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VAN WETENSCHAPPEN
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PROCEEDINGS OF THE SECTION OF SCIENCES

VOLUME XIX

JOHANNES MÜLLER :—: AMSTERDAM
:———: SEPTEMBER 1917 :———:

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CONTENTS.

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VOLUME XIX

N°. 6.

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CONTENTS.

Mathematics. — *"On an arithmetical function connected with the decomposition of the positive integers into prime factors."* I. By J. G. VAN DER CORPUT. (Communicated by Prof. J. C. KLUYVER).

(Communicated in the meeting of May 27, 1916).

Let u be any arbitrary integer > 1 and resolve u into prime factors; let e_u represent the smallest exponent of these factors and let a_u indicate how many times e_u occurs in the series of this exponents. Moreover we take $e_1 = 0$ and v_u represents the greatest divisor of u, for which $e_{v_u} > m$, m being any arbitrary positive integer. The object of this paper is to deduce a formula obtaining two general arithmetical functions F and f, satisfying four relations, n representing any positive integer, viz.

1st. for $e_u < m$ and also for $e_u = m$, $a_u > n$

$$F(u) = 0;$$

2nd. if $e_u \leq m$,

$$f(u) = 0;$$

3rd. for $e_u = m$, $a_u = n$,

$$F(u) = f(v_u):$$

4th.

$$F(u) = O(v_u^{\mu}),$$

μ having a constant value $< \dfrac{1}{m(m+1)}$.

The integers m and n are called the parameters of the function F and f the function corresponding to F.

This article, now, is intended to demonstrate the formula

$$\left.\begin{aligned}\sum_{\substack{u=2 \\ u \equiv l}}^{x} F(u) &= \frac{a x^{\frac{1}{m}}}{\log x} + O\left(\frac{x^{\frac{1}{m}}}{(\log x)^2}\right) \qquad \text{for } n = 1, \\ &= \frac{a x^{\frac{1}{m}} (\log \log x)^{n-1}}{\log x} + O\left(\frac{x^{\frac{1}{m}} (\log \log x)^{n-2}}{\log x}\right)\end{aligned}\right\} \quad . \quad (1)$$

for any arbitrary integral positive value of n and this proof will be given in § 2 for $n = 1$, in § 3 for the other case. The modulus of the congruences, for which this modulus has not been mentioned, is in this paper the arbitrary positive integer k, x represents a number > 1, l an integer, prime to k, a has a constant value, viz.

$$a = \frac{bm}{h \,.\, (n-1)!} \sum_{\substack{u=1 \\ uz^m \equiv l}}^{\infty} \frac{f(u)}{u^{\frac{1}{m}}};$$

h is the number of positive integers $\leq k$, prime to k, b is the number of incongruent roots z of the congruence

$$z^m \equiv 1$$

and the sum

$$\sum_{\substack{u=1 \\ uz^m \equiv l}}^{\infty} \frac{f(u)}{u^{\frac{1}{m}}}$$

is extended over all the positive integers u, for which the congruence

$$uz^m \equiv l$$

has roots in z.

§ 2. Lemma.

$$\sum_{v=1}^{\sqrt{x}} \frac{f(v) \log v}{v^{\frac{1}{m}}} = O(1),$$

$$\sum_{v=1}^{x} f(v) = O\left(\frac{x^{\frac{1}{m}}}{(\log x)^3}\right).$$

and

$$\sum_{v=1}^{\sqrt{x}} \frac{f(v)}{v^{\frac{1}{m}}} = \sum_{v=1}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left(\frac{1}{\log x}\right).$$

Proof. From the relations satisfied by the functions F and f, it follows, u being an integer, for which v_u has the value v, that

$$f(v) = f(v_u) = F(u) = O(v_u{}^{\mu}) = O(v^{\mu})$$

and for $\frac{1}{m} > s > \frac{1}{m+1} + \mu$ the left member of the identity

$$\prod_{p} \left\{1 + \frac{1}{p^{(m+1)(s-\mu)}} + \frac{1}{p^{(m+2)(s-\mu)}} + \ldots\right\} = 1 + \sum_{\substack{v=2 \\ e_v > m}}^{\infty} \frac{1}{v^{(s-\mu)}}$$

is a convergent product and consequently the right member a convergent sum, therefore

$$g(x) = \sum_{v=1}^{x} \frac{f(v)}{v^s} = f(1) + \sum_{\substack{v=2 \\ e_v > m}}^{x} \frac{O(v^{\mu})}{v^s}$$

$$= O(1) + O \cdot \sum_{\substack{v=2 \\ e_v > m}}^{\infty} \frac{1}{v^{s-\mu}} = O(1).$$

Hence

$$\sum_{v=1}^{\sqrt{x}} \frac{f(v)\, log\, v}{v^{\frac{1}{m}}} = \sum_{\substack{v=1 \\ e_v > m}}^{\sqrt{x}} O\left(\frac{1}{v^{s-\mu}}\right)$$

$$= O(1),$$

$$\sum_{v=1}^{x} f(v) = \sum_{v=1}^{x} v^s \{g(v) - g(v-1)\}$$

$$= [x]^s g[x] - \sum_{v=1}^{x-1} g(v) \{(v+1)^s - v^s\}$$

$$= O(x^s) \,.\, O(1) - \sum_{v=1}^{x-1} O(1) \{(v+1)^s - v^s\}$$

$$= O(x^s) + O \,.\, \sum_{v=1}^{x-1} \{(v+1)^s - v^s\}$$

$$= O(x^s) + O\{[x]^s - 1\}$$

$$= O(x^s)$$

$$= O\left(\frac{x^{\frac{1}{m}}}{(log\, x)^3}\right)$$

and

$$\sum_{v=x+1}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} = \sum_{v=x+1}^{\infty} \frac{g(v) - g(v-1)}{v^{\frac{1}{m}-s}}$$

$$= -\frac{g[x]}{[x+1]^{\frac{1}{m}-s}} + \sum_{v=x+1}^{\infty} g(v) \left\{\frac{1}{v^{\frac{1}{m}-s}} - \frac{1}{(v+1)^{\frac{1}{m}-s}}\right\}$$

$$= O\left(\frac{1}{x^{\frac{1}{m}-s}}\right) + \sum_{v=x+1}^{\infty} O(1) \,.\, \left\{\frac{1}{v^{\frac{1}{m}-s}} - \frac{1}{(v+1)^{\frac{1}{m}-s}}\right\}$$

$$= O\left(\frac{1}{x^{\frac{1}{m}-s}}\right) + O \sum_{v=x+1}^{\infty} \left\{\frac{1}{v^{\frac{1}{m}-s}} - \frac{1}{(v+1)^{\frac{1}{m}-s}}\right\}$$

$$= O\left(\frac{1}{x^{\frac{1}{m}-s}}\right) + O\left(\frac{1}{[x+1]^{\frac{1}{m}-s}}\right)$$

$$= O\left(\frac{1}{x^{\frac{1}{m}-s}}\right)$$

$$= O\left(\frac{1}{log\, x^2}\right),$$

consequently

$$\sum_{v=1}^{\sqrt{x}} \frac{f(v)}{v^{\frac{1}{m}}} = \sum_{v=1}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} - \sum_{v=\sqrt{x}+1}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}}$$

$$= \sum_{v=1}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left(\frac{1}{\log x}\right).$$

Identity. If ψ_1 and ψ_2 represent two arbitrary arithmetical functions, the sum

$$\sum_{d_1 d_2 \leqq x} \psi_1(d_1)\, \psi_2(d_2),$$

extended over all the positive integers d_1 and d_2, of which the product is not greater than x, is equal to

$$T_1 + T_2 - T_3 T_4$$

where

$$T_1 = \sum_{d_1=1}^{\sqrt{x}} \psi_1(d_1) \sum_{d_2=1}^{\frac{x}{d_1}} \psi_2(d_2)\,,$$

$$T_2 = \sum_{d_2=1}^{\sqrt{x}} \psi_2(d_2) \sum_{d_1=1}^{\frac{x}{d_2}} \psi_1(d_1)\,,$$

$$T_3 = \sum_{d_1=1}^{\sqrt{x}} \psi_1(d_1)$$

and

$$T_4 = \sum_{d_2=1}^{\sqrt{x}} \psi_2(d_2)\,.$$

Proof. A term $\psi_1(d_1)\, \psi_2(d_2)$, occurring in the sum in question, appears in the formula $T_1 + T_2 - T_3 T_4$

for $d_1 \leqq \sqrt{x}$ $\quad d_2 \leqq \sqrt{x}$ $\quad$ exactly $1 + 1 - 1 = 1$ times,
for $d_1 \leqq \sqrt{x}$ $\quad d_2 > \sqrt{x}$ $\quad$ exactly $1 + 0 - 0 = 1$ times,
for $d_1 > \sqrt{x}$ $\quad d_2 \leqq \sqrt{x}$ $\quad$ exactly $0 + 1 - 0 = 1$ times.

Lemma. If we take $n = 1$, the sum

$$\sum_{\substack{p^m v \leqq x \\ p^m v \equiv l}} f(v)\,,$$

extended over all the positive integers v and all the prime numbers p, for which the relations

$$p^m v \leqq x \quad \text{and} \quad p^m v \equiv l$$

exist, is equal to

$$\frac{a x^{\frac{1}{m}}}{\log x} + O\left(\frac{x^{\frac{1}{m}}}{(\log x)^2}\right),$$

where

$$a = \frac{bm}{h} \sum_{\substack{v=1 \\ vz^m \equiv l}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} .$$

Proof. Let l_1 and l_2 be two integers, prime to k; if the congruence

$$z^m \equiv l_1$$

has no roots in z, we have

$$\sum_{\substack{p^m v \leq x \\ p^m \equiv l_1 \\ v \equiv l_2}} f(v) = 0 ,$$

since it is then impossible to find a prime number p, satisfying the congruence

$$p^m \equiv l_1 .$$

Let us now, however, consider the case, that the congrence does possess roots and consequently has b incongruent roots $z_1, z_2, \ldots, z_b$. The preceding identity gives

$$\sum_{\substack{p^m v \leq x \\ p^m \equiv l_1 \\ v \equiv l_2}} f(v) = T_1 + T_2 - T_3 T_4 ,$$

where

$$T_1 = \sum_{\substack{v=1 \\ v \equiv l_2}}^{\sqrt{x}} f(v) \,.\, \sum_{\substack{p^m \leq \frac{x}{v} \\ p^m \equiv l_1}} 1 ,$$

$$T_2 = \sum_{\substack{p^m \leq \sqrt{x} \\ p^m \equiv l_1}} \sum_{\substack{v=1 \\ v \equiv l_2}}^{\frac{x}{p^m}} f(v) ,$$

$$T_3 = \sum_{\substack{v=1 \\ v \equiv l_2}}^{\sqrt{x}} f(v)$$

and

$$T_4 = \sum_{\substack{p^m \leq \sqrt{x} \\ p^m \equiv l_1}} 1 .$$

From the preceding lemma ensues

$$T_3 = O\left\{\frac{(\sqrt{x})^{\frac{1}{m}}}{(\log \sqrt{x})^3}\right\} = O\left(\frac{x^{\frac{1}{2m}}}{(\log x)^2}\right)$$

and for $p^m \leq \sqrt{x}$

$$\sum_{\substack{v=1\\ v\equiv l_2}}^{\frac{x}{p^m}} f(v) = O\left\{\frac{\left(\frac{x}{p^m}\right)^{\frac{1}{m}}}{\left(\log\frac{x}{p^m}\right)^3}\right\}$$

$$= O\left\{\frac{x^{\frac{1}{m}}}{p\left(\log\frac{x}{\sqrt{x}}\right)^3}\right\}$$

$$= O\left(\frac{x^{\frac{1}{m}}}{p\,(\log x)^3}\right),$$

hence

$$T_2 = \sum_{\substack{p^m\leq\sqrt{x}\\ p^m\equiv l_1}} O\left(\frac{x^{\frac{1}{m}}}{p\,(\log x)^3}\right)$$

$$= O\left(\frac{x^{\frac{1}{m}}}{(\log x)^3}\right)\sum_{\substack{p^m\leq\sqrt{x}\\ p^m\equiv l_1}}\frac{1}{p}$$

$$= O\left(\frac{x^{\frac{1}{m}}}{(\log x)^3}\right).\;O\sum_{n=1}^{x^{\frac{1}{2m}}}\frac{1}{n}$$

$$= O\left(\frac{x^{\frac{1}{m}}}{(\log x)^3}\right).\;O\left(\log x^{\frac{1}{2m}}\right)$$

$$= O\left(\frac{x^{\frac{1}{m}}}{(\log x)^2}\right).$$

From the inequalities

$$0\leq T_4\leq\sum_{\substack{p^m\leq\sqrt{x}\\ p\equiv l_1}}1\leq\sum_{n=1}^{x^{\frac{1}{2m}}}1\leq x^{\frac{1}{2m}}$$

ensues

$$T_4 = 0\left(x^{\frac{m}{1}}\right)$$

and now only the term T_1 is to be considered.

From the well-known proposition [1]) that the number of prime numbers $\leq x$ and congruent to l with regard to the modulus k, is equal to

$$\sum_{\substack{p \leq x\\ p \equiv l}} 1 = \frac{1}{h} \cdot \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right) \quad . \; . \; . \; . \; . \; . \quad (2)$$

ensues

$$\sum_{\substack{p^m \leq \frac{x}{v}\\ p^m \equiv l_1}} 1 = \sum_{\substack{p \leq \left(\frac{x}{v}\right)^{\frac{1}{m}}\\ p \equiv z_1,\, z_2, \ldots, z_b}} 1$$

$$= \sum_{b \geq \varrho \geq 1} \; \sum_{\substack{p \leq \left(\frac{x}{v}\right)^{\frac{1}{m}}\\ p \equiv z_\varrho}} 1$$

$$= \sum_{\varrho=1}^{b} \left\{ \frac{1}{h} \cdot \frac{\left(\frac{x}{v}\right)^{\frac{1}{m}}}{\log \left(\frac{x}{v}\right)^{\frac{1}{m}}} + O \cdot \frac{\left(\frac{x}{v}\right)^{\frac{1}{m}}}{\left\{\log \left(\frac{x}{v}\right)^{\frac{1}{m}}\right\}^2} \right\}$$

$$= \frac{bm\, x^{\frac{1}{m}}}{h v^{\frac{1}{m}} \log \frac{x}{v}} + O \left\{ \frac{x^{\frac{1}{m}}}{v^{\frac{1}{m}} \left(\log \frac{x}{v}\right)^2} \right\}.$$

For $v \leq \sqrt{x}$ we have

$$\frac{1}{\log \frac{x}{v}} = O\left(\frac{1}{\log \frac{x}{\sqrt{x}}}\right) = O\left(\frac{1}{\log x}\right),$$

$$\frac{1}{\log \frac{x}{v}} = \frac{1}{\log x} + \frac{\log v}{\log x \,.\, \log \frac{x}{v}}$$

$$= \frac{1}{\log x} + O\left\{\frac{\log v}{(\log x)^2}\right\};$$

therefore

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[1]) E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, I. p. 468.

$$T_1 = \sum_{\substack{v=1\\ v\equiv l_2}}^{\sqrt{x}} f(v) \left\{ \frac{bm\, x^{\frac{1}{m}}}{h v^{\frac{1}{m}} \log \frac{x}{v}} + O \cdot \frac{x^{\frac{1}{m}}}{v^{\frac{1}{m}} \left(\log \frac{x}{v}\right)^2} \right\}$$

$$= \sum_{\substack{v=1\\ v\equiv l_2}}^{\sqrt{x}} f(v) \left\{ \frac{bm\, x^{\frac{1}{m}}}{h v^{\frac{1}{m}} \log x} + O \cdot \frac{x^{\frac{1}{m}} \log v}{v^{\frac{1}{m}} (\log x)^2} \right\}$$

$$= \frac{bm\, x^{\frac{1}{m}}}{h \log x} \sum_{\substack{v=1\\ v\equiv l_2}}^{\sqrt{x}} \frac{f(v)}{v^{\frac{1}{m}}} + O \cdot \frac{x^{\frac{1}{m}}}{(\log x)^2} \sum_{\substack{v=1\\ v\equiv l_2}}^{\sqrt{x}} \frac{|f(v)| \log v}{v^{\frac{1}{m}}}$$

and according to the preceding lemma this is equal to

$$\frac{bm\, x^{\frac{1}{m}}}{h \log x} \left\{ \sum_{\substack{v=1\\ v\equiv l_2}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left(\frac{1}{\log x}\right) \right\} + O \left\{ \frac{x^{\frac{1}{m}}}{(\log x)^2} \right\}$$

$$= \frac{bm\, x^{\frac{1}{m}}}{h \log x} \sum_{\substack{v=1\\ v\equiv l_2}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O \left\{ \frac{x^{\frac{1}{m}}}{(\log x)^2} \right\}.$$

By substituting the values found for T_1, T_2, T_3 and T_4, we find the relation

$$\sum_{\substack{p^m v \leq x\\ p^m \equiv l_1\\ v \equiv l_2}} f(v) = T_1 + T_2 - T_3\, T_4$$

$$= \frac{bm\, x^{\frac{1}{m}}}{h \log x} \sum_{\substack{v=1\\ v\equiv l_2}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O \left\{ \frac{x^{\frac{1}{m}}}{(\log x)^2} \right\},$$

if the condition that the congruence

$$z^m \equiv l_1$$

has roots, is satisfied.

Write down a series, composed of h integers prime to k and not containing two numbers, which are congruent to each other, with regard to the modulus k; give to l_1 successively each value of this series, satisfying the condition, that the congruence

$$z \equiv l_1$$

has roots and determine for every value of l_1 a number l_2 by the congruence

$$l_1 l_2 \equiv l;$$

the relations thus obtained, added give in the left member

$$\sum_{\substack{p^m v \leq x \\ p^m v \equiv l}} f(v)$$

and in the right member

$$\frac{bmx^{\frac{1}{m}}}{h \log x} \sum_{v=1}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left\{\frac{x^{\frac{1}{m}}}{(\log x)^2}\right\},$$

v assuming all the positive integral values for which the congruences

$$v \equiv l_2 \qquad z^m \equiv l_1 \qquad l_1 l_2 \equiv l$$

are possible, i.e. for which the congruence

$$z^m l \equiv v$$

has roots and we conclude

$$\sum_{\substack{p^m v \leq x \\ p^m v \equiv l}} f(v) = \frac{ax^{\frac{1}{m}}}{\log x} + O\left\{\frac{x^{\frac{1}{m}}}{(\log x)^2}\right\},$$

where

$$a = \frac{bm}{h} \sum_{v=1}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}}.$$

$$z^m \; l \equiv v.$$

We have got on far enough now to proceed to proving formula (2) for $n = 1$; we observe that for $n = 1$

$$F(u) - \sum_{p^m \mid u} f\left(\frac{u}{p^m}\right)$$

is a finite function of u, which equals nothing for $e_u \leq m$ and which is equal to $O(u^{\mu_1})$ for $e_u > m$, μ_1 representing a constant number $< \frac{1}{m(m+1)}$.

In order to prove this, we distinguish four cases:

1. $e_u < m$;

if $\frac{u}{p^m}$ is resolved into prime factors, the smallest exponent of these factors is in this case smaller than m, hence

$$F(u) = 0 \qquad \text{and} \qquad f\left(\frac{u}{p^m}\right) = 0,$$

so that the formula considered is equal to nothing.

2. $e_u = m, \qquad a_u > 1$;

if $\frac{u}{p^m} = w$ is resolved into prime factors, the series of the exponents of these prime factors obtains at least one exponent $= m$, consequently

$$e_w \leqq m,$$

hence

$$f\left(\frac{u}{p_m}\right) = 0,$$

and n having the value 1,

$$e_u = m, \qquad a_u > n,$$

hence

$$F(u) = 0,$$

consequently

$$F(u) - \sum_{p^m/u} f\left(\frac{u}{p^m}\right) = 0.$$

3. $e_u = m, \qquad a_u = 1;$

consequently

$$u = p_1^m v \qquad e_v > m,$$

v being not divisible by the prime number p_1. In this case we have

$$F(u) = f(v).$$

As $\frac{u}{p^m}$ contains at least one prime factor, of whom the exponent is equal to m (viz. the prime factor p_1) except for $p = p_1$, we have

$$f\left(\frac{u}{p^m}\right) = 0, \text{ for } p \neq p_1,$$

$$= f(v), \text{ for } p = p_1,$$

consequently

$$F(u) - \sum_{p^m/u} f\left(\frac{u}{p^m}\right) = f(v) - f(v) = 0.$$

4. $e_u > m;$

Suppose $\mu < \mu_1 < \frac{1}{m(m+1)}$ and let

$$u = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_\sigma^{\alpha_\sigma},$$

be resolved into prime factors; hence

$$u \geqq 2 \,.\, 2 \dots . \,2 = 2^\sigma,$$

$$\sigma \leqq \frac{\log u}{\log 2} = O(\log u)$$

$$\sum_{p^m/u} 1 = \sigma = O(\log u)$$

The conditions

$$F(u) = O(v_u{}^{\mu}), \quad \text{and} \quad f(v) = O(v^{\mu}),$$

mentioned in § 1 and at the beginning of § 2 give the relations

$$F(u) = O(u^{\mu}) = O(u^{\mu_1})$$

and

$$\begin{aligned}\sum_{p^m/u} f\left(\frac{u}{p^m}\right) &= \sum_{p^m/u} O\left\{\left(\frac{u}{p^m}\right)^{\mu}\right\}\\ &= O(u^{\mu}) \sum_{p^m/u} 1\\ &= O(u^{\mu}) \,.\, O(log\, u)\\ &= O(u^{\mu_1})\,;\end{aligned}$$

hence it follows that the function considered is in this case equal to

$$O(u^{\mu_1}) - O(u^{\mu_1}) = O(u^{\mu_1}).$$

According to the first lemma we have

$$\sum_{\substack{u=2\\ u\equiv l}}^{x} \left\{F(u) - \sum_{p^m/u} f\left(\frac{u}{p^m}\right)\right\} = O\left(\frac{x^{\frac{1}{m}}}{(log\, x)^2}\right)$$

and consequently

$$\begin{aligned}\sum_{\substack{u=2\\ u\equiv l}}^{x} F(u) &= \sum_{\substack{u=2\\ u\equiv l}}^{x} \sum_{p^m/u} f\left(\frac{u}{p^m}\right) + O\left(\frac{x^{\frac{1}{m}}}{(log\, x)^2}\right)\\ &= \sum_{\substack{p^m v \leq x\\ p^m v \equiv l}} f(v) + O\left(\frac{x^{\frac{1}{m}}}{(log\, x)^2}\right)\end{aligned}$$

and according to the last lemma this may be modified to the formula sought

$$\sum_{\substack{u=2\\ u\equiv l}}^{x} F(u) = \frac{a x^{\frac{1}{m}}}{log\, x} + O\left(\frac{x^{\frac{1}{m}}}{(log\, x)^2}\right), \ldots \text{ for } n = 1.$$

§ 3. By starting from formula (1) by which the mean value of the function $F(u)$ has been given in the interval from 1 to x (the limits included), it is possible, as is known, to determine in an elementary way the mean value in the same interval of a number of other functions, connected with the function F; this we shall however only elaborate for some cases.

Lemma. From (1) ensues

$$\sum_{\substack{u=2\\u\equiv l}}^{x} \frac{F(u)}{u^{\frac{1}{m}}} = \frac{a}{mn} (log\ log\ x)^n + O\,(log\ log\ x)^{n-1} \quad . \quad . \quad . \quad . \qquad (3)$$

and

$$\sum_{\substack{a=2\\u\equiv l}}^{x} \frac{F(u)\ log\ u}{u^{\frac{1}{m}}} = O\,(log\ x\ .\ (log\ log\ x)^{n-1}).$$

Proof. Substituting

$$log\ log\ x = x_2,$$
$$log\ log\ u = u_2$$

and

$$g(x) = \sum_{\substack{u=2\\u\equiv l}}^{x} F(u) = \frac{a\, x^{\frac{1}{m}}\, x_2^{\,n-1}}{log\ x} + O\left(\frac{x^{\frac{1}{m}}\, x_2^{\,n-2}}{log\ x}\right)$$

we have

$$g_1(x) = \sum_{\substack{u=2\\u\equiv l}}^{x} \frac{F(u)}{u^{\frac{1}{m}}}$$

$$= \sum_{u=2}^{x} \frac{g(u) - g\,(u-1)}{u^{\frac{1}{m}}}$$

$$= \frac{g[x]}{[x]^{\frac{1}{m}}} + \sum_{u=2}^{x-1} g(u) \left\{ \frac{1}{u^{\frac{1}{m}}} - \frac{1}{(u+1)^{\frac{1}{m}}} \right\}$$

$$= O\left(\frac{x_2^{\,n-1}}{log\ x}\right) + \sum_{u=2}^{x-1} \left\{ \frac{a u^{\frac{1}{m}}\, u_2^{\,n-1}}{log\ u} + O\left(\frac{u^{\frac{1}{m}}\ u_2^{\,n-2}}{log\ u}\right) \right\} \left\{ -\frac{1}{m u^{\frac{1}{m}+1}} + O\left(\frac{1}{u^{\frac{1}{m}+2}}\right) \right\}$$

$$= O\,(x_2^{\,n-1}) + \frac{a}{m} \sum_{u=2}^{x-1} \frac{u_2^{\,n-1}}{u\ log\ u} + O \sum_{u=2}^{x-1} \frac{u_2^{\,n-2}}{u\ log\ u}$$

$$= O\,(x_2^{\,n-1}) + \frac{a}{m} \left\{ \frac{x_2^{\,n}}{n} + O\,(x_2^{\,n-1}) \right\} + O\,(x_2^{\,n-1})$$

$$= \frac{a x_2}{mn} + O\,(x_2^{\,n-1})$$

and

$$\sum_{\substack{u=2\\u\equiv l}}^{x} \frac{F(u)\ log.u}{u^{\frac{1}{m}}} = \sum_{u=2}^{x} \{g_1\,(u) - g_1\,(u-1)\}\ log\ u$$

$$= g_1\,[x]\ log\ [x] - \sum_{u=2}^{x-1} g_1\,(u)\ log\left(1 + \frac{1}{u}\right)$$

$$= \frac{a x_2^n \log x}{mn} + O(x_2^{n-1} \log x)$$

$$- \sum_{u=2}^{x-1} \left\{ \frac{a u_2^n}{mn} + O(u_2^{n-1}) \right\} \left\{ \frac{1}{u} + O\left(\frac{1}{u^2}\right) \right\}$$

$$= \frac{a x_2^n \log x}{mn} + O(x_2^{n-1} \log x) - \frac{a}{mn} \sum_{u=2}^{x-1} \frac{u_2^n}{u} + O \sum_{u=2}^{x-1} \frac{u_2^{n-1}}{u}$$

$$= \frac{a x_2^n \log x}{mn} + O(x_2^{n-1} \log x)$$

$$- \frac{a}{mn} \{x_2^n \log x + O(x_2^{n-1} \log x)\} + O(x_2^{n-1} \log x)$$

$$= O(x_2^{n-1} \log x).$$

Lemma. Suppose that the function F_1 has m and n_1 as parameters and f_1 as corresponding function and that F_2 has m and n_2 as parameters and f_2 as corresponding function. If the formula (1) holds good for $n = n_1$ and for $n = n_2$, we have

$$\sum_{\substack{d_1 d_2 \leq x \\ d_1 d_2 \equiv l}} F_1(d_1) F_2(d_2) = \frac{b m (n_1 + n_2) x^{\frac{1}{m}} x_2^{n_1+n_2-1}}{h n_1! n_2! \log x} \sum_{\substack{v=1 \\ v z^m \equiv l}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left(\frac{x^{\frac{1}{m}} x_2^{n_1+n_2-2}}{\log x} \right),$$

l being prime to k; in this relation $f(v)$ has been substituted for the formula

$$\sum_{d \mid v} f_1(d) f_2\left(\frac{v}{d}\right)$$

Proof. Let l_1 and l_2 be two integers, prime to k; it follows from the identity deduced in the preceding paragraph that

$$\sum_{\substack{d_1 d_2 \leq x \\ d_1 \equiv l_1 \\ d_2 \equiv l_2}} F_1(d_1) F_2(d_2) = T_1 + T_2 - T_3 T_4, \quad . \quad . \quad . \quad . \quad (4)$$

where

$$T_1 = \sum_{\substack{d_1=1 \\ d_1 \equiv l_1}}^{\sqrt{x}} F_1(d_1) \sum_{\substack{d_2=1 \\ d_2 \equiv l_2}}^{\frac{x}{d_1}} F_2(d_2),$$

$$T_2 = \sum_{\substack{d_2=1 \\ d_2 \equiv l_2}}^{\sqrt{x}} F_2(d_2) \sum_{\substack{d_1=1 \\ d_1 \equiv l_1}}^{\frac{x}{d_2}} F_1(d_1),$$

$$T_3 = \sum_{\substack{d_1=1 \\ d_1 \equiv l_1}}^{\sqrt{x}} F_1(d_1)$$

and $$T_4 = \sum_{\substack{d_2=1\\ d_2\equiv l_2}}^{\sqrt{x}} F_2(d_2)\,.$$

For $d_1 \leq \sqrt{x}$ we have

$$\begin{aligned}\log\log\frac{x}{d_1} &= \log\log x + \log\left(1-\frac{\log d_1}{\log x}\right)\\ &= x_2 + O(1)\,,\end{aligned}$$

if x_2 has been again substituted for $\log\log x$; consequently

$$\left(\log\log\frac{x}{d_1}\right)^{n_2-1} + O\left(\log\log\frac{x}{d_1}\right)^{n_2-2} = x_2^{\,n_2-1} + O\left(x_2^{\,n_2-2}\right),$$

$$\frac{1}{\log\dfrac{x}{d_1}} = \frac{1}{\log x} + \frac{\log d_1}{\log x\,.\,\log\dfrac{x}{d_1}} = \frac{1}{\log x} + O\left(\frac{\log d_1}{(\log x)^2}\right)$$

and

$$\frac{\left(\log\log\dfrac{x}{d_1}\right)^{n_2-1} + O\left(\log\log\dfrac{x}{d_1}\right)^{n_2-2}}{\log\dfrac{x}{d_1}} = \frac{x_2^{\,n_2-1}}{\log x} + O\left(\frac{x_2^{\,n_2-2}}{\log x}\right) + O\left(\frac{x_2^{\,n_2-1}\log d_1}{(\log x)^2}\right).$$

It has been assumed that formula (1) holds good for $n=n_2$, hence

$$\begin{aligned}\sum_{\substack{d_2=1\\ d_2\equiv l_2}}^{\frac{x}{d_1}} F_2(d_2) &= \frac{a_2\left(\dfrac{x}{d_1}\right)^{\frac{1}{m}}\left\{\left(\log\log\dfrac{x}{d_1}\right)^{n_2-1} + O\left(\log\log\dfrac{x}{d_1}\right)^{n_2-2}\right\}}{\log\dfrac{x}{d_1}}\\ &= \frac{a_2 x^{\frac{1}{m}}}{d_1^{\frac{1}{m}}}\left\{\frac{x_2^{\,n_2-1} + O\left(x_2^{\,n_2-2}\right)}{\log x} + O\left(\frac{x_2^{\,n_2-1}\log d_1}{(\log x)^2}\right)\right\},\end{aligned}$$

where a_2 has been substituted for

$$\frac{bm}{h\,(n_2-1)\,!}\sum_{\substack{v_2=1\\ v_2 z^m\equiv l_2}}^{\infty}\frac{f_2(v_2)}{v_2^{\frac{1}{m}}}\,.$$

If this result is substituted for the value found for T_1, we find

$$\begin{aligned}T_1 = \frac{a_2 x^{\frac{1}{m}} x_2^{\,n_2-1}}{\log x}\sum_{\substack{d_1=1\\ d_1\equiv l_1}}^{\sqrt{x}}\frac{F_1(d_1)}{d_1^{\frac{1}{m}}} + O\left(\frac{x^{\frac{1}{m}} x_2^{\,n_2-2}}{\log x}\right)\sum_{\substack{d_1=1\\ d_1\equiv l_1}}^{\sqrt{x}}\frac{|F_1(d_1)|}{d_1^{\frac{1}{m}}} + \\ + O\left(\frac{x^{\frac{1}{m}} x_2^{\,n_2-1}}{(\log x)^2}\right)\sum_{\substack{d_1=1\\ d_1\equiv l_1}}^{\sqrt{x}}\frac{|F_1(d_1)|\log d_1}{d_1^{\frac{1}{m}}}\,.\end{aligned}$$

It has been understood that formula (1) holds good for $n = n_1$, consequently for the functions F_1 and $|F_1|$ and according to the first lemma of this paragraph we have

$$\sum_{\substack{d_1=1\\ d_1\equiv l_1}}^{\sqrt{x}} \frac{|F_1(d_1)|}{d_1^{\frac{1}{m}}} = O\left(x_2^{n_1}\right),$$

$$\sum_{\substack{d_1=1\\ d_1\equiv l_1}}^{\sqrt{x}} \frac{|F_1(d_1)|\, log\, d_1}{d_1^{\frac{1}{m}}} = O\left(x_2^{n_1-1}\, log\, x\right)$$

and

$$\sum_{\substack{d_1=1\\ d_1\equiv l_1}}^{\sqrt{x}} \frac{F_1(d_1)}{d_1^{\frac{1}{m}}} = \frac{a_1 x_2^{n_1}}{mn_1} + O\left(x_2^{n_1-1}\right),$$

where

$$a_1 = \frac{bm}{h\,(n_1-1)\,!} \sum_{\substack{v_1=1\\ v_1 z^m \equiv l_1}}^{\infty} \frac{f_1(v_1)}{v_1^{\frac{1}{m}}}.$$

Hence

$$T_1 = \frac{a_1 a_2 x^{\frac{1}{m}}\, x_2^{n_1+n_2-1}}{mn_1\, log\, x} + O\left(\frac{x^{\frac{1}{m}}\, x_2^{n_1+n_2-2}}{log\, x}\right)$$

$$= \frac{b^2 mn_2 x^{\frac{1}{m}}\, x_2^{n_1+n_2-1}}{h^2 n_1!\, n_2!\, log\, x} \sum_{\substack{v_1=1\\ v_1 z_1^m \equiv l_1}}^{\infty} \sum_{\substack{v_2=1\\ v_2 z_2^m \equiv l_2}}^{\infty} \frac{f_1(v_1) f_2(v_2)}{(v_1 v_2)^{\frac{1}{m}}} + O\left(\frac{x^{\frac{1}{m}}\ {}_2^{n_1+n_2-2}}{log\, x}\right).$$

The value of T_2 is found by interchanging n_1 and n_2 in this formula and as according to our supposition relation (1) holds good for $n = n_1$ and for $n = n_2$, we have

$$T_3 = \sum_{\substack{d_1=1\\ d_1\equiv l_1}}^{\sqrt{x}} F_1(d_1) = O\left(\frac{(\sqrt{x})^{\frac{1}{m}}\, (log\, log\, \sqrt{x})^{n_1-1}}{log\, \sqrt{x}}\right)$$

$$= O\left(\frac{x^{\frac{1}{2m}}\, (log\, log\, x)^{n_1-1}}{log\, x}\right)$$

and

$$T_4 = O\left(\frac{x^{\frac{1}{2m}}\, (log\, log\, x)^{n_2-1}}{log\, x}\right).$$

By substituting these values for T, T_2, T_3 and T_4 in (4), we find the formula

$$\sum_{\substack{d_1 d_2 \leq x \\ d_1 \equiv l_1 \\ d_2 \equiv l_2}} F_1(d_1)\,F_2(d_2) = \frac{b^2 m(n_1+n_2) x^{\frac{1}{m}} x_2^{\,n_1+n_2-1}}{h^2 n_1!\, n_2!\, \log x} \sum_{\substack{v_1=1 \\ v_1 z_1^m \equiv l_1}}^{\infty} \sum_{\substack{v_2=1 \\ v_2 z_2^m \equiv l_2}}^{\infty} \frac{f_1(v_1)\,f_2(v_2)}{(v_1 v_2)^{\frac{1}{m}}} + $$

$$+\, O\left(\frac{x^{\frac{1}{m}} x_2^{\,n_1+n_2-2}}{\log x}\right).$$

Write again down a series consisting of h integers prime to k and not containing two numbers, which are congruent to each other with regard to the modulus k; give to l_1 successively each value of this series and determine for every value of l_1 a number l_2 by the congruence

$$l_1 l_2 \equiv l\,;$$

the h relations, thus obtained, added, give in the left member

$$\sum_{\substack{d_1 d_2 \leq x \\ d_1 d_2 \equiv l}} F_1(d_1)\,F_2(d_2)$$

and in the right member

$$\frac{b^2 m(n_1+n_2) x^{\frac{1}{m}} x_2^{\,n_1+n_2-1}}{h^2 n_1!\, n_2!\, \log x} \cdot c + O\left(\frac{x^{\frac{1}{m}} x_2^{\,n_1+n_2-2}}{\log x}\right),$$

where

$$c = \sum_{l_1} \sum_{\substack{v_1=1 \\ v_1 z_1^m \equiv l_1}}^{\infty} \sum_{\substack{v_2=1 \\ v_2 z_2^m \equiv l_2}}^{\infty} \frac{f_1(v_1)\,f_2(v_2)}{(v_1 v_2)^{\frac{1}{m}}}$$

$$= \sum_{\substack{v=1 \\ v z^m \equiv l}}^{\infty} \frac{1}{v^{\frac{1}{m}}} \sum_{l_1} \sum_{\substack{v_1 v_2 = v \\ v_1 z_1^m \equiv l_1}} f_1(v_1)\,f_2(v_2).$$

For every value of v_1 exactly $\frac{h}{b}$ incongruent values for l_1 are to be found for which the congruence

$$v_1 z_1^m \equiv l_1$$

has roots, hence

$$\sum_{l_1} \sum_{\substack{v_1 v_2 = v \\ v_1 z_1^m \equiv l_1}} f_1(v_1) f_2(v_2) = \frac{h}{b} \cdot \sum_{v_1 v_2 = v} f_1(v_1) f_2(v_2)$$

$$= \frac{h}{b} \sum_{d \mid v} f_1(d)\, f_2\left(\frac{v}{d}\right)$$

$$= \frac{h}{b}\, f(v),$$

$$c = \sum_{\substack{v=1\\ vz^m \equiv l}}^{\infty} \frac{1}{v^{\frac{1}{m}}} \sum_{l_1} \sum_{\substack{v_1 v_2 = v\\ v_1 z_1{}^m \equiv l_1}} f_1(v_1) f_2(v_2)$$

$$= \frac{h}{b} \sum_{\substack{v=1\\ vz^m \equiv l}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}}$$

and consequently

$$\sum_{\substack{d_1 d_2 \leq x\\ d_1 d_2 \equiv l}} F_1(d_1) F_2(d_2) = \frac{bm(n_1+n_2) x^{\frac{1}{m}} x_2{}^{n_1+n_2-1}}{hn_1! n_2! \log x} \sum_{\substack{v=1\\ vz^m \equiv l}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left(\frac{x^{\frac{1}{m}} x_2{}^{n_1+n_2-2}}{\log x}\right),$$

which was to be proved.

Mathematics. — "*On an arithmetical function connected with the decomposition of the positive integers into prime factors.*" II. (*Continued and concluded.*) By J. G. VAN DER CORPUT. (Communicated by Prof. J. C. KLUYVER).

(Communicated in the meeting of June 24, 1916).

Lemma. [1] The number of (positive integral) divisors of the positive integer v satisfies the relation

$$\sum_{d|v} 1 = O(v^{\mu}).$$

for every $\mu > 0$.

Proof. If $v \geq 2$ decomposed into prime factors be equal to

$$v = \prod_{p|v} p^{\alpha}$$

we have

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[1] This proposition occurs for the first time in RUNGE: Ueber die auflösbaren Gleichungen von der Form $x^5 + ux + v = 0$ [Acta mathematica, Bd. VII (1885), pages 173—186], pages 181—183, with a proof similar to this one. This proof has been borrowed of E. LANDAU. Ueber die Anzahl der Gitterpunkte in gewissen Bereichen [Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, mathematisch-physikalische Klasse (1912), pages 687—771], page 716. In his "Handbuch der Lehre von der Verteilung der Primzahlen," I. p. 220, he gives the by far sharper relation:

If δ be positive, $\xi = \xi(\delta)$ fitly chosen and x an integer $\geqq \xi$, we have

$$\sum_{d|x} 1 < 2^{\frac{(1+\delta)\log x}{\log\log x}}.$$

$$\sum_{d|v} 1 = \Pi_{p|v} (\alpha + 1),$$

$$\frac{\sum_{d|v} 1}{v^{\mu}} = \Pi_{p|v} \frac{\alpha+1}{p^{\alpha\mu}}.$$

The quantity $\frac{\alpha+1}{p}$ is limited (μ taken fixed!) for an invariable value of p and variable $\alpha = 1, 2, \ldots$, since it is equal to nothing for $\alpha = \infty$; for any $p \geqq 2^{\frac{1}{\mu}}$ and any value $\alpha \geqq 1$ it is even $\leqq 1$, existing in that case the inequalities

$$\frac{\alpha+1}{p^{\alpha\mu}} \leqq \frac{\alpha+1}{2^{\alpha}} < 1.$$

Therefore, if v contains one or more prime factors $\geqq 2^{\frac{1}{\mu}}$, we have

$$\Pi_{\substack{p|v \\ p \geqq 2^{\frac{1}{\mu}}}} \frac{\alpha+1}{p^{\alpha\mu}} \leqq 1$$

and as there exist only a finite number of prime numbers $p < 2^{\frac{1}{\mu}}$ $\frac{\sum_{dv} 1}{v^{\mu}}$ is limited, i.e. smaller than a number independent of v.

Lemma. Let n_1 and n_2 be two arbitrary positive integers, whose sum $n_1 + n_2$ is equal to n and suppose F to be an arbitrary function with the parameters m and n; three functions F_1, F_2, and F_3 may be found then in such a way that the parameters of F_1 are equal to m and n_1, of F_2 equal to m and n_2 and of F_3 equal to m and n--1 with the relation

$$F(u) = \frac{n_1! n_2!}{(n_1+n_2)!} \sum_{d|u} F_1(d) F_2\left(\frac{u}{d}\right) + O\{F_3(u)\} \quad . \quad . \quad . \quad (5)$$

Proof. Introduce the functions F_1, F_2, and F_3 by means of the following relations:

$F_1(u) = f(v_u)$ for $e_u = m$, $a_u = n_1$,
$= 0$ in the other cases,
$F_2(u) = 1$ for $e_u = m$, $a_u = n_2$, $v_u = 1$,
$= 0$ in the other cases,
$F_3(u) = v_u^{\mu}$ for $e_u > m$ and also for $e_u = m$, $a_u \leqq n-1$,
$= 0$ in the other cases, i.e. for $e_u < m$ and also for $e_u = m$, $a_u > n-1$.

From these definitions it appears that the parameters of F_1 are equal to m and n_1, of F_2 to m and n_2 and of F_3 to m and $n-1$, so that now only relation (5) is to be proved and in order to do this, we distinguish **5** cases:

1. $$e_u > m;$$

then we have

$$u = v_u$$

and the quantity

$$\sum_{d|u} F_1(d)\, F_2\left(\frac{u}{d}\right) = \sum_{d|v_u} O(1)$$
$$= O \sum_{d|v_u} 1$$

is according to the preceding lemma equal to $O(v_u{}^{\varepsilon})$ and therefore

$$F(u) - \frac{n_1!\, n_2!}{(n_1+n_2)!} \sum_{d|u} F_1(d)\, F_2\left(\frac{u}{d}\right) = O(v_u{}^{\varepsilon}) - O(v_u{}^{\varepsilon}). \quad . \quad (6)$$
$$= O(F_3(u)).$$

2. $$e_u = m, \qquad a_u \leq n-1;$$

then we have

$$u = p_1{}^m\, p_2{}^m \ldots p_{a_u}{}^m\, v_u$$

and

$$\sum_{d|u} F_1(d)\, F_2\left(\frac{u}{d}\right) = O \sum_{d|u} 1$$
$$= O \sum_{d|p_1{}^m p_2{}^m \ldots p_{a_u}{}^m} 1 \,.\, \sum_{d|v_u} 1$$
$$= O(1)\,.\, O(v_u{}^{\varepsilon}),$$

so that the relation (6) holds good in this case as well.

3. $$e_u = m, \qquad a_u = n;$$

in this case we have

$$u = p_1{}^m\, p_2{}^m \ldots p_n{}^m\, v_u$$

and

$$F(u) = f(v_u),$$

where at least one of the following conditions is satisfied, d representing any arbitrary divisor of u and d' being subsituted for $\frac{u}{d}$:

a) $e_d < m$,
b) $e_{d'} < m$,
c) $e_d = m, \quad a_d > n_1$,
d) $e_{d'} = m, \quad a_{d'} > n_2$,
e) $e_d = e_{d'} = m, \quad a_d = n_1, \quad a_{d'} = n_2$.

In the first four cases we have

$$F_1(d)\,F_2(d') = 0$$

and the last case appears only if

$$d = q^m v_d \qquad d' = q'^m v_{d'}$$

where q is a divisor of

$$P = p_1 p_2 \ldots p_n$$

consisting of n_1 prime factors, where $q' = \frac{P}{q}$ is therefore composed of $n - n_1 = n_2$ prime factors and where the product of the integers v_d and $v_{d'}$ is equal to v_u. In case (e) we have therefore

$$F_2(d') = 0, \qquad \text{except for} \qquad v_{d'} = 1,$$

consequently

$$\begin{aligned} F_1(d)\,F_2(d') &= f(v_d)\,F_2(d') \\ &= f(v_u) \quad \text{for} \quad v_{d'} = 1, \\ &= 0, \quad \text{for} \quad v_{d'} \neq 1, \end{aligned}$$

hence

$$\begin{aligned} \sum_{d|u} F_1(d)\,F_2(d') &= \sum_{q|P} f(v_u) \\ &= f(v_u) \sum_{q|P} 1. \end{aligned}$$

P containing exactly $\frac{(n_1+n_2)!}{n_1!n_2!}$ different divisors composed of n_1 prime factors, we have

$$\sum_{q|P} 1 = \frac{(n_1+n_2)!}{n_1!n_2!}$$

and therefore

$$\begin{aligned} \frac{n_1!n_2!}{(n_1+n_2)!} \sum_{d|u} F_1(d)F_2(d') &= f(v_u) \\ &= F(u), \end{aligned}$$

from which relation (5) ensues at once.

4. $$e_u = m, \quad a_u > n;$$

one of the following conditions at least is in this case satisfied

$a)$ $e_d < m,$

$b)$ $e_{d'} < m,$

$c)$ $e_d = m, \quad a_d > n_1,$

$d)$ $e_{d'} = m, \quad a_{d'} > n_2,$

so we have

$$F(u) = 0 \quad \text{and} \quad F_1(d)\,F_2\left(\frac{u}{d}\right) = 0$$

5. $$e_u < m;$$

in this case one of the numbers e_d and $e_{d'}$ at least is smaller than m, so that again the relations

$$F(u) = 0 \quad \text{and} \quad F_1(d)\,F_2\left(\frac{u}{d}\right) = 0$$

hold good.

These lemmas having been demonstrated, the proof of formula (1) for any arbitrary value of n will be easy, viz.: we shall demonstrate the proposition for $n = n_1 + n_2$, supposing that it has been proved for $n = n_1$, for $n = n_2$ and for $n = n_1 + n_2 - 1$, where n_1 and n_2 represent two arbitrary positive integers; as the proposition in § 2 has been proved for $n = 1$, the validity for $n = 2, 3, 4 \ldots$ etc. respectively, follows from this argument.

Let $F(u)$ be the function with parameters m and $n_1 + n_2$, for which relation (1) has to be proved; we introduce (and according to the preceding lemma this is possible) the function $F_1(u)$ with parameters m and n_1, the function $F_2(u)$ with parameters m and n_2 and the function $F_3(u)$ with parameters m and $n_1 + n_2 - 1$, so that we have

$$F(u) = \frac{n_1!\,n_2!}{(n_1+n_2)!} \sum_{d|u} F_1(d) F_2\left(\frac{u}{d}\right) + O\{F_3(u)\}$$

and consequently

$$\sum_{\substack{u=2\\ u\equiv l}}^{x} F(u) = \frac{n_1!\,n_2!}{(n_1+n_2)!} \sum_{\substack{dd' \leq x\\ dd' \equiv l}} F_1(d) F_2(d') + O \sum_{\substack{u=2\\ u\equiv l}}^{x} |F_3(u)|$$

As relation (1) holds good for $n = n_1 + n_2 - 1$, consequently for the function $|F_3(u)|$, we have

$$\sum_{\substack{u=2\\ u\equiv l}}^{x} |F_3(u)| = O\left\{\frac{x^{\frac{1}{m}}\, x_2^{\,n_1+n_2-2}}{\log x}\right\}$$

and as according to our proposition, (1) holds good also for $n = n_1$ and for $n = n_2$, i. e. for the functions F_1 and F_2, we have, according to the second lemma of this paragraph

$$\sum_{\substack{dd' \leq x\\ dd' \equiv l}} F_1(d) F_2(d') = \frac{bm(n_1+n_2)\,x^{\frac{1}{m}}\, x_2^{\,n_1+n_2-1}}{h\,n_1!\,n_2!\log x} \sum_{\substack{v=1\\ vz^m \equiv l}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left(\frac{x^{\frac{1}{m}}\, x_2^{\,n_1+n_2-2}}{\log x}\right),$$

so that we conclude

$$\sum_{\substack{u=2\\ u\equiv l}}^{x} F(u) = \frac{bm\,x^{\frac{1}{m}}\, x_2^{\,n_1+n_2-1}}{h(n_1+n_2-1)!\log x} \sum_{\substack{v=1\\ vz^m \equiv l}}^{\infty} \frac{f(v)}{v^{\frac{1}{m}}} + O\left\{\frac{x^{\frac{1}{m}}\, x_2^{\,n_1+n_2-2}}{\log x}\right\};$$

therefore formula (1) has been proved for all positive integers n.

§ 4. In this last paragraph we have to consider the proof and the significance of the formulae (1) and (3), which have been demonstrated in §§ 2 and 3. As to the proof, we see that relation (1) has been deduced from (2) in an elementary way and as has been observed at the beginning of the preceding paragraph, some other formulae, e. g. (3) may be proved by means of (1). Relation (3) may also be demonstrated directly, viz. without the round-about way along formula (1), by not starting from formula (2) but from the relation

$$\sum_{\substack{p \leq x \\ p \equiv l}} \frac{1}{p} = \frac{1}{h} \log \log x + O(1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)^{1)}$$

This proof is analogous to the one used in order to demonstrate relation (1); on executing it, it will appear that in that case the proof is even simpler. Yet, that proof has not been given in this paper, because (1) lies deeper that (3), i. e. (3) is to be deduced from (1) and the reverse is not possible, so that it would not do to prove formula (3) first, as it is not possible then to conclude to formula (1) and as will be seen it is principally this formula that we want. The question, however, is somewhat different for $k=1$, as (7) in that case is to be deduced 2) quite elementarily from the identity

$$\sum_{p \leq x} \log p \left(\left[\frac{x}{p} \right] + \left[\frac{x}{p^2} \right] + \ldots \right) = \sum_{u=2}^{[x]} \log u \qquad {}^{3)}$$
$$= x \log x + O(x)$$

so that relation (3) may be proved quite elementarily for $k=1$.

Formula (1) is also to be proved directly, i. e. without using (2); it is namely possible to prove (1) with propositions in the theory of functions in a way, analogous to the one, used to demonstrate formula (2); it is clear, however, that, in that case, an elementary proof is not to be thought of and we have succeeded in deducing (1) from (2) by means of elementary methods.

If in (1) and (3) μ is taken equal to nothing, we have this

Proposition. If the finite arithmetical function $F(u)$ is equal to nothing for $e_u < m$ and also for $e_u = m, a_u > n$, and the function $f(u)$ equals nothing for $e_u \overset{=}{<} m$, $F(u)$ being equal to $f(v_u)$ for $e_u = m, a_u = n$, the formulae (1) and (3) hold good, if l and k are prime to each other.

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1) E. Landau. Handbuch I. p. 450.

2) E. Landau. Handbuch I. p. 98—102.

3) E. Landau. Handbuch I. p. 77, (formula 4).

In order to bring out the significance of this proposition four applications are given as follows.

Application I. Any integer >1, resolved into prime factors, has a series of exponents and the question arises how many integers below a given limit are to be found with a given series of exponents and how many of these integers are to be met in a given arithmetical series, of which the first term and the difference are prime to each other. It is clear that the first question is a special case of the second. If the given series of exponents consists of one number and this number is equal to one, the second question is identical with the question how many prime numbers are to be found in that arithmetical series below a certain limit and the answer is given by formula (2); if the given series of the exponents is composed of one number $m>1$, it is sought how many numbers equal to the m^{th} power of a prime number occur in the arithmetical series, below a given limit and this is easy to calculate by means of formula (2). The question, however, becomes more intricate, as soon as the series of exponents consists of more than one number, but in that case the answer may be found by means of the proposition, for any series of exponents. Take e.g. the smallest number, occurring in the given series of the exponents, equal to m and suppose that this number occurs n times in this series, so that the given series of the exponents is equal to

$$\alpha_1, \alpha_2, \ldots, \alpha_\sigma, m, m, \ldots, m,$$

where

$$\alpha_\varrho > m \qquad \text{for} \qquad \sigma \geq \varrho \geq 1.$$

Take $F(u)=1$, if the integer u, resolved into prime factors, has a series of exponents, equal to the given series and take $F(u)=0$ in the other cases; take $f(u)=1$, if the series of the exponents of the prime factors of the integer u is equal to $\alpha_1, \alpha_2, \ldots, \alpha$ and $f(u)=0$ in the other cases. The conditions, laid down in the proposition are then satisfied, viz.

1. $F(u)=0,$ for $e_u < m$ and also for $e_u = m,\ a_u > n,$
2. $f(v)=0,$ for $e_u \leq m,$
3. $F(u)=f(v_u),$ for $e_u = m,\ a_u = n,$

for if $e_u = m,\ a_u = n$ and the given series of the exponents is (not) corresponding to that of u, the series $\alpha_1, \alpha_2, \ldots, \alpha_\sigma$ is (not) corresponding to the series of the exponents of v_n, so that both the functions $F(u)$ and $f(v_u)$ are in that case equal to one (nothing).

The proposition may therefore be applied and formula (1) gives the sums

$$\sum_{u=2}^{x} F(u) \qquad \text{and} \qquad \sum_{\substack{u=2 \\ u \equiv l}}^{x} F(u),$$

which exactly represent the numbers sought. So we find e.g.

The number of positive integers $\leq x$, composed of two different prime factors, occurring in these numbers respectively in the degree α and β, is for $\alpha > \beta$ equal to

$$\frac{x^{\frac{1}{\beta}}}{\log x} \sum_p \frac{1}{p^{\frac{\alpha}{\beta}}} + O\left(\frac{x^{\frac{1}{\beta}}}{(\log x)^2}\right).$$

The number of positive integers $\leq x$, composed of one quintuple and three double prime factors (these prime factors are thought different from each other) is equal to

$$\frac{2\sqrt{x}}{\log x} (\log\log x)^2 \sum_p \frac{1}{p^{\frac{5}{2}}} + O\left(\frac{\sqrt{x}}{\log x} . \log\log x\right)$$

and among these numbers

$$\frac{2\sqrt{x}}{\log x} (\log\log x)^2 \sum_{p \equiv l \,(\mathrm{mod.}\, 8)} \frac{1}{p^{\frac{5}{2}}} + O\left(\frac{\sqrt{x}}{\log x} . \log\log x\right)$$

integers are to be found, which are congruent to l, with regard to the modulus 8 ($l = 1$, 3, 5 or 7) and

$$\frac{\sqrt{x}}{\log x} (\log\log x)^2 \sum_{p \equiv \pm l \,(\mathrm{mod.}10)} \frac{1}{p^{\frac{5}{2}}} + O\left(\frac{\sqrt{x}}{\log x} . \log\log x\right)$$

integers, which are congruent with l, with regard to the modulus 10 ($l = 1$, 3, 7 or 9).

In the following application, viz. with the function $\pi_n(x)$ defined there, the case will be treated that the given series of exponents consists of n numbers, each equal to 1.

Application II. We introduce the following well-known notation[1]: $\pi_n(x)$ represents the number of squareless integers $< x$, composed of n prime factors, $\varrho_n(x)$ the number of integers $\leq x$, of which the number of different prime factors is equal to n, and $\sigma_n(x)$ the number of integers $\leq x$, for which the total number of prime factors equals n.

Gauss surmised in 1796

$$\pi_n(x) \sim \frac{1}{(n-1)!} \cdot \frac{x(\log\log x)^{n-1}}{\log x}.$$

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[1] See for this notation e.g. E. Landau, Handbuch I, pages 205, 208, 211.

This relation has been first proved by E. Landau; from the proposition

$$\sum_{p \leq x} 1 \sim \frac{x}{\log x}$$

he deduced viz. in an elementary way these relations [1] [2])

$$\pi_n(x) \sim \frac{x\,(\log\log x)^{n-1}}{(n-1)!\,\log x},$$

$$\sigma_n(x) \sim \frac{x\,(\log\log x)^{n-1}}{(n-1)!\,\log x},$$

and

$$\varrho_n(x) \sim \frac{x\,(\log\log x)^{n-1}}{(n-1)!\,\log x}.$$

By using the deeper lying relation

$$\sum_{p \leq x} 1 = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$$

he proves, also elementarily [1])

$$\pi_n(x) = \frac{x\,(\log\log x)^{n-1}}{(n-1)!\,\log x} + O\left(\frac{x\,(\log\log x)^{n-2}}{\log x}\right),$$

$$\sigma_n(x) = \frac{x\,(\log\log x)^{n-1}}{(n-1)!\,\log x} + O\left(\frac{x\,(\log\log x)^{n-2}}{\log x}\right)$$

and

$$\varrho_n(x) = \frac{x\,(\log\log x)^{n-1}}{(n-1)!\,\log x} + O\left(\frac{x\,(\log\log x)^{n-2}}{\log x}\right).$$

What I want to prove now is that these formulae are only special cases of the proposition. Take $F(u) = 1$, if u be equal to a squareless number composed of n prime factors and take $F(u) = 0$ in the other cases; then we have

$$\sum_{u=2}^{x} F(u) = \pi_n(x);$$

if $F(u)$ be equal to 1 or 0 according as the total number of prime factors of u is equal to n or not, we have

$$\sum_{u=2}^{x} F(u) = \sigma_n(x)$$

and finally, by giving to $F(u)$ the value 1 or 0, according as the number of different prime factors of u is equal to n or not, we have

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[1]) E. Landau. Sur quelques problèmes relatifs à la distribution des nombres premiers. [Bulletin de la Société mathématique de France. Vol. 28 (1900) pg. 25—28].

[2]) E. Landau. Handbuch I. p. 205—213.

$$\sum_{u=2}^{x} F(u) = \varrho_n(x).$$

In each of these three cases the function $F(u)$ satisfies the conditions stated, if in them

$$m = 1 \quad \text{and consequently} \quad b = 1,$$

$$f(1) = 1$$

and $$f(v) = 0, \quad \text{for} \quad v > 1$$

so that a possesses the value $\frac{1}{h.(n-1)!}$, and we conclude, that the relations (1) and (3) are modified to the formulae

$$\sum_{\substack{u=2 \\ u \equiv l}}^{x} F(u) = \frac{x(\log\log x)^{n-1}}{h\,(n-1)!\log x} + O\left\{\frac{x(\log\log x)^{n-2}}{\log\ \ x}\right\}$$

and

$$\sum_{\substack{u=2 \\ u \equiv l}}^{x} \frac{F(u)}{u} = \frac{(\log\log x)^{n}}{h.n!} + O\,(\log\log x)^{n-1}.$$

For $k = 1$ and consequently $h = 1$ the first of these relations passes into the formulae written down for $\pi_n(x)$, $\sigma_n(x)$ and $\varrho_n(x)$, and the second relation produces an asymptotical expression, not of the number but of the sum of the reciprocals of the integers considered, e.g. the sum of the reciprocals of all squareless numbers composed of n prime factors $\leq x$, is equal to

$$\frac{(\log\log x)^{n}}{n!} + O\,(\log\log x)^{n-1}$$

and the same holds good for the numbers that are mentioned in the definition of $\varrho_n(x)$ or $\sigma_n(x)$. These formulae concerning the sum of the reciprocals being special cases of formula (3), where k has the value 1, may be proved by means of a merely elementary reasoning, as has been observed at the beginning of this paragraph.

By giving an arbitrary value to k in the formula, however, we find that the number of squareless numbers $\leq x$, composed of n prime factors and congruent to l, with regard to the modulus k, is equal to

$$\frac{x(\log\log x)^{n-1}}{h.(n-1)!\log x} + O\left(\frac{x(\log\log x)^{n-2}}{\log x}\right)$$

and that the sum of the reciprocals of these numbers is equal to

$$\frac{(\log\log x)^{n}}{h.n!} + O\,(\log\log x)^{n-1},$$

while again for the integers that are mentioned with the definition of the functions $\varrho_n(x)$ and $\sigma_n(x)$, perfectly analogous formulae hold good.

For the very reason that the function $F(u)$ is general it will not be difficult to deduce other corresponding relations; so we find the same results if we consider the squareless numbers composed of not more than n prime factors, or the integers for which the total number of prime factors is $\leq n$, or the integers for which the number of different prime factors is not greater than n, etc.

Application III. In an arithmetical series, the difference of which is k and the first term of which is prime to k, occur

$$\frac{\pi^2 x}{6\, k \log x} \prod_{p|k} \left(1 + \frac{1}{p}\right) + O\left(\frac{x}{(\log x)^2}\right)$$

numbers $\leq x$, equal to a square multiplied by a prime number.

That this is again a special case of our proposition appears by taking $F(u)$ equal to 1 or 0, according to u being equal or not to a square multiplied by a prime number.

We have

$$m = 1 \quad \text{hence} \quad b = 1,$$
$$n = 1$$

and

$$\begin{aligned} f(v) &= 1, \text{ if } v \text{ is a square,} \\ &= 0, \text{ if } v \text{ is not a square,} \end{aligned}$$

consequently

$$\begin{aligned} \frac{b}{h} \sum_{\substack{u=1 \\ uz^m \equiv l}}^{\infty} \frac{f(u)}{u^{\frac{1}{m}}} &= \frac{1}{k \prod_{p|k} \left(1 - \frac{1}{p}\right)} \sum_{\substack{u=1 \\ (u,k)=1}}^{\infty} \frac{f(u)}{u} \;{}^{1)} \\ &= \frac{1}{k \prod_{p|k} \left(1 - \frac{1}{p}\right)} \sum_{\substack{v=1 \\ (v,k)=1}}^{\infty} \frac{1}{v^2} \\ &= \frac{1}{k \prod_{p|k} \left(1 - \frac{1}{p}\right)} \prod_{p|k} \left(1 - \frac{1}{p^2}\right) \sum_{v=1}^{\infty} \frac{1}{v^2} \\ &= \frac{\pi^2}{6k} \prod_{p|k} \left(1 + \frac{1}{p}\right) \end{aligned}$$

and we have only to substitute these values in (1), in order to find the relation sought.

Application IV. If all the prime factors of the positive integer

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1) (u, k) represents the greatest common divisor of u and k, so that the number **u** in this sum assumes respectively each integral positive value prime to k.

q are greater than the prime number p, and

$$w = p^{\alpha} q ,$$

the number of positive integers $\leq x$, congruent to l, with regard to the modulus k, (l and k prime to each other) for which the number of divisors is exactly equal to w, is given by

$$\frac{a x^{\frac{1}{p-1}}}{\log x} + O\left(\frac{x^{\frac{1}{p-1}}}{(\log x)^2}\right) \qquad \text{for } \alpha = 1$$

and by

$$\frac{a x^{\frac{1}{p-1}} (\log \log x)^{\alpha-1}}{\log x} + O\left\{\frac{x^{\frac{1}{p-1}} (\log \log x)^{\alpha-2}}{\log x}\right\}$$

for any arbitrary positive integral value of α, where

$$a = \frac{b\,(p-1)}{h.(\alpha-1)!} \sum_{u=1}^{\infty}{}' \frac{1}{u^{\frac{1}{p-1}}}$$

extended over all the positive integers u, of which the number of divisors is exactly equal to q and for which the congruence

$$u z^{p-1} \equiv l \qquad (\text{mod. } k)$$

has roots z; b represents the number of incongruent roots of the congruence

$$z^{p-1} \equiv 1 \qquad (\text{mod. } k).$$

In order to prove this, we take the number of divisors of u equal to τ_u, and

$$\begin{aligned} F(u) &= 1 \quad \text{for} \quad \tau_u = w , \\ &= 0 \quad \text{for} \quad \tau_u \neq w . \end{aligned}$$

We have to prove first that this function satisfies the conditions written in the proposition, if

$$\begin{aligned} m &= p - 1, \\ n &= \alpha , \\ f(v) &= 1 \quad \text{for} \quad \tau_v = q , \\ &= 0 \quad \text{for} \quad \tau \neq q ; \end{aligned}$$

In order to give this demonstration, we distinguish four cases:

1. Let e_u be smaller than $p-1$;

for

$$u = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_\sigma^{\alpha}$$

the number of divisors of u

$$\tau_u = (\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_\sigma + 1)$$

is divisible by $e_u + 1$, consequently by a number $< p$, hence

$$\tau_u \neq w\,,$$
$$F(u) = 0.$$

2. Take $e_u = p - 1$, $a_u > \alpha$;
τ_u is then divisible by
$$(e_u + 1)(e_u + 1) \,.\;.\, (e_u + 1) = (e_u + 1)^{a_u} = p^{a_u}\,,$$
therefore by $p^{\alpha+1}$, so that in this case too, we have
$$\tau_u \neq w\,,$$
$$F(u) = 0.$$

3. Take $e_u = p - 1$, $a_u = \alpha$;
we have then
$$u = p_1^{p-1}\, p_2^{p-1} \ldots p_\alpha^{p-1}\, v_u$$
and from
$$\tau_u = p^\alpha\, \tau_{v_u} \qquad\qquad w = p^\alpha q$$
it follows that there are only two possibilities, viz.
$$a) \qquad F(u) = 1, \qquad \tau_u = w, \qquad \tau_{v_u} = q, \qquad f(v_u) = 1,$$
$$b) \qquad F(u) = o, \qquad \tau_u \neq w, \qquad \tau_{v_u} \neq q, \qquad f(v_u) = 0,$$
hence in this case
$$F(u) = f(v_u)$$

4. Take $e_v \leq p - 1$;
as τ_v is divisible by $e_v + 1$, consequently by a number $\leq p$, τ_v is in this case unequal to q, hence
$$f(v) = 0.$$

Now that it has been proved that the conditions stated are satisfied, we are allowed to apply the proposition and formula (1) gives at once the relation sought.

Finally we observe: in application II some asymptotical expressions have been written for $\pi_n(x)$, $\sigma_n(x)$ and $\varrho_n(x)$, but LANDAU deduces still sharper formulae for these functions. He proves[1]) that for each positive integral value of q, constant numbers $A_{a,b}$, $B_{a,b}$ and $C_{a,b}$ are to be found, for which the relations

$$\left.\begin{aligned}
\pi_n(x) &= x \sum_{a=1}^{q} \sum_{b=0}^{n-1} A_{a,b} \frac{(\log\log x)^b}{(\log x)^a} + o\left(\frac{x}{(\log x)^q}\right),\\
\varrho_n(x) &= x \sum_{a=1}^{q} \sum_{b=0}^{n-1} B_{a,b} \frac{(\log\log x)^b}{(\log x)^a} + o\left(\frac{x}{(\log x)^q}\right)\\
\text{and}\qquad \sigma_n(x) &= x \sum_{a=1}^{q} \sum_{b=0}^{n-1} C_{a,b} \frac{(\log\log x)^b}{(\log x)^a} + o\left(\frac{x}{(\log x)^q}\right)
\end{aligned}\right\} \quad . \quad . \quad (8)$$

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[1]) E. LANDAU. Ueber die Verteilung der Zahlen, welche aus ν Primfaktoren zusammengesetzt sind. [Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Math.-physikalische Klasse. (1911). pages 361—381].

hold good. It is a matter of course that such a relation does not hold good for any function satisfying the condition stated in § 1. It appears, however, that we have only to modify this condition a little to be sure that such a relation does hold good, viz.:

If the arithmetical function $F(u)$ of the integer $u > 1$ satisfies the conditions:

1. for $e_u < m$, and also for $e_u = m$, $a_u > n$, we have
$$F(u) = 0;$$
2. for $e_u = m$, $a_u \leq n$ we have
$$F(u) = f(v_u, a_u),$$

where $f(v,a)$ represents an arithmetical function of the positive integers v and a, and

3. $$F(u) = O(v_u^{\mu}), \quad \text{where} \quad \mu < \frac{1}{m(m+1)};$$

then there are constant values $D_{a,b}$ for $q \geq a \geq 1$, $n-1 \geq b \geq 0$ to be found for any positive integral value of q, satisfying the relation

$$\sum_{\substack{u=2 \\ u \equiv l}}^{x} F(u) = x^{\frac{1}{m}} \sum_{a=1}^{q} \sum_{b=0}^{n-1} D_{a,b} \frac{(\log \log x)^b}{(\log x)^a} + o\left(\frac{x^{\frac{1}{m}}}{(\log x)^q}\right)$$

This proposition is again very general; this appears obviously by the observation that the functions which occur in the four applications of this paragraph and which have been substituted for $F(u)$ also satisfy this condition, so that the formulae deduced in those applications are also to be intensified with this proposition. And the formulae obtained in application II are exactly the formulae (8).

The proposition is elementarily, i. e. without using considerations belonging to the theory of functions to be deduced from the well-known relation

$$\sum_{\substack{p \leq x \\ p \equiv l}} 1 = \frac{1}{h} \int_{2}^{x} \frac{du}{\log u} + o\left(\frac{x}{(\log x)^q}\right),^{1)}$$

according to a reasoning somewhat similar to the one followed here in order to prove formula (1); it goes, however, without saying that the proof is not so simple.

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[1] E. Landau. Handbuch. I. p. 468.

Mathematics. — *"Some considerations on complete transmutation"*. (First Communication). By Dr. H. B. A. Bockwinkel. (Communicated by Prof. L. E. J. Brouwer).

(Communicated in the Meeting of June 24, 1916.)

1. In a paper "Sur les opérations en général et les équations différentielles linéares d'ordre infini", which appeared in the Ann. de l'Ecole Norm. of 1897, C. Bourlet considered a very general category of additive functional operations, called *transmutations* by him. The name "additive", or "distributive"[1]) they owe to the property that the transmuted function of a sum is equal to the sum of the transmuted functions. The transmutation is further called:

Uniform, if it makes a given function pass into only *one* other;

Continuous, if the limit of the transmuted function is equal to the transmuted limit of that function[2]);

Regular, if it transforms a regular function into another likewise regular function.

We have in this case always in view a certain circular domain with centre $x = x_0$, in which the functions u, to which the operation is to be applied, are regular. The meaning of the last definition is more exactly that the transmutation is called regular, if the result of it is a function v, which is also regular in such a domain.

The result of the operation may often be represented by a series of the form

$$Tu = a_0(x)\,u + \frac{a_1(x)}{1!}u' + \dots + \frac{a_m(x)}{m!}u^{(m)} + \dots, \quad . \quad . \quad (1)$$

in which $a_0(x), a_1(x) \dots, a_m(x), \dots$ represent functions of x perfectly determined by the given transmutation, and are regular in a domain with centre x_0, whereas $u', u'', \dots u^{(m)} \dots$ are the respective derivatives of the function u, to which the operation is applied. The series (1), which also occurs in Pincherle's paper, is to the theory of operations, what the series of Maclaurin is to the theory of functions.

We shall call a transmutation complete in a certain point $x = x_0$ of the complex plane, if a circle with centre x_0 and radius ρ is to be indicated, such that each function, which is regular within that circle and on its circumference, has a transmuted function

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[1]) The latter name is used by S. Pincherle in a paper on the same subject, Math. Ann. 49 (1897) p. 325—382.

[2]) Cf. for a complete explanation of this term N°. 9 (2nd Comm.).

which in the point x_0 is determined by a series of the form (1)[1].

In future we shall say of a function which is regular *within and on the circumference of* a certain circle, that it "belongs to that circle" (after PINCHERLE); its radius of convergence is in this case *greater* than the radius ϱ of that circle. We shall further, if we speak of the domain (r), mean by it the closed set of points within and on the circumference of a circle with centre x_0 and radius r; this circle we shall briefly indicate as the circle (r) and if we mean a centre other than x_0, we shall indicate this specially.

With regard to the complete transmutation thus defined BOURLET states his theorem XI, which follows here:

La condition nécessaire et suffisante pour que la transmutation (1) *fournisse une transmuée pour toute fonction régulière dans un domaine de rayon ϱ autour du point x_0, est que la série*

$$\psi(x_0, z) = a_0(x_0) + \frac{a_1(x_0)}{z - x_0} + \frac{a_2(x_0)}{(z - x_0)^2} + \quad . \quad . \quad . \quad . \quad . \quad . \qquad (A)$$

soit convergente pour toute valeur de z telle que $|z - x_0| = \varrho$.

That this condition is necessary is proved by BOURLET by applying the transmutation (1) to the function

$$u = \frac{1}{z - x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (B)$$

in which z is a constant, such that $|z - x_0| = \varrho$. This part of his proof, so far as I am aware, is correct.

I object to the second part of his proof. BOURLET says: If the condition is fulfilled T may be represented by the identity:

$$Tu = \frac{1}{2\pi i} \int_C \frac{u(z)}{z - x_0} \psi(x_0, z)\, dz,$$

in which the integral is taken along the circumference C of the circle (ϱ).

This, however, is incorrect, if the function u has ϱ *exactly* as radius of convergence. Yet, according to the theorem, such a function, being regular *within* (ϱ), ought to have a transmuted in x_0; for by the expression "régulière dans un domaine de rayon ϱ" BOURLET means, as he expressly states in a note before: "développable en une série . . .,

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[1] BOURLET's exact words are: "Complète dans un *domaine* de rayon ϱ". This may give rise to the misunderstanding as if was meant that a transmuted function existed *in the whole domain* (ϱ), which is not true in general. From the reasoning by which BOURLET arrives at the theorem XI to be mentioned directly follows only that, under the conditions mentioned there, there exists a transmuted in the single point x_0. I have therefore preferred to say "complete in a *point*", reserving the name "complete in a *domain*" for another case (cf. N⁰. 4).

pour toute valeur de x telle que l'on ait $(x-x_0) < \varrho$," (with *only* the symbol $<$, not the symbol $=$ with it). Besides, if **Bourlet**, in stating his theorem XI, had meant functions that are also regular on the *circumference* of (ϱ), the *first* part of his proof would have been wrong. For in that case the function (B) *need* not have a transmuted in the point x_0.

If we stick to the term "regular" as laid down by **Bourlet** himself and in frequent use, the second part of his *proof* is incorrect. But this ought not to be wondered at, if we observe that the corresponding part of the *theorem* is also false.

Let us consider the transmutation

$$Tu = \sum_{0}^{\infty}{}_{m} \frac{u^{(m)}}{(m+1)^2\,(m!)} \quad . \; . \; . \; . \; . \; . \; . \; . \qquad (2)$$

The series (A) is here, for a domain of the origin $(x_0 = 0)$,

$$\psi(0, z) = 1 + \frac{1}{2^2 z} + \frac{1}{3^2 z^2} + \ldots + \frac{1}{(m+1)^2 z^m} + \ldots$$

and converges for *all* values of z with modulus 1. For the function

$$u = \frac{1}{(1-x)^2},$$

that is regular within the circle with radius 1, the series (2) produces however no transmuted in the origin.

2. Although the inaccuracy in the stating of the theorem is slight, it seems proper to express it in the following more accurate form:

If the series (1) *is to produce for all functions belonging to a certain circle* (ϱ) *a transmuted in the centre* x_0 *of that circle, it is necessary and sufficient that the series*

$$\psi(x_0, t) = a_0(x_0) + \frac{a_1(x_0)}{t} + \frac{a_2(x_0)}{t^2} + \ldots \quad . \; . \; . \qquad (3)$$

converges for each value of t *with modulus greater than* ϱ.

Before proceeding to the demonstration it is convenient to make the following observation: From the shape of a power series which (3) has with regard to $\frac{1}{t}$, it is to be deduced that, if it converges for a certain value of t, it converges *absolutely* for any other value with *greater* modulus and, if it diverges for a certain value of t, it diverges for any other value with *smaller* modulus.

The necessity and sufficiency of the condition is now proved in this way:

1. The condition is necessary. Suppose that it were not fulfilled for a certain value $t = \varrho_1 > \varrho$, so that the series

$$a_0(x_0) + \frac{a_1(x_0)}{\varrho_1} + \frac{a_2(x_0)}{\varrho_1^2} + \ldots \quad \ldots \quad (4)$$

diverged. Then choose a positive number ϱ_2, such that

$$\varrho < \varrho_2 < \varrho_1 ,$$

and a function

$$u = \frac{1}{x_0 + \varrho_2 - x} ,$$

which evidently belongs to (ϱ). The series (1) gives for it in the point x_0,

$$Tu = \frac{1}{\varrho_2} \left[a_0(x_0) + \frac{a_1(x_0)}{\varrho_2} + \frac{a_2(x_0)}{\varrho_2^2} + \ldots \right] ,$$

which series, however, according to the previous observation, diverges, because (4) does so. In other words, if the condition mentioned in the proposition is not fulfilled, the series (1) does not produce a transmuted in the point x_0 for *all* functions belonging to (ϱ): the condition therefore is necessary.

2. The condition is sufficient. Consider a function u with a radius of convergence $r > \varrho$. Choose a number ϱ_1, such that

$$\varrho < \varrho_1 < r .$$

If the condition is fulfilled the series

$$\chi(x_0, z) = a_0(x_0) + \frac{a_1(x_0)}{z - x_0} + \frac{a_2(x_0)}{(z - x_0)^2} + \ldots$$

converges absolutely for any value of z for which $|z - x_0| = \varrho_1$. Moreover the last series, considered as a function of z, converges *uniformly* on the circle, determined by

$$z = x_0 + \varrho_1 e^{i\varphi} ;$$

for the moduli of its terms are *anywhere on that circle* equal to those of the corresponding terms, independent of z, in the absolutely converging series

$$a_0(x_0) + \frac{a_1(x_0)}{\varrho_1} + \frac{a_2(x_0)}{\varrho_1^2} + \ldots$$

From this the uniformity in question may be deduced according to a well-known reasoning.

The integral

$$\frac{1}{2\pi i} \int\limits_C \frac{u(z)}{z - x_0} \chi(x_0, z)\, dz ,$$

taken along the circumference of (ϱ_1), *exists* therefore, and may be found by term by term integration. The series (1) is the result then, and this series converges therefore, for the function considered, in the point x_0. The condition is thus sufficient 1).

3. With regard to the proposition we observe the following. There will evidently be a lower boundary a for the numbers ϱ, such that convergence of (3) takes place for any t with modulus greater than a, and divergence for any t with modulus less than a; this again follows from the shape of a power series, which (3) has with regard to $\frac{1}{t}$. At the same time it is to be deduced in the usual manner that the numbers $a_m(x_0)$ satisfy the following two conditions:

1. Corresponding to any arbitrarily small number ε there is an integer m_ε, such that

$$|a_m(x_0)| < (a+\varepsilon)^m, \text{ for } m \geqq m_\varepsilon .$$

2. Corresponding to any arbitrarily small number ε there are an infinite number of integers m, for which

$$|a_m(x_0)| > (a-\varepsilon)^m .$$

On the other hand these two conditions are necessary and sufficient to characterize the number a as the lower boundary mentioned above.

According to a well-known mode of expression we can also say that a is the *upper limit*, for $m = \infty$, of the expression $|a_m(x_0)|^{\frac{1}{m}}$, i.e. symbolically

$$a = \overline{\lim_{m=\infty}}\, |a_m(x_0)|^{\frac{1}{m}} \text{ 2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

BOURLET's theorem may now be expressed as follows:

If the series (1) *is to produce in a given point x_0 a transmuted for all the functions belonging to a certain domain* (ϱ) *with centre x_0, it is necessary and sufficient that ϱ is not smaller than the number a determined by* (5).

The circle (a) *is therefore the minimum circle with centre x_0, of which it holds that the series* (1) *produces a transmuted in x_0 for all functions belonging to it*; this statement is again equivalent to the theorem of BOURLET. It ought however not to be supposed, that a series of the form (1) *never* produces a transmuted in x_0 for some

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1) See the observation in note 1 of § 4 about this proof.

2) Enc. d. Math. Wiss. I A. 3, p. 71

function with a radius of convergence less than a. Let

$$a_0(x) = a_2(x) = \ldots = a_{2k}(x) = \ldots = 0,$$

and

$$a_{2k-1}(x) = c^{2k-1},$$

in which by c a constant is meant greater than 1. Here the number a is equal to c, hence, according to the supposition, greater than 1.

The function

$$u = \frac{1}{1-x^2},$$

for which, in the point $x = 0$, the radius of convergence is equal to 1, has, though this radius is smaller than the number a, in that point a transmuted, which is equal to zero.

4. **Bourlet**'s theorem informs us about the question when the series (1) is complete *in one definite point* x_0, but, as a matter of course, only those cases are interesting, in which a certain circle (ϱ) with centre x_0 is to be indicated, such that the series (1) produces for all functions belonging to it a transmuted not only in the point x_0, but in *all* points of a certain domain (α) round x_0. **Bourlet** however has not drawn attention to this. If the series (1) satisfies the new condition, we shall call the transmutation determined by it *complete in the domain* (α).

Let us suppose that for any point of a domain (α) a number a, as mentioned above, can be indicated, and that the number m_ε, in the condition 1 at the beginning of § 3, remains below a fixed number in the whole domain (α), which number we will also indicate by m_ε. This latter supposition we shall quote in future as "the uniformity supposition of N°. 4". As the number a will in general depend on the place of the point x, we prefer to represent it by a_x. According to this supposition the expression

$$a_x = \overline{lim}_{m=\infty} \left| a_m(x) \right|^{\frac{1}{m}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

has in all internal points and points on the circumference of (α) a *finite* value. We further suppose that the quantity a_x remains in the whole domain (α) below a fixed number G, in other words, that a_x in the domain (α), is a *limited*-function of the points of that domain (always the circumference being included). The numbers a_x have then in the domain (α) an upper boundary, which we will represent by $a(\alpha)$ or briefly by a, and we now assert that the upper boundary of the a_x-values for points of the *circumference* of (α) is equal to that for the *whole* domain (α). Suppose that this were not so, but that the upper limit of a_ε for the circumference of (a) was a certain number $a' < a$. According to the supposition made, the values which an arbitrary function $a_m(x)$, with rank $m \geqq m_\varepsilon$, assumes on the circumference of (a) would all have a modulus smaller than

$$(a' + \varepsilon)^m .$$

We now suppose ε chosen so small that

$$a' + \varepsilon < a.$$

Then an *infinite* number of functions $a_m(x)$ satisfy the condition that in a certain point of the domain (a)

$$| a_m(x) | > (a' + \varepsilon)^m .$$

Among this infinite number of functions we select one for which $m > m_\varepsilon$. But *this same* function on the *circumference* (a) satisfies the condition

$$| a_m(x) | < (a' + \varepsilon)^m .$$

Since the *maximum* modulus of a function in a domain (a), lying within the domain of regularity, is found on the *circumference* of (a), the preceding two inequalities are contradictory and the proposition has been proved.

If now the functions $a_m(x)$ satisfy the conditions mentioned above, the transmutation determined by the series (1), is complete in the domain (a), as the following generalisation of the theorem of BOURLET shows:

If the series (1) *is to produce in any point of a given circular domain* (a) *a transmuted for all functions belonging to a domain* (ϱ) *concentric with* (a), *it is necessary and sufficient that the radius* (ϱ) *is not smaller than the number* β *determined by the equality*

$$\beta = a + \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or equivalent to this:

The circle (β) *is the minimumcircle which has the property that the series* (1) *produces for all functions belonging to it, a transmuted in any point of the given circle* (a).

Proof 1°. The condition, mentioned in the theorem, is necessary. Let us suppose, in order to prove this, that ϱ is smaller than β; we have only to indicate a function belonging to (ϱ) and for which the series (1) does not produce a transmuted in a certain point of (a).

Choose a number r such that

$$\varrho < r < \beta,$$

and consider the function

$$u = \frac{1}{x_0 + re^{i\gamma} - x},$$

in which γ remains indefinite for the moment; this function has the radius of convergence r, and therefore belongs to (ϱ). Since a is the upper limit of the numbers a_x, for the domain (α), there is a point P, on the circumference of (α), for which a_x is greater than $a-(\beta-r)$, or, according to (7), than $r-\alpha$; let us suppose

$$a_x = r - \alpha + \delta.$$

Then there are an unlimited number of m-values, for which, if ε has a *given* value $< \delta$, we have in the point P

$$|a_m(x)| > (r-\alpha+\varepsilon)^m.$$

If φ is the argument of $x-x_0$ for P, we choose $\gamma=\varphi$, and then have in the same point

$$u = \frac{e^{-i\varphi}}{r-\alpha}, \quad \frac{u^{(m)}}{m!} = \frac{e^{-im\varphi}}{(r-\alpha)^{m+1}}.$$

For the above m-values, we have therefore, in P,

$$\left|\frac{a_m(x)\,u^{(m)}}{m!}\right| > \frac{1}{r-\alpha}\left(1+\frac{\varepsilon}{r-\alpha}\right)^m,$$

and these m-values being unlimited in number, the condition, necessary for convergence that the limit of the terms is zero, has not been satisfied in this case for the series (1); the series therefore diverges in P.

2°. The condition is sufficient. The quantity a_x, being in an arbitrary point x of the domain (α) not greater than $\beta-\alpha$, there is corresponding to any arbitrarily chosen number ε an integer m_ε, such that in the point x

$$|a_m(x)| < (\beta-\alpha+\varepsilon)^m, \text{ for } m \geqq m_\varepsilon.$$

If further the function u belongs to (β), it belongs also to a somewhat greater circle (ϱ); let us suppose

$$\varrho = \beta + \delta, \ (\delta > 0).$$

Let $M(\varrho)$ be the maximum modulus of u on the circumference of (ϱ); from the theory of functions it is known that in the whole domain (α) the condition

$$\left|\frac{u^{(m)}}{m!}\right| < \frac{\varrho M(\varrho)}{(\varrho-\alpha)^{m+1}} \quad \ldots\ldots\ldots \quad (8)$$

is satisfied.

We have thus in x, for $m \geqq m_\varepsilon$

$$\left|\frac{a_m\,u^{(m)}}{m!}\right| < \frac{\varrho M(\varrho)}{\beta-\alpha+\delta}\left(\frac{\beta-\alpha+\varepsilon}{\beta-\alpha+\delta}\right)^m$$

If we now suppose that for ε a number smaller than δ has been chosen, we have for the remainder $R_k(x)$ of the series (1), after k terms, provided $k \geqq m_\varepsilon$,

$$|R_k(x)| < \frac{\varrho M(\varrho)}{\delta-\varepsilon}\cdot\left(\frac{\beta-\alpha+\varepsilon}{\beta-\alpha+\delta}\right)^k \quad . \; . \; . \; . \; . \; . \; . \quad (9)$$

The amount on the righthand side of the inequality having zero for its limit, for $k=\infty$, the convergence of the series has been proved, and at the same time, as the amount in question is independent of x, that this convergence is *uniform*[1]).

5. Neither in the last case should it be held that a series of the form (1) *never* produces for some function not belonging to (β) a transmuted in the whole domain (α). Let us consider the transmutation

$$Tu = \sum_{0}^{\infty}{}_m \frac{(1-c-x)^m\, u^{(m)}}{m!},$$

in which c is a positive constant smaller than 1. Here $a_x = |1-c-x|$; the upper limit of a_x on the *circumference* of the circle (α) with the origin as centre, is therefore $1-c+\alpha$, and this is at the same time the upper limit for the *domain* (α). Hence we have

$$\beta = 1 - c + 2\alpha\,.$$

Choose for u the function

$$u = \frac{1}{1-x}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

For this we get

$$Tu = \frac{1}{1-x}\sum_{0}^{\infty}{}_m \left(\frac{1-c-x}{1-x}\right)^m,$$

in which the series apparently converges in the whole domain (α), if $\alpha < 1-\frac{1}{2}c$. Further we have $\beta > 1$, if $\alpha > \frac{1}{2}c$. Finally for

$$\tfrac{1}{2}c < \alpha < 1 - \tfrac{1}{2}c\,,$$

the corresponding β is greater than 1, and therefore the function (10), of which, in $x=0$, the radius of convergence is 1, does not belong to (β), while nevertheless the series (1) produces in the whole domain (α) a transmuted for that function. The reason for it is here to be

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[1]) The inequality (8) may also be used with more advantage to prove the unextended theorem of BOURLET the series (1) not needing to be first deduced then from an integral. We have retained the reasoning with the integral in our correction of the original theorem and proof in § 2, in order to remain in contact with the exposition of the writer, by which an easier criticism of the disputed points is made possible.

found in the fact that the point where, on the circumference of (a), the magnitude a_x assumes its upper limit value, does not coincide with the point where the greatest modulus of the function u and its derivatives are found. It will be clear from the considerations in N°. 4 that the lack of this coincidence may give rise to cases as considered here.

But it is not the only possible cause; even if the functions $a_m(x)$ and the function u with its derivatives have their greatest modulus in *the same* point of a domain (a), such a case may sometimes arise. Let a transmutation be given for which, c being a real positive constant

$$a_m(x) = c^m, \quad \text{if} \quad m = 2^{2n-1}, \quad (n = 1, 2, 3, \ldots)$$
$$a_m(x) = 0, \quad ,, \quad m \neq 2^{2n-1}$$

Here the function a_x is equal to the constant c. Consequently the upper limit a of a_x in a domain (a) is equal to c, and

$$\beta = a + c.$$

We take again $x = 0$ as centre and consider the function

$$u = \sum_{0}^{\infty}{}_{n} \; x^{2^{2n}}, \quad \ldots \ldots \ldots \quad (11)$$

which, just as its derivatives, has its greatest modulus in the point $x = a$ of the domain (a), so that, since a_x assumes its upper limit value c in each point of (a), the coincidence referred to above takes place here. The series (1) passes in this case into

$$Tu = \sum_{1}^{\infty}{}_{n} \frac{c^m \, u^{(m)}}{m!}, \quad m = 2^{2n-1}.$$

Now we have for $m = 2^{2n-1}$

$$| u^{(m)}(x) | \overline{<} | u^{(m)}(\xi) | < D^m_\xi \left(\frac{\xi^{2m}}{1-\xi} \right)$$

since the series representing the first member of this inequality consists of *part* of the terms of the series in which the last member may be developed.

If in general

$$y = \frac{x^{h+k}}{1-x},$$

we have for real positive x

$$\frac{y^{(k)}}{k!} < \frac{x^h}{(1-x)^{k+1}} \cdot \frac{(h+k)!}{h!\,k!} \,^{1)}.$$

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[1]) By means of the formula of LEIBNIZ we find

In our case we find, by putting $h = k = m$,

$$\left|\frac{u^{(m)}(x)}{m!}\right| < \left(\frac{\xi}{1-\xi}\right)^m \cdot \frac{2m!}{m!^2} \cdot \frac{1}{1-\xi},$$

always on the understanding that $m = 2^{2n-1}$. By means of the formula of Stirling:

$$k! = k^k e^{-k} \sqrt{2\pi k}\, e^{\frac{\theta}{12k}}, \; (0 < \theta < 1)$$

we can also write for it here

$$\left|\frac{u^{(m)}(x)}{m!}\right| < \left(\frac{4\xi}{1-\xi}\right)^m \cdot \frac{e}{(1-\xi)\sqrt{\pi m}}$$

so that

$$\left|\frac{a_m u^{(m)}}{m!}\right| < \left(\frac{4c\xi}{1-\xi}\right)^m \cdot \frac{e}{(1-\xi)\sqrt{\pi m}}.$$

The last inequality holds for any m, since, for the values of m, not equal to an odd power of 2, $a_m(x) = 0$. The terms of the series (1) therefore will be smaller in absolute value than those of a converging geometrical progression, if

$$4c\xi < 1 - \xi, \quad \text{or} \quad \xi < \frac{1}{1+4c}$$

and hence that series will converge in the whole domain (α) if

$$\alpha < \frac{1}{1+4c}.$$

As $\beta = \alpha + c$ here, β will be greater than the radius of convergence 1 of the function considered, if

$$\alpha > 1 - c$$

The preceding two inequalities may be fulfilled for a number of values of α, if the number c is chosen greater than $\frac{3}{4}$. In any domain (α) satisfying the inequality

$$1 - c < \alpha < \frac{1}{1+4c} < \frac{1}{4}$$

the series (1) produces therefore a transmuted for the function (11), although the corresponding domain (β) has a radius greater than the radius of convergence of that function; this being the case, though

$$\left(\frac{x^{h+k}}{1-x}\right)^{k} = \frac{(h+k)!\, x^h}{1-x} \sum_0^k {}_i \frac{k!}{i!\,(h+k-i!)} \left(\frac{x}{1-x}\right)^{k-i}$$

and since $(h+k-i! > h!\,(k-i)!$, the right hand member is less than

$$\frac{(h+k)!\, x^h}{h!\,(1-x)} \sum_0^k {}_i \binom{k}{i} \left(\frac{x}{1-x}\right)^{k-i} = \frac{(h+k)!\, x^h}{h!\,(1-x)^{k+1}}$$

the point, where, in the domain (α), that function has its maximum modulus, coincides with a point where a_x attains its upper limit value.

6. We will now say something about the dependence between the quantities α and β. The number α may vary from *zero* to an amount A, which is the upper limit of the radii of domains, in which the quantity a_x determined by (6) is a limited function. The number A cannot in any case be greater than the radius of convergence of one of the functions $a_m(x)$, but it may be less, since, even when all those functions are regular in a certain domain (α), it is possible that the upper limit (6) has not a finite value in some point of (α). It might also occur that the limit (6) did exist in all the points of (α), but was not bounded in that domain. On the other hand it may happen that the number A is infinite (e.g. if $a_m(x)$ = constant $= c^m$).

From the fact that the quantity a has been defined as the upper limit of the function a_x in the domain (α), it follows at once that a cannot decrease if α increases, in other words that a is a monotone function of α. Therefore according to (7), β is a *monotonely increasing* function of α, not smaller than α. (β may be *equal* to α, e.g. if $a_m(x) = 1 : m\,!$, for in that case $a_x =$ constant $= 0$).

Let b be the value of β for $\alpha = 0$, and B the one for $\alpha = A$; in many cases B will be infinite, but it need not be so. Every value β may assume lies, as β is a monotonely increasing function of α, in the interval (b, B), and corresponds to only *one* value of α. The number b, which, as a β-value, belongs to $\alpha = 0$, may be zero, if $a_{x_0} = 0$. In that case *any* function for which x_0 is an *ordinary* point, with arbitrarily small domain of regularity, has in x_0 a transmuted determined by (1). If a, as a function of α, is in that case continuous in $\alpha = 0$, the series (1) produces for *any* function, with arbitrarily small domain of convergence, a transmuted in a certain *domain* of x_0. The transmuting series in that case is, according to a name introduced by PINCHERLE, *of the first kind.*

Chemistry. — *"In-, mono- and divariant equilibria"*. XIII. By Prof. F. A. H. SCHREINEMAKERS.

(Communicated in the meeting of December 21, 1916).

21. *Ternary systems with two indifferent phases.*

In the previous communication we have deduced the four P,T-diagramtypes, which occur in ternary systems with two indifferent phases. Now we shall consider a case more in detail.

We take a ternary system with the components water and the two salts Z and A which are not volatile, in which of the salt Z yet also the hydrate $Z.n\,H_2O$ occurs, which we shall represent by Z_n (fig. 1).

Suppose, the invariant equilibrium

$$Z + Z_n + L_d + G$$

occurs in the binary system $W + Z$ at the temperature T_d and under the pressure P_d. The liquid L_d is represented in fig. 1 by the point d between W and Z_n; of course we might as well have taken d between Z and Z_n. When we add the salt A to this equilibrium, then the equilibrium $Z + Z_n + L + G$ arises; the liquid L proceeds then a curve $d\,h\,m$ (fig. 1). It is evident that T and P change along this curve $d\,h\,m$ from point to point.

Now we assume that in the point m the added salt A dissolves no more, so that at T_m and under P_m the invariant equilibrium:

$$Z + Z_n + A + L_m + G$$

is formed. A similar case is found e.g. in the system: *water* + $Na_2SO_4 + NaCl$. In the binary system: *water* + Na_2SO_4 viz. at 32°.5 the equilibrium

$$Na_2SO_4 + Na_2SO_4 \,.\, 10\,H_2O + L + G$$

occurs. On addition of $NaCl$ at 17°.9 arises:

$$Na_2SO_4 + Na_2SO_4 \,.\, 10\,H_2O + NaCl + L + G.$$

As the gas-phase G is represented in fig. 1 by the point W, the phases Z, Z_n and G are situated on a straight line. Z, Z_n and G are, therefore, the singular phases, A and L_m the indifferent phases of the equilibrium:

$$Z + Z_n + A + L_m + G.$$

Consequently from the invariant point start the singular equilibria:

$(M) = Z + Z_n + G$ [Curve (M) in fig. 2]
$(A) = Z + Z_n + L + G$ [Curve $(A) = md$ in fig. 2 and md in fig. 1]
$(L) = Z + Z_n + A + G$ [Curve $(L) = mt$ in fig. 2]

and further the equilibria:

$(Z) = Z_n + A + L + G$ [Curve $(Z) = rm$ in fig. 2 and rm in fig. 1]
$(Z_n) = Z + A + L + G$ [Curve $(Z_n) = mb$ in fig. 2 and mb in fig. 1]
$(G) = Z + Z_n + A + L$ [Curve (G) in fig. 2]

Let us first consider the binary system $W + Z$, in which at T_d and under P_d the invariant equilibrium

$$Z + Z_n + L_d + G$$

occurs. From the invariant point d (fig. 2) start the equilibria:

$Z + L + G$, represented by curve *da* (fig. 2)
$Z_n + L + G$, ... *do* (fig. 2)
$Z + Z_n + L$, ... *ds* (fig. 2)
$Z + Z_n + G$, ... *dmt* (fig. 2)

The solutions of the first equilibrium are represented in fig. 1 by points of *da*, those of the second equilibrium by points of *do*. Curve *ds* is drawn vertically in fig. 2; the little curve with the arrows indicates that it may proceed as well a little towards the right as to the left. [This little arc has the same meaning for the curve (G) in this and in the following figures].

It follows from the reaction: $Z_n \rightleftarrows Z + G$ which may occur between the phases of the equilibrium $Z + Z_n + G$, that curve $t\,m\,d$ is a curve ascending with the temperature. With this reaction from left to right viz. as well the volume as the entropy increases.

This curve $d\,m\,t$ is at the same time the (M)-curve of the ternary system $W + Z + A$. Consequently on this curve is situated somewhere the point m in which occurs the invariant equilibrium:

$$Z + Z_n + A + L_m + G$$

of the ternary system. The two other singular equilibria:

$$(A) = Z + Z_n + L + G \text{ and } (L) = Z + Z_n + A + G$$

coincide with this curve $t\,m\,d$. As the equilibrium (A) exists under higher pressures and at higher temperatures than the equilibrium (L), (A) is represented by curve $m\,d$ and (L) by curve mt in fig. 2. [Further we shall show this yet in another way].

The equilibrium $(G) = Z + Z_n + A + L$ goes, starting from m towards higher pressures and it may go as well towards higher as towards lower temperatures. [We shall refer to this later].

Now we have still to draw in fig. 2 the curves (Z) and (Z_n). For this we consider the concentration-diagram of fig. 1. In this are represented the solutions of:

$(A) = Z + Z_n + L + G$ by curve $m\,d$
$(Z) = Z_n + A + L + G$,, ,, $m\,r$
$(Z_n) = Z + A + L + G$,, ,, $m\,b$

We make the obvious supposition that the curves $m\,d$ and $m\,b$ go towards higher pressures, starting from m and that curve $m\,r$ goes towards lower temperatures, starting from m. [We shall refer to this later].

The dotted curves are the saturation curves of Z_n, Z and A under their own vapour-pressure; the little arrows indicate the direction, in which the pressure increases.

The regions, in which Z_n, Z and A occur as solid phases, are indicated by circumcircled letters.

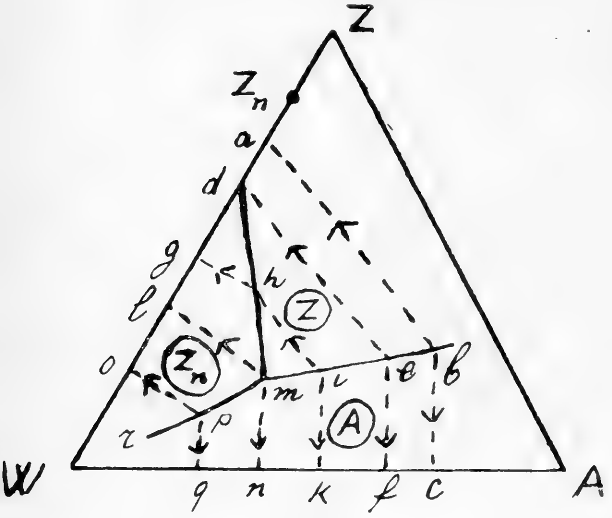

Fig. 1.

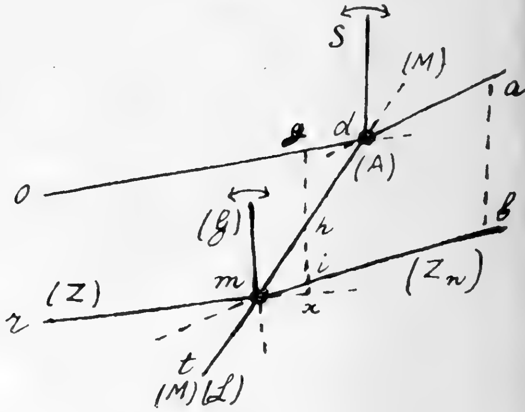

Fig. 2.

It appears from the direction of the little arrow on curve hi (fig. 1) that the vapour-pressure is higher in h than in i. This saturation-curve hi is represented in fig. 2 by a straight line hi parallel to the P-axis, the point h is situated, therefore, higher than the point i, so that curve mhd must be situated above curve mib.

Curve ab of fig. 1 is represented in fig. 2 by the straight line ab parallel to the P-axis; as, in accordance with fig. 1, the pressure is higher in a than in b, in fig. 2 point a must be situated above point b and consequently curve da above curve mb.

We have drawn in fig. 2 curve mb starting from m towards higher pressures, later on we shall see that this need not be always the case.

Now we have still to determine in fig. 2 the position of curve (Z) with respect to the other curves. We are able to do this in different ways, we shall show that the metastable prolongation mx of curve rm is situated below curve mb.

For this we imagine in fig. 1 the curves gh and ki to be prolonged till they intersect one another in a point x. This point of intersection is a point of the metastable prolongation of curve rm. As $T_x = T_h = T_g = T_i$, the points x, h, g and i are situated in fig. 2 on a straight line parallel to the P-axis. It is apparent from fig. 1 that the vapour-pressure is smaller in point x than in h and in i; in fig. 2 the point x is situated, therefore, below point i, so that curve mx is situated below curve mb.

We could also draw in fig. 2 still the P,T-curve of the equilibrium $A+L+G$ of the binary system $W+A$; it appears from fig. 1 that this curve must be situated in fig. 2 above the curves $r\,m$ and $m\,b$.

In our previous considerations we have followed for the deduction of the P,T-diagram the same way as in the deduction of the P,T-diagrams for some special cases in binary systems [Communication XI]. We have used viz. the concentration-diagrams and some of their properties. In the case which is discussed now, we used the property that the vapour-pressure increases along the saturation-curves in the direction of the little arrows. Further we have made the obvious supposition that in the concentration-diagram (fig. 1) the curves mb and md go starting from m towards higher temperatures and that curve mr goes, starting from m towards lower temperatures.

We may, however, follow also quite another way, in which we may deduce as well the P,T- as the concentration-diagram and in which we more plainly feel the suppositions which are assumed in the deductions.

For this we consider the different reactions which may occur in the invariant equilibrium:

$$Z+Z_n+A+L+G$$

With this we shall assume that the liquid L is represented in fig. 1 by a point m within the triangle Z_nAW. From the position of the five phases with respect to one another follow the reactions:

1. For the singular equilibrium $(M)=Z+Z_n+G$

$$Z_n \rightleftarrows x\,G+(1-x)\,Z \qquad (\Delta V)_M \quad ; \quad (\Delta H)_M$$

2. For the equilibrium $(Z)=Z_n+A+L+G$

$$L \rightleftarrows y\,G+u\,Z_n+(1-y-u)\,A \qquad (\Delta V)_Z \quad ; \quad (\Delta H)_Z$$

Herein $x, y, u, 1-x$ and $1-x-y$ have positive values, which may be determined when the compositions of the phases are known. $(\Delta V)_M$ and $(\Delta H)_M$ are the changes in volume and entropy when reaction (1) proceeds from left to right, so that the indicated quantities participate in the reaction. The same is true for $(\Delta V)_Z$ and $(\Delta H)_Z$. At the following reactions we shall indicate in the same way the changes in volume and entropy.

Now we may deduce, as has been discussed formerly, from 1 and 2 the reactions for the other monovariant equilibria and also the isovolumetrical and isentropical reaction. We find:

3. For the equilibrium $(G) = Z + Z_n + A + L$

$$(y + xu)\, Z_n + x\,(1-y-u)\, A \leftrightarrows x\, L + y\,(1-x)\, Z \quad (\Delta V)_G \;\; ; \;\; (\Delta H)_G.$$

Herein is:

$$(\Delta V)_G = y\,(\Delta V)_M - x\,(\Delta V)_Z \;\; ; \;\; (\Delta H)_G = y\,(\Delta H)_M - x\,(\Delta H)_Z$$

4. For the equilibrium $(Z_n) = Z + A + L + G$

$$L \leftrightarrows (y + ux)\, G + u\,(1-x)\, Z + (1-y-u)\, A \quad (\Delta V)_n \;\; ; \;\; (\Delta H)_n$$

Herein is:

$$(\Delta V)_n = u\,(\Delta V)_M + (\Delta V)_Z \;\; ; \;\; (\Delta H)_n = u\,(\Delta H)_M + (\Delta H)_Z$$

5. For the isovolumetrical reaction:

$$(\Delta V)_n \,.\, Z_n + (1-y-u)\,(\Delta V)_M \,.\, A + (\Delta V)_G \,.\, G \leftrightarrows$$
$$(1-x)\,(\Delta V)_Z \,.\, Z + (\Delta V)_M \,.\, L \qquad 0 \;\; ; \;\; (\Delta H)_V$$

Herein is: $(\Delta H)_V = (\Delta V)_Z \,.\, (\Delta H)_M - (\Delta V)_M \,.\, (\Delta H)_Z$

6. For the isentropical reaction:

$$(\Delta H)_n \,.\, Z_n + (1-y-u)\,(\Delta H)_M \,.\, A + (\Delta H)_G \,.\, G \rightleftarrows$$
$$(1-x)\,(\Delta H)_Z \,.\, Z + (\Delta H)_M \,.\, L \qquad (\Delta V)_H \;\; ; \;\; 0.$$

Herein is: $(\Delta V)_H = (\Delta V)_M \,.\, (\Delta H)_Z - (\Delta V)_Z\,(\Delta H)_M$

consequently $\qquad (\Delta V)_H = -(\Delta H)_V$

In order to express in another way the occurring changes in volume and entropy, we represent the volumes and entropies of the unity of quantity of the phases

$Z \;\; Z_n \;\; A \;\; L$ and G

by $\qquad V_Z \;\; V_n \;\; V_A \;\; V_L$ and V_G

and $\qquad H_Z \;\; H_n \;\; H_A \;\; H_L$ and H_G

With the aid of the reactions 1—6 we find:

$$(\Delta V)_M = x\, V_G + (1-x)\, V_Z - V_L$$
$$(\Delta H)_M = x\, H_G + (1-x)\, H_Z - H_L$$
$$(\Delta V)_Z = y\, V_G + u\, V_n + (1-y-u)\, V_A - V_L$$
$$(\Delta H)_Z = y\, H_G + u\, H_n + (1-y-u)\, H_A - H_L$$
$$(\Delta V)_G = x\, V_L + y\,(1-x)\, V_Z - (y+xu)\, V_n - x\,(1-y-u)\, V_A$$
$$(\Delta H)_G = x\, H_L + y\,(1-x)\, H_Z - (y+xu)\, H_n - x\,(1-y-u)\, H_A$$
$$(\Delta V)_n = (y + ux)\, V_G + u\,(1-x)\, V_Z + (1-y-u)\, V_A - V_L$$
$$(\Delta H)_n = (y + ux)\, H_G + u\,(1-x)\, H_Z + (1-y-u)\, H_A - H_L$$
$$(\Delta H)_V = (1-x)\,(\Delta V)_Z\, H_Z + (\Delta V)_M \,.\, H_L - (\Delta V)_n \,.\, H_n$$
$$- (1-y-u)\,(\Delta V)_M \,.\, H_A - (\Delta V)_G\, H_G$$
$$(\Delta V)_H = (1-x)\,(\Delta H)_Z\, V_Z + (\Delta H)_M \,.\, V_L - (\Delta H)_n \,.\, V_n$$
$$- (1-y-u)\,(\Delta H)_M \,.\, V_A - (\Delta H)_G\, V_G \,.$$

Now we have to examine whether those changes in volume and entropy are positive or negative. When we knew the values of $x, y, u, V_Z \ldots, H_Z \ldots,$ then those changes would be easy to calculate. When this is not the case, then we have to try to find in another way they are positive or not.

$(\Delta V)_M$ and $(\Delta H)_M$ are the increase of volume and entropy at the reaction

$$Z_n \rightarrow x\,G + (1-x)\,Z$$

consequently at the separation of the hydrate Z_n into anhydric salt Z and watervapour G. Consequently we are allowed to assume that $(\Delta V)_M$ and $(\Delta H)_M$ are positive.

$(\Delta V)_Z$ and $(\Delta H)_Z$. We write:

$$(\Delta V)_Z = y\,(V_G - V_A) + u\,V_n + (1-u)\,V_A - V_L$$
$$(\Delta H)_Z = y\,(H_G - H_A) + uH_n + (1-u)\,H_A - H_L.$$

Consequently both are positive for values of y which are not too small. For small values of y $(\Delta H)_Z$ becomes negative, for $y=0$ we find viz

$$(\Delta H)_Z = uH_n + (1-u)\,H_A - H_L$$

which is negative, when we assume that heat is wanted for the melting of solid substances.

$(\Delta V)_Z$ can become negative for very small values of y; for this is it necessary that $u\;V_n + (1-u)\;V_A - V_L$ is negative.

$(\Delta V)_G$ and $(\Delta H)_G$. It appears from the value of $(\Delta V)_G$ that this may be as well positive as negative. $(\Delta H)_G$ is the change in entropy at reaction 3, in which only solid substances and the liquid L participate. When we assume that heat is wanted for the formation of liquid, then $(\Delta H)_G$ is positive.

$(\Delta V)_n$ and $(\Delta H)_n$. $(\Delta V)_n$ is always positive on account of the large value of V_G. For $y=0$ becomes:

$$(\Delta V)_n = ux\;V_G + u\,(1-x)\;V_Z + (1-u)\;V_A - V_L\,.$$

When in fig. 1 the point m is not situated in the immediate vicinity of point A, so that u and consequently also ux does not become extremely small, then $(\Delta V)_n$ is still positive, even for $y=0$.

$(\Delta H)_n$ is positive; for small values of y it may, however, become negative, for this it is necessary that

$$ux\,H_G + u\,(1-x)\,H_Z + (1-u)\,H_A - H_L$$

is negative.

$(\Delta V)_H$ and $(\Delta H)_V$. It is apparent from the value mentioned for $(\Delta V)_H$ that this has the same sign as $-(\Delta H)_G$ on account of the large value of V_G. Hence it appears that $(\Delta V)_H < 0$ and $(\Delta H)_V > 0$.

We have seen above that the sign of $(\Delta V)_Z$, $(\Delta H)_Z$ and $(\Delta H)_n$ depends on the value of y, consequently of the position of the point m in fig. 1. In proportion as viz. the point m is situated more closely to the line AZ_n, y becomes smaller; when m is situated on AZ_n then $y = 0$.

I. First we consider the case that the point m is situated not too closely to the line AZ_n. Then we have:

$$(\Delta V)_M \text{ , } (\Delta V)_Z \text{ and } (\Delta V)_n > 0 \text{ ; } (\Delta V)_H < 0 \text{ ; } (\Delta V)_G \gtrless 0$$

$$(\Delta H)_M \text{ , } (\Delta H)_Z \text{ , } (\Delta H)_G \text{ , } (\Delta H)_n \text{ and } (\Delta H)_V > 0.$$

It follows from 5, when we omit the reaction coefficients, for the isovolumetrical reaction that:

$$Z_n + A \pm G \rightleftarrows Z + L \qquad 0 \text{ ; } (\Delta H)_V > 0$$

$$(Z)\,(L) \quad \Big| \quad (Z_n)\,(A)$$

Towards lower T | Towards higher T.

As $(\Delta V)_G$ may be as well positive as negative, we give in this reaction to the phase G as well the sign $+$ as $-$

It follows from this reaction that the curves (Z) and (L) go towards lower temperatures starting from the invariant point m and the curves (Z_n) and (A) towards higher temperatures. As the phase G may have as well the positive as the negative sign, the direction of curve (G) is undefined; it may go, starting from the invariant point as well towards higher as towards lower T.

When we omit the coefficients in the isentropical reaction, then follows from 6:

$$Z_n + A + G \rightleftarrows Z + L \qquad (\Delta V)_H < 0 \text{ ; } 0$$

$$(Z)\,(L) \quad \Big| \quad (Z_n)\,(A)\,(G)$$

Towards lower P | Towards higher P.

Hence it appears that the curves (Z) and (L) go towards lower pressures, starting from the invariant point and the curves (Z_n), (A) and (G) go towards higher pressures.

It is apparent from both these reactions that the curves must be situated as in fig. 2 as regards their direction of pressure and temperature. The curves (Z) and (L) must go viz. starting from m towards lower P and T, the curves (Z_n) and (A) towards higher P and T. Curve (G) must go towards higher P, starting from m, but it may go as well towards lower as towards higher T.

Now we have still to determine the position of the curves with respect to one another. We have viz. still to show that in fig. 2 curve (Z_n) is situated below curve (A) and above the metastable prolongation of curve (Z), etc.

As in the three singular equilibria (M), (A) and (L) the same reaction (1) occurs:

$$\left(\frac{dP}{dT}\right)_M=\left(\frac{dP}{dT}\right)_A=\left(\frac{dP}{dT}\right)_L=\frac{(\Delta H)_M}{(\Delta V)_M}.$$

For the equilibrium (Z_n) we have

$$\left(\frac{dP}{dT}\right)_n=\frac{(\Delta H)}{(\Delta V)_n}$$

Hence it follows:

$$\left(\frac{dP}{dT}\right)_A-\left(\frac{dP}{dT}\right)_n=\frac{(\Delta H)_M}{(\Delta V)_M}-\frac{(\Delta H)_n}{(\Delta V)_n}.$$

As $(\Delta V)_M$ and $(\Delta V)_n$ are positive, the second side has the same sign as

$$(\Delta V)_n\,(\Delta H)_M-(\Delta V)_M\,(\Delta H)_n$$

As, in accordance with (4)

$$(\Delta V)_n=u(\Delta V)_M+(\Delta V)_Z \text{ and } (\Delta H)_n=u(\Delta H)_M+(\Delta H)_Z$$

that form passes into:

$$(\Delta V)_Z\,(\Delta H)_M-(\Delta V)_M\,(\Delta H)_Z=(\Delta H)_V>0.$$

Consequently it is apparent from this:

$$\left(\frac{dP}{dT}\right)_A-\left(\frac{dP}{dT}\right)_n>0 \text{ or } \left(\frac{dP}{dT}\right)_A>\left(\frac{dP}{dT}\right)_n$$

Or curve (A) must be situated in fig. 2 above curve (Z_n).

Now we take:

$$\left(\frac{dP}{dT}\right)_n-\left(\frac{dP}{dT}\right)_Z=\frac{(\Delta H)_n}{(\Delta V)_n}-\frac{(\Delta\ )_Z}{(\Delta V)_Z}.$$

The second part has the same sign as:

$$(\Delta V)_Z\,(\Delta H)_n-(\Delta V)_n\,(\Delta H)_Z\,.$$

When we substitute in this again the values of $(\Delta V)_n$ and $(\Delta H)_n$ from 4, then it passes into $u\,(\Delta H)_V>0$. Hence it follows:

$$\left(\frac{dP}{dT}\right)_n-\left(\frac{dP}{dT}\right)_Z>0 \text{ or } \left(\frac{dP}{dT}\right)_n>\left(\frac{dP}{dT}\right)_Z$$

or curve (Z_n) must be situated in fig. 2 above the prolongation of curve (Z).

II. Now we let the point m in fig. 1 approach more closely to line AZ_n, so that y gets small values. As long as the changes in volume and entropy keep the same signs as in I, we obtain a P,T-diagram as in fig. 2.

For small values of y $(\Delta V)_Z$, $(\Delta H)_Z$ and $(\Delta H)_n$ may change their sign and become, therefore, negative; now we shall consider those cases more in detail.

The first of those three quantities which becomes negative when y diminishes, is

$$(\Delta H)_Z = y\,(H_G - H_A) + u\,H_n + (1-u)\,H_A - H_L.$$

When $(\Delta V)_Z = y\,(V_G - V_A) + u\,V_n + (1-u)\,V_A - V_L$ becomes negative, then this may however, only take place, on account of the large value of V_G, for very small values of y.

$(\Delta H)_n$ can only become negative, when $(\Delta H)_Z$ is negative: this follows from:

$$(\Delta H)_n = u\,(\Delta H)_M + (\Delta H)_Z$$

in which $u\,(\Delta H)_M$ is positive.

Consequently we distinguish four cases.

a. $(\Delta H)_Z < 0$; $(\Delta V)_Z > 0$; $(\Delta H)_n > 0$
b. $(\Delta H)_Z < 0$; $(\Delta V)_Z > 0$; $(\Delta H)_n < 0$
c. $(\Delta H)_Z < 0$; $(\Delta V)_Z < 0$; $(\Delta H)_n > 0$ $(\Delta V)_G > 0$
d. $(\Delta H)_Z < 0$; $(\Delta V)_Z < 0$; $(\Delta H)_n < 0$ $(\Delta V)_G > 0$.

In c and d at the same time $(\Delta V)_G$ is taken > 0; it follows from:

$$(\Delta V)_G = y\,(\Delta V)_M - x\,(\Delta V)_Z$$

that this must be the case.

Hence it appears viz. that for extremely small values of y [and only for those $(\Delta V)_Z$ may become negative] $(\Delta V)_G$ and $(\Delta V)_Z$ have opposite signs.

a. Now we have:

$(\Delta V)_M$, $(\Delta V)_Z$ and $(\Delta V)_n > 0$; $(\Delta V)_H < 0$; $(\Delta V)_G \gtrless 0$
$(\Delta H)_M$, $(\Delta H)_G$, $(\Delta H)_n$ and $(\Delta H)_v > 0$; $(\Delta H)_Z < 0$.

When we omit the coefficients in (5), then the isovolumetrical reaction becomes:

$$Z_n + A \pm G \rightleftarrows Z + L \quad 0 \quad ; \; (\Delta H)_v > 0$$

(Z) (L) | (Z_n) (A)
Towards lower T | Towards higher T

If follows from (6) for the isentropical reaction:

$$Z + Z_n + A + G \rightleftarrows L \qquad (\Delta V)_H < 0 \quad ; \quad 0$$

(L) | (Z) (Z_n) (A) (G)
Towards lower P | Towards higher P

It follows from both these reactions that the curves must be situated as in fig. 3, with respect to their directions of temperature and pressure.

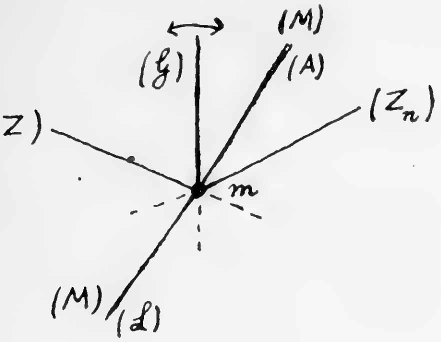

Fig. 3.

In the same way as in I we may show that curve Z_n must be situated below curve (A) and above the metastable part of curve (Z), etc, so that we obtain a partition of curves as in fig. 3.

Fig. 2 and 3 differ from one another only in this respect that curve (Z) goes, starting from m in fig. 2 towards lower and in fig. 3 towards higher pressures.

b. Now we have:

$$(\Delta V)_M,\ (\Delta V)_Z \text{ and } (\Delta V)_n > 0\ ;\ (\Delta V)_H < 0\ ;\ (\Delta V)_G \gtrless 0$$

$$(\Delta H)_M,\ (\Delta H)_G \text{ and } (\Delta H)_V > 0\ ;\ (\Delta H)_n \text{ and } (\Delta H)_Z < 0$$

The isovolumetrical reaction becomes:

$$Z_n + A \pm G \rightleftarrows Z + L \qquad 0 \quad ; \quad (\Delta H)_V > 0$$

(Z) (L) | (Z_n) (A)

Towards lower T | Towards higher T

The isentropical reaction becomes:

$$Z + A + G \rightleftarrows Z_n + L \qquad (\Delta V)_H < 0 \quad ; \quad 0$$

(Z_n) (L) | (Z) (A) (G)

Towards lower P | Towards higher P

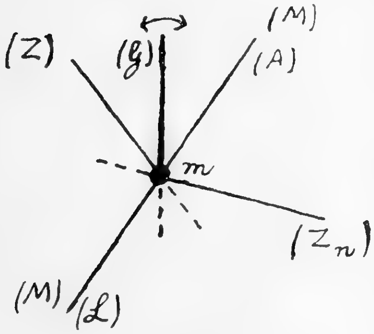

Fig. 4.

From both these reactions it follows that the curves must be situated as in fig. 4 with respect to their directions of temperature and pressure. In the same way as in *I* we may show now again that curve (Z_n) must be situated below curve (A) and above the metastable part of curve (Z), etc. so that we obtain a partition of the curves as in fig. 4.

Figs. 3 and 4 differ from one another only in that respect, that curve (Z_n) goes starting from m in fig. 3 towards higher and in fig. 4 towards lower pressures.

c. Now we have

$$(\Delta V)_M, (\Delta V)_n \text{ and } (\Delta V)_G > 0 \quad ; \quad (\Delta V)_Z \text{ and } (\Delta V)_H < 0$$
$$(\Delta H)_M, (\Delta H)_n, (\Delta H)_G \text{ and } (\Delta H)_V > 0 \quad ; \quad (\Delta H)_Z < 0$$

The isovolumetrical reaction becomes now:

$$Z + Z_n + A + G \rightleftarrows L \quad 0 \quad ; \quad (\Delta H)^V > 0$$

(L) | $(Z)\,(Z_n)\,(A)\,(G)$
Towards lower T | Towards higher T.

The isentropical reaction becomes:

$$Z + Z_n + A + G \rightleftarrows L \qquad (\Delta V)_H < 0 \quad ; \quad 0$$

(L) | $(Z)\,(Z_n)\,(A)\,(G)$
Towards lower P | Towards higher P

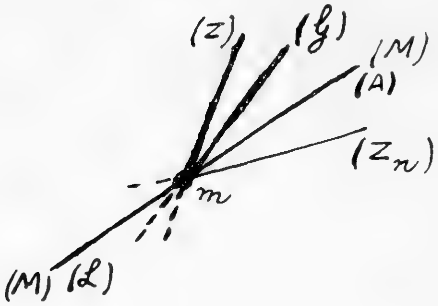

Fig. 5.

