end{aligned}$$

The coefficients x_{ij} are proportional to e_j , and also depend on the masses.

III. Inequalities with very long periods (exceeding 12 years). These also are negligible in the radii-vectores.

The latitudes of the satellites over the plane of Jupiter's orbit are given by the formula :

$$s_i = I_i \sin (v_i - N_i).$$

The inclinations and nodes¹⁾ are in SOUILLART'S theory determined by the formulas:

$$\begin{aligned} I_i \sin N_i &= \sum_j \sigma_{ij} \gamma_j \sin \theta_j + \mu_i \omega \sin \theta + \text{periodic terms} \\ I_i \cos N_i &= \sum_j \sigma_{ij} \gamma_j \cos \theta_j + \mu_i \omega \cos \theta + \text{periodic terms} \end{aligned}$$

Mr. COOKSON and I have in all our work on the satellites referred the latitudes to a fundamental plane, which is defined by its inclination and node referred to the ecliptic and mean equinox of date. For these MARTH'S values have been adopted, which are for 1900.0 :

$$J = 2^\circ 9' 3''.94 \quad \Omega = 336^\circ 21' 28''.4$$

The longitude of the node of this plane on LEVERRIER'S orbit of Jupiter, counted *in the orbit*, and the inclination on that orbit are :

$$\theta_0 = 315^\circ 25' 48''.4 \quad \omega_0 = 3^\circ 4' 4''.75.$$

The longitude of the node of the orbital plane on the fundamental plane, counted *in the fundamental plane*, is therefore

$$\theta'_0 = 135^\circ 24' 34''.3.$$

The longitudes in the fundamental plane have been counted from the point O , of which the longitude is ²⁾

$$O = 135^\circ 27' 2''.5.$$

If the inclination and *descending* node of the fundamental plane on the orbit of 1850.0 are represented by ω_0 and ψ_0 , (thus $\psi_0 = \theta_0 + 180^\circ$) and if the longitudes of the nodes Ω_i are reckoned from this descending node, we have :

$$\begin{aligned} p_i &= -i_i \sin \Omega_i = -I_i \sin (N_i - \psi_0) \\ q_i &= i_i \cos \Omega_i = I_i \cos (N_i - \psi_0) + \omega_0 \end{aligned}$$

from which

$$\begin{aligned} p_i &= \sum_j \sigma_{ij} \gamma_j \sin (\psi_0 - \theta_j) + \mu_i \omega \sin (\psi_0 - \theta) + \text{periodic terms} \\ q_i &= \sum_j \sigma_{ij} \gamma_j \cos (\psi_0 - \theta_j) + \mu_i \omega \cos (\psi_0 - \theta) + \omega_0 + \text{periodic terms.} \end{aligned}$$

¹⁾ By node I always mean ascending node, unless otherwise stated.

²⁾ MARTH evidently intended to adopt $O = \psi_0$. Probably he has applied the correction, needed to derive O from $\psi_0 + 180^\circ$, with the wrong sign.

Here γ_i and θ_i are the *own* inclinations and nodes. The angles θ_i vary proportionally with the time and the coefficients σ_i depend on the masses. We have again $\sigma_{ii} = 1$. ω and θ are the inclination and node of the mean equator of the planet on the orbital plane for 1900.0. In the discussions we have used the abbreviations:

$$x_0 = -\omega \sin(\psi_0 - \theta)$$

$$y_0 = -\omega \cos(\psi_0 - \theta) - \omega_0.$$

x_0 and y_0 thus determine the position of the equator. The adopted fundamental plane nearly co-incides with the equator, and the node ψ_0 has nearly the theoretical motion of the node θ .¹⁾ The angle $\psi_0 - \theta$ is therefore constant and very nearly equal to 180° .

In *Gron, Publ.* 17, Chapter IV, I have given the quantities $Q_i, p^2, \tau_{ij}, \frac{d\tilde{\omega}_i}{dt}, x_i, z_{ij}, \sigma_{ij}, \mu_i, \frac{d\theta_i}{dt}$ as functions of the masses, or rather, as functions of the small quantities \varkappa' and r_i , which are defined by

$$Jb^2 = 0.0219087 \quad (1 + \varkappa) \quad (b = 1 \text{ for } d = 39''.0)$$

$$= 0.0000\ 0000\ 530042 (1 + \varkappa) \quad (\text{astronomical units}).$$

$$\varkappa = \varkappa' = 0.055$$

$$m_1 = 0.0000\ 40 (1 + r_1) \quad m_2 = 0.0000\ 80 \quad (1 + r_2)$$

$$m_3 = 0.0000\ 22 (1 + r_3) \quad m_4 = 0.0000\ 424\ 751 (1 + r_4).$$

Of \varkappa', r_2 and r_3 only the first power was kept, of r_1 and r_4 all powers, which could be derived from SOUILLART'S formulas were taken into account.

For the reciprocal of the mass of the system the value

$$\mathfrak{M} = 1047.40.$$

was adopted.

The data to be derived from the observations can be divided into three groups.

A. The inclinations and nodes, represented by the quantities p and q_i , i.e. the quantities determining the latitudes.

B. The data determining the longitudes and radii-vectores. These are the mean longitudes, the equations of the centre and the coefficients of the great inequalities of the three inner satellites (l_i, h_i, k_i, x_i)

C. The mean distances a_i .

A. In determining the elements from eclipse-observations, the elements of group B are derived from the observed epochs of the middle of the eclipse, those of group A from the duration of the

¹⁾ The motion of ψ_0 actually used by MARTIN is not exactly the theoretical motion of θ . The difference is however negligible.

eclipse. This duration depends not only on those elements, but as well on the form of the shadow-cone, i.e. on the geometrical compression of the planet. This latter not being known with sufficient accuracy, it is impossible to determine the latitudes from observations of eclipses. The elements of the first group must therefore be derived exclusively from heliometric or photographic extra-eclipse-observations of the satellites.

B. For the determination of the elements of group *B*, however, the eclipses are very valuable. One eclipse-observation, which is a determination of *time*, provides a much more accurate determination of the longitude of the satellite than one extra-eclipse-observation. On the other hand the latter can be repeated as often as the weather and the available time of the observer permit, while eclipses only occur in a limited number. Another advantage of eclipse-observations is that their accuracy is independent of the distance of Jupiter from the earth, while the accuracy of extra-eclipse-observations is inversely proportional to that distance. Extra-eclipse-observations are thus generally combined in series extending over a few months on both sides of the epoch of opposition. It must not be forgotten, however, that away from the opposition the time during which Jupiter is above the horizon, and therefore the number of observable eclipses, diminishes rapidly.

For the first satellite, where eclipses are numerous, and micrometrical observations least accurate, the advantage is very probably on the side of the eclipse-observations; for the fourth, of which eclipses are rare and extra-eclipse-observations are most accurate,¹⁾ this ratio is reversed. So long as nothing is known about the results derived from the series of photometric eclipse-observations made at the observatory of Harvard College in the years 1878 to 1903, it is not possible to form a definite judgment regarding the relative value of the two kinds of observations. Anyhow the attempt is justified to derive also the elements of group *B* exclusively from extra-eclipse-observations.

C. The four mean distances represent only one unknown quantity, since the determination of their ratios from the mean motions (also taking into consideration the uncertainty of the perturbations which must be applied) is very much more accurate than the direct deter-

¹⁾ My meaning is, of course, that the determination of the jovicentric place of the satellite from extra-eclipse-observations is most accurate for IV. This is due only to the large mean distance, not to the observation itself. This latter, i.e. the determination of the relative geocentric place, seems to be slightly more accurate for II and III than for I and IV.

mination of these ratios from the observations. This one unknown — the scale-value of the system — from which the mass of the planet is derived, can naturally only be determined from extra-eclipse-observations. It has already been remarked that all series of such observations made before 1891, were made with a view to this determination.

The number of unknowns of the problem is thus 32, viz :

A. the "own" inclinations and nodes	$\gamma_i, \theta_i \dots$	8 unknowns
the position of the equator	$\omega, \theta \dots$	2 „
the dynamical compression	$Jb^2 \dots$	1 „
B. mean longitudes with one rigorous condition	$\varepsilon_i \dots$	3 „
„ motions „ „ „ „	$n_i \dots$	3 „
the amplitude and phase of the libration	$k, t_0 \dots$	2 „
the own excentricities and perijoves	$e_i, \tilde{\omega}_i \dots$	8 „
the mass of each satellite	$m_i \dots$	4 „
C. the reciprocal of the mass of the system	$\mathfrak{M} \dots$	1 „

32

The observations which have been used in the derivation of the results to be communicated below are the following :

1. Heliometer-observations made in 1891 at the Royal Observatory, Cape of Good Hope, by GILL and FINLAY. These have been reduced by me and were published in my inaugural dissertation¹⁾. After the publication some mistakes and errors of computation have been found, which have been already corrected in the results used here. The corrected results will soon be published in Cape Annals, Vol XII, Part. I.

The high order of accuracy of this series is due to the principle, introduced by GILL, to measure only distances and position-angles of the satellites relatively to each other, and not relatively to the planet²⁾. Thus large systematic errors are avoided.

2. Heliometer-observations made in 1901 and 1902 at the Cape Observatory by BRYAN COOKSON, M. A., reduced by himself, and published in Cape XII.2. Corrections to the values of the unknowns as published there were afterwards given in Cape XII.4, *Appendix*. In these series

¹⁾ Reduction of Heliometer-observations of Jupiter's satellites, made by Sir DAVID GILL, K. C. B. and W. H. FINLAY, M. A., by W. DE SITTER. Groningen, J. B. WOLTERS 1901.

²⁾ HERMANN STRUVE had previously used the same method in his observations of the satellites of Saturn.

also only relative positions of the satellites *inter se* were determined.

From these three series all elements were derived, and all have been used in the final discussion, the values being taken unaltered from the definitive publications already quoted. The only exception is the position of the fundamental plane for 1902, the inclination of which on the ecliptic is $2^{\circ}8'38''$, instead of $2^{\circ}11'38''$, as printed in Cape XII.2 page 191 ¹).

3. Photographic plates, taken at the Cape Observatory in 1891 and 1903, measured and reduced by me, and published in Cape XII.3. The quantities p_i and q_i alone were derived for each epoch. These have been taken unaltered from Cape XII.3.

4. Photographic plates, taken at the Cape in 1904, measured and reduced by me. From these plates were derived p_i and q_i , which are published in Cape XII.3, and l_1, l_2, l_3 , which are published in *Gron. Publ.* 17. The published results have been adopted unaltered.

In these last three series also only coordinates of the satellites relatively to each other were used. The planet was not measured by me.

5. Photographic plates, taken at the Cape in 1902, measured and reduced by COOKSON, and published in Cape XII.4. This series requires a closer consideration.

Mr. COOKSON has measured on the plates differences of *RA* and decl. of the four satellites and *Jupiter*. The pointing on the planet is, according to his own statement, "not very accurate" (Cape XII.4, p. 24). But, according to the author, high accuracy is not required, since it is eliminated in the reductions. This elimination, however, is very incomplete.

It is effected as follows. From the measured differences of *R. A.* $x_i - x_p$ a preliminary solution is made, the resulting values of the unknowns are substituted in the equations of condition, and residuals are formed, which may be called δx_i . The mean of these residuals for any one plate, say δx_0 , is then considered to be the correction δx_p to be applied to x_p , i. e. the error in the pointing on the planet with reversed sign. This mean is therefore subtracted from the observed co-ordinates $x_i - x_p$. Now this method only eliminates the *accidental* part of the correction δx_p . The systematic part is already in the first approximation included in the values of the

¹) The inclination and node referred to the equator are correct as printed, in the reduction to the ecliptic some mistake must have occurred. The consequence of this is that the inclination ω_0 of the fundamental plane on the orbit of *Jupiter* requires a correction of -0.0092 , instead of $+0.0375$, as would appear from the printed data of Cape XII.2.

unknowns Δh_i , Δk_i , Δr_i , and is not removed from them by the subsequent approximations. The coefficients of these unknowns consist of a constant and a periodic part, of which the former amounts on an average to three times the latter. (See e.g. COOKSON, Cape XII.4, p. 102). If this periodic part is neglected, the three unknowns cannot be separated, and they represent together only one unknown, which I have called F_i (see my dissertation, p. 69), for each satellite. Thus, if the systematic part of δv_p had been introduced as an unknown the equations of condition would have been :

$$\frac{dx}{dF} F - \delta v_p + \dots = O - C.$$

Thus it would not be possible to separate F and δv_p . Whether the unknown δv_p is actually written down in the equations or not, does not affect the result; in any case the value which is found for F is not F itself, but $F - \delta v_p \left/ \frac{dx}{dF} \right.$, and the residuals δv_i , and therefore also their mean δv_0 , do *not* contain the systematic part of the error of pointing on the disc of the planet.

If we assume that the values of F found from the simultaneous heliometer observations (see above, sub 2), are the true ones, then the differences $P-H$, which are given by COOKSON in Cape XII. 4, page 102 (where for $F_3 - 0.0295$ should be read instead of -0.0395) are proportional to this systematic error, and we have $\delta v_p = \frac{dx}{dF} (P-H)$.

We thus find for the four satellites :

$$\left. \begin{array}{l} \delta v_p = - 0''.19 \pm 0''.04 \\ \quad - 0.15 \pm .06 \\ \quad - 0.17 \pm .05 \\ \quad - 0.33 \pm .04 \end{array} \right\} \text{mean } - 0''.21 = - 0''.0035.$$

The agreement of the four values is remarkable. The probable errors, of course, would only be a true measure of the accuracy, if it could be assumed that the periodic parts of the coefficients of Δh_i etc. have been entirely without any influence on the final results, which is very far from being true, especially for the fourth satellite, of which only a small number of revolutions is included in the period of observations. The mean systematic error of pointing on the disc is of the same order of magnitude as the errors which I found to exist in the measures by RENZ (see below, sub 6). So there can be little doubt that this is the true explanation of the large and systematic

differences between the results from the photographs and those from the heliometer in 1902. Accordingly I have rejected all the results from the photographic series, with the exception of p_i and q_i , which depend almost exclusively on differences of declination, in which the unknowns Δh_i , Δk_i , Δx_i have small and not constant coefficients, and the elimination of δv_p is therefore much more complete. I have adopted the values derived from the solution in which the orientation was determined from the *trails*. The reason why this is to be preferred to the orientation derived from the standard stars has been explained by me in Cape XII. 3, *Appendix*. The values of Δq_i and Δp_i have been adopted unaltered from Cape XII. 4.

6. Photographic plates taken at the observatories of Helsingfors by Prof. DONNER and of Pulkowa by Dr. KOSTINSKY, measured by RENZ, and published in the Mémoires de St. Petersburg, VIIIth series, Vol. VII, N^o. 4 and Vol. XIII, N^o. 1.

From the measures by RENZ I have derived corrections Δl_1 , Δl_2 , Δl_3 to the mean longitudes, which have been published in Gron. Publ. 17. The values found there have been adopted unaltered.

RENZ measured the positions of the satellites relatively to Jupiter. I have commenced my discussion of these measures by rigorously eliminating the pointing on the planet. It appears that these pointings are indeed subject to very large systematic errors (Gron. Publ. 17, art. 9b).

7. Heliometer observations made by BESSEL in Königsberg from 1832 to 1839, published by himself in "Astronomische Untersuchungen, Band II"; re-reduced by SCHUR and published in Nova Acta Acad. Leop. Carol., Vol. 44, pages 101—180. Only the values of h_3 , k_3 , h_4 , k_4 are included in the discussion, and only h_4 and k_4 have contributed to the final result.

BESSEL has referred the satellites to the planet. His observations are affected by large systematic errors, as has been pointed out by SCHUR, in consequence of which their real accuracy cannot be assumed to be in accordance with the probable errors.

8. The values of the "own" nodes and perijoves in 1750. These have been determined by DELAMBRE and by DAMOISEAU. Regarding the accuracy of these determinations nothing definite is known. The agreement between the two results, which is very good, cannot be taken as a measure of the accuracy, since we do not know in how far DAMOISEAU is independent of his predecessor. It will be seen below that the rôle played by these data in the derivation of the final results is a very subordinate one.

If from a combination of the values found on different epochs for

the osculating elements we wish to derive not only the values of these elements, but also of the masses, it is necessary that the expression of the perturbations as functions of the masses be known. The masses which form the basis of SOUILLART'S theory probably require considerable corrections. In consequence of the mutual commensurability of the mean motions the perturbations of higher orders are very large —: in some cases larger than those of the first order. For these reasons the perturbations cannot be assumed to be linear functions of the masses. The formulas needed to compute the corrections to the perturbations corresponding to given corrections to the masses have been developed by me, on the basis of SOUILLART'S numerical theory. They have been published in *Gron. Publ.* 17, art. 17.

The data required for the determination of the masses are:

I. The motions of the nodes, especially of θ_2 . The inclination of satellite I is too small to allow the motion of its node to be determined with accuracy, and the motions of θ_3 and θ_4 are too slow to be of any importance for the determination of the masses, compared with θ_2 . The motion of θ_2 is the datum from which the constant of compression Jb^2 must be derived.

II. The motions of the perijoves, especially of $\tilde{\omega}_4$. The excentricities of I and II are again too small to allow a determination of the motion of the perijove to be made. The motion of $\tilde{\omega}_3$ on the other hand, if it could be accurately determined, would be of little value for the determination of the masses on account of the small coefficients of these masses. The motion of $\tilde{\omega}_4$, which owing to the large excentricity of this satellite can be very accurately determined, is used for the derivation of the value of m_3 .

IIIa. The great inequalities in the longitudes and radii-vectores of the first and third satellites. These depend chiefly on m_2 , and serve to determine this mass.

IIIb. The great inequality of the second satellite. This furnishes an equation involving m_1 and m_3 .

These data are those used by LAPLACE. To these I have added:

IV. The period of the libration. This depends on m_1 , m_2 and m_3 . Of these m_2 only has a small coefficient, consequently the observed period practically gives an equation between m_1 and m_3 , from which combined with IIIb these two masses can be found ¹⁾.

¹⁾ See "Over de libratie der drie binnenste satellieten van Jupiter, en eene nieuwe methode ter bepaling van de massa van satelliet I," door Dr. W. DE SITTER. Handelingen van het 10e Ned. Nat. en Geneesk. Congres. (Arnhem 1905), pages 125—128.

Finally I add for the sake of completeness:

V. The ratio of the two eccentricities of III, from which m_4 must be determined. It has not been possible to determine this ratio from the data at my disposal, and I have therefore been compelled to leave m_4 uncorrected.

The investigation can thus be divided into the following parts, or subordinate investigations:

I. The determination of the inclinations and nodes on the different epochs, and of the motions of the nodes. This discussion must at the same time give the position of the mean equator, since the major part of the motions of the nodes is due to the compression of the planet, and consequently the plane of the equator is the one to which the theoretical motions are referred, and on which the own inclinations are constant. This discussion has been made with preliminary values of p_i and q_i in *Cape XII. 3, Chapters XV—XXI.*

II. The determination of the equations of the centre and of their secular variations. This was done in *Gron. Publ. 17, Art. 19.*

III. The determination of the great inequalities. These have been adopted unaltered from the heliometer observations of 1891, 1901 and 1902.

IV. The determination of the libration. This was carried out in *Gron. Publ. 17.*

The determination of the masses from the equations of condition furnished by these various subordinate investigations was effected in *Gron. Publ. 17*, so far as it was possible with the data which were then at my disposal. I there found the masses:

$$\left. \begin{array}{ll} \alpha' = + 0.025 & v_2 = + 0.050 \\ v_1 = - 0.360 & v_3 = + 0.025 \end{array} \right\} \dots \dots \dots (A)$$

The equations of condition from which corrections to these values were derived, will be communicated below. I will now first describe the various subordinate investigations I to IV, to which I add V: the determination of the mean motions, and VI: the determination of the mass of the system.

I. *Inclinations and Nodes.*

The data are the values of p_i and q_i for the five epochs 1891.75, 1901.61, 1902.60, 1903.72, 1904.89. The unknowns are γ_i , Γ_{i0} , v_0 , y_0 and the motions of the nodes ¹⁾. These latter depend on α' and v_i , of which only α' has been introduced as unknown. The

¹⁾ In this investigation we put for abbreviation $\Gamma_i = \psi_0 - \theta_i$.

discussion is carried out in Cape XII. 3, based on the masses of SOUILLART'S theory. It must now be repeated with the masses (A). Further the following corrections must be applied.

a. The observed values of p_i and q_i must be reduced to one and the same fundamental plane for all epochs. At the time when the discussion of Cape XII. 3 was made, I had not at my disposal the data for carrying out this reduction for the epochs 1901 and 1902.

b. In the discussion of Cape XII. 3 I was compelled to reject the observations of the satellites III and IV in 1901 and 1902. COOKSON had found in the latitude of IV an empirical term, which had also influenced the results for III, and which could be demonstrated not to exist in the observations of 1891, 1903 and 1904. Mr. COOKSON has since then found the true explanation of this apparent periodic term, and has corrected his results accordingly. The corrected results must now be introduced into the discussion. It appears that now not only nothing must be rejected, but that also the representation of the observations generally is much improved.

c. The results of the photographs of 1902, which were not yet known when the discussion of Cape XII. 3 was made, must be taken into account.

It seems unnecessary to mention here all the details of the discussion. It will be published in Cape XII. 1, *Appendix*, and it will suffice here to state the results.

It may be remembered that in Cape XII. 3 two final solutions were made, of which Sol. VI was based exclusively on modern observations, while in Sol. VII the motion of θ_2 was derived from a comparison with DELAMBRE (1750), and the motions of the other nodes theoretically corresponding with this were adopted¹⁾. Thus α' was not introduced as an unknown in this solution. The agreement of the solutions VI and VII was very good, with the exception of α' and y_0 . The values (A) of α and v_i are chosen so that the corresponding motions of the nodes are about the means of those found in Sol. VI and Sol. VII.

The corrections (*a*), (*b*) and (*c*) were now applied, the quantities σ_{ij} and μ_i , which are used in the solution were altered so as to correspond with the masses (A), and a new solution was made (Sol. VIII) in which, similarly to Sol. VI, the unknowns were γ_i , I_{10} , x_0 , y_0 and $\delta\alpha'$. The method by which the solution was effected is the same as in Cape XII. 3, and has also been described

¹⁾ The correspondence was only approximate, the expressions of the motions of the nodes as functions of the masses (*Gron. Publ.* 17, art. 17), not yet being computed at that time.

in detail in these Proceedings (March 1906). The values found for γ_i and I_{i0} were very nearly equal to those found previously. The correction to α' was very small, viz.:

$$\delta\alpha' = + 0.0026 \pm .0058.$$

The masses now become

$$\left. \begin{aligned} \alpha' &= + 0.0276 & r_2 &= + 0.050 \\ r_1 &= - 0.360 & r_3 &= + 0.025 \end{aligned} \right\} \dots \dots (B)$$

The motions of the nodes were now made to correspond with these masses, the values found for γ_i , I_{i0} , x_0 and y_0 were introduced into the equations of condition, and residuals $\Delta\gamma_i$ and $\sin \gamma_i \Delta I_i$ were formed. From these latter I then derived for each satellite separately a correction to the motion of the node. These corrections are given below sub I. The values of the nodes in 1750.0 were next computed and compared with those determined by DELAMBRE. This comparison gave the corrections II to the motions of the nodes.

<i>Correction to the motion of</i>	I (modern)	II (DELAMBRE)	<i>Adopted</i>
θ_1	$+ 0^\circ.0094 \pm .0029$		
θ_2	$- 0.00001 \pm 00009$	$- 0^\circ.00042 \pm 0^\circ.00020$	$- 0^\circ.00010 \pm .00008$
θ_3	$- 0.00048 \pm 23$	$- 0.00034 \pm 20$	$- 0.00041 \pm 15$
θ_4	$- 0.00013 \pm 11$	$+ 0.00008 \pm 50$	$- 0.00010 \pm 10$

These corrections have been used as the right-hand members of equations of condition, from which, together with those derived from the other subordinate investigations, corrections to the values (B) of the masses have been determined. These equations will be given further on. It will be seen that the adopted values agree within the probable errors with those derived from the modern observations alone. If thus these latter were adopted, the final results could only be altered within their probable errors. The finally adopted masses are:

$$\left. \begin{aligned} \alpha' &= + 0.0326 \pm .0075 \\ r_1 &= - 0.350 \pm .030 \\ r_2 &= + 0.050 \pm .050 \\ r_3 &= + 0.005 \pm .020 \\ r_4 &= 0 \pm 0.25 \end{aligned} \right\} \dots \dots (C)$$

These were now introduced into the quantities σ_{ij} and μ_i and a new solution was made (Sol. IX), in which the motions of the nodes corresponding to the masses (C) were adopted, and accordingly $\delta\alpha'$ was not introduced as unknown. The result is:

Solution IX.

$$\begin{array}{ll}
\gamma_1 = 0.0272 \pm .0028 & \theta_1 = 60.2 \pm 7.0 - \{0.13614 \pm .00100\} t \\
\gamma_2 = 0.4683 \pm 16 & \theta_2 = 293.18 \pm 0.19 - \{0.032335 \pm .000240\} t \\
\gamma_3 = 0.1839 \pm 26 & \theta_3 = 319.73 \pm 0.52 - \{0.006854 \pm .000180\} t \\
\gamma_4 = 0.2536 \pm 23 & \theta_4 = 11.98 \pm 0.67 - \{0.001772 \pm .000030\} t
\end{array}$$

The time t is counted in days from 1900 Jan. 0, mean Greenwich noon. The nodes are reckoned from the first point of Aries. The motions contain the precession, for which NEWCOMB'S value was adopted. The probable errors of the motions of the nodes were computed from those of the masses (C). For the position of the mean equator referred to LEVERRIER'S plane of Jupiter's orbit for 1900.0 I find

$$\begin{aligned}
\omega &= 3.1153 \pm 0.0014 \\
\theta &= 315.800 \pm 0.025 \quad (1900 \text{ Jan. } 0.0)
\end{aligned}$$

Table I contains the observed corrections to SOULLART'S theory, their probable errors derived from the discussion of each series separately, and the residuals which remain after the substitution of the final values of γ_i , θ_i , ω and θ .

The probable error of weight unity, determined from these residuals is

$$\pm 0^{\circ}.0097.$$

Weights had originally been assigned, corresponding to a probable error of weight unity of

$$\pm 0^{\circ}.0100.$$

Comparing each residual with its probable error, we find the following distribution

	<i>actual</i>	<i>theoretical</i>
smaller than ϱ	30½	28.0
between ϱ and 2ϱ	16½	18.1
,, 2ϱ ,, 3ϱ	6	7.5
exceeding 3ϱ	3	2.4

Remembering that the corrections Δp_i and Δq_i are for each epoch the results of a series of observations, made for the different epochs by different observers and different instruments, and reduced absolutely independently of each other, we must consider this excellent agreement of the actual distribution with the ideal one according to the law of errors as a strong proof of the freedom of the observations from systematic errors. Accordingly the probable errors of the resul-

ting inclinations and nodes can confidently be regarded as a correct measure of the accuracy. How much better the observations are represented by these values than by those adopted in SOUILLART'S theory is evident at once by comparing the residuals with the observed corrections.

For 1750.0 I now find :

	<i>Sol. IX.</i>	DELAMBRE	DAMOISEAU
θ_2 ,	264.7 ± 13.2	283.3	282.0
θ_3 ,	335.2 ± 10.0	352.5	353.5
θ_4 ,	109.1 ± 1.8	105.0	98.3

The agreement with the values found by DELAMBRE and DAMOISEAU is now satisfactory. If the probable errors of θ_2 and θ_3 in 1750 are estimated at $\pm 5^\circ$ (see Cape XII. 3 page 111), the difference in both cases hardly exceeds the sum of the probable errors. As has been already said, I consider the probable errors of Solution IX as a true measure of the accuracy. This however they only remain for 1750 on the assumption that the theory, by means of which the elements have been carried back from 1900 to 1750, be correct. This, however, cannot be assumed without some qualification. It is well known that SOUILLART has integrated the equations of motion by two different methods. The difference between the motion of the node of II in 150 years according to the two methods is nearly $1^\circ.4$. It is thus quite possible that the terms of higher order in the masses, which are neglected in *both* methods, may also amount to a very appreciable quantity. In the interval of 150 years θ_2 has completed nearly five revolutions, while its motion is practically derived from the interval 1891—1904, during which the node has moved about 155° degrees. Remembering this, the agreement between the values carried back from 1900 to 1750, and those directly determined, is as good as could reasonably be expected.

In Cape XII. 3. I pointed out that the solutions VI (modern observations alone) and VII (motion of θ_2 from comparison with DELAMBRE) were in perfect agreement except for the motions of the nodes and for γ_0 . I then stated as my opinion, that the substitution of better masses for those of SOUILLART could be expected to reconcile the two solutions. This expectation has been entirely fulfilled. With regard to the motions of the nodes, (which are practically the same in Sol. VIII and Sol. IX) we have just seen that the agreement with 1750 is satisfactory. With regard to γ_0 the following comparison of the different solutions shows that

indeed the difference between Sol. VIII and IX is much smaller than between VI and VII, and now leaves nothing to be desired.

Values of y_0 .

Sol. VI + 0°.0388 ± °.0044	Sol. VIII + 0°.0454 ± °.0029
Sol. VII + .0490 ± 24	Sol. IX + .0473 ± 14.

For the other unknowns the differences between the solutions VIII and IX are entirely negligible. In addition to the improvement of the masses, also the reduction to one and the same fundamental plane, and the corrections applied by COOKSON to the values for 1901 and 1902 are largely responsible for this improvement in the agreement of the two solutions.

II. *Equations of the centre.* The values of the own excentricities and perijoves were derived by me from the heliometer observations of 1891, 1901 and 1902, in *Gron. Publ.* 17, *Art.* 19. (See also these Proceedings, June 1907). The discussion was there carried out for two sets of coefficients τ_{ij} , the results agreeing within their probable errors. It is therefore unnecessary to repeat it here with the coefficients corresponding to the masses (C), which are intermediate between the two sets there used. The reasons why the photographic results of 1902 must be rejected, have already been given above. The finally adopted values are thus the same as in *Gron. Publ.* 17, with only a few unimportant alternations in the last decimal places, viz:

$$\begin{array}{l}
 e_1 = 0°.0031 \pm °.0080 \quad \tilde{\omega}_1 = 155°.5 \pm \infty + \{0°.14703 \pm °.00144\} t \\
 e_2 = 0.0172 \pm 40 \quad \tilde{\omega}_2 = 62.7 \pm 10°.0 + \{0.038955 \pm .000455\} t \\
 e_3 = 0.0868 \pm 65 \quad \tilde{\omega}_3 = 338.3 \pm 3.0 + \{0.007032 \pm .000180\} t \\
 e_4 = 0.4264 \pm 20 \quad \tilde{\omega}_4 = 283.15 \pm 0.30 + \{0.001896 \pm .000021\} t
 \end{array}$$

The probable errors depend on judgment, and are probably estimated rather too large. The values of e_1 and $\tilde{\omega}_1$ were not derived from the observed values of h_1 and k_1 , but from the inequalities of group II, as will appear below when we treat of the libration. The adopted p. e. of e_1 is the largest value which can still be considered to be not improbable having regard to the observed values of h_1 and k_1 . This p. e. being larger than the value of e_1 itself, the p. e. of $\tilde{\omega}_1$ cannot be stated.

The motions have been computed by the masses (C) and their probable errors correspond to the probable errors of these masses.

These values of e_i and $\tilde{\omega}_i$, and the values of τ_{ij} corresponding to the masses (C) give the residuals contained in Table II, together

TABLE II. EQUATIONS OF THE CENTRE.

Series	Observed correction	Probable error	Residual									
	h_1			h_2			h_3			h_4		
1891	+ 0.031	+ .013	+ .031	+ 0.008	± .008	+ .001	+ 0.030	± .005	+ .003	+ 0.0617	± .0026	+ .0001
1901	+ .091	± 28	+ 86	- .015	± 19	+ 6	+ .007	± 14	+ 13	+ .0653	± 51	- 108
1902	- .008	± 34	- 44	- .040	± 18	- 27	- .022	± 11	- 13	+ .0847	± 34	- 16
	k_1			k_2			k_3			k_4		
1891	- 0.064	+ .021	- .067	- 0.055	± .016	± .005	- 0.053	± .008	+ .003	+ 0.0261	± .0038	- .0025
1901	- .102	± 54	- 96	+ .030	± 31	+ 1	- .069	± 17	- 23	+ .0390	± 59	+ 175
1902	- .060	± 25	0	+ .037	± 15	+ 4	- .017	± 10	+ 28	+ .0237	± 56	+ 35

with the observed corrections to SOULLART's theory and their probable errors.¹⁾ The residuals are very satisfactory, especially so if satellite I is left out of account. (See also *Gron. Publ.* 17, pages 92 and 115).

From the values of $\tilde{\omega}_4$ in 1900, 1836 and 1750 I have already in *Gron. Publ.* 17 derived the motion of $\tilde{\omega}_4$. The value found there requires however a small correction. The values which BESSEL, and following his example SCHUR also, gives for $E_4 \sin \Omega_4$ and $E_4 \cos \Omega_4$, i. e. for h_4 and k_4 , are in reality the values of $e_4 \sin \tilde{\omega}_4$ and $e_4 \cos \tilde{\omega}_4$. This was not noticed at first, and must now be corrected.

I now find for 1836

$$h_4 = - 0^\circ.704 \quad k_4 = - 0^\circ.395.$$

Using, as before, the most probable values of e_3 , $\tilde{\omega}_3$ and τ_{43} , we find from this:

$$e_4 \sin \tilde{\omega}_4 = - 0^\circ.351 \quad e_4 \cos \tilde{\omega}_4 = - 0^\circ.208$$
$$\tilde{\omega}_4 = 239^\circ.4 \pm 0^\circ.8.$$

We have now:

	$\tilde{\omega}_4$	<i>Residual</i>
1750.0	180°.4	+ 0°.1
1836.0	239.4	0 .0
1900.0	283.1	0 .0

from which:

$$\frac{d\tilde{\omega}_4}{dt} = 0^\circ.001872 \pm 0^\circ.000020 (\alpha)$$

If the probable error were derived from the residuals, or from the probable errors for the separate epochs, we should find a much smaller value. The larger value has been adopted chiefly on account of the possibility of systematic errors of BESSEL, which will be mentioned below.

COOKSON has already (*Cape XII.* 2. page 197) derived the motion of $\tilde{\omega}_4$ from the observations of 1836, 1879 (SCHUR) 1891, 1901 and 1902. He finds:

$$\frac{d\tilde{\omega}_4}{dt} = 0^\circ.001892 \pm 0^\circ 000024, (\beta)$$

The values (α) and (β) agree within their probable errors. So, if (β) were adopted instead of (α) , the final results could only be

¹⁾ In deriving these residuals the longitudes of the perijoves are, of course, counted from the point O, as was done in the tabular places.

altered within their probable errors. They would then be entirely independent of eclipse observations.

With the finally adopted elements we find for BESSEL the following residuals.

BESSEL 1836.0

	<i>Observed</i>	<i>Residual</i>		<i>Observed</i>	<i>Residual</i>
h_3	$-0^{\circ}.033 \pm ^{\circ}.010$	$+ ^{\circ}.008$	h_4	$-0^{\circ}.704 \pm ^{\circ}.007$	$+ ^{\circ}.028$
k_3	-0.188 ± 14	$+ .020$	k_4	$-.395 \pm 9$	$+ .026$

It thus appears that, although $\tilde{\omega}_4$ is well represented, h_4 and k_4 leave large residuals. It is remarkable that all four residuals are positive. This must probably be ascribed to systematic errors in the observations, which have already been proved to exist by SCHUR's discussion, and which probably are not entirely eliminated by the empirical corrections applied by SCHUR.

The theoretical values of h_3 and k_3 are :

$$\frac{1}{2} h_3 = \tau_{31} e_1 \sin \tilde{\omega}_1 + \tau_{32} e_2 \sin \tilde{\omega}_2 + e_3 \sin \tilde{\omega}_3 + \tau_{34} e_4 \sin \tilde{\omega}_4$$

$$\frac{1}{2} k_3 = \tau_{31} e_1 \cos \tilde{\omega}_1 + \tau_{32} e_2 \cos \tilde{\omega}_2 + e_3 \cos \tilde{\omega}_3 + \tau_{34} e_4 \cos \tilde{\omega}_4.$$

The two first terms are exceedingly small, but $\tau_{34} e_4$ is large, and this term has been used by LAPLACE to determine m_4 , with which the coefficient τ_{34} is roughly proportional. An attempt to derive τ_{34} from a comparison of the equations of the centre in 1836 and 1900 had to be given up, as will be easily understood by considering the residuals and probable errors stated above. Also a comparison with 1750 is not possible, for DELAMBRE and DAMOISEAU both state nothing but the values of the coefficients and the arguments, and it is not possible to derive from these the values of h_i and k_i as found directly from the observations. I have thus been compelled to leave m_4 uncorrected.

The values of $\tilde{\omega}_3$ and $\tilde{\omega}_4$ for 1750, computed from the final values for 1900.0 and the final motions, are :

		DELAMBRE	DAMOISEAU
$\tilde{\omega}_3$	313.0 ± 10.3	309.4	315.0
$\tilde{\omega}_4$	179.3 ± 1.2	180.3	180.4

The agreement is excellent, in fact better than could have been expected.

(To be continued).

Botany. — “*Contribution N^o. 1 to the knowledge of the Flora of Java*” by Dr. S. H. KOORDERS.

(Communicated in the meeting of February 29, 1908).

§ 1. On the oecological conditions, on the means of dissemination and on the geographical distribution of the species of *Myricaceae*, occurring wild in Java, especially in the higher mountains.

As has been shown by KOORDERS and VALETON in their critical systematic investigations of the *Myricaceae* of Java, contribution N^o. 9 to the knowledge of the Trees of Java, [in *Mededeelingen uit 's Lands Plantentuin N^o. LXI (1903) p. 99—105*] there are found on this island but two wild species, both of which generally become arborescent. These are:

1. *Myrica javanica* BL. (= *M. macrophylla* MIRB.) and
2. *Myrica longifolia* TEIJSM. & BINNENDIJK (= *M. integrifolia* ROXB. = *M. Lobbii* TEIJSM. & BINN.).

The botanical investigations of the alpine flora of Java, which I am now undertaking for the Dutch Ministry of the colonies in the Herbaria at Leiden and at Utrecht, often afford a not unwelcome opportunity of amplifying and correcting my previous publications on geographical distribution and oecology, which publications were mainly based on my botanical notes of numerous journeys in Java, Sumatra and North-East Celebes.

An example is presented by the order of the *Myricaceae*, since in a recent foreign publication¹⁾ observations appear to have been overlooked, which had already been published, partly in 1903 in the above-mentioned Contribution N^o. 9 to the knowledge of the Trees of Java, by KOORDERS & VALETON²⁾, and partly in a still earlier small Dutch publication,³⁾ which is very difficultly accessible abroad. We are here concerned with some observations on the geographical distribution and especially on the means of dissemination of an alpine tree of Java, namely *Myrica javanica* BLUME.

1) ERNST, Prof. Dr. A., Die neue Flora der Vulkaninsel Krakatau mit 2 Kartenskizzen und 9 Landschafts- und Vegetationsbildern. — Zürich 1907. — On p. 61 of this very interesting publication: “Auch bei fruchtessenden Tauben sollen sich im Kropfe und Magen häufig Samen von anschnlicher Grösse vorfinden und BECCARI gibt an, . . .” (l. c. p. 61).

2) KOORDERS & VALETON l.c. in *Mededeelingen uit 's Lands Plantentuin LXI (1903) p. 102*.

3) KOORDERS, Spontane en kunstmatige reboisatie op den Sendoro (Spontaneous and artificial reforestation on the Sendoro) in *Tijdschr. v. Nijverheid en Landbouw van Nederl. Indie*, Vol. 51 p. 241—287 (with a map).

§§ 1. MYRICA JAVANICA BL.

§§§ 1. GEOGRAPHICAL DISTRIBUTION AND OECOLOGY.

§§§§ 1. Distribution and oecology outside Java.

In 1895 I collected in the Sapoetan mountains of North East Celebes, at an altitude of 1400—1500 metres above sea-level, herbarium specimens of a small tree, which grew wild there. These specimens are now in the Mus. Botan. Hort. Bog. at Buitenzorg; I considered that they differed only so slightly from Javanese specimens of *Myrica javanica* BLUME, that I identified¹⁾ them with the latter and still regard them as conspecific. As is proved by a specimen which I saw in the Royal Botanical Museum at Berlin, this species was also collected by WARBURG in N. E. Celebes and was regarded by him as specifically different from the above-named Javanese species. Except from Celebes and from Java, no stations for *Myrica javanica* BLUME have been recorded in the literature. Nor have I seen any specimens collected outside Java, in the Herbaria at Leiden and at Utrecht.

Since I have not, at present at my disposal the specimens collected by me in Celebes and preserved in the Herbarium at Buitenzorg, I can give no further data regarding WARBURG's separation of his specimens from Celebes in connexion with the specific value of the differences, which I myself (*l. c.* p. 615) had already observed between the specimens of *Myrica javanica* BL., collected by me in Java and in Celebes. For the present I therefore continue to regard the arborescent *Myrica*, collected on the Sapoetan summit in N.E. Celebes, as identical with *Myrica javanica* BL. of the Javanese mountains.

§§§§ 2. Horizontal and vertical distribution
and oecology outside Java.

As appears from Herb. Kds. in Mus. Hort. Bogor., I made in the years 1888—1903 the following observations, which in part have already been published in KOORDERS and VALETON, contribution to the knowledge of the Trees of Java, IX (1903) p. 102.

In West- and in Central Java above 1500 metres. In the Preanger on the Gede at 3000 m. near the summit, on the Galoenggoeng by the lake of Telagabodas at 1650 m. and 1700 m. In Tegal on the

¹⁾ KOORDERS, S. H., Report of an official botanical journey through the Minahasa, being a first survey of the Flora of N. E. Celebes; with 10 maps and 3 plates. (In Mededeelingen uit 's Lands Plantentuin N^o. XIX (1898). Batavia and The Hague p. 615).

Slamat at 1800 m. and higher above Simpar. In the residency Banjoemas on the Prahoe and on the Dieng-plateau at 2500 m. In the residency Kedoe on the Merbaboe, Oengaran and Telemojo at 1800 m. and higher. Has hitherto not been found further East. Growing in Java mostly gregarious, and, together with about 15 other ever-green woody species, forming alpine forests. Proper to alpine regions, and found in lower regions only near *solfatare*, which are rich in mineral salts, hence exclusively on physiologically-dry soils. Prefers altitudes of 1800—3000 m. above sea-level; has not been observed in Java below 1500 m.

The following oecological conditions may be mentioned for *Myrica javanica*: the species withstands the low air-temperature, the intense insolation and low atmospheric humidity of JUNGHUN's alpine zone, but does not appear to resist the over-dry climate of East Java.

The species occurs on very arid and stony soils, which are presumably poor in soluble mineral constituents, in consequence of long-continued washing out, and also grows luxuriously in physiologically-dry situations, near *solfatare*, etc. In the hot plain, the species is altogether wanting. It can fairly well resist much sunlight, as well as shade, and likewise is proof against strong winds.

I have not yet found Phanerogamic parasites on this alpine tree; on the other hand I have observed parasitic Fungi on it, for instance in the residency Kedoe the following: *Myxosporium caudidissimum* RACIBORSKI, *Microcyclus Koordersii* HENNINGS and *Pestalozzia Myricae* KOORD.¹⁾

Even on the borders of the natural area of distribution of *Myrica javanica*, I have never observed these parasitic Fungi on the three in such quantity, that they alone would appear to limit the distribution of the adult-plants in Java. My experiments on infection with conidia of *Pestalozzia Myricae* do not, however, preclude the possibility, that this fungus imposes a natural limit on the development of the seedlings; the experiments showed that, with too much shade and too much moisture in the soil, most of the *Myrica* seedlings were killed off by the parasitic fungus, even in a district within the natural area of distribution.

At the same time these experiments showed, that *Myrica* seeds, when germinating in full sunlight (for instance on bare mountain-slopes, naked rocks and layers of rapilli), would not sustain serious injury from this fungus parasite which is often so harmful.

¹⁾ KOORDERS, Botanische Untersuchungen über einige in Java vorkommende Pilze, besonders über Blätter bewohnende parasitisch auftretende Arten [in Verhand. Koninkl. Akademie v. Wetenschappen, Deel XIII, Tweede Sectie (1907) No. 4, p. 183, 218 and 224].

A further study of parasitic fungi, especially of such as attack Phanerogam-seedlings, will probably afford a good explanation of some apparently insoluble problems of plant distribution, especially where other causes do not sufficiently account for the sudden absence of a species under apparently favourable oecological conditions.

Even in mountain regions, where the original vegetation has been completely destroyed by human interference, as for instance in central Java on the Sendoro, where in 1891 miles of country were laid waste by fire, extensive forests of *Myrica javanica* are formed by a natural process in a relatively short time. When I visited the volcano Sendoro in 1903, and therefore 12 years after the fire, the higher slopes, which in 1891 had been burned down to the rock, were covered with alpine bush, grown up naturally. These slopes, which were situated above the plantations of the Forestry Department, have been referred to in detail in a publication mentioned above.¹⁾ The woods extended for many thousands of acres, especially on the more humid S. W. side of the mountain, and in many places *Myrica javanica* was predominant to such an extent, that one might have spoken of almost pure *Myrica* forests. According to information obtained by me on the spot, there appeared in addition to *Myrica javanica*, as first tree-like pioneer on many of the most completely burned places, as soon as one year after the fire *Albizia montana* BENTH, growing in groups in the midst of an extensive grass wilderness. This latter species has also been found by me repeatedly elsewhere, in the alpine regions of Java as one of the very first pioneers of the forest on the burned-down slopes of volcanoes.

Brief reference may here be made to some results of investigations, which I have carried out in various parts of Java, Sumatra and N. E. Celebes during many years, regarding the characteristics of the Phanerogam pioneers on volcano slopes after complete denudation by fire, and on other lands in the interior of Java, Sumatra and Celebes, after a similar loss of vegetation, due to other causes (e. g. deserted arable lands). It would appear that, without reference to the height above sea-level nor to the systematic position of the pioneers, the new vegetation is characterised by the following properties, which are related to the abovementioned peculiar oecological conditions :

1. Without exception the new plants are, by structure and distribution *xerophytes*, which remain alive under extraordinarily unfavourable conditions of water-supply and transpiration.

¹⁾ Compare KOORDERS Spontaneous and artificial reforestation on the Sendoro l. c.

2. In addition to "poverty of water", nearly all these species can well resist direct sunlight, while many are not killed either by great poverty of light.

3. Nearly all grow rapidly or very rapidly, and all soon produce abundant seed. Many of the herbaceous pioneers of this vegetation already bear plentiful fruit within a few months, while several woody pioneers already flower and bear seed in two years.

4. The seeds are never large and are very easily distributed, either by wind or by animals (especially endozoically by birds).

5. The herbaceous species are mostly anemophorous, whilst the majority of the trees appear to be zoophorous. As might be expected *a priori*, species with seeds, which are only adapted for distribution by water, are altogether absent.

6. The new vegetation consists of herbaceous and erect woody plants; climbing plants only occur among the first pioneers exceptionally and in small numbers.

7. The woody pioneers, which in the first few months grow more slowly than the many herbaceous species (e. g. many gramineae and compositae), are nearly all characterized by a great power of resistance against shade, by an especially well developed root-system and by the possession of a foliage-crown, which by exclusion of light, causes the death of the herbaceous species beneath it, generally within one or two years of the closing of the crown of the young trees.

8. On account of their xerophytic nature, there need be no surprise, after what has been said above under no 1, that a few other plants may occasionally ¹⁾ occur in Java among the first pioneers of vegetation. Such plants, which elsewhere, under different oecological conditions, are temporary or permanent epiphytes, occur on bare lava and on deserted stone buildings (for instance in the deserted fortress of Noesakambangan). Similarly a few land-halophytes grow more or less abundantly on the soil [e. g. *Dodonaea viscosa* (LINS.) JACQ., in alpine districts of central and eastern Java].

¹⁾ Compare also l. c. p. 73 SCHIMPER Pflanzengeographie p. 90 and 102 and the literature cited in those publications.

§§§ 2. MEANS OF DISTRIBUTION.

The following observations were, in part, made by me in 1891, on a botanical journey in Central Java, and were written down in the same year, but were not published¹⁾ till some years later in a dutch article entitled: "dissemination of *Myrica* seeds by birds" in a memoir on spontaneous and artificial reforestation of the Sendoro.

"The above observations are of interest on account of the circumstance, that *Myrica javanica* REINW. (*Pitjisan, javanese*) has been found by experience to be one of the most important species of trees for the reforestation of the Sendoro, and especially on account of the fact, that as yet there are in the literature no numerical data as to the means and agents of distribution of this zoophorous species.

The drupaceous stone-fruits of *Myrica* trees being edible, and there always being a large number of birds, especially wild pigeons, in the *Myrica* reforestations, it seemed not improbable, that the fruits are eaten by the birds, and are disseminated by them.

The contents of the crop and gizzard of three birds, shot in the *Myrica* forests above Kledoeng (1450 m. above sea-level), were found to consist almost entirely of undamaged *Myrica* stones, some of which were still surrounded by the soft mesocarp; of these birds one green pigeon had 231, a second green pigeon 144, and one koetilan (*Ixos haemorrhous*) 4 seeds of *Myrica javanica*, still enclosed in the stone.

In the crop and gizzard of a single pigeon there were thus no fewer than 231 undamaged "seeds" of the *Myrica* tree. The gizzards of the two green pigeons contained only the red mesocarp with the seeds still enclosed in the stone, without other remnants of food, but the crop and the gizzard of the koetilan contained in addition some rests of insects.

Since all the stones which were examined, were quite undamaged and had only been freed from the surrounding fleshy portion of the fruit, it seems to me there is no doubt, that the strong endocarps, thrown out by the birds, will germinate extremely well, and it may be assumed, that the green pigeons especially contribute largely to the dissemination of the seeds of *Myrica javanica* on the G. Sendoro.

It may be added, that the above-mentioned stones, collected from the three bird's gizzards, were sown at my request by Mr. E. TOBI

¹⁾ KOORDERS, S. H., Spontaneous and artificial reforestation of the Sendoro in Java (in *Tijdschrift van Nijverh. en Landb. v. Nederl. Indie*, Dl 51, p. 241—287, with a map). — Compare also: VALETON, Th. The distribution of fruits by animals (in *Teijsmannia* IV, p. 219).

at Kledoeng, in order to see whether their germinative power had really not suffered by the sojourn in the crop and gizzard (KOORD. l. c. p. 45—47)".

To the above notes the following may now be added. When in 1903 I returned to the mountain in question, I learned verbally from the keeper, whom Mr. E. TOBI (now Chief Inspector, Head of the Forestry Service in the Dutch Indies) had kindly instructed at my request to sow the *Myrica javanica*-“seeds”, that the “seeds” from birds’ gizzards and crops, received by him, had all germinated excellently and had further developed well.

These observations may therefore be regarded as proving, that some birds, and especially a species of the large green wild pigeons referred to (probably *Vinago Capellei* or an allied species of the genus *Vinago*) may, in Java, contribute very largely to the dissemination of the alpine tree, here in question; they also show that in some cases 100% of the “seeds”, which have been ingested, germinate well, and that the number of *Myrica* fruitstones found in the crop of a single bird may amount to 231.

As yet no data are known to me regarding the dissemination of *Myrica* seeds in Java by other animals.

§§ 2. *MYRICA LONGIFOLIA* TEJSM. & BINNENDIJK.

§§§ 1. Distribution outside Java: unknown.

§§§ 2. Distribution and oecological conditions in Java. It is stated by MIQUEL (Flora Ind. Bat. I, 1, p. 872) that *Myrica longifolia* TEJSM. & BINN., which, according to KOORDERS and VALETON, Bijdr. Booms. Java IX (1903) p. 104, is synonymous with *Myrica Lobbi* T. & B., had been found on the Megamendoeng in the Preanger. This has certainly never been confirmed, although I have repeatedly collected herbarium material in this district. Only in one single place have I found this species wild, namely in Central Java, in the residency Semarang on the G. Telemojo above Sepakoeng, at an altitude of 1700 m. above sea level. The tree grew sporadically in an evergreen, mixed forest on fairly dry soil of volcanic grit, on a lateral summit of the G. Telemojo, called G. Pendil. In the same forest there also occurred among other plants: *Weinmannia Blumei* PLANCH. (Saxifragaceae) and *Wendlandia Junghuhniana* MIQ. (Rubiaceae). At the original above-mentioned station of *Myrica longifolia* TEJSM. & BINN., *Myrica javanica* BL. was however entirely absent.

§§§ 3. Means of distribution. The dissemination of this species very probably also takes place through the agency of birds, which eat the drupaceous stonefruits. Further data about the means of distribution of this rare species are wanting. The fruits resemble those of *Myrica javanica* BL., so closely however, that in any case I cannot consider the limited vertical and horizontal distribution of *Myrica longifolia* T. & B. as being due to difficulties of dissemination, but feel obliged to attribute it to other oecological conditions.

§ 2. On *Oreiostachys*. GAMBLE, a new genus of Gramineae-Bambuseae, collected by Dr. A. PULLE in Java at an altitude of 1600 m. above sea-level

When in 1907 I was engaged on the systematic and phyto-geographical investigation of the Alpine flora of Java, and was working in the herbarium of the Royal Botanical Museum at Dahlem near Berlin ¹⁾, Prof. Dr. F. A. F. C. WENT was so kind as to make me the offer, very highly appreciated by me, of giving me also for investigation a collection of Alpine plants, made in 1906 in Java by Dr. A. PULLE, and belonging to the Herbarium at Utrecht. In the preliminary naming of this collection, which proved to have been excellently collected, and preserved and labelled with great care, I found a fine flowering specimen of a *Bambusea*, the naming of which presented exceptionally great difficulties, here and at Berlin (*i. a.* by the absence of fruits, which are almost indispensable for the determination of *Bambuseae*). Thereupon I sent this herbarium

¹⁾ Most grateful mention should here be made of the assistance given me by Professor Dr. A. ENGLER, Director of the Royal Botanic Garden at Dahlem, near Berlin, and by Dr. D. PRAIN, Director of the Royal Gardens, Kew, near London. Prof. Dr. A. ENGLER was so kind as to lend me the plants, which he had personally collected in Java on his last "Forschungsreise", especially in the highest mountain regions of Western Java, and also in the Tengger; this loan was not limited to my stay at Dahlem-Berlin, but continued after my return to Leiden.

Dr. PRAIN had the extreme courtesy to send me, for further study and in response to my request, made in 1907, while I was in Berlin, all the duplicate specimens of the Phanerogams, collected by me in 1899 on the Tengger mountains of Eastern Java, and described by me in *Natuurk. Tijdschr. v. N. I* (Dl. 60 p. 242—280 and 370—374, Dl. 62 p. 213—266), which specimens had been presented to the Royal Herbarium at Kew, by the Botanic Gardens of Buitenzorg; Dr. PRAIN also granted me the loan of these specimens for further study, while I was in Holland. When it is remembered, that all these specimens had already been incorporated in the herbarium according to their specific name, and were scattered among hundreds of thousands of other specimens, at the time when I made my request, it is evident, that this action implies extreme scientific liberality and readiness to help on the part of the Director of the Royal Botanic Gardens, and of the scientific staff of the Royal Herbarium at Kew.

These collections are still the subject of study.

specimen to Mr. J. S. GAMBLE, F. R. S. in England. This botanist, the author of the excellent monograph on Indian *Bambuseae*, published in Vol. VII of the Annals of the Royal Gardens of Calcutta), succeeded in determining, that the plant belonged to a new genus of *Gramineae-Bambuseae-Arundinarieae*, near the genus *Sasa* SHIBATA; it might thus perhaps prove to be related to a plant, collected in Java by JUNGHUHN and only known to MIQUEL and to BÜSE in the sterile state. The latter plant was only briefly described by BÜSE sub n. 7 on p. 393 of his *Plantae Junguhnianae* and by MIQUEL sub n. 15 on p. 420 of Vol. III of his *Flora Indiae Batavae* under the name *Bambusacea spec. indet.* (without further addition). GAMBLE rightly supposed that the authentic specimen of JUNGHUHN's herbarium might be in the Herbarium at Leiden. A search, which I thereupon made in this herbarium, confirmed GAMBLE's surmise in a striking manner. For in the first place I succeeded in finding the authentic specimen of JUNGHUHN's plant, in a packet of JUNGHUHN's *Gramineae* of Java, returned long ago by BÜSE's heirs to the herbarium at Leiden; fortunately the specimen was not only in an excellent state of preservation and had the authentic determination label of BÜSE (1854), but also the original herbarium notes, which I presume to have been made in 1839 by JUNGHUHN, at the time of collection. In the second place I succeeded at Leiden, by a comparison of the leaves of the flowering branches of PULLE's herbarium specimen (which leaves were greatly reduced in size) with the sterile, normally developed leaves of JUNGHUHN's plant, in establishing the unspecific identity of the two plants an identity already surmised by GAMBLE. A comparison of the authentic labels of BÜSE and of JUNGHUHN, preserved at Leiden, further showed that a clerical error had arisen in BÜSE's text l. c. 393—394 (which MIQUEL l. c. 420 had cited without criticism and in an abbreviated form under "*Bambusacearum species dubiae*"). This error seems to me to have arisen from BÜSE's not having read with care JUNGHUHN's labels indicating the place of growth.

This was evident from the following:

1. On the above-mentioned labels of JUNGHUHN I could clearly read without difficulty: "J. Sunda-landschap. 3—6000' Bambu-ö-ö", while BÜSE l. c. 394 gives: *Habitat Javae sylvas intactas Pekalongan, altit. 3—6000'. JUNGHUHN. — Incolae hanc vocant Bambu öö, jide JUNGHUHN. — Species propria scandens aut ramis pendentibus?*" (Büse l. c.).

Pekalongan is however, not situated in the Sunda district ("Sundalandschap" = the present Preanger), but in Central Java.

2. Furthermore JUNGHUHN cannot have collected at Pekalongan at an altitude of 3—6000 feet, since Pekalongan lies in a low plain.

3. JUNGHUHN collected much herbarium material on and near the plateau of Pengalengan.

4. As is proved by authentic specimens preserved at Leiden, the plants collected on and near the Pengalengan plateau have often similar labels to the *Bambusacea spec. indet.* of JUNGHUHN, here in question.

5. The herbarium specimen of Dr. PULLE, no. 3173 was also collected near the Pengalengan plateau in the Preanger, and also at an altitude of 1600 m.; the species is not yet known from any other locality. Taking this into account, there seems little doubt, that MIQUEL's words "bij Pekalongan" etc., under JUNGHUHN's *Bambusacea*, should be deleted and should be replaced by: in the Preanger, at 3000—6000 feet (1000—2000 m.) probably on or near the plateau of Pengalengan discovered in 1839 by JUNGHUHN, and called there according to JUNGHUHN *Bambu-ö-ö* (Sundanese).

According to verbal information, which Dr. PULLE was kind enough to give me, his *Bambusacea* no. 3173 was called by the Sundanese *Awi-eueul*. Here I adopt the same transliteration as was used by the late Dr. J. BRANDES and most other authors, for the Sundanese *eu*-sound.

I now consider the word *Bambu-ö-ö* a less correct transcription ¹⁾

¹⁾ In Java all the species of Bamboe are invariably referred by the natives to one of the following genera: *Bambu* (Malay) = *Awi* (Sundanese); *Pring* (Low Javanese) = *Dëling* (High Javanese).

HASSKARL writes *Awi-ülül* (Sundanese) for his *Bambusa elegantissima*, HASSK. Plant. Jav. rar. (1848) 42; Miq. Fl. Ind. Bat. III (1885) 420 = *Becsha elegantissima* (HASSK.) KURZ = *Melocanna elegantissima* (HASSK.) KURZ = *Schizostachyum elegantissimum* (HASSK.) KURZ in Indian Forests I (1876) 347. — About an authentic specimen of *Schizostachyum elegantissimum* (HASSK.) KURZ Mr. J. S. GAMBLE was so kind as to give me the following valuable information: "I have here, belonging to my Herbarium and probably a duplicate from Calcutta, authentic pseudophylls — large things 12 inch long and 6 inch broad and very hairy, marked in KURZ' own handwriting "*Melocanna elegantissima*". They do not look at all as if they could belong to Dr. PULLE's plant (GAMBLE msc.). Lateron, when the additional from his plant asked for from Java by Dr. PULLE of one of his friends in the Preanger will be received here, I hope there will be an opportunity for giving more information on this interesting species.

for *Awi-eueul*, because among the Sundanese bamboo is not called "*bambu*" but "*awi*", and because the "specific" name *ü-ö* would probably be more correctly written *eu-eu* or perhaps better still *eueul*.

It is a remarkable fact, that during the course of more than half a century the natives have evidently used practically the same "specific" name *eueul* or *eueu* to designate a species, which for so long a time remained imperfectly known to science. This fact is also of importance, in that it will be possible, by means of this constant native name, to find the species on the spot, for further study and in order to obtain the seeds and fruits, as yet unknown to science.

The specimens of JUNGHUHN'S collecting number 143 are now registered in the Herb. Lugd. Bat. as H. L. B. n. 901, 7—617—618—619—620.

According to the description¹⁾ of his journey, JUNGHUHN botanized for some time in October 1839 in the same high mountain regions, where Dr. PULLE found his flowering *Oreostachys*. He does not, however, mention in that publication the plant which was later referred to by BÜSE l. c. and by MIQUEL l. c. as *Bambusacea spec. indet.* In any case, I have been unable to find anything in the description of the journey, which would give certainty regarding my surmise, mentioned above and still regarded as probable, that JUNGHUHN'S sterile specimen was collected in 1839.

Dr. JONGMANS, 2nd Conservator at the Herbarium at Leiden, was so courteous as to allow me to lend to Mr. GAMBLE a small fragment of JUNGHUHN'S sterile herbarium material of MIQUEL'S *Bambusacea spec. indet.* n. 15, which material I had examined.

Mr. GAMBLE was thus enabled to supplement his diagnosis of the leaves, and further to confirm the conspecific identity found by myself, of Herb. JUNGHUHN n. 143 with Herb. PULLE n. 3173.

I here add without alteration the generic and specific diagnoses of *Oreostachys Pullei* GAMBLE, which Mr. J. S. GAMBLE F. R. S. has kindly placed at my disposal, and beg to thank him heartily for his disinterested help, so highly appreciated by myself. A brief résumé as to locality, native, names etc. is appended.

¹⁾ JUNGHUHN. Uitstapje naar de bosschen van de gebergten Malabar, Wajang en Tiloe op Java, (in Tijdschr. voor Natuurl. Geschied. en Physiol. VIII (1841) 349--412.

²⁾ Mr. A. H. BERKHOUT in Wageningen, late of Dutch East Indian Forest Department, was so kind as to inform me, that he observed *Awi-eueul* (which he kept for *Bambusa elegantissima*) growing common on the slopes of the mountains Malabar and Patoeha, but never in flower. HASSKARL (l. c. 42) gives for his *Bambusa elegantissima*: "inter montes Tilu et Malabar provinciae Bandong in terra Preangereana copiosissime obviam venit". To HASSKARL the flowers were also unknown.

According to a letter received by me, the flowers and flowering branches, referred to in the following diagnoses, have been described by GAMBLE after the Utrecht herbariumspecimen (PULLE n. 3173) and the leaves after the above-mentioned fragment of the Leiden specimen (JUNGHUHN n. 143 = *H. L. B.* n. 901, 7—617—618—619—620).

“**Oreiostachys**, GAMBLE *gen. nov.* Spiculae 1 florae, ovato-oblongae, secus ramos paniculae in racemis brevibus dispositae; floribus hermaphroditis. Glumae subcoriaceae, mucronato-acuminatae, multinerves, dorso apicem versus pallide villosae; 4—6 vacuae inferiores, ab imo gradatim auctae; florens vacuis simillima; palea etiam glumis similis sed bimucronata, ecarinata, dorso interdum corrugata, quam gluma villosior, dorso basi interdum rachilla terminali munita. Lodiculae 3, breves, nunc obtusae, nunc spathulatae, pilis longis sericeis ciliatae. Stamina 6; filamenta longissima glabra; antherae elongatae, loculis inferne acutis. Ovarium glabrum, ovoideum vel cylindricum, apice incrassatum; stylus basi 3-fidus, stigmatibus plumosis. Caryopsis non visa.”

“Gramina suffruticosa, culmis maxime fistulosis; pseudophyllis scabris, apice fimbriatis, apiculo brevi. Folia petiolata, cum vaginis articulata, nervulis transversis nullis vel obscuris. Inflorescentia in culmis aphyllis; ramis longis vel brevibus verticillatim dispositis, sed vaginis et pseudophyllis munitis, quam maxime decomposita.” (J. S. GAMBLE *msc.* 28. I. 1908 in *Herb. Acad. Rheno-Traject et Herb. Acad. Lugd. Bat.*)

“**Oreiostachys Pullei**, GAMBLE *spec. nov.* Suffrutex scandens 10 m. altus (fide cl. PULLE); culmi inter nodos maxime fistulosi; nodi annulati; pseudophylla straminea, scabra, ore longe fimbriata, apiculo brevi. Folia tenuiter membranacea, lineari-lanceolata, apice longe setaceo-acuminata, basi inaequaliter cuneata, margine et apice scabra, supra laevia infra paullo asperula et ad costam prope basin villosula, 12—20 cm. longa, 1—3 cm. lata; costa subtus lucens, nervi utrinque 5—8 haud conspicui, nervulis transversis perobscuris; vaginae glabrae striatae, apice ciliis paucis albis rigidis munitae; ligula longiuscula puberula serrata. Inflorescentia in culmis florentibus; paniculae ad nodos verticillatae, ramis longis vel brevibus, ad 10—12 cm. longae; spicatae in ramulis racemosis pauciflorae, acutae, 10—15 cm. longae, 2—3 mm. latae, 1-florae; rachis angulata, sinuata, scabra; bractae multe foliosae foliis similes sed vaginis majoribus. Glumae vacuae 4—6 subcoriaceae, ovatae, longe mucronatae, 5—15-nerves,

nervulis transversis obliquis frequentioribus, dorso scabrae et sub apice albo-rillosae; gluma florens vacuis simillima, 12 mm. longa 5 mm. lata; palea etiam florenti similis et aequilonga, bimucronata, dorso rotundata, ecarinata. Lodiculae 3, 1—1.5 mm. longae, basi cuneatae, apice obtusae, longe albo-fimbriatae; interdum elongatae, spathulatae (an in floribus morbois?) 7—8 mm. longae, longe ciliatae. Stamina 6; filamenta longissima; antherae 7—7.5 mm. longae. Ovarium 1—2 mm. longum, glabrum; stigmatibus plumosis plerumque plus minus coalitis. Caryopsis non visa. (J. S. GAMBLE msc 28. 1. 1908 in Herb. Acad. Rheno-Traject. et Herb. Acad. Lugd. Bat.)

Geographical distribution: Outside Java: unknown. In Java: Only in Western Java, in the Preanger, at an altitude of 1000—2000 m. Here only represented by two collecting numbers 1) in virgin forest, probably in 1839 on or near the plateau of Pengalengan (JUNGHUHN n. 143, in Herb. Lugd. Bat. sub n. H. L. B. 908, 7—617—618—619—620. — branches which only bear leaves) and 2) in the Wajang-Windoe mountains, near the tea-plantation Malabar in virgin forest, at a height of 1600 m. above sea-level (A. PULLE n. 3173 in Herb. Rheno-Traject. Suffruticose, climbing, 10 m. high. Spikes dark violet. Flowering without leaves. This species of Bamboo has not flowered for a very long time, according to verbal information obtained locally. Flowering branches collected on June 25, 1906). — Native name: *Awi-eueu* or *Awi-eueul* (Sundanese) near Pengalengan.

This species is phyto-geographically very interesting, because 1) this monotype represents a genus, which according to GAMBLE (see above), is more nearly related to the genus *Sisa* SHIBATA, occurring in Japan, but absent from the Malay Archipelago, than to the other genera of *Gramineae-Bambuseae*, represented in the Archipelago by numerous wild species; 2) because in Java, according to my numerous journeys, nearly all wild-growing *Bambusaceae* only occur below 1600 m., while *Oreostachys Pullei* GAMBLE, was found growing wild by PULLE 1600 m. above sea-level, 3) because this species appears to be endemic in Java, and seems there to be localized in a few mountain regions of the Preanger.

Histology. — “*A method of cold injection of organs for histological purposes*”. By Prof. H. J. HAMBURGER.

For a considerable time the want has been felt of replacing at injections for histological purposes the warm substance, for which as a rule stained gelatine was taken, by a cold one; not only because when using a warm mass the technical difficulties, which are great already, are rendered more complicated still, owing to the care necessary to keep organ and mass at bodily temperature, but also because in a warm waterbath the structure of the tissues is frequently impaired. Therefore TAGUCHI proposed in 1888 ¹⁾ to use for this purpose a suspension of Japanese Indian ink in water, but GROSSER ²⁾ pointed out as a drawback that on further treatment of the sections the isolated grains not seldom drop, if not out of the smaller yet out of the larger vessels; whilst already at the cutting they are not seldom dispersed over the surface of the section. He therefore tried to find a fluid which could easily be solidified after the injection and found that the white of a hen's egg cut and afterwards filtrated answered this purpose very well.

When we too wished to apply this method the difficulty made itself felt, that in this manner we could not obtain the mass in a sufficiently fluid state. When according to the prescription we rubbed the piece of Indian ink over the plate of ground glass a membrane was always formed. Moreover it was found that the suspension thus prepared, when kept in a bottle, had become a solid mass after 24 hours, although evaporation was out of the question.

Probably this had to be attributed to the Indian ink of which, as is wellknown, many kinds are found in the trade. But we did not succeed in getting a better one.

We then tried to obviate this difficulty by mixing the egg white solution with *liquid Indian ink* as is to be obtained in the trade under the name of GÜNTHER-WAGNER'sche flüssige Perlusche, in the volumetric proportion of 1 to 1. The result was a thin liquid mass, which, when examined under the microscope, contained only extremely small particles, which were in BROWN's molecular motion.

After injection with this fluid the organ was fixed in sublimate-formol by which the injected egg white could be precipitated. After the usual washing with water containing iodine, pieces of the organs were stained with alumcochineal, and afterwards embedded in paraffin

¹⁾ Archiv. f. Mikrosk. Anatomie, **31**, p. 565, 1888.

²⁾ Zeitschr. f. Wissenschaftliche Mikroskopie, **17**, p. 187, 1900.

in the usual manner. On microscopic examination the blood-vessels were found to be filled now *with a perfectly homogeneous black mass.*

This method has an advantage over that of GROSSER in the fact that we need not fear the injection-fluid becoming solidified before the injection; besides the preparation of the suspension requires *much less time.*

In another direction too we have simplified the method, viz. by substituting blood-serum for egg white. A mixture of 3 parts of blood-serum with 2 parts of the above named Indian ink gave excellent results.

The blood-serum need not be derived from the same species of animal. For injections of caviae or rabbits we got good results by using horse-serum or cow-serum, fluids that are easily obtained.

Here too fixation was brought about by means of sublimate-formol.

As yet kidneys and liver were microscopically examined. But the injection fluid also penetrated skin, muscles and brain.

An attempt to prepare suspensions of carmine grains in serum suggested itself now, but these experiments failed as the carmine particles conglomerated. Perhaps, however, mixtures of dissolved carmine or of colloidal fluids may be prepared with serum, giving good results.

The above mentioned experiments were made in cooperation with Mr. A. F. DE BOER and Mr. G. A. KALVERKAMP, medical students.

Groningen, March 1908.

Mathematics. — “*The sections of the net of measure-polytopes M_n of space Sp_n with a space Sp_{n-1} normal to a diagonal.*”
By Prof. P. H. SCHOUTE.

1. In the first part of a communication on fourdimensional nets and their sections by spaces (*Proceedings*, Febr. 1908) we have i.a. transformed the net (C_3) into a net (C_{16}) and a net (C_{24}); so here the regular simplex, the fivecell C_5 , was not considered. Whereas the regular simplex of Sp_2 , the equilateral triangle, furnishes a plane-filling all by itself as well as in connection with some other regular polygons, and the regular simplex of Sp_3 , the tetrahedron, can fill the space in combination with the octahedron, it is impossible, as was shown in the quoted paper, to find for the regular simplex C_5 of Sp_4 other regular cells, which can together fill the space of Sp_4 .

This leads us gradually to the question, whether it is not possible

to point out one or more polytopes — if not quite regular ones — which with C_5 fill the fourdimensional space. We have here in view to give to this question an answer, emanating from the connection of a few results formerly arrived at.

2. We consider the net (M_5) of the measure-polytopes M_5 of space S_{p_5} and cut this by a space S_{p_4} normal to a diagonal. This work breaks immediately up into two parts. First the section of space S_{p_4} with a definite measure-polytope M_5 must be found, e. g. with the one, the centre of which has been taken for origin of a rectangular system of coordinates with axes parallel to the edges; we must next investigate how we can prove from this section in which way the intersecting space S_{p_4} affects the other measure-polytopes of the net.

The answer to the first part of this question can be found by means of one of the two diagrams 1 and 2, which we shall therefore discuss successively. Of these diagram 1 is what we arrive at when we project

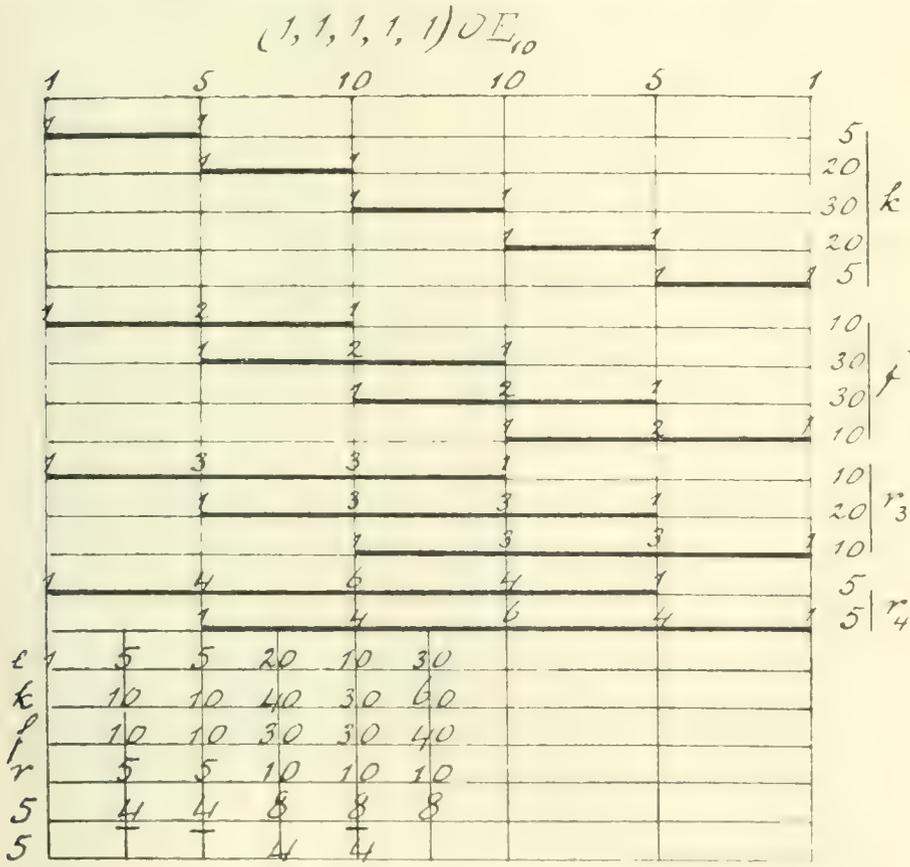


Fig. 1.

the bounding elements of M_5 on the diagonal: it is an extension of the second diagram $n = 5$ of the plate, added to the communication on the section of the measure-polytope M_n of the space S_n with a central space $S_{p_{n-1}}$ normal to a diagonal (*Proceedings*, Jan. 1908). Here, too, we restrict ourselves to a few sections, viz. to the transition forms and to those intermediary forms which bisect the distance of two adjacent transition forms; according to the notation introduced there, we distinguish the transition forms by the symbols $\frac{1}{5}M_5$, $\frac{2}{5}M_5$, $\frac{3}{5}M_5$, $\frac{4}{5}M_5$, the intermediary forms by the symbols $\frac{1}{10}M_5$, $\frac{3}{10}M_5$, $\frac{5}{10}M_5$, $\frac{7}{10}M_5$, $\frac{9}{10}M_5$. As these sections have been incidentally already found in the last quoted paper, we can suffice here by a mere enumeration; to be able to indicate relations in measure we again assume that we have taken half the edge of M_5 as measure-unit.

Transition forms. As two sections pM_5 and qM_5 of which the fractional symbols p and q complete each other to unity, form two oppositely orientated positions of the same polytope, we have here to deal with but two transition forms, viz. $\frac{1}{5}M_5 = -\frac{4}{5}M_5$ and $\frac{2}{5}M_5 = -\frac{3}{5}M_5$. Of these $\frac{1}{5}M_5$ is a regular fivecell $C_5^{(2\sqrt{2})}$, whilst $\frac{2}{5}M_5$ is formed (see *Proceedings*, page 488 under $n = 6$) by truncating a fivecell $C_5^{(4\sqrt{2})}$ at the vertices as far as halfway the edges and hence transforming it into a polytope (10, 30, 30, 10) with edges $2\sqrt{2}$; for the last form *Proceedings*, page 503, can be compared.

Intermediary forms. Of the three intermediary forms $\frac{1}{10}M_5 = -\frac{9}{10}M_5$, $\frac{3}{10}M_5 = -\frac{7}{10}M_5$, $\frac{5}{10}M_5$ the first is a $C_5^{(\sqrt{2})}$, the second (*Proceedings*, page 488 under $n = 7$) a fivecell $C_5^{(3\sqrt{2})}$ truncated as far as a third of the edges, passing by this proceeding into a polytope (20, 40, 30, 10) with edges $\sqrt{2}$, the third (*Proceedings*, page 487 under $n = 5$) a $C_5^{(5\sqrt{2})}$ truncated as far as three fifths of the edges, which has on account of this passed into a polytope (30, 60, 40, 10) with edges $\sqrt{2}$.

We shall now pass to diagram 2 where the plane through two

opposite edges $PQ, P'Q'$ intersecting the diagonal $P'Q$ has been taken as plane of projection. The projection of $M_5^{(2)}$ on this plane is the

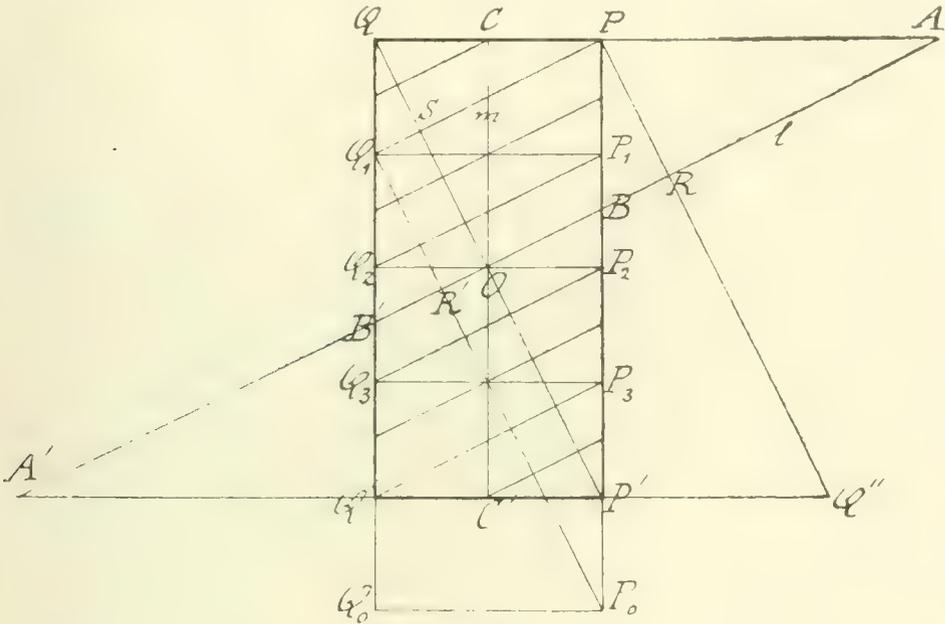


Fig. 2.

rectangle $PQQP'$ with edges $PQ = 2, PP' = 4$, which is divided by three lines parallel to PQ into four equal rectangles. The intersecting space Sp_4 passing through the centre O stands according to the perpendicular l , erected in O on the diagonal $P'Q$, normal to the plane of projection. If we suppose (*Proceedings*, page 491) a few measure-polytopes $M_5^{(2)}$, which are laid against each other in the direction

of the edge PQ on either side, to be united to a prism of which the basis is an $M_4^{(2)}$ and the edges normal to OA have the direction

PQ , then the section of the space Sp_4 through O with this prism is a rhombotope Rh_4 of which AA' — with a length of $4\sqrt{5}$ — represents the axis with the period 4. Comparison of this rhombotope with the measure polytope $M_4^{(2)}$ of $M_5^{(2)}$ lying in the space Sp_4 perpendicular according to m on the plane of projection shows us that the rhombotope can be obtained by stretching this polytope $M_4^{(2)}$ in the direction of the diagonal CC' to an amount of $OA : OC = \sqrt{5}$.

This rhombotope is truncated perpendicularly by the spaces Sp_4 , projecting themselves in the points of intersection B, B' of the axis AA' with the sides PP', QQ' of the rectangle. If again we make

use of the annotation $a(p, q)$ formerly introduced (*Verhandelingen*, vol. IX, N^o. 7, page 17) then the central section is a polytope $4\sqrt{5}\left(\frac{3}{8}, \frac{5}{8}\right)$ and we find, omitting the length of axis $4\sqrt{5}$ alike for all sections, for the transition forms and the intermediary forms described above the following rhombotope symbols:

$$\begin{array}{l} \frac{1}{10} M_5 = \left(0, \frac{1}{8}\right), \\ \frac{3}{10} M_5 = \left(\frac{1}{8}, \frac{3}{8}\right), \\ \frac{5}{10} M_5 = \left(\frac{3}{8}, \frac{5}{8}\right), \\ \frac{7}{10} M_5 = \left(\frac{5}{8}, \frac{7}{8}\right), \\ \frac{9}{10} M_5 = \left(\frac{7}{8}, \frac{8}{8}\right), \end{array} \left| \begin{array}{l} \frac{1}{5} M_5 = \left(0, \frac{1}{4}\right), \\ \frac{2}{5} M_5 = \left(\frac{1}{4}, \frac{2}{4}\right), \\ \frac{3}{5} M_5 = \left(\frac{2}{4}, \frac{3}{4}\right), \\ \frac{4}{5} M_5 = \left(\frac{3}{4}, \frac{4}{4}\right). \end{array} \right.$$

3. The second part of the question, viz. how the intersecting space $S\rho_4$ affects the other measure-polytopes can now be answered by means of analytical geometry as well as by descriptive geometry.

With reference to the system of coordinates assumed above the centres and vertices of all cells $M_5^{(2)}$ of the net have all nothing but integers as coordinates, the centres only even integers, the vertices only odd ones. From this follows in general that the distances from the centres to the central space $\sum_1^5 x_i = 0$ are multiples of fifth parts of the diagonal, those of the vertices to the same space odd multiples of tenth parts of the diagonal. In this way a space of intersection $\sum_1^5 x_i = p$ in general furnishes five different sections of which the fractions placed before M_5 differ respectively $\frac{1}{5}$. If the space of intersection passes through a vertex we find the transition sections; if it passes through a centre we find the intermediary forms.

We arrive at the same result by diagram 2. If we allow the same space $S\rho_4$ bisecting perpendicularly the diagonal $P'Q$ of the central cell to intersect the right adjacent cell with the diagonal PQ'' , then the segment QO cut from the diagonal of the central cell passes

into PR , which means a decrease of $QS = \frac{1}{5} QP'$, and this is repeated every time a cell is taken further to the right. If we exchange the central cell by an other one of which the projection $P_0P_1Q_1Q_0$ covers for three fourths that of the central one, then QO passes into Q_1R' , again a decrease of $\frac{1}{5}$, and this too is repeated every time the projection moves onward in the direction PP' to an amount of PP_1 . So here too we find five different symbols pM_5 , of which the fractions gradually increase with $\frac{1}{5}$. With the aid of the above table this result of the notation pM_5 can be transformed into that of the rhombotope symbols.

We have now answered the question put at the commencement. If we wish to fill Sp_4 with C_5 and a single other groundform, then the form (10, 30, 30, 10) with the same length of edges can do service; both forms appear then in two oppositely orientated positions. If by the side of C_5 we allow two other groundforms to fill Sp_4 , we can make use of the forms (20, 40, 30, 10) and (30, 60, 40, 10) of the same length of edges; if we take into consideration difference in orientation, then this space-filling demands five forms. And if one does not object to connecting more than two really different groundforms we can take the five forms

$$\frac{1}{20} M_6, \frac{5}{20} M_6, \frac{9}{20} M_6, \frac{13}{20} M_6, \frac{17}{20} M_6,$$

i. e.

$$\left(0, \frac{1}{16}\right), \left(\frac{1}{16}, \frac{5}{16}\right), \left(\frac{5}{16}, \frac{9}{16}\right), \left(\frac{9}{16}, \frac{13}{16}\right), \left(\frac{13}{16}, 1\right).$$

of which the first is a $C_6(4V^2)$; these appear in only one position.

4. Before passing on to the general case of Sp_n we indicate the shortest way, by which one can calculate the number of component parts when filling a fourdimensional block of one of the found forms but of k -times larger linear dimension. To prepare the general case of an arbitrary n we introduce a simpler notation. We distinguish the transition forms and the intermediary forms by the letters T and I and then indicate by exponent — this, to avoid rootsigns, in V^2 as new unit — the size, by a footindex the place of the section. We then represent the polytope, formed by truncating regularly a regular fivecell with a length of edges pV^2 at the five corners to the fraction q of the edge by the symbol $qS^{(p)}$. Thus each of the five different forms is represented by four different signs as follows:

$$\begin{array}{l}
 \frac{1}{10} M_5^2 = \left(0, \frac{1}{8} \right) = I_1^1 = S_1^1 \\
 \frac{3}{10} M_5^2 = \left(\frac{1}{8}, \frac{3}{8} \right) = I_2^1 = \frac{1}{3} S_1^3 \\
 \frac{5}{10} M_5^2 = \left(\frac{3}{8}, \frac{5}{8} \right) = I_3^1 = \frac{3}{5} S_1^5 \\
 \frac{1}{5} M_5^2 = \left(0, \frac{1}{4} \right) = T_1^2 = S_1^2 \\
 \frac{2}{5} M_5^2 = \left(\frac{1}{4}, \frac{2}{4} \right) = T_2^{(2)} = \frac{1}{2} S_1^4
 \end{array}$$

whilst the forms appearing past the middle $\frac{7}{10} M_5^{(2)}, \frac{9}{10} M_5^{(2)}$ and $\frac{3}{5} M_5^{(2)}, \frac{4}{5} M_5^{(2)}$ of opposite orientation are indicated by $I_{-2}^{(1)}, I_{-1}^{(1)}$ and $T_{-2}^{(2)}, T_{-1}^{(2)}$.

By considering the truncated fivecells $qS_{(p)}$ we find immediately :

$$\left. \begin{array}{l}
 T_1^{(2k)} = I_1^{(2k)} \\
 I_2^{(k)} = I_1^{(3k)} - 5 I_1^{(k)} \\
 T_2^{(2k)} = I_1^{(4k)} - 5 I_1^{(2k)} \\
 I_3^{(k)} = I_1^{(5k)} - 5 I_1^{(3k)} + 10 I_1^{(k)}
 \end{array} \right\} \dots \dots (1)$$

Of these relations e. g. the last one is deduced in the following way : The form $I_3^{(k)} = \frac{3}{5} S^{(5k)}$ appears by truncating the fivecell $S^{(5k)} = I_1^{(5k)}$ to $\frac{3}{5}$ of the edges. As each two of the five polytopes $S^{(3k)} = I_1^{(3k)}$, which are taken off by the truncation, have an $S^{(k)} = I_1^{(k)}$ in common, we subtract when diminishing $I_1^{(5k)}$ by $5 I_1^{(3k)}$ ten times $I_1^{(k)}$ too much.

Together the equations (1) lead to the relations of volume:

$$\frac{I_1^{(k)}}{1} = \frac{T_1^{(2k)}}{16} = \frac{I_2^{(k)}}{76} = \frac{T_2^{(2k)}}{176} = \frac{I_3^{(k)}}{230} = \frac{R^{(2k)}}{384},$$

where $R^{(2k)}$ is the rhombotope formed by the required stretching of an M_4^{2k} in the direction of a diagonal. If the number 384 is deduced from the remark that $T_1^{(2k)} = \frac{1}{4!} R^{(2k)}$, then the two relations

$$2(16 + 176) = 384 \quad , \quad 2(1 + 76) + 230 = 384,$$

which express that R^{2k} can be built up either out of the four forms $T_i^{(2k)}$ or out of the five forms $I_i^{(k)}$ can serve to control.

We shall now indicate at full length how the obtained relations will serve to get us over the entire difficulty of the determination of the demanded numbers. To this end we notice that the vertices of the k^5 measure polytopes $M_5^{(2)}$ forming together a block $M_5^{(2k)}$ project themselves on a diagonal of that block except in the ends in the $5k-1$ points dividing this diagonal into $5k$ equal parts. If we indicate (diagram 3) the $5k+1$ points obtained in this way on the diagonal by $A_0, A_1, A_2, \dots, A_{5k}$, then the segment A_0A_5 bears the projection of a single $M_5^{(2)}$, the segment A_1A_6 that of a group of five, the segment A_2A_7 that of a group of fifteen measure-polytopes,

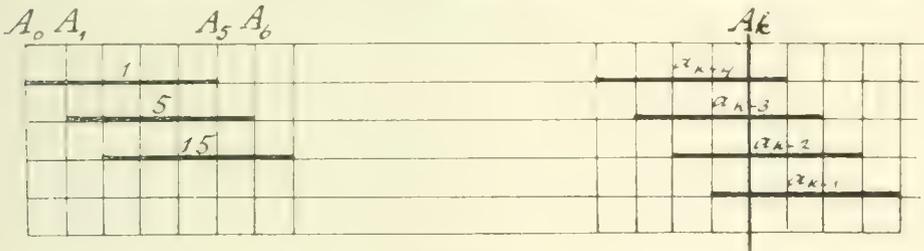


Fig. 3.

etc., where the numbers 1, 5, 15, etc. of the measure polytopes with the same projection are the coefficients a_p of the terms x^p in $(1 + x + x^2 + \dots + x^{k-1})^5$ for $p = 0, 1, 2$, etc. When determining the section $\frac{1}{5} M_5^{(2k)}$ we find that the intersecting space $S\rho_4$ hits the diagonal of projection in the point of division A_k , from which ensues that the groups of polytopes $M_5^{(2)}$ corresponding to the coefficients a_0, a_1, \dots, a_{k-5} are not yet cut, the groups corresponding to the coefficients $a_k, a_{k+1}, \dots, a_{5k-5}$ are no more cut, so that we have but to deal with the four groups shown to the right of the diagram:

$$a_{k-4} T_{-1}^{(2)}, \quad a_{k-3} T_{-2}^{(2)}, \quad a_{k-2} T_2^{(2)}, \quad a_{k-1} T_1^{(2)}.$$

Now for the coefficients a_p the particularity appears that for $p < k$ they can be represented as binominal coefficients viz. by the equation

$$a_p = (p + 4)_4,$$

whilst for greater values of p they are "gnawed" binominal coefficients. So we find here immediately

$$T_1^{(2k)} = I_1^{(2k)} = (k + 3)_4 T_1^{(2)} + (k + 2)_4 T_2^{(2)} + (k + 1)_4 T_{-2}^{(2)} + (k)_4 T_{-1, \dots}^{(2)}$$

and in quite the same way .

If we restrict ourselves to these forms and if again we do not take the transition form consisting of a single vertex into consideration, we have in both cases to deal with n different forms, namely for n even with $\frac{1}{2} n$ transition and $\frac{1}{2} n$ intermediary forms, for n odd with $\frac{1}{2}(n - 1)$ transition and $\frac{1}{2}(n + 1)$ intermediary forms. Thus we get again in $S\rho_{n-1}$ two more or less regular space-fillings in which the regular simplex of that space shares.

In connection with the symbols ${}_qS^{(p)}$ the relations hold here

$$\begin{aligned} I_2^{(1)} &= I_1^{(3)} - (n)_1 I_1^{(1)}, \\ T_2^{(2)} &= I_1^{(4)} - (n)_1 I_1^{(2)}, \\ I_2^{(1)} &= I_1^{(5)} - (n)_1 I_1^{(3)} + (n)_2 I_1^{(1)}, \\ T_3^{(2)} &= I_1^{(6)} - (n)_1 I_1^{(4)} + (n)_2 I_1^{(2)}, \\ &\vdots \end{aligned}$$

which leads for n even to

$$T_{\frac{1}{2}n}^{(2)} = I_1^{(n)} - (n)_1 I_1^{(n-2)} + (n)_2 I_1^{(n-4)} - \dots + (-1)^{\frac{1}{2}(n-1)} (n)_{\frac{1}{2}n-1} I_1,$$

for n odd to

$$I_{\frac{1}{2}(n+1)}^{(1)} = I_1^{(n)} - (n)_1 I_1^{(n-2)} + (n)_2 I_1^{(n-4)} - \dots + (-1)^{\frac{1}{2}(n-1)} (n)_{\frac{1}{2}(n-1)} I_1,$$

whilst the ratios of volume are determined by

$$\frac{I_1^{(1)}}{1} = \frac{T_1^{(2)}}{2^{n-1}} = \frac{I_2^{(1)}}{3^{n-1} - (n)_1} = \frac{T_2^{(2)}}{4^{n-1} - (n)_1 2^{n-1}} = \frac{I_3^{(1)}}{5^{n-1} - (n)_1 3^{n-1} + \dots} = \text{etc.}$$

Farthermore the formulae of reduction hold :

$$\begin{aligned} I_1^{(2k)} &= (k+n-2)_{n-1} T_1^{(2)} + (k+n-3)_{n-1} T_2^{(2)} + \dots + (k)_{n-1} T_{-1}^{(2)} \left\{ \dots (1) \right. \\ I_1^{(2k+1)} &= (k+n-1)_{n-1} I_1^{(1)} + (k+n-2)_{n-1} I_2^{(1)} + \dots + (k)_{n-1} I_{-1}^{(1)} \left. \right\} \end{aligned}$$

which enable us to calculate the number of the parts of different kinds, into which a block of $(2k)^n$ or $(2k + 1)^n$ measure-polytopes

$M_n^{(2)}$ can be cut up.

As an example, which gives something to calculate, we consider the case of the middle section perpendicular to the diagonal of a block of 10^{10} measure-polytopes $M_{10}^{(2)}$. We then find in connection with the relations

$$\frac{T_1}{1} = \frac{T_2}{502} = \frac{T_3}{14608} = \frac{T_4}{88234} = \frac{T_5}{156190} = \frac{R}{9!},$$

where R represents the rhombotope that is the sum of the nine forms

$$T_1, T_2, T_3, T_4, T_5, T_{-4}, T_{-3}, T_{-2}, T_{-1},$$

starting from

$$\frac{1}{2} M_{10}^{(20)} = T_5^{(20)} = I_1^{(100)} - 10 I_1^{(80)} + 45 I_1^{(60)} - 120 I_1^{(40)} + 210 I_1^{(20)},$$

by applying

$$I_1^{(20k)} = (10k + 8)_9 T_1^{(2)} + (10k + 7)_9 T_2^{(2)} + \dots + (10k)_9 T_{-1}^{(2)}$$

for $k = 5, 4, 3, 2, 1$ after some calculation the result

$$\begin{aligned} & 394713550 (T_1^{(2)} + T_{-1}^{(2)}) + 410820025 (T_2^{(2)} + T_{-2}^{(2)}) \\ & + 422709100 (T_3^{(2)} + T_{-3}^{(2)}) + 430000450 (T_4^{(2)} + T_{-4}^{(2)}) \\ & + 432457640 T_5^{(2)}, \end{aligned}$$

which after substitution of the relations given above leads back to the identity

$$T_5^{(20)} = 10^9 T_5^{(2)}.$$

Physiology. — *“The electric response of the eye to stimulation by light at various intensities”*. By W. EINTHOVEN and W. A. JOLLY. (Communication from the Physiological Laboratory of Leiden).

Although the electrical response of the eye to stimulation by light, which was discovered by HOLMGREN has since been studied by numerous observers, there has not so far been undertaken a systematic investigation of the electromotive changes which are caused by stimuli of very varying strength. Such an investigation, however, can as we hope to show, contribute not a little to our comprehension of the retinal processes.

We have in our work employed exclusively isolated frogs' eyes. We have been enabled on the one hand by means of the string galvanometer, which for the retinal currents may be regarded as the most sensitive instrument available, to record and measure very weak electromotive forces, such as are evoked by light of extremely low intensity; on the other hand we have tried by a suitable system of lenses to concentrate light of as great intensity as possible

upon the retina of the eye under observation. The rays proceeding from the crater of an arc lamp, which passed through a collimator slit in close proximity were dispersed by a spectroscopic arrangement and from the spectrum so obtained any desired portion could be isolated by a simple device.

If we made use of rays lying between the wave lengths $\lambda = 0,590 \mu$ and $\lambda = 0,497 \mu$, whose green central part — about $\lambda = 0,544 \mu$ — may be considered to have relatively a very strong effect on the eye¹⁾, we could by the aid of suitably chosen diaphragms vary the light intensities in the proportion of 1 to 10^3 , and with the weakest intensity could obtain galvanometric deflections of several centimetres. The arrangement of our experiments did not permit of our easily diminishing the light further in an accurately measurable manner, but we hope later to be able to do so.

In some experiments white light has been used, which of course could be taken stronger than the spectral green. In this case all the rays of the visible spectrum lie at our disposal, and the light may be further increased by widening the slit or by replacing it with the crater itself. According to a rough calculation the intensity of the white light used by us, that is to say of the combined rays lying within the limits of the visible spectrum, is about 10 times greater than our maximum green. The intensities of the weakest green and of the white light are thus in the proportion of about ²⁾ 1 to 10^{10} .

If the isolated eye, which has not shortly before been exposed to strong light, be illuminated by rays of intermediate strength a form of curve is obtained similar to that recorded by previous observers³⁾.

The current is led off from the cornea and the posterior surface of the bulbus. The current of rest is compensated in the usual way and the connections with the galvanometer are made in such a

1) Cf. F. HIMSTEDT and W. A. NAGEL. Die Verteilung der Reizwerte für die Froschnetzhaut im Dispersionsspectrum des Gaslichtes, mittels der Aktionsströme untersucht, Berichte der Naturforsch. Ges. zu Freiburg i. B., XI, 1901, p. 153.

2) The intensities of the light used will later be communicated in absolute measurement and at the same time the accurate proportion of the intensities of the green and white will be given.

3) Cf. for instance FRANCIS GOTCH, The Journal of Physiol. 29, p. 388, 1903. Ibid. 31, p. 1, 1904. HANS PIPER, Engelmann's Arch. f. Physiol. Suppl. 1905, p. 133. E. TH. VON BRÜCKE u. S. GARTEN, Pflüger's Arch. f. d. ges. Physiol. 120, p. 290, 1907, the latter of whom give a critical review of the literature dealing with the subject. The observers mentioned have all made use of a quickly recording measuring instrument.

manner that a current passing from the cornea through the instrument to the posterior surface of the eye deflects the image of the string in an upward direction. An action current in this direction may be termed positive, and in the reverse direction negative.

On momentary illumination of the eye there is observed a small preliminary negative deflection which is immediately followed by an upward movement of the string. After a somewhat acute peak the curve sinks, at first rapidly then more gradually, but while still distant from the zero line it mounts again. This latter ascent begins a couple of seconds after the beginning of the illumination, and the second summit, which is reached much later, often considerably exceeds the peak in height. Finally the curve gradually regains the zero line.

If the illumination be continued for some time, a new elevation occurs at the moment of darkening whose height is greater the longer the illumination has endured.

The complicated form of these curves and the striking fact that a deflection in the same direction takes place both on illumination and on darkening suggest that there are in the eye two or more different processes occurring partly simultaneously partly successively whose fusion determines the form of the electric reaction.

Further investigation confirms this suggestion, and if recourse is had to very weak or very strong light it seems even to be possible to bring about a separation of the supposed processes. The phenomena are explained in the simplest manner by the assumption that the processes are three in number, whether they are together dependent upon the same substance or each upon a separate one. For the sake of convenience we shall speak of three substances and as we do not intend in the meantime to attempt to define them anatomically in the eye, we prefer to try to describe their characteristics and to mention the conditions, under which their effects appear as pure as possible.

The first substance.

The substance which we have termed "the first" reacts more quickly than the other two. On lighting it displaces the image of the string downwards, on darkening upwards. Its effect can with difficulty be obtained pure but nevertheless it is very marked in a light adapted eye, — which for the sake of brevity we may call a light eye¹⁾ — and the more so the stronger the illumination has been.

In the nature of the case the darkening stimulation can be taken

¹⁾ An eye which is dark adapted may be called a dark eye. Both terms are analogous to "Lichtfrosch" and "Dunkelfrosch" which are commonly used.

very strong in a light eye, and accordingly an eye which has been illuminated strongly develops on darkening a huge positive potential difference. The upward deflection so evoked can however not be of long duration, because by the darkening the light eye is beginning to be changed into a dark eye and therefore the effect of our first substance is no longer so clearly indicated.

Although in the light eye the conditions are less favourable for the lighting than for the darkening stimulus it is nevertheless possible to apply the former in either of two ways. In the first place we may suddenly increase the intensity of the light that is radiating on the eye, and secondly we may darken the light eye for a short period, so that it has not yet become a dark eye and then suddenly illuminate it.

The second method gives better results than the first and we possess numerous curves where after a short darkening of a light eye a strong light stimulus was applied. The "on effect"¹⁾ is a steep downward deflection and attains the considerable amount of 120 to 130 microvolts. It is true that it is followed immediately by an upstroke, the latter however is but small in comparison with the strong upstroke which under similar conditions is evoked in a dark eye.

The second substance.

The second substance reacts less quickly than the first. On lighting it moves the string with moderate velocity upwards, and on darkening slowly downwards, thus on applying stimuli of the same kind it develops potential differences which are opposed to those of the first substance. Its effect appears almost unmixed in a dark eye which is illuminated for a short time by weak light.

If when illuminating with light of very low intensity, the darkening follows rapidly upon the lighting, in a similar way as in a momentary illumination, there is recorded a curve of simple form, with a steeper anacrotic part which is evoked by the lighting and a less steep katacrotic part evoked by the darkening. The top of the curve lies, *within certain limits* higher the more the energy of the illumination is increased either by using greater intensity or longer duration of the light. These limits are determined by the functioning of the other two substances, which when their effects become perceptible influence the form of the curve and considerably complicate it. If a strong momentary illumination be applied there appears a short negative preliminary deflection by the function of the first substance

1) A convenient expression introduced by GORCH.

and the very slow second elevation which follows must be attributed to the functioning of the third substance.

The third substance.

The third substance reacts in the same direction as the second substance but more slowly. On lighting it displaces the image of the string slowly upwards and on darkening still more slowly downwards. So much slower is the third substance than the other two that its effect in a recorded curve appears as a rule almost entirely isolated, and thus can be easily followed.

The effect of the third substance falls out under two conditions (1) In a fully light adapted eye and (2) in a dark eye submitted to very faint light for a short time.

Specially remarkable are the curves obtained if the duration of the lighting of a dark eye is systematically changed, and we wish to direct attention more particularly to the "off effect" in such cases. If the duration of the light is very short and the light is weak, then as already mentioned the effects of the second substance appear unmixed. The off effect here consists in the descent of the curve to the zero line.

If the duration of the light is taken a little longer, and the effect of the other two substances begin to become perceptible, the off effect is determined by the resultant of three forces: The first substance tends to displace the image of the string upwards. It is at first acting weakly but its strength increases regularly during illumination so that it soon surmounts the effect of the other substances. In the case of longer lighting the off effect therefore is always an upward movement which increases with the duration of the lighting.

The second substance tends to depress the image of the string, acts first with moderate strength but decreases gradually during lighting. As the second substance in particular is acting in a dark eye the conditions for its functioning grow during the illumination more unfavourable. A strong darkening effect can not be expected in a dark eye.

The third substance is so slow, that the darkening effects of the first and second take place usually at a moment when the third substance is still tending to displace the string upwards. The darkening effect of the third substance itself, consisting in a slow descent of the string, appears much later and fairly isolated.

The general result is that we can observe in a series of curves, — obtained from a dark eye where the light has been gradually lengthened in duration, — that the darkening effect, in the first

curves a negative deflection, becomes in the later ones a positive deflection. The latter, on further lengthening of the duration of light, gradually increases in size. In the conflict between negative and positive deflections, there is sometimes seen an upward movement, which is immediately preceded by a small downward one.

Of the various particularities which occur in the course of the experiments we shall only briefly mention the latent period. The duration of this period is dependent to so high a degree upon the intensity of the illumination, that it is possible to some extent to judge of the intensities of light used by previous observers from the latent periods recorded by them. With very weak lighting there appear latent periods of the second substance which may exceed two seconds.

In opposition to GOTCH and GARTEN WALLER¹⁾ also mentions latent periods as large in amount as we have observed and others much larger, but as WALLER in his experiments made use of a slow THOMSON galvanometer, there remained the possibility that there were two opposite forces which at first neutralised one another and then after an interval one obtained the mastery. The forces assumed by WALLER agree with our first and second substances.

A more detailed description of our experiments accompanied by a reproduction of some of our curves will appear elsewhere.

Geophysics. — *“The height of the mean sea-level in the Y before Amsterdam from 1700—1860”*. By Prof. H. G. VAN DE SANDE BAKHUYZEN.

Our section has been engaged in former years with an investigation of the subsidence of the land in the Netherlands, and it is especially to Dr. F. J. STAMKART, member of the committee for that investigation, that we owe several important communications on this subject.

Twenty years ago, when calculating the results of the precise levelling, I made some computations in order to determine the subsidence of the land but have not published them. The interesting paper on this subject of Mr. RAMAER, head-engineer, director of the hydrographic survey, has now induced me to re-examine my former notes and as they perhaps may contribute towards the solution of the problem, whether the land under Amsterdam has subsided since

¹⁾ AUGUSTUS D. WALLER, *Philosoph. Transact. of the Royal Soc. of London, Ser. B, vol. 193, p. 123, 1900.*

1700, I intend to publish them here as a continuation to the earlier reports of the "Comittee for the subsidence of the land."

My results have been chiefly derived from the heights of the water in the Y before Amsterdam, recorded from 1700 to 1860 at each hour of the day and at each half hour during the night in the town's tidal station situated at the present fishmarket near the "Nieuwe Markt." Part of them occur in two communications of STAMKART (*Verslagen en mededeelingen der Kon. Akad. van Wetenschappen Afd. Natuurkunde*, 15^e deel 1863, p. 59—69 and 17^e deel 1865, p. 261—303) and some in STAMKART's posthumous papers in keeping of the Academy.

The way in which these observations were made is described as follows by Dr. STAMKART in his paper in Vol. XVII p. 273. The tidal station was erected above the water; in the wooden floor of one of the rooms was a hole through which a gauging rod carrying a mark of the A.P. (zero of Amsterdam) was plunged vertically into the water so far until a notch of the rod caught on the wooden floor. The height to which the gauging rod was wetted showed the level of the water with regard to the zero on the gauge.

In order to draw reliable conclusions about the level of the North-sea on our coast based on the results of the water-level in the Y, it is necessary to investigate whether during the period under consideration variations have occurred in the influx and the outflow of the water of the Y before Amsterdam, owing to changes in the depth and width of the canals leading from the Northsea to the Y. It is very probable that these variations may produce opposed effects on the high and the low water and hence give rise to greater variations in the difference between high and low water than in the mean sea level. The variations of these differences in the succeeding years will therefore be a good standard of the changes in the canals.

From the tide tables of the town's tidal station we derive the following differences between high and low water during 58 years.

TABLE I.

Year	Difference between high and low water						
1700	309 mm.	1715	222 mm.	1805	303 mm.	1847	303 mm.
1701	323 "	1716	299 "	1806	342 "	1848	309 "
1702	331 "	1717	342 "	1807	345 "	1849	318 "
1703	320 "	1725	327 "	1808	331 "	1850	322 "
1704	320 "	1749	318 "	1809	327 "	1851	324 "
1705	312 "	1775	328 "	1810	338 "	1852	329 "
1706	319 "	1796	318 "	1811	330 "	1853	319 "
1707	314 "	1797	313 "	1812	316 "	1854	320 "
1708	308 "	1798	302 "	1813	323 "	1855	287 "
1709	286 "	1799	287 "	1825	341 "	1856	314 "
1710	218 "	1800	288 "	1843	325 "	1857	317 "
1711	325 "	1801	327 "	1844	313 "	1858	295 "
1712	332 "	1802	321 "	1845	310 "	1859	328 "
1713	322 "	1803	284 "	1846	328 "	1860	332 "
1714	332 "	1804	323 "				

If we assume that in each of the 3 periods of 18 years this difference has been constant, we find for it:

in the 1 st period	1700—1717	319 mm.
„ „ 2 ^d „	1796—1813	318 „
„ „ 3 ^d „	1843—1860	316 „

The mean error of a yearly mean is then $\pm 14,8$ mm.

If we suppose that from 1700—1860 the difference has remained constant, then the difference derived from all the 58 years amounts to 318 mm. and the mean error of a yearly mean to $\pm 14,4$ mm. Therefore we are justified in assuming that the difference has remained constant during the whole period 1700—1860 and was 318 mm. $\pm 1,9$ mm.

Moreover we derive from this table that between 1700 and 1860 no perceptible change has occurred in the influx and outflow of the water from the Northsea to the Y, no more than in the mean level of the Northsea with regard to the mean level of the Y.

As mean level of the Y before Amsterdam with regard to the zero adopted in the tidal station (zero of Amsterdam) we shall assume the half sum of the high and low water.

For the same 58 years we derive for that mean the following values.

TABLE II.

Year	Mean Sea level above A.P.						
1700	— 172 mm.	1715	— 166 mm.	1805	— 105 mm.	1847	— 79 mm.
1701	„ 169 „	1716	„ 163 „	1806	„ 69 „	1848	„ 102 „
1702	„ 148 „	1717	„ 159 „	1807	„ 69 „	1849	„ 64 „
1703	„ 187 „	1725	„ 154 „	1808	„ 147 „	1850	„ 53 „
1704	„ 146 „	1749	„ 134 „	1809	„ 112 „	1851	„ 66 „
1705	„ 179 „	1775	„ 89 „	1810	„ 90 „	1852	„ 59 „
1706	„ 199 „	1796	„ 84 „	1811	„ 99 „	1853	„ 76 „
1707	„ 160 „	1797	„ 115 „	1812	„ 103 „	1854	„ 12 „
1708	„ 153 „	1798	„ 96 „	1813	„ 114 „	1855	„ 76 „
1709	„ 193 „	1799	„ 136 „	1825	„ 51 „	1856	„ 48 „
1710	„ 167 „	1800	„ 134 „	1843	„ 20 „	1857	„ 101 „
1711	„ 144 „	1801	„ 55 „	1844	„ 15 „	1858	„ 96 „
1712	„ 126 „	1802	„ 123 „	1845	„ 54 „	1859	„ 64 „
1713	„ 149 „	1803	„ 132 „	1846	„ 40 „	1860	„ 75 „
1714	„ 106 „	1804	„ 92 „				

These values show that the mean sea level has not remained unchanged with regard to the adopted zero of Amsterdam. This becomes still more evident if we form the means of the 3 periods of 18 years. We then obtain:

1708,5	—160,3 mm.	± 5,9 mm.
1725	—154 „	± 25,1 „
1749	—134 „	± 25,1 „
1775	— 89 „	± 25,1 „
1804,5	—104 „	± 5,9 „
1825	— 51 „	± 25,1 „
1851,5	— 61 „	± 5,9 „

If we suppose that during each of the periods of 18 years the mean sea level has remained unchanged we derive from the deviations of the yearly means from the mean of 18 years a mean error for each year of $\pm 25,1$ mm. and in the mean of 18 years a mean error of $\pm 5,9$ mm.

If on the contrary we suppose that during each of the periods of 18 years the mean sea level with regard to the adopted Amsterdam zero has varied proportionally to the time, we get for the mean error of the yearly mean $\pm 24,3$ mm. and for the yearly variations:

from 1700—1717	+ 1,57 mm.	$\pm 1,10$ mm.
„ 1796—1813	+ 0,14 „	$\pm 1,10$ „
„ 1843—1860	- 2,30 „	$\pm 1,10$ „

Hence in the 1st and 2nd periods, in agreement with the general variation of the mean sea levels from 1700—1826, the mean sea level has apparently come nearer to the adopted A.P., but has retired thence in the 3^d period in agreement with the variation from 1825—1851,5. Nevertheless the mean errors of each of these yearly variations, $\pm 1,10$ mm., are so large with regard to the variations themselves, that we attach only a very small weight to the values found; only to the yearly variation in the 3^d period, more than twice the value of the mean error, we may attach a somewhat larger weight. If we adopt a uniform yearly variation between the years 1708,5 and 1804,5, this would amount to 0.58 mm.; in good harmony with this are the results for 1725 and 1749, but the result for 1775 shows a deviation of 32 mm.

We conclude that the elevation of the adopted A.P. above the mean sea level has gradually varied and that the variations can be considered as partly proportional to time; they cannot however be derived exactly from the observations.

The elevation of the A.P. in the tidal station above the mean sea level in the first and the last year of the series of observations, 1700 and 1860, are according to table II 162 mm. and 75 mm. each with a mean error of ± 25 mm. In order to obtain for these elevation values with a smaller mean error, we may use, upon supposition that no sudden variations have taken place in the zero of the gauging rod, the elevation observed in closely preceding or following years, which must be reduced to the year 1700 or to 1860 with an adopted yearly variation. Because the yearly variation is not known with great precision, as appeared above, it is desirable that these years should not be at a great distance from 1700 or from 1860; therefore

I have confined myself to the mean of the 5 years 1700—1704 and 1856—1860. To these means we must add the variations during a period of two years, which are probably smaller than 3 mm. and 5 mm. the values which would follow from the periods of 18 years; instead of these I adopt 1 mm. and 4 mm. and consequently:

adopted A.P. above mean sea level in 1700 = $164 + 1 = 165$ mm.
 „ A.P. „ „ „ „ „ „ 1860 = $76 + 4 = 80$ „

For the mean error of these values I have derived ± 12 mm.