From both these reactions it follows that the curves must be situated as in fig. 5 with respect to their direction of temperature and pressure.

Now we have still to show that curve (Z) is situated above (G), curve (G) above (A) and curve (A) above (Z_n); this latter appears again in the same way as in I.

In order to show that curve (Z) is situated above curve (G) we take:

$$\left(\frac{dP}{dT}\right)_Z - \left(\frac{dP}{dT}\right)_G = \left(\frac{(\Delta H)_Z}{(\Delta V)_Z} - \frac{(\Delta H)_G}{(\Delta V)_G}.\right.$$

As $(\Delta V)_Z$ is negative, the second part has the same sign as:

$$(\Delta V)_Z \cdot (\Delta H)_G - (\Delta V)_G \cdot (\Delta H)_Z.$$

When we substitute in this:

$$(\Delta V)_G = y(\Delta V)_M - x(\Delta V)_Z \text{ and } (\Delta H)_G = y(\Delta H)_M - x(\Delta H)_Z$$

then we find:

$$y[(\Delta V)_Z (\Delta H)_M - (\Delta V)^M (\Delta H)_Z] = y(\Delta H)_V > 0$$

Hence it is apparent that in fig. 5 curve (Z) must be situated above curve (G).

In order to show that curve (G) is situated above curve (A) we take:

$$\left(\frac{dP}{dT}\right)_G - \left(\frac{dP}{dT}\right)_A = \frac{(\Delta H)_G}{(\Delta V)_G} - \frac{(\Delta H)_M}{(\Delta V)_M}.$$

In the same way as above we find that the second part must have the same sign as $x(\Delta H)_V$, so that this is positive. In fig. 5 curve (G) must be situated, therefore, above curve (A).

d. Now we have:

$(\Delta V)_M$, $(\Delta V)_n$ and $(\Delta V)_G > 0$; $(\Delta V)_Z$ and $(\Delta V)_H < 0$
$(\Delta H)_M$, $(\Delta H)_G$ and $(\Delta H)_V > 0$; $(\Delta H)_Z$ and $(\Delta H)_n < 0$.

The isovolumetrical reaction becomes:

$$Z + Z_n + A + G \rightleftarrows L \qquad 0 \;;\; (\Delta H)_V < 0$$

(L) $(Z)\,(Z_n)\,(A)\,(G)$

Towards lower T Towards higher T

The isentropical reaction becomes:

$$Z + A + G \rightleftarrows Z_n + L \qquad (\Delta V)_n < 0 \;;\; 0$$

$(Z_n)\,(L)$ | $(Z)\,(A)\,(G)$

Towards lower P | Towards higher P

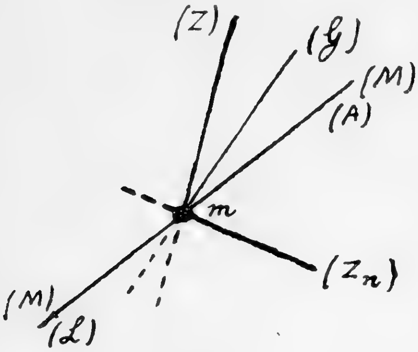

Fig. 6.

From both these reactions it appears that the curves must be situated as in fig. 6 with respect to their direction of temperature and pressure. It is apparent in the same way as in c, that, just as in fig. 5, also in fig. 6 curve (Z) must be situated above (G) and curve (G) above (A).

The only difference between fig. 5 and fig. 6 is this: curve (Z_n) goes, starting from m in fig. 5 towards higher, in fig. 6 towards lower pressures.

When we compare the P,T-diagrams deduced above, with one another, then we see that they belong to a same type, viz. that of fig 4 (XII). This must, of course, be the case, as the phases, G, Z_n, Z, L_m and A are situated with respect to one another in the same way as the five phases in fig. 3 (XII).

In a P,T-diagram we imagine a curve $X + Y + L + G$, to be drawn in which X and Y represent two salts. On this curve is situated a point of maximum-pressure, there may also be situated a point of maximum temperature. We call the part at the left of the point of maximum pressure the ascending branch, the part between the point of maximum pressure and the point of maximum temperature the descending branch and the other part the returning branch.

The difference between the figs 2—6 is dependent on the position of the invariant point m. In fig. 2 this point is situated on the ascending branch of each of the curves (Z) and (Z_n), in fig. 3 on the descending branch of curve (Z) and on the ascending branch of (Z_n), in fig. 4 on the descending branch of each of the curves (Z) and (Z_n) in fig. 5 on the returning branch of curve (Z) and on the ascending branch of (Z_n) and in fig. 6 on the returning branch of curve (Z) and on the descending branch of curve (Z_n).

As we have found now the P,T-diagrams, we may easily deduce the corresponding concentration-diagrams with the aid of those.

I shall not enter into this subject any further and leave this deduction to the reader.

Leiden, Inorg. Chem. Lab. *(To be continued.)*

Chemistry. — *"A New Method for the Passification of Iron."* By Prof. A. SMITS and C. A. LOBRY DE BRUYN. (Communicated by Prof. P. ZEEMAN).

(Communicated in the meeting of Dec. 21, 1916).

I. If iron is immersed in an electrolyte, we have to deal with the following complex equilibria:

$$\begin{array}{ccccc} \mathrm{Fe_S} & \rightleftarrows & \mathrm{Fe_S^{\cdot\cdot}} & + & 2\,\theta_S \\ \downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow \\ \mathrm{Fe_L} & \rightleftarrows & \mathrm{Fe_L^{\cdot\cdot}} & + & 2\,\theta_L \end{array} \quad \ldots\ldots\ldots \quad (1)$$

$$\begin{array}{ccccc} \mathrm{Fe_S} & \rightleftarrows & \mathrm{Fe_S^{\cdot\cdot\cdot}} & + & 3\,\theta_S \\ \downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow \\ \mathrm{Fe_L} & \rightleftarrows & \mathrm{Fe_L^{\cdot\cdot\cdot}} & + & 3\,\theta_L \end{array} \quad \ldots\ldots\ldots \quad (2)$$

from which follows:

$$\begin{array}{ccccc} \mathrm{Fe_S^{\cdot\cdot}} & \rightleftarrows & \mathrm{Fe^{S\cdot\cdot\cdot}} & + & \theta_S \\ \downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow \\ \mathrm{Fe_L^{\cdot\cdot}} & \rightleftarrows & \mathrm{Fe_L^{\cdot\cdot\cdot}} & + & \theta_L \end{array} \quad \ldots\ldots\ldots \quad (3)$$

Those of these equilibria that are indicated by vertical arrows except the equilibrium between the uncharged ironatoms in the solid phase and the electrolyte refer to that part of the heterogeneous equilibrium that governs the potential difference.

Now it has been pointed out before that the iron, which is in internal equilibrium, can be in electromotive equilibrium only with a solution which contains almost exclusively ferro-ions, so that under these circumstances the equilibrium:

$$\mathrm{Fe_L^{\cdot\cdot}} \rightleftarrows \mathrm{Fe_L^{\cdot\cdot\cdot}} + \theta_L$$

in solution lies almost entirely to the left.

If we now add ferri-ions, a consequence of this will be that ferro-ions and electrons from the iron go into solution, which disturbs the equilibrium in the iron surface.

This disturbance can now cease again, as a result of the reaction

$$\mathrm{Fe_S} \rightarrow \mathrm{Fe_S^{\cdot\cdot}} + 2\,\theta_S$$

taking place in the iron surface, and it will therefore entirely depend on the velocity with which this ionisation takes place, whether a disturbance, i. e. in this case an ennobling of the metal surface and attending it a decrease of the negative potential difference, occurs.

Now we had found already two years ago [1]) that it is indeed possible to disturb the internal equilibrium in the iron surface in the way indicated here, and it has further appeared that as was to be expected, the degree of the disturbance depends 1. on the velocity with which the liquid is stirred; 2 on the concentration of the ferri-ions, and 3 on the temperature.

When this had been ascertained, we have made attempts to carry the disturbance through a solution of a ferri-salt so far that the iron became passive. As a ferri-salt we chose ferri-nitrate, because it had appeared to us that the nitrate ion exercises a negative catalytic influence on the setting in of the internal equilibrium.

Experiments made at the ordinary temperature with iron electrodes cemented with sealing wax in a glass tube, at first gave a negative result with a single exception. In the meantime we heard from Messrs. ORNSTEIN and MOLL, who, induced by our research, had also occupied themselves for some time with the passivity of iron, that they had succeeded in making a thin iron wire fused into glass, passive by immersing it in a solution of $Fe(NO_3)_3$.

On continuation of the investigation it now appeared that our negative result up to then was probably owing to our cementing the iron electrode, and that when the cement does not perfectly exclude from all contact the part of the iron covered by this material including all capillary rifts and cracks, a passification probably fails to take place in a solution of $Fe(NO_3)_3$, in consequence of seeding originating from the non-passive parts which are all the same in contact with the solution. By bestowing the greatest care on the cementing we succeeded accordingly in obtaining iron-electrodes which become almost instantaneously passive on immersion in a solution of $Fe(NO_3)_3$ at the ordinary temperature.

This result may also be reached by fusing the iron in, as Messrs. ORNSTEIN and MOLL did. Then use must be made of enamel, because else no perfectly isolating fusing can be brought about on account of the great difference in expansion coefficient between glass and iron.

This method of passifying iron also succeeds with iron electrodes of greater dimensions. Then the iron is suspended on a platinum

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[1]) Versl. Nat. en Geneesk. Congres Amsterdam April 1915. Z. f. phys. chem. **40**, 723 (1915).

wire or, to make the experiment more rigorous, on a hook of glass[1]).

2. When we now apply our new views to what has been discussed here, we immediately see that the disturbance of iron brought about by a solvent, will depend on two circumstances. 1 of the velocity of attack, and 2 of the velocity of the reaction

$$Fe_S \rightarrow Fe_S^{\cdot\cdot} + 2\theta_S.$$

Hence it is clear that the disturbance will increase with the concentration of the $Fe(NO_3)_3$-solution and that when the same $Fe(NO_3)_3$-solution is used, a slighter disturbance will be found when the influence of the temperature on the homogeneous reaction $Fe_S \rightarrow Fe_S^{\cdot\cdot} + 2\theta_S$ is greater, than on the velocity of attack, which is a heterogeneous reaction.

The continued investigation has now proved that this is really the case.

The best way to realize this is by consulting the following tables, which show besides that the temperature at which the passivity occurs is the higher, the greater the concentration of the ferri-nitrate-solution is.

Active iron-electrode put in a solution containing 0.14 grammolecules of $Fe(NO_3)_3$ per litre.

Temp. of the solution	Condition of the iron	
20°	passive	
30°	passive	
35°	active	limit 31°—34°
31°	passive	
34°	active	
30°	passive	

Solution containing 0.11 grammol. $Fe(NO_3)_3$ per litre.

Temp. of the solution	Condition of the iron	
10°	passive	
19°	passive	
23°	active	limit 20.5°—22
20.5°	passive	
22°	active	
20°	passive	

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[1] When we take a platinum wire the experiment is not conclusive because then an element is formed, in which the iron forms the negative electrode, and this is continually polarized.

Solution containing 0.06 grammol. $Fe(NO_3)_3$ per litre.

Temp. of the solution	Condition of the iron	
9°	active	
3°	passive	
5°	passive	
7°	passive	limit 8°—9°
10°	active	
8°	passive	

3. Though the possibility of the here observed phenomenon had been predicted on the ground of our views, more experiments had to be made before we could see a firm confirmation of the said considerations in the results obtained.

It is namely known that $Fe(NO_3)_3$ is partially hydrolytically split up; hence it had to be examined if the disturbing influence could possibly have been exerted by the nitric acid present in the solution used.

To ascertain this the experiment was repeated at room temperature with a solution of nitric acid, which was a slightly stronger acid than the ferri-nitrate solution used. The result was that the iron remained active. To make the result still more pronounced the nitric acid concentration was increased to 32°/₀ by weight of HNO_3, but the result did not change, the iron remained active.

This indubious result showed therefore that the disturbing action of the ferri-nitrate solution with regard to iron is really owing to the ferri-ion, and that the obtained results may be accepted as convincing confirmations of the newer views about the electromotive processes and equilibria between a metal and an electrolyte.

SUMMARY.

Through the above described investigation it has therefore been proved with certainty that, in perfect agreement with the supposition pronounced already before, the disturbance in the iron surface by a solution of ferri-nitrate must be attributed to this that the unary iron can be in electro-motive equilibrium only with a solution which contains only exceedingly few ferri-ions by the side of ferro-ions, so that iron put in a ferri-solution will emit ferro-ions and electrons.

If the internal equilibrium in the metal sets in less rapidly than the metal goes into solution, the internal equilibrium in the iron-surface will be disturbed in the noble direction, i.e. the surface will become richer in ferri-ions, and poorer in electrons.

This case actually presents itself here, and the said disturbance increases with quicker stirring and also through increase of the ferri-ion concentration, and fall of temperature. Owing to the negative catalytic influence of the nitrate ion, the iron could easily be passified in this way.

It could be established with certainty that not the hydrolytically split off nitric acid effects this disturbance, for even a nitric acid solution of $32\,{}^{0}/_{0}$ by weight of NHO_3 was not able to make the iron passive.

Amsterdam, Dec. 16, 1916. *Anorg. Chem. Laboratory of the University.*

Physics. — EINSTEIN's *theory of gravitation and* HERGLOTZ's *mechanics of continua*". By GUNNAR NORDSTRÖM. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of November 25, 1916).

In a way somewhat different from that used by LORENTZ and HILBERT, EINSTEIN has recently deduced his gravitation theory from HAMILTON's principle[1]. In doing this he divides HAMILTON's function $\mathfrak{H}^*$ into two parts:

$$\mathfrak{H}^* = \mathfrak{G}^* + \mathfrak{M} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

in this way that the first part $\mathfrak{G}^*$ depends on the $g^{\mu\nu}$'s only and their first derivatives with respect to the coordinates $g^{\mu\nu}_{\alpha}$ while the second part $\mathfrak{M}$ contains the $g^{\mu\nu}$'s only and besides certain variables $q(\varrho)$, which determine the state, and the derivatives of these $q(\varrho)$'s.

By varying the $g^{\mu\nu}$'s EINSTEIN obtains the equations for the gravitation field in the form

$$\sum_{\alpha} \frac{\partial}{\partial x_{\alpha}}\left(\frac{\partial \mathfrak{G}^*}{\partial g^{\mu\nu}_{\alpha}}\right) - \frac{\partial \mathfrak{G}^*}{\partial g^{\mu\nu}} = \frac{\partial \mathfrak{M}}{\partial g^{\mu\nu}}, \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

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[1] A. EINSTEIN. HAMILTONsches Prinzip und allgemeine Relativitätstheorie, Berliner Sitzungsberichte 1916. p. 1111.

and by varying the $q(\varrho)$'s he finds a system of equations for the field of the matter on which however he does not dwell.

The assumptions EINSTEIN makes on the character of the quantities with respect to transformations are sufficient to determine the function $\mathfrak{G}^*$. As $\frac{\mathfrak{M}}{\sqrt{-g}}$ must be a scalar quantity we find that at transformations both parts of equation (2) behave as "covariant volume-tensors".

The quantity

$$\mathfrak{T}_\sigma^\nu = -\sum_\mu \frac{\partial \mathfrak{M}}{\partial g^{\mu\sigma}} g^{\mu\nu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which occurs in EINSTEIN's equation (21) as momentum and energy is therefore a mixed volume tensor.

In this paper will be shown that the same volume tensor $\mathfrak{T}_\sigma^\nu$ is obtained as tension-energy tensor if HERGLOTZ's mechanics for deformable bodies [1]) are extended in the way required for EINSTEIN's theory [2]).

For incoherent masses the problem has already been solved by Prof. LORENTZ [3]). The problem may however also be solved for arbitrary elastic bodies if HERGLOTZ's formulae for a system of coordinates, in which the $g_{\mu\nu}$'s have their normal values, are transformed to an arbitrary system of coordinates.

HERGLOTZ denotes the cartesian coordinates belonging to the point of matter, when the body was in its normal state, by ξ_1, ξ_2, ξ_3. Here the property of the normal state has to be added that the $g_{\mu\nu}$'s have their normal values $\pm$ 1 and 0. Then an arbitrary system of coordinates may be introduced in which the point of the matter (ξ_1, ξ_2, ξ_3) may have at the time $t = x_4$ the coordinates x_1, x_2, x_3 in space. If still an arbitrary time ξ_4 is introduced

$$\xi_4 = \xi_4 (\xi_1, \xi_2, \xi_3, t),$$

then the four equations

$$x_i = x_i (\xi_1, \xi_2, \xi_3, \xi_4) . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

describe the motion of the body. Further we put, just as HERGLOTZ did

$$a_{ij} = \frac{\partial x_i}{\partial \xi_j} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

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[1]) G. HERGLOTZ. Uber die Mechanik des deformierbaren Körpers vom Standpunkt der Relativitätstheorie. Ann. d. Phys. **36**, 1911, p. 493.

[2]) If there exists an electro-magnetic field, then $\mathfrak{T}$ contains of course still an electro-magnetic term, which must be treated separately.

[3]) H. A. LORENTZ, HAMILTON's principle in EINSTEIN's theory of gravitation. These Proc. XIX, p. 751.

Then the components of the velocity of the matter are:

$$u = \frac{a_{14}}{a_{44}} \quad , \quad v = \frac{a_{24}}{a_{44}} \quad , \quad w = \frac{a_{34}}{a_{44}} . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

If we consider a definite particle of the matter, then it is always possible to introduce a special system of coordinates, in which this particle is at rest, while at that point the $g_{\mu\nu}$'s have their normal values. Quantities determined with respect to this system of coordinates will be indicated by the index 0. Now

$$a^0{}_{14} = a^0{}_{24} = a^0{}_{34} = 0.$$

For a line element we have in general

$$ds^2 = - dx_1{}^{0^2} - dx_2{}^{0^2} - dx_3{}^{0^2} + dx_4{}^{0^2} = \sum_{\mu,\nu} g_{\mu\nu}\, dx_\mu\, dx_\nu . \quad . \qquad (7)$$

and for its projection ds_r on the space perpendicular to the world line

$$ds_r{}^2 = - dx_1{}^{0^2} - dx_2{}^{0^2} - dx_3{}^{0^2} = \sum_{\mu,\nu} \gamma_{\mu\nu}\, dx_\mu\, dx_\nu \ . \quad . \quad . \qquad (8)$$

$\gamma_{\mu\nu}$ being a covariant tensor. In the system of coordinates S^0 $\gamma^0{}_{11} = = \gamma^0{}_{22} = \gamma_{33}{}^0 = -1$, while all other components are zero. Because of these properties we easily find a general expression for $\gamma_{\mu\nu}$, if a certain tensor is subtracted from $g_{\mu\nu}$. In order to find this tensor we first form the contravariant tensor $a_{\sigma 4} a_{\tau 4}$ from the velocity vector $a_{\sigma 4}$ and then divide it by the scalar quantity $\sum\limits_{\alpha,\beta} g_{\alpha\beta}\, a_{\alpha 4}\, a_{\beta 4}$. Of the tensor

$$\frac{a_{\sigma 4}\, a_{\tau 4}}{\sum\limits_{\alpha,\beta} g_{\alpha\beta} a_{\alpha 4} a_{\beta 4}} = \frac{dx_\sigma}{ds} \frac{dx_\tau}{ds}$$

which we obtain in this way, only the component $(_{44})$ differs from zero; it is equal to 1 so that it can neutralize g^0_{44}. In order to render the tensor covariant it must still twice be multiplied by the covariant fundamental tensor. We then find for $\gamma_{\mu\nu}$:

$$\gamma_{\mu\nu} = g_{\mu\nu} - \sum_{\sigma,\tau} g_{\mu\sigma}\, g_{\nu\tau} \frac{a_{\sigma 4} a_{\tau 4}}{\sum\limits_{\alpha,\beta} g_{\alpha\beta} a_{\alpha 4} a_{\beta 4}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

For the scalar quantity in the denominator

$$\sum_{\alpha,\beta} g_{\alpha\beta}\, a_{\alpha 4}\, a_{\beta 4} = a^{0^2}_{44} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

Now HERGLOTZ's considerations from § 5 of his article may easily be generalized as much as is necessary for our case. HERGLOTZ supposed a kinetic potential $\iiiint \Phi d\xi_1 d\xi_2 d\xi_3 d\xi_4$ to exist. Let us do this too. First for the case of rest and no gravitation this potential must take the form $\iiiint \Omega d\xi_1 d\xi_2 d\xi_3 dt$ of the ordinary kinetic potential dependent on quantities that characterize the deformation and on the entropy. In the system of coordinates S^0 the equation

$$\Phi = \Omega(\varepsilon, e_{ij})\, a^c_{44} \quad \ldots\ldots\ldots \quad (11)$$

must therefore hold, if ε is the entropy per unit of the normal volume and if at rest the six deformations e_{ij} are determined by the following equations (equations (16) of HERGLOTZ):

$$\left.\begin{array}{c} 1 + 2e_1 = {a^0}^2_{11} + {a^0}^2_{21} + {a^0}^2_{31}, \\ 2e_{23} = 2e_{32} = a^0_{12}a^0_{13} + a^0_{22}a^0_{23} + a^0_{32}a^0_{33} \\ etc. \end{array}\right\} \quad \ldots \quad (12)$$

The quantities e_{ij} show how the form of an element of the body, when at rest, differs from the normal form.

Secondly Φ must be invariant with respect to arbitrary transformations. Thus the general expression for Φ is obtained by transforming equation (11) from S^0 to an arbitrary system.

At this transformation the connexion

$$dx^0_i = \sum_k \frac{\partial x^0_i}{\partial x_k} dx_k$$

must be used. We then obtain from (7) and (8).

$$g_{kl} = -\sum_{i=1}^{i=3} \frac{\partial x^0_i}{\partial x_k}\frac{\partial x^0_i}{\partial x_l} + \frac{\partial x^0_4}{\partial x_k}\frac{\partial x^0_4}{\partial x_l}, \quad \ldots\ldots \quad (13)$$

$$\gamma_{kl} = -\sum_{i=1}^{i=3} \frac{\partial x^0_i}{\partial x_k}\frac{\partial x^0_i}{\partial x_l} \quad \ldots\ldots\ldots \quad (14)$$

Further (5) gives

$$a^0_{ij} = \sum_k \frac{\partial x^0_i}{\partial x_k} a_{kj}, \quad \ldots\ldots\ldots \quad (15)$$

while for the expressions (12) may be written

$$1 + 2e_{11} = \sum_{k,l} a_{k1} a_{l1} \left\{ \frac{\partial x^0_1}{\partial x_k}\frac{\partial x^0_1}{\partial x_l} + \frac{\partial x^0_2}{\partial x_k}\frac{\partial x^0_2}{\partial x_l} + \frac{\partial x^0_3}{\partial x_k}\frac{\partial x^0_3}{\partial x_l} \right\} etc.$$

so that according to (14)

$$\left.\begin{array}{r} 1 + 2e_{11} = -\sum_{k,l} a_{k1}\, a_{l1}\, \gamma_{kl}, \\ 2e_{23} = -\sum_{k,l} a_{k2}\, a_{l3}\, \gamma_{kl}, \\ etc. \end{array}\right\} \quad \ldots\ldots\ldots \quad (16)$$

If the expression (9) for $\gamma_{\mu\nu}$ is introduced and if we put

$$A_{kl} = -\sum_{\mu,\nu} g_{\mu\nu}\, a_{\mu k}\, a_{\nu l} = -\tfrac{1}{2} \sum_{\mu,\nu} g_{\mu\nu} \left(a_{\mu k}\, a_{\nu l} + a_{\mu l}\, a_{\nu k}\right), \quad . \quad (17)$$

then the expressions for the deformations at rest are finally obtained in the form:

$$\left.\begin{aligned} 1 + 2e_{11} &= A_{11} - \frac{A_{14}^{\;2}}{A_{44}}, \\ 2e_{23} &= A_{23} - \frac{A_{24}A_{34}}{A_{44}} \\ &\textit{etc.} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (18)$$

These expressions look just like the corresponding equations (16′) of Herglotz. Here however the quantities A_{kl} have a wider meaning.

By introducing into (11) the expressions (18) for the deformations at rest and by taking into consideration that:

$$a_{44}^{\;0} = \sqrt{-A_{44}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (19)$$

we obtain for Φ a function of ε and the A_{kl}'s only. We may however also consider Φ as a function of the quantities ε, a_{ij}, $g_{\mu\nu}$.

$$\Phi = \Phi_1(\varepsilon, A_{kl}) = \Phi(\varepsilon, a_{ij}, g_{\mu\nu}) \quad . \; . \; . \; . \; . \; . \quad (20)$$

The a_{ij}'s and the $g_{\mu\nu}$'s occurring in the expressions A_{kl} only, there exists the expression

$$\sum_n a_{jn} \frac{\partial \Phi}{\partial a_{in}} = \sum_n g_{in} \left\{ \frac{\partial \Phi}{\partial g_{jn}} + \frac{\partial \Phi}{\partial g_{nj}} \right\} = 2 \sum_n g_{in} \frac{\partial \Phi}{\partial g_{jn}} \quad . \; . \quad (21)$$

which is of much interest for the following investigation. This equation is easily proved by taking on both sides the differential quotients of Φ first with respect to the A_{kl}'s and only then with respect to the a_{in}'s or g_{jn}'s. At this last differentiation g_{jn} and g_{nj} are not identical, if $j \neq n$, and in order that $\frac{\partial \Phi}{\partial g_{jn}}$ may be equal to $\frac{\partial \Phi}{\partial g_{nj}}$ the last expression (17) for A_{kl} has to be used. This gives:

$$\sum_n a_{jn} \frac{\partial A_{kl}}{\partial a_{in}} = -a_{jk} \sum_\nu g_{i\nu}\, a_{\nu l} - a_{jl} \sum_\mu g_{\mu i}\, a_{\mu k} = 2 \sum_n g_{in} \frac{\partial A_{kl}}{\partial g_{jn}},$$

and by means of this equation (21) is easily proved.

We must remark that the right-hand side of (21) represents the $\binom{j}{i}$ component of a mixed tensor. By dividing the equation by the determinant

$$D = |\, a_{ij} \,|, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (22)$$

volume tensors are obtained on both sides, for

$$D^0 = \sqrt{-g}\, D$$

is a scalar. According to equation (68) of HERGLOTZ the left-hand side of (21) divided by D represents the components of the tension-energy tensor for matter. If this formula of HERGLOTZ holds in the general theory of relativity too, the components of the tension-energy tensor of matter are[1]:

$$-\mathfrak{T}_i^j = \frac{1}{D}\sum_n a_{jn}\frac{\partial \Phi}{\partial a_{in}} = \frac{2}{D}\sum g_{in}\frac{\partial \Phi}{\partial g_{jn}} \quad . \quad . \quad . \quad . \qquad (23)$$

At all events this formula is valid in the system of coordinates S^0 and from the general covariancy of both expressions for $\mathfrak{T}_i^j$ can be concluded, that it holds for an arbitrary system of coordinates. In the following this will be proved directly by means of HAMILTON's principle.

Every function φ of the $g_{\mu\nu}$'s may also be regarded as a function of the contravariant $g^{\mu\nu}$'s and in general we have:

$$\sum_\tau g_{\mu\tau}\frac{\partial \varphi}{\partial g_{\nu\tau}} = -\sum_\tau g^{\nu\tau}\frac{\partial \varphi}{\partial g^{\mu\tau}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

In analogy with the assumption (69) of HERGLOTZ we put

$$\mathfrak{F} = -\frac{\Phi}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

($\mathfrak{F}$ = volume scalar). As D depends on the a_{ij}'s only and not on the $g^{\mu\nu}$'s, the second expression (23) for $\mathfrak{T}_i^j$ may then be written in the following way:

$$\mathfrak{T}_i^j = \frac{1}{D}\sum_n a_{jn}\frac{\partial \Phi}{\partial a_{in}} = 2\sum_n g^{jn}\frac{\partial \mathfrak{F}}{\partial g^{in}} \quad . \quad . \quad . \quad . \qquad (23a)$$

Comparing this equation with (3) we see that they are identical if

$$2\,\mathfrak{F} = \mathfrak{M} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

Thus we have found the connexion between EINSTEIN's theory of gravitation and HERGLOTZ's mechanics. Now we have still to deduct the first expression (23) for $\mathfrak{T}_i^j$ from HAMILTON's principle. This may be done in a way analogous to that used by prof. LORENTZ for the case of incoherent masses. The integral

$$\iiiint \mathfrak{F}\, dx_1\, dx_2\, dx_3\, dx_4$$

must then be thus varied that the world-points x_1, x_2, x_3, x_4 of the matter are shifted by the increments $\delta x_1, \delta x_2, \delta x_3, \delta x_4$ of their coordinates, these increments being zero at the boundary. As functions

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[1] The negative sign has to be added, in order that the tensions are taken with the same sign as in EINSTEIN's calculation.

of x_1, x_2, x_3, x_4 the $g_{\mu\nu}$'s must keep their values [1]). For the sake of simplicity changes of the entropy will not be considered, so that we have to do with adiabatic changes of the body only. HAMILTON's function $\mathfrak{H}$ must be regarded as a function of the a_{ij}'s and of the contravariant $g^{\mu\nu}$'s

$$\mathfrak{H} = \mathfrak{H}\,(a_{ij}\,,\, g^{\mu\nu}).$$

If the variation of $\mathfrak{H}$ in a fixed point (x_1, x_2, x_3, x_4) of the four-dimensional world is denoted by $\delta\,\mathfrak{H}$, and the variation in a point $(\xi_1, \xi_2, \xi_3, \xi_4)$ of the matter by $\Delta\,\mathfrak{H}$, then we have

$$\delta\mathfrak{H} = \Delta\mathfrak{H} - \sum_i \frac{\partial\mathfrak{H}}{\partial x_i}\,\delta x_i\ ,$$

$$\Delta\mathfrak{H} = \sum_{i,n} \frac{\partial\mathfrak{H}}{\partial a_{in}}\,\frac{\partial\delta x_i}{\partial \xi_n} + \sum_{\mu\nu} \frac{\partial\mathfrak{H}}{\partial g^{\mu\nu}} \sum_i \frac{\partial g^{\mu\nu}}{\partial x_i}\,\delta x_i\ .$$

Now the variation principle gives

$$0 = \delta \iiiint \mathfrak{H}\, dx_1\, dx_2\, dx_3\, dx_4 =$$

$$\iiiint \left\{ \sum_{i,n} \frac{\partial\mathfrak{H}}{\partial a_{in}} \sum_j \frac{\partial\delta x_i}{\partial x_j}\, a_{jn} + \sum_{\mu,\nu} \frac{\partial\mathfrak{H}}{\partial g^{\mu\nu}} \sum_i \frac{\partial g^{\mu\nu}}{\partial x_i}\, \delta x_i - \sum_i \frac{\partial\mathfrak{H}}{\partial x_i}\ \delta x_i \right\} dx_1 dx_2 dx_3 dx_4 .$$

The δx_i's being independent of each other, we obtain by partial integration of the first term in brackets

$$-\sum_j \frac{\partial}{\partial x_j} \sum_n a_{jn} \frac{\partial}{\partial a_{in}} - \frac{\partial\mathfrak{H}}{\partial x_i} + \sum_{\mu,\nu} \frac{\partial\mathfrak{H}}{\partial g^{\mu\nu}}\,\frac{\partial g^{\mu\nu}}{\partial x_i} = 0\ . \quad . \quad . \quad (27)$$

If we put

$$\mathfrak{T}^j_i = \sum_n a_{jn} \frac{\partial\mathfrak{H}}{\partial a_{in}} + \delta^j_i\ \mathfrak{H}\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where δ^j_i is 1 or 0 according as i and j are equal or not, it is easily proved that this expression (28) for $\mathfrak{T}^j_i$ is identical with the first expression (23), if $\mathfrak{H}$ and Φ are connected by equation (25).

If we put still

$$\mathfrak{T}_{\mu\nu} = \sum_\sigma g_{\nu\sigma}\, \mathfrak{T}^\sigma_\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

(comp. EINSTEIN p. 1116) then we have according to (23*a*)

$$\mathfrak{T}_{\mu\nu} = -\,2\,\frac{\partial\mathfrak{H}}{\partial g^{\mu\nu}}\ ^{2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

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[1] EINSTEIN proceeds in the opposite way. He varies the $g_{\mu\nu}$'s, while the coordinates of the matter are kept constant.

[2] According to (26) and (30) the right hand sid of (2) is therefore equal to $-\,\mathfrak{T}_{\mu\nu}$.

and equation (27) becomes finally

$$\sum_j \frac{\partial \mathfrak{T}_i^j}{\partial x_j} + \tfrac{1}{2} \sum_{\mu,\nu} \mathfrak{T}_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial x_i} = 0 \quad . \; . \; . \; . \; . \; . \; . \qquad (31)$$

This equation is identical with EINSTEIN's equation (22), which he obtained by variation of the $g^{\mu\nu}$'s. The variation performed in this paper does therefore not give new equations for the field. It shows however clearly that the expressions (23) for $\mathfrak{T}_i^j$ also hold for the general theory of relativity.

Finally the formulae may still be specialized for an adiabatic fluid. For such a fluid (comp. HERGLOTZ § 10) Ω is only a function of the quantity

$$\Delta = \frac{\sqrt{-g}D}{a^0_{44}} = \frac{\sqrt{-g}D}{\sqrt{\sum\limits_{\alpha,\beta} g_{\alpha\beta} a_{\alpha 4} a_{\beta 4}}}, \quad . \; . \; . \; . \; . \; . \qquad (32)$$

which gives the relation between the volume when in rest and the normal volume. We thus have according to (11)

$$\Phi = \Omega \left(\frac{\sqrt{-g}D}{\sqrt{\sum\limits_{\alpha,\beta} g_{\alpha\beta} a_{\alpha 4} a_{\beta 4}}} \right) \sqrt{\sum_{\alpha,\beta} g_{\alpha\beta} a_{\alpha 4} a_{\beta 4}} . \quad . \; . \; . \; . \qquad (33)$$

The tension-energy tensor may be calculated most simply in its contravariant form. By doing this we obtain.

$$\mathfrak{T}^{ij} = -\frac{2}{D} \frac{\partial \Phi}{\partial g_{ij}} = -p\, g^{ij} \sqrt{-g} + (p\sqrt{} + \mathfrak{F}) \frac{a_{i4}\, a_{j4}}{\sum\limits_{\alpha,\beta} g_{\alpha\beta} a_{\alpha 4} a_{\beta 4}}, \qquad (34)$$

where, as in HERGLOTZ's article,

$$p = \frac{\partial \Omega}{\partial \Delta} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (35)$$

denotes the scalar pressure. By performing once respectively twice a mixed multiplication with the covariant fundamental tensor we obtain the expressions for ${}_i^j$ and $\mathfrak{T}_{ij}$.

The quantity $p\sqrt{-g} + \mathfrak{F}$ may be regarded as the density of the mass when in rest. If therefore, we assume $\sqrt{-g} = 1$, the obtained expressions agree with those given by EINSTEIN [1]).

Leiden, Nov. 24 1916.

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[1]) A. EINSTEIN, Die Grundlage der allgemeinen Relativitätstheorie, § 19, Ann. d. Phys. **49**, 1916, p. 769.

Physics. — "*The equations of the theory of electrons in a gravitation field of* EINSTEIN *deduced from a variation principle. The principal function of the motion of the electrons.*" By J. TRESLING. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of November 25, 1916)

HILBERT [1]) derived the fundamental equation of the electro-magnetic field in the empty space from a variation principle, in which the principal function is considered as dependent on a four-dimensional vector potential and its differential quotients with respect to the four coordinates.

In this paper it will be shown that this method may be extended to a space in which electrons are moving. For this purpose a new term must be added to the principal function as it was used by HILBERT, by which term the influence of those electrons is represented. Only then the connexion between vector potential and the intensities of the field can be indicated, as the equations give us the influence of the potential on the charges, while it is just in this influence that the intensities of the field are expressed.

In another way Prof. LORENTZ [2]) had already deduced the fundamental equations from a variation principle. In my opinion however the generalization of HILBERT's method solves the problem in the finest possible form. Nearly the whole following way of calculation is taken from the mentioned paper of LORENTZ.

The components of the vector potential are indicated by q_s, a four-dimensional element of volume $dx_1\,dx_2\,dx_3\,dt$ by dS. We shall also write dx_4 for dt. Let the density of the electric charge be ϱ. Then the charge of a volume dV is $\varrho dV = e$. We put $\varrho \frac{dx_i}{dt} = w_i$ (i from 1 to 4).

These quantities w^i are no vector-components. They become so however when they are divided by $\sqrt{g}$.

For $\varrho \dfrac{dx_i}{dt} = \varrho \dfrac{dx_i dV\sqrt{g}}{dt\,dV\sqrt{g}} = \dfrac{e\,dx_i}{dV_0 ds}\sqrt{g} = \varrho_0 \dfrac{dx_i}{ds}\sqrt{g}$ when dV_0 and ϱ_0 represent the volume resp. the density of e when at rest.

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[1]) DAVID HILBERT. Die Grundlagen der Physik (Erste Mitteilung) Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen 1915, S. 395.

[2]) H. A. LORENTZ. HAMILTON's principle in EINSTEIN's theory of gravitation. These Proc. XIX p. 751.

The other quantities with indices will always be vector components and the letters will have the same meaning that is generally ascribed to them in the theory of gravitation.

With an expression as $q_a w^a$ in which some indices occur twice we denote the sum of a number of such terms, which are obtained by giving to those indices all values from 1 to 4.

In a four-dimensional extension S we consider an electro-magnetic field and moving electrons. Let the state be characterized by the values of the potential q_c and of the intensity of the current w^i and let a force K_i act on the charges per unit of volume. Then the principle may be formulated as follows: The equations which exist between these quantities are such that a virtual displacement of the charges and a virtual change of the potential cause a variation of certain two integrals over the volume S, which is equal to the negative work of the external forces during that change. The gravitation potentials are kept constant.

In a formula this may be expressed as follows

$$\delta\int L_1 \sqrt{g}\, dS + \delta\int L_2 \sqrt{g}\, dS + \int K_a\, \delta x_a \sqrt{g}\, dS = 0 \quad . \quad . \quad . \quad I$$

Here HILBERT's invariant occurs

$$L_1 = \tfrac{1}{4}\,(q_{mn} - q_{nm})\,(q_{\mu\nu} - q_{\nu\mu})\, g^{m\nu}\, g^{n\mu} \quad \text{where} \quad q_{mn} = \frac{\partial q_m}{\partial x_n}$$

and a new one

$$L_2 = \frac{w^a q_a}{\sqrt{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad II$$

Let the potentials be changed by the amounts δq_s and suppose the intensity of the current to be varied by a displacement of the charges over a distance δx_c. Then LORENTZ gives for the changes of the current

$$\delta w^a = \frac{\delta \chi_{ab}}{\delta x_b} \quad \text{where} \quad \chi_{ab} = w^b\, \delta x_a - w^a \delta x_b$$

This is easily found by considering the changes in density and velocity which are caused by the displacements δx_c.

We shall write

$$\delta L_1 \sqrt{g} = -\frac{\partial}{\partial x_k}\left(\frac{\partial L_1 \sqrt{g}}{\partial q_{ik}}\right) . \ \delta q_i + \frac{\partial}{\partial x_k}\left(\frac{\partial L_1 \sqrt{g}}{\partial q_{ik}}\, \delta q_i\right) \quad . \quad . \quad . \quad (1)$$

$$\delta L_2 \sqrt{g} = w^i\, \delta q_i - \chi_{ib}\, q_{ib} + \frac{\partial}{\partial x_b}\,(\chi_{ib}\, q_i) \quad . \quad . \quad . \quad . \quad (2)$$

At the limits the potentials are not changed and the charges are not displaced; so that in the integration over S the last term of both equations vanishes. Therefore it may omitted, and we obtain

$$\delta L_1 \sqrt{g} = -\frac{\partial}{\partial x_k}\left(\frac{\partial L_1 \sqrt{g}}{\partial q_{ik}}\right) \cdot \delta q_i$$

$$\delta L_2 \sqrt{g} = w_i \delta q^i + (q_{bi} - q_{ib})\, w^q \delta x_i$$

For $\frac{\partial L_1 \sqrt{g}}{\partial q_{ik}}$ is found $\sqrt{g}\,(q_{mn} - q_{nm})\, g^{mk}\, g^{ni}$

If we put

$$q_{ab} - q_{ba} = \overline{\psi}_{ab} \quad \ldots\ldots\ldots \quad (A)$$

$$g^{a\mu}\, g^{b\nu}\, \overline{\psi}_{ab} = \psi^{\mu\nu}$$

we have

$$L_1 = -\tfrac{1}{4}\, \psi^{mn}\, \overline{\psi}_{mn} \quad \ldots\ldots\ldots \quad III$$

$$\frac{\partial L_1 \sqrt{g}}{\partial q_{ik}} = \sqrt{g}\, \psi^{ki}$$

And the variation principle gives

$$K_i = -\frac{\overline{\psi}_{bi}\, w^b}{\sqrt{g}} \quad \ldots\ldots\ldots \quad (B)$$

$$w_i = \frac{\partial \sqrt{g}\, \psi^{ki}}{\partial x_k} \quad \ldots\ldots\ldots \quad (C)$$

From ψ_{ab} a new tensor can be derived:

$$\psi^{*\mu\nu} = \frac{\overline{\psi}_{\mu'\nu'}}{\sqrt{g}} \; \textit{when } \mu \neq \nu \qquad \psi^{*\mu\mu} = 0$$

The μ', ν' represent just the two other indices than μ, ν, in this way, that μ, ν, μ', ν' may be brought into the order 1, 2, 3, 4 by an even number of permutations.

According to (A) we have

$$\frac{\partial \sqrt{g}\, \psi^{*ab}}{\partial x_b} = 0 \quad \ldots\ldots\ldots \quad (D)$$

In order to find the tension-energy tensor we shall proceed as Prof. LORENTZ does. We calculate the left side of I for the case that the whole electro-magnetic field and the charges are displaced over a constant distance δx_c in the direction x_c. By using (B) and (C) we find from (1) and (2)

$$\left[\frac{\partial}{\partial x_b}\left(-\sqrt{g}\, \psi^{bi} q_{ic} + w^b q_c\right) - \frac{\partial}{\partial x_c}\left(w^i q_i\right)\right] \delta x_c \quad \ldots \quad (3)$$

The term $\frac{\partial}{\partial x_b}(w^b q_c)$ may be transformed into $q_c \frac{\partial w^b}{\partial x_b} + w^i q_{ci}$, the first part of which vanishes because of the undestructibleness of the charge

$$w^i q_{ci} = \frac{\partial \sqrt{g}\psi^{bi}}{\partial x_b} q_{ci} \text{ from } (C)$$

$$= \frac{\partial \sqrt{g}\psi^{bi} q_{ci}}{\partial x_b} - \sqrt{g}\psi^{bi} \frac{\partial q_{ci}}{\partial x_b}$$

Here the last term is equal to 0.
We thus substitute in (3)

$$\frac{\partial}{\partial x_b}(w^b q_c) = \frac{\partial \sqrt{g}\psi^{bi} q_{ci}}{\partial x^b}$$

Then (3) becomes

$$\left[\frac{\partial}{\partial x_b}(\sqrt{g}\,\overline{\psi}_{ci}\,\psi_{bi}) - \frac{\partial}{\partial x_c}(w^i q_i)\right] \delta x_c$$

this must be equal to

$$\left[K_c \sqrt{g} - \left[\frac{\partial L_1 \sqrt{g}}{\partial x_c} - \left(\frac{\partial L_1 \sqrt{g}}{\partial x_c}\right)_{w,q} + \frac{\partial L_2 \sqrt{g}}{\partial x_c} - \left(\frac{\partial L_2 \sqrt{g}}{\partial x_c}\right)_{w,q}\right]\right] \delta x_c$$

Here $\frac{\partial L_i \sqrt{g}}{\partial x_c}$ means the increase of the quantity $L_i \sqrt{g}$ in the case of a displacement in the direction x_c. Our variation is however such that after it only the q_s and w^s have a value as before at a point at a distance $-\partial x_c$ further in the direction of x_c. That is why the terms $\left(\frac{\partial L_i \sqrt{g}}{\partial x_c}\right)_{w,q}$ occur, which represent the increase of $L_i \sqrt{g}$ only caused by the variation of the g_{mn} while w^i and q are kept constant.

Taking into consideration (II) and (III), and equalizing the two last expressions, we obtain

$$K_c \sqrt{g} + \left(\frac{\partial L_1 \sqrt{g}}{\partial x_c}\right)_{wq} = \frac{\partial}{\partial x_b}\left[\sqrt{g}(\overline{\psi}^{bi}\,\overline{\psi}_{ci} - \tfrac{1}{4}\,\delta^b_c\,\psi^{mn}\,\psi_{mn})\right]$$

Here $\delta^b_c = 1$ or 0 according as $b = c$ or $b \neq c$.
Thus we have as energy tensor

$$T^b_c = \sqrt{g}\,(\psi^{bi}\,\overline{\psi}_{ci} - \tfrac{1}{4}\,\delta^b_c\,\psi^{mn}\,\overline{\psi}_{mn}) \quad . \quad . \quad . \quad . \quad . \quad (E)$$

Neglecting the gravitation we may use a system of coordinates where the $g_{\mu\nu}$ have their normal values and the intensities of the field may be determined by means of the equations:

$$K_1 = -\rho\,(\mathrm{d}_x + \mathrm{v}_y\,\mathrm{h}_z - \mathrm{v}_z\,\mathrm{h}_y) \text{ etc.}$$

By means of the scheme

$$\overline{\psi}_{mn} = \begin{vmatrix} 0 & -h_z & h_y & -d_x \\ h_z & 0 & -h_x & -d_y \\ -h_y & h_x & 0 & -d_z \\ d_x & d_y & d_z & 0 \end{vmatrix}$$

(B) is brought into the form $K = -\varrho\,(d + [v\,h])$

(C) becomes $curl\ h = \varrho \frac{dx}{dt} + \frac{\partial d}{\partial t}$

$$div\ d = \varrho$$

$$(D) \qquad curl\ d = -\frac{\partial h}{\partial t}$$

$$div\ h = 0$$

(E) gives as tension components the tensions of MAXWELL, as energy current POYNTING's Vector and also the right density of energy.

If the three-dimensional vector, the components of which are q_1, q_2 and q_3 is denoted by A and if we put

$$q_4 = \varphi$$

then equation (A) may be written in the form

$$h = curl\ A$$

$$d = grad\ \varphi - \frac{\partial A}{\partial t}$$

Using the notation of LAUE [1]) and writing $\psi^{mn} = -M_{mn}$ we obtain the equations

$$Div\ M = w \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (C')$$

$$Div\ M^* = o \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (D')$$

$$T_c^b = [\,[M,M]\,] \quad . \ . \ . \ . \ . \ . \ . \quad (E')$$

$$K = Div\ T_c^b \ . \ . \ . \ . \ . \ . \ . \ . \quad (B')$$

$$M = Rot\ q \quad . \ . \ . \ . \ . \ . \ . \ . \quad (A')$$

Physics. — "*Electric current measuring instruments with parabolic law of deflection*". By Prof. J. K. A. WERTHEIM SALOMONSON.

(Communicated in the meeting of November 25, 1916).

In current or voltage measuring instruments the deflection is either about proportional to the current strength passing through the instrument or it grows with the square of the current strength. In this way they constitute two different classes of instruments. The ordinary galvanometer with a movable coil or magnet belongs

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[1]) M. LAUE. Das Relativitätsprinzip.

to the first group the deflection in which obeys a linear law. The electrodynamometer is a prototype of the second class, in which a quadratic law prevails.

If the movable part of an instrument of the first group be mechanically coupled to the movable part of an instrument of the second type, the rotating axes of the two instruments being made to coincide, and we send the same current through both instruments, taking care that the constituents of the movable part try to move it in an opposite direction, we have got a new instrument type showing a few peculiarities which are not to be found in either of the constituent parts.

If I be the current strength, the deflection φ, the constant of the galvanometric part a, the dynamometric constant b, we can put

$$I = a\varphi \quad \text{and} \quad I = b\sqrt{\varphi}$$

if the constituent parts are separately considered.

For the two parts used together we find:

$$\varphi = \frac{I}{a} - \frac{I^2}{b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the equation for a parabola. Therefore we may call instruments formed by the combination of an instrument with a linear law of deflection with one obeying a quadratic law, instruments with a parabolic law of deflection.

From the equation 1) giving the deflection in terms of the current strength we immediately find, that there will be no deflection either with a current $I = 0$ or when

$$I = \frac{b^2}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we call positive the direction of the movement of the movable system when the dynamometric part has been short-circuited we find that a maximal positive deflection occurs when

$$I = \frac{b^2}{2a}$$

the deflection being

$$\varphi = \frac{b^2}{4a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Any current between $I = 0$ and $I = \frac{b^2}{a}$ gives a positive deflection; currents larger than $I = \frac{b^2}{a}$ cause negative deflections. We also get a negative deflection with currents smaller than $I = 0$ i. e. after commutation of the normal direction of the current.

For very small currents we have a sensibility $= a$, which means that a current $I = a$ gives a deflection of one division of the scale. Near the second position of equilibrium a current increment of $-a$ or a current decrement of $+a$ augments a positive or decreases a negative deflection by 1 part of the scale. The absolute sensibility of the instrument is the same either in the first or in the second zero position.

The relative sensibility i. e. the ratio of the current increment causing an increase of one part of the scale into the current strength approaches to 0 in the first zero position; in the second zero position it is $\frac{b^2}{2a^2}$ which will generally be a very large figure. Instead of in this way we preferably calculate the relative sensibility for the current causing the largest deflection at which the linear law still holds good. Expressed in this way we can say that the relative sensibility will generally be of the order of 200—500. But in the second zero position we can easily obtain a relative sensibility of 100000 or more. Even with a moderate ratio between a and b^2 we obtain very high figures for the sensibility near the second zero position.

As a matter of course we can only use very small parts of the parabola in most cases. We need scarcely point to the fact that formula (1) is only available for deflections for which both the linear law for the galvanometer and the square law for the dynamometer hold good.

In instruments of this kind the thing we aim at is a high relative sensibility near the second zero position. As an example we might take an instrument giving a full scale deflection at 10 milliamperes, and which has its second zero position at say 1 ampere; in this case a change in the current strength of only one percent would cause the spot to leave the scale.

Many years ago I had an instrument of this kind made for me in a rather simple form. I used it as an indicator that the current in a potentiometer remained constant during the measurements so as not to be obliged to correct the setting of the potentiometer by means of the Weston-element. For this purpose the parabolic indicator proved to be very useful. We can easily give such an instrument any relative or absolute sensibility. With a shunt to the galvanometric part we diminish the relative sensibility; with a shunt to the dynamometrical part we can increase it. A shunt to the complete instrument increases the current strength for the second zero-position.

One might expect a parabolic instrument to make an excellent standard- or normal instrument. But there are a few objections to using it for this purpose. If the dynamometrical part be constructed with two astatic movable coils we may consider this part as electrically constant and invariable. But we cannot succeed in rendering the galvanometer part electrically and mechanically constant and invariable. If this part be constructed with a movable coil and a permanent field magnet, this last cannot be considered as electrically constant for more than a few hours. With a movable magnet this forms one inconstant factor, the magnetism of the earth being another. The horizontal intensity of the earth-magnetism changes every year more than 0.01 to 0.03 percent and restricts the utility as a standard instrument to that value. But the variation in the magnetism of a permanent field magnet or a permanent movable magnet is probably many times greater. Consequently the instrument can only be expected to be of practical use during short consecutive periods in which no appreciable changes in the earth magnetism occur and when no stray fields are present. The influence of stray fields might perhaps be counteracted by judicially enclosing the whole instrument in a seamless cylindrical soft iron covering.

We have still to consider another possible use of instruments with a parabolic law of deflection i.e. for detecting or even measuring small changes in the horizontal intensity of the magnetism of the earth. An astatic electrodynamometrical system should be connected with a coil without iron core, moving in the earth field only, there being no field magnet. In that case the constant a depends on the horizontal intensity of the earth field. The galvanometric part should possess a sensibility many times greater than the dynamometrical part. For the exact measurement of the current strength a PELLAT or KELVIN standard instrument might be used, or perhaps a reliable potentiometer. I do not know whether a practical method might be worked out in this way; I do not even think it offers any advantage over the classic methods.

With the apparatus I had made for myself, I have been able to demonstrate the facts mentioned in this paper. In my instrument I found that, a and b having been measured as carefully as possible, a current $I = \frac{b^2}{a}$ generally failed to give an exact zero position. I was able to prove that this was to be expected; the difference being caused by two circumstances: 1. the electrodynamometer not being an astatic one; 2. the instruments reacting mutually. Both circumstances might have been partly eliminated by a construction which rendered the whole instrument perfectly symmetric about a horizontal plane passing through the centre of the movable magnet.

Physiology. — *"Fibrin-excretion under the influence of an electric current."* By E. HEKMA. (Communicated by Prof. H. J. HAMBURGER).

(Communicated in the meeting of October 28, 1916.)

Blood-clotting is based as we know upon the formation of a fibrin-gel. The gel or clotted substance obtained by adding blood-serum to a transsudate or to fluid blood-plasm free from elements formed, is likewise generally looked upon as fibrin. These fibrin clottings have until lately been considered as irreversible gels. Wrongly so, however, as my experiments showed. The fibrin-gel formed in the above mentioned way is indeed entirely insoluble in pure water, but by means of traces of alkali or acid it may be brought into a sole-state again, under the formation of optically empty fibrin-alkali- and acidhydrosoles. From these soles the fibrin may be excreted again with its properties unmodified. As regards the fibrinalkalihydrosoles it may for instance be done by weak acid (neutralisation), as regards the fibrinacidhydrosoles by weak alkali (neutralisation), while in both soles a number of reagents e.g. bloodserum, saturated neutralsalt solutions etc. effected an excretion of fibrin.

Further I have observed that the electric current also possesses this property. Both in artificial and in natural fibrinalkalihydrosoles (bloodplasm, transsudate) and likewise in fibrinacidhydrosoles an electric current may effect an excretion of fibrin. In the latter case the fibrin is formed at the negative, in the first two cases at the positive electrode.

For the experiments in question the fluid to be investigated was put into a U-shaped tube into the legs of which thin platinum electrodes were inserted (broad $\frac{1}{2}$ c.m.). As a rule the current was supplied by two accumulators.

In an artificial fibrinalkalihydrosole this experiment produces after some time a slight formation at the *positive electrode*, whilst after some hours a considerable clot has been formed round this electrode.

The fluid in the leg of the U-tube with the negative electrode remains clear; the only thing observable in it being the formation of gas beads. If the experiment is made with a weak fibrinalkalihydrosole prepared with a very weak alkali $\left(\text{e.g. } \frac{n}{500} \text{ NaOH}\right)$ a jelly like, filmy substance settles as a rule on the anode, which substance contains a great number of gas-beads and from whence thin fibres extend into the fluid. A microscopical investigation shows that this is a network of fibrin fibres, so that there is no question about a real jelly.

If, however, the experiment is made with a highly concentrated fibrinalkalihydrosole, the fibrin likewise settles round the anode, often though as a jelly-like mass in which a microscopical investigation can detect no fibres: in this case we have to deal with a real jelly. If the gel is removed by means of a curved spatula from the fluid, it breaks up into small lumps. If these are put into distilled water, they sometimes take the film- and fibre-shape, whilst in other cases they are further broken up.

In a fibrinacidhydrosole of moderate concentration, made for instance with $\frac{n}{50}$ HCl, the other conditions of the experiment being the same, a clot, which likewise encloses gasbeads, settles in the course of a few hours on the *negative* electrode. Mostly this clot can be carefully removed as a whole; in water it presents the appearance of a mass of threads, which on being examined microscopically are found to make up a network of fibrin fibres. In highly concentrated fibrinacidhydrosoles the fibrin secretion may assume the form of a jelly-like mass or of thin films; mostly, however, a connected film is formed also in this case, which, in water, breaks up into a mass of fibres. It should be mentioned that in these experiments the fluid in the leg of the **U**-tube with the positive electrode remains clear; here too a few gas beads may be observed.

If in the same manner a current is led through blood-plasm which has been kept fluid, the result will be after a few hours a connected jelly-like filmy clot with numerous gas beads, from whence slender fibres extend into the fluid at the *positive electrode,* whilst the fluid has remained clear in the other leg of the **U**-tube. When removed in water this clot mostly breaks up into thin flakes and fibres. On being examined microscopically the mass at first appears to be a dense amorphous granular mass. After one of the flakes or fibres has been unravelled a network of fibrin fibres is revealed, which had not been observed before on account of the numerous amorphous substances. These amorphous grains are undoubtedly other albumens, which have been precipitated with the fibrin at the anode.

As regards these experiments made with artificial fibrinalkali- and acidhydrosoles, it is by no means necessary to start from fibrin obtained by adding bloodserum to bloodplasm kept fluid or to a transsudate. The same results are arrived at with fibrin which has been obtained by adding to the above-mentioned natural coagulation-fluids some suitable reagent. For instance, weak acid (neutralisation) or a saturated neutralsalt solution (especially a saturated NaF solution). If alkali or acid hydrosoles, made from the fibrin gel thus obtained, are exposed to an electric current, they will be found to behave in a manner entirely analogous to that of the soles mentioned before.

How are we to account for the fibrin-secretion or coagulation under the action of an electric current? In my opinion the most obvious supposition is that the electric current renders inactive the alkali or acid of the fibrinalkali- and acidhydrosoles; this may be explained as follows:

If I am not mistaken it is assumed that the molecules of an aqueous alkali-solution e.g. a weak NaOH sol. are split up when acted upon by an electric current, so that the ions of OH are formed at the anode; these are subsequently rendered inactive under the formation of water and oxygen. If this is correct, the alkali in a fibrinalkali-hydrosole, under the influence of which the fibrin is in a sole-state, will be rendered inactive by the anode; it will so to speak disappear from the fluid, at least there. And since the fibrin cannot remain in a sole-state after alkali has been withdrawn, it will be secreted at the anode. It can easily be demonstrated that the part of the fluid which comes into contact with the positive electrode becomes much less alkaline, unlike that part which is in the leg of the U-tube containing the negative electrode. This holds good both for natural and for artificial fibrinalkalihydrosoles.

If on the other hand an electric current is led through a watery weak acid-solution, the acid molecules are, if I have not misinterpreted the current views on the subject, split up into electrically charged atoms (ions), whilst the ions of H with their positive charge become electrically neutral on being brought into contact with the negative electrode or disappear from the fluid there. If we assume this view to be correct, the acid in a fibrinacidhydrosole will suffer an electric dissociation and then become inactive. The consequence will be that the fibrin can no longer remain in a solestate and will be excreted at the cathode. This decreased acidity of the fluid-column which is in contact with the cathode can be easily demonstrated in the experiment. It should be noticed in passing that one gets an impression that the electrolytic dissociation of alkali and acid, or at least the disappearance of alkali and acid from the respective fluids, is restricted to the leg of the U-tube which contains the positive or negative electrode. That is to say if weak currents are applied; if very strong continuous currents are led through the fibrinalkali- and acidhydrosoles, everything is changed.

The observation that under the influence of the electric current fibrin in natural coagulation-fluids is secreted at the same (positive) electrode as fibrin in artificial fibrinalkalihydrosoles confirms, as it seems to me, the accuracy of a conclusion which I arrived at by another way before, viz. that fibrin in natural coagulation substances, hence also in blood, is present in a preformed state as an alkalihydrosole. That, therefore, the fibrin-secretion in natural coagulation fluids, and consequently the clotting of blood, is in principle based upon a transition from the alkalihydrosole- into the gel-state.

Elsewhere grounds have been adduced for the opinion that fibrin in its optically empty soles is not present in a simply dissolved state, but that the fibrin particles under the influence of electrolytes are expanded by water. That in other words the fibrin-particles in the alkali- and the acidhydrosoles contain so to say a charge of an electrolyte (alkali, acid) *and* water. Such an amicroscopical system: fibrin-substance—electrolyte—water I have denoted by the name of "micell". Taking the word micell in this sense, the optically empty fibrin-soles may be looked upon as micellular solutions. This view also makes it clear why fibrin may be secreted in one case as a real jelly, in another as a system of fibres, which facts we could establish again at the gel-formation under the influence of an electric current. In the first instance we have to deal with an agglutination of the fibrin micells still in a somewhat swollen state (by an imperfect loss of the electrolyte and consequently of the water in them), resulting in a real jelly. The second instance relates to an agglutination of fibrin particles which are no longer swollen; the micells have more completely lost their electrolyte and consequently their water, whilst the unswollen discharged micells (fibrin-particles) owing to a property peculiar to fibrin, agglutinate lengthwise into needles and then into fibres (micellular-crystallization process). From a more general point of view it is a remarkable fact that the fibrin-secretion under the influence of an electric current is entirely analogous to that which is occasioned by weak acid or alkali, by neutral salt solutions, by bloodserum etc. I shall, however, not dwell at present on the further significance of this fact.

Groningen, October 1916. *Physiological Laboratory.*

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
TE AMSTERDAM.

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CONTENTS.

Zoology. — *"The Gastrulation of Rana esculenta and of Rana fusca"*. By Dr. H. C. DELSMAN. (Communicated by Prof. J. BOEKE).

(Communicated at the meeting of November 25, 1916).

In my note of May 27, 1916, I was able to mention that similar pricking experiments to those described at that time for *Rana fusca,* were performed by me on the eggs of *Rana esculenta* also, an object, which in investigations on the earliest development of the frog egg we encounter much less frequently than the eggs of *Rana fusca,* which are to be obtained so much more easily. In some respects for pricking experiments like the present ones the eggs of *Rana esculenta* appeared to me to offer advantages over the eggs of the other species, but on the other hand certain disadvantages are to be noticed, which in the latter are at least less conspicuous. Among the advantages it may be noted that in pricking, which in this case too was performed with the point of a hedgehog's quill, one did not need to operate with nearly so much caution, to prevent the production of a voluminous extraovate, which has a disturbing influence on the further development. The egg content namely is in *Rana esculenta* far less liquid than in *Rana fusca*; indeed it is much more solid and tough, so that every prick not too clumsily made produces a little wound which in the last mentioned species can only be attained with the greatest caution and after several failures. Accordingly it was not difficult to apply to one egg several marks, e.g. one at the animal pole (*a*), and one or more at the crossing points of the third, equatorial cleavage furrow with the other, meridional ones, which, as in Fig. 1, we can indicate here again as *b* (dorsal), *c* (ventral) and *d* (the two lateral ones). Also the lighter colour of the egg has a great advantage, as it renders the surface images more distinct. On the reverse, the marks, so much more easily applied, also come off more easily, the wounds healing too soon. In not one of the eggs marked by me — all from one spawning — did it prove possible to rear them until the appearance of the medullary plate, without all the marks coming off beforehand. Next year therefore I hope to try and renew the marks in time during the development and thus to attain what this year was not reached. Yet the results reached until now seem to me sufficiently interesting to communicate them, and in completeness they are only a little behind those for *Rana fusca.*

To my surprise, I found that the external features of the gastrulation process and the behaviour of the dorsal and the ventral blastoporic rim in *Rana fusca* and *esculenta* differ from one another pretty considerably, so that a comparison of the two cases becomes especially interesting. Let us first consider the facts and afterwards look for an explanation.

The eight-celled stage of *Rana esculenta* agrees in the main with that of *Rana fusca,* as a comparison of Fig. 1 with the figure for

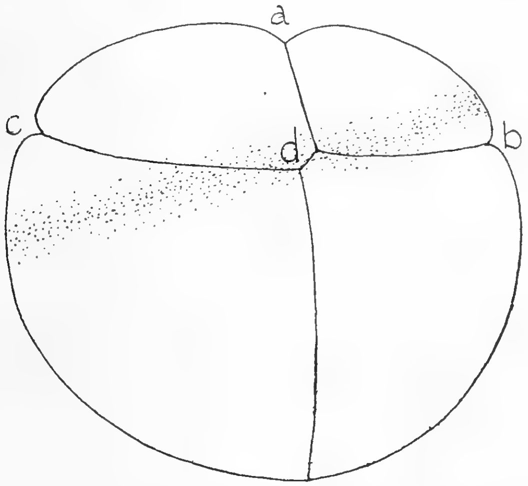

Fig. 1. Egg of *Rana esculenta,* 8 cells, from the side. The zone of demarcation between the darker and lighter area is defined by spots.

R. fusca of my former communication shows at once. The proportion of the size of the blastomeres in both cases is nearly the same. Nevertheless the distribution of the pigment points to a difference in the internal structure: the line of demarcation of darker and lighter hemisphere, in both figures indicated by a dotted band, not only lies much nearer to the animal pole in *Rana esculenta,* but it has also a much more horizontal situation. Now this boundary-line does not coincide in the least with the boundary of the future ecto- and entoderm, but it is apparently of importance in so far as in both frog species, as we will see, the border of the blastopore shortly after its appearance nearly runs parallel to it. We will revert to this in due course.

Turning to the figures 2—6, all drawn with a drawing-prism from the same egg, which was marked at the animal pole and at the point *b*, we see in fig. 3, how the first indication of the blastopore appears as a short, transverse slit, a little beneath the equator of the egg, at a place therefore which wholly corresponds to what we found in *Rana fusca.* It may be observed, that the boundary between the darker and lighter hemispheres has wandered downward

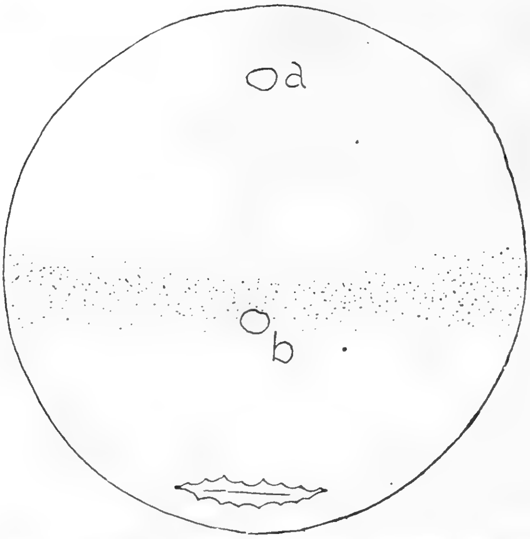

Fig. 2. Egg of *Rana esculenta*, marked on May 6 in the eight-celled stage at the points *a* and *b*. First appearance of the blastopore (*bl.*) From the dorsal side, May 7, 2.30 p.m.

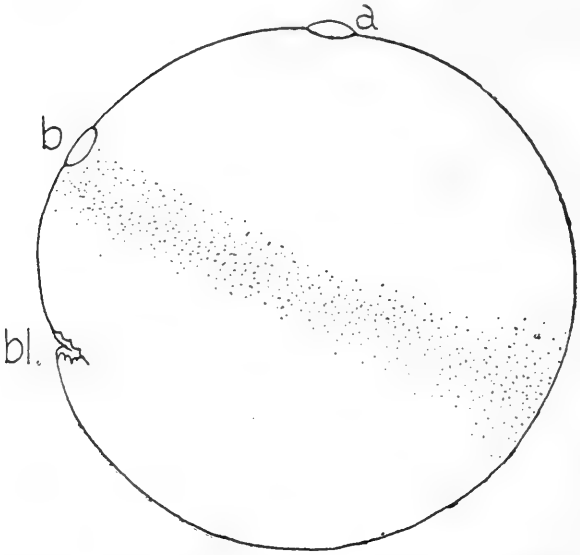

Fig. 3. The same egg, from the side, dotted zone as in fig. 1.

a considerable distance parallel to itself, away from the animal pole, as appears from a comparison with fig. 1, which as a matter of fact does not represent the same egg, a circumstance, which having regard to the great uniformity of the eggs in this respect, does not imply any difficulty. From this however one must in no way conclude, that the cells containing the pigment perform such a wandering downward themselves. The behaviour of the marks at *b*, *c*, and *d* in the different eggs tells us otherwise: their distance from the animal pole just as in *Rana fusca* increases only very slightly. Besides, former investigators have already pointed to the fact that during development the formation of new pigment goes on, especially at places of great cell-activity.

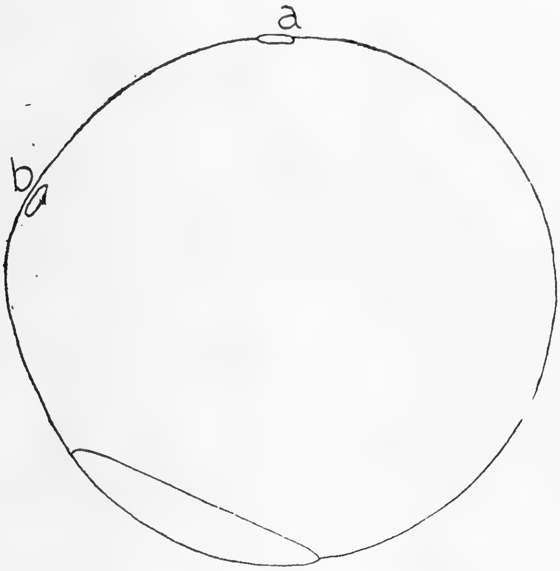

Fig. 4. The same egg, from the side, May 8, 6.30 a.m.

Unfortunately the next figure of the egg, was drawn much later (fig. 4), when the blastopore had already been contracting for some time. Other eggs however teach us, that, when the border of the blastopore has just closed at the rear side to a ring, this ring is much wider than in *Rana fusca.* While in the latter species the longitudinal diameter of the blastopore is about 60°, in the former it amounts to no less than 120°, about twice as much. So the exact situation of the anterior and posterior border in this stage in regard to the points *a* and *b* could not be made out and in the fig. 8, which is a composition of the other figures, I have accordingly indicated the border of the blastopore with a dotted line, as it will probably run. I have indicated the anterior border as lying a little in front of the place where the first trace of the invagination became visible, which accordingly would point to a primary backward movement of the dorsal rim. Such a primary wandering backward may be noticed in fact in such eggs, which to this end. have been provided with marks at a shorter or longer distance in front of and behind the blastopore border. Evidently it is the result of the forming of an invagination border at this place, where cells, lying originally

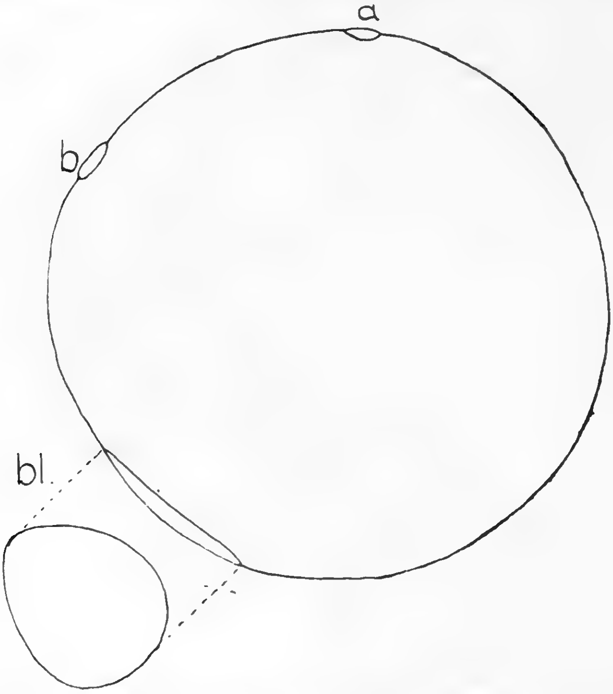

Fig. 5. The same egg, from the side, 8 May, 1.50 p m.

in front of the primary transverse rim of fig. 2 and 3, are carried inward. This however does not mean, that epiblast cells wander into the interior to participate in the construction of the archenteron roof. To me the view of Mac Bride[1]) seems to be preferable, according to which the first transverse slit does not appear at the border of the ecto- and entoderm area, but within the entoderm area, a little under the demarcation line. Thus the slit does not represent so much the first beginning of the blastoporic rim, as that of the archenteric invagination beneath it, and the cells in front of it, which disappear under the just forming blastoporic rim, are to be counted to the entoderm. Hence it is no wonder, that in a somewhat further advanced stage we find the blastoporic border a little in front of the rim of fig. 2 and 3, which is rendered the

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[1]) E. W. Mac Bride, 1909, The Formation of the Layers in Amphioxus etc. Quart. Journ. Vol. 54.

more intelligible when we see that during gastrulation the whole entoderm area performs a wandering forward. The assumption that the situation of the blastoporic rim at this stage as dotted in fig. 8, is right, is also favoured by the fact, that the line 1—1 thus runs parallel to the boundary between the darker and the lighter area of the egg, as indicated in fig. 3. This becomes evident, if we hold the figures 3 and 8 up to the light one upon the other, in such a way that the points *a* coincide. Just as in *Rana fusca* we also find in *Rana esculenta* that this line of demarcation in the different stages always runs parallel to the blastoporic border, approaching it gradually, until at last it reaches it.

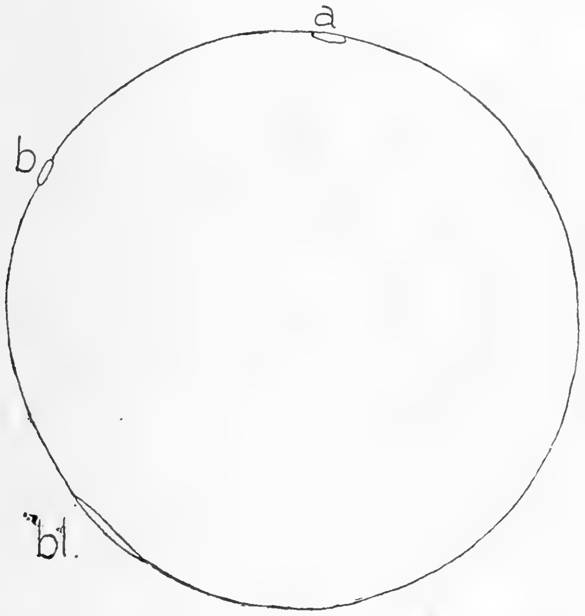

Fig. 6. The same egg, from the side, May 8, 9.15 p.m.

Now in holding up to the light one upon the other the figs. 4, 5 and 6, in such a way that the marks *a* each time coincide, it may be stated further that the distance of the points *a* and *b*, just as in *Rana fusca*, increases only very slightly, and moreover the way in which the blastopore contracts may be studied in detail. This same method was adopted again in composing the summary figure 8. While in *Rana fusca* the ventral blastopore border, which there appears approximately diametrically opposite the animal pole, does not make any forward movement, in *Rana temporaria* it not only does so, but the ventral border even progresses still more rapidly than the dorsal one!

Although some time after drawing fig. 6, I found the marks detaching themselves, yet it may be stated already that the closing of the blastopore here does not occur, as in *Rana fusca*, diametrically opposite the animal pole, but more to the dorsal side. So the appearance of the medullary plate in this egg was not observed anymore before the detachment of the marks, just as little as in the other eggs. Now, however, it does not rarely occur, that in eggs, where the blastopore has not yet quite closed, the first rudiment of the medullary plate becomes visible already. Such an egg is represented e.g. in fig. 7, where we see that the foundation of the embryo does not, as in *Rana fusca*, encircle nearly 180° of the egg, but is somewhat shorter. If now we hold up to the light this drawing together with that of fig. 7 and we transfer to fig. 7 the position of the marks *a* and *b* from fig. 6, it appears that they find themselves at exactly the same place as we stated in *Rana fusca*, i.e. respectively just in front of the transverse head-fold and at the transition of cerebral and medullary plate (fig. 7*). The objection might be raised that the possibility is not excluded, that during or before the appearance of the medullary plate there might still occur cell wanderings, which would raise doubts as to the correctness of the above conclusion. As we have seen, however, that in *Rana fusca* there is no question of anything of the kind, we may safely assume the same in this case. So this result for *Rana esculenta* again confirms the conclusions drawn from the theory which has engendered the present investigations.

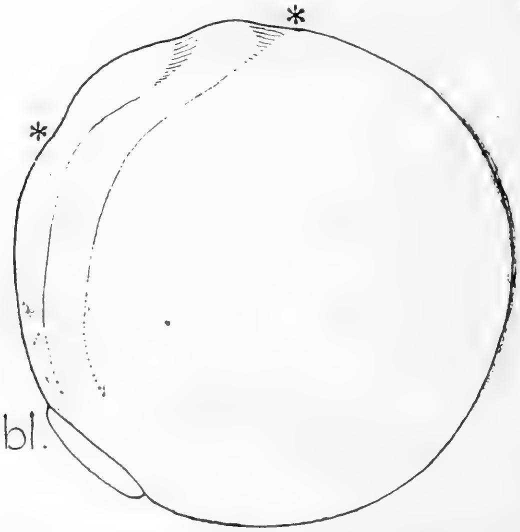

Fig. 7. Another egg, with foundation of medullary plate. * *a* and *b* as transferred from fig. 6.

For the sake of completeness in fig. 8 the position of the marks *c* and *d*, from another series, has also been indicated. As already

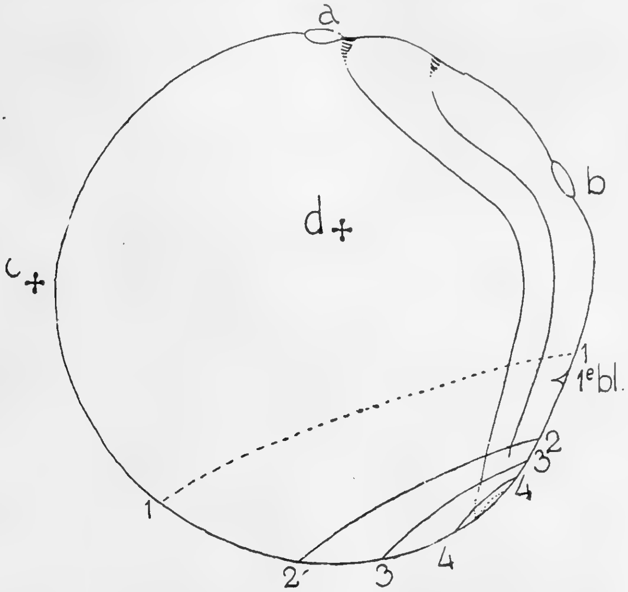

Fig. 8. Combination of figs. 3—7 and others, from which the situation of *c* and *d* and the extension of the blastopore in phase 1 are borrowed. 1e *bl* = first indication of the blastopore (fig. 2 and 3).

observed before, the mutual distance of the marks *a*, *b*, *c*, and *d*, just as in *Rana fusca*, changes but little during cleavage and gastrulation. Yet it could be stated that the distance *a*—*c* increases somewhat.

Very conflicting views have up to the present day been held as to the gastrulation of vertebrates. To many an adherent of one of these views the result of the recorded pricking experiments will be somewhat surprising. Who, after studying fig. 8, could maintain any longer that the foundation of the dorsal parts of the embryo originally lies as a ring round the border of the blastopore and is formed from it by concrescence? By far the greater part of the embryo is formed in front of the place, where the dorsal blastoporic rim first appears, and the contraction of the blastopore proceeds nearly concentrically. An explanation of the facts mentioned seems to me to be afforded by the views concerning the gastrulation, which follow from my theory on the derivation of vertebrates from annelids.

To this end let us first consider once more the movement of the ventral blastoporic border. Have we to deal here with a similar overgrowth of the yolk as at the dorsal lip? In that case we ought to find in sections under the ventral lip, just as under the dorsal one, an archenteric slit or cavity. Not, that this archenteric cavity under the dorsal lip owes its existence solely to the overgrowth of the yolk by the dorsal lip. In this case the cavity would not reach further forward than the place where this dorsal lip appeared first. As a matter of fact, however, it soon reaches considerably further forward, so that doubtless also an active enlargement of the archenteric cavity by dehiscence of the entoderm cells occurs, though it seems to me less suitable to assume a sharp demarcation of the parts of the archenteron formed in these two manners, and to distinguish these as archenteron and metenteron, as ASSHETON[1]) did. Only by the overgrowth of the ventral blastoporic lip however, should there be formed already an archenteric cavity or slit under it, reaching to the place of its first appearance. This now proves not to be the case, as shown in fig. 9; only a short slit is present under the ventral lip, not nearly reaching up to where this lip first appeared. So the conclusion must be drawn that not only the ventral blastopore lip, but also the whole entoderm area in front of it performs a wandering to the dorsal side, and that accordingly the entoderm is not only overgrown by the dorsal blastoporic lip in a backward direction, but also actively wanders forward to disappear under it. This reminds us of the controversy between ROUX and SCHULTZE, mentioned in my former

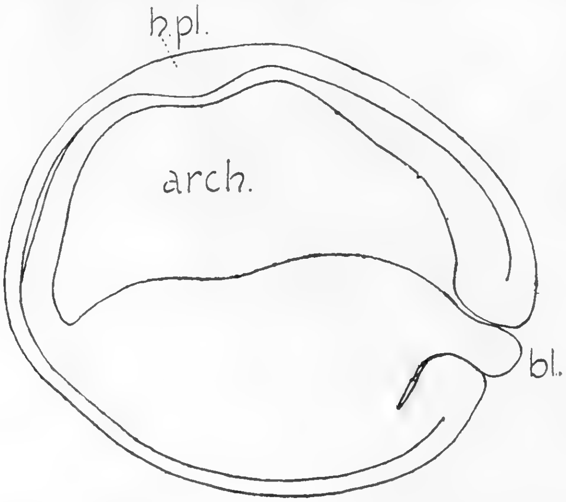

Fig. 9. Sagittal section of a gastrula of *Rana esculenta*. *h. pl.* cerebral plate, *arch.* archenteron, *bl.* blastopore with yolk plug

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[1]) R. ASSHETON, 1909, Professor HUBRECHT's Paper on the Early Ontogenetic Phenomena in Mammals. Quart. Journ. Vol. 54.

communication, on the wandering of the dorsal blastopore border. Roux's opinion was, that only the dorsal rim wanders over the yolk, which for *Rana fusca* proves to be right, though not 180°. Schultze on the contrary declared all movement of the dorsal border to be illusory and to be explained by a rotation of the egg. Actively, according to him, the entoderm wanders forward under the dorsal border, and it appears by our present results that Schultze's view, at least as far as *Rana esculenta* is concerned, is not quite erroneous either.

Now apparently we have in this wandering of the entoderm area during gastrulation in *Rana esculenta* the same dorsally directed movement before us, which in *Rana fusca* is performed immediately after fertilization, and which there causes in the eight-celled stage the demarcation line between the darker and lighter area of the egg surface to make a so much greater angle to the egg equator than in *Rana esculenta,* while for the blastopore border, just after it has closed to a ring, the same holds. All this is shown at once by a comparison of fig. 1 and 8 of the present paper with fig. 1 and 2 of the former.

Before looking now for the explanation of the phenomenon, a short discussion must precede of the views, to which my theory of the origin of vertebrates leads concerning the gastrulation of vertebrates, in the first place of anamnia. In studying this theory many a one will have wondered how from two in Protaxonia (Hatschek) diametrically opposite areas as the apical plate (round the animal pole) and the stomodaeum (round the blastopore) in craniote vertebrates an organ could arise, which so much gives the impression of a unity, as the cerebral and the medullary plate. A considerable displacement at any rate must have occurred, to bring together these two parts.

This approach we now see performed before our eyes in the ontogeny of annelids. While the entoderm, which remains after the production of the three quartets of ectomeres, originally lies diametrically opposite to the animal pole, we find the mouth, which is directly to be traced back to the blastopore, in the trochophora lying just under the prototroch, which forms the border of the apical plate. As discussed in my article on the development of *Scoloplos armiger,* the displacement is to be ascribed to three factors.

In the first place we observe a wandering of the whole entoderm area to the ventral side (Fig. 10*a*), a result of the active multiplication and extension of the ectoderm cells at the rear side, i.e. mainly the *d*-quadrant of the egg, whereas the cells of the anterior side,

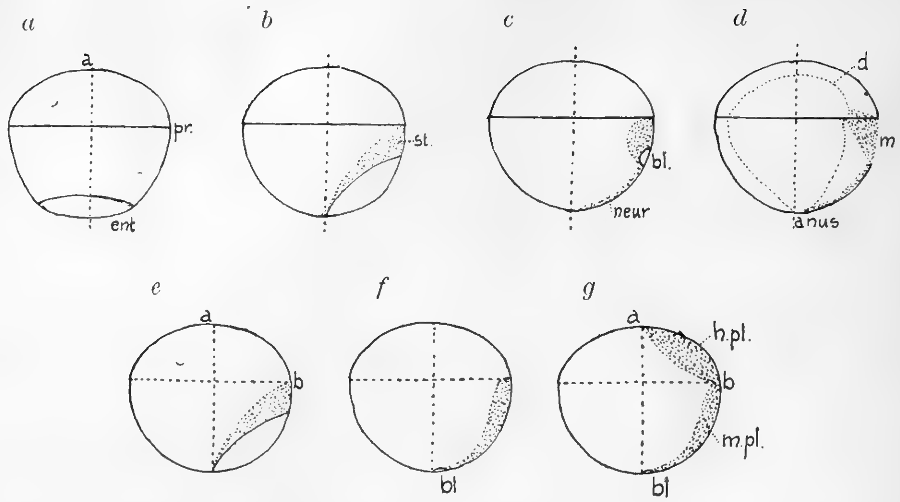

Fig. 10. Diagrammatic representation of the behaviour of the blastopore, see text. *a*, *b*, *c*, *d* in polychaete annelids, *e*, *f*, *g* in chordates. *bl.* blastopore, *d.* gut, *ent.* entoderm, *h. pl.* cerebral plate, *m.* mouth, *m. pl.* medullary plate, *neur.* neurotroch, *pr.* prototroch.

the *b*-quadrant, are backward in development. This causes the entoderm area to wander to the ventral side to such an extent, that no longer its centre but its hind border is found opposite the animal pole. In this region afterwards the anus is formed.

Secondly the blastopore does not close concentrically, but excentrically in a forward direction (Fig. 10*b*), be it with or without concrescence of the lateral borders. This depends on the relative speed with which either the lateral borders or the hind border move forward over the entoderm, and this again depends on the way in which the descendents of 2*d*, the so-called somatic plate, spread over the left and right side and over the posterior end of the embryo. Evidently concrescence here seems to be the rule and at the suture, where left and right blastopore borders have met, the neurotroch arises.

In the third place the foundation of the stomodaeum here does not any longer surround the blastopore as a ring of uniform breadth, as in Protaxonia, but lies more in the way of a crescent round the anterior border. For of the third quartet it is only the cells of the anterior two quadrants, 3*a* and 3*b*, of the second quartet only 2*a*—2*c*, which participate in the formation of the stomodaeum. After the sinking in of this crescentic rudiment to the formation of the stomodaeum-tube, which arises outside the final, narrowed blastopore, the mouth comes to lie just underneath the prototroch (Fig. 10*d*).

We shall see now what we find of these phenomena in the frog egg and to this end begin with the egg of *Rana fusca*. At once it appears that the first of the three above-mentioned processes, the wandering of the entoderm area to the ventral, c. q. to the dorsal side, is here performed very precociously, immediately after fertilization, and consequently is already finished in the unsegmented, fertilized egg. At least we find, as is shown by fig. 1 of my former communication, that the white area here does not lie at all diametrically opposite the animal pole, but much more to the future dorsal side. The boundary between the ecto- and entoderm areas probably runs parallel to the demarcation of the darker and lighter areas of the egg, as may be also concluded from the place, where afterwards the dorsal and ventral borders of the blastopore appear (fig. 2, ibid.). Evidently we have to deal here with a case of precocious segregation, though it concerns here more a wandering than a segregation.

The second process mentioned above, the rostrad-excentrical closure of the blastopore, we do not find in *Rana fusca*; on the contrary, the closure proceeds caudad-excentrically. As mentioned already in my former communication and elsewhere, I see in this caudad-excentrical closure a result of the interference of the contraction of the blastoporic border with a backward movement of the blastopore, following directly from my theory on the homology of stomodaeum and epichordal neural tube in annelids and vertebrates. As a result of the strong elongation which we must assume that the stomodaeum of annelids undergoes to be transformed into the medullary tube of vertebrates (cf. the scheme in my article in Anat. Anz. Bd. 44, p. 493), the entrance to the stomach (Schlundpforte, HATSCHEK), into which the blastopore passes, must perform a wandering over nearly the whole length of the body to become the neurenteric canal (also resulting from the blastopore). This backward wandering now in chordates is performed in anticipation of the formation of a tube, already during the contraction of the blastopore border. By this process the final, narrowed blastopore is carried back to the place where it was originally found in Protaxonia, viz. diametrically opposite the animal pole. Whether this caudad-excentrical closure of the blastopore is performed by concrescence or not, is here of no importance; as stated earlier, I do not believe that concrescence, at least in amphibians, occurs to any considerable extent. The medullary plate, of which the foundation in stage 10*e*, just as the foundation of the stomodaeum in fig. 10*b*, surrounds as a crescent the anterior border of the blastopore — a conclusion reached for Amphioxus also, e. g. by KORSCHELT and

HEIDER in the last edition of their "Lehrbuch" — during the contraction undergoes a change in shape as indicated in fig. 10*c* and discussed already in my former paper. We see in this the backward growing out of the stomodaeum of annelids into the epichordal neural tube of chordates, projected as it were on a plane. In fig. 10*g* it has been indicated, how in craniotes, in addition to the epichordal neural plate, the praechordal cerebral plate is now added, while in acrania the condition of fig. 10*f* continues (Anat. Anz. T. 44.)

How are now our statements for *Rana esculenta* to be brought into accordance with those for *Rana fusca*, how are they themselves to be interpreted and what are the points of difference from the latter species? Simply in this way, that 1 in *Rana esculenta* the egg contains more yolk or at least is less isolecithal in structure, and 2 that the wandering of the entoderm area, shown in fig. 10*a* and *b*, here occurs later.

Let us revert firstly once more to the annelids. In my article on the development of the annelid *Scoloplos*, published this year (1916), I have tried to show that among the eggs of polychaete annelids three types are to be distinguished. In the first place we have the small, poorly yolked eggs of *Polygordius*, *Hydroides* etc., in which the cleavage results in a very equal coeloblastula (Fig. 11*a*). Now in the larger eggs of other species two types of polarity may very early be recognized, which exert their influence on the here very determinate cleavage. In the first place the polar or radially symmetrical polarity, expressing itself in accumulation of yolk at the vegetative pole, which again causes the entoderm cells to be much larger than the cells of the three quartets of ectomeres. In the second place the bilateral

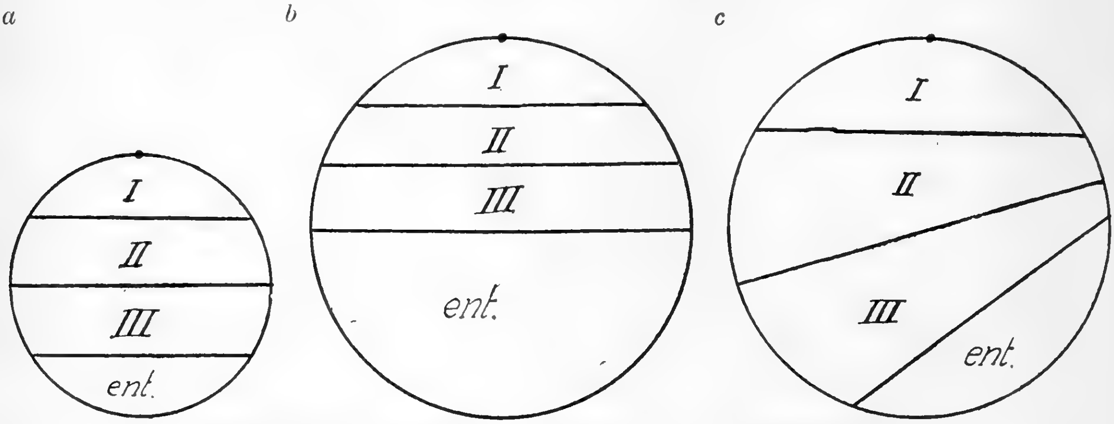

Fig. 11 *a*, *b*, *c*. Diagrammatic representation of the 3 types of polychaete eggs. *I*, *II*, *III* = 1st, 2nd and 3d quartet of ectomeres, ent. = entomeres.
a. minute, yolkless egg. *b*. egg with pronounced polar polarity. *c*. egg with pronounced bilateral polarity.

polarity, which expresses itself in that the cells of the rear side (*d*-side) from the beginning are much larger than the corresponding cells at the anterior side (*b*-side), so that the entoderm area from the beginning does not lie diametrically opposite the animal pole. The scheme of fig. 11 may serve to illustrate this. As a rule we see at the same time both kinds of polarity in the larger eggs exerting their influence on the cleavage, but in one case the first predominates, in the other the second prevails. As an example of the prevalence of polar polarity, I mentioned *Nereis* where the macromeres (entoderm) are especially large in regard to the ectomeres, which lie over them as a little cap, while on the other side the bilateral polarity is only slightly expressed, the cells of the rear side not being much larger than those of the anterior side. In the reverse this last condition prevails very strongly in *Scoloplos*, which accordingly can serve as an example of the predominance of the bilateral polarity (fig. 10*c*). Especially 2*d* is of extraordinary size, while the entoderm cells are not at all remarkable for special bulk. So the entoderm area is displaced here from the beginning to the ventral side.

Hence the eggs of *Rana fusca* and *esculenta* evidently are in the same relation to each other as *Scoloplos* and *Nereis*. In the first species a precocious displacement of the entoderm area and less yolk, as appears e. g. from the extension of the blastopore. In *Rana esculenta* a later wandering of the entoderm area and a greater amount of yolk, at least a less isolecithal structure of the egg, as appears from the large blastopore together with the fact, that the foundation of the embryo encircles considerably less than 180° of the egg circumference; the belly accordingly is relatively more swollen than in *Rana fusca*. Originally the entoderm area in *Rana esculenta*, though not perfectly, yet lies much more diametrically opposite the animal pole than in *Rana fusca*, as appears from the fact, that the demarcation line of the lighter and darker hemispheres of the egg and later the border of the blastopore make a much smaller angle with the egg equator than in the last mentioned species. So in *Rana esculenta* the polar or radial symmetry is originally more strongly pronounced, in *Rana fusca* the bilateral symmetry.

In conclusion, attention may be drawn to the fact of how little this difference in the internal constitution of the egg influences the cleavage. Were things as in annelids, with their determinate cleavage, we might expect, that in the eight-celled stage in *Rana esculenta* the four upper cells would be relatively smaller than the four lower ones as compared to *Rana fusca*, and, reciprocally that in the latter species the four ventral cells would be larger than the four dorsal ones. Nothing of the kind proves to be true: the eight-celled stages in *Rana fusca* and *esculenta* are nearly uniform. Besides, we saw in the foregoing communication, how relatively independent the direction of the first cleavages is of the internal constitution of the egg.

Chemistry. — *"Röntgen-investigation of allotropic forms"*. (Preliminary communication). By Dr. J. Olie Jr. and Dr. A. J. Bijl. (Communicated by Dr. Ernst Cohen).

(Communicated in the meeting of January 27, 1917.)

Debye and Scherrer have in the "Nachrichten der Königlichen Geschellschaft der Wissenschaften zu Göttingen" [1]) published their investigations about "Interferenzen an regellos orientierten Teilchen in Röntgenlicht". Led by theoretic considerations Debye [2]) had come to the conclusion that secondary Röntgenlight, emitted by a body shone upon by Röntgenlight is not equally strong in every direction. The arrangement of the electrons in the atoms must necessarily give to that light a maximum of intensity in certain definite directions. Even if the atoms should not be arranged regularly, the resultant of all secondary light-emission will be a definite division of the light in space into maxima and minima of intensity.

Debye expected and actually obtained in his investigations with Scherrer results, which clearly proved the existence of such a division of lightrays (interference). But by the side of the phenomenon he expected he noticed in several cases, whenever crystalline material had been used for the investigation, a much more striking phenomenon. Besides the diffuse maxima, which were visible in the photos as so many spots with vague outlines, there were to be seen some rather distinct lines, which made one think of a spectrum. Debye and Scherrer pointed out that this should not be explained from the interference of the Röntgen rays in the electron-complex of the atom, but in an analogous way from the interference of those rays falling upon the crystalline structure that are to be formed in the macroscopically unarranged mass. [3]) This is contrary to the generally accepted opinion that Röntgen-interference images can only be obtained with large and properly-shaped crystals. From the theoretical considerations as well as from experiments it becomes

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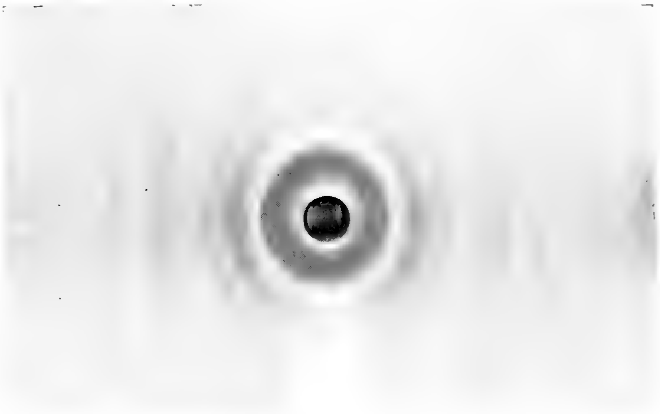
[1]) Mathem-physikal. Klasse 1916 Heft 1. See also Phys. Zeitschr. **17**, 277 (1916).

[2]) Nachr. der K. Ges. der Wissensch. in Göttingen math.-physikal Klasse 1915.

[3]) Fine crystalpowder or quasi amorphous material.

J. OLIE Jr. and A. J. BIJL. "Röntgen-investigation of allotropic forms". (Preliminary communication).

Graphite.



Diamond

clear that the maxima of emission-intensity can produce only cone-shaped surfaces, of which the top can be in the secondary source of emission only. The angles at the vertex of these cone-surfaces may be of different size and the direction of the secondary emission may be adverse to that of the primary rays. If a thin bar, composed of compressed crystal powder, on which Röntgen light falls vertically in the direction of the axis and which has been placed cylindrically in such a way that the axis of the bar [1]) and that of the cylinder are in the same place, and only an opening remains in the film to admit the primary light-rays, the photographic image will show a profile section of the cylinder with the different cone-surfaces of maxima-emission-intensity.

The vertex angles of the cones, and consequently also the distances between the lines in the film depend upon the crystal-form of the material in which the light falls, and of the nature (wave-length) of the (homogeneous) primary light-emission. A material when shone upon by homogeneous Röntgen-rays of small wave-length will show a different interference-figure from that which we obtain when it is shone upon by Röntgen-light of greater wave-length i. e. when the lightwaves are longer the lines will not be so close together.

Debye and Scherrer were now able, whenever the wave-length of the primary homogeneous Röntgenlight was known, to infer the crystal-form of the material from the interference-figure obtained through the light falling in a bar of crystal powder or quasi amorphous material.

Besides the many interesting, purely physical questions to which this highly important discovery will lead, the question offering itself in the first place from a chemical point is: What will be the result when allotropic modifications of the same material are under the light of homogeneous X rays of the same wave-length; — and will it be possible to bring to light the different modifications that have been observed in the same material in a purely physical way.

Here we think especially of the dynamical allotropies.

The questions which we asked ourselves as soon as we had read Prof. Debye's article, we have laid before the professor and it appeared that Prof. Debye had also considered this question, but that, owing to the want of suitable material, he had not made any investigation in that direction. He, too, expected that in this manner it would be found possible to distinguish between allotropic forms. With the greatest willingness he left the investigation in this direc-

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[1]) The shape is not of much importance.

tion entirely to us, for which we hereby tender him our best thanks, and also for the many valuable hints we received from him as regards the technicalities of the investigation.

The allotropy of carbon, as was most natural so to say, was, after some experimenting investigations examined Röntgenographically, the accompanying photo is a reproduction of the interference-figures obtained by letting Cu-rays [1]) fall upon a bar of compressed graphite [2]) and upon a bar of diamond-powder [3]).

From them we notice, qualitatively too, how different the interference-figures of these two allotropic forms are, according to the quite different crystal-systems (diamond is regular, graphite is monoclinic) in which they are met with. We therefore expect that this method will bring light in many cases in which it is doubtful whether there is really allotropy, or where two materials that cannot be chemically separated are present side by side. Also in those cases in which it is doubtful whether we have to do with the amorphous or with the crystalline state, the Röntgen-investigation will, as DEBYE himself declares, enable us to make the matter clear.

At the same time we direct the attention to the possibility of making a qualitative Röntgen analysis of, say, a mixture or an alloy without any loss of material.

About the particulars of this investigation and the further results of it, we hope to be able to say something at some other time.

Chemistry. — "*Amygdalin as nutriment for Aspergillus niger.*" By Dr. H. J. WATERMAN. (Communicated by Prof. J. BÖESEKEN.)

(Communicated in the meeting of January 27, 1917).

PURIEWITSCH [4]) has proved that the extract of the cells of *Aspergillus niger* splits up amygdalin into glucose, benzaldehyde and hydrogen cyanide, whereas the *living mycelium* of this species of mould behaves in quite a different manner towards amygdalin.

In the latter case benzaldehyde and hydrogen cyanide are not

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[1]) For the αCu-line $\lambda = 1.549 \times 10^{-8}$.

[2]) As made by MOISSAN and prepared from pure C in the electrical furnace.

[3]) Average diameter of the parts 2 à 3 μ.

[4]) H PURIEWITSCH, Ueber die Spaltung der Glykoside durch die Schimmelpilze. Ber. d. deutsch. Bot. Ges. **16**, 368 (1898); Also compare : F. CZAPEK, Biochemie der Pflanzen, 1ster Band, 2e Aufl. 1913, p. 363—365; F. LAFAR, Handb. d. techn. Mykologie, Bd. IV, Spezielle Morphologie u. Physiologie der Hefen und Schimmelpilze, 1905—07, p. 250—251.

formed and FEHLING-reducing substances do not appear in the liquid containing amygdalin. It was demonstrated that in this case amygdalin was absorbed and utilized by the mycelium because as the quantity of mycelium increased the quantity of amygdalin in the solution at the same time diminished.

These apparently contradictory results of PURIEWITSCH and of other investigators in analogous cases have frequently been a subject of discussion. HÉRISSEY [1]) for instance says: "If during the metabolism amygdalin and corresponding glucosides, in a similar way as in vitro by emulsin, are first split up into compounds which are easily assimilated such as glucose, on one hand and into noxious compounds on the other hand, it may be expected that these poisons will be converted at once into other chemical compounds." But this is a matter of uncertainty, HÉRISSEY says and he does not give a definite opinion.

W. KRUSE [2]) is surprised at the said experiments of PURIEWITSCH and points to the fact that other investigators have not obtained the same results. The uncertainty about this subject made me take it up in order to try to clear it up.

Solutions containing 2 °/$_0$ amygdalin and the necessary inorganic nutriment were inoculated with spores of *Aspergillus niger.* The temperature during cultivation was 33°.

Many times in the course of development the quantity of dry substance on one hand and the polarisation of the solution on the other hand were determined from which the assimilated amygdalin could be calculated. The mould layer, after being washed with distilled water, was therefore dried at 105° to constant weight. (Tab. I p. 924)

My experiments confirmed the observation of PURIEWITSCH that amygdalin is assimilated by the living mycelium, whilst the production of young mycelium occurs at the cost of the assimilated amygdalin. (Table I). This table shows too that amygdalin is a better nutriment than glucose at least with regard to the dry weight of mould obtained.

This conclusion agrees with results which I obtained before, viz. that the presence of a benzolnucleus in the assimilated organic chemical compound increases the quantity of mould formed at the cost of this nutriment. [3])

Formerly I demonstrated with great probability that in a special

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[1]) E. H. HÉRISSEY, Recherches sur l'émulsine. Thèse Paris 1899.

[2]) W. KRUSE, Allgemeine Mikrobiologie, 1910, p. 458.

[3]) H. J. WATERMAN, Zeitschr. f. Gärungsphysiologie, Bd. **3**, Heft 1 (1913).

TABLE I.

Glucose as exclusive organic food			*Amygdalin* as exclusive organic food		
Composition of the culture liquid: 50 cm.^{3} of tapwater, in which dissolved 0,15 % NH_4NO_3, 0,15 % KH_2PO_4, 0,1 % magnesiumsulfate (crystallised). Temperature 33°.					
A. 2 % glucose (1000 mgr.)		Number of days after inoculation	B. 2 % amygdalin (1000 mgr.)		
Assimilated glucose (mgr.)	Obtained dry weight of mould (mgr.)		Assimilated amygdalin (mgr.)		Obtained dry weight of mould (mgr.)
1000	320	6	670	In the nutrient solution neither glucose nor benzaldehyde were found.	315
		12	670, 710		298, 215
		16	635		251
1000	242, 264	38	750		271
		42	680		200
		95	not determined		214, 237, 203

case sucrose can be assimilated without preceding dissociation into glucose and fructose.[1])

In this case, too, it might be supposed that the assimilation of amygdalin will not be preceded by a conversion into glucose, benzaldehyde and HCN outside the organism.

Benzaldehyde and to a small degree HCN too, especially in high concentrations, diminish or stop the development of *Aspergillus niger* in nutrient liquids containing glucose. (Table II).

The solutions *p* and *q* were prepared as follows:

p. 42,5 mgr. KCN dissolved in distilled water and filled up to 100 cm^3. Added 10 cm^3. of $0,98 \times {}^1/_{10}$ Normal sulfuric acid.

q. 100 cm^3. of distilled water, added 10 cm^3 of $0,981 \times {}^1/_{10}$ Normal sulfuric acid.

Both solutions were used *immediately after their preparation.*

The purpose of the experiments 7,8 and 9 was only to demonstrate that the quantity of sulfuric acid added in N°. 4, 5, and 6 could have no retarding influence. The phenomena of growth which were observed in N°. 2 after six days and in N°. 3 after ten days could not be attributed to the total evaporation of the benzaldehyde because the nutrient liquid of N°. 3 distinctly smelled of benzaldehyde even after 10 days.

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[1]) Zur Physiologie der Essigbakterien, Centralbl. f. Bakteriologie, 2e Abt. Bd. **38**, 451 (1913).

TABLE II.

Retarding influence of benzaldehyde and hydrogen cyanide.
Culture liquid: 50 cm.^{3} tapwater, in which dissolved 0,15 % NH_4NO_3, 0,15 % KH_2PO_4, 0,1 % magnesiumsulfate (crystallised) and 2 % glucose. Temperature 33°.

N0.	Added	Course of development.					
		After 1	3	4	6	7	10 days
1		beginning growth	vigorous growth, many spores	vigorous growth, many spores	vigorous growth, many spores	vigorous growth, many spores	vigorous growth, many spores
2	1 drop of pure benzaldehyde [1])	?	no growth	?	vigorous growth, hardly any spores		
3	3 drops of pure benzaldehyde [1])	no growth	no growth	no growth	no growth	no growth	beginning growth { The liquid has the smell of benzaldehyde
4	0,5 cm.^{3} of a HCN containing liquid (p)	?	vigorous growth, rather many spores	vigorous growth, many spores	vigorous growth, many spores	vigorous growth, many spores	vigorous growth, many spores
5	2 cm^3. of a HCN containing liquid (p)	?	vigorous growth, many spores				
6	5 cm.^{3} of a HCN containing liquid (p)	no growth	beginning growth	rather vigorous growth, few spores			
7	0.5 cm.^{3} of a H_2SO_4 containing liquid (q)	beginning growth	vigorous growth, many spores	vigorous growth, many spores			
8	2 cm.^{3} of a H_2SO_4 containing liquid (q)						
9	5 cm.^{3} of a H_2SO_4 containing liquid (q)						
10		beginning growth	vigorous growth, many spores	vigorous growth, many spores			

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[1]) The substance present in the laboratory collection was purified by washing with distilled water and distillation.

From other experiments it has become evident that benzaldehyde when used in very slight concentrations, may serve as nutriment for *Aspergillus niger*.

From the above results it may also be expected that in amygdalin containing liquids to which emulsin has been added *Aspergillus niger* will not develop.

This follows too from the experiments which are united in Table III.

TABLE III.

50 $cm.^3$ tapwater, in which dissolved 0,15 % NH_4NO_3, 0,15 % KH_2PO_4, 0,1 % magnesiumsulfate (crystallised). Temp. 33°.

N°.	Dissolved	Development	
		After 3	5 days
1, 2, 3	2 % glucose	rather vigor. growth, beginning formation of spores	vigorous growth, many spores
4, 5	2 % glucose + 0,04 % emulsin	vigorous growth, many spores	
6	2 % glucose + 0,1 % emulsin		
7, 8, 9	2 % amygdalin	rather vigor. growth, rather many spores	
10, 11	2 % amygdalin + 0,04 % emulsin	no growth, the smell of the liquid resembles benzaldehyde or (and) HCN.	just as after 3 days[1]
12	2 % amygdalin + 0,1 % emulsin		

The emulsin used (MERCK) had no retarding influence on the development of *Aspergillus niger* with glucose as source of carbon.

The retarding influence of emulsin stated in solutions containing amygdalin could therefore only be ascribed to the products of hydrolysis of the amygdalin viz benzaldehyde and HCN.

The above experiments prove that when important quantities of amygdalin, before being assimilated, are already dissociated into glucose, benzaldehyde and HCN outside the organism *Aspergillus niger* will not develop. It is proved, too, that this is caused especially by the retarding action of benzaldehyde.

The strong retarding action of benzaldehyde can on one side be explained by its strong solubility in fats (benzaldehyde in every proportion is miscible with olive oil) and on the other side by the possibility of a rapid conversion into benzoic acid. Benzaldehyde

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[1] After 30 days Nos. 10, 11 and 12: no growth, but the smell of benzaldehyde or (and) HCN can no more be stated.

will immediately overburden cells[1]). Only if care is taken that this does not happen benzaldehyde can be used as nutriment.

As follows from the above this is the case when we use very slight concentrations of benzalhyde. Amygdalin which is among the substances that do not cause overburdening phenomena can be converted *in* cells into glucose, benzaldehyde and HCN without any slackening influence on the growth[2]).

In the communications mentioned we have demonstrated that generally speaking it is not the nature of the substance absorbed but in the first instance the quantity, that causes overburdening of cells and the accompanying retarding of growth. In this way we have at the same time a specific for bringing narcotic substances into the organism without any harm to the latter.

For this purpose the narcotic substance should be combined with one or more other chemical compounds, so that a complex chemical compound results, which can not overburden cells, but from which the desired active substance may be formed *within* the cell.

Dordrecht, December 1916.

Chemistry. — *"In-, mono- and divariant equilibria."* XIV. By Prof. F. A. H. SCHREINEMAKERS.

(Communicated in the meeting of January 27, 1917).

22. The occurrence of three indifferent phases; the equilibrium M is constant singular.

In the previous communications we have discussed the occurrence of two indifferent phases; now we shall briefly consider the occurrence of three indifferent phases.

Again we take the two reaction-equations:

$$a_1F_1 + \ldots + a_pF_p + a_{p+1}F_{p+1} + \ldots = 0 \quad . \quad . \quad . \quad (1)$$

and

$$\mu_1 a_1 F_1 + \ldots + \mu_p a_p F_p + \mu_{p+1} a_{p+1} F_{p+1} + \ldots = 0 \quad . \quad (2)$$

in which a_1 and μ_1 are positive and at the same time:

$$\mu_1 > \ldots > \mu_p > \mu_{p+1} > \mu_{p+2} > \ldots \quad . \quad . \quad . \quad (3)$$

When we put:

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[1]) J. BÖESEKEN and H. J. WATERMAN, These Proceedings, January 24, 1912 p. 608; H. J. WATERMAN, Dissertation Delft, 1913.

[2]) When at the same moment any conversion into the just mentioned substances occurs too outside the cell of course retarding of growth will all the same be stated.

$$\mu_p = \mu_{p+1} = \mu_{p+2} = \mu$$

then (2) passes into:

$$\mu_1 a_1 F_1 + \ldots + \mu a_p F_p + \mu a_{p+1} F_{p+1} + \mu a_{p+2} F_{p+2} + \ldots = 0 \quad (4)$$

In order to find the reaction between the phases of the equilibrium (F_p), we have to eliminate F_μ from (1) and (4); with this not only F_p disappears, but also F_{p+1} and F_{p+2}. Consequently we do not get a reaction between $n+1$ phases, but between the n—1 phases

$$F_1\ F_2 \ldots F_{p-1}\ F_{p+3} \ldots F_{n+2}.$$

For the reaction between the phases of the equilibria (F_{p+1}) and (F_{p+2}) we find the same relation between those n—1 phases. In each of the other reaction-equations for the monovariant equilibria, however, $n+1$ phases occur.

The phases F_p, F_{p+1} and F_{p+2} are, therefore, the indifferent phases, the n—1 other phases are the singúlar ones.

We now have four singular equilibria, viz.:

$$(M) = F_1 + \ldots + F_{p-1} + F_{p+3} + \ldots + F_{n+2}$$
$$(F_p) = (M) + F_{p+1} + F_{p+2}$$
$$(F_{p+1}) = (M) + F_p + F_{p+2}$$

and

$$(F_{p+2}) = (M) + F_p + F_{p+1}.$$

The three indifferent phases may have in (1) the same sign or not. (In the first case $+++$ or $---$, in the second case, $++-$, $--+$, $+--$, $-++$, $+-+$ or $-+-$). Just as in Comm. X we are able to show now: when in a reaction-equation two indifferent phases have the same (or opposite) sign, then they have also in all other reaction-equations the same (or opposite) sign.

Just as in Comm. X we are able to show: when the three indifferent phases have the same sign, then the singular equilibrium M is transformable, when they have not the same sign, then the equilibrium (M) is not transformable.

In the same way as in Comm. X it now follows:

1. The three indifferent phases have the same sign or in other words: the singular equilibrium M is transformable. Curve (M) is monodirectionable; the four singular curves coincide in the same direction.

2. The three indifferent phases have not the same sign or in other words the singular equilibrium M is not transformable. Curve (M) is bidirectionable; of the 3 other singular curves, 2 cur-

ves (viz. those, which have the same sign) coincide with the one direction of the (M)-curve, the third curve coincides with the other direction of the (M)-curve.

With the aid of those rules we may deduce again, just as in Comm. X, the main-types of the P,T-diagrams; we leave this, however, to the reader and we shall consider more in detail one single example only.

We take a ternary system with the components W ($W =$ water), A and B. Let occur in the invariant point the equilibrium:

$$A + B + Ice + L_q + G$$

in which L_q represents the liquid q (fig 1) and G represents the vapour. When G consists of water vapour only, then in the equilibrium (5) also the reaction $Ice \rightleftarrows G$ may occur; A, B and L_q are then the indifferent phases, Ice and G the singular phases. Then we have the singular curves:

$(M) = Ice + G$ [Curve (M) in fig 3]
$(A) = B + Ice + L + G$ [qb in fig. 1; $qb = (A)$ in fig. 3]
$(B) = A + Ice + L + G$ [qa in fig. 1; $qa = (B)$ in fig. 3]
$(L) = A + B + Ice + G$ [Curve (L) in fig. 3]

and further the curves

$(Ice) = A + B + L + G$ [qc in fig 1; (I) in fig. 3]
$(G) = A + B + Ice + L$ [Curve (G) in fig. 3]

With the aid of the previous considerations we may deduce the type of P,T-diagram; first, however, we shall do this in another way.

Let us consider viz. the case that the vapour G does not consist of water only, but that it contains also a little of the components A and B. Then we have the equilibrium:

$$A + B + Ice + L_q + G_{q_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which G_{q_1} represents the vapour q_1 (fig. 1). The point q_1 is situated in the vicinity of the point W. The five phases of equilibrium (6) now form a type of concentration-diagram as in fig. 5 (II), consequently the type of P,T-diagram must be as in fig. 6 (II). [We have to bear in mind that the figs. 4 (II) and 6 (II) have to be changed inter se]. As q_1 is situated in fig. 1 in the vicinity of W, the line qq_1 intersects either WB and AB or WA and BA. It is apparent from fig. 6 (II) that the curves (I), (A) and (B) must form now a three-curvical bundle, as in fig. 2. When we assume that the line qq_1 intersects the lines WB and AB, then curve (B) must be situated between the curves (A) and (I). We now easily see (amongst others yet also from the diagonal succession of the curves)

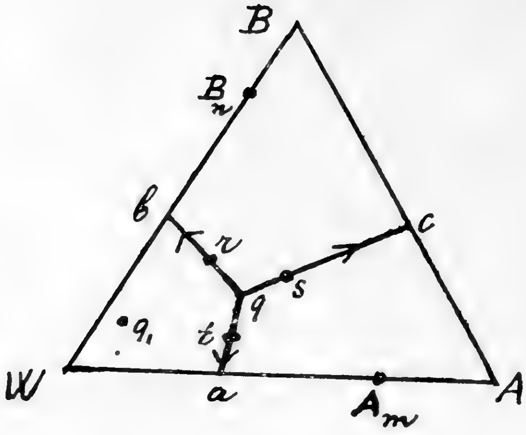

Fig. 1.

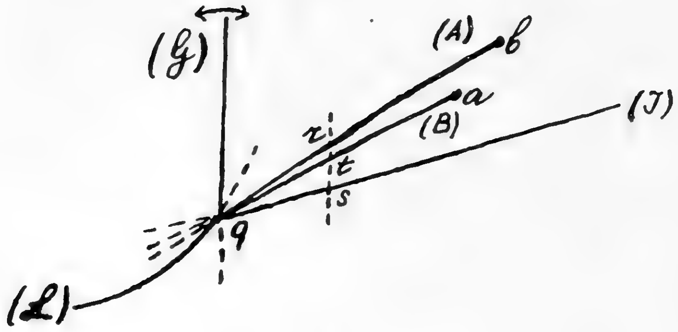

Fig. 2.

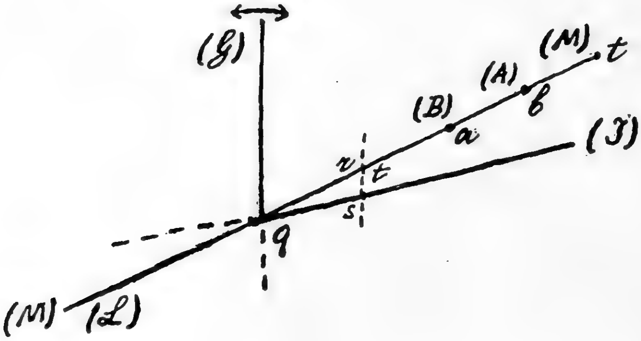

Fig. 3.

that we obtain a *P*,*T*-diagram, as is drawn in fig. 2. [The points *a* and *b* are the finishing points of the curves (*B*) and (*A*) and they are in accordance with the points *a* and *b* of fig. 1; the finishing-point *c* of curve (*I*) has not been drawn in fig. 2).

It appears from fig. 2 that at the same temperature the vapour-tension of $(A) = B + ice + L + G$ is larger than that of $(B) =$

$A + ice + L + G$ and this is larger again than that of $(Ice) = A + B + L + G$; this follows from the assumption that the line qq_1 intersects the line WB. [This also appears in the following way. We take in fig. 1 the 3 points r, s and t in such a way, that $T_r = T_s = T_t$ and further 3 points r_1, s_1 and t_1 (those are not drawn in fig. 3), which represent the vapours belonging to r, s and t. Then rs is the saturation-curve under its own vapour-pressure of B, $r_1\ s_1$ is the corresponding vapoursaturationcurve. From the change in pressure along this curve it follows $P_r > P_s$. When we also consider the other curves, then we find $P_r > P_t > P_s$].

When we now consider the case that the vapour Gq_1 in (6) contains watervapour only, then equilibrium (6) passes into (5). Then in fig. 1 q_1 coincides with W, so that the singular equilibrium $(M) = Ice + G$ occurs. As A, B and L now become indifferent phases, (A), (B) and (L) become, therefore, singular curves, which consequently have to coincide. It appears from fig. 2 that this coincidence may take place only in such a way that the stable parts of (A) and (B) coincide and that (L) coincides with the metastable parts of (A) and (B). Then we obtain fig. 3, in which the (M)-curve is therefore bidirectionable.

The position of the curves in fig. 3 is in accordance with the rules, which we have deduced in the general considerations. As we are not able to transform the singular equilibrium $(M) = Ice + G$ into the invariant equilibrium (5), (M) is, therefore, not transformable, so that (M) must be bidirectionable.

When we take a reaction, in which occur the 3 indifferent phases A, B and L, e.g.

$$L \rightleftarrows A + B + G \quad \text{consequently} \quad A + B + G - L = 0$$

then it appears that the 3 indifferent phases have not the same sign. Hence it follows again that curve (M) must be bidirectionable. As A and B have the same sign, the curves (A) and (B) have to coincide in the one direction — and the curve (L) in the other direction with the (M)-curve. All this is in accordance with fig. 3, which we might have found reversally also from those data.

We may deduce fig. 3 yet in another way, which we shall indicate briefly. We draw firstly in a P,T-diagram the curve $(M) = Ice + G$; this terminates in the triplepoint t (fig. 3) of the pure water. The curves $(A) = B + Ice + L + G = (M) + B + L$ and $(B) = A + Ice + L + G = (M) + A + L$ go starting from q towards higher T and they have to coincide with the (M)-curve.

Curve $(L) = A + B + Ice + G = (M) + A + B$ has to coincide also with the (M)-curve, but it goes, starting from q towards lower temperatures.

Curve $(I) = A + B + L + G$ goes starting from q, also towards higher temperatures, but it must be situated below the curves (A) and (B). In order to show this latter we take again the three points r, s and t in fig. 1. As the vapour-pressure increases, starting from s, along the isotherms rs and ts, the curves (A) and (B) must be situated, therefore, in fig. 3 above curve (I).

Those considerations are also valid when we replace the components A and B by their hydrates A_m and B_n, provided that solution q is situated within the triangle $W A_m B_n$ and not too close to the line $A_m B_n$. When this is really the case, then we are able to define the directions of the curves in the same way as e.g. in Comm. XIII.

Leiden, *Inorg. Chem. Lab.* (*To be continued*).

Physics. — *"On the equation of state of water and of ammonia"* By G. Holst. Supplement N°. 41*f* to the Communications from the Physical Laboratory at Leiden. (Communicated by Prof. H. Kamerlingh Onnes).

(Communicated in the meeting of January 27, 1917).

In an investigation published some time ago on the equation of state of methylchloride and ammonia[1]), it was shown that the sign of the coefficient C of Kamerlingh Onnes's[2]) equation of state

$$pv = RT\left(1 + \frac{B}{v} + \frac{C}{v^2} + \frac{D}{v^4} + \frac{E}{v^6} + \frac{F}{v^8}\right)$$

was different for the two substances; for ammonia it was negative; for methyl chloride, as for other normal substances, C was found to be positive. At that time I ventured the hypothesis C would be found negative for other associating[3]) substances. Following this idea I have calculated B and C for water vapour, starting from the data given by M. Jacob[4]) in tables 7 and 8 of his paper on

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[1]) G. Holst. Comm. Leiden No. 144.

[2]) See for instance H. Kamerlingh Onnes and W. H. Keesom. Enc. d. Math. Wiss. Art. V 10 p. 728. also Comm. Leiden Suppl. 23.

[3]) Comp. Enc. p. 722, where it is pointed out that besides the associating (polymerized) substances, others occur (deviating substances) which show similar deviations as the first.

[4]) M. Jacob. Zeitschr. Ver. D. Ing. 1912 p. 1980.

the specific heat and specific volume of water vapour. Calling the pressures given by JACOB in K.G./cm² p_j, the specific volumes in m³/K.G. v_j and the residual term of the equation of state $1000\, v_j = \frac{4{,}706\, T}{p_j} - R_j$: R_j in dm³/KG. we find:

$$B + \frac{C}{v} + \ldots = -\frac{R_j\, p_j\, v_j}{T}\, 0{,}17080$$

where $v = \frac{v_j}{1{,}2440}$.