As yet it remains undecided whether the variation from 165 mm. to 80 mm. is due to a slow variation in the mean level of the North sea on our coast, or to a variation of the adopted A.P. in the tidal station either caused by the sinking of the whole station or of the wooden floor, or by accidental or perhaps intentional changes in the height of the A.P. on the gauging rod which during the period from 1700 to 1860 has certainly been renewed several times.

Some data towards the solution of this dilemma may be borrowed from the elevations of the bench marks in the 5 sluices: Oude Haarlemmersluis, Nieuwebrugsluis, Kraansluis, Westindischesluis and Kolksluis; these bench marks have been established in 1682, and consist of grooves cut in stones indicating the elevation of the A.P. The good mutual agreement between the heights of the grooves in the year 1875 which appeared from the levelling made by our member Dr. LELY (the largest difference between them amounted to only 8 mm.) proves that those grooves have been placed with the greatest care, and makes us confident that in 1700, when the first observations in the tidal station were made, the zero on the gauging rod agreed well with that on the stones placed in the sluices some years earlier.

We are therefore entitled to assume with a high degree of probability that in 1700 the A.P. on the 5 sluices was 165 mm. above the mean level of the Y.

In 1860 STAMKART by a levelling has compared the height of the A.P. in the tidal station at that time with the heights of two bench marks in the tower of the St. Anthoniewaag. He found:

lower bench mark 3208,4 mm. above A.P. in the tidal station
 higher „ „ 3705,4 mm. „ „ „ „ „ „

In the same year Dr. STAMKART and Mr. V. D. STERR have also determined by means of levelling the difference in height between the higher bench mark in the St. Anthoniewaag and the grooves in the 5 sluices (Versl. en meded. XVII p. 277—284). From these observa-

tions we derive the following values for the height of the higher bench mark above the A.P. according to the mean of the 5 sluices in 1860 :

STAMKART	V. D. STERR	mean
3628 mm.	3624 mm.	3627 mm.

In the derivation of the mean value we have, with regard to the mean errors, accorded a greater weight to STAMKART'S result.

In 1875 our colleague Dr. LELY by means of a still preciser leveling under direction of COHEN STUART has derived 3622 mm. for the same difference in height.

The differences between the results of 1860 and those of 1875 may be explained very well by errors of observation, so that we may accept with a high degree of accuracy that the bench mark in the St. Anthoniewaag between 1860 and 1875 has not varied with regard to the 5 sluices and that in 1860 the height of the mark above the A.P. of the sluices was 3623 mm. with a mean error of ± 2 mm.

If from this value we subtract 3705, i. e. the height of the mark above the A.P. in the tidal station found by Dr. STAMKART in 1860 we find :

height of the A.P. according to the mean of the 5 sluices above the A.P. in the tidal station in 1860 = 82 mm.

The mean error of this result is about ± 3 mm.

As in 1860 the height of the A.P. in the tidal station was elevated 80 mm. above the mean level of the Y, it follows that in 1860 the A.P. derived from the mean of the 5 sluices above the mean sea level is :

$$80 + 82 = 162 \text{ mm. } \pm 13 \text{ mm.}$$

If we compare this value with the corresponding value of the year 1700, i. e. 165, we may conclude that the height of the mean sea level in the Y, and hence the mean level of the Northsea on our coast has not perceptibly varied with regard to the ground in which the foundations of the 5 sluices are built.

The uncertainty of this conclusion may be expressed by a mean error of ± 18 mm.

The 5 sluices are not in close neighbourhood of each other, the extreme ones are separated by a distance of one kilometre; hence it is over a fairly extensive part of the ground on which Amsterdam is built that the level of the land with regard to the level of the Northsea has remained unchanged during more than one century and a half.

With the same degree of probability with which we have derived

this invariability we may derive from the observations the subsidence of the A.P. in the tidal station with regard to the A.P. derived from the marks in the 5 sluices, amounting to $165 - 80 = 85$ mm. between 1700 and 1860.

The method by which the height of the water in the tidal station was obtained and the possible causes of the subsidence of the zero on the rod added to the invariability of the 5 grooves in the sluices and hence of a fairly large part of the ground of Amsterdam with regard to the sea, render the idea very probable that this subsidence has a pure local character and that we are not entitled to derive any results with regard to the subsidence of a larger part of the ground of Amsterdam.

It has often been asked what the Amsterdam zero represents. Our colleague Dr. VAN DIESEN has devoted to this subject an interesting study in which he has gathered from old documents everything which may help us to find how this zero has been established. With certainty nothing can be derived from it. But the observations show: 1 that in 1700 the A.P. was 165 mm. \pm 12 mm. above the mean sea level in the Y, 2 that the height of the mean high water was $\frac{318}{2} = 159$ mm. \pm 1 mm. above the same mean sea level, and we conclude thence that both in 1700 and 1860 the A.P. within the limits of the errors of observation agreed with the mean high water in the Y.

Astronomy. — “*On the masses and elements of Jupiter’s satellites, and the mass of the system* (continued), by Dr. W. DE SITTER. (Communicated by Prof. J. C. KAPTEYN).

III. *The great inequalities.*

The values of these, derived from the heliometer-observations of 1891, 1901 and 1902, have been collected in Table III, together

TABLE III. GREAT INEQUALITIES.

Authority	x_1	x_2	x_3
1891	$0^{\circ}509 \pm 0^{\circ}018$	$1^{\circ}021 \pm 0^{\circ}013$	$0^{\circ}059 \pm 0^{\circ}007$
1901	$0.481 \pm .47$	$1.089 \pm .30$	$0.049 \pm .20$
1902	$0.372 \pm .34$	$1.171 \pm .19$	$0.034 \pm .12$
DAMOISEAU	0.455	1.074	0.073
SOUILLART’S theory	0.432	1.026	0.063
Masses (C)	$0.430 \pm .020$	$0.988 \pm .017$	$0.064 \pm .003$

with their probable errors. The photographic determination of 1902 has been rejected for the reason which has already been explained.

From these values of x_i have been derived the equations of condition, which will be given below.

The arguments of these inequalities are $l_i + v$, where

$$v = l_2 - 2l_3 = l_1 - 2l_2 + 180^\circ.$$

Their periods are thus nearly the same as those of the equations of the centre, and in a short series of observations, such as those used here, the great inequalities are not well separated from the equations of the centre. This is the reason of the bad agreement of the results from the three series of observations.

In the eclipses the period of the great inequalities is the same for the three satellites, viz: 438 days¹). The periods of the equations of the centre in the eclipses have between 10 and 19 times this length, and the two classes of unknowns are thus well separable by eclipse observations. Here however, there arises a new complication, which did not exist in the case of extra-eclipse observations. The periods of the inequalities of group II, which are between 406 and 486 days, are nearly the same as the period of the great inequalities, and therefore the reliability of the determination of x_i from eclipse observations will depend in a large measure on the accuracy of our knowledge of the inequalities of group II. Thus *e. g.* with the masses (C) the coefficient of the inequality in the longitude of satellite II, which has a period of 463 days, is 0.038. This inequality is entirely neglected by DAMOISEAU (being proportional to e_2), and it is probable that his value of x_2 — which, according to the introduction to his tables, was derived directly from the observations — will be more or less affected by this circumstance. The same thing is true in a somewhat lesser degree of the corresponding terms in the longitudes of I and III.

The uncertainty which still reigns supreme with regard to the values of the great inequalities, is disappointing. We may hope that the reduction of the photometric eclipse observations of the Harvard observatory will contribute to diminishing this uncertainty.

IV. *The Libration.*

The mean longitudes l_1, l_2, l_3 , have been derived from the observations of 1891 (GILL, heliometer), 1892—93, 1893—94, 1894—95, 1895—96, 1897, 1898 (Helsingfors and Pulkowa, plates), 1901, 1902

¹) See LAPLACE. *Mécanique Céleste*, Tome IV, Livre VIII, Chapitre II.

(Cookson, heliometer) and 1904 (Cape, plates). The reduction has been carried out in Gron. Publ. 17. The masses (A) are the result of this discussion. The period of the libration being independent of α' , it is the same for the masses (B) as for (A). Also the transition from (B) to (C) does not affect this period. It is thus only necessary to investigate in how far the change from (A) to (C) affects the inequalities of group II, and what is the effect of this on the libration. This effect was found to be so small that a new determination of the libration appeared superfluous. The finally adopted libration is thus the same as in Gron. Publ. 17, viz :

$$\vartheta = 0^{\circ}.158 \sin \frac{T-1895.09}{7.00},$$

where the time T is expressed in years.

The probable error of the period corresponding to the adopted probable errors of the masses (C) is ± 0.13 .

The corrections to the mean longitudes on 1900 Jan. 0.0 also have been adopted unaltered from Gron. Publ. 17.

Table IV contains the observed corrections to the mean longitudes, with their probable errors as derived directly from the observations, and the residuals remaining after substitution of the final values of the inequalities of group II and the libration. The last two columns contain the p.e. of the quantity $\Delta l_1 - 3 \Delta l_2 + 2 \Delta l_3$, and the residuals for this same quantity.

In determining the libration from extra-eclipse observations we find the mean longitudes for epochs, which approximately co-incide with the epoch of opposition, and which therefore are on the average separated by intervals of 400 days. This interval differs but little from the periods of the inequalities of group II. These latter thus present themselves as inequalities with apparent periods between 6 and 8 years, and are therefore not well separable from the libration. In the eclipses this difficulty does not exist.

The method of successive approximations, which has been used in Gron. Publ. 17, to derive from the observations the most probable values of the libration and of the inequalities of group II, need not be explained here. It must suffice to refer the reader to that publication (see also these Proceedings, June 1907). The residuals of Table IV are practically the same as those found in Gron. Publ. 17, and they also need not be considered in detail here. Those of the satellites I and III are not very satisfactory, as has been pointed out there. On this point also the results derived from extra-

TABLE IV. MEAN LONGITUDES AND LIBRATION.

Series	Δl_1			Δl_2			Δl_3				
	Observed correction	p. e.	Residual	Observed correction	p. e.	Residual	Observed correction	p. e.	Residual		
1891	+ 0.100	\pm .006	- .034	+ 0.065	\pm .003	- .007	- 0.031	\pm .002	- .013	\pm .012	- .039
1892-3	+ .073	\pm 8	- .22	+ .051	\pm 5	- 17	- .023	\pm 3	- 4	\pm 14	+ 21
193-4	+ .128	\pm 14	+ 31	+ .019	\pm 9	- 26	- .034	\pm 5	- 16	\pm 34	+ 76
194-5	+ .131	\pm 11	+ 11	+ .012	\pm 6	+ 9	- .029	\pm 3	- 17	\pm 23	- 49
195-6	+ .152	\pm 6	- 8	- .026	\pm 4	0	- .004	\pm 2	+ 13	\pm 44	+ 18
197	+ .112	\pm 41	- 64	+ .019	\pm 9	+ 6	- .014	\pm 4	+ 9	\pm 30	- 59
198	+ .163	\pm 10	+ 2	+ .120	\pm 5	+ 21	- .002	\pm 3	+ 26	\pm 19	- 10
1901	+ .136	\pm 9	+ 23	+ .020	\pm 6	- 1	- .037	\pm 4	- 26	\pm 23	- 24
1902	+ .134	\pm 7	- 9	- .025	\pm 4	+ 2	- .023	\pm 3	- 7	\pm 17	- 29
1904	+ .231	\pm 42	+ 64	+ .063	\pm 7	+ 45	- .006	\pm 4	+ 26	\pm 27	+ 71

eclipse observations need confirmation from observations of eclipses.¹⁾

V. *Mean longitudes and mean motions.*

The corrections to the mean longitudes on 1900 Jan. 0.0 of the three inner satellites have been determined together with the libration, and the residuals have already been given in Table IV. For the fourth satellite the adopted correction is $-0^{\circ}.030$, and the residuals are given in Table V.

TABLE V Δl_4 .

Epoch	Observed correction	p. e.	Residual
1891	$-0^{\circ}.0248$	$\pm 0^{\circ}.0010$	$+0^{\circ}.0035$
1901	$-0^{\circ}.0361$	± 48	-58
1902	$-0^{\circ}.0342$	± 46	-37

If the corrections are added to the values adopted in computing the tabular places, and then referred to the first point of Aries by adding the adopted longitude of the point O , we find for 1900 Jan. 0, mean Greenwich noon, the values which are given below, sub I.

In the introduction to his tables DAMOISEAU states the mean longitudes for 1750 Jan. 0.5, mean time of Paris. If we consider these as being derived directly from the observations, they require a small correction, since DAMOISEAU has used the value $493^{\text{s}}.2$ of the light-time, while in the reduction of the modern observations the value $498^{\text{s}}.46$ was adopted. If DAMOISEAU had adopted this latter value, he would have found the same longitudes for an epoch which is $5^{\text{s}}.26 \times \Delta$ earlier, Δ being the mean distance of Jupiter. The observed mean longitudes, in order to correspond correctly to the tabular epoch, therefore require the correction²⁾:

$$: \frac{5.26}{86400} \cdot \Delta \cdot n_i = + 0.000317 n_i$$

¹⁾ It has also been pointed out in Gron. Publ. 17 that the series of extra-eclipse observations from which the libration was derived, not being made for this special purpose, does not in every respect fulfil the conditions necessary for a good determination of the libration.

²⁾ In Gron. Publ. 17 I assumed, on the authority of COOKSON, Cape XII. 3, page 56, that MARTH's longitudes for 1750.0 were identical with DAMOISEAU's. This, however, they are not, MARTH having applied the correction for the change in the adopted constant of aberration with the wrong sign. This was pointed out to me by Mr. BANACHEWICZ.

Applying this correction, and carrying the longitudes forward to 1900 Jan. 0.0, Greenwich M. T., we find the values II below.

Mean longitudes for 1900 Jan. 0.0.

<i>I (modern)</i>	<i>II (DAMOISEAU)</i>
$l_1 = 142^\circ.604 \pm 0^\circ.010$	$142^\circ.645 \pm 0^\circ.004$
$l_2 = 99.534 \pm .007$	$99.569 \pm .006$
$l_3 = 167.999 \pm .007$	$168.028 \pm .008$
$l_4 = 234.372 \pm .002^s$	$234.360 \pm .010$

The estimated probable errors for DAMOISEAU do *not* contain the p. e. of the mean motions used for carrying the longitudes forward from 1750 to 1900. The uncertainty of DAMOISEAU'S mean motions has been estimated by the late Prof. OUDEMANS in these Proceedings (October 1906). He finds for the four mean motions, in units of the eighth decimal place:

$$\pm 73 \qquad \pm 55 \qquad \pm 37 \qquad \pm 24$$

Comparing the values I and II we find the following corrections to DAMOISEAU'S mean motions:

$$\begin{aligned} \delta n_1 &= -0.0000\ 0075 \pm 0.0000\ 0020 \\ \delta n_2 &= -0.0000\ 0064 \pm 16^s \\ \delta n_3 &= -0.0000\ 0053 \pm 20 \\ \delta n_4 &= +0.0000\ 0022 \pm 18 \end{aligned}$$

It is noticeable that these corrections are very nearly of the magnitude of the uncertainties estimated by OUDEMANS. If these corrections are applied, the resulting values do not satisfy the condition

$$n_1 - 3n_2 + 2n_3 = 0.$$

If, however, we apply the further corrections

$$\delta n_1 = -2 \qquad \delta n_2 = +3 \qquad \delta n_3 = -3$$

to the eighth decimal place, then the condition is rigorously satisfied. The mean motions thus derived are those finally adopted. They are

$$\begin{aligned} n_1 &= 203^\circ.4889\ 9261 & n_3 &= 50^\circ.3176\ 4587 \\ n_2 &= 101.3747\ 6145 & n_4 &= 21.5711\ 0965 \end{aligned}$$

These are the mean motions relatively to the point Aries. If the sidereal mean motions are required, they must be diminished by $0^\circ.0000\ 3822$.

VI. The mass of the system.

The determination of the mass of the system of Jupiter by NEWCOMB¹⁾,

¹⁾ Astronomical papers of the American Ephemeris, Vol. 5, Part. 5.

which has now become a classic in astronomy, was based on observations of satellites, on perturbations in the motion of comets, and of the planets Themis, Polyhymnia and Saturn. It seems to me advisable to retain of these only the determinations from the three planets. Of the older observations of the satellites the uncertainty of the scale-value (which is increased threefold in the mass of the planet) is such that their weight, compared with the modern observations, and with the determinations from the perturbations of planets, is absolutely negligible. NEWCOMB has also, for this same reason, assigned a very small weight to these observations of the satellites.

The use of observations of comets seems to me very dangerous. It is very uncertain, if not improbable, that the observed centre of light should retain the same relative position with respect to the centre of gravity throughout one apparition of the comet, and *a fortiori* in different apparitions. NEWCOMB also points out that the results based on observations of comets are unreliable for this reason. Nevertheless he assigns a large weight to the determination by VON HAERDTL from WINNECKE'S comet, on the ground that the normal places of this comet are so well represented by VON HAERDTL'S results. It appears to me that this good representation does not diminish the stringency of the argument stated above, and in my opinion it is advisable to reject also this determination, together with those from other comets.

There remain the determinations from the three planets, which I adopt with the same weights assigned to them by NEWCOMB, and the modern observations of satellites, which were only made, or at least reduced, after NEWCOMB'S discussion was published. For these latter the scale-value is determined in an entirely satisfactory manner by simultaneous observations of standard stars. Nevertheless I have assigned to these observations a relatively smaller weight than to the determinations from the planets, to allow for the possibility of small systematic errors in transferring the scale-value from the distance of the standard stars to the mutual distances of the satellites.

In my reduction of GILL'S observations of 1891 I have included in the probable error of Δ^2 the effect of the uncertainty of the standard stars used for the determination of the scale-value. The probable errors stated by COOKSON do not include this uncertainty. The distances of the stars used by COOKSON are not so accurately known as of the stars used in 1891. I have for these reasons assigned a smaller weight to COOKSON'S two determinations than to GILL'S. The several determinations and their probable errors and adopted weights are given in Table VI.

TABLE VI. RECIPROCAL OF THE MASS OF THE SYSTEM.

Authority	Observed values	Weight	Residual
KRÜGER, perturbations of Themis	1047.54 ± 0.19	5	+ 0.14
HILL, „ „ Saturn	$.38 \pm .12$	7	- .02
NEWCOMB, „ „ Polyhymnia	$.34 \pm .06$	20	- .06
GILL—DE SITTER, Satellites, 1891	$.50 \pm .06$	10	+ .10
COOKSON, „ 1901	$.46 \pm .09$	4	+ .06
COOKSON, „ 1902	$.25 \pm .06$	6	- .15

The mean by weights is $1047.394 \pm .026$. The simple mean is 1047.412 . The mean of the determinations from the planets alone is 1047.380 , and the mean of the determinations from the satellites is 1047.417 . The value which I propose to adopt is

$$\mu = 1047.40 \pm 0.03.$$

The probable error was derived from the residuals. The distribution of these residuals, each compared with its own probable error as stated by the observers, is in excellent agreement with the theoretical distribution according to the law of errors. The adopted p.e. can therefore be considered to be a trustworthy measure of the real accuracy.

I may be allowed to state as my conviction that it will not be possible in the near future materially to improve the value here adopted. In order to attain from observations of satellites a smaller probable error than ± 0.03 , or $1/35000$, the scale-value must be known within less than $1/130000$. It thus appears useless to attempt a new determination of the mass from observations of the satellites, until we are in the possession of means as well of fixing the distance of a pair of standard-stars with this accuracy, as of transferring the scale-value determined therefrom to other (smaller) distances without the possibility of systematic errors. Investigations of modern heliometers point to the conclusion that the transferring of the scale-value from a distance of, say, $7000''$ to one of $700''$ is still subject to uncertainties, which may reach an amount equivalent to an error of $0''.1$ in the larger distance, and which therefore may amount to $1/70000$ of the scale-value. On the other hand it seems a high demand on our present observational means to fix a distance of about 2° of two stars with an uncertainty smaller than $0''.07 = 0.005^1$.

¹) The accuracy of the distance of the standard stars used in 1891 was $\pm 1/60000$. (See my dissertation, page 8).

NEWCOMB has already pointed out that oppositions of Polyhymnia, as favourable as the one used in his work, will not recur till the end of the twentieth century, and a similar statement is true for Themis. HILL has pointed out ¹⁾ that Jupiter produces in the motion of certain minor planets (those of the Hecuba type) perturbations of long periods, which amount to several degrees. Thus e. g. Freia is subject to a perturbation, whose geocentric amplitude is 12^o.7 with a period of 121 years. The length of the period makes it impossible to derive an improved value of the mass by this method in the near future.

Derivation of the final masses.

The right-hand-members of the equations of condition, which have served to determine the corrections to the values (*B*) of the masses, have been derived, as explained above under I to IV, from:

I. the motions of the nodes θ_2 and θ_3 (those of θ_1 and θ_4 I leave out of consideration, as having too small weights),

II. the motion of the perijove $\bar{\omega}_4$,

III. the great inequalities x_1, x_2, x_3 ,

IV. the period of the libration.

The equations are:

I.

$$\begin{aligned} &-.0266 \delta v' - .0030 \delta v_1 - .0001 \delta v_2 - .0040 \delta v_3 - .0002 v_4 = -.00010 \pm .00008 \\ &-.0051 \quad -.0003 \quad -.0007 \quad 0 \quad -.0007 = -.00041 \pm 15 \end{aligned}$$

II.

$$+.00077 \delta v' + .00004 \delta v_1 + .00007 \delta v_2 + .00082 \delta v_3 - .00005 v_4 = -.000036 \pm .000020$$

These three equations depend in part on the values of the elements in 1750, which were determined from eclipse-observations. It has already been pointed out above that practically the same results would be found from extra-eclipse observations alone.

III.

	1891	1901	1902	Adopted
$-.003 \delta v_1 + .403 \delta v_2 - .014 \delta v_3$	= +.080	+ .051	-.058	+.020 \pm .040
+ .195	-.008	+.816	= +.019	+.087
-.001	+.060	-.006	= -.004	-.014
			-.023	-.009 \pm 10

The probable errors of the separate determinations have been given in Table III. The p.e. of the adopted values were estimated according to the agreement of the separate values.

IV.

$$+2.40^s \delta v_1 + 0.24^s \delta v_2 + 1.35^s \delta v_3 = 0.000 \pm 0.18$$

¹⁾ Collected works I, page 105.

If now we reduce all these equations to the same weight, so that the p. e. of their right-hand members becomes ± 0.10 , we find, if also the signs of I are reversed:

						<i>Res.</i>	<i>SOULL.</i>	
I.	}	$+33.3\delta x'$	$+3.8\delta r_1$	$+0.1\delta r_2$	$+5.0\delta r_3$	$+0.2r_4$	$= +0.12^5$	$+0.02 - .78$
		$+ 3.4$	$+0.2$	$+0.4^5$	0	$+0.5$	$= +0.27$	$+0.25 +.16$
II.	}	$+ 3.8^5$	$+0.2$	$+0.3^5$	$+4.1$	-0.2^5	$= -0.18$	$-0.12 - .82$
			0	$+0.6$	0		$= +0.05$	$+0.05 +.05^5$
III.	}		$+0.5$	0	$+2.0^5$		$= +0.12$	$+0.16 +.06^5$
			0	$+0.1^5$	0		$= -0.09$	$-0.09 - .09$
IV.		$+1.4$	$+0.1^5$	$+0.8$		$= 0.00$	$0.00 +. 7$	

The finally adopted corrections are:

$$\delta x' = + 0.005 \pm .0075$$

$$\delta r_1 = + 0.010 \pm .030$$

$$\delta r_3 = - 0.020 \pm .020$$

$$\delta r_2 = 0 \pm .050$$

$$r_4 = 0 \pm 0.25$$

The corresponding values of the masses are:

$$Jb^2 = 0.0214 180 \pm .0001543 \quad (b = 1 \text{ for } d = 39''.0)$$

$$= 0.0000 0000 518169 \pm 3975 \quad (\text{astronomical units})$$

$$m_1 = 0.0000 260 \pm .0000 012$$

$$m_2 = 0.0000 231 \pm \quad 11$$

$$m_3 = 0 0000 804 \pm \quad 16$$

$$m_4 = 0.' 000 424 751 \pm .0000 106$$

}

(C)

Substituting these corrections, there remain the residuals stated above. If SOULLART's masses are substituted there remain the residuals given in the last column.

The equations II and III are contradicting each other: II demands a negative value δr_3 , III a positive value. On account of the bad agreement of the different determinations of x_2 I have assigned a very small weight to the equation III. It is to be noticed that the large negative correction δr_3 could have been partly avoided by assuming a large positive value of r_4 , e. g. $r_4 = +0.5$. Even then, however, it would not be possible to bring about a satisfactory agreement of II and III without spoiling the representation of I and IV.

The probable errors stated for the corrections $\delta x'$ and δr_i as well as the values of these corrections themselves, depend largely on judg-

ment¹). In estimating the probable errors I have taken into account as accurately as I could the imperfections as well of the theory on which the left-hand members of the equations of condition depend as of the observations from which the right-hand members are derived. It has been my aim to estimate true *probable* errors, i. e. the masses (C) are those which with our present knowledge of the system I consider the most probable, and I consider it equally probable that the deviation of the values (C) from the truth is smaller than the stated p. e., as that it exceeds this quantity.

The above contains all that can be derived from modern extra-eclipse observations. The resulting values of the inclinations and nodes, and of the mass of the system, i. e. the groups A and C of unknowns, must be considered as final, so far as the observational data at present available go. The results for the other unknowns (those of group B) cannot be accepted as final until they are confirmed by the reduction of the photometric eclipse observations of the Harvard observatory. With regard to the inclinations and nodes, I have already pointed out in *Cape XII. 3* (page 121) that a new determination about the year 1920 is desirable. For the determination of m_4 it will be necessary, as was pointed out by me in my dissertation, p. 82 and 85, to supplement the modern observations by a determination of h_3 and k_3 about 1790 from a re-reduction of old eclipses. Of these an amply sufficient number exists. Between the years 1772 and 1799 I have found in the literature of the epoch records of 63 eclipses of which the immersion and emersion have been observed by the same person; and about one third of these have been observed by more than one observer.

In order to derive entirely satisfactory results it will also be necessary to revise SOUILLART's analytical theory, as pointed out by me in *Gron. Publ. 17*, page 118.

The masses and elements derived in the above, though not to be considered as final, still doubtlessly are much nearer to the truth than those used in SOUILLART's theory. It therefore seemed desirable to introduce them into the expressions for the latitudes, longitudes and radii-vectores as given by that theory. To take account of the uncertainties of the masses I give the coefficients as functions of the small quantities ϱ and λ_i , which are defined by

¹) "The probable error arising from the uncertainty of such judgments must be included among the possible unavoidable sources of error." NEWCOMB, *Astronomical Papers of the American Ephemeris*, Vol. 5, Part 4, page 398.

[Note added in the English translation].

$$Jb^2 = (Jb^2)_0 (1 + \varrho)$$

$$m_i = (m_i)_0 (1 + \lambda_i),$$

where $(Jb^2)_0$ and $(m_i)_0$ represent the values (C). The squares and products of ϱ , λ_1 , λ_2 and λ_3 will be neglected. These developments are based entirely on those of Gron. Publ. 17, and what was there said about their accuracy and reliability also applies here.

The semi-major-axes corresponding to the adopted mean motions and the adopted mass of the system have been computed by the formula ¹⁾:

$$n_i^2 a_i^3 = f \frac{1 + m_i}{\mathcal{M}(1 + \Sigma m_i)} \left(1 + \frac{Jb^2}{a_i^2} \right).$$

Their logarithms are

$$\log a_1 = 7.450\ 1443 + 0.000\ 101\ \varrho$$

$$\log a_2 = 7.651\ 8277 + .000\ 040\ \varrho$$

$$\log a_3 = 7.854\ 6197 + .000\ 016\ \varrho$$

$$\log a_4 = 8.099\ 8338 + .000\ 005\ \varrho$$

The values of the coefficients τ_{ij} , which occur in the expressions for the equations of the centre, are

$$\tau_{21} = +0.0280 - 031\ \varrho + .027\ \lambda_1 - .002\ \lambda_2 + .055\ \lambda_3$$

$$\tau_{31} = -0.0053 - .003\ \varrho - .005\ \lambda_1 - .004\ \lambda_2 - .001\ \lambda_3$$

$$\tau_{41} = 0.0000$$

$$\tau_{12} = -0.0320 + .058\ \varrho + .027\ \lambda_1 - .011\ \lambda_2 - .061\ \lambda_3$$

$$\tau_{32} = -0.0447 + .022\ \varrho + .003\ \lambda_1 - .042\ \lambda_2 + .006\ \lambda_3$$

$$\tau_{42} = 0.0000$$

$$\tau_{13} = +0.0171 - .013\ \varrho + .002\ \lambda_1 + .014\ \lambda_2 + .015\ \lambda_3$$

$$\tau_{23} = +0.1619 - .098\ \varrho - .005\ \lambda_1 + .019\ \lambda_2 + .116\ \lambda_3 + .0019\ \lambda_4$$

$$\tau_{43} = -0.1173 + .112\ \varrho + .006\ \lambda_1 + .024\ \lambda_2 - .142\ \lambda_3 + .0163\ \lambda_4$$

$$\tau_{14} = +0.0016 - .002\ \varrho \qquad \qquad \qquad +.001\ \lambda_2 + .001\ \lambda_3 + .0014\ \lambda_4$$

$$\tau_{24} = +0.0139 - .018\ \varrho - .001\ \lambda_1 - .001\ \lambda_2 + .010\ \lambda_3 + .0112\ \lambda_4$$

$$\tau_{34} = +0.0828 - .072\ \varrho - .001\ \lambda_1 - .017\ \lambda_2 + .009\ \lambda_3 + .0726\ \lambda_4$$

The daily motions of the own perijoves (referred to the first point of Aries) are :

$$(\tilde{\omega}_1) + 0.14703 \quad +.1295\ \varrho \quad +.0070\ \lambda_1 \quad +.0166\ \lambda_2 \quad +.0007\ \lambda_3 \quad +.0001\ \lambda_4$$

$$(\tilde{\omega}_2) + 0.038955 \quad +.02590 \quad - .00371 \quad +.00406 \quad +.01974 \quad +.00019$$

$$(\tilde{\omega}_3) + 0.007032 \quad +.00530 \quad +.00024 \quad + .00100 \quad \qquad \qquad +.00066$$

$$(\tilde{\omega}_4) + 0.001896 \quad +.00075 \quad +.00003 \quad +.00007 \quad +.00082 \quad - .00005$$

¹⁾ It will be seen that I adopt here LAPLACE'S definition of the mean distances. All other constant terms of the radius-vector will be included in $\rho_i = r_i/a_i$. These ratios ρ_i must not be confounded with the small quantity ρ representing a possible correction to the adopted value of Jb^2 .

The great inequalities are :

$$\begin{aligned} x_1 &= 0.4303 - .0024 \lambda_1 + .4228 \lambda_2 - .0145 \lambda_3 \\ x_2 &= 0.9875 + .1273 \lambda_1 - .0090 \lambda_2 + .8188 \lambda_3 \\ x_3 &= 0.0636 - .0010 \lambda_1 + .0629 \lambda_2 - .0063 \lambda_3 \end{aligned}$$

The coefficients of the inequalities of group II are :

$$\begin{aligned} x_{11} &= \{-2.49 \quad -.04 \quad \varrho + .04 \quad \lambda_1 - 2.46 \lambda_2 + .17 \lambda_3\} e_1 \\ x_{12} &= \{+0.98 \quad -.19 \quad \varrho - .13 \quad \lambda_1 + .97 \lambda_2 - .10 \lambda_3\} e_2 \\ x_{13} &= \{+0.083 \quad -.03 \quad \varrho - .02 \quad \lambda_1 + .02 \lambda_2\} e_3 \\ x_{14} &= \{+0.0062 \quad -.003 \quad \varrho - .002 \lambda_1 \lambda_2 \quad .005 \lambda_3 + .002 \lambda_4\} e_4 \\ x_{21} &= \{+2.26 \quad -.05 \quad \varrho + 2.20 \lambda_1 - .03 \lambda_2 + .03 \lambda_3\} e_1 \\ x_{22} &= \{+2.19 \quad +.16 \quad \varrho - .74 \lambda_1 + .08 \lambda_2 + 2.93 \lambda_3\} e_2 \\ x_{23} &= \{-0.535 \quad -.27 \quad \varrho - .03 \lambda_1 + .15 \lambda_2 - .02 \lambda_3\} e_3 \\ x_{24} &= \{-0.0368 \quad +.005 \varrho - (.002 - .017 \lambda_1) \lambda_2 + .022 \lambda_3 + .045 \lambda_4 \lambda_2 \quad .046 \lambda_4\} e_4 \\ x_{31} &= \{-0.01 \quad -.01 \quad \lambda_1\} e_1 \\ x_{32} &= \{-0.67 \quad -.65 \quad \lambda_3\} e_2 \\ x_{33} &= \{+0.109 \quad +.07 \varrho - .01 \lambda_1 + .07 \lambda_2\} e_3 \\ x_{34} &= \{+0.0078 \quad +(.002 + .009 \lambda_1) \lambda_2 + .011 \lambda_3\} e_4 \end{aligned}$$

The quantities determining the libration are :

$$\begin{aligned} Q_1 &= + \{.003440 - .00022 \lambda_1 - .00050 \lambda_2 - .00142 \lambda_3\} (1 + \lambda_2) (1 + \lambda_3) \\ Q_2 &= - \{.005161 - .00021 \lambda_1 - .00045 \lambda_2 - .00135 \lambda_3\} (1 + \lambda_1) (1 + \lambda_3) \\ Q_3 &= + \{.000452 - .00002 \lambda_3\} (1 + \lambda_1) (1 + \lambda_2) \\ \beta^2 &= Q_1 - 3Q_2 + 2Q_3 \quad \psi = \beta (t - t_0) \\ \vartheta &= 0^\circ.158 \sin \psi \\ \vartheta_1 &= + 0.1735 \vartheta \quad \vartheta_2 = - 0.2603 \vartheta \quad \vartheta_3 = + 0.0228 \vartheta \end{aligned}$$

The position of the orbital planes of the satellites is in SOUILLART'S theory referred to the orbit of Jupiter, of which the inclination and node¹⁾ referred to the ecliptic and mean equinox are (according to LEVERRIER, but with NEWCOMB'S precession):

$$\begin{aligned} \varrho &= 1^\circ 18' 31''.1 - 0''.2051 T \\ \vartheta &= 99 \quad 26 \quad 36 \quad + 36.396 T, \end{aligned}$$

where T is the time counted in tropical years from 1900 Jan. 0.0 Greenwich M. T.

It is preferable, however, to refer the latitudes of the satellites to the mean equator of the planet. The inclination and node of this mean equator referred to the orbit (the node being counted "in the orbit") are:

$$\begin{aligned} \omega &= 3 \quad 6' 55''.1 - 0''.0243 T \\ \theta &= 315 \quad 48 \quad 0 \quad + 50.158 T \end{aligned}$$

The inclination and node of the mean equator referred to the ecliptic are thus :

1) Unless otherwise stated, node stands for ascending node.

$$= 2^{\circ} 12' 8'' 7 + 0''.4231 T$$

$$\mathcal{N} = 336 24 24 + 48.916 T$$

The inclination and node of Jupiter's orbit referred to the mean equator are therefore (the node being counted "in the equator"):

$$\omega = 3^{\circ} 6' 55''.1 + 0''.0243 T$$

$$\theta' = 135 46 44 + 50.155 T$$

The position of the orbital planes of the satellites — excluding periodic, but including secular perturbations — referred to the mean equator, are given by the formulas:

$$i_i \sin(\theta' - \mathcal{N}i) = p_i = \sum_j \sigma_{ij} \gamma_j \sin \Gamma_j$$

$$i_i \cos(\theta' - \mathcal{N}i) = q_i = \sum_j \sigma_{ij} \gamma_j \cos \Gamma_j + (1 - \mu_i) \omega$$

Referred to the orbit of Jupiter they are¹⁾

$$I_i \sin N_i = \sum_j \sigma_{ij} \gamma_j \sin \theta_j + \mu_i \omega \sin \theta$$

$$I_i \cos N_i = \sum_j \sigma_{ij} \gamma_j \cos \theta_j + \mu_i \omega \cos \theta$$

where we have²⁾

$$\Gamma_i = 180^{\circ} + \theta - \theta_i$$

If the periodic perturbations are represented by δp_i , δq_i , δs_i , we have for the latitude of the satellite referred to the mean equator

$$\beta_i = (q_i + \delta q_i) \sin(v_i - \theta') + (p_i + \delta p_i) \cos(v_i - \theta')$$

and referred to the orbital plane of Jupiter

$$s_i = I_i \sin(v_i - N_i) + \delta s_i.$$

Here v_i is the true orbit-longitude of the satellite. In both formulas quantities of the third order in the inclinations are neglected. The neglected terms in β_i are thus of the order of magnitude of $0^{\circ}.00002$ and in s_i of the order of $0^{\circ}.01$.

The values of the coefficients σ_{ij} and μ_i are:

$$\sigma_{31} = -0.019 + .012 \varrho - .019 \lambda_1$$

$$\sigma_{31} = -0.001 + .001 \varrho - .001 \lambda_1$$

$$\sigma_{41} = 0.000$$

$$\sigma_{12} = +0.0203 - .020 \varrho + .020 \lambda_2$$

$$\sigma_{32} = -0.0347 + .028 \varrho + .002 \lambda_1 - .035 \lambda_2 + .005 \lambda_3 - .0005 \lambda_4$$

$$\sigma_{42} = -0.0010 - .001 \varrho - .001 \lambda_2 + .001 \lambda_3$$

1) Rigorously these formulas are true with reference to the *fixed* orbit of Jupiter, and a correction must be applied to derive the latitude referred to the moving orbit. It is, however, sufficiently accurate to use the same formulas for the latitude referred to the moving orbit, provided we take for ω and \mathcal{N} the inclination and node of the mean equator referred to this same moving orbit (as was done here). For the motion of the node \mathcal{N} referred to the moving orbit I adopted $-0''.0979$ instead of $-0''.0710$ (SOUILLART II page 166). This is the value which results if SOUILLART's final value of b_4 is used instead of the approximate value used by SOUILLART himself.

2) The meaning of Γ_i is thus here slightly different from what it was in the subordinate investigation I.

$$\begin{aligned}
\sigma_{1,3} &= + 0.0056 - .013 \varrho + .003 \lambda_2 + .010 \lambda_3 - .0001 \lambda_4 \\
\sigma_{2,3} &= + 0.1488 + .132 \varrho - .011 \lambda_1 + .005 \lambda_2 + .125 \lambda_3 + .0026 \lambda_4 \\
\sigma_{4,3} &= - 0.1772 + .176 \varrho + .008 \lambda_1 + .028 \lambda_2 - .211 \lambda_3 + .0282 \lambda_4 \\
\sigma_{1,4} &= - 0.0018 - .003 \varrho + .001 \lambda_3 + .0018 \lambda_4 \\
\sigma_{2,4} &= + 0.0183 - .034 \varrho - .002 \lambda_1 - .002 \lambda_2 + .017 \lambda_3 + .0207 \lambda_4 \\
\sigma_{3,4} &= + 0.1203 - .110 \varrho - .005 \lambda_1 - .016 \lambda_2 + .021 \lambda_3 + .1064 \lambda_4 \\
\mu_1 &= 0.99944 + .0009 \varrho - .0002 \lambda_3 - .0002 \lambda_4 \\
\mu_2 &= 0.99428 + .0095 \varrho + .0002 \lambda_1 + .0001 \lambda_2 - .0022 \lambda_3 - .0023 \lambda_4 \\
\mu_3 &= 0.97271 + .0294 \varrho + .0012 \lambda_1 + .0040 \lambda_2 - .0010 \lambda_3 - .0088 \lambda_4 \\
\mu_4 &= 0.86245 + .0555 \varrho + .0018 \lambda_1 + .0045 \lambda_2 + .0503 \lambda_3 - .0056 \lambda_4
\end{aligned}$$

The daily motions of the nodes θ_i are:

$$\begin{aligned}
(\theta_1) &= -0.13614 - .1327 \varrho - .0023 \lambda_2 - .0010 \lambda_3 - .00008 \lambda_4 \\
(\theta_2) &= -0.032335 - .02602 \varrho - .00198 \lambda_1 - .00013 \lambda_2 - .00399 \lambda_3 - .000191 \lambda_4 \\
(\theta_3) &= -0.006854 - .00493 \varrho - .00021 \lambda_1 - .00071 \lambda_2 - .00004 \lambda_3 - .000695 \lambda_4 \\
(\theta_4) &= -0.031772 - .00077 \varrho - .00003 \lambda_1 - .00007 \lambda_2 - .00075 \lambda_3 + .000098 \lambda_4
\end{aligned}$$

and for the angles I_i we have:

$$\frac{dI_i}{dt} = 0^\circ.000038 - \frac{d\theta_i}{dt}.$$

The quantities p_i are thus:

$$\begin{aligned}
p_1 &= +0.02720 \sin I_1 + 0.00951 \sin I_2 + 0.00103 \sin I_3 - 0.00046 \sin I_4 \\
p_2 &= - 0.0052 \quad + .46830 \quad + .02734 \quad + .00464 \\
p_3 &= - .00003 \quad - .01625 \quad + .18390 \quad + .03051 \\
p_4 &= .00000 \quad - 0.0047 \quad - .03259 \quad + .25360
\end{aligned}$$

In q_i we have the same coefficients, and again in $I_i \sin N_i$ and $I_i \cos N_i$. The constant terms $(1-\mu_i)\omega$ of q_i and the coefficients of $\sin \theta$ and $\cos \theta$ in $I_i \sin N_i$ and $I_i \cos N_i$ respectively are:

$$\begin{aligned}
(1-\mu_1)\omega &= 0.00174 & \mu_1 \omega &= 3.1136 \\
(1-\mu_2)\omega &= 0.01792 & \mu_2 \omega &= 3.0974 \\
(1-\mu_3)\omega &= 0.08502 & \mu_3 \omega &= 3.0303 \\
(1-\mu_4)\omega &= 0.42851 & \mu_4 \omega &= 2.6868
\end{aligned}$$

The position of the true equator referred to the mean equator is defined by its inclination ω_1 and node ψ_1 , which are determined by the formulas

$$\begin{aligned}
\omega_1 \sin(\theta' - \psi_1) &= \sum_j \sigma_{0j} \gamma_j \sin I_j \\
\omega_1 \cos(\theta' - \psi_1) &= \sum_j \sigma_{0j} \gamma_j \cos I_j.
\end{aligned}$$

The inclination Ω and node Ψ of the true equator referred to the orbit of Jupiter are then:

$$\begin{aligned}
\Omega \sin \Psi &= \sum_j \sigma_{0j} \gamma_j \sin \theta_j + \omega \sin \theta \\
\Omega \cos \Psi &= \sum_j \sigma_{0j} \gamma_j \cos \theta_j + \omega \cos \theta,
\end{aligned}$$

where we have:

$$\begin{array}{ll}
 \sigma_{01} = -0.00097 (1 + \lambda_1) & \sigma_{01} \gamma_1 = -0.00003 \\
 \sigma_{02} = -0.00094 (1 + \lambda_2) & \sigma_{02} \gamma_2 = -0.00044 \\
 \sigma_{03} = -0.00441 (1 + \lambda_3) & \sigma_{03} \gamma_3 = -0.00081 \\
 \sigma_{04} = -0.00363 (1 + \lambda_4) & \sigma_{04} \gamma_4 = -0.00092
 \end{array}$$

Before giving the expressions for the perturbations I will first state the values of the arguments. For brevity I put

$$\tau = l_2 - l_1 \quad v = l_2 - 2l_1 \quad \varphi_i = v + \tilde{\omega}_i$$

L = the mean longitude of Jupiter

M = „ „ anomaly „ „

$W = 5V - 2W' - 16^\circ 31'$ } in LEVERRIER'S notation.

$W_1 = W - 2V - 130$ }

$V = 2L - 2\theta' + 180^\circ$

$V' = 2L - \theta$.

The values of the arguments then are, if t is the time counted in days from 1900 Jan. 0, Mean Greenwich Noon (J. D. 2415020) :

$$l_1 = 142.604 + 203.48899261 t$$

$$l_2 = 99.534 + 101.37476145 t$$

$$l_3 = 167.999 + 50.31764587 t$$

$$l_4 = 234.372 + 21.57110965 t$$

$$\tau = 291^\circ.535 + 51^\circ.0571166 t \quad v = 123^\circ.5 + 0^\circ.73947 t$$

$$\psi = 252^\circ.4 + 0^\circ.14081 t$$

$$\tilde{\omega}_1 = 155.5 + 0.14703 t \quad \varphi_1 = 279^\circ.0 + 0.88650 t$$

$$\tilde{\omega}_2 = 62.7 + 0.03896 t \quad \varphi_2 = 186.2 + 0.77843 t$$

$$\tilde{\omega}_3 = 338.3 + 0.00703 t \quad \varphi_3 = 101.8 + 0.74650 t$$

$$\tilde{\omega}_4 = 283.15 + 0.001896 t \quad \varphi_4 = 46.7 + 0.74137 t$$

$$\Gamma_1 = 75.6 + 0.13618 t \quad \theta_1 = 60^\circ.2 - 0.13614 t$$

$$\Gamma_2 = 202.64 + 0.032373 t \quad \theta_2 = 293.16 - 0.032335 t$$

$$\Gamma_3 = 176.09 + 0.006892 t \quad \theta_3 = 319.71 - 0.006854 t$$

$$\Gamma_4 = 123.84 + 0.001810 t \quad \theta_4 = 11.96 - 0.001772 t$$

$$\theta = 315^\circ.800 + 0^\circ.0000381 t$$

$$\theta' = 135.779 + 0.0000381 t$$

$$L = 238^\circ.0 + 0^\circ.08313 t \quad M = 225^\circ.3 + 0.08308 t$$

$$W = 117.9 + 0.00112 t \quad W_1 = 64.2 + 0.01617 t$$

$$V = 24.5 + 0.16608 t \quad V' = 160.3 + 0.16612 t$$

The periodic perturbations in the latitudes are of the form:

$$\begin{array}{l}
 \delta p_i = x \sin \alpha \\
 \delta q_i = x \cos \alpha
 \end{array}
 \quad
 \begin{array}{l}
 \delta s_i = x \sin (v_i - \alpha - \theta')
 \end{array}$$

All coefficients being very small, we may in the arguments replace v_i by l_i , and neglect the difference of θ' and $180^\circ + \theta$. The coefficients and arguments are :

	<i>coefficient</i>	<i>argument</i> $\delta p_i, \delta q_i$	<i>argument</i> δs_i
Sat. I	$+ 0.00042$	$\Gamma_2 - 2v - 2\theta'$	$l_1 + 2v + \theta_2$
	$+ 0.00025$	V	$l_1 - V'$
Sat. II	$- 0.00099$	$\Gamma_2 - 2v - 2\theta'$	$l_2 + 2v + \theta_2$
	$+ 0.00010$	$V + \Gamma_2$	$l_2 + \theta_2 - 2L$
	$+ 0.00078$	V	$l_2 - V'$
Sat. III	$+ 0.00010$	$V + \Gamma_3$	$l_3 + \theta_3 - 2L$
	$+ 0.00177$	V	$l_3 - V'$
Sat. IV	$+ 0.00032$	$V + \Gamma_4$	$l_4 + \theta_4 - 2L$
	$+ 0.00380$	V	$l_4 - V'$

The expressions for the longitudes and radii-vectores are given below. The inequalities are arranged in three groups, according to the periods, as explained in the beginning of this paper. Inequalities which are smaller than $1''$ in longitude and 0.000005 in radius-vector have been neglected. The developments in powers of the small quantities ϱ and λ_i of the great inequalities (arguments 4τ , 2τ and τ for the satellites I, II, and III respectively), of the inequalities of group II and of the libration have already been given above, and only the values of the coefficients are repeated here. The more important of the smaller inequalities are here also given as functions of ϱ and λ_i . Where no development is given the coefficients were taken from SOUILLART'S theory, corrected for the adopted values of the eccentricities (and inclinations) but not for the masses. The multipliers of ϱ and λ_i are given in units of the last decimal place of the coefficients to which they belong.

The true orbit-longitudes are :

$$\begin{aligned} r_1 &= l_1 + 0.0276 \sin \psi + \delta v_1 \\ r_2 &= l_2 - 0.0411 \sin \psi + \delta v_2 \\ r_3 &= l_3 + 0.0036 \sin \psi + \delta v_2 \\ r_4 &= l_4 + \delta v_4 \end{aligned}$$

The radii-vectores are :

$$r_i = a_i \varrho_i$$

$$\varrho_1 = 1.000\ 0066 + \delta \varrho_1$$

$$\varrho_2 = 1.00\ 0549 + .000\ 014\ \lambda_1 + .000\ 084\ \lambda_2 + \delta \varrho_2$$

$$\varrho_3 = 1.000\ 0155 + .000\ 009\ \lambda_1 + .000\ 011\ \lambda_2 - .000\ 002\ \lambda_4 + \delta \varrho_3$$

$$\varrho_4 = 1.000\ 0755 + .000\ 008\ \lambda_1 + .000\ 008\ \lambda_2 + .000\ 034\ \lambda_3 + \delta \varrho_4$$

The inequalities δv_i and $\delta \varphi_i$ are:

Ia. Equations of the centre.

$$\delta v_i = a_{i1} \sin(l_i - \tilde{\omega}_1) + a_{i2} \sin(l_i - \tilde{\omega}_2) + a_{i3} \sin(l_i - \tilde{\omega}_3) + a_{i4} \sin(l_i - \tilde{\omega}_4)$$

$$a_{11} = +0.0062 \quad a_{12} = -0.0011 \quad a_{13} = +0.0030 \quad a_{14} = +0.0014$$

$$a_{21} = -0.0002 \quad a_{22} = +0.0344 \quad a_{23} = +0.0281 \quad a_{24} = +0.0118$$

$$a_{31} = 0.0000 \quad a_{32} = -0.0015 \quad a_{33} = +0.1736 \quad a_{34} = +0.0706$$

$$a_{41} = 0.0000 \quad a_{42} = 0.0000 \quad a_{43} = -0.0204 \quad a_{44} = +0.8528$$

$$\delta \varphi_i = a'_{i1} \cos(l_i - \tilde{\omega}_1) + a'_{i2} \cos(l_i - \tilde{\omega}_2) + a'_{i3} \cos(l_i - \tilde{\omega}_3) + a'_{i4} \cos(l_i - \tilde{\omega}_4)$$

$$a'_{11} = -0.000054 \quad a'_{12} = +0.000010 \quad a'_{13} = -0.000026 \quad a'_{14} = -0.000012$$

$$a'_{21} = +0.000002 \quad a'_{22} = -0.000300 \quad a'_{23} = -0.000245 \quad a'_{24} = -0.000103$$

$$a'_{31} = 0.000000 \quad a'_{32} = +0.000013 \quad a'_{33} = -0.001516 \quad a'_{34} = -0.000616$$

$$a'_{41} = 0.000000 \quad a'_{42} = 0.000000 \quad a'_{43} = +0.000178 \quad a'_{44} = -0.007445$$

The inequalities of the groups Ib and Ic are of the form:

$$\delta v_i = \kappa \sin a \quad \delta \varphi_i = \kappa' \cos a.$$

They are:

<i>Argument</i>	<i>coefficient</i> in δv_i	<i>coefficient</i> in $\delta \varphi_i$
<i>Satellite I.</i>		
2r	+0.0034 (1 + λ_2)	-0.000 017 (1 + λ_2)
3r	+0.0016 (1 + λ_3)	-0.000 011 (1 + λ_3)
4r	+0.4303	-0.003 755 (x_1 , see above)
8r	+0.0014 + 23 λ_2	-0.000 012 - 20 λ_2

<i>Satellite II.</i>		
r	-0.0123 (1 + λ_3)	+0.000 061 (1 + λ_3)
2r	+0.9875	-0.008 617 (x_2 , see above)
3r	+0.0052 (1 + λ_3)	-0.000 058 (1 + λ_3)
4r	+0.0051 + 1 λ_1 - 1 λ_2 + 109 λ_3	-0.000 034 + 8 λ_1 + 1 λ_2 - 83 λ_3
5r	+0.0004 (1 + λ_3)	-0.000 006 (1 + λ_3)
6r	+0.0005 + 3 λ_1 + 2 λ_2	-0.000 008 - 5 λ_1 - 2 λ_2
$l_3 - l_4$	-0.0006 (1 + λ_4)	+0.000 004 (1 + λ_4)
$2(l_2 - l_4)$	+0.0005 (1 + λ_4)	-0.000 006 (1 + λ_4)
r + φ_1	-0.0005	+0.000 002
2r + φ_2	-0.0003	+0.000 006
2r + φ_3	+0.0026	-0.000 021
2r + φ_4	+0.0010	-0.000 008
$l_1 - \tilde{\omega}_2$	-0.0004	+0.000 003
$l_1 - \tilde{\omega}_3$	-0.0004	+0.000 002

<i>Argument</i>	<i>coefficient in δv_i</i>	<i>coefficient in $\delta \varrho_i$</i>	
<i>Satellite III.</i>			
τ	$-\overset{\circ}{0}.0636$	$+.000\ 555$	(x_3 , see above)
2τ	$-0.0011 (1+\lambda_2)$	$+.000\ 015 (1+\lambda_2)$	
3τ	$-0.0008 - 6\lambda_1 - 2\lambda_2$	$-.000\ 006 - 11\lambda_1 + 4\lambda_2$	
$l_3 - l_4$	$-0.0041 (1+\lambda_4)$	$+.000\ 022 (1+\lambda_4)$	
$2(l_3 - l_4)$	$+0.0138 (1+\lambda_4)$	$-.000\ 132 (1+\lambda_4)$	
$3(l_3 - l_4)$	$+0.0010 (1+\lambda_4)$	$-.000\ 012 (1+\lambda_4)$	
$\tau + \varphi_3$	-0.0008	$+0.000\ 007$	
$\tau + \varphi_4$	-0.0003	$+0.000\ 003$	
$l_3 - 2l_4 + \tilde{\omega}_3$	$+0.0004$	$-.000\ 000$	
$l_3 - 2l_4 + \tilde{\omega}_4$	-0.0004	$+.000\ 001$	
$2l_3 - 3l_4 + \tilde{\omega}_4$	-0.0004	$+.000\ 003$	
$l_3 - 2L + \tilde{\omega}_3$	$+0.0006$	$-.000\ 005$	

Satellite IV.

$l_4 - l_1$	$-\overset{\circ}{0}.0003 (1+\lambda_1)$	$+.000\ 005 (1+\lambda_1)$
$l_4 - l_2$	$-0.0005 (1+\lambda_2)$	$+.000\ 008 (1+\lambda_2)$
$l_3 - l_4$	$-0.0023 (1+\lambda_3)$	$+.000\ 101 (1+\lambda_3)$
$2(l_3 - l_4)$	$-0.0012 (1+\lambda_3)$	$+.000\ 018 (1+\lambda_3)$
$2l_4 - 2L$	$+0.0012$	$-.000\ 015$
$l_3 - 2l_4 + \tilde{\omega}_3$	-0.0006	$+.000\ 002$
$l_3 - 2l_4 + \tilde{\omega}_4$	$+0.0007$	$-.000\ 006$
$l_4 - 2L + \tilde{\omega}_4$	$+0.0064$	$-.000\ 056$
$2l_4 - 2\tilde{\omega}_4$	$+0.0040$	$-.000\ 028$

Inequalities of group II. (The expressions as functions of ϱ and λ_i have already been given above).

<i>Argument</i>	<i>Coefficients in δv_i</i>		
	<i>Sat. I</i>	<i>Sat. II</i>	<i>Sat. III</i>
φ_1	$-\overset{\circ}{0}.0077$	$+\overset{\circ}{0}.0070$	$-\overset{\circ}{0}.0000$
φ_2	$+0.0169$	$+0.0377$	-0.0115
φ_3	$+0.0072$	-0.0464	$+0.0095$
φ_4	$+0.0026$	-0.0157	$+0.0033$

In the radii-vectores these inequalities can be neglected, with the exception of the following term in Satellite II:

$$\delta \varrho_2 = +.000\ 006 \cos \varphi_3.$$

Inequalities of group III. These also are negligible in the radii-vectores. The largest of them is:

$$\delta \varrho_3 = +.000\ 001 \cos (\tilde{\omega}_3 - \tilde{\omega}_4).$$

In the longitudes we have:

Argument	Coefficients in δv_i			
	Sat. I	Sat. II	Sat. III	Sat. IV
M	+ 0.0006	- 0.0102	- 0.0135	- 0.0320
W		- 0.0008	- 0.0012	- 0.0029
W_1		- 0.0001	- 0.0001	- 0.0003
Γ_2	+ 0.0099	+ 0.0028	+ 0.0001	
Γ_3		+ 0.0016	+ 0.0024	- 0.0018
Γ_4	- 0.0001	+ 0.0019	+ 0.0029	+ 0.0010
$\Gamma_2 - \Gamma_3$	- 0.0027	- 0.0011	- 0.0005	
$\Gamma_2 - \Gamma_4$	- 0.0005	- 0.0002	- 0.0001	
$\Gamma_3 - \Gamma_4$	+ 0.0011	- 0.0005	- 0.0013	- 0.0010
$\tilde{\omega}_3 - \tilde{\omega}_4$	- 0.0005	+ 0.0002	+ 0.0009	- 0.0011

It should be kept in mind that all the above developments are based on SOUILLART's analytical theory. New values of the masses and elements were introduced into his formulas, and a few numerical mistakes were corrected, but the analytical formulas were not altered. The only exception is the expression for the period of the libration, which was computed to terms of the third order in the masses inclusive, while SOUILLART rested content with those of the second order (See Gron. Publ. 17, art. 18).

Physics. — “*An auto-collimating spectral apparatus of great luminous intensity*”, by Prof. H. E. J. G. DE BOIS, G. J. ELIAS and F. LÖWE. (Communication from the Bosscha-Laboratory).

In optical work an illuminator is often wanted, which combines great brightness with monochromatic purity of the light, if possible of the order 0,1 μ . In spite of the almost boundless variety of available spectral apparatus¹⁾, such an appliance is wanting. For this purpose WÜLFING²⁾ it is true, constructed a monochromator and investigated the luminosities obtainable by means of different sources of light, but its aperture was only $\frac{1}{8}$, the dispersion also rather slight. An auto-collimator lately described by FABRY and JOBIN³⁾ has an aperture of only $\frac{1}{16}$. Recently one of us has described three spectral apparatus with constant deviation (parallel or at right angle)⁴⁾:

1) H. KAYSER, Handb. d. Spectroscopie 1, p.p. 489 et seq. gives a survey leading up to 1900.

2) E. A. WÜLFING, N. Jahrb. f. Mineralogie Beil. 12, p. 343, 1898. — C. LEISS, Zeitschr. für Instr. kunde 13 p. 209, 1898. S. NAKAMURA, Ann. d. Phys. (4) 20 p. 811, 1906.

3) CH. FABRY & A. JOBIN, Journ. de Phys. (4) 3 p. 202, 1904.

4) F. LÖWE, Zeitschr. f. Instr. Kunde 26, p. 330, 1906 and 27, p. 271, 1907.

here however, it was not necessary either to pay attention to great brightness.

The combination of the last mentioned condition with a considerable value of dispersion naturally leads to the application of the principle of auto-collimation, preferably with 2 "halfprisms". In using the instrument as a secondary spectral source of light immobility of the entrance slit cannot be dispensed with, and for the exit slit it is also required — perhaps with the exception of small sources of light, easily moved. We are, however, aware of the drawbacks which attend this principle, viz. a higher degree of false reflexes, and the so-called "vignettation" of the beam of light due to the necessity of placing the slits at different heights.

The utmost brightness is specially required for experiments on polarisation, in which a nearly crossed position of the nicols allows only a very small fraction of the light to pass. It follows that in such cases polarisation by the apparatus itself does not give rise to difficulties; it may even prove advantageous that refraction should in each case take place at the angle of polarisation; for then there is absolutely no loss by reflection of light polarized parallel to the refracting edges of the prisms. So in its usual position the apparatus would allow light to pass which shows a strong partial polarisation along the vertical.

From BREWSTER'S law it follows that the angle of refraction of the whole prisms must then amount to $(180^\circ - 2 \text{ arc tg. } n)$; we shall by preference choose prisms of 60° (resp. 30°), corresponding to $n = \sqrt{3} = 1,732$. For this case, with 2 whole and 2 half prisms, the simple scheme of fig. 1 is naturally evolved, where evidently all the angles of incidence amount to 60° .

Now the glass must meet the following principal requirements: 1) index of refraction for a mean colour about 1,73; 2) no strong absorption of violet light; 3) homogeneity and absence of bubbles; 4) resistance against atmospheric influence; 5) sufficient dimensions of the rough blocks. In spite of the present ample choice it proved impossible as yet to satisfy all these 5 conditions to a sufficient degree. In the instrument constructed in the spring of 1907 by C. ZEISS we therefore used heavy flint N^o. 1771 of the firm of SCHOTT & Co. at Jena, for which $n_D = 1,794$; according to what precedes an angle of refraction of somewhat more than 58° (or 29°) corresponds to this. The value of dn between $C-F$ amounts to 0,0309; from this follows a dispersion for every whole prism of $4^\circ 4'$; hence for the whole course of the rays $2 \times (\frac{1}{2} + 1 + 1 + \frac{1}{2}) \times 4^\circ 4' = 25^\circ$ nearly.

In order to keep the system in the minimum of deviation, every

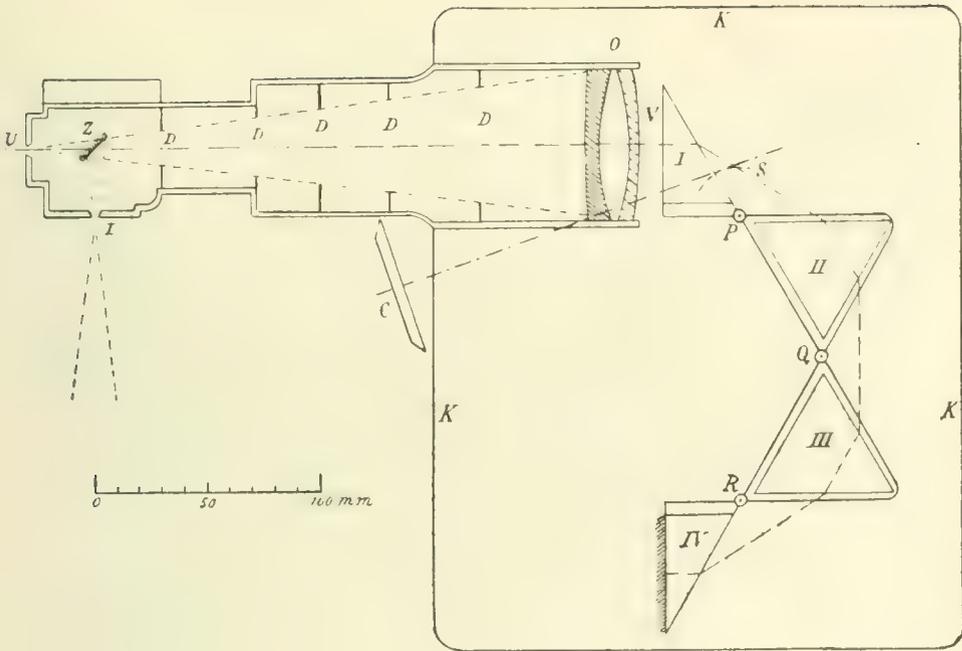


Fig. 1. — $\frac{3}{10}$ natural size.

prism must evidently be subjected to an equal rotation with respect to the preceding one, round the points R , Q and P . Prism I remains rigidly connected with the collimator tube; now let every point of II describe an arc α of a circle round P ; then the points of III describe cycloids, those of IV higher cycloids, in which the total rotation of III and IV with respect to the ground plate amounts to 2α and 3α respectively, apart from their simultaneous translation.

In a similar case one of us (*F.L.*) successfully constructed a toothed-wheel mechanism for a quartz-monochromator years ago, which was now also chosen. The old arrangement for cutting the rather intricate forms of the teeth was now again adopted. An analogous mechanism was, moreover, lately described by HAMY¹⁾ and executed by JOBIX.

For the sake of simplicity prism II is primarily rotated by means of a worm-wheel arc S roughly represented in the diagram, which could be effected with a bamboo rod from the observer's place. The reading takes place on the circle C ; the prism tables are provided with german silver feet, which slide on a glass plate; prism IV is

¹⁾ M. HAMY, Journ. de phys. (4) 7, p. 52, 1908; Zeitschr. f. Instr. Kunde 28, p. 122 1908.

silvered on the back side.¹⁾ The angle of I amounts to $30^{\circ}40'$; the plane of entrance V — 52×52 mm. large — forms an angle of about $40'$ with the wave-front, which causes the disturbing reflex to be thrown aside. The object-glass, consisting of three lenses, has a diameter of 67 mm., a focal distance of 260 mm., so that the aperture amounts to $\frac{1}{4}$. The lateral spherical aberration is according to calculation of the order 0,01 mm. A small part of the convex front surface has been blackened to prevent reflection; square diaphragms D have been placed in the collimator tube for the same purpose. The whole system of prisms is placed under a closed metal cover K ; inside this the necessary chemical substances are supplied in order to protect the sensitive glass surfaces against the action of water-vapour, carbonic acid, hydrogen sulphide etc. Whether these measures will prove effectual remains, as a matter of course, to be seen after a considerable lapse of time.

At the end of the collimator tube the "slit holder" is arranged so as to rotate round its axis. The bilateral entrance slit I , which is provided with a prism for comparison, is 3.5 mm. long, it is slightly curved (radius of curvature 70 mm.), and can rotate a little so that the slope and curvature of the spectral lines is compensated for a mean colour; the slit is focussed by means of a spiral groove. A mirror silvered at the front side directs the rays towards the lens, which on their way back pass along its upper or lower side so as to reach the exit-slit U ; the latter is also bilateral, 3,5 mm. long, but rectilinear. It may be exchanged for monocentric non-reflecting eye-pieces with a focal distance of 9 or 25 mm., or for a normal camera 60×90 mm., by means of which only a small spectral region can of course be photographed at the same time.

The whole apparatus is constructed without any iron, and mounted very compactly on a marble slab. The adjusting screws form a right-angled triangle, one of the catheti lying under the optical axis, whose height above the plane of the table is 125 mm.

For measurements in the ultraviolet the object-glass is replaced by a quartz-fluorite achromatic lens ($\varphi = 33$ mm., $f' = 260$ mm., apert. $\frac{1}{8}$); it would of course be too expensive to fill the whole aperture; a couple of quartz-half prisms according to CORNU is also provided.

¹⁾ In many respects it may be preferable to fix a metal mirror with glycerin to the back plane, for it is easy to remove it, and also to adapt the apparatus for transmitted light; in this case a telescope or a spectrograph with camera has to be added; the same mirror may be used, if necessary, to give the desired direction to the light. Besides, some alloys reflect considerably better than silver in the ultraviolet (about $320 \mu\mu$).

Some time ago one of us¹⁾ applied to a spectrograph a peculiar graduation according to wave-lengths from $5 \mu\mu$ to $5 \mu\mu$, which proved very convenient. For instruments of great dispersive power, however, this principle hardly works well; more accurate results are obtained with a calibration curve, though this takes more time. For this calibration the lines of the gas spectra of hydrogen, helium, and those of a mercury arc lamp may be used; also those of the spark spectrum of copper and of the flame spectrum of potassium: in this way a sufficiently uniform distribution of lines is obtained between 410 and $770 \mu\mu$. The accuracy of the readings is of the order $0.05 \mu\mu$.

Though from the outset we had been intent upon preventing end play in the mechanism of motion, it proved as yet impossible to avoid this altogether, so that it was necessary for the readings to have the motion take place always in the same sense. We hope, however, to remedy this defect by further improvements.

With the apparatus used as a spectrometer a very satisfactory resolution of neighbouring spectral lines could be brought about, the theoretical dissolving power of the set of prisms in the usual sense amounting to 65000 . Thus with the strong eye-piece the yellow helium line is seen resolved into its two components, whose distance apart amounts to about $0,035 \mu\mu$.

FABRY and JOBIN (loc. cit. p. 208) give a comparative table of the breadth occupied in the spectrum by a wave-length interval of $1 \mu\mu$ in the violet at about $434 \mu\mu$; in the red the dispersion is of course much less:

APPARATUS	DISPERSION
BRUCE (Yerkes-Observatory)	1,4 mm per $\mu\mu$
MILLS (Lick-Observatory)	0,8 " " "
FABRY and JOBIN	2,0 " " "
DU BOIS, ELIAS and LÖWE	1,96 " " "
ROWLAND-grating of Berlin University, 1st order (5684 lines per cm; radius 390 cm),	2,18 " " "

FABRY and JOBIN's fourfold focal distance is therefore all but compensated by our greater dispersion.

Though particular care was taken to prevent reflexes, yet it proved

¹⁾ F. Löwe, Zeitschr. f. Instr. Kunde **26**, p. 332, 1906.

impossible entirely to exclude diffuse light—probably due to the diffusion on the faces of and inside the prisms, so that we shall always have to take account of its presence, even though it be only to a very slight degree. In fact, we have not investigated any apparatus or prism, in which the disturbing influence of this phenomenon was not more or less felt. The question whether a certain diffusion still occurs with a really macro-homogeneous, optically “empty” refracting medium, is difficult to solve, and must for the present be considered a pending problem.¹⁾

The “vignettation” amounts on an average to 25 %, as may be observed by accommodating on the square objective diaphragm. When the apparatus was used as a monochromator the intensity of the light came up to what we expected; with sunlight it is still from 5 to 10 times higher (according to meteorologic circumstances) than with an arc-lamp crater projected on the entrance slit. Accordingly with monochromatic light of great purity even polarisation apparatus of very slight transmittivity may be used. When thus applying the instrument to illuminative purposes the entire path of the beam from the source of light on to the retina, and especially its divergence, ought to be carefully adapted to that part which lies within the apparatus, if all the possible benefit is really to be derived from it.

Physics. — “*The influence of temperature and magnetisation on selective absorption spectra*”, II. By Prof. H. E. J. G. DU BOIS and G. J. ELIAS. (Communication from the Bosscha Laboratory).

§ 12. Since our former communication (These Proc. Febr. p. 578) the cryomagnetic arrangement was further improved in some respects in order to obtain a stronger field, and to diminish the inconvenient formation of rime. The truncated end-planes of the conic polar pieces had a diameter of 6 mm., the split cores²⁾ a diameter of 3.5 mm.: the width of the slit at the end was from 0.4 to 0.6 mm., the slit being wedge-shaped so as to fit the convergence and divergence of the beam of rays between two lenses; it was arranged in such a way that the whole surface of the grating was illuminated, so that the theoretical dissolving power, — amounting to about 100,000 — had its full effect. Subsequent in the direction of the rays was a doubled quarter-wave plate with horizontal demarcation-

¹⁾ C. A. LOBRY DE BRUYN and L. K. WOLFF, *Rec. d. Trav. Chim.* **23**, p. 155, 1904; L. MANDELSTAM, *Physik. Zeitschr.* **8**, p. 608, 1907; M. PLANCK, *ibid.* **8**, p. 906, 1907.

²⁾ H. DU BOIS, *Zeitschr. für Instr. Kunde* **19** p. 360, 1899.

tion adjusted at the laboratory according to CORNU and W. KÖNIG¹⁾. On account of the considerable astigmatic difference in the images of horizontal and vertical lines formed by a concave grating, the plate was placed near the focus of a third lens in order to enable us to cancel this astigmatism for different parts of the spectrum by comparatively small displacements. The line of demarcation could then be adjusted sufficiently sharply in the spectrum, which KÖNIG had not succeeded in doing. A nicol followed the mica plate, and then came the principal slit. With this arrangement a normal doublet is known to appear in the spectrum as a broken line e. g. thus $\frac{1}{2}$; and on rotation through 90° of the nicol round the direction of the rays or of the $\frac{1}{4}$ plate round its *vertical* diameter $\frac{1}{4}$ at once appears.

§ 13. As a rule the samples were mounted in a copper frame-piece and clasped between the polar end-planes; it is desirable to have an airtight fitting so as to prevent cold currents of air with formation of rime. The level of the liquid air may now rise above the openings so that the sample is quite immersed. The air stagnating in the bores is effectually dried by the preliminary cooling with solid carbon dioxide. With thin samples we obtain in this way a field of 40 kilogauss, which is quite essential for the proper resolution of the quadruplets etc to be described later. With sunlight and a width of 0.05 mm. of the principal slit there was still plenty of light even in the violet; the FRAUNHOFER lines, however, proved so troublesome in many cases that the much weaker arc light had to be used. The spectrum was measured by means of a magnifying glass and a graduated glass scale, the divisions of which amounted to 0.225 mm., exactly corresponding to 0,1 μ in the spectrum of the first order. The auto-collimator, which we also used has been described since our first communication (see the preceding paper).

All the following experiments were made with a longitudinal field, in other words with an axial direction of the rays; many new adjustments would be required after turning round the heavy electro-magnet, so that we hope to extend the observations to an equatorial direction of rays later on.

§ 14. Third series. Of the large number of coloured com-

¹⁾ A. CORNU, Compt. Rend. **125** p. 555, 1897. — W. KÖNIG, Wied. Ann. **62** p. 242, 1897. We found it safer not to place this arrangement at the end of the beam near the magnifying glass, on account of polarisation by the grating; cf. P. ZEEMAN, These Proc, Oct. 1907.

pounds of trivalent titanium and vanadium we investigated some without, however, having found anything noteworthy as yet. The selective properties in this series culminate for chromium; we shall therefore restrict ourselves to a closer investigation of some chromic compounds already discussed in our former paper.

Chromium alum.

From the well-known regular crystals plates of a thickness of about 2 and 3 m.m. were cut. At 18° a rather intense band 669,8—671,6 is seen in the red; at -193° it becomes considerably narrower, viz.: 668,6—669,4, the centre shifting $1.7 \mu\mu$ towards the violet; moreover another rather strong line 670,2 appears; between 619 and 716 no less than 21 fainter and sharper bands and lines are actually visible.

In a field of 34 kilogauss the two principal lines appeared broken; the horizontal distance of the corresponding edges of their upper and lower halves, henceforth briefly called the *break*, amounted to about $0,10 \mu\mu$; the sense was *opposite*¹⁾. Band 668.6—669.4 shows one fine narrow satellite on the red side, towards the violet two of them; the former disappeared in the field; the two latter ones became very vague, and seemed, as seen with sunlight, to join in the break of the principal band.

Ruby.

§ 15. With the square plate ($7 \times 7 \times 3$ m.m.) mentioned in our preceding paper a long edge contained the optical axis. From the same ruby cone a small quadratic prism ($1,5 \times 1,5 \times 4$ m.m.) was now ground, the axis being parallel to a short edge. With the slight thickness of 1,5 m.m. sufficient absorption is shown even with grating dispersion. We must now distinguish the cases that the optical axis is \parallel or \perp with respect to the direction of the field.

I. Optical axis \parallel direction of field:

A. *Pair of bands in the blue at -193° .* Besides the two bands in the red already described, a pair in the blue are rather striking among the other 8; we shall briefly call these B_1 and B_2 . At -193° their situation is: $B_2 = 474,2—474,9$, and $B_1 = 476,1—476,5$ (at 18° they lie $474,9—475,7$ and $476,5—477,1$, more towards the red). The distance of the central lines measured in the grating spectrum, amounted to $1,63 \mu\mu$. In a field of 36 kgs ($=$ kilogauss) the break for B_1 amounted to $0,04 \mu\mu$ and for B_2 to $0,055 \mu\mu$, the sense being

¹⁾ I. e. with respect to that which has been found up to now for all vapours; such an opposite sense was also observed by J. BECQUEREL in most cases.

opposite; an asymmetry in the break of the bands towards both sides — with respect to their position with field off — appeared to exist, but could not be measured with sufficient certainty. At a temperature considerably exceeding that of liquid air, the blue bands are no longer to be determined in the grating spectrum.

§ 16. *B. Pair of bands in the red; we call them R_1 and R_2 .*

1) At -193° we have $R_2 = 691,7$ and $R_1 = 693,1$, the distance measured in the grating spectrum being $1,38 \mu\mu$.

Line R_1 : Width with field off $0,065 \mu\mu$. With 23 kgs. a *triplet* begins to appear, which is not yet clearly visible with 18 kilogauss; lefthand line (red side) not sharply divided from middle line, forming together a strong line, $0,10 \mu\mu$ wide; righthand line (violet side) divided from middle line at a distance of $0,09 \mu\mu$. With 26,5 kgs. the triplet further resolves, the distance on either side becoming $0,11 \mu\mu$.

With 36 kgs. the lefthand line is strong, the middle line perhaps stronger still, not sharply divided, distance $0,165 \mu\mu$; the righthand line faint, at a distance of $0,14 \mu\mu$ from the middle line.

Line R_2 : Width with field off $0,055 \mu\mu$. With 23 kgs. triplet: lefthand line not separated from middle line, forming together broad line $0,075 \mu\mu$ wide; righthand line separated from middle line at a distance of $0,07 \mu\mu$. With 26 kgs. the triplet further resolves: distance $0,08$ and $0,09 \mu\mu$ respectively.

With 36 kgs. the lefthand line is rather strong, not quite detached from the middle line, at a distance of $0,115 \mu\mu$; the righthand line faint, more clearly separated from the middle line, at a distance of $0,15 \mu\mu$.

In all these cases the lateral components were circularly polarised in the opposite sense; as the middle line vanished at neither of the two positions of the $\frac{1}{4}$ plate, it could not be circularly polarized; linear polarisation was not observed and quite excluded on account of axial field-symmetry. It is not yet the moment here to enter into an explanation of this highly remarkable phenomenon; it may perhaps simply be due to imperfect resolution of the inner lines of a quadruplet¹⁾. A magnetic displacement of the middle line with respect to its position with field off²⁾ could not be ascertained; at all events it never amounted to more than 1 or 2 hundredths of $\mu\mu$.

There is no reason in this case to doubt of the proportionality of the resolution with the intensity of the field.

¹⁾ Cf. P. ZEEMAN, These Proc. Febr. 1908.

²⁾ Cf. H. KAYSER, Handb. d. Spectroscopie, 2 p. 655, Fig. 52. Something similar was also sometimes observed for the sextuplet of D_2 .

2) At -79° the bands were already considerably widened and faded so that the thicker ruby plate had to be investigated through which the light proceeded 7 mm. in the direction of the axis.

Heating from -193° to -79° displaced R_1 by $0,62 \mu\mu$, R_2 $0,58 \mu\mu$ towards the red so that their distance now became $1,42 \mu\mu$. In a field of 18,5 kgs. R_1 exhibited a lefthand break of $0,12, \mu\mu$, a righthand one of $0,065 \mu\mu$, and R_2 deviated $0,04$ on the left, $0,07 \mu\mu$ on the right.

3) At $+18^\circ$ and a field of 18,5 kgs. R_1 exhibited a break of $0,07 \mu\mu$ towards both sides, R_2 one of $0,055 \mu\mu$. Heating from -193° to $+18^\circ$ shifted R_1 $0,76 \mu\mu$, R_2 $0,69 \mu\mu$ towards the red, so that their distance now became $1,45 \mu\mu$ ¹⁾.

4) At $+200^\circ$ the phenomenon was rather vague. By estimation the two lines showed a symmetrical break of $0,04 \mu\mu$ with 18,5 kgs. Heating from 18° to 200° moved both R_1 and R_2 $1,1 \mu\mu$ towards the red, their distance therefore not being changed. As yet we have not heated the ruby any higher.

In general we may perhaps conclude from the rather intricate course of the phenomenon that the influence of magnetisation slightly decreases with increase of temperature. The distance between R_1 and R_2 , on the other hand, seems to become a little larger.

§ 17. We now proceed to the second case:

II Optical axis \perp direction of field, where we must distinguish the ordinary and the extraordinary spectrum. In this case only the nicol, no longer the double $\lambda/4$ plate was used, because circular polarisation does not come in here.

1. Ordinary spectrum; plane of polarisation horizontal:

A. Pair of bands in the blue at -193° . The width with field off amounted to $0,17$ for B_1 , to $0,14 \mu\mu$ for B_2 , the distance of the central lines being $1,68 \mu\mu$; the lines looked about equal: In a field of 36 kgs. the width increased to $0,26 \mu\mu$ for the two lines; half the increase in width amounted therefore for B_1 to $0,045$, for B_2 to $0,06 \mu\mu$.

B. Pair of bands in the red at -193° . We have (cf. § 7) $R_2 = 691,8$ and $R_1 = 693,2$. The width with field off amounted to $0,08$ for R_1 , to $0,07 \mu\mu$ for R_2 , their distance in the grating spectrum being $1,41 \mu\mu$.

With a field of 20 kgs. R_1 became widened, and seemed shaded

¹⁾ We gave up the idea of reproducing a photograph, because the reproduction in our former paper is greatly inferior in distinctness to our own prints. Moreover, where measurement proves possible, reproduction appears almost superfluous.

The sufficiently good agreement of the ratios proves the proportionality of the resolution with the intensity of the field, at least as a first approximation; it is rather improbable that weaker fields should exhibit any deviations from this proportionality.

§ 18. Almost analogously behave the lines in the

2. *Extraordinary spectrum*; plane of polarisation vertical.

A. *Pair of bands in the blue* at -193° . The width with field off amounted to 0,10 for B_1 , to 0,15 $\mu\mu$ for B_2 , the distance of the middle lines was 1,70 $\mu\mu$; the lines appear somewhat displaced compared with the ordinary spectrum, viz. B_1 0,025 $\mu\mu$ towards red, and B_2 0,007 towards violet; moreover B_2 was vaguer and paler than B_1 .

In a field of 36 kgs. the widths became 0,18 and 0,22 $\mu\mu$; so for B_1 and B_2 respectively half the increase in width amounted to 0,04.

B. *Pair of bands in the red* at -193° . The width with field off amounted to 0,07 for R_1 , to 0,06 $\mu\mu$ for R_2 , their distance being 1,41 $\mu\mu$.

They both seem to have shifted 0,02 $\mu\mu$ towards the violet, compared with their position in the ordinary spectrum; R_1 is fainter.

With 36 kgs. R_1 exhibits a quadruplet of 4 lines about equally strong, at apparently equal distances, too indistinct, however, to be measured; distance of the extreme limits 0,49 $\mu\mu$; the middle appeared to have moved 0,02 $\mu\mu$ towards the violet with respect to the position with field off.

For R_2 the inner lines of the 4 were probably slightly stronger than the outer ones; the determinations were rather uncertain; the distance of the limits about 0,4 $\mu\mu$.

§ 19. *Fifth series*. Of this we now investigated a few sulphates of the material used in 1899, which crystallise monoclinically as octohydrates; they do so in plates containing both optical axes. As a matter of course no circular polarisation occurs; in this respect uniaxial and even more so cubic crystals, e. g. chromium alum, are to be preferred.

Neodymium sulphate [$\text{Nd}_2 (\text{SO}_4)_3 \cdot 8 \text{H}_2\text{O}$]. Rosy-red plate 0,8 mm. thick at -193° . Two narrow bands in the yellow, and three in the green exhibited an increase in width of from 0,05 to 0,08 $\mu\mu$ in a field of 40 kgs; two of the last mentioned became brighter in the middle, and so began to look like doublets.

Samarium sulphate [$\text{Sm}_2 (\text{SO}_4)_3 \cdot 8 \text{H}_2\text{O}$]. Light yellow semi-transparent plate of crystal, 2,8 mm thick at -193° . Two narrow bands in the yellow-green exhibited an increase in width in a field of 28 kgs. the amount of which ought to be determined with a sample of better transparency.

Physics. — “*Isotherms of monatomic substances and their binary mixtures. II. Isotherms of helium at — 253° C. and — 259° C.*”, by Prof. H. KAMERLINGH ONNES. Communication N°. 102^c from the Physical Laboratory at Leiden.

§ 1. *Survey of the determinations.* The measurements were made in the same way as those of Comm. N°. 102^a (Dec. '07). The whole of the piezometer had a four times larger content, viz. about 2 liters, the piezometer reservoir on the other hand was more than four times smaller, it was, namely, somewhat more than 2 cm³. Accordingly the densities to which the measurements refer, are considerably larger, and lie between 591 and 794 times the normal one. The temperatures at which the determinations were made, are measured on the hydrogen thermometer of Comm. N°. 95^c.

$$t = -252^{\circ}.84 \text{ C. and } t = -258^{\circ}.94 \text{ C.}$$

from which by extrapolation by means of table XXV of Comm. N°. 101^b (Dec. '07) see § 3 of Comm. N°. 102^b follows for the temperatures below 0° C. measured on the absolute scale

$$\theta = -252^{\circ}.84 + 0^{\circ}.12 = -252^{\circ}.72$$

$$\text{and } \theta = -258^{\circ}.94 + 0^{\circ}.12 = -258^{\circ}.82$$

The determination of the mean temperature of the gas in the capillary stem of the piezometer reservoir, with regard to the part that extends above the bath in the cryostat, required here greater accuracy than before, because compared with the quantity of the gas in the smaller reservoir that in the stem was of more importance. With a view to the determination of this mean temperature a cylindric reservoir of the same height as the capillary was placed by the side of and on a level with the capillary, which reservoir was filled with helium, and provided with an appliance to read the pressure in it ¹⁾. By means of this pressure it is easy to derive with the required accuracy what mean density for the gas in the capillary of the piezometer must be taken. At 0° the pressure in this auxiliary apparatus was 118.3 cm. of mercury. With the measurement at — 253° C. it varied between 33.1 and 51.1 cm., at — 259° C. between 31.8 and 48.1 cm.

¹⁾ A similar contrivance has been applied by different observers in the determination of the mean temperature of the capillary of a gas thermometer (TRAVERS, SENTER and JACQUEROD, Ph. Tr. Royal Soc. London Ser. A. vol. 200 p. 143 (1902)).

§ 2. *Results for pv_A .*

The subjoined table contains the results of the determinations in the same way as table I of Comm. N^o. 102^a.

TABLE I. Helium. Values of pv_A .

N ^o .	θ	p	pv_A	d_A
1	— 252°.72	53.848	0.09120	591.53
2		60.716	0.09533	626.92
3		65.997	0.09867	668.87
4	— 258°.82	40.012	0.06150	650.65
5		46.222	0.06559	704.71
6		53.326	0.07063	754.97
7		59.797	0.07531	794.00

The corresponding values for $pv_{A,d=0}$ are:

$$\begin{aligned} \text{for } -252^\circ.72 & \quad pv_A = 0.07455 \\ \text{for } -258^\circ.82 & \quad pv_A = 0.05222 \end{aligned}$$

§ 3. *Further results.* The number of points on every isotherm is too small, and the densities are too large to allow already now the derivation of the first individual virial coefficients of the polynomial of state (cf. § 4 of Comm. N^o. 102^a). If, however, we give a graphical representation, it shows that the isotherm pv_A for -259° must exhibit a minimum and hence B_A must be negative at this temperature. Further follows from the isotherm of -253° , that the intersection of this line with the axis $d = 0$ lies near the BOYLE-point. Probably B_A is also already negative at -253° though only slightly. All this agrees very well with what was derived in § 5 of Comm. N^o. 102^a, and speaks for the validity of the extrapolation applied there with a view to the calculation of the critical temperature of the helium.

In conclusion I gladly express my thanks to Mr. C. BRAAK for his assistance in this investigation.

Physics. — “*On the measurement of very low temperatures. XX. Influence of the deviations from the law of BOYLE-CHARLES on the temperature measured on the scale of the gas-thermometer of constant volume according to observations with this apparatus.*” By Prof. H. KAMERLINGH ONNES and C. BRAAK. Comm. N^o. 102^d from the Physical Laboratory at Leiden.

§ 1. In Comm. N^o. 97^b (Jan. '07) under XV the formula of CHAPPUIS (see Comm. N^o. 95^e (Oct. '06) form. (3)) for the calculation of the temperatures according to the hydrogen thermometer of constant volume was compared with formula (6) of XIV of the same Communication, in which formula attention has been paid to the deviations from the law of BOYLE, whereas they are neglected in CHAPPUIS' formula. As the result of this comparison we stated there that for a dead space of $\frac{1}{100}$ the mean relative coefficient of pressure between 0° and 100 is to be increased with 2 units of the 7th decimal, and the coefficient of pressure of the hydrogen thermometer at 1090 mm. zero point pressure was, therefore, to be put at 0,0036629 instead of at 0,0036627, a modification which is, however, so slight, that it just coincides with the limit of the errors of observation. We have just found out that for this calculation inaccurate values of $B_{0_0}^{(p)}$ and $B_{10_0}^{(p)}$ have been used. New calculations have revealed that the difference is much smaller than was stated just now, so that it is to be taken into account only for much higher values of the dead space and, with the exception of carbonic acid, has no influence even on CHAPPUIS' last decimal (the 8th). That the use of the incorrect $B^{(p)}$ was not detected, was due to the fact that the calculation of neglects indicated in XV had accidentally led to the same result, here, however, because the four corrections, as has been mentioned in XV, had been erroneously taken with the same sign, whereas they almost entirely cancel each other. We shall therefore in future keep to the unchanged coefficient of pressure 0.0036627.

A consequence of the improved calculation is also that table XVIII of Comm. N^o. 97^b (Jan. '07) can be dispensed with. The first two corrections derived in XIV § 3 of the Communication mentioned, now become so small that they fall outside the region of observation. The correction calculated at the end of § 3 becomes somewhat smaller for CHAPPUIS' carbonic acid thermometer than has been given there, viz. — 0.22×10^{-6} , to which another correction of — 0.8×10^{-7} is to be added, if also the expansion by the pressure of the gas is to be taken into consideration.

§ 2. The restoration of our former value 0.0036627 further involves the following modifications, which are all of no importance as they do not exceed the errors of observation, but should be applied to make the agreement in the calculations complete :

1. that in table XVI of Comm. N^o. 97^b (Jan. '07) in the first column the values of table XII are restored, and so all the numbers in the last decimal are increased by a unit. The latter holds also for the values of the second column of table XVI,

2. that in table XVII of the same Communication the values of the first column, except the last two, are increased by a unit in the last decimal.

3. that no further corrections are required for the temperatures in table XVI of Comm. N^o. 99^a (June '07) and table XX of Comm. N^o. 100^a (Dec. '07) (see conclusion of § 14 of Comm. N^o. 99^a and of § 18 of Comm. N^o. 100^a).

4. that in § 3 of Comm. N^o. 100^b (Dec. '07) the value for $pv_{.1100^{\circ}.2}$ and the corresponding virial coefficients are subjected to small changes, which, however, are of no importance,

5. that the last line of Comm. N^o. 101^a (Dec. '07) must be left out,

6. that in § 1 of Comm. N^o. 101^b (Dec. '07) $\alpha_{AV} = 0,0036619$ changes into 0,0036617, and $T_{0^{\circ}C.} = 273^{\circ}.08$ into $273^{\circ}.10$, while $T_{0^{\circ}C.} = 273^{\circ}.07$ of note ¹) in the § mentioned changes into $T_{0^{\circ}C.} = 273^{\circ}.09$ and that in § 2 $t = -273^{\circ}.08$ C. becomes $-273^{\circ}.10$ C., the changes in B'_{100} and in the values of table XXV being imperceptible,

7. that the numerical values in §§ 1 and 3 of Comm. N^o. 102^b (Dec. '07) require the emendations which have been applied in the translation in the Proceedings (Febr. 29 '08) (See footnote 1 there).

Physics. — “*On the condensation of helium.*” By Prof. H. KAMERLINGH ONNES. Communication N^o. 105 of the Physical Laboratory at Leiden.

(Not communicated here, see next communication).

Physics. — “*Experiments on the condensation of helium by expansion.*” By Prof. H. KAMERLINGH ONNES. Communication N^o. 105 of the Physical Laboratory at Leiden.

In the last session I communicated what I had observed in expanding helium, which at a temperature of -259° C. had been strongly compressed. I made the experiment in consequence of my determinations of the isotherms of helium at different temperatures i. a. also

at -253° C. and -259° C., from which I had calculated nearly 5° K. for the critical temperature of helium ¹). It thence followed that it would be possible by rapid expansion of helium compressed at 100 atm. at the meltingpoint of hydrogen to pass below the critical temperature and to cause a mist to appear in the gas ²). It was to put this conclusion to the test, that I compressed nearly 7 liters of helium, purified by burning with copperoxyde and leading over charcoal at the temperature of liquid hydrogen (so that I could trust to have a gas with only very small admixtures) in a thick walled tube placed in a non silvered vacuum glass with liquid hydrogen, and provided with a stopcock through which the helium could be let off from the tube into a gasholder, a gasbag or a vacuum. The liquid hydrogen round the tube was exhausted at such a pressure that hydrogen crystals just appeared at the surface of the liquid. The vacuumglass with hydrogen was surrounded by a second non silvered vacuumglass with liquid air. In the thickwalled tube, leaving only a small clearance, there was placed an extremely thin walled beaker ³) for protecting the gas which was cooled by expansion against conduction of heat from the walls, the layer of gas, between the beaker and the walls of the tube, though it was very thin, being a bad conductor.

At the expansion of the helium a dense gray cloud appeared from which separated out solid masses floating in the gaseous helium, resembling partly cotton wool, partly also denser masses, as if floating in a syrupy liquid, adhering to the walls and sliding downward while at the same time vanishing rapidly (20"). There was no trace of melting.

As far as I could judge then from the experiments I considered it probable that this solid substance was for the greater part helium.

If helium passed immediately to the solid state then the position of the vapour line in respect to the adiabatics would be more favourable for condensation than was to be expected according to the formula of VAN DER WAALS. The voluminous aspect of the solid mass was in harmony with this. By the above and also by other observations

1) OLSZEWSKI from expansion experiments has deduced that the critical temperature of helium lies below 2° K. DEWAR estimates the *boiling* point according to the absorption in charcoal at higher than 5° K. (This would agree with a critical temperature of 8 K. Note added in the translation).

2) Liquefaction by making use of the JOULE KELVIN process would also be possible. (Note added in the translation).

3) This device has been used by OLSZEWSKI in his experiments on the expansion of hydrogen (Note added in the translation).

which afterwards gave rise to doubt or proved incorrect, I had for some time the conviction that I had seen solid helium rapidly giving off vapours of the pressure shown by the gas (once more than 15 atm. was observed).

The continuation of my experiments has shown that they must be explained in quite a different way. By a not sufficiently explained cause the gas proved to be not so pure as was to be expected considering the method of purification. In analysing what was absorbed by charcoal at the temperature of boiling hydrogen till the charcoal did no more absorb hydrogen, (so that the gas could only contain traces of hydrogen) it could be proved that in one case the gas has contained only 0.45 and in another only 0.37 volume percents of hydrogen at most¹). But this small admixture must have had a very great influence.

For at a repetition of the experiment with the helium subjected to the new treatment no cloud at all was observed. The experiment is not decisive as the velocity of expansion had been too small, but it is difficult before further investigation to find in the difference of velocity of expansion the cause that the helium in the tube remained now perfectly clear.

The explication of the previous observations is to be found in solution phenomena of solid hydrogen in gaseous helium. The phenomena which made the impression of being the giving off of vapour had been the solution of deposited solid hydrogen in the gaseous helium, the latter rapidly returning from the lower temperature to that of melting hydrogen, and the pressure increasing in consequence. Helium at the temperatures, that come into account here can according to the theory of mixtures take up at every temperature a percentage of hydrogen determined by that temperature in such a way that it is not deposited at any pressure. On plausible suppositions one can deduce that at temperatures above the melting point of hydrogen this percentage can be considerable and that at this melting point itself it can be more than one percent. From mixtures with smaller percentage the hydrogen is only deposited at lower temperatures e.g. by expansion. By the smallness of the quantity of hydrogen present it is also explained that after prolonged blowing off of the helium no solid hydrogen was left. For the quantity left was so small that it could evaporate in the space which it found at its disposal.

It remains remarkable that as small a quantity of admixture as the gas contained has been able to give the total phenomenon of a

¹, About a small possible quantity of neon I could not yet be certain.

substance condensing to a solid and reevaporating, though the rapid evaporation, in which even denser masses were seen to be blown away sometimes, is in harmony with the smallness of this quantity of substance. There cannot have been much more than 1 mgr. or 15 cubic millimetres of hydrogen in round numbers in the tube — probably there was less in it — and yet the tube of nearly 7 cubic centimetres was over its whole length for almost a quarter filled with a dense flaky substance.

As far as the experiments on the expansion of helium at the melting point of hydrogen are now advanced they show the curious forms that the solution phenomena of a solid in a gas take in the case of helium and hydrogen. They further point to the possibility of realising with mixtures of hydrogen and helium the rising or falling of the solid substance according to the pressure exerted on the gas, the barotropic phenomenon for a solid and a gas. But the question of condensing helium is to be considered yet as an open one, which will ask an extensive investigation.

POSTSCRIPTUM.

I have had the occasion to repeat the experiment with the gas that remained perfectly clear in the last expansion experiment; and which also according to the spectroscopic test contained only traces of hydrogen. I now used a greater velocity of expansion. A thin cloud appeared and vanished extremely rapidly (in 1" nearly). The mist now had another aspect.

It is possible that the traces of hydrogen left in the gas will prove sufficient to cause this mist. But it is also possible that the mist has been a liquid cloud and the changed aspect seemed to point to this. If this might prove to be the case then the critical point would be nearly as I calculated it from the isothermals and helium would follow tolerably well the laws of VAN DER WAALS. The tube broke and I could not attain more certainty about the nature of the cloud.

The preceding experiments show very strikingly how careful one has to be in making conclusions from the appearing or not appearing of a cloud by expansion. A decision about the critical temperature of helium is therefore only to be obtained by a prolonged systematical investigation which will take much time.

(April, 24, 1908).

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Friday April 24, 1908.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige
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Mathematics. — “On the cyclic minimal surface”. By Prof. J. C. KLUYVER.

(Communicated in the meeting of January 25, 1908).

ENNEPER (Zeitschr. Math. Phys. 14) pointed to the existence of a minimal surface containing a system of circles lying in parallel planes, with centres situated on a plane curve. Let us suppose that this curve passes through the origin of the rectangular coordinates, that it is situated in the XZ -plane and that the variable circle with the centre $(\xi, 0, \xi)$ and the radius R , generating the surface, lies always in a plane parallel to the XY -plane.

The rectangular coordinates x, y, z of a point of the surface are given by the equations

$$x = \xi + R \cos \alpha, \quad y = R \sin \alpha, \quad z = \xi,$$

so that they are expressed in the two parameters α and ξ . We find that the differential equation of the minimal surfaces is satisfied when

$$R^2 (\xi'' R \cos \alpha + R R''') - R^2 (1 + \xi'^2 + R'^2 + R'^2 + 2 \xi' R' \cos \alpha) = 0,$$

in which equation the dashes denote the differentiations with regard to ξ .

The equation breaks up into

$$\xi' R = 2 \xi' R'$$

and into

$$R R'' = 1 + \xi'^2 + R'^2.$$

The first equation furnishes

$$\xi = \frac{A R^2}{b^2},$$

where A denotes a positive constant and b the minimum value of R .

The second equation now passes into

$$\frac{d}{d\xi} \left(\frac{R'}{R} \right) = \frac{1}{R^2} + \frac{A^2 R^2}{b^4}$$

and the integration furnishes

$$R'^2 = \frac{1}{b^2} (R^2 - b^2) \left(1 + \frac{A^2 R^2}{b^2} \right),$$

so that finally we can express ξ en ζ in R by means of elliptic integrals.

We find

$$\xi = \frac{A}{B} \int_0^R \frac{R}{\sqrt{(R^2 - b^2) \left(1 + \frac{A^2 R^2}{b^2} \right)}} dR, \quad \zeta = b \int_0^R \frac{R^2 dR}{\sqrt{(R^2 - b^2) \left(1 + \frac{A^2 R^2}{b^2} \right)}}$$

Here an elliptic argument can be introduced. We put

$$R = \frac{b}{cn u},$$

$$k = \sin \theta = \frac{1}{\sqrt{1 + A^2}},$$

and we find

$$\xi = bk' \int_0^u \frac{dw}{cn^2 w}, \quad \zeta = bku.$$

By allowing u to vary from $-K$ to $+K$ the centre M with the coordinates ξ, ζ in the XZ -plane describes completely the locus of the centres and the equation

$$R = \frac{b}{cn u}$$

indicates how the radius of the circle changes during the motion.

We notice that the minimal surface depends on two constants b and k , that the smallest circle ($u = 0$) is found in the XY -plane, that with respect to the origin there is symmetry, and that for $u = K$, $\zeta = bkK$ the radius R has become infinite whilst at the same time the centre M is at infinite distance.

As however

$$\lim_{u=K} (\zeta - R) = b \lim_{u=K} \left[k' \int_0^u \frac{dw}{cn^2 w} - \frac{1}{cn u} \right] = \frac{b}{k'} (k'^2 K - E)$$

and $\zeta - R$ retains therefore a finite value the surface contains two right lines

$$z = \pm bkK,$$

$$x = \pm \frac{b}{k'} (k'^2 K - E).$$

For $k = 1$ the elliptic integrals degenerate. We have

$$\xi = 0, \quad \zeta = bu, \quad R = b \tanh u,$$

and the surface has passed into a catenoid. The smaller k is, the more the surface deviates from the catenoid and the more oblique it becomes. For, we find for the coefficient of direction of the tangent to the locus of the centres M :

$$\frac{d\zeta}{d\xi} = \frac{k cn^2 u}{k'}$$

and the greatest value of this coefficient $k : k'$, which is arrived at in the origin, tends to zero when k tends to zero. The surface is then altogether in the XY -plane.

I shall now endeavour first to investigate in the following when it is possible to bring through two equal circles placed in parallel planes a cyclic minimal surface and then to calculate the part of the minimal surface extended between those circles.

When for both circles the radius R is taken equal to 1, the centres $M(\xi, \zeta)$ and $M'(-\xi, -\zeta)$ are situated in the XZ -plane symmetrically with respect to the origin and their planes are parallel to the XY -plane, the question is whether the two equations:

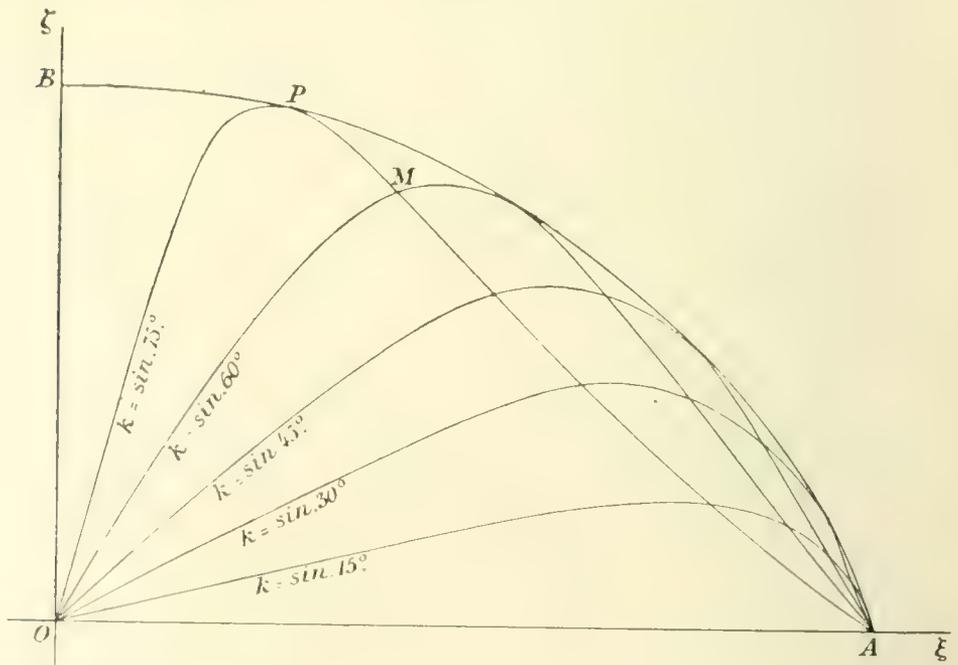
$$\xi = k' \operatorname{cn} u \int_0^u \frac{dw}{\operatorname{cn}^2 w}, \quad \zeta = k u \operatorname{cn} u$$

admit of suitable solutions for k and u . If these are found, we have $b = \operatorname{cn} u$ and both parameters b and k of the minimal surface are known.

In order to investigate the indicated equations we regard for the present in the $\xi\zeta$ -plane ξ and ζ as variables and we consider the curve which is described by point (ξ, ζ) , when for constant k the variable u describes the range of values from 0 to K . We have:

$$\begin{aligned} \xi(0) &= 0 & \zeta(0) &= 0, \\ \xi(K) &= 1 & \zeta(K) &= 0. \end{aligned}$$

So for all values of k the curve will run from the origin O to point A on the ξ -axis (see the diagram).



Farther we have :

$$\xi = k' cn u \int_0^u \frac{d(tn w)}{dn w} < k' cn u \int_0^u \frac{d(tn w)}{dn u},$$

$$\xi < \frac{k' sn u}{dn u},$$

so that from :

$$\frac{d\xi}{du} = \frac{1}{cn u} (k' - sn u dn u \xi)$$

follows :

$$\frac{d\xi}{du} > k' cn u.$$

We conclude that for increasing u the variable ξ grows regularly from 0 to 1. So the curve OA is intersected but once by a line $\xi = \text{constant}$.

At the same time :

$$\frac{d\xi}{du} = k (cn u - u sn u dn u) = k cn u \left(1 - u \frac{sn u}{sn(u + K)} \right).$$

For small u we find $\frac{d\xi}{du}$ to be positive, it keeps on decreasing, becomes one time zero and is then negative. So the variable ξ reaches somewhere a maximum and the curve OA is either not cut by a line $\xi = \text{constant}$ or in two points. The form of the curve $k = \text{constant}$ is therefore as is indicated schematically in the diagram. In order to be able to compare the curves belonging to different values of k we can determine the values which the differentialquotient $\frac{d\xi}{d\xi}$ assumes in the points O and A .

We have

$$\left(\frac{d\xi}{du} \right)_{u=0} = k, \quad \left(\frac{d\xi}{du} \right)_{u=0} = k,$$

$$\left(\frac{d\xi}{du} \right)_{u=K} = E - k^2 K, \quad \left(\frac{d\xi}{du} \right)_{u=K} = -kk'K,$$

from which ensues

$$\left(\frac{d\xi}{d\xi} \right)_{\xi=0} = \frac{k}{k'}, \quad \left(\frac{d\xi}{d\xi} \right)_{\xi=1} = - \frac{kk'K}{E - k^2 K} = - \frac{k'K}{k \int_0^K cn^2 u du}. \quad (O)$$

From this is apparent that in O the value of $\frac{d\xi}{d\xi}$ increases with k ,

that on the other hand the absolute value of $\frac{d\tilde{\xi}}{d\xi}$ decreases in A for increasing k . For, if k becomes greater $k'K$ decreases, but the denominator $k \int_0^k cn^2 w dw$ increases.

Taking into consideration the form just sketched of a curve OA belonging to a definite value of k we find that a second suchlike curve belonging either to a larger value or to a smaller value of k will certainly intersect the first curve somewhere. So as soon as a cyclic minimal surface passes through two equal circles placed in parallel planes we shall be able to bring a second cyclic minimal-surface through these circles.

We must now investigate when the two cyclic minimal surfaces coincide, i. o. w. we must find the envelope of the curves OA .

If we put $c = k^2$, $c' = k'^2$, then the system of curves is given in the equations

$$\tilde{\xi} = \sqrt{c'} cn u \int_0^u \frac{dw}{cn^2 w}, \quad \xi = \sqrt{c} u cn u;$$

we regard c as the parameter of the curve, $\varphi = am u$ as the parameter determining a point on a given curve, so that the coordinates $(\tilde{\xi}, \xi)$ of a point of the envelope satisfy the condition

$$\frac{D(\tilde{\xi}, \xi)}{D(c, \varphi)} = 0.$$

If here and in future we put for shortness' sake

$$A(u) = \int_0^u \frac{dw}{cn^2 w}, \quad B(u) = \int_0^u \frac{dw}{dn^2 w}$$

and we take into account that for constant $\varphi = am u$ we have

$$\frac{\partial u}{\partial c} = \frac{1}{2c} (B(u) - u),$$

we find

$$\frac{\partial \tilde{\xi}}{\partial c} = -\frac{1}{2\sqrt{c'}} cn u B(u), \quad \frac{\partial \xi}{\partial c} = \frac{1}{2\sqrt{c}} cn u B(u),$$

$$\frac{\partial \tilde{\xi}}{\partial \varphi} = \sqrt{c'} sn u (c B(u) - Q(u)), \quad \frac{\partial \xi}{\partial \varphi} = -\sqrt{c} sn u (c' B(u) + Q(u)),$$

where $Q(u)$ is given by the equations

$$Q(u) = u - E(u) - \frac{dn u cn u}{sn u},$$

$$\begin{aligned}
&= K - E - \int_u^K \frac{dw}{sn^2 w} \\
&= \frac{1}{sn^2 u} (u - cn^2 u A(u) - dn^2 u B(u)), \\
&= A(u) + k^2 B(u) - \frac{1}{sn u cn u dn u}.
\end{aligned}$$

From this ensues

$$\frac{D(\xi, \zeta)}{D(c, \varphi)} = -\frac{1}{2\sqrt{cc'}} cn u sn u B(u) Q(u),$$

and so the points of the envelope of the curves OA are determined by the equations

$$Q(u) = K - E - \int_u^K \frac{dw}{sn^2 w} = 0 \quad 1).$$

As when c is given, the first member of the equation increases regularly from $-\infty$ for $u=0$ to $K-E$ for $u=K$, the equation $Q(u)=0$ admits of one solution u_0 . By differentiating we find

$$\frac{du_0}{dc} = \frac{1}{2c} \int_0^{u_0} dw \left[\frac{dn^2 u_0}{dn^2 w} - 1 \right],$$

i. e. a negative value; therefore the greater c is, the smaller is the argument u_0 , which I call the critical argument. This argument moves finally between rather narrow limits. For $c=0$ we find

$K=E=\frac{\pi}{2}$ and so also $u_0=\frac{\pi}{2}=1.5708$. For $c=1$ we find

$$Q(u) = u - E(u) - \frac{dn u cn u}{sn u} = u - \frac{1}{sn u} = u - \frac{Ch u}{Sh u}.$$

So the critical argument u_0 satisfies the equation

$$u_0 = \frac{Ch u_0}{Sh u_0}.$$

From this ensues

$$\begin{aligned}
u_0 &= 1.1997, \\
\varphi_0 &= am u_0 = 56^\circ.28', \\
cot \varphi_0 &= u_0 cn u_0 = 0.6627.
\end{aligned}$$

¹⁾ G. JUGA. (Ueber die Constantenbestimmung bei einer cyklischen Minimalfläche, Math. Ann. Bd. 52) gives this equation in the form

$$cnu dnu + (E(u) - u) snu = 0.$$

For values of c between 0 and 1 it is easy to solve u_0 out of the equation

$$Q(u_0) = K - E - \int_{u_0}^K \frac{dv}{sn^2 v} = 0$$

by means of the tables of LEGENDRE. If u'_0 is an approximate value of the critical argument, the calculation of NEWTON furnishes

$$u'_0 - Q(u'_0) sn^2 u'_0$$

as following approximation. In this way the critical argument is calculated in the following table for some values of $k^2 = c$

$k = \sqrt{c}$	$\varphi_0 = am u_0$	u_0	$b = cn u_0$	ξ_0	ζ_0	φ'_0	ξ'_0	ζ'_0
$\sin 0^\circ$	90°	1.5708	0.	1.	0.	90°	1.	0.
15°	$87^\circ 4'$	1.5442	0.0520	0.9966	0.0208	$87^\circ 0'$	0.9954	0.0245
30°	$79^\circ 17'$	1.4701	0.1859	0.9498	0.1367	$79^\circ 23'$	0.9427	0.1423
45°	$70^\circ 3'$	1.3708	0.3412	0.7930	0.3308	$70^\circ 16'$	0.7916	0.3325
60°	$62^\circ 31'$	1.2801	0.4614	0.5573	0.5116	$62^\circ 35'$	0.5549	0.5133
75°	$57^\circ 57'$	1.2198	0.5306	0.2813	0.6251	$57^\circ 57'$	0.2776	0.6265
90°	$56^\circ 28'$	1.1997	0.5524	0.	0.6627	$56^\circ 28'$	0.	0.6627

and moreover are indicated in it the coordinates ξ_0, ζ_0 of the point P , in which the curve OA belonging to each value of k touches the envelope of that system of curves.

By the equations

$$\xi_0 = \sqrt{c'} cn u_0 A(u_0) \quad , \quad \zeta_0 = \sqrt{c} x_0 cn u_0$$

we now find in connection with the condition

$$Q(u_0) = 0$$

that ξ_0 and ζ_0 are given as functions of c only. We can deduce out of it

$$\frac{d\xi_0}{dc} = -\frac{1}{2\sqrt{c}} du_0 B(u_0) [cn u_0 du_0 + u_0 c sn^3 u_0],$$

$$\frac{d\zeta_0}{dc} = \frac{1}{2\sqrt{c}} du_0 B(u_0) [cn u_0 du_0 + u_0 c sn^3 u_0],$$

$$\frac{d\zeta_0}{d\xi_0} = -\frac{k'}{k}$$

From this appears that for increasing k or \sqrt{c} the coordinate ξ_0 decreases regularly and the coordinate ζ_0 increases regularly. In connection with the numbers inserted in the table it follows that the

envelope of the curves OA has about the shape of a quadrant of ellipse BA of which half of the great axis $OA = 1$ and half of the small axis $OB = 0.6627$.

Moreover it is clear that the tangent to any curve $k = \text{constant}$, in the point P where the latter touches the envelope, is normal to the tangent in the origin O drawn to this same curve. The preceding calculations now lead to the conclusion that through the two equal circles with radius $k = 1$ placed parallel and symmetrically with respect to the origin two cyclic minimal surfaces will pass, when the centre $M(\xi, \zeta)$ of the upper circle is situated inside the curve BA of the diagram, that the two surfaces coincide when M has arrived on the curve BA and that the circles cannot be connected by a minimal surface when M falls outside the curve BA .

If M lies inside the curve BA two curves OA pass through M . One of these touches the envelope in P , a point on curve OA between O and M . So the argument u belonging to M is greater than the critical argument u_0 in P and so the minimal surface belonging to it and extended between the circles M and M' would contain the two circles along which this minimal surface is cut by a second minimal surface with an infinitesimal slight difference. So this minimal surface is unstable. For the second minimal surface laid through the circles an argument u corresponds to M smaller than the critical argument u_0 ; this surface is therefore stable and can be realized in a proof of PLATEAU.

If two surfaces can be laid through the circles the most oblique surface (the surface belonging to the smaller value of k and with the greater value of the radius b of the mean section) is therefore always stable, the other is unstable.