When $-\frac{R_j\, p_j\, v_j}{T}\, 0{,}1708$ is drawn as a function of $\frac{1{,}2440}{v_j}$ a series of straight lines is obtained. From this diagram B and C can be immediately read as a function of the temperature. In this way the values were found given in the following table. (p. 934)

In the first place it will be seen that for water, as for ammonia, C is negative, and increases strongly with decreasing temperature. It is further clear that it will not be a simple matter to find a formula which represents C as a function of the temperature, all the more that there is nothing to guide us in the choice of the correct form of the function. As W. H. KEESOM told me that he and Miss VAN LEEUWEN had undertaken the deduction of a function of the kind required, I thought it advisable to await the result of this calculation before venturing upon the calculation of a purely empirical formula for myself.

For the other coefficient, B, there is something to go by: water, like ammonia, has a large dielectric constant, which is a temperature function.

We may therefore assume, with P. DEBYE[1]), that the water molecule has an electric moment. For spherical molecules with an electric bipole at the centre, W. H. KEESOM[2]) has calculated the coefficient B as a function of the temperature. I will therefore compare the experimental values with those which KEESOM calculated. For this purpose, as suggested in Comm. Leiden Suppl. 25, we will draw F as a function of $\log h\nu$ and $\log B$ as function of $\log T$.

If the curves are shifted until they coincide over a fairly large range, we find for instance that $\log B = 7{,}35 - 10$ coincides with $F = 0{,}065$ and $\log T = 2{,}828$ with $\log h\nu = 0{,}358$.

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[1]) P. DEBYE. Phys. Zeitschr. (13), 97, 1912. Comp. also J. KROO. Ann. d. Phys. (42), 1383, 1913.

[2]) W. H. KEESOM. Comm. Leiden Suppl. 24*b*.

TABLE 1.

t	B	C
110	0.0157	—0.00120
120	—0.0146	—0.00095
130	—0.0135^{5}	—0.00070
140	—0.0125^{5}	—0.00053^{5}
150	—0.0117^{5}	—0.00039
160	—0.0111	—0.00023
170	—0.0103^{5}	—0.00016^{5}
180	—0.0097^{5}	—0.00011^{5}
190	—0.0091^{0}	—0.00008^{0}
200	—0.0085^{5}	—0.00006^{5}
220	—0.0075	—0.000036
240	—0.0065^{5}	—0.000024
260	—0.0058	—0.000016
280	—0.0051	—0.000014
300	—0.0045	—0.000015
350	—0.0032	—0.000013
400	—0.0022^{5}	—0.000012
450	—0.0015	—0.000010
500	—0.0010^{5}	—0.000008
550	—0.0006^{8}	—0.000006

As in ammonia, here also deviations show themselves at the lower temperatures (below 250° C.).

From these data, according to Keesom's calculations, the radius and the dielectric moment can be derived for the water molecule, when assumed spherical.

In this way we find $\sigma = 3.21.10^{-8}$ cm. and $m_e = 2.62.10^{-18}$ in e.s. units.

Calculating these quantities for ammonia also, in the same way, I found $\varrho = 3.54.10^{-8}$ cm. $m_e = 2.36.10^{-18}$ e.s. units.

The dimensions of the molecule, correspond, therefore, as regards the order of magnitude, with those determined in other ways.

For liquid water DEBYE has calculated the electric moment, and gives $m_e = 5{,}7.10^{-19}$ e.s. units. The correspondence is not altogether what might be desired. I have therefore recalculated the electric moment from the measurements made by BAEDEKER [1]), who determined the electric constant for water vapour and for ammonia. For water vapour the range of temperatures examined is very small, so that not much reliance was be put on the conclusions to be drawn.

According to DEBYE, the following formula applies to the dielectric constant

$$\varepsilon = \varepsilon_0 + \frac{a}{T}$$

where $a = \frac{4\pi}{3}\frac{m_e^2 N}{k}$ [2]). N represents the number of molecules in 1 cm^3., k PLANK's constant $1{,}346.10^{-16}$ erg.

The first term ε_0 is due to the quasi-elastic electrons, the second to the bipoles. I have calculated the first from the index of refraction for which I took $n = 1{,}000255$ for water, and $n = 1{,}000377$ [3]) for ammonia. These values apply, it is true, to the visible spectrum, but the uncertainty introduced by this cannot be great, as ε_0 itself is small. In the following table the calculated values of $(\varepsilon - \varepsilon_0)\frac{\varrho_0}{\varrho}T$ will be found. The factor $\frac{\varrho}{\varrho_0}$ is introduced so as always to work with the same number of molecules.

The last column in the above table shows that for that part of the dielectric constant which is due to the bipoles, the same law holds as given by CURIE for the magnetic susceptibility, at least in the case of ammonia. The correspondence is not so good for water. At the same time, in order to be able to continue the calculation, I have assumed that the law applied to water also [4]), using the mean constant in the calculation. In this way we find for the electric

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[1]) K. BAEDEKER Z. f. phys. Chem. (36), 305, 1901.

[2]) See P. LANGEVIN. Ann. Chim. Phys. (5), 70, 1905.

[3]) Recueil de Constantes Physiques.

[4]) Whether deviations actually occur in water, as in magnetic substances, must be settled by further experiments.

Further, the question arises, why the value of the electric moment calculated for water vapour deviates from that calculated by DEBYE for the liquid. I thought the deviation might be accounted for by the fact that DEBYE has assumed in his

TABLE 2.

Dielectric constant for ammonia.

t	ε	$\frac{\varrho}{\varrho_0}$	ε_0	$\frac{\varrho_0}{\varrho}\left(\varepsilon-\varepsilon_0\right)$	$\frac{\varrho_0}{\varrho}\left(\varepsilon-\varepsilon_0\right)T$
18.4	1.00730	0.934	1.00070	0.00707	2.06
19.0	704	931	70	681	1.99
59.4	547	814	61	597	1.98
62.1	538	808	61	591	1.98
83.8	482	757	57	562	2.00^5
95.3	453	733	55	543	2.00
108.4	434	707	53	539	2.05^5

mean: 2.01

Dielectric constant for water vapour.

t	ε	$\frac{\varrho}{\varrho_0}$	ε_0	$\frac{\varrho_0}{\varrho}\left(\varepsilon-\varepsilon_0\right)$	$\frac{\varrho_0}{\varrho}\left(\varepsilon-\varepsilon_0\right)T$
140.0	1.00765	0.645	1.00033	0.0115^5	4.7
142.2	767	641	33	114^5	4.7^5
143.2	736	640	33	110	4.6
145.8	694	636	32	104	4.4
148.6	648	632	32	097^5	4.1

mean: 4.5

moment of the water molecule $m_e = 2{,}3\ 10^{-18}$ e.s. units, and for ammonia $m_e = 1\ 52\ 10^{-18}$ e.s. units.

The order of magnitude is the same as the electric moment calculated from the equation of state. The numerical correspondence, however, still leaves something to be desired. This is not surprising,

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calculation that the density of the liquid remained constant. His formula runs

$$\frac{\varepsilon-1}{\varepsilon+2}\frac{T}{\varrho}\varrho_0=\frac{4\pi}{9}\frac{m^2{}_eN}{k}+\frac{4\pi}{3}\Sigma\frac{N_\mu e^2}{f_p}T=a+1T$$

in which for the calculation $\varrho = \varrho_0$ was assumed. If the necessary correction is introduced for the density, a becomes negative for water, so that no real value is found for the electric moment. So that in fact, like BOGUSLAWSKI (Phys. Zeitschr. 1914 p. 283) I could not find any agreement between theory and experiment for liquid water.

The form $\frac{\varepsilon-1}{\varepsilon+2}$, also, is not a very suitable one as soon as ε becomes much greater than 1. If, as in water, ε varies between 60 and 80, this fraction varies very little with changing ε.

as the measurements of the dielectric constant lie partially in the field of temperature, where the values for B calculated on the assumption of bipoles deviate from those determined experimentally, and moreover the supposition upon which the calculations are based will not entirely correspond to the facts.

Finally, I should like to draw attention to the conclusions, which follow from these calculations, for the determinations made by PH. A. GUYE[1]) and his fellow-workers of the molecular weight of gases from the weight of a litre under normal circumstances and the compressibility. These measurements, which have been made with the greatest care, have not always led to a satisfactory agreement between the molecular weight determined in this and in other ways, especially in the case of easily compressible gases. From our calculations it follows 1*st* that for an accurate determination of the compressibility the measurements must be so arranged that they enable us to determine[2]) with the necessary accuracy not only B but also C; and 2*nd* that when it is not established in another way that a given gas behaves as a normal substance, the compressibility for that gas must be specially determined.

Our calculations demonstrate that the deviations from the law of corresponding states, which in various substances may be very important as regards the value of B, may be even greater for C, so much so that the sign[3]) for substances with and without bipoles may be different. The fact that for some gases including ammonia a correct value for the molecular weight was obtained by making use of the law of corresponding states, even where this was not to be expected according to the preceding discussion, must therefore be regarded as due to accident. And it is not to be expected that at other temperatures an equally good agreement would be found.

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[1]) See for instance Mém. de la Soc. de Phys. de Genève (35) 1905—1907. and further Journ. d. Chem. Phys. various volumes.

[2]) This conclusion was also drawn by H. KAMERLINGH ONNES and W. H. KEESOM Enc. Math. Wiss. V 10 p. 902. They also point out here (p. 900) the influence of the deviations from the law of the corresponding states upon the determinations of the molecular weight.

[3]) Whereas B changes its sign for all substances examined, a possible difference of sign at equal reduced temperatures may thus be ascribed to the choice of the critical quantities as corresponding, this is not the case with C. For normal substances (see H. KAMERLINGH ONNES Comm. Leiden N⁰. 74, p. 10) C is everywhere positive and increases with falling temperature.

Anatomy. — *"The development of the Fossa Sylvii in embryos of Semnopithecus."* By Dr. D. J. Hulshoff Pol. (Communicated by Prof. C. Winkler.)

(Communicated in the meeting of January 27, 1917).

During the last years of my stay at Lawang-Java, I had the opportunity to shoot a great number of semnopithecus maurus in the woods on the slopes of the Tenger mountains.

The uteri, which were taken out of the monkeys, which were shot, were preserved in the beginning in formaline, but afterwards were brought in alcohol for the voyage to Europe. Later on too, after the brains were dissected from the skulls, they remained in alcohol.

As a rather complete series of the brain of a monkey embryo is not often found, it is comprehensible that the material which I collected, has given rise some new points of view.

On the ape fissure and plis de passage I communicated already previously.[1]) In this communication the development of the fossa Sylvii is going to be described, which totally differs in certain points from what is found in human embryos.

Therefore it is sufficient to describe the monkey embryos with the serial numbers 15—14—13*a*—13*b*—12—11 and 10, because the sulcus in N°. 10 has already reached the full development and there totally agrees with what is found in adult specimens.

Before I pass on to this, I will first recall to memory in short, how the development takes place in the human being.

In the description given by Ekker[2]) was pointed out that in the 3rd month, in which the brain has become 1,9— 2,6 cm., a curvature commences, as found in a bean, between the frontal and parietal part. This curvation continues at the base towards the lateral surface, and Ekker describes it on page 208 as follows: "stellt anfangs eine ganz flache etwa dreiseitige Grube dar, deren Spitze nach abwärts gegen die Hirnbasis, deren Basis nach Auswärts sieht."

In the 4th month the fossa Sylvii *is distinctly marked*, as the brain matter begins to bulge alongside its rim: "welche nach unten und vorn gegen die Schädelbasis flach ausläuft."

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[1]) D. J. Hulshoff Pol. The fissura simialis in embryos of semnopitheci. Royal Acad. of Science. Amsterdam Febr. 1916. The relation of the plis de passage of *gratiolet* to the ape fissure R. A. of S. April 28, 1916, and the ape fissure — Sulcus Lunatus — in man. May 27, 1916.

[2]) A. Ekker. Zur Entwicklungsgeschichte der Furchen und Windungen der Grosshirn hemisphären im Foetus des Menschen. Archiv f. Anthropologie 1868.

The development of the rims above the fossa Sylvii, which lies a little lower, progresses in the 5th month.

A distinct separation between an anterior and a posterior part of this sulcus one gets in the following, therefore in the sixth month. Now too this fossa is nearly totally open, only the posterior part has closed to a fissure.

During the following months the process of operculisation continues, so that in the 9th month only the most inferior part is still opened.

Retzius [1]) gives in his treatise firstly an account of the description, made by Cunningham, who states that the fissura lateralis begins as an almost round furrow, which later on becomes a triangle. Retzius himself describes its appearance in the middle of the 3rd month in the shape of a half-moon or kidney, also as a sharp point. (Pl. I fig. 28—29). At first however one does not notice much of an insula. In the beginning of the 5th month the marked area becomes broader.

Although from the above-mentioned may follow, that the form in which the fossa Sylvii begins is not always the same, yet on one point agreement exists, that is to say, this furrow begins at the lateral edge and from the very first commencement is opened towards the lower end.

Limiting myself, in reference to the foetal brains of anthropoids, to the investigations of the latest periods, therefore those of Anthony [2]), I may state he then found in a foetus of a gorilla of 6—8 months: "Le circulair superieur de Reil s'étend, au côté gauche, comme chez un foetus humain du même age, jusqu'au sillon limite antérieur de l'insula . . . A droite le circulaire superieur de Reil est conforme au type habituel observé chez les singes, c'est-à-dire, qu'il n'atteint pas le sillon limite antérieur de l'insula".

As to the foetal brains of a chimpanzee [3]), corresponding with human brains of the 7th and 8th foetal month, these should differ with regard to the "complex Sylvin". For this communication, however, it is of importance that this furrow in this period is already totally closed.

In embryos of Semnopitheci the relation is quite different from that in human beings.

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[1]) D. Retzius. Das Menschenhirn. Stokholm 1896.

[2]) Anthony, R. Sur un cerveau de foetus de gorille. Comptes rendus des séances de l'Académie des Sciences, t. 161, p 153 séance du 9 Aout 1915, Paris.

[3]) Id. Sur un cerveau de foetus de chimpansé. Comptes rendus des séances enz t. 162, p. 604, séance 17 Avr. 1916, Paris.

Thus we see in N°. 15 (fig. 1) in which the embryo has a length of 9 c.m. and the brain a weight of 2 grams, that on the lateral

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2.

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4.

5.

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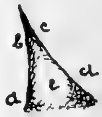

surface, therefore not on its lateral edge, a small curvation is visible *b*. It is found above the place where, on the edge, the frontal and temporal part come together. This curvation is small, but plainly visible. If one examines the surroundings of this small line, then another thin line is seen, passing from the first mentioned one towards the lateral edge. It forms as it were a junction between the line *b* and that edge.

The curvation *b* has developed (fig. 2) in N°. 14 (length 11 c.m., brain weight 3 grams) into a distinct furrow. In this embryo too, the junction between *b* and the lateral edge is not more than the lengthening of a furrow in the form of a thin line *a*. One sees however on the posterior part already a beginning of a bulging rim.

As to the form of this furrow *b*, this is totally different from what will be found later on in the other sulci, because the rims are not placed opposite each other, but the anterior is flattened.

Fig. 3 shows us this furrow transversally sectioned. In it we see that the posterior wall goes straight downward to the bottom of the furrow, but that it passes on the anterior part along a sloping surface to the frontal part.

In N°. 13*a* (length 13.5 c.m., brainweight 9 grams) which is nearly of the same size as 13*b*, the direction of the sulcus has totally altered. Where it possessed in fig. 2 a direction somewhat from inferior-posterior to superior-anterior, the direction in this embryo (fig. 4) has totally changed and runs from inferior-anterior to superior-posterior. The posterior wall of the future fissura Sylvii is now totally formed and protrudes over the anterior part. This border is remarkable, because it has become deeper at the lower portion therefore near *a*. Moreover in this embryo too, there has been formed a part of the anterior wall of the fissura Sylvii *c*, whereas the lower part *d* is formed by the frontal lobe. The total sulcus therefore, is not much more than a hook, of which the longest side is formed by *ab*, and the shortest by *c*.

It is remarkable however, that whereas the top part *b* of the posterior wall of the fissura Sylvii is not deep, this on the other hand is the case with the part *c*. This would give the impression as if this latter part corresponds with *b* in fig. 2. Against this would plead that one sees in fig. 1 as well as in fig. 2 the existing sulcus *b* connected by a superficial curvation with the lateral edge, what must be the sulcus *a* in fig. 4. Yet the first view remains possible, if one accepts, that during the following development the fold *a* (fig. 2) is continued upward from the corner in *b* and that it uplifts the latter sulcus as it were at its posterior end, moving it also upward.

The point of junction between *a* and *b* (fig. 2) in that case would be the same as between *b* and *c* (fig. 4), with this difference, that it is removed in postero-superior direction. When in the last figure *a* is the oldest part and *b* the youngest, then it is comprehensible that *a* is deeper than *b*. This conception would be supported by the fact, that the direction of *b* in fig. 1 is a more perpendicular one than in fig. 2, so that the position of this fold in these three periods of development presents itself as *b* in fig. 1 and 2, and *c* in fig. 4.

As however the link is missing between fig. 2 and fig. 4, which might solve this riddle, we must be contented to conclude, that:

in fig. 2 the part *b* is deeper than the junction *a* with the lateral edge;

in fig. 4 the reverse is found, hence *a* is deeper than *b* and *c* too is deeper than *b*.

In embryo 13*b*, probably of a more advanced growth than 13*a* we find the anterior wall of the fissura Sylvii totally formed (fig. 5), although the border in *d* may not yet be strongly developed. The image which is now formed, totally agrees with that which is also found in human embryos. (Retzius plate I fig. 33 and 35, plate III fig. 3) i. e. the form of the triangle with the top upward and the opened base directed downward.

In embryo N°. 12 (length 15 cm. and brainweight 12 grams) the fossa Sylvii is completely closed and the insula is operculised. If one opens the walls of the sulcus (fig. 6) then it appears that the image, given in fig. 5, in large features has remained. The operculum temporale is still most developed, much more than the operculum frontale in *d*. On the other hand it is seen, contrary to fig. 5, that the junction *bc* has sunken into the depth, while the insula rises as a round little ball (*e*) but is totally covered by the opercula.

During the further development, (embryo N°. 11, length 18 c.m., brainweight 21 grams) it is seen that the part of the sulcus, corresponding with *a* in fig. 6 has been pushed more to the bottom and lies deeper than the part *bc*.

During a still later period of foetal life (embryo N°. 10, length 19 c.m., brainweight 21 grams) there is not much difference between the opercula, corresponding with *ab* and *cd* in fig. 6. The only thing remarkable is the large prominence of the insula as a small round elevation, above its surroundings. It is more distinctly visible than one finds in adult monkeys.

From the above said is proved the great difference in development of the fossa Sylvii in mankind and in semnopithecus.

In the former the development commences at the base, where this is bordered by the lateral edge and moreover in the form of a circle or triangle, which later on extends upward, on the lateral surface.

In Semnopithecus one finds exactly the reverse: one sees first a sulcus on the lateral surface (fig. 1 and 2), which develops in downward direction. Afterwards a second separating sulcus, though also on the lateral surface, is added to it (fig. 4 sub *c*), which too develops in downward direction (fig. 5*d*), by which then only the form of a triangle is formed, with its base downward opened.

A second important difference is the period of complete operculisation.

The length of embryo n°. 12 [1]), in which the fissura Sylvii was quite closed, measured 15 cm. As a new-born Semnopithecus has a length of 27 cm., this is a little more than the half of it. The whole length of a full term human foetus is ± 53 cm., so that a little more than its half comes to 29,5 cm., which therefore would make out the *beginning of the sixth month.*

As however it is known, that in man the whole operculisation only takes place at the end of the 9th month or shortly after birth, this is also therefore an important difference.

A third point of interest is the period in which this sulcus shows itself for the first time on the brain surface.

I pointed out that the first indication of it was found in embryo

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[1]) The length of the monkey embryo is measured from the middle of the head to the root of the tail. The length of a human foetus is taken from head to heel. The difference does not alter in any way the above mentioned calculation, as the relations remain limited to monkeys reciprocally and also between human foetus.

n°. 15, with a length of 9 cm. As a new-born monkey measures 27 cm., that would be just a third part of the length. Now the third part of the length of a full term human foetus is 17 à 18 cm., which, according to RETZIUS, corresponds with the end of the 4th month.

When the ratios in length between the monkey embryos are thought similar to those which consist inter se between the human foetus, then with adequate development one should find the first indication of the origin of the fissura Sylvii, also in the embryos of man, only at the end of the 4th month.

Now the records of ECKER and RETZIUS are rather similar and they point out the third month as the period in which the first indication of this sulcus is found in human embryos. This should be therefore a month sooner than the analogous period in Semnopithecus.

The conclusions to which I think I may come, are therefore the following:

1. The fissura Sylvii in Semnopithecus commences on the lateral surface, and develops towards the lateral edge, which is the reverse in man.
2. The first that one sees of this fossa is a sulcus, to which later on, at the anterior side a second is added, both of which bordering the insular area temporal and frontalward.
3. The first indication of the commencement of this sulcus is found a month later than after the calculation in human foetus.
4. The total operculisation of the insula is found in Semnopithecus at the stage which would be reached for human foetus in the beginning of the 6th month.

Physiologie. — "*The Olfactology of the Methylbenzol Series*". By Dr. E. L. BACKMAN from Upsala (for the present at Utrecht). (Communicated by Prof. Dr. H. ZWAARDEMAKER.)

(Communicated in the meeting of January 27, 1917).

What may be defined as an homologous series, is an arrangement of substances in the order of their atomicity which changes progressively and in a particular way in the same straight line. We know that such substances, when odorous, give a scent which also varies gradually and almost continuously, and evoke smell-sensations representing points upon an intensive scale like the atomic compositions. The question whether intermediate compensations occur between the terms of this scale is of general importance for physiology. Hitherto the stimulus-limina of the terms of only few homologous series have been established (HAYCRAFT [1]), J. PASSY [2]), ZWAARDEMAKER [3]), and quite recently the electrical phenomenon (ZWAARDEMAKER [4])).

The present communication deals with the liminal stimuli of a series not yet investigated, the possible combinations within the series, the diffusion-rate, the adsorption to electrically charged metallic plates and the intensity of the vapour-electricity of the terms. We generally made use of ZWAARDEMAKER's experimental method [5]).

The just noticeable smell was determined in the smell-chamber (l. c. p. 56) by evaporating extremely small quantities of odorous matter in aqueous solution at a carefully maintained temperature in the space of 64 Litres. The determination did not take place before 3 or 4 minutes after complete volatilization. Between two determinations the chamber was aired and the adhering scent was removed from the walls by means of chalk and a towel. In the same chamber, now provided with a back-wall of filterpaper, impregnated with the saturated aqueous solution, we measured the diffusion. By repeated determinations we established the time required for a distinct sensation of smell in the centre of the opposite wall.

For a quantitative determination of the vapour-electricity produced by spraying the several terms, we employed a glass sprayer and a circular aluminium plate 20 cm. in diameter, connected to an EXNER-electroscope. The sprayer as well as the cap of the electroscope were earthed, the aluminium plate, however, had been carefully insulated by mounting it on a block of paraffin. Invariably 10 c.c. were sprayed under an overpressure of two atmospheres through compressed air. By establishing the capacity of the electroscope with receiving plate and the magnitude of the deflections at successive voltages the number of coulombs, obtained per c.c. as a charge on the plate, could easily be ascertained. Under these experimental conditions no charge was obtained by spraying pure water. Before performing the measurements we first searched for the optimal distance between sprayer and aluminium disc at which we could be sure of the maximal charge. The tables show the averages of the charges in coulombs per c.c. of sprayed liquid, calculated from the deflections of the electroscope.

The adsorption of the odorous substances was examined for elec-

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[1]) HAYCRAFT. Brain II 1888 p. 166.

[2]) J. PASSY. Compt. rendus. Mai 1892, Mai 1893.

[3]) ZWAARDEMAKER. Die Physiologie des Geruchs. Leipzig, 1894 p. 238.

[4]) ZWAARDEMAKER. Proceedings Kon. Ak. v. Wetensch. May 27 1916. Vol. 25, p. 3.

[5]) Cf. TIGERSTEDT's Hdb. d. physiol. Methodik. Bd. 3 p. 46.

trically charged metallic plates. To this end the full scent, furnished by an olfactometric cylinder pushed off to its full length (l.c. p. 65), was sent for five minutes through a nickel-plated tube 10 cm. in length and 0,8 cm. in bore, while the insulated tube was charged to 220 volts. The metallic tube consisted of two parts insulated from each other, which, therefore, could be charged oppositely. When the five minutes had elapsed we tried to find out if any odour adhered to the wall and how long it remained there. Such was the rapidity at which the air was drawn through the olfactometric cylinder and the nickel tube that as much as 6 Litres passed every minute.

The determination of the odorimetric coefficients as well as the compensation- and combination-tests were performed with some of ZWAARDEMAKER's precision-olfactometers. The rapidity of the airstream was the same (6 Litres per minute); besides the smelling took place after the air had been streaming for precisely $^1/_4$ minute. At that moment the connection with the suction-pump was broken, the communication with the olfactometrical cylinder was arrested, smelling took place through a short side-tube at the reservoir, the latter of a capacity of about 100 c.c. Every experimental sitting comprised a number of tests of the same day together with those of previous days. Only the averages have been tabulated. The olfactometrical cylinders were invariably filled the previous day with a saturated aqueous solution of the terms of the homologous series.

Liminal Stimuli of the methyl-benzol series.

Benzol	5,3	10^{-6}	Grams per Litre of air
Toluol	2,0	10^{-6}	,, ,, ,, ,, ,,
Xylol	0,8	10^{-6}	,, ,, ,, ,, ,,
Pseudocumol	0,2	10^{-6}	,, ,, ,, ,, ,,
Durol	0,087	10^{-6}	,, ,, ,, ,, ,,

These values expressed in gram-molecules per Litre of air

Benzol	6,80	10^{-8}	gram-molecules per Litre of air
Toluol	2,17	10^{-8}	,, ,, ,, ,, ,,
Xylol	0,76	10^{-8}	,, ,, ,, ,, ,,
Pseudocumol	0,18	10^{-8}	,, ,, ,, ,, ,,
Durol	0,07	10^{-8}	,, ,, ,, ,, ,,

These determinations fully bear out HAYCRAFT's rule that in the homologous series of organic chemistry the smell-intensity at first gradually runs up as we pass from the lowest to the higher terms. By further addition of methyl-groups the amount of matter, required to produce a just noticeable smell-sensation, gradually diminishes. The increment of smell-intensity is even approximately proportional to the number of methyl-groups. The smell-intensity of toluol is about three times greater than that of benzol, that of xylol about three times greater than that of toluol and that of pseudocumol about four times greater than that of xylol. The smell-intensity of Durol is only about twice that of Pseudocumol: the increment of intensity therefore, is smaller than might be expected from the ratios found previously. Most likely this is why on further methylation there is an absolute decrease of smell-intensity.

Diffusion-rate over a distance of 40 cm. at 19° C.

The time required for a perceptible smell-sensation set up by the terms of the series at a distance of 40 cm. is:

Benzol	1 min.	15 sec.
Toluol	1 ,,	15 ,,
Xylol		60 ,,
Pseudocumol		60 ,,

Needless to say that, in order to determine the precise diffusion-rate also the tension of saturated vapour and the smell-intensity has to be taken into account.

Electrical charge by spraying.

First of all the charge of sprayed saturated solutions was determined. True, such solutions do not admit of easy comparison, the solubility in water of the members of the methyl-benzol series being widely different; still, the values serve our purpose technically.

I found the following:

	Optimal distance	Electric charge in coulombs per c.c.
Benzol	35 cm.	116.4×10^{-11}
Toluol	25 „	$92.1 \times 1.^{-11}$
Xylol	40 „	74.5×10^{-11}
Pseudocumol	35 „	32.3×10^{-11}
Durol	33 „	24.1×10^{-11}

The charges of sprayed equimolecular solutions can be readily compared. The two highest terms, however, are so little soluble, that the first term would not yield a charge in that extreme dilution. For this reason only groups should be examined in which the first three terms can indeed be taken equimolecular

	Conc.		Optimal distance	Electrical charge in coulombs per c.c.
Benzol	$\frac{1}{1200}$	normal	30 cm.	18.9×10^{-11}
Toluol	$\frac{1}{1200}$	"	35 "	18.9×10^{-11}
Xylol	$\frac{1}{1200}$	"	43 "	28.7×10^{-11}
Pseudocumol	$\frac{1}{4800}$	"	35 "	18.4×10^{-11}
Durol	$\frac{1}{200000}$	"	35 "	18.0×10^{-11}

It will be seen that on further methylation the electrical charges of an equimolecular solution rise slowly at first, later on very rapidly. Parallel to this runs a progressive insolubility in water.

Adsorptivity.

The capacity of the terms of the methyl-benzol series of being adsorbed to charged or uncharged metallic plates is remarkably slight. Benzol is the only one that adheres to the metal wall in some measure, the other substances not at all.

Olfactometry (at the stimulus-threshold).

The olfactory values appeared to be the following:

Benzol	0.3 cm.	corresponding to the liminal values in gram-molecules found in a previous experiment (stimulus-threshold).
Toluol	0.5 ,,	
Xylol	0.7 ,,	
Pseudocumol	0.8 ,,	

The olfactometrical coefficients may readily be computed from them. They are for benzol 3.3; for toluol 2.0; for xylol 1,4 and for pseudocumol 1.3.

These liminal values vary distinctly as the series advances. Each term has its peculiar odour, which enables us to tell the one from the other by smelling. The odours of benzol and toluol, however, are somewhat alike. Both are empyreumatic, that of toluol is the stronger of the two. Xylol, on the other hand, has lost the empyreumatic character almost entirely, and exhibits an aromatic quality. This quality is more obvious still in pseudocumol, so that in this respect it bears some resemblance to xylol. It seems to me that difference between xylol and pseudocumol is greater than that between benzol and xylol. Finally durol follows with an exclusively aromatic, phenol-like odour.

The Combination-test.

I feel convinced that a combination of the terms of the methyl-benzol series, without perceptible antagonism though weakening each other, may produce mixed smells, forming unmistakably a new unitary smell, though we still may trace in it the empyreumatic, respectively the aromatic quality of the component parts. It is rather

Combination	Cylinder-length	Sensation
Benzol-toluol	0.30 and 0.50 cm.	toluol-like
„ „	0.15 „ 0.25 „	„
„ „	0.07 „ 0.12 „	(none)
Benzol-xylol	0.15 „ 0.35 „	faintly benzol-xylol-like
„ „	0.07 „ 0.17 „	(none)
Benzol-Pseudocumol	0.20 „ 0.60 „	distinctly pseudocumol-like
„ „	0.15 „ 0.40 „	weak, doubtful odour
Toluol-xylol	0.40 „ 0.50 „	xylol-like
„ „	0.25 „ 0.35 „	„
„ „	0.15 „ 0.25 „	(none)
Toluol-Pseudocumol	0.35 „ 0.60 „	distinctly pseudocumol-like
„ „	0.25 „ 0.40 „	„ „
„ „	0.15 „ 0.30 „	(none)
Xylol-Pseudocumol	0.50 „ 0.60 „	distinctly pseudocumol-like
„ „	0.45 „ 0.50 „	weak mixture of pseudocumol and xylol
„ „	0.35 „ 0.40 „	(none)

immaterial whether the new sensation is looked upon as a mixed sensation or as a novel, individual one. Psychologically it seems to me e.g. that a combination of gaseous benzol and gaseous xylol yields one simple, unanalysable scent, though it contains an empyreumatic as well as an aromatic component.

From these experiments it follows in the first place, that the combination of two odours, belonging to an homologous series, yields a smell-sensation even when each separate odour is subliminal. Distinct smell-sensations were generally obtained by me with two half stimulus-limina consequently by adding two half liminal values. Here then we have to do with a summation-smell. Two subliminal stimuli, belonging to the same homologous series are added and build up together a smell-sensation. This phenomenon may be looked upon as an analogue to summation-actions occurring in another field in physiology. The present writer e.g. observed some time ago, that some rest nitrogen-compounds of various chemical composition exert a distinct influence upon the heart and the blood-pressure through their analogous physiological action, even when injected in such small quantities that of themselves they are inactive [1]).

The results of our combination-tests were to the following effect:

Combination	Cylinder-length	Sensation
Benzol-toluol-xylol	0.15—0.25—0.35 cm.	distinctly xylol-like
„ „ „	0.07—0.12—0.17 „	(none)
Benzol-toluol-pseudocumol	0.15—0.25—0.40 „	distinctly pseudocumol-like
„ „ „	0.07—0.12—0.20 „	doubtful, indistinct
Toluol-xylol-pseudocumol	0.35—0.45—0.60 „	distinctly pseudocumol-xylol-like
„ „ „	0.25—0.35—0.40 „	faintly pseudocumol-like
„ „ „	0.20—0.45—0.50 „	„ „

These combinations also substantiate our previous statement. In my judgment stimuli smaller than $^1/_2$ olfact do not build up a clear sensation.

Lastly we subjoin some combination-tests with four smell-stimuli: (See following table p. 950).

Also with these combinations every sensation is unitary. Still, in the first three groups the xylol-pseudocumol-scent supersedes. Sub-

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[1]) E. L. Backman. Thesis. Upsala 1917.

Combination	Cylinder-length	Sensation
Benzol-toluol-xylol-pseudocumol	1.80—3.00—4.20—4.80 cm.	Mixed scent of benzol-toluol and xylol-pseudocumol. The latter superseding.
Benzol-toluol-xylol-pseudocumol	1.20—2.00—2.80—3.20 cm.	Mixed scent of benzol-toluol and xylol-pseudocumol. The latter superseding.
Benzol-toluol-xylol-pseudocumol	0.15—0.25—0.35 -0.40 cm.	Mixed scent of benzol-toluol and pseudocumol. The latter superseding.
Benzol-toluol-xylol-pseudocumol	0.07—0.12—0.17—0.20 cm.	(none).

liminal stimuli, when combined, build up an accumulative sensation here also. We conclude therefore: *The combination of two or more subliminal quantities of substances of an homologous series gives rise to a distinct accumulative odour.*

It is very difficult to say whether a similar summation occurs also with superliminal quantities, though to me it does not seem improbable.

The Compensation tests.

Initially one of the olfactometrical cylinders of the double olfactometer was moved out a little to obtain a stimulus of 6 olfacts. Subsequently a small amount, followed by a larger one, of another substance was added and the smell-sensation was observed. The determinations were made at intervals of various lengths and were repeated several times on the same and on successive days. In the interval between two tests the cylinders were pushed hard up against the screen to obviate excessive volatilization or to balance the diffusion-difference within the cylinder where, otherwise, the intensity of the upper layers of the odorous substance would be reduced. We still wish to call attention to the fact, that the results obtained one day were invariably found back on the following day.

Generally speaking we found that by keeping the amount of one substance constant, and allowing the other to increase, also the smell-sensation passes gradually from one odour into the other, but that there will always be a field in which the two odours weaken, nay even cancel each other. On the whole, therefore, complete compensation of the sensations is obtained by mixing carefully apportioned quantities of odorous substances.

Cylinderlengths	Sensation
1.8 cm. Benzol + 2.3 cm. Toluol	benzol-like
1.8 „ „ + 2.4 „ „	faint, indistinct odour
1.8 „ „ + 2.5 „ „	„ „ „
1.8 „ „ + 2.6 „ „	toluol-like
0.9 „ „ + 0.9 „ „	benzol-like
0.9 „ „ + 1.0 „ „	„
0.9 „ „ + 1.1 „ „	faint, indistinct odour
0.9 „ „ + 1.2 „ „	„ „ „
0.9 „ „ + 1.3 „ „	„ „ „
0.9 „ „ + 1.4 „ „	toluol-like
3.6 „ „ + 4.7 „ „	benzol-like
3.6 „ „ + 4.8 „ „	faint, indistinct odour
3.6 „ „ + 4.9 „ „	„ „ „
3.6 „ „ + 5.0 „ „	toluol-like
1.8 „ „ + 3.7 „ Xylol	benzol-like
1.8 „ „ + 3.8 „ „	(none)
1.8 „ „ + 3.9 „ „	(none)
1.8 „ „ + 4.0 „ „	(none)
1.8 „ „ + 4.1 „ „	xylol-like
0.9 „ „ + 1.5 „ „	benzol-like
0.9 „ „ + 1.6 „ „	faint, doubtful odour
0.9 „ „ + 1.7 „ „	(none)
0.9 „ „ + 1.8 „ „	(none)
0.9 „ „ + 1.9 „ „	faint, doubtful odour
0.9 „ „ + 2.0 „ „	xylol-like
3.6 „ „ + 7.7 „ Toluol	benzol-like
3.6 „ „ + 7.8 „ „	faint, doubtful odour
3.6 „ „ + 7.9 „ „	(none)
3.6 „ „ + 8.0 „ „	(none)
3.6 „ „ + 8.1 „ „	(none)
3.6 „ „ + 8.2 „ „	faint, doubtful odour
3.6 „ „ + 8.3 „ „	xylol-like

Cylinderlengths	Sensation
1.8 cm. Benzol + 1.5 cm. Pseudocumol	benzol-like
1.8 " " + 1.6 " "	(none)
1.8 " " + 1.7 " "	(none)
1.8 " " + 1.8 " "	faint, doubtful smell
1.8 " " + 1.9 " "	pseudocumol-like
0.9 " " + 0.6 " "	benzol-like
0.9 " " + 0.7 " "	(none)
0.9 " " + 0.8 " "	(none)
0.9 " " + 0.9 " "	pseudocumol-like
3.6 " " + 3.1 " "	benzol-like
3.6 " " + 3.2 " "	faint, doubtful smell
3.6 " " + 3.3 " "	(none)
3.6 " " + 3.4 " "	(none)
3.6 " " + 3.5 " "	pseudocumol-like
3.0 " Toluol + 1.9 " Xylol	toluol-like
3.0 " " + 2.0 " "	faint, doubtful smell
3.0 " " + 2.1 " "	(none)
3.0 " " + 2.2 " "	(none)
3.0 " " + 2.3 " "	(none)
3.0 " " + 2.4 " "	xylol-like
1.5 " " + 0.8 " "	toluol-like
1.5 " " + 0.9 " "	faint, doubtful smell
1.5 " " + 1.0 " "	" " "
1.5 " " + 1.1 " "	(none)
1.5 " " + 1.2 " "	(none)
1.5 " " + 1.3 " "	faint, doubtful smell
1.5 " " + 1.4 " "	xylol-like
6.0 " " + 4.3 " "	toluol-like
6.0 " " + 4.4 " "	faint, doubtful smell
6.0 " " + 4.5 " "	(none)
6.0 " " + 4.6 " "	(none)
6.0 " " + 4.7 " "	faint, doubtful smell

Cylinderlengths	Sensation
6.0 cm. Toluol + 4.8 cm. Xylol	xylol-like
3.0 „ „ + 1.9 „ Pseudocumol	toluol-like
3.0 „ „ + 2.0 „ „	faint, doubtful smell
3.0 „ „ + 2.1 „ „	(none)
3.0 „ „ + 2.2 „ „	(none)
3.0 „ „ + 2.3 „ „	faint, doubtful smell
3.0 „ „ + 2.4 „ „	pseudocumol-like
1.5 „ „ + 0.9 „ „	toluol-like
1.5 „ „ + 1.0 „ „	(none)
1.5 „ „ + 1.1 „ „	(none)
1.5 „ „ + 1.2 „ „	(none)
1.5 „ „ + 1.3 „ „	pseudocumol-like
6.0 „ „ + 4.1 „ „	toluol-like
6.0 „ „ + 4.2 „ „	faint, indistinct smell
6.0 „ „ + 4.3 „ „	(none)
6.0 „ „ + 4.4 „ „	(none)
6.0 „ „ + 4.5 „ „	pseudocumol-like
4.2 „ Xylol + 2.8 „ „	xylol-like
4.2 „ „ + 2.9 „ „	faint, indistinct smell
4.2 „ „ + 3.0 „ „	(none)
4.2 „ „ + 3.1 „ „	(none)
4.2 „ „ + 3.2 „ „	(none)
4.2 „ „ + 3.3 „ „	(none)
4.2 „ „ + 3.4 „ „	faint, doubtful smell
4.2 „ „ + 3.5 „ „	pseudocumol-like
7.0 „ „ + 5.0 „ „	xylol-like
7.0 „ „ + 5.1 „ „	(none)
7.0 „ „ + 5.2 „ „	(none)
7.0 „ „ + 5.3 „ „	(none)
7.0 „ „ + 5.4 „ „	(none)
7.0 „ „ + 5.5 „ „	faint, doubtful smell
7.0 „ „ + 5.6 „ „	pseudocumol-like

The compensation is indistinct only for benzol and toluol. We also ascertained in every test that in the neighbourhood of the compensation-point smell-sensation grows weaker.

We invariably found the ratios of the cylinderlengths of two antagonistic substances to be the same for a particular compensation-pair. They appear again when reducing the cylinderlengths to olfaction-values.

Number of olfacts				Sensation
Benzol	Toluol	Xylol	Pseudocumol	
3	2.4			(0)
6	4.9			(0)
12	9.7			(0)
3		2.5		0
6		5.6		0
12		11.4		0
3			1.0	0
6			2.0	0
12			4.2	0
	3	1.7		0
	6	3.2		0
	12	6.5		0
	3		1.4	0
	6		2.7	0
	12		5.5	0
		6	3.9	0
		10	6.6	0

This investigation, then, confirms the rule previously found by Zwaardemaker 1) for other compensation-pairs, viz. when *a* olfacts

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1) H. Zwaardemaker, die Physiologie des Geruchs, Leipzig 1895 S. 194. Cf. Zwaardemaker the restriction to the domain of cardinal values in Tig. Hdb d. Physiol. meth. Bd. III 1 p. 89. Beyond the domain of the cardinal values the log. of *a* and *b* will have to be multiplied.

of one odorous substance are neutralized by b olfacts of another, $n \times a$ and $n \times b$ olfacts will similarly neutralize each other. What strikes us as novel and remarkable, is that a typical compensation was found with odorous substances of one and the same homologous series.

Complete compensation may of course also be obtained among four homologues. We subjoin the results of an experiment with the quadruple olfactometer

Number of olfacts				Sensation
Benzol	Xylol	Toluol	Pseudocumol	
12	11.4	12	5.5	(none)
12	11.4	10	5.5	pseudocumol-like
12	8.6	12	5.5	benzol-like
12	13.6	12	5.5	xylol-like
12	0	12	0	benzol-toluol-like

Another similar compensation is e.g. benzol 12, pseudocumol 4.2, toluol 12 and xylol 6.5 olfacts. This combination also gives complete compensation. Likewise toluol and pseudocumol can compensate xylol alone, e.g. 14 olfacts of toluol and 3.9 olfacts of pseudocumol with 13.5 (7.4 + 6.0) olfacts of xylol.

SUMMARY.

1. The liminal stimulus of the odorous substances of the methylbenzol series exhibits for the first four terms a fairly proportional decrease with increasing methylation. Consequently the liminal smells intensify as the series advances. Moreover they pass from a marked empyreuma (benzol-toluol) to a predominating aroma (xylol-pseudocumol) until ultimately a phenol-like odour (durol) comes to the front.

Meanwhile the smell-intensity lessens, so that a prolonged methylation may perhaps give rise to inodorous compounds.

2. The rate at which the odour is spread by diffusion is nearly the same for the four homologues examined.

3. The electric charge produced by spraying an equimolecular solution of the substances augments slowly at first, afterwards more rapidly as the series advances.

4. The adsorption of the substances examined to electrically charged metallic plates is next to none.

5. The four terms examined may evoke unitary mixed sensations, without a trace of antagonism.

6. Combinations of two or more subliminal stimuli yield (anyhow in the proportions of our tests) a distinct accumulative effect. It seems, however, that to obtain this result no less than $^1/_2$ olfact should be taken.

7. The combination of two or more fairly strong stimuli of the homologous series examined by us effectuates with careful apportionment a complete compensation of the mixed odours to the zero-point of the sensitive scale. This takes place without antagonism. Only when the odorous substances are closely allied, as in the case of benzol and toluol, the compensation is less complete.

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CONTENTS.

Chemistry. — *"Ultramicroscopic investigation of very thin metal and saltfilms obtained by evaporation in high vacuum".* By Prof. W. REINDERS and L. HAMBURGER. (Communicated by Prof. J. BÖESEKEN).

(Communicated in the meeting of October 28, 1916).

It is a well-known fact, that the blackening of incandescent lamps must be attributed to the slow sublimation of the material used for filament, which settles on the bulb as a very slight deposit, and becomes darker proportional to the time of incandescence.

Thin deposits on the bulb may be obtained in this manner, not only of carbon, tantalum and tungsten, but also of other metals such as silver and gold [1]) and — as one of us has described [2]) — even of metal-compounds ($NaCl$, Na_2O, $NaOH$, Na_2WO_4 etc.), when they are brought to a lower or higher rate of incandescence in the high vacuum of the incandescent lamp.

Whereas the films of carbon, tantalum and tungsten are black, those of silver and gold are coloured and the salts form an absolutely colourless, clear deposit, which is invisible to the eye, as long as it remains in the same condition.

We have subjected these films to an ultramicroscopic investigation, the results of which we shall communicate in the following pages.

Rock-salt. This deposit was obtained by heating in high vacuum (0.0003 to 0.0010 mm.) a tungsten filament, which had been fixed in an incandescent lamp in the usual manner and had been partially covered with solid $NaCl$. Before the connection with the air-pump had been broken by melting off, the lamp had been heated to 380° C. in order to exhaust the gases from the bulb.

As long as the vacuum is maintained, the salt-deposit is perfectly clear and colourless, so that it is imperceptible to the eye. When the lamp is opened, so that the moist air can enter, the deposit soon becomes opalescent and especially after the lapse of some time this phenomenon becomes more pronounced. When a bit of the bulb was brought under the ultramicroscope [3]), as soon as possible

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[1]) M. FARADAY. Phil. Trans. Roy. Soc. London **147**, 145 (1857).

G. T. BEILBY. Proc. Roy. Soc. London **72**, 226, (1903).

L. HOULLEVIGUE. Ann. chem. et phys. (8) **20**, 138 (1910).

[2]) L. HAMBURGER. Chem. Weekbl. **13** (1916), 535.

[3]) Cardioid condenser, Special object glass V of Zeiss, Compensation ocular 18, glycerine-immersion.

after the opening of the lamp, the deposit showed a network of small, radiant ultramicrons. In course of time this network became distinctly coarser and after some hours separate particles could be discerned, while after a still longer period some of them clearly revealed the cubic form of rock-salt crystals. Hence a slow *crystallization* or *re-crystallization* of the rock-salt takes place.

This change caused by the addition of the moist air made it impossible to observe the layer of rock-salt in its initial state. In order to do this it should be shut off from the gasphase, both in the course of and after its formation, and after the opening of the balloon it should remain shut off as well.

Closure with oil, obtained by opening the balloon under oil, so that the latter entered instead of the air, proved unsuccessful. A better result was obtained by using Canada Balsam, which had been freed from dissolved gases by exposure to high vacuum under gentle heating, by which process the balsam got thickened. During this operation it had been present in a side-tube, which had been fused to the balloon. After the formation of the NaCl-film the Canada Balsam is made into a thin fluid by cautious heating and is spread out over the layer of salt, so that it was partly covered by the balsam. When the lamp was then opened, the layer of salt underneath the Canada-Balsam remained perfectly clear and transparent, whereas in the uncovered places it turned white.

In the ultramicroscope the layer covered with Canada Balsam was optically empty.

This became quite obvious when adjusting at the boundary-line Canada Balsam-air. Where the air had been allowed to operate the field was strongly lighted by the network of small ultramicrons, where the Canada Balsam had protected, the field was dark. The two fields were separated by a very distinct line of demarcation.

So the rock-salt is deposited on the bulb as an optically homogeneous phase, crystals are altogether absent, the molecules are absolutely disordered, and we are dealing here with the amorphous, vitreous condition, that may be compared to the undercooled fluid.

This condition is metastable and gradually passes into the crystallised, stable condition. Various circumstances may start or accelerate this conversion:

1. *Access of air.* Water vapour especially plays an active part here, very dry air has hardly any effect, breathing upon it makes the conversion take place very rapidly.

2. *A Rise of Temperature.*

3. *Increase of thickness of the layer by more prolonged sublimation.*

Silver. Instead of the tungsten filament a thin wire of very pure silver had been fixed in an incandescent lamp. The lamp had been carefully exhausted with the help of liquid air, during which process it had been heated to 380° C, and during a short time the wire itself was kept at a low dull-red heat by means of an electric current. Thus any impurities that might be found on the bulb and the wire were removed as well as possible. Hereupon the lamp was melted off. If now the filament was brought to a deep-red heat with the aid of the electric current a deposit soon appeared against the bulb. With a prolonged sublimation the colour of this deposit changed from an original pale yellow into orange-yellow, red, violet and finally into blue.

On opening the lamp we observed as a rule a deepening of the colour, i.e. a change of colour trending to yellow-red-blue. In some cases this change was quite pronounced and the colour became yellowish-red, almost blue, in other cases it was less marked and the yellow became only darker or more reddish-violet. The colours and particularly the order in which they occur during their formation and conversion are quite similar to those we observe with colloidal silver and photohaloïds.[1]) Hence we cannot but think that likewise they all must be attributed to one and the same cause, viz. to the presence of small dispersed particles of the pure metal.

The ultramicroscopic investigation has quite corroborated this view.

We also find this similarity in the case of gold. As early as 1857 FARADAY[2]) explicitly pointed this out in his classical experiments with extremely thin layers of gold and colloidal solution of gold. In both cases he accepted as an explanation that the colours were caused by small separate particles of the pure metal.

Ultramicroscopic examination. We once more observed a splinter of the balloon with the aid of the ultramicroscope (Cardioidcondenser).

The red and the blue deposits and in many cases the yellow ones as well were optically quite soluble. They proved to consist of a closely connected network of ultramicrons, on the whole varying in colour, viz. blue, yellowish-brown, or green, the *principal* colour of which being complementary to the colour observed macroscopically; hence macroscopic: blue, ultramicroscopic: yellowishbrown; macrosc.:

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[1]) W. REINDERS. Chem. Weekbl. 1910, 971 and 1001; Zeitschr. f. phys. Chem. **77**, 213, 356.

[2]) M. FARADAY. Bakerian Lecture, Phil. Trans. Roy. Soc. London, **147** (1857), 145.

red, ultramicr: green etc. With a very rare exception exhibiting Brownian motion, they all were immovable, therefore tied fast to the back-ground.

The apparent size of these ultramicrons, hence their mutual distance was almost equal in the several compounds, there was at any rate no obvious connection between the variations in this apparent dimension and the macroscopically investigated colour of the deposit. So this proves that the *distance* between the particles is about equal, if, at any rate the thickness of the film is not such, that more layers are lying one above the other. But the intensity of the light of the particles varies; with the red and the blue films it is fairly strong; with the yellow films it is on the whole but slight.

Associating this with the theory of RAYLEIGH [1]) we draw the conclusion: *that the dimensions of the yellow ultramicrons are smaller than those of the red and the blue ones.*

There is this essential difference between the coloured silver-film and the clear layer of salt that the latter is optically homogeneous, the former distinctly heterogeneous.

The question arose whether this heterogeneousness was already present in the silver deposit in its original form, or whether it proceeded from the effect of the air as is the case with rock-salt.

To obtain a solution to this question we again fused a side-tube with Canada Balsam to the silver-lamp as we had previously done with the salt-lamp. By gentle heating in high-vacuum the balsam was freed from the dissolved gases and when the deposit of silver had been formed it was carefully spread over it, so that part of it was covered by the Canada Balsam.

On opening the lamp the colour of the deposit of silver that had been covered with Canada Balsam remained the same, whereas in the uncovered part a distinct change of colour set in.

In *ultramicroscopic investigation* it appeared that the part covered with Canada Balsam also consisted of a network of ultramicrons. There was no marked difference between the apparent size of these ultramicrons and those of that part that had been exposed to the air. A sharp demarcation line between the two fields, such as had been observed with the rocksalt and which here too, was macroscopically quite perceptible on account of the variation in colour, was looked for in vain. The two parts imperceptibly passed into each other: the network itself showed no difference and only in the total-colour of the ultramicroscopic field, which had more of brown in it

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[1]) Phil. Mag. (4), 41, 107, 274, 447 (1871).

with the blue deposit, more of green with the red deposit, could any difference be observed.

The difference was quite obvious however, in a few isolated cases, with a lamp where the evaporation of the silver had been effected by a very slow process and where the deposit had assumed a pale yellow colour, which by the influence of the air passed into violet. The field that had not been covered with Canada Balsam produced the brilliantly lighted mosaic of connected ultramicrons, while the protected field was optically hardly soluble, and but very faintly showed a similar network. The line of demarcation between the two fields was very marked.

With the deposit of silver too, access of air results in a coarsening of structure, as we noticed before in the case of rocksalt.

We have not been able to observe it utterly structureless, such as the layer of salt. Yet the possibility remains that the heating to 60°, necessary to equally spread the considerably thickened Canada-Balsam, or even the mere contact with Canada Balsam is sufficient to prepare the passing into a more stable, granular condition and that the primarily formed greenish yellow deposit is structureless indeed, hence molecular-dispersed.

It further appears that with an increasing thickness of the layer, even without access of air, the discontinuity, the construction from separate particles becomes more obvious. In this respect too, there is analogy between the deposit of silver and that of salt.

The difference between the silver-film and the salt-film is a more gradual and a less essential one than seemed at first sight. In either case there arises primarily a thin layer of a great homogeneity, which, however, is unstable and shows a tendency to contract into separate particles. This tendency grows proportional to the increase of thickness and is also promoted by the presence of catalysers as vapour. With silver, however, the instability is much greater than with rocksalt; therefore we always find that with a very slight thickness deposits of silver are no longer homogeneous, but have separated into accumulations of small particles.

In connection with KNUDSEN's [1]) experiments on the influence of the temperature of condensation on the nature of the matter condensed, we also modified the temperature of the bulb during the entire time of the burning. From his experiments KNUDSEN draws the conclusion that when metal-vapour molecules strike against a bulb, the chance of their being reflected is extremely slight, if

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[1]) Ann. d. Physik, (4) **50**, 472 (1916).

only the temperature of that bulb is kept below a certain critical value, varying for each metal. For mercury he fixed this temperature at —135°, for silver he estimates it at 575°. Though the room temperature at which the glass was kept during our experiments, is far below this critical temperature of the silver, so that we may assume that the silver-molecules striking against the bulb do not reflect, yet it seems unacceptable, that they resume their position of rest immediately after the collision. The glass- and silver-molecules situated near the point of impact will be disturbed in their state of equilibrium and get into a state of motion; small separate particles of silver come into each other's sphere of attraction and find an opportunity of uniting into greater conglomerates. This motion will be so much the stronger and therefore the chance of the agglomeration of silver-molecules so much the greater, in proportion as the temperature of the bulb that is struck, is higher. By maintaining a low temperature of the bulb during the entire duration of the burning of the lamp, the possibility of acquiring an entirely structureless, amorphous deposit is heightened.

Hereupon some experiments were made in which the silver was sublimated, while the lamp had been cooled down to a temperature of liquid air, this temperature being maintained during the whole time of burning The sublimate showed pretty nearly the same colours and succession of colours as those which arose in room temperature, any appreciable difference could not be observed. Thus a lowering of temperature to below 20° has no material effect on the nature of the film, when observed with the naked eye.

If however, the temperature is raised after the deposit has formed a change of colour sets in, in high vacuum as well. By heating to 260° for 20 minutes it passed back from reddish-violet to yellowish-brown, and yellowish-brown to a faint yellowish-green.

Ultramicroscopically this deposit showed larger particles, being better discernible by themselves. Several of them had loosened from the bulb and freely moved in the immersion-liquid.

Faraday [1]) and Beilby [2]) also experimenting on the much thicker deposits of gold and silver which they had prepared by lamination of compact metal or precipitation by chemical process, both observed a retrogression of the tint and an agglomeration into larger particles as the effect of heating. When we remember that as a rule an enlargement of the particles is accompanied by a deepening of the

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[1]) l. c. page 1.

[2]) l. c. page 2.

tint (e.g. coagulation of gold sol) the connection between these two changes is not very clear, unless we take for granted that the totalcolour of the film is in the main that of a connected film, which both FARADAY and BEILBY have noticed between the larger conglomerations. Our own observations are not contrary to this statement. The conversion of colour quite makes the impression as if the film had become thinner. But we have not been able to ascertain a discrete and thinner film [1]).

Gold. Similarly as with silver in this case too, thin deposits could be obtained, the colour of which first became pink, with prolonged sublimation blue and then green.

Here again access of the air of the room resulted in a deepening of colour.

When observed ultramicroscopically a mosaic of brownish-red, greenish or more bluish coloured ultramicrons could again be discerned. The primary colour of this mosaic was complementary to the tint that was observed microscopically. So we see that on the whole this image bears a perfect resemblance to that of the deposit of silver.

Tungsten. The deposit of tungsten differs considerably from the silver-deposit.

Firstly it shows no colours, but immediately turns a muddy gray even with the thinnest layers.

Secondly these films, even the quite dark ones are ultramicroscopically utterly insoluble.

The investigation of the tungsten deposits which had been obtained by cathodic atomizing proves that the power of reflection of tungsten is in itself no impediment for the discrete particles being made perceptible, if only the latter are large enough (vide infra). So we must accept that the tungsten particles are much smaller than those of silver and gold. This admission is not a very improbable one. For in a normal temperature the vapour tension of tungsten is infinitely lower than that of gold or silver. Whereas a silverfilament glowing at a dull-red heat (600) produces a distinct deposit on the bulb, even after half an hour, we require a heat of nearly 3000° [2])

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[1]) J. C. M. GARNETT (Phil. Trans. Roy Soc. London A. Pos. 237 1905)) tries to account for this change of colour by accepting that by heating the density of the layer decreases while consequently the volume of air, enclosed by the metal-particles increases. But such an assumption is not confirmed by any observation.

[2]) With our tests we required for the formation of the tungsten deposit about 6 hours, for the silver deposit from 30 minutes to 2 hours.

to obtain a similar effect with tungsten. Therefore the difference between the temperature of evaporation and the temperature of condensation is for silver only 600°, for tungsten nearly 3000°. The "refrigeration" of the tungsten-vapour is thus exceedingly strong and therefore we may expect the state of agglutination of the tungsten-condensate on the bulb to differ but immaterially from that in the state of vapour. From the behaviour of tungsten towards nitrogen we deduce that the tungsten-vapour is monoatomic[1]). From KNUDSEN's[2]) measurements of the accommodation-coefficient I. LANGMUIR[3]) too draws the conclusion that a reflection of the tungsten-atoms against a cold surface is highly improbable (c. f. besides p. 963). So we may expect that the tungstenatoms are immediately fixed by the bulb they collide against and will not find any opportunity for the formation of large conglomerations.

If the mere conglomeration of molecules is difficult, an arrangement as required by the crystallised state will surely not take place. Hence the state of the tungsten-deposit will agree with that of the rocksalt-deposit, with the amorphous-vitreous.

The thickness of this layer is extrememely slight. A bulb of 120 c.m². covered with 0.12 m.gr. tungsten was already decidedly grey-coloured. If we fix the specific weight of tungsten at 20, then the thickness of that layer is calculated 0.5 $\mu\mu$. But layers of a fourfold thickness were quite insoluble (optically) as well.[4])

Deposits of metal obtained by means of Cathodic Atomizing.

By cathodic atomizing as well, very thin deposits of metal may

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[1]) I. LANGMUIR. Journn. Amer. Chem. Soc. **35**, 931 (1913).

[2]) Ann. d. Physik, **34**, 593 (1911).

[3]) Phys. Review 1913², 332. C. f. also, Ibid 1916² 149.

[4]) This difference in dimension of the particles of silver and tungsten is also manifested in the conductivity of thin films of metal, on which S. WEBER and E. OOSTERHUIS have just now published very careful measurements. (Proc. of the the Koninkl. Akad. v. Wetensch. Amsterdam **25** (1916), 606).

For the appearance of a perceptible conductivity it will be necessary that the separate particles coalesce or that their distance has fallen at any rate below a certain minimum value. With a forming layer of metal this condition will be satisfied so much the sooner (i.e. with a lesser average thickness) the smaller the dispersed particles are. W. and O. discovered that the conductivity of tungsten becomes perceptible with a thickness of 0,5 $\mu\mu$ and that it shows a marked increase at 2,5 $\mu\mu$; for silver these figures were respectively 65 and 25 $\mu\mu$.

be obtained on the bulb[1]). We have prepared such deposits so that we could compare them with the deposits obtained by evaporation.

As a cathode we used a loop-shaped or spiral-shaped curved wire of the metal to be atomized, which was connected *inside* the glass with the platinum feeding-wires. It had been placed in a pear-shaped balloon, the latter being united by a narrow tube to the space of the anode, in which the platinum-anode was. The latter had been entirely fused into a quartz-tube, with the exception of the extreme point of some mm^2 surface. The atomizing took place under a pressure of $^1/_{15}$ mm. mercury, in a dry, oxygenless, current of hydrogen, which ran from the space of the cathode to the space of the anode to wash away any gases that might come from the anode.

Silver. The development of heat during the atomizing was such that the filament partially melted. As the vapour-tension of silver is rather high at the melting-point, part of the silver will evaporate in the cathode-vacuum and sublimate against the bulb. The deposit thus obtained will therefore be formed partly by atomizing, but partly by sublimation as well.

The colour of the deposit was blue, violet, red or yellow, proportional to the shorter or longer distance from the bulb to the cathode.

Ultramicroscopically it again showed in the first place a mosaic of very small ultramicrons, of equal dimensions, and perfectly similar to the deposit obtained by evaporation. By the side of these ultramicrons, however, or strictly speaking in the background were much larger particles of about 1 μ diameter and separated from each other by a distance of 6—12 μ.

As with the experiments on the sublimation deposit, here too part of the layer had been covered with Canada Balsam, which being softened by careful heating, had been extended over the film, when still in the vacuum. By this process the twofold nature of the particles of silver clearly came to light. Whereas the Canada Balsam had no effect on the mosaic of the underground, the coarser particles above were loosened by the balsam and had accumulated on the rim of the drop. The line of demarcation of the Canada Balsam which otherwise would be imperceptible was now very clearly marked by this line of radiant larger particles.

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[1] M. Faraday, l.c. pag. 1.

L. Houllevigue, Ann. chim. et phys. (8) **20,** 138 (190); **21,** 197 (1911).

H. Fritze, Ann. der Physik, (4) **47,** 763 (1915).

B. Pogany, Ann. d. Physik. (4) **49,** 481 (1916).

A. Riede, Ann. d. Physik. (4) **45,** 881 (1914).

Evidently the silver of the underground is due to evaporation and the larger particles proceeded from atomizing.

Consequently evaporation produced a much finer deposit than atomizing.

The results which H. FRITZE[1]) obtained with cathode-atomizing of silver agree with this statement. For he explicitly states that the colour of his deposits was never yellow, though he devoted much attention to the exhausting and the removing of vapour of water and other gases absorbed by the bulb. As a rule the deposit was blue, with a decrease of thickness it became red, while finally with still thinner layers, no distinct colour can be identified, it results in "jede Andeutung einer ausgeprägten Farbtönung zu verlieren".

Such a vague, non-characteristic colour as indicated by FRITZE may also be found in the film proceeding from the effect of air on our thin greenish-yellow silver-deposits. It is, as we have seen, distinctly heterogeneous and much coarser of granule than the greenish-yellow layer formed by slow sublimation.

Tungsten. Again the colour of the film was black, like that obtained by evaporation. It appeared ultramicroscopically that by rapid atomizing very course particles were formed (2 to 5 μ diameter); by slow atomizing small particles arose, radiant without colour and — also with an extremely thin black deposit — so numerous that they filled the whole field. So here again it appears that coarser particles are obtained by cathode atomizing than by evaporation [2]).

SUMMARY.

1. The clear, colourless condensate of rock-salt that settles on the bulb during evaporation in high vacuum is, also under ultramicroscopic investigation, optically homogeneous and must be considered as a salt in the amorphous-vitreous state.

2. The opalizing, which this deposit undergoes, by the influence of moist air, originates in the formation of separate crystals, whose growth could be followed ultramicroscopically.

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[1]) Ann. der Physik (4) **47**, 763 (1915).

[2]) Of course, the coarseness of the particles in cathode-atomizing depends on the temperature the material attains during the process. When the temperature is high coarser particles are torn off than when the cathode-atomizing is conducted in such a manner (e.g. by repeated rests) that the temperature of the material remains lower.

3. During evaporation in vacuum silver develops a deposit against the bulb. With increasing thickness the colour of this deposit passes through greenish-yellow, orange, red, violet and blue.

4. The red, the violet and the blue films are distinctly heterogeneous. They consist of a network of very small ultramicrons. The yellow deposit shows a hardly perceptible heterogeneousness and approaches in structure the amorphous-vitreous state.

5. The deposits are not proof against the influence of moist air. The colour changes in the direction yellow → red → blue and the structure becomes coarser. Heating likewise causes a coarsening of structure.

6. Gold forms — in a similar way as silver — coloured deposits, which are ultramicroscopically heterogeneous.

7. Tungsten forms a black deposit, ultramicroscopically it is not soluble.

8. Deposits obtained by cathode-atomizing consist as a rule of coarser particles than the evaporation-deposits.

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Works Ltd.

Physics. — "*The virtual displacements of the electro-magnetic and of the gravitational field in applications of* HAMILTON's *variation principle*", By Dr. A. D. FOKKER. (Communicated by Prof. LORENTZ).

(Communicated in the meeting of January 27, 1917.)

In some papers on EINSTEIN's theory of gravitation Prof. LORENTZ[1]) recently applied HAMILTON's principle to the deduction of the principal equations of this theory from one single variation law. Starting from an invariant equation he was able to reach conclusions which again were represented by invariant equations. It was however not necessary to keep the equations invariant during the whole deduction. On the contrary, an artifice, consisting in the choice of a specially defined virtual displacement (without taking into consideration the conditions of invariancy), proved very useful.

Now it is possible to let the invariancy exist continually during

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[1]) H. A. LORENTZ, *On* HAMILTON's *principle in* EINSTEINS *theory of gravitation, Proceedings*, Kon. Ak. v. Wet. Amsterdam, XIX, p. 751. *Over* EINSTEINS *theorie der zwaartekracht*, I, II, III, Verslagen, Kon. Ak. v. Wet. XXIV, p. 1389, 1759, XXV, p. 468.

the deductions; and that in a way which fully appreciates the fact that the tensor of the ten gravitation potentials and the tensor of the four electrodynamical potentials, being directed quantities, have a geometrical character (§ 12 etc.). Moreover the tensors of stress, momentum and energy appear in a new way from the variation calculation. In the following paragraphs this will be shown. Thanks to the cited papers and to some others, a short indication will often suffice.

The variation principle.

1. For a material particle, falling under the influence of a force, Hamilton's principle takes the form:

$$0 = \delta \int_1^2 - m\, ds + \int_1^2 \Sigma(p)\, k_p\, \delta r^p\, ds,$$

where m is the coefficient of mass of the particle, ds the arc-length of the world-line run by the particle in the world referred to a system of four space-time parameters x_1, x_2, x_3, x_4. Further $k_p (p = 1, 2, 3, 4)$ represents the four-vector of the force acting on the particle, while δr^p $(p = 1, 2, 3, 4)$ denote the components of the virtual displacements. In the variation of the motion there corresponds to each point-instant x_m $(m = 1, 2, 3, 4)$ of the unvaried path a point-instant $x_m + \delta r^m$ $(m = 1, 2, 3, 4)$ of the varied path. The final points of the path remain unvaried. As usually we assume

$$ds^2 = \Sigma(ab)\, g_{ab}\, dx_a\, dx_b,$$

where g_{ab} $(a, b = 1, 2, 3, 4, g_{ab} = g_{ba})$ are the gravitation potentials.

If the particle has an electric charge, so that it is influenced by an electro-magnetic field this may be taken into consideration by writing

$$0 = \delta \int_1^2 (- m ds + \lambda \Sigma(l)\, e \varphi_l\, dx_l) + \int_1^2 \Sigma(p)\, k_p\, \delta r^p ds.$$

Here φ_l $(l = 1, 2, 3, 4)$ represent the electro-dynamic potentials, four quantities changing from point to point and determining the field. λ is a constant determined by the choice of the units of mass and charge in which m and e are expressed. Now k_p no longer contains the electric forces.

2. Applied to a limited extension of the four-dimensional world Hamilton's principle is represented by the equation:

$$0 = \delta \int L \, dx_1 \, dx_2 \, dx_3 \, dx_4 + \int \Sigma^{(p)} K_p \, \delta r^p \, dx_1 \, dx_2 \, dx_3 \, dx_4 \quad . \quad . \quad (1)$$

Here K_p denotes the p^{th} component of the force acting on the system per unit of volume. $\sqrt{-g} \, dx_1 \, dx_2 \, dx_3 \, dx_4$ being a scalar (if g is the determinant of the g_{ab}), $K/\sqrt{-g}$ and not K must be a covariant vector, which further will be denoted by k. For the same reason not L, but $L/\sqrt{-g}$ must be a scalar, if the variation law shall be expressed invariantly. We suppose the function of LAGRANGE L to consist of different separate parts for the gravitational field, for the matter, for the electro-magnetic field and for the electric convection-current.

Structure of the function of LAGRANGE.

3. The contribution of the *gravitational field* to L will be denoted by $\sqrt{-g} \, H$. It will be known, that for H must be taken $G/2\varkappa$, where G is a scalar indicating the curvature of the field figure and $\varkappa$ the gravitation constant. By means of RIEMANN's symbol, G may be expressed as follows:

$$G = \Sigma(im) \, g^{im} \, G_{im},$$

$$G_{im} = \Sigma(kl) \, g^{kl} \, (ik, lm),$$

$$(ik, lm) = \tfrac{1}{2} \left(g_{im,kl} + g_{kl,im} - g_{il,km} - g_{km,il} \right) +$$

$$+ \Sigma(ab) \, g^{ab} \left\{ \begin{bmatrix} im \\ a \end{bmatrix} \cdot \begin{bmatrix} kl \\ b \end{bmatrix} - \begin{bmatrix} il \\ a \end{bmatrix} \cdot \begin{bmatrix} km \\ b \end{bmatrix} \right\}.$$

The quantities g^{ab} $(a, b = 1, 2, 3, 4)$ are the algebraic complements of the g_{ab}; $g_{im,kl}$ is written for the second derivative of g_{im} with respect to x_k and x_l; and CHRISTOFFEL's symbols mean:

$$\begin{bmatrix} im \\ a \end{bmatrix} = \tfrac{1}{2} \left(g_{ia,m} + g_{ma,i} - g_{im,a} \right).$$

Further the notation g^{ab}_{c} and g^{ab}_{cd} for the first, respectively second derivative of g^{ab} with respect to x_c and x_d will be used from time to time.

4. The contribution of the *matter* to L will be denoted by $\sqrt{-g} \, R$. In order to find out what has to be put for $\sqrt{-g} \, R$ we must investigate how the element $-m \, ds$, which occurs in the variation law for the motion of a single material particle, can be extended to $\sqrt{-g} \, R \, dx_1 dx_2 dx_3 dx_4$ for the matter we are considering. LORENTZ has indicated ¹) what $\sqrt{-g} \, R$ becomes for a con-

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¹) l. c. XIX p. 754, XXV, p. 478.

tinuously varying current of incoherent material points or for a more general case in which there are acting certain molecular forces between the points.

For an ideal gas $\sqrt{-g}\, R$ will be the sum of the elements of the world-lines described per unit of time by the molecules present in a unit of volume, each element multiplied by $-m$, if m is the mass of the molecule that describes the element.

Now it is known that for a molecule with the mass m the momentum is given by

$$i_a = - m \Sigma (b) g_{ab} \frac{dx_b}{ds}$$

for $a = 1, 2, 3$, and that the energy is $-i_4$. For an ideal gas the expressions for the stresses, the momentum and the energy per unit of volume and the energy-current can be written down directly. Without entering into details by introducing a distribution function I only give the table of notations

$$\begin{array}{cccccc} \sqrt{-g}\, T_1^1 & \sqrt{-g}\, T_1^2 & \sqrt{-g}\, T_1^3 & \sqrt{-g}\, T_1^4 & & X_x \; X_y \; X_z - I_x \\ \sqrt{-g}\, T_2^1 & \sqrt{-g}\, T_2^2 & \sqrt{-g}\, T_2^3 & \sqrt{-g}\, T_2^4 & & Y_x \; Y_y \; Y_z - I_y \\ \sqrt{-g}\, T_3^1 & \sqrt{-g}\, T_3^2 & \sqrt{-g}\, T_3^3 & \sqrt{-g}\, T_3^4 & (=) & Z_x \; Z_y \; Z_z - I_z \\ \sqrt{-g}\, T_4^1 & \sqrt{-g}\, T_4^2 & \sqrt{-g}\, T_4^3 & \sqrt{-g}\, T_4^4 & & S_x \; S_y \; S_z \; E. \end{array} \qquad (2)$$

Here the coordinates x, y, z and t 1) are supposed to be used. T_a^m is a mixed tensor. It may be called the *dynamical* tensor. It is not symmetrical. The covariant tensor

$$T_{ab} = \Sigma (m) g_{mb} T_a^m ,$$

on the contrary is symmetrical.

It may be remarked that the sum of the diagonal components is equal to

$$\Sigma (a) \sqrt{-g}\, T_a^a = - \sqrt{-g}\, R.$$

5. The contribution to L of the *electric current* and the electromagnetic field may be divided into two parts, $\lambda \sqrt{-g}\, S$ and $\lambda \sqrt{-g}\, M$, λ being the same constant as in § 1.

For $\sqrt{-g}\, S \; dx_1 dx_2 dx_3 dx_4$ we take the extension of the element $\Sigma (l)\, e\, \varphi_l dx_l$ that occurred in the variation law for a single charged particle. If the extension is effected in such a way that we pass to a continuous electric convection-current, we find

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1) X_x, Y_x, Z_z are the forces, exerted in the direction of X, Y, or Z by the surroundings of a unit cube, on a face for which the outwardly directed normal has the direction of the axis indicated by the index.

$$\sqrt{-g}\, S = \Sigma\,(m) \sqrt{-g}\, W^m \varphi_m.$$

$\sqrt{-g}\, W^m$ $(m = 1, 2, 3, 4)$ denotes what is usually indicated by ϱv_x, ϱv_y, ϱv_z and ϱ. Here, as in other places, the factor $\sqrt{-g}$ occurs because we take the different quantities per units of time and volume, expressed in the coordinates and not in natural units. It is to be noted that at a change of the g_{ab}, $\sqrt{-g}\, W^m$ remains unchanged. This corresponds to the fact that for a single charged particle the term $\Sigma\,(m)\, e\varphi_m dx_m$ is independent of the gravitation potentials.

For the *electro-magnetic field* the scalar may be constructed in the following way. From the potentials the covariant field-intensities are derived:

$$f_{pq} = \frac{\partial \varphi_q}{\partial x_p} - \frac{\partial \varphi_p}{\partial x_q}.$$

From these we form the contravariant intensities of the field:

$$F^{ab} = \Sigma\,(mn)\, g^{am} g^{bn} f_{mn}.$$

Finally we form the scalar:

$$M = -\tfrac{1}{4} \Sigma\,(abmn)\, g^{am} g^{bn} f_{ab} f_{mn},$$

$$= -\tfrac{1}{4} \Sigma\,(ab)\, F^{ab} f_{ab}$$

Further it may be remarked here, that

$$\frac{\partial M}{\partial f_{ab}} = -\tfrac{1}{2} F^{ab}, \text{ and } \frac{\partial M}{\partial g^{am}} = -\tfrac{1}{2} \Sigma\,(bn)\, g^{bn} f_{ab} f_{mn}.$$

SCHWARZSCHILD [1]) has already used the integrand $\sqrt{-g}\, S$ in the variation law. Recently TRESLING [2]) has communicated to the Academy of Sciences how this term may be used in HAMILTON's principle.

Except as to the sign, the term $\sqrt{-g} M$ corresponds to the term used by LORENTZ, who writes ψ_{ab} for what has been called here $\sqrt{-g} F^{ab}$ and ψ_{ab} for f_{ab}.

Variations of the field quantities.

6. In the first place we shall consider the variation which is obtained by varying the *electric field* in such a way, that everywhere the potentials φ_m are changed to an amount $\delta\varphi_m$.

The $\delta\varphi_m$ $(m = 1, 2, 3, 4)$ will be infinitesimal continuous functions of the coordinates.

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[1]) K. SCHWARZSCHILD, *Zur Elektrodynamik. I.* K. Ges. Wiss. Göttingen, Math. phys. 1903.

[2]) J. TRESLING, *The equations of the theory of electrons in a gravitation field of* EINSTEIN *deduced from a variation principle. The principal function of the motion of the electrons.* These Proceedings. XIX p. 892.

The variation becomes[1]:

$$\delta \int L\, dx_1\, dx_2\, dx_3\, dx_4 = \lambda \int dx_1\, dx_2\, dx_3\, dx_4\, \Sigma\,(mq) \left[\frac{\partial}{\partial x_q} (\sqrt{-g}\, F^{mq}\, \delta\varphi_m) + \right.$$

$$\left. + \delta\varphi_m \left\{ \sqrt{-g}\; W^m - \frac{\partial}{\partial x_q} (\sqrt{-g}\, F^{mq}) \right. \right].$$

If at the boundaries of the four-dimensional extension the $\delta\varphi_m$ are chosen equal to 0, while within this extension they have arbitrary values, then HAMILTON's principle demands that

$$\sqrt{-g}\; W^m = \Sigma\,(q) \frac{\partial}{\partial x_q} (\sqrt{-g}\, F^{mq}), \qquad (m = 1,\ 2,\ 3,\ 4) \quad . \quad (3)$$

These are the four *equations of the field* in an invariant form.

7. The second variation to be considered is a variation of the *gravitational field*. At each point-instant of the extension it may be determined by the changes δg^{ab} of the quantities g^{ab}.

If we have to do with an ideal gas, we may deduce directly that now the variation of $\sqrt{-g}\, R$ is:

$$\delta\,(\sqrt{-g}\, R) = \Sigma\,(abm)\, \tfrac{1}{2} \sqrt{-g}\, g_{ma}\, T^m_b\, \delta g^{ab}\;{}^{2)} \quad . \quad . \quad . \quad (4a)$$

Taking into consideration that,

$$\delta \sqrt{-g} = -\, \Sigma\,(ab)\, \tfrac{1}{2} \sqrt{-g}\, g_{ab}\, \delta g^{ab},$$

$$\delta M = -\, \Sigma\,(abdn)\, \tfrac{1}{2}\, g^{dn} f_{ad} f_{bn}\, \delta g^{ab} = -\, \tfrac{1}{2}\, \Sigma (abcdmn)\, g_{am}\, g^{cm}\, g^{dn} f_{cd} f_{bn}\, \delta g^{ab},$$

$$\delta M = -\, \Sigma\,(abmn)\, \tfrac{1}{2}\, g_{am}\, F^{mn} f_{bn}\, \delta g^{ab},$$

we easily find for the variation of $\sqrt{-g}\, M$

$$\lambda\, \delta\,(\sqrt{-g}\, M) = \Sigma\,(abm)\, \tfrac{1}{2} \sqrt{-g}\, g_{ma}\, E^m_b\, \delta g^{ab} \quad . \quad . \quad . \quad (4b)$$

where we have put

$$E^m_b = -\, \lambda \Sigma\,(n)\, F^{mn} f_{bn} - \lambda \delta^m_b\, M.$$

δ^m_b is a mixed tensor, the components of which are 1 or 0 according as $m = b$ or $m \neq b$. We shall also introduce the notation $E_{ab} = \Sigma (m)\, g_{am}\, E^m_b$.

We shall see further on that $\sqrt{-g}\, E^m_b$ are the stresses etc. in the electro-magnetic field in the same way as $\sqrt{-g}\, T^m_b$ are those in the matter.

For the above mentioned reason the variation of $\sqrt{-g}\, S$ will be zero.

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[1] It should be kept in mind that $\delta f^{mq} = \dfrac{\partial \delta\varphi_q}{\partial x_m} - \dfrac{\partial \delta\varphi_m}{\partial x_q}$ and that $F^{mq} = -\, F_{qm}$

Comp. for the deduction TRESLING, l.c.

[2] Comp. LORENTZ, l.c. XXV, p. 476, form. (63).

8. When g^{ab} changes by δg^{ab}, then g_c^{ab} and g_{cd}^{ab} change by $\frac{\partial \delta g^{ab}}{\partial x_c}$ and $\frac{\partial^2 \delta g^{ab}}{\partial x_c \partial x_d}$. If we consider $\sqrt{-g}\, H$ as a function of the g^{ab} and their derivatives, the variation of $\sqrt{-g}\, H$ becomes

$$\delta(\sqrt{-g}\, H) \quad \Sigma(abcd) \left[\delta g^{ab} \left\{ -\tfrac{1}{2} \sqrt{-g}\, g_{ab} H + \sqrt{-g} \frac{\partial H}{\partial g^{ab}} - \frac{\partial}{\partial x_c} \left(\frac{\sqrt{-g} \partial H}{\partial g_c^{ab}} \right) + \right.\right.$$

$$\left.+ \frac{\partial^2}{\partial x_c \partial x_d} \left(\frac{\sqrt{-g}\, \partial H}{\partial g_{cd}^{ab}} \right) \right\} + \frac{\partial}{\partial x_c} \left(\sqrt{-g}\, \frac{\partial H}{\partial g_c^{ab}} \delta g^{ab} \right) +$$

$$\left. + \frac{\partial}{\partial x_d} \left(\frac{\sqrt{-g}\, \partial H}{\partial g_{cd}^{ab}} \delta g_c^{ab} \right) - \frac{\partial}{\partial x_c} \left\{ \delta g^{ab} \frac{\partial}{\partial x_d} \left(\frac{\sqrt{-g}\, \partial H}{\partial g_{cd}^{ab}} \right) \right\} \right] . \quad (4c)$$

If $H = G/2\varkappa$, it can be proved [1]) that

$$\frac{\partial H}{\partial g^{ab}} - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x_c} \left(\frac{\sqrt{-g}\, \partial H}{\partial g_c^{ab}} \right) + \frac{1}{\sqrt{-g}} \frac{\partial^2}{\partial x_c \partial x_d} \left(\frac{\sqrt{-g}\, \partial H}{\partial g_{cd}^{ab}} \right) = \frac{1}{2\varkappa} G_{ab} \quad (4d)$$

Summarizing and choosing the variations δg^{ab} arbitrarily with only this condition that both they and their first derivatives vanish at the boundaries of the extension, HAMILTON's principle requires that

$$0 = \int dx_1\, dx_2\, dx_3\, dx_4\, \Sigma(abm) \left[\left\{ \tfrac{1}{2} \sqrt{-g}\, g_{am} (T_b^m + E_b^m) + \right.\right.$$

$$\left.\left. + \frac{1}{2\varkappa} \sqrt{-g}\, (G_{ab} - \tfrac{1}{2} g_{ab} G) \right\} \delta g^{ab} \right] . \quad . \quad (5)$$

Hence we find the well-known *equations for the gravitational field*

$$G_{ab} - \tfrac{1}{2} g_{ab} G = -\varkappa (T_{ab} + E_{ab}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The origin of the second term of the left hand side is apparent; it appears by the variation of $\sqrt{-g}$ in the principal function.

Virtual displacement of the matter.

9. The third variation we shall consider will be caused by giving to the *molecules* of our gas *virtual displacements.* We do not choose these displacements different for each individual molecule, but to all molecules which at a certain moment are present in a definite element of volume we give the same virtual displacement, characterized by the infinitesimal vector δr^p (comp. § 1), which may be an arbitrary function of the coordinates. The variation gives directly

$$\int dx_1\, dx_2\, dx_3\, dx_4\, \Sigma(almp) \left[\frac{\partial}{\partial x_m} \left\{ \sqrt{-g}\, (-\delta_a^m R - T_a^m)\, \delta r^a \right\} + \right.$$

$$\left. + \delta r^p \left\{ \sqrt{-g}\, k_p + \frac{\partial}{\partial x_m} (\sqrt{-g}\, T_p^m) - \tfrac{1}{2} \sqrt{-g}\, g^{al} \frac{\partial g_{am}}{\partial x_p} T_l^m \right\} \right] \quad . \quad (7)$$

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[1]) Comp. LORENTZ, l.c. XXV, p. 472.

If the δr^p are zero at the boundary, then HAMILTON's principle demands that the integral always vanishes, so that

$$\sqrt{-g}\,k_p + \Sigma\,(aml) \left\{ \frac{\partial}{\partial x_m} (\sqrt{-g}\,T_p^m) - \tfrac{1}{2} \sqrt{-g}\,g^{al} \frac{\partial g_{am}}{\partial x_p} T_l^m \right\} = 0 \quad . \quad (8)$$

These are the *equations of motion of the matter* in an invariant form. k_p is a covariant vector and the form between brackets is $\sqrt{-g}$ times the covariant divergency of the mixed tensor T_p^m.

10. Consider now the *virtual variation* of the *electric current.* If each electric particle undergoes a displacement δr^p, then the variation of the intensity of the current at a definite point-instant, is

$$\delta (\sqrt{-g}\,W^m) = \Sigma\,(a) \frac{\partial}{\partial x_a} (\sqrt{-g}\,W^a\,\delta r^m - \sqrt{-g}\,W^m\,\delta r^a)\,, \text{ }^{1})$$

so that the integral is varied by:

$$\int dx_1\,dx_2\,dx_3\,dx_4\,\Sigma(map) \left[\frac{\partial}{\partial x_m} \lambda \left\{ \sqrt{-g}\,(W^m \varphi_a - \delta_a^m S)\,\delta r^a \right\} + \right.$$

$$\left. + \,\delta r^p \left\{ \sqrt{-g}\,k_p + \lambda \sqrt{-g}\,W^m \left(\frac{\partial \varphi_m}{\partial x_p} \right) - \frac{\partial \varphi_p}{\partial x_m} \right) \right\} \right] \quad . \quad (9)$$

If δr^p vanishes at the boundary of our extension, we must have therefore

$$\sqrt{-g}\,k_p + \Sigma_{(m)} \sqrt{-g}\,W^m \left(\frac{\partial \varphi_m}{\partial x_p} - \frac{\partial \varphi_p}{\partial x_m} \right) = 0. \quad . \quad . \quad . \quad (10)$$

This may be called the "*equation of motion*" *for the electric current.* The second term may be said to represent the force exerted by the electric field on the carrier of the charge.

Virtual displacements of the fields.

11. Before calculating the variation which is obtained by a virtual displacement of the electro-magnetic field or of the gravitational field, we have to state what will be meant by this.

Doubtlessly we can say: to give a virtual displacement to the electro-magnetic field means to assume that the four potentials which originally occur at the point-instant x_p $(p = 1, 2, 3, 4)$ will be found after the displacement at the point-instant $x_p + \delta r^p$ $(p = 1, 2, 3, 4)$. From this follows that there will be at one and the same point-instant a variation $\delta \varphi_m$

$$\delta \varphi_m = - \Sigma(p) \frac{\partial \varphi_m}{\partial x_p} \delta r^p \,.$$

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[1] Comp. LORENTZ, l.c. XXIII, p. 1077.

It is, however, immediately evident that $\delta\varphi_m$ is no covariant vector though φ_m is one, so that we should compare with HAMILTON's invariant integral another, which is no longer invariant.

The same difficulty arises if a virtual displacement of the gravitational field is defined as the shift of a set of values g_{ab} from the point-instant x_p to another next to it $x_p + \delta x^p$. By so doing we do not obtain a covariant variation

$$\delta g_{ab} = - \Sigma(p) \frac{\partial g_{ab}}{\partial x_p} \delta x^p .$$

12. A closer examination of the geometrical meaning of the tensor components g_{ab} teaches us that in virtue of the equation $\varepsilon^2 = \Sigma(ab)\, g_{ab}\, dx_a\, dx_b$ they form together an infinitesimal quadratic three-dimensional extension, the "indicatrix" around each point-instant of the field figure.

The whole gravitational field may be said to be represented by the totality of the indicatrices described around the different point-instants, in the same way as in elementary considerations an electric field is described by FARADAY's lines of force. A virtual displacement of the gravitation field must therefore mean a displacement of all these indicatrices, in such a way, that it does not disturb the configuration and intersections of the indicatrices.

Let us consider two neighbouring indicatrices h and j, which intersect in the figure i. We may give the displacements to the indicatrix h and the indicatrix j separately and also to the figure i. We then demand that the shifted figure i shall again be the intersection of the shifted indicatrices h' and j'.

This cannot be managed by the variation specified in the preceding paragraph. There all point-instants of an indicatrix were supposed to undergo one and the same virtual displacement, equal to that which belongs to the centre. Now on the contrary we require that the virtual displacements of the point-instants of an indicatrix be defined by the values of δx^p at the different point-instants themselves.

If the δx^p are not constant, the virtual displacement will generally consist not only in a certain translation, but also in a rotation of the indicatrices. Analogous considerations may be applied to the virtual displacement of the electro-magnetic field. The potentials which together form a covariant tensor of the first order, represent at each point-instant a trivector multiplied by $\sqrt{-g}$, i. e. (in infinitesimal dimensions) a certain linear three-dimensional extension.

13. In order to find what has to be put for δg_{ab} and $\delta \varphi_m$, if they are to represent a virtual displacement of the fields in agreement with the geometrical character of the potentials g_{ab} and φ_m just discussed, we shall proceed in the following way. First we shall describe the world by means of somewhat altered coordinates. We introduce the transformation

$$x_m = x'_m - \delta r^m \, (m = 1, 2, 3, 4),$$

where δr^m represent the infinitesimal components of the displacement, the squares of which will be neglected, so that in differentiating a quantity which contains this δr^m or is to be multiplied by it, we need make no difference between partial differentiations with respect to x_m and to x'_m.

After the transformation of the coordinates we shall deform the net of coordinates together with the field in such a way that the surfaces $x'_m = a_m$ come at the place where originally were found the surfaces $x_m = a_m$. This is evidently reached by a virtual displacement of the field characterized everywhere by δr^m. In order to find what we have after the displacement we have only to omit the accents.

For the indicated transformation we have

$$dx_m = dx'_m - \Sigma(p) \frac{\partial \delta r^m}{\partial x'_p} dx'_p .$$

The geometrical character of the g_{ab} implies that the form

$$\Sigma\,(ab)\, g'_{ab}\, dx'_a\, dx'_b = \Sigma\,(ab)\, g_{ab}\, dx_a\, dx_b =$$
$$= \Sigma\,(ab)\, g_{ab} \left(dx_a' - \Sigma\,(p) \frac{\partial \delta r^a}{\partial x'_p} dx'_p \right) \left(dx'_b - \Sigma\,(p) \frac{\partial \delta r^b}{\partial x'_p} dx'_p \right)$$

is invariant.

Hence we deduce easily that

$$g'_{ab} = g_{ab} - \Sigma\,(p)\, g_{pb} \frac{\partial \delta r^p}{\partial x'_a} - \Sigma\,(p)\, g_{ap} \frac{\partial \delta r^p}{\partial x'_b} .$$

Here g_{ab} is the same function of $(x'_m - \delta r^m)$ which g_{ab} was of x_m. Therefore

$$g'_{ab} = g_{ab}\,(x'_m) - \Sigma\,(p) \left\{ \frac{\partial g_{ab}}{\partial x_p} \delta r^p + g_{pb} \frac{\partial \delta r^p}{\partial x'_a} + g_{ap} \frac{\partial \delta r^p}{\partial x'_b} \right\} .$$

If, omitting the accents, we now express that the new net of coordinates can be made to coincide with the original one, when the field is displaced at the same time, we find for the variation at a definite point-instant:

$$\delta g_{ab} = - \Sigma\,(p) \left\{ \frac{\partial g_{ab}}{\partial x_p} \delta r^p + g_{pb} \frac{\partial \delta r^p}{\partial x_a} + g_{ap} \frac{\partial \delta r^p}{\partial x_b} \right\} . \quad . \quad . \quad (11)$$

In the same way we find by a virtual displacement of the electromagnetic field

$$\delta\varphi_m = -\Sigma\,(p)\left\{\frac{\partial\varphi_m}{\partial x_p}\,\delta r^p + \varphi_p\,\frac{\partial\,\delta r^p}{\partial x_m}\right\} \quad . \quad . \quad . \quad . \quad (12)$$

These variations δg_{ab} and $\delta\varphi_m$ are really covariant tensors. Tensors formed in an analogous way are mentioned without commentary in a paper by HILBERT.¹)

14. For the case of a *virtual displacement of the electric field* we have

$$\delta f_{mn} = -\Sigma\,(p)\left\{\frac{\partial f_{mn}}{\partial x_p}\,\delta r^p + f_{pn}\,\frac{\partial\,\delta r^p}{\partial x_m} + f_{mp}\,\frac{\partial\,\delta r^p}{\partial x_n}\right\} \quad . \quad . \quad . \quad (13)$$

We can now easily calculate the variation of HAMILTON's integral. We find

$$\lambda\int dx_1\,dx_2\,dx_3\,dx_4\,\Sigma\,(mnp)\Bigg[\frac{\partial}{\partial x_m}\left(-\sqrt{-g}\,W^m\,\varphi_p\,\delta r^p + \sqrt{-g}\,F^{mn} f_{pn}\,\delta r^p\right) +$$

$$+\,\delta r^p\left\{\frac{1}{\lambda}\sqrt{-g}\,k_p + \frac{\partial}{\partial x_m}\left(\sqrt{-g}\,W^m\varphi_p\right) - \sqrt{-g}\,W^m\,\frac{\partial\varphi_m}{\partial x_p}\right.$$

$$\left. -\,\frac{\partial}{d x_m}\left(\sqrt{-g}\,F^{mn} f_{pn}\right) + \tfrac{1}{2}\sqrt{-g}\,F^{mn}\,\frac{\partial f_{mn}}{\partial x_p}\right\}\Bigg].$$

Using the equation of continuity of the electric current

$$\Sigma\,(m)\,\frac{\partial}{\partial x_m}\left(\sqrt{-g}\;W^m\right) = 0$$

and transforming with

$$\frac{\partial g^{kl}}{\partial x_p} = -\,\Sigma\,(am)\,g^{ka}\,g^{lm}\,\frac{\partial g_{am}}{\partial x_p},$$

$$\Sigma\,(mn)\,\tfrac{1}{2}\,F^{mn}\,\frac{\partial f_{mn}}{\partial x_p} = -\,\frac{\partial M}{\partial x_p} + \tfrac{1}{2}\,\Sigma\,(almn)\,F^{mn} f_{ln}\,g^{al}\,\frac{\partial g_{am}}{\partial x_p},$$

we find, introducing the symbol E^m_a, for the variation

$$\int dx_1\,dx_2\,dx_3\,dx_4\,\Sigma\,(almp)\Bigg[\frac{\partial}{\partial x_m}\{\sqrt{-g}(-\lambda\,W^m\,\varphi_a - \lambda\,\delta^m_a\,M - E^m_a\,\delta r^a\} +$$

$$+\,\delta r^p\left\{\sqrt{-g}\,k_p + \lambda\sqrt{-g}\,W^m\left(\frac{\partial\varphi_p}{d x_m} - \frac{\partial\varphi_m}{\partial x_p}\right) +\right.$$

$$\left.\frac{\partial}{d x_m}\left(\sqrt{-g}\,E^m_p\right) - \tfrac{1}{2}\sqrt{-g}\,g^{al}\,\frac{\partial g_{am}}{\partial x_p}\,E^m_l\right\}\Bigg] \quad . \quad . \quad . \quad . \quad (14)$$

For a virtual displacement which is zero at the boundaries of the

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¹) DAVID HILBERT. *Die Grundlagen der Physik*, *I.* K. Ges. Wiss. Göttingen, Math. Phys. 1915.

extension, HAMILTON's principle requires that

$$0 = \sqrt{-g}\, k_p + \lambda\, \Sigma\, (m)\, \sqrt{-g}\; W^m \left(\frac{\partial \varphi_p}{\partial x_m} - \frac{\partial \varphi_m}{\partial x_p} \right) +$$

$$+ \Sigma\, (lma) \left\{ \frac{\partial}{\partial x_m} (\sqrt{-g} E_p^m) - \tfrac{1}{2} \sqrt{-g}\, g^{al} \frac{\partial g_{am}}{\partial x_p} E_l^m \right\} \quad . \quad . \quad (15)$$

These may be called the *equations of motion for the field.* We see that the acting external force and the force which the carrier of the charges exerts on the field[1]) must be opposite to the covariant divergency of a tensor multiplied by $\sqrt{-g}$. The equations correspond exactly to those which we found for the matter. For that reason we are justified in considering the tensor E_p^m as the *dynamical tensor* of the *stresses, momenta and energy* in the electro-magnetic field.

15. For the *virtual displacement of the gravitational field* it is easy to find the variation of the part of the integral containing $\sqrt{-g}\, H$. The integral being a scalar, we have

$$\int \sqrt{-g}\, H\, dx_1 dx_2 dx_3 dx_4 = \int \sqrt{-g'} H' dx_1'\, dx_2'\, dx_3'\, dx_4',$$

for the transformation of § 13. H being a scalar, we also have

$$H' = H (x'_p - \delta r^p).$$

$$\sqrt{-g'} = \sqrt{-g} \frac{\partial (x_1 \ldots x_4)}{\partial (x_1' \ldots x_4')} = \sqrt{-g(x'_p - \delta r^p)} \left\{ 1 - \Sigma\, (p) \frac{\partial \delta r^p}{\partial x_p} \right\}.$$

So that after the displacement we find (by omitting the accents)

$$\delta \int \sqrt{-g}\, H\, dx_1\, dx_2\, dx_3\, dx_4 = \int dx_1\, dx_2\, dx_3\, dx_4\, \Sigma\, (p) \frac{\partial}{\partial x_p} (-\sqrt{-g} H \delta r^p). \quad (16)$$

In what follows we shall use the results of § 8. With

$$\delta g^{ab} = - \Sigma\, (mn)\, g^{am} g^{bn} \delta g_{mn}\, , \quad . \quad . \quad . \quad . \quad . \quad (17)$$

we apply the formulae ($4a$, $4b$, 11) and find after a short transformation for the total variation

$$\int dx_1\, dx_2\, dx_3\, dx_4\, \Sigma\, (almp) \left[\frac{\partial}{\partial x_m} \{ \sqrt{-g} (- \delta_a^m H + T_a^m + E_a^m)\, \delta r^a \} + \right.$$

$$\left. + \delta r^p \left\{ \sqrt{-g} k_p - \frac{\partial}{\partial x_m} (\sqrt{-g} T_p^m + \sqrt{-g} E_p^m) + \tfrac{1}{2} \sqrt{-g} g^{al} \frac{\partial g_{am}}{\partial x_p} (T_l^m + E_l^m) \right\} \right] \quad (18)$$

As in the preceding cases HAMILTON's principle now teaches us that, whenever the displacement vanishes at the boundary of the extension, we must have

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[1]) Per unit of volume.

$$0 = \sqrt{-g}\,k_p + \Sigma(alm)\left\{\frac{\partial}{\partial x_m}\left(\sqrt{-g}\,Z_p^m\right) - \tfrac{1}{2}\sqrt{-g}\,g^{al}\frac{\partial g_{am}}{\partial x_p}\,Z_l^m\right\} \quad . \quad (19)$$

where

$$Z_p^m = -\left(T_p^m + E_p^m\right) = \frac{1}{\varkappa}\,\Sigma\,(b)\,g^{mb}\left(G_{pb} - \tfrac{1}{2}\,g_{pb}\,G\right) \quad . \quad . \quad (20)$$

These might be called the *"equations of motion" for the gravitational field.* Comparing this with our former result, we are induced to consider the tensor Z_p^m as the *dynamical tensor of the stresses, momenta and energy in the gravitational field.* We see that it is just equal and opposite to those of the matter and of the electro-magnetic field taken together.

16. By formula (16) we can prove, that the *covariant divergency of* Z_p^m must be *identically zero.* The variation of $\int \sqrt{-g}\,H\,dx_1\,dx_2\,dx_3\,dx_4$ may also be calculated by means of the formulae of § 8. If we choose the δr^p and their first and second derivatives equal to zero at the boundary, then according to (16) the variation must vanish. From 4*c* and *d* together with (17) and (11) we find

$$\delta\int \sqrt{-g}\,H\,dx_1 dx_2 dx_3 dx_4 = \Sigma(ab)\int \frac{1}{2\varkappa}\sqrt{-g}\,\left(G_{ab} - \tfrac{1}{2}g_{ab}G\right)\delta g^{ab}\,dx_1 dx_2 dx_3 dx_4 =$$

$$= \int dx_1 dx_2 dx_3 dx_4\,\frac{1}{\varkappa}\,\Sigma(abmkl)\left[\frac{\partial}{\partial x_m}\left\{\sqrt{-g}\,g^{bm}\left(G_{ab} - \tfrac{1}{2}\,g_{ab}\,G\right)\,\delta r^a\right)\right\} -$$

$$- \partial r^a\left\{\frac{\partial}{\partial x^m}\left(\sqrt{-g}\,g^{bm}\left(G_{ab} - \tfrac{1}{2}\,g_{ab}\,G\right)\right) - \tfrac{1}{2}\sqrt{-g}\,g^{kl}\,\frac{\partial g_{km}}{\partial x_a}\,g^{mb}\left(G_{lb} - \tfrac{1}{2}g_{lb}\,G\right)\right]\right\}$$

This can only be equal to zero if the coefficient of δr^a, i.e. $\sqrt{-g}$ times the covariant divergency of Z_a^m is zero, so that

$$\Sigma(bklm)\frac{1}{\varkappa}\,\frac{\partial}{\partial x_m}\left\{\sqrt{-g}\,g^{bm}\left(G_{ab} - \tfrac{1}{2}\,g_{ab}\,G\right)\right\} -$$

$$- \frac{1}{2\varkappa}\sqrt{-g}\,g^{kl}\,\frac{\partial g_{km}}{\partial x_a}\,g^{mb}\left(G_{ib} - \tfrac{1}{2}\,g_{lb}\,G\right) \equiv 0 \quad . \quad . \quad (21)$$

17. This identity, which implies four connexions between the components of $(G_{ab} - \frac{1}{2}\,g_{ab}\,G)$, is important because it shows that the ten differential equations

$$G_{ab} - \tfrac{1}{2}\,g_{ab}\,G = 0$$

which determine the gravitational field at those places of our extension where there is neither matter nor an electro-magnetic field, are not independent of each other. In such extensions void of matter the gravitation potentials may therefore be subjected arbitrarily to four additional connexions. EINSTEIN has shown that this indefiniteness in the extensions void of matter can never give rise to an indefiniteness in the observations that can be made with material instruments.

The identity further confirms that in the absence of an external force the laws of conservation of energy and momentum hold for the matter. Indeed, from the field equation (6), which is given in (20) in another form, together with (21) it is evident that

$$0 = \Sigma(klm) \frac{\partial}{\partial x_m} \{\sqrt{-g}\,(T_a^m + E_a^m)\} - \tfrac{1}{2} \sqrt{-g}\, g^{kl} \frac{\partial g_{km}}{\partial x_a} (T_l^m + E_l^m). \quad (22)$$

We may even conclude that no other force can be exerted on the matter and the electro-magnetic field by any agency if this does not change the gravitation field at the same time.

18. The second term on the left-hand-side of (21) can be transformed. We may write for it

$$\Sigma(lb) \frac{1}{2\varkappa} \sqrt{-g}\, \frac{\partial g^{lb}}{\partial x_a} (G_{lb} - \tfrac{1}{2} g_{lb}\, G).$$

According to (4*d*) this comes to the same as

$$\Sigma(lbcd)\, g_a^{lb} \left[\frac{\partial \sqrt{-g}\, H}{\partial g^{lb}} - \frac{\partial}{\partial x_c}\left(\frac{\partial \sqrt{-g}\, H}{\partial g_c^{lb}}\right) + \frac{\partial^2}{\partial x_c\, \partial x_d}\left(\frac{\partial \sqrt{-g}\, H}{\partial g_{cd}^{lb}}\right)\right].$$

The same may also be expressed as follows

$$\Sigma(lbcd) \frac{\partial}{\partial x_a} (\sqrt{-g}\, H) - \frac{\partial}{\partial x_c} \left\{ g_a^{lb} \frac{\partial \sqrt{-g}\, H}{\partial g_c^{lb}} - g_a^{lb} \frac{\partial}{\partial x_d}\left(\frac{\partial \sqrt{-g}\, H}{\partial g_{cd}^{lb}}\right) + \right.$$
$$\left. + g_{da}^{lb} \frac{\partial \sqrt{-g}\, H}{\partial g_{cd}^{lb}} \right\}.$$

If now we put

$$\sqrt{-g}\, z_a^c = \Sigma(lbd)\, g_a^{lb} \frac{\partial \sqrt{-g} H}{\partial g_c^{lb}} + g_{ad}^{lb} \frac{\partial \sqrt{-g} H}{\partial g_{cd}^{lb}} - g_a^{lb} \frac{\partial}{\partial x_d} \frac{\partial \sqrt{-g} H}{\partial g_{cd}^{lb}} - \delta_a^c \sqrt{-g} H,$$

then we have according to the preceding equation and (21), (20):

$$\Sigma(m) \frac{\partial}{\partial x_m} \{\sqrt{-g}(T_a^m + E_a^m)\} + \Sigma(c) \frac{\partial}{\partial x_c} (\sqrt{-g}\, z_a^c) = 0 \quad . \quad . \quad (23)$$

So we find in $\sqrt{-g}\, z_a^c$ a complex, the "quasi-divergency" (no invariant) of which is the opposite of the quasi-divergency of the dynamical tensor of matter and electro-magnetic field. LORENTZ ¹) and DE DONDER ²) have deduced another similar complex

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¹) l. c. XXV, p. 473.

²) TH. DE DONDER, *Les équations différentielles du champ gravifique* d'EINSTEIN *créé par un champ electromagnétique de* MAXWELL-LORENTZ. Verslagen, Kon. Ac. Wet. Amsterdam, XXV, p. 150.

$$\sqrt{-g}\, s_a^c = \Sigma(lbd)\, g_{lb,a} \frac{\partial \sqrt{-g}H}{\partial g_{lb,c}} + g_{lb,ad} \frac{\partial \sqrt{-g}H}{\partial g_{lb,cd}} - g_{lb,a} \frac{\partial}{\partial x_d} \frac{\partial \sqrt{-g}H}{\partial g_{lb,cd}} - \delta_a^c \sqrt{-g}H,$$

which is found as easily as $\sqrt{-g}z_a^c$ by transformation of the second term of the identity (21).

If we wish we may take the components of one of these complexes for the stresses, momenta etc. in the gravitation field. According to (21) we have however identically

$$\Sigma\,(m) \frac{\partial}{\partial x_m} (\sqrt{-g}\; Z_a^m) \equiv \Sigma\,(c) \frac{\partial}{\partial x_c} (\sqrt{-g}\; z_a^c) \equiv \Sigma\,(c) \frac{\partial}{\partial x_c} (\sqrt{-g}\; s_a^c)\,,$$

so that we have also

$$\Sigma\,(m) \frac{\partial}{\partial x_m} \left\{ \sqrt{-g}\,(T_a^m + E_a^m) \right\} + \frac{\partial}{\partial x_m} (\sqrt{-g}\; Z_a^m) = 0\,. \quad . \qquad (24)$$

Now it is quite a matter of taste and, as to the calculations one of opportunity, which of the three equations (22), (23) or (24) will be regarded as the expression of the laws of conservation of energy and of momentum and whether z_a^c, s_a^c will be regarded as a dynamical quasi-tensor, or Z_a^m as the dynamical pure tensor of the gravitation field; or finally whether it is better not to introduce a dynamical tensor in the gravitational field at all.

Connexion with LORENTZ's *theory of electrons.*

19. Finally we shall shortly show how the deduced formulae are connected with the classic formulae of the theory of electrons. For this purpose we must treat the case of constant gravitation potentials having the values

$$g_{ab}\,(=)\;\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0\,. \\ 0 & 0 & 0 & c^2 \end{matrix}$$

To these corresponds the value $g = -c^2$ and the values of the algebraic complements

$$g^{ab}\,(=)\;\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0\,. \\ 0 & 0 & 0 & \frac{1}{c^2} \end{matrix}$$

Our formulae are based on HAMILTON's principle for the motion of a point which falls freely. In the case now under consideration it takes the form

$$0 = \delta \int_1^2 - m \, ds = \delta \int_1^2 - m \sqrt{c^2 - v^2} \, dt .$$

Comparing this with what we were used to write in the old mechanics of relativity

$$0 = \delta \int_1^2 - mc^2 \sqrt{1 - v^2/c^2} \, dt ,$$

we see that in our formulae the function of LAGRANGE has been taken c times smaller. Correspondingly definite forces, energy, stresses etc. will have to be represented by numbers which are c times smaller than they were formerly.

If for instance the unit of electric charge is left as it is defined in the theory of electrons (in these units an electron has e.g. a charge $-\sqrt{4\pi} \times 4{,}65 \times 10^{-10}$), and if at the same time the unit of the intensity of the electric field d and that of the magnetic field h are left unchanged, we shall have to write for the force per unit of volume

$$f = \varrho \left\{ \frac{1}{c} d + \frac{1}{c^2} [v \, . \, h] \right\} .$$

If we wish our equations (3) for the electric field

$$\Sigma (b) \frac{\partial}{\partial x_b} (\sqrt{-g} \, F^{ab}) = \sqrt{-g} \, W^a ,$$

in which the components of the *current* $\sqrt{-g} \, W_a$ are ϱv_x, ϱv_y, ϱv_z, ϱ, to agree with the well known relations

$$c \frac{\partial h_z}{\partial y} - c \frac{\partial h_y}{\partial z} - \frac{\partial}{\partial t} d_x = \varrho v_x , \quad etc.,$$

the components of the *contravariant field tensor* must be

$$F^{ab} \, (=) \begin{array}{cccc} 0 & h_z & -h_y & -\frac{1}{c} d_x \\ -h_z & 0 & h_x & -\frac{1}{c} d_y \\ h_y & -h_x & 0 & -\frac{1}{c} d_z \\ \frac{1}{c} d_x & \frac{1}{c} d_y & \frac{1}{c} d_z & 0. \end{array}$$

Hence it follows for the components of our covariant field tensor

$$\Sigma (ab) \, g_{ap} \, g_{bq} F^{ab} = f_{pq} \, (=) \begin{array}{cccc} 0 & h_z & -h_y & c \, d_x \\ -h_z & 0 & h_x & c \, d_y \\ h_y & -h_x & 0 & c \, d_z \\ -cd_x & -cd_y & -cd_z & 0 \, . \end{array}$$

We know that $f_{pq} = \frac{\partial \varphi_q}{\partial x p} - \frac{\partial \varphi_p}{\partial x_p}$. Hence it is evident how the scalar potential φ and the vector potential a of the theory of electrons are connected with our *potentials*:

$$\varphi_1 \quad \varphi_2 \quad \varphi_3 \quad \varphi_4 \quad (=) \quad a_x \quad a_y \quad a_z \quad -c\varphi \, .$$

For the components of the force acting on the charge per unit of volume we found in our formula (10):

$$- K_p = - \sqrt{-g}\, k_p = \lambda \, \Sigma \, (m) \sqrt{-g} \, W^m f_{pm} \, .$$

To make this agree with the above, we must, with a view to the choice of units, give the value $\lambda = 1/c^2$ to the coefficient λ. The formula thus becomes

$$- \sqrt{-g} \, k_p = \frac{1}{c^2} \, \Sigma (m) \sqrt{-g} \, W^m f_{pm} \, .$$

It keeps this form when we pass to a system of coordinates in which the unit of time is c times smaller and in which the velocity of light becomes equal to 1 (c remains 3.10^{10}). It may be remarked in passing that in the papers of LORENTZ [1]) and TRESLING the factor $1/c^2$ is failing. It is thus seen that they have silently used a unit of charge c times larger than the usual one.

The scalar for the field becomes

$$M = - \tfrac{1}{4} \, \Sigma \, (ab) \, F^{ab} f_{ab} = \tfrac{1}{2} \, (d^2 - h^2) \, ,$$

and the principal function $\lambda \sqrt{-g} M = \frac{1}{2c} \, (d^2 - h^2)$. In agreement with what has been said at the beginning of this paragraph this expression is c times smaller than the one we were accustomed to.

The *stresses*, the *negative momenta*, the *energy* and the *energy-currents* become

$$\sqrt{-g} \, E_a^m = - \lambda \Sigma (b) \sqrt{-g} \, F^{mb} f_{ab} - \lambda \sqrt{-g} \delta_a^m \, M \, ,$$

$$\begin{array}{llll}
\frac{1}{2c} (2h_x^2 - h^2 + 2d_x^2 - d^2), & \frac{1}{c} (h_x h_y + d_x d_y), & \frac{1}{c} (h_x h_z + d_x d_z), & - \frac{1}{c^2} (d_y h_z - d_z h_y), \\
\frac{1}{c} (h_x h_y + d_x d_y), & \frac{1}{c} (2h_y^2 - h^2 + 2d_y^2 - d^2), & \frac{1}{c} (h_y h_z + d_y d_z), & - \frac{1}{c^2} (d_z h_x - d_x h_z), \\
\frac{1}{c} (h_x h_z + d_x d_z), & \frac{1}{c} (h_y h_z + d_y d_z), & \frac{1}{2c} (2h_z^2 - h^2 + 2d^2 - d^2), & - \frac{1}{c^2} (d_x h_y - d_y h_x), \\
(d_y h_z - d_z h_y), & (d_z h_x - d_x h_z), & (d_x h_y - d_y h_x), & \frac{1}{2c} (h^2 + d^2) \, .
\end{array}$$

We see that all these components become c times smaller than formerly, as has been remarked already in the beginning of this paragraph.

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[1]) For the comparison with the papers of LORENTZ it may be remarked that $\sqrt{-g} \, F_{ab} = \mathfrak{f}_{ab}$ and $f_{ab} = \overline{\mathfrak{f}}_{ab}$. Further that $\sqrt{-g} \, W_m = \mathfrak{w}_m$.

Physics. — *"On the Energy and the Radius of the Electron".* By J. D. VAN DER WAALS Jr. (Communicated by Prof. J. D. VAN DER WAALS).

(Communicated in the meeting of February 24, 1917).

It is well known that the contracting electron (the so called LORENTZ-electron) has its electric energy U and its magnetic energy T, but has moreover still a quantity of energy of another kind. POINCARÉ [1]) has calculated the amount of this energy, which we will indicate by E. He has found [2]):

$$E = E_1 + \frac{e^2}{8\pi a^4} \times \frac{4}{3}\pi a^3 k. \quad \ldots \ldots \ldots \quad (1)$$

E_1 represents an integration constant, a the radius of the electron, e its charge, c the velocity of light, $k = \sqrt{1-\beta^2}$, $\beta = \frac{\mathfrak{w}}{c}$, and $\mathfrak{w}$ the velocity of the electron. The energy therefore can be interpreted as the work done by an internal constant negative pressure $\frac{e^2}{8\pi a^4}$ when the volume of the electron changes. If we calculate the mass of the electron from its electromagnetic momentum $\mathfrak{G}$ in the usual way followed by LORENTZ, then it appears that this mass satisfies the equation

$$m = \frac{1}{c^2}(T + U + E) \quad \ldots \ldots \ldots \quad (2)$$

provided we put $E_1 = 0$. At least this equation is satisfied for the electron with surface charge, to which case I will confine myself. If $E_1 = 0$, the energy E is the work required to increase the volume of the electron from the value 0 to the actual value. According to equation (2) we do not ascribe to the electron any other mass than that which follows from its energy according to the wellknown principle.

I will here consider the calculation of E a little more closely. In calculating this quantity POINCARÉ took as basis the condition that the contracted form of the electron will be its equilibrium-form, and that E therefore is the work of a force which is in equilibrium with the electromagnetic forces. He has not brought it in connection with the dynamics of the electron in an accurate way. For he writes:

$$\mathfrak{G} = \frac{\partial}{\partial \mathfrak{w}}(T - U) \quad \ldots \ldots \ldots \quad (3)$$

which equation however is not satisfied.

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[1]) H. POINCARÉ, Rend. del circ. mat. di Palermo. Tomo XXI, Ad. d. 23 Luglio 1905.

[2]) The here used notation is another than that of POINCARÉ.

ABRAHAM had calculated E in another way. He started from the supposition that equation (3) would not hold, but that we should have to correct it into:

$$\mathfrak{G} = \frac{\partial}{\partial \mathfrak{w}}(T - U - E) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

As E in this equation is the only unknown quantity it may be derived from it. This yields again the value of equation (1) for E. In this calculation as well as in that of POINCARÉ the energy E has been introduced as an amount of potential energy. Moreover ABRAHAM assumes that $\mathfrak{G}$ is the total momentum of the electron, i.e. that the electron has only its electromagnetic momentum, which is determined by $\frac{1}{c^2} \times$ POYNTINGS' vector and not any momentum of another kind.

This, however, seems a priori little plausible. For according to the principle of the mass of the energy we should expect, that the electron would possess an amount of momentum $E \times \frac{\mathfrak{w}}{c^2}$, which was not of electromagnetic nature. Moreover we should expect, that in the moving electron POINCARÉ's pressure would give rise to a transfer of energy, which would be accompanied with still another amount of momentum. We will therefore denote the total momentum by $\mathfrak{G}_{tot}$, a quality which we will leave undetermined for the present. If we do not put a priori $\mathfrak{G}_{t.t} = \mathfrak{G}$, ABRAHAM's way of calculating E loses its applicability. We must therefore follow a somewhat different way for the calculation of E. For this purpose we will assume concerning $\mathfrak{G}_{tot}$ that it satisfies the equation:

$$\mathfrak{G}_{tot} = \frac{\partial}{\partial \mathfrak{w}}(T - U - E) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3b)$$

which contains the unknown quantities $\mathfrak{G}_{tot}$ and E, A second equation is therefore required in order to determine them both, for which purpose the equation

$$\mathfrak{G}_{tot} = \frac{\mathfrak{w}}{c^2}(T + U + E) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

can be used.

So we find for E the differential equation:

$$\frac{\partial E}{\partial \mathfrak{w}} + \frac{\mathfrak{w}}{c^2} E = -\frac{\partial (T - U)}{\partial \mathfrak{w}} + \frac{T + U}{c^2}\mathfrak{w} \quad . \quad . \quad . \quad . \quad , \quad (4)$$

As $T - U = -\frac{e^2}{2a}k$ and $T + U = \frac{e^2}{2ak}\left(1 + \frac{1}{3}\frac{\mathfrak{w}^2}{c^2}\right)$ we find as solution of (4)

$$E = \frac{e^2 k}{6a} + C e^{-\frac{\mathfrak{w}^2}{2c^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where C is an arbitrary integration constant. For the LORENTZ electron we have an $m = \frac{m_0}{k}$. If we postulate that the corresponding equation for the energy will also hold good, namely $T + U + E = \frac{1}{k}(T_0 + U_0 + E_0)$, then we must put $C = 0$ and we find again for E the value calculated by POINCARÉ and ABRAHAM.

Now it appears a posteriori that

$$\mathfrak{G}_{tot} = \frac{\partial}{\partial \mathfrak{w}}(T - U - E) = \mathfrak{G}$$

From this follows that the two corrections which we mentioned above need not be applied to $\mathfrak{G}$ in order to find $\mathfrak{G}_{tot}$. The two corrections appear to cancel each other. This result is remarkable. It proves that the energy inside the electron is stationary in space. At the frontside Poynting's vector is directed towards the electron. Hence when the energy reaches the frontside of the electron it is suddenly stopped in its motion and it remains at rest inside the electron till it is reached by the backside of the electron. Then it is again put into motion as radiating energy, but now the motion is directed away from the electron.

I will remind that the usual way of calculating the radius of the electron depends on the supposition that $\mathfrak{G}_{tot} = \mathfrak{G}$. For the applicability therefore of this calculation it is decisive whether or no the suppositions assumed above are correct.

1st. that the supplementary energy E must be introduced in the function of LAGRANGE as potential energy, in agreement with the assumptions of POINCARÉ and of ABRAHAM.

2nd. that the mass of the electron is equal to its energy $\times \frac{1}{c^2}$.

Chemistry. — "*Amygdalin as nutriment for Aspergillus niger.*" II. By Dr. H. I. WATERMAN. (Communicated by Prof. Dr. J. BÖESEKEN.)

(Communicated in the meeting of February 24, 1917).

In an earlier communication [1]) I have shown that amygdalin is a better nutriment for *Aspergillus niger* than glucose, at least with regard to the weight of mycelium obtained.

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[1]) These Proceedings January 26, 1917 p. 922.

TABLE. The influence of amygdalin on the development of Aspergillus niger with benzaldehyde as organic nutriment. Temperature: 33°.

	A_1	B_1	C_1	D_1	E_1	F_1	G_1	H_1
Composition of the nutrient liquid	45 cM.^{3} P 5 cM.^{3} H_2O	25 cM.^{3} P 25 cM.^{3} H_2O	20 cM.^{3} P 30 cM.^{3} H_2O	10 cM.^{3} P 40 cM.^{3} H_2O	5 cM.^{3} P 45 cM.^{3} H_2O	2 cM ^{3} P 48 cM.^{3} H_2O	1 cM.^{3} P 49 cM.^{3} H_2O	50 cM.^{3} H_2O
Development after one day	—	—	?	?	?	?	—	—
Development after two days	—	—	—	—	?	+ (slight)	+ (slight)	—
	A_2	B_2	C_2	D_2	E_2	F_2	G_2	H_2
Composition of the nutrient liquid	45 cM.^{3} P 5 cM.^{3} Q	25 cM.^{3} P 5 cM ^{3} Q 20 cM.^{3} H_2O	20 cM ^{3} P 5 cM.^{3} Q 25 cM.^{3} H_2O	10 cM.^{3} P 5 cM.^{3} Q 35 cM.^{3} H_2O	5 cM.^{3} P 5 cM.^{3} Q 40 cM.^{3} H_2O	2 cM.^{3} P 5 cM.^{3} Q 43 cM.^{3} H_2O	1 cM.^{3} P 5 cM.^{3} Q 44 cM.^{3} H_2O	5 cM.^{3} Q 45 cM.^{3} H_2O
Development after one day	—	—	—	?	—	+ (very slight)	+ (slight)	?
Development after two days	—	—	—	?	?	++	++	—
	A_3	B_3	C_2	D_3	E_3	F_3	G_3	H_3
Composition of the nutrient liquid	50 cM.^{3} H_2O	25 cM.^{3} P 25 cM.^{3} Q	20 cM.^{3} P 25 cM ^{3} Q 5 cM.^{3} H_2O	10 cM.^{3} P 25 cM.^{3} Q 15 cM.^{3} H_2O	5 cM.^{3} P 25 cM.^{3} Q 20 cM.^{3} H_2O	2 cM.^{3} P 25 cM.^{3} Q 23 cM.^{3} H_2O	1 cM.^{3} P 25 cM.^{3} Q 24 cM.^{3} H_2O	25 cM.^{3} Q 25 cM.^{3} H_2O
Development after one day	—	?	—	?	—	?	+ (slight)	?
Development after two days	—	—	—	—	+	++	++	—

The solution *P* was prepared as follows: 1,5 gr. NH_4NO_3, 1,5 gr. KH_2PO_4, 1,5 gr. $MgSO_4 . 7 H_2O$ and ± 0,3 gr. $CaCl_2$ were dissolved in boiling distilled water; after cooling ± 1,85 gr. benzaldehyde was dissolved and the clear solution was filled up with H_2O to 1 Liter.

Solution *Q* was obtained by dissolving 2,5 gr. amygdalin in distilled water, this solution was boiled during a short time and after cooling filled up with H_2O to 250 cM^3.

The possibility was considered that *within* the cell benzaldehyde might be formed, whilst at the same time it was proved that *outside* the cell important quantities of amygdalin by no means should be converted into glucose, benzaldehyde and hydrogen cyanide.

This result was confirmed in another way by a new series of experiments, from which it is to be concluded with certainty that *amygdalin without any preceding conversion* into glucose benzaldehyde and hydrogen cyanide, is *absorbed* by the cells.

The referential experiments are united in the table.

From these experiments it follows that the addition of amygdalin diminishes the noxious influence of benzaldehyde. Compare E_1,F_1 and G_1 on one side with E_2,F_2,G_2 and E_3,F_3,G_3 on the other side.