It is worth mentioning that whilst here the quantities φ_0, ξ_0, ζ depend in rather an intricate way on $k = \sin \theta$, we can find by approximation out of simple formulae very accurate values for these quantities.

If we call the critical amplitude $56^{\circ}28'$ of the catenoid β , we shall be able to assume with great accuracy the following relations:

$$\cos \varphi_0 = \cos \beta \sin^2 \theta \left(1 + \frac{1}{9} \cos^2 \theta \right),$$

$$\xi_0^2 = 1 - \left(\frac{\cos \varphi_0}{\cos \beta} \right)^2,$$

$$\zeta_0 = \cot \beta \left(\frac{\cos \varphi_0}{\cos \beta} \right)^5,$$

from which ensues for the equation of the envelope BA

$$\xi_0^2 + \left(\frac{\xi_0}{\cot \beta} \right)^{\frac{10}{7}} = 1.$$

In the table the values of g_0 , ξ_0 and ξ_3 calculated in this way are added in the three last columns, to be compared.

To conclude with we give a computation of a part of a given cyclic minimal surface with given parameters b and k , situated between two equally large circles corresponding to the arguments $+u$ and $-u$.

The coordinates x, y, z , of a point of the surface are again determined by the equations:

$$x = bk' A(u) + \frac{b}{cn u} \cos \alpha, \quad y = \frac{b}{cn u} \sin \alpha, \quad z = bk u,$$

out of which we can find for the line-element on the surface the expression

$$\frac{ds^2}{b^2} = \frac{P du - i cn u da + i k' \sin \alpha du}{cn^2 u} \times \frac{P du + i cn u da - i k' \sin \alpha du}{cn^2 u},$$

in which P is determined by the equation

$$P^2 = (k' \cos \alpha + sn u dn u)^2 + k^2 cn^4 u.$$

We introduce for α an imaginary argument v .

We substitute

$$tg \frac{1}{2} \alpha = i \frac{tg \frac{1}{2} am v}{tg \frac{1}{2} am (u - K)}$$

and we find

$$\sin \alpha = \frac{i sn v sn (u - K)}{cn v - cn (u - K)},$$

$$\cos \alpha = \frac{1 - cn v cn (u - K)}{cn v - cn (u - K)},$$

$$\frac{da}{\sin \alpha} = \frac{dn v}{sn v} dv - \frac{dn (u - K)}{sn (u - K)} du,$$

$$P = \frac{cn^2 u dn v dn (u - K)}{k' (cn v - cn (u - K))^2}.$$

and finally

$$\frac{ds^2}{b^2} = \frac{dn^2 v dn^2 (u - K)}{k'^2 (cn v - cn (u - K))^2} (du - dv) (du + dv).$$

From this ensues that $u + v$ and $u - v$ are the parameters of the lines of length zero, so that v is the parameter of the greatest incline.

According to the general properties of the minimal surfaces we have for the superficial element $d\Omega$ the expression

$$\frac{d\Omega}{b^2} = \frac{dn^2 v \, dn^2 (u-K)}{k'^2 (cn v - cn (u-K))^2} du \frac{dv}{i},$$

and we find for that part of the surface limited by the two circles with the arguments $+u$ and $-u$:

$$\frac{\Omega}{4b^2} = \int_0^u du \int_0^{2iK} \frac{dv}{i} \frac{dn^2 v \, dn^2 (u-K)}{k'^2 (cn v - cn (u-K))^2}.$$

To perform the integration we start from the identity

$$\begin{aligned} f(u) &= - \int_0^{2iK} \frac{dv \, sn(u-K) \, dn(u-K)}{i \, cn v - cn(u-K)} = 2K' Z(u-K) + \frac{\pi u}{K} = \\ &= 2u(E' - K) + 2k'^2 K' B(u), \end{aligned}$$

which furnishes first

$$\int_0^{2iK'} \frac{dv}{i} \frac{dn^2 (u-K)}{cn v - cn(u-K)} = \frac{k' j(u)}{cn u}.$$

Moreover

$$\int_0^{2iK'} \frac{dv \, dn^2 v - dn^2 (u-K)}{i \, cn v - cn(u-K)} = k^2 \int_0^{2iA'} \frac{dv}{i} cn v + 2k^2 K' cn(u-K).$$

A dash before the integral sign indicates that the path of integration does not pass through point $v = iK'$.

Out of the two last equations follows by means of addition

$$\int_0^{2iK'} \frac{dv}{i} \frac{dn^2 v}{cn v - cn(u-K)} = \frac{k' j(u)}{cn u} + k^2 \int_0^{2iK'} \frac{dv}{i} cn v + 2k^2 K' cn(u-K),$$

an equation which, if we differentiate with regard to u and then divide by $k' cn u$, passes into

$$\begin{aligned} \int_0^{2iK'} \frac{dv}{i} \frac{dn^2 v \, dn^2 (u-K)}{k'^2 (cn v - cn(u-K))^2} &= \frac{1}{cn u} \frac{d}{du} \left(\frac{j(u)}{cn u} \right) + \frac{2k^2 K'}{dn^2 u} \\ &= \frac{1}{2} \frac{d}{du} \left(\frac{j(u)}{cn^2 u} \right) + \frac{2k^2 K'}{dn^2 u} + \frac{k^2 K}{cn^2 u \, dn^2 u} + \frac{E-K'}{cn^2 u}. \end{aligned}$$

Now integrating according to u between the limits 0 and u we find finally

$$\frac{\Omega}{4b^2} = \frac{u}{cn^2 u} (E' - K) + E' A(u) + K \frac{dn^2 u}{cn^2 u} B(u).$$

If the given circles have the radius $R = 1$ then b is equal to $cn u$ and we can write

$$\frac{\Omega}{4} = uE' + \xi \operatorname{cn} u \frac{E' - K'}{k'} - \operatorname{sn}^2 u K' Q(u),$$

where ξ again represents the x -coordinate of the centre M of the upper circle.

If this centre M moves on the envelope BA of the diagram, then u becomes equal to the critical argument u_0 , $Q(u)$ equal to zero and we have obtained the greatest possible minimal surface Ω_0 for the given value of k . So

$$\frac{\Omega_0}{4} = u_0 E' + \xi_0 \operatorname{cn} u_0 \frac{E' - K'}{k'}.$$

We can now put the question where we have to put M on the envelope BA , that is what value must be given to k for Ω_0 , to obtain the greatest possible value. To answer that question we substitute $c = k^2$ and $\varphi_0 = \operatorname{am} u_0$; then Ω_0 is a function of c , whilst φ_0 and ξ_0 are connected with c by means of the equations

$$Q(u_0) = K - E - \int_{u_0}^K \frac{dw}{\operatorname{sn}^2 w} = 0,$$

$$\xi_0 = \sqrt{c} \operatorname{cn} u_0 A(u_0).$$

By differentiation we find

$$\frac{d\varphi_0}{dc} = -\frac{1}{2} \operatorname{sn}^2 u_0 \operatorname{dn} u_0 B(u_0),$$

$$\frac{du_0}{dc} = -\frac{1}{2c} \operatorname{cn}^2 u_0 A(u_0),$$

$$\frac{d\xi_0}{dc} = -\frac{1}{2\sqrt{c}} \operatorname{dn} u_0 B(u_0) (\operatorname{cn} u_0 \operatorname{dn} u_0 + u_0 c \operatorname{sn}^2 u_0),$$

and finally by means of these results

$$\frac{d}{dc} \left(\frac{\Omega_0}{4} \right) = \frac{K' - E'}{c'} \operatorname{cn} u_0 \operatorname{dn} u_0 B(u_0) (\operatorname{cn} u_0 \operatorname{dn} u_0 + u_0 c \operatorname{sn}^2 u_0).$$

As the right member of the last equation is always positive, Ω_0 always increases with c or with k . The greatest possible surface between the two circles is obtained by placing M in B ; we have then a part of the catenoid, of which half the height is equal to $\cot \beta = 0.6627$.

$$\text{Now} \quad K' = E' = \frac{\pi}{2},$$

$$\frac{\Omega_0}{2\pi} = u_0 = 1.1997.$$

The smallest value Ω_0 obtains for $k = 0$. Then $\xi_0 = 0$, $\xi_0 = 1$; the minimal surface consists only of the surface of the circles M and C' placed side by side in the XY -plane. We have

$$\frac{\Omega_0}{2\pi} = 1.$$

So also the surface Ω_0 keeps moving between rather narrow limits. Although the value of Ω_0 depends again in rather an intricate way on k we can put pretty accurately, if once the critical argument u_0 or the amplitude φ_0 has been calculated,

$$\frac{\Omega_0}{2\pi} = \frac{1}{\operatorname{sn} u_0}.$$

This is evident from the following table, in which have been inserted for some values of k the corresponding values of $\frac{\Omega_0}{2\pi}$ and of $\frac{1}{\operatorname{sn} u_0}$.

k	$\frac{\Omega_0}{2\pi}$	$\frac{1}{\operatorname{sn} u_0}$
$\sin 0^\circ$	1.	1
15°	1.0002	1.0001
30°	1.0111	1.0176
45°	1.0556	1.0639
60°	1.1241	1.1271
75°	1.1795	1.1795
90°	1.1997	1.1997

As we have $b = cn u_0$, where b represents again the radius of the mean section we can in any case put with great approximation

$$\Omega = \frac{2\pi}{\sqrt{1-b^2}},$$

and in this way we obtain for the greatest possible just stable part of an arbitrary cyclic minimal surface that can be extended between two circles with radius $R = 1$ the same expression as for the catenoid.

Botany. — “*Contribution N°. 1 to the knowledge of the Flora of Java.*” By Dr. S. H. KOORDERS. (Continuation¹).

(Communicated in the meeting of March 28, 1908).

§ 3. On the geographical distribution, oecological conditions and means of dissemination of the *Aceraceae*, growing wild in the highest mountain regions of Java.

§§ 1. Synonyms and geographical distribution.

This order, which in BENTHAM and HOOKER's *Genera Plantarum* and in BOERLAGE *Handleid. Flora N. I.* forms part of the *Sapindaceae*, consists of two genera; only one of these (*Acer*, LINN.) occurs wild in Java. Of the genus *Acer* about 50 species are known; only one of these (*Acer niveum* BL.) belongs to the flora of Java, and has frequently been found there, growing wild in the higher mountain regions (up to 2550 m. above sealevel).

Some authors, e. g. PAX l. c., distinguish two varieties in Java, which were regarded by BLUME as species *Acer niveum* BL. *genuinum* PAX and *A. niveum* var. *cassiaefolia* (BL.) PAX. According to PAX l. c. the former of these has broad elliptical or ovate leaves with rounded base and a snowy white under surface, the latter oblong leaves with an acute base and a blue-grey under surface. The type is represented at Buitenzorg in Herb. Kds. by specimens from the G. Gedé (Herb. Kds. 12645 β) and the variety by specimens from Takóka (Herb. Kds. 7251 β). By far the greater number of specimens (e. g. many from the Gedé), however belong to neither of these two forms, as they combine various properties in a number of ways. We therefore consider the two varieties to be merely the extreme forms of one and the same, more or less varying²) type. Some specimens in Herb. Kds., should further be noted, in which the under surface of the leaf (in the dried state) appears to be green, e. g. Kds. 7265 β from the G. Slamát; by this character and also by the incipient serration of the leaf margin, these specimens approach to *A. laevigata* WALL. Kds. 7267 β from Pringombo should also be considered; the leaves, which, in the living state are pale blue-grey cannot be distinguished from those of *A. oblongum*. The colour of

¹) Continued from These Proc., Febr. 29th 1908 p. 687.

²) In his last monograph of the *Aceraceae* PAX l. c. (1902) 31 also, however, already says, that the variety *cassiaefolium* (BL.) PAX, which he formerly separated off, scarcely differs from the type.

dried specimens in general, and of this species in particular, depends according to KOORDERS and VALETON Bijdr. Booms. Java IX (1903) p. 256), very largely on the manner and rate of drying of the herbarium.

Acer niveum BL. Rumphia III (1847) 193 t. 167 B. f. 1; HIERN in Hook. Fl. Br. Ind. I, 693; PAX Monogr. d. Gattung Acer in ENGL. Botan. Jahrb. VII, 207; PAX in ENGLER Pflanzenreich Heft 8 IV, 163 (1902) 31; KOORD. et VALETON l.c. 254; — *A. laurinum* HASSK. in Tijdschr. v. Nat. Gesch. en Physiol. X (1843) 138 (nomen tantum); MIQ. Fl. Ind. Bat. I, 2 (1859) 582; — *A. javanicum* JUNGH. in Tijdschr. Nat. Gesch. en Physiol. VIII (1841) 391 (nomen tantum); — *A. cassiaefolium* BL. l. c. f. 2.

Geographical distribution outside Java: India or.: "Assam, hills of Martaban and Tenasserim" (BRANDIS, Indian Trees, 181). "Assam and Burma" (according to PAX l.c.) Malay Archipelago: Sumatra (JUNGH. in Herb. Lugd. Bat.); in N. E. Celebes in the Minalasa on the Lolomboelang mountains (Herb. Kds in Mus. H. Hort. Bogor; comp. KOORD. Verslag botan. reis N. O. Celebes (1898) p. 409). Has also been collected in Celebes by WARBURG (comp. PAX l.c. 31).

Geographical distribution and oecological conditions in Java: Has been collected, according to Herb. Kds, in Western and Central Java, and also in Eastern Java, at an altitude of 700—2550 m. at the following points. Hitherto (according to Herb. Kds.) it has been found in the following places in Java: In the res. Bantën on the G. Karang at 1000 m. above Tjimanoeck, and on the G. Poelasari at 1050 m. near bivouac Kihodjan (both in the division Pandeglang). In the res. Preanger: 1) on the G. Gèdé near and above Tjibodas at 1450 m., 1600 m., etc. and also at 2200 m. above sea level; 2) near Takoka at 1200 m. the Djampang; 3) near Pangentjongan in the Galoenggoeng (in the div. Limbangan at 1250 m., 1400 m., and at 1800 m. above sea level); 4) near Tjigenteng in the Kendeng-Patoeha mountains at 1450 m. and 1600 m. above sea level. In the res. Tegal-Pekalongan on the G. Slammat above Simpar at 1400 m. and above Soerdjã on the N.-W. Prahoe at 1400 m. In the res. Kedoe at 2200 m. on the G. Këmbang above Bédaka and at 2500 m. on the highest summit of the Prahoe-Diëng mountains. In the res. Banjoemas on the Midangan mountains near Pringãmbã 800 m. above sea level In the res. Semarang on

the G. Oengaran and the G. Telémajā at about 1400 m., e.g. above Sēpa-koeng In the res. Madioen on the Wilis-mountains above Ngeběl between 1400 m. and 2000 m. (not collected there at a greater height). In the res. Prābālingā-Pasaroehan on the Tēngger-mountains at 2000 m. near Ngadisari. In de res. Bēsoeki on the Idjenplateau near bivouac Oengoeop-oengoeop at 1700 m. Up to the present this species is therefore known from the res. Bantōn (in Western-Java) to the res. Bēsoeki (in Eastern-Java) from 700 m. to 2550 m. above sealevel. — Occurrence: Does not grow socially in Java, but occurs fairly plentifully in some mountain forests e.g. in Western-Java on the G. Gédé. Ecological conditions: This species has not yet been observed by me in Java on soils, where there is a great, permanent dearth of water nor where there is physiological drought resulting from a large saline content, nor on soils rich in lime and common salt; neither does the species grow on soils which are periodically liable to strong desiccation. It grows almost exclusively on permanently damp, fertile, volcanic soils, rich in humus, in close shady mountain forests of high trees and consisting of a great number of species. In the hot plain, even in permanently humid districts, the species does not occur. The lowest station is in a ravine in Eastern-Java at about 700 m., the highest is at nearly 2550 m. above sealevel in Central-Java. I feel obliged to consider the possibility of the occasional, be it very exceptional, occurrence of *Acer niveum* in physiologically dry, saline soils, in consequence of a herbarium note of JUNGHUN, found by me in 's Rijks Herbarium, and referring to a specimen, collected by this naturalist on the Diēng-plateau at about 2000 m. near the Kawah-Tjondro-dimoeko. I have here as yet no other data at my disposal, which would show with certainty, whether this species does not only occur in Java "near", but also "on" such soils. — Leaf fall: At the same moment there stood in the same locality (in the same forest, in close proximity to each other) two individuals of apparently the same age. On the 2nd of June 1898 one of these was in full (old) leaf, while the adjoining specimen was practically without leaves, except one branch which bore young foliage. On March 23rd 1893, near Takóka, one of the trees (of this species), which had been numbered for the purpose of the investigation, was completely without leaves, although it stood in the midst of tree species, which were then nearly all in full foliage. — Time of flowering and fruiting: Flowers were collected in June and in July, August, Sept. and Nov. — Habitus: A forest giant, which immediately reveals its presence, even in the thickest virgin forest,

by its fallen leaves on the ground, and sometimes by its characteristic winged fruits; the leaves are noticeable on account of the colour of their lower surface, which remains greyish white for a fairly long time. This greyish white or bluish grey colour is also rather striking in the living plant. In the flowering period this giant of the forest further attracts attention by its almost leafless condition in the midst of evergreen trees. In alpine regions, at 2000 m. above sealevel, in Western and Central-Java, (e.g. Preanger, Bagelen), this species stands out by its dimensions, which are rather considerable for a high altitude; so, for instance, at 2200 m. sealevel, on the G. Kembang near Bédaka, a specimen was 20 m. high, with a trunk $\frac{1}{2}$ m. in diam. The above data about Java, relating to oecological conditions and geographical distribution, have been taken from observations, made by me in Java 1888—1903, and mostly published in KOORDERS and VALETON l.c. 257—258. — In the National Herbaria at Leiden and at Utrecht I found with the specimens, collected in Java by JUNGHUHN, BLUME, REINWARDT, etc., and now examined by me, no special data about oecological conditions: in most cases there was only written on the labels “Java” without further indications.

§§ 2. Means of dissemination.

The only means of dissemination is the fruit, known as *samara*, which is primarily intended for distribution by wind, but which seems, in addition, to have a certain capacity for being transported by water, according to an experiment of mine. At least, if the fruits are quite dry, they remain floating for some days on a $3\frac{1}{2}$ % solution of common salt. In this species the fruits are produced in Java, as far as is known, only once a year, but then mostly in great numbers. Although the winged fruits are fairly heavy (when dry they weigh about 100 milligrams, the wings, which are often 5 cm. long and 2 cm. broad, being included), and although I never found in Java any indication, that the fruits are distributed by animals, distribution must nevertheless take place easily, as is proved by the large number of localities, cited above, where the tree is found. As the occurrence of the species is limited to the higher regions of several active volcanoes, at places which are more than 40 kilometres apart, and which are separated by hot plains, in which the species has never been found wild in Java at the present time, it would appear, that the force of the wind on the higher mountains of Java is sufficient for transport over a distance of 40 kilometres, even of such large samaræ as those of *Acer nireum*.

I think it however more probable, that in the case of this species, as in that of the next one, (*Dodonaea viscosa*) such large winged fruits have been and are still, only transported in stages. It may have been, that in former times other climatological conditions enabled these two species to grow wild in the 30—40 kilometres of intervening low lands, in such places where growth can no longer take place at the present time. It may also be that even under the present conditions of climate, isolated specimens have escaped notice and might be found between the two places so far apart. Finally we may suggest, that transport by wind does not primarily take place through the air direct, but chiefly in stages, in such a way, that the fruits remain for a longer or shorter time on the ground, or floating on the surface of water; in the latter case of course, till they are washed ashore and are then carried further by the wind.

The original occurrence across the sea of this *Acer* growing wild in Java and provided with fruits, which are apparently only adapted for wind transport, may, it seems to me, be readily explained by a combination of wind transport in stages with transport by water, but not exclusively by so called direct wind transport. This explanation possibly also applies to other species, growing in Java and belonging to other genera or orders, with physiologically similar fruits or seeds, which have hitherto only been regarded as anemophilous.

It should further be noted, that this species only bears fruit at an advanced age, when the crown has already attained a considerable height. This character is perhaps useful, since the tree generally occurs scattered in dense ever-green heterogeneous mountain forests, composed of high trees. For this species, which is obviously in the main dependent on wind distribution, the above-mentioned character is probably connected with the ecological conditions determining the original occurrence of the tree, and the character referred to, has arisen through natural selection. For in the damp Javanese mountain forests, which are generally very dense, only those species have a good chance of being disseminated by the wind, whose fruiting branches protrude above the dense leaf covering, formed by the crowns of the surrounding trees.

In connexion with the obvious relation between the conditions of growth, the fruiting period and the means of dissemination of *Acer nireum*, we may quote what has been said by VÖGLER¹⁾ on a similar

¹⁾ VÖGLER, P., Ueber die Verbreitungsmittel der Schweizerischen Alpenflanzen in Flora oder allg. botan. Zeitung 89 (1901) p. 2.

relation in the case of other species likewise having large winged fruits, such as those of *Acer*.

“... Derartige Arbeiten erhielten einen viel grösseren Werth, wenn sie einem Zusammenhang oder auch nur Paralellismus zwischen den ähnlichen Verbreitungsmitteln und anderen durchgehenden biologischen Verhältnissen der betreffenden Arten nachgingen. Eine ganz kleine Untersuchung dieser Art bietet LUBBOCK¹⁾ in dem er nachweist, dass von 30 Gattungen, ““figured as having seeds or fruits with a long wing, known as a *Samara*””, alle zu den Bäumen oder Klettersträuchern gehören, keine einzige zu den niedrigen Kräutern”
VOGLER l.c. .

§ 4. On the geographical distribution, oecological conditions and means of dissemination of the Sapindaceae, growing wild in the highest mountain regions of Java.

The *Sapindaceae*, as defined by RADLKOFER in ENGLER and PRANTL's, *Natürliche Pflanzenfamilien*, consist of about 73 genera with over 600 species. Of these only a single species occurs in Java, growing wild in the highest mountain regions, namely *Dodonaea viscosa* (LINN) JACQ.

§§ 1. Synonyms.

Dodonaea viscosa (LINN.) JACQ. Enum. Pl. Carib. 19, non SIEBER, non Mart.; HIERN. in HOOK. Fl. Br. Ind. I, 697; KURZ For. Flora I, 287; BRANDIS Indian trees (1906) 186; HASSK. Pl. Jav. var. 292; KOORD. en VALETON Bijdr. Booms. Java IX (1903) 226; — *D. angustifolia* BLANCO Fl. Filip. ed. I, 312; — *D. angustifolia* LINN. F Suppl. 218; — *D. Burmanniana* DC. Prod. I, 616; — *D. Candollei* BL. ! msc. in Herb. Lugd. Bat. = *D. Candoleana* BLUME ! Rumphia III, 190; — *D. divica* ROXB. Hort. Beng. (28); Fl. Ind. II, 256; — *D. Dombeyana* BL. ! in Rumphia III, 189; — *D. ferrea* JUNGH. ! msc. forma 1, 2 et 3 in Herb. Lugd. Bat.; — *D. jamaicensis* DC. Rod. I, 616; — *D. Kingii* G. DON, Syst. I, 674; — *D. latifolia* SALIB. Prod. 276; — *D. microcarpu* DC. Prod. I, 617; — *D. montana et littoralis* JUNGH. in Java I, ed. II 267; — *D. nerifolia* A. CUNN. ex A. GRAY Bot. U. St. Expl. Exped. I. 262; — *D. oblongifolia* LINK. Enum. Hort. Berol. I, 381; et in Bot. Reg. t. 1051; — *D. ovata* DUM.-COURS. Bot. Cult. ed. II, 7, p. 327; — *D. pallida* MIQ. ! Anal. Bot. Ind. III, 7; — *D. pen-*

¹⁾ LUBBOCK, Flowers, fruits and leaves. London (1886) p. 79 (quoted by VOGLER l.c.).

tandra GRIFF. Notul. IV, 548; — *D. salicifolia* DC. Prod. I, 617; — *D. Schiedeana* SCHLECHT. in Linnaea XVIII (1844) 33 (err. typ. 49); — *D. senegalensis* BLUME! nosc. in Herb. Lugd. Bat.; — *D. spatulata* SM. in REES Cycl. XII n. z.; — *D. triquetra* JUNGH. in Natuurk. en Geneesk. Arch. Neêrl. Indië II (1845) 36; *non* ANDR.; — *D. viscosa* ROYEN ex BLUME!, Rumphia III, 191; — *D. Wightiana* BLUME in Rumphia III, 189; — *D. Waitziana* BLUME! l. i.; — *D. Zollingeri* TURCZ. in Bull. Soc. Nat. Mosc. XXXVI (1863) I, p. 587; — *Caryophyllanthus littoreus* RUMPHIUS Herb. Amb. IV, t. 50; — *Ptelea viscosa* LINN. Spec. ed. I, 108.

For the very numerous synonyms of this polymorphic species, which has extremely wide vertical and horizontal distribution, I have chiefly relied on the most recent literature as regards these species, which occur outside the Dutch East Indies, but have checked them as far as possible by the very rich material in the National Herbaria at Leiden and at Utrecht. The Dutch East-Indian synonyms are chiefly based on my own examination of the above collections, and on KOORDERS and VALETON Bijdr. Booms. IX l. c. From various facts it appears that this tree (at least the littoral form) was already known to RUMPHIUS, and that it has been described as separate species by a large number of authors under more than 25 different specific names.

According to an unpublished note of REINWARDT, found by me with a herbarium specimen collected on the sandy beach of Ternate, this observer has the credit of having already realized, that the coast and the mountain forms of the specimens of *Dodonaea viscosa* from Malay Archipelago belong to one and the same species.

§§ 2. Geographical distribution and oecological conditions of *Dodonaea viscosa* outside Java.

According to the literature (e. g. RADIKOFER) and the herbaria consulted by me at Leiden and at Utrecht, *Dodonaea viscosa* is generally distributed in tropical and subtropical regions of the whole world, and is known outside Java from sandy sea shores as well as from inland localities up to an altitude of 1400 meters. BRANDIS [Indian Trees (1906) 187] states: "Trans Indus, Afghanistan and Beluchistan. Common locally, often covering extensive tracts in the drier regions of North-West and Central India as well in the Deccan. Also on the seacoast" (BRANDIS l. c.). In the National Herbarium at Leiden I saw an authentic herbarium specimen of *Dodonaea arabica*

HOCHST and STEUD. According to the attached label, this specimen was collected on Dec. 8th, 1835 by W. SCHIMPER (the father of the phytogeographer F. W. SCHIMPER) at 4000 feet (1330 meters) above sea level on the summit of the mountain Kara in Hedschas (Arabia). According to HOOKER Flora Brit. India l.c. this specimen is identical with the widely distributed *Dodonaea viscosa* (L.) JACQ. HOOKER'S view is undoubtedly correct. It seems to me that the occurrence of the littoral *D. viscosa* (L.) JACQ. on the above-mentioned mountain can easily be explained, by assuming that the locality, where SCHIMPER collected his *Dodonaea*, was extremely poor in water. In 's Rijks Herbarium at Leiden I also saw a specimen of *Dodonaea viscosa* L. (det. P. HENNINGS) from Herb. SCHLAGINTWEIT N^o. 80846, which was collected in the Panjab in North-West India between November 15th to 28th, 1855 at 650—850 meters above sea-level, and finally a specimen from Herb. FIEBRIG N^o. 2501, correctly named *Dodonaea viscosa*, which was collected in 1903—1904 in Eastern Bolivia (South America) at a height of 1400 meters. As proved by a herbarium specimen from British India, due to HOOKER and THOMSON, and seen by me in Rijks Herb. at Leiden, *Dodonaea Burmanniana* D. C. which is synonymous with *D. viscosa*, grows there at a height of 0—600 meters above the sea. In 's Rijks Herbarium at Leiden I further saw a herbarium specimen, which according to the label, had been collected in 1841 by FORSTEN "on extensive beds of lava" in Ternate (Spice Islands); this specimen had been determined by BLUME as *Dodonaea Candollei* Bl. var. *minor* BLUME. In my opinion there is no doubt, that this is merely a form (from an arid locality) of the ordinary *Dodonaea viscosa* (L.) JACQ.

§§ 3. Geographical distribution and oecological conditions of *Dodonaea viscosa* in Java.

The following data regarding the vertical and horizontal distribution, and the oecological condition, of *Dodonaea viscosa* (LINN.) JACQ. which, in part have already been published in KOORDERS and VALETON l.c., can now be communicated; they are based on observations made by myself in Java 1885—1906, and on herbarium specimens collected by me.

In Western and Central Java, as well as in Eastern Java on sandy sea-shores, further in Central and in Eastern Java at 1450 m. above sea-level and higher, especially above 1800 m. and still at 2600 m. According to Herb. Kds. it has been collected in Java in the following localities: In Western Java: near Tjemara in S. W.

Banten, growing on the flat sandy beach. In the Southern Preanger near Palaboehanratoc, also on the sandy beach. In Central Java : on the G. Prahoe at 2000 m. on the Prahoe-Diëng mountains along the path from Soerdjã to the Diëng plateau in the res. Tegal-Pekalongan. Near Sepakoeng (res. Semarang) on the G. Telemãjã at about 1700 m. and also in the res. Semarang on the G. Merbaboeh above Andongtjemoro at about 1600 m. In the res. Kedoe on the G. Sendarã near Kledoeng at about 1600 m. In the res. Madioen on the G. Wilis above Ngebël at 1450 m. and higher up the mountain to 2000 m. In the res. Pasoeroehan-Probolinggo on the G. Ardjoenã above Malang at about 2100 m. and on the Tengger mountains above Tosari and Ngadisari still at 2600 m. above sea-level. In the res. Besoeki on the Idjen plateau near the bivouac Oengoep-oengoep at 1700 m. and on the Kendëng ridge above Pantjoer at 1700 m.; also on the sandy beach of Gradjagan and on the sandy beach of Poegër (on the South coast of the divisions Banjoewangi and Djember respectively). Completely absent from the regions between the above alpine stations and those in the beach. On the other hand where this *Dodonaea* (*D. viscosa*) appears, it generally either grows socially forming smaller or larger woods, or it occurs at least in very large numbers. — **Oecological conditions.** It is completely restricted (at least when growing wild) to physiologically dry localities, namely either to the dry alpine regions of Central and Eastern Java above 1400 m. or on to the sea-beach, which is physiologically dry in consequence of its richness in salts. On the beach this species has been observed by me in W., as well as in Eastern Java. (Compare also under "Means of dissemination", and further K. & V. l. c. 229.

§§ 4. Means of dissemination of *Dodonaea viscosa*.

The inflated, thin-walled, light, winged fruits are not only eminently adapted for wind distribution, but (as has already been mentioned by some authors, and has been confirmed by me experimentally), they are also extremely well suited for transport by water. Of some fruits, which I placed in a $3\frac{1}{2}\%$ solution of common salt, 80% still floated after 25 days.

In Java the plant bears a large number of fruits at an early age, e. g. before it is 2 years old.

As I have observed in Central Java, this species occurs wild on two volcanoes which are more than 40 kilometres apart, in a straight line, and on these only above an altitude of 1400 m., whereas it

is completely wanting in the intervening plain, except on the sea-beach, 30 kilometres off. Since moreover no argument has been advanced in favour of dissemination by animals, it would appear, that the winds of Central Java are capable of transporting the fruits of *Dodonaea viscosa* over a distance of more than 30 kilometres although these fruits weigh 0,040 grams, and have a surface of 2½ square centimeters.

There is, however, scarcely need, to point out here, that great care is necessary¹⁾ in drawing conclusions as to transport by wind. I only refer to what has been said above, regarding the wind distribution of *Acer niveum*. Notwithstanding the apparent possibility of a direct transport by wind over large distances, I consider that also in the case of *Dodonaea* windtransport in stages is much more probable.

Its general occurrence on the tropical shores of the whole world is sufficient evidence of the extreme suitability for transport by water over very great distances, so that no more need be said on this point.

The extraordinary power of resistance, which I have repeatedly observed, against drought of the air and of the soil, against direct sunlight, against the saline contents of the soil and also against strong winds, together with the property of bearing numerous fruits at an early age, which fruits are well adapted to transport by wind and by water (also by sea water) — all these characteristics fully explain, why this tree appears in Java, as the pioneer of new vegetation not only in alpine regions, but also on sandy sea beaches.

According to what has been said above, the almost complete absence of the species from the broad belt between the beach and the mountains, is probably due, to the crowding out by other plants of such seedlings as may arise from fruits, which doubtless frequently fall in the intervening zone.

Summarising, it appears to me, that the apparently whimsical distribution of this characteristic Javanese Sapindacea can be readily deduced, with a large degree of probability, from the properties mentioned above, and especially from those properties, which are connected with the edaphic condition of the species.

§ 5. Note on some incompletely known species of *Quercus*, in 's Rijks Herbarium, at Leiden.

In KOORD. and VALETON Bijdr. Booms. Java X, 65 there are mentioned at the end of the description of 25 species of *Quercus*, growing

¹⁾ Compare also VÖGLER in SCHROETER l. c. 740.

wild in Java, five further species as "doubtful and incompletely known"; the latter were included on the authority of BLUME Mus. Lugd. Bat. I, 294—304; we were unable at the time at Buitenzorg to refer to the authentic specimens of these.

As I have now been able to examine the authentic specimens of BLUME in 's Rijks Herbarium at Leiden, I append my observations regarding these species.

1. *Quercus Pinanga* BLUME Mus. Lugd. Bat. I (1850) 303.

I completely agree with the view of KING, quoted in KOORD. and VALETON l. c. 65. The remark, published by BLUME l. c., that the above-mentioned species occurs "in Java in the mountain forest" must therefore be regarded as not wholly accurate because BLUME evidently prepared his diagnosis from a few leaves of *Quercus glabra* THUNB. (from Japan) an old tree of which was observed by me in a cultivated state in Hort. Bogor. as late as 1903.

Q. Pinanga BLUME should therefore be erased from the Flora of Java and be considered synonymous with *Q. glabra* THUNB.

2. *Quercus litoralis* BLUME l. c. 303.

On the authentic herbarium label there was written i. a.: "*Quercus litteralis* Bl., Java, leg. BLUME, Pasang-laut (Sund)".

Since the native name is Sundanese, this species cannot come from Eastern Java, as BLUME l. c. incorrectly remarks, but must come from Western Java, probably from the Preanger or Banten, where most of the specimens, collected by BLUME, were obtained.

The authentic specimen I regard as beyond doubt synonymous with *Quercus spicata* SM. var. *gracilipes* KING (comp. KOORD. and VALETON l. c. 42). This species of BLUME's must also therefore be deleted.

3. *Quercus glutinosa* BLUME l. c. 304.

According to the authentic herbarium label of REINWARDT this species was named by REINWARDT in manuscript *Quercus micans* REINW., and was afterwards renamed by BLUME *Quercus glutinosa* BL. moreover, it was not collected "in the mountain forests of Western Java" but found by REINWARDT near Tondano in N.E. Celebes, in the year 1821. This species can therefore also be deleted from the flora of Java. It is not, as MIQUEL incorrectly thought, identical with *Quercus induta* BL., to which it shows a superficial resemblance; the species is specifically distinct from *Q. induta* BL., as was indeed already correctly surmised by DE CANDOLLE and by KING (comp. KOORD. and VALETON l. c. 65).

4. *Quercus sphacelata* BLUME l. c. 304.

The authentic specimen of this species consists of a branch with

leaves, but without flowers. I consider it a large leaved shoot (for instance, from a latent bud of the trunk) of *Quercus spicata* SM. var. *gracilipes* KING.

On the authentic label is written: "*Quercus sphaeelata* Bl., Pasang, Java, in montanis Moeriah, Herb. Waitz."

5. *Quercus nitida* Bl. l.c. 294.

The view, already expressed in KOORD. and VALETON l.c. 65, that this species, which so far has only been recorded with certainty from Sumatra, does not yet belong to the flora of Java, is confirmed in my opinion, by the material in 's-Rijks Herbarium at Leiden.

Leiden, March 1908.

(To be continued).

Geophysics. — "*The Starting Impulse of Magnetic Disturbances.*"

By DR. W. VAN BEMMELEN.

(Communicated in the Meeting of March 28, 1908).

Last year¹⁾ I communicated the compilation of a statistical list of the magnetic disturbances which the magnetograph at Batavia has recorded during the period 1880—1899. I drew the attention to the phenomenon of the starting impulse i.e. the suddenly appearing change of the magnetic elements, which very often accompanies the beginning of a magnetic storm.

This phenomenon appearing in like manner at Batavia and at other places, I ventured a supposition on the manner in which we can represent to ourselves the appearance of magnetic disturbances. To obtain a closer knowledge of this in my opinion very instructive phenomenon, I requested at the end of 1906 all Magnetic Observatories to give me their data for a number of cases selected by myself.

With great readiness those data were forwarded to me from several observatories and it is an agreeable duty for me to express at this place my thanks for it.

Besides this material received from many sides I have worked out all cases registered at Batavia and at Buitenzorg and have also been able to watch the nature of the electric earth-current during the phenomenon. I wish to communicate here of the results of this material what is most important, commencing with Batavia.

¹⁾ Proceedings 29 September 1906.

Also: Observations made at the R. Magn. and Met. Observatory of Batavia, Vol. XXVIII, App. III.

BATAVIA.

Out of the diagrams obtained in the period 1882—1899, I measured for 131 cases the amount and the duration of the initial movements of the three components (Horizontal Intensity = H , Declination = D , Vertical Intensity = Z).

Direction of the initial movement.

ΔH was without exception positive.

ΔD was with a few exceptions West; but 12% of the number of cases was introduced by a slight Easterly movement.

ΔZ was negative; but in 6% of the number the movement was introduced by a slight positive movement.

Duration.

Here I have *not* taken into consideration the duration of the slight introductory movement.

124 cases furnished :

$$\angle H = 4.5 \text{ min.} \quad \angle D = 3.2 \text{ min.} \quad \angle Z = 12.0 \text{ min.}$$

The duration of the Z movement is in general difficult to determine, as the decrease of the vertical force keeps on mostly much longer.

It is important to notice that the initial movement of D stops or is inverted, whilst of H the increasing movement keeps on.

Amount.

The average amplitude of the movement, arranged according to the different parts of the day in which it took place, expressed in 0.00001 C. G. S. ($= \gamma$):

h	ΔH	ΔD	ΔZ
0—6 a.m.	+ 45	7 W	— 11
6—12 „	+ 41	10 „	— 16
0—6 p.m.	+ 52	7 „	— 16
6—12 „	+ 40	8 „	— 11

Of a characteristic inequality of the vector during the day little is noticeable. The amount ΔH arranged according to the duration of the movement is:

Duration	ΔH	Number of cases
0— 2 min.	53 γ	15
2— 4 „	43	45
4— 6 „	42	35
6— 8 „	33	20
8— 15 „	46	6

So the amount of the movement is fairly well independent of the duration, from which results inversely that *the shorter the increase of H is, the quicker it is.*

Let us finally observe that the appearing of the slight easterly movement did not show any preference for certain times of day or year.

BUITENZORG.

Since May 1906 a TÖPFFER-SCHULZE magnetograph has registered at Buitenzorg, which gives the curves of the three elements on the same registering-strip; this circumstance besides that of giving finer lines, greater sensibility and wider measure of time, is very suitable for the study of the initial-movement.

For the period May 1906 — Nov. 1907 I measured 29 cases and from that material it was clearly evident that in most cases the movements of the elements display a certain independence of each other and *do not always begin at the same moment.*

I calculated the azimuth of the horizontal component of the vector for the first part of the movement, so *before* the movement of D is inverted, and I found in 20 cases directions between the extremes N and $N 58^{\circ} W$, an average of

$$N 21^{\circ} W.$$

Vertical Intensity.

The results for ΔZ were surprising; for the vertical component showed, different from that of Batavia, an introductory positive movement followed by the slow negative movement known of Batavia.

Here no instrumental cause had anything to do with the matter: both magnetographs (of ADIE and SCHULZE) registered this pre-movement at one time at Batavia very rarely, but at Buitenzorg regularly. Luckily registrations have been made for more than a year at Batavia and at Buitenzorg at the same time and from those registrations it was evident that the introductory movement at Buitenzorg precedes that of Batavia. The introductory movement at Buitenzorg commences (according to the average of 29 cases) 0.3 minute after the H -movement begins and lasts about 1 to 3 minutes, after which the Z -lines of both places show simultaneously a decrease.

ANSWERS TO THE QUESTIONS.

The data on the initial movement were asked for in my letter

for a number of cases, in which that movement had made its appearance under different circumstances at Batavia. The answers were of a very different nature and therefore I made of each case a summary diagram in which in an equal manner under each other was noted down the registration image of H , D and Z for each station. In many respects it would be useful to reproduce these diagrams, but the difficulties connected with it and the numerous imperfections of the material have made me set it aside.

These imperfections are chiefly caused by the measure of time of the diagrams of the various observatories not being taken ample enough to be able to fix the simultaneity of the different movements, which take place within a few minutes.

From the notation of the TÖFFER-SCHULZE magnetograph at Buitenzorg where the circumstances were pretty favourable I could deduce with certainty that the commencement of the movement of the three components is often not simultaneous. Accuracy down to parts of minutes cannot be demanded of most magnetic diagrams, where 1 hour takes up 10, 15 or 20 mm. It was therefore impossible to draw trustworthy maps, on which the vector of disturbance during the initial movement was represented in its varying magnitude and direction, so I must restrict myself to the following.

In the following table the direction of the movement has been given for all the cases.

In the case of an introductory movement of slight amplitude this is indicated for H by $+/-$ or $-/+$, for D by w/E or e/W .

In some cases the introductory movement was of the same order of amplitude with the following movement and this is accordingly indicated by $+/-$ or $-/+$ and W/E or E/W .

We see in this table the movements of *one* or more elements for different constant direction. Further information is furnished by the annotation of the Rev. P. DE MOIDREY in the "Bulletin des Observations de l'Observatoire de Zi-Ka-Wei, T. XXXI" and the copies of disturbances for Greenwich, Parc st. Maur and Samoa. The former states for Zi-Ka-Wei the frequency of the positive H and Z movement at 95 pCt. and of the E movement of D at 90 pCt.; whilst out of the Samoa curves the H and Z movement proves to be $+$ in 9 cases.

Hence the following movements appear pretty constant:

Station	<i>H</i>	<i>D</i>	<i>Z</i>	Geogr. Latitude
Batavia	+	<i>W</i>	—	—6°
Buitenzorg	+	<i>W</i>	+ / —	—7°
Manila	+		+	15°
Zi-Ka-Wei	+	<i>E</i>	—	31°
Samoa	+		+	—13°
Honolulu	+		no reg.	20°
San Antonio			+	29°
Coïmbra	+		no reg.	40°
Greenwich			+	51
St. Maur			—	49
Perpignan	+			43°
Bombay	+	no reg.	no reg.	19
Mauritius	+		+	—20°
Melbourne			—	—38°

So it seems that the constancy in the appearance of a definite sense of movement decreases with the geographical latitude and that it is furthermore for *H* the greatest and for *D* the smallest, moreover for *H* always with a *positive* sense of movement.

EXTENSION OF THE MOVEMENT ABOUT THE EARTH.

The nature of the initial movements, which took place simultaneously on different points of the surface of the earth could be studied by means of the above mentioned summary diagrams.

It was quite evident that for *H* and *D* but not for *Z* places lying close to each other show about the same image, but that for places in other parts of the world this is often quite different.

The small number of stations only allowed a closer investigation of that difference for North-America and Europe. I therefore give below a survey of the movement for *H* and *D* in Europe and North-America, where however the latter for the years 1892—94 is represented only by two stations (Toronto in Canada and San Antonio in Texas).

Date	Europe		N.-America	
	ΔH	ΔD	ΔH	ΔD
May 18 th 1892	+/-	w/E	-/+	e/W
July 16 th „	-	E and W	are missing	
Aug. 12 th „	-	E and W	+	E and W
„ 18 th 1893	+	e/W	are missing	
Sept. 25 th „	+	W	„	„
Jan. 2 nd 1894	+	w/E	-/+	E/W (only San Antonio)
Febr. 20 th „	-/+	w/E	-/+	w/E
July 20 th „	+	e/W	+	e/W
Aug. 20 th „	+	e/W	+	w/E
Nov. 13 th „	-/+	e/W	+/-	w/E
Aug. 16 th 1902	+	e/W	+	w/E
April 5 th 1903	+	e/W	-/+	w/E
Febr. 3 rd 1905	+	e/W	+	w/E

We see here repeatedly that the European group and the American one are the reverse of each other, and in Europe we find mostly a *N.W.* vector, in America a *N.E.* one, which points to a centre of disturbance near Greenland, thus situated near the magnetic pole and that of the Aurora-Borealis or pole of disturbance.

For some cases I have tried to obtain a survey by means of a map with the simultaneous vectors of disturbance, but here the almost insurmountable difficulty presents itself, that one cannot make up simultaneous values of ΔH and ΔD , which are really trustworthy. Above I pointed to the fact with reference to the measurements on the Buitenzorg magnetograms, that this requires an ample and trustworthy time-measurement. Though it was impossible for me to calculate exact values for the azimuth of those vectors, yet I could about fairly well find the direction.

For the initial movement of the disturbance on Febr. 3^d, 1905 the vectors of the introductory movement pointed to a centre at the West coast of North America and that of the main movement to a centre near Greenland. For the disturbances on Aug. 16th, 1902 and April 5th, 1903 I found about the same ¹⁾.

VERTICAL FORCE.

It is a striking fact, that places lying at a slight distance from each other show an opposite change in the vertical component. Above

¹⁾ Writing this I see from a paper by Dr. BRÜCKMANN (*Meteor. Zeitschrift*, 1907, No. 12) on the same subject, that he also arrives at a centre of disturbance appearing in the vicinity of the magnetic pole.

I already mentioned that Batavia and Buitenzorg are different in this respect and now I discovered that Greenwich and Paris (Pare St. Maur) show regularly an opposite movement. Out of the reproductions of diagrams of disturbances published for several years I made the following list.

St. Maur and Greenwich.

Date	Hour	St. Maur			Greenwich		
		ΔH	ΔD	ΔZ	ΔH	ΔD	ΔZ
1891 March 2	2	+	<i>e/W</i>	-	+	<i>e/W</i>	+
June 14	9	+	<i>E</i>	-	+/-	<i>e/W</i>	+/-
1892 January 4	19	+	<i>E</i>	-	+	<i>w/E</i>	
February 13	5	+	<i>E W</i>	-	-/+	<i>E/W</i>	+
„ 20	19	+	<i>E/W</i>	-	+	<i>W</i>	+
March 11	23	+	<i>E/W</i>	-	+	<i>W</i>	+
May 16	22	+	<i>E</i>	-	+	<i>E</i>	+
„ 18	8	+	<i>w/E</i>	-	+/-	<i>w/E</i>	+/-
June 27	5	+	<i>E</i>	-	+	<i>W</i>	+
July 12	18	+	<i>W</i>	-	+	<i>W</i>	+
July 16	13	+	<i>e W</i>	+/-	+	<i>E/W</i>	-/+
Aug. 3	14	+	<i>W</i>	-	+	<i>W</i>	+
Sept. 5	23	-/+	<i>E/W</i>	-	-/+	<i>E/W</i>	+
1893 March 25	4	+	<i>E</i>	-	+	<i>E</i>	+
April 26	16	-/+	<i>W</i>	-	-/+	<i>e/W</i>	-/+
June 9	13	+	<i>W</i>	-	-/+	<i>e/W</i>	-/+
August 6	4	+	<i>e/W</i>	-	+	<i>e/W</i>	+
September 8	1	+	<i>W</i>	-	-/+	<i>w/E</i>	+
1894 January 11	20	+	<i>W</i>	-	+	<i>W</i>	+/-
February 20	20	+	<i>E</i>	-	-/+	<i>W/E</i>	+
„ 22	22	+	<i>W</i>	+/-	+	<i>W</i>	+
„ 28	15	+	<i>W</i>	-	/+	<i>e/W</i>	-/+
July 20	6	+	<i>E</i>	-	+	<i>E</i>	+
1895 May 29	15	-/+	<i>e/W</i>	-	-/+	<i>e/W</i>	-/+
1896 July 23	18	-/+	<i>W</i>	-	-/+	<i>e/W</i>	+
Aug. 29	17	-/+	<i>W</i>	-	-/+	<i>e/W</i>	-/+
1898 March 15	1	+	<i>W</i>	-	+	<i>W</i>	+
„ „ 13		+	<i>W</i>	-	+	<i>W</i>	+
September 9	14	+	<i>W</i>	-	+	<i>W</i>	+ ¹⁾
1899 January 28	19	+	<i>W</i>	-	+	<i>W</i>	+
June 28	22	+	<i>W</i>	-	+	<i>W</i>	+ ¹⁾

¹⁾ During a disturbance.

It is seen that ΔH and ΔD generally correspond; only for H the introductory movement is more frequent at Greenwich; ΔZ is regularly opposite, for Greenwich positive and for Paris negative.

It is remarkable to notice out of the reproduced registering lines how the oscillations following upon the initial movement correspond again for the two places; a single striking quick movement amid the disturbance, as it were a new starting impulse, is then again opposite. This repetition seems to be a real phenomenon. Thus the initial shock on Oct. 30th, 1903 was a clear initial movement amid a disturbance going on already for hours. At places with higher latitude it lost itself in the oscillations of that older disturbance. The phenomenal violence of the second part of the disturbance is perhaps owing to two disturbances being placed one above the other. At the violent disturbances of Febr. 13th, March 6th and June 27, 1892, as well as of Aug. 6th, 1893 two initial impulses appeared.

THE CAUSES OF THE INITIAL MOVEMENT.

The remarkable inequality of the movement in the vertical force, so constant for places situated close to each other, offers us perhaps a means to clear up what is puzzling in this phenomenon.

If we attribute the appearance of those vectors of disturbance to that of electric currents, as is more than probable, then it is impossible to assume that the movement of electricity which generates these vectors would have in the free atmosphere such a distribution limited to the place. The cause of this must be in the appearance of the electric earth-current. We must assume that, when suddenly a disturbance arises, the earth-current then generated selects fixed paths through the earth-crust.

That the earth-current for different places of the earth situated close to each other may be different, is highly probable; at least for the surface-current I have found it lately¹⁾ for North- and South-Java. The inequality was, that as the corresponding magnetic variations became shorter the earth-current variations increased more in amplitude for the volcanic southern part than for the alluvial and diluvial northern part.

This great difference in earth-current must become much less for the deeper strata; proof of this is found in the equality of the magnetic variations at Batavia situated on the Northcoast of Java and at Buitenzorg on the edge of the volcanic part.

But the possibility for a difference when a current is suddenly

¹⁾ See the following paper: Earth-current registration at Batavia, 2nd communication.

generated is made possible by these results for deeper strata too. On the diagrams obtained at Batavia when the earth-current is registered, there are a few cases of an initial impulse and as the time unit was very ample (1 millim. = 1 min.) and moreover as the magnetic component too was registered with great sensitiveness, these cases are very instructive.

Date and Hour.	Initial movement of the <i>E-W</i> earth-current commences before that of the magn. North-component.
14 May 1906, 4 ^h a.m. Bat. T'	0.0 min. Commenc. gradual
30 July " 3 " "	0.6 " " "
3 September " 7 p.m. " "	0.0 " " pretty sudden
22 " " 8 " " "	0.0 " " sudden
10 November " 12 " " "	0.0 " " pretty sudden
26 " " 1 am " "	0.0 " " sudden
26 December " 11 p.m. " "	0.0 " " "
8 January 1907 12 " " "	0.0 " " "
15 " " 3 am. " "	0.0 " " pretty sudden
14 February " 3 " " "	0.0 " " sudden
27 January 1908 9 p.m. " "	0.0 " " "

With the exception of *one* case, where indeed the determination of the time was less accurate on account of the gradual commencement of the initial movement, we thus find simultaneousness for the initial movement for earth-current and magnetic vector.

As has often happened, we must change a hypothesis of explanation formed on first getting acquainted with the facts, when later on we have arrived at a more extensive knowledge of the facts by extension of the material.

This is the case here too.

Though I at first thought to find the seat of the current of electricity which is supposed to generate the initial impulse in the highest layers of the atmosphere, the nature now revealed of the vertical component induces me to look for the seat rather in the earth itself.

At the outset the current must be in general an East-West current of positive or West-East of negative electricity, because everywhere the horizontal magnetic component increases. The situation and form of that current seems to be variable and to undergo a great influence

of a proper magnetism of the earth; it also seems to change during the increase in intensity and situation. For the magnetic disturbance itself following immediately upon the initial impulse we must assume that especially extra terrestrial currents are the cause; at least for the magnetic after-disturbance as well as for the part that shows a regular daily variation I have made this probable¹). Moreover the Aurora Borealis points to this. The magnetic vector of after-disturbance is the mean vector of disturbance deprived of its greater and smaller oscillations during the disturbance. It increases rapidly after the initial impulse and then slowly decreases.

As here the horizontal intensity just decreases we must conclude to a likewise W-E. current of negative electricity in these higher atmospheric layers. It remains an open question why the intra-terrestrial current at the outset and the extra-terrestrial current during the further course of the disturbance have both a constant East-West direction.

Geophysics. — "*Registration of the earth current at Batavia.*"

2nd part. By Dr. W. v. BEMMELEN.

In my first paper on the registration of electric earth-currents at Batavia, to investigate the connection between the oscillations in earth-current and magnetic force, I had to point to several unanswered questions.

First of all the fact that the earth-current between Anjer and Batavia is four times greater than the one between Batavia and Cheribon. I hope soon to be able to measure the current between Batavia and a place E. and S. of Anjer to try to shed light on this abnormality.

Further more it remained a mystery why that connection with the magnetic force showed such a characteristic difference for the current between Semarang and Batavia with that for the current between Batavia and places closer by. That difference consisted chiefly in the fact, that when the duration of a magnetic oscillation becomes shorter, the amplitude of the earth-current increased much more for the long line than for the short one.

I pointed out, that perhaps an influence of the distance might

¹) Met. Zeitschrift 1895. p. 321. T. M. VIII p. 153.

have something to do with it and that a registration of the current at Semarang going over the distances Semarang—Cheribon and Semarang—Soerabaya would probably be able to enlighten us in that respect.

This idea I have, indeed, been able to realize by a visit to Semarang in the month of December 1907. At the Post- and Telegraph Office they kindly accommodated me for some days with a room where I could place the instruments used at Batavia. Though I had some delay by a slight accident, yet I could get excellent diagrams during two nights.

The result was definite, viz. *the current between Semarang—Soerabaya and Semarang—Cheribon corresponds in character and intensity to that between Batavia—Cheribon.*

Oscillation of the magnetic North component.		Amplitude of the earth current in Volt per K.M. Amplitude magnetic component in dynes.			
Half oscillation	Amplitude	Sem.-Cheribon	Sem.-Soerabaya	Batavia-Cheribon	
		18—19 December 1907.		Found formerly.	Number of cases.
0.5 min.	1.2 γ	22.6	19.3	21.2	14
1.0 "	1.5 "	20.9	16.9	20.5	11
13.5 "	3.0 "	16.2	14.4	16.0	8
19—20 December 1907.					
0.6 min.	0.5 γ	26.3	23.6	23.0	33
1.4 "	1.1 "	23.4	20.0	20.5	16
9.5 "	2.4 "	20.3	16.7	16.5	19

So :

Duration of half an oscillation	Earth-current Batavia-Cheribon		Earth-current Batavia-Cheribon	
	Earth-current Semarang-Cheribon		Earth-current Semarang-Soerabaya	
$\frac{18-19}{XII}$ '07	0.5 min.	0.94	1.10	
	1.0 "	0.98	1.21	
	13.5 "	0.99	1.11	
$\frac{19-20}{XII}$ '07	0.6 min.	0.87	0.97	
	1.4 "	0.88	1.02	
	9.5 "	0.81	0.99	

The registration of both nights together gives :

Duration of half an oscillation	Amplitude	Sem.-Cher.	Sem.-Soerabaja	Number of cases
0.36 min.	0.6 γ	24.3	21.4	20
0.65 "	0.9	26.3	22.8	20
0.87 "	0.9	26.5	22.7	20
1.08 "	1.2	23.7	19.7	10
3.65 "	1.0	22.9	18.3	10
13.10 "	3.2	22.0	18.2	11

From these numbers we also find the initial increase of the earth-current amplitude for the (half) duration smaller than about 0.7 min. just as it was found for the currents Batavia—Cheribon and Batavia—Anjer.

For the difference in phase was found out of 14 cases, Semarang—Cheribon $17^{\circ}.5$, Semarang—Soerabaja $16^{\circ}.7$, whilst for Batavia—Cheribon formerly 22° was found.

So the registration at Semarang furnished a highly important confirmation of the results found for Batavia—Cheribon. And yet no conclusion could as yet be drawn for an influence of the distance on the amplitude.

And indeed, new observations made at Batavia soon offered another view upon the subject. I heard that a connection with Semarang was possible at the same time along lines through the Northcoast plain and along the line already used round the South by the railroad.

Registration with these lines running between the *same* earthplates at Semarang and Batavia (observatory) gave the remarkable result that the current in both lines was unequal.

The Northline corresponded with results found before on the Batavia—Cheribon line, the Southline gave again the heightened increase of the earth-current amplitude when the duration of the oscillation decreased.

Duration of half an oscillation X.	Amplitude Earth-current in Volt per K.M.		Number of cases
	Amplitude Southline	Amplitude Northline	
0.50 min.	62	28	20
0.77	48	29	20
1.53	50	34	20
5.53	24	27	20
8.60	23	27	10
21.77	13	20	11

DIFFERENCE IN PHASE

Duration of half oscillation X	South-line	Number of cases	Duration of half oscillation X	Northline	Number of cases
0.8 min.	31°	39	1.3 min.	16°	68
6.8 "	31	18	6.5 "	18	40
16.4 "	31	20	17.0 "	26	23

So the difference in phase is different for the two lines. Extraordinary is here the increase of the difference in phase for the Northline, which is not found on the other lines.