If a conversion into glucose, benzaldehyde and HCN should precede the absorption of amygdalin just the contrary should be stated.

Physiology. — "*Experimental researches on the permeability of the kidneys to glucose*" ¹). By Prof. H. J. HAMBURGER and R. BRINKMAN.

I. THE PROPORTION BETWEEN K AND CA IN THE CIRCULATING FLUID.

(Communicated in the meeting of January 27, 1917).

1. *Introduction.*

No solution has been offered to the question of importance to physiologists as well as to clinicists, viz. why the urine of a normal person is entirely or all but entirely free from sugar as long as the sugar-percentage of the blood serum does not rise above a certain concentration, and why as a rule glucosuria only sets in when accompanied by hyperglycaemia.

Two explanations suggest themselves:

It may be supposed that the normal glomerulus epithelium is proof against ± 0.1 °/₀ of glucose without becoming permeable to it, but cannot keep back all the glucose of a higher concentration. Not much can be said in favour of this view, for it is not very likely that cells which are permanently exposed to a 0.1 °/₀ solution of a physiological non-electrolyte such as glucose should be changed by a 0.2 °/₀ solution ²).

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¹) A more detailed account will be published elsewhere.

²) We shall not discuss here the hypothesis of a glomerulus-epithelium absolutely permeable to glucose, with back-resorption of it through the kidney-ducts, nor the oxidation of the glucose in the kidney.

The second explanation, which has found many advocates, is that the blood sugar in the serum is not present in a free state, but is in normal circumstances only met with as a colloidal compound (LÉPINE's sucre virtuel), which cannot pass the glomerulus membrane. If the serum does not contain a quantity of this substance sufficient to bind the glucose, then part of the glucose remains circulating in a free state and can pass the glomerulus-epithelium, in other words glucosuria sets in. Several colloidal glucose compounds have been suggested already (jecorin, lecithin glucose, globulin-glucose).

Objections have been raised, however, against this retention of sugar by a substance, present in the serum. Serum has been made to dialyze against glucose-solutions (ASHER, RONA and MICHAELIS) and it was found that the percentage of glucose became equal on both sides of the membrane. A retention of sugar in a colloid form was, therefore, manifestly impossible. This statement has produced a considerable impression, and the result seems to be that matters have come to a dead stop.

We have asked ourselves, however, if the results obtained in experiments with parchment membranes might be applied to glomerulus epithelium. Obviously there is a possibility of compounds of glucose with some serumsubstance diffusing through a parchment membrane, but not through a membrane of glomerulus epithelium. BECHHOLD's experiments have amply demonstrated that certain colloidal particles diffuse through one membrane, but not through another with smaller pores.

We experimented, therefore, with celloidin membranes of various celloidin-percentages and ultrafiltrated under a pressure of 4 or 5 atmospheres serum through it to which known quantities of glucose had been added, but the reduction-power of the ultrafiltrates did not warrant us to conclude that a colloidal glucose-compound had been kept back by the ultrafilter.

Pursuing the same line of thought it seemed advisable now to investigate systematically whether in spite of the results of these diffusion- and ultrafiltration-experiments the second view was not the right one after all.

In the first place it would have to be ascertained, which had not been done previously, that *free* glucose diffuses through the kidney. To investigate this, RINGER-fluid, to which sugar had been added, would have to be transmitted through the vascular system of the kidney. If it was found then that the fluid flowing from the ureters contained the same concentration of sugar as the transmission-fluid, and if further it became evident that a RINGER-fluid containing sugar, to which serum had been added, produced an artificial urine *free* from sugar, then it was, as we thought, conclusively shown that the serum contains a substance retaining sugar in a form which cannot diffuse through the glomerulus epithelium, and then further researches might be made as to the nature of this substance.

Before entering upon the description of the experiments we wish te make a few observations of a technical nature.

Some remarks of a technical kind.

The experiments were exclusively carried out with frogs, viz. with large male specimens of the Rhine frog. The spinal marrow was destroyed with a needle, and all organs except kidneys, testicles and bladder were removed at once. Then a thin injection-needle was inserted into the aorta communis and a canula into each ureter. The fluid which circulates through the vascular system must be amply provided with oxygen. The pressure amounted to 60—80 centimetres of water. In this way from 300 to 800 cubic c.m. of fluid flows through the kidneys per hour. The amount of fluid passing through the ureter is 0.5 c.c. or less. This fluid must be looked upon as a glomerulus product, for if at the same pressure fluid is transmitted through the vena Jacobsonii, then no fluid is secreted in the ureters. At a higher pressure some fluid is formed but very slowly.

The glucose-percentage of the urine is not affected by the vena Jacobsonii being tied off. This makes it probable that the kidney ducts have little to do with the glucose-motion.

The glucose-determination of transcirculating-fluid and kidney-product was carried out by means of the excellent method of I. Bang [1]). It enables one to determine the glucose-percentage in 0.1 c.c. of fluid to within 0.006 °/₀.

2. *The permeability of the frog's kidney to glucose which has been dissolved in* Ringer-*fluid.*

1st Series of Experiments.

As we said before the fundamental problem to be solved in the first place was whether at a transmission of Ringer-fluid containing

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[1]) J. Bang. Methoden zur Mikrobestimmung einiger Blutbestandteile. Wiesbaden, J. F. Bergmann. 1916.

glucose, the glucose-percentage of the urine would become equal to that of the circulating fluid. Repeated experiments showed that the glucose concentrations were exactly the same in both experiments. We shall give some of the values obtained. Each experiment was repeated at least three times and all gave the same results.

Jan. 20 to Jan. 26. 1916.

1. Ringer-solution containing 0,1 °/₀ of glucose transmitted from the aorta through the kidneys under a pressure of ± 50 c.m. Reduction of transmission-fluid 0,098 °/₀. Reduction (expressed in glucoseconcentration) of the left kidney 0,095 °/₀, right kidney 0,095 °/₀.

2. Circulation of Ringer-solution containing exactly 0,05 °/₀ of glucose, from the aorta. Pressure 60 c c. of water. Reduction of urine to the left 0,05 °/₀, to the right 0,045 °/₀.

3. Circulation from the aorta with pure Ringer-solution. The urine shows no reduction.

These results formed, as it seemed, a reliable foundation for further researches. It was now expected that on serum being added to the Ringer-fluid containing glucose, the free glucose would be entirely or partially bound; in other words that the reduction capacity of the ureter-fluid would be smaller than that of the circulation-fluid.

3. *The permeability of the kidneys to glucose when it is dissolved in a mixture of serum and* Ringer-*fluid.*

2nd. Series of Experiments.

For these experiments horse's or neat's serum was diluted with a 2-, 3-, 4- and 5-fold quantity of Ringer-fluid to which mixtures in every instance a known quantity of glucose was added. The secretion of ureter-fluid took place very slowly, but could be promoted by the addition of urea.

We subjoin a few of the many experiments.

1. Frog's kidney through which flows a fluid consisting of 50 c.c. of horse's serum + 150 c.c. of Ringer + glucose + urea. Reduction of this mixture 0,17 °/₀, reduction urine 0,086 °/₀. Consequently 0,09 °/₀ of glucose has been kept back (= the quantity of glucose in normal horse's serum).

2. The fluid consists of 75 c.c. of neat's serum + 225 c.c. of Ringer + glucose + urea. Reduction of the transmission fluid 0,21 °/₀, reduction of the urine only 0,12 °/₀, to the right, 0,105 °/₀. to the left.

3. The fluid consists of 60 c c. of horse's serum + 240 c.c. of Ringer + glucose + urea. Reduction of the transmission-fluid 0,14 °/₀. Reduction of the urine to the right 0,03 °/₀, to the left 0,028 °/₀. Hence at a 5-fold serum dilution 0,11 °/₀ of glucose in still kept back.

In the same way 0.07 °/₀ of glucose was kept back at a 6-fold, 0.06 °/₀ at a 7-fold dilution but at an 8-fold dilution *next to nothing.*

Evidently a considerable quantity of sugar is retained as long as the dilution of the serum is not an 8-fold one (0.17—0.086, 0.21—0.11, 0.21—0.105, 0.14—0.03, 0.14—0.028).

In stronger dilutions the retention of sugar grows less, and in an 8-fold dilution it is 0.

It was now attempted to trace the cause of this rather abrupt turning point, but in the midst of this somewhat elaborate investigation, which we shall not discuss here, the stock of RINGER-fluid gave out, and a fresh quantity had to be prepared. It soon became evident now that the retention power of the kidney for glucose in the serum-RINGER-mixtures was entirely different from what it had been in the previous experiments.

The possibility had to be taken into account that the RINGER-fluid was not identical with the one formerly used. Was the Ca-percentage different perhaps? We often read of a $CaCl_2$-solution of a given concentration without there being added if it has been made of anhydrous $CaCl_2$ or of $CaCl_2$ 6 aq. It was indeed found that an addition of some $CaCl_2$ strongly affected the glucose-excretion, for now the concentration of the ureter fluid was equal to that of the transmission-fluid. This observation, confirmed by parallel-experiments, induced us to determine whether the circulation of the new serumless RINGER-solution would cause all glucose to be diffused, as had been the case with the original RINGER-fluid.

To our surprise we discovered that when the new RINGER-fluid was transmitted, glucose was retained by the kidneys.

Under these circumstances it became necessary to institute a systematic investigation of the way in which a change in the composition of the RINGER-*fluid affects the permeability of the kidney.* The present paper confines itself to this investigation. We shall afterwards revert to the effect of serum being added.

4. *Change in the proportion of the quantities of K and Ca in the* RINGER-*fluid.*

3rd Series of Experiments.

It appears from the following table that only the amount of $CaCl_2$ was modified, the KCl remaining the same.

Each of the four experiments was repeated 3 times with exactly the same results.

Evidently no glucose is retained if a solution of $CaCl_2$ 0,005 % is used; when however, the solution contains 0,0075 %, 0.095—0.065=

	$^0/_0$ of NaCl	$^0/_0$ of $NaHCO_3$	$^0/_0$ of KCl	$^0/_0$ of $CaCl_2$ (without crystallisation water)	$^0/_0$ of Reduction	
					Circul. fl.	Urine
1)	0.7	0.02	0.01	0.005	0.09	0.09
2)	0.7	0.02	0.01	0.0075	**0.095**	**0.065**
3)	0.7	0.02	0.01	0.010	0.09	0.08
4)	0.7	0.02	0.01	0.015	0.09	0.09

0,03 $^0/_0$ of glucose is retained. If the Ca-perc. is raised to 0,010 $^0/_0$ only 0,01 $^0/_0$ is retained, at 0,015 $^0/_0$ nothing again.

Hence the most favourable proportion between the concentrations of KCl and $CaCl_2$ is 4:3, which, expressed in the number of atoms, results in: K : Ca = 2 : 1. The following table demonstrates that the proportion between K and Ca and not the absolute amount of Ca is the important thing, because a slight increase of the K-perc. necessitates a corresponding increase of Ca.

$^0/_0$ NaCl	$^0/_0$ $NaHCO_3$	$^0/_0$ KCl	$^0/_0$ $CaCl_2$	$^0/_0$ Reduction	
				Circul. fl.	Urine
0.7	0.02	0.01	0.0075	0.095	**0.065**
0.7	0.02	0.015	0.0075	0.08	0.08
0.7	0.02	0.014	0.011	0.10	**0.065**

It should be noticed that the glucose perc. in the blood of the winter-frog (these were used for the experiments) amounts to 0.03 $^0/_0$. In the summer-frog it is 0.05 $^0/_0$.

In accordance with this difference it was found that the kidneys of the summer-frog, when treated with Ringer-*fluid containing glucose, retained* 0.05 $^0/_0$ *glucose,* which is partly due to the temperature. We shall have occasion to mention the influence of the temperature again.

5. *The proportion of Na, K, Ca.*

4th Series of Experiments.

In order to continue the experiments a new consignment of frogs had to be used. The Ringer-fluid, used for circulation, consisted again of NaCl 0.7 $^0/_0$, $NaHCO_3$ 0.02 $^0/_0$, KCl 0.01 $^0/_0$, $CaCl_2$ 0.075 $^0/_0$, glucose 0.09 %. To our surprise little or no glucose was retained now. Then the possibility was debated whether perhaps the amount of Na might tell upon the results. The answer is supplied by the following table.

% NaCl	% $NaHCO_3$	% KCl	% $CaCl_2$	% Reduction	
				Circul. fl.	Urine
0.7	0.02	0.01	0.0075	0.09	0.09
0.7	0.02	0.01	0.010	0.102	0.085
0.7	0.02	0.01	0.012	0.105	0.085
0.7	0.02	0.01	0.0075	0.085	0.085
0.6	0.02	0.01	0.0075	0.085	0.060
0.6	0.02	0.01	0.0075	0.010	0.070
0.6	0.02	0.01	0.005	0.09	0.070
0.6	0.02	0.01	0.0025	0.085	0.075
0.6	0.02	0.01	0.010	0.12	0.115

Obviously these results differ from those in the preceding table in which KCl : $CaCl_2$ = 4 : 3 and where 0.03 % of glucose was retained. Something indeed is retained when we use the new frogs, but not 0.03 %. 0.02 % is retained when we use KCl: $CaCl_2$ = = 4 : 4 or 1 : 1.

Why did the new consignment of frogs behave in a way different from the first? The temperature at which they were kept might be the cause. *It was ± 8° C.; formerly it had been higher.* That the difference is indeed due to the temperature is made manifest by the fact that, in order to obtain the same results some $CaCl_2$ must be added to the circulating fluid when the kidney is cooled down by ice. If we wish to maintain the proportion KCl : $CaCl_2$ = 4 : 3, then the NaCl must be reduced from 0.7 to 0.6 %.

It follows that with every condition of the glomerulus epithelium must correspond, if it is to keep back a maximum amount of glucose, a certain proportion between Na, K and Ca.

It is not improbable that the anions too play a part in the equilibrium, but at any rate one gets an impression that the proportion of cations preponderates.

The fact that a disturbance in the equilibrium of the cations strongly affects the permeability of the kidney to sugar may explain two important observations, which have hitherto not been understood.

One relates to an experiment by UNDERHILL and CLOSSEN[1]), who injected into the ear-vena of a rabbit a solution of $CaCl_2$ and discovered that the hypoglycaemia is attended with glucosuria. The most obvious explanation is that we have to deal with a disturbance in the equilibrium between Na, K and Ca.

Secondly it has been known for many years that uranium may also cause glucosuria[2]). Now ZWAARDEMAKER and FEENSTRA discovered[3]) that in the RINGER-fluid which sustains the beating of the frog's heart, the K may be replaced by the likewise radioactive uranium. It will appear from the ensuing communication that in the physiological circulation-fluid of the kidney the K may likewise be replaced by an equiradioactive quantity of uranium. Hence it is not assuming too much if we consider the uranium-glucosuria as being caused by an equilibrium-disturbance occasioned by a disturbance of the normal K-percentage.

It should be noticed that the glucosuria caused by $CaCl_2$- and by uranium-injections are the only two of which it may be stated with certainty that they are of a renal kind. Thus with warmblooded animals an equilibrium-disturbance in the relative cation-percentage of the circulating-fluid (here bloodplasm) might also be the cause of a modified permeability of the glomerulus epithelium to sugar.

SUMMARY.

1. When a RINGER-fluid in which the atoms of K and Ca are as 2 to 1 and which contains glucose is transmitted through the frog's kidney at 7°—10° C, then a comparison of the glucose concentrations of circulating-fluid and ureter-fluid shows us that 0.03 °/₀ of glucose is retained by the kidneys.

2. If the proportion between K and Ca is somewhat modified, the glucose retention will decrease, further modification reducing it to 0; in other words the urine contains then as great a concentration of glucose as the circulating-fluid.

3. Evidently we have to deal here with a variable permeability of the glomerulus epithelium to glucose, not only depending on

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[1]) UNDERHILL and CLOSSON. Americ. Journal of Physiol. **5**, p. 321, 1916. Quoted from BANG. Der Blutzucker 1913 p. 103.

[2]) POLLACK, Arch. für exp. Path. u. Pharmakol. **64** p. 415, 1911. See also BANG, Der Blutzucker.

[3]) ZWAARDEMAKER and FEENSTRA. These reports 1916, 28 April, 27 May, 30 September.

slight variations of the chemical composition of the circulation fluid, but also on slight differences of temperature. Thus this simple membrane forms a nice object for quantitative studies on permeability under various physiological conditions.

January 1917. *Physiological Laboratory, Groningen.*

Physiology. — *"Experimental researches on the permeability of the kidneys to glucose"*. By Prof. H. J. Hamburger and R. Brinkman.

II. THE POTASSIUM REQUIRED IN THE CIRCULATING-FLUID IS REPLACED BY URANIUM AND RADIUM.

(Communicated in the meeting of January 27, 1917)

From our preceding paper it appeared that if a Ringer-fluid containing glucose and composed of NaCl 0.7 °/₀, $NaHCO_2$ 0.02 °/₀, KCl 0.01 °/₀, $CaCl_2$ 0.0075 °/₀ was circulated through a frog's kidney, 0.03 °/₀ of glucose was retained. Now Zwaardemaker and Feenstra availing themselves of the conclusions arrived at by N. R. Campbell that potassium is the only radio-active element found in the body, have discovered that in the Ringer-fluid which maintains the beating of the heart, potassium may be replaced by uranium, radium and thorium and that in equivadioactive doses[1]. It seemed of importance to us to determine whether in the above-mentioned circulating-fluid this substitution may likewise be effected with regard to the kidney. Can here too uranium and radium take the place of potassium and if so in what proportion, *in a molecular or in a radioactive one?*

Hence the KCl in the Ringer-fluid which contained 100 milligrammes of KCl per litre was replaced by the equiradio-active quantity; viz. 15 milligrammes of $U(NO_3)_4$ per litre. And it was indeed found that here too the maximum quantity of glucose was retained. If, however, instead of 15 milligrammes of nitrate of uranium, 25 milligrammes are added, only very little glucose is retained. If the litre of Ringer-fluid without K, contains 35 mGr. of $U(NO_3)_4$, no glucose is retained at all.

Now 100 m.Gr. of KCl are chemically equivalent with 112 m.Gr.

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[1] These Proceedings Vol. XIX p. 99, XX 341 and 633.
Compare also Zwaardemaker, Feenstra and Benjamins, ibid. Nov. 10 1916.

of the uranium salt, consequently considerably more than 15 m.Gr. which was found by the experiments.

The same substitution could be carried out with bromide of radium solution.

1 capsule of the Allgem. Radiogen Gesellschaft contains $\frac{1}{5} \times 10^{-3}$ milligrammes of $RaBr_2$. It was dissolved by heating in 100 cc. of distilled water, which had been acidified with some HCl. This solution having been neutralized with some NaOH sol. containing no K, $2\frac{1}{2}$ cc. of the fluid obtained was put into 1 litre of RINGER-solution which contained no K. Some glucose having been added, the fluid thus obtained could indeed retain 0.03 °/₀ of glucose. The fluid contained 5×10^{-6} mGr. of RBr_2 per litre, the same quantity as that which was found by ZWAARDEMAKER and FEENSTRA to keep up the contractions of the frog's heart.

The use of 7×10^{-6} instead of 5×10^{-6} mGr of $RaBr_2$ per litre, causes the retention of glucose to decrease; this is also the case when $3\frac{1}{2} \times 10^{-6}$ is used.

If the $RaBr_2$ had acted in chemical equivalency the quantity of it being necessary to replace the 100 mGr. of KCl would have been $\frac{168}{149} \times 100 = 259$ mGr. of $RaBr_2$ per Litre instead of 5×10^{-6} mGr.

Hence it follows that potassium, uranium and radium affect the retentive power of the glomerulus membrane for glucose, in equiradioactive doses, and not in chemically equivalent doses.

Further experiments will probably show that the limits can be determined more closely than it has been done here.

These investigations throw a light upon the uranium-glucosuria, which has hitherto not been explained. It is most likely caused by a disturbance in the relative percentages of the metal atoms in the bloodserum, the potassium of which has been increased by the addition of a metal which is, in a certain sense, related to it (uranium or radium).

The fact that glucosuria is caused by the injection of some nitrate of uranium is not contradicted by the salutary therapeutic effect of uranium in diabetes (HUGHES and WEST) [1]; but then it would have to be assumed that the potassium percentage had decreased. As to this, however, all data are wanting.

Groningen, January 1917. *Physiological Laboratory.*

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[1] Quoted from CAMMIDGE. Glucosuria and allied conditions. London, EDWARD ARNOLD 1913, p. 339.

Chemistry. — *"In-, mono- and divariant equilibria"*. XV. By Prof. F. A. H. SCHREINEMAKERS.

(Communicated in the meeting of February 24, 1917).

The occurrence of two indifferent phases; the equilibrium M is variable singular.

Now we consider the case that the singular equilibrium (M) is no more constant, but variable; one or more phases of M have, therefore, a variable composition. (Comm. X).

When (M) is constant singular, then, as we have deduced in communication X, the following propositions hold:

1. When the two indifferent phases have the same sign, then M is transformable.
2. When the two indifferent phases have opposite sign, then M is not transformable.

It is evident that the same rules are valid also when M is a variable singular equilibrium.

In order to examine what P,T-diagrams can occur now, we take an invariant point with the phases $F_1 \ldots F_{n+2}$, in which F_p and F_{p+1} are the indifferent — and consequently the other ones are the singular phases. Then we have the singular equilibria:

$$(M) = F_1 + \ldots + F_{p-1} + F_{p+2} + \ldots F_{n+2}$$

$$(F_p) = (M) + F_{p+1} \text{ and } (F_{p+1}) = (M) = F_p$$

in which (M) now contains one or more phases of variable composition.

When (M) is constant singular, then curve (M) is monodirectionable [fig. 1 (X)] or bidirectionable [fig. 2 (X)]; in the first case the 3 singular curves coincide in the same direction, in the second case (F_p) and (F_{p+1}) coincide in opposite direction.

When (M) is however variable singular then the three singular curves can no more coincide. Let viz. P_0 and T_0 be the pressure and temperature of the invariant equilibrium and let us assume that in (M) and consequently also in (F_p) and (F_{p+1}) the phases F_x, F_y etc. of variable composition occur. Under P_0 and at T_0 F_x and F_y have then the same composition in (M) and (F_p). Now we take a temperature T_1. When we bring (M) to the temperature T_1 and under the corresponding pressure, then F_x and F_y get another composition F_x' and F_y'. Those compositions are of course such compositions that between the phases of (M) the phases-reaction is still always possible.

When we bring (F_p) to the temperature T_1 and under the corresponding pressure, then F_x and F_y do not get the composition F_x' and F_y', but another composition F_x'' and F_y''.

When we take away at T_1 the phase F_{p+1}, from $(F_p) = (M) + F_{p+1}$, then we do not obtain the equilibrium (M), but, as F_x'' and F_y'' have another composition than F_x' and F_y', an equilibrium different from (M). Consequently curves (M) and (F_p) do not coincide. The same is true for (M) and (F_{p+1}) and for (F_p) and (F_{p+1}); the singular curves do not coincide, therefore. They form, as is drawn in the figs. 1—5, three separate curves. Now we can show:

1. the three singular curves touch one another in the point i.
2. (F_p) and (F_{p+1}) are situated on the same side of the (M)-curve.

The first follows immediately from the relation

$$T\frac{dp}{dT} = \frac{\Delta W}{\Delta V}.$$

In the point i viz. the reaction, which occurs in the three singular equilibria, is the same, so that in the point i $\frac{dP}{dT}$ is the same also for the three curves.

In order to show the second, we consider the bivariant equilibrium:

$$(F_p F_{p+1}) = F_1 + \ldots + F_x + F_y + \ldots + F_{p-1} + F_{p+2} \ldots F_{n+2}. \quad (1)$$

This region has a turning-line (M), which is defined by the fact that in (1) the variable phases $F_x, F_y, \ldots$ have such a composition that a phases-reaction is possible between those n phases. The singular curve (M) is, therefore, the same as the turning-line of the region $(F_p\, F_{p+1})$; consequently we have here the special case, which we have already mentioned in (VIII) viz. that the point i in fig. 5 (VIII) is situated on the turning-line $xyzu$ of the region $(F_p\, F_{p+1})$. As (F_p) and (F_{p+1}) must be situated within the turning-line of this region, they are situated, therefore, on the same side of the (M)-curve.

In order to deduce the P,T-diagrams, we are able to apply again the rules of the isovolumetrical and isentropical reaction to the curves (M), (F_p) and (F_{p+1}). In this application with respect to the (M)-curve we have, however, to bear in mind the following.

When we have a constant singular curve (M), then we are able to realise always a whole series of equilibria (for instance between the temperatures T_1 and T_2) of the (M)-curve with the aid of one single complex K of definite composition. When (M) is however variable singular, then this not always remains possible. Then we may have the case, that we can obtain only one single equilibrium of the (M)-curve (e.g. that of a temperature T) with each definite complex K; in order to realise the equilibrium of a temperature $T+dT$, we have to take then a complex of another composition.

When the latter is the case and when T_0 is the temperature and P_0 the pressure of the invariant point, then we can not obtain with a same complex K an equilibrium of the temperature T_0 and T_0+dT or of the pressures P_0 and P_0+dP; the rules of the isovolumetrical and of the isentropical reaction, therefore, are then not applicable.

In the first case we have:

the two indifferent phases have the same sign; the equilibrium (M) is, therefore, transformable.

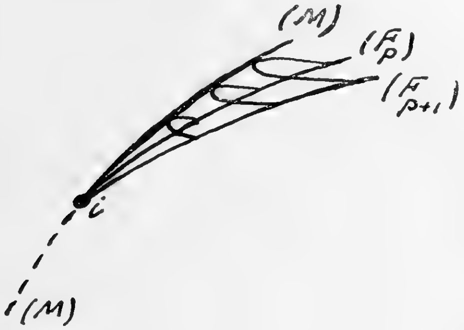

Fig. 1.

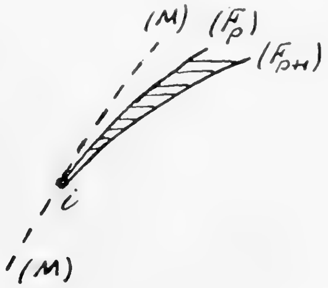

Fig. 2.

The stable parts of the curves (F_p) and (F_{p+1}) go in the same direction, starting from the point i; then we obtain P,T-diagrams as in figs. 1 and 2.

[In those and the following figures only the stable part of the curves (F_p) and (F_{p+1}) is drawn; the metastable part of the (M)-curve is dotted.] In fig. 1 the one part of the (M)-curve is stable, the other part metastable; in fig. 2 the (M)-curve is only stable in the point i.

In the second case we have:

the two indifferent phases have opposite sign, the equilibrium (M) is therefore, not transformable.

The stable parts of the curves (F_p) and (F_{p+1}) proceed in opposite direction, starting from the point i; then we obtain P,T-diagrams as in the figs. 3, 4 and 5. In fig. 3 the (M)-curve is bidirectionable, in fig. 4 monodirectionable, in fig. 5 it is metastable, except in the point i.

[In a following communication we shall show that the (M)-curve can also have a turning-point. When this is casually situated in the point i, then the diagrams under consideration will be changed by this in some respect.]

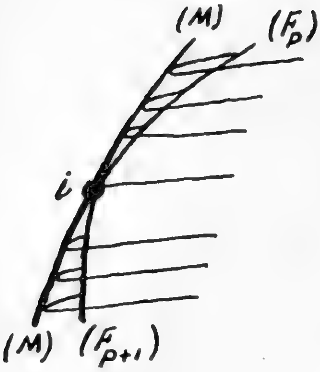

Fig. 3.

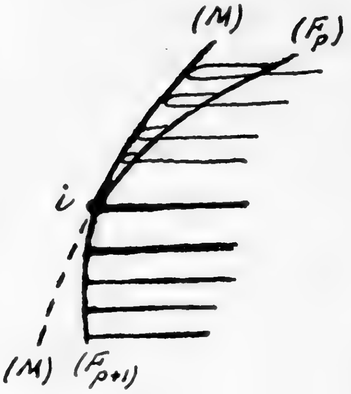

Fig. 4.

Fig. 5.

The stable part of the region $(F_p F_{p+1})$ extends itself between the curves (F_p) and (F_{p+1}). This region is indicated in the figures by some horizontal lines and little arcs.

In fig. 1 it extends from (F_p) and (F_{p+1}) up to the (M)-curve; the stable part of the region $(F_p\, F_{p+1})$ consists, therefore, of two leaves, which cover one another partly.

In fig. 2 the stable part of the region $(F_p\, F_{p+1})$ cannot extend as far as the part of the (M)-curve, which is situated in the vicinity of the point i. It may be situated, as is drawn in fig. 2 and then it has one leaf.

I leave to the reader the deduction of the regions in the figs. 3—5.

Now we shall consider some cases, which we can easily deduce from fig. 1 (VIII) and the corresponding fig. 2 (VIII). We imagine in fig. I (VIII) the liquid L on the line GZ_2, so that L and d coincide. Then we have the variable singular equilibrium:

$$(M) = Z_2 + L + G$$

which is transformable. This equilibrium (M) is represented in fig. 1 (VIII) by the line $GdZ_2 = GLZ_2$, the turning-line of the region Z_2LG, the stable part of which is situated between the curves La and Lb. Now we distinguish two cases:

I. Curve La is situated at the left and curve Lb at the right side of GZ_2 (viz. when we go from G towards Z_2). The part $dZ_2 = LZ_2$ of the equilibrium (M) is, therefore, stable, the part $dG = LG$ is metastable.

Let us imagine in fig. 2 (VIII) the (M)-curve to be drawn also, which starts from i in accordance with fig. 1 (VIII) and which must be situated above the curves ia and ib. The three singular curves (M), (Z_1) and (Z_3) must then touch one another in i. The three curves are then situated with respect to one another as in fig. 1.

We are able to deduce the position of the three curves also from fig. 3 (VIII). Curve $(M) = dg$, which touches ia in d, represents the turning-line of the region $(Z_1 Z_3) = Z_2\, LG$. When we let coincide d with i, then ig, ia and ib must touch one another in i. Hence we see also that the position of the three singular curves and that of the region $(Z_1 Z_3) = Z_2\, LG$ is in accordance with fig. 1.

As long as point L is situated in fig. 1 (VIII) at the right side of the line GZ_2, in figs. 2 (VIII) and 3 (VIII) curve ia is situated above ib. When, however, in fig. 1 (VIII) L falls on GZ_2, then in the figs. 2 (VIII) and 3 (VIII) ia and ib touch one another in i, but ia may be situated as well above as below ib. This appears at once from fig. 1 (VIII).

We may consider the position of L on the line GZ_2 as a transition case viz. between the case that L is situated at the right [fig. 1 (VIII)] and that L is situated at the left of the line GZ_2. In the first case ia is situated above ib [fig. 2 (VIII)], in the second case ib must be situated above ia.

[When we wish to consider this transition more in detail, then we have to bear in mind the following. When L is situated as in fig. 1 (VIII), then in fig. 2 (VIII) curve (Z_3) must be situated above (Z_1). This is only true, however, in so far as we consider points of those curves in the vicinity of point i. It is apparent from fig. 1 (VIII) that this is certainly true for points on Ld and Lm. At a further distance from i the curves (Z_1) and (Z_3) in fig. 2 (VIII) may, however, intersect one another. It appears viz. from the direction of the little arrows e.g. on curve agb in fig. 1 (VIII) that the pressure in a and b might be the same. When this is the case, then in fig. 2 (VIII) the points a and b must coincide and consequently the curves (Z_1) and (Z_3) have a point of intersection.]

II. Both the curves La and Lb are situated in fig. 1 (VIII) at the right side of he line GZ_2. The equilibrium (M) is, therefore, metastable, except in the point L.

Now we imagine in fig. 2 (VIII) to be also drawn the metastable (M)-curve. It appears from fig. 1 (VIII) that the (M)-curve must be situated above curve (Z_3) and this curve above curve (Z_1). Those three curves must then touch one another in i. The position of the three singular curves and of the region $(Z_1 Z_3) = Z_2 LG$ is then in accordance with fig. 2.

Now we imagine in fig. 1 (VIII) the liquid L on the line GZ_3, so that L and e coincide. Then we have the variable singular equilibrium:

$$(M) = Z_3 + L + G$$

which is, however, no more transformable now. This equilibrium (M) is represented in fig. 1 (VIII) by the line $GeZ_3 = GLZ_3$, the turning-line of the region Z_3LG, the stable part of which is situated between the curves Lb and Lc. According to the position of the curves La, Lb and Lc with respect to the line GLZ_3 in the P,T-diagram of fig. 2(VIII) different cases follow which are in accordance with the figs. 3—5.

At the deduction of the figs. 1—5 we have assumed the following. When we bring the equilibria (M), (F_p) and (F_{p+1}) from T_0 and P_0 to the temperature T_1 and corresponding pressures, then the variable phases (e. g. F_x) get other compositions in each of the three equilibria [e. g. F_x', F_x'' and F_x''']. This is however not always the case. Let us assume viz. that in the invariant point the phases of the singular equilibrium:

$$(M) = F_3 + F_4 + \ldots + F_x + \ldots + F_{n+2}$$

contain together only n—1 of the components, in the equilibrium (M) then one of the n components is missing; we call this component K.

The variable phase F_x contains, therefore, also only n—1 components (or less) and this is not only the case at T_0 and under P_0, but also at other T and P; this is not only the case in the equilibrium (M), but also in the other equilibria.

This is e. g. the case when F_x is a gas and K a substance which is not volatile or when F_x is a mixed-crystal and K not miscible with this.

In the equilibrium:

$$(F_1 F_2) = F_3 + F_4 + \ldots$$

we have now n—1 components in n phases, consequently it is not bivariant, but monovariant; in the P,T-diagram it is, therefore, not represented by a region, but by a curve. The equilibria (M), (F_1) and (F_2) coincide, therefore, with this curve. As the equilibrium (M) of course is not transformable (viz. the substance K is missing), the (M)-curve is bidirectionable and the curves (F_1) and (F_2) coincide, therefore, in opposite direction. Then we obtain fig. 2 (X). A similar case shall occur e.g. in a ternary system with the components A, B and C, when in the invariant point exists the equilibrium

$$A + B + C + L + G$$

in which the gas-phase G contains only two substances e.g. B and C.

Summary of the P,T-diagramtypes.

When we take an invariant point with the phases $F_1 \ldots F_{n+2}$, then different cases may occur.

I. Reactions are possible, in which all phases of the invariant point may participate. We write those reactions:

$$a_1 F_1 + a_2 F_2 + \ldots . + a_{n+2} F_{n+2} = 0. \quad . \quad . \quad . \quad (1)$$

and

$$\mu_1 a_1 F_1 + \mu_2 a_2 F_2 + \ldots . + \mu_{n+2} a_{n+2} F_{n+2} = 0 \quad . \quad . \quad (2)$$

Now we distinguish the following cases.

A. $\mu_1, \mu_2, \ldots$ are all different. Consequently there are no indifferent phases, then we obtain the general P,T-diagramtypes.

B. $\mu_1 = \mu_2 = \mu$. Consequently there are two indifferent phases viz. F_1 and F_2 and three singular curves viz. (M), (F_1) and (F_2). In the equilibrium $(F_1 F_2)$ may occur the reaction:

$$(\mu - \mu_3) a_3 F_3 + (\mu - \mu_4) a_4 F_4 + \ldots . = 0 \quad . \quad . \quad . \quad (3)$$

This equilibrium $(F_1 F_2)$ may be mono- or bivariant (not invariant). When $(F_1 F_2)$ is monovariant, then it is represented in the P,T-diagram by a curve, then the singular curves coincide [Figs. 1 (X) and 2 (X)].

When $(F_1 F_2)$ is bivariant, then it is represented in the P,T-diagram by a region, the 3 singular curves touch one another in the invariant point [figs. 1—5 (XV)].

C. $\mu_1 = \mu_2 = \mu_3 = \mu$. Consequently there are three indifferent phases viz. F_1, F_2 and F_3 and four singular curves viz. (M), (F_1), (F_2) and (F_3). In the equilibrium $(F_1 F_2 F_3)$ may occur the reaction:

$$(\mu - \mu_4) a_4 F_4 + (\mu - \mu_5) a_5 F_5 + \ldots = 0. \quad . \quad . \quad . \quad (4)$$

This equilibrium $(F_1 F_2 F_3)$ is tri-, bi- or monovariant.

When it is monovariant, then the singular curves coincide. An example is discussed in Communication XIV.

D. $\mu_1 = \mu_2 = \ldots = \mu_r = \mu$; in which $r \leq n$. Consequently there are r indifferent phases and $r+1$ singular equilibria. In the equilibrium $(F_1 F_2 \ldots F_r)$ may occur the reaction:

$$(\mu - \mu_{r+1}) a_{r+1} F_{r+1} + \ldots = 0. \quad . \quad . \quad . \quad . \quad (5)$$

This equilibrium may be from r- to monovariant.

E. $\mu_1 = \ldots = \mu_k = \mu$ and $\mu_{l+1} = \ldots = \mu_{l+m} = \mu'$, in which $K < l$. Consequently there are two groups of indifferent phases. To the first group belong $K+1$, to the second group $m+1$ singular curves.

For $K = l$ E passes into D.

Also three and more groups of indifferent phases may occur. We find an example in the system water + a salt A + a salt B, when in the invariant point occur the phases $G + A + B + A_m + B_n$, in which A_m and B_n represent hydrates.

II. No reaction is possible, in which all phases of the invariant point may participate.

When e.g. the phase F_1 cannot take part into one single reaction, then in (1) and (2) a_1 becomes $= 0$. Then we have an invariant point with $n+1$ phases, for which the considerations sub I are true.

Leiden, *Inorg. Chem. Lab.* *(To be continued)*.

Physics. — "*Experimental Inquiry into the Laws of the Brownian Movement in a Gas.*" By Miss A. SNETHLAGE. (Communicated by Prof. P. ZEEMAN).

(Communicated in the meeting of Feb. 24, 1917).

1. In a former paper[1]) some objections have been advanced to EINSTEIN's formula for the Brownian movement by Prof. VAN DER WAALS JR. and me. According to this formula:

$$\overline{\Delta^2} = \frac{2RT}{N} B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $\overline{\Delta^2}$ represents the mean square of the displacement which a "Brownian particle" obtains per second in a definite direction. Equation (1) has been derived on the supposition that the particle meets in its movement with a resistance of friction. Accordingly B is the inverse value of the factor of resistance which is found when the particle travels with constant velocity under influence of an external force. Statistical mechanics, however, teaches that a particle, in equilibrium with the surrounding molecules, does not experience a force dependent on its velocity, hence no ordinary friction. We have written the equation of motion in the form:

$$\ddot{u} = -p\dot{u} + q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and derived a value for $\overline{\Delta^2}$, which does not lay claim to great accuracy, but leads, at least for the Brownian movement in a gas, to $\overline{\Delta^2}$ being proportional with $\frac{1}{a^2}$, when a represents the radius of the particle.

According to STOKES' formula with CUNNINGHAM's correction:

$$\frac{1}{B} = 6\pi\zeta a k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which ζ represents the coefficient of friction of the medium and

$$k = \left(1 + A\frac{\lambda}{a}\right)^{-1}$$

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[1]) These Proc. **18**, 1916, p. 1322.

λ is the mean length of path of the molecules, A is a constant determined at 0,873 in GUYE's laboratory at Geneva [1]). Different investigators have performed measurements of $\overline{\Delta^2}$, however almost always from the mutual differences of the times in which a particle travels a definite distance under the influence of an external force. It is, however, the question whether the thus determined values of $\overline{\Delta^2}$ are the same as those of equation (1). For with the movement under an external force the distribution in space of the molecules of the medium is disturbed, and it moves for a part with the particle. The chance to a Brownian displacement upward or downward will no longer be symmetrical. Only one investigation is known to me, that by FLETCHER [2]), in which no external force acted on the particle. The data obtained in this way, are, however, not numerous.

It seemed therefore not superfluous to me, to start another inquiry into the validity of equation (1). In my experiments, carried out in the Physical Laboratory at Amsterdam (Director Prof. ZEEMAN), the displacement of a particle was measured while the gravity and the electric force were in equilibrium with each other. This can be established with pretty great accuracy; in order, however, not to be disturbed by a small residual force I observed the movement in horizontal instead of vertical direction.

2. I made use of the well-known method of MILLIKAN [3]) and EHRENHAFT [4]).

When v_v represents the velocity of fall of the particle with mass M, v_v' the velocity under influence of gravity and an electric force of equal direction, v_s the velocity of rising, when this electric force is reversed, the following equations hold:

$$Mg = \frac{1}{B} v_v \quad \text{. . . . .} \quad (4a)$$

$$e\mathfrak{E} - Mg = \frac{1}{B} v_s \quad \text{. . . . .} \quad (4b)$$

$$e\mathfrak{E} + Mg = \frac{1}{B} v_v' \quad \text{. . . . .} \quad (4c)$$

$e\mathfrak{E}$ is the absolute value of the electric force, e the charge of the particle.

From (4*a*) and (4*b*) follows:

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[1]) A. SCHIDLOF et Mlle J. MURZYNOWSKA, Arch. de Genève 4,40,1915, p. 386 and 486.

[2]) H. FLETCHER, Phys. Rev. 33, 1911, p. 81.

[3]) R. A. MILLIKAN, Phys. Rev. 29, 1909, p. 560.

[4]) See for a full description: F. EHRENHAFT. Wien. Sitz. ber. **IIa**, 123, 1914, p. 53.

$$e\mathfrak{E} = \frac{1}{B}(v_v + v_s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If B is known, then $\mathfrak{E}$ can be calculated.

According to (3) we may write ($4a$) as:

$$\tfrac{4}{3}\pi a^3 (\varrho - d)\, g = 6\, \pi \zeta\, a k v_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

ϱ is the density of the particle, d that of the medium, g the acceleration of gravity.

From (6) we can calculate a.

Schrödinger [1]) showed that:

$$v_v = \frac{L}{\overline{t_v}}$$

in which $\overline{t_v}$ is the mean of the times required by the particle to fall over the distance L. The Brownian movement namely causes the measured times to differ somewhat inter se.

The measurement of $\overline{\Delta^2}$ took place in the following way: I observed a great many times the time in which the particle covers a certain distance in horizontal direction, when the gravity is neutralized by an electric force. To find $\overline{\Delta^2}$ from these times of displacement we ask: what is the chance that the particle after a time t crosses for the first time a dividing line at a distance l, no matter on which side? We confine ourselves to the X-movement. I have made use of the method which Schrödinger [1]) uses for a similar problem.

When at a time $t = 0$ a great number N of particles start from the point $O_{(x=o)}$, the number with coordinates between x and $x + dx$ at the moment t will be:

$$N \sqrt{\frac{\alpha}{\pi t}}\; e^{-\frac{\alpha x^2}{t}}\, dx\, .$$

We see the meaning of α by calculating the mean value of x^2, It then appears that:

$$\alpha = \frac{1}{2\,\overline{\Delta^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Calling the points that lie at a distance l on the right and on the left of O, A and B (fig. 1), we shall calculate how many particles have passed neither of the points in the time t.

Fig. 1.

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[1]) E. Schrödinger. Phys. Zeitschr. 16, 1915, p. 289.

It is easy to say how many *have* passed over A or B. Let us confine ourselves first of all to A alone. When a particle reaches A, the chances that it will lie on the leftside or on the rightside of A some time later, will be equally great. Hence the number of particles N_1 that reaches A, is equal to double the number on the righthandside of A.

$$N_1 = 2N\sqrt{\frac{a}{\pi t}}\int_{l}^{\infty} e^{-\frac{\alpha x^2}{t}}\,dx$$

The number of particles that has passed B is of course equally great. The required number M, which has neither reached A nor B, is however not equal to $N - 2\,N_1$, for we have counted the particles that have passed over the path OAB among the two groups N_1, as also the particles OBA. (The meaning of this way of writing is clear.) The number N_2 that has travelled the path OAB is equally great as the number that has reached a point C at a distance $3\,l$ from O. This is of course again:

$$N_2 = 2N\sqrt{\frac{a}{\pi t}}\int_{3l}^{\infty} e^{-\frac{\alpha x^2}{t}}\,dx\,.$$

Thus we find:

$$M = N - 2\,N_1 + 2\,N_2\,.$$

Now, however, the particles $OABA$ and $OBAB$ have again been counted among each of the groups N_2 etc. Continuing in this way we find:

$$M = N - 2N_1 + 2N_2 - 2N_3 + \ldots \text{ ad inf.} \quad \ldots \quad (8)$$

in which:

$$N_m = 2N\sqrt{\frac{a}{\pi t}}\int_{(2m-1)l}^{\infty} e^{-\frac{\alpha x^2}{t}}\,dx\,.$$

In order to find the chance $P(t)\,dt$ that a particle for the first time passes one of the points A or B between t and $t + dt$, we must differentiate (8) with respect to t:

$$P(t)\,dt = -\frac{1}{N}\frac{dM}{dt}\,dt\,.$$

$$M = N - 4N\sqrt{\frac{a}{\pi t}}\left\{\int_{l}^{\infty} e^{-\frac{\alpha x^2}{t}}\,dx - \int_{3l}^{\infty} e^{-\frac{\alpha x^2}{t}}\,dx + \int_{5l}^{\infty} \ldots \text{ ad inf.}\right\} \quad (9)$$

This series is convergent for all finite values of t.

If we now introduce a new variable y, so that:

$$y^2 = \frac{\alpha x^2}{t}$$

then (9) is transformed to:

$$M = N - \frac{4N}{\sqrt{\pi}} \left\{ \int\limits_{l\sqrt{\frac{\alpha}{t}}}^{\infty} e^{-y^2}\, dy - \int\limits_{3l\sqrt{\frac{\alpha}{t}}}^{\infty} e^{-y^2}\, dy + \ldots \text{ ad inf.} \right\}$$

hence:

$$P(t)\, dt = 2l \sqrt{\frac{\alpha}{\pi}}\, t^{-\frac{3}{2}} \left\{ e^{-\frac{\alpha l^2}{t}} - 3e^{-\frac{9\alpha l^2}{t}} + \ldots \text{ ad inf.} \right\} dt \quad . \quad (10)$$

This series is convergent for all values of t.

If we now put:

$$\frac{\alpha l^2}{t} = z_1^2 \quad , \quad \frac{9\alpha l^2}{t} = z_3^2 \quad , \quad \frac{25\alpha l^2}{t} = z_5^2 \text{ etc.}$$

then we get:

$$P(t)\, dt = -\frac{4}{\sqrt{\pi}} \{ e^{-z_1^2}\, dz_1 - e^{-z_3^2}\, dz_3 + \ldots \text{ ad inf.} \} . \quad . \quad (11)$$

Our purpose is to determine α.

From (10) we can calculate the mean value of $\frac{1}{t}$.

We find for this:

$$\overline{\frac{1}{t}} = \frac{1}{\alpha l^2} (1 - \tfrac{1}{9} + \tfrac{1}{25} - \ldots \text{ ad inf.})$$

If we represent the sum of the series between brackets by f, and put:

$$\overline{\frac{1}{t_\Delta}} = \frac{1}{t^*_\Delta}$$

then

$$\alpha = \frac{f}{l^2}\, t^*_\Delta$$

and

$$\overline{\Delta^2} = \frac{l^2}{2f} \frac{1}{t^*_\Delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The index Δ annexed to t denotes that we mean the times of displacement. f is to be approximated with an arbitrary degree of accuracy. For our purpose suffices $f = 0.916$.

The observations give t^*_Δ, hence $\overline{\Delta^2}$ can be calculated.

3. I shall briefly describe the apparatus which I used for my observations.[1] I assume the method of MILLIKAN and EHRENHAFT to be known.

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[1] Compare for a full description my thesis for the doctorate, which will shortly appear.

The *condenser* consisted of 2 square brass plates placed horizontally. I have worked with 2 different condensers C_1 and C_2 of the following dimensions:

	C_1	C_2
sides of the plates	20 mm.	14 mm.
thickness	2 ,,	8 ,,
distance about	3 ,,	2 ,,

C_2 has been used most.

I observed through a *microscope* placed horizontally. A micrometer was adjusted between the two lenses of the eye-piece. It consisted of two sets of lines drawn normal to each other, 0.1 mm. apart. The magnification with respect to this micrometer amounted to from 4 to 5, the total magnification to from 80 to 100.

For the *illumination* I used first an arc lamp of 8 amp., later a so-called reductor lamp. This lamp burns 14 volt and has a very small incandescent body, hence a very great brightness per unit of area. I worked with a lamp of 100 candles.

The *electric circuit* was arranged in such a way that of *one and the same particle* I could successively measure the movement under influence of a constant force and of an alternating force. Fig. 2 schematically represents the course of the electric current. By reversing the double-pole double-throw switch O_1, I could successively insert the condenser C into the continuous-current circuit and into the alternate-current circuit. In the fig. the two condenser plates are in connection with the points M and N of the adjustable resistances W_{1-3}, which short-circuit the battery G. B_2 is an alternate-current voltmeter. By opening S_1 and throwing over O_1, C was brought into connection with the poles of the secondary winding of the transformator T, which converted the 110 volt of the municipal electric current supply to ± 2000 volt. B_1 is a voltmeter of Braun, L is a lamp resistance. V_1 and V_2 are lequid resistances, S_1 and S_2 breakers of the current.

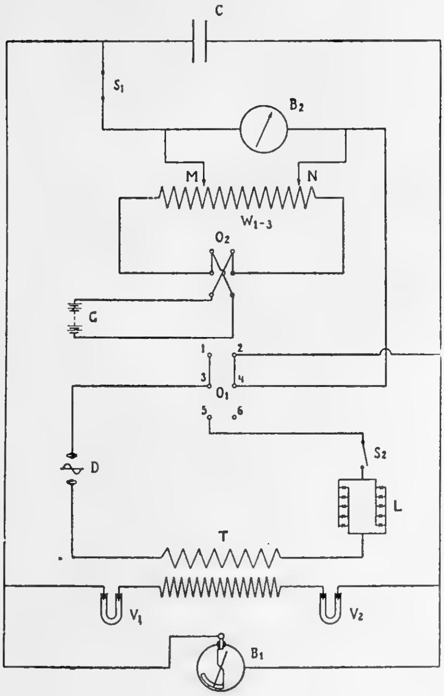

Fig. 2.

For the *registration of the movement* I had at my disposal a MORSE registering instrument, the paper ribbon of which was moved by a three-phase motor of $\frac{1}{4}$ H.P. A MORSE-key served as signal instrument. This arrangement appeared very accurate when tested by means of a chronometer.[1]

The mercury particles were obtained by **EHRENHAFT**'s method. For the other substances I made use of an oil spray which is sold for medical purposes.

4. After the condenser had been carefully adjusted horizontal, I proceeded to the measurement of t_v, t_s, t'_v, and t_Δ of a definite particle. This was brought above in the field of vision, and at the moment that it, falling, crossed one of the 2 horizontal lines, which served as marks, the MORSE-key was pressed down. Then the field was excited, and the time of rising t_s was measured in the same way. This happened several times in succession, the time of falling t'_v also being noted down, when the electric field was reversed. Then gravity was cancelled by an electric force, and the indicator was pressed down when the particle in a horizontal direction passed a following vertical dividing line on the left or on the right of the preceding one. The sense of the displacement was indicated by different signs.

When for a particle the observation was over, the distances between the dots on the paper were measured. These distances, expressed in cm., which are proportional to the times of falling and rising, are indicated by τ. The factor of reduction of τ to t was determined repeatedly with an accurate chronometer.

Equation (6) only holds for spherical particles. I used the following criterion to test the spherical shape. When the particle has different dimensions in different directions, it will be orientated under influence of an electric force, and experience another resistance than in falling. Now follows from equations (4*a*, *b*, *c*):

$$\frac{2v_v}{v_v'-v_s}=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If, however, the factor of resistance in the falling is $\frac{1}{B}$, in the movement under an electric force $\frac{p}{B}$, equation (13) holds no longer, but instead:

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[1] This had already appeared before in experiments by Prof. **ZEEMAN** in an optical determination of the current velocity in a cylindrical tube. These Proc. **18**, 1916, p. 1240.

$$\frac{2v_v}{v_v'-v_s}=p \qquad (13a)$$

Now in measurements with ammonium chloride I actually found values of p departing as much as 50°/₀ from unity. In this p was always smaller than 1, which points to a position of the particle with its length in the direction of the electric force.

For the particles with which I performed my experiments, equation (13) was always sufficiently fulfilled.

They were: *a.* electrically sprayed mercury (which will be discussed later),

b. ice oil, density 0.87,

c. potassium mercury iodide, density 2.56.

I used the last substance in order to get a length of radius lying between that of the mercury and oil particles. The time of falling, namely, depends besides on the radius, also on the density (equation 6). In the observations t_v must not be too small, the measurements becoming too inaccurate in this case, and not too large, because then the deviations owing to the Brownian movement have too much influence and the number of times of falling required to determine v_v, then becomes very great. For this reason I could observe particles with smaller radius of a heavier substance than of a lighter one.

For the measurement of $\overline{\Delta^2}$ I used only series for which at least 100 times of displacement were observed. SCHRÖDINGER[1]) calculates the relative accuracy of the results, obtained in such a way, for an analogous problem at $\sqrt{\frac{2}{n}}$, when n represents the number of elements of the series. In our case the accuracy will not differ much from this.

The distribution of the times of displacement that is to be expected follows from equation (11). The chance that t lies between t_1 and t_2 is:

$$\int_{t_1}^{t_2} P(t)\,dt = -\frac{4}{\sqrt{\pi}}\left\{\int_{(z_1)_1}^{(z_1)_2} e^{-z_1^2}\,dz_1 - \int_{(z_3)_1}^{(z_3)_2} e^{-z_3^2}\,dz_3 + \ldots\right\} \qquad (14)$$

in which:

$$(z_1)_1^2 = f\,\frac{t_\Delta^*}{t_1} \quad etc.$$

The distribution of the τ's is the same.

---

1) E. SCHRÖDINGER, l.c.

In fig. 3*a* and fig. 3*b* I show for two particles in how far the observed and the calculated distribution agree with each other.

Fig. 3*a* refers to the mercury particle N°. 123, fig. 3*b* to the oil particle N°. 158.

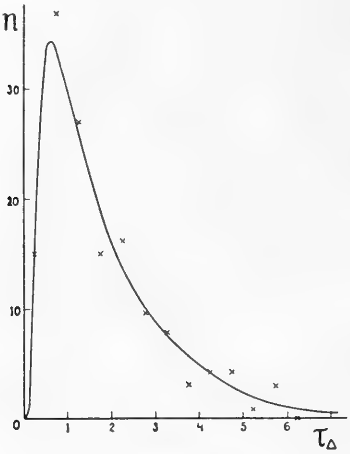

Fig. 3*a*.

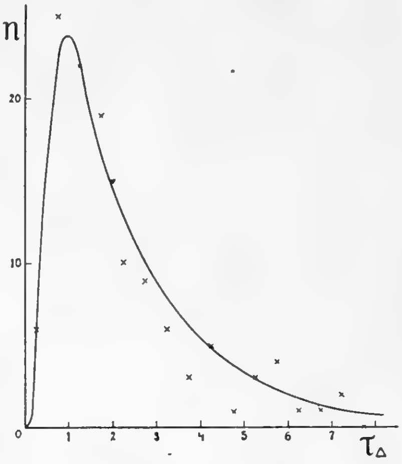

Fig. 3*b*.

I have obtained the curved lines by calculating the to be expected number of times of displacement between $\tau = 0$ and $\tau = 0.5$ by means of equation (14), starting from the measured value of τ^*_Δ, and by drawing this number as ordinate of the point $\tau = 0.25$. In the same way the ordinate of $\tau = 0.75$ gives the number of times of displacement between $\tau = 0.5$ and 1.0 etc.

The crosses give the corresponding values found from observation.

5. I will now proceed to the discussion of the results. I used for this 13 series obtained with oil, 13 with potassium mercury iodide, and 14 with mercury. For some series the time of fall, hence also the radius, proved the same, e.g. for N°. 152 and 153. Such series I have combined. Everything was recalculated to 17° C.

For most experiments $l = 1.87\ 10^{-3}$, $L = 2.24\ 10^{-2}$.

Table I (p. 1015) gives the results obtained with oil and mercury iodide, arranged in descending values of a.

I will first try to determine whether $\frac{1}{\Delta^2}$, hence also t^*_Δ, is proportional to a^2, to ak or to a.

The circles in fig. 4 (p. 1016) represent observations with oil, the crosses observations with potassium mercury iodide. For the present we shall leave the series with mercury out of consideration.

TABLE I.

Oil.					Potassium mercury iodide.				
Number	sign of the charge	number τ_Δ	$a \times 10^5$	t^*_Δ	Number	sign of the charge	number τ_Δ	$a \times 10^5$	t^*_Δ
154	+	114	5.85	4.50	204	+	104	5.99	4.06
172	−	133	5.79	4.25	200	−	167	4.13	3.42
170	−	199	4.88	3.61	189—190	+	207	3.91	2.73
163		113	4.12	2.80	198	−	127	3.41	2.79
156	−	106	4.03	2.32	194—205	−−	272	3.31	2.12
157—173	+−	222	3.86	2.87	187	−	118	2.99	1.94
161	−	111	3.35	2.17	192	+	242	2.89	2.07
152—153	++	202	3.28	2.51	184	−	146	2.78	1.91
169	+	139	3.19	2.37	191	−	185	2.56	1.77
171	−	202	2.97	2.35	203	−	109	2.42	1.77
158	+	121	2.87	1.88	206	+	111	2.29	1.70

We inquire which of the three lines determined by the equations:

$$t^*_\Delta = Ca^2 \quad \text{(15}a\text{)}$$

$$t^*_\Delta = C'ak. \quad \text{(15}b\text{)}$$

$$t^*_\Delta = C''a \quad \text{(15}c\text{)}$$

agrees best with the observations.

To settle this point it is required that we choose a point through which we can lay those curves.

A curve drawn at sight across the points will run very close along point P with coordinates 4.88 and 3.61. We shall choose this as starting point and lay through it the curves 1, 2, and 3 agreeing with equations (15 *a*, *b*, and *c*).

It appears that 1 represents the observations very imperfectly,

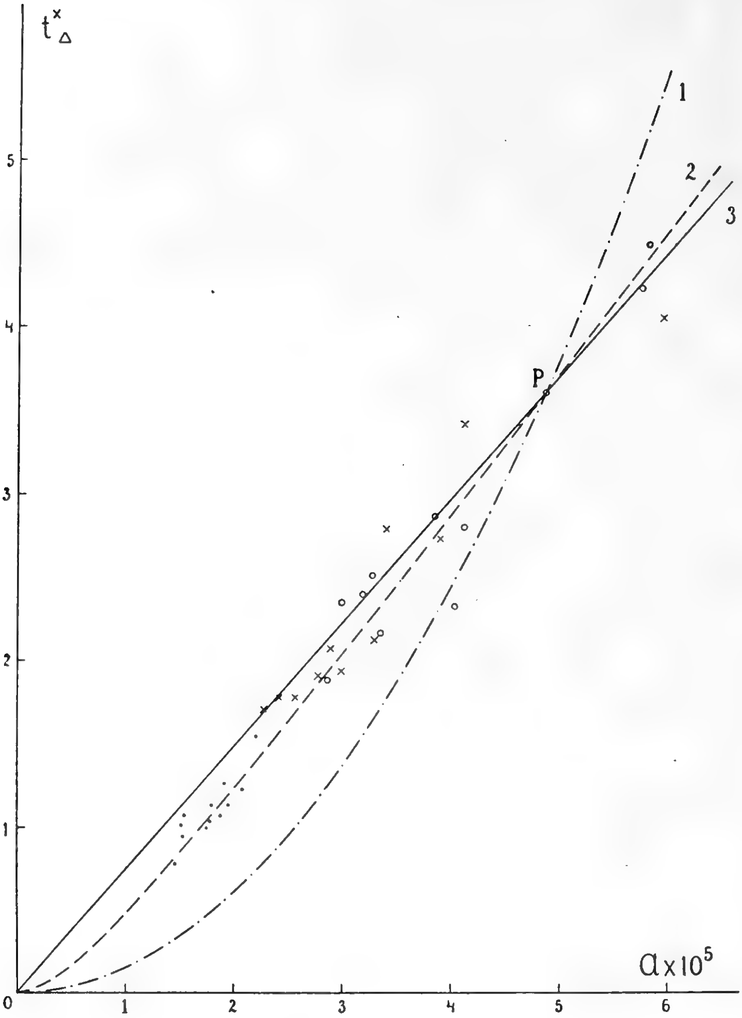

Fig. 4.

and it is no better when we choose another point instead of *P*. The curves 2 and 3 however are in good agreement with the situation of the points, taking into consideration that the deviations are naturally considerable, quite, apart from errors.

It cannot be decided with certainty whether 2 or 3 should be preferred. It seems to me that 2 is slightly more satisfactory. At any rate the supposition that $\overline{\Delta^2}$ should be proportional to $\frac{1}{a^2}$ is contradicted by the experiment, whereas Einstein's formula, at least as far as the connection between $\overline{\Delta^2}$ and a is concerned, is confirmed.

6. In this we have not yet made use of the electrically sprayed mercury particles. These have given rise by their behaviour to the question of the subelectrons. EHRENHAFT [1]) thought he had to conclude from his experiments with this substance: electricity is not divided into quanta, or if it is, the quantum is much smaller than the electron assumed up to now. Among the opponents of this thesis especially TARGONSKI [2]) has tried to give an explanation of the phenomenon by assuming that the particles possess a much slighter density than that of mercury. This would result in a charge that was calculated much too small. TARGONSKI determines the spec. grav. of the grey layer which covers the wall of the vessel and the surface of the mercury after repeated spraying, and finds for it 7.3. He derives from this that the mean density of the sprayed particles is much smaller still. He does not determine, however, the density of the drops themselves. I think I have found a means in my experiments to determine it directly, though it be not with very great accuracy. The particles were sprayed in my experiments in ordinary air, in those by EHRENHAFT and TARGONSKI in dry nitrogen.

With the aid of the known value of t^*_Δ the corresponding value of a can be read from fig. 4 when we assume that the curve 2 is

TABLE II.

Number	sign of the charge	t_v	t^*_Δ	a from fig.	ρ calc.
146	—	3.22	1.55	2.35	9.7
94	+	3.49	1.22	1.95	12
116—120	—	3.99	1.13	1.85	12
102—137	+—	4.08	1.17	1.90	11
141	—	4.28	1.07	1.80	12
150	—	4.52	1.12	1.85	10
110	—	4.67	1.04	1.75	11
138	+	4.77	0.97	1.65	12
117	+	5.80	1.07	1.80	8.6
123	+	6.01	0.93	1.60	10
124	—	6.02	1.01	1.70	9.1
149	—	6.46	0.77	1.40	12

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[1]) F. EHRENHAFT l. c.

[2]) A. TARGONSKI, Arch. de Genève, 4, 41, 1916, p. 207.

the correct one. Then the density follows from equation (6). I have carried out the calculations, and collected the results in table II. The density appears to be about 11 on an average, much higher, therefore, than according to TARGONSKI. Probably a layer of a lighter substance forms at the surface, but the chief component remains mercury.

In fig. 4 the points indicate the places that the observations occupy in the whole of the experiments, when I start from the supposition that the density is 11. In fig. 5 I have enlarged the first part of the preceding figure in order to show where the points would lie when we had to do with pure mercury. This has been indicated by points. The crosses give the place for $\varrho = 7$, which is still too high according to TARGONSKI.

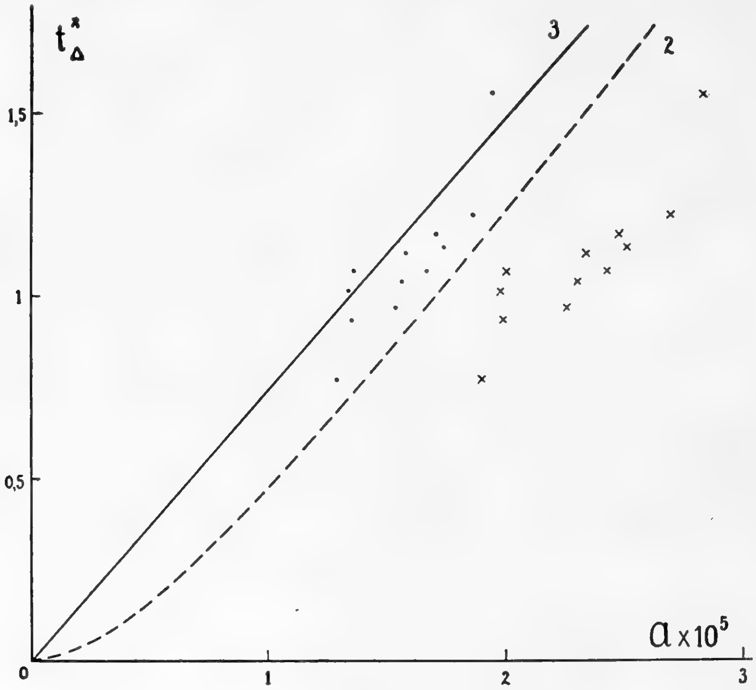

Fig. 5.

7. To examine whether the formula of EINSTEIN-CUNNINGHAM holds also numerically, we can calculate the value of N from curve 2 of fig. 4 with the formula:

$$\overline{\Delta^2} = \frac{1}{3\pi\zeta a k} \frac{RT}{N}.$$

I find for it $N = 6.38 \; 10^{23}$.

Among the different calculations of N that of SOMMERFELD, from the theory of quanta of the spectral lines, may be considered as the most accurate one. SOMMERFELD finds $N = 6.08\ 10^{23}$, a value that differs 5 °/₀ from that calculated by me.

8. Now also **e** can be calculated from equation (5). The measurement of this is, however, not so very accurate, the reading of the voltmeter being uncertain to some percentages. In table III I record the results for a number of particles. Let us assign to every particle, the number of electrons given in the third column, then this, with the total charge of the second column, for the electron gives

TABLE III.

Oil						potassium mercury iodide						electrically sprayed mercury $\varrho = 11$			
Number	charge $\times 10^{10}$	number of electrons	$e \times 10^{10}$	$m_{calc.}$	$m_{meas.}$	Number	charge $\times 10^{10}$	number of electrons	$e \times 10^{10}$	$m_{calc.}$	$m_{meas.}$	Number	charge $\times 10^{10}$	number of electrons	$e \times 10^{10}$
156	5.4	1	5.4	0.51	1.0	177	9.1	2	4.5^5	1.3	1.7	116	19.6	4	4.9
157	9.7	2	4.8^5			178	17.4	4	4.3^5	1.8	2 2	117	15.7	3	5.2
158	4.1 11.7	1 2	4.1 5.8^5	0.58 1.6	1.0 2.0	183	27.8	6	4.6	3.0	4	124	15.7	3	5.2
161	4.7	1	4.7	0.53	0.8	184	47.2	10	4.7			137	22.5	5	4.5
170	14.7	3	4.9			187	11.0	2	5.5			138	21.0	4	5.2^5
171	4.1	1	4.1	0.61	1.1	190	39.6	9	4.4			146	14.4	3	4.8
172	19.6	4	4.9	1.3	1.9	191	11.6	2	5.8	1.7	2	149	16.2	4	4.0^5
173	9.6	2	4.8	1.1	1.3	192	21.7	5	4.3						
						194	10.1	2	5.0^5						

the values of the fourth column. These hood lie around MILLIKAN's value 4.77 10^{-10}.

I shall discuss the meaning of m in the following §.

9. I have stated in § 3 that I also observed the particle in an alternate field. It then executed a vibrating movement, and made the impression of a luminous line, clearest at the extremity where the velocity was smallest, so that as a whole it resembled a dumb-bell. Accordingly I shall speak of the dumb-bell movement. I have tried to measure the length of this dumb-bell by comparing the falling luminous line with the distance of the dividing lines. This was very difficult, particularly because I had only a few seconds time. Then the constant field had again to be excited by quick throwing over of a number of switches, so that I could make the particle rise again before it disappeared out of the field of vision. Hence the measurements are only estimations with a considerable mean error. I wanted to try and get an answer to the following question: does STOKES-CUNNINGHAM's formula sufficiently express the resistance also for this rapid movement?

Then the movement must satisfy the equation:

$$M\ddot{x} = e\mathfrak{E}_0 \sin 2\pi \frac{t}{T} - 6\pi\zeta a k\dot{x} \; . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$\mathfrak{E}_0$ is the maximum intensity of field, T the period of the alternate current[1]). It is easy to calculate that we find for the length of the dumb-bell from this equation:

$$2A = 2\frac{e\mathfrak{E}_0}{M}\left(\frac{T}{2\pi}\right)^2 \frac{1}{\sqrt{1+\frac{K^2T^2}{4\pi^2}}}$$

in which $K = \frac{6\pi\zeta ak}{M}$.

Now $K\frac{T}{2\pi}$ is large with respect to 1, so that we may write:

$$2A = e\mathfrak{E}_0 \frac{T}{6\pi^2\zeta ak} \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

In table III $m_{calc.}$ gives the value of $2A$ calculated from equation (17) expressed in multiples of the distance of 2 lines; $m_{meas.}$ gives the measured lengths. It appears that $m_{calc.}$ is always smaller than $m_{meas.}$. This suggests that the resistance for the vibrating movement

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[1]) Only in approximation is the intensity of the alternate field represented by a sinus function.

would be smaller than for the movement under constant force. The observations are too rough for quantitative calculations, but the differences of $m_{calc.}$ and $m_{meas.}$ are too great and too much in one direction to be attributed to errors of observation.

Amsterdam. *Physical Laboratory.*

Chemistry. — *"Current Potentials of Electrolyte solutions."* (Second Communication). By Prof. H. R. KRUYT. (Communicated by Prof. ERNST COHEN).

(Communicated in the meeting of January 27, 1917.)

1. In a former paper [1]) I communicated a series of measurements with respect to the influence of dissolved salts on the current potential, after having made investigations with solutions of the chlorides of potassium, barium and aluminium. These salts were chosen, because they are electrolytes with resp. a monovalent, a bivalent and a trivalent cation. In Tables 2 and 3 similar results are given for investigations made with hydrochloric acid and the chloride of *p*-chloro-anilene. A standard solution of HCl was prepared by conducting gaseous hydrochloric acid in "conductivity water"; to get the solution of $p\,ClC_6H_4NH_2 . HCl$ KAHLBAUM's $p\,ClC_6H_4NH_2$ was dissolved in water containing the equivalent quantity of HCl from the solution first mentioned.

The results given in Tables 2 and 3 show the decrease of the current potential to be here much larger than in the case of potassiumchloride (cf. Table 1, columns 1 and 2). This result is in perfect agreement with the investigations on electric endosmosis (for literature, see my first communication), and it can be easily understood when we suppose, as FREUNDLICH does, that these phenomena are in close relation with the adsorption of the ions: the H-ion, and also the organic ions (especially aromatic ones) are adsorbed in a greater amount than those of the light metals. A comparison of Tables 2 and 4 shows, that the monovalent H-ion and the bivalent Ba-ion bring about nearly the same lowering of the current potential.

2. A comparison between the electric charges of the capillary tube is still of more importance than that of the current potentials especially with regard to the problems of colloid-chemistry [2]).

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[1]) These Proceedings **17**, 615 (1914).

[2]) See H. R. KRUYT, These Proceedings **17**, 623 (1914).

According to HELMHOLTZ' theory the electric charge is proportional to the product of the current potential and the conductivity of the flowing liquid [1]). This electric charge ε is dependent on the electric

TABLE 1.

Potassiumchloride (K)			
Conc. μ Mol p. L.	$\frac{E}{P}$	$\varkappa \cdot 10^5$	$\varepsilon \cdot 10^3$
0	ca. 350	—	—
50	102	0.65	19.9
100	57	1.3	22.4
250	23	3.2	23.0
500	12	6.4	23.3
1000	4.0	12.7	15.4

TABLE 2.

Hydrochloric acid (H)			
Conc. μ Mol p. L.	$\frac{E}{P}$	$\varkappa \cdot 10^5$	$\varepsilon \cdot 10^3$
0	ca. 350	—	—
25	77	0.95	22.1
50	43	1.9	24.8
100	22	3.8	25.4
250	7.7	9.5	22.1
500	3.1	18.8	17.5
1000	1.2	37.6	13.6
50000	0	—	—
	no change of pole		

TABLE 3.

p Chloro-Anilenechloride (An)			
Conc. μ Mol p. L.	$\frac{E}{P}$	$\varkappa \cdot 10^5$	$\varepsilon \cdot 10^3$
0	ca. 350	—	—
31	114	0.4	13.9
62	65	1.0	19.6
124	26	2.1	16.6
310	12.2	3.9	14.5
500	4.9	8.0	11.8
1000	1.8	14.7	9.7

TABLE 4.

Bariumchloride (Ba)			
Conc. μ Mol p. L.	$\frac{E}{P}$	$\varkappa \cdot 10^5$	$\varepsilon \cdot 10^3$
0	ca. 350	—	—
10	139	0.24	10
25	79	0.60	14.2
50	44	1.20	16.0
100	25	2.4	18.1
200	9	4.7	12.7
1000	1.1	23.1	7.6

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[1]) I seize this opportunity to make a slight emendation to p. 625 of the paper mentioned above. In the equation (2) I confounded the letters *b* and *k*; in the conclusion, printed in italics, "inversely" should be omitted (or "conductivities" should be read instead of "resistances"). The calculations in the paper are however correct in respect to this alteration.

moment M of the double layer, according to:

$$\varepsilon = 4\pi M$$

and

$$M = \frac{E}{P} \cdot \eta \cdot \varkappa$$

where η is the viscosity and $\varkappa$ the specific conductivity of the flowing liquid. We can consider the viscosity of the liquids used to be equal to that of water; the conductivity $\varkappa$ has been measured in each case. The results showed a discrepance with those calculated from KOHLRAUSCH' data of only 1 to 2×10^{-6} Ohm^{-1}; as I measured the conductivity of the water I found it to be from 9×10^{-7} to 2×10^{-6} Ohm^{-1}. In the Tables 1, 2, and 4 the conductivities are those calculated according to KOHLRAUSCH; only in Table 3, for $p\,ClC_6H_4NH_2HCl$, I give the results of my own measurements.[1])

To calculate ε, I reduced all the values to c.g.s.-units; therefore I had to multiply:

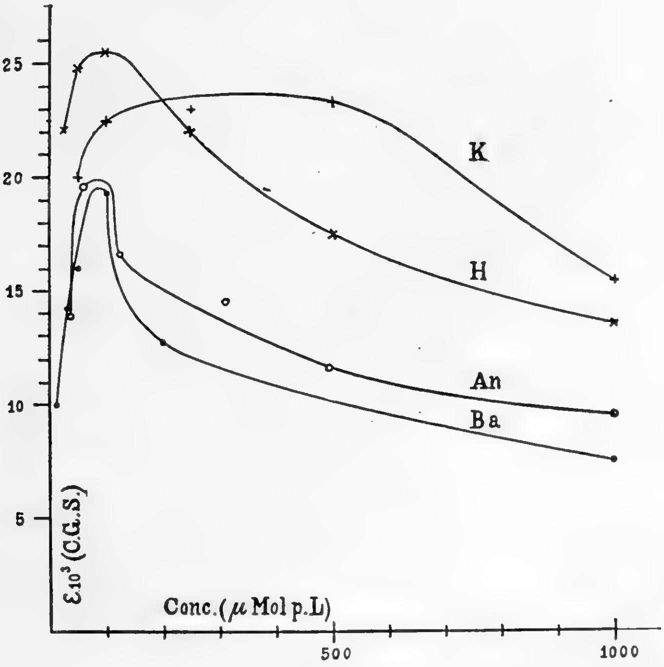

Fig. 1.

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[1]) There would be some reasons not to use KOHLRAUSCH' results and to give no correction for the conductivity of the water used. For probably, this conductivity is not without some influence on the current potential. Still it seems better for a correct comparison to reduce all values to the conductivity of the added electrolytes only.

$$\frac{E}{P} \text{ in } \frac{\text{millivolt}}{\frac{\text{cm.Hg}}{\text{cm}^2}} \text{ by } \frac{10^{-3} \times \frac{1}{3} \times 10^{-2}}{13.6 \times 981}$$

$$\varkappa \text{ in Ohm}^{-1} \text{ by } 9 \times 10^{11}$$

η was put to 0.0108 dyne.

In the last column of the Tables 1--4 the electric charge of the capillary tube is given per cm² in c.g.s.-units. Fig. 1 represents the relations between this charge and the concentration of the liquid flowing through the capillary tube.

3. From these results we may infer that each of these four electrolytes can effect a greater charge to the capillary tube than pure water can; only when an optimum charge is reached, higher concentrated solutions lower the charge.

In the chemistry of colloids much attention has been paid until now to the fact that electrolytes lower the potential of contact; this now appears to be true only for solutions of higher concentration. It is remarkable that in all four cases, mentioned in the Tables 1– 4, the current potentials are lowered by the electrolytes, but that the contact potentials are modified in the peculiar way with an optimum value.[1])

A short time after the publication of my previous paper on this subject, FRANK POWIS[2]) of DONNAN's laboratory communicated (Nov. 1914) a most interesting investigation about the influence of electrolytes on the cataphoresis of oil-emulsions. The similarity of our results is striking; therefore we came in many respects to the same conclusions[3]). When calculating the contact-potential for oil and water resp. for glass and water, POWIS found an optimum only in the case of the monovalent cation of potassium, but it is clear from our Fig. 1 that he could not have observed such a value as regards Ba as he has made no measurements of solutions with a concentration below 200 μ Mol $BaCl_2$. We may now draw the conclusion that the difference between a monovalent and a bivalent cation is only quantitative and not qualitative.

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[1]) It is impossible to make out if the optimum is present in the case of $AlCl_3$. As $\frac{E}{P}$ has reached the value zero at 0.8 μ Mol, the optimum should appear at a still smaller concentration. I regard the data to be insufficient to decide whether we are dealing here with an optimum or not.

[2]) Z. f. physik. Chem. 89, 91 (1915).

[3]) We gave a.o. the same criticism on the theory of irregular series of floculation. With regard to the way in which the final value is reached in each case, I found the same progress, as POWIS describes (l.c p. 179).

The less recent researches of Ellis[1]) do not clash with my results either, he found an optimum only in the case of NaOH, but if we pay attention to the concentrations of his experiments it is clear again that it was impossible for him to have observed the other maxima.

The relation between the four curves in our, fig. 1 is most remarkable from the point of view of colloid chemistry. It is well known that all anorganic ions are adsorbed in nearly the same molecular concentration with exception only of the ions H and OH (and those of the heavy metals). Considering that the Cl-ions have a tendency to give a higher charge to the adsorbing surface, we can easily understand that the ion of potassium causes much less resistance to the increase of the negative charge than the highly absorbative ions of H and $pClC_6H_4NH_3$ and the Ba-ion, which is absorbed in a normal way, but bears the double electric charge. Therefore we find with KCl a plain optimum, which is noticeable even in rather strongly concentrated solutions; with the other three ions the optimum is reached at a concentration from 50—100 μ Mol and the curve is rather pointed, especially with the anilene- and the Ba-ion. Powis' results regarding the influence of the anions and Ellis' regarding that of the OH-ion are in agreement with these conclusions.

It is worth observing that the charges of contact for the different electrolytes at a certain concentration show the same sequence as the corresponding limit values with colloids, but only at concentrations somewhat higher than those of the maxima. The descending branches e. g. at a concentration of 1000 μ Mol. p. L. show the sequence $K^{\cdot}$, $H^{\cdot}$, $An^{\cdot}$ and $Ba^{\cdot\cdot}$; the limit values[2]) for the floculation of the As_2S_3 sol are (m. Mol p. L.)

KCl50 HCl 31 $pClC_6H_4NH_2HCl$ 1.08
$BaCl_2$ 0.69.

To test a quantitative relation between electric charge and limit value (as I tried to do in a previous paper) it is necessary to make use only of the values of the descending branches of fig. 1. For that purpose a close investigation of these branches will be necessary but cannot be made with the apparatus I have used till now. I hope that I shall be able before long to continue these researches with an apparatus fit for experiments at higher pressures.

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[1]) Z. f. physik. Chem. I **78**, 321 (1912); II **80**, 597 (1912); III **89**. 145 (1915).

[2]) Freundlich, Kapillarchemie p. 351 (Leipzig 1909).

4. In the chemistry of colloids one often meets with facts which can be explained by the result of these investigations, viz. that the contact potential in a very diluted solution is higher than in pure water.

The stability of suspensoids is a function of the electric change of the suspended particles [1]). In literature "stabilising ions" are often mentioned; it is well known that a suspensoid sol free from all electrolytes has but little stability, a long dialyzing often causes floculation. [2]) The importance of these traces of electrolytes can now easily be understood in connexion with this investigation: though greater amounts of electrolyte lower the potential of contact and so cause floculation, when the critical potential is reached, extremely low concentrations increase the electric charge and at the same time the stability. Of course they are absolutely necessary when the contact potential in pure water is lower than the critical potential of the sol and when the optimum in the potential curve is higher than that critical value. As the presence of some electrolyte generally seems to be necessary for a stable sol, we may conclude the potential in pure water to be ordinarily rather low. The current potential, it is true, may be high, this is only a consequence of the enormous electrical resistance of this liquid. In the Tables above-mentioned we have not calculated ε for pure water as the value $\frac{E}{P}$, ca 350 m.V., is not exact, the enormous resistance in the cell lowering the exactness of the measurement too much. When we put $\varkappa$ to 9×10^{-7} (this was indead the case with the water when freshly prepared) the electric charge of the tube would be less than 0.01 c.g.s units.

5. The result of this investigation, that an optimum in the curve for the contact potentials may be generally received, draws our attention to the change of our knowledge about the iso-electric phenomenon. In fig. 2 A, B, and C the concentrations are given as abscissae, the contact potentials being ordinates. The axis of ordinates is erected in the iso-electric point, so we have the alcalic liquids on the left, acid liquids on the right. Fig. 2 A

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[1]) I think Powis [Z. f. physik. Chem. **89**, 186 (1914)] is quite right when he concludes that spontaneous floculation does not occur at the potential zero but at a certain "critical" value. In fig. 1 of my communication [these Proceedings **23**, 623 (1914)] the same supposition is represented graphically.

[2]) A great many examples may be found in Svedberg's work: Methoden zur Herstellung kolloider Lösungen anorganischer Stoffe (Dresden 1909).

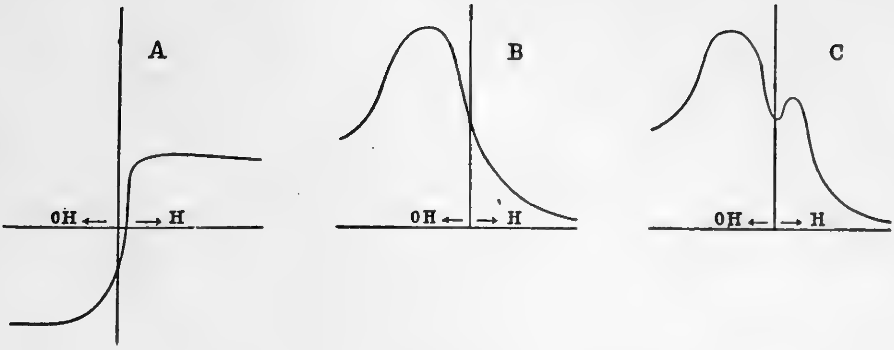

Fig. 2.

illustrates Perrin's [1]) results, as he measured the electro endosmosis of a naphtalene membrane. Fig. 2B shows the results of Ellis' investigations [2]) and Fig. 2C completes that representation by adding the results of this communication, It is still difficult to give a decision whether the minimum lies exactly at the iso-electric point or not.

6. Finally we must inquire into the cause of the increase of the electric charge in diluted solutions and we have to look for points of agreement with the theory of selective adsorption of ions. Powis [3]) too discussed this question.

The adsorption of ions by a surface with an electric charge of the same sign as that of the ions does not form an exceptional case. Lately Freundlich and Poser [4]) mentioned a similar phenomenon when studying the adsorbents with positive and negative electric charge. [5])

They found bolus (negative charge) did not adsorb dye-anions but that aluminium oxide did adsorb cations, some of them (chrysoidin, malachit green) even to a high degree.

"Dies erklärt sich unserer Meinung nach einfach damit, dass die adsorbierende Oberfläche keineswegs mit dem aktiven Elektrolyten, der die Ladung bedingt — gesättigt zu sein braucht; sie kann also sehr wohl neben diesem Stoff noch weitere adsorbieren, genau wie etwa Kohle, die eine nicht zu grosse Menge Benzoesäure adsorbiert hat, noch Oxalsäure aufzunehmen vermag" (p. 318).

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[1]) Journ. de chim. phys. **2** 601 (1904).

[2]) loc. cit. spec. I p. 348 and II p. 606.

[3]) Z. f. physik. Chem. **89**, 103—105, (1915).

[4]) Koll. Beih. **6**, 297 (1914).

[5]) Michaëlis and Lachs, Z. f. Elektrochemie **17**, 1 (1911), as well as Kruyt and van Duin, Koll. Beih. **5**, 269 (1914) found that negatively charged charcoal adsorbed more Cl than K out of a KCl solution.

The same could be the case with the surface of the glass tube. In the iso-electric point only a minimal concentration of anions (the OH ions of water) is present; the concentration of anions becomes larger by adding alkali, chlorides, even acids to water. Evidently the charge given to the tube by the OH ions and the silicic acid of glass is not so large that it cannot increase any more.

The potential increases more as the added anion is better adsorbed; this explains why we meet with such a great increase towards the side of the OH-ions in Fig. 2C and why POWIS found that the optimum on the right side is higher when the anion is more adsorbed or polyvalent [1]). Of course a cation always accompanies the anion and the former resists a continuous increase of the electric charge, as it is adsorbed itself and gets in a more favourable condition for electro-adsorption because it bears an electric charge of opposite nature. Consequentely a lowering of the charge must occur at higher concentration.

Still it is curious that the concentration of anions *really* present plays an important part; for *potentially* there is in water an amount of OH-ions, which is nearly unlimited and we are accustomed in questions of this sort to consider as decisive the concentration of potential ions. The phenomenon of hydrolysis (e.g. with solutions of $AlCl_3$) is usually of no importance [2]). These results warn us however to be prudent with all theory on this account, though on the other hand the condition of the ions of water is not always quite comparable with that of salts in solutions.

POSTSCRIPT.

Just now a paper appeared of HERSTAD, Koll. Beih. **8**, 399 (1916). His investigations on the influence of dialysis of goldsols and its limit values (cf. fig. 15 and 16) are in striking accordance with this communication. The paper induces us to measure the potentials in the case of $HgCl_2$ solutions. Still more amazing is an investigation of BEANS and EASTLACK [Journ. of the Americ. chem. Soc. **37**, 2667 (1915)] on the best conditions for the preparation of highly dispersed gold sols. It gives the impression that the concentration of electrolytes necessary to get red sols coincide with those of the optimum charge of the glass capillary tube. I hope to revert to this subject before long.

Utrecht, VAN 'T HOFF-*Laboratory*. December 1916.

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[1]) POWIS it is true made investigations on KCl and $K_4Fe(CN)_6$, but his conclusions probably hold as well for acids as for salts.

[2]) Cf. FRENDLICH, Z. f. physik. Chem. **44**, 136 (1903).

Physiology. — *"Contribution to the knowledge of the influence of digitalis on the frog's heart. Spontaneous and experimental variations of the rhythm."* By Dr. S. DE BOER.[1] (Communicated by Prof. Dr. G. VAN RIJNBERK).

(Communicated in the meeting of November 25, 1916.)

I. *Introduction.*

It has been proved by a long series of investigations into the influence of digitalis on the frog's heart, that as a first result of the poisoning a slackening of the palpitation sets in, which is not a consequence of a stimulation of the Vaguscentra, as paralysis of the extremities of the Vagus through atropine does not prevent this slackening. It was moreover found that an increase of the size of the systoles sets in after the poisoning. In the second stage of the poisoning an irregular activity of the heart occurs, followed by a stagnation of the ventricle in a maximal condition of contraction. Afterwards the stagnation of the auricles follows.

BÖHM[2]) discovered that after the poisoning of the frog's heart with digitalis the systolic emptying increases in completeness, so that at the highest point of the systole the ventricle is white, a proof that the contents have been removed to the last drop. Suddenly the number of palpitations can be reduced to half the usual number, a halving of the rhythm of the ventricle, which is still repeated once or twice, till at last the ventricle stands still in systole. Then the auricles still continue to pulsate for a considerable time. BÖHM ascertained moreover that the irregularities of the heart-rhythm caused by poisoning with digitalis, disappear by stimulation of the Vagus.

WYBAUW[3]) obtained similar phenomena after poisoning with helleboreine: decrease of the frequency of palpitation, increase of the volume of palpitation and the activity of the heart in the first stage of poisoning; during the second stage of this process irregularities took likewise place, till at last the heart stood still in systole.

After washing the poison out the phenomena of poisoning could recoil again, whilst repeated poisoning occasioned a repetition of these phenomena.

By HEDBOM[4]) and STRAUB[4]) similar disturbances of rhythm of the

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[1]) This investigation was made in the Physiological Laboratory at Amsterdam.

[2]) PFLÜGERS Archiv. Bd. 5.

[3]) Archiv f. exper. Pharmakol. Bd. 44.

[4]) Archiv f. exper. Path. u. Pharm. Bd. 45, 1901, Seite 317.

[5]) Archiv f. exper. Path. u. Pharm. Bd. 45, 1901, Seite 346.

frog's heart were found after poisoning with antiarine. **Straub** ascertained moreover that the duration of the refractory stage of the ventricle increases after the poisoning with antiarine. **Straub** indicates this prolongation of the refractory stage as the cause of the halving of the rhythm.

The prolongation of the refractory stage after poisoning with digitalis was ascertained by **Brandenburg.** [1])

The action of the specimens of the digitalisgroup corresponds consequently in many regards with that of veratrine. Both with digitalis- and veratrine-poisoning we find a decrease of the frequency of palpitation, an increase of the size of the systole, a prolongation of the refractory stage, which causes the disturbances of rhythm. The image of poisoning of the two poisons shows however still important differences.

II. *My own experiments.*

A. Method.

Specimens of Rana Esculenta served as trial-objects. The heart was suspended in the usual manner by attaching the point to a lever. Care was taken that during the stripping and the preparation of the heart the frog lost as little blood as was possible. The oscillations of the heart were registered by the lever on an endless smoked paper (circumference 2 m.) and enlarged 15 times. Under the curves of the heart a line was drawn through the stimulation-signal. This indicated the moment at which one of the partitions of the heart was stimulated. A downward movement of the signal was brought about by closing the primary circuit of the induction-apparatus. The closing induction strokes were blended off. The opening of the primary circuit caused an upward movement of the signal. The opening induction-strokes were conducted only to the preparation. Under the line of the stimulation-apparatus the time was registered in seconds. Three series of experiments were made. In the first series the irritability of the auricle before and after the injection of digitalis was ascertained. In the second series the same experiments were applied to the point of the ventricle, in the 3rd series to the basis of the auricle. About 200 systoles were always registered before the injection with digitalis. Then 10—15 drops of digitalis dialysatum Golaz were injected under the skin of the thigh.

After the poisoning the curves were registered during several hours on the smoked paper. In this way I obtained a survey of

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[1]) Archiv. f. Physiol. Jahrg. 1904. Suppl.

the process of the poisoning, and at the same time I could study the heart in the various stages of poisoning by applying extra-stimulations.

B. The image of poisoning with digitalis.

If we speak of the image of poisoning that we observe with a frog's heart after injection of digitalis, then we understand by it the reaction of the heart on such a dose as occasions disturbances of the rhythm and in the end stagnation of the heart. We can arrange this image of poisoning into 3 stages.

1. The beginning of the poisoning in which the undisturbed normal rhythm still continues, i.e. every impulse of the sinus venosus is answered by all the partitions of the heart with a contraction. 2. Stage of the disturbances of the rhythm. This stage often begins with an alternation of the ventricle which thereupon is converted into halving of the rhythm of the ventricle or formation of groups, afterwards often alternation of the halved ventricle-systoles, then further halving of the ventricle-rhythm. The halving of the auricle-rhythm sets in later than the halving of the ventricle-rhythm. 3. Stage of the groups of Luciani, usually converting into separate ventricle-systoles e. g. to about 16 auricle-systoles 1 systole of the ventricle. Then follows a stagnation of the ventricle. These are the 3 stages of the image of poisoning, as it shows itself in the ventricle. After the stagnation of the ventricle the auricles still continue to pulsate either in the normal or in the halved rhythm or in bigeminus-groups, whilst frequently variations in these rhythms occur.

1. *First stage.* The frequency of the palpitation of the heart slowly decreases. The systolic emptying of the ventricle becomes more complete. We see the ventricle contract during the systoles to a small white ball. The duration of the *a—v* interval increases, towards the end of this stage the height of the systoles decreases remarkably. We observe at the same time a distinct decrease of the irritability of the muscle of the ventricle. Stimulations that before the injection, in the beginning of the diastole, caused an extra-systole, must after 15 minutes be either fortified or be applied later in the heartperiod in order to have the same result. Fig. I represents the suspension-curves of a frog's heart in the first stage of the poisoning. At 1 an extra-stimulation is applied to the basis of the ventricle towards the end of the diastole. No extra-systole of the ventricle takes place, but the auricle shows an extra-systole. During the compensatory pause the stimulation is repeated at 2, but at a moment at which

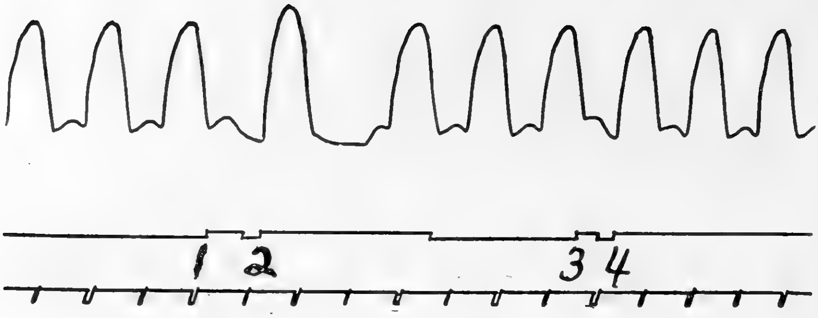

Fig. 1.

the ventricle continuing to pulsate when undisturbed, would not have passed into systole. Therefore this extra-systole is followed by a compensatory pause.. When at 3 and 4 I repeat the same experiment, but apply now at 4 the 2nd extra-systole with a slight scope at a moment, when in normal circumstances the ventricle would likewise have produced a systole, an extrasystole occurs that is not followed by a compensatory pause. As before the injection with digitalis a slighter stimulation on the basis of the ventricle in the beginning of the diastole promptly caused an extra-systole of the ventricle, this fact proves clearly the decrease of the irritability of the muscle of the ventricle. At the same time this experiment teaches us, that an extra-systole of the ventricle is only followed by a compensatory pause, when the extra-systole falls entirely beyond the physiological period of stimulation.

In different ways the first stage can pass into the 2nd. As a rule the rhythm of the normal equally high systoles passes into alternation. The large systole of an alternation-pair is then greater, the little one smaller than the systoles of the normal rhythm. In Fig. 2 such a transition is represented. Fig. 3 shows an alternation, in which the little systole sets in retardedly, on account of a distinct prolongation of the *a—v* interval. This causes the distance between the beginning of a great ventricle-systole and the beginning of the next following little ventricle-systole to become considerably greater than the distance between a little systole of the ventricle and the next following great one.

Now the alternation lasts very short, now longer, and passes then into halving of the rhythm of the ventricle. It occurs likewise that the alternation does not set in at all, so that then the normal rhythm of the ventricle passes directly into the halved rhythm or into formation of groups, as I described circumstantially after poisoning with veratrine.

2. *Second stage.* Consequently the 2nd stage begins usually with

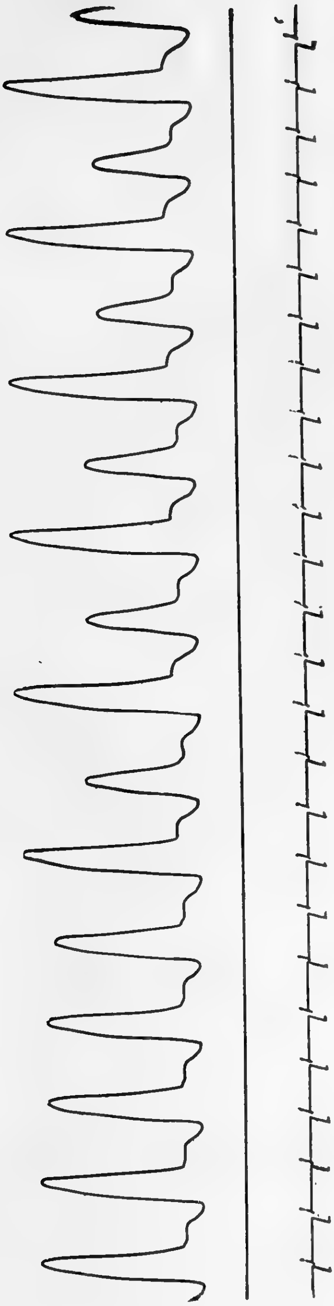

Fig. 2.

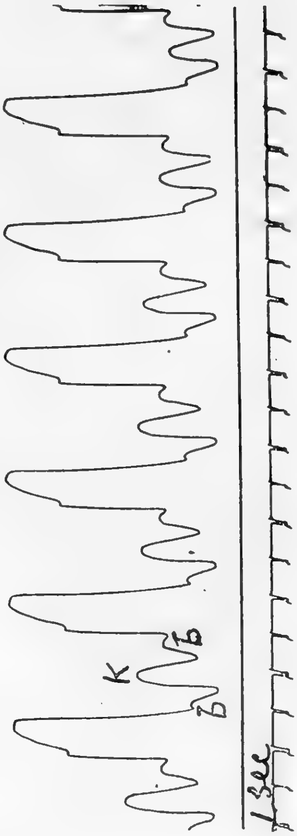

Fig. 3.

alternation of the ventricle. This alternation was likewise described by Muskens 1) and Brondgeest 2). Gradually the height of the little

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1) These Proc. X p. 78.

2) Nederl. Tijdschr. v. Geneesk. 2e reeks jaarg. 39. Vol. I, 1903, p. 1294.

curves of the alternation decreases, after which the halved rhythm of the ventricle sets in. In this halved rhythm of the ventricle alternation followed by a further halving can likewise set in again. The rhythm of the auricle halves later than that of the ventricle. The halved rhythm of the ventricle is often not complete. After the systoles of the auricle, succeeding the systoles of the ventricle, the ventricle is entirely filled with blood, and at the same time repeatedly

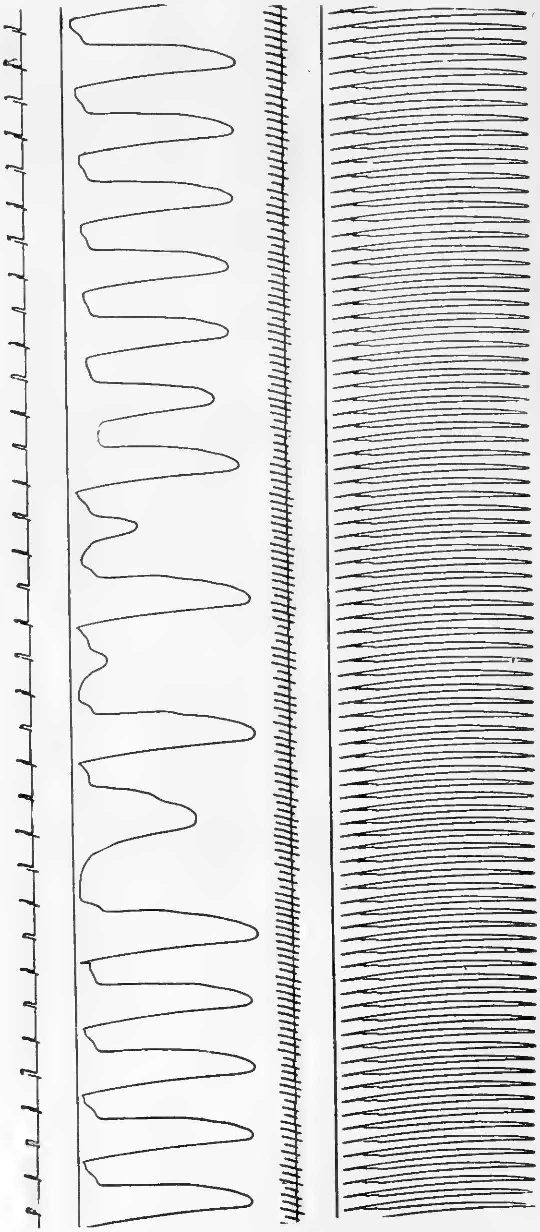

Fig. 4.

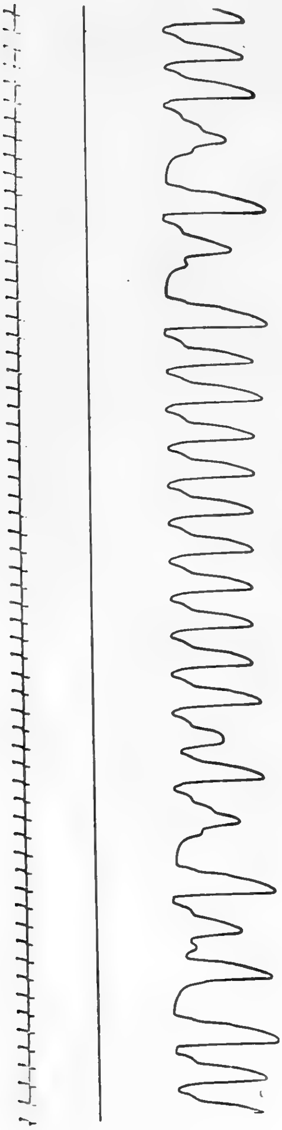

Fig. 5.

very small contractions, abortive systoles of the muscle of the ventricle, take place (vide Fig. 11 at the figures 2). I observed this phenomenon so often that I should naturally be inclined to suppose a connection between the occurrence of the small contractions of the ventricle and the considerable filling with blood of the ventricle. My supposition in this respect was supported by the fact, that I did not observe this phenomenon after the poisoning with veratrine, when the filling of the ventricle with blood decreases considerably [1]).

Besides the mentioned disturbances of the rhythm of the ventricle we observe in this stage still considerable disturbances in the conductivity of the ventricle. This can cause the systole to increase considerably in width (vide Fig. 4th lower row of curves). But it can likewise cause a great difference in the shape of the curves of the systoles and make them deviate entirely from the normal ones.

Many types of these can be observed, some of them I shall describe here. In the first place the ascending line of curves can show a distinct inclination (Fig. 5). Then the top can be split (Fig. 6) and finally a new ascent of the curves can occur in the dilatation (Fig. 7). This 2nd ascent can obtain a greater height than the first top (Fig. 8 and Fig. 13). The postulation could be made that extra-systoles are at work with these curves, but several data tell against this conjecture.

In the first place the fact that the new ascent can occur during the stage of contraction, tells against it, and at the same time the fact that the 2nd ascent during the diastole can by far exceed the first in height. We have here consequently no coordinated systoles of the ventricle.

We often see that these deformed systoles exercise a regulating influence on the rhythm of the systoles of the ventricle. A previously existing alternation afterwards often passes into systoles of the ventricle of equal height. (vide Fig. 4, 5, 6 and 7). We must find the explanation of this phenomenon in the prolonged pause, following after these deformed systoles. This one prolonged pause restores the muscle of the ventricle so much, that during some time normal systoles can follow. The prolonged pause after the deformed systoles owes its existence to the fact that on account of the increase of duration of these systoles the next following impulse coming from the auricle, reaches the ventricle during the refractory stage so that one systole of the ventricle falls out.

The following experiment shows distinctly that one prolonged

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[1]) The increase of the filling of the heart after injection of digitalis is evidently caused by the concommitting vasoconstriction.

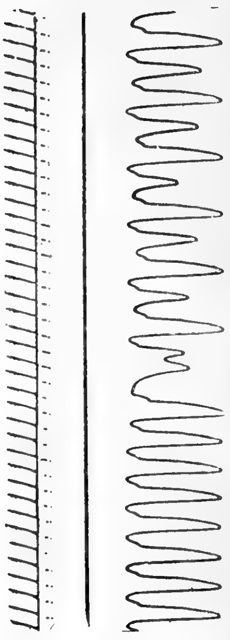

Fig. 6.

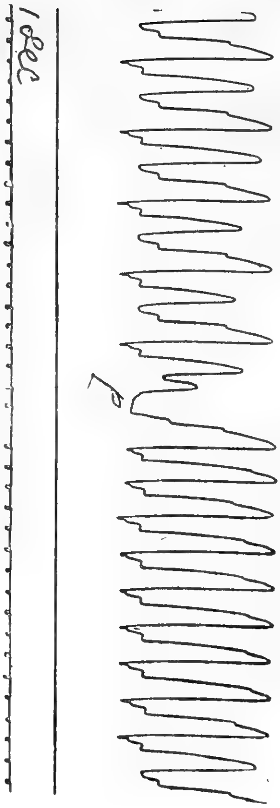

Fig. 7.

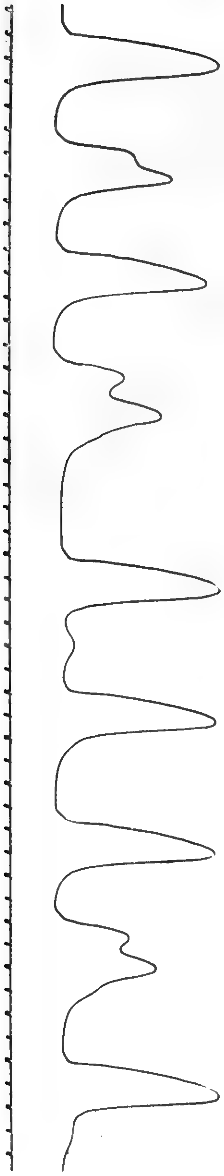

Fig. 8.

pause can regulate an alternation to the normal rhythm with equally high systoles. When a heart pulsates in alternation, I apply an extra-stimulation to the auricle or to the basis of the ventricle at the end of the diastole (Fig. 9). We can then obtain an extra-pause without extra-systole of the ventricle. This extra-pause is then followed by systoles of the ventricle, which obtain an equal height. In this way in Fig. 9 at 1 an extra-stimulation was applied to the basis of the ventricle at the end of the diastole. Because the muscle of the ventricle is still refractory, no extra-systole of the ventricle takes place, but by current-loops the auricle is incited to extra-contraction. In this way the extra-pause occurs without extra-systole of the ventricle, and thereupon we see 9 systoles of equal height. Then the alternation sets in again, which is in the same manner transferred again into the normal rhythm of the ventricle by an extra-stimulation.

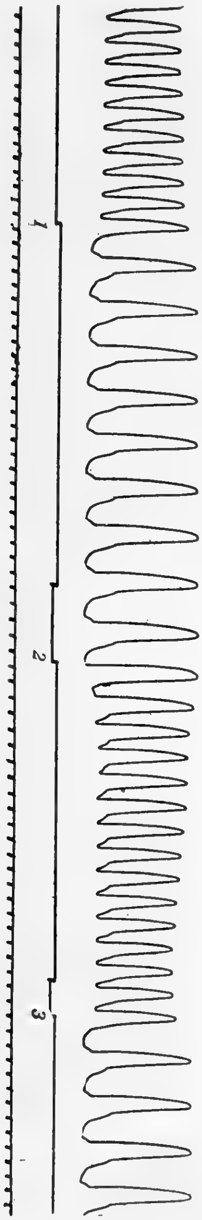

Fig. 10.

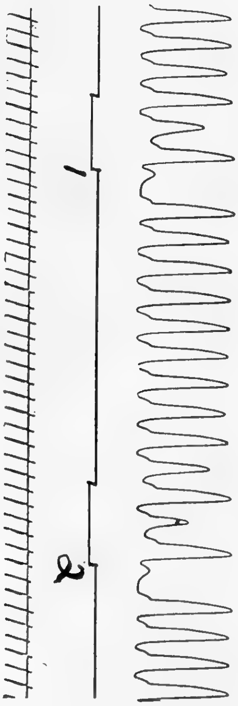

Fig. 9.

The alternation of the ventricle is usually followed by the halved rhythm of the ventricle.

We see this last rhythm repeatedly return spontaneously to the normal twice as quick rhythm. Other variations of rhythm occur likewise e. g. between the normal rhythm of the ventricle and bigeminusgroups (in which every third systole of the ventricle has fallen

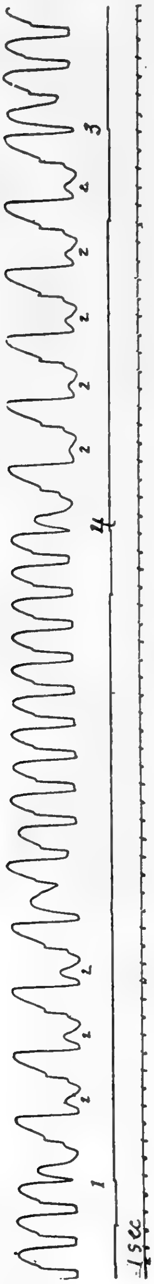

Fig. 11.

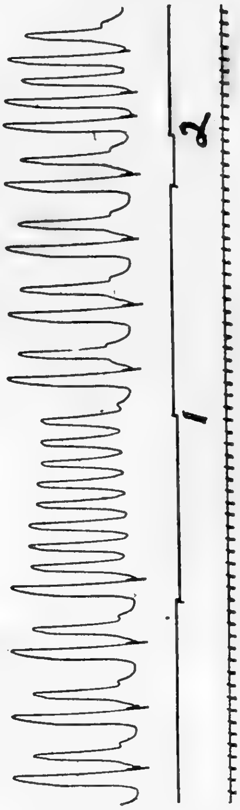

Fig. 12.

out) further between bigeminusgroups and the halved rhythm of the ventricle. *These variations of rhythms, which can occur spontaneously, I could likewise bring about experimentally.* A few examples may follow here.

In Fig. 10 I apply at 1 during the 10_{th} systole an extra-stimulation to the basis of the ventricle at a moment at which the ventricle is still refractory. By current-loops the auricle is incited to an extra-contraction, after which an extra-pause follows. The next following systole of the ventricle has considerably increased after the prolonged pause, by which the ventricle is fastened in the halved rhythm. By means of a succeeding extra-stimulation at 2 at the end of the pause this halved rhythm of the ventricle is changed again into the normal one. At 3 I change this normal rhythm of the ventricle again into the halved one.

In fig. 11 we see the representation of the curves of another poisoned heart. During the first 3 curves of the figure the heart pulsates in the normal rhythm of the ventricle. At 1 I apply an extra-stimulation to the point of the ventricle, which causes a little extra-systole. The postcompensatory systole is much enlarged. Then 3 more of these enlarged systoles follow, but each of these enlarged systoles is followed by a very slight contraction of the ventricle which varies a little in size (indicated by the figures 2). We have here consequently the halved rhythm of the ventricle with this reservation, that every systole of the ventricle is still followed by a slight contraction of the muscle of the ventricle. This rhythm of the ventricle passes spontaneously into the normal one. At 4 I modify this normal rhythm, in the same way as at 1, by an extra-stimulation into the halved rhythm of the ventricle (at the figures 2 again slight contractions of the ventricle occur). At 3 I apply again at the end of the diastole an extra-stimulation to the point of the ventricle. A little extra-systole with a refractory stage of short duration is the result. The normal rhythm of the ventricle is restored by it.

Fig. 12 represents curves of a frog's heart after poisoning with digitalis. In the figure we see first 3 bigeminusgroups occasioned by the falling out of every 3rd systole of the ventricle. These bigeminousgroups pass spontaneously into the normal rhythm of the ventricle. At 1 an extra-systole of the auricle not followed by a systole of the ventricle occurs after an extra-stimulation of the auricle. Under the influence of the prolonged pause the next following systole of the ventricle is now considerably enlarged, the consequence of this is, that now bigeminousgroups set in. By an extra-stimulation to the auricle in the pause between 2 groups at 2 this bigeminy is modified into the normal rhythm of the ventricle. A more explicit explanation of these artificial modifications of rhythm can be found in the communication dealing with the artificial modification of rhythm after the poisoning of a frog's heart with veratrine [1]).

If now we restrict our discussion to the ventricle, we have during the 2nd stage of the poisoning the following modifications:

1. Halvings of rhythm
2. Formation of groups.
3. Alternation.
4. Deformed systoles (often with more than 1 top).
5. Abortive systoles.

The image of the curves in this stage is often a very irregular one, and it can only be analysed by taking into account the deviations of shape mentioned above. In Fig. 13 such an irregular image of curves is represented. Now a systole of the ventricle falls out, now an enlarged systole of the ventricle occurs; after such an enlarged systole a little abortive one can follow, as I described above. Between these deformed systoles occur, on account of disturbances in the conductibility in the ventricle. In this way the play of the curves that was at first incomprehensible, becomes clearer. When the ventricle functions so irregularly, then the last stage is certainly reached which in literature is indicated as the toxical

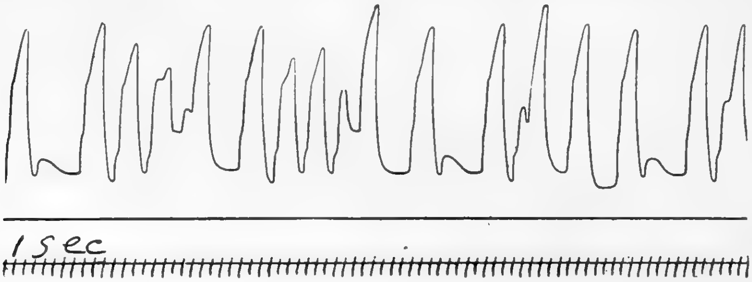

Fig. 13.

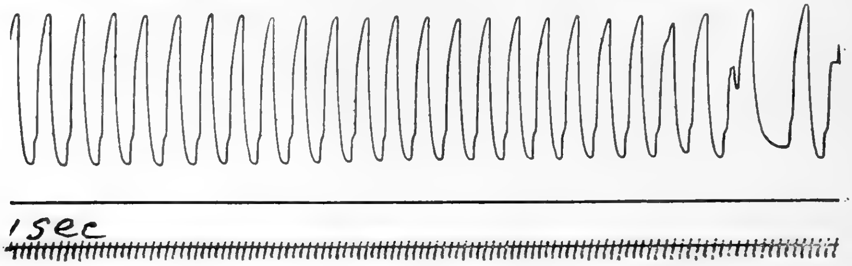

Fig. 14.

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[1]) These Proceedings Vol. XVIII N⁰. 10 pag. 1588, 1916.

stage. It is remarkable that this toxical stage can promptly be reduced again to the normal rhythm in which every systole of the auricle is followed by a powerful systole of the ventricle.

In Fig. 14 we see a representation of the same frog's heart of Fig. 13, but now after the sinus venosus has two minutes previously been refrigerated during a short time. After this refrigeration the rhythm of the ventricle became quite normal again, so that every systole of the auricle was followed by a systole of the ventricle. After the refrigeration the sinus venosus assumes gradually again the temperature of its surroundings, by which the tempo of the pulsations of the heart is quickened again. The consequence of this is, that the toxical stage shows again irregular curves of the ventricle. On the diastolic line the last but one curve of the figure shows a second large elevation, which proves that at that moment the conductibility in the ventricle is considerably disturbed. This deformed curve is, as always is the case, followed by a prolonged pause and a succeeding enlarged systole. From this moment the former irregularities in the curves of the ventricle appear again. This experiment is very instructive. If we ask why through this simple intervention we could make the toxical stage return to the therapeutic one, the answer to this question can easily be found. The irregularities described above were brought about by the circumstance that the refractory stage of the ventricle or of part of the ventricle lasted longer than 1 sinus-period. Consequently the ventricle or part of it was at a given moment not able to react upon the "Erregung" that reached it from the auricle. The refrigeration of the sinus venosus prolongs the duration of the periods of the heart, so that the wrong proportion does not exist any longer and the normal rhythm is restored. This affords us a new indication that the irregularities we have described are caused by a wrong proportion between the duration of the refractory stage and the duration of the sinus-periods. This experiment teaches us likewise that in the pharmacy of the digitalisgroup a dose is then toxical, when the frequency of the heart is a definite one. At high frequencies the toxical dose is much smaller than at low frequencies. This holds likewise for veratrine.

3. *Third stage.* During the third stage the periods of Luciani occur. The auricles continue to pulsate during the pauses in the normal tempo, so that we have here to do with periods of the ventricle (consequently of the contracting extreme organ). The ascending-stairs in the beginning and the descending-stairs towards the end of a group

occur often, but not always (vide Fig. 15). As a transition to the stage of the periods of Luciani I found a periodical descent and

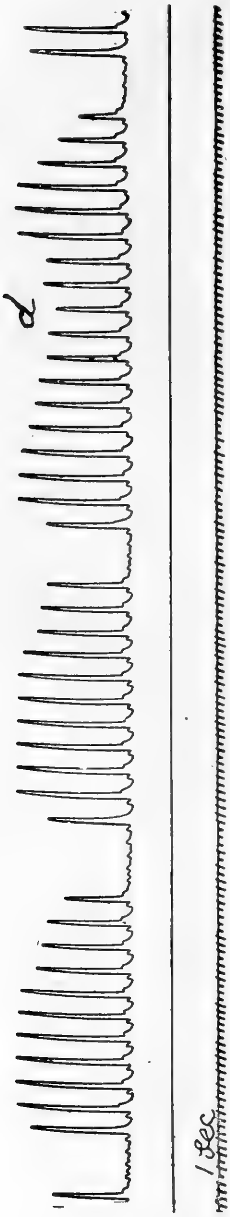

Fig. 15.

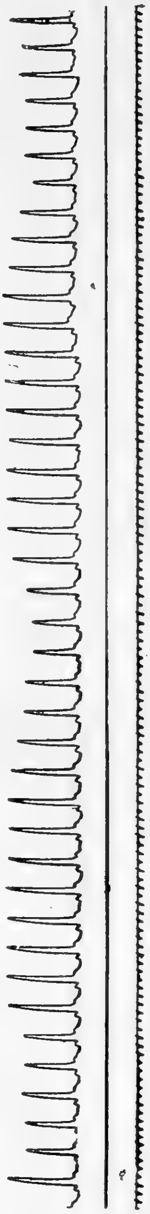

Fig. 16.

ascent of the curves of the ventricles as is represented in Fig. 16. These two figures originate from the same frog's heart; the curves of Fig. 16 were registered somewhat less than half an hour before these of Fig. 15. In Fig. 15 we see in the course of the 3rd group at *d* the curves of the ventricle descending and ascending again. It seems as if this group was dividing itself. This 3rd stage was treated by me more elaborately in the Biological Section of the Society for promoting Physiology, Physic, and Surgery (Genootschap ter bevordering van Natuur-, Genees- en Heelkunde). Here I observe only that analogous observations were made by me after the poisoning of the frog's heart both with veratrine and antiarine.

The groups of Luciani were often followed by separate systoles of the ventricle (crisis of Luciani). There upon stagnation of the ventricle. It can likewise occur that the crisis does not set in at all.

LANGENDORFF and ÖHRWALL are of opinion that the variations of rhythm form the transition to the periods of Luciani. I can by no means share this view. It is true that in my method variations of rhythm proceed the periods of Luciani, but they do not occur in the latter part of the 2nd stage and, consequently, they cannot form the transition to the 3rd stage. They occur exactly *in the beginning* of the 2nd stage.

They are indeed variations between the 2nd and the 1st stage of poisoning [1]).

Amsterdam, 13 Nov. 1916.

Physiology. — "*On the Analogy between Potassium and Uranium when acting separately in contradistinction to their antagonism when acting simultaneously.*" By Prof. H. ZWAARDEMAKER.

(Communicated in the meeting of February 24, 1917.)

Physiologically there is some analogy between potassium- and uranium-salts. Both are strong poisons, the first for the heart, the second for the kidneys [2]). Again, in small doses they very largely aid the functions of these organs [3]). The automatically beating frog's heart, when fed artificially is a very suitable object to watch this useful action, as it is not fed along capillaries, but through lacunae penetrating everywhere from the cavity into the wall.

For artificial circulating fluids salt-solutions are used, so diluted that they possess the normal osmotic pressure and, moreover, contain next to each other 2 molecules of calcium chloride 1 or 2 molecules of

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[1]) A more explicit paper follows in the Archives Neerlandaises de Physiology de l'homme et des animaux. Tome I. 3e livraison p. 502 (1917).

[2]) WOROSCHILSKY, Arb. a. d. pharmakol. Inst. in Dorpat. Bd, **5**, 1890.

[3]) Moreover, in the concentration of $^1/_{1000}$n. uranium is conducive to alcoholic fermentation as well as to the growth of the tubercle-bacillus and the Bacillus pyocyaneus. (Vide BECQUEREL and others in C. R. t. 154 ff.).

potassium chloride and 100 molecules of sodium chloride; also oxygen and a buffer to obviate a change of the reaction through lactic acid and similar acids, which has proved to be highly deleterious[1]).

A few months ago I demonstrated with Mr. FEENSTRA[2]) that in such circulating fluids potassium chloride can be replaced by uranium-, thorium-, or radium-salts. The same statement was formerly made regarding rubidium by SYDNEY RINGER[3]). I am now in a position to add that radium-emanation initially to the quantity of 100 MACHE-units is also one of the elements, which, as far as their physiological action is concerned, can be substituted for potassium. These facts induced us to ascertain whether radiation of mesothorium or radium could be applied for the same purpose. About 34 experiments made with the assistance of Messrs. BENJAMINS and FEENSTRA[4]) confirmed this supposition to the full. The same result has since been obtained in a number of subsequent experiments.

Quite unexpectedly an analogy thus reveals itself between radiation and the salts of potassium, rubidium, uranium, radium (and its emanation), elements belonging to widely differing groups of the periodic system. The radio-activity, common to all, is no doubt the true cause of the similarity in action. The question immediately arises, what is the amount of energy concerned? This may be determined when the artificial circulating fluid contains, instead of 100 mgrm. of potassium chloride, 5.10^{-6} mgrm. of a radium-salt per litre. Assuming $^1/_{10}$ cc. to flow through the beating heart every second, 4.10^{-7} erg of radio-active energy per second is carried into the organ. It may be supposed that part of this amount is adsorbed by the muscle cells and acts physiologically. Then, however, we approach the energy quanta, which are known to stimulate the sense-organs, when they are responded to by an end-organ specially intended for it[5]). Also the quanta of energy transmitted by artificial circulating fluids containing uranium- or thorium-salts, appeared in our experiments to be of the same order[6]).

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[1]) G. R. MINES, J. of the Marine Biol. Assoc. Vol. 9. Oct. 1911, p. 171.

[2]) T. P. FEENSTRA, These Proc. XIX p. 99, XX 341 and 633.

[3]) S. RINGER, Journ. of Physiol. Vol. 4, p. 370.

[4]) H. ZWAARDEMAKER, L. E. BENJAMINS and T. P. FEENSTRA, On Radium-radiation and cardiac action. Ned. Tijdschr. v. Gen. 1916 II, p. 1923.

[5]) H. ZWAARDEMAKER, Erg. d. Physiol. Bd. 4, p. 452. 1905.

[6]) By far the greater number of our experiments were performed on the so-called Kroneckered heart, but in many cases also the whole heart was experimented upon.

Apparently the amount of energy applied with radiation is much larger, but there is no saying how much of it is adsorbed by the muscular elements, and this, of course, is the essential thing.

The amount of energy in the case of the normal element, potassium, is still more difficult of determination. The photo-chemic effect of potassium salts shows itself only after 56 days on the photographic plate and the ionizing influence upon the air is 1000 times smaller than that of uranium-oxide. The total radio-activity may be deduced from it by a round-about way with some allowance for the absorptivity. The doses of the two elements to be present in the artificial circulating fluids are empirically in the ratio of 53 : 15 mgrm. per litre [1]).

The analogy in action, described just now of all known radioactive elements, of small (*K* and *Rb*) as well as large (*U*, *Th*, *Ra*) atomic weight, is not restricted to the heart. The vascular endothelium is also affected by it as far as *K*, *Rb*, *U*, and *Th* are concerned; likewise the striated muscles according to experiments performed in the last few months by Dr. GUNZBURG in our laboratory. Furthermore the statements made by Prof. HAMBURGER and his pupil BRINKMAN in the previous meeting about the phenomenon that the epithelium of the kidney becomes permeable, when either the potassium- or the uranium-salt is wanting in the artificial circulating fluid, lend support to our view.

Between potassium on the one hand and uranium and thorium on the other, there is, moreover another correspondence, viz. their relation to calcium and strontium. The first and the second group of these salts counteract each other, i.e. their dosage may be varied within certain limits, on this understanding, however, that if the

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[1]) The distribution of the radio-active energy supplied with the artificial circulating fluid takes place in widely different ways: with the lighter elements it is distributed over a large number of atoms, every atom carrying only a very minute quantum of energy. With the heavier elements, however, it is accumulated in a few atoms, most of all in the case of radium. There is, therefore, much less chance for an atom contained in the circulating fluid to adhere to or to enter a muscle cell, with the heavier elements than with the lighter ones. It suffices, however, that some few cells are rendered automatic, for, when once begun, the excitability of the cardiac muscle spreads automatically; still, those few cells must be worked upon. We might in some way make allowance for the chance, alluded to here, by dividing the quanta of energy, transmitted per $^1/_{10}$ c c., by the atomic weight of the carrier. This would bring out the inferiority of the energy borne by the heavier atoms. It really will be seen then that the energy-values for the various elements approximate each other (see the Table. Proc. K. A. v. Wet. Vol. XX p. 636).

amount of a salt of the first group be raised the fluid should also contain a greater amount of the salt of the second group.

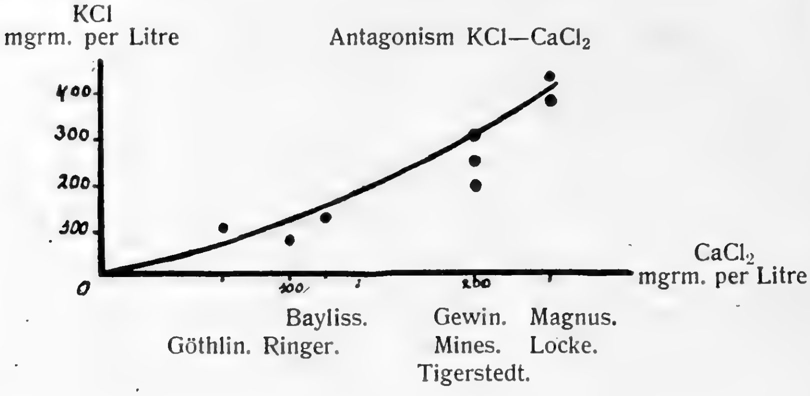

Fig. 1.

This counteraction is represented in two graphs. The data of the first are taken from the literature, in which are recorded a great number of widely different and efficient combinations of salts as generally accepted circulating fluids. The second illustrates the results of experiments purposely performed. In both graphical representations a well nigh straight curve indicates the ratios in which either the potassium-salt or the uranium-salt must be combined with the calcium-salt. Potassium-chloride and uranyl-nitrate also admit of an equilibrium with strontium-chloride; for their graphical representation I lack the necessary experimental data [1]).

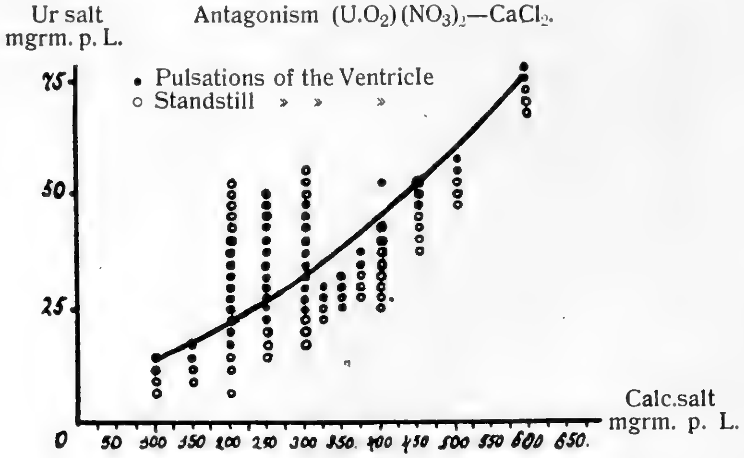

Fig. 2.

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[1]) Vide W. H. JOLLES, Thesis. Utrecht 1916, p. 39.

In contradistinction to the remarkable analogy between potassium and uranium (present in the fluids respectively as potassium-ion and uranyl-ion) there is a no less remarkable antagonism. When a heart, that beats well when fed with a potassium-containing fluid, is supplied with a circulating fluid containing per litre 25 mgrms of uranyl-nitrate instead of 100 mgrms of potassium-chloride, it stops suddenly. And conversely when a potassium-containing fluid is given to a heart which beats normally with a uranium-containing fluid, it is also brought to a standstill. Not before many minutes later do the automatic contractions recommence. The very same takes place when the normal potassium-containing fluid is administered after a fluid containing thorium or radium or emanation. Not, however, when the heart is supplied with first rubidium and then potassium or vice versa. Nor when a uranium-containing fluid succeeds one containing emanation or the reverse. It is evident, therefore, that, when applied in succession the lighter elements compensate each other, just as the heavier ones do. On the other hand the two groups are mutually antagonistic. When applied successively, they arrest the cardiac action; when applied singly, each sustains it for an indefinite space of time.

A mixture of potassium and an equal amount of rubidium causes the heart to beat; the same holds for an equal apportionment of uranium and thorium; but potassium combined with uranium or thorium, or a combination of rubidium and uranium stop the heart's pulsations. Thus the lighter and the heavier elements are reciprocally antagonistic not only when acting successively but also simultaneously. The following graph shows the ratios of the potassium-uranium antagonism when acting simultaneously.

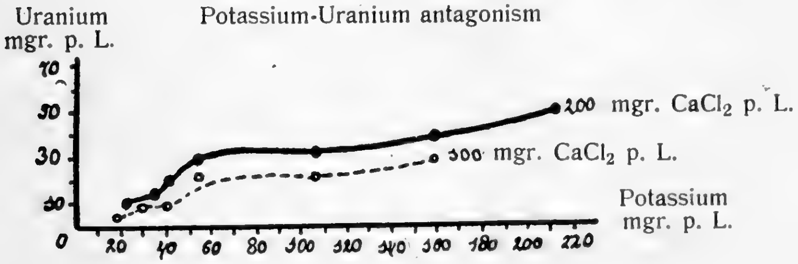

Fig. 3.

The points indicate the combinations with which the heart was reduced to a standstill. In the mixtures lying above the continuous line the cardiac action was restored by the influence of an excess of potassium; in the mixtures below the continuous line pulsation recommenced under the influence of an excess of uranium. The continuous line marks the results of a series of experiments with fluids containing besides the antagonistic salts, the sodium chloride and the buffer also 200 mgrms of calcium chloride per litre. Under it is seen a dotted line, illustrating the same for circulating fluids containing only 100 mgrms of calcium chloride.

Also the antagonism of uranium for radium-radiation (through a thin micawall) respectively mesothorium-radiation (through a thin glass wall) is easy of determination. From the balance-point of the antagonists potassium-uranium the heart may, through exposure to radiation, resume its normal beats in a wonderfully short time, in a few minutes. When adding some uranyl-nitrate a standstill will ensue from which the heart will recover again on exposure to radiation [1]). In the condition of equilibrium the radiation is evidently on the side of the potassium, opposed to the uranium.

It is especially the last experiments, which we made repeatedly, that show distinctly that nothing but the radioactivity of the uranium counterpoises the influence of the radiation (through a mica- or a glass-wall). If so the potassium-uranium antagonism must be entirely ascribed to these causes.

Obviously we feel inclined to deduce from the detected antagonism an index for the biological radioactivity of potassium and to place it alongside of the photochemic respectively the electric; but as yet an impediment is met with in the complicate influence (counteracting) of the calcium, which (in Fig. 3) originated two curves of a similar character, one for the simultaneous addition of 100 and another for that of 200 mgrms of calcium chloride per litre.

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[1]) A prolonged exposure brings about another standstill which will be followed by pulsation after the addition of uranium, not after that of potassium. Gradually, however, a condition manifests itself whose nature I have not been able to make out, and which I will provisionally term the secondary condition of radio-activity. Its peculiar feature is that the heart will beat only with a Ringer-solution containing neither potassium nor uranium. When criticizing these experiments due regard should be paid to the presence of the minute amounts of uranium X in addition to the uranium.

Physics. — "*Methods and apparatus used in the cryogenic laboratory.* XVII. *Cryostat for temperatures between* 27° *K. and* 55° *K*". By Prof. H. KAMERLINGH ONNES. (Communication 151*a* from the Physical Laboratory at Leiden).

(Communicated in the meeting of June 24, 1916).

1. *Introduction.* In section 1 of Comm. XVI of this series (Comm. N°. 147*c* Proc. XVIII, I, p. 507 I pointed out the importance of arrangements by which it would be possible to obtain constant and uniform temperatures in the range from about 27° K. to about 55° K. and I mentioned that a cryostat had been constructed suitable for this region of temperatures, in which for accomplishing this purpose a current of hydrogen warmed to the desired temperature was made to pass through the experimental chamber [1]). The degree of constancy and uniformity of the temperatures which was obtained have exceeded our expectations, at least when it is possible to adapt the arrangement of the measurements to the requirements of the apparatus, as happened to be the case in the investigations which have so far been carried out with it. It is true that we have not succeeded in obtaining as easy and certain a regulation of the temperature with the hydrogen-vapour cryostat as would be available, if substances existed suitable for liquid baths between 55° K. and 27° K. [2]). But the deviations very often remained below 0.01 of a degree for a considerable time [3]) (a fuller account is given below in section 3). We may therefore say that the gap in the series of constant and uniform temperatures which still existed between the two regions which are easily governed by liquid oxygen and liquid hydrogen respectively [4]), has now also been filled

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[1]) The principle of this arrangement was already used by A. Perrier and H. Kamerlingh Onnes in their research on the magnetic properties of solid oxygen above 20° K (Comm. No. 139 *c*. Proc. XVI, 2, p. 894).

[2]) The possibility of using neon under pressures above the normal in special experiments — as will probably be practically realisable between 27° K and 34° K — is here left out of account.

[3]) Compare the measurements of the vapour-pressure along the heterogeneous isothermals for different values of T in the investigation of the critical data of hydrogen (Comm. N°. 151 c).

[4]) Besides for the range from 27° K—55° K the hydrogen-vapour cryostat is also suitable for temperatures lower than 27° K; in many experiments it will thus for instance be able to replace the neon-cryostat for the range from 25° K—27° K; this may be of some importance considering that the dimensions of the experimental space may have to be kept smaller in the neon-cryostat than in the

up in a satisfactory manner [1]. In its present construction the hydrogen-vapour cryostat is not yet suitable for experiments in which the phenomena in the experimental space have to be followed by the eye, as this space is completely surrounded by copper walls. But we hope to remove this objection by a modification of the apparatus.

Since the hydrogen-vapour cryostat has proved to fulfil its object, a helium-vapour cryostat will be built on the same principles, in order also to bridge the other gap which still remains in the series of low temperatures for which appliances are available which guarantee the constancy and uniformity of the temperature necessary for experimental work, viz. the very important [2] interval from 14° K. to 4°,25 K. (freezing point of hydrogen to boiling point of helium).

§ 2. *Description of the apparatus.* The cryostat (see fig. 1) [3] consists of the evaporator V and the cryostat-glass B, which latter contains the experimental chamber E. The air-tight german-silver caps V_N and B_N, by which the two parts are closed immovably, are connected together by means of strong tinned iron strips g_1, g_2, g_3 (see fig. 2) and clamping rings g_0 and g_4.[4]

A continuous current of superheated hydrogen-vapour is needed to keep the walls of the experimental chamber as well as the gas and measuring apparatus inside at a constant and uniform temperature. This current is supplied by the evaporator.

The unsilvered lower part of the vacuumglass of this evaporator V_0 contains liquid hydrogen. The hydrogen is transferred to the

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hydrogen-vapour cryostat in view of the difficulty of providing large quantities of the gas.

(In Proc. XVIII, 1, p. 508 l. 1 from below, insert after "most experiments": "at temperatures between 25° K and 27° K"). Perhaps we shall find that it will be possible with the hydrogen-vapour cryostat to descend almost to the boiling point of hydrogen and thus to embrace the region where otherwise — as mentioned l. c. — it might be possible to use a bath of liquid hydrogen boiling under enhanced pressure.

[1] The proper working of the vapour cryostat is much impeded when experiments are carried out with it, in which heat-actions take place inside the experimental chamber.

[2] We need only point out the desirability of this interval being filled up for accurate determinations of the critical data of helium and for deciding whether lead becomes supra-conducting continuously or, in the same way as tin and mercury, suddenly.

[3] The section of this figure is taken along the line shown in fig. 2.

[4] This firm connection is necessary because of two glass syphon-tubes connecting the two parts. With a view to the expansion by change of temperature of the tubes and the strips, the latter were made of iron.

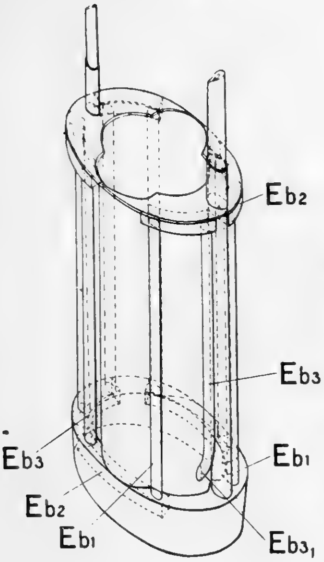

Fig. 5.

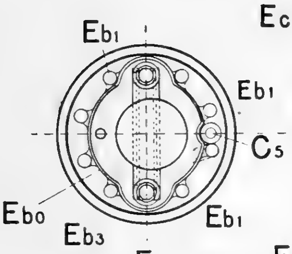

Fig. 4.

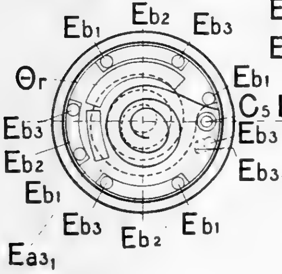

Fig. 3.

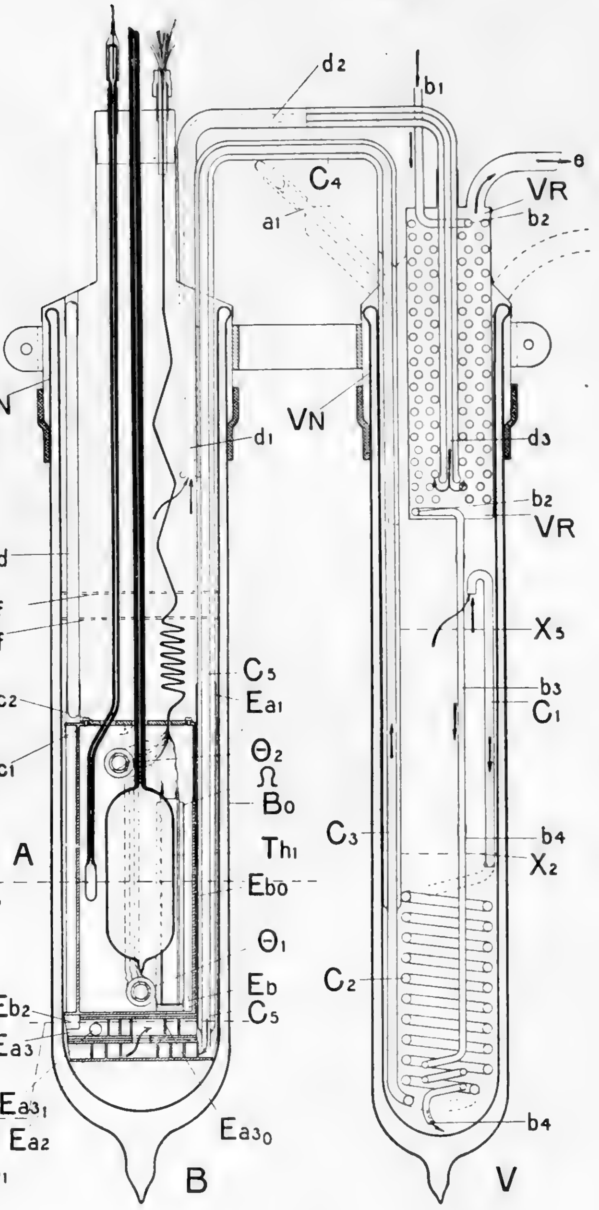

Fig. 1

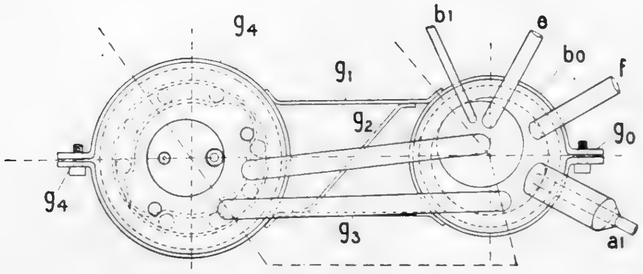

Fig. 2.

evaporator in the manner commonly used in the laboratory (Comm. N°. 94*f*, Proc. IX p. 156 comp. also Pl. 1 Comm. N°. 103 Proc. X p. 592) from a supply-bulb through the tube a_1, which is closed by a small rubber tube with glass stopper. In the beginning the evaporator is filled up to X_5; when the liquid surface has sunk to X_2 a fresh supply is put in. Through the copper tube b_1 gaseous hydrogen is led in from a high-pressure supply cylinder; this gas undergoes a preliminary cooling in b_2 and is then carried into the liquid hydrogen by the tube b_3 (which is made of german silver in order to reduce heat-conduction to the liquid hydrogen) and the copper tube b_4; this causes a continuous evolution of hydrogen vapour, which is carried to the cryostat-glass B_0 — a silvered vacuum glass — by the glass tube C_1. On its way it passes the glass spiral C_2 and the syphon-like twice bent silvered vacuum-tube C_3, C_4 C_5 which is sealed to C_2. Its end-piece C_5 is sealed into the supply-tube Ea_1 of the heating-chamber Ea_2 which is the lower one of two adjoining flat horizontal copper boxes, the upper one Ea_3 serving as regulating and adjusting chamber, the two together being attached to the hollow bottom of the experimental chamber. The two boxes are isolated from each other and similarly the upper one from the bottom of the experimental chamber by means of paper; inside each of the boxes is provided with a vertical partition running round as a spiral Ea_{20}, Ea_{30} by which they are made into spirally wound tubes of rectangular section. Inside the spiral of the heating box is a heating wire of constantan of 100 Ω, insulated with silk and wound round a flat spirally wound band (the wire is shown diagrammatically in fig. 6 as Ea_{21}). After passing through the heating tube the superheated hydrogen-vapour, which is now brought to the desired temperature, flows into the regulating and adjusting chamber Ea_3, where it follows again the spiralshaped path shown it by the partition Ea_{30}. In doing so it passes along a tin wire insulated with silk and arranged as Ea_{21} the resistance of which is measured to 0.001 on a commercial WHEATSTONE-bridge. According to the indication of the resistance of this wire the temperature is approximately adjusted. The same adjusting chamber also contains the bulb of the regulating thermometer Θ_r (see fig. 3), which will be discussed further down.

After having passed the adjusting and regulating chamber at the bottom of the experimental chamber, the gas passes (zie figs. 3, 4, 5) a copper exchange tube *Eb*, which consists of eight tubes alternately running up (Eb_1) and down (Eb_3) coupled by horizontal chambers (Eb_2), the whole being intimately united [1]) with the vessel which is

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[1]) The tubes are soldered to the side-wall of the experimental chamber, the

formed by the side-wall Eb_0 and the bottom of the experimental chamber and is of high conductivity and comparatively large heat-capacity (the vessel with its lid weighs 1.2 k.g.) Finally the gas emerges in the experimental space immediately above the bottom at Eb_{31} and finds its way to the protecting space in the cryostat-glass above the experimental chamber through small apertures [1]) in the copper lid Ec (fig. 1), which closes the experimental chamber at the top.

The copper vessel Eb with its lid Ec which encloses the experimental space, together with the box attached to the bottom, occupies the lower part of the cryostat-vessel (see fig. 1), and hangs, without touching the inner wall of this vessel, by means of the vacuum-tube C_6 and the glass rod Ed from the air-tight cap B_N, which closes the vacuum-vessel B_0 in the manner commonly used in the laboratory (see previous Communications of this series). Supply of heat by conduction to the walls of the experimental space is therefore practically excluded [2]).

Besides the measuring-apparatus, the necessary electric wires and the supply-tube of the superheated hydrogen-vapour C_4, the cap of the cryostat-vessel B_N (see fig. 1) transmits air-tight a second doubly bent syphon-like silvered vacuum-tube $d_1d_2d_3$, through which the hydrogen flows back to the evaporator. Here it passes through the regenerator V_R, which serves to effect a preliminary cooling of the hydrogen of ordinary temperature by which the evaporator is fed [3]); ultimately (see fig. 6) by way of e and a tap K_1 it finds its way to

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wall being shaped for the tubes to fit it as nearly as possible. Moreover the conduction between tubes and wall is promoted by a thick filling of solder.

[1]) The lid closes the experimental chamber as nearly as possible, but is not made air-tight. The side-wall Eb_0 (see fig. 1) is provided at the top with a horizontal ring-shaped rim Ec, which is soldered to it. On to this rim a number of small copper covering-plates are screwed, 2 mm. thick and fitting the rim, of such profiles, that when the measuring apparatus are in their proper places the plates cover up the experimental chamber as completely as possible and complete the lid until only a few interstices and small holes remain, through which gas may escape while at the same time the measuring apparatus in the experimental space are protected from radiation.

[2]) In order to prevent a radiation from the cap to the lid of the experimental chamber, screens Ef (shown as dotted lines in fig. 1) can be fitted in the protecting space, which are cooled down by the gas which emerges from theexperimental space.

[3]) The dimensions of the apparatus do not admit of more than a moderate degree of regeneration, as the tube b_2 cannot be very narrow in connection with a proper regulation of the supply.

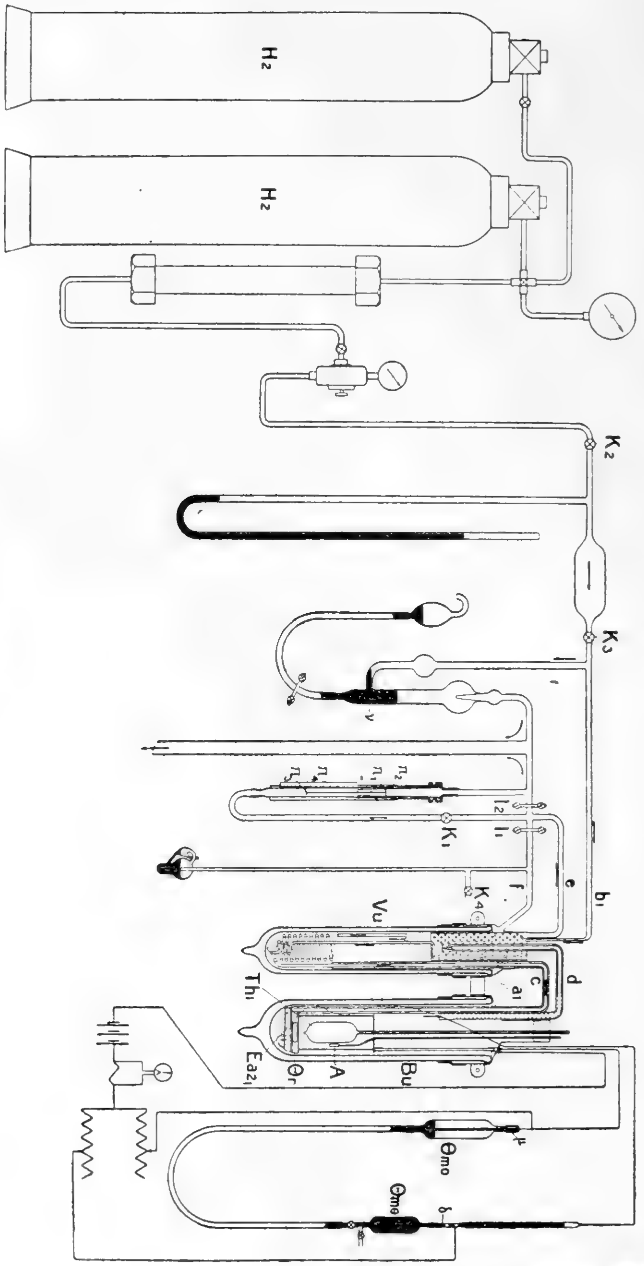
H₂
H₂
K₂
K₃
v
n₁
n
n₁
n₂
l₂
l₁
K₁
K₄
f
e
b₁
Vu
d
c
a₁
Th₁
A
Bu
Θr
Ea₂₁
Θmo
Θmo
μ
δ

Fig. 6.

Fig. 7.

the gasometer from which it is pumped back into high-pressure supply-cylinders [1]).

The uniformity of the temperature in the experimental space is checked by means of two resistance-thermometers θ_1 and θ_2 (fig. 1), consisting of platinum wires which are loosely wound on small porcelain cylinders with screwshaped grooves and are provided with two pairs of conducting wires [2]). The axes of the cylinders are placed horizontally (see fig. 4) [3]).

The regulation of the temperature to a constant value is conducted by means of a hydrogen thermometer, the german silver bulb of which Θ_r (volume 5,2 cc, see fig. 3) is placed in the regulating and adjusting chamber. At the ordinary temperature the larger part of the quantity of the gas required is contained in the wide tube of the manometer-part $\Theta_{m\theta}$ (fig. 6) of the thermometer: when the thermometer-bulb has been cooled to the low temperature, the gas is transferred to it by forcing up the mercury in the manometer. For this purpose the open tube Θ_{mo} of the manometer is connected with the closed part attached to the thermometer by means of an india-rubber tube of sufficient length. The mercury is driven up, until it lifts a small glass float σ (see figs 6 and 7), which is provided with a small platinum plate and a platinum contact-wire passing through the float and brings it close to a platinum point which is sealed into the capillary. [4]). The fine adjustment can be accomplished by means of a micrometer screw μ. If the temperature in the adjusting chamber of the cryostat, which we may take to be the same as that of the experimental space, falls, the float makes contact with the platinum point [5]), and in this manner switches in a shunt-con-

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[1]) As the figure shows, two other tubes are connected with e, viz. a leading-off-tube from the evaporator with safety tube and a tube connecting the evaporator with the gasometer independently of the "spedometer" (see section 3): both these tubes are closed by pinching screws l_1, l_2, when the cryostat is in use, they both serve in filling the evaporator.

[2]) So far we have not had an opportunity to exchange the thermometers and thus obtain a definite opinion as to the uniformity of the temperature, as in one of them a small change of the zero occurred, our statement that a constancy down to 0.00° has been reached is also to be taken as a provisional one founded on an estimation.

[3]) The further apparatus which the experimental chamber will be seen to contain in the drawings are a vapour-pressure apparatus, a helium thermometer and a resistance thermometer to be examined: these are connected with measurements which will form the subject of later communications.

[4]) The adjustment at this point corresponds to an initial pressure of about 8 atmospheres, the thermometer being taken as one of constant volume.

[5]) This contact works best, when the small plate is amalgamated and covered

nection parallel to the main circuit (see fig. 6) by which means a rise of temperature is started and an automatic regulation of the temperature is brought about.

§ 3. *Remarks concerning auxiliary apparatus, details of working and the action of the cryostat.*

Both the evaporator and the cryostat-vessel are immersed in vacuum-glasses with liquid air; the one surrounding the cryostat B_u (fig. 6) is completely silvered in order to reduce the radiation to the experimental chamber as much as possible; in silvering the vacuum-glass in which the evaporator is placed, V_u, a strip along a generating line of the cylinder is left transparent, through which the evaporation of the hydrogen may be followed.

In starting the cryostat it is first — with a view to saving liquid hydrogen — cooled down by blowing hydrogen of ordinary temperature from a supply-cylinder [1]) through a cooling coil immersed in liquid air into the evaporator.

When the tin wire thermometer in the regulating and adjusting chamber indicates, that the temperature has gone down to about —100° C., liquid hydrogen is brought into the evaporator and the supply of hydrogen of ordinary temperature is then started.

The velocity of the hydrogen flowing through the experimental space is regulated according to the indication of a "spedometer" which is joined in on the way to the gasometers; it consists of a small horizontal plate π_1 floating on the vertical gas-stream in a very slightly conical tube π_2, (length 15 cms, diameter at the top 1.62 cms., at the bottom 1.50 cms) the height to which the plate is raised being read by means of the small horizontal ring π_3 which serves as an index on a scale which is placed along the lower part of the measuring tube π_4.

The current of hydrogen of ordinary temperature which is supplied from high-pressure supply-cylinders $H_2 H_2$ through a reducing valve is further reduced in the manner shown in fig. 6 by the stopcocks K_2 and K_3 in such a manner, that a regular stream of gas-bubbles (escaping to the gasometer) bubbles through a mercury column ν of an adjustable height. As an instance (applying to the measurements of which the subsequent Communications N°. 151*b* and N°. 151*c* treat) about

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with a thin layer of mercury. When the contact failed to act, the shunt connection could also be closed by hand in accordance with the indication of the position of the float.

[1]) It will thus be seen, that use is made of pure hydrogen throughout (distilled or purified, see Comm. N⁰. 94*f* l. c. and 109*b*. Proc. XI, 2, p. 883.

60 cc. of gas measured under normal conditions is made to flow through the experimental chamber only $^{1}/_{7}{}^{\text{th}}$ of which is accounted for by the supply of hydrogen at ordinary temperature through b_1, the remainder being supplied by the evaporation of the liquid hydrogen.

When the adjusting-thermometer (resistance of the tin wire) indicates that the temperature has been reduced to a value slightly below the desired one, the heating current is put in action [1]). According to the reading of the two checking thermometers in the experimental space, the adjustment of the automatic regulating-thermometer is then modified, until the desired temperature in the experimental space has been attained. A rise of .1 mm of the float corresponds to about .003 degree. The micrometer-screw μ thus affords a high sensitiveness of adjustment of the temperature.

The high degree of uniformity and constancy of the temperature of the measuring apparatus in the experimental space which is obtained with the apparatus and the method of working above described may be considered to be due to the following circumstances: *a*) the access of heat by radiation and conduction to the copper enclosure of the experimental space has been reduced to an extremely small amount [2]); *b*) the interchange of heat between the gas supplied from the heating space and the walls of the experimental chamber is much promoted by the long winding path followed by the gas in the side-walls, the exchange taking place over a large surface of highly conducting material which is moreover distributed as uniformly as possible; *c*) the difference of temperature between the gas in the experimental space and the walls has been reduced to a very small value; *d*) the speed of the gas supplied from the heating space is sufficient to prevent quantities of heat which are supplied having an influence on the temperature of the experimental space; *e*) the constancy of the velocity of the

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[1]) In the adjustment to 29°.5 K the heating current was 0.06 amp., when the float was not making contact, and 0.14 amp. when it did. At. 55° K. these currents were 0.114 amp. and 0.264 respectively.

[2]) Comp. section 1 note 4 page 1050. In using the cryostat for experiments, care has to be taken that galvanic generation of heat and supply of heat by conduction along experimental wires are reduced to a minimum; in the experiments to which the figures refer the conducting wires were taken very long and were wound in the cryostat in a manner which excluded heat-conduction to the experimental chamber. In experiments on condensation and expansion it is necessary to wait a long time before it may be assumed that temperature equilibrium has been reestablished.

gas-current in question is such as not to give rise to capricious modifications of the temperature of the experimental space; *f*) the heat-capacity of the walls of the experimental space is sufficient to efface the rapidly alternating deviations from the mean value of the temperature of the gas-current in question, which are due to the changes in the heat-development in the heating-wire, the consequence being that the walls only follow the changes of the mean value; *g*) the gas in the experimental space owing to its low temperature has a very much higher heat-capacity than under normal circumstances and finally *h*) the gas-current emerging from the heat-exchange tube in the experimental chamber keeps the gas in continual motion [1]) along the walls and the apparatus.

In the experiments which have been made with the cryostat so far, it was noticed that capricious disturbances from time to time interrupted the periods of constant temperature [2]). But when the measurements were continued for a long time, generally periods of more than half an hour or longer were repeatedly found in which the temperature of the experimental apparatus and thermometers remained constant to .01 of a degree, whereas these periods are preceded by even longer ones during which the temperature did not vary by more than .02 of a degree, so that the measuring apparatus during this time were able to assume the desired temperature with very near approximation.

Physics. — "*Isothermals of mon-atomic substances and their binary mixtures.* XVIII. *A preliminary determination of the critical point of neon.*" By H. KAMERLINGH ONNES, C. A. CROMMELIN and P. G. CATH. (Communication N°. 151 *b* from the Physical Laboratory at Leiden).

(Communicated in the meeting of June 24, 1916).

1. *Introduction.* The chief reason why the critical data of neon are not known yet with any degree of accuracy — notwithstanding their great importance for the comparison of its thermal properties with those of other, especially monatomic substances — is doubtlessly the fact, that so far it had been impossible to obtain temperatures

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[1]) In cryostats with baths of liquefied gas strong stirring is necessary on other grounds.

[2]) Each time after a fresh adjustment of temperature it is necessary to wait some time for the experimental space and the measuring apparatus to arrive at the new temperature.

in the neighbourhood of 45° K. sufficiently constant to make reliable measurements of the critical temperature. Since in the hydrogen-vapour cryostat [1]) we have obtained an apparatus by which it is possible to govern the temperatures in the range between the melting point of oxygen and the boiling point of hydrogen, this difficulty has disappeared and we could now attempt the long-desired determination of the critical condition of neon with every chance of success. The reason why our results must still be looked upon as preliminary ones is not due to a want of constancy in the temperature of observation or to other defects in the method adopted, but to the fact that the neon on which we have experimented was not absolutely pure. Small as the admixtures were, their influence showed itself very clearly in a gradual increase of pressure during condensation. [2]) The difference between initial and final pressures in the vapour-pressure measurements immediately below the critical point amounted to .2 of an atmosphere. [3]) In the determination of the vapour-pressure of hydrogen in the immediate vicinity of the critical point which was carried out with the same apparatus (comp. the next Communication N°. 151*c*) where, on account of the purification of hydrogen by distillation, the purity of the experimental gas was completely guaranteed, differences of that kind did not occur.

If the pressure rises during condensation, the determination of the critical data becomes uncertain. [4]) Our result for the critical temperature may therefore differ from the true value by a few tenths of a degree; a similar uncertainty applies to the critical pressure. The circumstance, that observations on the critical temperature of neon are so far completely lacking and that it will take some time before the more accurate measurements [5]) aimed at will be completed, justify sufficiently the publication of our present results.

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[1]) Comp. the preceding Comm. N°. 151*a*.

[2]) Previous investigations, in the first place by KUENEN (Comm. N°. 8 Meeting of Oct. 1893 and Comm. N°. 11 Meeting of May and June 1894), have sufficiently shown the great influence which even very small admixtures produce on the phenomena in the critical region.

[3]) In the table the pressure at the beginning of condensation is given as the vapour pressure.

[4]) Instead of the critical temperature of the pure substance the experiment gives the plaitpoint temperature of the mixture.

[5]) In these determinations we hope to be able to utilize a visual method by a modification of the hydrogen vapour cryostat (comp. Comm. N°. 151*a*) which will allow us to follow the phenomena inside the experimental chamber by eye.

2. *Apparatus and method.* The measurements were carried out with a vapour-pressure apparatus which will be described in a future paper on the vapour-pressures of neon and hydrogen. The small bulb in which the gas is liquefied is shown at A in fig. 1 of the preceding communication. It is placed in the experimental chamber E of the hydrogen-vapour cryostat half way between bottom and top side by side with a helium-thermometer Th_1 and a resistance-thermometer Ω. The vapour-pressure apparatus is so arranged, that the quantities of gas which were liquefied at a given temperature between the beginning and the end of condensation could be measured.

Using the values obtained in that manner at different temperatures in the neighbourhood of the critical temperature it was possible in connection with temperature and pressure by means of an extrapolation over a small range to derive the critical temperature and pressure within the limits of accuracy given above.

The value to be ascribed to the critical pressure can be checked by means of the pressure at the point of inflexion of an isothermal immediately above the critical temperature, which was determined specially for this purpose.

The manner in which the extrapolation was carried out will be elucidated by means of a diagram in the next communication dealing with the critical point of hydrogen.

We mention in this connection that owing to the impurity of the neon referred to, small though it was, the heterogeneous isothermals in a pressure-density diagram did not run exactly parallel to the density-axis, whereas they did with hydrogen.

Owing to these pressure-differences along the heterogeneous isothermal it was more difficult than in the case of hydrogen to arrive at an exact calculation of the critical constants.

As regards the preparation of neon it may be mentioned that the impure gas forming our stock was first freed from hydrogen after the addition of oxygen by explosion, it was then frozen a number of times at the air pump and ultimately repeatedly distilled over carbon cooled in liquid air. Although often repeated and carefully carried out these operations have evidently not been sufficient to free the neon completely from admixtures.

The pressure measurements were made by the aid of the closed hydrogen-manometer M_{60} which has been often mentioned in previous communications of this series (see for instance Comm. N°. 146*c*). The temperatures were measured with the constant-volume helium gas-thermometer Th_1 referred to above; the bulb had a volume of 110 cc., the "waste space" was .7 °/₀ of the volume of the bulb;

the zero-point pressure was 1000 mm. and the temperatures were calculated using .0036614 as the pressure-coefficient. For the calculation of the temperatures we may refer to a previous communication [1]).

3. *Results.*

The results of our observations are contained in the following table:

T	θ	$p_{\text{coëx}}$ (intern. atm.)	quantity of gas (cc)
43°.83 K	—229°.26 C	24.305	670
44°.43	—228°.66	26.049	416

Above t_k the following point was established:

T	θ	p (intern. atm.)
44°.94 K	—228°.15 C	27.462

From these data we have derived:

Critical constants		
T_k	θ_k	p_k
44°.74 K	−228°.35 C	26.86

We are glad to record our thanks to Mr. J. M. BURGERS phil. cand., assistant in the Physical Laboratory, for his assistance in checking the automatic temperature-regulation during the experiments by means of the resistance-thermometer Ω and the thermometers θ_1 and θ_2 (cf. fig. 1 of the previous communication).

4. *Discussion.*

In a previous communication [2]) two of us had drawn some preliminary conclusions as to the critical temperature of neon from

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[1]) H. KAMERLINGH ONNES and G. HOLST, Proc. XVII, 1, p. 501. Comm. N°. 141*a*.

[2]) H. KAMERLINGH ONNES and C. A. CROMMELIN, Proc. XVIII, p. 515. Comm. N°. 147*d*.

a comparison of the net of isothermals of neon with that of argon.

The values then found by a comparison with argon viz. — 228°.2 C. and — 227°.9 C. were, however, obtained using the result of a somewhat rough determination of the critical pressure of neon, viz. 29 atm.[1]). Repeating these calculations utilising the value now found for the critical pressure, the results come out a little lower, namely —228°.9 C and —228.°6 C, which values appear to agree very satisfactorily with the experimental value. Our supposition expressed at the time, which was rendered probable by the course of the vapour-pressures in connection with that of the isothermals, that argon and neon, looked upon from the point of view of the law of corresponding states, differed but little from each other, is thereby confirmed in a very satisfactory manner.

The estimate of the critical temperature of neon obtained at the time by a comparison with hydrogen (—231°.2 C) deviated much more from the observed value. In this comparison use was made, however, of the critical temperature of hydrogen, as determined by BULLE[2]), —241°.14 C and, moreover, of our rough determination of the critical pressure of neon above referred to: these values have now to be replaced by those found by ourselves (for hydrogen as will be shown in Comm. N°. 151*c* we have found $T_k = 33°.18$ K, $\theta_k = -239°.91$ C, $p_k = 12.80$ atm.).

The calculation when corrected in this way gives — 230°.2 C for the critical temperature of neon, a value which deviates much less from the result of direct experiment than before.

Physics. — "*The viscosity of liquefied gases.* VI. *Observations on the torsional oscillatory movement of a sphere in a viscous liquid with finite angles of deviation and application of the results obtained to the determination of viscosities.*" By J. E. VERSCHAFFELT. (Communication N°. 151*d* from the Physical Laboratory at Leiden). (Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated in the meeting of February 24, 1917).

1. In a previous communication[3]) the theory of the oscillatory rotation of a sphere in a viscous liquid was developed on the supposition of the rotation taking place with such small angular velocities,

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[1]) H. KAMERLINGH ONNES, Proc. XII, 1, p. 175. Comm. N°. 112.

[2]) F. BULLE, Phys. Zeitschr. 14, p. 860. 1913.

[3]) Comm. N°. 148*b*. Proc. XVIII 2. p. 840.

that the influence of the centrifugal force may be disregarded. This simplification can also be expressed by saying that in the integration of the general differential equations all terms are left out which are of a higher order of magnitude than the first with respect to the velocity which is treated as infinitely small[1]. From a theoretical point of view there is no objection to this supposition, but where the object is to use the theoretical results in the experimental determination of viscosities of liquids or gases, it is necessary to know for a given liquid the limit of velocity (or rather of amplitude of oscillation, which limit may also change with the time of swing) beyond which the theory is no longer applicable in practice: in other words what the error is with a given amplitude caused by the simplifying supposition.

The results of the approximate theory were used in the determination of the viscosities of mixtures of oxygen and nitrogen[2]. The velocities occurring in these experiments (not higher than .04 cm. p. sec.) would seem to be sufficiently small and a definite indication, that the deviation from the simplified theory could not be considerable in these experiments, was given by the fact, that over a fairly large range of angles of deviation (between 4° and 1.5°) the logarithmic decrement δ of the amplitudes appeared to be very nearly independent of the amplitude itself, whereas the opposite might be expected, if a deviation from the theory existed. Moreover the method, when applied to water, appeared to yield satisfactory results[3].

A perfectly trustworthy proof, that the velocities could actually be looked upon as small, was, however, not available. From a consideration of the order of magnitude of the terms neglected in the differential equation one might even be inclined to conclude that such was not the case. Indeed a development of the equation shows that neglecting the terms of the second order $\left(u\frac{\partial u}{\partial x} \text{ etc.}\right)$ as compared to the terms of the first order such as $\frac{\partial u}{\partial t}$, comes to the same as neglecting ω^2 with respect to $\frac{\partial \omega}{\partial t}$ (ω being the angular velocity of the oscillating body) and this leads to the simple conclusion, that the approximate theory is only applicable, if the angle of

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[1] Comp. e.g. G. Kirchhoff, Vorlesungen über mathematische Physik, n⁰. 26.

[2] Comm. N⁰. 149*b*. Proc. XVIII, 2, p. 1659.

[3] Comp. for instance Comm. N⁰. 148*b*, § 16.

deflection α remains a small fraction of a radian. Seeing that in the experiments the angles reach a value of .07 the condition was obviously not satisfied, at least not to a degree sufficient to guarantee a sufficient accuracy of the results; in that case it would have been necessary to go down to amplitudes as much as a hundred times smaller. However this is only a rough estimate which does not exclude the possibility of the accuracy having after all been higher than was to be expected on the above ground, seeing that no account is taken of the numerical factors the value of which can only be given by a further approximation in the theoretical treatment of the problem [1]). For this reason it seemed desirable to make a further investigation into the dependence of δ on the amplitude: this investi-

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[1]) Comp. G. Zemplén, Ann. d. Physik, 38, 84, 1912.

The condition given in the text only holds moreover in the limiting case, where the time of oscillation is very small. In that case we have (see Comm. N°. 148*b*, § 17) $\omega = \omega_R \frac{R^2}{r^2} e^{-b(r-R)}$, where $b = \sqrt{\frac{\mu}{\eta} k}$ is a large number, and thus $\frac{\partial \omega}{\partial r} = -b\omega$ and $\frac{\partial^2 \omega}{\partial r^2} = -b \frac{\partial \omega}{\partial r} = b^2 \omega$, from which it follows, that $\frac{\partial \omega}{\partial r}$ is small as compared to $\frac{\partial^2 \omega}{\partial r^2}$ and $\frac{\eta}{\mu} \frac{\partial^2 \omega}{\partial r^2} = k\omega = \frac{\partial \omega}{\partial t}$, so that the term $\frac{\partial \omega}{\partial t}$ actually gives the order of magnitude of the terms of the first order. If on the other hand the oscillatory movemement is very slow (*b* very small; see Comm. 148*b*, § 18), we have $\omega = \omega_R \frac{R^3}{r^3} \frac{R'^3 - r^3}{R'^3 - R^3}$, hence $\left(\frac{\partial^2 \omega}{\partial r^2}\right)_R = -\left(\frac{4}{r} \frac{\partial \omega}{\partial r}\right)_R = 12 \frac{\omega_R}{R^2} \frac{R'^3}{R'^3 - R^3}$, whereas the term with $\frac{\partial \omega}{\partial t}$ practically disappears; in that case ω^2 must be small with respect to $\frac{\eta}{\mu} \frac{\partial^2 \omega}{\partial r^2}$, and we thus obtain, except for a numerical factor, the same condition as given by Lamb and Rayleigh (Lamb. Hydrodynamics, 1906, p. 547) for a uniform rotation viz. $\omega_R R \ll \frac{uR}{\eta}$ (the symbol $\ll$ standing for: much smaller than). Whereas for very rapid oscillations the condition was $\alpha \ll 1$, very slow oscillations require $\alpha \ll \frac{12}{2\pi} \frac{\eta T}{\mu R^2} \frac{R'^3}{R'^3 - R^3}$, and this limit can rise indefinitely with increasing *T*. In the general case (*b* neither specially large, nor specially small) the three terms $\frac{\partial^2 \omega}{\partial r^2}$, $\frac{\partial \omega}{\partial r}$ and $\frac{\mu}{\eta} \frac{\partial \omega}{\partial t}$ are of the same order and undoubtedly a much more complicated condition holds, probably expressing, that α must be very small as compared to a number which depends on the value of bR and, as far as I have been able to ascertain, is larger the smaller the value of bR, i.e. the larger η, δ and T and the smaller μ and R (comp. Comm. 148*b*, equation 20).

gation was carried out both experimentally and theoretically; in this paper are given the results of the experimental investigation.

2. In order to settle experimentally, whether in the experiments previously described the velocities could be regarded as infinitely small, i.e. whether within the limits of the experimental errors the oscillation of the sphere, with the amplitudes then used, was a damped harmonic vibration with is logarithmic decrement independent of the amplitude, the limits of the amplitudes were in the first place widened as far as was possible with the arrangement for mirror-reading used. The scale had a length of 60 cms. and was at a distance of 153.7 cms. from the axis of the oscillating system, so that to right and left elongations could be observed to a maximum-amplitude of 5.5° (0.1 radian). The absolute accuracy of the readings was .00003 of a radian (0.1 mm. on the scale); amplitudes of .2° could therefore still be read with a relative accuracy of 1 $^0/_0$; this accuracy was, moreover considerably raised by all elongations being observed both to right and left.

The apparatus was the same as described in Comm. N°. 149*b* and made use of in the determination of viscosities of mixtures of oxygen and nitrogen. The oscillating system was usually loaded with the aluminium cylinder, occasionally with the copper one. Various liquids were used in the vessel, in the first place water, afterwards liquids with smaller viscosity: benzene, carbon disulphide, ether[1]) and liquid air, and finally a mixture of water and glycerine, with a view to obtaining a set of observations with a liquid of higher viscosity.

In contrast with what was found between the limits previously chosen (comp. Comm. N°. 149*b* IV § 3 and V § 1), with the wider limits admitted this time the line representing $log\ a$ as a function of t showed a distinct curvature especially between 20 and 30 cms. This fact seems to show that the angles of deviation used in the previous investigation were accidently near the limit of the range, where they may be looked upon as practically infinitely small.

3. The oscillation of the sphere was thus not a pure harmonic damped oscillation and it was surmised[2]) that the movement would be a compound damped harmonic one, to be represented by the

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[1]) The benzene, carbon disulphide, and ether were commercial liquids; for the object of the above experiments it was unnecessary to work with perfectly pure liquids.

[2]) This was afterwards confirmed by the mathematical treatment of the problem (see next Comm.).

real part of a series of the form:

$$a = a_1 e^{kt} + a_3 e^{3kt} + a_5 e^{5kt} + \dots \text{ 1)}, \quad \dots \quad (1)$$

where $k = k' + k''i$ with $k'' = \frac{2\pi}{T}$ and $k' = -\frac{\delta}{T}$ (see Comm. N°. 148*b*, § 4), so that the successive terms represent pure harmonic damped vibrations of 3, 5 etc. times the frequency of the main vibration and 3, 5 etc. times more rapid damping. 2)

This turned out to be actually the case, at least as far as this could be inferred from the observations of the extreme amplitudes 3).

It was found, that within the limits of accuracy of the observations the extreme amplitudes could be represented as follows: 4)

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1) By a suitable choice of the zero of the time the coefficient a_1 may be made real, but owing to the possible phase-differences the remaining coefficients are in that case not necessarily real.

2) For reasons of symmetry the terms with even powers of k must be absent form this series: in fact, a difference between deflections to right and to left cannot be made, so that a change of phase of π (increase of kt by $(2n+1)\pi$) must bring about a change of sign in all the terms.

3) It may be proved that when the oscillatory motion satisfies equation (1), the extreme amplitudes may also be represented by a similar formula, this time with a real value of k (the real part of the complex k) and with real α's (which, however, are not the real parts of the complex α's). Conversely, if the extreme amplitudes can be represented by a formula of the form (1), this will very probably also be the case for the complete motion.

4) This result may be looked upon as a proof, that within the limits of amplitude of the present experiments the limit was actually reached, below which the velocities may be regarded as practically infinitely small. It might perhaps be objected that it is only natural that a *limited* portion of the line α, t may be represented by a series of that kind, and that the accuracy which is reached only depends on the number of terms introduced; in fact the same would be the case with an algebraical series. To this may be answered, that we did not assume equation (1) with a definite number of terms a priori and then determined the value of the coefficients in the usual way: on the contrary the method of calculation actually was such as to show itself the *necessity* for the use of the formula. The method was as follows: a graphical treatment showed that the curve $\log \alpha, t$ was pretty nearly a straight line which only showed a distinct curvature towards the large amplitudes; from the part corresponding to the low α's a first term α_1 could thus be determined with considerable accuracy. The differences $\alpha - \alpha_1$ having been drawn up and the line $\log(\alpha - \alpha_1), t$ being plotted, it appeared that the line obtained, at the lowest limit where it was trustworthy, very distinctly showed a direction-coefficient three times as high as the line $\log \alpha, t$. A first approximate value was thus obtained for α_3, and this was subtracted from α; in consequence of this the line $\log(\alpha - \alpha_1), t$ became straight over a greater distance than the line $\log \alpha, t$ and thus a more accurate value for α_1 could be found and then also for α_3. If then $\log(\alpha - \alpha_1 - \alpha_3), t$ was plotted, the resulting line was found to have a five time higher direction coefficient than $\log \alpha, t$, etc.

It should be mentioned that in the calculation the time-intervals between the *a.* *Water* at 9°.6 C. ($\mu = 1,000$, $\eta = 0,01319$ ¹)

$$\alpha = 0,01\, x + 6,73 \,.\, 10^{-6}\, x^3 - 3,06 \,.\, 10^{-8}\, x^5, \text{ ²)}$$

where $x = e^{-\frac{\delta}{T}t}$, with $T = 20,95$ and $\delta = 0,1272$.

b. *Benzene,* at 9,°8 ($\mu = 0,890$, $\eta = 0,00773$)

$$\alpha = 0,01\, x + 7,83 \,.\, 10^{-6}\, x^3 - 3,00 \,.\, 10^{-8}\, x^5 \,,$$

with $T = 20,86$ and $\sigma = 0,0898$.

c. *Carbon disulphide,* at 10°,8 ($\mu = 1,277$, $\eta = 0,003839$)

$$\alpha = 0,01\, x + 18,57 \,.\, 10^{-6}\, x^3 - 13,65 \,.\, 10^{-8}\, x^5 + 4,0 \,.\, 10^{-10}\, x^7,$$

with $T = 20,83$ and $\delta = 0,0705$.

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moments at which two successive extreme elongations were reached $\left(\frac{d\alpha}{dt} = 0\right)$, were all taken equal to $\frac{T}{2}$; this is not absolutely correct: a mathematical investigation shows, that the moment at which $\frac{d\alpha}{dt} = 0$ does not lie exactly halfway between the moments at which $\alpha = 0$ and that the small shift of the extreme points depends upon the amplitude. However in the present experiments — the damping being comparatively small — the shift of the extreme points was within the limits of the errors of observation. We may also put it in a different way by saying, that the elongations were read at the moments $(2n+1)\,\frac{T}{2}$, which can also be represented by a series of the form (1), and these elongations did not differ perceptibly from the extreme values.

[1] The viscosities were calculated from the data T, δ, $T_0 = 20,65$ at the ordinary temperature and 20.61 in liquid air, $K = 573.5$ at the ordinary temperature and 571.0 in liquid air (see Comm. N°. 149*b*), and taking into account that the atmospheric air itself by its action on the part of the system which is not immersed in the liquid (mainly the cylinder) and the internal friction of the wire together accounted for a decrement $\delta_2 + \delta_3 = 0,00606$ (see further down).

[2] It is clear that the coefficients in these equations are not exactly the same as would hold for a single spherical body; this can only be true in a very rough approximation, seeing that they refer to the complete oscillating system, of which the sphere only is immersed in the liquid. Moreover the coefficients (indirectly) undergo a modification through the influence of the internal friction of the wire, owing to k also depending upon it (directly according to Guye c.s. the internal friction of the wire does not give any higher terms than the first; comp. for instance Arch. de Genève, (4), 26, 136 and 263, 1908). However, it is impossible from the formulae to derive those which would hold for a single sphere, because there is no additivity for the three sources of friction to a higher approximation than the first (see Comm. N°. 149*b*, IV, 5); it is obvious that, if the system could be subjected to the influence of the three frictions separately, formulae would be obtained with different exponents k and that these formulae would not be capable of being combined to a single one with one definite value of k.

d. Ether, at 11°,2 ($\mu = 0{,}725$, $\eta = 0{,}002785$)

$$\alpha = 0{,}01\, x + 19{,}95 \,.\, 10^{-6}\, x^3 - 18{,}66 \,.\, 10^{-8}\, x^5 + 8{,}5 \,.\, 10^{-10}\, x^7,$$

with $T = 20{,}76$ and $\delta = 0{,}04840$.

e. Liquid air, at 80°,8 K = — 192°,3 C. ($\mu = 0{,}956$, $\eta = 0{,}001718$)

$$\alpha = 0{,}01\, x + 15{,}82 \,.\, 10^{-6}\, x^3 - 12{,}88 \,.\, 10^{-8}\, x^5 + 5{,}3 \,.\, 10^{-10}\, x^7,$$

with $T = 20{,}72$ and $\delta = 0{,}04194$.

The angles of deflection are expressed in radians. The formulae are drawn up in such a manner that the time is reckoned from the moment at which $\alpha = 0{,}01$ ($x = 1$; scale-deflection 3.074; at these very small amplitudes the influence of the higher terms practically disappears); they are derived from observations between the limits $\alpha = .1$ and $\alpha = .003$.

Considering the absolute accuracy of the scale-readings (see above) it follows, that the influence of the higher terms does not make itself felt with water till $x = 1{,}6$ or $\alpha = 0{,}016$, with benzene a little sooner, with carbon disulphide and ether at $\alpha = 0{,}011$ and with liquid air at $\alpha = 0{,}012$. These few data are sufficient to show, as was to be expected (see § 1, note), that for one and the same oscillating system the limit for practical infinite smallness of the velocities lies in general the lower the smaller the viscosity of the liquid.

In order to confirm this result by another experiment with a liquid of higher viscosity an observation was made with a mixture of water and glycerine:

f. Mixture of water and glycerine, 11° C. ($\mu = 1{,}087$, $\eta = 0{,}02825$)

$$\alpha = 0{,}01\, x + 3{,}27 \,.\, 10^{-6}\, x^3 - 1{,}01 \,.\, 10^{-8} \,.\, x^5,$$

with $T = 21{,}10$ and $\delta = 0{,}2623$. The deviation actually begins here at a higher limit than with pure water, viz. at $\alpha = 0{,}021$.

In the computation of the viscosities account had to be taken as formerly (Comm. N°. 149*b*, V) of the friction of the air for the not-immersed parts of the apparatus and of the internal friction of the wire. For this purpose an experiment was made in ordinary air, but without the sphere [1]) and with a copper cylinder instead of the aluminium one (see Comm. N°. 149*b*); in this experiment the observations were again extended between the same wide limits as in the previous experiments.

g. Air $t = 13°{,}4$, $p = 765$ mm., $T = T_0 = 20{,}65$, $\delta = 0{,}00606$,

$$\alpha = 0{,}01\, x + 98{,}8 \,.\, 10^{-6}\, x^3 - 16{,}8 \,.\, 10^{-8}\, x^5.$$

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[1]) This experiment cannot therefore be compared with the previous ones, where the friction acted chiefly on the sphere.

In this case the second term will be seen to exert a very high influence, which makes itself felt from $a = 0{,}006$ onwards.

Finally another experiment was made to ascertain the influence of a lengthening of the time of oscillation by an increase of the moment of inertia. For this purpose the copper cylinder was used instead of the aluminium cylinder, by which the oscillating system, including the sphere, obtained a moment of inertia of 945 C.G.S. ($T_0 = 26.70$). With this system we found for

h. Water, at 9°,8 ($\mu = 1{,}000$, $\eta = 0{,}01310$)

$$a = 0{,}01\,x + 8{,}50\,.\,10^{-6}\,x^3 - 3{,}24\,.\,10^{-8}\,x^5,$$

with $T = 26.96$ and $\delta = 0.0924$. The deviation is thus somewhat larger than in the former experiment with water, notwithstanding the increase of T; but δ has become smaller, which for an equal value of T corresponds to a diminution of η.

4. On account of practical difficulties involved no attempt was made to extend the observations to angles of deviation much smaller than .01 of a radian [1]). On the other hand a few more observations were made in which the deviations were very large (up to 6π). For this purpose on the small copper tube which is soldered to the lower end of the small glass tube B_1 of the oscillating system (comp. Comm. N°. 149*b*, IV, 2) a small aluminium disc was placed (radius 3 cms; mass 23,7 grms; moment of inertia 144) with a cylindrical copper edge, the external cylindrical surface of which was provided with a scale-division in degrees. A small telescope was focussed on this scale, by which one tenth of a degree could be read by estimation.

To the wooden board which covers the stone pillar at the level of the steel pin *St* which carries the sphere a small clip was fastened, by means of which the pin could be gripped and in this manner the oscillating system after a rotation through a definite angle (about three complete rotations) could be arrested. By removing the spring which held the clip closed, the oscillating system was released and the amplitudes of the oscillations performed by the system were observed, first on the divided disc down to about 5°, and then as

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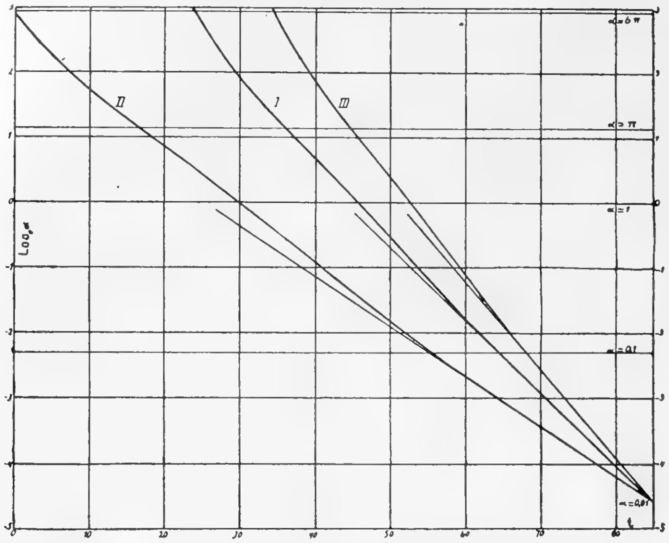
[1]) In order to descend to angles ten times smaller the reading arrangement would have had to be placed at a ten times larger distance (i.e. 15 meters). In itself this would not have been an insurmountable difficulty, but much higher demands would then have been made on the rigidity of the mounting; even now, although apparatus and reading-arrangement were placed on pillars built in the ground (comp. Comm. N°. 149*b*, IV, 2), the vibrations occasioned by the passing traffic were often troublesome.

formerly on the glass scale of the reading-arrangement. The liquid was contained in a vessel placed on a small table which was mounted on the stone pillar of the apparatus.



The results of these observations are graphically represented in the adjoining figure. The three curves represented show, how in three different experiments the logarithm of the elongation diminished with the time. The ordinates represent $log_e\, \alpha$, and the abscissae the time expressed in the time of oscillation as unit; the direction-coefficient of the asymptote, also shown in the figure, gives the logarithmic decrement δ with the opposite sign. The three curves have been shifted in such a manner as to make them intersect in one point corresponding to an amplitude of 0,01 radian.

The three curves refer:

I. to water at 10°,0 (aluminium cylinder + disc, $K = 717$, $T_0 = 23{,}12$, $T = 23{,}42$, $\delta = 0{,}1118$),

II. to benzene at 11°,0 ($K = 717$, $T = 23{,}32$, $\delta = 0{,}0771$),

III. to water at 10°,2 (without the aluminium cylinder, $K = 544$, $T_0 = 20{,}14$, $T = 20{,}35$, $\delta = 0{,}1355$).

All three bear the same character: with increasing elongation the logarithmic decrement first increases, then diminishes and finally increases again, in accordance with the sign of the coefficients a_3, a_5 and a_7 (§ 3). [1]

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[1] On account of the small accuracy of the observations in the range from $\alpha = \frac{1}{2}\pi$ to $\alpha = 0.1$ an analysis of the curves similar to the one followed in § 4 was not really possible.

5. It follows from the foregoing discussion that in order to observe under such circumstances that the logarithmic decrement of the oscillations becomes independent of the amplitude within the limits of the errors of observation,[1] it would be necessary to go down to amplitudes about ten times smaller than those used so far, a method involving the difficulties just mentioned.

It is not necessary, however, to look for a solution in that direction, as it has been found possible by a reliable extrapolation from a range, situated just above the limit of the practically infinitely small velocities, to derive the logarithmic decrement of actually infinitely small velocities with sufficient accuracy. In this manner the method of the observation of the damping of oscillations becomes a practical method for the determination of viscosities of liquids.

By the aid of the data now obtained it becomes possible to correct the values for the viscosities of liquid mixtures of oxygen and nitrogen which were formerly found in Comm. N°. 149*b*, V. Although the elongations observed between the limits $a = .07$ and $a = .02$ exhibited a practically constant logarithmic decrement, the formula (*e*) now found for liquid air shows that the mean value of the logarithmic decrement between those limits (i.e. the value of $T\frac{d\log a}{dt}$ at $a = .05$ about or $x = 5$) is equal to $1.046\,\delta_0$, δ_0 being the value for infinitely small vibrations. Similarly it follows from the equation (*g*) found for gaseous air, that for $a = .06$ (the mean amplitude at which the logarithmic decrement for the gaseous phase was determined in the experiments of Comm. N°. 149*b*) $T = \frac{Td\log a}{dt} = 1.47\,\delta_0$.

Using these results we find for the mixtures of boiling point τ°:

$\tau = 79.57$	$\delta = 0.03802$	$\delta_2 + \delta_3 = 0.00310$	$\mu = 0.909$	$\eta = 0.001658$
82.34	0.04137	,,	1.003	0.001806
77.91	0.03632	,,	1.841	0.001615
89,62	0.04431	,,	1.143	0.001858

with a relative accuracy which may be estimated at $\frac{1}{1000}$.

6. In plotting the elongations to right and left as read in the experiments of sections 3 and 4, the two lines containing the two

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[1] Unless very large times of swing are used (comp. § 1), which, however, in order to obtain moderate decrements would require a very high moment of inertia (comp. Comm. N°. 148*c*). The latter circumstance and also the extremely long time which each experiment would require in that case, render a large time of oscillation undesirable.

series of points were found not to be perfectly symmetrical with respect to a straight line $a =$ const. : the diameter of the set of two lines was slightly curved and approached the zero of the deflections asymptotically. This may be interpreted by assuming that this zero (a_0) changes during the experiment; the shift is very small, however: 1 mm. about on the scale in the experiments of section 3, 1° at the utmost in those of section 4.

To begin with this shift was looked upon as due to an elastic time-effect of the wire, connected with the initial torsion given to it at the beginning of each experiment; but when the phenomenon was found to be very regular, independent of the original torsion of the wire, and the return to the position of equilibrium appeared to happen more slowly in liquids of smaller viscosity, some phenomenon in the liquid was thought of, the possibility being contemplated, that by *starting* the motion of the oscillating system the liquid acquired a one-sided rotational motion which carried the sphere along, and which would naturally be a damped motion, so that the sphere would gradually return to the position of equilibrium. A calculation showed, however, that an influence of that kind could not make itself felt during such a long time.[1]) Moreover the change of a_0 was found always to occur in the same direction, independently of the direction of the initial rotation of the system. and lastly a_0 was the same function of the amplitude with all the liquids, viz. proportional to the square of the amplitude: $a_0 = 0.04\ a^2$, that is: the logarithmic decrement of a_0 was twice that of a.

The conclusion was drawn, that the shift of the zero was only an apparent one, and what was really observed was the effect of a

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[1]) The damping of this motion of the liquid can easily be calculated. The motion is bound to be aperiodic and may thus be represented by equation (56) of Comm. N⁰. 148*d*; if the sphere is practically at rest, we must have $D = 0$, in order that u need not be zero; hence:

$$tg\, b'' (R'-R) = \frac{b''(R'-R)}{1 + b''^2 R'R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

an equation which allows an infinite number of solutions for b'', and thus for k, but of which only the first solution which is not zero has to be considered. In our case $R'=3$, and $R=2$; the solution thus becomes $b''=3{,}2$ and therefore $-k=10\frac{\eta}{\mu}$.

In the case of a hollow sphere ($R' = 0$, comp. Comm. N⁰. 148*b*, § 21) equation (2) becomes $tg\, b''R = b''R$, whence $b''R = 4{,}5$ and $-k = 20\frac{\eta}{\mu R^2}$ (Comp. LAMB Hydrodynamics p. 577).

small term $\alpha_2 = a_2 e^{2kt}$ in equation (1) a term which could not, however, originate in the motion of the liquid (an asymmetry of the oscillating system was out of the question and, moreover, a_2 was independent of the nature of the liquid), but probably had to be ascribed to a want of symmetry in the wire [1]).

Physics. — "*The viscosity of liquefied gases.* VII. *The torsional oscillatory motion of a body of revolution in a viscous liquid.*" By J. E. Verschaffelt. (Communication N°. 151*e* from the Physical Laboratory at Leiden). (Communicated by Prof. H. Kamerlingh Onnes.)

(Communicated in the meeting of February 24, 1917).

1. In Comm. N°. 148*b* the theory of the torsional oscillatory motion of a sphere in a viscous liquid was developed to a first approximation; the results of the experimental investigation described in the previous part (VI, Comm. N°. 151*d*) render it advisable to develop the theory to a higher degree of approximation. The present paper is an attempt to a solution of the problem, not only for a sphere but for an arbitrary body of revolution. This attempt was in so far successful as a method of solution is given, in which the motion of the liquid and of the body is put into the form of a series; the terms of these series, however, contain functions of the coordinates which in the mean time owing to the difficulties of the integration remain determined by differential equations, and coefficients the numerical value of which cannot yet be given. In form these series agree with those which were found experimentally (Comm. N°. 151*d*).

The motion of the liquid.

2. We start from the well-known hydrodynamical equations [2])

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[1]) An asymmetry of this kind is not improbable, as the wire owing to the method of preparation showed a permanent twist: on the tension being taken off it curls up spirally and the zero changed with the weight suspended from it. During the oscillation of the system the wire obtains a higher twist in the one direction and is untwisted in the other, which might involve a small deviation from Hooke's law to be expressed by a term $N\alpha^2$ in the equation of motion of the oscillating body (Comm. N°. 148*b*, equation 23).

[2]) u, v, w are the components of the velocity at a point x, y, z; p is the pressure in the liquid, its density being ρ and its viscosity η; X_0, Y_0, Z_0 are the components of the external field of force, in which the liquid is placed, in our

$$\left.\begin{array}{c}-\frac{\partial p}{\partial x}+\eta\,\Delta\,u+\mu\,X_0=\mu\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\text{ ,etc}\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\,,\end{array}\right\}\;.\quad(1)$$

the last of which we shall replace by a different one which follows from the whole set of four viz.:

$$\Delta p=2\mu\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\frac{\partial u}{\partial x}-\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}\right).\quad(1')$$

If the motion of the liquid is the result of the friction of an immersed body of revolution rotating about its axis (the z-axis) and if, moreover, the boundary of the liquid, if it exists, is also the same in alle meridian planes, we may put

$$u=\varepsilon x-\omega y\quad,\quad v=\varepsilon y+\omega x\,,\;.\;\;.\;\;.\;\;.\;\;.\quad(2)$$

where ε and ω are functions of the cylindrical coordinates $\varrho=\sqrt{x^2+y^2}$ and z and of the time.

For small velocities we may write:

$$\varepsilon=\varepsilon_1+\varepsilon_2+\varepsilon_3+\ldots\quad,\quad\ldots\qquad,\quad p=p_1+p_2+p_3+\ldots\quad(3)$$

where each successive term of the series is considered as infinitely small with respect to the preceding one. We therefore treat the motion of the liquid as the result of a composition of a series of conditions of motion, the velocities of which diminish very rapidly, the further we go down the series [1]). Consequently the equations (1) can now be separated into a series of sets each of which determines a condition of motion. Putting in the n^{th} set:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-X_n\text{ , etc.}\;.\;\;.\;\;.\;\;.\;\;.\;\;.\quad(4)$$

X_n, Y_n, Z_n are to be looked upon as the components of a force generated by the inertia of the liquid; they are completely determined by the preceding approximations. Also by (1') in each successive approximation the distribution of pressure is determined by the preceding approximation.

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case the field of gravity ($X_0=0$, $Y_0=0$, $Z_0=g$). Δ is the symbol for LAPLACE's operator.

It will be supposed, that neither μ nor η are functions of the coordinates or of the time. In a piece of apparatus of ordinary dimensions this condition is practically fulfilled, even for a gas, when no greater differences of pressure occur than are occasioned by gravity. (Comp. on this point ZEMPLÉN, Ann. d Phys., 38, 81, 1912).

[1]) This method of treatment of the problem was given by A. N. WHITEHEAD (Quarterly Journ. of pure and applied Mathem., 23, 78, 1889) for the purpose of finding a second approximation: he applied it to the case of a uniform rotation of a sphere. Comp. also ZEMPLÉN, Ann. d. Phys., 38, 74, 1912.

3. In the *first approximation* (velocities infinitely small of the *first* order) we have

$$\Delta p_1 = 0\ ^{1)} \quad , \quad \eta \Delta u_1 = \mu \frac{\partial u_1}{\partial t} \text{, etc.} \quad . \quad . \quad . \quad (5)$$

If the motion of the body is an oscillation of the damped harmonic kind, the angle of deviation of which can be represented by the real part of

$$\alpha = a e^{kt} \text{ , }^{2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

we must put

$$p_1 = 0 \quad , \quad w_1 = 0 \quad , \quad \varepsilon_1 = 0 \quad , \quad \omega_1 = k a \varphi_1 \text{ ,} \quad . \quad . \quad . \quad (7)$$

where φ_1 is now only a function of ϱ, z and $b = \sqrt{\frac{\mu}{\eta} k}$ determined by the differential equation

$$\frac{\partial^2 \varphi_1}{\partial \varrho^2} + \frac{3}{\varrho} \frac{\partial \varphi_1}{\partial \varrho} + \frac{\partial^2 \varphi_1}{\partial z^2} - b^2 \varphi_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and by the boundary conditions, that at the surface of the body $\varphi_1 = 1$ and at the external boundary of the liquid (at infinity, if the liquid is infinite) $\varphi_1 = 0$)[3]).

4. In the *second approximation* one finds:

$$\left. \begin{aligned} & -\frac{\partial p_2}{\partial x} + \eta \Delta u_2 + \mu X_2 = \mu \frac{\partial u_2}{\partial t} \text{, etc.} \\ \text{where} \quad & \\ & X_2 = \omega_1{}^2 x \ , \ Y = \omega_1{}^2 y \ , \ Z_2 = 0 \ . \end{aligned} \right\} \quad . \quad . \quad . \quad (9)$$

This represents the first approximation to a motion in the field of the centrifugal force[4]). Moreover:

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[1]) In the first approximation the distribution of pressure is the same as in condition of rest.

[2]) k is a complex imaginary quantity; a may be taken as real. (Comp. Comm. N⁰. 148*b*).

[3]) If the body of revolution is a sphere, φ_1 becomes a function of $r = \sqrt{x^2 + y^2 + z^2}$ only, and equation (8) reduces to equation (11) of Comm. N⁰. 148*b*.

[4]) In general this field of force has no potential; that is why it causes a movement of circulation in the liquid. (Comp. Comm. N⁰. 148*d*, Proc. XVIII, 2, p. 1038). For that reason it is also impossible to put in general $\frac{\partial p_2}{\partial x} = \mu X_2$, etc., as in the distribution of the pressures under the influence of the external field of force. Only in the case of an infinitely long cylinder, where φ_1 is merely a function of ϱ, the field of force of the centrifugal force has a potential; a movement of circulation is absent in that case and the motions of higher order disappear at the same time.

$$\Delta p_2 = \frac{\mu}{\varrho} \frac{\partial (\varrho \omega_1)^2}{\partial \varrho} \, . \qquad \qquad (9')$$

These equations are satisfied by putting:

$$\varepsilon_2 = 2\,k\,a^2\,\psi_2 \quad , \quad \omega_2 = 0 \quad , \quad w_2 = 2\,k\,a^2\,\gamma_2 \quad , \quad p_2 = 2\,k\,a^2\,\pi_2 \, . \, . \qquad (10)$$

where ψ_2, γ_2, π_2 are new functions of ϱ, z, and b. A motion of circulation is thus obtained in the meridian planes; this motion is a damped pulsating one, with twice as high a frequency and degree of damping, as the oscillation in first approximation.

5. In *third approximation* we have again $\Delta p_3 = 0$, or $p_3 = 0$, and

$$\eta\,\Delta\,u_3 + \mu\,X_3 = \mu\,\frac{\partial u_3}{\partial t}, \text{ etc.}$$

where this time

$$X_3 = \boldsymbol{\Phi}_3\,y \quad , \quad Y_3 = -\,\boldsymbol{\Phi}_3\,x \quad , \quad Z_3 = 0 \, , \qquad . \; . \qquad (11)$$

with

$$\boldsymbol{\Phi}_3 = 2\,k^2\,a^3 \left[2\,\varphi_1\,\psi_2 + \varrho\,\psi_2\,\frac{\partial \varphi_1}{\partial \varrho} + \gamma_2\,\frac{\partial \varphi_1}{\partial z} \right] .$$

In a third approximation we thus have a motion caused by a periodic damped field of force at right angles to the meridian planes and containing the time in the factor e^{3kt}. It follows, that this motion like the one in first approximation consists of an oscillatory rotation of the liquid in shells, each with its own amplitude and phase, but with the same period and degree of damping; i. e. the equations can be satisfied by putting:

$$w_3 = 0 \quad , \quad \varepsilon_3 = 0 \quad , \quad \omega_3 = 3\,k\,a^3\,\varphi_3 \, ,$$

φ_3 being a new function of ϱ, z, and b, determined by the differential equation:

$$\frac{\partial^2 \varphi_3}{\partial \varrho^2} + \frac{3}{\varrho}\frac{\partial \varphi_3}{\partial \varrho} + \frac{\partial^2 \varphi_3}{\partial z^2} - 3b^2 \varphi_3 = 2b^2 \left(2\varphi_1\psi_2 + \varrho\psi_2\frac{\partial \varphi_1}{\partial \varrho} + \gamma_2\frac{\partial \varphi_1}{\partial z} \right) \qquad (12)$$

and by the condition, that $\varphi_3 = 0$ at the boundaries of the liquid.

6. As one would be inclined to expect and as, moreover, can be easily proved, further approximations yield alternately circulation in meridian planes and oscillations, about the axis, with frequencies and degrees of damping which increase in an arithmetical series. By putting

$$-\,X_n = \boldsymbol{\Psi}_n\,x - \boldsymbol{\Phi}_n\,y \quad , \quad -\,Y_n = \boldsymbol{\Psi}_n\,y + \boldsymbol{\Phi}_n\,x \, , \, . \quad . \qquad (13)$$

one finds, using a well known method [1])

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[1]) If the relations hold for $n = 1$ to m, it may be proved, that they also hold for $n = m + 1$.

$$\left.\begin{array}{l}\Psi_{2n+1}=0, \Phi_{2n}=0, Z_{2n+1}=0\,, \varepsilon_{2n+1}=0\,, \omega_{2n}=0\,, w_{2n+1}=0\,, p_{2n+1}=0 \\ \varepsilon_{2n}=2nka^{2n}\psi_n\,, \omega_{2n+1}=(2n+1)ka^{2n+1}\varphi_{2n+1}\,, w_{2n}=2nka^{2n}\gamma_{2n}\,, \\ \hfill p_{2n}=2nka^{2n}\pi_{2n}\end{array}\right\} \quad (14)$$

From the foregoing discussion it appears, that, when a body of revolution in a liquid oscillates about its axis in a simple harmonic damped motion (how the motion is sustained, is of no account), the liquid will assume a motion which consists partly of a compound harmonic damped oscillation of liquid shells, where the amplitude may be represented by

$$\alpha = \alpha_1 + \alpha_3 + \alpha_5 + \ldots = a\varphi_1 e^{kt} + a^3\varphi_3 e^{3kt} + a^5\varphi_5 e^{5kt} + \ldots {}^{1)} \quad . \quad (15)$$

and for the rest of a motion of circulation in meridian planes.

The above reasoning still holds, if the motion of the oscillating body itself is a compound harmonic one of the form:

$$\alpha = a\,e^{kt} + \sigma_3\,a^3\,e^{3kt} + \sigma_5\,a^5\,e^{5kt} + \ldots ; \quad . \quad . \quad . \quad . \quad (16)$$

the functions φ_{2n+1} are, however, not then zero at the surface of the body, but equal to σ_{2n+1}.

The motion of the body.

7. The question now arises: of what nature will the motion be which the body assumes in the liquid, when without friction it would perform a simple harmonic oscillation? Certainly not a simple damped motion, for, even if by some artifice the body was for some time made to swing exactly in the simple damped motion, which it must assume according to the first approximation, the higher terms of the liquid motion would still by friction give rise to forces which would try to disturb the simple motion and which would certainly create this disturbance, as soon as the body was left to itself. It is obvious that they would impart to the body a composite motion, corresponding to equation (16) where the even terms would not occur, seeing that the liquid motions of even order only give friction along the meridians and thus cannot have any influence on the oscillation²).

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1) The quantities φ are functions of ρ, z and b which are exclusively determined by the boundary conditions. In the case, when the body is an infinitely long cylinder, all φ's are zero, with the exception of φ_1. At the solid boundaries of the liquid the φ's become zero (except $\varphi_1 = 1$). If the liquid is partly bounded by a free surface, a special condition will hold there.

2) The friction along the meridians can only produce an imperceptible deformation of the body. It might seem as if the circulational motion in the liquid, although it is kept up by the body and damped by friction in the liquid, did not occasion a loss of energy of the body. The explanation of this seeming contradiction may be found in the circumstance, that the motions of different order are not mutually independent and a loss of energy of even order is provided by products of velocities of uneven order.

In this manner it becomes intelligible that the oscillating body, also when it is made to swing freely in the liquid, will assume a motion corresponding to equation (16)[1], as was first revealed by experiment (comp. Part VI).

In equation (16) the exponent k which contains the time of oscillation T and the logarithmic decrement $\left(k = -\frac{\delta}{T} + \frac{2\pi i}{T}\right)$, as also the coefficients σ_3, σ_5, etc. (a is arbitrary) are determined by the interactions between liquid and body. By the friction to which the body is subject, moments act on it which can be calculated, as was actually done in the first approximation in Comm. N°. 148*b* for the case of a sphere, as soon as the functions ω_n and γ_n are known; if the moments of the couples caused by the successive sets of motion are represented by C_1, C_3, C_5 etc. (the moments of even order are all zero), the equation of motion of the body is:

$$K\frac{d^2\alpha}{dt^2} - C + M\alpha = 0\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

(comp. 23, Comm. N°. 148*b*), where $C = C_1 + C_3 + C_5 + \ldots$ The quantities C are given by

$$\left.\begin{aligned} &C_n = -\int_{z_1}^{z_2} \varrho F ds = \eta \int_{z_1}^{z_2} \varrho^2 \frac{\partial \omega_n}{\partial N}\, ds\ ^{2)} = \eta n k \alpha_n \int_{z_1}^{z_2} \varrho^2 \frac{\partial \varphi_n}{\partial N}\, ds = -L_n \frac{d\alpha_n}{dt}\,, \\ &\text{where} \\ &\qquad\qquad L_n = -\eta \int_{z_1}^{z_2} \varrho^2 \frac{\partial \varphi_n}{\partial N}\, ds = -\eta A_n\,; \end{aligned}\right\} \quad (18)$$

A_n is a numerical quantity (of the dimensions of a volume) which depends on the shape of the body and further on the quantity b, i.e. on k (time of swing and decrement), on the constants η and μ of the liquid and finally on the coefficients σ up to and including σ_n[3]. Equation (17) can thus be separated into a series of equations

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[1] That is to say after the disturbances, which are due to the starting of the motion, have subsided: these disturbances are not gone into here (comp. Comm. N°. 148*b*, § 4, note).

[2] As in Comm. N°. 148*b*, F represents the tangential force per unit area in the direction of the motion; ds is the area of a circular strip round the body of revolution; $\frac{\partial \omega_n}{\partial N}$ is the gradient of the angular velocity of the n^{th} order in the liquid close to the body; z_1 and z_2 are the z-limits of the body.

[3] If different parts of the oscillating system are surrounded by different fluids (e.g. a part by a liquid and the other part by air, as was the case in the experiments) L_n itself has to be divided into parts, each of which refers to one of the fluids.

$$\left.\begin{array}{l} K\dfrac{d^2 a_n}{dt^2} + L_n\dfrac{da_n}{dt} + M a_n = 0\,, \\ \text{or} \\ n^2 k^2 K + n k L_n + M = 0\,, \end{array}\right\} \quad . \quad . \quad . \quad . \quad (19)$$

the first of which ($n = 1$) is the same as equation (26) of Comm. N°. 148*b*, by which k is determined. The other equations determine the quantities σ.[1]).

Herewith the problem is formally completely solved. Numerical application would, however, only be possible, if one succeeded in finding the functions ω_n and γ_n [2])

Physics. — "*The viscosity of liquefied gases.* VIII. *The similarity in the oscillatory rotation of a body of revolution in a viscous liquid*". By J. E. Verschaffelt. (Communication N°. 151*f* from the Physical Laboratory at Leiden). (Communicated by Prof. H. Kamerlingh Onnes).

(Communicated in the meeting of February 24, 1917).

1. In Comm. N°. 148*c* the conditions were derived under which similarity would exist between two different modes of motion of an oscillating sphere in a viscous liquid. The discussion was at that time entirely based on the first approximation of the problem: but even then it was anticipated that the conclusions would prove to hold in general (Comm. N°. 148*c* § 5), not only in nearer approximations, but also for bodies of different shape to the sphere; this will now be shown to be the case.

Returning once more to the general hydrodynamical equations (equation (1) of the previous communication, Comm. N°. 151*e*), we will inquire whether it is possible to introduce units of length, mass and time such that everything specific disappears from the equations. The *external* similarity of the liquid motion of course requires in the first place similarity of the body oscillating in the liquid (the latter condition is of itself satisfied in the case of a sphere); let R be a

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[1]) In all this the supposition is retained that the moment of the torsional couple is proportional to the angle of torsion and that the ordinary laws of friction remain valid.

[2]) Not till then would it be possible to settle the exact condition for infinite smallness of the velocities (comp. preceding communication § 1 note), i.e. the condition for α_3 (for the body) to remain below a definite fraction of α_1 or, as would be even more useful for our purpose, the condition for the decrement δ not to deviate by more than a definite amount from the limiting value δ_0.

characteristic length of the body (in the case of the sphere the radius). Let T represent a time characterising the motion: in the case of an oscillating moment we naturally select as such the time of oscillation which, as we have seen, is also the periodic time of the liquid, at least for a body of revolution.[1]

These quantities we shall take, in each special case, as the units of length and time and shall put $t = T\mathfrak{t}$, $l = R\mathfrak{l}$, so that $\mathfrak{t}$ and $\mathfrak{l}$ will now represent the reduced time and the reduced length. As unit of mass we shall further take the mass of the (new) unit of volume of the liquid. The equations then retain their form[2]

$$-\frac{\partial \mathfrak{p}}{\partial \mathfrak{x}} + \eta' \Delta \mathfrak{u} = \frac{\partial \mathfrak{u}}{\partial \mathfrak{t}} + \mathfrak{u}\frac{\partial \mathfrak{u}}{\partial \mathfrak{x}} + \mathfrak{v}\frac{\partial \mathfrak{u}}{\partial \mathfrak{y}} + \mathfrak{w}\frac{\partial \mathfrak{u}}{\partial \mathfrak{z}}, \text{ etc.}; \quad . \quad . \quad (1)$$

The only difference is, that η' in the new system of units has a different numerical value from before, namely in view of its dimensions $(L^{-1} M T^{-1})$:

$$\eta' = \frac{\eta T R}{\mu R^3} = \frac{\eta T}{\mu R^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1')$$

The viscosity has not one definite value in the new system any more than in the old units; the coefficient η' in equation (1) may thus assume various values and a first condition therefore for *internal* similarity of the liquid motion is, that this coefficient has the same value (e.g. in the C.G.S. system) in all the cases considered; in other words: by ascribing to η' all possible values from 0 to ∞ we characterize an infinite series of different liquid motions.

2. We have thus found that for two similar conditions of motion η' must have the same value. If the motion of the body were an undamped harmonic one (or a uniform rotating motion) this condition would be the only one for internal similarity of two motions; but when the motion is a damped harmonic motion, the other condition is to be added, that the logarithmic decrement δ of the swings (a dimensionless number, and therefore independent of the units chosen) must obtain the same value in similar cases; and as all possible values from 0 to ∞[3] may be assigned to the decrement, the quantity δ characterizes a second infinite series of different liquid motions.

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[1] This is probably true in the general case too. With a uniform rotation the time of revolution would be taken for T.

[2] The action of gravity as having no influence on the motion is here left out of account.

[3] Even from $-\infty$ to $+\infty$, if one chose to also consider motions with artificially increasing amplitudes.

Assuming now that the motion of the body (and of the liquid) is a composite harmonic one and damped, corresponding to equation (15) and (16) of the previous communication, there are still a great number of different conditions of motion possible, which only differ by the coefficients σ; in this case similarity requires, that those coefficients have the same value in all cases.

3. When the bodies assume such motion not artificially, but by friction in the liquids, it will be necessary — in order that the two conditions of similarity: equal η' and equal σ may be satisfied — that the elements M and K which determine the friction-free motion of the bodies also satisfy definite conditions; according to equations (17) and (18) of the previous paper these conditions are, that the expressions $\frac{M}{K}T^2$ and $\frac{L_n}{K}T$ have the same values in all cases ($kT = -\delta + 2\pi i$ is in all cases the same number). The former condition expresses the fact, that the time of swing of the friction-free body $\left(T_0 = 2\pi\sqrt{\frac{K}{M}}\right)$, measured in the time of swing in the liquid as unit, must be the same in all cases (conversely -- and this is simpler — the time of swing T_0 can in each case be chosen as unit of time). As regards the second condition, according to which each expression $\frac{L_n T_0}{K}$ must have the same numerical value in all similar cases, it will be seen, having regard to the dimensions of L_n, that this condition requires, that the expression $\eta\frac{R^3 T_0}{K}$ has the same numerical value in all cases; in that case all quantities L_n have also equal values in the various cases, if each time the specific units are introduced, because the only things left are integrations of functions which possess identical values in corresponding points, and finally, according to the first equations (18), the values σ_n are also the same in all cases.

Taken together, the conditions for similarity are therefore, that the quantities:

$$\left.\begin{array}{lll} c_1 = \dfrac{\mu R^2}{\eta T_0} & \text{and} & c_2 = \dfrac{\eta R^3 T_0}{K} \\ & & \\ c_3 = \dfrac{\mu R^5}{K} & \text{and} & c_4 = \dfrac{\eta^2 R}{\mu M} \end{array}\right\} \quad \text{or} \quad \ldots\ldots \quad (2)$$

have equal values (in C. G. S. units) in all cases; the same conditions as were also found in Comm. N°. 148*c*.

4. By the aid of the above considerations it is possible to represent the results of the experiments described in Comm. N°. 151*d* somewhat more systematically. As there are only two infinite series of modes of motion, everything may be reduced to the change of two of the five elements μ, η, T_0, K and R [1]). In the first place, seeing that in most of the experiments the apparatus remained unchanged (T_0, K and R thus remaining the same), and only the liquid was changed, the most natural procedure is to consider η and μ as the variables and accordingly to reduce equation *h* of section 3 (Comm. N°. 151*d*) which was found with a modified oscillating system to the same values of K and T_0 which applied to the equations *a* to *f*. Equation *h* itself does not change thereby, considering that α and x are dimensionless numbers; δ does not change either and the conditions (2) show, that the same equation *h* ought to have been found, if the sphere, connected with the aluminium cylinder, had oscillated in a liquid for which $\mu = .607$ and $\eta = .01028$. The value of $\psi = \frac{T - T_0}{T_0} = .010$ would also have remained unaltered.

It is preferable, however, to reduce everything to one and the same liquid, for instance with $\mu = 1$ and $\eta = .01$ (water at 20°), and allow K and T_0 to change [2]). In that case the adjoining table is obtained.

This table may be utilised for the purpose of deriving the most favourable conditions, under which the experiments with a given liquid can be conducted; we have only to reduce it to a different liquid by multiplying all values of T_0 and K by definite factors. As an instance, taking for liquid hydrogen the values: $\mu = .07$ and $\eta = .00013$, the latter of which was calculated in Comm. N°. 148*b* section 13 by the application of the law of corresponding states, it will be seen that for this liquid the values of T_0 have to be taken about 5 times higher and those of K about 15 times smaller; if we wish, therefore, to experiment with liquid hydrogen under conditions about corresponding to those which existed in the experiments with ether, it would be necessary to take $K = 60$ and $T_0 = 40$.

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[1]) At least, if we use the rough approximation of confining the oscillating body to the sphere. Of course it would be possible to extend the theory of similarity to the case where the oscillating system is partly in one liquid and partly in another; it is easily seen, that in that case the ratios of the densities of the two liquids must be equal in all cases.

[2]) This was also the method followed in Comm. N°. 148*c*. The advantage is that K and T_0 can be changed over a much wider range than η and μ.

	$T_0 = 3.55$	6.21	7.94	17.93	27.24	34.98	53.65
$K = 945$						$\psi = 0.010$ $\delta = 0.0924$ $a_3 = +8.50 \,.\, 10^{-6}$ $a_5 = -3.24 \,.\, 10^{-8}$	
$K = 792$			$\psi' = 0.005$ $\delta = 0.04840$ $a_3 = +19.95 \,.\, 10^{-6}$ $a_5 = -18.66 \,.\, 10^{-8}$ $a_7 = +8.5 \,.\, 10^{\ 10}$				
$K = 645$				$\psi = 0.010$ $\delta = 0.0898$ $a_3 = +7.83 \,.\, 10^{-6}$ $a_5 = -3.00 \,.\, 10^{-8}$			
$K = 600$	$\psi = 0.005$ $\delta = 0.04194$ $a_3 = +15.82 \,.\, 10^{-6}$ $a_5 = -12.88 \,.\, 10^{-8}$ $a_7 = +5.3 \,.\, 10^{-10}$						
$K = 574$					$\psi = 0.014$ $\delta = 0.1272$ $a_3 = +6.73 \,.\, 10^{-6}$ $a_5 = -3.06 \,.\, 10^{-8}$		
$K = 528$							$\psi = 0.021$ $\delta = 0.2623$ $a_3 = +3.27 \,.\, 10^{-6}$ $a_5 = -1.01 \,.\, 10^{-8}$
$K = 449$		$\psi = 0.007$ $\delta = 0.0705$ $a_3 = +18.57 \,.\, 10^{-6}$ $a_5 = -13.65 \,.\, 10^{-8}$ $a_7 = +4.0 \,.\, 10^{-10}$					

5. The above application of the theory of similarity also shows, that in making experiments with various liquids, as described in section 4 of Comm. N°. 151*d*, by changing K and T_0 in all possible ways a double set of curves would be obtained forming a net-work which would be identically the same for all liquids[1]). By the equations (2) the elements belonging to the curves in one net could be calculated from those belonging to the corresponding curves in the other; conversely having obtained the nets for two liquids the values of μ and η might be composed and in this manner these quantities (more especially η) might be determined for one of the liquids.

Possibly this method may find its practical application some time.

Physics. — "*The viscosity of liquefied gases.* IX. *Preliminary determination of the viscosity of liquid hydrogen.*" By J. E. VERSCHAFFELT and CH. NICAISE. (Communication N°. 151*g* from the Physical Laboratory at Leiden). (Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated in the meeting of February 24, 1917).

1. The measurements were made with the same apparatus as was used for the determination of the viscosity of liquid air (see Comm. N°. 149*b*, IV), into which, however, some improvements were introduced. In the first place, some nickel plated paper screens were placed under the cap to intercept the radiation of the cap; in the second place a small tinned hand-pump was introduced[2]) into the liquid, the rod of which passed through a small metal tube in the cap and could be worked up and down by means of an india-rubber tube which made an air-tight connection with the metal tube, by which means the liquid could be stirred previously to each experiment, and any slight differences of temperature or density could be equalized. The thermometer and the syphon tubes were taken away, as being unnecessary in these experiments; the temperature was deduced from the atmospheric pressure (the liquid, which in this case possessed

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[1]) As an instance, with the same restriction as in the experiments of section 3, curve II would also be found for water with $K = 807$ and $T = 15.21$.

[2]) Of the pattern generally used in the cryostats of the laboratory (see for instance Comm. N°. 123).

a sufficiently high degree of purity, was boiling [1]) under a pressure which was kept constant to a few mms of mercury by the laboratory pumps, and was only a little higher than the atmospheric pressure [2])); the emptying of the apparatus could be very quickly accomplished by the evaporation of the liquid, after removing the external vacuum glass, which, as in the experiments with liquid air, contained liquid air. For the internal vacuum-vessel, which contained the liquid hydrogen, a completely silvered glass was taken, in which only two opposite windows were left open, so as to enable us to observe the height of the liquid when filling the glass; as the external vessel was silvered with only a transparent strip left open, it was only necessary to turn this outside glass a little way in order to protect the liquid hydrogen practically completely from external radiation.

2. Technical difficulties in connection with the use of liquid hydrogen and the much lower temperature (± 20° K.) did not arise; the only thing was, that, as was to be expected, the damping of the oscillations was very small (about 6 times less than in liquid air) in consequence of which the internal friction of the suspension wire acquired a very high and unpleasant degree of importance. This friction, in fact, proved to be not only comparatively large, but to depend to a high degree upon accidental circumstances, difficult to estimate and control; consequently, although for each experiment separately an accurate logarithmic decrement could be deduced, only a very moderate agreement could be found between the various experiments [3]). In the first place it was found that unstretching and re-stretching of the wire (by exchanging the cylinders C) altered the viscosity of the wire very greatly (usually increasing it); in the second place the viscosity was a function of the time, which only decreased slowly, in an approximately exponential manner, and required some days to become constant; in the third place the gas in which the wire was placed proved to have a great influence upon its viscosity; pumping a vacuum, filling with air, replacing

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[1]) The boiling, which was entirely superficial, without the formation of bubbles, took place very slowly, thanks to the screens.

[2]) The small difference of pressure between the vapour pressure in the apparatus and the air pressure outside, was read on the small open manometer *Ma* (see figure in Comm. N⁰. 149*b*).

[3]) This is clearly also the cause of the differences which were observed in the experiments with liquid air (see Comm. N⁰. 149*b* V). This instability in the internal friction in the wire has given similar difficulties to previous investigators: see for instance ZEMPLÉN, Ann. d. Phys. 19, 802, 1906.

air by hydrogen (this especially) or the reverse, caused a great increase in the friction[1]), which, as we have said, gradually become less again[2]).

Under these unfavourable circumstances, in order to be able to arrive at provisional results with the apparatus as it was constructed, we were obliged to change our method of working to some extent and to demand a much smaller degree of accuracy from the results. The determination of the times of oscillation by registration, in particular (see Comm. N°. 149*b*, IV, § 4), was a complication disproportional to the accuracy, and could be quite adequately replaced by a purely chronometric determination, by means of a stop-watch, which showed $^1/_5$th of a second[3]). Further, the sensibility of the wire to changes of condition made it necessary that the apparatus should remain unchanged during a whole series of experiments, that is, that the cylinders should not be exchanged; as this did away with the use of the cylinders altogether (see Comm. N°. 149*b*, IV, § 5) they could just as well be left out[4]). We, therefore, continued the work with a constant oscillating system; in consequence of which the friction of the gas upon that part of the apparatus not immersed in liquid had to be eliminated in a different way. We did this in the following manner: besides the experiments in which the sphere oscillated in the liquid, we also made experiments with the sphere just above the liquid oscillating in vapour at a low temperature; from the knowledge of the density and viscosity of this vapour, by means of the formulae (24′) and (28) in Comm. N°. 148*b*, the retarding couple could be determined which the sphere experienced by the friction in the vapour; this couple could be subtracted from the total moment in the experiment in the vapour; the difference we considered might be taken as giving the couple which the oscillating system experiences by friction in the experiments in the liquid.

3. After several unsatisfactory attempts, we succeeded in carrying out in one day (July 12th 1916) a series of reliable, and

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[1]) Presumably a consequence of occlusion of gases by the metal wire. We have not used quartz wires yet, which probably would not possess this unpleasant peculiarity.

[2]) To avoid further trouble from these changes we left the apparatus always filled with hydrogen.

[3]) By determining the duration of ten oscillations T could still be determined to within about ·01 sec.

[4]) The moment of inertia of the oscillating system was thus, at ordinary temperature, $K = 372{\cdot}5 + 27{\cdot}8 = 400{\cdot}3$. Hereby the period of oscillation becomes smaller than before (17·22 sec.) it is true, but that was not a decisive objection.

as we think, mutually comparable observations. The logarithmic decrements δ as observed are given in the following table, which also contains the mean times at which the observations took place (they lasted about 4 minutes; each time 20 full oscillations were observed [1]), as well as the physical state of the substance which surrounded the sphere.

1.	$11^{u}25^{m}$	a.m.	vapour	$\delta = 0{\cdot}00393$
2.	11 40	,,	,,	379
3.	11 50	,,	,,	370
4.	12 15	,,	liquid	682
5.	12 25	,,	,,	672
6.	12 40	,,	,,	663
7.	2 15	p.m.	vapour	340
8.	2 35	,,	,,	317
9.	2 50	,,	,,	317

We shall now give a short description of the course of the observations. After the apparatus had been filled with hydrogen gas for a few days, we began to cool it in the morning of July 12th at about 10 a.m. and about 11 a.m. liquid was poured in, only so far that the sphere did not yet touch the liquid, when three observations were taken in the vapour; it appears that the decrement decreased rapidly, which seems to indicate a disturbance in the wire caused by the filling of the apparatus. About 12 o'cl. more liquid was poured in until the sphere was entirely immersed; again three observations were made, which gave a much larger decrement, and this in the same way decreased in the course of time, and to about the same degree as in the vapour. About 1 p.m. the external vacuum glass was removed, so that the liquid hydrogen could boil away pretty quickly; at about 2 p.m. the liquid had so far boiled away that the sphere projected completely out of the liquid; then the vacuum glass with liquid air was again put round the internal one, and three more observations were made in the vapour.

From our observations it follows, that the transference of the sphere from the vapour into the liquid involves an increase of the logarithmic decrement by a mean value of 0·00319, while for the

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[1] As we did not expect a greater accuracy than 1 $^0/_0$ about, we considered it useless to raise the degree of accuracy for the separate results by lengthening the series.

friction of the sphere in vapour 20° K. with $\mu = 0{\cdot}0012$ [1]) and $\eta = 0{\cdot}000011$ [2]) a decrement $\delta = 0{\cdot}00015$ was calculated. We conclude from this that the decrement caused by the friction of the liquid alone was $\delta_1 = 0{\cdot}00334$. The liquid was under a mean pressure of 766 mms. mercury; the temperature was thus 20°·36K. [3]) and the density 0·0708 [4]). From this it follows that $\eta = 0{\cdot}000117$.

This determination was made before the investigation of the suitability of the method was completed (see Comm. N°. 151*d*). From that investigation it appears probable, that the value found for η was a few percent too high, but the data are wanting by which the necessary correction might be estimated. We therefore give as the approximate value of the viscosity of liquid hydrogen

$$\eta = 0{\cdot}00011.$$ [5])

Physics. — "*Critical point, critical phenomena and a few condensation-constants of air*". By J. P. Kuenen and A. L. Clark. (Communication N°. 150*b* from the Physical Laboratory of Leiden).

(Communicated at the meeting of February 24, 1917).

The critical temperature and pressure of air have been determined by Olszewski [6]), Wroblewski [7]), and Witkowski [8]). Their results do not agree amongst each other as well as might be desired:

	Olszewski	Wroblewski	Witkowski
t_k	— 140°	about — 140.°5	— 141°
p_k	39 atm.	„ 37 à 37.5	39

The main object of our investigation was to obtain reliable values for the critical constants, including the critical density, which involves a detailed study of the condensation-phenomena in the critical region. Wroblewski noticed that air behaves differently near its critical point

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[1]) The vapour still behaves approximately as an ideal gas.

[2]) According to H. Kamerlingh Onnes, C. Dorsman and Sophus Weber. Comm. N°. 134*a*.

[3]) Normal boiling point: 20°,33 K., $\frac{dp}{dt} = 200 \frac{\text{m.M}}{\text{degree}}$ (see Comm. N°. 137*d*).

[4]) See Comm. N°. 137*a*.

[5]) This value agrees satisfactorily with that calculated in Comm. N°. 148*b* on the basis of the law of corresponding states.

[6]) K. Olszewski. C. r. 99, p. 184, 1884.

[7]) S. v. Wroblewski. Wied. 26, p. 134, 1885.

[8]) A. W. Witkowski. Phil. Mag. (5) 41, p. 288, 1896.

from other substances: this he correctly ascribed to the circumstance that air is a mixture, but the special phenomena which he describes were largely due to insufficient mixing. It was therefore necessary to repeat the investigation with all those precautions which in previous investigations on mixtures have proved necessary and amongst others to try and realise "retrograde" condensation [1]), which is characteristic of mixtures. So far an investigation of that sort had never been carried out but at the ordinary and at higher temperatures.

As expected the investigation proved to be beset with great experimental difficulties. Generally speaking these difficulties all originate in the circumstance that the mixture cannot as a whole be cooled down to a low temperature, at least, if the possibility must be left open — and this is an essential condition in the experiments here contemplated — of changing the volume of the substance gradually. A substance which at the very low temperatures could play the part which is otherwise fulfilled by mercury, viz. that of enclosing a fixed quantity of the substance in a variable volume, is unfortunately not known. It is therefore necessary to compress the mixture in a small tube which is closed at the bottom and cooled to the low temperature, by means of a piezometer, thus using the same method as followed with pure substances, so that every time a different fraction of the total quantity of substance is present in the observation-tube. With pure substances this does not involve any fundamental difficulty; by measuring the quantity of gas present in the part of the piezometer which is outside the cryostat, the quantity in the observation-tube can at each measurement be derived by subtraction from the total quantity, even when the substance is partly liquefied. With mixtures this is different: in the condensation of a mixture new mixtures are formed each time of different composition; in a series of observations in which the mixture is alternately compressed and expanded the mean composition of the mixture in the observation-tube will thereby very soon become different from that of the whole and the observations lose all definite meaning as referring to mixtures of varying and unknown composition. Taking air, the mixture dealt with in our investigation, as an instance, when it has been partly liquefied and is now re-evaporated, in the beginning the nitrogen will principally boil away and disappear from the tube, whereby the mixture, which remains behind, becomes richer and richer in oxygen and no longer has the same mean composition as air.

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[1]) J. P. Kuenen. Comm. Leiden 4. 1892.

It is therefore necessary in the experiments to lay down the general rule when partial condensation has once set in not again to increase the volume[1]) and, if the observations have to be repeated or a new series has to be started at a different temperature, it is necessary — in order to obtain complete homogeneous mixing — several times in succession to lower the pressure in the apparatus to normal and recompress to the high pressure. Otherwise in consequence of the very slow diffusion in the capillary connecting tubes with the supply-tube of the piezometer a mixture of higher boiling point remains behind in the small experimental tube, so that the succeeding observations are bound to be incorrect and amongst others the condensation will set in too soon. The importance of all this was not sufficiently realized in our experiments to begin with, so that a great number of our earlier observations had to be rejected later on.

The electro-magnetic stirring also involves greater difficulties at low temperature than otherwise, in consequence of the great width of the cryostat which brings with it a corresponding size of the electromagnet surrounding the cryostat. The observation-tube, capillary and stirrer were the same as used by CROMMELIN[2]) in his investigation of argon. In this tube the difficulty just referred to is got over by the small piece of iron on which the electro-magnet acts being placed in an enlargement above the glass capillary of the experimental tube; to this piece of iron by means of a long glass thread running down through the capillary the stirrer inside the observation-tube is attached. By this means the stirrer can be moved up and down by means of a small hand-electromagnet.

This arrangement involved the difficulty for our purpose, that the glasscapillary referred to has to be comparatively wide in order to leave room for the glassthread and that this is connected with greater danger of mutual diffusion between the different mixtures inside and outside the observation-tube and that greater uncertainties arise in the determinations of the densities. It also happened more than once that the stirrer got stuck in consequence of the flexibility of the glassthread and perhaps of microscopic deposits of solid substances. It goes without saying that the air had been freed as well as possible of water-vapour and carbon-dioxide: possibly the sticking of the glassthread indicates that traces of these substances had remained after all; in any case the disturbance usually became

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[1]) In the neighbourhood of the critical point, where the two phases have nearly the same composition, small expansions cannot be objected to.

[2]) C. A. CROMMELIN. Comm. 115 § 2. See also Comm. 83. Plate IV.

worse as more air was condensed and therefore had passed through the capillary. Usually the disturbance could be reduced or obviated entirely by operating the stirrer while the air was being compressed into the observation-tube.

The uncertainty in the computation of the density arises in the following manner: beside the observation-tube a long piece of the glass capillary is inside the cryostat, in our experiments about 35 cms with a volume of a quarter of the tube. Only a part of this length is immersed in the cold liquid; as previous experiments have shown, the temperature of the tube above the liquid increases pretty rapidly and near the top approaches the normal. A considerable correction has to be applied to the measurements for the gas inside this capillary. Assuming that the experiments are exclusively conducted with compression, the air which enters the capillary will retain its composition in the incompletely cooled part and will have to be taken into account as air. In the lower part on the other hand the air will separate into liquid and vapour and, if the liquid flows down properly, this part will finally contain saturated vapour. Still, owing to the circumstance, that thorough stirring can only take place in the observation-tube at the bottom, there is no guarantee, that the vapour in the capillary has the correct composition, while on the other hand the stirring in the capillary has the disadvantage of a partial mixing of the gases in the cold and the warmer parts of the capillary. As the extent to which these factors come into play is unknown, it is impossible to take them into account and the upper portion has to be taken as air, the lower portion as saturated vapour. In our case there was the additional difficulty, that the density of the saturated vapour was not yet accurately known, as it can only be found by interpolation from measurements of the vapour-density of a number of mixtures. In the mean time we had to be satisfied with an estimate. It may be added, that in a determination of a vapour-density, i.e. with only a trace of liquid in the tube, the entire cooled portion (observation-tube + part of capillary) has to be considered as containing saturated vapour; similarly in a measurement near the critical point, when on stirring the liquid surface flattens out and disappears, the same volume has to be assumed in the calculation as being filled homogeneously. The various uncertainties arising from the sources mentioned show themselves in small irregularities in the results which were obtained.

On the above grounds it is our intention in future experiments to return to the ordinary method of stirring notwithstanding the clumsy dimensions of the electro-magnet which it involves: the glass capillary can then be taken very narrow, so that the influence of the diffusion will be imperceptible and the determinations of the volume will obtain greater accuracy. But the rule, that in a set of readings, after the condensation has once begun, no expansion must be applied, will always remain valid.

Finally it may be remarked that, whereas air is, properly speaking, a ternary mixture of nitrogen, oxygen and argon, the amount of the latter gas is so small and its properties differ so little from those of the other two components, that a perceptible influence on the phenomena cannot be assumed and our mixture may thus be actually looked upon as being a binary mixture.

Apparatus. The apparatus which we have used agree in the main with former apparatus in use in the cryogenic laboratory: we may therefore refer to previous communications (for the piezometer compare Comm. 69). Between the observation-tube and the compression-apparatus a steel three-way stopcock was inserted by means of which the apparatus could be connected with a separate reservoir (pipette) filled with pure air: by this means, if required, measured quantities of air could be introduced into or removed from our piezometer; this arrangement was chiefly made with a view to density-determinations of liquid air at temperatures far below the critical, where the quantity present in the compression-tube would not have been sufficient. But as we were obliged to confine ourselves to experiments in liquid ethylene, there was after all no necessity for drawing on the pipette.

The pressures were measured on a closed hydrogen-manometer (Comm. 78), a metal gauge being used as a control during the measurements. Two platinum-thermometers (Comm. 141*a*) served for reading the temperatures. The cryostat was described in Comm. 83.

Critical point and critical phenomena. As shown by the theory of mixtures a distinction has to be made between two different critical points: the "plait-point", where the two coexisting phases become identical and where thus the critical phenomena will be most conspicuous and secondly, corresponding to a somewhat higher temperature, the "critical point of contact", i.e. the limit for the separation into two phases. In the temperature-range between those two points the condensation is "retrograde", in this case of the first kind. We have succeeded in confirming these several theoretical conclusions for air and have thus been able to show, that at these very low temperatures the phenomena are no other than what theory leads to expect. The experiments were far from easy, as the two points lie very close together, which is connected with the circumstance that the condensation-loop in the $p-T$ figure turned out to be comparatively narrow. The plait-point was found at

$$t = -140^\circ.73 \qquad p = 37.25 \ atm.$$

the critical point of contact at

$$t = -140^\circ.63 \qquad p = 37.17 \ atm.$$

The special kind of condensation which is characteristic of mixtures is thus confined to a range of .1°, and it is therefore necessary to make the compression proceed extremely slowly, if the observation is to succeed. High demands as regards constancy during a long period are thus made on the temperature and the success of our endeavours was no doubt due to the excellent appliances and arrangements which are available for this purpose in the cryogenic laboratory.

In order not to extend this paper unduly we shall not describe our observations in all detail. An exception may be made for a phenomenon which was observed at — 140°.64 and a pressure of 37.26 atmospheres and which shows very clearly the extreme sensitiveness of the substance in the critical region to changes of temperature and pressure. At the above pressure the surface between liquid and vapour was just no longer visible ; by lifting the stirrer — as in other corresponding states in this temperature-range — a mist was produced which, however, in this instance did not simply disappear, but automatically disappeared and reappeared again a few times in succession with a period of about one second. Evidently by the upward motion of the stirrer, the air under it becomes slightly expanded and thus cooled, while at the same time some gas is driven into the higher, warmer part of the capillary. The increase of pressure produced thereby assists in driving the gas back into the observation-tube, by which motion the mist becomes dissolved. The conditions must have been such this time, that the phenomenon repeated itself a few times in succession. The slow periodic time makes it probable, that a movement up and down of the mercury-surface in the wide compression-tube was at the bottom of the phenomenon. ¹)

It is worth mentioning, that more than once a blue opalescence

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¹) A vibration of the air in the small tube under the influence of its own elasticity would take place much faster and is, moreover, improbable, because the air is cooled during the compression and warmed during the expansion, whereas the condition for a vibration being sustained by supply and withdrawal of heat is exactly opposite. The closed reservoir at the end of the tube should be the warm part of it, as in the case of the well-known singing which sometimes takes place in blowing a bulb at the end of a tube.

was noticed, when the pressure was somewhat lower than that required to produce the cloudy condensation, and sometimes persisted during the stirring. By a rapid movement of the stirrer the opalescence could as a rule be changed into a mist. Pure oxygen and argon do not seem to show opalescence. [1])

Condensation-constants. Our results as regards temperature and pressure are given in Table I and in Figure I.

TABLE 1.

t	p_b	p_e
—140.63	37.17	37.17 P. of C.
—140.64	37.12	37.24
—140.69	37.02	37.26
—140.73		37.25 P. P.
—140.74	36.99	37.42
—140.75	36.86	
—140.80	36.65	37.20
—140.80	36.70	
—140.83^5	36.68	
—140.85	36.75	37.18
—140.89	36.57	
—140.99	36.35	
—141.35	35.24	36.49
—141.99	34.58	
—142.35	33.91	35.31
—143.14	32.06	
—143.34	31.85	33.79
—144.12	31.06	33.12
—144.35		32.52
—146.32		29.83
—150.12	23.68	25.04

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[1]) See Comm. 145*b*.

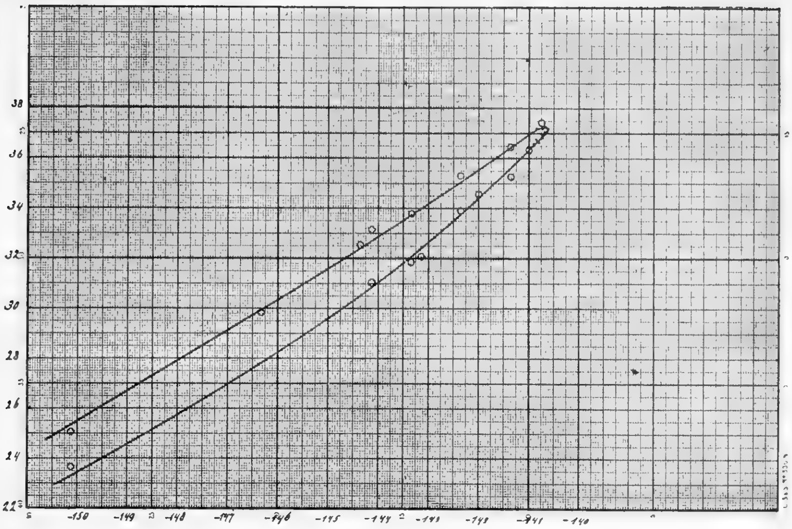

Fig. 1.

TABLE 2.

Densities.

Temperature	Vapour	Liquid
—140.63	0.31	0.31 P. of C.
—140.69		0.323
—140.70		0.328
—140.73		0.35 P. P.
—140.75	0.277	
—140.80	0.265	0.365
—140.84	0.269	
—140.85		0.359
—140.89	0.262	
—140.99	0.253	
—141.34		0.439
—141.99	0.217	
—142.35		0.461
—143.34	0.188	
—143.35		0.488
—144.35		0.503
—146.32		0.523

The points of beginning and of completed condensation which are contained in the Table were all determined with compression, as explained before; for the sake of accuracy the change of volume was conducted as slowly as possible. All the same the results show very distinct irregularities, which must find their explanation in the various grounds mentioned.

The densities expecially must be accepted with some reserve: they are given in Table 2 (p. 1095) and Figure 2. The densities at the plaitpoint and the point of contact were read from the figure.

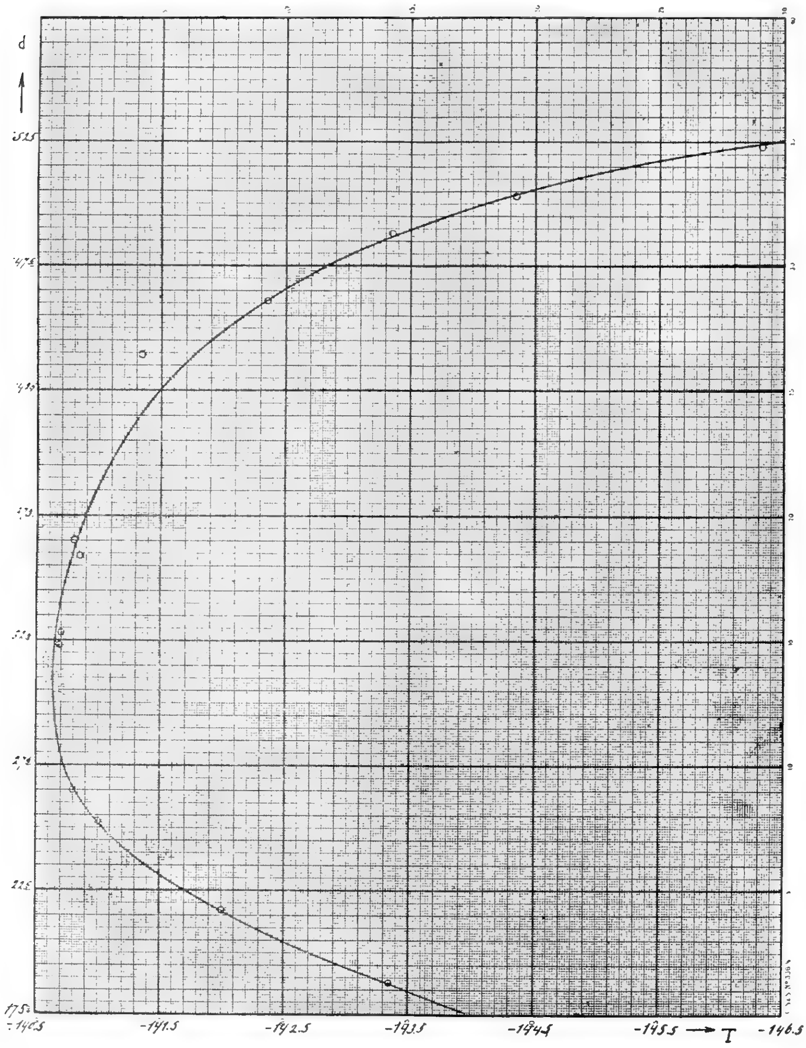

Fig. 2.

In comparing the plait-point of air with the critical points of its constituents (Comm. 145*b*), viz. —118°.82, 49.73 atm. for oxygen and —147°.13, 33.49 atm. for nitrogen, it appears, that the critical curve for the oxygen-nitrogen mixtures differs very little from a straight line. On a straight line a temperature of —140°.73 would correspond to a pressure of 37.16 atm. whereas the plait-point pressure is 37.25 atm. The critical point appears to change approximately proportionally to the composition in weight: t_k calculated according to the composition would be —140°.56. PAWLEWSKI's rule thus holds approximately for mixtures of oxygen and nitrogen.

In connection with the very small distance between plait-point and point of contact, already referred to, a further conclusion may be drawn from this result. The latter fact involves that the critical temperature which air would have, if it condensed without change of composition, deviates very little from the two other critical points and that this temperature — the so-called critical temperature of the undivided mixtures — thus also changes proportionally to the composition, with a certain degree of approximation. If this proportionality held for the molar composition, we might, as the theory of mixtures shows, infer from this, that at low temperatures the vapour-pressure of these mixtures would change about linearly with the composition of the liquid [1]). Evidently with mixtures of oxygen and nitrogen this is not the case and in agreement with this BALY's experiments [2]) gave a vapour-pressure line which is very distinctly curved.

The critical density of air calculated from those of oxygen and nitrogen, 0.43 and 0.31 respectively, according to the simple rule of mixtures by weight, is 0.34, a value which as the table shows is intermediate between the density at the critical point of contact 0.31 and that of the plait-point 0.36. Assuming that the rule of mixtures would hold approximately for the mixtures of constant composition and that therefore the critical density of undivided air would be 0.34, a somewhat higher density might actually be expected at the plait-point. For the connodal curve lies entirely outside the saturation-curve for the mixtures of constant composition and on that curve the plaitpoint lies in this case on the side towards the smaller volumes or higher densities. The somewhat smaller density at the critical point of contact is also as might be expected.

To Professor H. KAMERLINGH ONNES our sincere thanks are due for his continual interest in our research, many helpful suggestions

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[1]) Comp. e.g. J. P. KUENEN, Handb. ang. Phys. Ch. IV p. 126 1906.

[2]) E. C. C. BALY, Phil. Mag. (5) 49 p. 517 1900.

and for the way in which he enabled us to complete our investigation in the short time at our disposal.

We also wish to thank Mr. J. M. Burgers for looking after the temperature readings, Mr. A. T. van Urk for assistance during the whole work and finally him and Mr. Ch. Nicaise for help in the calculations.

Chemistry. — *"On the distinction between methylated nitro-anilines and their nitrosamines by means of refractometric determinations"*. (II). By Dr. J. D. Jansen (Communciated by Prof. Ernst Cohen).

(Communicated in the meeting of February 24, 1917).

In a former communication[1]) I called attention to the difference in optical properties between coloured nitro-compounds, as nitro-anilines and nearly colourless ones as dinitro-benzenes. In the first mentioned substances, the molecular-refractions of the isomerides showed great differences, whereas the molecular-refractions of the isomerides of the colourless compounds were nearly the same.

This phenomenon proved to be closely related to the light absorption. The first group of compounds (the coloured) showed absorption-bands in the neighbourhood of the kind of light chosen for the determination of the refraction. The refractive-indices and in connection therewith, the molecular-refractions of those substances were raised. This rise is not the same for the different nitro-compounds, because it depends on the place and the depth of the absorption-bands (anomalous dispersion). The colourless substances, the absorption-bands of which are situated far outside the visible part of the spectrum, showed no rise at all of the molecular-refractions or only a very slight one.

By stating this fact we directed the attention to the molecular-refractions of the 2.5- and 2.3-dinitro-dimethyl-p-toluidines and the respective nitrosamines.

	Spec. Refr.	M.-R.	Δ
2.5-dinitro-dimethyl-p-toluidine	0,2730	61,4	1,8
2.3- ,, ,, ,, ,,	0,2649	59,6	
2.5-dinitro-tolyl-methyl-nitrosamine	0,2391	57,4	0,5
2.3- ,, ,, ,, ,,	0,2370	56,9	

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[1]) Proc. Roy. Akad. Amst. 564 (1916).

The difference in molecular-refraction of the two coloured dinitro-dimethyl-p-toluidines nearly disappeared after substitution of a NO-group for a CH_3-group in consequence thereof almost colourless nitrosamines are formed.

Moreover the molecular-refractions of the nitro-toluidines prove to be considerably raised. That this is really the case, becomes more evident, if we pay attention to the fact that the molecular-refraction is obtained by multiplication of the specific refraction with the molecular weight, the latter being greater with the nitrosamines than with the dimethyl-compounds.

Though the difference in molecular-refractions of nitro-toluidines lies far above the experimental errors, it was not exceedingly high. Therefore I thought it of some interest to determine the molecular-refractions of nitrosamines, derived from isomeric nitro-compounds which showed a great difference in molecular-refraction.

These nitro-compounds were chosen out of some dinitro-dimethyl-(diethyl) anilines mentioned already in a former communication.