Formerly was found for the difference in phase on the line Semarang—Batavia 36°. Whether the difference with the difference in phase now found of 31° is real must still be called doubtful.

These new results led to the conclusion that the difference in character found formerly might not be attributed to the greater distance, but to a peculiarity of the line itself. As the two lines round the South and the North were between the same earth-plates and possessed about the same resistance, I had to conclude to an appearance of electromotive force in the line itself. There are two possibilities for this:

1st. induction immediately in the wire;

2nd. contact with the ground.

Now it is very well possible that both causes are very different in the diluvial and alluvial plain of the Northcoast and the volcanic Southern regions.

To separate these two causes I have taken the following double experiment.

Batavia and Cheribon are connected by two parallel brass wires of the intercommunal telephone; there are likewise two telegraph wires on the same poles between Batavia and Soerabaja. Such a double line I connected with my galvanometer and switched on between galvanometer and earth a resistance which was great compared to that of the wire. (For Cheribon that resistance was 5000 Ohm, for Soerabaja 40.000 Ohm). I then left both wires connected with the galvanometer for some hours and then broke off the connection with *one* of the wires.

After a few hours I switched this wire on again, but broke the connection with the other one, and then finally I connected both wires again.

If now the earth-current were only a current from groundplate to groundplate then during these changes it might change but slightly in intensity, as the total resistance changed so little.

On the other hand, if the earth-current were for a part not originating from the plate, but was immediately caused by induction or an other influence (e.g. the catching of electrons moving in the atmosphere) then that part when connected with *one* wire would be half of that when connected with two wires and so a considerable difference in intensity of the current would be noticeable.

It might be possible that this influence differed with the duration of the oscillation of the magnetic component and were different in the coastregion from that in higher volcanic regions in South-Java; in this way the difference in character found above might be explained.

Before mentioning the results of this experiments I wish to consider what the influence is of the loss by isolation.

The loss by isolation will chiefly take place along branches and poles accidentally touching the wire. The first influence will be irregular and in general for both lines alike.

With the second each telegraphpole will give an earth connection with great resistance for both wires at the same time, as the wires run across the same yokes.

Along this earth connection a current will run if the earth-potential at that place differs from that in the wire.

That current will then feel little influence of the fact whether *one* or both wires are connected with the galvanometer.

The result of the experiment for the lines to Cheribon as well as for those to Soerabaja was not ambiguous, as the figures below indicate.

Duration of half an oscillation		Amplitude earth-current in m.m. reading.		
		Amplitude	X	in m.m. reading.
Both lines	One line	both lines	one line	
1.0 min.	1.2 min	2.5	2.5	} Batavia- Cheribon
5.4 „	6.6 „	2.0	2.0	
0.6 min.	0.8 min.	6.1	5.7	} Batavia- Soerabaja
8.3 „	19.3 „	1.4	1.5	

This simple experiment is in my opinion of fundamental importance, as it shows *that no electromotoric force is roused in the line itself*, a fact that a priori cannot be called so improbable.

Nothing remained now but to assume that the difference in character of the current in the North-line and in the South-line is caused by the fact that by the loss by isolation the current is partly taken up out of the ground over which the line runs and that that current

was different in the Northcoastplain to that in the mountains. If this were so, then the earth-current in the Preanger country, where the Southline runs in a niveau of ± 600 M. between numerous volcanoes would have to show the same peculiarity. To prove this it was fortunately not necessary for me to remove with my instruments, but I could suffice by making the following connection.

Earth at Buitenzorg—Galvanometer—Observatory—Batavia—Buitenzorg—Tasikmalaja—Earth at that point. Buitenzorg is situated at the N.W.-foot of the mountains and Tasikmalaja at the East foot. Both places lie still at a height of ± 300 M. above the sea.

Loss by isolation along the poles on the there-and-back line Buitenzorg—Batavia—Buitenzorg could not bring the earth-current out of the plain between Buitenzorg and Batavia into the line, and could only cause a part of the Preangercurrent to flow away. That loss could thus not falsify the result.

The current between Buitenzorg and Tasikmalaja really proved to possess the above mentioned character, i.e. it showed a much stronger increase when the duration of the oscillation decreased than the

Northcoast lines Duration of half an oscillation	Batavia-Tasikmalaja	
	Ampl. Earth-current in Volt per K.M.	X in dynes
0.4 min.		60
0.8 "		56
1.2 "		55
9.0 "		15

If loss by isolation is the cause of the inequality of the current between Batavia and Semarang round the North and the South, then that loss will be smaller in the dry season than in the wet one. And, indeed, I found that this was the case as the figures below will indicate.

18—21 Juni 1907				December 1907			
Duration of half an oscillation	Ampl. Sem.-Bat. current in Volt. p. K.M.			Duration of half an oscillation	Ampl. Sem.-Bat. current in Volt. per K.M.		
	Ampl.	X	in dynes		Ampl.	X	in dynes
0.6 min.		43		0.5 min.		62	
1.0 "		35		0.8 "		48	
1.4 "		31		1.5 "		50	
5.0 "		18		5.5 "		24	
11.5 "		13		8.6 "		23	
				21.8 "		13	

These characteristic differences treated above can perhaps afford an occasion to find an explanation of the nature and the cause of the

earth-currents, but more observations under other circumstances will undoubtedly be necessary.

It seemed important to me to investigate whether that great difference in earth-current is always incidental to difference in amplitude of the magnetic variations.

For, Buitenzorg lies on the edge of the volcanic Southernpart of Java, and Batavia lies in the Northeastplain; moreover simultaneous registrations of the magnetic component are for both places available.

I have used the registration of the X-component at Batavia on the earth-current diagrams and of the TÖPFER-Unifilar of the X-component at Buitenzorg.

In January, February, March, July, August, September the magnetic variation-instrument registered on the earth-current diagrams the magnetic component perpendicular to the direction Batavia—Anjer, i. e. N4°E. The difference in direction with that on TÖPFER's instrument with which the X-component was registered, can be neglected.

On each diagram I compared the amplitude of a variation of short and of long duration, as much as possible at an equal distance from the basis. In this way I was independent of differences in values of the scale division and other differences.

I got as average case in 30 cases in the months of January—March '07 and 24 cases in the months of June—September '07 :

Average Amplitude of the			
Variations of short duration.		Variations of long duration.	
Buitenzorg	Batavia	Buitenzorg	Batavia
1.44 m.m.	13.21 m.m.	2.68 m.m.	21.27 m.m.
1.01 „	8.71 „	2.93 „	24.03 „
1.25 m.m.	11.21 m.m.	2.79 m.m.	22.49 m.m.

Whilst thus the longer variations give a proportion $\frac{22.49}{11.21} = 2.01$,

the short pulsations give $\frac{2.79}{1.25} = 2.24$.

That difference of 10 % I believe must be ascribed to the following circumstance :

According to the image of the earth-current diagrams, on which the pulsations are large and easy to see, the points of reversion are pointed. On the Buitenzorgdiagrams on a \pm ten times smaller scale those sharp points are blunted and we obtain a too small amplitude.

That shortening can be estimated at a tenth millimeter, i. e just 10 % of the amplitude.

With the oscillations of long duration that inaccuracy in the registration does of course not appear.

So we come to the conclusion *that the oscillations of short and of long duration of the magnetic force at Batavia and at Buitenzorg have the same ratio of amplitude and that they therefore cannot be caused, or only for a small part, by the current running through the outer crust of the earth.*

By far the greater part of the influence of the earth-currents must therefore come from currents at greater depths and of greater extension, and more equal in intensity.

Chemistry. — “*On the Tri-para-Halogen-Substitution-Products of Triphenylmethane and Triphenylcarbinol.*” By Dr. F. M. JAEGER. (Communicated by Prof. A. P. N. FRANCHIMONT).

(Communicated in the meeting of March 28, 1908).

§ 1. Some years ago, I investigated ¹⁾ crystals of *Tri-p-Chloro-Triphenylmethane*, from different preparations which had been obtained by Dr. P. J. MONTAGNE in two different ways, namely from *p*-leucaniline by diazotation and subsequent introduction of the three chlorine atoms and from tetrachlorobenzopinacoline by intramolecular rearrangement.

I then gave a detailed description of the remarkable optical behaviour of the compound in convergent polarised light and endeavoured to elucidate the same by a coloured figure.

Wishing to extend this research also to the other halogen-derivatives, I have first of all prepared the tribromoderivative of *p*-leucaniline by the method proposed by O. FISCHER and W. HESS. ²⁾ Afterwards I received from Prof. FISCHER a small quantity of each of the three halogen derivatives, which enabled me to prepare the three corresponding trihalogen-carbinols by oxydation with chromic acid in acetic acid solution, so that these three substances could be included also in this investigation. I will not omit to thank this savant once more for his kind assistance.

Of *Tri-p-Bromotriphenylmethane* ³⁾ I gave a description a short time ago in the *Zeits. f. Kryst.* **44**, 57—58. (1907). The habit of the crystals is quite analogous to that of the chloro-compound; they are more compact of form and generally much larger, but at the same time they cannot be measured so accurately, owing to a curving

¹⁾ *Receuil* **24**, 124, 131. (1905).

²⁾ O. FISCHER und W. HESS, *Berl. Ber.* **38**, 336. (1905).

³⁾ F. M. JAEGER, *Zeits. f. Kryst. und Miner. Bd.* **44**, 57. (1907).

of the planes. Nevertheless the complete isomorphism with the chloro-compound may be clearly shown; of course the differences are somewhat larger than in the case of isomorphous substitution products in which one atom only is replaced by another and not three at the same time, as is the case here.

§ 2. In the following the crystalforms of the diverse substitutionproducts are described.

Tri-p-Jodotriphenylmethane.

$(C_6H_4J)_3 \cdot CH$; m. p. $132^\circ C$.

This compound was kindly presented to me for investigation by prof. O. FISCHER of *Erlangen*.

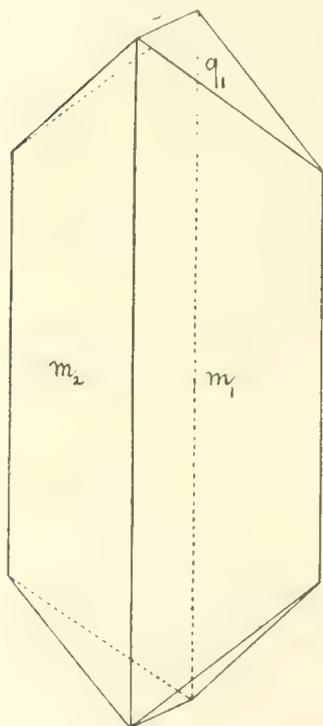


Fig. 1.

Tri-p-Jodotriphenylmethane.
 narrow and dim; $q = \{011\}$,
 yielding good reflexes; $p = \{130\}$, very
 narrow and dull.

From ligroïn it crystallises in small, refractive, pale yellow needles which are readily measurable. From benzene, however, a *double compound* containing benzene crystallises in large transparent prisms. The chloro- and bromo-compounds, however, do *not* unite with benzene; from the benzene solution the crystals of the pure compounds are always deposited.

A. *Tri-p-Jodotriphenylmethane*, from ligroïn.

The symmetry is *rhombic-bipyramidal*; the axial ratio is calculated as:

$$a : b : c = 0,5765 : 1 : 0,8798.$$

Evidently this substance is directly isomorphous with the *Cl* and the *Br*-compound although here the differences are again more considerable than usual on account of the simultaneous substitution of *three* isomorphogeneous atoms.

Forms observed: $m = \{110\}$, well developed and lustrous; $q = \{100\}$, very narrow and dim; $q = \{011\}$, yielding good reflexes; $p = \{130\}$, very narrow and dull.

Angular values:	Measured:	Calculated:
$m : m = (110) : (\bar{1}\bar{1}0) =$	$59^\circ 55\frac{1}{2}'$	—
$m : q = (110) : (011) =$	$70^\circ 30\frac{1}{2}'$	$70^\circ 44\frac{1}{2}'$
$q : q = (011) : (011) =$	$82^\circ 41'$	—
$m : q = (110) : (130) =$	$30^\circ 15'$	$30^\circ 30'$

Distinctly cleavable along m .

The optical axial plane for *all* rays is {001}; the *a*-axis is the first bisectrix with positive character. Average strong, rhombic dispersion, with $\rho > v$; the apparent axial angle in cedar-oil (1,54) is about 68° .

The sp. gr. of the crystals is 2,141 at 15° ; the equivalent volume 290,64.

Topic parameters : $\chi : \psi : \omega = 4,7883 : 8,3061 : 7,3077$.

B. Tri-p-Jodotriphenylmethane + 1 Benzene.

Large, very lustrous and transparent crystals.

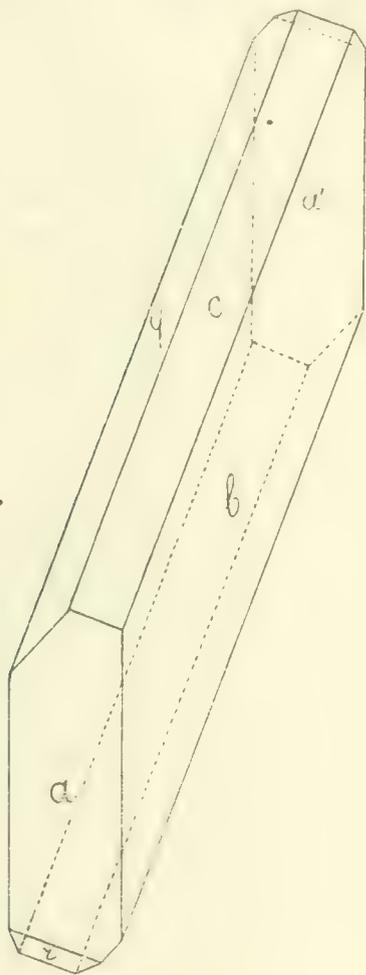


Fig. 2.

Tri-p-Jodo-Triphenylmethane + 1 Benzene.

When taken out of the motherliquor they keep transparent for a fairly long time but after a few hours they lose all their benzene while retaining their form; sometimes there is only a partial loss. It is not improbable that the amount of benzene varies with the temperature and pressure.

The symmetry is *triclino-pinacoidal*. Axial ratio:

$$a : b : c = 0,5719 : 1 : 1,4298.$$

$$A = 101^\circ 12' \quad \alpha = 109^\circ 8'$$

$$B = 123 45 \quad \beta = 126 21$$

$$C = 98 5 \quad \gamma = 107 32'$$

Forms observed: $b = \{010\}$, very predominant and lustrous; $c = \{001\}$ and $a = \{100\}$, well developed and yielding sharp reflexes; $q = \{0\bar{1}1\}$, also rather largely outgrown; $r = \{\bar{1}02\}$, narrow but readily measurable.

The habit is flattened towards $\{010\}$ but elongated along the a -axis. Perfect cleavage parallel $\{010\}$.

Angular values:	Measured:	Calculated:
$a : b = (100) : (010)$	$= 98^\circ 5'$	—
$c : b = (001) : (010)$	$= 78^\circ 48'$	—
$c : a = (001) : (100)$	$= 56^\circ 45'$	—
$a : r = (100) : (10\bar{2})$	$= 50^\circ 10'$	—
$b : q = (010) : (0\bar{1}1)$	$= 44^\circ 4'$	—
$a : q = (100) : (0\bar{1}1)$	$= 106^\circ 8\frac{1}{2}'$	

The ratio of the axes a and b in the two derivatives is quite analogous.

In accordance with the supposition of a varying benzene percentage the angular values of the individual crystals vary rather considerably.

§ 3. When we compare the three *para*-substituted trihalogen-compounds of triphenylmethane with each other, there can be no doubt as to the analogous molecular structure of the derivatives in the solid condition. Only in *optical* orientation the *chloro*-compound distinctly differs:

Tri-p-Chloro-compound:
Rhombic-bipyramidal.

Tri-p-Bromo-compound:
Rhombic-bipyramidal.

Tri-p-Iodo-compound:
Rhombic-bipyramidal.

Forms:	Forms:	Forms:
$\{110\}; \{011\}; \{010\}; \{130\}; \{012\}; \{102\}$	$\{110\}; \{011\}; \{010\}; \{102\}$	$\{110\}; \{011\}; \{130\}$
$a : b : c = 0,5904 : 1 : 0,9261.$	$a : b : c = 0,5896 : 1 : 0,9003.$	$a : b : c = 0,5765 : 1 : 0,8798.$
Cleavable towards $\{110\}$.	Cleavable towards $\{110\}$.	Probably cleavable towards $\{110\}$.

Thick-prismatic towards the c -axis.	Short-prism. tow. the c -axis	Elongated prisms towards the c -axis
$(110) : (1\bar{1}0) = 61^\circ 7'$	$(110) : (1\bar{1}0) = 61 3$	$(110) : (1\bar{1}0) = 59^\circ 55\frac{1}{2}'$
$(110) : (011) = 69^\circ 47\frac{1}{2}'$	$(110) : (011) = 70^\circ 8'$	$(110) : (011) = 70^\circ 44\frac{1}{2}'$
$(011) : (011) = 85^\circ 36'$	$(011) : (0\bar{1}1) = 83^\circ 59'$	$(011) : (0\bar{1}1) = 82^\circ 41'$
Sp. Gr. = 1,435;	Sp. Gr. = 1,752;	Sp. Gr. = 2,141;
Equiv. Vol. 242,16.	Equiv. Vol. 274,54.	Equiv. Vol. : 290,64.

Chief dimensions of the crystal structure.	Chief dimensions of the crystal structure $\chi : \psi : \omega =$	Chief dimensions of the crystal structure $\chi : \psi : \omega =$
$\chi : \psi : \omega = 4.5004 : 7.6225 : 7.0593.$	$= 4.7327 : 8.0270 : 7.2267.$	$4.7883 : 8.3061 : 7.3077.$

Optical orientation :

Optical orientation :

Optical orientation :

The axial plane for <i>orange</i> and <i>green</i> is {001} but for the <i>orange</i> and <i>red</i> rays it is, however, {010}. The first diameter for all colours is the <i>a</i> -axis of - character. The axial angle for <i>violet</i> is nearly 0°.	For <i>all</i> colours the axial plane {001}. The first bisectrix is <i>a</i> -axis of + character. Weak dispersion $q > v$. The apparent axial angle in cedar-oil is about 50°.	For <i>all</i> colours the axial plane is {001} with the <i>a</i> -axis as first diameter of + character. Middlemost dispersion: $q > v$. The apparent axial angle in cedar-oil is about 70°.
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It should, however, be remarked that this *Tri-p-Chlorotriphenylmethane* exhibits also a very interesting optical variability as will be noticed from the subjoined observations :

a. Crystals from O. FISCHER; the compound is recrystallised from petroleum-ether (b.p. 40°—60°).

For *all* colours the optical axial plane was: {010}. Very strong dispersion: $q > v$; the *a*-axis, is the 1st diameter and possesses a negative character. The apparent axial angle in olive-oil is very small and amounts to about 5°.

With other crystals, particularly the thicker prisms, I found that the axial plane for violet and blue rays is {001} but for *all* other colours {010}; the *a*-axis, is the first bisectrix, but now of a positive character; the very strong dispersion was: $q > \text{green}$.

Other little crystals only exhibited the *violet* in {001}, and the *blue*, *green*, *red*, *yellow* etc. in the plane {010}.

b. Crystals from the collection of the Organic Chemical Laboratory at *Leiden*, prepared by P. J. MONTAGNE. *They were optically perfectly identical with the crystals which I examined previously.*¹⁾ Some of the crystals had become opaque but had retained their form. This fact is already mentioned by MONTAGNE²⁾ who observes also that the meltingpoint remains practically unaltered.

At my request Dr. MONTAGNE forwarded me some powder of *Tri-p-Chlorotriphenylmethane* from tetrachlorobenzopinacoline, which after recrystallisation from petroleum ether showed the following properties :

The axial plane for *all* colours is now {001}. Very strong dispersion: $q > v$; the *a*-axis is the 1st bisectrix; the apparent axial angle

¹⁾ Zie Recueil d. Trav. d. Chim. d. Pays-Bas, 24, 124, 131. (1905).

²⁾ loco cit. p. 122.

in olive-oil is much larger than in the first case and amounts to about 10^2 .

Recrystallisation from petroleum-ether does not alter the properties of a definite crystal species; all preparations, however have the same meltingpoint and a complete identical crystalform.

We are therefore confronted with the fact that the compound $\text{CH}(\text{C}_6\text{H}_4\text{Cl})_3$, m. p. 92° occurs, under varying circumstances, in forms which cannot be distinguished by chemical and crystallographical means, but whose *optical orientation is very different*. Sometimes, the crystals show a positive, sometimes a negative double refraction; one crystal shows a crossing of the axial planes for diverse colours, another for only a single colour; others again for no colour whatever, the axial plane then being either $\{001\}$ or $\{010\}$ whilst the dispersion is sometimes: $\rho > \nu$, sometimes $\rho < \nu$.

Of course, the possibility is not excluded that exceedingly small traces of foreign impurities cause this change of the so sensitive optical orientation. The result of the investigation of *Tri-p-Chlorocarbino*l showing its complete isomorphotropic relation to the said derivative, renders it not improbable that a trifling admixture of this oxidationproduct is the cause of the phenomenon.

In accordance with this is the fact, communicated to me privately by Dr. MONTAGNE, that a turbidity of the transparent crystals *never* occurs with the thin *rapidly* formed needles, but always with the thick and short crystals of *Tri-p-Chlorotriphenylmethane*, obtained by *slow* crystallisation.

But it is also conceivable that such large molecules as that of *Tri-p-Chlorotriphenylmethane* might in different circumstances suffer small deviations of their average atomistic configuration, which cannot be demonstrated chemically or crystallographically, but which can be shown optically.

Of late years numerous investigations have been carried out which must lead to the conclusion, that many properties of crystallised matter such as the growth- and cohesion-phenomena must be contributed to the regular molecular aggregation, whereas other ones such as the *optical* properties would have their origin, at least to a great extent, in the properties of the molecules themselves. This view is strengthened by different observations made with the so-called liquid crystals and doubly-refracting liquids; also by some experiments made by WALLÉRANT a.o. on the optical behaviour of deformed solid crystals. And phenomena like those observed here with *Tri-p-Chloro-Triphenylmethane* may show that it is possible that the spacial configuration of the chemical molecules is variable within

narrow limits. I believe I have noticed something similar some time ago with a specimen of the *Dibromide* of 1-3-5 *Hexatriene* presented to me by Prof. VAN ROMBURGH¹⁾. Notwithstanding the identical crystal form the preparation made by addition of bromine to the hydrocarbon showed slight optical differences with that prepared from divinylglycol by means of PBr_3 .

And although I will not as yet venture to give a decision one way or other, I fancy that on account of the phenomena described here the matter is of sufficient importance to be brought to the notice of chemists.

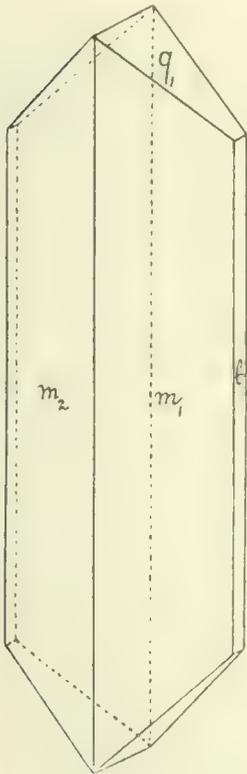


Fig. 3.
Tri p-Chloro-Triphenyl-
carbinol.

§ 4. Tri-p-Chloro-Triphenylcarbinol.

$(C_6H_4Cl)_3 : C-OH$: m.p. $98^\circ C$.

Crystallises from ethyl alcohol in colourless, strongly refracting needles, also from ether + ligroin. The crystals possess great lustre and are well constructed.

Rhombic-bipyramidal.

$a : b : c = 0,6009 : 1 : 0,9781$.

Forms observed: $m = \{110\}$, yields ideal reflexes; $q = \{011\}$, also yielding irreproachable images; $b = \{010\}$ and $p = \{210\}$, narrow but easily measurable. The habit of the crystals is elongated towards the c -axis. Crystals from ethylalcohol are short prismatic and still exhibit the forms $o = \{133\}$ and $s = \{102\}$, generally reflecting badly.

Measured :	Calculated :
$m : q = (110) : (011) = 68^\circ 53\frac{1}{2}'$	—
$m : b = (110) : (010) = 59^\circ 0'$	—
$m : m = (110) : (\bar{1}\bar{1}0) = 62^\circ 1'$	$62^\circ 0'$
$q : b = (011) : (010) = 45^\circ 46\frac{1}{2}'$	$45^\circ 38'$
$q : q = (011) : (0\bar{1}\bar{1}) = 88^\circ 36\frac{1}{2}'$	$88^\circ 44'$
$b : p = (010) : (210) = 42^\circ 27'$	$42^\circ 16\frac{2}{3}'$
$p : m = (210) : (110) = 16^\circ 47\frac{1}{3}'$	$16^\circ 43\frac{1}{3}'$

¹⁾ Compare Trans. Chemic. Soc. (1908) p. 517—524.

No distinct cleavage was found.

The optical axial plane is $\{001\}$, with the a -axis as a first bisectrix of positive character. Weak dispersion: $q > v$. The apparent axial angle in olive oil amounts to about 55° .

The sp. gr. of the crystals is: 1,423; the equivalent volume 255,44. Topic parameters. $\chi : \psi : \omega = 4,5516 : 7,5748 : 7,4089$.

A comparison with *tri-p-chlorotriphenylmethane* shows that the morphotropic relations of both compounds are of such a nature that they border on *isomorphism*. In fact, both compounds form mixed crystals with each other.

§ 5. Tri-p-Bromo-Triphenylcarbinol.

$(C_6H_4Br)_3 : C.OH$; m.p. $133^\circ C$.

Crystallises from ethylalcohol in small colourless, clear crystals possessing a high lustre and a good geometrical construction.

Rhombic-bipyramidal.

$a : b : c = 0,8407 : 1 : 0,8081$.

Forms observed: $m = \{110\}$, predominant and yielding sharp reflexes; $a = \{100\}$ narrow but easily measurable; $q = \{011\}$, gives excellent reflexes and is well developed; $r = \{101\}$ small and somewhat dull. The crystals from alcohol are shown in fig. 4.

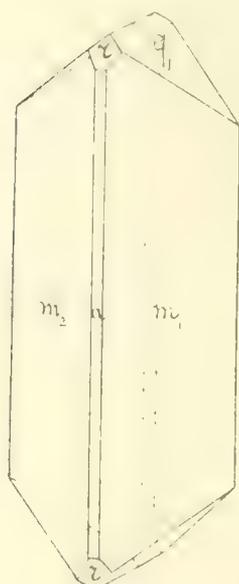


Fig. 4.
Tri-p-Bromo-Triphenyl-
carbinol.

Angular Values :	Measured :	Calculated :
$m : a = (110) : (100) =$	$40^\circ 3\frac{1}{4}'$	—
$m : q = (\bar{1}\bar{1}0) : (0\bar{1}1) =$	$66^\circ 8\frac{1}{2}'$	—
$q : q = (011) : (0\bar{1}1) =$	$77^\circ 52'$	$77^\circ 53'$
$m : m = (110) : (1\bar{1}0) =$	$80^\circ 7\frac{1}{2}'$	$80^\circ 7\frac{1}{2}'$
$m : r = (110) : (101) =$	$57^\circ 59'$	$57^\circ 58'$
$r : q = (101) : (0\bar{1}1) =$	$55^\circ 50\frac{1}{2}'$	$55^\circ 53\frac{1}{2}'$
$r : r = (101) : (101) =$	$87^\circ 45'$	$87^\circ 44'$
$r : a = (101) : (100) =$	$46^\circ 7\frac{1}{2}'$	$46^\circ 8'$

No distinct cleavage.

The optical axial plane is $\{001\}$; the b -axis is the first bisectrix and of a negative character. The apparent axial angle in olive-oil is about 65° . No strong dispersion: $q > v$.

The sp. gr. of the crystals is 1,847; the equivalent volume 269,08. Topic parameters: $\chi : \psi : \omega = 6,1739 : 7,3439 : 5,9346$.

In contrast to what was found with both chloroderivatives, tribromocarbinol shows no distinct form-relationship with tribromotriphenylmethane¹⁾. The substitution of H by $-OH$, however, appears to exert an influence on the equivalent volume which is of a nature opposite to that which causes the same substitution in the chloro-derivative.

§ 6. Tri-p-Iodo-Triphenylcarbinol.

$(C_6H_4I)_3 : C-OH$; m.p. : $155^\circ C$.

Crystallises from ethylalcohol in fairly large yellowish crystals which, however, contain either no terminal planes at all or else strongly curved ones.

In any case the isomorphism with the previous compound may be easily proved.

Rhombic-bipyramidal.

$$a : b : c = 0,8543 : 1 : 0,817.$$

Forms observed: $m = \{110\}$, predominant and highly lustrous; $a = \{100\}$ narrow and generally absent but always giving a good reflexion; $q = \{011\}$ distinctly developed but in most cases curved and only approximately measurable; $r = \{101\}$ was observed once or twice.

	Measured:	Calculated:
$m : m = (110) : (\bar{1}\bar{1}0) =$	$*81^\circ 1'$	—
$q : q = (011) : (0\bar{1}1) =$	$*78 29$	—
$a : m = (100) : (110) =$	$40^\circ 36\frac{1}{2}'$	$40^\circ 30\frac{1}{2}'$
$m : q = (110) : (011) =$	$66 3\frac{1}{2}$	$65 44\frac{1}{3}$
$m : r = (110) : (101) =$	$58 10$	$58 18$

No distinct cleavage.

The optical axial plane is $\{001\}$ with the b -axis as first bisectrix. Particularly large dispersion: $q > v$. The apparent axial angle in olive-oil amounts to about 80° .

The compound crystallises from benzene in combination with the solvent.

B. Tri-p-Iodo-Triphenylcarbinol + Benzene.

This occurs in large, yellowish needles having a strong lustre but generally possessing no terminal planes. In the case of one single individual however a few angles were measured. No trace of efflorescence was noticed in the crystals.

¹⁾ There is no question of a direct isomorphism. By exchanging the a - and b -axis we can find $a' : b' : c' 1,189 : 1 : 0,9612$; which (with double a -axis) somewhat resembles the values for the bromoderivative.

Triclino-pinacoidal.

$$a : b : c = 1,3991 : 1 : 1,6135$$

$$A = 94^\circ 12'$$

$$a = 109^\circ 16'$$

$$B = 123^\circ 10'$$

$$\beta = 117^\circ 36'$$

$$C = 70^\circ 41'$$

$$\gamma = 62^\circ 52'$$

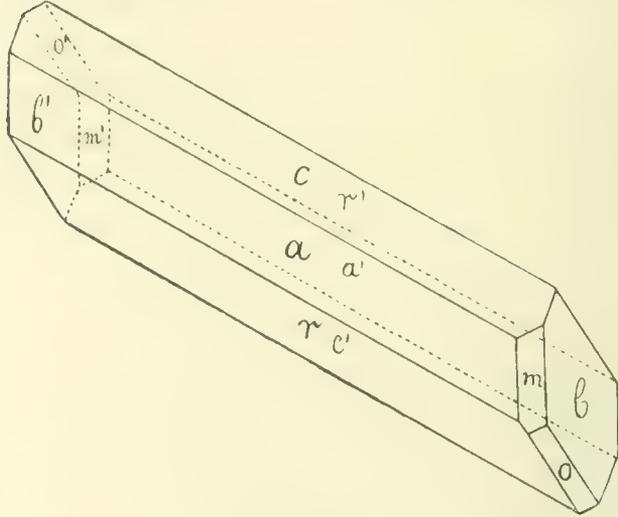


Fig. 5.

Tri-*p*-Jodo-Triphenylcarbinol + 1 Benzene.

Forms observed: $c = \{001\}$ and $a = \{101\}$, equally strongly developed; $r = \{\bar{1}01\}$, broader than a and c and very lustrous; $b = \{010\}$, well developed; $o = \{11\bar{1}\}$ and $m = \{110\}$, about equally large and giving a good reflexion.

The habit is elongated towards the b -axis. The ratio $b : c$ is practically twice that of the *Tri-p*-Jodotriphenylcarbinol itself.

$a : c = (100) : (001) =^* 56^\circ 50'$	—
$c : r = (001) : (\bar{1}01) =^* 72 \quad 6$	—
$c : b = (001) : (0\bar{1}0) =^* 94 \quad 12$	—
$a : b = (100) : (0\bar{1}0) =^* 70 \quad 4\frac{1}{2}$	—
$c : o = (001) : (1\bar{1}\bar{1}) =^* 80 \quad 22$	—
$m : o = (\bar{1}\bar{1}0) : (\bar{1}\bar{1}\bar{1}) = 39 \quad 22$	39.23 $\frac{1}{2}$
$m : c = (\bar{1}\bar{1}0) : (00\bar{1}) = 60 \quad 2$	59.45

No distinct cleavage.

On $\{\bar{1}01\}$ the extinction amounts to about $32\frac{1}{2}^\circ$ in regard to the b -axis. Sp. Gz. 2,079 at 17° ; Equiv. Vol. = 344,39.

Topic axes: $\chi : \psi : \omega = 8,4070 : 6,0090 : 9,6950$.

Groningen, March 1908.

Geophysics. — *“On the analysis of frequency curves according to a general method.”* By Dr. J. P. VAN DER STOK.

§ 1. In working out meteorological data statistically (climatology), frequencies of all descriptions are found. No doubt the majority are between indefinite limits as most other frequencies of different origin, but it also happens that the limits are sharply defined as in the case of observations upon the degree of cloudiness, where they lie between 0 and 10.

An intermediate form is found in the frequencies of rain showers arranged according to duration or quantity; on the one hand they are rigidly limited by the zero value, on the other hand the heavy showers are without definite limits, so that the curve gradually approaches the axis of abscissae.

The elaboration of wind-observations requires the treatment of frequencies in two dimensions, and produces curves, which differ in character from other frequency curves according to the nature of their origin.

The development in series according to the formula of BRUNS¹⁾ and CHARLIER, appears to be the method indicated for frequencies with indefinite limits; but the deduction of this formula is based upon a generalisation in the use of definite integrals as already pointed out by BESSEL and therefore not quite free from premises, which may be applicable to the theory of probability but have no connection with the problem in question which may be defined as the analysis of an arbitrary function between given limits. Besides, this method of deduction can hardly be applied in the case of definite limitation.

The formulae of PEARSON, as also those of CHARLIER, are entirely based upon the premises of the theory of probability and, as they are not given in series form, they only contain a definite number of constants which, in some cases, is too limited to allow a complete characterisation of the curve, particularly in the working out of frequencies of the cloudiness, as will be shown in an example in another communication.

Besides, the constants, which partly appear in exponential form,

¹⁾ H. BRUNS. *Wahrscheinlichkeitsrechnung und Kollektivmasslehre*, Berlin, 1906. Idem. *Beiträge zur Quotenrechnung*. Kön. Sächs. Gesellsch. d. Wiss. Bnd. 58. Leipzig, 1906.

G. V. L. CHARLIER. *Researches into the theory of probability*. Meddel. Lunds astr. observ. Ser. II. n^o. 4. 1906.

Idem. *Ueber das Fehlergesetz*. Ark. för Matem. Astron. och. Fys. Bnd. 2. n^o. 8, 1905.

give no clear indication of the part they play in the construction of the curve, and it is not well possible to describe their function in a simple manner either verbally or graphically.

The object of this communication is to propose a general and simple method by which a curve may be found, which being integrated between certain limits, defined by the distribution of the data, will give the sums characteristic of this distribution, and that for frequencies of different kinds, as far as this is possible owing to the elements of uncertainty proceeding from the imperfection of the data which, of course, always remain.

This curve, representing the law which the phenomenon follows, should be called the frequency-curve; the curve of the aggregate values, obtained by grouping the original data within definite limits, may then be called the curve of distribution according to BRUNS. Its form depends upon the degree of condensation of the original data (Abrundung after BRUNS), but approximates more to that of the frequency curve as the condensation becomes less extensive and consequently the number of observations is greater.

Such a development of an arbitrary function can evidently be made in an infinite number of ways; it is therefore necessary to postulate some general principles.

The following premises apply to the method of development selected :

1. That the development takes place according to polynomials of an ascending degree.

2. that for the determination of the constants, the calculation of means of different orders is used, in relation to an origin favourably selected according to the requirements of the various cases.

The expression "moments" which is frequently employed, has been avoided as an unnecessary analogy with mechanical problems.

§ 2. DEVELOPMENT BETWEEN DEFINITE LIMITS.

a. No given values of the function at the limits.

The polynomials, the degree of which is indicated by a suffix, are represented by Q_n , and the series by :

$$u = A_0 Q_0 + A_1 Q_1 + A_2 Q_2 + \dots \text{etc.} \dots \dots \dots (1)$$

The simplest form which can be given to the polynomials is :

$$Q_n = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n$$

In this case the most practical choice for the origin of coordinates is evidently the mean between the limits as then, on integrating between the limits, all odd terms vanish: hence a separation between

even and odd polynomia becomes necessary, and the general expression is :

$$\begin{aligned}
 Q_n &= x^n + a_2x^{n-2} + a_4x^{n-4} + \dots + a_n && n \text{ even} \\
 &= x^n + a_1x^{n-2} + a_3x^{n-4} + \dots + a_{n-2} && n \text{ odd}
 \end{aligned}$$

A simplification of the formulae can then be obtained by altering the scale value in such a way that the limits become ± 1 , which is always possible; for the sake of convenience these limits have been omitted in the following expressions.

The means of different order are indicated by :

$$u_n = \int ux^n dx .$$

In order to enable us to calculate from the infinite series (1) the *A*-coeff. in a finite form, the unique and sufficient condition is that the *a*-coeff. be determined so that the condition :

$$\int Q_n x^m dx = 0 \dots \dots \dots (2)$$

is satisfied for all values of $m < n$ as then all integrals beyond the $m + 1^{\text{th}}$ term vanish and, at the same time, the *a*-coeff. are entirely fixed, but for an arbitrary constant factor.

If this operation has been performed, it is at once evident from (2) that :

$$\int Q_m Q_n dx = 0$$

for all values of m different from n and, further, that :

$$A_n = a \int u Q_n dx \dots \dots \dots (3)$$

where :

$$a^{-1} = \int Q_n Q_n dx = \int Q_n x^n dx .$$

The $n/2$ (n even) or $n-1/2$ (n odd) constants of the polynomium Q_n are calculated from the $n/2$ or $n-1/2$ equations :

$$\left. \begin{aligned}
 \int Q_n dx &= 0 \\
 \int Q_n x^2 dx &= 0 \\
 \dots \dots \dots \\
 \int Q_n x^{n-2} dx &= 0
 \end{aligned} \right\} (n \text{ even}) \quad \left. \begin{aligned}
 \int Q_n x dx &= 0 \\
 \int Q_n x^3 dx &= 0 \\
 \dots \dots \dots \\
 \int Q_n x^{n-2} dx &= 0
 \end{aligned} \right\} (n \text{ odd})$$

in calculating uQ_n in (3), we have to deal with the unnecessary factor k_n .

However the relation (5), where:

$$k_n = \frac{(2n)'}{2^n \cdot n! \cdot n'}$$

so that:

$$Q_n = \frac{2^n \cdot n! \cdot n'}{(2n)'} P_n \dots \dots \dots (6)$$

is useful in deriving from the well known properties of the P -functions those of the Q -functions.

They satisfy LEGENDRE'S equation as well as the zonal harmonics:

$$(x^2 - 1) \frac{d^2 Q_n}{dx^2} + 2x \frac{dQ_n}{dx} - n(n + 1) Q_n = 0.$$

The recurrent formula becomes:

$$Q_{n+1} - xQ_n + \frac{n}{(2n+1)(2n-1)} Q_{n-1} = 0,$$

and

$$Q_n = \frac{n!}{(2n)'} \frac{d^n (x^2 - 1)^n}{dx^n} \dots \dots \dots (6a)$$

Hence, we find:

$$\alpha^{-1} = \int Q_n Q_n dx = \frac{1}{k^2} \int P_n P_n dx = \frac{2}{k^2 (2n+1)} = \frac{2^{2n+1} n! n! n! n!}{(2n+1)! (2n)!}$$

and for A_n :

$$A_n = \alpha \left[\mu_n - \frac{n(n-1)}{2 \cdot (2n-1)} \mu_{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} \mu_{n-4} - \text{ect.} \right] (7)$$

b. Given $u = 0$ for $x = \pm 1$.

The case discussed sub *a*, where nothing is supposed to be known concerning the function to be developed, will seldom occur in practice and, as all adaptation is due to the accomodating power of the A -constants the application would, in such a case, necessitate the calculation of many terms and, therefore, hardly be profitable.

Now, in dealing with observations of the degree of cloudiness, the case presents itself, that a curve has to be found, which is characterized by the limiting values mentioned above.

The observations of serene sky (cloudiness zero) and of an entirely

overcast sky (cloudiness ten) ought to be considered separately from the other observations as they constitute climatological factors of peculiar importance for the description of the climate (principally in northerly latitudes). Moreover they are to be regarded rather as discrete quantities, which do not show any continuous transition to a cloudiness resp. of degree 1 or 9.

The other degrees of cloudiness may then be regarded as observations of continuous quantities subject to the above mentioned conditions.

In this case we may easily cause all terms of the series (1) to suit these conditions by simply multiplying the series by a factor that vanishes for $x = \pm 1$ e. g. $x^2 - 1$, and then applying to the new functions, which we shall call R , the same reasonings as sub a .

The degree of the polynomia is then increased by two, so that we have to start with R_2 .

The general expression becomes:

$$\begin{aligned} R_{n+2} &= (x^2 - 1) R'_n = (x^2 - 1) [x^n + a_2 x^{n-2} + \dots a_n], \quad n \text{ even} \\ &= (x^2 - 1) [x^n + a_1 x^{n-2} + \dots a_{n-2}], \quad n \text{ odd.} \end{aligned}$$

The result of this operation is evidently that the surface enclosed by the curve, as determined by the first term of the series, is not represented by a rectangle of base 2 and height 0.5 as in the case of the Q -functions, but by a parabola of base 2 and height 0.75, which makes again the surface equal to unity.

By alternately asymmetrical and symmetrical deformations the shape of this parabola is then altered by means of the next terms in such a manner as to make it approach more and more to the frequency curve corresponding to the given data.

It may be noticed here that in the case of fixed limits, there is no reason to choose for the origin of coordinates the point corresponding to the arithmetical mean; for logical and practical reasons the point intermediate between the limits is then indicated.

The condition, which has to be satisfied by the a -coeff. of the R -function, and by which they are fully determined, is now that:

$$\int R_{n+2} x^m dx = \int R'_n x^m (x^2 - 1) dx = 0, \quad m < n$$

or

$$\int x^{m+2} R'_n dx = \int x^m R'_n dx. \quad \dots \dots \dots (8)$$

The a -coeff. are calculated from the equations:

$$\left. \begin{aligned} \frac{1}{(n+3)(n+1)} + \frac{a_2}{(n-1)(n-1)} + \frac{a_4}{(n-1)(n-3)} + \dots + \frac{a_n}{3.1} &= 0 \\ \frac{1}{(n-5)(n+3)} + \frac{a_2}{(n+3)(n+1)} + \frac{a_4}{(n+1)(n-1)} + \dots + \frac{a_n}{5.3} &= 0 \\ \dots &\dots \\ \frac{1}{(2n-1)(2n-1)} + \frac{a_2}{(2n-1)(2n-3)} + \frac{a_4}{(2n-3)(2n-5)} + \dots + \frac{a_n}{(n+1)(n-1)} &= 0 \end{aligned} \right\} n \text{ even}$$

and

$$\left. \begin{aligned} \frac{1}{(n+4)(n+2)} + \frac{a_1}{(n+2)(n)} + \frac{a_3}{(n)(n-2)} + \dots + \frac{a_{n-2}}{5.3} &= 0 \\ \frac{1}{(n+6)(n+4)} + \frac{a_1}{(n+4)(n+2)} + \frac{a_3}{(n+2)(n)} + \dots + \frac{a_{n-2}}{7.5} &= 0 \\ \dots &\dots \\ \frac{1}{(2n+1)(2n-1)} + \frac{a_1}{(2n-1)(2n-3)} + \frac{a_3}{(2n-3)(2n-5)} + \dots + \frac{a_{n-2}}{(n+2)(n)} &= 0 \end{aligned} \right\} n \text{ odd}$$

By successive elimination of $a_2, a_4, \dots, a_1, a_3, \dots$ we find from these equations for the general form of the R functions:

$$R_{n+2} = x^{n+2} - \frac{(n+2)(n+1)}{2 \cdot (2n+1)} x^n + \frac{(n+2)(n+1)(n)(n-1)}{2 \cdot 4 \cdot (2n+1)(2n-1)} x^{n-2} - \text{etc.} \quad (9)$$

and from this expression by dividing it by $x^2 - 1$:

$$R'_n = x^n - \frac{n(n-1)}{2 \cdot (2n+1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n+1)(2n-1)} x^{n-4} - \text{etc.} \quad (10)$$

The recurrent formula for both R and R' is:

$$R'_{n+1} - x R'_n + \frac{n(n+2)}{(2n+3)(2n+1)} R'_{n-1} = 0$$

and the functions are solutions of the diff. equations

$$(x^2 - 1) \frac{d^2 R_{n+2}}{dx^2} - (n+2)(n+1) R_{n+2} = 0$$

$$(x^2 - 1) \frac{d^2 R'_n}{dx^2} + 4x \frac{dR'_n}{dx} - (n+3)n R'_n = 0.$$

On comparing the expression for R'_n with that for Q_n it is readily seen that the R' functions may be found by differentiation of the Q_{n+1} -function, so that:

$$R' = \frac{1}{n+1} \cdot \frac{dQ_{n+1}}{dx} \dots \dots \dots (11)$$

This might have been expected as the value:

$$R' = k_n \frac{dQ_{n+1}}{dx}$$

satisfies the condition (8)

$$\int x^{m+2} \frac{dQ_{n+1}}{dx} dx = \int x^m \frac{dQ_{n+1}}{dx} dx, \quad m < n$$

which is easily proved by partial integration.

Therefore the series discussed here :

$$u = \sum A_n R_{n+2} \quad u = 0 . 1 . 2 \dots$$

might also (but for a constant factor) be written thus :

$$u = (x^2 - 1) \sum A_n \frac{dQ_{n+1}}{dx} \quad u = 0 . 1 . 2 \dots$$

The calculation of the A -constants is based upon the evident property of the R functions that :

$$\int R_{n+2} R'_m dx = 0, \quad m \text{ different from } n$$

hence

$$A_n = \beta \int u R'_n dx$$

where :

$$\beta^{-1} = \int R_{n+2} R'_n dx = \int R_{n+2} x^n dx = \int x^n (x^2 - 1) R'_n dx = 0$$

or, by (11)

$$\beta^{-1} = \frac{1}{n+1} \int x^n (x^2 - 1) \frac{dQ_{n+1}}{dx} dx$$

From the diff. equation of the R -function follows :

$$\frac{d}{dx} \left[(x^2 - 1) \frac{dQ_{n+1}}{dx} \right] = (n+2)(n+1) Q_{n+1}$$

thence :

$$\beta^{-1} = \frac{n+2}{n+1} \int x^{n+1} Q_{n+1} dx$$

or by (8) :

$$\beta^{-1} = - \frac{2^{2n+1} (n+2)! n! n! n!}{(2n+3)(2n+1)!(2n+1)!}$$

and A_n is calculated by the expression :

$$A_n = \beta \left[u^n - \frac{n(n-1)}{2 \cdot (2n+1)} u^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n+1)(2n-1)} u^{n-4} - \dots \text{etc.} \right] \quad (12)$$

The negative sign of β is due to our having chosen as general

factor $x^2 - 1$, a quantity which, by the definition of the limits, is always negative.

As well as the Q -functions, the R -functions might be multiplied by an arbitrary, constant factor, such that any peculiar development becomes possible or also with a view of simplifying some expressions. In our case e. g. k_n might be chosen so that $\beta = 1$; practically however this would hardly afford any advantage.

c. Given $u = \frac{u_1}{u_0}$ for $x = \pm 1$.

As has been remarked above, in working out observations of cloudiness the case presents itself that the frequencies for the extreme limits vanish; if, however, we have to deal, not with the original observations, but with average values as, e. g. daily means, the frequencies of serene and overcast sky, although still of peculiar interest for the knowledge of the climate, cannot be regarded as discrete values because, owing to the operation of taking the means, a continuous transition of these extreme values into the intermediate values must be assumed.

In this case, when the curves assume peculiar forms quite different from the well known curves generally met with, we can take care that the conditions for the extreme limits are bound to the first term of the series whilst all other terms remain as they are in the case discussed sub b .

Now the first term must contain three constants, two for the extreme values and one for the fixing of the area.

In the expression

$$a_0 + b_0x + c_0x^2 \dots \dots \dots (13)$$

the constants must satisfy the three conditions

$$u_1 = a_0 + b_0 + c_0$$

$$u_0 = a_0 - b_0 + c_0$$

$$2a_0 + \frac{2c_0}{3} = 1$$

hence:

$$4a_0 = 3 - (u_1 + u_0)$$

$$2b_0 = u_1 - u_0$$

$$4c_0 = 3(u_1 + u_0) - 3.$$

The reasoning as well as the application then remain the same as sub b ; again

$$\int R_{n+2} R'_m dx = 0, \quad m \text{ different from } n$$

with the exception however of the first term of the series which now assumes the form (13). In calculating A_n we have therefore to apply a correction to the expression for A_n which is easily found by remarking that :

$$\begin{aligned} (n+1) \int x^m R'_n dx &= \int x^m \frac{dQ_{n+1}}{dx} dx \\ &= \left(x^m Q_{n+1} \right)_{-1}^{+1} - m \int x^{m-1} Q_{n+1} dx. \end{aligned}$$

For $m < n+2$ the last integral vanishes and, R_n being of the second degree, we have to consider this case only.

We have, therefore :

$$(n+1) \int x^m R'_n dx = \left(x^m Q_{n+1} \right)_{-1}^{+1}, \quad m < 3$$

By (6) we find :

$$\left(Q_{n+1} \right)_{-1}^{+1} = \frac{2}{k_{n+1}} = \frac{2^n (n+2)! n!}{(2n+1)!} \quad (n \text{ even})$$

whilst for n odd the expression vanishes.

Hence also :

$$\left(x^m Q_{n+1} \right)_{-1}^{+1} = \frac{2}{k_{n+1}} \quad (m+n \text{ even})$$

and equal to zero for $m+n$ odd; in calculating the constant A_n we have, therefore, only to apply a correction such that, instead of (12), now is used, for n odd :

$$A_n = \beta \int u R'_n dx - \frac{2^{n+1} b_0 n! n!}{(2n+1)!} = \beta \int u R'_n dx - \frac{2^n (u_1 - u_0) n! n!}{(2n+1)!} \quad (14)$$

and for n even :

$$A_n = \beta \int u R'_n dx - \frac{2^{n+1} (a_0 + c_0) n! n!}{(2n+1)!} = \beta \int u R'_n dx - \frac{2^n (u_1 + u_0) n! n!}{(2n+1)!} \quad (15)$$

This example of adaptation, of which many variants might be given, will suffice to demonstrate the applicability of the method to special cases.

§ 3. DEVELOPMENT BETWEEN DEFINITE LIMITS ON THE ONE SIDE AND INDEFINITE LIMITS ON THE OTHER.

a. No given value for the limit.

As has been noticed above, frequencies of duration and quantities

of rainshowers lie between the asymmetrical limits: zero for the smallest and ∞ for the largest values.

Frequencies of this kind, therefore, offer an example of a transition between the case of fixed limits and infinite limits on both sides. As here there exists no symmetry in the limits, the zero-point cannot be chosen so that, on integrating, the odd functions vanish, hence a separation between even and odd functions would have no sense, and we are obliged to employ complete polynomia of ascending degree.

Here, as in the case discussed in § 2, there is no advantage in making the origin of coordinates coincide with the arithmetical mean and, from a logical as well as a practical standpoint, the zero-limit is indicated.

In order to develop the function between the limits ∞ and zero, the only thing to do is to multiply the series of polynomia with a suitable factor e. g. e^{-x} , so that the equation of the frequency curve becomes :

$$u = e^{-x} (A_0 S_0 + A_1 S_1 + \dots etc.) \\ = A_0 \psi_0 + A_1 \psi_1 + \dots etc.$$

where :

$$S_n = x^n + a_1 x + a_2 x^{n-2} + \dots a_n.$$

The conditions to be satisfied by the a coeff. are then :

$$\int_0^\infty e^{-x} S_n dx = 0, \int_0^\infty e^{-x} x S_n dx = 0 \dots \int_0^\infty e^{-x} x^{n-1} S_n dx = 0$$

and as :

$$\int_0^\infty e^{-x} x^n dx = n!,$$

the general conditional equations are :

$$n! + (n-1)! a_1 + (n-2)! a_2 + \dots 1! a_{n-1} + 0! a_n = 0 \\ (n+1)! + n! a_1 + (n-1)! a_2 + \dots 2! a_{n-1} + 1! a_n = 0 \\ \dots \dots \dots$$

$$(2n-1)! + (2n-2)! a_1 + (2n-3)! a_2 + \dots n! a_{n-1} + (n-1)! a_n = 0$$

Hence we find for the general expression :

$$S_n = x^n - \frac{n^2}{1!} x^{n-1} + \frac{n^2(n-1)^2}{2!} x^{n-2} - \dots (-1)^n n! \dots (16)$$

The method of calculating A_n is the same as in the former cases as here too :

$$\int_0^{\infty} e^{-x} S_m S_n dx = \int_0^{\infty} \psi_m S_n dx = 0, \quad m < n$$

and

$$A_n = \gamma \int_0^{\infty} u S_n dx$$

where :

$$\gamma^{-1} = \int_0^{\infty} e^{-x} S_n S_n dx = \int_0^{\infty} \psi_n x^n dx$$

but :

$$\int_0^{\infty} \psi_n S_n dx = - (\psi_n S_n)_0^{\infty} + 2 \int_0^{\infty} \psi_n \frac{dS_n}{dx} dx$$

or, as the last integral vanishes according to the conditions :

$$\gamma^{-1} = - (\psi_n S_n)_0^{\infty} = n! n!$$

because, by (16), only the last term has to be taken into account.

The expression for A_n then becomes :

$$A_n = \frac{\mu_n}{n! n!} - \frac{n}{1!} \frac{\mu_{n-1}}{n! (n-1)!} + \frac{n(n-1)}{2!} \frac{\mu_{n-2}}{n! (n-2)!} - \dots \frac{(-1)^n}{n!} \quad (17)$$

by which the problem is solved.

The application to special cases will be simplified by a brief summary of the relations existing between the different quantities introduced which are analogous to those holding for zonal harmonics.

We remark that for S_n and ψ_n we can also write :

$$S_n = (-1)^n \left(\frac{d}{dx} - 1 \right)^{(n)} x^n, \quad \psi_n = (-1)^n \frac{d^n}{dx^n} (e^{-x} x^n) \quad (18)$$

hence :

$$S_n = -n S_{n-1} + \frac{x}{n} \frac{dS_n}{dx} \quad \text{and} \quad S_n = (x-n) S_{n-1} - x \frac{dS_{n-1}}{dx}$$

from which the recurrent formula :

$$S_{n+1} + (2n+1-x) S_n + n^2 S_{n-1} = 0, \quad \dots \quad (19)$$

can be derived, wherein for S_n as well ψ_n may be written.

Further the functions satisfy the diff. equ. :

$$x \frac{d^2 S_n}{dx^2} + (1-x) \frac{dS_n}{dx} + n S_n = 0$$

$$x \frac{d^2 \psi_n}{dx^2} + (1+x) \frac{d\psi_n}{dx} + (n+1) \psi_n = 0.$$

b. Given $u = 0$ for $x = 0$.

In the same manner as the Q -series has been made to suit the zero-condition of the function at the limits, the ψ -series can be made fit for the case that the function assumes the zero value for the lowest limit by multiplication with x . This case presents itself e.g. for frequencies of wind-velocity, the curve of which originates at the zero-point as absolute calms do not occur.

By this operation the degree of the polynomia is increased by one and we can write down at once the new T function from (16) by multiplication with x and, at the same time, substituting $n + 1$ for n except in the binomial factors which remain the same.

The condition for the determination of the a -coeff. is now:

$$\int_0^{\infty} e^{-x} x^n T_{n+1} dx = 0, \quad m < n$$

and the general expression:

$$T_{n+1} = x^{n+1} - \frac{n}{1!} \cdot \frac{(n+1)!}{n!} x^n + \frac{n(n-1)}{2!} \cdot \frac{(n+1)!}{(n-1)!} x^{n-1} \dots (-1)^n (n+1)! x \quad (20)$$

From this evidently:

$$S_n = \frac{1}{n+1} \cdot \frac{dT_{n+1}}{dx} \dots \dots \dots (21)$$

a similar relation as is shown by (11) between the Q and R functions.