	Spec. Refr.	M.-R.	△
3.4-dinitro-dimethyl-aniline	0,2975	62,8	6
3.6- „ „ „	0,2693	56,8	
3.4-dinitro-diethyl-aniline	0,3060	73,1	7,8
3.6- „ „ „	0,2730	65,3	

The refractometric values of the respective nitrosamines appear to be as follows:

	Spec. Refr.	M.-R.	△
3.4-dinitro-phenyl-methyl-nitrosamine[1])	0,2444	55,2	2,4
3.6- „ „ „ „ [1])	0,2337	52,8	
3.4-dinitro-phenyl-ethyl-nitrosamine	0,2483	59,6	2,4
3.6- „ „ „ „	0,2380	57,2	

The difference in molecular-refractions of the isomeric nitrosamines is, just as in the above mentioned case, much smaller than that of the dimethyl-compounds. The molecular-refractions of the dinitro-dimethyl-anilines are much more raised than those of the dinitro-dimethyl-toluidines. Here, too, this rise is prominent with the specific refractions. The specific refractions of the nitrosamines are between 0,2483 and 0,2337 and those of the dinitro-dialkyl-anilines between 0.3060 and 0.2693. In relation with these high values of the specific refractions of nitrated dialkyl-anilines and the low, little differing values of the nitrosamines, the refractometer (according to my opinion)

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[1]) The description of these compounds, and of some others will soon appear.

may be used as a valuable instrument (not yet put into practice) to determine if we have to do with alkylated-nitro-anilines (toluidines) or with their nitrosamines. Just as in the case of the 3.4-dinitrophenyl-methyl-nitrosamine — a compound of whose characteristic behaviour we hope soon to give a description — the refractometric determination furnishes valuable data, because this compound has a yellow colour — nitrosamines are nearly colourless — and is moreover not easily to be identified as a nitrosamine by chemical researches.

Utrecht. *Org. Chem. Univ. Lab.*

Mathematics. — "*Some Considerations on Complete Transmutation.*" (Second Communication.) By Dr. H. B. A. Bockwinkel. (Communicated by Prof. L. E. J. Brouwer.

(Communicated in the meeting of October 28, 1916.)

7. If we say that a transmutation of the form (1)

$$Tu = a_0(x)\, u + \frac{a_1(x)}{m!} u' + \ldots + \frac{a_m(x)}{m!} u^{(m)} + \ldots, \quad . \quad . \quad . \quad (1)$$

is complete in a certain circular domain (α) with centre x_0, it does not imply that this domain (α) *belongs* to the given transmutation as an *invariable* field; indeed, if T is complete in (α), it is certainly also complete in a domain $(\alpha') < \alpha$, and probably in a domain $(\alpha'') > (\alpha)$. Only *the aggregate of functions* which have in such a domain a transmuted determined by (1), is different for any new domain, such that the aggregate diminishes if the domain increases. If we want to make this more prominent, we shall say more fully: "the transmutation is complete in the domain (α) *with corresponding domain* (β), or something of the kind. As β further increases and decreases monotonely with α, it is clear that functions with *exactly* the radius of convergence β have a transmuted determined by (1) in any domain $(\alpha') < (\alpha)$; the series converges for this *uniformly* in the domain (α') (final paragraph N°. 4).

Special attention should be paid to the fact that we call a transmutation only then "complete in a domain (α)" if it *is in that domain determined by a series of the form* (1) for *all* functions belonging to a certain circle (ϱ). If this is not the case, the transmutation, which is then of course *defined* in another way than by the series (1), may produce a transmuted in the whole domain (α) for all functions belonging to (β), *but this does not in itself give a reason to call it complete in* (α). In order to make the difference clearer we give a few examples which, at the same time, are proper to dissolve another misunderstanding that may have arisen after the reading of the consideration in N°. 5 on exceptional cases. These exceptional cases consist in the fact that the series (1) for some functions not belonging to (β) produces a transmuted in the whole domain (α), or, as we may say as well, that that series produces for some functions, which have exactly (β) as their circle of convergence, a transmuted in a domain greater than (α). It occurs however frequently that the transmuted of a function u with a radius of convergence β *exists* anywhere within a certain circle greater than (α), without being determined there by a series of the form (1).

Consider the transmutation

$$Tu = \sum_{0}^{\infty}{}_{m} \frac{(-1)^m x^{m+1}}{(m+1)!} u^{(m)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

For domains (α) with the origin as centre we have evidently $a = \alpha$, hence,

$$\beta = 2\alpha , \qquad \text{or} \qquad \alpha = \tfrac{1}{2}\beta .$$

Yet the radius of convergence of $v = Tu$ is not half of that of u but exactly equal to it, which becomes clear it is known that the transmutation (12) represents the operation D^{-1}, the point $x = 0$ being taken as the lower limit of integration. The operation D^{-1} is therefore a transmutation of such a kind that it produces a transmuted in the whole domain (α) for any function belonging to (α), but it is in that domain determined by a series of the form (1) only in so far as regards functions belonging to the circle $(\beta) = (2\alpha)$. The fact is that, with regard to the last mentioned functions, the circle $(\frac{1}{2}\beta)$ for the series (12), which is not a *power series* in the letter x, forms the greatest *circular* domain of convergence, but not for the *power series* into which (12) may be transformed; this, however, is not at all unusual. From the transmutation

$$Tu = \sum_{0}^{\infty}{}_{m} \frac{x^{m+1}}{(m+1)!} u^{(m)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which is obtained from (12) by cancelling the alternation of signs, it is at once to be seen that it cannot produce this phenomenon so generally; it is clear that, in this case, for those functions u, which are identical with their natural majorants 1) it holds that $r' = \frac{1}{2} r$ and r are the corresponding radii of convergence of $v = Tu$ and u. If we further take into account that for the transmutation (13) we have

$$Tu = 2D^{-1}\, u\,(2x) - D^{-1}\, u\,(x)\,,$$

it is to be seen that the same may be said of *all* functions u.

The operation

$$Tu = \sum_0^\infty {}_m \frac{x^{2m}}{(m+1)!}\, u^{(m)}, \quad \ldots\ldots \quad (14)$$

in which $a\,(\alpha) = \alpha^2$, and therefore

$$\beta = \alpha + \alpha^2, \qquad \text{or} \qquad \alpha = \tfrac{1}{2}\,(\sqrt{4\,\beta + 1} - 1),$$

gives for *majorant* functions transmuted for which the corresponding radii of convergence r' and r depend on each other as α and β, but for the function

$$u = \frac{1}{1+x}\,,$$

for which $r = 1$, a transmuted

$$v \equiv Tu = \frac{1}{1+x} - \frac{x^2}{(1+x)^2} + \frac{x^4}{(1+x)^3} - \ldots\ldots \quad (14')$$

or

$$v = \frac{1}{1+x+x^2}\,,$$

which also has $r' = 1$ as its radius of convergence, while, however, the series (14') only converges within a circle with radius

$$\alpha = \tfrac{1}{2}\,(\sqrt{4 \times 1 + 1} - 1) = \tfrac{1}{2}\,(\sqrt{5} - 1) = 0{,}6\ldots$$

Here again the example becomes clearer by the *meaning* of the transmutation, which in this case appears to be a *substitution,* viz.

$$T(u) = S_{x+x^2}(u)\,. \quad \ldots\ldots\ldots \quad (15)$$

If in general

$$T = S_{\omega(x)}\,,$$

we find in the memoirs of BOURLET and PINCHERLE

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1) J. TANNERY, Introduction à la théorie des fonctions II, N^0. 343, understands by the natural majorant of another function, the one that is deduced from the first by replacing the coefficients in the development of its power series, *in the neighbourhood of the origin,* by their modulus. We shall, if there is question of the neighbourhood of an *arbitrary* point x_0, understand by the natural majorant of u the function obtained from u by replacing the coefficients *for the point* x_0 by their modulus.

$$S_{\omega(x)}(u) = \sum_{0}^{\infty}{}_{m} \frac{(\omega(x)-x)^m}{m!} u^{(m)}(x),$$

a formula, whose correctness ensues at once from the ordinary theorem of TAYLOR for the theory of functions [1]). Evidently we have here $a(\alpha) =$ the maximum modulus of $\omega(x) - x$ on the circumference of α. If for the neighbourhood of $x = 0$, $\omega(x) - x$ is a majorant function we have

$$a(\alpha) = \omega(\alpha) - \alpha$$

consequently

$$\beta = \omega(\alpha).$$

All this holds *independently of* the function to which S is to be applied. But the question, what is going to be the new domain of convergence does depend on it. For to that purpose the equation

$$\omega(x) = re^{i\theta}$$

has to be solved for all values of θ that are arguments of singular points of u on the circumference of (r) and to choose that solution for which $x - x_0$ has the smallest modulus ξ; any circular domain round x_0 smaller than (ξ) in so far as it does not contain a singular point of the substitution function $\omega(x)$ itself, is of such a nature that S *exists* in the whole of that domain; this domain therefore varies with the function considered. Thus for the transmutation (15), applied to the functions

$$u = \frac{1}{1-x}, \text{ and } u = \frac{1}{1+x}$$

we have to solve resp. the equations

$$x^2 + x = 1, \text{ and } x^2 + x = -1$$

the former gives as radius of convergence of $v = Tu$ the number $\frac{1}{2}(\sqrt{5} - 1)$, in agreement with the fact that this u is a majorant function, the other the number 1, equal to the radius of convergence of the corresponding u itself.

It may also occur that the transmuted of a function is not determined in any neighbourhood of x_0 however small by a series of the form (1), though it *exists* in all the points of such a neighbourhood. This happens when the circle of convergence of the function is smaller than the *minimum* circle (b), which still has the property that the series (1) produces a transmuted function in the point x_0 for all functions belonging to it. Take in the neighbourhood of $x = 0$

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[1]) Cf. No. 17.

$$Tu = S_{x+1}(u).$$

Here $a_x = \omega(x) - x = 1$, consequently $a = 1$. For the function

$$u = \frac{1}{1+x}, \text{ we have } Tu = \frac{1}{2+x};$$

the resulting function exists in every domain round $x = 0$ with a radius smaller than 2, while it is not determined in any small neighbourhood whatever by a series of the form (1), which is here

$$-\sum_0^\infty {}_m \left(\frac{-1}{1+x}\right)^{m+1}.$$

That it is quite common if the transmuted function does *exist* in some domains, but is not represented in it by a series of the form (1), appears moreover quite clearly as follows: For a definite, given function the domain may be constructed once for all, where the series (1) produces *anywhere* a transmuted for that function as well as the domain where it produces *nowhere* a transmuted. Consider for this purpose all singular points s of the function and construct to each the set of points with the property that the value which the magnitude a_x has there is *smaller* than their distance to s, and also the set of points where that value is *greater* than the distance in question. The latter set of points shall as a rule consist of continua round the singular points; in none of those continua the series will converge, but it is clear that in general the transmuted of the function considered may be continued within those continua, except in a few points which *may* but *need* not coincide with the singular points. Thus in the transmatution (12), which answers to D^{-1}, and where $a_x = |x|$, the domain of points where a_x is greater than the distance to the point $x = 1$, is the continuum on the right of the straight line $x = \frac{1}{2}$, and the series (12) does not converge in any point of that half plane for functions which only have the singular point $x = 1$. The operation D^{-1}, however, which is represented by the series in the half-plane on the left of that line, exists moreover anywhere in the first mentioned continuum, except in a line starting from the singular point. In the example of the substitution S_{1+x} which we gave last, we have $a_x = 1$; hence a function with the singular point $x = -1$ has not in any point of a circular domain with that point as centre and 1 as radius, a transmuted *determined by* (1). Nevertheless it has a transmuted function, except in the point $x = -2$ of the circle mentioned (here the singular point has been displaced by the transmutation). For the substitution $S_{\omega(x)}$ in general we have for the point $x = 0$, $a_x = |\omega(0)|$. If therefore the radius of convergence of u is smaller than $|\omega(0)|$, the substitution is not in any arbitrarily small neighbourhood of O determined by a series of the form (1) [1]. The substitution S_{x+x^2}, however, treated above, for which $\omega(0)=0$, is expressed by such a series for any function u, with O as ordinary point in a certain domain round O. For functions with the singular point $x=+1$, the domain of the series (1) belonging to S_{x+x^2} consists of the continuum within an oval which is symmetrical with regard to the real axis, and cuts it in the points

$$x=-\tfrac{1}{2}(\sqrt{5}+1) \text{ and } x=\tfrac{1}{2}(\sqrt{5}-1)\,.\quad .\quad .\quad .\qquad (16)$$

This oval is obtained as the locus of the points where a_x or $|x^2|$ is equal to $|x-1|$. The circle with radius

$$\tfrac{1}{2}(\sqrt{5}-1)=0{,}6\ldots$$

is therefore the greatest circular domain with O as centre where the series in question converges. The singular point $x=1$ is moved here to the two points (16), from which it follows that also for S_{x+x^2} *itself*, the greatest circular domain of operation around O is the one with the radius $\frac{1}{2}(\sqrt{5}-1)$, in agreement with what has been said above with regard to the function $\dfrac{1}{1-x}$. If, however, the function u has the singular point $x=-1$, the domain of the series (1) is the continuum within an oval into which passes the first mentioned when it rotates 180° round the imaginary axis, so that *the same* circle as considered just now, with radius $\frac{1}{2}(\sqrt{5}-1)$ forms the *circular* domain of that series round O as centre. But the singular point $x=-1$ is moved to the intersections of the oval with the circle with radius 1, so that now the circular domain of operation of S_{x+x^2} round O, is that same circle; we observed this above with regard to the function $\dfrac{1}{1+x}$. In general the point or the points towards which a singular point s of a function u is transposed, always lies, for the operation of substitution, on the circumference of the domain where the corresponding series converges. For, that circumference is determined by the equation

$$|\omega(x)-x|=|s-x|,$$

which is among others satisfied by the points for which

$$\omega(x)=s$$

which exactly form the removed singular points. If x_0 lies in the domain where $|\omega(x)-x|<|s-x|$, the circular domain of operation, with centre x_0, of the series of the form (1) corresponding to $S_{\omega(x)}$, has a radius equal to the minimum distance of that point to the circumference; hence, if one of the removed singular points lies at

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[1] Cf. further treatment of the substitution in N⁰. 17.

that minimum distance from x_0, the circular domain of $S_{\omega(x)}$ will be exactly the same as that of the corresponding series, at least in so far as regards functions with no other singular points than s.

If we consider a group of functions belonging to the circle (r) with x_0 as centre, each point of this circle must in turn be considered as a singular point, and, if x_0 always lies in the domain in which $a_x < |s-x|$, the minimum distance r' from x_0 to the aggregate of all the circumferences as mentioned above has to be determined; (r') is then the circular domain of operation of the *series* corresponding to T for the functions in question, while the circular domain of operation round x_0 for T *itself* is in general greater, having a radius equal to the minimum distance of x_0 to the (possibly moved) singular points.

8. In the considerations of the preceding number it was supposed that the transmutation T had been *defined* in another way than by the series (1). It may, however, occur that we *start* from a series (1) as the definition of a transmutation. The latter then gives for the moment, for functions with a circle of convergence (β), a transmuted function only in the interior of the corresponding domain (α). But the analytical continuation of the function $v = Tu$, now furnishes so to say at the same time the *analytical continuation of the transmutation*, so that the latter is also determined *outside* the domain (α). The continuation thus considered however has to be repeated for each new function, while it would be desirable to have an *analytical expression*, which represents the results of the operation, at least for a complete sub-group of the functions considered, in a domain outside (α).

In order to carry this out, a point x_1 inside the circle with centre x_0 may be taken as a new centre and to that purpose a correspondence between quantities α' and β' as was explained in N°. 6, may be established; the functions $a_m(x)$, from the *given* series supply the means for it. Not for all the functions, however, belonging to (β) the series will converge in the new domain (α'), so that this proceeding, in anticipation of an expression that we shall introduce in the next number, will always furnish an extension of the *numerical* field at the cost of the *functional* one. We observed the same in considering concentric fields: if α increases β does, i.e. the functional group shrinks.

The consideration in the preceding number teaches, however, that in this way, even if only functions with one and the same singular point are taken in view, a whole field of points cannot be reached where the transmuted of those functions really *exist*.

This domain is, generally speaking, the same for all functions with the same singular points; exceptions apply to cases for which BOURLET's theorem, docs not hold as was indicated at the end of N°. 3. If, however, the result $v = Tu$ has been continued analytically for *one* function u of the group, and if the same has been done with the results of certain operations that may be called the *derived operations* of T, it is possible to indicate for all the functions having the *same kind of singularity* as u, a series containing the results mentioned as coefficients, and converging in parts of the excluded domain. If conceived in an opposite direction, this process produces an *extension* of the *functional* field with *conservation* of the *numerical one*; it forms the proper analogon for the functional calculus of the "prolongement analytique" in the ordinary theory of functions. (See further N°. 20).

We observe, that a *regular transmutation* may only be continued in *one* way, as this is the case with a *regular function.*

9. We will now discuss the question in how far the complete transmutation is *continuous.*

BOURLET has called a transmutation T continuous "si la limite de la transmuée d'une fonction est la transmuée de la limite de cette fonction" He explains this further by adding: "En d'autres termes si une fonction $u(x, h)$, dépendant d'un paramètre h, tend vers une certaine limite, lorsque h tend vers une certaine valeur, $Tu(x, h)$ a aussi une limite, et l'on a

$$lim\ [Tu(x, h)] = T\,[lim\ u(x, h)]".$$

In this it is implied that all the functions to be considered may be obtained by attributing a definite value to the parameter h in a certain expression $u(x, h)$, this value belonging to a certain complex domain, in which h varies. It is however useful in connection with the character of the complete transmutation to make a somewhat more general supposition about the group of functions to which the operation is to be applied, more in accordance with (though not exactly equal to) the one to be found in the paper by M. FRÉCHET "Sur quelques points du calcul fonctionnel" (Rendic. d. Circ. Mat d. Palermo, 22 (1906) p. 45 (N°. 70).

Before, however, proceeding to a more precise statement of the nature of the group of functions in question we will make a few general observations, in which we have not specially in view the complete transmutation, but an arbitrary additive operation.

That it is necessary to indicate very exactly the group of functions over which the operation is to be extended, is not mentioned by

Bourlet, but Pincherle continually draws attention to it, though the *notion of continuity* is not expressly treated by him. The latter speaks of a "champ fonctionnel"; we will also use this term and indicate the group of functions to which the operation is to be applied — *the domain of the independent variable* (*function*), as we might call it — by the name of *functional* field, abbreviated to F.F. By way of contrast we will sometimes call a domain of the independent variable *number x* a *numerical field*, abbreviated N.F. Two such fields, an N.F. and an F.F. in which the transmutation has been defined, we call *associated*. Now the F.F. will naturally always be of such a kind that for the functions belonging to it a *common* domain of the independent variable number x is to be indicated in which they are holomorphic; it will even often occur that their being determined in a common domain serves as defining predicate of the group of functions considered. But the *transmutation* need not be defined in this whole domain, as appeared already in the preceding pages for the complete transmutation. We have therefore to distinguish between the domain of x-values in which the *functions* are determined, and that in which the *operation* is defined. The latter we shall call: the *numerical field of the operation* (transmutation); abbreviated N.F.O.; the former we call the *numerical field* of the *function*, abbreviated N.F.F. In most cases first the numerical field of operation will be fixed, in which the results of the transmutation are to be considered, and then an associated *functional field* of functions having a transmuted function in the first mentioned field.

As appears from the paper by Fréchet it is sufficient in order to arrive at the notion of continuity of an operation, that in the class of the elements to which it is to be applied, the notion "écart" of two elements may have been defined; we will use the word *distance* for it. We agree that we shall take into consideration only such domains as a N.F.F. as lie entirely in finite space. By this agreement the cases in which *all the* functions of the F.F. have an infinite circle of convergence are not excluded from the considerations; the agreement only means that in such a case an N.F.F. must be *fixed* that lies entirely in finite space, and to which our statements will refer. We further suppose that each function separately is *limited* in the N.F.F. This contrasts with what is found in the memoir of Fréchet, who considers as F.F. an aggregate of functions which are holomorphic within the same surface, but not limited (l.c. N°. 70); it is, however, in agreement with our preceding considerations, in which we did not in general give statements about functions that have the same *circle of convergence,* but about such as *belong* to the same circle. Moreover we need not use now the artificial definition of FRÉCHET for the distance of two functions; we shall simply understand by it the *maximum* modulus of their difference in the N.F.F. In the same way we mean by the distance of the *results* which the operation produces for two functions the *maximum* modulus of the difference of those results in the N.F.O.

If we understand by a *point* of the F.F., a *function* $u_1(x)$ with which the independent variable function $u(x)$ may be identified we can say that the notion of continuity only appears in *limiting* points, i.e. in "points" $u_1(x)$, which have the property that at an arbitrarily small distance of them — or in an arbitrarily small vicinity as we might say — still "points" $u(x)$ of the F.F. are found. As a rule all "points" of the F.F. have that property, in other words, there are no *isolated* "points" in the F.F. or: the F.F. is *dense in itself.*

According to the classic definition of continuity an operation will be continuous, if, generally speaking, the distance between the *results* which it produces for two functions of the F.F., becomes *small with* the distance of these *functions themselves.* More amply and exactly our definitions of continuity read as follows:

1. A transmutation T determined in a pair of associated fields is called continuous in a "point" $u_1(x)$ of the F.F. if there is corresponding to each arbitrarily given number τ, however small, a number σ, such that for all the values of x in the N.F.O.

$$| T u(x) - T u_1(x) | < \tau$$

under the single condition that

$$| u(x) - u_1(x) | < \sigma$$

for *all* the values of x in the N.F.F.

2. A transmutation T, defined as before, is called *continuous in the* F.F., if, according to the preceding definition, it is continuous in *any* point of the F.F.

3. A transmutation T, defined as before, is called *uniformly continuous in the* F.F., if corresponding to any arbitrarily given amount τ, however small, there is an amount σ such that for *any* "pair of points" $u_1(x)$ and $u_2(x)$ of the F.F. and for all x-values of the N.F.O.

$$| T u_1(x) - T u_2(x) | < \tau$$

under the single condition that

$$| u_1(x) - u_2(x) | < \sigma$$

for *all* the values of x in the N.F.F.

If in statements about the continuity of a transmutation we want to draw at the same time our attention to its *numerical* field, we shall say for the sake of completeness: "continuous *for,* or *with regard to* or even *in* the N.F.O." In using the preposition "in" it should be borne in mind that there is no question about a comparison of the results of a transmutation, for *one and the same function,* in a point of the N.F.O. and a *numerical neighbourhood* of that point; the notion of continuity of a transmutation merely refers to a comparison between the results for *one and the same* point of the N.F.O. in a "point" of the F.F. and a *functional* neighbourhood of that "point".

The form given here for the definition of the continuity of a *transmutation* corresponds to that of CAUCHY for the continuity of *functions.* Equivalent to it is the following form corresponding to that of HEINE for the continuity of functions:

A transmutation is *continuous* if the result $v(x) = Tu(x)$ in the N.F.O. *approaches* to $v_0(x) = Tu_0(x)$, if the function $u_0(x)$ in the N.F.F. *approaches* to $u_0(x)$. It is meant by it that if $u(x)$ *as the variable function* is identified successively with the funcions of a convergent sequence

$$u_1(x), u_2(x), \ldots \quad u_n(x), \ldots,$$

which all belong to the F.F. and in the N.F.F. *uniformly* approach to $u_0(x)$, the transmuted $v(x) = Tu(x)$ is identified successively with the functions of a sequence

$$v_1(x), v_2(x), \ldots \quad v_n(x) \ldots.$$

which converges in the N.F.O. *uniformly* towards $v_0(x)$, and that this happens for *any* suchlike fundamental series of functions $u_n(x)$.

10. It is of importance to observe now that the difference between the three cases of continuity mentioned in the preceding section is superfluous for *additive* operations, since the following proposition holds, which shows analogy with the well-known theorem of HEINE from the theory of functions:

An additive operation, which is continuous in only one point $u_1(x)$ of the functional field, is uniformly continuous in that field.

Let M_u be the upper limit of the modulus of the function u in the N.F.F. It is to be derived then from the hypothesis that, corresponding to an arbitrarily given amount τ, there is an amount δ, such, that in an *arbitrary* point x of the N.F.O.

$$| Tu(x) - Tu_1(x) | < \tau, \text{ if only } M_{u-u_1} < \delta.$$

Suppose

$$u(x) = u_1(x) + \delta(x)$$

then we have, from the additive property of T

$$|T\,\delta(x)| < \tau, \text{ if } M_{\partial(x)} < \delta,$$

and this last result in which the initial function $u_1(x)$ occurs no more, says: Corresponding to any arbitrarily little amount τ there is an amount δ, such that the absolute value of the *transmuted* of a function, in the arbitrary point x of the N.F.O. is smaller than τ if the absolute value of that *function itself* is anywhere in the N.F.F. smaller than δ, provided that function belongs to the F.F. considered. Hence: if the distance between two functions u_1 and u_2 of the F.F. is smaller than δ, we have in the whole N.F.O.

$$|T(u_1(x) - u_2(x))| < \tau$$

or, according to the additive property

$$|T\,u_1(x) - T\,u_2(x)| < \tau.$$

The proposition has thus been established.

The expressions "continuous in a point of the F.F."; "continuous in the F.F."; „uniformly continuous in the F.F." can consequently be substituted for each other; as a rule we shall make use of the middle one. Or, if we want to direct our attention at the same time ot the *numerical* field, we shall use an expression like the following: *The transmutation is continuous in the pair of fields considered.*

11. The mode of reasoning followed in the preceding proof suggests the observation that the discussions about the continuity of an *additive* operation may be simplified by using the following proposition:

If an additive operation is to be continuous, it is necessary and sufficient that there is corresponding to any arbitrarily chosen number τ a number δ, such that in the whole N.F.O.

$$|T\,u(x)| \leqq \tau,$$

under the single condition that

$$M_u \leqq \delta.$$

Here M_u denotes again the maximum modulus of u in the N.F.F.

The condition contained in the proposition is sufficient. For, if it is fulfilled, and if the *distance* between two functions u_1 and u_2 of the F.F. is at most equal to δ, the transmuted of their difference will, in absolute value, in the whole N.F.O. be at most equal to τ and the same will therefore, according to the additive property of T, be the case for the absolute difference of the two transmuted.

The condition is necessary. If it is not fulfilled, this means that there exists a positive number τ such that there is corresponding to any arbitrarily given number δ, a function $\delta(x)$ in the F.F. for which *at the same time*

$$M_\delta \leq \delta \quad \text{and} \quad | T\delta(x_1) | \geq \tau,$$

the latter for some point x_1 of the N.F.O.

The operation T can never be continuous now; from whatever initial function $u_1(x)$ we may start, corresponding to any arbitrarily given number δ there is a function $u(x)$ such that *at the same time*

$$M_{u-u_1} \leqq \delta \qquad \text{and} \qquad | Tu(x_1) - Tu_1(x_1) | \geqq \tau.$$

In order to see this we need only choose for u the function $u_1(x) + \delta(x)$ and apply the additive property of T.

We may also express the proposition of this number in connection with the *second* definition of continuity as follows: *If an additive transmutation is to be continuous, it is necessary and sufficient that the function* $v(x) = Tu(x)$ *converges in the N.F.O. to zero, if this is the case with* $u(x)$ *in the N.F.F.*

After these general considerations we return to the *complete* transmutation.

12. We called a transmutation *complete in a circular domain* (α) of centre x_0, if there exists a circle (ϱ), concentric with (α) and therefore also a minimum circle (β) determined by the formula (7), such that the series (1) by which the transmutation is determined, produces for all functions belonging to that circle a transmuted function in the *whole* domain (α). Thus, a transmutation, of which we say that it is *complete in a domain* (α) is in consequence of this without more determined in a *numerical field of operation,* viz. (α), while we may take as associated *functional field*: any group of functions belonging to a domain (ϱ) as meant just now, not smaller than (β). The numerical field of the *functions* is in this case the circle (ϱ); the upper limit in this field of the modulus of a function u of the F.F., which we have represented by M_u in the general considerations, is now equal to the maximum modulus $M(\varrho)$ of u on the *circumference* of (ϱ).

For the complete transmutation the following proposition of continuity holds:

A transmutation T, which is complete in a numerical field of operation (α), is continuous in any F.F. formed by the functions belonging to a circle (ϱ) greater than the domain (β) corresponding to (α).

We suppose $\varrho = \beta + \sigma$, $(\varrho < 0)$. In the second part of the proof of the proposition in N°. 4 it was found that in an arbitrary point x of the domain (α) the remainder $R_{m_\varepsilon}(x)$ of the series (1), after m_ε terms, satisfies the condition

$$|R_{m_\varepsilon}(x)| < \frac{\varrho M(\varrho)}{\delta - \varepsilon} \cdot \left(\frac{\beta - \alpha + \varepsilon}{\beta - \alpha + \delta}\right)^{m_\varepsilon}.$$

For the meaning of the letters we refer to No. 4. We can also write for this

$$|R_{m_\varepsilon}(x)| < D \times M(\varrho)$$

in which

$$D = \frac{\varrho}{\delta - \varepsilon} \cdot \left(\frac{\beta - \alpha + \varepsilon}{\beta - \alpha + \delta}\right)^{m_\varepsilon}$$

is an amount not depending on the chosen function u nor on the special point x of the numerical field of operation; as to the latter we remind of the uniformity supposition of No. 4, according to which corresponding to the arbitrarily given number ε, an integer m_ε may be chosen, which is *the same* for all points x of the N.F.O. Let further G be an integer below which all quantities $a_m(x)$ with index m smaller than m_ε remain in absolute value, then we have for each of those m-values:

$$\left|\frac{a_m u^{(m)}}{m!}\right| < G \times \frac{\varrho M(\varrho)}{(\varrho - \alpha)^{m+1}},$$

and consequently

$$\left|\sum_{0}^{m_\varepsilon - 1}{}_{m} \frac{a_m u^{(m)}}{m!}\right| < E \times M(\varrho),$$

in which E is again a number that does not depend on the function chosen nor on x. We finally have in all points x of the N.F.O.

$$|T u(x)| < (D + E)\, M(\varrho).$$

From this it follows that a number δ, corresponding independently of x to a *given arbitrarily small* number τ — of which mention is made in the proposition of continuity of No. 11 — can really be indicated; apparently we may write for it,

$$\delta = \frac{\tau}{D + E}.$$

The proposition has thus been proved.

13. We observe that in the foregoing we have not proved the continuity of the operation in the F.F. formed by all the functions belonging to (β), but only this continuity with regard to the F.F. of functions belonging to a somewhat *larger* circle. But then for the first mentioned F.F. the proposition does not generally hold. Let us consider the operation for which

$$a_0 = a_1 = 0,$$

and further

$$a_m(x) = m(m-1)a^{m-2}, \quad \text{for} \quad m \geqq 2,$$

if a is a certain positive constant. The value of a_x is here equal to a, consequently constant, and we have

$$\beta = a + \alpha.$$

Let us consider, in the neighbourhood of $x = 0$, a function u of the form

$$u = c\left[\frac{x}{\beta+\eta} + \left(1 - \frac{x}{\beta+\eta}\right) log\left(1 - \frac{x}{\beta+\eta}\right)\right], \quad . \quad . \quad (17)$$

in which c and η are positive constant values which are at our disposal. However we may dispose of them, the function u always belongs to (β) and its maximum modulus $M(\beta)$, on the circumference of (β), satisfies the condition

$$M(\beta) < c \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

For from the development of u in a power series

$$u = c\left[\frac{y^2}{1.2} + \frac{y^3}{2\,3} + \ldots\right], \quad \left(y = \frac{x}{\beta+\eta}\right),$$

which has merely positive coefficients, follows that u attains its maximum modulus on the circumference of (β) for $x = \beta$, and in that point the form between brackets in the righthand member of (17) is smaller than 1. For the transmuted function of u, in the point $x = \alpha$ we now find after some calculation

$$Tu(\alpha) = \frac{c}{\eta(\beta+\eta)}$$

or, if we take $\eta < 1$

$$Tu(\alpha) > \frac{c}{\eta(\beta+1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

There now exists a certain positive number τ such that there is corresponding to any arbitrarily given small number σ a function u, belonging to (β), and for which the conditions

$$M(\beta) < \sigma \quad \text{and} \quad |\, Tu(\alpha) \,| > \tau$$

are at the same time satisfied.

For, according to (18, and (19) we need only take in (17)

$$c = \sigma, \qquad \eta = \frac{\sigma}{(\beta+1)\,\tau},$$

if the latter amount is smaller than 1, and else for some proper fraction. The condition of continuity, occurring in the proposition of N°. 11, is therefore not satisfied in the whole domain (α), and we observe moreover that for τ even an *arbitrarily great* number may be taken.

Mathematics. — *"On the nodal-curve of an algebraic surface".* By Dr. J. WOLFF. (Communicated by Prof. HK. DE VRIES).

(Communicated in the meeting of September 30, 1916).

1. We consider a surface F of order n with a nodal curve Δ, and without any other singularity. Suppose that we represent F by means of a birational transformation on another surface F^*, in such a way that Δ passes into a non-singular curve Δ^* of F^*. Δ^* may then be *one* single curve or consist of two parts. The former occurs if the developable surface Ω of the pairs of tangent planes along Δ forms one whole, the latter if Ω consists of different parts. We shall occupy ourselves with the first case. The deficiency π^* of Δ^* is then equal to that of Ω, for the points of Δ^* correspond *one* for *one* with the planes of Ω. In whatever way F is birationally transformed into a surface in which Δ gets a non-singular curve as image, that image will always have the the same deficiency π^*. The value of π^* has been calculated by CLEBSCH in case of F being a rational surface, in other words, may be birationally represented in a plane. He finds $\pi^* = d(n-4) + 1$, in which d is the order of Δ [1]). This is deduced analytically. By means of a geometrical wording the proof is to be simplified. We shall start with this and then prove the proposition for an arbitrary surface, consequently also if it is not rational.

2. Let $F^n(x_1 x_2 x_3 x_4) = 0$ be a rational surface of order n_2, which, by means of the formulae

$$\begin{aligned} \varrho x_1 &= f_1(\xi_1\, \xi_2\, \xi_3) \\ \varrho x_2 &= f_2(\xi_1\, \xi_2\, \xi_3) \\ \varrho x_3 &= f_3(\xi_1\, \xi_2\, \xi_3) \\ \varrho x_4 &= f_4(\xi_1\, \xi_2\, \xi_3) \end{aligned}$$

is represented in a plane F^*, in which ξ_1, ξ_2, ξ_3 stand for the homogeneous coordinates of a point, while the f_i are homogeneous functions of a certain degree ν. Let F have no other singularities but a nodal curve Δ of order d, and let its image on F^* be *one* single curve Δ^*. The plane sections C of F are represented as curves C^* of order ν, forming a linear system on F^*. The sections of F with the ∞^2 planes passing through a point P represent themselves as the ∞^2 curves C^* of a net. The Jacobian J^* of that net, locus of the nodus of the ∞^1 curves provided with them contained in the net, is the image of the curve of contact J of the cone of contact

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[1]) Math. Ann. Bd. 1, bl. 270.

laid out of P at F. J is the section of F with the first polar surface of P, apart from Δ. From this it ensues that the sections Σ of F with *arbitrary* surfaces of order $n-1$ represent themselves as a system of curves Σ^*, individuated by the compound curve Δ^*+J^*. As the f_i are of order v, the order of Δ^*+J^* is equal to $(n-1)v$. J^* (as jacobian of a net of curves of order v) is of order $3(v-1)$. Hence Δ^* is of order

$$(n-1)\,v-3\,(v-1)=v(n-4) \qquad 3.$$

The curves C^* may have base points. If B_h is an h-fold base point, in such a way that all C^* pass h-times through B_h, the Jacobian J^*, as is known, passes $3h-1$ times through B_h. The section Σ of F with an arbitrary surface of order $n-1$ is represented on F^* as a curve Σ^*, which is represented by a homogeneous equation of order $n-1$ in the f_i, so that such a curve passes $(n-1)h$ times through B_h. Hence Δ^* passes $(n-1)h-(3h-1)=h(n-4)+1$ times through B_h. The deficiency π^* is easy to calculate now. We have viz.

$$\pi_*=\tfrac{1}{2}\,\{v\,(n-4)+2\}\,\{v\,(n-4)+1\}-\Sigma\,\tfrac{1}{2}\,h\,(n-4)\,\{h\,(n-4)+1\},$$

in which the summation extends over the various base points B_h. If we consider that the deficiency of a plane section C of F is equal to that of its image C^* on F^*, in other words that we have

$$\tfrac{1}{2}\,(v-1)\,(v-2)-\tfrac{1}{2}\,\Sigma\,h\,(h-1)=\tfrac{1}{2}\,(n-1)\,(n-2)-d,$$

we find

$$\pi^*=d\,(n-4)+1.$$

3. The above reasoning can be of no service if F is not rational, so that F^* is not a plane. Let F^n be a surface of order n, rational or not, with a double curve Δ of order d and without any other singularity. Let μ be the class of the developable surface Ω formed by the pairs of tangent planes along Δ and let k be the number of points of Δ, where the two tangentplanes coincide (pinch-points).

Suppose that F has been transformed into another surface F by a birational transformation into another surface F^*, in such a way that Δ passes into *one* single curve Δ^*. The plane sections C of F are represented by curves C^*, which form a linear system on F^*. These C^* may have base points B_h so that they all pass h times through B_h. The sections of F with the ∞^2 planes passing through a point P are represented by the curves of a net (C^*). The curve of contact J of the cone of contact laid at F out of P is transformed into the Jacobian J^* of (C^*), from which it follows again that Δ^*+J^* is a curve belonging to the linear system $|\Sigma^*|$ formed by the images of the sections Σ of F with *arbitrary* surfaces of order n—1. Let us now for a moment suppose that Δ^* belongs to a linear system on F^* of which all curves pass as often though the different points B_h as Δ^*, and let $\Delta_1{}^*$ be a curve of that linear system. In that case $\Delta_1{}^* + J^*$ is also a curve of $|\Sigma^*|$. Let us for convenience' sake represent the number of intersections of two curves *outside the points* B_h by placing the letters we have chosen for those curves, between brackets, we have

$$[\Sigma^*,\ \Delta^*] = [\Delta_1,\ \Delta^*] + [J^*,\ \Delta^*].$$

$[\Delta_1,\ \Delta^*]$ is called the "*degree*" g of the linear system to which Δ^* belongs.[1]) J rests in $k+\mu$ points on Δ, consequently

$$[J^*,\ \Delta^*] = k + \mu$$

Σ has $d\,(n-1)$ nodes on Δ, therefore $[\Sigma^*,\ \Delta^*] = 2d\,(n-1)$. Hence,

$$2d\,(n-1) = g + k + \mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We obtain a second relation if for a moment we make a particular supposition: let there exist surfaces φ^{n-4} passing through the nodal curve Δ, consequently *adjuncts* of order n—4 of F. They intersect F apart from Δ in so-called *canonical curves* K, which have the property of being represented on F^* as *canonical curves* K^*, consequently as sections of F^* with adjuncts of order n^*—4.

Two properties of the canonical curves K we have to apply here. An adjunct φ^{n-4} forms with 3 planes an adjunct φ^{n-1} of order n—1. To φ^{n-1} belong also the 1st polar surfaces of arbitrary points of space. So a K forms together with 3 plane sections C a curve of $|J|$. Consequently a K^* forms together with 3 curves C^* a J^*, so that

$$[K^*, \Delta^*] + 3\,[C^*, \Delta^*] = [J^*, \Delta^*]\,.$$

The second property we want, we find by observing that an adjunct φ^{n-4} forms with 1 plane an adjunct φ^{n-3}. A φ^{n-3} intersects the plane of a C in a curve φ^{n-3}, which passes through the d nodes of C so that outside it, it has moreover $2p-2$ points in common with C, where p is the deficiency of C. Hence the canonical curves K intersect C in $2p-2-n$ points, where n is the *degree* of the linear system of the C. But this holds good for any linear system of curves.[2]). Let us apply this to the system to which Δ^* on F^* belongs, we have then $[K^*, \Delta^*] = 2\pi^*-2-g$.

Further is $[C^*, \Delta^*] = 2d$, because C has d nodes on Δ, and

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[1]) Cf. e.g. F. ENRIQUES, *Introduzione alle Geometria sopra le superficie algebriche.* (Memorie di mat. e di fis. d. Soc. It. d. Sc., Serie 3, volume 10, p. 14)

[2]) F. ENRIQUES, *Introduzione*, p. 64.

$[J^*, \Delta^*] = k + \mu$, because J has with Δ $k + \mu$ points in common.
We find therefore

$$2\pi^* - 2 - g + 6d = k + \mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2) it ensues at once

$$\pi^* = d(n - 4) + 1.$$

§ 4. The two particular suppositions we have made are superfluous. Let Δ^* *not* belong to a linear system of which all the curves in the points B_h have the multiplicity h. For such an *isolated curve* Δ^* a positive or negative integer g is always to be defined, which is called the *virtual degree* [1]) of Δ^*. An infinite number of linear systems may be construed, in such a way that Δ^* is a part of curves belonging to it. Let $|E^*|$ be such a system and let R be a curve that completes Δ^* into an L^*. It may be seen to that there are an indefinite number of such curves R^*. They form then a linear system $|R^*|$, the restsystem of Δ^* with regard to $|L^*|$. Let g_1 be the degree of $|L^*|$ in other words the number of variable intersections of two L^*, and g_2 the degree of R^*. If now the Δ^* also formed a linear system $|\Delta^*|$, then we should of course have, g being the degree of it: $g_1 = g + g_2 + 2i$, where i represents $[\Delta^*, R^*]$. For g_1 is $[\Delta^* + R^*, \Delta_1^* + R_1^*]$, in which Δ_1^* and R_1^* are arbitrary curves of $|\Delta^*|$ and $|R^*|$.

If Δ^* is isolated then its virtual degree, by definition, is the number g determined by the equation $g_1 = g + g_2 + 2i$.

This virtual degree g is independent of the choice of $|L^*|$. We may further prove that we may calculate with it as if g were "the number of intersections of Δ^* with itself", independent of its positive or negative sign. The formula (1) holds good if Δ^* is isolated: in that case g represents its virtual degree.

The same holds true of the formula (2), not only if Δ^* is isolated, but also if no canonical curves exist. In all cases $2\pi^* - 2 - g$ is called the *immersion constant* of Δ^* and is $[J^*, \Delta^*] - 3[C^*, \Delta^*]$ [2]).

§ 5. If Ω consists of two different developable surfaces Ω_1 and Ω_2, Δ^* consists of two different curves Δ_1^* and Δ_2^*, which both have the same deficiency π as Δ. Δ_1^* and Δ_2^* intersect each other in the k images of the pinch-points on Δ. Without nearer determinations it cannot be said that the formula $\pi^* = d(n-4) + 1$ holds good, because Δ^* is degenerated. But it may be supposed that the

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[1]) F. Enriques, *Introduzione*, p. 28.

[2]) F. Severi, *Il genere aritmetico ed il genere lineare*, (Atti della R. Acc. d. Sc. di Torino, vol. 37, 1901—2).

curve $\Delta_1^* + \Delta_2^*$ belongs to a continuous system and in that case the curves of that system are of the deficiency

$$\pi^* = \pi + \pi + k - 1 = 2\pi + k - 1.$$

And also if $\Delta_1^* + \Delta_2^*$ does not belong to such a system $2\pi + k - 1$ is called the *virtual deficiency* of this degenerate curve [1]). Let μ_1 be the class of Ω_1 and μ_2 the one of Ω_2. A Σ intersects Δ in its $d(n-1)$ nodes. They are represented in $d(n-1)$ pairs on F^* and of each pair *one* point lies on Δ_1^*, the other on Δ_2^*. Hence

$$[\Sigma^*, \Delta_1^*] = [\Sigma^*, \Delta_2^*] = d(n-1).$$

And as $|\Sigma^*| = |\Delta_1^* + \Delta_2^* + J^*|$, we have

$$d(n-1) = [\Delta_1^* + \Delta_2^* + J^*, \Delta^*_1] = [\Delta_1^* + \Delta_2^* + J^*, \Delta_2^*].$$

Consequently

$$d(n-1) = g_1 + k + k + \mu_1 = g_2 + k + k + \mu_2, \quad . \quad . \quad . \quad (1')$$

where g_1 and g_2 are the virtual degrees of Δ_1^* and Δ_2^*.

The immersion constant $2\pi - 2 - g$ of Δ_1^* is equal to

$$[J^*, \Delta_1^*] - 3\,[C^*, \Delta_1^*],$$

hence

$$\begin{array}{ll} & k + \mu_1 - 3d = 2\pi - 2 - g_1 \\ \text{and} & k + \mu_2 - 3d = 2\pi - 2 - g_2 \end{array}\Bigg\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2')$$

From (2') it ensues

$$2\pi + k - 1 = \tfrac{1}{2}(g_1 + \mu_1 + g_2 + \mu_2 + 4k) - 3d + 1.$$

Consequently with regard to (1')

$$2\pi + k - 1 = d(n-4) + 1.$$

The formula $\pi^* = d(n-4) + 1$ holds consequently good if Ω degenerates, provided the virtual deficiency is taken for π^*.

So we have this general proposition:

The order of an algebraic surface that has no other singularity but a nodal curve Δ of order d, along which the pairs of tangent planes form a developable surface Ω of deficiency π^*, is

$$n = 4 + \frac{\pi^* - 1}{d}.$$

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[1]) Cf. e.g. E. Picard "*Théorie des fonct alg. de 2 var.*" vol. 2, page 106.

Zoology. — *"On an eel, having its left eye in the lower jaw"*. By Mrs. C. E. Droogleever Fortuyn—van Leyden. (Communicated by Prof. J. Boeke).

(Communicated in the meeting of Februry 24, 1917).

Through the kindness of Dr. H. C. Redeke I obtained an eel in which the left eye was lacking in the ordinary place, while on the lower side of the head, somewhat to the left of the medial line, an eye was visible which externally was quite normally shaped.

In order to find out whether this submaxillary eye was the left one and if so, how it had come to occupy such a curious position, and further whether it was also internally of normal structure, two series of transverse sections were made, one of the lower jaw and one of the remainder of the head.

It appeared that the left eye had indeed been shifted downward, that the structure was quite normal and that a well developed optic nerve and strong muscles, attached to the sclerotic in the usual way, rendered it possible and even very likely that the eye had functionated. These nerve and muscles originated from the upper part of the head; the nerve came forth from the brain in the usual manner, perfectly symmetrically with the nerve of the right eye; the muscles proceeded caudally quite symmetrically with the muscles of the right side. Nerve and muscles however followed the normal way over a short distance only, they soon bent downward and descended through the head to the lower jaw, right through the buccal cavity along a stalk connecting the upper and lower jaws and situated just before the tongue. The nerve was surrounded by the four straight eye-muscles; the two oblique ones were situated orally of the first-mentioned complex of muscles and nerve.

From this stalk the whole complex proceeded downward right through the lower jaw to the place where the eye was found. Besides nerve and muscles also a blood-vessel descended, which entered the eye together with the nerve.

Of the bony roof of the mouth, which this complex of muscles and nerve had passed, the entopterygoid, lying between the parasphenoid and the palatine, was laterally and posteriorally shifted, so that it no longer bordered on the parasphenoid. A muscle, the arco-palatine adductor muscle, was much lengthened and behind the muscle-nerve complex bent from the entopterygoid to the parasphenoid.

For the rest little change had occurred in the upper part of the head. The place where the eye should have been, was filled up with connective tissue, except a small pit. The tongue was shortened and the copula of the hyoid arch strongly compressed, probably on account of the stalk which proceeded exactly in front of the tongue. In the lower jaw the genio-hyoid muscle was also strongly compressed on the left side; otherwise here also little change was noticed.

What may have been the cause of the abnormal growth and how can this condition have developed? On the former point we must remain entirely in the dark. As to the second we may start from two suppositions: 1. the eye has descended in a full-grown condition; 2. the eye-vesicle already deviated from the ordinary course when evagination from the brain took place and has developed to an eye in an abnormal place. In my opinion the first supposition is impossible. For the changes brought about in the head point out that it is *the eye* which chose its course and that the shifted bones and muscles adapted themselves to the abnormal condition which they found when being formed. If it were the eye that had deviated after the bones had developed, not the entopterygoid would have been displaced, but the muscle-nerve complex would have grown along the bone. Moreover it is not likely that the tongue would have been compressed *after* developing, but that it developed after the eye-stalk had formed and so was impeded in its growth. Finally it is difficult to understand how with a full-grown eye the cornea would have participated in the descent.

Assuming the second supposition, namely that the eye-vesicle already deviated from its normal course when it evaginated from the brain, we must, in order clearly to understand the process, consider how the condition of the head was when the eye first originated. For Muraena Prof. BOEKE gives us important data on this point. (Die Entwicklung der Muraenoiden, PETRUS CAMPER, Vol. II, 1903). At the time of the evagination of the eye-vesicles also the infundibulum is evaginated ventrally. Before it lies the so-called anterior mesodermic mass, a coalescence of mesoderm and entoderm, according to BOEKE. It consists of a thickened cell-mass, proceeding in two wings on both sides of the brain, and of a one-layered tongue bordering on the periblast. At a later stage the thickened mesodermic mass coalesces with the ectoderm, while the lower tongue curves round and coalesces with the intestinal epithelium.

The ectoderm invaginates and grows towards the entoderm of the intestine; afterwards the buccal cavity is formed in it.

The two lateral mesoderm streaks of the head originally form a solid mass ventrally of the evaginating eye-vesicles. Later in these streaks cavities arise and according to BOEKE they are transformed into true somites. After the stalks of the eye-vesicles have been formed, cells grow from the wall of these somites against the capsule of the eye-vesicles in order to form the eye-muscles. BOEKE observed that the musculus obliquus superior and the musculus rectus externus originate from the wall of these somites. The same has been observed by MISS PLATT in Selachians for all eye-muscles.

How shall we imagine now that all this took place in our abnormal eel, related to the Muraena? When the eye-vesicle evaginated it probably did not grow laterally, but forward and downward. It reached the anterior mesodermic mass, which it pierced in growing in a forward and downward direction. It passed the place where entoderm and ectoderm grow towards each other and finally came to lie against the ectoderm, more particularly the ectoderm from which later the skin of the lower jaw is formed. This latter reacted on it by forming a lens and a cornea, which is in itself very remarkable but not impossible, since the experiments of SPEMANN, LEWIS and others have shown that at any rate in Amphibians lenses may be formed from the ectoderm in very unusual places.

It still remains to be explained how the eye-muscles found their way towards the eye in the lower jaw. This will also have happened at a very early stage in the development of the eye, immediately after the formation of the stalk of the eye-vesicle. This latter had then been little shifted aside yet. The cells derived from the wall of the somites of the head then laid themselves, as in ordinary cases, against the capsule of the eye-vesicle and were carried along its unusual course through the anterior and inferior part of the head, while as in a normal case they developed to muscles.

Leyden, February 1917. *Histological Department of the Anatomical Cabinet.*

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

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CONTENTS.

Mathematics. — *"Two null systems determined by a net of cubics".* By Prof. JAN DE VRIES.

(Communicated in the meeting of January 27, 1917).

§ 1. A net $[c^3]$ of cubics determines on an arbitrary straight line f an involution I_3^2 of the third order and the second rank. This involution possesses three groups, in which the three points have coincided; f is therefore stationary tangent for three curves c^3. If the three points of inflection are associated to f as null points F, a null system $N_{3,3}$ arises. For in the pencil (c^3), which has a point F as base point, three curves occur, on which F is point of inflection; each point has therefore three null rays. In this null system we shall indicate the null rays by i, their null points by I.

The above mentioned I_3^2 has further a neutral pair of points, consequently two points forming with any point of f a group of the I_3^2. This pair is of course formed by two base points of a pencil included in $[c^3]$. If these two points are considered as null points B of $f \equiv b$, a null system $N_{8,2}$ arises; for to any point B are associated in that case the remaining eight base points B^* of the pencil (c^3) determined by B, so that B is null point of eight null rays [1]).

§ 2. If i is made to revolve round a point P, the three null points I describe a curve $(P)^6$, which passes three times through P. Through P pass 18 tangents t, which touch the curve elsewhere. The I_3^2 on t has moreover a neutral double point in the point of contact D; for the coincidence of two triple points always goes together with the coincidence of the points of the neutral pair [2]).

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[1]) If c^3 has 7 base points this null system is replaced by an $N_{1,2}$. Cf. my paper "*Plane linear null systems*". (These Proceedings XV, 1165).

[2]) If the involution is represented by

$$x_1 x_2 x_3 + a(x_1 x_2 + x_2 x_3 + x_3 x_1) + b(x_1 + x_2 + x_3) = 0 ,$$

As D represents two coinciding base points, there is a c^3, which has D as node; the locus of the points D is the curve of Jacobi of the net, Δ^6.

The 18 tangents t are at the same time tangents out of P to the curve $(P)^{10}$, determined by the null system $N_{8,2}$ and possessing an octuple point in P.

The curves $(P)^6$ and Δ^6 have besides the 18 points of contact of the straight lines t, moreover 18 points D^* in common. Evidently PD^* is one of the tangents d,d' in D^* at the c^3, which has D^* as node. The nodal tangents of the rational curves of the net envelop therefore a curve of the 18th class [1]) (curve of Zeuthen).

The pairs of tangents d,d' determine on a straight line l a symmetrical correspondence [18], which has double coincidences in the 6 points D lying on l. The remaining coincidences arise from tangents in cusps; the net, therefore, possesses 24 curves with a cusp.

Let us moreover consider the correspondence (36, 18), which is determined on l by the straight lines t and d. Here too the 6 points D lying on l are double coincidences; the remaining 42 arise from straight lines t, which have coincided with one of the nodal tangents d. In the corresponding point $D \equiv B^{(3)}$ the curves of a pencil (c^3) have evidently three coincided points in common.

The net consequently contains 42 *pencils, the curves of which osculate each other.*

§ 3. If a point I is made to describe the straight line p, its null rays envelop a curve $(p)^6$ of class six, which has p as triple tangent. The two remaining null points of i will then describe a curve π, the order of which we can determine by investigating how many points it has in common with p. To them belong in the first place the 6 points D lying on p, for on the base tangent t belonging to D, the point D represents two points I. π has further evidently nodes in each of the three null points of p; it is consequently of order 12.

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the triple points are found from

$$x^3 + 3ax^2 + 3bx = 0,$$

the neutral points from

$$x_1x_2 + a(x_1 + x_2) + b = 0 \text{ and } ax_1x_2 + b(x_1 + x_2) = 0.$$

They coincide if $b = 0$; but in that case two triple points have coincided in $x = 0$.

[1]) If to each point D the two tangents d,d' are associated a correspondence (1,2) arises between the points of Δ^6 and the tangents of $(d)^{18}$. From the formula of correspondence of Zeuthen we find then that $(d)^{18}$ is of genus 31.

From this it ensues that the curves $(p)^6$ and $(q)^6$, indicated by the straight lines p and q, have 12 tangents i in common, on which every time one null point lies on p, the other null point on q. Moreover the three null rays of the point pq are common tangents. The remaining 21 common tangents can only arise from figures c^3 composed of a conic c^2 and a straight line s. The number of those figures amounts therefore to 21. The 21 straight lines s are *singular rays* of $N_{3,3}$; for each point of s is to be considered as point of inflection I, consequently as null point of s.

The curve $(p)^6$ is of order 24, is therefore intersected in 18 points by its triple tangent. For each of those points two null rays i coincide; the points that have this property form therefore a curve γ^{18}. On this curve lie of course the 24 cusps and the 42 triple base points $B^{(3)}$ (§ 2).

§ 4. The straight lines s are at the same time singular null rays for the null system $N_{8,2}$. For on s the net curves determine a cubic involution, of which each group contains three base points belonging to one and the same pencil. For each of the four coincidences D of this I_3 s is a base tangent t; these points, therefore, lie on Δ^6. The remaining intersections of s and Δ^6 are found in the nodes of the figure (c^2,s).

If the base point B describes the straight line p its eight null rays envelop a curve $(p)^{10}$, of class 10, with bitangent p. At the same time the base points B^* associated to B describe a curve of order 8, π^8, which has the two null points of p and 6 points D in common with p. From this it ensues that the curves $(p)^{10}$ and $(q)^{10}$ have eight tangents in common, which each possess one null point on p and the second null point on q. Those curves have moreover the eight null rays of the point pq in common. The remaining common tangents are procured by the 21 singular *null rays* s; they are consequently *bitangents* of the curve $(p)^{10}$.

From this it ensues that $(p)^{10}$ is of order $90-22 \times 2$ or 46, so that p contains 42 points B, for which two of the associated base points B^* have coincided in a point D. The groups of seven base points, which are associated to the double base points D, lie therefore, on a curve of order 42; it is the *branch curve* β^{42} of the involution, the groups of which consist of 9 base points of a pencil (c^3). This result may also be arrived at by the following consideration. The curve π^8 has with Δ^6 six points of the line p in common; the remaining 42 intersections of those curves are double base points D, for which one of the associated base points lies on p.

The curve β^{42} touches Δ^6 in each of the 42 triple base points $B^{(3)}$; for such a point may be considered in two ways as coincidence of a double base point $B^{(2)}$ with one of the base points associated to it. The remaining intersections of β^{42} and Δ^6 form 84 pairs of double base points. The net contains therefore 84 *pencils, each possessing two base tangents* t, of which the curves c^3, therefore, touch each other in two base points.

§ 5. The curve $(t)^{18}$, enveloped by the base tangents t, is, as well as Δ^6, of genus 10; its singular tangents must therefore be equivalent to 126 bitangents. They are evidently represented by the 21 singular rays s, which are quadruple tangents of $(t)^{18}$. The *order* of $(t)^{18}$ is consequently 54.

If a point I is made to describe the singular straight line s, its null rays i envelop a curve $(s)^6$; for the intersections of s with a curve $(P)^6$ send each a null ray through P. But through each point of s pass but two other null rays, as s is null ray to each of its points. Consequently s is quadruple tangent of $(s)^6$, and s contains four points S, for which two null rays coincide with s.

Analogously is s quadruple tangent of the curve $(s)^{10}$, which is enveloped by the groups of six null rays b belonging to the points B of s (two of them always coincide with s). The four points of contact of s are easily indicated: they form the two pairs of base points, which are associated in the I_8 to the nodes of the figure (c^2, s). For through each of those nodes D passes a base tangent t, for each of the base points lying on s and belonging to D *three* null rays b have consequently coincided with s.

The two base tangents t just mentioned are at the same time common tangents of $(s)^6$ and $(s)^{10}$; the remaining ones are represented by s (which replaces 16) and by the remaining 20 singular null rays which are bitangents of $(s)^{10}$.

§ 6. We shall now suppose that all the curves of $[c^3]$ pass through a point S. The net then contains a curve σ^3, which has a node in S and determines with every other c^3 of the net a pencil, the curves of which touch each other in S. Any straight line passing through S is therefore a base tangent of a pencil, and σ^3 is the locus of the groups of seven base points belonging to S. Δ^6 too has a node in S.

The variable base points B now form a null system $N_{7,2}$, which has a *singular point* in S. For, any straight line b passing through S contains two null points: the point S and the third intersection of b with σ^3.

At the same time S is *singular point* for the null system $N_{3,3}$, for any straight line passing through S is a stationary tangent for a c^3, which has S as point of inflection.

If a straight line b is made to rotate round P, its null points B describe now a curve $(P)^9$, with septuple point P, which evidently passes through S. Through P pass now only 16 base tangents t; they are also tangents at the curve $(P)^6$, which is determined by $N_{3,3}$. But P must lie on 18 tangents of $(P)^6$ (§ 2); hence PS replaces two of those tangents, and is consequently *stationary tangent*, with S as point of inflection of $(P)^6$. This is confirmed by the observation that $(P)^6$ and $(P)^9$ have in P 21, in the points of contact of the 16 straight lines t 32 points in common, so that they must intersect in S. In consequence of this the possibility that $(P)^6$ should have a node in S is excluded.

The null rays b of the points of a straight line p now envelop a curve of class 9, which has p as bitangent. Let us consider the tangents it sends through S. Three of them are indicated by the points that p has in common with δ^3. The remaining six must be component parts s^* of compound figures c^3. Of the 21 straight lines s, *six* pass consequently through S. On each of those six straight lines the net determines an involution I_2 of associated base points B, B^*; such a singular straight line is consequently *simple tangent* of $(p)^9$, while the remaining singular straight lines are now also *bitangents*.

The curves $(p)^9$ and $(q)^9$ have consequently in common the 7 null rays of the point pq, the 8 null rays, which each have *one* null point on p, the other on q, the 6 singular null rays s^* and the 15 singular null rays which are bitangents for the two curves.

§ 7. If the net has two base points S_1 and S_2, their connector is really component part of a figure (c^2, s), consequently singular for $N_{3,3}$, but *not* a singular null ray of $N_{6,2}$. Each of the two *singular null points* S_1, S_2 bears 5 singular null rays s, and the null systems $N_{6,2}$ and $N_{3,3}$ have moreover 10 singular null rays s.

Let us now suppose that the net has k base points S. The variable base points B of the pencils (c^3) determine a null system $N_{8-k,2}$. Each singular point S bears $(7-k)$ singular null rays s^*; for of the $(10-k)$ tangents which the curve $(p)^{10-k}$ sends through S, three are again indicated by the intersections of p with the curve c^3, which has a node in S. The straight lines that connect the points two by two, are *not* singular for $N_{8-k,2}$ (they are for $N_{3,3}$). The number of singular null rays s, therefore, amounts to

$$21 - k\,(7-k) - \tfrac{1}{2}\,k\,(k-1) \quad \text{or} \quad \tfrac{1}{2}\,(7-k)\,(6-k).$$

These straight lines are bitangents of the curve $(p)^{10-k}$.

The following table contains for the null system $N_{8-k,2}$ the number of *singular null points*, the number of *singular null rays* s (bearing an I_3 of null points) and the number of *singular null rays* s^* (containing an I_2 of null points).

k	s	s^*
0	21	0
1	15	6
2	10	10
3	6	12
4	3	12
5	1	10
6	0	6
7	0	0

The curve $(P)^{10-k}$ has an $(8-k)$-fold point in P, consequently lies on $2\,(9-k)$ of its tangents t. The *base tangents*, therefore, envelop a curve of class $2\,(9-k)$.

The curve $(P)^6$, belonging to $N_{3,3}$, has in each of the k singular points S a point of inflection, with *stationary tangents* PS (§ 6).

§ 8. The net $[c^3]$ distinguishes itself from a general net $[c^n]$ in this, that in the latter no figures appear composed of a straight line and a c^{n-1}. In connection with this the null system $N_{3,3(n-2)}$, which is determined by the points of inflection, has in general *no singular rays*.

If the point I is made to describe the straight line p, its null rays i envelop a curve of class $3\,(n-1)$. The curves belonging to p and q have besides the three null rays of the point pq, moreover $(9\,n^2 - 18 + 6)$ tangents in common; they are here the null rays i, which have one null point in p and another in q. Their number is therefore at the same time the order of the curve π described by the null points of the straight lines i, of which a null point lies in p.

The intersections of π with p form three groups. In the first place each of the $3\,(n-2)$ null points of p is a $(3\,n-7)$-fold point of π. A second group consists of the intersections of p with the curve Δ of JACOBI, which is of order $3(n-1)$. The third group consists of $(18n-33)$ points, where a c^n has *four* consecutive points in common with its tangent. From this it ensues that the *points of undulation* of a net form a curve of order $(18n-33)$. [1])

The curve (P) is of order $3(n-1)$ and has a triple point in P; through P pass consequently $(9n^2-21n)$ of its tangents. They now form two groups: the first consists of base tangents t, the second of tangents u in points of undulation.

(P) now intersects the curve Δ in $3(n-1)(2n-3)$ points D, of which one of the two tangents passes through P (class of the curve of ZEUTHEN) [2]), consequently in $9(n-1)^2 - 3(n-1)(2n-3)$ or $3n(n-1)$ points D, for which the base tangent t passes through P.

From this it then ensues, that P lies on $(6n^2-18n)$ tangents u. The *four-point tangents*, therefore, envelop a curve of class $6n(n-3)$. [3])

Mathematics. "*On a Representation of the Plane Field of Circles on Point-Space*". By Dr. K. W. WALSTRA. (Communicated by Prof. JAN DE VRIES).

(Communicated in the meeting of January 27, 1917).

§ 1. The circles in the plane XOY are represented by

$$C \equiv X^2 + Y^2 - 2aX - 2bY + c = 0.$$

If we consider a, b, and c as the co-ordinates x, y, z of a point, a correspondence (1, 1) is obtained between the circles of a plane and the points of space. The image of a circle is obtained by placing a perpendicular in the centre on the plane and by taking on it as co-ordinate the *power* of the point O with regard to the circle.

For the radius we have $r^2 = a^2 + b^2 - c$.

Circles with equal radii are therefore represented by the points of a paraboloid of revolution, with equation $x^2 + y^2 - z = r^2$.

The images of the point-circles lie on the *limiting surface* G,

$$x^2 + y^2 = z,$$

a *paraboloid of revolution*, touching the plane XOY in O.

§ 2. A pencil of circles is indicated by $C_1 + \lambda C_2 = 0$. For the circle λ we have

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[1]) Another deduction of this number is to be found in my paper: "Characteristic numbers for nets of algebraic curves". (These Proceedings XVII, 937).

[2]) Cf. my paper "On nets of algebraic plane curves". (These Proc. VII, 633).

[3]) These Proc. XVII, 936.

$$(1+\lambda)\,a = a_1 + \lambda a_2,\ (1+\lambda)\,b = b_1 + \lambda b_2,\ (1+\lambda)\,c = c_1 + \lambda c_2.$$

From this we find for the images

$$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2} = \frac{z-z_1}{z_1-z_2}.$$

A pencil of circles is therefore represented by a straight line.

Its intersections with G are the images of the point-circles of the pencil. The point at infinity of the line represents the axis of the pencil.

A tangent at G is the image of a pencil of circles of which the limiting points have coincided; *any two points of a tangent are therefore the images of two touching circles.*

This may be confirmed as follows. Let d be the distance of the centres of *two circles* with radii r and r'; we have then $d = r \pm r'$ or $\sqrt{(a-a')^2 + (b-b')^2} = \sqrt{a^2 + b^2 - c} \pm \sqrt{a'^2 - b'^2 - c'}$.

After some reduction we find for the images

$$\left(xx' + yy' - \frac{z+z'}{2}\right)^2 = (x'^2 + y'^2 - z')\,(x^2 + y^2 - z),$$

which relation expresses that the images lie on a tangent of G.

§ 3. A net of circles is represented by $C_1 + \lambda C_2 + \mu C_3 = 0$.

From this it ensues for the images

$(1 + \lambda + \mu)\,x = x_1 + \lambda x_2 + \mu x_3$ etc. consequently

$$\begin{vmatrix} x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ z & z_1 & z_2 & z_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

A net of circles is therefore represented by a plane.

Plane sections of G have circles as horizontal projections. For the section of $x^2 + y^2 = z$ with $z = \alpha x + \beta y + \gamma$ has as projection the figure represented by $x^2 + y^2 - \alpha x - \beta y - \gamma = 0$.

The point-circles of a net of circles lie therefore on a circle; this proposition is reversible.

The net that corresponds to $z = \alpha x + \beta y + \gamma$, has as equation

$$X^2 + Y^2 - 2aX - 2bY + (\alpha a + \beta b + \gamma) = 0,$$

where a and b are variable parameters. If we write for this

$$X^2 + Y^2 + a\,(\alpha - 2X) + b\,(\beta - 2Y) + \gamma = 0,$$

it appears that all circles have in the point $(\frac{1}{2}\alpha, \frac{1}{2}\beta)$ equal power viz. $\frac{1}{4}(\alpha^2 + \beta^2) + \gamma$; this point is the centre of the circle that contains the point-circles of the net.

To a tangent plane of G corresponds a net of circles that pass through a fixed point. For, to $2x_1x + 2y_1y = z + z_1$ corresponds a net of which all the circles have in (x_1,y_1) the power $x_1{}^2 + y_1{}^2 - z_1 = 0$.

Two pencils of circles are in general represented by two skew straight lines. If, however, they have a circle in common their images lie in a plane and their four point-circles lie on a circle; the pencils belong to a net.

§ 4. For two orthogonal circles we have $d^2 = r_1{}^2 + r_2{}^2$, so

$$(a_1 - a_2)^2 + (b_1 - b_2)^2 = (a_1{}^2 + b_1{}^2 - c_1) + (a_2{}^2 + b_2{}^2 - c_2)$$

or

$$2a_1a_2 + 2b_1b_2 = c_1 + c_2.$$

For the images we have consequently $2x_1x_2 + 2y_1y_2 = z_1 + z_2$, i. e. *the images of two orthogonal circles are harmonically separated by the limiting surface.*

To the connection between pole and polar plane corresponds the fact that all circles intersecting a given circle orthogonally form a net.

To the relation between two associated polar lines corresponds the fact that pencils of circles may be arranged in pairs, so that any circle of a pencil is intersected orthogonally by any circle of the other.

To a polar tetrahedron corresponds a group of four circles that are orthogonal in pairs. (Of them only three are real).

§ 5. If the circle C intersects the circle C_1 diametrically we have $d^2 = r^2 - r_1{}^2$ or

$$(a_1 - a)^2 + (b_1 - b)^2 = (a^2 + b^2 - c) - (a_1{}^2 + b_1{}^2 - c_1).$$

We consequently have for the images

$$2x_1x + 2y_1y - z = 2x_1{}^2 + 2y_1{}^2 - z_1.$$

The circles that intersect a given circle diametrically form a net.

According to § 3 this net has as radical centre $\frac{1}{2}\alpha = x_1$, $\frac{1}{2}\beta = y_1$, i. e. the centre of C_1 (which was to be expected), and in that point the power $z_1 - x^2{}_1 - y^2{}_1 = - r^2{}_1$.

§ 6. The circles touching at a given C_1, have their images on the enveloping cone of G, which has the image of C_1 as vertex (§ 2). Three enveloping cones have eight points in common; they are the images of eight circles which touch at three given circles.

The circles touching at two circles C_1 and C_2 are represented by a twisted curve ϱ^4 of the fourth degree, a net of circles consequently contains four circles that touch at C_1 and C_2. The enveloping cones that have the images of C_1 and C_2 as vertices touch at G along conics that have two points in common, viz. the images of the intersections of C_1 and C_2.

The intersections of ϱ^4 with a tangent plane of G are the images of four circles passing through a given point and touching at C_1, C_2 (§ 3).

The circles touching at a given straight line are represented by a cylindrical surface that envelops G and of which the straight lines are perpendicular to the given straight line consequently parallel to the plane XOY.

Mathematics. — *"A Quadruply Infinite System of Point Groups in Space"*. By Dr. CHS. H. VAN OS. (Communicated by Prof. JAN DE VRIES).

(Communicated in the meeting of January 27, 1917).

Let a pencil (a^3) be given, consisting of cubic surfaces a^3. An arbitrary straight line l is touched by four surfaces a^3 of the pencil. As the space contains ∞^4 lines l, there are ∞^4 groups of four points of contact. We shall indicate this system of groups of four points by S^4.

§ 1. If we take for the line l a line g lying on *one* of the surfaces a^3, the four surfaces mentioned coincide with this surface a^3, while the points of contact become indefinite. These straight lines g are therefore *singular lines* of S^4. They form a ruled surface R, of which we shall determine the order.

A line g intersects a second surface a^3 in three points lying on the base-curve ϱ^9 of the pencil (a^3); the lines g are therefore *trisecants* of the curve ϱ^9. If on the other hand we consider a trisecant of ϱ^9, the surface a^3, which passes through an arbitrary point of this trisecant will have four, consequently an infinitely great number of points in common with it, so that the trisecant is a straight line g.

Through an arbitrary point pass 18 bisecants of ϱ^9 [1]), the genus of ϱ^9 amounts consequently to $\frac{1}{2} \times 8 \times 7 - 18 = 10$. If we therefore project the curve ϱ^9 out of one of its points, we get as projection a curve of order eight with $\frac{1}{2} \times 7 \times 6 - 10 = 11$ nodes. Through the said point pass therefore 11 trisecants of ϱ^9, so that the surface R has the curve ϱ^9 as 11-fold curve.

A surface a^3 intersects the surface R along the curve ϱ^9 and according to the 27 straight lines g lying on a^3, the *order* of R amounts to 42.

§ 2. Any line l passing through a given point P contains one

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[1]) Cf. e.g. ZEUTHEN, *Lehrbuch der abzählenden Geometrie*, page 46.

group of S^4; these groups of four points form a surface Π. If we take for the line l a line that touches the surface a^3 passing through P in P, *one* of the points of the associated group will lie in P. The surface Π passes therefore through P and touches there at the surface a^3 passing through P, because the tangents of Π in P are also the tangents of a^3 in P. The surface Π has therefore a single point in P. Any line l passing through P has therefore with Π five points in common. This surface is therefore of order five. It is easy to see that it is the *polar surface* of P with regard to the pencil (a^3).

If the line l passes through a point Q of ϱ^9, two of the surfaces a^3, which touch at l, will coincide into the surface that touches at l in the point Q; the associated intersections of l and Π consequently coincide also. The surface Π therefore passes through ϱ^9 and touches along this curve at the cone which projects ϱ^9 out of P.

The lines l, for which one of the points of the group of four points lying on it lies in P, are the tangents in P of the surface a^3 passing through P. The locus of the remaining points of these groups is obviously the intersection of the surface Π with the tangent plane in P, so a *curve of order five*, which has a *node* in P.

§ 3. If a line l intersects the curve ϱ^9 in a point P, the two surfaces a^3, which touch at l, will coincide into the surface that touches at l in P. If we therefore cause the line l to rotate round P, 2 points of the group lying on l will lie in P, so that the line l intersects the surface Π^5 belonging to P only in two points outside P. This surface Π^5 has consequently in P a *triple point*.

The points of ϱ^9 are therefore *singular points* of S^4; for each of them belongs to ∞^2 groups, while an arbitrary point belongs to ∞^1 groups.

Let us now take for the point P the conical point of one of the 32 nodal surfaces a^3. For any of the lines l passing through P, this surface belongs to the surfaces a^3, which touch at l, so that one of the points of the group lying on l lies in P. This point P therefore is also a singular point of S^4. Any straight line passing through P intersects the surface Π^5 belonging to P in three points lying outside P; this surface therefore has a *conical point* in P.

§ 4. We shall now consider the *coincidences* of S^4. If two of the surfaces a^3 which touch at a line l, coincide, two of the coincidences of the involution that is determined by the pencil (a^3) on the line l will coincide. This may 1 take place on account of three associated points coinciding in one of the coincidences of this involution. The straight line l is then *principal tangent* of one of the surfaces a^3.

The lines bearing the coincidences formed in this way, being the principal tangents of the surfaces a^3, form a line complex of order 9; for the rays of this complex, which lie in a plane, are inflectional tangents of a pencil of cubics along which this plane intersects the pencil (a^3); and these inflectional tangents envelop a curve of class 9.

It appeared in § 2 that an arbitrary point P belongs to ∞^1 groups of S^4. The remaining points of these groups lie on a plane curve of order five, which has a node in P. These groups are formed by the intersections of the c^3 with the lines passing through P. If we now consider the tangents at the branches of c^3 passing through P, each of these two tangents has in P three coinciding points in common with c^3. Therefore P is a coincidence of the two groups of S^4 lying on these lines. An arbitrary point P belongs therefore to 2 coincidences of S^4.

At the c^5 mentioned $5 \times 4 - 2 - 4 = 14$ tangents may be drawn out of P. To them belong the lines connecting P with the 9 intersections of the plane of c^5 with the base-curve ϱ^9 [1]).

If Q is the point of contact of one of the remaining 5 tangents, two of the intersections of the line $P\,Q$ with the curve c^5 coincide in Q, so that Q is a coincidence. An arbitrary point P belongs therefore to *five* groups, which have a coincidence Q lying outside P.

Between the points P and Q exists evidently a correspondence (5, 4); for to each coincidence Q belong two points P, and each point Q belongs to 2 coincidences.

§ 5. If the point P describes a plane V, the points Q will describe a surface ψ, if the points Q describes a plane V, P describes a surface Φ.

In order to find the orders of these surfaces we inquire their intersections with the plane V. If the point P describes the plane V and Q also lies in it, the line PQ lies in this plane. As this line bears the coincidence lying in Q, it is an inflectional tangent of one of the curves of the pencil, along which the plane V intersects the pencil (a^3), while Q is the associated point of inflection. The locus of these points of inflection Q is a curve ι^{12} of order twelve.

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[1]) As will afterwards appear these lines also bear coincidences which, however arise in a different way from those considered in this §.

In order to find the locus of the associated points P we observe that, if a line l describes a plane pencil, the points of the group of S^4 lying on l will describe a curve of order five. The inflectional tangents PQ envelop a curve of class 9; the points of the group lying on PQ consequently describe a curve of order $9 \times 5 = 45$. To them belongs the curve ι^{12}, twice counted, as in Q two points of a group coincide. The rest curve, i.e. the locus of the points P, is therefore of order 21.

This curve is the intersection of the plane V with the surface Φ. This surface is consequently of *order* 21.

The curve ι^{12} is the intersection of the plane V with the surface ψ. Now, however, an arbitrary point Q of the surface ψ belongs to *one* point P of the plane V, while the point Q of the curve ι^{12} belongs to two points P. The curve ι^{12} is therefore a nodal curve of the surface ψ. This surface is therefore of order 24.

The order 21 of the surface Φ gives the number of times that the point Q lies in a plane V and the point P on an arbitrary line l. It consequently also gives the order of the curve described by the point Q if the point P describes a straight line l.

In the same way, if the point Q describes a line l, the point P will describe a curve of order 24.

§ 6. If a line l passes through a point Q of the base-curve ϱ^9, two of the surfaces a^3, which touch at l, will coincide into the surface a^3, which touches l in Q. Any secant of ϱ^9 therefore also bears a coincidence of S^4.

Such a secant is touched outside ϱ^9 by two surfaces a^3; the points of contact are associated to Q by S^4. If *one* of these points of contact coincides with Q, *three associated points* of the S^4 will coincide in Q. The surface a^3 belonging to this point of contact has in Q 3 coinciding points in common with l in that case. The principal tangents passing through a point Q of the curve ϱ^9, form a cone of order three; for, a plane V passing through the point Q, intersects the pencil (a^3) along a pencil that has a base-point in Q, and the curve ι^{12}, which is the locus of the points of inflection of the curves of this pencil has a triple point in Q.

On each generatrix of this cone lies another point S, which is associated to Q by S^4; these points form a curve σ, passing *once* through Q. For let us consider the tangent t in Q at the curve ϱ^9. An arbitrary surface a^3 intersects the tangent t apart from Q only in *one* point, so there is not a single surface a^3 that touches at t outside Q. The four associated points of S^4 lying on t coincide therefore with Q and we see that the curve σ passes through Q and here touches at the line t.

A plane V passing through the point Q, intersects the above mentioned cubic cone along three generatrices, which each contain *one* point S. The point Q and these three points S are the intersections of the plane V with the curve σ; this curve is consequently of order *four*.

§ 7. It appeared in § 3 that, if T is a node of a surface a^3, this point must be a singular point of S^4, for if l is an arbitrary straight line passing through T, the said surface a^3 will have two points in common with the straight line l in T. If we take for the line l one of the tangents of the surface a^3 in the point T, two of the surfaces touching at l will coincide with the said surface a^3 and T is consequently a coincidence. The two other points of the associated group are the intersections of the line l with the surface Π^5, which belongs to the point T. These tangents l form a quadratic cone, which intersects the surface Π^5 along a curve of order ten. To this curve, however, belong as may be easily seen the 6 straight lines passing through T and lying on the surface a^3. The rest-section, i.e. the locus of the points of the above mentioned groups, is therefore a curve of order *four*.

§ 8. The points that belong with an arbitrary point P to the same group S^4, form a curve c^5 of order five. If now the point P describes a line l; these curves will describe a surface Λ of which we shall determine the order.

For this purpose we investigate the intersections of Λ with the surface Π^5, belonging to a point P of the line l.

These surfaces Π^5 form a pencil. For, through an arbitrary point X passes *one* surface a^3, and the tangent plane in X at this surface intersects the line l in *one* point P, which with X belongs to a same group of S^4. Through this point only *one* surface Π passes.

The last reasoning does not hold good if X is chosen on the base-curve ϱ^9; it then lies viz. on ∞^1 tangent planes of surfaces a^3. The curve ϱ^9 is therefore a part of the base-curve of the pencil (Π^5).

Neither does that reasoning hold good if the said tangent plane passes through the line l. The rest of the base-curve of the pencil (Π^5) is therefore the locus of the points of contact of the tangent planes at surfaces a^3 passing through the line l. This curve must be of order 16, as it forms, together with ϱ^9, the base-curve of the pencil (Π^5). This is really so, for a plane V passing through l intersects the curve mentioned in the four points, in which l is touched by surfaces a^3, and in the 12 points, in which the plane V is touched by surfaces a^3.

The planes π, in which the curves c^5 lie which belong to the points of the line l, envelop a developable surface of class five. These planes are the tangent planes in the points P of the straight line l at the surfaces a^3 passing through these points. Four of these tangent planes pass through l, because l touches at four surfaces a^4; through an arbitrary point of l pass therefore altogether five of these planes.

Through an arbitrary point of one of the above mentioned curves ϱ^9 and ϱ^{16} pass therefore five planes π, consequently five curves c^5. These curves are therefore 5-fold curves of the surface A. A surface Π^5 now intersects the surface A along these fivefold curves and along the curve c^5 lying on Π^5, consequently, together, along a curve of order $5 \times 9 + 5 \times 16 + 5 = 130$; the order of A is therefore 26.

Any point of A belongs evidently to a group of S^4, of which one of the points lies on the line l. A second line m, intersects the surface A in 26 points. *There are consequently* 26 *groups of* S^4, *of which two points lie on two given straight lines.*

§ 9. A plane V intersects the surface A^{26} along a curve c^{26} of order 26. Any point of this curve belongs to a group, of which one of the points lies on the line l; the other points of these groups form a curve λ, the order of which we shall determine.

To this end we try to find the intersections of this curve λ with the plane V. They are the following:

1. The straight line l intersects the plane V in a point P. The curve c^5, belonging to this point P, has a node in P, and further intersects the plane V in three points that lie on the curve c^{26}. The line connecting one of these points with the point P contains two points of the curve λ, which points lie in the plane V; so 6 intersections of λ with the plane V are found.

2. If a point Q describes the plane V, two coincidences of S^4 will lie in Q; the remaining points belonging to these coincidences, describe, as appeared in § 5, a surface of order 21; it is intersected in 21 points by the line l. The coincidences belonging to one of these intersections, are evidently points of the curve c^{26}, which have coincided with one of the associated points of the curve λ. In this way 21 intersections of the curve λ with the plane V are found.

3. The plane V intersects the base-curve ϱ^9 in 9 points Q.

Through each of these points pass five curves c^5, so that this point belongs to five points P of the line l. The lines connecting these points with P bear each a coincidence lying in Q, so that every time a point of c^{26} coincides in Q with an associated point of λ. Each of these 9 points of Q being a fivefold point of the curve λ, 45 intersections of λ and V are found.

The total number of intersections of λ and V amounts therefore to $6 + 21 + 45 = 72$; this therefore is the order of λ.

A second plane V' intersects the curve λ in 72 points; *there are consequently* 72 *groups of* S^4, *of which two points lie in two given planes, whilst a third lies on a given straight line.*

§ 10. As appears from the preceding there are ∞^2 groups of S^4, of which two points lie in two given planes V and V'. The remaining points of these groups form a surface of order 72, for a line l contains 72 of these points.

Among these groups there are ∞^1 that have a coincidence lying outside the planes V and V'. The locus of these coincidences is a curve ϱ, the order of which we shall determine.

With a view to this we try to find the number of intersections of the curve ϱ with the plane V.

The plane V' intersects the plane V along a line l. The latter contains 21 points P, to which belong a coincidence Q lying in the plane V and a second point of the plane V, the 21 points are the intersections of the straight line l with the surface Φ, which belongs to the plane V. The 21 associated points Q are evidently intersections of the plane V with the curve ϱ.

The plane V intersects the curve ϱ^9 in 9 points Q. There are ∞^1 groups of S^4, of which three points coincide in Q; as appeared in § 6, the locus of the remaining points of these groups is a biquadratic twisted curve. It intersects the plane V' in four points. There are consequently four groups of which a point lies in V', while the three other points have coincided in the intersection of the support with the plane V. This may evidently be considered in such a way that a point of the plane V has coincided with a coincidence associated to it; each of these groups produces therefore an intersection of the plane V with the curve σ. The number of these groups amounts evidently to $9 \times 4 = 36$.

The order of σ therefore amounts to $21 + 36 = 57$.

A third plane V'' intersects the curve σ in 57 points. *There are consequently* 57 *groups of* S^4, *which have in a given plane* V'' *a coincidence, while the two other points lie in two given planes* V *and* V'.

§ 11. The surface of order 72 formed by the remaining points of the groups of which two points lie in two given planes V and V', is intersected by a third plane V'' along a curve c^{72}. There are consequently ∞^1 groups of S^4 of which three points lie in three given planes. The fourth points of these groups form a curve μ, of which we shall determine the order. To this purpose we try to find the intersections of the curve μ with the plane V.

The plane V intersects the planes V' and V'' along two lines l' and l''. The surface A^{26}, which belongs to the line l', is intersected by the line l'' in 26 points. Two of these points lie on the line l', which is a nodal line of A^{26}; the 24 remaining ones determined 24 groups of S^4, of which two points are respectively lying on the two lines l' and l''.

The supports of these groups lie in the plane V, and the remaining two points of each of these groups are intersections of the plane V with the curve μ. In this manner 48 intersections are found.

There are 57 groups of S^4 that have a coincidence in V, while the two other points of those groups lie in the planes V' and V''. In each of these coincidences a point of V has coincided with the associated point of μ; in this way 57 coincidences of V and μ are found.

The plane V intersects the curve ϱ^9 in 9 points. Each of these points Q bears ∞^2 coincidences of S^4; the remaining points of these groups lie on the polar surface Π^5 of the point Q. This surface intersects the plane V' along a curve γ^5; among the groups mentioned there are consequently ∞^1, of which *one* point lies in the plane V'; the remaining points of these groups form a curve η. This curve η intersects the plane V' in the 9 points, in which V' intersects the curve ϱ^9; for, in each of these intersections the corresponding group has a coincidence, so that there a point of V coincides with the corresponding point of η. The curve η is therefore of order 9.

The plane V'' intersects the curve η^9 in 9 points. With each of the 9 intersections of V and ϱ^9 9 groups are consequently found, which have a coincidence in the intersection mentioned, while the two other points lie in the planes V' and V''. It is easy to see that these coincidences are in their turn intersections of the plane V with the curve μ; in this way 81 intersections are found.

The total number of intersections of V and μ, amounts therefore to $48 + 57 + 81 = 186$. This, therefore is the order of μ.

A plane V''' intersects the curve μ in 186 points. *There are consequently* 186 *groups of* S^4, *of which four points lie in four given planes.*

Zoology. — *"The colourpattern on Diptera wings."* By Prof. J. F. VAN BEMMELEN.

(Communicated in the meeting of March 31, 1917).

The investigation of the colour patterns on the wings of Lepidoptera brought me to the conviction, that in their markings original and modified motives of design could be distinguished, the first being arranged in strict dependency on the nervural system and in regular repetition over the whole of the wing-surface. This result foreshadowed the probability, that on the wings of other orders of insects similar arrangements of pattern might be met with, which would allow of a similar distinction between a primordial pattern and its later or secondary modifications: the primitive pattern in the same way being directed by the course of the veins in its distribution over the wing-surface. And as the comparative investigation of the nervural systems in different orders of insects had led to the final conclusion that all of them represent modifications of one common groundplan, the supposition that a similar fundamental connection might exist between the primitive colour-markings, occurring between those nervures, became extremely probable [1]).

Starting from this supposition, J. BOTKE [2]) argued, that the similarity between the colour-pattern on the wings of Cossids, Micropterygids, Hepialids and other Lepidopterous families, and those of Trichoptera and Panorpata, should not be considered as a merely accidental resemblance, but depended on a real homology. Evidently it is worth while to extend this investigation to the remaining Insect-orders.

That I take the Diptera as a starting point, is in no way due to the opinion, that a near relationship exists between this order and Lepidoptera, nor is it because I should consider the Diptera as especially primitive insects; it is in consequence of a recent (1916) publication by J. H. DE MEIJERE: Zur Zeichnung des Insekten-, im besonderen des Dipteren- und Lepidopteren-flügels, in which he has treated these two orders in succession, and noted equivalent features in them, though he has carefully abstained from drawing comparisons between them in details.

Now DE MEIJERE does not acknowledge my distinction between primary and secondary wing-markings, nor does he accept the

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[1]) Vide NEEDHAM and COMSTOCK, The wings of Insects, American Naturalist, 1898.

[2]) J. BOTKE, Les motifs primitifs des ailes des Papillons et leur signification phylogénétique. Onderzoekingen verricht in het Zoöl. Lab. der Rijks-Universiteit Groningen V, 19.6, Tijdschr. d. Ned. Dierk. Ver. 2de Ser. Dl. XV.

assumption of a similarity rooted in community of origin. On the contrary, as he considers absence of colour to be the primitive condition in Diptera, he believes in "the probability of an independent origin of colour in many different points of the group, because we meet coloured wings in so many and such different families."

But though he thinks that these different colour-patterns have arisen independently of each other, he accepts a connection between them in so far that he believes special regions of predilection for colour-formation to be indicative. These regions being either the nervures themselves, or the spaces between these nervures, his observation may be regarded as confirmation of my opinion, that the colour pattern is originally bound to the nervural system. The same may be said of the evident predilection for pigment-accumulation along the wing-margins and at its root.

I likewise fully agree with DE MEYERE, where he ascribes the formation of coloured transversal bars in many cases to the broadening of colour-seams along transverse veins, as well as when he attributes the cloudy "fumigation" of wing-areas to an extension of spots or blotches (which therefore originally must have been smaller).

All these phenomena may be considered as manifestations of the different manner, in which an original pattern can become modified and differentiated. The same is the case with the transformation of spots into transverse stripes, or the coalescence of two spots on either side of a nervure into one single blotch, which consequently will become divided by the vein.

The final proof that the more complicated patterns on coloured Dipterous wings may justly be considered as differentiations of one common primitive design bearing a simpler and more regular character, can only be obtained by showing that the formation of the definite pattern is preceded by the temporary presence of a preliminary pattern, possessing the above mentioned more primitive character; that is the same proof as I was able to obtain for Lepidoptera. But in expectation of this ontogenetic proof, it is allowable to heighten the probability of the supposition by adducing arguments founded on the comparison of fullgrown forms, which in the mean time may furnish us with a reliable image of this primitive pattern. For this purpose we have to start from the comparison of species belonging to the same genus, or to nearly related genera, and, having come to a conclusion about their ancestral form, to compare this with a similar one of various genera belonging to another family.

It might reasonably be supposed, that the chance of encountering a more primitive design would be greater in forms provided with a more original nervural system, which in Diptera is equivalent to a more complete one. However I hope to be able to argue that this is not necessarily the case.

Notwithstanding this I think it is preferable to start from forms with a less modified nervural system, e.g. the genus Haematopota, containing a number of species, whose wings show a rich but at the same time regular ornamentation.

Comparing the four species: italica, tuberculata, pluvialis and maculata, it is beyond doubt, that in all of these the colour-markings are arranged according to the same groundplan, which in italica, pluvialis and tuberculata is more fully and regularly developed than in maculata, the latter showing the proximal half of its wings almost destitute of markings, except a few irregular patches, while the distal part contains two transverse rows of dark markings, those of the outer row being smaller and more isolated, those of the inner (submarginal) broader and connected to a continuous band. In both parts however the extension of the markings is limited by

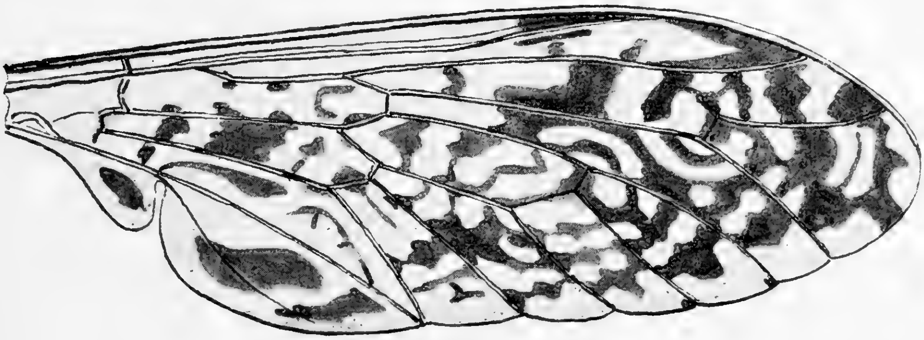

Fig. 1. Haematopota pluvialis.

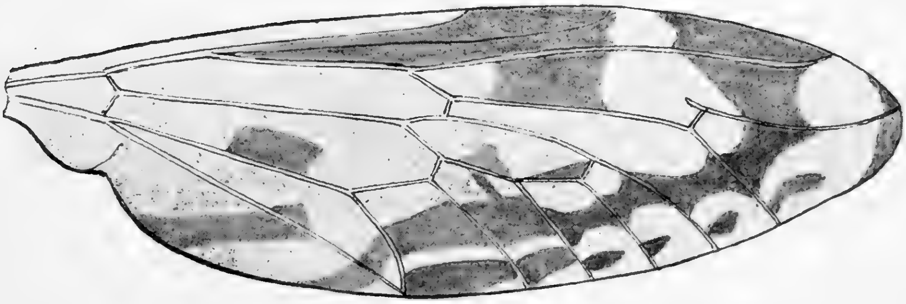

Fig. 2. Haematopota maculata (De Meijere).

the nervures and whenever they seem to pass these limits, the real cause lies in the meeting of two independent markings of similar extent along the course of a vein.

Fundamentally therefore the features on Haematopota-wings are identical with those we could remark on the wings of Hepialids amongst Lepidoptera: viz. an almost perfectly regular alternation of dark and light patches, filling out the areas (cells) between the nervures, but not passing over their borders, and occurring in like numbers in successive internervural cells, which necessarily brings about their arrangement in transverse rows parallel to the external wing-margin.

In those wingparts, where the regularity of the pattern is interrupted, we clearly see the modification by which this is brought about, e.g. where two neighbouring transverse markings are coupled together by a longitudinal bar, or where they curve over into each other.

Though I hesitate in ascribing importance to the configuration of the single markings, I will not abstain from pointing out the remarkable similarity between the dumbbell-shaped light markings of Haematopota italica (corresponding to paired triangular markings in H. tuberculata), and the hourglasses of Hepialids.

The comparison of these four species of the genus Haematopota therefore leads to the conclusion, that the original condition of their wings is not the uncoloured state, but on the contrary that of a complete pattern extending over the whole wing surface, and consisting of light and dark patches in regular alternation, arranged between the nervures, alike in size and placed at equal distances, so as to compose zigzag-ranges of markings, transversely running in a direction parallel to the external wing-margin. In all these features therefore the pattern corresponds to that of Hepialids, Zeuzerids, Trichoptera and Panorpata.

Judging by v. d. Wulp's figure (Tijdschrift voor Entomologie Vol. 17, Pl. 8), the wing of Poecilostola angustipennis satisfies the above mentioned criteria for a primitive colour-pattern, in still higher degree than that of the Haematopotas; viz. in strict dependency on the course of the nervures and regular repetition of the same motive of wing-design. For here all internervural cells contain longitudinal series of numerous small spots, arranged all along both sides of the nervures. It is only in the third cell from behind, viz. that situated between first cubital and first anal nervure, that a third range of spots is seen along the middle-line of the cell, where the 2^d^ cubital vein would be found, had not that nervure obliterated.

In Poecilostola punctata (fig. by GRÜNBERG, Diptera, in BRAUER's Süsswasserfauna Deutschlands, p. 57) a similar median row of small spots is seen between first and second anal nervure, thus giving ground for the supposition that here also a vein may have been obliterated, viz. An_2. But at the same time amongst the small spots, which are arranged in rows, we find a number of larger ones distributed, some of them showing an irregular configuration, as if they had arisen by the coalescence of a certain number of smaller ones. These bigger spots lie in the first place at the end of the longitudinal veins, in the second place on forkings and junctions.

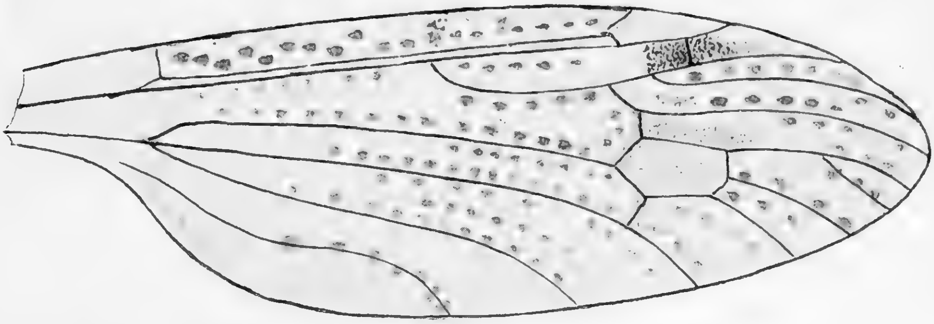

Fig. 3. Poecilostola angustipennis, (VAN DER WULP).

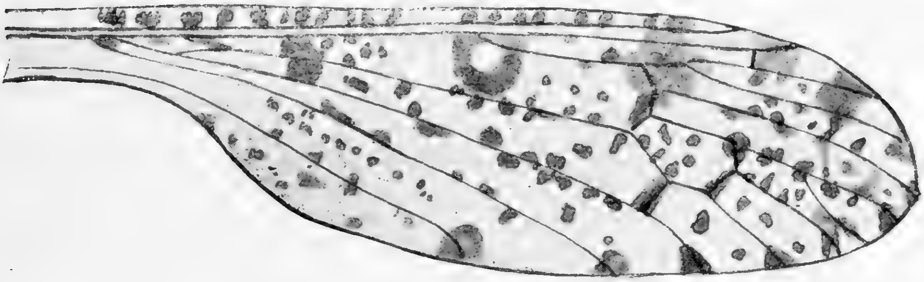

Fig. 4. Poecilostola punctata, (MEIG).

Extending our comparison to Acyphona maculata (GRÜNBERG p. 29) we find here all the above mentioned bigger spots, arranged in the same way, while the smaller ones with a very few exceptions, are absent. This suggests the conclusion, that the absence of the small spots is caused by their obliteration.

Further extending our investigations to forms with a reduced nervural system, e.g. members of the genera Tephritis, Sciomyza and Traginops, we see the same feature as in the Haematopotas, viz. regular alternation of light and dark patches, arranged in rows along the veins, but modified and complicated in so far as either a dark or a light streak extends over the middle of several internervural cells, with which in the first case the dark nervural spots, in the second the light ones may be connected. This distribution of the colour therefore brings about two different effects: when the median is dark the uncoloured areas present themselves as light spots on a coloured background, when it is light the pigmented areas form dark spots on a light field. Yet both patterns are varieties of the same groundplan, as may be seen by comparison of different species belonging to the same genus; e.g. Tetanocera vittigera, showing dark spots on a clear wing-surface and T. umbrarum, with light spots on a dark ground. But the same conclusion can be drawn from the consideration of different areas of the same wing: one internervural cell containing a dark median bar with light spots at either side, a neighbouring one the opposite arrangement, or even the proximal half of one such cell differing in this respect from the distal one.

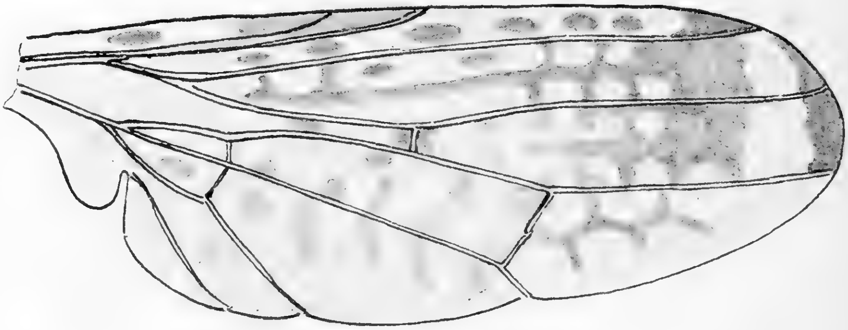

Fig. 5. Sciomyza javana (de Meyere).

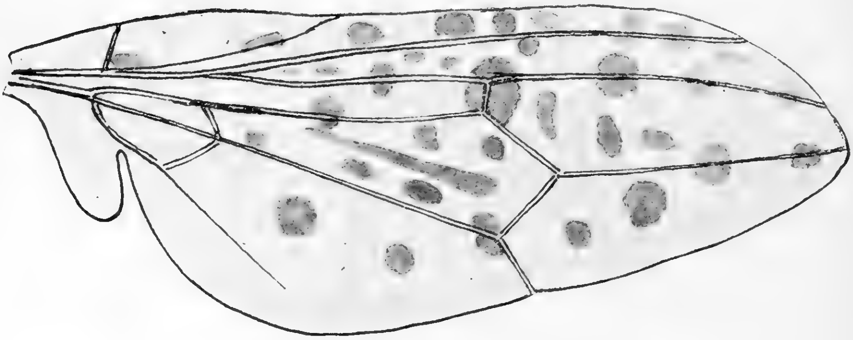

Fig 6. Traginops orientalis.

Though number as well as size of the spots is found to be inconstant, I am inclined to assume, that also in these respects a fundamental condition exists, while the deviations must be explained either by coalescence of dark spots to larger coloured areas, or on the contrary by reduction of the pigmented parts, bringing the light areas into predominance. Also in these animals the proximal half of the wing generally is the less coloured, showing fewer and smaller spots, resulting in their total absence in the departments of lobus and alula.

Turning to those cells, which contain either a dark or a light median streak, these are seen to be precisely those which may be supposed to have been formed by coalescence of two neighbouring internervural areas, by the obliteration of the nervure originally separating them. In Sciomyza javana, which shows these dark median bars to perfection, they stretch along the spaces, which in Diptera with a more complete nervural system are occupied by the fourth radial and the first median vein. We therefore are justified in assuming, that we have here the same phenomenon which is seen in Lepidoptera-wings (esp. Papilionids and Danaids), where the course of obliterated veins in the discoidal and cubito-anal cells is marked either by black or by light streaks of pigment.

De Meyere also has given his attention to the pigmentation of the median streaks in internervural cells, for he says on p. 58: ¹) In the cells of Diptera-wings median rows of spots occur relatively seldom in typical array. As examples may be given: Sciomyza Schönherri, Hydrophorus nebulosus (between the 1st and 2nd and the 2nd and 3rd longitudinal vein respectively, while pigmentation of the end of the nervures, seams of colour along transverse veins, and traces of a double row of spots in the upper part of the middle of the hindmarginal cell also occur), Ilythea spilota, Scatella quadrata, furthermore some Pterocallines. In several species related to the first mentioned ones especially among Tetanocerinae, the median spots are well marked in the hindmarginal cell, a double row of spots occurring in the remaining longitudinal cells. As in Schönherri the spots often already appear as transverse streaks, and as in other cases there is an evident connection between two spots lying one above the other in the same cell, as indicated by their position, it is my opinion, that such a double row of spots must be considered as resulting from the division of a median row. I may cite a typical example in Tetanocera (Pherbina) punctata, which at the same time shows a further feature, viz. *a median longitudinal streak in the cells*".

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¹) Translated from the German original by me.

And on p. 69: "Interesting features are shown by the Sciomyzidae. Many of their species show colourseams of the transverse veins; pigmentation of the longitudinal veins is found in Tetanocera elata, Elgiva lineata, of the anterior margin in T. elata. Marked punctuation occurs in numerous forms, e. g. T. punctata; the points usually being arranged in one row in the hindmarginal cell, in two rows in the remaining cells. These rows lie along the side of the nervures, or in other words, each nervure lies between two rows of spots: the spots to either side often corresponding to each other, though not always; also the spots placed along the two borders of the same cell frequently form pairs. On page 58 I argued that I consider these two lateral rows of spots as derived from the division of one median row. In T. punctata an accessory median bar is only slightly developed, but in other species this bar and the lateral spots are intimately connected; coryleti and unguicornis already show this connection more clearly than punctata and marginata, fumigata etc.; it leads to preponderance of the dark colour, only two rows of hyaline spots being left free. In punctulata and umbrarum the scheme of the colour pattern is still further differentiated by the more specialised character of the spots in certain transverse bands."

"Among Sciomyzines Sc. albocostata, cinerella, fumipennis show a marked striation of the longitudinal veins, leading in the last named species to almost complete vanishing of the ground colour. Spots on the transverse veins are found in griseola amongst others, transverse bars are developed in bifasciella, a median row of spots in the hind marginal cell is characteristic of Schönherri. By bipartition of these spots I should be inclined to explain amongst others the condition of Tetanocera punctata, possessing double rows of spots. The same degree of differentiation has been reached by Sciomyza javana, which in Sc. albocostata is accompanied by a white discoloration of the margins. In this family also therefore, we see several different motives of markings among nearly related forms."

De Meijere therefore, though acknowledging the fundamental equivalency of dark spots on a light ground and light spots on a dark one, does not arrive at the conclusion which I have drawn from the presence of a median bar; on the contrary he considers the two paramedian rows of spots in a cell as derivatives of one median row. A case like Poecilostola, where in one single cell a median row occurs between two lateral ones, and it is exactly this cell which belongs to those internervural spaces that may be supposed to have originated by the coalescence of two neighbouring cells by obliteration of the separating vein, remains unexplained on DE MEIJERE's hypothesis.

Yet it must be conceded, that it would not be reasonable to assume a coalescence of cells in every case where nervures are accompanied at both sides by rows of light or dark spots, however far the inference, founded on the comparison of Sciomyza to Traginops, seems justified to me, that a dark median bar can be replaced by a light one, and in this way all traces of the original duplicity of the cell can be obliterated. In the case of e.g. Poecilostola such an assumption is out of the question, except for the cells Cu_1—An_2 and An_2—An_1. The same feature is seen in Hepialids among Lepidoptera, which also often show the hourglass- or dumbbell-shaped spots separated into two halves, adjoining opposite nervures, and so forming paramedian rows of independent spots. Black median bars may therefore also be simple remnants of the general dark ground colour.

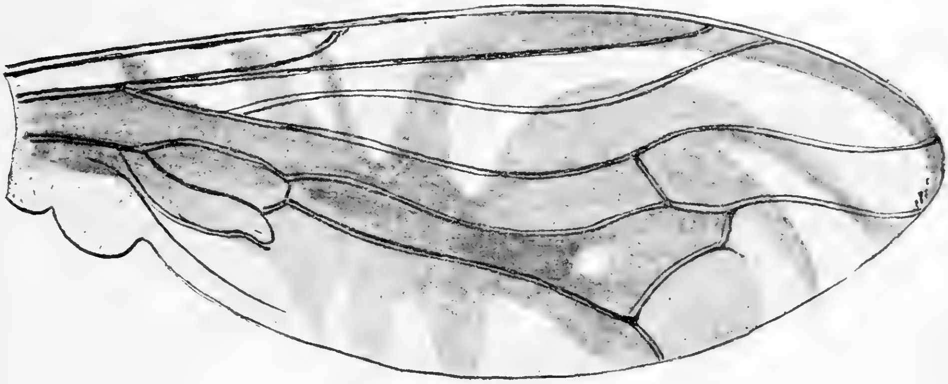

Fig. 7. Cleitamia astrolabei ♀ (Boisd).

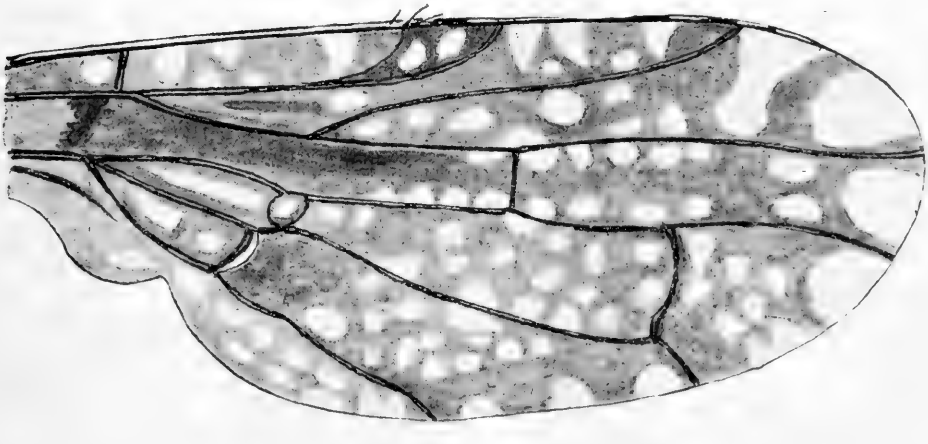

Fig. 8. Trypeta cribrata ♀ (v. d. Wulp).

Should this supposition, as explained in the foregoing, of a regular and simple but complete, original colour-design, common to all Diptera, be true, then it must be possible to bring even the most complicated and variegated patterns occurring in this order, into connection with this fundamental design.

The first test I was able to make in this direction, immediately gave the surprising result, that this comparison proved remarkably easy for a pattern so capricious as that of Cleitamia astrolabei,

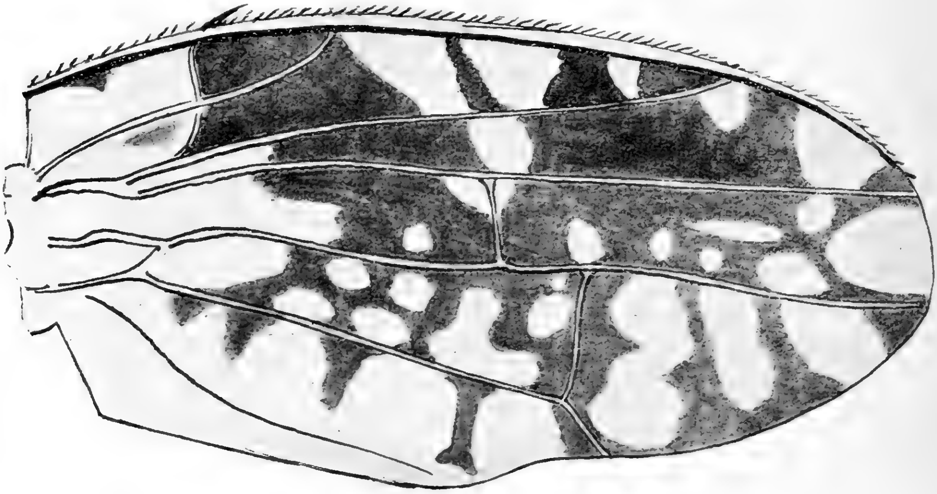

Fig. 9. Tephritis pantherina.

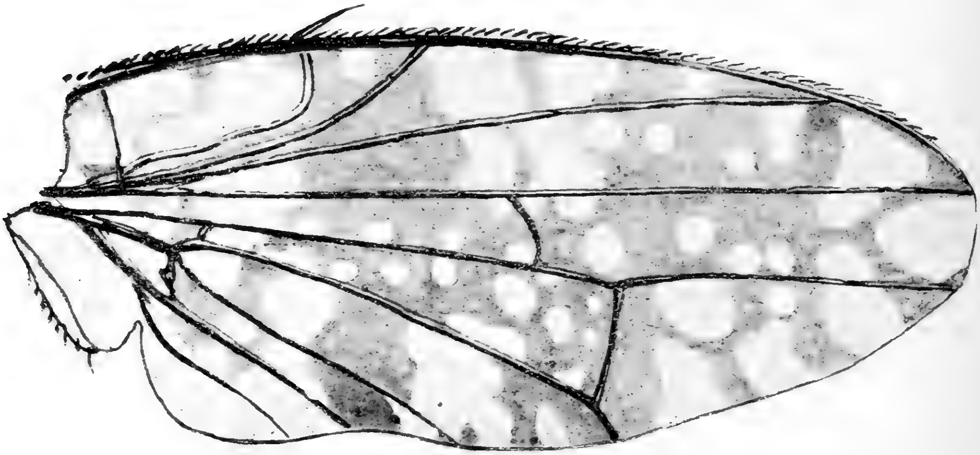

Fig. 10. Tetanocera (Dictya) umbrarum.

consisting of an irregular central dark area, from which ten bars radiate to the circumference in different directions, some of them being straight and short, others long and curved, but all apparently without any regard to the course of the nervures. For comparing this spider-like colour-pattern with that of Tephritis pantherina [which down to minute details agrees with that of Tetanocera (Dictya) umbrarum and consists in both of a considerable number of light spots on a dark ground, strictly keeping within the limits of the nervures, but often corresponding and fusing in adjoining cells] we become convinced that the lighter areas of Cleitamia exactly correspond in arrangement and size to marginal light spots of Tephritis. The single assumption we have to make is, that along the distal part of the front margin six of these spots have coalesced, and morover have become separated from that margin by a narrow rim of dark pigment, at the same time that a number of spots, especially at the proximal part of the front margin, form connections with more centrally placed ones and so constitute transverse light bars, which extend over the radial nervure. The number of these spots is not absolutely constant, though showing a certain regularity, as becomes evident by the comparison of the wings of Trypeta cribrata with those of Tetanocera umbrarum, showing that spots, which in the latter have coalesced, still remain independent in the former, T. cribrata therefore probably representing a still more primitive state.

As in the case of the species of the genus Haematopota, here also it is possible to arrange a number of nearly related forms in a series, showing a regular transition from the primitive condition: numerous similar light spots in rows on both sides of the nervures, and in certain cells also along the median axis of the internervural space, larger light patches occurring along the wing-margins [e.g. Trypeta cribrata] — then continuing through forms like Tetanocera umbrarum and Tephritis pantherina, in which the number of the spots is diminished, in consequence both of coalescence and of obliteration by pigment-ingression (obscuration) — until we culminate in a form like Cleitamia astrolabei, with its large but less numerous light areas, which differ considerably amongst each other, and do not seem to respect the limits of the nervures.

In the opposite direction Tetanocera umbrarum may be compared in detail (e. g. the number of the spots) with Tetanocera vittigera, on the simple assumption, that the light areas of the former have enlarged to such an extent, that they have coalesced in the middle of the cells and in so doing have cut up the dark background into fragments which in their turn now give the impression of spots. Traces of the original extension of that dark background may still be seen in cell R_5—M_1 and M_1—Cu_1 in the form of faint dark middle-bars.

But besides Cleitamia astrolabei a number of other species of the same genus occur,[1]) showing a wing-pattern of apparently less complication and capriciousness, which makes them appear simpler and consequently more original. On more exact comparison however, with each other as well as with the patterns in nearly related genera, we become convinced that we should read the series from the other end, starting with astrolabei (or better still with osten sackeni, in which the number of light spots along the front margin is one more than in astrolabei, and so reaches to five) passing along C. biarcuata (Fig. 136), similis (Fig. 134), amabilis (Fig. 128), liturata (Fig. 129), to arrive at kertészi (Fig. 135), whose distal wing-half is almost filled up by a single, broad, dark bar, extending from fore- to hind margin, and only leaving a narrow hyaline halfmoon of white at the wingtip, the proximal wing-half in the mean time not showing an elaborate pattern, but only one dark longitudinal patch in the middle, accompanied along its fore- and hindside by a light streak, and separated from the distal colour-field by a light transverse bar.

As in so many other cases, here also the simplest colour-design is in reality the most modified; it comes nearest to general unicolourism selfcolour).

Starting again from Cl. ostensackeni, we may also go by rivelloides (Fig. 131) and similis (Fig. 132) — while in passing we remark the similarity with the colour-pattern in the genus Bothrometopsa, belonging to the fam. Pterocallinae —, and so come to the species gestroï (Fig. 130), which, though vastly differing in design from kertészi, still exceeds this in extent and uniformity of the dark area.

Besides the observation, that it is possible to find a connection between patterns of widely different appearance, and that one pattern may be derived from the other, a second remark may be made viz. that in different genera, and even in subfamilies and families, the same series of pattern is seen constantly to return, showing the identical interrelation between the different links of the chain. To represent this phenomenon in a marked way, it is desirable to designate the different types of pattern with special names.

For these names the chief motives of the pattern might be used, e. g. that of the paranervural rows of spots (as in Sciomyzas), that of the median series of spots (Scatellas), that of the straight transverse bars (Pterocerines), of the curved bars (Cleitamia astrolabei), of the numerous small spots (Coremacera) etc.

Yet on second thoughts this principle of nomenclature does not

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[1]) Compare: Genera Insectorum, Hendel, Platystominae Taf. 7.

seem advisable, as the majority of patterns are composed not only of a chief motive, but at the same time of one or more accessory motives, the latter often representing the remnants of the original design. For instance in the pattern of Cleitamia the light areas between the dark bars may, as argued above, be considered as coalesced light spots, and precisely these spots constitute the original pattern.

Therefore it seems preferable simply to adopt the names of species, genera and families for the patterns which are shown in special clearness and completeness by them, e. g. the astrolabei-pattern, the scatella-design, the Pterocerine-system of bars. It should however never be forgotten, that these patterns (modified in sundry details), may equally well occur in other genera and even families of Diptera, and are also found in other orders of insects, it thus becoming a matter of chance, in what group of insects a characteristic type of pattern is first remarked and named.

As it becomes evident, that between these different types of design a genetic connection really exists, so that they can be arranged in a series, leading from the most primitive and regular to the farthest modified and most capricious, and that this series is the same for different interrelated genera and families, the conclusion, that this correspondence roots in relationship, is a natural one. I accept it in this sense, that the common ancestors of genera and wider groups of interrelated forms already possessed these different patterns, which passed into the hereditary predisposition of their descendants.

In this way the study of Diptera-wings has led me to the same general conclusion, as I was brought to by the intercomparison of Hepialid-wings, "that the motives and patterns of the colour-design are older than the genera and families which display them."

De Meyere has also noticed the phenomenon of the corresponding series of patterns, as is shown by his remark in the opening sentence of his paper: "It is only necessary to look over any tolerably extensive collection of Diptera, to become convinced, that in those families, where coloration occurs, the design may be widely different in the several forms belonging to them, a family-character therefore not being presented, *while on the contrary different families often show the identical patterns*" (the italics are mine).

And again on page 75: "therefore the various motives often return in the different families".

In this feature de Meijere however does not see anything more than a proof, that wing design in different families has developed in similar ways and so should be considered as a case of parallelism.

Of any interrelationship between patterns there could be no question, even within the limits of the same family: "Even where the nervural system is the same, we find either a striation of the transverse nervures or of the longitudinal ones, either a colouration of the wing-root or of the tip. Therefore it would be a mistake to try to arrange the patterns even of one family into one single evolutionary series."

On p. 70 DE MEYERE remarks: "In the group of Trypetines we meet with a number of patterns, which cannot be brought into connection with each other."

It follows from my remarks in the foregoing paper, that I have come to the opposite conclusion.

On p. 63 DE MEYERE calls attention to the fact that: "the broad-winged Trypetine fly Platensina ampla carries some spots which differ from the common hyaline ones in their hue, which seen under a certain angle is dead-brown." The difference, according to his view, is only due to a lighter staining of the chitinous layer and of the hairs arising from it.

By the kindness of my colleague DE MEYERE I was able to investigate a specimen of this fly, and was in the first place impressed by the fact, that these apparently dead spots did not occur over the whole wing-surface, but left the margins free, where, in the usual places also occupied in other fly-species, hyaline spots occurred at regular distances from each other. It furthermore awakened my curiosity, that these central spots, though evidently dead, i.e. not-diaphanous, did not show a brown, but on the contrary a light blue shade, and were slightly lustrous. Noting their position in relation to the hyaline, it became evident, that a bluish spot was situated in the prolongation of each hyaline marginal spot, with the exception of the fifth (adjoining the extremity of the subcostal vein), which was distinguished from the more distal marginal spots by greater length (in a transverse direction) and by a constriction in the middle. This difference might be also expressed by saying that the fifth spot bears a bead-like appendix, by which it extends farther towards the centre of the wing than its companions. It is this appendix which occupies the same position in relation to the peripheral part of the spot, as the blue spots do to the remaining hyaline spots. And furthermore these blue spots occur in exactly the same positions, where in other Trypetines hyaline ones are seen.

I therefore came to the conclusion, that the blue spots are nothing but vanishing hyaline ones, which become obliterated by penetration of the ground colour. Their occurrence is in my opinion a proof for the assumption, that unicolourism (self colour) is a secondary feature, originating from the effacing of a pattern of spots. Of the way in which this is effected Platensina ampla gives us a good idea.

As remarked above, DE MEIJERE calls the shade of the abnormal spots light brown, when seen in a certain direction, but he leaves undetermined, what direction this is. Now I found, that the colour is very different according to its being observed in reflected or transmitted light. Seen in the latter, the spots are actually brown, but with the first mentioned illumination they are light blue with a hazy lustre. Besides this there is a difference, if the transmitted light is made to pass straight at full strength, or obliquely and in moderate quantity. Only in the latter case the spots stand out clearly against their dark surroundings, showing a light-brown shade, and are clearly seen to transmit more light than the rest of the wing-surface. In strong and directly transmitted light on the contrary they hardly contrast with the surrounding dark wing-membrane, and can only be distinguished from it by a somewhat lighter ring round a darker core. By means of all three methods of observation, however, we may establish the hairs in the area of the blue spots to be colourless, just as above the hyaline spots.

I therefore agree with DE MEIJERE, that the absence of colouring matter in the hairs within the precincts of the dead spots, contributes to their lighter shade, and that this shade furthermore proceeds from a scantier quantity of brown pigment in the wing-membrane. Still I wish to make a distinction between these two causes in so far that I ascribe the whitish-blue lustre of the spots more especially to the first, their hazel-brown shade in weak and obliquely transmitted light on the contrary to the second.

The occurrence of these dead spots therefore provides us with a new argument for asserting that in the order of Diptera the different colour-patterns stand in genetical interrelation, and for opposing DE MEIJERE's inference, that they are absolutely independent of each other.

On p. 70 (at the bottom) DE MEIJERE again mentions a case of two different kinds of spots in a Trypetine fly, viz.: "Tüpfelflecke" in the dark transverse bars, which in several species (especially those related to the genus Tephritis) should be distinguished by shade as well as by localisation from the common hyaline spots situated between these bars, which are considered by DE MEIJERE as remnants of the original unbroken hyaline wing-surface.

The author does not mention the species he means, so that I cannot tell precisely which cases he has in view. But if, as I presume, he means the brownish tint of part of the light spots, as seen for example in Oxyna parietina, — where a certain number of spots, localised in the broad dark transverse bands, differ from the rest of the spots between these bars in a yellow brown shade as well as in smaller size, — I cannot agree with his distinction of two kinds of spots. For in other species of the genus Oxyna, I see in the same localities similar spots, differing only in so far as they possess the usual hyaline aspect and are not smaller than, or in any other way distinct from their companions in the interspaces between the bars. Therefore I cannot consider the "Tüpfelflecke" themselves as an addition to the pattern, but only their hue and diminutive size as secondarily acquired properties, which, far from leading to the evolution of new spots, on the contrary contribute to the disappearance of existing spots.

Groningen, March 30 1917.

Zoology. — *"On the Setal Pattern of Caterpillars."* II. By Dr. A. Schierbeek. (Communicated by Prof. J. F. van Bemmelen).

(Communicated in the meeting of March 31, 1917).

In a former communication [1] I called attention to the constant arrangement of setae on the body of caterpillars, and I gave the reasons which induced me to propose a new nomenclature for these setae, representing them in a set of schematical figures.

My investigations led me to the following conclusions:

1. setae (bristles), tubercula (eminences bearing setae), verrucae (warts), scoli (spines) and maculae (pigment-spots) are all of them homological structures.
2. the abdominal segments possess a setal pattern of more primitive arrangement than the thoracic.
3. the system of bristles which I have designated as type **I** is the most primitive, the remaining types (**I***a*, **I***b*, **II**) may be derived from it.
4. changes occurring in type **I** possess a definite systematic value.
5. stripes have developed at a later date than pigment-spots.
6. the design of the pupa shows the nearest resemblance to that of the first larval instar, but often deviates considerably from that of the last one.

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[1] A. Schierbeek, *On the Setal Pattern of Caterpillars.* Proc. Roy. Acad. Sc. Amsterdam. Sect. 2, Vol. XXIV. p. 1710—1723; March 25, 1916.

Since then I have continued my investigations and published a paper containing the complete descriptions, the necessary illustrations and the list of literature as announced in my preliminary communication. [1])

The remaining conclusions to which my further investigations have led me, may be summed up here:

In the first place I have reconsidered the question as to the exact position of the rudimentary stigma on the thoracic segments and have come to agree with Boas, who asserts the metathoracic stigma to have moved forward unto the intersegmental membrane between meta- and mesothorax, while the so called prothoracic stigma is in reality that of the mesothorax.

A spot or some other mark of similar character, occurring in the place where the stigma might be expected, in reality corresponds to the rudiment of the wing. [2])

In the second place I devoted my attention to the number of the abdominal segments.

In accordance with Poulton (1890) and Spuler (1910) I think I am able to trace an eleventh abdominal segment in certain caterpillars, during their first instar; viz. in *Hepialus hecta* L., *H. c.f. lupulinus* L., *Phalera bucephala* L., *Sphinx ligustri* L., *Pieris brassicae* L. and *P. napi* L.

The impossibility of demonstrating the presence of this eleventh segment in the majority of Lepidopterous larvae can be explained by taking into consideration the fact, that the moment of piercing of the egg-shell by different insects is not a fixed point in the course of development, but is dependent on the quantity of food-yolk etc. (Henneguy 1904). This explanation of the difference in the stage of development at the moment of emergence from the egg is the more probable, as Chapman (1896) was able to show that a great deal of variety exists between the eggs of different *Lepidoptera*. [3])

Furthermore I made a closer comparison between the markings of caterpillars with those of other insect-larvae.

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[1]) A. Schierbeek, *On the Setal Pattern of Caterpillars and Pupae*, Inaug. Dissertation. Groningen. 20 Jan. 1917. This study is also published in: Onderzoekingen verricht in het Zoölogisch Laboratorium der Rijks Universiteit te Groningen. Part VI, and in Tijdschr. Ned. Dierk. Ver. 2e Ser. Vol. XV.

[2]) See also Paul Mayer, Jenaïsche Zeitschr. f. Naturw. 1876.

[3]) This most accurate investigator writes me kindly that he has solved the problem of the difference between Packard and myself in 1887. He was able to show that there are two races of *Orgyia antiqua* with a different number of moultings. The sexes of these races also were unequal in this respect. (♂3, ♀4 and ♂4, ♀5; *Ent. Month. Mag.*)

In my former paper I called attention to the fact that within the limits of a single family different modifications of the setal pattern have taken place, the final impression therefore being that the families have differentiated themselves independently and parallel to each other.

According to HANDLIRSCH (1903, '06, '10) the splitting up of the *Lepidoptera* into their different families took place in the Cretacean period and after it; while the differentiation of the orders of Hexapoda belongs to the Inferior-Carbonic time. A close similarity is therefore not to be expected a priori. Moreover a monophyletic origin of the Holometabola certainly cannot be considered as an indisputable fact.

When therefore we meet with any similarity in pattern, we may explain it as a remnant of the markings on the primitive insects, or of the first Holometabola, but we could equally well imagine the pattern to be of such high biological value, that it has developed in a corresponding manner in several ordines independently of each other.

Notwithstanding this I am of opinion that, however important the possession of setae may be, their special arrangement cannot possibly be of any biological account, the correspondence in pattern therefore probably is a consequence of community of descent. For want of material of other insect-larvae than *Lepidoptera* I had to rely exclusively on illustrations in the entomological literature, which for the greater part do not excell in accuracy and clearness. The youngest instars especially, which are by far the most important, are generally wanting.

According to HANDLIRSCH the *Panorpids* closely approach the ancestors of the *Lepidoptera.* Their larvae have received an accurate investigation from BRAUER (1851—'63). In the beginning they wear setae, afterwarts verrucae of highly interesting form. Judging from BRAUER's figures three setae occur on either side above the stigma on each segment.

The *Tenthredinidae* are frequently considered as very primitive *Hymenoptera.* DYAR (1894) thought that he could deduce the markings of caterpillars from those of the *Tenthredinid* larvae. On every segment three vertical rows occur on either side, each row counting three setae.[1])

Even if we accept a monophyletic origin of the *Holometabola, Coleoptera* cannot be taken as near relatives of the *Lepidoptera.*

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[1]) This agrees very well with the three subsegments of JANET (1901).

It is therefore the more remarkable that TOWER (1906) met with two vertical ranges of spots on the abdominal segments of *Leptinotarsa*-larvae, each consisting of three spots.

Comparing to this the setal pattern Type **I** [which I described in my former communication as consisting of a row of setae above the stigma, composed of s.s. dorsalis, dorsolateralis and suprastigmalis], a great similarity in design between these larvae, otherwise so different, may be remarked.

But only an extensive investigation of the greatest possible numbers of larvae of different orders, especially the youngest instars, will lead to conclusive evidence on the question whether the pattern, distinguished by me, really possesses a general meaning for all holometabolic insects. It will be important to compare it to fossil remains of insects, supposing this to be possible. As far as I can see, HANDLIRSCH's doubts about a monophyletic origin of *Holometabola* are well founded.

On one point, which in my previous publication I only touched slightly, I wish to dwell a little longer.

In 1912 J. F. VAN BEMMELEN called attention to the correspondence in design between pupae of different *Rhopalocera* and a fullgrown caterpillar of *Pieris brassicae* L. Though this similarity between the markings of larva and pupa had already been remarked in a single case by POULTON (1890), he had explained it by assuming the larval pigment to remain for a certain period unaltered in the pupal skin, and thus giving origin to a temporary or a lasting design. VAN BEMMELEN objects to this that such an explanation will not do in cases where the larvae are very different from each other, while the pupae show an almost identical pattern, e. g. *Pieris brassicae* and *P. napi* and *Euchloe cardamines*, especially as he was able to show that this same pattern also occurred in other *Rhopalocerous* pupae. He therefore took this phenomenon to be an affirmation of the opinion, which sees in the pupae a subimaginal stage reduced to immobility.

Moreover I wish to call attention to LAMEERE's assertion (1900), that the hypodermis is totally renewed during the pupal period.

For me VAN BEMMELEN's opinion possesses greater probability than that of POULTON, the pupal design being of real morphological importance. As stated in my former note, the pupal pattern is almost completely similar to that of the first instar of the larva.

Starting from the observation that in *Rhopalocerous* pupae provided with a colour pattern this pattern is almost identical in a great many genera and taking into consideration, that the first deposition of pigment takes place around the base of the setae, the arrangement of pigmental spots may be identified with the setal pattern. If this view be correct, the opinion of EIMER and his followers, who attribute a high value to longitudinal stripes as a primitive element in design, becomes untenable and that of J. F. VAN BEMMELEN (1912) and TOWER (1905), to which DE MEIJERE has recently concurred, gains greatly in probability.

Supposing the homology between colour-design and setal pattern to exist in fact, it follows that not only those chrysalids which exhibit coloured markings should be examined, but likewise those without colours, though provided with setae. In this respect it is highly remarkable that as early as 1670 SWAMMERDAM perceived the significance of "hairs" (bristles) on the pupa for the explanation of the chrysalid stage.

Numerous uncoloured pupae possess a setal pattern, nearly always corresponding to my type **I**. Therefore I am convinced that the original chrysalid-pattern consists of setae arranged according to this type and most probably provided with pigmental accumulations at the base of the hairs.

Now CHAPMAN (1893—'96) showed that *Lepidopterous* pupae generally are not so immobile as is commonly supposed and that the more active pupae belong to the most primitive groups of Lepidoptera.

Combining these two facts, we are led to conceive the primitive chrysalis as being provided with a tolerable power of motion and decorated with a setal pattern in the same way as the larva. This conception completely harmonizes with the view that the pupa represents an immobilized subimaginal stage. It is generally acknowledged that many other facts point in the same direction, e.g. the preliminary colour-pattern on the wings (J. F. VAN BEMMELEN 1889), the external sexual differences, already described in 1843 by RATZEBURG and rediscovered independently by JACKSON and POULTON in 1890, the smaller degree of difference in size of antennae and wings between male and female during pupal than during the imaginal stage, as pointed out by POULTON in 1890. With J. F. VAN BEMMELEN I feel justified in adding to these arguments the existence of a pupal colour-pattern.

As the pupal pattern generally agrees with that of the first larval instar, but only with that of the full-grown larva in those cases in which the latter has retained the primitive design, I feel justified in drawing the conclusion that the pupal stages as well as the first larval instar bear a primitive character in distinction to the later larval instars, which represent secondarily introduced phases of development.

On quite different grounds the same assertion has been defended by DEEGENER (1909).

Considering the regular occurrence of a macula dorsolateralis on the pupa situated between m. dorsalis and m. suprastigmalis (even when a seta dorsolateralis is wanting on the abdominal segments of the larva), the pupa may be said to have best preserved the original hexapodal colour-design of the insect, at least in this instance and in the ornamentation also the thoracic segments which in other respects have been so profoundly modified.

The Hague, March 1917.

Physiology. — *"Distance-relations in the Effects of Radium-radiation on the isolated Heart"*. By Prof. H. ZWAARDEMAKER.

(Communicated in the meeting of March 31, 1917).

The results of 34 initial experiments [1]), justified the present writer in establishing that an isolated frog's hart, fed after KRONECKER's method with RINGER's mixture deprived of potassium chloride, resumes its beats again after, a standstill when exposed to the radiation of mesothorium or radium. [2]) We used 6 mgrms of mesothorium enclosed in a glass bulb and 3 mgrms of radiumbromide under mica. On an average an exposure of half an hour was required for the restoration of the pulsations. [3])

Then however a good and regular contraction recommenced, the rhythm being about the same as when the heart was not yet freed from the circulating potassium. The mesothorium-tests were also successful when the rays had to pass through an aluminium screen 0.2 mm. thick.

In the meeting of February I could also demonstrate that potassium and rubidium, when either of them is contained in the

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[1]) H. ZWAARDEMAKER, C. E. BENJAMINS and T. P. FEENSTRA, Radium-radiation and cardiac action. Ned. Tijdsch. v. Geneesk. 1916 II p. 1923.

[2]) The previous removal of potassium from the circulating fluid is very essential, for exposure of the heart in situ or perfused with normal RINGER's mixture, does not alter the actual beats materially.

[3]) The intervals between the commencement of the exposure to radiation and the recurrence of pulsation differ very much. They depend on the velocity of perfusion, the natural frequency of pulsations, the time the inactive circulating fluid needs to cause a standstill, etc.

circulating fluid, are neutralised by uranium, thorium, radium, or emanation. To this series the radiation may be added, on the side of potassium and rubidium. Its effect may be neutralised when uranium or thorium are contained in the circulating fluid.

This led me to investigate the quantitative relations coming to the front when placing a mesothorium preparation of 5 mgrms in a glass bulb at various distances and when to a neutral mixture of 40 mgrms of potassium chloride and 10 mgrms of uranylnitrate the quantum of uranyl-salt, required for each distance, was added. [1])

Let us consider a frog's heart through which after the method of KRONECKER has been sent first RINGER's mixture and subsequently, for some time, a potassium-free circulating fluid composed of 7 grms sodium chloride, 200 mgrms of calcium chloride and 200 mgrms of sodium bicarbonate per litre. At a certain moment this inactive circulating fluid is replaced by one similarly composed to which has been added 40 mgrms of potassium-chloride and 10 mgrms of uranylnitrate. The heart thus supplied will soon lose its contractility. Next the preparation of 5 mgrms of mesothorium (in a glass bulb) is placed at a distance of precisely 8, 7, 6, 5, 4, 3, 2, 1 mm and we wait for the recommencement of the heart's beats, which on the average will take place after 13 minutes (minimum 1, maximum 60 minutes). Now some uranyl nitrate in excess is added to the circulating fluid in a solution containing 1 mgrm of uranium salt per c.c. This procedure is continued for some time along with the radiation until the heart stops again. In this case a slight touch will engender one systole. Should there be a short series at first then the wanted equilibrium is not yet obtained and a little more uranium has to be added. At length a standstill will ensue and may be maintained for 5—10 minutes (we took 5 minutes with a comparatively quick flow and 10 minutes with a tardy one).

The looked for equilibrium between radiation on the one hand and uranium in excess in the circulating fluid on the other was found at the following distance (See table p. 1163)

The equilibruim once established the relations may be altered again. The mesothorium may be moved nearer to the heart, or a larger quantum of uranium salt may be added. Either process is responded to by the heart's contraction and a second equilibrium may be looked for. In this way we often succeeded in finding even three successive equilibria.

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[1]) I have to thank Mr. T. P. FEENSTRA assistant in the Phys. Lab. for his painstaking assistance in these experiments.

Distance between mesothorium and the heart (mm).	Uranyl nitrate added to the primary mixture (mgrms).
8	2 (?)
6	5,5
5	7
4	8
3	12
2	20
1,5	30
1	40

When representing the data of the table graphically we get a curved line reminding one of an exponential curve. I, therefore, plotted the distances along the axis of the abscissae and along the axis of the ordinates the logarithms of the mgrms of uranium salt present in the successive equilibria over and above the 10 mgrms that from the outset were present along with 40 mgrms of potassium chloride per litre of circulating fluid. The curve thus originating is represented in the graph by a firm line.

As the figure points out the curve of the logarithms of the uranium dosage, counterbalancing the radiation, is approximately straight. A slight deviation is noticeable only for the greater distances.

This result is not what could be expected: alpha-rays do not come into play here, as the restoring influence is exerted through an aluminium screen of 0,2 mm and even a leaden screen of 0.1 or 0.2 mm. Of the remaining beta-rays much is absorbed in the heart, as was shown when the heart was contained in the leaden roof of a small ionising-chamber. The gamma-rays passed almost unchecked.

There are additional reasons for stating that the beta-rays in casu are biologically active. From the experiments just described it follows that the essential part is to a great extent absorbed by the air.

Already at the distance of 9.5 mm the quantum liable to be compensated by uranium is spent. At 6 mm only an equivalent of an extra addition of 5 mgrms of uranyl nitrate per litre, is present: the very quantum transmitted through a 0,2 mm aluminium screen placed at close distance ¹).

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¹) A radium preparation of 3 mgrm. has a uranium equivalent through an aluminium screen of 0,1 mm. In considering such values we should of course also

The excitation of fresh contractions of a heart, fed with a totally inactive circulating fluid (potassium-free RINGER's mixture) or with a fluid containing correctly proportioned antagonistic quanta of potassium and uranium is, therefore, to be ascribed to a broad bundle of beta-rays of very low penetrating power. That this non-homogeneous mass of weak rays should be absorbed in the air according to an almost exponential law is not in the least surprising. Nor is it to be wondered at that subsequently what comes through the air from various distances is completely absorbed in the organ. The foregoing experiments also show that the quanta of uranium, which at the several distances counteract the restoring influence of the radiation so that the heart's contractions excited by the mesothorium cease again, are subject to the same quantitative law the absorption of the radiation has to obey. As the distance increases these uranium quanta should be diminished in such a sense that their logarithm remains inversely proportional to the distance.

The qualitative relation induces me to assume that the biologically active quantum of energy shot out from the mesothorium in its weak beta-radiation, and the uranium dosage, required for the equilibrium, and arresting the contractility of the heart, are antagonistic. The radiation as well as the radio-active substance sent into the heart are to derive their activity from the mutually compensating energies carried along with them.

The character of the weak beta-rays, in which lies the whole gist of the matter, seems to be rather well defined by the above, at least when, as is the case in our experiments, the restoring influence is considered. An obliterating influence is effected also by the more penetrating rays and perhaps it may be for the very reason that we cannot get rid of them, that the recovery, obtained through artificial radiation, was always transitory, never persistent.

The following equilibria are obtained when working with a radium-preparation of 3 mgrms in the same way as has been described above for the mesothorium-radiation. This preparation yields the advantage of a more even area of radiation. (See table p. 1165).

We will also graphically represent these data by plotting the distance along the axis of the abscissae to the logarithms of the uranium doses as ordinates.

This gives a curve represented by the dotted line which again may be considered as approximately straight.

It is obvious that by extrapolating mentally towards the ordinate

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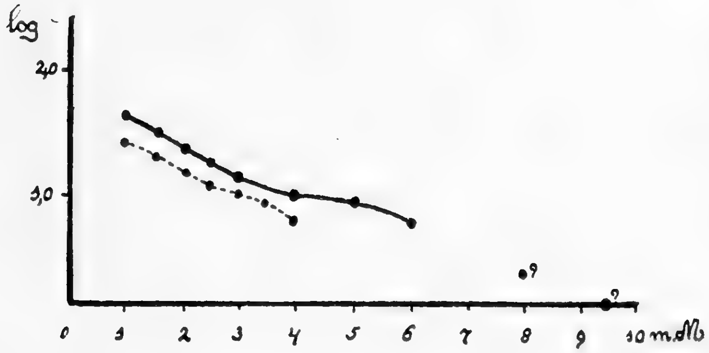
take into account the secondary rays of very low penetrating power that are generated in passing through the screen.

Distance between radium and heart (mm.)	Uranyl-nitrate added to the primary mixture (mgrms.)
4	5
3,5	6,5
3	9
2,5	11,5
2	15
1,5	18
1	22



——— Mesothorium-equilibria — — — — Radium-equilibria.

0 · a method may be found to determine the biological value of a radio-active preparation (in a recovering sense). But it is out of place here to deal with this practical application.

Botany. — "*The influence of light- and gravitational stimuli on the seedlings of Avena sativa, when free oxygen is wholly or partially removed*". By Dr. U. P. v. Ameijden. (Communicated by Prof. F. A. F. C. Went).

(Communicated at the meeting of February 24, 1917).

§ 1. *Introduction.*

Our conception of the influence of oxygen removal on geotropism and phototropism is mainly due to Correns[1]). His method of working

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[1]) C. Correns. Ueber die Abhängigkeit der Reizerscheinungen höherer Pflanzen von der Gegenwart freien Sauerstoffes. Flora 75. 1892.

consisted in placing the experimental plants in complete or partial vacuum, and was that employed by most investigators before him. The dependence of the geotropic stimulation process on oxygen was examined by partially exhausting the vessel containing the seedlings and then placing them in a horizontal position. The seedlings were then observed for 6—12 hours, to see whether curvature took place. Thus he found for instance that seedlings of *Helianthus* were still capable of reaction, when the oxygen was reduced to traces; *Sinapis* seedlings on the other hand required at least 4—5°/₀ of oxygen to develop a curvature. When no curvature took place, there was also no after effect in ordinary air. Correns concludes from these experiments that oxygen is necessary for the execution of a geotropic process.

His heliotropic experiments were so arranged that the seedlings were continuously exposed in the receiver to unilateral day light, and he concludes that oxygen is necessary also for heliotropic stimulation. The quantities of oxygen which just permit of heliotropic curvature differ, however from the minimum quantities allowing a geotropic reaction. Thus for a phototropic reaction of *Sinapis* seedlings the oxygen could not be reduced below 6 percent.

Geotropic curvatures are therefore, according to Correns, executed by the same objects at a lower pressure than phototropic ones. In my opinion Correns is not justified in drawing this conclusion from his experiments since he compares stimulation intensities, the result of which perhaps are curvatures of very different degree. Geotropic and phototropic stimuli can only be compared, if we employ stimuli of such intensity that they produce maximal curvatures of the same strength.

Árpád Paál[1]) published a paper dealing exclusively with the influence of rarefaction of the air on the geotropic stimulation process and therein considered separately the perception and the reaction, the former by determining the presentation-times under normal pressure and after evacuation to various extents, the latter by causing perception to take place at normal pressure and allowing the reaction to occur under reduced pressure. His experiments led him to the view that with diminution of pressure the presentation-times and the reaction-times are lengthened.

It should be noted in this connection that he of course still adhered to the old conception of presentation- and reaction-times and

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[1]) Árpád Paál. Analyse des geotropischen Reizvorgangs mittels Luftverdünnung Jahrb. f. wiss. Bot. L. pag. 1. 1912.

that he did not yet attach to them the meaning which was afterwards given them by ARISZ[1]). I have always used both terms in the sense in which ARISZ uses them, hence the apparent contradiction between the results of ÁRPÁD PAÁL and my own.

§ 2 *Methods.*

The experiments were all carried out at the same temperature by placing the boxes with seedlings in an electrically heated thermostat, in which the temperature was kept constant by means of a thermoregulator. In the middle of the back wall of the thermostat there was an opening through which the axle of the clinostat passed, enclosed in an oil packing, so that no air could enter from outside; this also secured the easy rotation of the axle. The end of the axle in the thermostat could be screwed into the clamp intended for holding the boxes with seedlings. The latter remained in the thermostat throughout the duration of the experiment and could in this way be stimulated geotropically as well as phototropically since the front and side walls were of glass, so that the plants could be rotated on the horizontal clinostat axis immediately after stimulation. All experiments were carried out with a single box of seedlings. The clamp was arranged for and held two boxes, but the second merely acted as a counterpoise in order to obtain as far as possible a uniform rotation of the clinostat.

All experiments were carried out under a total pressure of one atmosphere and therefore the air in the thermostat was gradually replaced by nitrogen diffusing in from the commercial metal cylinders. Since the latter contain 4—5 percent of oxygen, the gas was first passed through washing bottles containing alkaline pyrogallol, in order to remove the oxygen. Since, however, not all the oxygen was absorbed and CO was moreover formed, the gas was passed through a red-hot tube containing copper, in order to absorb the rest of the oxygen, and a little CuO, in order to oxidize the CO formed to CO_2. The gas treated in this manner was allowed to enter the thermostat and the air contained in the latter was thus gradually driven out through an exit. It took 1½ to 2 hours to wash out all traces of oxygen, as was shown by estimations with a phosphorus pipette.

In order to trace the influence of oxygen deprivation on the geotropic and phototropic stimulation processes I first carried out a

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[1]) W. H. ARISZ. Untersuchungen über den Phototropismus. Recueil des Travaux Botaniques Néerlandais. Vol. XII, 1915.

large number of experiments in air in order to obtain a standard, from which possible deviations might be measured. In these experiments I used stimuli of arbitrary intensity. Thus the geotropic stimulation consisted in placing the seedlings during a quarter of an hour in a horizontal position, which therefore means an intensity of stimulus of 900 mgs. Phototropic stimulation took place by exposure during eight seconds to a lamp, placed at a metre's distance from the middle of the box, with the seedlings arranged so that intensity of the light falling on them was 5 metre candles. This therefore corresponds to a stimulus of 40 metre-candle-seconds. I now determined the maximal curvatures corresponding to the two stimuli and the intervals of time between the beginning of stimulation and the attainment of maximum curvature, i.e. the reaction times. The result is, that for both stimuli the maximal curvature is 2 mm. and that the geotropic reaction time is 65 minutes, the phototropic 75 minutes.

§ 3. *Influence of oxygen deprivation on perception.*

In order to see whether *Avena* seedlings are able to perceive a stimulus in an oxygen-free atmosphere, I first left them for some time in the thermostat, while a current of nitrogen was passing. When the objects had in this way been deprived of oxygen for some time the stimulus was administered, when they were still in a nitrogen atmosphere; immediately afterwards the nitrogen current was stopped and ordinary air was sucked through the thermostat by means of an aspirator. In considering the length of the preliminary sojourn (fore-period) in nitrogen, given below, it must be remembered that this includes the 1½—2 hours, necessary to free the thermostat completely from oxygen. During the reaction time the seedlings were therefore in air; when this time was up the seedlings were removed from the thermostat and their curvatures were measured.

I. Geotropic experiments.

Two or three hours of preliminary sojourn in nitrogen had not the slightest effect. A subsequent stimulus of 900 mgs. expressed itself by a maximal reaction of 2 mm. in the air. A fore-period in nitrogen of 5 hours was clearly evident by a diminished response, and after 6 hours there was no reaction at all.

TABLE 1.

Strength of stimulus 900 mgs. Temperature 20° C. Reaction time 65 minutes.

Fore-period in nitrogen	Number of seedlings	Amount of curvatures in mm.
5 hours	5	1 1 ½ 0 0
	7	1 1 1 ½ 0 0 0
	5	1 ½ ½ ½ 0
	4	½ ½ 0 0
6 hours	8	all without curvature
	5	id.
	5	id.

2. Phototropic experiments.

In these experiments also a preliminary stay of three hours was without the slightest effect on the curvatures, even six hours in nitrogen were not quite sufficient to prevent the reaction entirely, but an eight hours stay in nitrogen before stimulation was enough.

TABLE 2.

Strength of stimulus 40 M.C.S. Temperature 20° C. Reaction time 75 minutes.

Fore-period in nitrogen	Number of seedlings	Amount of curvature in mm.
6 hours	5	1½ 1 ½ ½ ½
	8	1½ ½ ½ ½ 0 0 0 0
	7	1 1 ½ ½ 0 0 0
	4	½ ½ ½ ½
	6	1 ½ ½ ½ 0 0
8 hours	6	all without curvature
	6	id.
	8	id.

It is evident therefore, that the seedlings must be deprived of oxygen for a long time in order to lose their irritability altogether.

Since the possibility existed, that especially in the space between the cotyl and the first leaf, sufficient oxygen remained for a long time to account for the prolonged irritability, I repeated the experiments with seedlings of *Sinapis alba* and obtained with them mutatis mutandis the same results; we must therefore assume that the seedlings, as a result of intramolecular respiration, have sufficient energy at their disposal to perceive stimuli for a considerable time, albeit in lessening degree. We may not, however, conclude at once that no perception of stimulus can occur in the absence of oxygen, for it might quite well be that the stimulus is indeed perceived, but that the processes in the plant, which cause the reaction, have already been so influenced by the want of oxygen, that no curvature was possible. For these reasons I carried out geotropic experiments, in which the objects had a six hours' fore-period in nitrogen and phototropic ones, in which this period was 8 hours; in both cases perception took place in the air, this being therefore the sole point of difference from the previous set of experiments.

1. Geotropic experiments.

TABLE 3.

Strength of stimulus 900 mgs. Temperature 20° C. Reaction time 65 minutes.

Fore-period in oxygen	Number of seedlings	Amount of curvatures in mm.								
6 hours	8	2	1½	1	1	1	1	1	½	½
	7	1½	1½	1	1	1	1	½		
	7	2	1	1	1	1	1	½		
	8	1½	1½	1½	1	1	1	½	½	

2. Phototropic experiments.

TABLE 4.

Strength of stimulus 40 M.C.S. Temperature 20° C. Reaction time 75 minutes.

Fore-period in oxygen	Number of seedlings	Amount of curvatures in mm.								
8 hours	9	1½	1½	1	1	½	½	½	½	0
	8	1½	1½	1	1	½	½	½	0	
	6	1	1	½	½	½	½			
	8	1½	1½	1	1	1	½	½	0	

On comparison of the above experiments with those in tables 1 aud 2, it at once follows that since in those of tables 3 and 4 there were curvatures, and not in those of tables 1 and 2, seedlings are unable to perceive a geotropic or phototropic stimulus in the absence of oxygen.

The fact that the curvatures obtained in the later experiments are smaller than those obtainable under normal conditions proves, that the seedlings have undergone a harmful influence from the prolonged want of oxygen, which still makes itself felt after the normal conditions have been reestablished.

§ 4. *The influence of oxygen deprivation on the reaction.*

In order to study the influence of an oxygen-free atmosphere on the reaction, I gave the seedlings a preliminary stay of 3 hours in nitrogen, administered the stimulus in this gas, and left them without oxygen also during the reaction time. The earlier experiments had shown that after a fore-period of 3 hours in nitrogen, the stimulus is still perceived normally in this gas. In a few experiments I watched the seedlings for a considerable further time in nitrogen in order to see whether a curvature occurred later. In that case we should have to postulate a lengthening of the reaction time owing to absence of oxygen. In the other experiments I admitted oxygen at once after the normal reaction time had elapsed, in order to see whether there was any after effect in this gas.

1. Geotropic experiments.

The plants remain the whole time in nitrogen.

TABLE 5.

Strength of stimulus 900 mgs. Temperature 20° C.

Fore-period in nitrogen	Time elapsed since beginning of stimulation in minutes	Number of seedlings	Amount of curvatures in mm.
3 hours	65	7	all without curvature
	100		id.
	125		id.
	150		id.
	65	8	id.
	100		id.
	130		id.
	65	8	7 without curvature, 1 with asymm. apex
	100		id.
	130		id.
	65	7	all without curvature

2. Phototropic experiments.

The seedlings remain the whole time in nitrogen.

TABLE 6.

Strength of stimulus 40 M. C. S. Temperature 20° C.

Fore-period in nitrogen	Time elapsed since beginning of stimulation in minutes	Number of seedlings	Amount of curvature in mm.
3 hours	75	7	all without curvature
	100		id.
	140		id.
	170		id.
	75	8	id.
	105		id.
	130		id.
	75	8	id.
	100		id.
	135		id.
	75	9	id.

If I replaced the nitrogen by air after 65 minutes, or respectively after 75 minutes there was always a slight after effect, which was plainly visible about one hour after air had begun to be sucked through. This is further evidence, that the stimulus had indeed been perceived, but that without oxygen no reaction could show itself. That these curvatures were so slight, is a proof that the stimulus was already passing off, and therefore we cannot speak of a lengthening of the reaction time as a result of the absence of oxygen. These experiments show, that a perceived geotropic or phototropic stimulus is unable to give a reaction in the absence of oxygen; further that there are no indications in favour of a lengthening of the reaction time.

§ 5. *Influence of an atmosphere with low oxygen content.*

By passing the gas from the nitrogen cylinder straight into the thermostat, without passing it first through the washing bottles with pyrogallol and the tube with red-hot copper, the plants were in an atmosphere containing 4—5 $^0/_0$ of oxygen. I only investigated the influence on the perception, by giving the seedlings a fore-period in this mixture and allowing perception also to take place in it, and then letting any possible reaction occur in ordinary air.

1. Geotropic experiments.

TABLE 7.

Strength of stimulus 900 mgr. Temperature 20° C. Reaction time 65 minutes

Fore-period in 4.3 % oxygen.	Number of seedlings.	Amount of curvature in mm.								
6 hours	9	2½	2	2	2	2	1½	1½	1½	1
8 „	9	3	2½	2	2	2	2	1½	1½	1½
24 „	8	2	1½	1	1	1	1	½	½	

2. Phototropic experiments.

TABLE 8.

Strength of stimulus 40 M. C. S. Temperature 20° C. Reaction time 75 minutes.

Fore-period in 4.3 % oxygen.	Number of seedlings.	Amount of curvature in mm.								
10 hours	9	2½	2	2	2	2	1½	1½	1½	0
24 »	9	2	1½	1½	1½	1	1	1	0	0

After a stay of 24 hours in the mixture of nitrogen and oxygen an influence on the perception is noticeable in both cases. The seedlings therefore remain able for a long time to perceive a geotropic or phototropic stimulus in an atmosphere containing a relatively low amount of oxygen. Here also there is no indication of a difference in the reaction to these two kinds of stimuli, contrary therefore to the opinion of CORRENS, according to which a geotropic curvature can be executed in a lower percentage of oxygen than a phototropic one.

Utrecht, February 1917.

Physics. — *"On the diffraction of the light in the formation of halos"*. By Dr. S. W. Visser. (Communicated by Prof. J. P. Kuenen).

(Communicated in the meeting of March 31, 1917.)

I. *Introduction.*

The halos which originate by refraction are often seen distinctly coloured. This fact is best known as regards the circumzenithic arc, the parhelia and the tangential arcs, but the large circle is also often coloured and in the ordinary circle of 22° too, distinct colours occur.

In this paper the last circle will be principally dealt with. According to the common refraction-theory of halos, as developed by Pernter[1]) amongst others in his well-known work, the red on the inside of the circle should be distinctly visible, whereas the green and blue would already be very pale. The observations on the other hand show, that the colours are often practically invisible, but that sometimes they appear with great brightness, and the same holds for the other halo-phenomena mentioned above. These bright colours are explained by Pernter as regards the parhelia and the tangential arcs by the presence of a large number of ice-crystals, whereby the intensity of the light would be increased and the colour become visible[2]). But if there is nothing but white light or nearly white light, an increase of intensity cannot produce anything but more white: the distribution of colour cannot undergo any change by a greater intensity of light.

A remark by Pernter himself shows, how little the colours fit into the usual theory[3]):

"It should be mentioned, that in a description of the phenomenon, as observed on September 4th 1900 at Aix la Chapelle by Sieberg I find the following statement as to the colours of the smaller circle: "In addition it was distinctly coloured red, yellow, green and blue from inside to outside". As this observation would contradict all others, if it referred to the complete ring, I assume, that the colours were observed, where the parhelia were situated on the ring, in which these colours have also been observed by others".

But the colours of the parhelia are not in accordance with the theory either and moreover Sieberg's observation is not at all in

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[1]) J. M. Pernter and F. M. Exner, Meteorologische Optik, Wien 1910 pag. 319.

[2]) Pernter. l.c. p. 318, 320, 321.

[3]) Pernter, l c. p. 228.

contradiction with those of others. This will all be shown further on.

In this paper an attempt will be made by a modification of the theory in which diffraction will be taken into account to explain the colour-phenomena as observed.

We shall begin by tabulating a number of the various colours that have been recorded, chiefly taken from the publication of the "Koninklijk Nederlandsch Meteorologisch Instituut": "Thunder storms, Optical Phenomena etc. in Holland according to voluntary observations 1901—1914"[1]). This will be followed by a discussion of the simple refraction-theory for ice-crystals with a refracting angle of 60° in order to arrive at the colours which might occur in the ordinary circle. A diffraction-theory will then be developed and finally the colours will be deduced for a specially well developed halo of 22°, which will appear to agree very well with the observations.

II. *Survey of some of the colours observed in halos.*

I shall confine myself to those records, in which colours are mentioned by name. Lyrical rhapsodies like: "brilliantly, very intensely, strongly, magnificently coloured", especially numerous for the parhelia, cannot be utilized. In the fourteen volumes of "Thunderstorms etc." which I consulted the colours of the parhelia are named in only five cases!

With regard to the circumzenithic arc Besson expresses himself as follows [2]):

Les couleurs sont souvent remarquablement pures: on distingue en bas le rouge, puis le jaune en passant par l'orangé, puis le vert, puis mais pas toujours le bleu et le violet. Cette dernière couleur est fréquemment absente, mais il n'est pas rare, qu'elle soit visible très nettement. Fait important à noter: le violet, quand il existe, est très pur, il n'est pas surmonté ou mêlé de blanc, comme dans les arcs tangents au halo de 22°. La coloration offre une intensité des plus variables: faible parfois au point d'être à peine perceptible, elle est d'autres fois aussi éclatante que celle du plus bel arc-en-ciel.

These references, incomplete though they are, are sufficient to show the great variety of colouring in the halo-phenomena. Especially important from this point of view are those cases in which different observers mention the same colours.

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[1]) Onweders, optische verschijnselen, enz. in Nederland naar vrijwillige waarnemingen 1901—1914. (Cited in the following table as: "Onw.").

[2]) L. Besson, Sur la Théorie des Halo's, Paris 1909, p. 53.

CZA = circumzenithic arc; UTA = upper tangential arc; CH = circumscribed halo; C 22° = circle of 22°; C 46° = circle of 46°; P = parhelion (paraselene).

	Date	Colours	Accompanying halos	Place of observation	Reference
	a. Circle of 22°.				
1	1903, Apr. 17	strongly coloured	CZA brown, blue; UTA; P	Vrijenban	Onw.
2	Sep. 23	brown-red, violet		Zoutkamp	»
3	1904, Apr. 2	golden brown, yellow, green, violet	UTA uncoloured; B coloured	Nymegen	»
4	Aug. 14	faintly orange	lateral tangential arcs uncoloured	Valkenburg	»
5	1905, March 28	red predominant	UTA red predominant; C 46° green predominant	Zutfen	»
6a	Oct. 7	yellow, violet	C 46° red, green	Zutfen	»
b	»	red, yellow, green, violet	P; faint UTA	Nymegen	»
7	1906, Feb. 12	red predominant	UTA very brightly coloured; C 46° all colours	Zutfen	»
8	1914, Apr. 11	orange, dark-green, white (moon)	P faint, white	Renesse	»
9a	1886, May 3	very brilliant, rainbow hues, except green	TA same colours, fainter	Boulogne s. Seine	La Nature 14, p. 379, 1886
b	»	strongly coloured	CH; C 46° red, blue, faint	Angers	»
c	»	red, yellow, green, blue, violet, inside violet	CH in the same colours	Argentan	»
10a	1887, Jan. 28	red, orange, yellow, green, blue, indigo, violet	C 46°; CZA; UTA	Pithiviers	La Nature 15, p. 161, 1887
b	»	magnificent rainbow	P; double inferior tangential arc	Souppes	»
c	»	prismatic colours	C 46° prismatic colours; P bright pink; CZA	Fontainebleau	»
d	»	spectrum-colours	UTA; P	Orleans	»
	b. Parhelia on the circle of 22°.				
11	1901, Jan. 29	red, orange, yellowish white	two above one another	Zutfen	Onw.

	Date	Colours	Accompanying halos	Place of observation	Reference
12	1904, Sep. 28	white	C 22° brown-red; C 46° red, blue	Zoutkamp	Onw.
13	1905, Sep. 19	uncoloured	UTA red, yellow, blue (broad), violet (narrow)	Nymegen	„
14	1910, Sep. 7	red specially bright, remaining colours very lively, blue and violet, also distinguishable	C 22°; CZA; LOWITZ's arc	Zutfen	„
15	1911, Nov. 3	red orange, light-green	C 22°, C 46°; UTA and CZA in the same colours	Renesse	„
	Compare also 10c.				
	c. Upper tangential arc				
16	1903, May 5	red to violet	C 22°; CH	Zutfen	„
17a	1904, Apr. 26	goldish brown, green, blue	P; C 22°; C 46°	Vrijenban	„
b	„	brown-red, yellow, green, violet		Delft	„
18	Oct. 6	red, green, some blue		Zutfen	„
19	1905, Nov. 2	goldish brown, clear white, blue	C 22°; C 46°; P	Vrijenban, Delft	„
20	1907, Oct. 3	red, violet	C 46° red, green; C 22; P faint; CZA	Several	„
21	1910, Jan. 30	red-orange, light green	C 22°; C 46°	Renesse	„
22	Feb. 8	red to violet	C 46° pale red	Groningen	„
23	May 18	brown-red, violet (moon)	C 22°, corona	Zutfen	„
	Comp. also Nrs. 3, 5, 7, 9, 13, 15, 25.				
	d. Inferior tangential arc.				
24	1904, May 21	red-orange, yellow, green, blue, violet	C 22°; parhel. circle	Zoutkamp	„
25	1909, Apr. 22	yellow predominant	UTA very brightly coloured; C 22°	Zutfen	„
	e. Circle of 46°.				
26	1901, March 22	red, yellow, green		Munnikeburen	„

	Date	Colours	Accompanying halos	Place of observation	Reference
27	1906, Oct. 8	conspicuously lively colours	C 22°	Zutfen	Onw.
28	1911, March 8	faint red	C 22°	Renesse	"
29	Oct. 1	red		de Bilt	"
30	1914, Dec. 29	red	CZA red, blue; P; pillar	de Bilt	"
31	1890, March 3	all the colours of the rainbow	C 22°; P; CZA rainbow colours	Parc Saint Maur	La Nature, 18, p. 238, 1890

Compare also: Nrs. 5, 6a, 7, 9b, 12, 20, 22.

f. Circumzenithic arc.

32	1911, Oct. 1	all colours to violet	UTA	Zutfen	Onw.
33	Nov. 15	red, yellow, green	C 22°	Munnikeburen	"

Compare also: 1, 15, 30, 31.

III. *The refraction of light in ice-crystals of a refracting angle of* 60°.

We may confine ourselves to the phenomenon as it presents itself in one plane brought through the eye and the sun. By a rotation of this plane about the line eye-sun the circular phenomenon will be generated. It will also be allowable to consider exclusively those crystals whose refracting edge is at right angles to the plane chosen, seeing that the colours can only take rise on the inner edge of the halo, where the light which moves perpendicularly to the axis of the crystals chiefly contributes to the formation of the circle.

In order to deduce the colour which will be seen in a given direction it is necessary to determine the intensities of the various spectral colours in that direction: from these the resulting colour can be calculated.

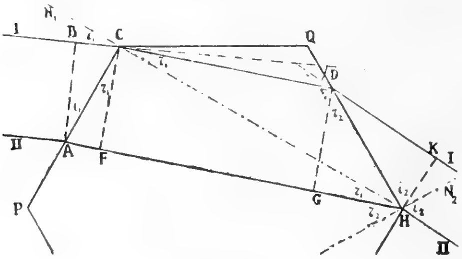

Fig. 1.

For an angle of incidence i_1 in the hexagonal prism *PCQH* (fig. 1; only the faces required for the construction are shown) the passing beam is confined within the rays I and II. In the direction of a given angle of deviation *D* there are two different beams of widths *AB* and HK respectively. The sum of the intensities of these emerging beams determines the intensity in the given direction. These two intensities may be replaced by those of the incident beams HK and AB. It is true that some light is lost by reflection, but no great error will be introduced by assuming that — on the inner side at any rate — all colours are weakened approximately in the same ratio by the reflections. The colour to be observed is not influenced by these losses: only the total intensity will be lessened. The intensities of the incident beams may be taken proportional to their widths and, as in the end we are only concerned with the ratios, the intensity may be put equal to the sum of the widths $AB + HK$ itself.

Calling the width of the side of the prism a, the angles of incidence and refraction i_1, r_1 and $i_2\, r_2$ (see fig.), we find:

$$AB + HK = \left(\frac{\cos i_1}{\cos r_1} + \frac{\cos i_2}{\cos r_2}\right) \sin r_1 \,.\, a \sqrt{3}.$$

$$\Bigg[HK = EH \cos i_2 = EG \frac{\cos i_2}{\cos r_2} = CF \frac{\cos i_2}{\cos r_2} = CH \sin r_1 \frac{\cos i_2}{\cos r_2} =$$

$$= \sin r_1 \frac{\cos i_2}{\cos r_2} . a\sqrt{3}; \; AB = AC \cos i_1 = CH \operatorname{tg} r_1 \cos i_1 = \sin r_1 \frac{\cos i_1}{\cos r_1} . a \sqrt{3} \Bigg].$$

For a given value of a the intensity L, leaving out a constant factor, may be put equal to:

$$L = \left(\frac{\cos i_1}{\cos r_1} + \frac{\cos i_2}{\cos r_2}\right) \sin r_1 \quad . \; . \; . \; . \; . \; . \quad (1)^1).$$

This function can be calculated for all values of i and n. Computing the deviation D corresponding to a given value of i_1 the intensity for the direction determined by D is found by substituting in (1) that value of i_1 and the corresponding values of r_1, i_2 and r_2.

The various refractive indices n of ice which are required were derived from measurements by PULFRICH[2]) by graphical interpolation (with a small extrapolation, utilizing a remark of PULFRICH's, that the dispersion of ice is equal to that of water) taking for n the mean of the values for the ordinary and extraordinary rays (for the whole spectrum the difference in the minimum deviation between the two amounts to only 6′).

The values required for the purpose are as follows (P refers to

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[1]) For angles of incidence greater than that of the symmetrical case the indices 1 and 2 interchange.

[2]) C. PULFRICH, Wied. Ann. 34, p. 336, 1888.

a measurement by PULFRICH; D_0 = angle of minimum deviation):

	λ	n	D_0		λ	n	D_0
B	0.687	1.3071(P)	21°37′	x	0.494	1.3136	22°7′
C	0.656	1.3079(P)	21°41′	F	0.486	1.3140(P)	22°8′
D	0.589	1.3098(P)	21°50′	y	0.449	1.3157	22°16′
E	0.527	1.3121(P)	22°0′	G	0.431	1.3168	22°21′

x and y are two special wave-lengths which are of importance for the deduction of the colour-effect following further on.

For these eight colours and for angles of incidence between about 41° (corresponding to the angle of minimum deviation, which differs for the various colours) and 62° the expression (1) was computed. In this set angles of deviation up to 25° are included. Larger angles of incidence were found to be unnecessary; and, moreover, the loss of light by reflection probably begins to exercise its influence in this region.

The results are contained in the accompanying table. A special calculation for the line F which differs very little from x was not carried out. Fig. 2 gives the dependence of the intensity on the deviation.

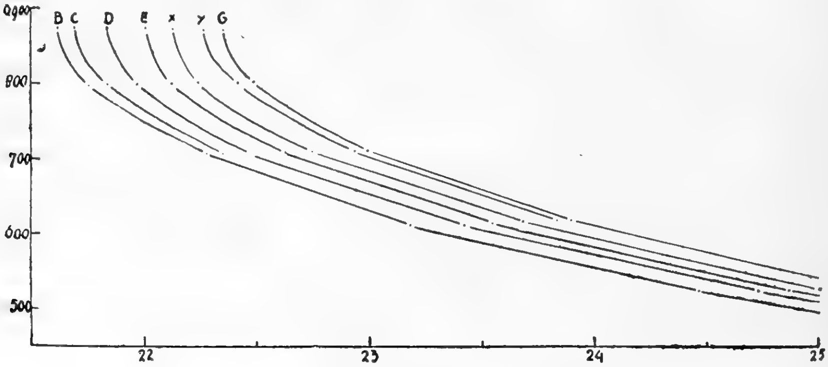

Fig. 2.

By means of these curves the intensity was determined in directions from 21°30′ upwards ascending by 15′. It is necessary in this computation to bring into account the finite extension of the sun. The same approximation was applied as used by PERNTER in the case of the rainbow; in fact PERNTER's method of computing the colours was followed throughout[1]): the curves are shifted in three steps of 5′ both to the left and to the right and read each time. The figures thus obtained, corresponding to seven points of the sun at intervals of 5′, are added up. The practical execution of the method comes to reading the curves from 5′ to 5′ and each time

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[1]) PERNTER, l.c. pag. 529 sqq.

i	*B*		*C*		*D*		*E*		*x*(*F*)		ν		*G*	
	L	*D*	*L*	*D*	*L*	*D*	*L*	*D*	*L*	*D*	*L*	*D*	*L*	*D*
about 41°	0.875	21°37′	0.874	21°41′	0.873	21°50′	0.872	22° 0′	0.871	22° 7′	0.870	22°16′	0.870	22°21′
45° 0′	798	45	798	50	799	58	800	7	800	15	801	25	801	29
50 0	704	22 17	705	22 21	705	22 29	706	38	707	45	708	56	708	23 0
55 0	608	23 11	609	23 17	610	23 26	611	23 33	612	23 41	614	23 50	615	54
60 0	521	24 28	522	24 33	522	24 44	522	24 52	523	25 0	523	25 8	524	25 13
62 0	483	25 9	483	25 14	485	25 22								

combining seven readings: the sum represents the intensity of the light at the middle point.

The sums thus arrived at are then reduced to the intensities with which the eight colours concerned occur in the light of the sun by multiplying each by a special coefficient.

The final calculation of the resulting colour by PERNTER's method, which is based on MAXWELL's colour-equations, consists in dividing the intensity found for each of the eight colours over the three primary colours red, green and violet (.630 μ, .528 μ, and .475 μ) and thence to deduce the colour-equations which yield the final colours, each with the percentage of white with which it is mixed.

	S	*R*	*G*	*V*
B	23	1.000	0.000	0.000
C	94	0.904	0.011	0.085
D	262	0.557	0.446	—0.003
E	153	—0.006	0.993	0.013
x	118	—0.068	0.602	0.466
F	130	—0.061	0.346	0.715
y	152	0.020	0.007	0.973
G	68	0.000	—0.059	1.059
W		240	383	377

Column *S* gives the proportional numbers for sunlight, *R*, *G* and *V* those for the primary colours. The bottom row gives the proportion of the primary colours in white (*W*).

The following table contains the intensity as obtained for a few directions:

	B	*C*	*D*	*E*	*x*	*F*	*y*	*G*
21°30′	37.4	78	0	0	0	0	0	0
22°0′	121	505	1244	496	192	213	0	0
22°30′	110	454	1296	782	618	684	707	264
23°0′	101	418	1195	715	564	623	748	339
24°0′	89	367	1046	620	489	540	642	290
25°0′	79	327	928	551	433	479	567	257

The numbers for F were obtained from those for x by a shift of 1′ (the difference in the minimum deviation for the two wave-lengths).

The final results are contained in the following table:

D	L	W %	C %	G	Colour	
21°30′	12	1.7	98.3	29.5	red (weak)	
21°45′	116	5.5	94.5	3.7	orange	
22°0′	279	29	71	4.7	yellow	
15′	424	69	31	5.9	green-yellow	
30′	491	96	4	8.4	green	white
45′	491	96	4	17.0	blue	
23°0′	471	96	4	16.9	"	
30′	437	96	4	16.9	"	
24°0′	408	97	3	16.8	"	
25°0′	362	98	2	16.8	"	

In this table the letters have the following meaning:

D angle of deviation; L intensity of the light; W % and C % percentages of white and of colour; G the colour-number in PERNTER's (MAXWELL's) colour-triangle, the last column gives the corresponding colour.

Fig. 3 shows the dependence of L on D; from which it appears that the refraction-theory gives a maximum at a distance of more than 22°30′. The observations on the other hand show, that the

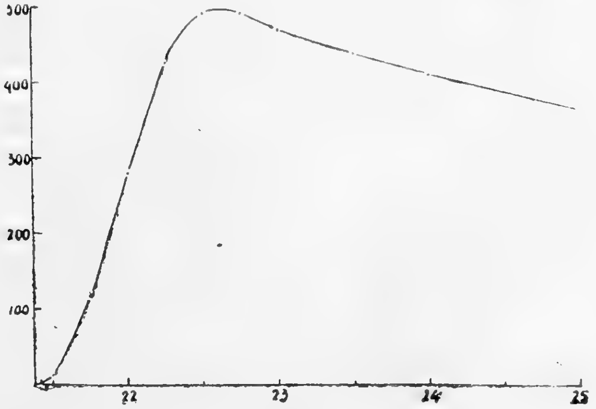

Fig. 3.

maximum is nearer to the sun. PERNTER[1]), taking the mean of a large number of trustworthy measurements, finds 21°50′. This difference of more than $^1/_2$° is much too large. Moreover, the refraction theory leads to the conclusion, that the inside of the circle would be red-orange, yellow, green-yellow and that blue can hardly appear and violet not at all. No increase of the intensity of the light can improve this disagreement with observation; the white always remains 24 times as intense as the blue; specially directed crystals are not capable of producing colours which are not there before. The colouring of the parhelia and tangential curves are not explained. It may also be noticed that on the underlying refraction-theory the size of the crystals which is determined by a cannot have an influence on the colour-phenomenon either, seeing that the width of the beam changes in the same ratio as a for all colours and that the light is parallel.

The conclusion to be drawn is simply this: the refraction theory is not able to give a complete explanation of the halo-phenomena.

I have not carried out a similar calculation for crystals with a refracting angle of 90°, but the circumstance, that the minima of deviation lie further apart in this case[2]), cannot be sufficient to explain the presence of differently coloured rings of 46°.

IV. The diffraction theory.

We may again confine ourselves to the phenomenon as occurring in one plane sun-crystal-eye, with the refracting edge at right angles to this plane. We shall also assume the special case of all crystals having the same size, which must be looked upon as a limiting case which is specially favourable to a development of colour.

As we have seen a beam of definite width emerges from the crystal after refraction, but this beam will be subject to diffraction. A large number of crystals of the same size, irregularly distributed, with parallel edges, all give similar beams, which will give rise to interference phenomena: the light source is seen as it were through

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[1]) PERNTER, l.c. 230. PERNTER's proof that a ring must be formed at 21°50 is by no means conclusive. He only shows that the yellow light has a minimum deviation of that magnitude (p. 313). The maximum intensity lies further out: about at 22°21′ (min. dev. of violet) + 16′ (sun's radius) = 22°37′, the place where all the colours of the complete spectrum are fully developed, in accordance with what the above more elaborate calculation gives. The light circle seen against the dark background will appear narrower to the eyes. Thereby the difference between observation and calculation becomes smaller, but it is more than doubtful, whether this effect could cause the difference to disappear altogether.

[2]) PERNTER, l.c. p. 354, 357.

a very large number of rectangular apertures, slits, spread at random. Diffraction fringes will be formed parallel to the refracting edges of the crystals and a rotation of the plane about the line eye-sun produces diffraction circles round the sun.

The places of the diffraction minima are given by the equation

$$sin\ \theta = m\frac{\lambda}{a},$$

where θ is the angle of diffraction for the minimum, m the order, λ the wave-length and a the width of the slit.

The first maximum for each colour is formed in its minimum deviation. Towards the outside of the ring each direction represents a maximum for each colour accompanied on both sides by diffraction fringes. The resulting colour cannot be anything there but white and the only colours which can be seen will have to be looked for on either side of the first maximum.

In each direction two beams *AB* and *HK* (fig. 1) of different width cooperate. Considering that these beams approach each other in width the nearer they are to the minimum-deviation and that in its neighbourhood the deviation of the rays changes very slowly with the angle of incidence, it follows that the diffraction lines very nearly cover each other in the neighbourhood of the minimum-deviation and hence that the colours must be more prominent there than elsewhere. In order to elucidate this effect I have computed the relative widths of the slits *AB* and *HK* (equation 1) on both sides of the minimum for angles of incidence between 37° and 50°

i_1	i_2	a_1	a_2	max.—min.		D
				a_1	a_2	
40°55′	40°55′	0.4363	0.4363	100′	100′	21°50′
42 0	39 51	4228	4306	103	101	51
43 0	38 53	4104	4250	106	103	53
44 0	37 55	3980	4190	110	104	55
45 0	36 59	3856	4130	113	106	59
46 0	36 3	3734		117		22 3
47 0	35 8	3612		121		8
48 0	34 14	3490		125		14
49 0	33 22	3370		130		22
50 0	32 30	3250		134		30

for $n = 1{,}310$ (yellow); further for the same angles the deviation which gives the place of the central maximum and finally the place of the first diffraction-minimum on both sides of the central maximum, assuming at the minimum-deviation a distance between central maximum and first minimum of 100′ (a value about equal to the one found in the special case to be dealt with further on corresponding to an absolute width of the slit a of 20.24 μ.)

The positions of the maxima and minima for the *D*-line derived from this are as follows

i	1st min.	1st max.	1st min.	i	1st min.	1st max.	1st min.
36°59′	20°13′	21°59′	23°43′	44°0′	20°5′	21°55′	23°44′
37 55	11	55	39	45 0	5	59	
38 53	9 50″	53	35	46 0	6	22 3	
39 51	9 38	51	32	47 0	8	8	
40 55	10 22	50	30	48 0	9	14	
42 0	8	51	34	49 0	12	22	
43 0	6	53	39	50 0	16	30	

The peculiar movement of the inside minimum is due to the cooperation of the change of the minimum deviation and of the angle of diffraction of the first minimum.

The results show that for angles of incidence between 39° and 48° the inside minima do not deviate by more than 5′ from the smallest value, that the same is true for the outside minima between 39° and 42°, and finally for the central maximum from 38° to 44°. What was found for yellow, also holds mutatis mutandis for the other colours.

By the superposition of these maxima and minima the development of colour will be much promoted in a manner, which is impossible on the ordinary refraction-theory, and by the presence of the diffraction minima the resulting colour is completely modified. This is particularly true for the first minimum on both sides of the central maximum. The theory taken generally shows the possibility of the formation of diffraction rings on both sides of the central maximum; but it goes without saying that these circles have a better chance of becoming visible on the light-free inside of the halo than on the outside which is covered with non-minimal light.

The resulting colours and the intensity of the light in each direction will again have to be found by a calculation similar to the one applied above in the refraction theory.

The fundamental formula shows that the phenomenon depends on the width of the slit *a*, that is on the size of the crystals. A possible procedure would thus be to calculate the colour for a number of different values of *a* chosen at random and in this manner try to reproduce the various observations. We shall, however, confine ourselves to a special case in which the observations themselves give an indication as to the size of the crystals which were operative.

V. *The halos of May* 19 1899 *and of September* 19 1905.

On two occasions Hissink at Zutfen observed very interesting halos which are described in "Onweders etc." as follows.

May 19 1899. "At 10.10 a.m. the small arc and the complete circumscribed halo became visible. For some time clouds prevented the observation, but when it cleared the circle became visible once more. At 11.52 a.m. an additional ring *b*, also circular, appeared, principally inside the upper half of the main ring and at 12.15 p.m. another circle *c* inside the former, whereas at 12.2 p.m. a further one *d* showed itself again nearer the sun. The two rings *b* and *c* were red on the side of the sun and showed round the red a greenish-yellowish tint, surrounded by violet. The small circle *d* had its outer edge coloured like the former, and its red on the side of sun was also similar, but the space on the inside of the circle was dark blue with a dull-brown hue."

By estimation Hissink determined the radii at: $d = 7°.5$; $c = 17°.5$; $b = 19°.5$ (putting the ordinary circle at 22°).

Sept. 19. 1905. "The halo observed on this day at Zutfen was a very rare one. It included *a* the large circle, *b* the upper tangential arc, *c* the small circle, *d* a circle with a radius of about 19°30′, *e* a circle with a radius of about 18° and *f* the left parhelion.

As regards the colour of the various parts it should be principally mentioned, that the large circle was comparatively brightly coloured and that the violet of the tangential arc near the point of contact was particularly striking.

The circles *d* and *e* are the most interesting, the radii being determined by Hissink by measuring the radius of the small circle with an octant, which gave 22°, and subsequently the distances between it and the arcs *d* and *e*. The latter were found to be 2°32′ and 4°2′, which would give 19°28′ and 17°58′ for the radii of these circles. Direct measurement of the radii gave 19°32′ and 18°2′

respectively. The means 19°30′ and 18°0′ must therefore have a comparatively high degree of certainty".

Similar circles have been observed on other occasions. Burney on June 9 1831 saw a ring of a radius of 20°[1]). Hissink himself saw one on Sept. 5 1899 of a radius estimated at 19°. On the ordinary theory all such circles are explained by means of specially shaped crystals with refracting angles which produce a circle at the distance required. The following crystal-faces come into consideration[2]) for the above cases:

Refracting angle		D_0 (yellow)
50°28′	two pyramidal faces at the same end of the crystal	17°26′
53°50′	two pyramidal faces at opposite ends of the crystal	18°56′
54°44′	a base face with a pyramidal face at the other end or a prism face with a pyramidal face exactly opposite.	19°20′

The distances of the rings for yellow are then 2°30′, 2°54′ and 4°24′ respectively.

The first one agrees exactly with Hissink's measurements, whereas the last is too large. The colours give difficulties which are not solved in this manner.

Starting from the supposition, that the rings of 18° and 19°30′ are nothing but secondary diffraction rings, I have made a calculation of the colours in the following manner.

Pernter[3]) gives the positions of the maxima and minima for the diffraction through slit-shaped apertures in connection with the theory of coronae, as follows:

		position	intensity
1st maximum		0.0000	1.000000
	1st minimum	1	0
2nd maximum		1.4303	0.047191
	2nd minimum	2	0
3rd maximum		2.4590	0.016480.

Applying these results to Hissink's measurements in 1905, where

$$a = 21°50' \qquad b = 19°30' \qquad c = 18°0'$$
$$a - b = 2°20' \qquad a - c = 3°50'$$

the angle of diffraction θ of the first minimum is found to be

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[1]) Pernter, l.c. page 266.

[2]) Onweders etc. 26 p. 83. 1905.

[3]) Pernter, l. c. pag. 452.

as calculated from a—b $\quad \frac{1}{1.43} \times 2.33 = 1.630^\circ = 1^\circ 38'$

as calculated from a—c $\quad \frac{1}{2.46} \times 3.83 = 1.557^\circ = 1^\circ 34'$

the mean 1°36′ having an uncertainty of 2′. This agreement seems to support the underlying supposition as to the nature of the subsidiary circles.

	B	*C*	*D*	*E*	*x*	*F*	*y*	*G*
23° 0	17.84	77.5	224.2	169.7	171 4	191.3	302.1	160.4
22 45	41.10	179.0	561.7	408.2	377.7	422.8	605.9	308.2
30	71.90	315.5	990.8	691 0	606.9	680.6	887.4	422.7
15	105.3	457.0	1416	925.5	761.8	846.4	1003	435.4
0	135.3	573.0	1700	992.3	765.5	841.1	870.9	331.7
21 45	152.8	633.0	1743	926.7	617.3	665 5	575.9	212.6
30	154.2	614.0	1531	690.4	391.1	402.4	272.6	72.8
15	136.9	533.5	1140	407.9	181.3	174.7	74.60	15.31
0	109.6	349.0	700.5	167.9	52.4	46.5	16.62	10.43
20 45	72.95	248.5	329.7	41.18	10.53	13.10	30.13	17.64
30	42.65	124.0	103.9	9.92	20.57	25.78	41.67	16.23
15	19.40	44.50	20.88	26.97	33.30	37.13	29.87	7.19
0	5.56	9.15	30.88	44.16	29.25	29.33	10.92	2.27
19 45	0.91	6.35	67.90	38.71	14.14	12.32	4.94	3.95
30	2.33	17.60	79.71	19.27	4.21	3.70	11.11	6.39
15	5.19	27.10	61.30	5.68	5.05	6.62	14.79	5.16
0	7.09	27.40	28.85	4.56	10.17	12.26	10.34	2.01
18 45	6.64	19.30	8.20	11.37	10.83	11.97	3.82	0.93
30	4.28	9.10	6.60	15.51	7.60	6.37	2.46	2.36
15	1.97	2.45	17.35	12.36	2.91	2.00	5.67	3.44
0	0.56	1.85	26.32	5.71	1.85	2.51	6.40	2.50
17 45	0.42	4.90	25.50	1.72	4.13	5.48	4.83	0.99
30	1.23	8.80	15.98	1.76	6.03	6.42	1.71	0.44
15	2.14	9.75	6.01	3.03	5.28	4.34	1.75	1.06
0	2.25	6.95	1.35	8.40	2.60	1.54	3.65	1.87

Assuming, that the maxima arise under the influence of the strong yellow light, the width of the slit a is found to be

$$a = \frac{\lambda}{\sin \theta} = \frac{0.539}{\sin 1°36'} = 21.10\ \mu.$$

The side of the base-face thus becomes 0.279 mm. which appears to be quite a possible size [1]).

Using this value of a the colour-effect can now be derived in the following manner. The expression $\frac{a \sin \theta}{\lambda}$ is first calculated ascending by 15′ for the same eight colours as used before. SCHWERD [2]) has computed the intensity as a function of $\frac{a \sin \theta}{\lambda}$ from $\frac{a \sin \theta}{\lambda} = 0$ up to $\frac{a \sin \theta}{\lambda} = 6$. By means of a graphical representation of these results the intensities for all the above values of the expression can be arrived at and the results are then reduced to the relative intensities in sunlight. They are then drawn graphically for the eight colours taking as abscissae the distance from the sun, at which the fringes are formed, the central maximum being taken at the minimum deviation. The curves are further used as explained before with a view to the dimension of the sun.

The results are contained in the foregoing table. (See p. 1189).

The figures in the horizontal rows give the relative intensities for the directions contained in the first column.

These data are then reduced to colour-equations and finally the colour-numbers in the colour-triangles as well as the percentage of white are computed. Fig. 4 gives the change of the intensity.

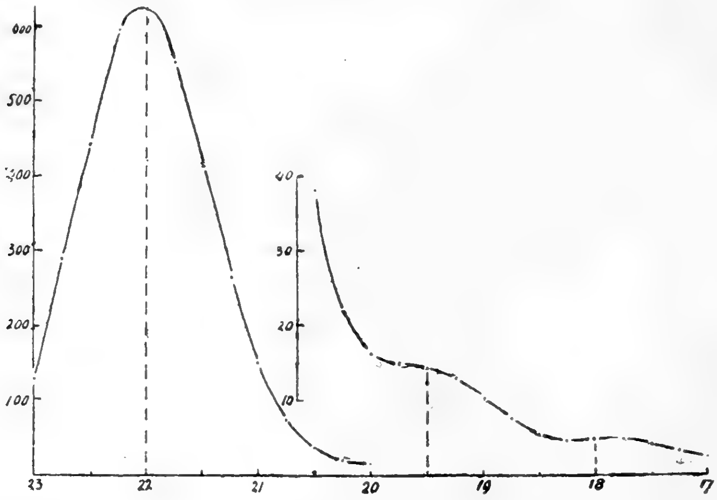

Fig. 4.

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[1]) PERNTER l. c. pag. 287. [2]) PERNTER l. c. pag. 453.

The results of the calculation will be found in the following table, where the symbols have the same meaning as in the corresponding table on the refraction-theory.

D	L	W %	C %	G	Colour	Observations
23° 0′	131	62	38	17.6	violet	violet N
22 45	290	68	32	17.3	blue—violet	
30	467	75	25	17.2	blue	blue N
15	595	86	14	16.0	blue	
0	621	62	38	4.3	yellow	yellow N
21 45	553	77	23	5.1	yellow	
30	414	56	44	4.7	yellow	
15	266	22	78	4.7	yellow	
0	150	17	83	2.8	red	red N
20 45	76	30	70	2.1	red	
30	38	54	46	29.3	red	
15	22	83	17	26.3	purple	violet Z
0	16.1	72	28	11.0	green-blue	green-yellow Z
19 45	14.9	45	55	6.7	green-yellow	
30	14.5	44	56	4.2	yellow	
15	13.1	59	41	1.6	red	red Z
0	10.3	73	27	29.9	red	violet Z
18 45	7.3	74	26	2.7	red	
30	5.4	68	32	6.3	green-yellow	green-yellow Z
15	4.6	67	33	6.3	green-yellow	
0	4.8	62	38	5.2	yellow	
17 45	4.8	68	32	2.8	red	red Z
30	4.2	64	36	2.8	red	
15	3.3	74	26	29.6	red	
0	2.9	73	27	6.8	green-yellow	

In the column under "Observations" Z refers to the circles observed at Zutfen by HISSINK in 1899, N to the colours of an upper tangential arc seen at Nymegen (N°. 13 of the Table) on

September 19 1905, the same afternoon as HISSINK's second observation. The application of the calculation to the upper tangential curve is allowable, at least in the neighbourhood of the point of contact, as the refraction takes place at that point in exactly the same manner as in the ordinary circle; the colours may thus be looked upon as belonging to the latter.

The agreement with the colours as observed at Nymegen and Zutfen is nearly complete: only the second violet is absent in the calculated set. The observer at Nymegen reports: red, yellow, blue (wide), violet (narrow). The calculation for 22°10′ and 22°5′ gives green-blue (colour-numbers 13.1 and 13.2) very near blue. Green is absent and blue has a width of 40′. The violet is nearly exhausted at 23°0′ and does not exceed a width of 15′. This agreement in the colours gives a strong support to the diffraction-theory as above developed.

A circle and a tangential curve without green are also reported from Boulogne sur Seine (N°. 9*a*).

As regards the agreement with HISSINK's circles: the colours given in the table are those observed on May 19 1899 and these need not be identical with those of 1905. Indeed, the characteristic feature of diffraction-rings is that their distance is variable, depending as it does on the dimensions of the refracting crystal. Perhaps the small remaining differences with the results of calculation may herein find their explanation. Intermediate calculations gave:

20°10′	*G* 13,2	green-blue
20°5′	12,7	green-blue
18°40′	4,2	yellow
18°35′	5,7	yellow

The intensities of the maxima are small and the maxima are but little prominent. They can only become visible by the differences in colour and only with a very high intensity of the main circle.

Professor VAN EVERDINGEN when asked for further information replied:

"that in the observations at Zutfen, both in 1899 and 1905, the colours of the small circle were described as *very bright*, as also those of the surrounding (circumscribed) halo or upper tangential curve".

It seems to me, that the above results render it extremely probable, that HISSINK's circles have to be taken as diffraction-rings; but in that case other similar rings must also arise by diffraction (compare the two cases mentioned on page 1188).

It is not impossible, that similar diffraction-rings may also occur outside the main circle. Pernter mentions two observations of that kind [1], but the data are too incomplete for a calculation to be based on them. In this connection the observation at Souppes (Nr 10 of the table) is important: in this case two concentric arcs are reported, the wider one of which is the inferior tangential arc. The other one may, as it seems to me, be looked upon as an external diffraction-ring of this arc.

As regards the main maximum, the theory gives it as lying at 22°0′ in complete agreement with the observations which give 21°50′ as the mean.

It is very probable, that by a calculation of the system of colours for other values of the width a the other observations may also be reproduced. In this connection the fact should be noted that in the various reports some combinations of colours occur repeatedly and will probably have to be ascribed to crystals of the same size. Some instances may be given here:

“Spectral colours”:	Circle of 22°:	9 and 10
	Circle of 46°:	7 and 31
	Parhelion	14
Red, yellow, green, violet:	Circle of 22°:	3 and 6
	Upper tangential arc:	17*b*
Red, violet:	Circle of 22°:	2
	Upper tangential arc:	20 and 23
Red, blue:	Circle of 46°:	9*b*
	Circumzenithic arc:	1 and 30
Red, green:	Circle of 46°:	6*a* and 23
	Upper tangential arc:	21

The case of red, yellow, blue, violet (circle of 22°, 9*a* and upper tangential arc, 13) is dealt with above.

The very lengthy calculations which would be required for the further testing of the theory, have not been carried out so far and we shall confine ourselves to some general remarks.

1. As in the rainbow we have in the colours a means of determining the size of the refracting particles. In order to obtain say a well developed violet it is necessary that the maximum intensity of violet coincides about with the extinction of red and green. A very rough approximation to the dimensions in this case is arrived at as follows.

Supposing the colours B, C, D and E to have their first minimum

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[1] Pernter, l.c. p. 260. Greshow's halo, Oct. 20 1747, radius 26°; and an observation by Whiston, radius 29°.

in the direction of the central maximum of G, the corresponding angles of diffraction are (see table on page 1182)

$$B\ 44' \quad C\ 40' \quad D\ 31' \quad E\ 21'$$

respectively and the widths of the slit

$$B\ 53.7\,\mu, \quad C\ 56.4\,\mu, \quad D\ 65.2\,\mu, \quad 86.1\,\mu.$$

The mean width $65.4\,\mu$ may be looked upon as giving a close approximation to the correct value. This gives .075 mm for the width of the side-face of the prism and .15 mm for the diagonal of the base.

Very small crystals will produce very broad maxima, in comparison to which 44′ — the difference between the maxima of red and violet — may be looked upon as small, in consequence of which the various colours will cover each other and nothing will be seen but white with a red inner edge.

2. It may be useful to point out the analogy with the rain-bow. In that case large drops give narrow diffraction-maxima and distinct colours, small drops broad maxima, diluted colours and the rare white rainbow. Similarly with the halo: the larger the ice-crystals, the more distinct the spectral colours will be. The "white halos" are by far the most common.

Still there are some very fundamental differences between rainbow and halo. Whereas in the former case the wave-front becomes curved, it remains flat in the latter case. Whereas in the rain-bow the maxima are strongly developed, though only on one side by which the extremely common secondary bows on the side of the violet are formed, these maxima are comparatively weak in the halo and possible on both sides. They will have the best chance of being seen in the dark region inside the red, but in the white on the outside they will but seldom succeed in making themselves visible.

3. In connection with the colours of halos the shape of the crystal is of some importance. Let us consider a crystal plate with a broad side-face but of small height. The width of the side-face determines the width of the slit which plays a part in the formation of the ordinary ring, the height determines the width of the slit for the circle of 46°, as this halo is formed by a refracting angle of 90°. A plate of the above shape is specially suited to the production of colours in the ordinary circle, but unsuitable as regards the large circle. With an elongated prism the colour-production in the circle of 22° is again dependent on the width of the side face, but for the circle of 46° the determining dimension is now the short diagonal

of the base which is $\sqrt{3}$ times or 1.7 times longer than the width of the side-face. A crystal of that kind is therefore more suited to the production of colour in the circle of 46° than in that of 22°. And as a matter of fact a number of halos enumerated on pages 1176 to 1178 show striking differences in the degree of colouring in the two circles: 9*b* belongs to the former kind, 7, 12, 27 and 31 to the latter.

4. A further important conclusion seems to me justified, although I have not tested it in detail. As we have found (p. 1185), in the neighbourhood of the minimum deviation we can turn the incident beam or, what comes to the same, the crystal over a comparatively large angle before its having any influence on the diffraction-fringes. But if that is true, the difficulty disappears which lies in the necessity of having to assume a constant, vertical axis in the usual explanation say of the circum-zenithic arc. [1])

The "strikingly pure colours", the "pure violet" of which BESSON speaks, are a consequence of diffraction, but not of a constant direction of the refracting edge.

5. We also found (page 1185) that in the external minimum a much smaller variation was admissible. The same will hold with regard to the next maximum: another ground, therefore, to expect, that diffraction-rings outside the main circle will be very great exceptions.

6. In the large circle of 46° the difference in the minimum for red and violet is 2°6′ [2]): the spectrum is thus spread out over an angle three times as wide as in the circle of 22°. But the slit becomes smaller in the ratio $\frac{\cos i_{46^0}}{\cos i_{22^0}} = \frac{\cos 67°51'}{\cos 40°55'} = \frac{1}{2}$. The colour-effect is thus enhanced in the ratio $^3/_2$. With a favourable shape of the crystal, the effect may be increased another 1.7 times and the conditions so become $2\,^1/_2$ times more favourable as regards production of colour in the circle of 46°. This agrees with the fact, that in this circle striking colours have been seen comparatively frequently.

7. In the formation of halos where the light no longer passes the crystal at right angles to the refracting edge, which corresponds to a broadening of the beam, the diffraction pattern agrees with that of a larger crystal with the light moving in a plane at right angles to the refracting edge. The chance of colour is increased. In agreement with this the tangential curves to the circles of 22° and of 46° (circum-zenithic curve) are pretty frequently distinctly coloured.

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[1]) See a.o. L. BESSON, l c.

[2]) PERNTER, l. c. p. 354.

8. The diffraction tells us something of the size of the crystals and by this means possibly of the temperature at which they have been formed: "with falling temperature the size of the crystals diminishes" [1]). In that way the halo-colours, which have been too much neglected, may possibly contribute to a better knowledge of the higher atmosphere.

VI. Conclusions.

The above investigation seems to me to justify the following conclusions:

1. The simple refraction-theory cannot explain the halo-phenomena completely, in particular as regards the great variety of the colours.
2. The diffraction-theory gives a simple explanation of the colours which appear and allows special conclusions to be drawn regarding the influence of the size and the shape of the crystals. It alone gives the ordinary circle its correct place of 22°.
3. The rings which have been observed in the neighbourhood of 22° are secondary diffraction-rings: their radii are not constant.
4. The diffraction-theory will probably be able to afford a better insight into the formation of the circumzenithic arc.
5. It is necessary that the colours be accurately recorded by each observer in order to permit a further testing of the theory and a complete deduction of the origin of the observed phenomenon.

Chemistry. — *"In-, mono- and divariant equilibria"*. XVI. By Prof. F. A. H. SCHREINEMAKERS.

(Communicated in the meeting of March 31, 1917).

The regions in the P,T-diagram.

In communication VIII we have already briefly discussed those regions; now we shall consider them more in detail. When the equilibrium

$$E = F_1 + F_2 + \ldots + F_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

consists of n components, then it is generally divariant; consequently it is generally represented in the P,T-diagram by a region. We shall consider this region E in its whole extensity, viz. without taking into consideration that some parts may become metastable by the occurrence of other phases.

With a definite equilibrium E we may distinguish:

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[1]) PERNTER, l. c. p. 289.

1. the total composition of E.

2. the composition of each of the phases, of which the equilibrium E consists.

We shall say that two equilibria have the same phases-composition when the phases of both equilibria have the same composition.

We now take a definite point x of the region E (consequently the equilibrium E under P_x and at T_x). Then the equilibrium E has either only one definite phase-composition E_x, or two phase-compositions E_x and E'_x or three viz. E_x, E'_x and E''_x etc. We may express this by saying that either one, or two or more equilibria E belong to the point x of the region E.

When only one single equilibrium E_x belongs to each point x of the region E, then we call the region one-leafed; when in a part of the region two equilibria (E_x and E'_x) belong to each point x, then we call that part two-leafed etc.

As the equilibrium E_x, which belongs to a definite point of the region E, may be as well stable as unstable, the region E may consist, besides of stable, yet also of unstable leaves.

When the point x traces the region E of the P,T-diagram or in other words, when we give to the equilibrium E all possible phase-compositions, then equilibria may occur, which show something particular.

1. The equilibrium E of n components in n phases passes into an equilibrium E_0 of $n-1$ components in n phases. [The index 0 indicates that the quantity of one of the components has become zero].

2. Between the n phases of the equilibrium E a phase-reaction

$$\lambda_1 F_1 + \lambda_2 F_2 + \ldots + \lambda_n F_n = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

may occur. We call this equilibrium E_R. [The index R indicates that a reaction may occur].

3. Critical phenomena occur between two phases; we call this equilibrium E_K.

The first case occurs when the quantity of one of the components e.g. K_1 may become zero in all phases. It is evident that the phases with constant composition are not allowed to contain this component K_1, therefore.

The equilibrium E_0 contains $n-1$ components in n phases and is, therefore, monovariant; consequently it is represented in the P,T-diagram by a curve, which we shall call curve E_0. This curve E_0 is, therefore, nothing else but a monovariant curve of a system with $n-1$ components. Consequently it is defined by:

$$\frac{dP}{dT} = \frac{\Delta H}{\Delta V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Herein ΔH represents the increase of entropy and ΔV the increase of volume with the reaction, which may occur in the equilibrium E_0.

As on curve E_0 the quantity of one of the components becomes zero, the region E must terminate (or begin) in curve E_0; for this reason we call E_0 the limit-line of the region E. We shall refer to this later. In fig. 1 $a\,b$ and $c\,d$ are the limit-lines of a region $a\,b\,c\,d$; on curve $a\,b$ one of the components is missing e.g. K_1, on curve $c\,d$ an other component e.g. K_2 is missing in the equilibrium E. When we go, starting from a point h of a limit-line towards a point l or m within the region, then the equilibrium E_0 passes into the equilibrium E.

Let us take now the second case, viz. that an equilibrium E_R occurs. The equilibrium E_R consists of n components in n phases, between which the phase-reaction (2) may occur. E_R is, therefore, a monovariant equilibrium and it may be represented by a curve in the P,T-diagram. It is defined by (3) in which ΔH and ΔV relate now to reaction (2). In order to examine the position of the region in the vicinity of this curve, we use the property: when in a system of n components in n phases a phases-reaction may occur, then at constant T the pressure and under constant P the temperature is maximum or minimum[1]).

Let ef be in fig. 2 a curve E_R. When we trace the region along a horizontal line (P constant) then in the point of intersection of this line with ef the temperature must be maximum or minimum. Let g be this point of intersection and let us assume that T_g is a maximum, then consequently the region must be situated at the left of curve ef. At $T_g + dT\,(dT > 0)$ then viz. no equilibrium E exists, at $T_g - dT$ two different equilibria E exist, however; consequently the region is two-leafed in the vicinity of curve E_R. In fig. 2 the one leaf of the region is dotted, the other leaf is striped. When we trace the region along a vertical line, then the pressure on ef is a minimum.

Consequently curve E_R is also a limit-line of the region E, but in connection with the property of the region in the vicinity of this curve, we call it "turning line" of the region E.

Also on the turning-line E_R the concentration of one of the components may become zero at a definite T (and corresponding P);

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[1]) F. A. H. SCHREINEMAKERS, Die heterogenen Gleichgewichte von BAKHUIS ROOZEBOOM. III[1]. 285.

then we obtain an equilibrium $E_{R.O}$, which belongs as well to the turning-line E_R as to the limit-line E_0. Turning-line and limit line touch one another in the point $E_{R.O}$.

In the case mentioned under 3 critical phenomena appear between 2 phases. This is the case when in the equilibrium E two liquids L_1 and L_2 get the same composition or when a liquid and a gas become identical. Then we obtain an equilibrium E_K of n components in n phases, of which 2 phases are in critical condition. This equilibrium E_K is represented in the P,T-diagram by a curve E_K which we call the critical curve of the region. In the vicinity of this curve E_K the region is one-leafed.

Consequently it is apparent from the previous that a bivariant region is one-leafed in the vicinity of a limit-line or critical-line, in the vicinity of a turning-line it is two-leafed. We shall refer to this later.

One- and two-leafed regions.

A one-leafed region may be limited by limit-lines and critical lines, but it may also be unlimited. When the equilibrium E contains e. g. only phases of invariable composition, then neither limit-line, nor critical line, nor turning-line exists. Consequently the region E is unlimited. [Of course a part of this region becomes metastable at higher T, because another phase is formed e. g. a liquid by melting or transformation of solids. When we leave out of consideration however the occurrence of other phases, then the region extends itself unlimited]. The region may also be unlimited when in the equilibrium, besides invariable phases also variable phases occur, which do not contain all components [e. g. mixed crystals or a gas].

We take an equilibrium $E = L + G$ of a binary system with the components A and B, which occur both in the vapour G. Then the region E has two limit-lines E_0. When in L and G the component A is missing, then we have the limit-line $E_{A=0}$, when B is missing, then we have the limit-line $E_{B=0}$. Consequently curve $E_{A=0}$ is the boiling-point-line of the substance B, curve $E_{B=0}$ that of the substance A.

When L and G have always different composition, then the region $E = L + G$ has no turning-line; then it may be represented by fig. 1 in which ab and cd are the limit-lines. When L and G may get the same composition, so that a reaction $L \rightleftarrows G$ may occur, then also a turning-line E_R exists. Then the region may be repre-

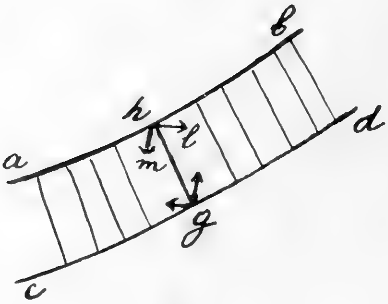

Fig. 1.

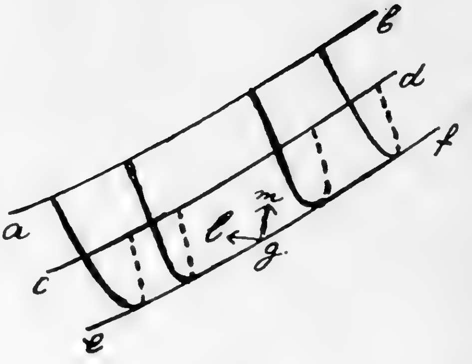

Fig. 2.

sented by fig. 2, in which $a\,b$ and $c\,d$ are the limit-lines and $e\,f$ the turning-line.

The same is true for an equilibrium $E = M + L$ or $M + G$ of a binary system $A + B$ [M represents mixed crystals].

The region E exists in fig. 2 of two leaves, viz. $a\,e\,f\,b$ and $c\,e\,f\,d$. On the one leaf the liquid contains more A, on the other leaf more B than the vapour.

Let us assume that in the binary system $A + B$ a compound F occurs. The region $E = F + L$ has then no limit-line E_0, but a turning-line E_R; this is the melting-line of the compound F. The region $E = F + L$ is, therefore two-leafed, in the one leaf are situated the liquids, which contain a surplus of A with respect to F, in the other leaf are situated the liquids, which contain a surplus of B.

The region $E = F + G$ of the binary equilibrium $A + B$ has also no limit-line, but a turning-line E_R; this is the sublimation-curve of the compound F.

We take a ternary system with the three volatile components A, B, and C, in which occurs a binary compound F of B and C. We now take the equilibrium $E = F + L + G$, in which consequently G contains also the 3 components. [Compare also "Equilibria in ternary systems XI"; in fig. 6 of this communication the arrow in the vicinity of point F on the curve going through the point F has to point in the other direction].

This region E has a limit-line $E_{A=0}$; consequently this represents the equilibrium $F + L + G$ of the binary system $B + C$ and it is indicated in fig. 3 by curve $a\,c\,d$; it has in b a maximum of pressure and in c a maximum-temperature.

When no equilibrium E_R occurs, then the region E is one-leafed and consequently it must be situated in fig. 3 within curve $a\,b\,c\,d$.

[Therefore, it does not extend itself, as is drawn in fig. 3 over af].

When an equilibrium E_R exists [this is the case when the 3 phases in the concentration-diagram are situated on a straight line] then also a turning-line E_R exists, this is represented in fig. 3 by ef. This point of contact f represents the equilibrium $E_{R_{A=0}}$.

The region E is now two-leafed, afe is the one, $dcfe$ is the other leaf.

When we consider the equilibrium $E = F + L + G$ at a constant T, lower than T_f, then the pressure on the turning-line ef is a maximum; when the turning-line was represented by gh, then the pressure would be a minimum.

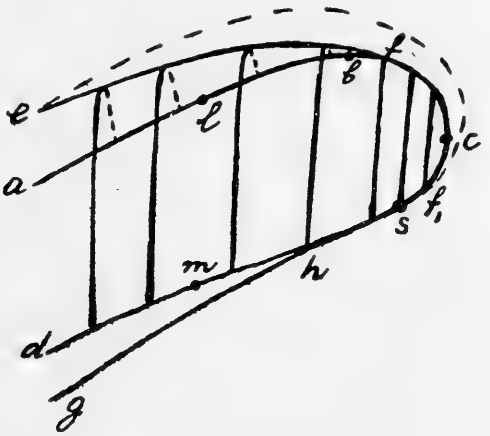

Fig. 3.

On curve acd is situated in the vicinity of c a solution s, which has the same composition as the compound F. When F melts with increase of volume, then s is situated on branch dc, as in fig. 3.

It is apparent from formula 17 of the communication on "Equilibria in ternary systems XI": when we enter at constant T starting from the point s the region $E = F + L + G$, then the pressure must increase.

Hence it follows, that the point of contact h of curve gh must always be situated on branch ds and that of curve ef always on branch as. In the latter case the point of contact may, therefore, also be situated between s and c e.g. in f_1; then we get a limit-curve like ef_1. The equilibrium $E = F + L + G$ then still exists at temperatures above T_c, the highest temperature at which the equilibrium $E_{A=0}$ may occur.

Let us now consider the equilibrium $E = B + L + G$ of the ternary system $A + B + C$. [Compare also "Equilibria in ternary systems XIII" February 1914]. The region E has then two limit-lines $E_{A=0}$ and $E_{C=0}$. The first represents the monovariant equilibrium $B + L + G$ of the binary system $B + C$; the second the same monovariant equilibrium of the binary system $A + B$. Each of those curves may either have a point of maximum-pressure or not, so that we may distinguish three cases. When in the equilibrium E does not occur an equilibrium E_R, then the region E is situated completely within the limit-lines and it is, therefore, one-leafed; when an equilibrium E_R occurs, then also a turning-line exists and the region is, therefore, two-leafed.

Two limit-lines $a\,b$ and $c\,d$ may intersect one another in a point s (fig. 4); this means that the two equilibria E_0 have the same pressure P_s at the temperature T_s. In this case there is always a limit-curve $e\,f$ (fig. 4), which may be situated as well above as below the point s. The turning-line $e\,f$ may touch the curves $c\,s$ and $s\,b$ in fig. 4.

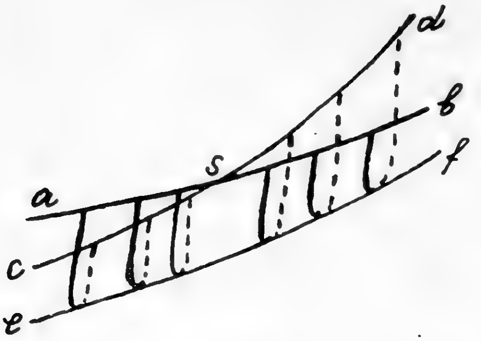

Fig. 4.

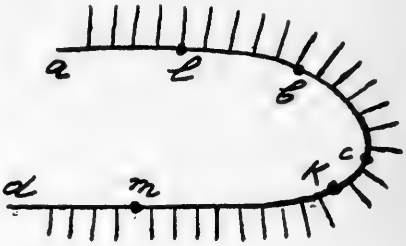

Fig. 5.

Let us now consider the equilibrium $E = L_1 + L_2 + G$, in which L_1 and L_2 represent two liquid-phases. [In a similar way we may also discuss the equilibria $L_1 + L_2 + F$, $M_1 + M_2 + L$ and $M_1 + M_2 + G$, etc., in which M_1 and M_2 represent mixed crystals]. When in the equilibrium $E = L_1 + L_2 + G$ the two liquids become identical, then a critical equilibrium exists: $E_K = L_K + G$. Curve E_K may have a form, like curve $a\,c\,d$[1]) in fig. 3. When in the equilibrium E_K the quantity of one of the components e.g. of A, approaches to zero, then curve E_K has a terminating-point $E_{K.A=0}$.

When $a\,c\,d$ represents in fig. 3 the critical curve E_K, then the region $E = L_1 + L_2 + G$ is situated either completely within curve

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[1]) Compare also F. A. H. SCHREINEMAKERS, Archives Néerl. Serie II. VI. 170 (1901).

$a\,c\,d$ or it is partly two-leafed with the turning-line $e\,f$ or $g\,h$. Also in fig. 5 a critical curve E_K is represented by $a\,c\,d$; the region E is situated here, however, completely outside the critical curve and it may have a turning-line also in this case.

We take in figs. 3 and 5 two points l and m on a vertical line; consequently we have $T_l = T_m$. At the temperature $T_l = T_m$ two equilibria E_K exist, therefore, the one $[E'_K = L'_K + G']$ under the pressure P_l, the other $[E''_K = L''_K + G'']$ under the pressure P_m.

The critical liquids L'_K and L''_K may now belong either or not to the same region of un-mixing under its own vapour-pressure of the temperature $T_l = T_m$. When they belong to the same region of un-mixing, then the region E is situated as in fig. 3; when they belong to different regions of un-mixing, then the region E is situated as in fig. 5; in both cases either a turning-line may occur or not.

We might think that in point c of figs. 3 or 5 two critical liquids get the same composition, so that L_c should be a critical liquid of the 2nd order. This is, however, not the case in the point c, but in another point K of the curve; this is drawn in fig. 5 on branch $d\,c$. Curve $a\,c\,d$ touches in this point a curve KK_1 (not drawn in the figure); the points of this curve KK_1 represent critical liquids of the 2nd order. Of all those liquids only the liquid K can be in equilibrium with vapour.

More-leafed regions.

Besides one- and two-leafed regions, of which we have considered above some examples, also more-leafed regions may occur. This may take place e.g. when in the region E occur two turning-lines. We shall consider a definite case for fixing the ideas. For this we take the equilibrium $E = B + L + G$ of a ternary system with the three volatile components A, B, and C. This equilibrium E has two limit-lines $E_{A=0}$ and $E_{C=0}$; these are represented in the concentration-diagram (fig. 6) by the sides BC and BA of the triangle $A\,B\,C$, in the P,T-diagram (fig. 7) by the curves $a\,e\,i\,l$ and $d\,h\,k\,n$. When we imagine in fig. 7 those two curves to be prolonged towards higher T, then both curves terminate in a point B, which represents the P and T of the melting-point under its own vapour-pressure of the substance B. Above we have already said that these curves may have a maximum of pressure or not.

The equilibrium $E = B + L + G$ consists at a temperature T_a of a series of solutions, which are saturated with solid B and a series of corresponding vapours. This series of solutions forms the saturation

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curve under its own vapour-pressure of B, the corresponding vapours form the vapoursaturationcurve. [Compare also: Equilibria in ternary systems XIII, February 1914]. In fig. 6 curve $a\,b\,c\,d$ represents a saturation-curve of B under its own vapour-pressure, the corresponding vapoursaturationcurve has not been drawn. Now we assume that on curve $a\,d$ occurs a point of minimum-pressure b and a point of maximum-pressure c; then the pressure increases along $a\,d$ in the direction of the little arrows.

Now we imagine in the P,T-diagram (fig. 7) a vertical line, which corresponds with the temperature T_a. It appears from fig. 6 that the points a, b, c and d must be situated in the P,T-diagram with respect to one another as in fig. 7; of course those four points must be situated on the same vertical line; for the sake of clearness a small deviation from the true position has been allowed in fig 7.

In accordance with fig. 6, therefore also in fig. 7 at the temperature T_a the pressure first decreases starting from a as far as in b, afterwards it increases starting from b up to c and further it decreases again starting from c as far as in d. The points b and c are drawn in fig. 7 within both the limit-lines; it is apparent, however, that b might be situated also below curve dn and that c might be situated also above curve $a\,l$.

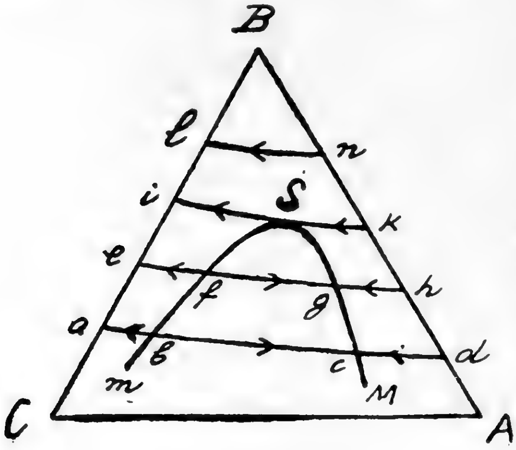

Fig. 6.

Now we take a temperature T_e; the saturation-curve under its own vapour-pressure is represented in fig. 6 by curve eh; it has a point of minimum-pressure in f, or point of maximum-pressure in g. We find the corresponding points in fig. 7.

Now we assume that on increase of T the point of minimum- and the point of maximum pressure of the saturation-curve under its own vapour-pressure come nearer to each other and that they coincide at T_i in the point S. Then the pressure increases along curve iSK (figs. 6 and 7) starting from K towards i. In the P,T-diagram the points i, S and K must then be situated with respect to one another, as in fig. 7; it is evident that the point S must be situated between the points i and K.

At temperatures above T_i, e.g. at T_l, the saturation-curves under

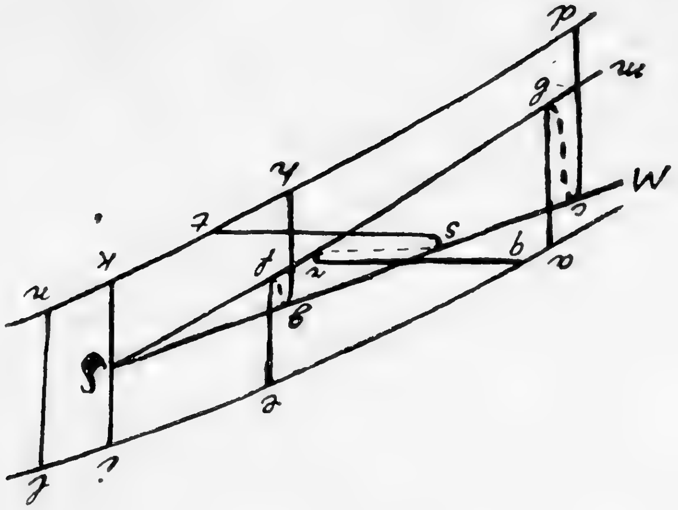

Fig. 7.

its own vapour-pressure have no more a point of minimum- or maximum-pressure, the pressure increases from n towards l (figs. 6 and 7).

The point of minimum pressure follows therefore, in figs. 6 and 7 a curve mS, the point of maximum-pressure follows a curve MS. The equilibrium E_R consists, therefore, of two branches, which meet in S; we may, however, also say that only one single turning-line exists $E_R = mSM$, which has a singular point in S.

Later we shall show in general that the two branches mS and MS of a turning-line E_R touch one another in the singular point S and that the tangent in S is situated between the two branches.

The region E in fig. 7 is now one-leafed, except in the part, situated within the turning-line, which is three-leafed. Of course this is only true in so far as this part is situated between the limit-lines.

Leiden, *Inorg. Chem. Lab.* (*To be continued*).

Chemistry. — "*In-, mono- and divariant equilibria*" XVII. By Prof. F. A. H. Schreinemakers.

(Communicated in the meeting of April 28, 1917).

Equilibria of n components in n phases.

Now we shall consider more in detail the equilibrium:

$$E = F_1 + F_2 + \ldots + F_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which we have already discussed in the previous communication. We represent the composition of:

F_1 by $x_1\, y_1\, z_1 \ldots 1 - x_1 - y_1 - z_1 \ldots$
F_2 ,, $x_2\, y_2\, z_2 \ldots 1 - x_2 - y_2 - z_2 \ldots$

The ζ, the entropy and the volume of F_1 we call Z_1 H_1 and V_1; those of F_2 we call Z_2 H_2 and V_2; etc.

Then we may write the conditions for equilibrium:

$$\left.\begin{aligned} Z_1 - x_1 \frac{\partial Z_1}{\partial x_1} - y_1 \frac{\partial Z_1}{\partial y_1} \ldots &= K \\ Z_2 - x_2 \frac{\partial Z_2}{\partial x_2} - y_2 \frac{\partial Z_2}{\partial y_2} \ldots &= K \end{aligned}\right\} \quad \ldots \ldots \quad (2)$$

viz. n equations (2) of which we only have written two. Further we have:

$$\left.\begin{aligned} \frac{\partial Z_1}{\partial x_1} = \frac{\partial Z_2}{\partial x_2} = \ldots = \frac{\partial Z_n}{\partial x_n} &= K_x \\ \frac{\partial Z_1}{\partial y_1} = \frac{\partial Z_2}{\partial y_2} = \ldots = \frac{\partial Z_n}{\partial y_n} &= K_y \end{aligned}\right\} \quad \ldots \ldots \quad (3)$$

The corresponding equations for the variables $z_1\, z_2 \ldots u_1\, u_2 \ldots$ etc. have still to be added to (3).

We find in (2) n, in (3) $n\,(n-1)$, consequently in total n^2 equations. Besides the $n\,(n-1)$ variables $x_1\, y_1 \ldots x_2\, y_2 \ldots$ etc. we have still the $n+2$ variables $T\ P\ K\ K_x\ K_y \ldots$ consequently in total n^2+2 variables. The equilibrium E has, therefore, two degrees of freedom and consequently it is bivariant.

We have assumed in (2) and (3) the general case that all phases have a variable composition and that each phase contains all components. When this is not the case, then we are able to make at once the necessary alterations in (2) and (3). When e.g. F_1 has a constant composition $x_1 = \alpha$, $y_1 = \beta$, etc., then the first equation (2) passes into:

$$Z_1 - \alpha \frac{\partial Z_i}{\partial x_i} - \beta \frac{\partial Z_i}{\partial y_i} \ldots = K \quad \ldots \ldots \quad (4)$$

in which the index i relates then to a phase F_i of variable composition. Then Z_1 is only still a function of P and T; in (3) then $\frac{\partial Z_1}{\partial x_1}$, $\frac{\partial Z_1}{\partial y_1}$ etc. .... disappear.

When we give to $P\ T\ x\ y \ldots$ the differentials $\Delta P\, \Delta T\, \Delta x\, \Delta y \ldots$: then we have:

$$\Delta Z = VdP - HdT + \frac{\partial Z}{\partial x}\Delta x + \frac{\partial Z}{\partial y}\Delta y + \ldots + \tfrac{1}{2}\, d^2 Z + \tfrac{1}{6}\, d^3 Z + \ldots$$

$$\Delta\left(x \frac{\partial Z}{\partial x}\right) = \frac{\partial Z}{\partial x}\Delta x + (x + \Delta x)\left[d\frac{\partial Z}{\partial x} + \tfrac{1}{2}\, d^2 \frac{\partial Z}{\partial x} + \ldots\right]$$

$$\Delta\left(y\frac{\partial Z}{\partial y}\right)=\frac{\partial Z}{\partial y}\,\Delta y+(y+\Delta y)\left[d\frac{\partial Z}{\partial y}+\tfrac{1}{2}\,d^2\frac{\partial Z}{\partial y}+\ldots\right]$$

Herein the sign d indicates that we have to differentiate according to all variables, which the function contains. Further is:

$$d^2Z=d\frac{\partial Z}{\partial P}\cdot\Delta P+d\frac{\partial Z}{\partial T}\cdot\Delta T+d\frac{\partial Z}{\partial x}\cdot\Delta x+d\frac{\partial Z}{\partial y}\cdot\Delta y+\ldots$$

$$d^3Z=d^2\frac{\partial Z}{\partial P}\cdot\Delta P+d^2\frac{\partial Z}{\partial T}\cdot\Delta T+d^2\frac{\partial Z}{\partial x}\cdot\Delta x+d^2\frac{\partial Z}{\partial y}\cdot\Delta y+\ldots$$

When we neglect in d^2Z and d^3Z the terms which are infinitely small with respect to ΔP and ΔT, then we may write:

$$d^2Z=d\frac{\partial Z}{\partial x}\cdot\Delta x+d\frac{\partial Z}{\partial y}\cdot\Delta y+\ldots$$

$$d^3Z=d^2\frac{\partial Z}{\partial x}\cdot\Delta x+d^2\frac{\partial Z}{\partial y}\cdot\Delta y+\ldots$$

From a form:

$$Z-x\frac{\partial Z}{\partial x}-y\frac{\partial Z}{\partial y}-\ldots=K$$

it follows, therefore:

$$\left.\begin{aligned}-V\Delta P+H\Delta T+(x+\Delta x)\left(d\frac{\partial Z}{\partial x}+\,..\right)+(y+\Delta y)\left(d\frac{\partial Z}{\partial y}+\,..\right)-\\ -\tfrac{1}{2}\,d^2Z-\tfrac{1}{6}\,d^3Z-\ldots=-\Delta K\end{aligned}\right\}\quad(5)$$

Now is

$$\Delta x\left[d\frac{\partial Z}{\partial x}+\tfrac{1}{2}\,d^2\frac{\partial Z}{\partial x}+\ldots\right]+\Delta y\left[d\frac{\partial Z}{\partial y}+\tfrac{1}{2}\,d^2\frac{\partial Z}{\partial y}+\,..\right]=d^2Z+\tfrac{1}{2}\,d^3Z+\ldots$$

so that we may write for (5)

$$\left.\begin{aligned}-VdP+HdT+x\left[d\frac{\partial Z}{\partial x}+\ldots\right]+y\left[d\frac{\partial Z}{\partial y}+\ldots\right]+\ldots\\ +\tfrac{1}{2}\,d^2Z+\tfrac{1}{3}\,d^3Z+\ldots=-\Delta K\end{aligned}\right\}\quad.\quad(6)$$

Now we apply this to the n equations (2) and we differentiate further also the $n\,(n-1)$ equations (3). First, however, we shall introduce the following notation; we put viz:

$$\frac{\partial Z_1}{\partial x_1}=(x)_1\ ;\ \frac{\partial Z_2}{\partial y_2}=(y)_2\ ;\ \frac{\partial^2 Z_3}{\partial x_3{}^2}=(x^2)_3\ ;\ \frac{\partial^3 Z_4}{\partial x_4{}^2\partial y_4}=(x^2y)_4\ \textit{enz.}$$

The index outside the parentheses indicates, therefore, which of the functions $Z_1\ldots Z_n$ has to be differentiated; the letters within the parentheses indicate according to which variables we have to differentiate.

Then it follows from the n equations (2):

$$\left.\begin{array}{l} -V_1\Delta P + H_1\Delta T + x_1\left[d(x)_1 + \ldots\right] + y_1\left[d(y)_1 + \ldots\right] + \ldots \\ \qquad + \frac{1}{2}\, d^2Z_1 + \frac{1}{3}\, d^3Z_1 + \ldots = -\Delta K \\ -V_2\Delta P + H_2\Delta T + x_2\left[d(x)_2 + \ldots\right] + y_2\left[d(y)_2 + \ldots\right] + \ldots \\ \qquad + \frac{1}{2}\, d^2Z_2 + \frac{1}{3}\, d^3Z_2 + \ldots = -\Delta K \end{array}\right\} . \quad (7)$$

and still n—2 other equations. It follows from (3)

$$\left.\begin{array}{l} d(x)_1 + \frac{1}{2}\, d^2(x)_1 + \ldots = d(x)_2 + \frac{1}{2}\, d^2(x)_2 + \ldots = \Delta K_x \\ d(y)_1 + \frac{1}{2}\, d^2(y)_1 + \ldots = d(y)_2 + \frac{1}{2}\, d^2(y)_2 + \ldots = \Delta K_y \end{array}\right\} . \quad (8)$$

etc. In accordance with our notation is e.g.

$$d(x)_1 = (Px)_1\,\Delta P + (Tx)_1\,\Delta T + (x^2)_1\,\Delta x_1 + (xy)_1\,\Delta y_1 + \ldots$$

or
$$\frac{\partial V_1}{\partial x_1}\Delta P - \frac{\partial H_1}{\partial x_1}\Delta T + \frac{\partial^2 Z_1}{\partial x_1^{\,2}}\Delta x_1 + \frac{\partial^2 Z_1}{\partial x_1 \partial y_1}\Delta y_1 + \ldots$$

$$d(y)_1 = (Py)_1\,\Delta P + (Ty)_1\,\Delta T + (xy)_1\,\Delta x_1 + (y^2)_1\,\Delta y_1 + \ldots$$

or
$$\frac{\partial V_1}{\partial y_1}\Delta P - \frac{\partial H_1}{\partial y_1}\Delta T + \frac{\partial^2 Z_1}{\partial x_1 \partial y_1}\Delta x_1 + \frac{\partial^2 Z_1}{\partial y_1^{\,2}}\Delta y_1 + \ldots$$

When a phase e.g. F_1 has a constant composition, then for this (4) is true; instead of the first equation (7) we find then:

$$-V_1\Delta P + H_1\Delta T + \alpha\left[d(x)_1 + \ldots\right] + \beta\left[d(y)_1 + \ldots\right] = -\Delta K.$$

Consequently in the first equation (7) are missing then the terms d^2Z_1, d^3Z_1 etc.

Equilibria of n components in n phases under constant pressure.

When we keep the pressure constant, then we have to omit in (7) and (8) all the terms with ΔP; the sign d indicates then that we have to differentiate according to all variables, except P.

Now we have in (7) and (8) n^2 equations and n^2+1 differentials ΔT, $\Delta x_1 \ldots$, so that their relations are defined. Consequently to each definite differential of one of the variables e.g. Δx_1 belongs a definite differential of each of the other variables, therefore, e.g. also of ΔT. On change of x_1 (or one of the other variables) the equilibrium E follows, therefore, in the P,T-diagram a straight line, parallel to the T-axis.

Now we shall put the question: when will the temperature be maximum or minimum?

For this it is necessary that ΔT is of the second order; then it follows from (7) and (8) that it must be possible to satisfy:

$$\left.\begin{array}{l} x_1\, d(x)_1 + y_1\, d(y)_1 + \ldots = \Delta K \\ x_2\, d(x)_2 + y_2\, d(y)_2 + \ldots = \Delta K \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and

$$\left.\begin{array}{l} d(x)_1 = d(x)_2 = \ldots = d(x)_n = \Delta K_x \\ d(y)_1 = d(y)_2 = \ldots = d(y)_n = \Delta K_y \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \quad \ldots\ldots \quad (10)$$

in which d indicates now that we have to differentiate according to all variables except P and T.

It must be possible to solve the ratios between the n^2 differentials $\Delta x_1\ \Delta x_2 \ldots \Delta y_1\ \Delta y_2 \ldots \Delta K,\ \Delta K_x, \ldots$ from the n^2 equations (9) and (10); this is only then possible, when a relation exists between the coefficients. With the aid of (10) we write for (9):

$$\left.\begin{array}{l} x_1\, \Delta K_x + y_1\, \Delta K_y + \ldots = \Delta K \\ x_2\, \Delta K_x + y_2\, \Delta K_y + \ldots = \Delta K \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \quad \ldots\ldots \quad (11)$$

so that we must be able to satisfy (10) and (11). Here this is the case when we are able to satisfy (11).

When we add the n equations (11) after having multiplied the first one by λ_1, the second one by λ_2, etc., then we obtain:

$$\Sigma\,(\lambda x)\, \Delta K_x + \Sigma\,(\lambda y)\, \Delta K_y + \Sigma\,(\lambda)\, \Delta K \quad \ldots\ldots \quad (12)$$

Hence it is apparent that we are able to solve the ratios between $\Delta x_1\ \Delta x_2 \ldots$ from (9) and (10), when

$$\left.\begin{array}{l} \Sigma\,(\lambda) \;\; = \lambda_1 + \lambda_2 + \ldots + \lambda_n = 0 \\ \Sigma\,(\lambda x) = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = 0 \\ \Sigma\,(\lambda y) = \lambda_1 y_1 + \lambda_2 y_2 + \ldots + \lambda_n y_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \quad \ldots \quad (13)$$

can be satisfied.

We might also satisfy (12) by putting equal to zero ΔK_x, ΔK_y, ...; now, however, we leave this case out of consideration and we shall refer to this later; then we shall see that the equilibrium is situated on the limit of its stability.

[Mr. W. van der Woude has drawn my attention to the fact that we can easily express the condition that (9) and (10) can be satisfied in a determinant. It appears that this can be written like the product of different other determinants, so that we know at once all the conditions looked for.

We have in (13) n equations between the $n-1$ ratios of $\lambda_1\ \lambda_2 \ldots \lambda_n$; consequently (13) can only be satisfied when a ratio exists between the variables. We may find it by eliminating from the equations (13) $\lambda_1 \ldots \lambda_n$; we may also write this equation in the form of the following determinant;

$$\begin{vmatrix} 1 & 1 & 1 & 1 \ldots \\ x_1 & x_2 & x_3 & x_4 \ldots \\ y_1 & y_2 & y_3 & y_4 \ldots \\ z_1 & z_2 & z_3 & z_4 \ldots \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \end{vmatrix} = 0$$

When we bear in mind, however, the compositions of the phases $F_1 \ldots F_n$ then it appears that (13) expresses, that it must be possible that between the phases a reaction of the form:

$$\lambda_1 F_1 + \lambda_2 F_2 + \ldots + \lambda_n F_n = 0$$

consequently a phase-reaction occurs. Then the equilibrium is an equilibrium E_R and consequently it is situated in the P, T-diagram on a curve E_R, viz. on a turning-line of the region E.

Therefore we find:

"in an equilibrium of n components in n phases under constant P the temperature is maximum or minimum, when between the phases a phase-reaction can occur".

Consequently in a binary system T is maximum or minimum when the two phases have the same composition; in a ternary system when the 3 points which represent the phases, are situated on a straight line; in a quaternary system when the 4 phases may be represented by 4 points of a plane; etc.

Now we have still to examine when T is a maximum and when it is a minimum. For this we have to determine ΔT. We take the equations (7) in which all the terms with ΔP must be omitted now. When we add the equations after having multiplied the first by λ_1, the second by λ_2, etc., then we find with the aid of (8) and (13):

$$\Sigma (\lambda H) . \Delta T + \tfrac{1}{2} \Sigma (\lambda d^2 Z) + \tfrac{1}{3} \Sigma (\lambda d^3 Z) + \ldots = 0 \quad . \quad . \quad (14)$$

or at first approximation:

$$\Sigma (\lambda H) . \Delta T = - \tfrac{1}{2} \Sigma (\lambda d^2 Z) \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Herein is:

$$\Sigma (\lambda H) = \lambda_1 H_1 + \lambda_2 H_2 + \ldots + \lambda_n H_n$$

consequently the increase of entropy which occurs at the reaction:

$$\lambda_1 F_1 + \lambda_2 F_2 + \ldots + \lambda_n F_n = 0$$

Further is:

$$\Sigma (\lambda d^2 Z) = \lambda_1 d^2 Z_1 + \lambda_2 d^2 Z_2 + \ldots + \lambda_n d^2 Z_n \quad . \quad . \quad . \quad (16)$$

or, as it follows from the values of $d^2 Z_1$ etc.:

$$\left.\begin{aligned}\Sigma(\lambda d^2 Z) = \; & \lambda_1 [d(x)_1 \Delta x_1 + d(y)_1 \Delta y_1 + \dots] \\ & + \lambda_2 [d(x)_2 \Delta x_2 + d(y)_2 \Delta y_2 + \dots] \\ & \dots\dots\dots\dots\dots\dots\dots \\ & + \lambda_n [d(x)_n \Delta x_n + d(y)_n \Delta y_n + \dots]\end{aligned}\right\} \quad . \; . \quad (17)$$

or also:

$$\left.\begin{aligned}\Sigma(\lambda d^2 Z) = \; & \lambda_1 [(x^2)_1 \Delta x_1^2 + (y^2)_1 \Delta y_1^2 + \dots + 2(xy)_1 \Delta x_1 \Delta y_1] \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + \lambda_n [(x^2)_n \Delta x_n^2 + (y^2)_n \Delta y_n^2 + \dots + 2(xy)_n \Delta x_n \Delta y_n]\end{aligned}\right\} \quad (18)$$

When one of the phases e.g. F_1, has a constant composition, then in (16) $d^2 Z_1$ disappears when there are more phases with constant composition, then in (16) and consequently also in (17) and (18) the corresponding terms disappear.

When the equilibrium E is stable (or, which comes to the same for our considerations "metastable") then $d^2 Z_1$ $d^2 Z_2$ —— are positive; when however the equilibrium is unstable, then one or more of the forms $d^2 Z_1$ —— may be negative.

Now it follows from (15) when the temperature is a maximum and when it is a minimum.

When $\Sigma(\lambda H)$ and $\Sigma(\lambda d^2 Z)$ have the same sign, then $\Delta T < 0$ and consequently T is a maximum

When $\Sigma(\lambda H)$ and $\Sigma(\lambda d^2 Z)$ have opposite sign, then $\Delta T > 0$ and consequently T is a minimum.

When $\Sigma(\lambda d^2 Z) = 0$, then T is neither maximum nor minimum.

In some cases it is easy to define this. Let us take e.g. an equilibrium

$$E = L_1 + F_2 + F_3 + \dots + F_n$$

in which L_1 is a liquid and $F_2 \dots F_n$ phases of invariable composition e.g. solids.

We cause the phase-reaction

$$\lambda_1 L_1 + \lambda_2 F_2 + \dots + \lambda_n F_n = 0$$

to proceed in such a way that λ_1 quantities of L_1 must be formed and we take λ_1 positive. In the equilibrium E therefore, a reaction occurs [melting or conversion of solid substances] at which liquid is formed. As, in general, heat is to be added at this reaction, $\Sigma(\lambda H) > 0$.

As $F_2 \dots F_n$ are phases with invariable composition $\Sigma(\lambda d^2 Z) = \lambda_1 d^2 Z_1$.

Consequently we have:

$$\Sigma(\lambda H)\, \Delta T = -\tfrac{1}{2} \lambda_1 d^2 Z_1 \quad . \; . \; . \; . \; . \; . \quad (19)$$

in which $\Sigma(\lambda H) > 0$ and $\lambda_1 > 0$. $d^2 Z_1$ is positive when the equilibrium is stable, but it may be negative when the equilibrium is unstable. Consequently T is a maximum when the equilibrium is stable, but it may be a minimum when the equilibrium is unstable.

When we summarize the previous considerations, then we find the following:

In an equilibrium of n components in n phases under constant P the temperature is maximum or minimum when a phase-reaction can occur between the phases.

When one of the phases is a liquid and when the $n-1$ other phases are solids with invariable composition, then T is a maximum when the equilibrium is stable (or metastable); T can be a minimum when the equilibrium is unstable.

We may apply those general considerations to special cases; with this we assume that the equilibrium is stable (or metastable).

In the binary equilibrium $E = L_1 + F_2 \dots L_1$ represents the liquid, saturated with solid F_2. In a T-concentration-diagram L_1 follows, therefore, the saturation-curve under its own vapour pressure. Consequently this curve must have its maximum-temperature in the point, in which L_1 has the same composition as F_2; this is, therefore, in the melting-point of F_2.

In the ternary equilibrium $E = L_1 + F_2 + F_3 \dots L_1$ is a liquid, saturated with $F_2 + F_3$. In the concentration-diagram L_1 follows, therefore, the saturationcurve of $F_2 + F_3$ under its own vapour-pressure. T changes along this curve from point to point. It will be necessary that T is a maximum in the point of intersection of this curve with the line $F_2 F_3$.

Similar considerations are true for systems with 4 and more components.

In a following communication we shall refer to unstable conditions.

Equilibria of n components in n phases under constant pressure and at a temperature which differs little from the maximum- or minimum temperature.

As between the n phases of an equilibrium E_R a phase-reaction may occur, (13) may be satisfied. The ratios between Δx_1 $\Delta y_1 \dots$ Δx_2 $\Delta y_2 \dots$ are then defined by (9) and (10). When we imagine $\Delta x_2 \dots$ Δy_1 $\Delta y_2 \dots$ to be expressed in Δx_1 and this to be substituted in (18) then it appears that we may write for $\Sigma(\lambda d^2 Z)$ a form like $A \Delta x_1^2$. Herein A has a definite positive or negative value. Then follows from (15):

$$\Delta x_1 = \pm \sqrt{\frac{2\Sigma(\lambda H)}{-A}}\,\Delta T. \quad . \quad . \quad . \quad . \quad (20)$$

Hence it appears that to each definite value of ΔT two values of Δx_1 (and consequently also of $\Delta x_2 \ldots \Delta y_1\, \Delta y_2 \ldots$) belong, which differ from one another in sign only. When the form under the root in (20) is positive, then $\Delta x_1\, \Delta x_2 \ldots \Delta y_1\, \Delta y_2 \ldots$ have, therefore, each two real values; when this form is negative, then $\Delta x_1 \ldots$ are imaginary.

Consequently we distinguish two cases.

When $\Sigma(\lambda H)$ and A have the same sign, then we must take ΔT negative in order to obtain real values for $\Delta x_1 \ldots$; the temperature T_R is, therefore, a maximum. At T_R there exists therefore one single equilibrium E_R; at $T_R + \Delta T$ (we take $\Delta T > 0$) no equilibrium E exists; at $T_R - \Delta T$, however, two different equilibria exist, which we shall call E' and E''.

When $\Sigma(\lambda H)$ and A have the same sign, then we have to take ΔT positive, in order to obtain real values for $\Delta x_1 \ldots$ Consequently T_R is a minimum. Then at $T_R + \Delta T$ two different equilibria E' and E'' exist, at $T_R - \Delta T$ no equilibrium E exists.

We may also express the previous in the following way.

When under constant P the temperature is a maximum on the turning-line E_R, then two leaves of the region go, starting from this turning-line, towards lower T and not a single leaf towards higher T.

When under constant P the temperature is a minimum on the turning-line E_R, then two leaves of the region go towards higher T starting from this turning-line, and no single leaf towards lower T.

With our considerations on the region E in the previous communication XVI, we have already applied these results. The figs. 2 (XVI) and 4 (XVI) in which *ef* represents a turning-line, are in accordance with this. In fig. 7 (XVI) *ql* and *dm* are the limit-lines, *MSm* a turning-line. In order to show that also this diagram is in accordance with those results, we consider the horizontal line *qrst*; in order to show more distinctly the situation of those leaves, it is partly dotted and curved in *r* and *s*. In *r* T is a maximum, this corresponds to the fact that two leaves go, starting from curve *mS* towards lower and no leaf towards higher T. In *s* T is a minimum, two leaves go towards higher- and no leaf towards lower temperatures.

We have seen that at the temperature T_R only one single equilibrium $E_R = F_1 + F_2 + \ldots$ exists and that a phase-reaction may occur between the phases of this equilibrium. At $T_R + \Delta T [\Delta T < 0$

when T_R is a maximum; $\Delta T > 0$ when T_R is a minimum] two equilibria exist, viz.

$$E' = F'_1 + F'_2 + \dots \quad \text{and} \quad E'' = F_1'' + F_2'' + \dots$$

No phase-reaction can occur between the phases of E'; no more between those of E''. The invariable phases have of course the same composition in the three equilibria; the compositions of the variable phases differ only little from one another in the three equilibria. Now we shall show:

a. The concentration-regions of E_R, E' and E'' are situated in the concentration-diagram outside one another.

The three equilibria have, therefore, such compositions that none of them can be converted into one of the two other equilibria.

b. The concentration-region of E_R is situated between those of E' and E''.

c. The corresponding phases of the three equilibria (e. g. F_1 F_1' and F_1''; F_2 F_2' and F_2''; etc.) are situated on a straight line; this is divided into equal parts by the phase of the equilibrium E_R.

Before showing this, we shall first elucidate the meaning by some examples.

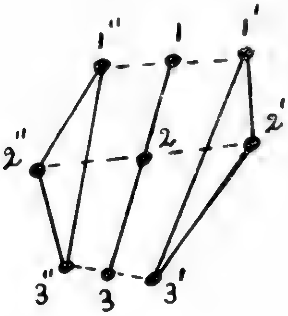

Fig. 1.

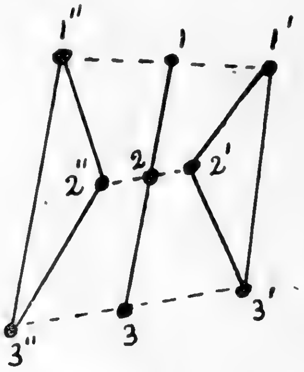

Fig. 2.

For this we choose the ternary equilibrium

$$E = F_1 + F_2 + F_3.$$

When we represent those phases in the concentration-diagram by the points 1, 2 and 3, then at T_R those three points are situated on a straight line (line 123 in figs. 1 and 2). The concentration-region of E_R is, therefore, the line 123.

At the temperature $T_R + \Delta T$ exist the equilibria:

$$E' = F_1' + F_2' + F_3' \quad \text{and} \quad E'' = F_1'' + F_2'' + F_3''$$

First we shall assume that each of the phases of the equilibrium E has a variable composition; the phases of E' are then represented in figs. 1 and 2 by the points $1'$, $2'$ and $3'$; those of E'' by $1''$, $2''$ and $3''$. The points $1'$, $2'$ and $3'$ are situated in the immediate vicinity of the points 1, 2 and 3; they form the anglepoints of a triangle $1'\ 2'\ 3'$, which represents the concentration-region of the equilibrium E'. Triangle $1''\ 2''\ 3''$ represents the concentration-region of E''.

In accordance with a the line 123 and the triangles $1'2'3'$ and $1''2''3''$ must not have one single point in common, in accordance with b the line 123 must be situated between the two triangles; in accordance with c $1'11''$, $2'22''$ and $3'33''$ are straight lines and is $11' = 11''$, $22' = 22''$ and $33' = 33''$. Consequently we obtain a diagram as in figs. 1 and 2.

Consequently at $T_R + \Delta T$ two triangles arise from the straight line which occurs at the temperature T_R; reversally the two triangles, which occur at $T_R + \Delta T$ coincide at T_R into a straight line.

The transitions, discussed for figs 1 and 2 will occur when the ternary equilibrium E consists of 3 liquids or of 2 liquids and vapour, or of 3 kinds of mixed crystals, or of a mixed crystal + liquid + vapour etc.

When one of the phases e. g. F_1 has an invariable composition, then we obtain figs 3 or 4; when two phases e. g. F_1 and F_2 have an invariable composition, then we obtain fig 5.

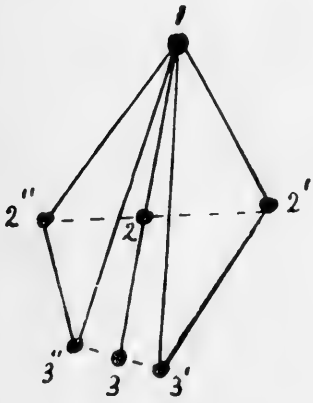

Fig. 3.

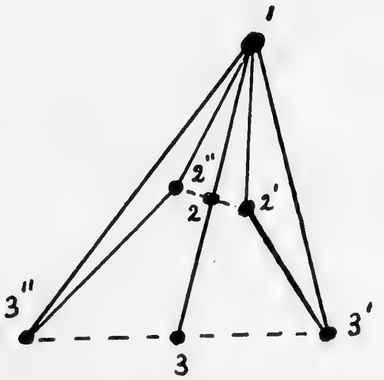

Fig. 4.

Fig. 5.

In order to show the rules, mentioned above, we represent of the equilibrium E_R the composition

of F_1 by $x_1\ y_1\ z_1 \ldots$
,, F_2 ,, $x_2\ y_2\ z_2 \ldots$

etc.