Hence, if we put:

$$T_{n+1} = x T'_n$$

$$A_n = \gamma' \int_0^{\infty} u T'_n dx$$

where:

$$\gamma'^{-1} = \int_0^{\infty} e^{-x} T_{n+1} T'_n dx = \int_0^{\infty} e^{-x} x^n T_{n+1} dx =$$

$$\int_0^{\infty} e^{-x} x^n \frac{dT_{n+1}}{dx} = (n+1) \int_0^{\infty} e^{-x} x^n S_n dx = (n+1)! n!$$

so that:

$$A_n = \frac{t_n}{0! (n+1)! n!} - \frac{t_{n-1}}{1! n! (n-1)!} + \frac{t_{n-2}}{2! (n-1)! (n-2)!} - \dots \frac{(-1)^n}{n!} \quad (22)$$

If we call the series discussed here, the ψ'_{n+1} series, so that:

$$\psi'_{n+1} = e^{-x} T_{n+1} = e^{-x} x T'_n$$

we find the following relations:

$$\begin{aligned} \psi'_{n+1} &= (-1)^n \frac{d^n}{dx^n} (e^{-x} x^{n+1}) = (-1)^{n+1} x \frac{d^{n+1}}{dx^{n+1}} (e^{-x} x^n) = -x \frac{d\psi_n}{dx} \\ & \quad - x \frac{d^2 T'_n}{dx^2} - (2-x) \frac{dT'_n}{dx} - n T'_n = 0 \\ & \quad x \frac{d^2 T'_{n+1}}{dx^2} - x \frac{dT'_{n+1}}{dx} + (n+1) T'_{n+1} = 0 \\ & \quad x \frac{d^2 \psi'_{n+1}}{dx^2} + x \frac{d\psi'_{n+1}}{dx} + (n+1) \psi'_{n+1} = 0. \end{aligned}$$

In exactly the same manner as the R -series could be expressed in diff. quot. of the Q series:

$$u_R = (x^2 - 1) \sum A_n \frac{dQ_{n+1}}{dx},$$

so the ψ' series might be expressed in diff. quot. of the ψ series:

$$u\psi = -x \sum A_n \frac{d\psi_n}{dx}.$$

In dealing with this kind of frequency curves an alteration of the scale value offers great advantages as well as in the case of fixed limits.

In the case discussed in § 2 it was possible by this artifice to simplify the limits; here such an alteration has no influence upon the limits which remain 0 and ∞ if we write hx for x , but we are able by this means to accommodate the first term of the series, by which the area is determined, according to the form of the curve, so that the task of the A -coefficients is lightened.

By the factor h , which by its nature is a positive quantity, no complication in the calculation of the constants is introduced: the series is now:

$$u = e^{-hx} [A_0 S_0(hx) + A_1 S_1(hx) + \dots \text{etc.}] \quad \dots \quad (23)$$

and, because:

$$\begin{aligned} \gamma^{-1} &= \int_0^\infty e^{-hx} S_n(hx) S_n(hx) dx = \frac{1}{h} \int_0^\infty e^{-t} S_n(t) S_n(t) dt \\ A_n &= h \left[\frac{h^n u_n}{n! n!} - \frac{n}{1!} \cdot \frac{h^{n-1} u_{n-1}}{n!(n-1)!} + \dots \dots \frac{(-1)^n}{n!} \right] \quad \dots \quad (24) \end{aligned}$$

We might also omit the coeff. h in (24) and write (23):

$$u = h e^{-hx} [A_0 S_0(hx) + A_1 S_1(hx) + \dots \text{etc.}] \quad \dots \quad (23^a)$$

The scale value h may, of course, be chosen quite arbitrarily; it is however desirable to do this in accordance with the nature of the curve and, therefore, to calculate it methodically from the given data.

This can be done by suppressing one of the A -constants in (23a) so that the mean value corresponding with this constant can be made use of to define h .

If then we put:

$$A_1 = 0$$

we find, as $A_0 = 1$,

$$u_1 = A_0 h \int_0^\infty a^{-hx} x dx = \frac{1}{h}.$$

§ 4. DEVELOPMENT BETWEEN THE INDEFINITE LIMITS $\pm \infty$.

By reasons of symmetry, in this case it is logical to take e^{-x^2} for the factor by which the limits are determined and when, for the same reason, the arithmetical mean is chosen as the origin, the polynomial can, as in the case of fixed limits, be separated into even and odd functions, because then, on integrating between the limits, the odd functions vanish.

The series becomes then:

$$\begin{aligned} u &= e^{-x^2} [A_0 U_0 + A_2 U_2 + A_4 U_4 + \dots \text{ etc.}] \\ &= A_0 \mathcal{F}_0 + A_2 \mathcal{F}_2 + A_4 \mathcal{F}_4 + \dots \text{ etc.} \end{aligned}$$

as, by the choice of the origin, the A_1 -term has to be omitted. The conditional equation for the determination of the a constants is:

$$\int_{-x}^x x^m \mathcal{F}_n dx = 0 \quad m < n \dots \dots \dots (25)$$

or, generally, for n even:

$$\begin{aligned} &[(n-1)(n-3)\dots 1] + 2a_2 [(n-3)(n-5)\dots 1] + 2^2 a_4 [(n-5)(n-7)\dots 1] + \\ &\quad \dots + 2^{n/2} a_n = 0 \\ &[(n+1)(n-1)\dots 1] + 2a_2 [(n-1)(n-3)\dots 1] + 2^2 a_4 [(n-3)(n-5)\dots 1] + \\ &\quad \dots + 2^{n-2/2} a_n = 0 \\ &\dots \dots \dots \\ &[(2n-3)(2n-5)\dots 1] + 2a_2 [(2n-5)(2n-7)\dots 1] + \\ &\quad \dots + 2^2 a_4 [(2n-7)(2n-9)\dots 1] + \dots + 2a_n [(n-3)(n-5)\dots 1] = 0 \end{aligned}$$

and, for n odd:

this might have been expected as this value satisfies the condition (25), which can be easily proved by successive partial integration. If we put: $k_n = 1$,

$$g_n = \frac{d^n e^{-x^2}}{dx^n} = (-2)^n U_n e^{-x^2}$$

and the expression for the A -coefficients becomes equal to that given by BRUNS.

Therefore $\frac{d g_n}{dx^n}$ may be substituted for g_n for the same reasons as, instead of the Q -functions, zonal harmonics might be employed; in practice however no labour is saved by this substitution as then the polynomials are charged with superfluous coefficients. After what has been said in § 3 about a change of the scale value, it will be sufficient to remark that in this case also the great advantage which can be derived from the introduction of a scale factor is the adaptation by means of the first term of the series to the shape of the curve, the surface remaining equal to unity.

The equation of the curve then becomes :

$$u = e^{-h^2 x^2} [A_0 U_0 (hx) + A_2 U_2 (hx) + etc. \quad . \quad . \quad (28)$$

and :

$$A_n = \frac{2^n h}{\sqrt{\pi}} \left[\frac{h^n u_n}{n!} - \frac{h^{n-2} u_{n-2}}{2^2 \cdot 1!(n-2)!} + etc. \right] : \quad . \quad . \quad (29)$$

The choice of the scale factor is of course quite arbitrary, but, in order to determine it in accordance with the nature of the curve, it is desirable to put $A_2 = 0$, then the average of the second order can be used for the definition of h and it is easily seen that :

$$h^2 = \frac{1}{2k^2}.$$

The coeff. of (29) in so far as they are independent of n may further be omitted and written before (28), then the equation of the curve becomes :

$$u = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2} [A_0 U_0 + A_2 U_2 + A_4 U_4 + etc.]$$

If we take into consideration only the first term in the development, we find the exponential law in its simplest form as $A_0 = 1$.

$$u = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2}.$$

§ 5. INDEFINITE LIMITS, TWO VARIABLES.

The treatment of wind observations now offers no difficulties as, in calculating the means of different order, the two variables (projections upon two axes arbitrarily chosen) can always be separated and* the method remains in all other respects quite the same. Only, instead of one mean of each order, we can now dispose of $p + 1$ means of order p .

If by V_n be denoted the same function of y as U_n is of x , the equ. of the curve assumes, as $U_0 = V_0 = 1$, the form :

$$u(x,y) = e^{-x^2-y^2} [A_0 + A_{1,0} U_1 + A_{0,1} V_1 + A_{2,0} U_2 + A_{1,1} U_1 V_1 + A_{0,2} V_2 + A_{3,0} U_3 + A_{2,1} U_2 V_1 + A_{1,2} U_1 V_2 + A_{0,3} V_3 + \text{etc.}] \quad (30)$$

The general expression for the polynomia is :

$$U_n V_m$$

and as, evidently :

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2-y^2} (U_n V_m) (U_n V_q) dx dy = 0$$

for all values of p different from n and of q different from m , we find for the A -coeff. :

$$A_{nm} = \varepsilon \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2-y^2} u(U_n V_m) dx dy$$

where :

$$\varepsilon^{-1} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2-y^2} (U_n V_m)^2 dx dy = \frac{n!m!}{2^{n+m}} \pi \quad (31)$$

From the considerations of § 4 it follows that the function :

$$\Phi_{nm} = e^{-x^2-y^2} U_n V_m$$

may as well be given the form :

$$\Phi_{nm} = k_{nm} \frac{d^n \dots}{dx^n dy^m} \Phi_0 = k_{nm} \frac{d^{n+m}}{dx^n dy^m} e^{-x^2-y^2}$$

as this satisfies the premised condition; then the series (30) assumes the form of a sum of diff. quot. like the series of BRUNS and

$$\Phi_{nm} = (-2)^{n+m} U_n V_m e^{-x^2-y^2}$$

in accordance to which (31) has to be modified. If it is possible to remove the origin of coordinates to the arithmetical mean by a correction of the projections for their average value, then the terms with the coeff. $A_{1,0}$ and $A_{0,1}$ vanish from (30).

If we wish to alter the scale values according to the nature of the

data, we have to write everywhere, hx and $h'y$, instead of x and y , whence

$$\varepsilon^{-1} = \frac{n!m!}{2^{n+m}} \frac{\pi}{hh'}.$$

The scale factors h and h' can then be determined by putting

$$A_{2,0} \text{ en } A_{0,2} = 0$$

and the two unmixed means of the second order can be disposed of for the determination of these constants:

$$\mu_2(x) = \frac{1}{2h^2} \text{ en } \mu_2(y) = \frac{1}{2h'^2}$$

If, further, we make the axes rotate about the origin so that they coincide with the principal axes of inertia, then also $A_{1,1}$ has to be put equal to zero and the corresponding mean

$$\mu_2(x, y)$$

enables us to calculate the direction of the principal axes.

The series (30) then becomes:

$$\begin{aligned} u = e^{-x^2-y^2} [& A_0 + A_{2,0}U_2 + A_{2,1}U_2V_1 + A_{1,2}U_1V_2 + A_{0,3}V_3 + \\ & + A_{4,0}U_4 + A_{3,1}U_3V_1 + A_{2,2}U_2V_2 + A_{1,3}U_1V_3 + \\ & + A_{0,4}V_4 + \text{enz.} \end{aligned}$$

where all terms except the first represent the deviations from the normal exponential law, the terms of odd degree being a measure of the different kinds of skewness, the terms of even degree of the different kinds of symmetrical deviations.

Chemistry. — “*Equilibria in quaternary systems.*” By Prof. F. A. H. SCHREINEMAKERS.

Let us first take the system with the components: *water, ethyl alcohol, methyl alcohol* and *ammonium nitrate*; we then have at the ordinary temperature one solid substance and three solvents which are miscible in all proportions so that the resulting equilibria are very simple. The equilibria occurring in this system at 30° have been investigated and are represented in the usual manner in Fig. 1; the angular points *W, M, A* and *Z* of the tetrahedron indicate the components: water, methyl alcohol, ethyl alcohol and the salt: ammonium nitrate.

The curve *wa* situated on the side plane *WAZ* represents the solutions consisting of water and ethyl alcohol and saturated with solid salt; the curve *wm* represents the solutions of water and

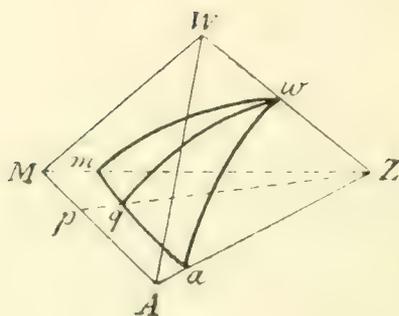


Fig. 1.

methyl alcohol mixtures saturated with solid salt, whilst ma indicates the solutions of mixtures of ethyl alcohol and methyl alcohol, also saturated with solid salt.

The quaternary equilibria, namely the solutions of mixtures of water, methyl alcohol and ethyl alcohol saturated with solid salt are represented by the surface wma which we may call the saturation surface of the solid salt Z .

If we introduce through one of the sides for instance through WZ a plane such as the plane WZp all points of that plane then represent phases containing the components A and M in the same proportion. This plane intersects the saturation surface along the curve wq ; this, therefore, indicates solutions saturated with solid salt in which the relation between methyl alcohol and ethyl alcohol is constant.

The points of such a curve are easy to obtain; the two alcohols are first added together so as to yield a mixture represented by p for instance; on adding varying quantities of water we obtain the points of the line pW and on saturating these solutions with the salt the points of the curve qw are indicated.

In this manner different sections of the saturation surface with planes passing through the side WZ have been obtained.

In the system, water, methyl alcohol, ethyl alcohol and potassium nitrate perfectly analogous equilibria occur; the saturation surface for 30° in this system has been determined by Miss C. DE BAAT.

In the system: *water, ethyl alcohol, ammonium nitrate* and *silver nitrate* the relations are somewhat less simple, for at 30° we have two solid components and one double salt: $\text{AgNO}_3 \cdot \text{NH}_4\text{NO}_3$; the equilibria occurring at 30° are represented in Fig. 2. Whereas Fig. 1 is a perspective representation of the tetrahedron, Fig. 2 is a pro-

jection on a plane parallel to two sides crossing each other, in this case the sides: WA and AgN , so that in the projection, these stand perpendicular to each other and divide each other in two.

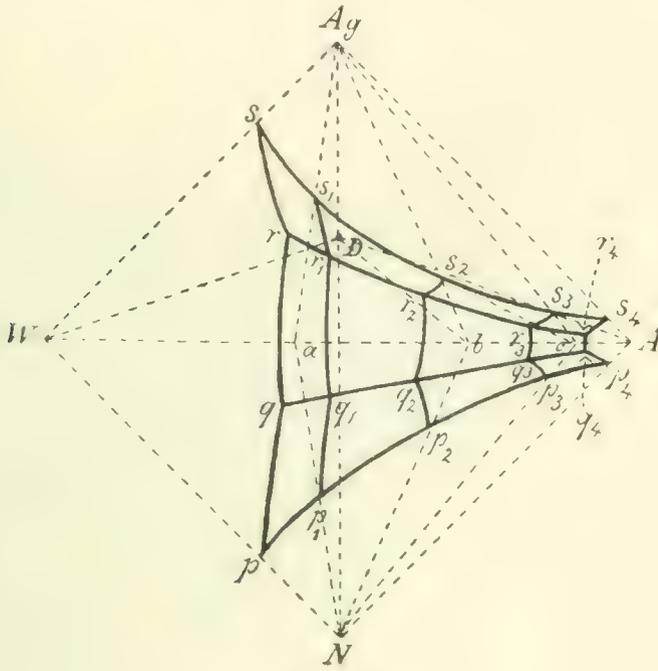


Fig. 2.

The angular points W , A , Ag and N indicate the components water, ethyl alcohol, silver nitrate and ammonium nitrate. The projection of an arbitrary point within the tetrahedron on the projection plane is easily indicated. If we take the line WA as X -axis and the line NAg as Y -axis of a co-ordinate system and if we take as positive directions those towards A and Ag we find:

$$X = \frac{A-W}{2} \qquad Y = \frac{Ag-N}{2}$$

when A , W , Ag and N represent the quantities of alcohol, water, silver nitrate and ammonium nitrate indicated by the said point within the tetrahedron.

In this manner Fig. 2 has been deduced and it is readily noticed that the equilibria are represented by three surfaces, namely ss_4r_4r , rr_4q_4q and qq_4p_4p . The first surface is the saturation surface of silver nitrate, the second that of the double salt and the third that of the ammonium nitrate.

The double salt is represented in the figure by the point D which,

of course, must be situated on the line AgN . If the compositions of the phases were expressed in mol. % D would fall in the origin of the co-ordinate system; this is however, not the case as the compositions are expressed in percentages by weight. The curve $s s_1 s_2 s_3 s_4$, situated on the side surface $WAgA$ is the saturation line of silver nitrate in water-alcohol mixtures; the solubility of this salt in water (point s) gradually becomes less on addition of alcohol; the solubility in absolute alcohol is represented by s_4 .

The saturation line of ammonium nitrate in water-alcohol mixtures is represented by $p p_1 p_2 p_3 p_4$. It will be noticed that the solubility of ammonium nitrate in water is much lessened by alcohol. The equilibria in the ternary system water, silver nitrate and ammonium nitrate are represented by the three saturation lines sr , rq and qp , situated on the side surface $WAgN$; sr indicates the solutions saturated with silver nitrate, qp those saturated with ammonium nitrate and rq those saturated with the double salt. On drawing the line WD this will be seen to intersect the saturation line rq of the double salt; this is therefore soluble in water without decomposition.

In order to study the equilibria in the quaternary system I operated as follows. Instead of water, I took a water-alcohol mixture containing 41,8 % of alcohol and in this determined the saturation lines of the silver nitrate, ammonium nitrate and the double salt. As the solutions all contained water and alcohol in constant proportion they must lie in a plane passing through the side AgN of the prism and intersecting WA in a point a indicating a 41,8 % alcohol. In this manner I found the three saturation lines $s_1 r_1$, $r_1 q_1$ and $q_1 p_1$ which therefore are all situated in the surface $aAgN$: if the line aD is drawn it will be noticed that this branch intersects $r_1 q_1$ showing that the double salt is also soluble without decomposition in dilute alcohol.

In a similar manner I determined the saturation line in water-alcohol mixtures containing 71,23 and 91,3 % of alcohol; I always found three branches in the figure; they are represented by $s_2 r_2$, $r_2 q_2$ and $q_2 p_2$ and by $s_3 r_3$, $r_3 q_3$ and $q_3 p_3$.

As the line bD intersects the saturation line $q_2 r_2$, the double salt is soluble without decomposition in 71,23 % alcohol; with the line cD it is different; this no longer intersects the saturation line $q_3 r_3$ of the double salt but only that of the silver nitrate $r_3 s_3$ showing this is decomposed by 91,3 % alcohol with separation of silver nitrate.

As the solubility of the components in absolute alcohol amounts to a few percent only, I have not investigated the ternary system alcohol — silver nitrate — ammonium nitrate but there is hardly

any doubt that the solubility lines will give something as represented by $s_1 r_4 q_4 p_1$ and the double salt is bound to be decomposed by absolute alcohol with separation of silver nitrate.

From the preceding it is obvious that the following equilibria occur in the quaternary system:

$L + AgNO_3$	of which L is represented by the surface: $s r r_4 s_4$
$L + NH_4NO_3$	" " " " " " " " $q p p_4 q_4$
$L + AgNH_4(NO_3)_2$	" " " " " " " " $r q q_4 r_4$
$L + AgNO_3 + AgNH_4(NO_3)_2$	" " " " " " " " curve: $r r_4$
$L + NH_4NO_3 + AgNH_4(NO_3)_2$	" " " " " " " " " $q q_4$

On looking at these equilibria several questions arise one of which I will mention. If, for instance we know that in the ternary system water, silver nitrate, ammonium nitrate, of which both salts are anhydrous, an anhydrous double salt occurs at 30° we may ask ourselves what equilibria will occur if the water is substituted by another solvent such as aqueous or absolute alcohol.

It is impracticable to answer this question in its entirety; if, however, we argue that no solid phases are formed which crystallise with the new solvent it becomes a fairly easy one. As a rule we can demonstrate that the same three saturation lines will occur also in the new solvent so that a solution saturated with the two components or solutions saturated with another double salt cannot be formed.

Therefore, although the same double salt must appear in both solvents, its behaviour in regard to the two pure solvents, may however, be quite different and various cases may occur; it may, for instance be soluble in both solvents without decomposition or it may be that, as in the case mentioned, it is soluble in the one solvent without and in the other with decomposition; or it may dissolve in both solvents with decomposition. In the latter event we may meet with two more cases; it may be that the same component is deposited from both solvents or it may be that one of the components is deposited from the one and the other from the other solvent.

Similar equilibria occur also at 30° in the systems:

water — alcohol — silver nitrate — potassium nitrate

and water — alcohol — benzoic acid — ammonium benzoate.

In the first system occurs a double salt of silver nitrate and potassium nitrate; in the latter, which is being investigated by Dr. H. Filippo, a combination of benzoic acid and ammonium benzoate is formed.

In the system: water, alcohol, ammonium sulphate and manganese sulphate quite different equilibria occur. The results of this investigation for 50° are represented in fig. 3; this is again the projection of the tetrahedron on a plane parallel to the sides WA and MnN . The angular points W , A , N and Mn indicate the components: water, alcohol, ammonium sulphate and manganese sulphate.

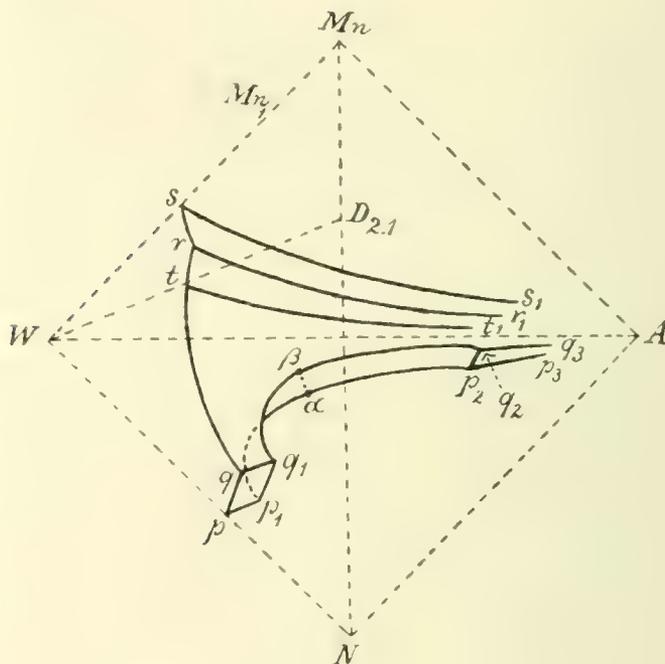


Fig. 3.

In this system an anhydrous compound $(Mn SO_4)_2 (NH_4)_2 SO_4$ occurs at 50° which is represented in the figure by the point D_2 . The $Mn SO_4$ gives at that temperature the compound $Mn SO_4 \cdot H_2O$, represented by the point Mn_1 .

On the side plane $MnWN$ we find the equilibria in the ternary system water, manganese sulphate, ammonium sulphate. Three saturation lines are found: sr is that of the $Mn SO_4 \cdot H_2O$, rtq that of the double salt D_2 , and qp that of the $Mn SO_4$. As the line WD_2 , intersects the saturation line of the double salt in t this is soluble in water without decomposition; t represents this saturated solution.

The isotherm ss_1 which indicates the equilibria in the ternary system water, alcohol, manganese sulphate consists of two saturation lines of which only one ss_1 has been determined. This indicates the solution saturated with $Mn SO_4 \cdot H_2O$; to this should join a saturation

line with the anhydrous MnSO_4 as solid phase which however, has not been determined.

The equilibria occurring in the ternary system: water, alcohol, ammonium sulphate are represented in the plane WAN by the isotherm $pp_1ap_2p_3$; this consists of the saturation lines pp_1 and p_2p_3 of the ammonium sulphate and of the branch p_1ap_2 of a binodal line with the critical liquid a . The points p_1 and p_2 therefore represent two ternary conjugated liquids saturated with ammonium sulphate.

The quaternary equilibria are represented by four surfaces:

rr_1s_1 is the saturation surface of the $\text{MnSO}_4 \cdot \text{H}_2\text{O}$

$rtqq_1\beta q_2q_3r_1$ is the saturation surface of the $D_{2.1}$

qqp_1q_1 and $q_2p_2p_3q_3$ are the saturation surface of the ammonium sulphate

$p_1ap_2q_2\beta q_1$ is the binodal surface.

The latter surface is divided in two parts by the critical line $a\beta$; with each point of the one part, a point of the other is conjugated; a similarly conjugated pair of points represents a pair of quaternary conjugated liquid phases. The binodal surface, therefore, represents the equilibria liquid + liquid. The sections of the four saturation surfaces give three saturation lines.

rr_1 represents the solutions saturated with $\text{MnSO}_4 \cdot \text{H}_2\text{O} + D_{2.1}$

qq_1 and q_2q_3 represent „ „ „ „ $(\text{NH}_4)_2\text{SO}_4 + D_{2.1}$

$q_1\beta q_2$ represents the conjugated liquid pairs saturated with ammonium sulphate. The point β is the critical solution saturated with ammonium sulphate.

If a plane is brought through the side WA and the point $D_{2.1}$ this intersects as far as has been determined the saturation surface of $D_{2.1}$ in the curve tt_1 ; the double salt is therefore not only soluble in water but also in dilute alcohol without decomposition.

At 25° quite different equilibria occur in this system; on the side plane WMA a new area of immiscibility is developed. At the same time the double salt $D_{2.1} = (\text{MnSO}_4)_2(\text{NH}_4)_2\text{SO}_4$ disappears in order to make room for the double salt $\text{MnSO}_4 \cdot (\text{NH}_4)_2\text{SO}_4 \cdot 6\text{H}_2\text{O}$. The resulting equilibria much resemble those mentioned previously which occur at 30° in the system: water, alcohol, ammonium sulphate and lithium sulphate.

I will therefore not discuss these equilibria any further.

Botany. — “*The development of the ovule, embryo-sac and egg in Podostemaceae*”. By Prof. F. A. F. C. WENT.

During my voyage to the West-Indies I had an opportunity of visiting in Surinam some of the rapids where *Podostemaceae* grow, namely the Armina falls of the Marowyne river. There I collected material of these remarkable plants, and at a later date I received an abundant supply obtained by the various expeditions, which of late years have investigated the interior of the colony. This material, preserved in alcohol, has suggested to me an investigation of the above order. I hope soon to publish the results *in extenso*, but wish in this place to deal briefly with one point, namely the development of the ovule, the embryo-sac and the egg.

As was mentioned above, the material was fixed in alcohol, but the fixation nevertheless proved to be good enough to allow of many cytological details being made out with a sufficient degree of certainty in stained preparations. In this preliminary communication I do not propose to discuss the method of treatment of the preparations, but merely record, that Messrs. A. H. BLAAUW and J. KUYPER have assisted me. A complete developmental series could only be obtained in the case of a few species, namely of *Oenone Imthurni* Goebel and *Mourera glaucatilis* Aubl. Of eight other species only a few stages of the development were examined, and of *Tristicha hypnoides* Spr. I only had the ripe seeds.

It soon became evident that the whole development of the ovule in this order departs very widely from the ordinary type of *Angiosperms*, but that within the limits of the order there is an extraordinary degree of uniformity, so that the differences between the species, which have been investigated, are so slight, that they may be passed over in silence in this preliminary notice. The description which follows, therefore applies to all the species.

The ovules are anatropous; in the youngest stage examined, the curvature had already taken place. In this stage the nucellus was still alone present and consisted of a central row of four cells surrounded by a single layer of peripheral cells. Of the central row the uppermost cell, which is therefore still surrounded by a cap of epidermal cells, becomes the spore mother-cell. Accordingly this cell is not only soon distinguished from all the other cells of the nucellus by its size, but also by its dense protoplasmic contents and by its large nucleus. The subsequent behaviour of this spore mother-cell will be further discussed below.

We may now consider how the integuments are formed. The

outer one arises *first* and here we find the first deviation from the normal course of development in Angiosperms. This integument simply arises as an annular fold on the nucellus, with which it remains connected by the chalaza, while for the rest it grows round pretty loosely. Finally there remains at the point, where its borders meet, a very narrow micropyle, which can only be seen properly in truly medial sections.

After the outer integument has already surrounded half the ovule, the inner one begins to develop. Cell divisions are seen to occur in a few epidermal cells of the nucellus, immediately above the point of attachment of the outer integument. These divisions take place in such a manner, that a wall arises in one of the basal cells of each longitudinal row of the epidermis: this wall forms an angle of 45° with the longitudinal direction of the ovule, so that each of the cells is divided into two. The upper half remains an epidermal cell of the nucellus, while the lower half develops to form the inner integument. In a transverse section the number of epidermal cells, counted on the periphery, is seen to be 5, occasionally 6 or 7. At first the inner integument will therefore show in transverse section an equal number of cells. Dividing walls soon arise, however, which make this inner integument two cells thick. More than two layers do not develop, as no further tangential walls are formed, but other walls, both radial and transverse to the long axis of the ovule are developed. Especially the number of radial walls is very different in the two cell layers: it is large in the outer layer, but on the other hand small in the inner layer. As a result, the number of cells of the inner layer of the inner integument is generally little more than five, when counted in transverse section. When afterwards the cells of the inner integument increase in size and often acquire dimensions, which make them very noticeable, it is the inner cells which are especially large. This growth is often accompanied by strong thickening of the walls.

The transverse walls, which arise in the cells of the inner integument, enable the latter to grow longitudinally. In this process the top of the nucellus remains free however, and is only surrounded by the outer integument, so that it lies in the endostomium; the strong longitudinal growth of the inner integument is chiefly directed downwards. At its base, near the chalaza, it of course remains connected with the nucellar tissue.

Now it is very remarkable, that the nucellar tissue does not participate by cell division in this strong longitudinal growth of the ovule. The portion of the nucellus, which projects beyond the inner integument, remains unaltered, except for certain changes, which

the spore mother-cell undergoes, and which will be discussed below. We may however at once point out, that in the formation of embryo-sac and egg-cell, the whole apparatus remains in the same place, and is therefore never surrounded by the inner integument.

The portion of the nucellus lying below this, is now elongated by the extreme stretching of a single cell (or in some cases perhaps two cells) in the central and in each of the 5, 6 or 7 peripheral rows of cells, of which it consists. The nuclei often also assume an extended shape, so that one gets the impression that a passive stretching has taken place. At the same time a digestion of the longitudinal walls occurs, and finally the protoplasts also coalesce more or less. In this way a great cavity arises, containing protoplasm, often in a peripheral layer and with 6, 7 or 8 nuclei, perhaps sometimes even more in consequence of nuclear fragmentation, which seems to occur.

If an ovule is examined in this stage, without the history of its development having been traced, this cavity is inevitably regarded as the embryo-sac, and the real embryo-sac, which lies above it, is then taken for the egg-apparatus. It is in this way that WARMING, who, for want of the necessary material could only trace part of the development of the ovule, has regarded things.¹⁾ This pseudo-embryo-sac remains in existence during the further development of the ovule to the seed, and is only compressed more or less in some cases by the large increase in size of the cells of the inner integument, which has already been dealt with above. When the embryo begins to develop it grows out into this pseudo embryo-sac, in the same way as would happen with a true embryo-sac.

We may now pass on to consider the fate of the spore mother-cell. At a certain period its nucleus shows a clear synapsis stage. In the division, which follows this, the reduction of the number of chromosomes therefore probably takes place. The fixation was not sufficient to allow one to conclude with certainty that a hetero-typic division of the nucleus occurs (the nuclei are moreover extremely minute); such observations as were made, leave very little doubt, however, when considered in connection with the preceding synapsis, that the haploid generation begins here. This nuclear division is followed by a cell division and the formation of a dividing wall. The upper of the two cells, which are thus formed, gradually degenerates and becomes more and more flattened by compression; remnants of it

¹⁾ EUG. WARMING. Familien Podostemaceae. II. Afhandling. Kgl. Danske Vidensk. Selsk. Skr. 6te Raekke, naturv. og math. Afd. 2det Bd. III. Kjöbenhavn, 1882. Compare e.g. p. 65 (107).

may nevertheless still be observed for a long time. In some cases the nucleus of this cell divides once more, in a plane perpendicular to that of the previous division, so that the equatorial plane of the second division is in the longitudinal direction, with respect to the ovule. Perhaps this division also takes place in other cases, in which the two nuclei cannot be seen on account of the unfavourable direction of the section and in consequence of the rapid degeneration of the cell. Only in a single instance I have thought that I observed a cell division following the division of the nucleus in the upper cell.

The lower of the above-mentioned two cells is the embryo-sac. Having regard to the size of the pseudo-embryo-sac, it is remarkable, that the real embryo-sac increases but little in size, and always remains situated in that upper part of the nucellus, which projects beyond the inner integument; it remains of course surrounded by the layer of epidermal cells, which later are only compressed and flattened more and more, so that they become scarcely visible.

The nucleus of the embryo-sac soon divides again. Only a single division was observed, and then the fixation did not allow many details to be made out; it can hardly be doubted, however, that this must be a homoiotypic division of the nucleus. The axis of this spindle is longitudinal with respect to the ovule and therefore also with respect to the embryo-sac. The lower of the two nuclei, which are formed, is seen to degenerate in the anaphases of the division, by a strong clumping of the chromatin masses, so that the latter come to lie at the base of the embryo-sac as a structureless chromatin-like clump, which stains deeply. This is evidently all, that can here be seen of the antipodal apparatus and of the lower polar nucleus. I shall call this nucleus the antipodal nucleus of the embryo-sac.

In contra-distinction to the last-named, the other nucleus assumes a normal shape and is prominent on account of its size. Soon afterwards there follows another division, of which I have been able to see the various stages. The axis of the spindle is this time also longitudinal to the embryo-sac and ovule. This division is not at first followed by a cell division, but afterwards each of the two daughter nuclei divides again. The actual process of division I have not observed, but have only found four nuclei; the second division evidently takes place very rapidly, for I have looked through hundreds of preparations of about this age, without getting the actual stage of division. This second division takes place in such a manner, that the axes of division are perpendicular to each other; for the upper

pair of nuclei the axis is at right angles to the length of the embryo-sac, and for the lower pair it is parallel to it.

Before this last division has taken place, the embryo-sac is still seen to be a single cell, as was already stated above; after this division four cells, each with its nucleus, may be observed. It is of course possible, since I have not seen the actual nuclear division, that the latter is preceded by a cell-division, in such a way, that each cell contains a nucleus, and that afterwards each of these two cells divides again, after its nucleus has divided. However this may be, there are finally four cells, which, it should further be noticed, are not separated by cell-walls — four naked protoplasts therefore. Of these four two, the synergids, lie at the top, next to each other; then follow the other two, one under the other, the upper one of the pair being the egg and the lower one all that remains of the embryo-sac with the upper polar nucleus.

Considering this lower cell first, we observe, that it remains small and that pretty soon its nucleus clumps to a little ball of chromatin, in which structure can no longer be discerned; often the antipodal nucleus may be seen at the same time. In other cases no remnants of it can be observed; I imagine that in such cases it has so far degenerated, that it can no longer be rendered visible. Yet another hypothesis might be suggested, namely, that these nuclei fuse like two polar nuclei. I regard this, however, as extremely improbable, for the very reason that the two nuclei are so clearly in a state of degeneration. Indeed, all the rest of the embryo-sac does not come to much; endosperm is not formed; the cell is still seen for some time, until it disappears with the developing embryo.

For some time the egg and the synergids undergo no further changes, and are ready for fertilisation. This process I have only been able to follow accurately in *Mourera fluviatilis* Aubl.; in a few other cases I found a young embryo, or sometimes pollen-grains, which had germinated on the stigma and had developed pollen-tubes. In a new species of *Apinagia*, still to be described, there occur, in addition to the normal hermaphrodite flowers, others, which have abortive stamens, and which remain inside the closed spathella, at least as far as I have been able to observe in the material at my disposal. Whether the latter flowers can also furnish ripe seeds, without fertilisation, I cannot say, as they had not developed beyond the stage, here described. In the numerous preparations of various *Podostemaceae* which I have examined, I found moreover many ovules, which were degenerating at the above-mentioned stage, evidently because no fertilisation had taken place. It seems to me, that the chance of

regular pollination among these plants is probably not so very large, and that in consequence of this so many ovules ultimately abort.

I now pass on to describe what I have seen of the fertilisation itself, and must remark, that I have but rarely observed anything of the penetration of the pollen-tubes: to some extent this is probably a result of the process of fixation, during which such tender, thin structures readily shrivel up; at the same time the staining does not succeed well. In any case I can however state, that the pollen-tube penetrates through the micropyle, and then reaches the egg-apparatus by passing between two epidermal cells of the nucellus. In one case I observed two nuclei in the top of the pollen-tube, one of which appeared to be a generative and the other a tube-nucleus. In another case I saw a nucleus, which had a much elongated appearance, and was constricted in the middle, so that there might have equally well been two generative nuclei. Taking all the cases, which I have seen, into account, I am led to the view, that the conditions in the top of the pollen-tube are normal, so that there are two generative nuclei and one tube-nucleus. In the actual process of fertilisation, the top of the pollen-tube unites with one of the synergids; the synergid and especially also the egg undergo at the same time peculiar changes in shape, somewhat resembling amoeboid movements. What further happens in the synergid cannot readily be made out, because its contents stain very strongly and become highly refractive. I nevertheless also succeeded in this case in observing the main features of the process. At least one nucleus of the pollen-tube penetrates into the synergid and assumes, in so doing, a more or less vermiform shape. Thereupon a fusion of the synergid with the egg takes place, so that the protoplasts communicate with each other at least at one spot. This communication does not last long, but during it one of the generative nuclei evidently penetrates into the egg-cell; anyhow stages are found later, in which two nuclei lie close to each other in the egg. Still a little later these are found in contact, and afterwards they are found fused in such a manner, that the origin from two nuclei can still be seen.

The fertilized ovum now rapidly enlarges, while all other cells in its neighbourhood are crowded out. As the epidermal cells of the nucellus have generally aborted, this large cell lies more or less by itself in the endostomium, almost filling it up. By the first division wall there is formed a bladder-like basal cell, which remains in the cavity, and a smaller one, which is gradually pushed forward into the pseudo-embryosac. This cell now undergoes some divisions, in which the walls are formed perpendicular to the long axis of the

young seed. When a row of four cells has thus arisen, the three which are turned towards the micropyle become a suspensor, while the fourth divides by a wall at right angles to the previous ones and becomes the embryo proper.

I have not traced the further development of the embryo, partly for want of sufficient material, but especially because WARMING has already furnished an excellent treatise dealing with this subject, and illustrated with figures. Considering the many new facts, which WILLIS has discovered about the germination of the *Podostemaceae* of Ceylon, an investigation of the American forms in this direction would certainly repay, since through Goebel we have only learned in detail of a single case. For this an investigation on the spot is necessary, and as will appear from the full paper, I have not been able to find much that is new in this direction.

What was hitherto known about the ovules of *Podostemaceae* we owe almost exclusively to WARMING. As was said above, this author described in detail the first development of the ovules of *Mniopsis Weddelliana* Tul., and it was only owing to the want of the exact stages, that the meaning of certain organs did not become clear to him. The development proper of the embryo-sac was completely left out of account, but the development of the embryo of this plant, beginning with the two-celled stage, was treated very thoroughly. It is quite clear from his letter-press and from his figures, that the whole development takes place in the same way as in the species examined by myself. The same can be said of the other cases, in which he has stated or figured something regarding the ovules of *Podostemaceae* namely *Castelnavia princeps* TUL. et WEDD.¹⁾ *Hydrobryum olivaceum* GARDN.²⁾ and *Tristicha hypnoides* SPRENG³⁾. On the last named CARIO⁴⁾ had already made observations which seemed to indicate an agreement with the other *Podostemaceae* as regards the development of the ovule. This is of some little importance, because this plant deviates in the structure of its flowers from the majority of the species of the order. If the development of the ovule here corresponds to what I found in the species examined by me this agreement constitutes an additional reason for supposing, that the order is extremely uniform in its embryogeny, in which it differs so widely from the other Angiosperms. I have already

¹⁾ WARMING, l.c. Plate XIV. Fig. 9—21.

²⁾ WARMING, Ibid. 6 Raekke, Nat. og math. Afd. VII. 4. 1891, p. 37, fig. 34.

³⁾ WARMING, Ibid. 6 Raekke, Nat. og math. Afd. IX. 2. 1899. p. 113, fig. 6.

⁴⁾ R. CARIO. Anatomische Untersuchung von *Tristicha hypnoides* Spreng. Botan. Zeitung. 1881 S. 73, Taf. I. Fig, 20—24.

remarked, that much to my regret, I have only ripe seeds of *Tristicha*, but no younger stages. In the 78th *Versammlung Deutscher Naturforscher und Aerzte* in 1906 at Stuttgart R. von WETTSTEIN made a communication : "Ueber Entwicklung der Samenanlagen und Befruchtung der Podostemonaceen". So far he has not published anything about this, however. I have indeed found an abstract of the communication in "Naturwissenschaftliche Rundschau" of 1906, Bd. XXI, p. 615, and in it several statements occur which agree completely with what I have observed, but in other respects there are such differences, that I must assume, that the reporter did not completely understand the meaning of the reader of the paper ; I dare not therefore rely on this abstract.

The *Podostemaceae* differ on the following points from the ordinary arrangement in *Angiosperms*, as regards the development of the ovule :
 1. The inner integument begins to develop after the outer ; this is perhaps connected with the fact, that the top of the nucellus remains free in the endostomium, a phenomenon, which has been observed in other plants. 2. The peculiar development of a pseudo-embryosac by the stretching and dissolution of the cell-walls of a layer of the nucellus. I am not acquainted with anything in the vegetable kingdom corresponding to this. One could only point out, in explanation, that in many cases the developing embryo-sac exercises a solvent action on the surrounding tissue of the nucellus, and that in the present case a similar action is exerted on those cells of the nucellus which are turned towards the chalaza ; these cells only disappear completely, when the embryo proceeds to develop there.

The phenomenon also suggests, that, to a certain extent, it is comparable to that of nucellar embryos. By this I mean, that these nucellar embryos prove the existence of causes, acting in the embryo-sac, which determine a developing cell to become an embryo. What these causes are, we do not know, but it is by no means inconceivable, that some day we may know them completely and even be able to imitate them, so that we may be able to produce an embryo at will. Similarly this phenomenon in *Podostemaceae* seems to me to prove, that there are causes acting in the ovule, which favour the development of such a large cavity as the embryo-sac, so that in those cases, in which the embryo-sac itself does not develop greatly, because it is enclosed and separated off in the upper part of the ovule, the cavity is formed by other cells, lying underneath the embryo-sac.

3. The development of the embryo-sac departs widely from the normal, in that no antipodal cells and no antipodal polar nucleus

are formed, on account of the early d egeneration of the nucleus, which, by its divisions should have given rise to these nuclei. Further more, after the egg-apparatus has been formed, the remaining portion of the embryo-sac is only very slightly developed, so that there is no question of the formation of endosperm (what happens to the second generative nucleus, if indeed present, I have not been able to make out). It is much clearer here than in most cases, that this portion of the embryo-sac and the egg-cell are sister-cells. This agrees with the view of PORSCH¹⁾, according to whom the egg-apparatus of the higher plants is a reduced archegonium, the synergids being the neck canal-cells and the upper part of the embryo-sac with the upper polar nucleus being the ventral canal-cell. The latter hypothesis is however specially difficult in this case, for here the positions of egg-cell and of ventral canal-cell would be exactly reversed. A reduction in the antipodal apparatus, similar to that which occurs here, is found in *Helosis guyanensis*, according to the investigations of CHODAT and BERNARD²⁾, and a still further reduction exists in *Cypripedium*, where, according to the researches of Miss PACE³⁾, the lower portion of the embryo-sac has not even been laid down at all. It need scarcely be argued, that we are here concerned with a progressive differentiation, and not with the recurrence of ancestral peculiarities. Perhaps it may not be amiss to point out, in conclusion that we cannot here fall back for "explanation" on a parasitic or saprophytic mode of life of *Podostemaceae*.

Mathematics. — "*On twisted curves of genus two*". By Prof. J. DE VRIES.

1. A curve of genus *two* bears one and only one involution of pairs of points I^2 . On the nodal biquadratic plane curve it is determined by a pencil of rays, having the node as vertex; its coincidences are then the points of contact of the six rays touching the curve. If we could arrange the points of the curve in a second I^2 then this I^2 would be projected out of the node by a system of rays with correspondence [2], and the above six tangents would furnish six rays of ramification whilst a [2] can have four only.

1) O. PORSCH. Versuch einer phylogenetischen Erkl arung des Embryosackes und der doppelten Befruchtung der Angiospermen. Jena 1907.

2) R. CHODAT et C. BERNARD. Sur le sac embryonnaire de l'*Helosis guyanensis*. Journal de Botanique. T. XIV. 1900. p. 72.

3) LULA PACE. Fertilization in *Cypripedium*. Botanical Gazette. XLIV. 1907. p. 353.

2 We shall now consider the fundamental involution of pairs of points, F^2 , on a twisted curve q^n of genus two. It can be generated by a pencil of cones of order $(n-3)$. For, through an arbitrary point P pass $\frac{1}{2}(n-1)(n-2)-2$ bisecants of q^n , and the cones of order $(n-3)$ through these $\frac{1}{2}(n-3)n-1$ right lines intersect q^n in two variable points more.

This F^2 arranges the planes through the arbitrary right line a in an $[n]$ -correspondence. If a is cut by a bisecant b bearing a pair of F^2 , then the plane ab is a double coincidence of $[n]$, for it corresponds to $(n-2)$ planes with which it does not coincide. On the other hand each coincidence of F^2 determines a single coincidence of $[n]$. The number of double coincidences amounts thus to $\frac{1}{2}(2n-6) = n-3$, so that a is cut by $(n-3)$ bisecants b . In other words:

(The right lines bearing the pairs of the fundamental involution form a scroll of order $(n-3)$.)

To determine the genus of this scroll ϕ^{n-3} we make use of a wellknown formula of ZEUTHEN. When there is between the points of two curves c and c' such a relation that to a point P of c correspond \varkappa' points P' of c' and to a point P' correspond \varkappa points P , whilst it happens y' times that two points P' and y times that two points P coincide, then the genus p and the genus p' of the curves are connected with the numbers mentioned before by the equation ¹⁾

$$2\varkappa'(p-1) - 2\varkappa(p'-1) = y - y'.$$

If now the points P and P^* of a pair of F^2 correspond to the point of intersection P' of the line connecting them and a fixed plane, then $p = 2$, $\varkappa' = 1$, $\varkappa = 2$, $y' = 0$, $y = 6$, so $2-4(p'-1) = 6$ and $p' = 0$.

So the scroll ϕ^{n-3} is of *genus zero* and possesses therefore a *nodal curve* of order $\frac{1}{2}(n-4)(n-5)$.

For a ϕ^5 this involutory scroll is quadratic, so it is a *hyperboloid* or a *cone*.

In the former case one of the systems of generatrices consists of trisecants, the other of the bisecants bearing the pairs of F^2 . The points of support of the trisecants are then arranged in the triplets of an involution which is likewise fundamental (i. o. w. given with the curve). That the latter has eight coincidences is easy to see from the (2,3)-correspondence between the two systems of generatrices.

By central projection we find a quadrinodal plane curve c^5 , on

¹⁾ See ZEUTHEN, Math. Ann. III, 150. A simple proof has been given by KLUYVER (N. Archief v. W, XVII, 16).

which F^2 is cut by the conics containing the four nodes, whilst the lines connecting the pairs envelop a conic and at the same time bear the groups of a fundamental I^3 1).

If the involutory scroll of F^2 is a quadratic cone then every two pairs of F^2 lie in a plane through the vertex, which is at the same time a point of φ^5 . This special φ^5 is evidently the section of a cubic surface and a quadratic cone, having a right line in common 2).

4. We shall now consider a φ^6 of genus two. The involutory scroll of F^2 is now of order three (Φ^3). Let q be the double line, e the single director of Φ^3 . As φ^6 lies on Φ^3 and a plane through q contains but one right line of Φ^3 which line bears a pair of F^2 , we find that q has four points in common with φ^6 , so it is a *quadriseccant*. So the fundamental involution is described by the pencil of planes having the quadriseccant as axis. From this is at the same time evident that φ^6 cannot have a second quadriseccant.

Each plane through e bears two pairs of F^2 , so e is a chord of φ^6 , and the pairs of F^2 are connected in pairs to form the groups of a particular I^4 .

The planes connecting e with the two torsal right lines of \mathfrak{I}^3 are evidently double tangential planes of φ^6 . On e therefore rest besides the tangents in the 6 coincidences of F^2 still the 4 tangents situated in those double tangential planes and the tangents to be counted double in the points of support of the chord e . The developable *surface of tangents* of φ^6 is therefore of order 14. This is evident also from the fact that the quadriseccant besides by the tangents in its points of support is intersected only by the six tangents of the coincidences.

By central projection out of a point of e we find a special c^6 with eight nodes of which the pairs of F^2 lie two by two on rays through a node which is at the same time the point of intersection of two nodal tangents.

5. The scroll β of the bisecants resting on a trisecant t is, like φ^6 , of genus *two*. For, if the points B_1, B_2, B_3 of φ^6 lie with t in one plane then we can make each point B_k to correspond to the chord

1) A number of properties of c^5 are to be found in my paper: "Ueber Curven fünfter Ordnung mit vier Doppelpunkten" (Sitz. Ber. Akad. Wien, 1895, CIV, 46—59). The curves r^5 and c^5 are treated by H. E. TIMMERDING "Ueber eine Raumcurve fünfter Ordnung" (Journal f. d. r. u. a. Math., 1901, CXXIII, 284—311).

2) The central projection of this r^5 has been treated in my paper quoted before page 63. It is generated by stating a projective correspondence between the rays of a pencil and the pairs of an involution, formed of the conics of a pencil.

$B_i B_m$, by which a (1, 1)-correspondence is determined between the points of φ^6 and the points of a plane section of the scroll β .

As each point of t evidently bears 5 bisecants, whilst a plane through t contains 3, the scroll β is a scroll of order 8. A plane section must now show singularities equivalent to 19 nodes. Now the intersection of t is a 5-fold point whilst the 6 intersections of φ^6 furnish as many nodes; the missing three nodes are evidently substituted by a threefold point which is the intersection of a trisecant resting on t .

So on the scroll τ of the trisecants these are arranged in pairs of an involution.

Furthermore follows from this that the scroll τ is of order 12. For, if x is the order of τ , then one of the $(x-1)$ points which t has in common with the remainder section in a plane laid through t is to be regarded as intersection of t ; the remaining $(x-2)$ are derived from multiple curves. Now t is cut outside φ^6 by one trisecant and in each of its points of support by three trisecants, so $x-2=10$ and $x=12$.

6. Out of a point C of φ^6 we find F^2 projected on the curve in the triplets of an involution C^3 .

For, if P is a point of φ^6 then the right line CP cuts the scroll φ^3 in an other point F , and the plane through C , F and the point conjugate to it in F^2 determines on φ^6 two points P' and P'' more, forming with P an involutory group.

The planes $\pi \equiv PP'P''$ envelop a quadratic cone, namely the tangential cone of φ^3 having C as vertex. A right line l through C is thus cut by two triplets of chords PP' situated in the two planes π through l ; but moreover by the two chords connecting C with the two connecting points C' and C'' . The involutory scroll of C^3 is therefore of order eight.

As we conjugate P to the chord $P'P''$ this scroll is also of genus two. In a plane section the point of intersection with φ^6 are nodes. From this ensues that there must be (see § 5) a nodal curve of order thirteen.

The central projection of φ^6 out of C is a quadrinodal c^5 upon which each group of C^3 is collinear to a pair of F^2 . If we regard c^5 as central projection of a φ^5 then C^3 originates from the I^3 on the trisecants; consequently C^3 has like the last mentioned I^3 eight coincidences.

7. If we bring a cubic surface ψ^3 through 19 points of φ^6 , this

curve lies on ψ^3 , so it is the partial section of ψ^3 with the involutory scroll Φ^1 . As q is nodal line of Φ^3 and single right line of ψ^3 , the two surfaces have another line r in common. This r cannot coincide with the single directrix e , for then each right line of Φ^3 would have four points in common with ψ^3 , viz: its points of intersection with ϱ^6 , q and e ; the surface ψ^3 would then however coincide with Φ^3 .

Inversely we can regard ϱ^6 as section of a cubic scroll Φ^3 with nodal line q and a cubic surface ψ^3 having with Φ^3 the right line q in common and a right line r resting on the former one. A plane π through q cuts Φ^1 in a right line, ψ^3 in a conic, so it contains besides q two points of the curve of intersection, from which is evident that q is a quadriseccant; its points of support are coincidences of the (1,4)-correspondences between the points of contact of π with the two surfaces; one of the five coincidences is the point of intersection of q and r . That the single directrix of Φ^3 is a chord of ϱ^6 , is evident from the fact that it cuts ψ^3 on r , thus two times on ϱ^6 .

8. If Φ^3 is replaced by a scroll of CAYLEY so that q is single directrix and at the same time generating line, then the conic of ψ^3 , lying in the torsal tangential plane of Φ^3 determines on q two points each of which replaces in each plane π through q two points of intersection with ϱ^6 ; so they are *nodes* of ϱ^6 . On this special curve the groups of F^2 are not arranged in pairs; for e coincides with q .

We obtain an other special ϱ^6 by taking instead of Φ^3 a cone with nodal edge q . The conics of ψ^3 situated in the planes touching Φ^3 along the nodal edge cut q in the points of support of the quadriseccant. Each edge of Φ^3 bears a pair of F^2 , so that a plane through the vertex T contains three pairs.

The tangential cone out of T to ψ^3 has q and r as nodal edges; the six single edges which it has in common with Φ^3 are evidently tangents of ϱ^6 and contain the coincidences of F^2 .

Through an arbitrary point O pass four tangential planes to Φ^3 ; the central projection of ϱ^6 furnishes a plane curve c^6 with four nodal tangents meeting in a single point C . The six single tangents out of C contain the coincidences of the fundamental involution, each ray of which through C bears three pairs. These are separated if we describe on F^2 a pencil of cubic curves having the eight nodes of c^6 as base-points.

Mathematics. — “*On algebraic twisted curves on scrolls of order n with $(n-1)$ -fold right line.*” By Prof. JAN DE VRIES.

1. If we intersect a cubic scroll Φ^3 by a pencil of planes having a generatrix a of Φ^3 as axis, we get a system of conics ϱ^2 , all passing through the point O , where a meets the double right line d . If we take a (p, q) -correspondence between this pencil of planes and the pencil of planes with axis d , then in this way to each ϱ^2 are assigned p right lines r of Φ^3 and to each right line r evidently q conics ϱ^2 . The locus of the points of intersection of the lines r and ϱ^2 corresponding to each other is a twisted curve of order $m = p + q$; for the points of the rational cubic curve which Φ^3 determines on an arbitrary plane are arranged in a (p, q) -correspondence, of which each coincidence is the point of intersection of a ϱ^2 with a right line r corresponding to it.

The twisted curve ϱ^m has the right lines r as q -fold secants, whilst it is intersected by each of the ∞^2 conics of Φ^3 in p points.

2. If Φ^3 is represented by central projection out of O on a plane τ cutting a and d in A and D , then the systems (r) and (ϱ^2) are transformed into the pencils (D) and (A) which are now likewise arranged in a (p, q) -correspondence. The curve c^m generated in this way has in D a p -fold point, in A a q -fold one. But it has moreover a q -fold point in the point of intersection B of the right line b of Φ^3 , which still passes through O ; for b is q -fold secant of ϱ^m . From this ensues that the correspondence (p, q) in τ cannot be taken arbitrarily.

The curve ϱ^m is completely determined by its central projection c^m . For, the cone projecting c^m out of O has a p -fold edge along d and q -fold edges along a and b ; so its section with Φ^3 consists of $2p + 2q = 2m$ right lines and a twisted curve of order m having p points in common with d and q points with a .

As the singular points of c^m are equivalent to $\frac{1}{2}p(p-1) + q(q-1)$ nodes, the genus of c^m is indicated by

$$g = \frac{1}{2}(p+q-1)(p+q-2) - \frac{1}{2}p(p-1) - q(q-1) = \\ = (p-1)(q-1) - \frac{1}{2}q(q-1).$$

or by

$$g = (m-1)(q-1) - \frac{3}{2}q(q-1).$$

This is at the same time the genus of Q^m . It is evident that p may not be smaller than $(\frac{1}{2}q + 1)$. For the smallest values of p and q we have

m	p	q	g
2	1	1	0
3	2	1	0
4	3	1	0
4	2	2	0
5	4	1	0
5	3	2	1
6	5	1	0
6	4	2	2
6	3	3	1

The above considerations may be extended by taking instead of scroll Φ^3 a scroll Φ^n with $(n-1)$ -fold right line d . Out of a point O of d now start $(n-1)$ right lines a_1, a_2, \dots, a_{n-1} . A (p, q) -correspondence between the pencils of planes (a_1) and (d) determines again a twisted curve of order $p + q = m$, having as central projection a c^m with p -fold point D and q -fold points in A_1, A_2, \dots, A_{n-1} ; and inversely Q^m is again entirely determined by c^m . For the genus of Q^m (and c^m) we now find

$$g = \frac{1}{2} (p + q - 1)(p + q - 2) - \frac{1}{2} p(p - 1) - \frac{1}{2} q(q - 1)(n - 1)$$

or

$$g = (q - 1)(m - 1) - \frac{1}{2} q(q - 1)n.$$

To obtain general twisted curves we shall not be allowed to take p larger than 4, q larger than 3. For $n = 2$ we find evidently the well known considerations concerning curves on an hyperboloid.

5. If we substitute in τ for the curve c^n a curve passing p times through D , q times through A_k and moreover cutting the right line DA_k in s points, then this curve is evidently the central projection of a curve on Φ^n , having a multiple point in O . For, each of the $(n-1)$ tangential planes in O will now contain s right lines, touching the twisted curve in O .

Physics. — “*The influence of temperature and magnetisation on selective absorption spectra.*” III. By Prof. H. E. J. G. DU BOIS and G. J. ELIAS. (Communication from the Bosscha-Laboratory).

§ 20. Since our former communication (These Proc. March p. 734) we have obtained a number of samples, the crystallisation of which in reasonable sizes from solutions in water or amylic acetate was brought about only after many failures and many weeks of patience. Notices concerning the influence of the anion or the temperature only on the absorption spectrum, must be laid aside as being too extensive, though incidentally some details may appear about it. With respect to the ZEEMAN-effect we shall also confine ourselves to a choice from the profuse material, which for the present can be little more than an enumeration of the many ways in which the influence of the magnetisation may manifest itself; it must be reserved for further investigation to impart more order and regularity to the present rather unsystematic series of results.

As a rule we worked again in the spectrum of the first order; in some cases we had recourse to the second order, in which some special effect may sometimes be better judged, at least from a qualitative point of view; for measurements the first order proved preferable on the whole. As we have never to do with very fine lines, too great a dispersion is of no use here, and certainly of much less importance than a strong magnetic resolving power. Very thin crystal chips — some tenths of a mm. thick — already exhibiting jet-black absorption bands, particularly for neodymium salts, we could use these, and expose them to very strong fields, mostly of 38—42 kilogauss. The distinctness of the spectral image depends to some extent on the choice of the proper thickness for every salt. The fields were measured with a bismuth spiral; the disturbance by the narrow slits is certainly less than with the usual round bores; the increase of the — saturated — magnetisation values of the polar end-pieces on cooling them down to -190° is probably slight; it would be desirable to obtain further information concerning these points. We consider, however, the accuracy of our field-measurements of the same order as that of the readings in the spectrum. We again preferred the latter to a photographic reproduction; for with visual observation the identification of the lines with field on and field off, especially with erbium compounds, proved to be decidedly easier.

§ 21. **Third series.** We have investigated a few organic double salts of chromium and potassium with a view to a possible

ZEEMAN-effect, the absorption spectra of which were fully described at ordinary temperature by LAPRAIK¹). The so-called "blue" (dichroitic red-blue) chromium-potassium oxalate $Cr_2K_6(C_2O_4)_6 + 6H_2O$] mentioned in our first paper exhibited in liquid air a strong band 696,4—701,4 (cf. § 5), evidently still too broad to be taken into account. This oxalate may not be mistaken for the so-called "red" compound:

Chromium-potassium oxalate $[Cr_2K_2(C_2O_4)_4 + xH_2O$; different authors consider $x=8, 10, 12$]; this was obtained by CROFT in 1842, and its absorption-spectrum was investigated by BREWSTER²). Strongly dichroitic (claret hue-bluish grey) probably monoclinic crystals. At -190° a number of fine bands and lines in the red are seen with the spectrometer, the most striking of which are a rather strong band 680,0, and a strong band 692,5 between the red ruby bands $R_2 = 691,8$ and $R_1 = 693,2$ (comp. §§ 7, 17).

A plate, 1.5 m.m. thick had to be examined with sunlight on account of its strong absorption; for the same reason the crystallographic orientation could not be determined. At -193° line 692,5 had a width with field off of 0,14 $\mu\mu$ with non-polarized light; in a field of 36,5 kgs. the widening amounted to about 0,05 $\mu\mu$.

Chromium-potassium malonate $[Cr_2K_6(C_3H_2O_4)_6 + 6H_2O]$, is evidently homologous with the "blue" oxalate. This could only be obtained as an interlaced dark crystal magma with irregular orientation; dichroitic (grass green-sky blue). In the red at -193° we find a strong band, the middle of which 693,3 coincides pretty nearly with the red ruby band $R_1 = 693,2$; and a broader rather faint band 698,3. A sample of a thickness of only 0,15 mm. exhibited band 693,3 with a width with field off of 0,8 $\mu\mu$ with unpolarized light; moreover it appears to have shifted 0,8 $\mu\mu$ towards the red with respect to the corresponding band of the oxalate above mentioned. In a field of more than 40 kgs. the band became distinctly vaguer and almost disappeared. We had no opportunity as yet to examine a malonate homologous with the "red" oxalate; perhaps the phenomenon would appear more clearly still in this case.

§ 22. Fifth series. We have now made a closer examination of some salts of the four metals *Pr*, *Nd*, *Sm* and *Er*, such as had been used in 1899.

Praseodymium sulphate $[Pr_2(SO_4)_8 \cdot 8H_2O]$. Light green plate, containing both optical axes, 0,6 mm. thick. Exhibits several not

¹) W. LAPRAIK, Journ. f. prakt. Chemie (2) 47 p. 307, 1893.

²) A. ROSENHEIM, Zeitschr. f. anorg. Chemie 11 p. 196, 1896; and 28 p. 337, 1901.

very narrow bands at -193° in the violet and blue; in the orange some heavy broad bands, moreover a strong band 599,0—599,3, a pretty faint band 600,9—601,4. The plate was now investigated with the median line (dividing the acute angle formed by the axes into two equal parts) vertical in a field of 40 kgs.

With vertically polarized light band 599,0—599,3 appeared to be subject to a distinct widening of 0,1 $\mu\mu$; the other band also became wider and vaguer. With horizontally polarized light the phenomenon was analogous, but less distinctly to be seen; on the other hand some of the wider bands then show an unmistakable widening¹⁾.

Neodymium sulphate [$Nd_2(SO_4)_3 \cdot 8H_2O$].

§ 23. As a supplement to what was communicated in § 19 a number of plates of different thickness were more fully examined; they again contained both optical axes; the line dividing the acute angle was again placed in a vertical position.

Group of bands in the blue at -193° ; 8 of these bands were measured. For the sake of brevity we have been obliged to draw up the results in a table, where λ denotes the wave-length, β_0 the width with field off, β_x the width in a field of x kgs., $d\beta$ the widening; in case a multiplet is formed, the distance of the centre-lines of the extreme components is denoted by $d\lambda$; the value of $d\lambda/\lambda^2$ is expressed in cm^{-1} , as is now usually done.

41 Kilogauss. — Plane of polarisation horizontal — Thickness 0.3 mm.

	I	II	III	IV	V	VI	VII	VIII	
λ	469.5	472.8	474.0	474.5	475.3	476.2	477.0	477.4	$\mu\mu$
β_0	0.26	0.26	0.14	0.05	0.035	0.16	0.105	0.09	„
β_{41}	0.26	0.35	—	0.105	—	0.195	0.21	0.23	„
$d\beta$	0	0.09	—	0.055	—	0.035	0.105	0.14	„
$d\lambda$	—	—	0.22	—	0.18	—	—	—	„
$d\lambda/\lambda^2$	—	—	9.8	—	8.0	—	—	—	cm^{-1}

¹⁾ From a copy that Prof. KAMERLINGH ONNES kindly sent us of the paper by himself and Mr. J. BECQUEREL (These Proc. X, p. 592) we now infer that the results given for the silicates of *Pr* and *Nd* really apply to the sulphates; we had then nearly finished our observations; as, moreover, these were made at -193° instead of -253° and in a much stronger field, the two series of results are not directly comparable; but they may serve to complete each other.

Here III formed an asymmetric doublet, of which the component on the red side was the narrower; V a faint doublet; for VI the middle became somewhat lighter.

With a vertical plane of polarisation all the bands became vaguer, most of them slightly shifting towards the red; in the field the bands widened or further faded away; now only III gave a symmetrical doublet.

Band in the blue-green at -193° : with a horizontal plane of polarisation $\lambda = 511,9$, $\beta_0 = 0,13 \mu\mu$; in a field of 42 kgs. a doublet appeared: width of the lefthand line 0,13, of the righthand line 0,18, of the light interval 0,09 $\mu\mu$; the whole made the impression of perhaps being a quadruplet. With a vertical plane of polarisation the phenomenon was analogous but less clear.

Group of bands in the green at -193° : 6 bands were measured.

42 Kilogauss. — Plane of polarisation horizontal — Thickness 0,3 mm.

	I	II	III	IV	V	VI	
λ	521.2	523.0	523.9	525.3	526.0	527.5	$\mu\mu$
β_0	0.49	0.105	0.355	0.10	0.13	0.195	>
β_{32}	0.58	—	—	0.45	0.275	0.275	>
$d\beta$	0.09	—	—	0.05	0.145	0.08	>
$d\lambda$	—	0.26	0.29	—	—	—	>
$d\beta^2$	—	9.5	10.5	—	—	—	cm^{-1} .

Band II gave an ordinary doublet; that of III remained rather dark in the middle, so that we may infer a more complex structure in this case also.

With a vertical plane of polarisation all this was less clearly visible, band II still gave a clear doublet, for III only a trace of this could be perceived.

Group of bands in the yellow at -193° . Two rather sharply defined bands 576,0 ($\beta_0 = 0,3$) and 586,0 ($\beta_0 = 0,14$) exhibited a distinct widening of 0,05 $\mu\mu$ in a field of 42 kgs. The intermediate bands are too wide for this kind of observation.

Group of bands in the orange-red at -193° : 5 bands were measured.

38 Kilogauss. — Plane of polarisation horizontal — Thickness 0,6 mm

	I	II	III	IV	V	
λ	623.2	624.1	625.6	627.2	628.3	
$\hat{\rho}_0$	0.31	0.13	0.18	0.13	0.08	„
$\hat{\rho}_{33}$	—	0.26	—	0.15	—	„
$d\beta$	—	0.13	—	0.02	—	„
$d\lambda$	0.22*	—	0.40*	—	0.34	„
$d\lambda/\lambda^2$	5.7	—	10.2	—	8.6	cm ⁻¹ .

Here I and III yielded very blurred doublets, of which the distances * of the extreme limits are given; V a distinct doublet with a shade between the components, perhaps a quadruplet.

With a vertical plane of polarisation the phenomenon was analogous and was confirmed with a thinner plate: I was a doublet, III was invisible here, with IV some light appeared in the middle with greater widening, structure probably complicated, V was again a very distinct doublet.

Group of bands in the red at — 193°: Four bands, among which two rather sharp ones 674,4 and 676,2 showed a widening or a fading away in the field; the last-mentioned became a doublet, perhaps even a quadruplet, with plane of polarisation vertical.

Neodymium nitrate [Nd(NO₃)₃ · 6 H₂O].

§ 24. It appeared important to investigate crystals besides the amorphous nitrate (§ 9): with a much slighter thickness crystals show intense and narrow absorption bands; the natural monoclinic plates were directed perpendicularly to one of the optical axes, so that in this case a nicol could be done away with. The wave-lengths of the bands are on the whole slightly less — down to 3 μ — for the nitrate than for the sulphate.

Band in the blue-green at — 193°. The wave-length now was $\lambda = 511,3 \mu$; in a field of 41 kgs. a doublet appeared, the components of which had a distance of 0,22 μ .

Group of bands in the green at — 193°. 5 bands were measured in the spectrum of the second order.

41 Kilogauss.						Thickness 0.2 mm.	
	I	II	III	IV	V		
λ	521.3	522.3	523.1	524.6	525.0	"	
β_0	0.155	0.11	0.18	0.09	0.045	"	
β_{41}	0.265	0.18	0.22	—	0.11	"	
d_β	0.11	0.07	0.04	—	0.065	"	
d_i	—	—	—	0.22	—	"	
d_i/λ^2	—	—	—	8.0	—	cm ⁻¹ .	

The doublet IV showed a shade in the middle.

Group of bands in the yellow at —193°. Two rather sharp bands 581,9 and 583,1 exhibited a widening of 0.05 $\mu\mu$ in a field of 42 kgs.; the others were too broad and too hazy.

Group of bands in the orange-red at —193°. 3 bands were measured.

40 Kilogauss.				Thickness 0.45 mm.	
	I	II	III S.	III	
λ	624.2	625.2	(626.7)	626.9	$\mu\mu$
β_0	0.265	0.18	(0.05)	0.14	"
d_i	0.5	0.5	—	0.5	"
d_i/λ^2	12.8	12.8	—	12.8	cm ⁻¹

Doublets I and III were normal, II on the other hand was asymmetric, the component on the red side being weaker; the satellite IIIS was no longer visible in the field.

Group of bands in the red at —193°: 8 bands were measured.

40 Kilogauss.								Thickness 0.45 mm.	
	I	II	III	IV	V	VI	VII	VIII	
λ	671.0	672.0	673.3	674.3	675.2	675.8	676.6	677.2	$\mu\mu$
β_0	0.31	0.26	0.25	0.22	0.25	0.26	0.22	0.31	"
β_{40}	v.	0.52	0.45	0.32	v.	v.	—	—	"
d_β	—	0.26	0.20	0.10	—	—	—	—	"
d_i	—	—	—	—	—	—	0.80	0.90	"
d_i/λ^2	—	—	—	—	—	—	17.5	19.6	cm ⁻¹

Band I, V and VI vanished in the field; with the field off II showed a dark core with shades on either side, which disappeared in the field. The two doublets VII and VIII gave the greatest resolution measured as yet — $1,5 \times (D_1 - D_2)$; with the field used the components facing each other happened to coincide, so that the pair of doublets looked like a very wide triplet with a heavy middle band. It ought to be possible to observe a phenomenon of this order of magnitude with every good spectroscope.

Finally we mention that the neodymium-magnesium nitrate of § 9 also occurs in hexagonal crystals; such optically uniaxial crystals are of great interest (§ 19); there also exists an isomorphous series of salts, which contain manganese, cobalt, nickel or zinc. Measurements on this subject have been made, partly they are still in preparation.

Samarium sulphate [$Sm_2(SO_4)_3 \cdot 8 H_2O$].

§ 25. We now examined a more transparent sample (cf. § 19), which again contained both optical axes, and was placed like the other sulphates; 4 bands in the green were measured at -193° .

40 Kilogauss — Plane of Polarisation horizontal — Thickness 0.8 mm.						
	I	II	III	IV		
λ	—	—	558.2	559.1		$\mu\mu$
β_0	0.09	0.09	0.105	0.18		"
β_{40}	0.48	0.11	0.18	0.31		"
$d\beta_{40}$	0.09	0.02	0.075	0.13		"

The effect was apparently small here; the widened bands were vague. With a vertical plane of polarisation the phenomenon scarcely changed.

Erbiumyttrium sulphate [$(Er, Y)_2(SO_4)_3$].

§ 26. We also examined an impure product obtained by treatment of the original minerals with sulphuric acid, in which erbium and yttrium occur in variable percentages, and the latter preponderates; the crystals were monoclinic. The group of bands in the green, yellow-green, and red showed peculiar and intricate effects of mag-

netisation; among others some bands which were hardly visible with field off were much more pronounced with the field on, in contrast to most of the other cases observed. Further measurements on this point are in progress.

It is quite probable that this product also contains other rare earth-metals (e.g. dysprosium and holmium); this is, moreover also rather likely for the other erbium salts.

Erbium nitrate [$Er(NO_3)_3 \cdot 6H_2O$].

§ 27. Here too, besides the amorphous salts (§ 10) monoclinic crystal plates of an average thickness of 0.6 m.m. were examined, containing both optical axes. The bands were finer than for any sample examined before. On account of the very complicated resolutions it was often somewhat difficult to ascertain to what bands the different components belonged; on exciting the field a sudden confusion was observed from which the single bands slowly emerged again on breaking the current. These observations were all made with unpolarized light.

The results are best arranged in a table in a way somewhat different from the above.

Group of bands in the green at — 193°.

λ	β_0	Influence of a field of 39 Kilogauss.
516.4	0.17	increases in width (not measured).
517.2	0.13	gives a quadruplet, the outer lines of which are very fine, the middle ones (from violet to red) resp. 0.12 and 0.15 $\mu\mu$ wide, the distances of the middles amounting respectively to 0.08 ; 0.27 ; 0.08 $\mu\mu$, while the middle of the outer components seems to have shifted about 0.01 $\mu\mu$ towards the red with respect to the line with field off.
517.6	very narrow	no more visible.
518.0	0.06	give a very complex set of lines, which ought to be further investigated.
518.3	0.11	
518.6	0.07	

λ	β_0	Influence of a field of 39 Kilogauss.
519.1	0.10	gives a doublet (not measured).
519.7	0.16	gives an asymmetric quadruplet, the extreme lines of which are very fine; those on the violet side are very faint; the middle ones are resp. (from violet to red) 0.11 wide (this one very faint) and 0.135 $\mu\mu$; the distances of the middles are resp. 0.07, 0.255, 0.13 $\mu\mu$, the middle between the extreme components seeming to have been displaced 0.05 $\mu\mu$ towards violet with respect to the line with field off.
520.2	0.17	gives an asymmetric doublet, of which the component on the violet side is 0.05 $\mu\mu$ wide, the other very narrow; distance of the middles 0.175 $\mu\mu$; the mean of these middles is not sensibly displaced with respect to the line with field off.
520.7	0.14	gives an asymmetric triplet (doublet with satellite on violet side); not measured.
521.3	0.11	gives an asymmetric quadruplet; outer component on violet side rather strong, on red side feeble; middle components stronger.

§ 28. *Group of bands in the yellow-green at — 193°*

λ	β_0	Influence of a field of 39 Kilogauss.
534.8	0.13 (faint)	widens and fades away, not measurable.
535.5	0.13	gives an asymmetric doublet, consisting of (from violet to red) first a shade of a width of 0.29 $\mu\mu$, then a strong band 0.18 $\mu\mu$ wide, then a shade, and at last a faint undefined band 0.26 $\mu\mu$ wide; the middle of the first component has shifted 0.17 $\mu\mu$ towards the violet side with respect to the original band 535.5, the place of the middle of the second component being 535.8.
shade ¹⁾ 535.8	0.09	

¹⁾ With a somewhat thicker crystal they form together a heavy band.

λ	β	Influence of a field of 39 Kilogauss.
536.95	narrow	give together, seen in the first order, a triplet, the components of which (from violet to red) are resp. 0.10; 0.05; 0.07 $\mu\mu$, wide, the first very faint, the last two stronger, and connected by a shade; the situation of the middles of the first two is shifted resp. 0.44 and 0.14 $\mu\mu$ towards the violet, that of the last 0.05 $\mu\mu$ towards the red with respect to the band 537.15 with field off.
537.15	satellite 0.05	
537.35	narrow	together give a triplet, the components of which (from violet to red) are resp. 0.08; 0.08; 0.10 $\mu\mu$, wide, the first two strong, and connected by a shade, the last very faint; the place of the middle of the first component has moved 0.02 $\mu\mu$ towards the violet, that of the two following ones resp. 0.07 and 0.34 $\mu\mu$ towards the red with respect to the original line 537.6.
537.6	feeble satellite 0.06	
537.8	narrow satellite somewhat stronger than the former	
538.5	narrow	gives a doublet (not measured).
539.15	0.09	gives a doublet, with components each 0.08 $\mu\mu$ wide, and distance of the middles 0.30 $\mu\mu$; on the violet side another shade is seen, where possibly a third component is found; the middle of the two lines of the doublet seems to have shifted 0.03 $\mu\mu$ towards the violet with respect to the original line 539.15.
539.7	0.08	gives an asymmetric sextuplet of which the four outer components are faint, and very narrow, the two middle ones heavy, and resp. (from violet to red) 0.06 and 0.03 $\mu\mu$ wide, the distances of the middles amounting resp. to : 0.045; 0.05; 0.11; 0.17; 0.035, total 0.41 $\mu\mu$. The middle of the two outer components coincides with the line with field off. The two com-

λ	β_0	Influence of a field of 39 Kilogauss.
		ponents on the violet side are connected by a shade.
540.3	0.07	gives a somewhat vague band, $0.22 \mu\mu$ wide, in which lines could not be distinguished with certainty: it may, however, be a triplet. The middle seems to have been displaced $0.03 \mu\mu$ towards the violet with respect to the line with field off.
540.8	0.07	gives an asymmetric quadruplet, of which the three components lying on the violet side are rather strong, the fourth weak: the mutual distances of the components are very nearly the same, and amount to $0.18 \mu\mu$.

§ 29. *Group of bands in the red at -193° .*

λ	β_0	Influence of a field of 40 Kilogauss.
640.3	not measured	gives an asymmetric doublet, (not measured).
640.9	not measured	gives an asymmetric doublet, (not measured).
642.2	0.09	gives a doublet, consisting of a faint, thin line on the violet side, and a strong line, $0.11 \mu\mu$ wide, the middle of which is at a distance of $0.09 \mu\mu$ from the other faint line, and has moved $0.18 \mu\mu$ towards the red with respect to the line with field off. Probably on the red edge of this strong line another faint line is found, which is connected with it, so that the whole would form a triplet.
642.8	0.11	gives a triplet, the extreme components of which are weak, and very narrow, the middle ones strong, and $0.125 \mu\mu$ wide: the distance of the middles (from violet towards red) amount resp. to 0.11 and $0.16 \mu\mu$; the middle of the inner components has been displaced $0.035 \mu\mu$ towards the red with respect to the line with field off.

λ	β_0	Influence of a field of 40 Kilogauss.
650.5	0.09 (some- what vague)	is widened; the width amounts to 0.35 $\mu\mu$, the band on the violet side is darker than the one lying towards the red; (possibly a quadruplet is formed, which, however, is uncertain); a displacement, however, was not observed.
651.3	0.09 (some- what vague)	increases in width and fades away; the width amounts to 0.40 $\mu\mu$; a shift towards violet seems to take place, but could not be ascertained.

§ 30. Seventh series. We now investigated.

Uranyl potassium sulphate [$UO_2 K_2 (SO_4)_2 + 2 H_2O$].

A rhombic plate containing both axes, 0.7 m.m. thick, was examined in a field of about 40 kgs. with unpolarized light at -193° . The well-known bands appeared to be much more numerous and narrower than for uranyl nitrate (c.f. § 11), among others 487.8, 488.2, 488.8 and 490.5 in the blue. These bands seemed to fade somewhat in the field, but the phenomenon was uncertain here, and in any case the widening did not amount to more than 0.02 $\mu\mu$. For the many bands in the violet no action of the field could be perceived.

Physics. — “*The value of the self-induction according to the electron-theory.*” By Prof. J. D. VAN DER WAALS Jr. (Communicated by Prof. J. D. VAN DER WAALS).

Many physicists refer to the existence of self-induction in order to make the existence of kinetic energy of electrons more intelligible. To a certain extent there is no objection to this, provided we keep in view, 1st that the kinetic energy consists for a large part electrical energy whereas for the calculation of the self-induction only the magnetical energy is taken into account, and 2nd that from a theoretical point of view it is the self-induction which is to be explained from the kinetic energy of the electrons, and not vice-versa. It is this second point which occasions me to make the following remarks.

Let us imagine a piece of metal which contains a great number

of positively and negatively charged particles with a total charge zero, the positive charge of the positive particles being exactly equal to the negative charge of the negative ones. When we put this piece of metal into motion we must ascribe to it a certain amount of electromagnetic mass which is equal to the sum of the electromagnetic masses of the positive and negative particles. If, however, we send a current through the piece of metal, then the energy of this current is *not* equal to $\frac{1}{2} \sum mv^2$, m being the mass of a particle and v the mean velocity which is imparted to them by the electromotive force.

This difference can be explained in the following way. In the case that the piece of metal moves, the positive and negative particles move in the same direction and then in all points of space as well the electrical as the magnetical forces exerted by the different electrons have a different direction and nearly cancel each other, in such a way that we find only sensible forces in the points which are so near one of the electrons that the forces exercised by that electron strongly preponderate over those exercised by all the other electrons, and need only to be taken into account. In the case that a current passes through the metal on the other hand the magnetic forces of a great part of the electrons will act in the same direction, and in a point at some distance, where the force ϕ exerted by a *single* electron is negligibly small the magnetic force exercised by *all* the electrons contained in a unit of volume of the metal (which number we will call N) together will nearly amount to $N\phi$, in consequence of which the energy will be of the order $N^2\phi^2$. This energy proves not at all to be equal to the sum, but rather to N times the sum of the amounts of energy which the single electrons would occasion at that point. From this we may deduce that the energy of the current is much larger than $\frac{1}{2} \sum mv^2$.

Though it might be worth while to try and calculate the amount of this energy more accurately, it seems to me that there can be little doubt but we should find for it:

$$\frac{1}{2} (L + L') i^2$$

where L represents the coefficient of self-induction as it is usually calculated from the magnetic energy alone, and

$$L' i^2 = \frac{1}{2} \sum mv^2.$$

If we assume that the current is transferred by only one kind of electrons then we may write for $\sum \frac{1}{2} mv^2$ for each unit of volume $\frac{1}{2} N mv^2$.

If we now assume the "piece of metal" to be a circular circuit

of metal wire with a radius R and the wire to have a circular section with a radius r , then we have $l = \pi r^2 N e v$ and

$$L' = \frac{\pi r^2 2\pi R N m v^2}{(\pi r^2 N e v)^2} = \frac{2R}{r^2} \frac{m}{N e^2}.$$

Supposing r to be small compared with R we may calculate L for this circuit from the formula of KIRCHHOFF:

$$L = 2 \pi R \left\{ l \left(\frac{8 R}{r} \right) - 1,75 \right\}.$$

The number N being probably different for different metals, L' also appears to depend on the kind of metal of which the circuit consists, whereas L only depends upon the geometrical properties of the circuit. On the other hand we see that L' has a constant value for a given wire, independent of the way in which the wire is wound to a coil, whereas L depends in a high degree upon the way of coiling.

The ratio $\frac{L'}{L}$ will therefore in different cases have a very different value. We shall try to get an idea of what order this quantity can be, and inquire whether we are always justified in neglecting L' compared with L . To this purpose we can make use of the value of $N e$, which has been calculated by J. J. THOMSON¹⁾ for bismuth. From the value of the resistance and from the variability of the resistance in the magnetic field THOMSON deduces that the value of $N e$ for bismuth amounts to about 0,11. If we put $\frac{e}{m} = 1,865 \cdot 10^7$ we find

$$L' = \frac{R}{r^2} \cdot 10^{-6}.$$

Metals with a greater conductivity will probably have a higher value for N . THOMSON estimates the value of N for copper or silver to be several thousands of times larger than for bismuth.

We obtain the same result by starting from the values

$$N_1 = 0,69 \cdot 10^{19} \quad N_2 = 0,46 \cdot 10^{19}$$

for the numbers of positive and of negative particles per cm^3 , which have been derived by DRUDE²⁾ from the behaviour of bismuth in

¹⁾ J. J. THOMSON. Rapports présentés au congrès de physique à Paris, III. p. 145. 1900.

²⁾ DRUDE. Ann. der Phys. IV Folge. 3. p. 388. 1900.

³⁾ EDW. B. ROSA and LOUIS COHEN. Bulletin of the Bureau of Standards. Vol. 4. No. 1. Reprint No. 75.

other respects. Putting $e = 10^{-20}$ we arrive at a value of Nv which does not deviate very much from that of THOMSON.

ROSA and COHEN²⁾ calculate for the self-induction of a circle with $R = 25$ cm. and $r = 0,05$ cm.

$$L = 654,40496.$$

For this same circle we find

$$L' = 0,01,$$

so if we neglect L' we make in this case only an error of $\pm 0,002\%$. This value applies to bismuth, for other metals the correction is probably much smaller still. The correction is also relatively smaller when we have not one circle but many windings. On the other hand the correction is much greater if we take a thinner wire.

Notwithstanding the perfect agreement between the numbers of THOMSON and of DRUDE these values do not seem very reliable to me. It is therefore not superfluous to inquire whether we can find another way in which we might evaluate L' . Perhaps this might be done as follows.

HAGEN and RUBENS¹⁾ have shown that the reflective power of metals for infra-red light of large wave-length can be explained by ascribing to these metals the same conductivity for electric vibrations of the considered frequencies as for stationary currents. This seems to indicate that the mean free path between two collisions of an electron against the atoms of the metal is small compared with those wave-lengths²⁾. As the same does not apply to light of a wave-length smaller than one micron, we should be inclined to deduce from these optical properties that the mean free path is not much smaller than one micron.

We cannot deny that this value of the free path is remarkably great, as we find a value for the free path for the molecules of the air at a pressure of one atmosphere, which is about 10 times smaller. But let us notwithstanding assume this value of the path-length to be correct, then it yields a new method to calculate L' . We find, namely, for the conductivity of a metal:

$$\sigma = n \frac{2Nv^2l}{mu}$$

where u is the mean velocity of the heat-motion of the electrons,

¹⁾ HAGEN and RUBENS. Berl. Sitzungsber. 1903, p. 269. Ber. d. deutschen phys. Gesellsch. 1903, p. 145.

²⁾ Comp. H. A. LORENTZ. These Proceedings V, p. 666, 1903.

l the path-length, and n a constant, which according to DRUDE amounts to $\frac{1}{4}$, according to LORENTZ¹⁾ to $\sqrt{\frac{2}{3\pi}}$.

So we get:

$$L' = \frac{2R}{r^2} \frac{m}{Ne^2} = \frac{2R}{r^2} \frac{nl}{\sigma u}.$$

At $T = 300$ we may put $u = 1,75 \cdot 10^8$. Moreover we have $\sigma = 6,14 \cdot 10^{-4}$ for silver, and we shall assume $l = 10^{-4}$. This yields:

$$L' = \frac{R}{r^2} \cdot 5 \cdot 10^{-8}.$$

This is about $\frac{1}{20}$ of the value found for bismuth, and not less than $\frac{1}{1000}$, as we expected. The error caused by neglecting L' amounts therefore for a circle of silver wire of $R = 25$ cm. and $r = 0,05$ cm. to $\pm 0,0001$ „.

It seems to me as yet impossible to compute for L' a value which may be trusted to be accurate. Yet I think that the above calculations make it probable that for coils which are wound in such a manner that they have a large self-induction, the value of L' may be neglected compared with L ; but that on the other hand L' may not always be neglected if the coil is wound in such a way that L is as small as possible. In the latter case it might perhaps be possible to determine L' experimentally, at least if L can then be calculated with a sufficient degree of accuracy. And if it should prove to be possible to determine the different values of L' for different metals, this would be a valuable datum for the extension of our knowledge of the motion of the electrons in metals.

Finally it may be remarked that we shall also find a considerable value for L' for a current which does not pass a metallic wire, but for instance a Röntgen-tube. The high value of the velocity of the electrons in this case, gives rise to a high amount of kinetic energy, and this "energy of the kathode-rays" will no doubt reveal itself as an increase of the self-induction of the circuit in which the tube has been inserted.

¹⁾ H. A. LORENTZ. These Proceedings VII, p. 448. 1904

Chemistry. -- "*The action of concentrated sulphuric acid on glycerol esters of saturated monobasic fatty acids.*" Preliminary communication. By B. W. VAN ELDIK THIEME. (Communicated by Prof. S. A. HOOGWERFF).

As is well known, the saponifications may generally be represented by the equation :



that is we shall always obtain an equilibrium between the reacting molecules which is dependent on the temperature, on the medium and on the nature of the ester.

The velocity of the saponification is moreover very low and is vigorously accelerated by hydrogen ions; so long, however, as the quantity of the acid added does not considerably modify the nature of the medium the equilibrium will not be changed thereby.

In the technics of fat-saponification dilute sulphuric acid is used as catalyst, for instance in the Twitchell process; from the above it follows that we must not expect the process to complete its course; it is considered satisfactory when the fat is resolved to 94 à 96% of free fatty acids.

If we use a stronger acid the process becomes modified. Firstly, we are dealing with another medium (in practice where the quantity of acid is small the medium itself is changeable during the process), secondly we have besides the first process also the following :



which means the expulsion of one of the acid residues by the other one.

Here also, however, we may expect the reaction to be reversible so that it will be completely to the right only when :

a. The sulphuric acid added, is anhydrous (100%),

b. An excess of acid is added to dry fat,

c. The temperature at which the action takes place, is kept within definite limits. From this it follows, that the statement of BÜXTE¹⁾ that butterfat is completely saponified by sulphuric acid of sp. gr. 1.8355 (corresponding with 93.5% of H₂SO₄) cannot possibly be correct. 5 grams of butterfat are heated in an Erlenmeijer flask of one litre capacity to 100°, 10 c.c. of 93.5% sulphuric acid are added and the whole heated for 10 minutes in a waterbath at 30—32°. 150 c.c. of water are added next.

¹⁾ Chem. Zeit. N^o. 12 1894 pg. 204, also KREIS Chem. Zeit. N^o. 76 1892 pg. 1394.

Moreover, the high temperature at which the action of the acid takes place is unsuitable for the purpose of a complete saponification, because, as will be seen, it is just the increase in temperature which causes the shifting of the equilibrium in equation (2) towards the left. On repeating BÜNTE's method I obtained the following figures: With 93.5% acid the butter fat was resolved to 81.0% of free fatty acid

..	93.5%	89.7%
..	100.0%	92.2%

From these figures, the influence of the concentration of the sulphuric acid is very obvious; also the imperfection of the method so that it cannot be a matter of surprise that it has been entirely abandoned.

In order to get a better insight into the action of concentrated sulphuric acid on fats I chose as starting material pure trilaurin prepared from Tangkailak fat obtained from the fruits of *Cylicodaphne Litsaea*, a tree growing in West Java. The fat consists of trilaurin and triolein so that it is easy to prepare trilaurin from the same by recrystallisation from ether.

The sulphuric acid employed was 100.0% as determined by titration. Experiment *a* took place at a temperature of 18°, experiment *b* and *c* at 1—2°. Time of action 30 minutes.

<i>a</i>	1 mol. of trilaurin to	6.5 mol. of H_2SO_4	gave	86.6%	free fatty acid
<i>b</i>	1	26.0 H_2SO_4 .. 95.5%
<i>c</i>	1	52.0 H_2SO_4 .. 100.0%

the reaction:

trilaurin + sulphuric acid = glyceroltrisulphuric acid + lauric acid seems, therefore, only practically complete with a very large excess of sulphuric acid and at a low temperature, for if experiment *c* is repeated and then again heated at 60° for 1½ hour, a shifting towards the left takes place and trilaurin is regenerated. The course of the investigation is briefly as follows:

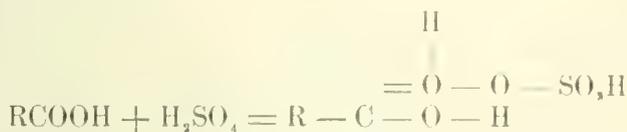
The 100.0% sulphuric acid is weighed in a flask which is then corked and placed in ice water. The weighed trilaurin is now added in small quantities. As by the action of the acid on trilaurin heat is generated no fresh portion of trilaurin must be added until the previous lot has dissolved¹⁾. When all the trilaurin has dissolved and the time of action has expired the contents of the flask are

¹⁾ From this evolution of heat with saturated compounds it follows that no undue importance should be attached to MAUMENÉ's experiment (Compt. rend. 1882. 35 pg 572) where this evolution of heat is made use of to detect unsaturated compounds.

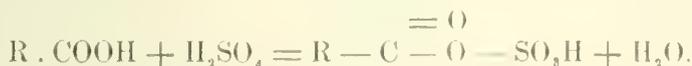
poured on to pounded ice in order to prevent as much as possible a rise in temperature and consequent saponification. Sufficient alcohol is now added so as to obtain a 60% alcohol mixture and this is then shaken with a mixture of ether and petroleum ether. After washing with water the ether is evaporated. In experiment *c* a substance was left with ester number 0 and acid number 280.5, which points to pure lauric acid. This proves that all the trilaurin was decomposed.

On repeating experiment *c* with subsequent heating at 60° for 1½ hour and removing the sulphuric acid in the manner described a substance was obtained with acid number 246.8 and saponification number 280.9; ester number 34.1. As trilaurin possesses an ester number 263.8, 12.9 % of trilaurin has been regenerated.

As regards the lauric acid which in the previous equation occurs together with glyceroltrisulphuric acid it must be remarked that this unites with H₂SO₄ to molecular compounds which are more or less soluble in benzene. Now if trilaurin is dissolved in 100 % acid (experiment *c*) and if this is shaken with dry benzene both lauric acid and sulphuric acid may be detected in that solvent. Compounds of a similar character have been described by HOOGWERFF and VAN DORP¹⁾. In these additive compounds the oxygen is sometimes taken as quadrivalent such as:



Others, H. MEIJER²⁾, believe in the existence of a kind of mixed acid anhydrides:



The latter is improbable as then we should want in all these compounds exactly 1 mol. of water of crystallisation for 1 mol. of the two acids. We already noticed in the saponification of butter fat that the concentration of the acid plays an important part; this is also the case with trilaurin. If now at a temperature of 1—2° we allow 52 mols. of 94.6% sulphuric acid to act on 1 mol. of trilaurin for 30 minutes a substance was obtained, after removal of the sulphuric acid, consisting of 80% of lauric acid and 20% of undecomposed glyceride. This glyceride was separated from the

¹⁾ Recueil XVIII 1899 bl. 211.

²⁾ Monatshefte für Chemie 24 p. 840.

lauric acid and its ester number found to be 244.0. The ester numbers of trilaurin, dilaurin and monolaurin are respectively 263.8, — 246.1 and 204.7 so that the separated glyceride is a mixture.

The mono- and dilaurin are probably formed here from compounds like $C_3H_5(OR)(O \cdot SO_3H)_2$ and $C_3H_5(OR)_2(O \cdot SO_3H)$ by decomposition with water $R = C_{11}H_{23}CO$.

Similar compounds are still under investigation. In the action of concentrated H_2SO_4 on nitroglycerol analogous reactions occur. NATHAN and RINTOUL.¹⁾ in an article on: Nitro-glycerine und seine Darstellung write:

“Die Absorption des Nitroglycerins durch die Abfallsäure ist nicht nur ein Lösungsvorgang. Es findet noch eine zweite Reaktion statt, zwischen der Schwefelsäure und dem Nitroglycerine, unter Bildung von Sulfoglycerin und Salpetersäure. Diese umkehrbare Reaktion gelangt schnell in den Gleichgewichtszustand, so dass bei einer normalen Abfallsäure eine Hälfte des gesamten absorbierten Nitroglycerin als Sulfoglycerin vorhanden ist; während der Rest tatsächlich als Nitroglycerin in Lösung geht.”

The reverse reaction (2) which still takes place at 60° even in the presence of a large excess of acid: glyceroltrisulphuric acid + lauric acid = trilaurin + sulphuric acid is to a certain extent comparable to the synthesis of glycerides according to GRÜN and SCHACHT²⁾. They, however, write:

“Die Esterificirung des Glycerins durch Schwefelsäure bleibt — auch bei Anwendung von grossen Überschüssen an Säure — bei der quantitativen Bildung von Glycerindischwefelsäure ($C_3H_5(OH)(O \cdot SO_3H)_2$) stehen, dementsprechend treten auch bei der Einwirkung der organischen Säuren auf diese Verbindungen nur zwei Acyle in das Glycerinmolekül; man gelangt zur Diglyceriden.”

“Die Bildung von Mono- und Triglyceriden konnte beim Einhalten der unten angegebenen Bedingungen nicht constatirt werden; ebenso wenig die Bildung anderer Nebenproducte.”

It seems to me that this conclusion cannot conform to theory: it is also in conflict with my own observations. First of all, glycerol-disulphuric acid is never formed quantitatively in the esterification of glycerol by sulphuric acid, secondly byproducts are formed in their synthesis from diglycerides.

If one part of glycerol is dissolved in four parts of 98.3% sulphuric acid there is formed chiefly a mixture of glyceroldi- and trisulphuric

¹⁾ Chemiker Zeitung No. 20, 1908, p. 246.

²⁾ Berichte 38 p. 2284 (1905) see also Berichte 40 p. 1778 (1907).

acid, also a small proportion of the mono-acid. If to this mixture is added palmitic acid dissolved in H_2SO_4 , a substance is obtained with an ester number of 203.1; the ester numbers of tripalmitin and dipalmitin are respectively 208.8 and 197.6. By one single recrystallisation from absolute alcohol, nearly chemically pure tripalmitin with an ester number of 208.1 and m.p. $64-65^\circ$ could be isolated. Therefore, a mixture of dipalmitin and tripalmitin has been the main product. The barium salt prepared by me according to their method possesses another composition as stated by them; it should, however, be observed that $C_3H_5O_2S_2Ba + 2H_2O$ does *not* require 7.63% of H_2O but 8.50%.

One part of chemically pure glycerol D 1.261 was dissolved in 4 parts of 98.5% sulphuric acid. After 15 minutes an equal volume of water was added, the liquid was neutralised with barium carbonate and after removal of the barium sulphate by filtration the liquid was evaporated in vacuum. After adding a little alcohol it is again evaporated so as to get rid as much as possible of the water.

If now, an excess of absolute alcohol is added, a thick white precipitate of syrupy consistence is formed, which is shaken several times vigorously with alcohol to remove any free glycerol. The precipitate solidifies after a while and is then dried in vacuum over P_2O_5 to constant weight.

3.132 grams of the dried salt gave on evaporation with sulphuric acid 1.7380 grams of $BaSO_4 = 55.49\%$ of sulphate or 32.65% of barium 0.7740 grams gave 0.4295 grams of sulphate = 55.49% or 32.65% of barium.

Calculated for the Ba salt of the anhydrous di-acid 60.24% $BaSO_4$.
 " " " " " " " " " mono-acid 48.67%

On heating the dried compound for $1\frac{1}{2}$ hour at 105° in an air-bath it turns brown and evolves acrolein. In this operation 1.059 grams lost 0.011 grams or 1.03%.

Therefore, a mixture of barium salts has formed which may be readily explained by the fact that on diluting the mixture of glycerol and sulphuric acid, the tri-acid already formed passes into lower acids.

CLAESSON¹⁾, who was the first to prepare glycerolsulphuric acid also observed this conversion of the tri-acid into the lower acids.

He prepared the tri-acid from anhydrous glycerol and chlorosulphonic acid; his statement that this tri-acid, on boiling with water or dilute acids, is readily and completely resolved into glycerol and sulphuric acid is, however, incorrect: at least after boiling for one

¹⁾ Journal für praktische Chemie [2] Bd. 20, p. 1. 1879.

hour there still remains a portion of the acid combined with glycerol in the form of a mono-acid.

Glyceroltrisulphuric acid was prepared by me according to CLAESSEX from anhydrous glycerol and chlorosulphuric acid. 1.619 grams of the acid was dissolved in water and boiled for an hour, the solution was neutralised with barium hydroxide and the resulting barium sulphate weighed. If the sulphuric acid had been eliminated completely 3.411 grams of barium sulphate ought to have been formed but only 2.121 grams were found; therefore 0.542 grams of sulphuric acid was left in combination with glycerol.

The above experiments, therefore, throw a little more light on the sulphuric acid saponification of fats. Further communications will follow shortly.

Gouda, 5 April 1908.

Laboratory Candle Works.

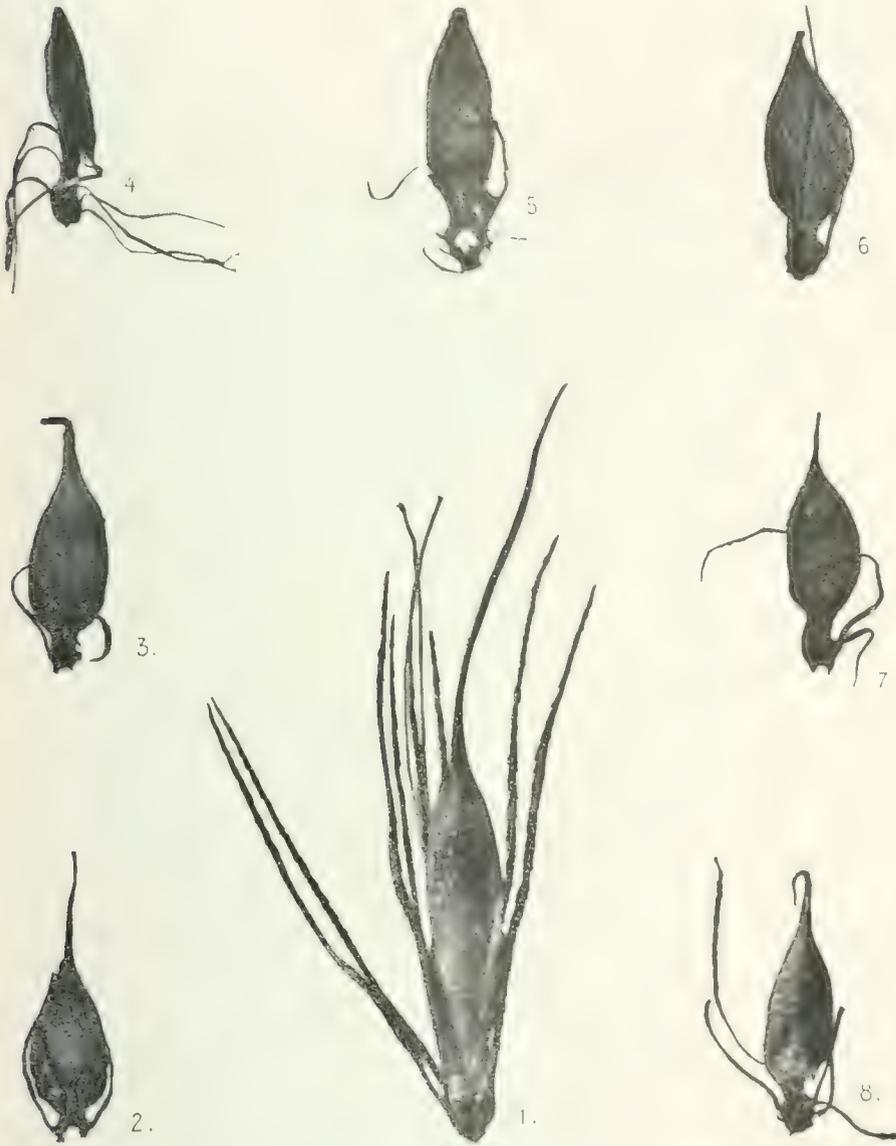
Palaeontology. — “*On Dulichium respiforme* sp. nov. from the brick-earth of Tegelen.” By Mr. CLEMEND REID F. R. S. and Mrs ELEANOR M. REID B.Sc. (Communicated by Prof. G. A. F. MOLENGRAAF).

In our paper on the Fossil Flora of Tegelen published in 1907¹⁾ we figured a fruit provisionally referred to *Rhynchospora*, though it did not possess the articulate beak of that genus. All the specimens then available were so much distorted and injured by germination that it was difficult to determine what the character of the perfect fruit would be. In addition to this, the most perfect specimen appeared to possess a quadrate base and 8 setae, characters unknown in *Dulichium*, to which genus the fruit was in other respects comparable.

Since the publication of our paper we have obtained more material, thanks to the kindness of Dr. LORRÉ and Baron L. GRENDL. This new material and a closer examination of the specimens before collected, enables us now to describe the fruit as a new species belonging to *Dulichium*, a genus now confined to America, though already recorded by Dr. N. HARTZ as occurring in an interglacial peat-moss in Denmark²⁾. Dr. HARTZ's specimens are referred, we think correctly, to the only living species, *Dulichium spathaceum*; our fruits are very different.

¹ Verhand. Kon. Akad. Wetensch. (Tweede Sectie). Deel XIII, No. 6, fig. 105.
² Dansk. geol. Forening 10, 1904, p. 13.

CLEMEND REID and Mrs ELEANOR M. REID. "On *Dulichium vespiforme*
sp. nov. from the brick earth of Tegelen."



1. *Dulichium spathaceum* L. C. RICH. Recent, America. 2—8. *Dulichium vespiforme* sp. nov. Fossil, Tegelen. All the figures are magnified to the same scale — 12 diameters.

Fruetus dimidio brevior eo *D. spathacei* tamen latior, long. (rostro incluso) circiter 3—5 mm.; setae 7 vel 8 (forsitan 9), longitudinaliter complanatae, canaliculatae, striatae; nux ovata subito in stipitem coarctata, in rostrum longum gracile attenuata, paulo triangularis vel plano-convexa, praesertim rostrum versus; superficies foveolata, multangula, eâ *D. spathacei* crassior; long. (rostro excluso) 2.0—2.5 mm., lat. 1 mm.

Figs. 1—8, photographed on the same scale, show the differences between the recent and fossil forms. In the living species the nut is oblong, not ovate, and is much narrower in proportion to its length. The long stalked nut with oblique attachment, inarticulate style, setae more than 6 with recurved hooks, are generic characters common to the two species. In section the nut of *D. vespiiforme* is somewhat triangular near the base, becomes plano-convex in the middle, but loses its convexity as it passes into the beak; the beak at its base is flattened triangular, but becomes terete above.

As all the specimens we have yet seen (about 20) have apparently germinated, it is impossible to describe the exact shape of the nuts or the complete setae; we have therefore figured several actual specimens, without attempting to restore them. The setae are more or less broken, but we cannot find clear evidence of more than 8 in any of the specimens; they differ from those of *D. spathaceum* in their flattening, and they are longitudinally channelled instead of showing a midrib. The fruits undoubtedly belong to the genus *Dulichium*; and as the living species has a wide range in latitude we thought it possible that some form (several different ones have been recorded) might agree with our fossil. We find, however, that the fruits of the recent forms in the Kew herbarium vary only slightly; they are always much larger than our fossil, and the nut is long, narrow, and parallel-sided.

Whether the genus *Dulichium* originated in Europe or in North America there is nothing to show. It has now only one living species, confined to America; but this species has been found also in a fossil state in Denmark. Now, in an older deposit in the province of Limburg, we discover an extinct form. Very little is yet known as to the geological history of the Cyperaceae, and *Dulichium* will probably turn out to have been widely distributed and to have had many species. The genus is at present very isolated and the new fossil form makes no approach to any other genus.

Physics. — “*Change of wavelength of the middle line of triplets.*”
(Second Part). By Prof. P. ZEEMAN.

6. We will now return to the observations of § 4. Arranging these according to strength of field it appears that the distance $a' - a''$ changes considerably with increasing magnetic intensity. The displacement of line 5791 is not a linear function of the strength of the field but increases more rapidly than would follow from this simple relation. However it is impossible without further consideration to deduce the law of displacement, because, as remarked in § 4, the distance of the lines of comparison does not remain invariable. This is the reason why somewhat different values of $a' - a''$ are obtained, when these are calculated from the change of $a - a'$, than when the change of $b - a'$ is considered.

The direction however of the displacement of 5791 is easily determined. It is towards the *red* end of the spectrum. A shift towards the side of increasing wavelengths corresponds in the figure of § 3 to a displacement in the direction from a' towards a'' . The less refrangible side of line 5791 is easily distinguished upon the negatives by the observation of the two weak less refrangible companion lines and the one weak more refrangible companion line¹⁾.

7. The shift of the middle line of the triplet may be demonstrated also by our method of the non-uniform field, if an echelon-spectroscope is made use of. A curvature of the middle line will be the immediate effect of the shift. If we use ROWLAND'S grating such a curvature would be invisible nor have I observed it in that case.

The visibility of the curvature will be much increased by taking care that in the image points corresponding to very different intensities of field lie closely together. In order to attain this an eleven times reduced image of the vacuum tube, charged with mercury and placed into the field, was projected on the slit of the auxiliary spectroscope. The lens used was a photographic objective of 10 cm. focus.

The Plate gives somewhat enlarged reproductions of negatives relating to line 5791 resp. line 5770. The middle line is given in two succeeding orders. Between these the other components of the triplets are seen. With increasing magnetic force the components deviate further and further from their own middle line. In the central part of the field of view the maximum distance is reached.

¹⁾ JANICKI. Feinere Zerlegung der Spektrallinien von Quecksilber u.s.w. Inaugural. Diss. Halle a. S. 1905, Annalen der Physik. Bd. 19, 36. 1906.

P. ZEEMAN. "Change of wavelength of the middle line of triplets."
(Second part).



Hg.	5770	5791	1 mm. = 0.12 A.E.
resolution:	symmetrical.	asymmetrical.	
middle lines:	straight.	curved.	

The component towards the red in the figures is always at the left of its middle line, being concave to it in the central part; the second manifestly curved line is the component towards the violet belonging to the other order.

The curvature of the middle lines, the demonstration of which is the object of our present experiment, is undoubtedly visible in the figure for 5791. It is still more easily seen by comparison with a straight bit of paper.

In the figure for 5770 this kind of curvature is absent.

The asymmetry of the magnetic resolution of line 5791 is at once evident by the fact that one of the middle lines is approached more nearly by the outer component than the other.

If we denote by a_v and a_r the distances of the components to their middle lines, then what I called on a former occasion¹⁾ the amount of the asymmetry is equal to $a_v - a_r$. This difference is also equal to the difference of the distances separating the plainly curved lines from the middle lines to which they do not belong, and to which they are convex.

The two negatives were taken with the same field intensity of about 34000 Gauss.

The question now arises whether the difference $a_v - a_r$ is equal to twice the shift of the middle line or not. In the first case the asymmetry is brought about solely by the motion of the middle line towards the less refrangible wavelengths, the outer components having undergone a symmetrical displacement relatively to the unmodified line. The other, more general case one would rather expect without hypothesis or without the results of measurements.

8. In order to test the question by experiment, I have taken on the same negative as well the figures described in § 7 as the unmodified lines. It appeared however rather soon that, in the case of line 5791, only in the most intense fields the separation of the middle lines, taken with field on and with field off, was sufficient to allow measurements.

I therefore refrain from communicating these experiments. Only one detail of the vacuum tube, charged with mercury and used in all my experiments with strong fields, may perhaps be mentioned. This vacuum tube of the form indicated by PASCHEN, has a rather wide capillary. That part however of the capillary which is placed in the magnetic field is drawn out. Only over this short distance the

¹⁾ ZEEMAN, These Proceedings 30 November 1907-

capillary has a small diameter. Now the gap-width of the electromagnet may be considerably diminished; at the same time the electrical resistance of the vacuum tube is moderate.

9. Measurements were made in the following manner. The slit of the auxiliary spectroscope performing the preliminary analysis of the light, was widened in order to obtain in the echelon spectroscope light of the *two* yellow mercury lines simultaneously. The steps of the echelon were placed parallel to the slit of the auxiliary spectroscope. The image of the vacuum tube projected on the slit was now chosen in such a manner that only light from the uniform part of the field was analyzed. By means of a suitable small screen placed before the photographic plate its middle part could be exposed first to light under magnetic influence; then the unmodified lines were taken in the upper and lower parts of the plate. The plates taken confirmed the result obtained in § 6 as to the shift of line 5791 towards the red.

As to line 5770 the amount and even the existence of the shift is not quite certain at present¹⁾. All these measurements were not further pursued however, because after the publication of the first part of this paper²⁾ and during my measurements there appeared a communication by GMELIN in the 1 April number of the *physikalische Zeitschrift*.

Independently of my paper and in another way our present subject was taken up by GMELIN. The enormous resolving power of the echelon spectroscope used by GMELIN apparently permits of greater accuracy in the measurements than would have been possible for me.

It may be remarked finally that, so far as the present results can settle the question, the observation of the asymmetry in a direction parallel to the lines of force,³⁾ which first induced me to this investigation, but which was given with some reserve, must have been correct.

¹⁾ The first part of this communication contains an error, which I only noticed after the printing off. In the last division of § 4 the change of wavelength has been calculated from the measured displacement in the same manner as must be done, when the distance of two adjacent orders with the field off, is compared with the distance separating the components towards red and towards violet with the field on. Of course this proceeding is faulty in the case of § 4. Hence the last sentence of § 4 and the last column of the table in § 5 have lost significance. The further consideration of the peculiar change of the distance of orders noticed in § 5 must be reserved for a future paper.

) ZEEMAN, These Proceedings 29 Febr. 1908.

³⁾ ZEEMAN, § 7 in *New Observations etc.* These Proceedings 29 Febr. 1908.

ERRATA.

In the Proceedings of the meeting of December 1907:

In Pl. I belonging to the communication of Prof. H. KAMERLINGH ONNES and C. BRAAK (p. 413) the numbers I and II are to be interchanged.

- p. 422 to footnote I add: In this communication the resistance thermometer of Comm. No. 95 *c* (Sept. '06), which is called Pt_I , was used.
- p. 423 to footnote 1 add: The thermometer till now called Pt_I was named Pt'_I after the breaking of the wire.
- p. 447 l. 20 from the top: for 79 *read* 78.

In the proceedings of the meeting of February 1908:

Pl. II belonging to the communication of JEAN BECQUEREL and H. KAMERLINGH ONNES in the subscript of Fig. 1 for 1.71 mM. in 1, 2, 3, 4 *read* 1.71 mM. in 1, 3, 4.

- p. 597 l. 5 from the bottom: for *we read* they.
- p. 604 l. 1 " " " " 106 and 107 *read* 147.
 l. 15 " " " " on *read* of
- p. 606 l. 19 " " " " observations *read* deviations
- p. 610 l. 1 " " top, for down to *read* as far as.
 l. 7 " " bottom: for 170 *read* 117.

In the proceedings of the meeting of Februari 1908.

- p. 591 l. 14 from the bottom: for 0°.10 *read* 0°.06.
- p. 522 l. 1 from the top: for 0.0000013 *read* 0.0000009.
 l. 2 " " " " 0.0036614 *read* 0.0036617.

In the paper by Dr. DE SITTER "On Jupiter's Satellites" (Meeting of March 28).

- p. 721 the value of $\log a_4$ should *read*

$$\log a_4 = 8.0998360,$$
- p. 727 the signs of a_{21} and a_{21}' should be inverted.

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In the Proceedings of the meeting of February 1908:

- p. 591 l. 17 from the bottom: *for* $-0^{\circ}.002$ *read* $+0^{\circ}.002$.
 l. 14 " " " *for* $-0^{\circ}.02$ *read* $+0^{\circ}.02$.
 p. 592 l. 2 from the top: *for* 0.0036614 *read* 0.0033613,
 (These errata supersede the errata given Proc. April '08 p. 865
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