Then the composition of the equilibrium E' is:

of F_1' $\quad x_1 + \Delta x_1 \quad y_1 + \Delta y_1 \quad z_1 + \Delta z_1$
,, F_2' $\quad x_2 + \Delta x_2 \quad y_2 + \Delta y_2 \quad z_2 + \Delta z_2$
. . . .

and the composition of the equilibrium E'' is:

of F_1'' $\quad x_1 - \Delta x_1 \quad y_1 - \Delta y_1 \quad z_1 - \Delta z_1$
,, F_2'' $\quad x_2 - \Delta x_2 \quad y_2 - \Delta y_2 \quad z_2 - \Delta z_2$.
. . . .

Herein $\Delta x_1\ \Delta x_2 \ldots$ are defined by (9) and (10); it is apparent that they may be as well positive as negative,

In order to be able to convert the equilibrium E_R into E', it must be possible to satisfy

$$a_1F_1 + a_2F_2 + \ldots = b_1 F_1' + b_2F_2' + \ldots \qquad (21)$$

in which all coefficients must be positive.

It follows from (21):

$$a_1 + a_2 + \ldots = b_1 + b_2 + \ldots$$
$$a_1x_1 + a_2x_2 + \ldots = b_1(x_1 + \Delta x_1) + b_2(x_2 + \Delta x_2) + \ldots$$
$$a_1y_1 + a_2y_2 + \ldots = b_1(y_1 + \Delta y_1) + b_2(y_2 + \Delta y_2) + \ldots$$

etc. When we put $a_1 - b_1 = c_1$; $a_2 - b_2 = c_2$; etc. then the previous equations pass into:

$$\left.\begin{aligned} &c_1 + c_2 + \ldots + c_n = 0 \\ &c_1x_1 + c_2x_2 + \ldots = b_1\Delta x_1 + b_2\Delta x_2 + \ldots \\ &c_1y_1 + c_2y_2 + \ldots = b_1\Delta y_1 + b_2\Delta y_2 + \ldots \end{aligned}\right\} \qquad (22)$$

etc. We can eliminate $c_1 \ldots c_n$ from the n equations (22). We add them viz. after having multiplied the 1st by μ_1, the 2nd by μ_2, etc. As $x_1\ y_1 \ldots$ viz. satisfy (9), they also satisfy:

$$\left.\begin{aligned} &\mu_1 + \mu_2x_1 + \mu_3y_1 + \ldots = 0 \\ &\mu_1 + \mu_2x_2 + \mu_3y_2 + \ldots = 0 \end{aligned}\right\} \qquad (23)$$

etc. (22) passes then into:

$$\left.\begin{aligned} 0 = b_1\,[\mu_2\Delta x_1 + \mu_3\Delta y_1 + \ldots] + b_2\,[\mu_2\Delta x_2 + \mu_3\Delta y_2 + \ldots] \\ + \ldots + b_n\,[\mu_2\Delta x_n + \mu_3\Delta y_n + \ldots] \end{aligned}\right\} \qquad (24)$$

Also it appears from (9) that we may satisfy (23) by taking $\mu_2 = a\,d(x)_1 = a\,d(x)_2 = \ldots$, $\mu_3 = a\,d(y)_1 = a\,d(y)_2 = \ldots$, etc. Therefore (24) passes into:

$$0 = b_1\,[d(x)_1\Delta x_1 + d(y)_1\Delta y_1 + \ldots] + b_2\,[d(x)_2\Delta x_2 + d(y)_2\Delta y_2 + \ldots] + \ldots$$

for which we may also write:

$$0 = b_1\,d^2 Z_1 + b_2\,d^2 Z_2 + \ldots + b_n\,d^2 Z_n \qquad (25)$$

Is must be possible to satisfy (25) by giving positive values to $b_1\ b_2 \ldots$ When we consider only equilibria in stable (or metastable) condition, then d^2Z_1, $d^2Z_2, \ldots$ are positive; it is, therefore, not possible to satisfy (25) and consequently also not (21).

Hence it follows, therefore, that E_R and E' cannot be converted into one another; as we are able to deduce this in the same way for E_R and E'' and also for E' and E'', the property mentioned sub *a* is proved. At the same time it appears from the deduction that it need not be true for equilibria in unstable condition.

For the equilibria E' and E'' this property follows also at once without calculation viz. from the condition that under constant P and at constant T ξ must be a minimum.

The properties, mentioned sub *b* and *c* follow now at once from property *a* and formula (20).

Leiden, Inorg. Chem. Lab. *(To be continued).*

Mechanics. — *"On the relativity of inertia. Remarks concerning* EINSTEIN'S *latest hypothesis"* [1]) By Prof. W. DE SITTER.

(Communicated in the meeting of March 31, 1917).

If we neglect the gravitational action of all ordinary matter (sun, stars, etc.), and if we use as a system of reference three rectangular cartesian space-coordinates and the time multiplied by c, then in that part of the four-dimensional time-space which is accessible to our observations, the $g_{\mu\nu}$ are very approximately those of the old theory of relativity, viz.:

$$\left.\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{array}\right\} , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The part of the time-space where this is so, I shall call "our neighbourhood". In space this extends at least to the farthest star, nebula or cluster in whose spectrum we can identify definite lines [2]).

How the $g_{\mu\nu}$ are outside our neighbourhood we do not know, and any assumption regarding their values is an extrapolation, whose uncertainty increases with the distance (in space, or in time, or in both) from the origin. How the $g_{\mu\nu}$ are at infinity of space or of time, we will never know. Nevertheless the need has been felt to

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[1]) A. EINSTEIN, Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie, Sitzungsber. Berlin, 8 Febr. 1917, page 142.

[2]) W. DE SITTER, On EINSTEIN'S theory of gravitation and its astronomical consequences (second paper), Monthly Notices R.A.S. Dec. 1916, Vol. LXXVII, p. 182. This limit refers to g_{44} only.

make hypotheses on this subject. The extrapolation, which offers itself most naturally, and which is also tacitly made in classical mechanics, is that the values (1) remain unaltered for all space and time up to infinity. On the other hand the desire has arisen to have constants of integration, or rather boundary-values at infinity, which shall be the same in all systems of reference. The values (1) do not satisfy this condition. The most desirable and the simplest value for the $g_{\mu\nu}$ at infinity is evidently *zero*. Einstein has not succeeded in finding such a set of boundary values [1]) and therefore makes the hypothesis that the universe is not infinite, but spherical: then no boundary conditions are needed, and the difficulty disappears. From the point of view of the theory of relativity it appears at first sight to be incorrect to say: the world *is* spherical, for it can by a transformation analogous to a stereographic projection be represented in a euclidean space. This is a perfectly legitimate transformation, which leaves the different invariants ds^2, G etc. unaltered. But even this invariability shows that also in the euclidean system of coordinates the world, in natural measure, remains finite and spherical. If this transformation is applied to the $g_{\mu\nu}$ which Einstein finds for his spherical world, they are transformed to a set of values which at infinity degenerate to

$$\left.\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right| \quad \ldots\ldots\ldots\ldots \quad (2A)$$

It appears, however, that the $g_{\mu\nu}$ of Einstein's spherical world [and therefore also their transformed values in the euclidean system of reference] do not satisfy the differential equations originally adopted by Einstein, viz:

$$G_{\mu\nu} = -\varkappa(T_{\mu\nu} \;\; \tfrac{1}{2} g_{\mu\nu} T) \, . \quad \ldots\ldots\ldots \quad (3)$$

Einstein thus finds it necessary to add another term to his equations, which then become

$$G_{\mu\nu} - \lambda g_{\mu\nu} = -\varkappa(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T) \, . \quad \ldots\ldots \quad (4)$$

Moreover it is found necessary to suppose the whole three-dimen-

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[1]) l.c. page 148. It will appear below that Einstein's hypothesis is equivalent to a determined set of values at infinity, viz: the set (2*A*). It is, in fact, evident that, if the universe measured in natural measure be finite, then, if euclidean coordinates are introduced the $g_{\mu\nu}$ must necessarily be zero at infinity, and inversily if the $g_{\mu\nu}$ at infinity are zero of a sufficiently high order, then the universe is finite in natural measure.

sional space to be filled with matter, of which the total mass is so enormously great, that compared with it all matter known to us is utterly negligible. This hypothetical matter I will call the "world-matter".

EINSTEIN only assumes *three*-dimensional space to be finite. It is in consequence of this assumption that in (2A) g_{44} remains 1, instead of becoming zero with the other $g_{\mu\nu}$. This has suggested the idea [1]) to extend EINSTEIN's hypothesis to the *four*-dimensional time-space. We then find a set of $g_{\mu\nu}$ which at infinity degenerate to the values

$$\left.\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}\right\} \quad \ldots\ldots\ldots \quad (2B)$$

Moreover we find the remarkable result, that now no "world-matter" is required.

In order to point out the analogy of the two cases I give the two sets of formulae together. The formulae A refer to EINSTEIN's (*three*-dimensional) hypothesis, the formulae B refer to the assumption here introduced (*four*-dimensional). I shall use the indices i and j, when they take the values 1, 2, 3 only; μ and ν take the values from 1 to 4. Further Σ is a sum from 1 to 4 and Σ' from 1 to 3; and $\delta_{\mu\mu} = 1$, $\delta_{\mu\nu} = 0$ if $\mu \neq \nu$.

I first take the system of reference used by EINSTEIN. In case A we take $x_4 = ct$, in B I take, for the sake of symmetry [2]), $x_4 = ict$. In both cases R is the radius of the hypersphere. The $g_{\mu\nu}$ for the two cases are

A	B
$g_{ij} = -\delta_{ij} - \dfrac{x_i x_j}{R^2 - \Sigma' x^2_i}$	$g_{\mu\nu} = -\delta_{\mu\nu} - \dfrac{x_\mu x_\nu}{R^2 - \Sigma x^2_\mu}$
$g_{44} = 1$	

In order better to show the spherical character I introduce hyperspherical coordinates by the transformations:

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[1]) The idea to make the four-dimensional world spherical in order to avoid the necessity of assigning boundary-conditions, was suggested several months ago by Prof. EHRENFEST, in a conversation with the writer. It was, however, at that time not further developed.

[2]) We can also take $x_4 = ct$. Then the four-dimensional world is hyperbolical instead of spherical, but the results remain the same.

$$\begin{array}{l|l}
x_1 = R \sin\chi \sin\psi \sin\vartheta & x_1 = R \sin\omega \sin\chi \sin\psi \sin\vartheta \\
x_2 = R \sin\chi \sin\psi \cos\vartheta & x_2 = R \sin\omega \sin\chi \sin\psi \cos\vartheta \\
x_3 = R \sin\chi \cos\psi & x_3 = R \sin\omega \sin\chi \cos\psi \\
 & x_4 = R \sin\omega \cos\chi
\end{array}$$

The expression of the line-element then becomes

$$A: \quad ds^2 = -R^2[d\chi^2 + \sin^2\chi(d\psi^2 + \sin^2\psi\, d\vartheta^2)] + c^2dt^2,$$

$$B: \quad ds^2 = -R^2[d\omega^2 + \sin^2\omega\{d\chi^2 + \sin^2\chi(d\psi^2 + \sin^2\psi d\vartheta^2)\}].$$

Finally I perform the "stereographic projection", and at the same time I introduce again rectangular coordinates, by the transformations:

$$\begin{array}{l|l}
\qquad A & \qquad B \\
r = 2R \tan\tfrac{1}{2}\chi & h = 2R \tan\tfrac{1}{2}\omega \\
x = r \sin\psi \sin\vartheta & x = h \sin\chi \sin\psi \sin\vartheta \\
y = r \sin\psi \cos\vartheta & y = h \sin\chi \sin\psi \cos\vartheta \\
z = r \cos\psi & z = h \sin\chi \cos\psi \\
 & ict = h \cos\chi \\
x^2 + y^2 + z^2 = r^2 & x^2 + y^2 + z^2 - c^2t^2 = h^2
\end{array}$$

It need hardly be pointed out that in A x, y, z and in B x, y, z, ict can be arbitrarily interchanged. I put further

$$\sigma = \frac{1}{4R^2}.$$

The $g_{\mu\nu}$ for the variables x, y, z, ct then become [1])

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[1]) In the system B all $g_{\mu\nu}$ are infinite on the "hyperboloid"

$$1 + \sigma h^2 = 0 \text{ or } 4R^3 + x^2 + y^2 + z^2 - c^2t^2 = 0 \quad . \quad . \quad . \quad (a)$$

This discontinuity is however only apparent. The four dimensional world, which we have for the sake of symmetry represented as spherical, is in reality hyperbolical, and consists of two sheets, which are only connected with each other at infinity. The formulae embrace both sheets, but only one of them represents the actual universe. The hyperboloid (a) is the limit between the two parts of the euclidean space x, y, z, ct corresponding to these two sheets. It is intersected by the axis of t at the points $ct = \pm 2R$, the distance of which from the origin is, in natural measure, $\int_0^{2R} \frac{c\,dt}{1 - \sigma c^2t^2} = \infty$. The length in natural measure of the half-axis of x is, in both systems, $\int_0^{\infty} \frac{dx}{1 + \sigma x^2} = \pi R$.

$$\left.\begin{array}{c|c} A & B \\ g_{ij} = -\dfrac{\delta_{ij}}{(1+\sigma r^2)^2} & g_{ij} = -\dfrac{\delta_{ij}}{(1+\sigma h^2)^2} \\ g_{44} = 1 & g_{44} = \dfrac{1}{(1+\sigma h^2)^2} \end{array}\right\} \quad . \; . \; . \; . \; . \quad (5)$$

All $g_{\mu\nu}$ outside the diagonal are zero. If σ is very small the $g_{\mu\nu}$ for moderate values of r and h have very approximately the values (1). At infinity they degenerate to the values (2 A) and (2 B), which have already been given above.

In order to find the relation between σ and λ we must substitute [1]) the values (5) in the equations (4). We must, in doing this, allow for the possibility that it may be found necessary to introduce a "world-matter". We neglect all ordinary matter, and we will suppose the world-matter to be uniformly distributed [2]) over the whole of space, and at rest, so that $T'_{44} = g_{44}\varrho$, and all other $T_{\mu\nu} = 0$. The field-equations then become

$$G_{ij} - (\lambda + \tfrac{1}{2}\varkappa\varrho)\, g_{ij} = 0,$$
$$G_{44} - (\lambda + \tfrac{1}{2}\varkappa\varrho)\, g_{44} = -\varkappa\varrho.$$

For the quantities $G_{\mu\nu}$ we find in the two systems

$$\begin{array}{c|c} A & B \\ G_{ij} = 8\sigma g_{ij} & G_{\mu\nu} = 12\sigma g_{\mu\nu} \\ G_{44} = 0, \; g_{44} = 1 & \end{array}$$

From which

$$\left.\begin{array}{c|c} \lambda = 4\sigma & \lambda = 12\sigma \\ \varrho = \dfrac{8\sigma}{\varkappa} & \varrho = 0 \end{array}\right| \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

The result for A is the same as found by Einstein. For B we have $\varrho = 0$: the hypothetical world-matter does not exist.

Which of the three systems is to be preferred: A with world-matter, B without it, both with the field-equations (4) and at infinity the $g_{\mu\nu}$ (2A) or (2B); or the original system without world-matter, with the field-equations (3) and the $g_{\mu\nu}$ (1), which retain the same values at infinity?

From the purely physical point of view, for the description of

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[1]) We can, of course, as well take the values in any other system of reference.

[2]) The meaning is a distribution in which ϱ is constant, ϱ being the density in *natural* measure. The density in coordinate-measure then, of course, is not constant, but (in the system x, y, z, ct) becomes zero at infinity.

phenomena in our neighbourhood, this question has no importance. In our neighbourhood the $g_{\mu\nu}$ have in all cases within the limits of accuracy of our observations the values (1), and the field-equations (4) are not different from (3). The question thus really is: how are we to extrapolate outside our neighbourhood? The choice can thus not be decided by physical arguments, but must depend on metaphysical or philosophical considerations, in which of course also personal judgment or predilections will have some influence.

To the question: If all matter is supposed not to exist, with the exception of one material point which is to be used as a test-body, has then this test-body inertia or not? the school of MACH requires the answer *No.* Our experience however very decidedly gives the answer *Yes,* if by "all matter" is meant all ordinary physical matter: stars, nebulae, clusters, etc. The followers of MACH are thus compelled to assume the existence of still more matter: the world-matter. If we place ourselves on this point of view, we must necessarily adopt the system A, which is the only one that admits a world-matter. [1]

This world-matter, however, serves no other purpose than to enable us to suppose it not to exist. Now the formula (6) shows, that if it does not exist ($\varrho = 0$), the field-equations are not satisfied: supposing it not to exist thus appears to be a logical impossibility; in the system A, the world-matter *is* the three-dimensional space, or at least is inseparable from it.

We can also abandon the postulate of MACH, and replace it by the postulate that at infinity the $g_{\mu\nu}$, or only the g_{ij} of three-dimensional space, shall be zero, or at least invariant for all transformations. This postulate can also be enounced by saying that it must be possible for the whole universe to perform arbitrary motions, which can never be detected by any observation. The three-dimensional world must, in order to be able to perform "motions", i.e. in order that its position can be a variable function of the time, be thought movable in an "absolute" space of three or more dimensions (*not* the time-space x, y, z, ct). The four-dimensional world requires for its "motion" a four- (or more-) dimensional absolute space, and moreover an extra-mundane "time" which serves as independent variable for this motion. All this shows that the postulate of the

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[1] The hypothesis formerly held by EINSTEIN, and denied by me, that it would be possible, with the equations (3) and by means of very large masses at very large distances, to get values of $g_{\mu\nu}$ which would degenerate to an invariant set at infinity, has now been shown to be untenable by EINSTEIN himself (l. c. page 146).

invariance of the $g_{\mu\nu}$ at infinity has no real physical meaning. It is purely mathematical.

The system A, with at infinity the values (2 A) of the $g_{\mu\nu}$ satisfies this postulate, if it is applied only to the three-dimensional world, and if we do not require invariance for all transformations, but only for those which at infinity have $t' = t$[1]). If the postulate is applied to the four-dimensional world, and to *all* transformations, then the system B is the only one that satisfies. We thus find that in the system A the time has a separate position. That this must be so, is evident a priori. For speaking of *the* three-dimensional world, if not equivalent to introducing an absolute time, at least implies the hypothesis that at each point of the four-dimensional space there is one definite coordinate x_4 which is preferable to all others to be used as "time", and that at all points and always this one coordinate is actually chosen as time. Such a fundamental difference between the time and the space-coordinates seems to be somewhat contradictory to the complete symmetry of the field-equations and the equations of motion (equations of the geodetic line) with respect to the four variables.

Some features of the systems A and B may still be pointed out. In A the velocity of light is variable[2]), at infinity it becomes infinite. In B it is always and everywhere the same. From the facts that we can identify lines in the spectra of the most distant objects known to us such as the Nubeculae, and that the parallaxes of these objects are not negative, we can conclude that at these distances we have still approximately $g_{ij} = -\delta_{ij}$, $g_{44} = 1$ and consequently that for A σr^2, and for B σh^2 must be very small. In the case A we can derive in this way an upper limit for σ. In B on the other hand we have, in consequence of the constancy of the velocity of light, $h^2 = 0$ for all purely optical observations (if we neglect the influence of matter).

As to the effect of σ on planetary motions: in both cases the

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[1]) Thus e. g. an ordinary LORENTZ-transformation:

$$x' = \frac{x - qct}{\sqrt{(1-q^2)}}\ ,\quad ct' = \frac{ct - qx}{\sqrt{(1-q^2)}}$$

is not allowed in the system A, but must be replaced by

$$x' = \frac{x - q\,c\,t}{\sqrt{\left(1 - \dfrac{q^2}{(1+\sigma r^2)^2}\right)}}\ ,\quad ct' = \frac{ct - \dfrac{qx}{1+\sigma r^2}}{\sqrt{\left(1 - \dfrac{q^2}{(1+\sigma r^2)^2}\right)}}.$$

[2]) In the system x, y, z, ct; in the system $\chi, \psi, \vartheta, ct$ it is constant.

orbital plane is not disturbed. In case A there are no secular terms depending on σ.

In B the terms produced by σ are of a lower order, in consequence of the fact that all $g_{\mu\nu}$ depend explicitly on the time. The motion of the perihelion is

$$\delta\tilde{\omega} = \frac{3\sigma a^3}{\lambda_0^2} nt - 2\sigma c^2 t^2,$$

and also the other elements have terms with c^2t^2; thus e.g. the parameter of the elliptic orbit is

$$p = p_0 \, e^{-2\sigma c^2 t^2},$$

where $\lambda_0^2 = \varkappa m / 8\pi$, m being the sun's mass, and $e = 2.718\ldots$ These "perturbations" [1]) being insensible according to our experience, we can here also assign an upper limit to σ.

I shall not attempt to determine this upper limit with any accuracy. For both cases we will be safe in taking e.g. $\sigma < 10^{-24}$ in astronomical units, or $2\sigma < 10^{-50}$ in centimeters [2]). We can, however, do no more than assign an upper limit to σ. To make possible a *determination* of the value of this constant, it would be necessary that it had a measurable effect on some physical or astronomical phenomenon. Now it cannot, of course, be excluded a priori that at some future time observations will be made, or phenomena will be discovered which can be explained with the aid of the constant σ, but so far no such phenomena are known, and there are no indications of anything in that direction. The constant σ only serves to satisfy a philosophical need felt by many, but it has no real physical meaning, though it can be mathematically interpreted as a curvature of space.

Finally we can also reject both systems A and B, and stick to the original field-equations (3) and the values (1) of the $g_{\mu\nu}$, which

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[1]) The terms of the lowest order in the "perturbing forces" are for the two cases:

$$In\ A: \quad S = -3\sigma + \frac{2\sigma}{\lambda_0^2} r(\dot{r}^2 - r^2\dot{\vartheta}^2) \quad , \quad T = \frac{4\sigma}{\lambda_0^2} r^2 \dot{r} \dot{\vartheta}, \quad W = 0,$$

$$in\ B: \quad S = \frac{2\sigma}{\lambda_0^2} r - \frac{2\sigma}{\lambda_0^2} ct\dot{r} \quad , \quad T = -\frac{2\sigma}{\lambda_0^2} ctr\dot{\vartheta}, \quad W = 0.$$

(For the notation see e. g. DE SITTER, *On* EINSTEIN's *theory of gravitation*, M.N. Vol. LXXI, pages 724 sqq.).

The terms with c^2t^2 in the case B arise through the fact that the units both of time and space (in coordinate-measure) depend on the time.

[2]) The density of the world matter in the system A then becomes $\varrho < 3.10^{-17}$ (astronomical units), or $\varrho < 2.10^{-23}$ (C. G. S. units). This corresponds to one star (of the same mass as the sun) in a sphere of one parsec radius.

are not invariant at infinity. Then, of course, inertia is not explained: we must then prefer to leave it unexplained rather than explain it by the undetermined and undeterminable constant λ. It cannot be denied that the introduction of this constant detracts from the symmetry and elegance of EINSTEIN's original theory, one of whose chief attractions was that it explained so much without introducing any new hypothesis or empirical constant.

Postscript.

Prof. EINSTEIN, to whom I had communicated the principal contents of this paper, writes (March 24, 1917): "Es wäre nach meiner Meinung unbefriedigend, wenn es eine denkbare Welt ohne Materie gäbe. Das $g_{\mu\nu}$-Feld soll vielmehr *durch die Materie bedingt sein, ohne dieselbe nicht bestehen können.* Das ist der Kern dessen, was ich unter der Forderung von der Relativität der Trägheit verstehe". He therefore postulates what I called above the logical impossibility of supposing matter not to exist. We can call this the "material postulate" of the relativity of inertia. This can only be satisfied by choosing the system A, with its world-matter, i. e. by introducing the constant λ, and assigning to the time a separate position amongst the four coordinates.

On the other hand we have the "mathematical postulate" of the relativity of inertia, i. e. the postulate that the $g_{\mu\nu}$ shall be invariant at infinity. This postulate, which, as has already been pointed out above, has no real physical meaning, makes no mention of matter. It can be satisfied by choosing the system B, without a world-matter, and with complete relativity of the time. But here also we need the constant λ. The introduction of this constant can only be avoided by abandoning the postulate of the relativity of inertia altogether.

Astronomy. — *"On the Theory of Hyperion, one of Saturn's Satellites."*. By J. WOLTJER JR. (Communicated by Prof. W. DE SITTER).

(Communicated in the meeting of April 27, 1917).

1. Among the peculiar disturbances, which the satellites of Saturn undergo by their mutual attraction, those, produced by Titan in the motion of Hyperion, are of much importance. In this paper I intend to give a short development of the theory of the latter satellite; my dissertation will contain more extensive calculations on this subject.

The ratio of the masses of Hyperion and Titan is only a very small quantity. The inclinations of the orbital planes of Hyperion and Titan to the aequator of Saturn are also small. To simplify the problem I shall neglect these inclinations as well as the influence of the mass of Hyperion, the sun, the other satellites, the ellipticity of Saturn and the rings. I shall, therefore, suppose Hyperion to be a particle with the mass zero, moving in the orbital plane of Titan, while the latter describes an undisturbed elliptical motion around the centre of a sphere with the mass of Saturn.

Notation:

$a =$ semi-major axis, $e =$ excentricity;

$l =$ mean anomaly, $g =$ longitude of pericentre;

$\mathrm{M} =$ mass of Saturn, $m' =$ mass of Titan;

$\mathrm{L} = \sqrt{a\mathrm{M}}$; $\mathrm{G} = \mathrm{L}\sqrt{1-e^2}$.

The accented letters refer to Titan, those without accents to Hyperion. The units are chosen so, that the constant of attraction $= 1$; x and y are coordinates in a system of axes in the orbital plane of Titan, the origin coinciding with the centre of Saturn.

From the fundamental equations, which Delaunay has used in his lunar theory [1]), the following differential equations for the motion of Hyperion result:

$$\frac{d\mathrm{L}}{dt} = \frac{\partial \mathrm{R}}{\partial l}, \quad \frac{d\mathrm{G}}{dt} = \frac{\partial \mathrm{R}}{\partial g},$$

$$\frac{dl}{dt} = -\frac{\partial \mathrm{R}}{\partial \mathrm{L}}, \quad \frac{dg}{dt} = -\frac{\partial \mathrm{R}}{\partial \mathrm{G}},$$

$$\mathrm{R} = \frac{\mathrm{M}^2}{2\mathrm{L}^2} - m'\frac{xx'+yy'}{r'^3} + \frac{m'}{\sqrt{(x'-x)^2+(y'-y)^2}}.$$

The function R is, with regard to the angular elements, a function of $l+g-l'-g'$, l, l' only. The following new quantities are introduced:

$$l + g - l' - g' = \Phi,$$
$$4l - 3l' + 3g - 3g' - 180° = \theta,$$
$$g - g' = \Omega,$$
$$g' = \chi.$$

One sees at once that R, with regard to the angular elements, is a function of Φ, θ, Ω only. The reason why these three quantities are introduced is this: from the observations the mean motion of the argument θ appears to be zero and θ to perform a libration on each side of the value $\theta = 0°$ with an amplitude of about 36°;

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[1]) Théorie du Mouvement de la Lune I, 13.

this argument, therefore, is of much importance; Ω is a secular argument, its introduction is therefore obvious; the argument Φ has a short period and thus leads to the terms in the development of the perturbative function, which are of little importance. From R therefore all terms with arguments containing Φ or a multiple of Φ are omitted; thus instead of R one uses the function $\overline{R} = \frac{1}{2\pi}\int_0^{2\pi} R\,d\Phi$.

For convenience sake in the rest of this paper the line above the letter R is omitted.

Putting

$$L - G = \Lambda,$$
$$3L - 4G = \Gamma,$$

and n' for the mean motion of Titan, the equations for θ, Φ, Λ, Γ become:

$$\frac{d\Lambda}{dt} = \frac{\partial R}{\partial \theta} - \frac{\partial R}{\partial \Omega}, \quad \frac{d\theta}{dt} = -\frac{\partial R}{\partial \Lambda} - 3n',$$
$$\frac{d\Gamma}{dt} = -4\frac{\partial R}{\partial \Omega}, \quad \frac{d\Phi}{dt} = \frac{\partial R}{\partial \Gamma} - n'.$$

2. The excentricity of Titan being a small quantity, I shall try to develop the solution of these equations in powers of this excentricity. Thus, first I put $e' = 0$; then R appears to be independent of Ω and to be a function of θ only. The equations then are:

$$\frac{d\Lambda}{dt} = \frac{\partial R\,(e'=0)}{\partial \theta}, \quad \frac{d\theta}{dt} = -\frac{\partial R\,(e'=0)}{\partial \Lambda} - 3n',$$
$$\frac{d\Gamma}{dt} = 0, \quad \frac{d\Phi}{dt} = \frac{\partial R\,(e'=0)}{\partial \Gamma} - n'.$$

Putting $R_0 = \frac{M^2}{2(4\Lambda - \Gamma)^2}$, $R_1 = (R - R_0)_{e'=0}$, $R = R_0 + R_1 + R_2$, the development of R_1 is:

$$R_1 = m' \sum_0^{\infty}{}_p\, A_p \cos p\theta,$$

where $A_0, \ldots A_p, \ldots$ are functions of Λ and Γ.

$R\,(e'=0)$ being independent of Φ and Ω, the equations for Λ, Γ and θ form a system apart; after the integration of this system Φ is determined by a quadrature. This system admits the solution: $\theta = 0$, $\Lambda =$ const., $\Gamma =$ const.. However, between the constant values of Λ and Γ, which are called Λ_0 and Γ_0, a relation must exist, which is a consequence of the condition that $\frac{d\theta}{dt} = 0$ for $\theta = 0$. This relation is:

$$\frac{4M^2}{(4\Lambda_0 - \Gamma_0)^3} - 3n' = m' \sum_{0}^{\infty}{}_p \left[\frac{\partial A_p}{\partial \Lambda}\right]_{\Lambda_0, \Gamma_0}.$$

Let a_0 and e_0 be the values of a and e belonging to $\Lambda = \Lambda_0$, $\Gamma = \Gamma_0$. The relation becomes, with regard to the equation $n' = \frac{\sqrt{M+m'}}{a'^{\frac{3}{2}}}$:

$$4\left(\frac{a'}{a_0}\right)^{\frac{3}{2}} - 3\sqrt{1+\frac{m'}{M}} =$$

$$= \frac{m'}{M} \sum_{0}^{\infty}{}_p \, a'^{\frac{3}{2}} \left[8\sqrt{a_0}\,\frac{\partial A_p}{\partial a} + \frac{4(1-e_0{}^2)-3\sqrt{1-e_0{}^2}}{e_0\sqrt{a_0}}\,\frac{\partial A_p}{\partial e}\right]_{a_0, e_0}.$$

From this equation the following value of $\frac{a'}{a_0}$ results:

$$\frac{a'}{a_0} = \sum_{0}^{\infty}{}_p \, \alpha_p \left(\frac{m'}{M}\right)^p,$$

where α_p is a function of e_0; the value of α_0 is $(\frac{3}{4})^{\frac{2}{3}}$ and thus: $\alpha_0 = 0.825$.

To investigate the nature of this solution of the differential equations, the adjacent solutions are to be examined. Putting $\Lambda = \Lambda_0 + \delta\Lambda$, $\theta = \delta\theta$ and taking account of the first power of these quantities only, the differential equations become:

$$\frac{d\delta\Lambda}{dt} = \frac{\partial^2 R_1}{\partial\theta^2}\,\delta\theta + \frac{\partial^2 R_1}{\partial\theta\,\partial\Lambda}\,\delta\Lambda,$$

$$\frac{d\delta\theta}{dt} = -\frac{\partial^2 (R_0+R_1)}{\partial\Lambda^2}\,\delta\Lambda - \frac{\partial^2 R_1}{\partial\Lambda\,\partial\theta}\,\delta\theta,$$

whence, by elimination of $\delta\Lambda$:

$$\frac{d^2\delta\theta}{dt^2} = -\frac{\partial^2 (R_0+R_1)}{\partial\Lambda^2}\,\frac{\partial^2 R_1}{\partial\theta^2}\,\delta\theta.$$

I have developed certain portions of the perturbative function numerically for the values $e = 0.1043$ and $\frac{a'}{a} = 0.8250634$. The first value is that which H. STRUVE [1]) has derived from observations, the

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[1]) Beobachtungen der Saturnstrabanten. Publications de l'Observatoire Central Nicolas. Série II. Vol. XI, pg. 290 and 267.

second is the same he has used in the computation of a few coefficients from the perturbative function. From my developments I deduce:

$$\frac{\partial^2 R_1}{\partial \theta^2} = + \frac{m'}{a'} \times 0.0728.$$

Neglecting $\frac{\partial^2 R_1}{\partial \Lambda^2}$ by the side of $\frac{\partial^2 R_0}{\partial \Lambda^2}$ (this is allowed, the first term having m' as factor, the second not), the differential equation for $\delta\theta$, taking account of the relation $\frac{\partial^2 R_0}{\partial \Lambda^2} = + \frac{48}{a_0{}^2}$, becomes:

$$\frac{d^2\delta\theta}{dt^2} + 2.38 \frac{m'}{a'^3}\, \delta\theta = 0,$$

or, taking account of the relation $n'^2 a'^3 = M + m'$ and neglecting the higher powers of m':

$$\frac{d^2\delta\theta}{dt^2} + 2.38\, n'^2 \frac{m'}{M}\, \delta\theta = 0,$$

hence:

$$\delta\theta = q \sin (\nu t + \gamma),$$

q and γ being the constants of integration and

$$\nu = + 1.54\, n' \sqrt{\frac{m'}{M}}.$$

Thus, the stability of the solution $\theta = 0$, $\Lambda =$ const., $\Gamma =$ const. is evident and oscillations about these values are possible. In reality these oscillations are very considerable. STRUVE derives the value 36°.64 for the amplitude of the libration in θ (l. c. pg. 287). However, the value of ν is already a close approximation, as appears from a comparison with observation: taking for $\frac{m'}{M}$ SAMTER's [1]) value $\frac{1}{4125}$, the above-mentioned formula for ν gives: $\nu = 0°.542$, while STRUVE gets (l. c. pg. 287): 0°.562.

3. Starting from the solution for $e' = 0$, viz. $\theta = 0$, $\Lambda =$ const., $\Gamma =$ const., I will construct the development of the solution in powers of e'. Putting $\delta\theta$, $\delta\Lambda$, $\delta\Gamma$ for the first-order terms in θ, Λ, Γ, the differential equations for these quantities are:

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[1]) Die Masse des Saturnstrabanten Titan. Sitz. Ber. der K. Preussischen Akad. der Wissenschaften 1912, pg. 1058.

$$\frac{d\delta\Lambda}{dt} = \frac{\partial^2 R_1}{\partial\theta\partial\Lambda}\delta\Lambda + \frac{\partial^2 R_1}{\partial\theta\partial\Gamma}\delta\Gamma + \frac{\partial^2 R_1}{\partial\theta^2}\delta\theta + \frac{\partial R_2}{\partial\theta} - \frac{\partial R_2}{\partial\Omega},$$

$$\frac{d\delta\Gamma}{dt} = -4\frac{\partial R_2}{\partial\Omega},$$

$$\frac{d\delta\theta}{dt} = -\frac{\partial^2(R_0+R_1)}{\partial\Lambda^2}\delta\Lambda - \frac{\partial^2(R_0+R_1)}{\partial\Lambda\partial\Gamma}\delta\Gamma - \frac{\partial^2(R_0+R_1)}{\partial\Lambda\partial\theta}\delta\theta - \frac{\partial R_2}{\partial\Lambda}.$$

In R_2 the terms of an order higher than the first with respect to e' are to be omitted. Taking account of the solution for $e'=0$, these equations become:

$$\frac{d\delta\Lambda}{dt} = +\frac{\partial^2 R_1}{\partial\theta^2}\delta\theta + \frac{\partial R_2}{\partial\theta} - \frac{\partial R_2}{\partial\Omega},$$

$$\frac{d\delta\Gamma}{dt} = 4\frac{\partial R_2}{\partial\Omega},$$

$$\frac{d\delta\theta}{dt} = -\frac{\partial^2(R_0+R_1)}{\partial\Lambda^2}\delta\Lambda - \frac{\partial^2(R_0+R_1)}{\partial\Lambda\partial\Gamma}\delta\Gamma - \frac{\partial R_2}{\partial\Lambda}.$$

Eliminating $\delta\Lambda$ and $\delta\Gamma$, one gets:

$$\frac{d^2\delta\theta}{dt^2} + \frac{\partial^2(R_0+R_1)}{\partial\Lambda^2}\frac{\partial^2 R_1}{\partial\theta^2}\delta\theta = -\frac{\partial^2(R_0+R_1)}{\partial\Lambda^2}\frac{\partial R_2}{\partial\theta} - \frac{d}{dt}\frac{\partial R_2}{\partial\Lambda} + \left[\frac{\partial^2(R_0+R_1)}{\partial\Lambda^2} + 4\frac{\partial^2(R_0+R_1)}{\partial\Lambda\partial\Gamma}\right]\frac{\partial R_2}{\partial\Omega}.$$

The development of R_2 (taking account of the terms of the first order with respect to e' only) is this:

$$R_2 = \frac{m'e'}{a'}\left[\cos\Omega\sum_{0}^{\infty}{}_{p} B_p\cos p\,\theta + \sin\Omega\sum_{1}^{\infty}{}_{p} C_p\sin p\,\theta\right],$$

B_p and C_p being functions of Λ and Γ.

From my development of certain portions of the perturbative function I deduce:

$$\left[\frac{\partial R_2}{\partial\theta}\right]_{\theta=0} = -0.574\frac{m'e'}{a'}\sin\Omega.$$

From the solution of the differential equations for $e'=0$ we have: $\frac{d\Omega}{dt} = 4\frac{\partial R_1}{\partial\Gamma} + \frac{\partial R_1}{\partial\Lambda}$; thus $\Delta\Omega$, the mean motion of Ω, is of the order of m'. A consequence of this and of the equation $\frac{\partial^2 R_0}{\partial\Lambda^2} + 4\frac{\partial^2 R_0}{\partial\Lambda\partial\Gamma} = 0$ is, that only the term in $\frac{\partial R_2}{\partial\theta}$ in the right member of the differential equation gives a contribution of the order of m'; the remaining terms only give contributions of the order of m'^2. The coefficient of $\delta\theta$ in the left member is the square of the mean motion of the argument of libration and of the order of m'; thus the divisor, which appears at the integration, is of the order of m'. In this divisor neglecting $\overline{\Delta\Omega^2}$, which is only of the order of m'^2, the solution of the differential equation, taking account only of the term in $\frac{\partial R_2}{\partial\theta}$ from the right member, becomes:

$$\delta\theta = -\frac{1}{\frac{\partial^2 R_1}{\partial\theta^2}} \frac{\partial R_2}{\partial\theta}.$$

Substituting the numerical values of the various quantities, we get:

$$\delta\theta = +\ 7.89\, e' \sin \Omega$$

and, for e' taking the value 0.0272 (Struve, l. c. pg. 172) and expressing the result in degrees:

$$\delta\theta = +\ 12^\circ.3 \sin \Omega.$$

The value from observation is (Struve, l. c. pg. 290): $\delta\theta = +\ 14.^\circ 0 \sin \Omega$; thus the agreement is very satisfactory, considering the simplifications admitted for the deduction of the theoretical value.

The value of $\delta\Gamma$ is to be determined by a quadrature from the equation:

$$\frac{d\delta\Gamma}{dt} = -\ 4\frac{\partial R_2}{\partial\Omega},$$

while $\delta\Lambda$ results from the equation for $\frac{d\delta\theta}{dt}$, without any integration. Considering the fact that R_2 as well as the mean motion of Ω are of the order of m', the values of $\delta\Gamma$ and $\delta\Lambda$ are seen at once to be of the order zero with respect to m'.

The value of $\delta\Phi$ results from the equation:

$$\frac{d\delta\Phi}{dt} = \frac{\partial^2(R_0+R_1)}{\partial\Gamma\partial\Lambda}\delta\Lambda + \frac{\partial^2(R_0+R_1)}{\partial\Gamma^2}\delta\Gamma + \frac{\partial R_2}{\partial\Gamma}.$$

Subtracting the equation for $\frac{d\delta\theta}{dt}$ from four times this equation, we get:

$$4\frac{d\delta\Phi}{dt} - \frac{d\delta\theta}{dt} = \left[4\frac{\partial^2 R_0}{\partial\Gamma\partial\Lambda} + \frac{\partial^2 R_0}{\partial\Lambda^2}\right]\delta\Lambda + \left[4\frac{\partial^2 R_0}{\partial\Gamma^2} + \frac{\partial^2 R_0}{\partial\Lambda\partial\Gamma}\right]\delta\Gamma +$$

$$+\left[4\frac{\partial^2 R_1}{\partial\Gamma\partial\Lambda} + \frac{\partial^2 R_1}{\partial\Lambda^2}\right]\delta\Lambda + \left[4\frac{\partial^2 R_1}{\partial\Gamma^2} + \frac{\partial^2 R_1}{\partial\Lambda\partial\Gamma}\right]\delta\Gamma + 4\frac{\partial R_2}{\partial\Gamma} + \frac{\partial R_2}{\partial\Lambda}.$$

Taking into account the relations:

$$4\frac{\partial^2 R_0}{\partial\Gamma\partial\Lambda} + \frac{\partial^2 R_0}{\partial\Lambda^2} = 4\frac{\partial^2 R_0}{\partial\Gamma^2} + \frac{\partial^2 R_0}{\partial\Lambda\partial\Gamma} = 0,$$

$\delta\Phi$ is seen to be of the order zero with respect to m'.

4. From the preceding developments there is seen to be every reason for the expectation, that the development of the solution in powers of the excentricity of Titan, supposing the free libration of θ to be zero, will meet with no difficulties. This conclusion is at variance with NEWCOMB's opinion in his paper: "On the motion of Hyperion. A new case in Celestial Mechanics". There he reaches the conclusion, that the development in powers of e' is not possible. The incorrect performance of this development by NEWCOMB is the reason of this difference of opinion; he omits the terms in the differential equation, which arise from the part of the perturbative function that does not contain e'; thus he gets a divisor of the order of m'^2 instead of one of the order of m'. In this respect the theory of Hyperion appears to present no difficulty.

In my dissertation I hope to extend the preceding developments by taking into account the amplitude of the free libration, as well as by giving more accurate results as regards the number of decimals.

Chemistry. — "*Vapour pressures in the system: carbon disulphide-methylalcohol*". By Dr. E. H. BÜCHNER and Dr. ADA PRINS. (Communicated by Prof. A. F. HOLLEMAN).

(Communicated in the meeting of March 31, 1917).

With regard to the vapour of partially miscible liquids, we find in many textbooks the following consideration for the case that the composition of the vapour lies between that of the liquid phases. When, on altering the temperature, the concentrations of the two liquids tend to the same value and, finally, become identical in a critical solution point, the vapour also, it is argued, must have the same composition at that temperature. It is then, however, tacitly assumed that the vapour, which lies at any low temperature between the liquids L_1 and L_2, remains between them at all other temperatures. This is, however, not at all the case, as KUENEN [1]) already showed some years ago with the help of VAN DER WAALS' theory. In an analytical way he proved, on the contrary, that at the critical point the vapour must have a different composition.

Also from general considerations it is easily seen that a vapour lying within the region of the two liquids must pass without, before the critical point is reached. If it did not, there would exist a point where three phases had the same composition. Now, it is already

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[1]) These Proc. **6**, Oct. 1903.

a particularity, when two phases of a binary system have the same concentration; this only occurs, when the components satisfy definite conditions. To allow of a third phase having also the same composition, still more special conditions, hardly ever to be expected, must be fulfilled. Although this does not prove the absolute impossibility of such a point, we see at once that it is highly improbable, and in any case that it cannot occur in general.

Until now, no single example has been experimentally investigated; it seemed, therefore, important to prove with some system the exactness of the above considerations. We chose the system carbon disulphide-methylalcohol.

The principal thing is to show that in the t, x-diagram the curve indicating the concentration of the vapour, runs as is drawn in figure 1. This gives the change of composition, which the three phases — permanently remaining in equilibrium — undergo by rise of temperature, or, as we may also express it, the t,x-projection of the triple curve. At low temperatures, the order of the phases is L_1GL_2; G and L_1 become equal at a certain temperature after which the order is GL_1L_2; at this peculiar temperature, the G-curve cuts the L_1-curve.

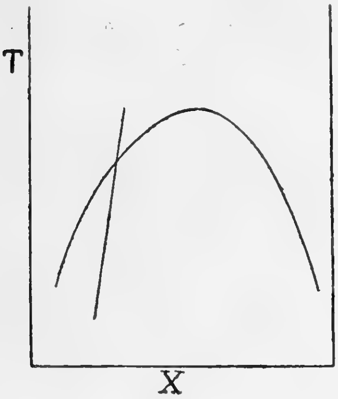

Fig. 1.

This is the cause of another particularity, which may also be considered as a characteristic of these systems. The p,x-diagram must have the form of fig. 2 at low temperatures — when the order is L_1GL_2 —, and that of fig. 3 at higher ones, as is easily seen graphically. The transition between these is formed

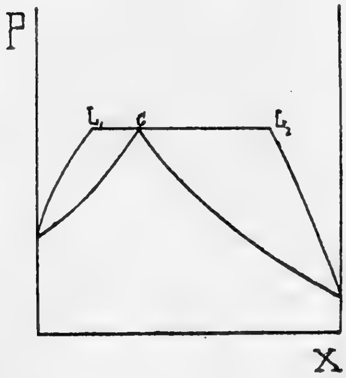

Fig. 2.

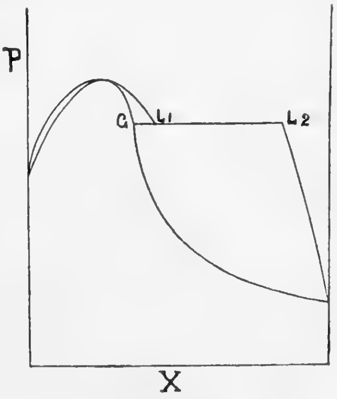

Fig. 3.

by a peculiar figure, in which G and L_1 coincide exactly at the triple pressure, where also the maximum-point (at which the liquid and vapour curves touch each other) is situated. We tried to study both figures in the system chosen.

In the first place, we investigated the p,x-diagrams at several temperatures. With a tensimeter the vapour pressures of different

TABLE 1.

CS$_2$		CH$_3$OH		Triple curve	
t	p	t	p	t	p
21.4°	307 mm	33.4°	188 mm	0°	154 mm
24.3	344	36.2	210	8.6	227
25.0	357	42.3	280	9.6	239
26.5	376	45.3	319	10.1	245
28.9	414	48.0	370	12.8	272
29.5	420	50.8	410	13.0	280
32.4	467	53.5	463	14.8	303
33.3	480	56.1	529	13.8	313
36.0	530	58.6	590	18.3	345
37.2	550	60.0	627	19.6	372
38.3	573	64.6	760	20.8	389
40.4	616			21.4	403
43.2	676			22.0	410
43.6	685			26.0	484
44.8	715			26.3	491
46.5	761			28.2	523
				29.0	550
				29.4	556
				31.0	588
				33.6	654
				34.1	668
				34.5	679
				36.4	725
				36.6	734

mixtures were determined at a set of temperatures. So long as there are three phases together, we find of course always the triple pressure; as soon, however, as one of the liquids has disappeared, we leave the triple curve, and every mixture will exhibit its own pres-

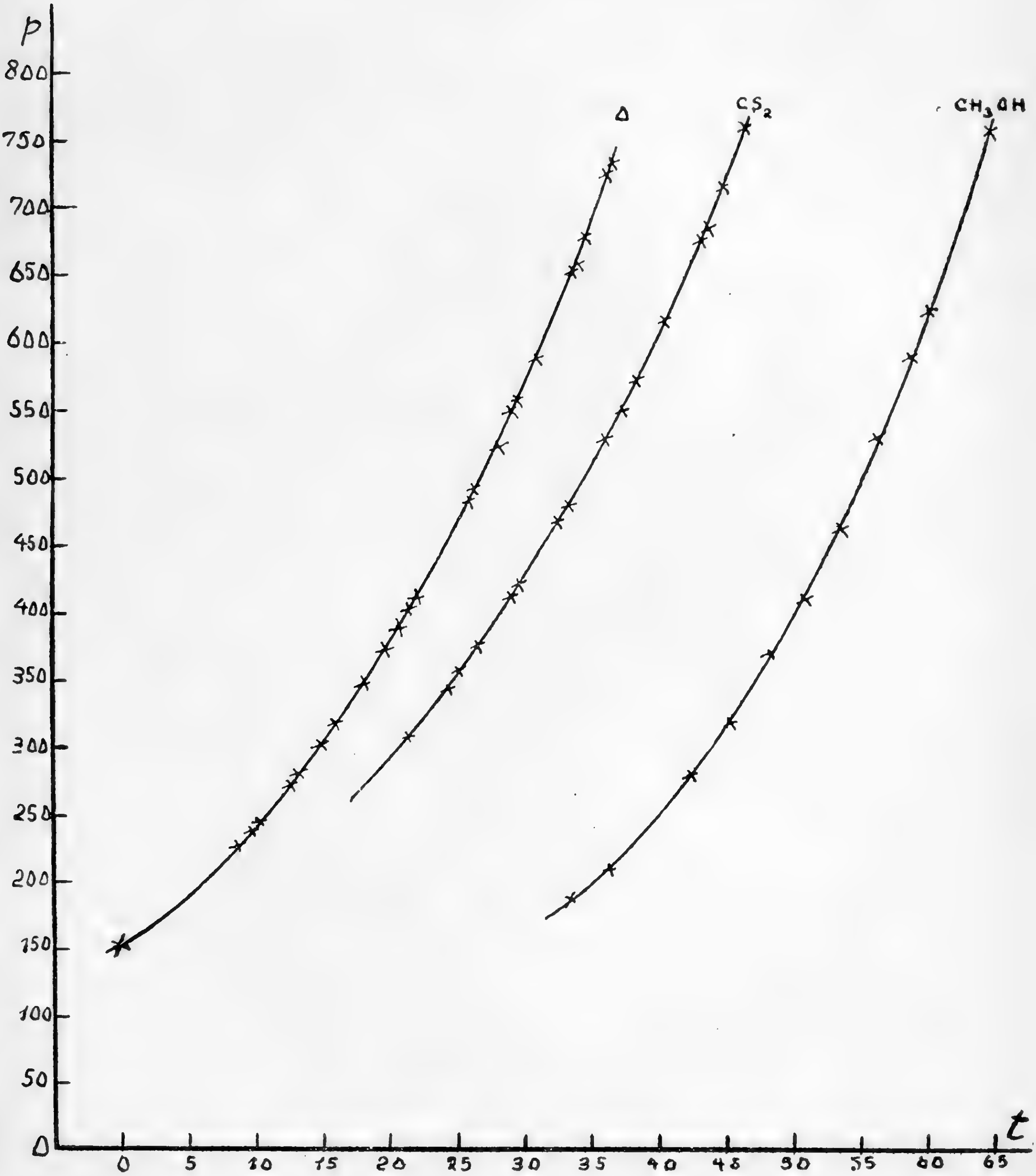

Fig. 4.

sure. On the p,t-curves thus found, we interpolated the pressures at definite temperatures and construed from these the p,x-curves. The tensimetrical method has the great advantage, that the mixtures come into contact with no other substance than mercury, the appa-

TABLE 2. Concentration in mol.proc. methylalcohol.

4.66 %		11.72 %		19.43 %		26.4 %		30.66 %		52.9 %		72.17 %		90.3 %	
t	*p*	*t*	*p*	*t*	*p*	*t*	*p*	*t*	*p*	*t*	*p*	*t*	*p*	*t*	*p*
17.8°	339 mm	34.4°	669 mm	39.9°	828 mm	40.2°	841 mm	39.8°	826 mm	37.7°	759 mm	17.2°	333 mm	0°	128 mm
21.0	388	38.2	767	43.2	936	41.0	865	41.2	877	38.6	792	22.6	416	14.3	240
23.1	422	41.5	874	46.8	1072	43.4	944	42.5	915	38.8	799	27.4	507	18.6	289
23.9	438	41.9	883	50.0	1193	45.8	1031	44.0	966	40.0	835	32.8	626	19.0	291
28.8	536	46.1	1033			48.3	1128	46.0	1037	42.0	896	39.2	790	19.4	296
34.8	672	50.2	1190			49.6	1178	46.8	1073	43,2	943	44.9	975	27.2	405
38.1	767							47.4	1090	43.6	954	45.1	979	27.8	416
44.4	959							47.9	1106	45.0	1001	50.0	1168	34.0	526
48.2	1092							49.6	1179	47.0	1075			39.2	639
50.4	1176									50.0	1196			45.4	798
52.4	1259													51.3	976
														54.6	1097
														58.3	1238

ratus being sealed as soon as it is filled. Much preliminary work showed us the importance hereof; rubber stoppers are attacked by carbon disulphide, corks cannot be used, because the methylalcohol absorbs water from them; when using ground stoppers or taps, one experiences difficulties with the lubricants.

The substances used were carefully purified. The purest commercial carbon disulphide was shaken with mercury, left for some time on quick-lime and distilled off. Finally it was distilled once more, all these operations being performed at red light. It was kept continually in the dark. The boiling point was 46.4° at 760 mm.; it did not stick to the condensor and had hardly any, in any case only a not disagreeable ethereal odour.

The methylalcohol, also the purest commercial product, was treated with iodine and natron, boiled afterwards with quick-lime, left for some time on sodium and distilled off. At last it was fractionated, and the fraction, which distilled between 64.5° and 64.6° (760 mm.) was separately received. It was absolutely free from water, as was proved by the well-known reaction with anhydric coppersulphate. The salt itself remained perfectly white, while part of it dissolved slowly with a light-green colour.

In table 1 (p. 1234) and fig. 4 (p. 1235) the results as to the components themselves and the triple curve are given. The points of the latter are determined with different mixtures; they all fit very well into the curve.

The measurements, relating to different mixtures above their solution point are joined in table 2. It appears that the pressures of most mixtures are nearly equal, and that the pressure curves run so closely to each other that we had to refrain from joining them in one figure. Instead thereof, we have construed for a number of temperatures the p,x-diagrams, by interpolating on the different curves the pressure values corresponding to those temperatures.

The result showed it to be impossible to realize in this system the transition sought for. Until just below the critical point, which we found at 37.4°, the p,x-curve retains the shape of fig. 2, in which the triple pressure is the highest (fig. 5, (p. 1238) that is given as an example for 35°). The form is very remarkable: the liquid branches change almost imperceptibly into the horizontal line, as is already known indeed for other systems with partial miscibility; a maximum however is not to be observed.

The curves for temperatures above the critical point might happen to give some decisive answer to the question.

If there had been a distinct transition from the curve of fig. 2

into that of fig. 3, we should have found above the critical point a curve with a maximum and an inflexion point. The curves however appeared to run very flat (fig. 6); it was therefore impossible to ascertain, where the maximum lies, while the inflexion point has already disappeared immediately above the critical point. [1])

It still remained to investigate, if a satisfactory result might be obtained by determinations of the vapour concentrations. We heated therefore a two layer system to boiling, distilled off a part of the vapour and determined the solution point of the distillate in order to find the concentration thereof.

First of all, we wanted the determination of the whole solubility curve, which we carried out by the well-known synthetic method in sealed tubes. Our results (table 3 and fig. 7) differ from those obtained formerly by ROTHMUND; our critical point lies 2.5° lower.

Doubtless this difference is to be explained by difference in purity of the substances; in the first place we think of traces of water. [2])

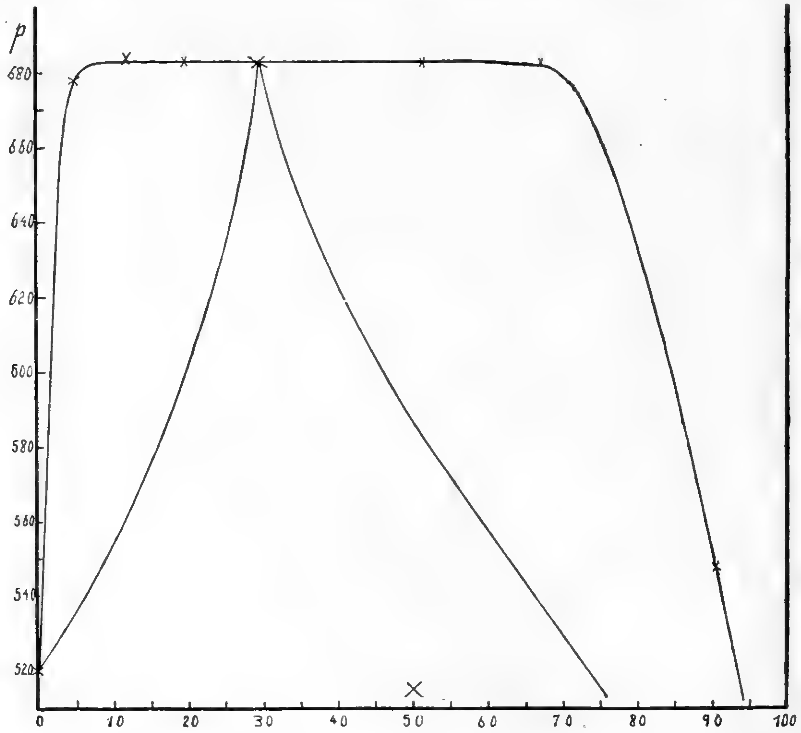

Fig. 5.

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[1]) Cf. the system phenol-water investigated by SCHREINEMAKERS, Z. f. phys. Ch **35**, 459, 1900.

[2]) We found experimentally, that the solution point is increased 0.7° by addition of 1 °/₀₀ water.

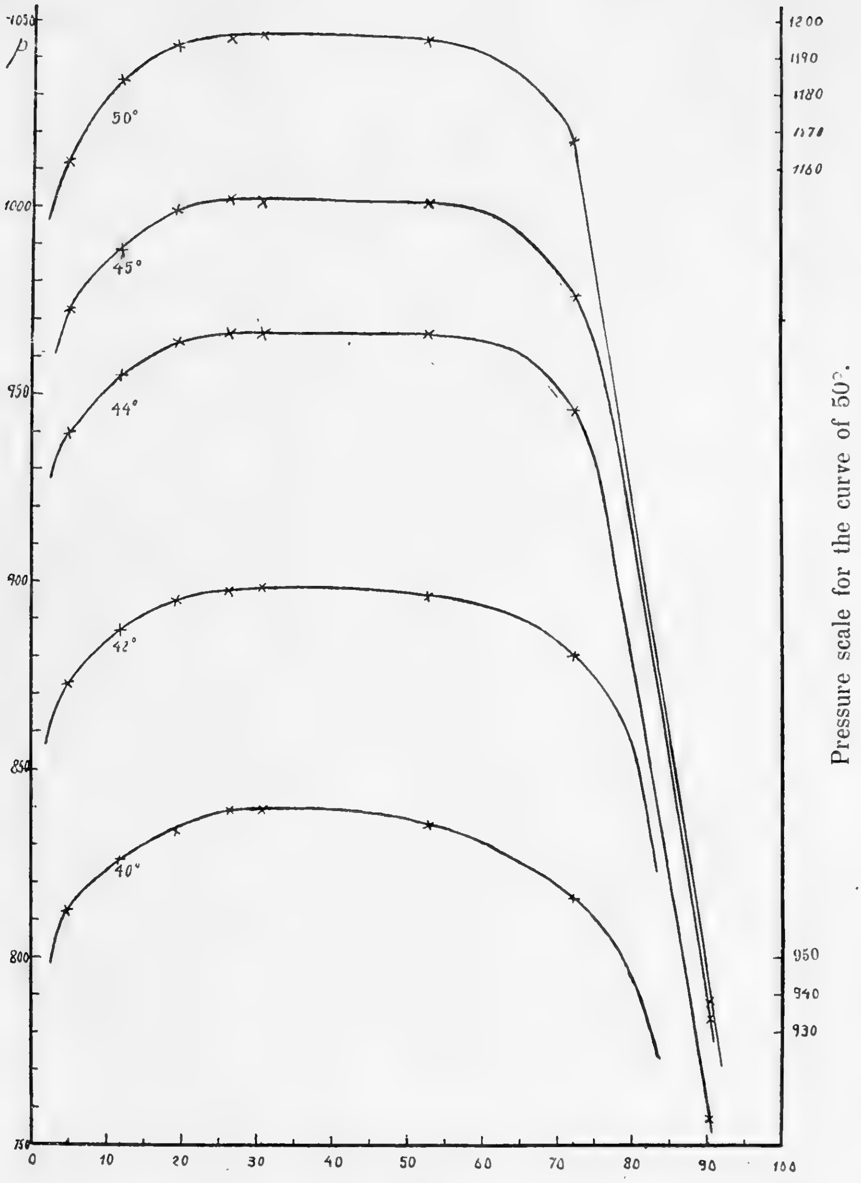

Fig. 6.

We too found in our first experiments critical points of 40° or even higher. It is sufficiently known that small impurities have an enormous influence, especially on the critical point; they may cause it to rise, for instance, 100 times as fast as the boiling point. This is the reason why the triple curve is not nearly so sensitive to impurities and why our former determinations with substances, which showed the critical point 39.5°, as well as the more recent ones fit perfectly into one line.

TABLE 3.

Conc. in mol.proc.	*t* solution point	
8.1	21.8	
14.1	30.8	
21.2	35.9	
24.4	37.4	
29.4	37.1	
34.6	37.2	The critical point is situated between these two concentrations.
42.4	37.4	
50.8	35.3	
52.9	34.7	
61.0	28.7	
72.8	11.2	
81.0	—15	

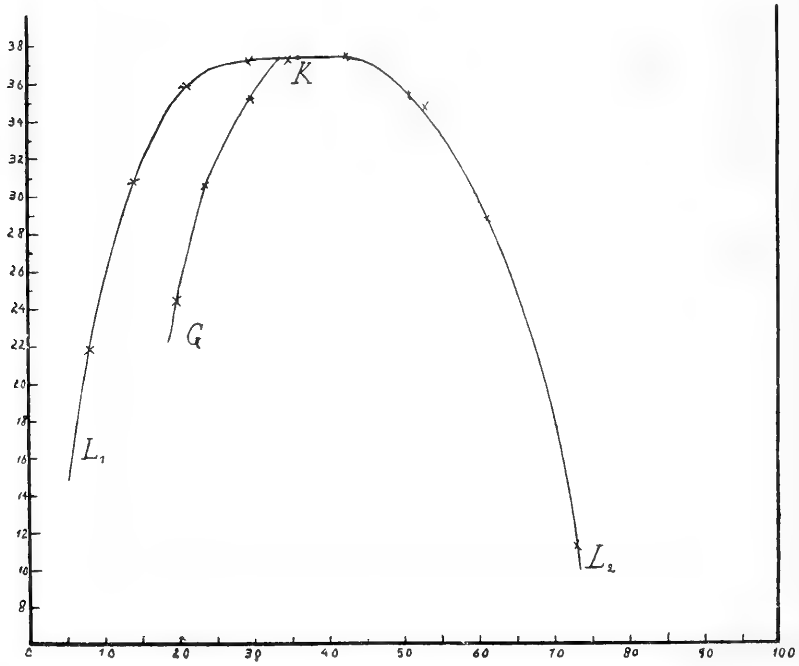

Fig. 7.

We used for the boiling experiments an apparatus, constructed some time ago by Prof. SMITS. At first, we had boiled the liquids by electric heating by means of a platinum wire in the liquid. It appeared, however, that in this case decomposition occurred, as was immediately proved by an abnormally high solution point of the distillate. Prof. SMITS then directed our attention to the apparatus represented in fig. 8. To the flat bottom of a glass tube 20 cm long and having a diameter of 2 cm a little, narrow tube *A*, 2 cm long, is sealed, which is wrapped with a sheet of asbestos paper, round which nickelin wire is wound. By an electric current the liquid in this short tube is then so strongly heated, that it brings the whole mass very quickly to boiling. Nevertheless, when making experiments under lower pressures, it was desirable to throw in some little capillary tubes, as the substances are very highly liable to superheating. The whole apparatus was packed with cottonwool in an asbestos mantle. It is provided with a ground stopper, from which a thermometer (Anschütz) is hanging, and with two side tubes: *B*, leading to a reflux condenser, and *C*, which — enveloped by nickelin wire and sufficiently heated during the experiment — leads the vapour to the receiver. This consisted of a wide glass tube cooled by carbondioxide and alcohol, in which one or two small tubes, provided with a constriction, were placed, which might contain about 1 c.c. of liquid. When the moment of receiving the distillate has arrived, the wide tube is turned in such a way that one of the narrow tubes comes to stand under the end of *C*. When it is sufficiently filled, the experiment is finished, and the tube is sealed. By a determination of the solution point, we knew at the observed boiling temperature the concentration of the vapour, while the composition of the liquids was found from the solubility curve. We made this experiment at three different pressures with the results given in table 4 and drawn in fig. 7.

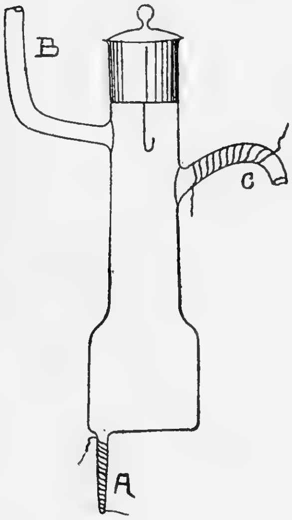

Fig. 8.

TABLE 4.

p	t	x_G
444 mm	24.4°	19%
600	31.5	24
688	35.2	30

Whereas the two first determinations were easily made, we met in the third with a serious difficulty, which made us refrain from further experiments above 35°.2. The solution point of the last distillate lies namely at 37°.2, so on the almost horizontal part of the solubility curve. Thereby, the determination of the concentration in this manner becomes inexact, which would become still worse at the higher temperatures. We have checked it for this distillate by adding a weighed quantity of carbon disulphide, altering thereby the composition and the solution point so as to bring them on to a part of the solubility curve, which is more easily determined. No important difference was found.

The experiments carried out show, however, clearly, that this system does not afford a plain proof of the theory. Although we see that the vapour line after extrapolation cuts the solubility curve at 32 molproc., whilst the critical point lies at $36^0/_0$, the temperatures of the intersection point and the critical point cannot be distinguished. The course of the curves being so unfavourable for our purpose, we decided to take no more experiments with this system. Our result is remarkable in this point: although the theory proves, that the vapour branch does not leave the region of limited miscibility in the critical point, the opinion previously expressed that this had to be the case, is not very far from the truth.

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Experimental Psychology. "— *Intercomparison of some results obtained in the Investigation of Memory by the Natural and the Experimental Learning-Method*". By Dr. F. Roels. (Communicated by Prof. C. Winkler).

(Communicated in the meeting of March 31, 1917.)

In the investigation of memory psychologists have always had recourse to learning-experiments, with the purpose to ascertain, under definite experimental conditions, the retentive capacity of the memory with regard to the material impressed upon it. Whatever method was employed (the learning-, or the saving- or the hitting-, or the helping-method) the imprinting occurred invariably in the same way. The material to be learned, by preference meaningless, was presented to the observer at a certain rate of succession, and more or less frequently, according to the object in view. Psychologists did not always take into account the learning-method peculiar to every individual so that now and again the rate of succession of the terms corresponded little with the time required by the observer to spontaneously take in the material presented. The difficulties arising from this, which are felt in individual psychological experiments much more strongly than in general investigations, do perhaps not render the results, achieved in this way, totally invalid. Nevertheless, viewed more closely, they appear to me weighty enough to justify an intercomparison of the results obtained by the natural and the experimental method.

The results reported in this paper have been obtained in a series of experiments performed in the Psychological Laboratory of the Utrecht Clinic for Psychiatry and Neurology. The course of the experiments was regulated as follows:

Three observers (M, R and D) committed to memory 40 series of 12 nonsense-syllables. For the first twenty (Group I) the observer was at liberty to choose his own rate of succession, to group the syllables, to determine the interval between two successive repetitions etc. all in his own way. The only restrictions he had to submit to were that in the successive repetitions he was allowed to pronounce a syllable only once, and that when once his attention had averted from a syllable, it should on no account return to it again. The other 20 series (Group II) were exhibited by means of a mnemometer of our own construction. It consisted of a drum, rotating evenly and at a carefully tested speed about an horizontal axis by the help of a HELMHOLTZ electromotor. On this drum was wound a strip of paper printed with the syllables at equal distances. Before the drum there was a screen with a slit in the centre past which the syllables flitted in succession when the drum was turned round. Thus the time of exposure was the same for each syllable, so were the intervals between two successive syllables, so also were those between two successive repetitions.

In the experiments of Group I as well as those of Group II the observer spoke through a voice-key, consisting of the diaphragm of a gramophone, to which a platinum disc had been attached. On this disc rested the platinum-covered point of a V-shaped arm, which was turning about an horizontal axis, and easily adjustable by the help of a sliding weight. The deflections of this arm broke the electric current flowing through the instrument even with the slightest intensity of the spoken sound on the diaphraghm. These breaks were registered by a marking magnet upon the drum of a kymographion. A second magnet drew a time-line ($^1/_5$ sec.) with the aid of KAGENAAR's chronoscope. We were thus enabled to determine by the natural and the experimental method the duration of every repetition, the time relation between the successive terms of one series, and its modification with the progress of the process of learning etc.

The determination of the time required for every repetition and for the whole learning-process involved some difficulty as our voice-key, though it indicated distinctly the moment when the observer started the first syllable of a series, did not precisely report the moment when the reading of the last syllable was completed. However, we have ignored this source of experimental error in our calculations, seeing that the moment at which the last syllable is begun is easy of determination and only a minimal time (at the most 0,2 sec.) is required to pronounce a syllable consisting of two consonants with a vowel or a diphthong between them. This may the more readily be done since it equally affects the time-values in both groups (I and II).

In the experiments of Group II we had to look out for the moment the first syllable appeared in the slit as it need not coincide with the moment when the observer reads it. We, therefore, fitted to one side of the drum of the mnemometer a button, which, whenever the drum had come round again to its starting point, came in contact with a spring. With this contact we made the appearance of the first syllable coincide. The breaking of the circuit brought about by the contact was registered by means of a marking magnet on the drum of a kymographion.

If the observer supposed he knew the series, he said it by heart. In case he broke down the experimenter presented the rest of the series once more. Close upon the recitation the observer told how he had proceeded in learning the syllables, how he had grouped his material, what associative connections he had made between the syllables.

Every day four series were committed to memory. Precisely 24 hours later we ascertained by the saving-method how much of the impressed material of the previous day had stuck. Group I was gone through unintermittently; not before this was got through did we start the second group.

The subjoined table shows the mean number of repetitions which the several observers required to learn a series by the natural- (I), and by the experimental (II) method. For each observer the first and the third horizontal row shows the results of the learning-experiments (*l*) of the first day; the second and the fourth those of the repetition-experiments (*r*) 24 hours later. Alongside of the arithmetical mean we also tabulated the mean deviation and the median, The last column illustrates the gain (expressed in percentages) realised after 24 hours by the natural- and by the experimental method. We also add a column for the number of series learned by heart.

TABLE I.

Observers	*Number of series*			*Arithm. mean*	*Mean deviation*	*Median*	*Gain after 24 hours*
M.	20	I	1	8.45	1.50	8	52.77
			r	4	0.46	4	
	20	II	1	9.25	1.41	9	47.03
			r	4.90	0.95	5	
R.	20	I	1	4.50	1.04	4	45.56
			r	2.45	0.78	2	
	20	II	1	7.60	1.50	7.50	59.21
			r	3.10	0.74	3	
D.	20	I	1	9.66	2.08	9	49.28
			r	4.90	0.80	5.50	
	8	II	1	10.50	3.25	9.50	46.43
			r	5.60	1.92	4.50	

The order of the observers relative to the number of repetitions in group I is maintained in group II. For each of them the number of the repetitions increases; for M. and D. in about the same degree (respectively 9,47 and 8,69 percent); for R the increase is much greater (68.89 perc.). A similar process is observed in the *r*-rows. Here also the increase is greatest for R (26.53 perc.), much less for D than for M (respectively 14.28 and 22.5 perc.).

The percentage of repetitions saved after 24 hours is for M and D higher in I than in II (respectively 52.77; 47.03 and 49.28; 46.43 perc.). The reverse is observed in the case of R, for whom II yields a considerably larger gain (59.21 and 45.56 perc.).

The second Table gives the average time required for getting a series by heart in group I and II. After what we said about the preceding table we need not enter into further details about its construction. The time-values are expressed in seconds.

With II the time of the learning-experiments decreases, for M and D respectively 5.28 and 14.74 perc. R, however, requires more

TABLE II.

Observers	*Number of series*			*Arithm. mean*	*Mean deviation*	*Median*	*Gain after 24 hours*
M.	19	I	l	92.40	15.67	87.45	53.05
	20		r	43.38	6.91	44.55	
	20	II	l	87.52	11.83	88.12	47.44
	20		r	46	8.48	43.40	
R.	20	I	l	61.43	16.15	57.97	52.89
	20		r	28.94	9.55	26.97	
	19	II	l	70.23	11.38	71.85	60.17
	19		r	27.97	6.54	26.75	
D.	17	I	l	114.40	32.36	86.85	51.70
	20		r	55.26	13.54	55.47	
	8	II	l	97.54	21.79	89.23	48.64
	8		r	50.10	14.21	43.20	

time with II (increase 14.32 perc.). The learning-times of the *r*-rows do not differ very much. For R and D they decrease with II (respectively 3.35 and 9.34 perc.) for M the increase is 6.04 perc. For M and D the time saved after 24 hours is greater with I (respectively 53.05; 47.44 and 51.70; 48.64 perc.); for R, however, considerably greater with II (52.89 and 60.17 perc.).

When summarising these data we see that the number of repetitions needed to learn the series by heart is larger with II than with I, in the learning- as well as in the repetition-experiments. The increase of the number of repetitions in the learning-experiments does not keep pace with that of the repetition-experiments, so that for two of our observers the gain after 24 hours is largest with I, for the third with II. Again, with II the learning does not only require less time, the gain effected after 24 hours is also greater. The few exceptions may be accounted for by the unequal increment in the number of repetitions in the learning- as well as in the repetition-experiments.

The third Table gives the mean duration of the recitation-times (seconds) in the learning- and the repetition-experiments with I and II along with the gain effected after 24 hours in percentages.

For all observers the recitation-time of the learning-experiments is longer than that of the repetition-experiments, with I as well as

TABLE III.

Observers	*Number of series*			*Arithm. mean*	*Mean deviation*	*Median*	*Gain after 24 hours*
M.	20	I	l	13.61	3.31	12.37	13.74
	20		r	11.74	2.24	10.95	
	20	II	l	11.49	2.19	10.47	4.96
	20		r	10.92	2.25	10.07	
R.	20	I	l	11.95	3.73	11.25	2.43
	20		r	11.66	2.21	11.65	
	19	II	l	13.92	4.57	12.63	19.04
	20		r	11.27	2.62	10.65	
D.	17	I	l	11.54	1.98	11.10	14.30
	20		r	9.89	1.79	9.30	
	8	II	l	11.15	2.98	10.78	3.14
	8		r	10.75	1.95	10.45	

with II. M. and D recite quicker with II. R, on the contrary quicker with I. This at least is the case in the learning-experiments. The recitation of the repetition-experiments lasts longer with II than with I only in the case of D. The column of percentages of repetitions saved after 24 hours with I and II clearly shows that the gain is greater with I for M and D; for R however, with II. — Consequently the recitation-time with I as well as with II is longer in the learning-experiments than in the repetition-experiments. Again, as to furthering a quick recitation the experimental method seems to have the advantage over the natural, whereas the latter yields a greater gain.

The mean rate of succession of the presentations of the series of the second group, measured from the moment when the first syllable appeared in the slit of the mnemometer to the appearance of the last was 9,5 seconds. The next table shows how the observers themselves determined spontaneously the rate of succession in the learning- and the repetition-experiments. We determined accordingly the mean duration of a repetition in the learning- and in the repetition-experiments. The first column gives the number of repetitions, from which the time-values have been calculated.

For R. and D. the mean duration of a repetition is markedly

TABLE IV.

Observers	*Number of Repetition*		*Arithm. mean*	*Mean deviation*	*Median*
M.	167	l	11.26	1.03	10.87
	80	r	11.31	1.21	11.47
R.	86	l	13.44	1.40	12.95
	49	r	11.79	1.08	11.66
D.	173	l	11.16	1.40	10.44
	110	r	9.86	1.39	9.71

greater for the learning- than for the repetition-experiments; for M. they are almost equal. For learning as well as for repeating a repetition requires on the average more time with I than with II. Only in the case of repeating does the average duration of a repetition for M. approximate to that of I.

The following tables illustrate how the observers modified the rate of succession spontaneously according as they were getting more familiar with the material. For every series we divided the repetitions necessary to learn the material by heart (learning and repetition) into three groups of successive repetitions. For each group we calculated the mean duration of its repetitions. A comparison of the time-values of each group shows the changes in the rate of succession in learning and repetition concurring with the greater familiarity on the part of the observer with the material to be impressed. It should be observed that, when the number of repetitions was not divisible by three, the first and the last group always contained the same number of repetitions which made the middle group longer or shorter by one repetition. Though the time of exposure of the syllables, as established by the mnemometer was always the same with II the observer had ample opportunity to lengthen or to shorten the duration of the repetitions to a certain extent, as he was at liberty to read the first and the last syllable of the series at any moment of the period during which they remained visible in the slit. So with II there may also be a tendency to shorten or to lengthen the duration of the repetitions as the learning-process advances.

Apart from a few exceptions for D, the duration of the repetitions increases according as the observer is getting more familiar with the material to be impressed, in the learning- as well as in

TABLE V. *Observer M.*

	Groups		*Arithm. mean*	*Mean deviation*	*Median*
I (20)	1st	l	9.79	0.63	9.72
		r	9.34	0.68	9.70
	2d	l	10.92	0.86	10.93
		r	10.78	0.92	10.72
	3d	l	12.57	1.66	11.82
		r	12.73	2.03	12.27
II (20)	1st	l	9.62	0.51	9.37
		r	9.52	0.60	9.31
	2d	l	9.57	0.52	9.47
		r	9.52	0.51	9.30
	3d	l	9.50	0.49	9.47
		r	9.49	0.55	9.40

TABLE VI. *Observer R.*

	Groups		*Arithm. mean*	*Mean deviation*	*Median*
I (20)	1st	l	12.42	0.71	12.57
		r	10.91	1.33	11.20
	2d	l	13.33	1.33	13.55
		r	11.10	1.78	10.50
	3d	l	14.13	2.41	13.56
		r	13.28	1.78	12.90
II (19)	1st	l	9.22	0.78	9.22
		r	9.11	0.44	9.10
	2d	l	9.09	0.99	9.08
		r	9.52	0.84	8.80
	3d	l	9.56	1.02	9.83
		r	9	1.02	9.35

TABLE VII. *Observer D.*

Groups			*Arithm. mean*	*Mean deviation*	*Median*
I (20)	1st	l	11.09	1.33	10.62
		r	9.82	1.49	9.70
	2d	l	10.70	1.33	10.35
		r	10.12	1.74	9.55
	3d	l	11.49	1.89	11.13
		r	10.11	1.58	10.25
II (8)	1st	l	9.76	0.53	9.76
		r	9.53	0.22	9.51
	2d	l	9.69	0.59	9.58
		r	9.22	0.37	9.22
	3d	l	9.67	0.67	9.50
		r	9.33	0.42	9.31

the repetition-experiments. Again with one exception for D the increase is greater in the repetition- than in the learning-experiments (for M. 3.39 and 2.78; for R. 2.77 and 1.71 sec.); this proves again that with at least two of our observers there is a tendency to lengthen the learning-time, when the knowledge of the material has increased in consequence of the repetition-experiments of the previous day. It is also proved by the fact that with a few exceptions, the lengthening of the learning-time, in the learning- as well as in the repetition-experiments is greater when passing from the IId to the IIId than from the Ist to the IId group. (for M. 1.13 and 165; 1.44 and 1.95 sec.; for R. 0.91 and 0.80; 0.19 and 2.81; sec.; for D.: — 0.39 and 0.79; 0.30 and — 0.01 sec.) It seems advisible to conclude, therefore, that with I for two of our three observers the time required for succession-repetitions increases as well in the learning- as in the repetition-experiments, when the observer gets more familiar with the material. With II there is no gradual increase at all with a fuller knowledge of the material. As with I, the time-values are indeed, generally smaller in the repetition- than in the learning-experiments, but where, as e. g. with I the mean duration of the repetition of the last group is always the longest, it is always the shortest with group II, with a few exceptions only. For the rest a comparison of the time-values of the several groups does not reveal any similarity.

TABLE VIII.

Observers		*Groups*		*Arithm. mean*	*Mean deviation*	*Median*
M.	I	1st	l	8.90	1.70	8.50
			r	3.90	0.36	4
		2d	l	8	1.30	8.50
			r	4.10	0.56	4
	II	1st	l	10.40	1.32	11
			r	5.80	1.30	5
		2d	l	8.10	1.50	8.50
			r	4	0.60	4
R.	I	1st	l	5.40	1.40	5.50
			r	2.60	0.80	2.50
		2d	l	3.60	0.68	4
			r	2.30	0.76	2.50
	II	1st	l	8.20	1.40	7.50
			r	3.10	0.74	3
		2d	l	7	1.60	7.50
			r	3.10	0.74	3
D.	I	1st	l	10.62	2.37	10
			r	5.80	1.16	6
		2d	l	8.70	1.80	8
			r	4	0.44	4

The tables VIII and IX illustrate the influence that practice exerts upon the number of repetitions and upon the learning-time with I and II in the learning- as well as in the repetition-experiments. In tabulating the data obtained in the learning- and repetition-experiments as regards the number of repetitions and the learning-time, they have been arranged in the order in which they were acquired and have then been split into two equal groups. For every group we calculated the arithmetical mean, the mean deviation and the median. The figures of table VIII refer to number of repetitions, those of table IX to the time-values, expressed in seconds. The data for D with II, being too small numerically could not be tabulated.

Table VIII clearly shows that the influence of practice reveals itself in a decrease of the number of repetitions. With I the number of repetitions required to learn the series by heart is smaller in the second group than in the first, (for M 10 °/₀; R 33 °/₀ and D 17.93 °/₀.) With II a similar gain is effected (for M 22 perc. for D 15 pct.). For the former, therefore, the influence is greater, for the latter, smaller with II than with I.

The values obtained in the repetition-experiments are not suitable for comparison as their significance depends for the greater part on their relation to those of the learning-experiments. We, therefore,

TABLE IX.

Observers		*Groups*		*Arithm. mean*	*Mean deviation*	*Median*
M.	I	1st	l	90.25	16.54	86.95
			r	36.02	5.89	41.15
		2d	l	94.55	14.80	94.37
			r	50.75	7.93	49.92
	II	1st	l	97.11	10.48	100.37
			r	53.31	11.23	49.62
		2d	l	77.99	13.18	80.47
			r	38.69	5.74	39.92
R.	I	1st	l	75.43	21.16	66.65
			r	28.56	8.80	26.80
		2d	l	47.44	11.15	51.82
			r	29.32	10.31	27.33
	II	1st	l	78.83	12.13	74.25
			r	31.21	8.84	28.90
		2d	l	61.64	10.64	62.85
			r	24.74	4.25	23.60
D.	I	1st	l	138.19	42.22	133.60
			r	68.70	12.74	65.02
		2d	l	90.61	22.50	79.70
			r	41.81	14.35	33.02

calculate the gain effected with I and II in the repetition-experiments of the first and the second group. This gain is expressed in percentages in the following table.

TABLE X.

Observers	*Groups*	I	II
M.	1st	56.15	44.33
	2d	48.75	51.62
R.	1st	51.85	62.20
	2d	36.11	55.72
D.	1st	45.29	—
	2d	54.05	—

For M and D the gain lessens with I. The learning of the series of the second group requires, it is true, fewer repetitions but the decrease of the number of repetitions in the repetition-experiments does not run parallel to it, so that after all the gain turns out to be smaller. D, who sat down for the first time to an experimental investigation of the memory, learns the series of the second group not only with fewer repetitions, but also furnishes a greater gain in the repetition-experiments, a phenomenon due to his inexperience, which made him more susceptible than the others to the favourable influence of practice and of the repetition of the experiments.

With II the influence of practice is noticeable for R in a fall of the percentage of repetitions saved; for M however, this percentage rises. Most likely the difference between those two observers is due to the fact that with II R tried to translate the rate of succession, which did not suit him, into his own, in which, of course, he succeeded only after some training. M, on the other hand, scrupulously stuck to the experimental rate all through the experiments with II.

The influence of practice on the learning-time (Table IX) appears for R and D generally in a decrease of the latter. This applies to I as well as to II, to the learning- as well as to the repetition-experiments. Whereas for R with I the number of repetitions in the learning-experiment decreases (33 perc.), the decrease in time is 37,11 perc., for D the decrease is respectively 17,03 perc. and 34 perc. For D the number of repetitions with I decreases in the repetition-experiments 31,04 perc., the learning-time 39,14 perc., so that here also the influence of practice is shown in a shorter learning-time;

for R however, the number of repetitions lessens 2,66 perc., so that under the influence of practice the learning-time increases. With I the learning-time for R in the second group increases in the learning-experiments as well as in the repetition-experiments; the number of repetitions and the time increase respectively 15 and 21,81 and 0 and 21,05 perc. The influence of practice reveals itself for M invariably in a longer learning time. The decrease of the number of repetitions and in the learning-time is in the learning- and in the repeating-experiments with I and II respectively 10 and + 4.76 perc., + 5.13 and + 40.89 perc., 22 and 19,69 perc., 31.04 and 27.43 perc.

The striking difference between M and the other two observers in relation to the influence of practice upon the learning-time is due to the fact that M proceeds in the learning-experiment in a different way from R and D. Whereas the latter on getting more familiar with the material, go on reading, M directly starts his recitation when he is able to do so. It is not that R and D do not recite the familiar syllables, they even like to begin, however not with the same energy as is the case with M. Under the influence of practice the observer familiarizes himself sooner with the syllables, which, given the tendency to recite as quickly as possible, soon induces him to alternate reading with reciting. The consequence, however, is that the learning-time is lengthened.

It is worthy of notice that, as shown by M's percentages, the natural method is more adapted to M's way of learning than the experimental. This is easy to understand, if we consider that the observer, if he will not run the risk of disturbing the learning-process, is compelled by the experimental rate to give up looking for a syllable, when, at the appearance of the following in the mnemometer, it has not yet been brought to consciousness.

Summarizing then, the data of the last three tables yield the following results: The influence of practice reveals itself in the learning-experiments in a decrease of the number of repetitions required for the process of learning. For M it is greater with II; for R, on the other hand, with I. In the repetition-experiments the gain lessens for M. and for R. D, however, saves repetitions, which is due to this observer being a novice in psychological experimentation. The lower percentage of repetitions saved with II for R under the influence of practice is probably due to the fact that with I this observer tried to translate the experimental rate which did not suit him, into the rate peculiar to his own method of learning, in which he succeeded only after sufficient practice. The influence of practice upon the learning-time is generally shown for R and D in a shorter time required for learning. For M however, the learning-time is lengthened, which is to be ascribed to the strong tendency to recite the familiar syllables. With I this tendency is more persistent than with II.

CONCLUSIONS.

1. The number of repetitions required for learning the material by heart in the learning- as well as in the repetition-experiments is larger with the experimental method than with the natural. The increase in the number of repetitions in the learning-experiments does not, however, run parallel to that of the repetition experiments, so that for two of our observers the gain effected after 24 hours is greatest with the natural method; for the third, however, with the experimental.

2. With the experimental method the learning of the material does not only require less time, also more time is saved after 24 hours. Some exceptions are accounted for by the differing increases of the number of repetitions required for the learning in the learning- and the repetition-experiments.

3. The recitation time, whether the natural or the experimental method be employed, is longer in the learning than in the repetition-experiments. The experimental method seems to be more adapted to a quick recitation than the natural. The latter, on the other hand is more economising.

4. As a rule the mean duration of a repetition is longer in the learning- than in the repetition-experiments. The natural rate of succession of our observers appeared to be considerably slower than the experimental.

5. With the natural method the rate of succession with two of our three observers, in the learning- as well as in the repetition-experiments is slowing gradually when they get more familiar with the material. With the experimental method this slowing process is entirely out of the question. True, here also the rate of succession is generally quicker in the repetition- than in the learning-experiments.

6. The effect of practice is shown in a decrease of the number of repetitions required for learning the material by heart. For one observer it is greatest with the experimental method, for the other with the natural.

7. The effect of practice upon the learning-time with the natural as well as with the experimental method, generally manifests itself in a shorter learning-time. A lengthening in the case of one of our observers must be ascribed to a strong tendency to recite the familiar syllables, which persists more readily with the natural than with the experimental method.

Zoology. — *"On the relation of the anus to the blastopore and on the origin of the tail in vertebrates"*. By Dr. H. C. DELSMAN. (Communicated by Prof. J. BOEKE).

(Communicated in the meeting of Feb. 24, 1917).

Both the foregoing communications (May 27 and November 25, 1916) being mainly dedicated to the mode of contraction of the blastopore border of amphibians, in this third one I should like to give some facts and considerations concerning the ultimate fate of the blastopore and its relation to the anus.

The statements made by the numerous investigators on this subject are so divergent that it must be very difficult for any one who cannot judge from personal experience to form a sound opinion. I will try to show that the application of the principles of my theory on the origin of vertebrates will once more serve to furnish us with the solution of an old problem which — especially by GROBBEN's (1900) classification of the animal kingdom — has been resuscitated. In the first place the different views and results of former investigators may be very briefly reviewed. We will confine ourselves mainly to the amphibian egg, in which a relation between anus and blastopore was for the first time noticed. Anurans and Urodelans will be treated separately, because, as I can confirm from my own investigations on *Rana esculenta* and *Amblystoma tigrinum,* these two groups in the relation of the anus to the blastopore exhibit a notable difference. We will begin with that group, on which the first observations were made, the Anurans.

BALFOUR (1881) in his Text-book gives a description of the origin of the anus, based mainly on the figures of GOETTE (1875) for *Bombinator igneus* and his own investigations on *Rana temporaria,* where the anus breaks through somewhat earlier than appears to be the case in toads generally. The blastopore passes into the neurenteric canal and the anus eventually arises at the bottom of a diverticulum of the alimentary tract, which meets an invagination of the skin. Perforation according to GOETTE's well-known representation of a longitudinal section in *Bombinator* only occurs when the growth of the tail is well advanced, in *Rana temporaria* according to BALFOUR somewhat earlier.

SPENCER (1885), on the contrary, comes to the conclusion that the blastopore in *Rana temporaria* remains open and passes directly into the anus. The blastopore is not enclosed by the medullary folds, and thus there is no neurenteric canal. The first conclusion is shared by Durham (1886), but secondarily, according to the latter, a neurenteric canal is formed, independent of the blastopore. Kupffer (1887), dealing with the same subject, comes to the conclusion that the blastopore remains open as the anus; so, too, Perenyi (1888).

Schanz (1887) also operated on *Rana temporaria*, together with *Triton.* In *Rana* he concludes that the medullary folds rather close over the blastopore, that there is indeed a neurenteric canal, though the lumen is not evident, and that the anus arises by perforation at the bottom of a little groove behind it. As regards the facts Sidebotham (1888) quite agrees with him. According to him Balfour's description is the right one, he too sees in sections the "diverticulum from the hind end of the mesenteron, dipping down towards a distinct pit in the epiblast below the blastopore and quite separate from it". Eventually perforation ensues. Similarly by Morgan (1890) in *Rana halecina* and *Bufo lentiginosus* the anus is seen to arise at the bottom of a little groove in the ectoderm behind the blastopore.

Goette (1890) after a renewed investigation on *Bombinator igneus* and some other Anurans reaches the conclusion that the anterior half of the slit-like blastopore is transformed into the neurenteric canal, the posterior half into the anus. Yet in *Pelobates* he claims that this posterior half first closes and that the anus is formed only later.

As is apparent from the foregoing, during this period nearly every year brought forth a new investigation on this subject. In 1890 that of Erlanger on *Rana esculenta* appeared; in 1891 that of Robinson and Assheton on *Rana temporaria*; in the same year a small treatise by Erlanger in reply to some observations made by the two English critics on his work. All agree however that in both cases the anus arises by perforation.

In later years the fate of the blastopore is alluded to only in a few investigations, e.g. by Bles (1905), who for *Xenopus laevis*, and by Seemann (1907), who for *Alytes obstetricans* shows that the blastopore is not enclosed by the medullary folds and passes directly into the anus, there being accordingly no neurenteric canal.

Most of the investigators who have paid special attention to the question thus come to the conclusion (which after my own examination of *Rana esculenta* I can support without reservation) that the anus arises by perforation a little distance behind the blastopore, which is transformed into the neurenteric canal. A short description may be given here in addition to the figures for *Rana esculenta*.

After the yolk-plug has disappeared from the surface the blastopore presents itself as a short longitudinal split (textfig. 1*a*). A median section through this egg is reproduced in Fig. 1 of the plate. In a

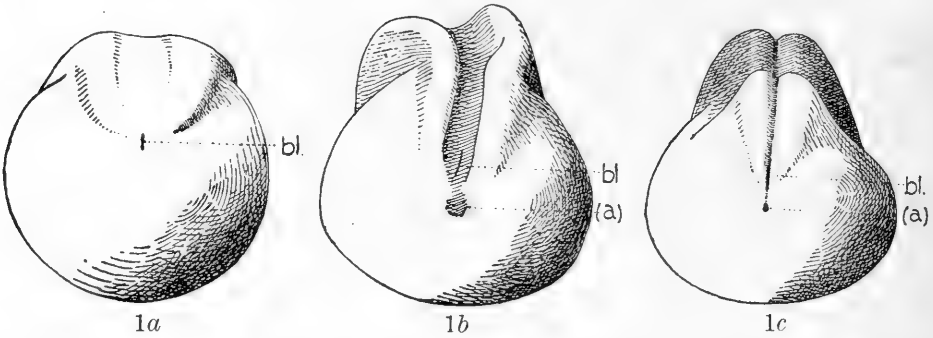

Fig. 1. Three eggs of *Rana esculenta* during the closure of the medullary folds (*a*) anal pit. *bl.* blastopore.

similar longitudinal series one succeeds better than might be expected in getting the blastopore as an opening (*bl.*), though of course this is only the case in one or two sections. The ventral blastopore lip is well developed and includes between itself and the yolkmass in the archenteron the anal diverticulum (Afterdarm, *a.d.*), which however is nothing but the intersection of a circular incision surrounding the mass of yolk-cells.

In a somewhat further advanced stage appears on the surface of the egg (textfig. 1*b*) behind the slit-like blastopore a shallow impression in the ectoderm (*a*), also clearly visible in a longitudinal section, as in fig. 2 of the plate. Underneath this impression a thickening of the ectoderm occurs, of which the beginning is already visible in fig. 1 (*). Opposite the invagination of the ectoderm a similar one is found in the entoderm at the bottom of the anal diverticulum.

In an egg as represented in textfig. 1*c* we see at the bottom of the shallow invagination of the ectoderm mentioned above a little pit, as yet not very deep, from which a still more shallow groove, the anal groove, runs forward to the blastopore-slit. The longitudinal section of this egg is given in fig. 3 of the plate. It bears a close relation to fig. 2, the anal membrane however has become thinner.

In a slightly further advanced stage, not represented here, the greatest part of the slit-like blastopore has been overgrown by the medullary folds, only at the hindmost extremity is there still a little opening, from which the anal groove runs to the anal pit. This anal groove, with a deeper depression at its anterior (rest blastopore) and at its posterior end (anal pit) appears to have been confused by several authors with the slit-like blastopore of fig. 1*a* and *b*, which they accordingly imagine to have closed in the middle by coalescence of the opposite borders, leaving only a passage at the anterior and at the rear end, the future neurenteric canal and the anus, while the rudiment of the tail arises as a double knob at the right and the left side of the place of coalescence, these knobs fusing afterwards over the middle of the blastopore. Thus ZIEGLER (1892) in his little article on the surface-views of *Rana*-embryos writes: "Etwas später sieht man an Stelle des Spaltes eine Rinne, welche vorn in den Canalis neurentericus, hinten in die Aftergrube übergeht; es sind nämlich jetzt die seitlichen Blastoporuslippen median zur Vereinigung gekommen". In the same way things are represented by HERTWIG in his Lehrbuch. Already a close examination of surface views however teaches us that the anal groove is not at all identical with the slit-like blastopore, but that its anterior end coincides with the rear end of the latter. The study of median sections excludes every possibility of doubt. In the present article I could not insert any more some figures of a surface-view and of median sections of this stage, in a more detailed account elsewhere I will do so.

The step to fig. 4 (plate) seems fairly large, yet this is only apparent. Already in fig. 3 we see the cerebral plate curving in. Especially notable is the opposition between the praechordal cerebral plate and the epichordal medullary plate, which as a matter of fact in this stage is no longer a flat plate, but curved into a groove between the medullary folds. Fig. 3 however is realized only in one or two sections, which are exactly median, to the right or the left side immediately one of the medullary folds is intersected, as indicated in fig. 3 with a dotted line. A paramedian section in this series thus offers a much greater resemblance to fig. 4 where the medullary folds have coalesced than the median one of fig. 3.

Fig. 4 is also of interest in that here apparently for the first time the neuropore in Anurans is represented. In his treatise on "Die Morphogenie des Centralnervensystems" in HERTWIG's Handbuch, KUPFFER (1906) says in regard to Anurans: "Der Neuroporus ist im letzten Momente vor seinem Schlusse noch nicht zur Beoachtung gekommen"; neither in investigations published since is there anything to be found on this subject. KUPFFER accordingly only represents a longitudinal section of a somewhat further advanced stage than in my fig. 2 and further stages later than my fig. 4, where the place of the neuropore is still recognisable by the presence of a conical thickening of the ectoderm or of a recessus neuroporicus in the anterior wall of the brain vesicle. It is evident that the curving backward of the transverse cerebral fold plays as great a role in the closing of the cerebral plate as the overgrowth of the lateral ridges.

There is yet another circumstance I should like to emphasize. Not only the ectoderm of the cerebral plate but also that which is situated in front of the transverse cerebral fold and which according to my theory is equivalent to that part of the apical plate of the Annelid trochophore which in Craniotes is not incorporated into the cerebral plate, is considerably thickened, and as for example in fig. 1 (*pr. cer.*) it exhibits an equally clear separation between the upper and lower layers of the ectoderm as the cerebral plate. Also in fig. 2 this agreement between cerebral plate and the part of the apical plate in front of it, which we might call the praecerebral part is evident. In the course of further development, however, a difference between the two parts of the apical plate evidences itself. In the cerebral, just as in the medullary plate, an intimate union of the upper and lower layers occurs, the demarcation between them disappears, and the upper layer, as ASSHETON (1909) has already observed, is incorporated in the wall of the brain and the medullary canal. In the praecerebral part of the apical plate however the coherence between the upper and lower layers becomes less and less, which no doubt is connected with the circumstance that this part of the ectoderm has to overgrow the cerebral plate. The lower layer finally lies as a compact cell-mass under the upper layer, which acts as ectoderm, and quite dissociated from it (fig. 4 *pr. cer.*). Judging from KUPFFER's (1906) figures of the later stages, it is this cell-mass which moving under the brain vesicle, ultimately gives rise to the hypophysis. A possible relation between the origin of the hypophysis and the animal pole in vertebrates would no doubt be worth closer examination.

If now we revert to the bottom of the body we see that here too the median sections of figs. 3 and 4 differ more from each other than paramedian ones do. The anus has broken through, the ventral blastopore lip accordingly seems to have vanished at once. The blastopore itself has been overgrown by the medullary folds. In the posterior part of the medullary tube the latter have applied themselves so closely one to the other, that the lumen of the tube is not continued between them and only a virtual neurenteric canal can be spoken of. Later, judging from the diagrams of other investigators, a lumen seems to reappear and thus a real neurenteric canal. SIDEBOTHAM and ERLANGER give diagrams of median sections of eggs in which the anus is just on the point of breaking through. From the study of whole eggs it appears quite evident that the medullary folds unite over the blastopore and that somewhat behind it at the bottom of the little depression indicated in fig. 1*c* (text) the anus breaks through.

I should like to emphasize a peculiarity which has only been pointed out by ERLANGER (1890), especially in relation to what we shall find in Urodelans. In the short time that passes between the stages of fig. 1 and fig. 3, the distance between blastopore and future anus diminishes a little; in other words, if we take the place of the future anus as a fixed point, the slit-like blastopore moves a little backwards towards it. So the ventral blastopore lip in median sections is not only getting thinner owing to the appearance of the groove between blastopore and anus, but also somewhat shorter. To this point we will revert later.

Let us pass now to the Urodelans. Characteristic in the early stages of development is here the little extension of the ventral ectoderm and the strong development of the dorsal parts, the foundation of the embryo accordingly encircling the egg over considerably more than 180°. This peculiarity the Urodelans have in common with the Dipnoans and Petromyzontes, of which the earliest stages of development, externally as well as in sections, exhibit a striking similarity to those of Urodelans.

According to SCOTT and OSBORNE (1879) the blastopore of *Triton* is overgrown by the medullary folds and becomes the neurenteric canal. SEDGWICK (1884) in his well-known article on the origin of metamerism writes concerning *Triton cristatus*: "in this animal the blastopore appears not to close, but to persist as the anus" and his pupil ALICE JOHNSON (1884) verified this by sections. A neurenteric canal, as described by SCOTT and OSBORNE, was never observed by her. SCHANZ (1887) in *Triton punctatus* comes to the conclusion that the blastopore is constricted in the middle, the anterior opening becoming the neurenteric canal, the posterior opening the anus. HOUSSAY and BATAILLON (1880) on the contrary find in the axolotl: "qu'il n'y pas de canal neurentérique, que le blastopore demeure toujours ouvert et qu'il devient l'anus définitif." Next comes the accurate investigation of MORGAN (1889, 1890) for the axolotl. He too finds that the hindmost part of the blastopore passes into the anus, the anterior part being overgrown by the medullary folds. Since my conclusions are closely akin to those of MORGAN, I will revert to them in detail presently.

GOETTE (1890) similarly sees in some Anurans (*Triton*, *Siredon*) the rear end of the blastopore pass into the anus.

A few further observations of recent times as to the fate of the blastopore may be touched on, thus those of DE LANGE (1907, 1912) and Ishikawa (1908) concerning *Megalobatrachus maximus*, of Kunitomo (1911) concerning *Hynobius*, and of Smith (1912) concerning *Cryptobranchus alleghaniensis*. All agree in this that the hind part of the slit-like blastopore remains open as the anus, the anterior part being overgrown by the medullary folds, except Ishikawa, who thinks this course of events to occur only exceptionally, the anus as a rule springing up as an independent formation, which is denied by de Lange (1912).

For *Petromyzon* and Dipnoans most investigators hold that either the whole blastopore or its hind end passes into the anus.

My own investigations concerning the axolotl all go to confirm the conclusions already reached by most of my predecessors, viz. that the rear part of the blastopore passes into the anus. If then I give a brief survey of my observations, it is with the express object of emphasizing some few circumstances which were not noticed by former investigators and seem to me of importance in giving a right interpretation.

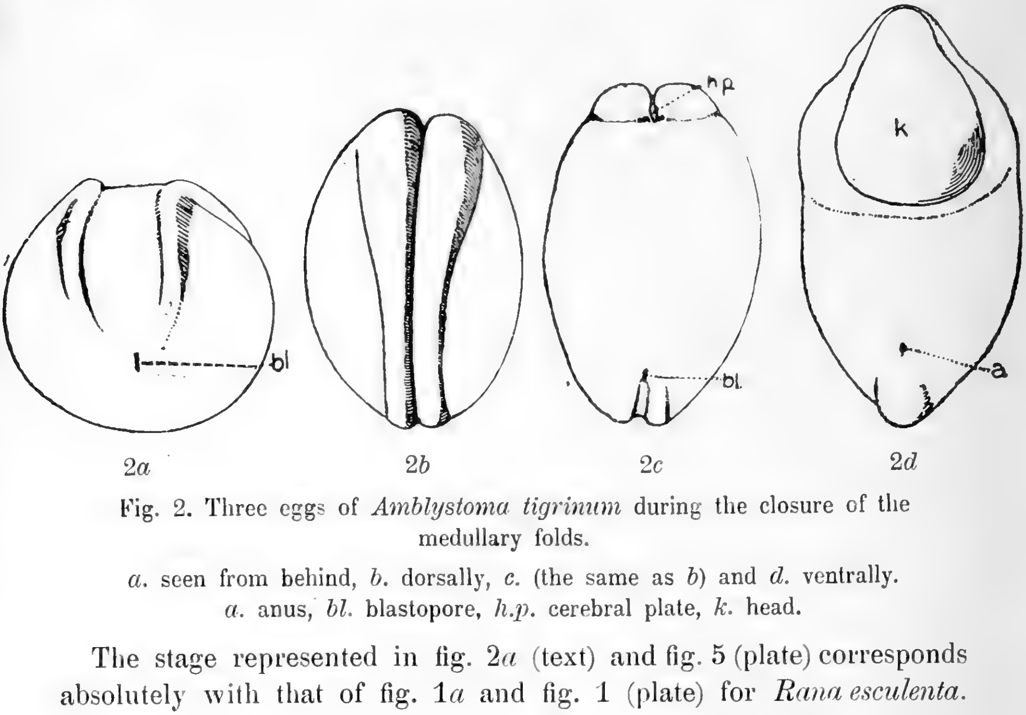

Fig. 2. Three eggs of *Amblystoma tigrinum* during the closure of the medullary folds.

a. seen from behind, *b.* dorsally, *c.* (the same as *b*) and *d.* ventrally.
a. anus, *bl.* blastopore, *h.p.* cerebral plate, *k.* head.

The stage represented in fig. 2*a* (text) and fig. 5 (plate) corresponds absolutely with that of fig. 1*a* and fig. 1 (plate) for *Rana esculenta*. Here too the medullary folds begin to appear and the blastopore has contracted to a short longitudinal slit. Already in fig. 5 it is evident, how much more the dorsal side is developed than the ventral side, the distance from the animal pole (which according to Eycleshymer, 1895, here too is to be found back just in front of the transverse cerebral fold) to the slit-like blastopore measured ventrally being much less than 180°. In accordance with this the dorsal blastopore lip, as fig. 5 (plate) compared to fig. 1 (plate) shows, and the archenteron are developed very strongly, the ventral blastopore lip and the so-called anal diverticulum very little. Yet both the latter are still easily recognisable and on the outside of the ventral lip, a little distance behind the blastopore, a small depression of the ectoderm (*a*) may even be noted, where the future anus might be expected, if things happened in the same way as in Anurans. Immediately behind that shallow depression we find here again the same thickening of the ectoderm (✱) as noted in *Rana* (cf. figs. 1, 2, 3, plate). So there is no fundamental difference, on the contrary agreement in every respect with what we found in *Rana*.

Now in *Rana* we stated that the blastopore, after becoming slit-like, continues to move backward a small distance, approaching the future anus, which manifests itself in longitudinal sections in that the little lip which represents the ventral blastopore border becomes a little shorter. This now we see happening also in somewhat further advanced stages of the axolotl-egg: on sections the ventral lip gets shorter and soon, being here already small, it disappears altogether. In the egg shown in fig. 2*b* and *c* (text) the medullary folds are on the point of coalescing, except at the fore and the rear end. The blastopore still appears as a slit. The longitudinal section (fig. 6) shows that the ventral blastopore lip has nearly disappeared: as a result of the backward movement the rear end of the slit-like blastopore has arrived at the spot where the anus must break through!

Especially interesting is next the egg shown in fig. 2*d*, where the medullary tube has just closed, except at the hindmost extremity, where the anterior part of the slit-like blastopore has just been overgrown by the medullary folds. Whilst in *Rana* the whole blastopore is in this way enclosed, in the axolotl the medullary folds leave an opening over the rear end of the blastopore, which is the anus (*a*).

Only one egg in this stage was found by me among my material. This was cut into longitudinal sections. Morgan studied a similar egg in transverse sections. I reproduce here the outline of his excellent figures which wholly confirm my way of presenting things. Fig. 3*a* represents a section through the medullary tube just in front of the blastopore. Under it the anal diverticulum has been intersected. The medullary folds just meet. Figs. 3*b* and *c* show the blastopore in its anterior half, as is of course the case in many succeeding sections. The medullary folds meet over the blastopore, the latter itself constituting the neurenteric canal. Figs. 3*d* and *e* are still further back, the medullary folds are less developed, and leave an opening, the anus. Comparing my description with that of former investigators it will be noted that, keeping strictly to the facts, I yet present them in a somewhat different way: I do not let the medullary folds finish halfway the length of the blastopore slit, but only in closing leave an opening over the rear end of the blastopore, the anus. Accordingly one can, retracing the medullary canal, not only pass through the neurenteric canal into the archenteron, but also through the anus to the outside, this being nowhere prevented by a coalescence of the two medullary folds across the middle of the blastopore, as many investigators are inclined to assume.

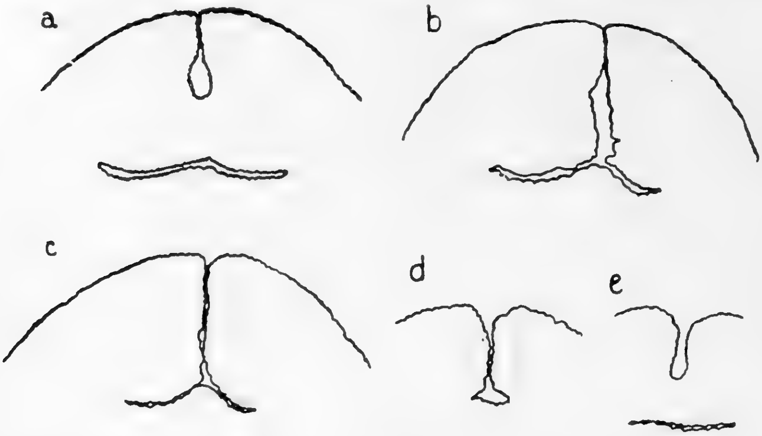

Fig. 3. Transverse sections through the blastopore of an egg of *Amblystoma punctatum*, where the medullary folds just close over it, after MORGAN (1890).

a in front of the blastopore, *b* and *c* through anterior half, *d* and *e* through rear end (anus).

Now in a longitudinal section (fig. 7, plate) the blastopore (*bl.* = *p. neur.*) and the anus (*a*) are easily distinguishable from one another. The blastopore becomes the neurenteric canal or, perhaps better, the neurenteric pore (porus neurentericus), as I prefer to call it henceforward. Entering the anus, one can pass through the neurenteric pore into the archenteron. The anterior part of the neurenteric pore however becomes — and is already in fig. 7 — virtual, the medullary folds applying themselves behind so closely to one another, that the lumen of the medullary canal is not continued any further between them, as MORGAN has already remarked. Hence the opinion of many investigators that the medullary folds do not reach to the blastopore and that there is no neurenteric canal. The hindmost part remains open as the internal opening of the anus. The result is really that the hindwall of the hindmost part of the medullary tube is perforated by the anus, which in Anurans arises directly behind it, and this is caused by the circumstance that the neurenteric pore, the former blastopore, in Urodelans has travelled back so far, that its rear end has reached the place where in Anurans the anus breaks through. This is at the same time the solution of the apparent contradiction between Anurans and Urodelans in this respect.

The interpretation which until now has been pretty generally adopted is that of SCHANZ (1887), MORGAN (1890), ERLANGER (1890) and ROBINSON and ASSHETON (1891), who contend that the place where the anus in Anurans breaks through really represents the rear end of the original wide blastopore, which has narrowed down by concrescence of the lateral borders not only at the anterior end, as postulated by HIS's concrescence theory, but also at the posterior end. The longitudinal groove between the blastopore and the anal depression in fig. 1 seemed to be an indication of a raphe. Thus the anus in Amphibia would be closed only temporarily and would not arise as an independent formation. In this way ERLANGER assumed concrescence at the dorsal as well as at the ventral blastopore border, ROBINSON and ASSHETON only at the ventral border. The line of concrescence in both cases is compared to a primitive streak, which, as ROBINSON and ASSHETON in accordance with BALFOUR's views on this point remark, can be expected only behind the blastopore: wrongly enough the adherents of the doctrine of concrescence call primitive streak the concrescence-seam assumed by them in front of the blastopore. To me it seems that one ought to add that a primitive streak is to be expected only in yolk-laden eggs with a germinal disc or in eggs that are to be derived from yolk-laden ones.

I will not absolutely deny that concrescence ever plays a part in vertebrate gastrulation, especially in yolk-laden eggs. But that its rôle is a much more subordinate one than the well-known doctrine of HIS assumes, seems to me beyond doubt. Even by students of the development of teleosteans, which seemed to afford the most acceptable confirmation of it, HIS' doctrine is rejected, as for example by SUMMER (1904). For amphibians the pricking experiments described in both my former communications have shown that there cannot be any question about the whole dorsal side of the embryonic rudiment arising by concrescence of the blastoporic lips.

It is quite true that in the amphibian egg a fine median line is often seen running from the blastopore forward, which strongly suggests a concrescence-raphe. Only, as ROBINSON and ASSHETON remark, this line continues to the fore-end of the cerebral-plate, the animal pole, where the blastopore has never been. For concrescence at the hind border of the blastopore still less evidence can be adduced. The groove between the slit-like blastopore and the anal pit does not become gradually longer, as might be expected in this case, the anal pit removing from the ventral border of the blastopore, but on the contrary it only gradually becomes more distinct and at the same time shorter, the blastopore approaching the anal pit. Evidently it is not to be considered as a concrescence-seam, perhaps it may be compared to the groove joining the two impressions made by two fingers pressed near one another into a soft cushion.

Concerning the relation between blastopore and anus in vertebrates three suppositions may be made:

1. there is a primary relation
2. there is no relation
3. there is a secondary relation.

The first supposition mentioned above is now the most widely accepted, even where in Anurans 2. seems to prevail yet it is assumed that this is to be traced back to 1. since what is found in Urodelans must be valid for Anurans. Thus MAURER (1906) in HERTWIG's Handbuch tries to trace back all the results for chordates to 1, though the evidence adduced is not always equally convincing. Already in *Amphioxus* no relation between the anus and the blastopore has as yet been discovered.

The possibility of 1. is in no way excluded by my theory, which derives chordates in opposition to GROBBEN from Protostomia, as long as the possibility of a relation between the anus and the blastopore in the latter group exists, as might be expected from SEDGWICK's well-known theory (1884), which derives the mouth and the anus of Bilateria from the anterior and the posterior extremity of a slit-like actinian mouth of which the borders coalesce in the middle. The concrescence-seam joining mouth and anus, which according to this theory should run over the ventral side of annelids, ought to be able to be traced in vertebrates too then in the groove between anus and blastopore, that is in the so-called "Afterrinne", the "primitive streak" of ROBINSON and ASSHETON (see above) — not in the hypothetical concrescence-raphe in front of the blastopore, the "primitive streak" of the theory of concrescence, as LAMEERE (1891) and HUBRECHT (1905) assume in their application of SEDGWICK's theory on Vertebrates. Thus the presence of a primary relation between the anus and the blastopore in Vertebrates would in no way oblige us to derive them with GROBBEN (1908) from the Deuterostomia, as long as the possibility of a similar relation in Protostomia exists.

However the theory of SEDGWICK finds in the development of Protostomia just as little support as I hope to show is the case in Tritostomia (Vertebrates). A process of so fundamental phylogenetical significance as assumed by SEDGWICK's theory might be expected to have left more distinct traces in the ontogenetic development than are demonstrated by the most careful research of recent investigators. Again and again we see the anus arise as a new formation, by perforation. In Annelids, where primarily we might expect to find evidence of a common origin of mouth and anus, a direct transformation of the rear end of the blastopore into the anus has never been demonstrated. Even in the primitive *Polygordius*, where as a matter of fact the blastopore is divided into two halves by a median constriction, the posterior opening nevertheless closes and the anus arises by perforation behind the two teloblasts, which lay at the rear end of the blastopore. To me the most probable conception of the origin of the anus seems to be this, that in a larva of the protrochula-type (MÜLLER's larva of Polyclad, pilidium of Nemerteans) the entodermal pouch, which is already turned in a backward direction, has applied itself to the ventral body-wall and is broken through by perforation, in the same way as occurs in Deuterostomia, and that thus the trochophore-larve has originated.

So I think the idea of a primary relation between the anus and the blastopore for Proto- as well as for Tritostomia should be abandoned. The anus in Proto- as well as in Tritostomia arises by perforation, independent of the blastopore.

Of the three above mentioned possibilities regarding the relation of the anus and the blastopore the second then seems to me, both for Proto- and Tritostomia, the right one. The third possibility however we find exemplified in Urodelans and apparently also in Dipnoans and Petromyzontes, which in their early development so closely agree with the former. Let us now invoke the aid of my theory for further interpretation.

According to this theory (DELSMAN, 1913) the vertebrate is to be derived from the Annelid by the stomodaeum growing out backwards so strongly that it extends, as the medullary tube, over the whole length of the soma, and, as we shall see, even further still (formation of the tail!). For the entrance of the stomodaeum into the entodermal part of the gut I propose the name porus cardiacus, this being the former blastopore. Already during the development of Annelids we see this cardiac pore by the lengthening of the stomodaeum travelling backwards into segments situated ever further to the rear. In Vertebrates this backward movement goes so far that finally the cardiac pore, as neurenteric pore, comes to lie absolutely at the rear extremity of the soma, just in front of the anus. This backward movement is evidently produced by a growing zone which has entered into activity at the inner end of the stomodaeum, round the porus cardiacus and which causes the stomodaeum to extend more and more to the rear. This growing zone I should like to call the periporal growing zone. The longitudinal growth of the soma of Annelids on the contrary is produced by a perianal growing zone. Both these growing zones now exert their influence as I hope to show, in the earliest development of Vertebrates, and things are still further complicated by the fact that the activity of both, ontogenetically anticipated, interferes with the gastrulation. Further researches (pricking experiments, counting of the mitoses) will have to test the correctness of the conclusions reached by the application of the above principles. They are as follows.

The ectoderm, which afterwards has to invest the whole soma, — dorsally too — in a stage as in figs. 1*a* and 2*a* (text) lies principally at the ventral and lateral sides, and only afterwards, by the closing of the medullary tube, extends over the dorsal side as well. The production of this somatic ectoderm now must evidently issue from the perianal growing zone: in the neighbourhood of the future anus, a short distance behind the ventral blastopore lip mitoses may to be expected to be most frequent. When however the blastopore is closed (figs. 1*a*, 2*a*), the rearward extension of this ventral ectoderm comes to an end. If now the perianal growing zone continues to be active, a ring-shaped thickening of the ectoderm round the anal pit will result. This being observed, it appears to me that it is here we have to look for the explanation of the ectodermal thickening, which in the figs. 1, 2 and 3 (plate) we see developing in an increasing degree just under the anal pit (*****), and which, as paramedian sections teach us, reach forward, also at the left and the right of it. In the axolotl, where the extension of the ventral ectoderm is so slight, this ectodermal thickening too, though present, is yet of very little importance (5*****). The activity of the perianal growing-zone soon afterwards seems to die down and the ectodermal thickening in the ensuing stages gradually disappears again. Somatogenesis has closed simultaneously with gastrulation. If it continued also after the end of the gastrulation, the anus would eventually lie somewhere between the yolk-cell-mass and the extremity of the tail. In fishes this case is pretty generally found. As an example may be mentioned the sturgeon (fig. 5, text), but many teleosteans might also be mentioned here, in whose larvae the place of the anus varies much and is of importance in determining the species.

Let us now turn to the periporal growing zone, which causes the growing out of the stomodaeum, resp. the medullary tube, resp. the medullary plate, together with the backward movement of the cardiac pore (Annelids), resp. the blastopore, resp. the neurenteric pore (Chordates). Organs or processes that are of much importance for the structure of the adult animal, in ontogeny often appear precociously. In Lamellibranchia e.g. the shell-gland invaginates already during gastrulation, though the latter process phylogenetically is no doubt much older. Thus also the activity of the periporal growing zone, and the backward movement of the cardiac pore associated with it begins very precociously, viz. already during gastrulation, when the future cardiac pore is still the blastopore. The interference of the contraction of the blastoporic rim with the backward movement of the blastopore causes the caudadly excentric closure of the blastopore, which is typical for chordates. The activity of the periporal growing zone, as long as the tubeformation has not set in, results not in the production of a stomodaeal viz. medullary tube, as is the case afterwards during the urogenesis, but provisorily in the formation of the medullary plate. The growing out of the stomodaeum to the medullary tube is thus in its first, somatogenetic part to be imagined projected on a plane, the dorsal plane of the embryo. When the blastopore has narrowed to a slit and the tube-formation sets in in the form of the medullary folds, the caudad wandering of this slit-like blastopore, as stated above, continues nevertheless, truly only over a little distance — indeed in view of the short duration of this stage nothing else could be expected — and so probably with undiminished speed. Further than the anus however this backward movement cannot go, phylogenetically: the stomodaeum of the Annelid, growing out backwards, at last reaches the anus. If now the movement stops a little in front of the anus, there will be no relation whatever between neurenteric pore (blastopore) and anus (fig. 4*a*, text), as we stated in the frog. If the movement continues yet a little further (fig. 4*b*), a secondary relation between neurenteric pore (blastopore) and anus results. [1] The anus now opens to the exterior through the hindmost extremity of the medullary tube, from the medullary canal one can pass through the anus to the exterior as well as through

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[1] In a longitudinal section as in fig. 4 the constellation at first sight might appear in fig. 4*b* radically different from that in 4*a*. If however one imagines things in space, the agreement between them will be evident.

the neurenteric pore into the archenteron, and from the archenteron through the neurenteric pore and the anus to the exterior. In ontogeny this will result in the medullary folds not closing over

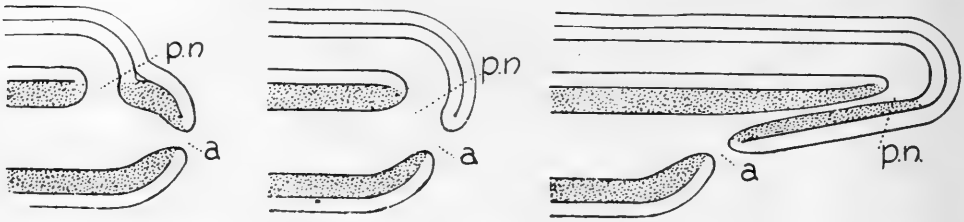

Fig. 4. Diagram of the relation between anus blastopore, and of the tail-forming
a. at the moment of the closure of the neural folds in Anurans,
b. „ „ „ „ „ „ Urodelans.
c. formation of the tail.
a. anus, *p.n.* neurenteric pore; the entoderm is dotted.

the rear end of the slit-like blastopore, but leaving an opening, the anus. Perhaps they will develop slightly at both sides of the rear part of the blastopore under the influence of the formation of the anus at this point, or they may not. If we imagine things very much enlarged and we look through the anus into the interior, we shall see the slit-like blastopore (neurenteric pore) in the distance, though its rear end, under the influence of the formation of the anus, will probably be widened a little. With this conception the facts stated by us in Urodelans so perfectly agree, that it seems hardly possible to doubt the correctness of this interpretation. We see in the axolotl the blastopore move backwards to the place where in Anurans the anus breaks through. We see over that place, that is over the rear end of the blastopore, the medullary folds not, as in Anurans, unite, but leave an opening. We have seen that we can pass from the medullary tube as well through the anus to the exterior as through the neurenteric pore into the archenteron. What makes things later less clear is that the medullary folds caudally so closely apply themselves one to another, that there is no lumen, no medullary canal (fig. 7, plate) — just as in the frog (fig. 4) — and that accordingly as in the frog the neurenteric pore would become virtual, if the rear part did not remain open as the anus. So only the anterior part of the slit becomes virtual, and hence the statement of several authors concerning Urodelans, Dipnoans and Petromyzontes, that the blastopore passes into the anus and a neurenteric canal is wanting, is to be explained. The apparent contrast between Anurans and Urodelans c.s. has thus found a solution. It would cause us no surprise if in an Anuran a state of things were observed such as in Urodelans seems to be the rule, or the reverse, the difference between them not being fundamental, but only gradual. It would not be impossible that in one species at one time the first,

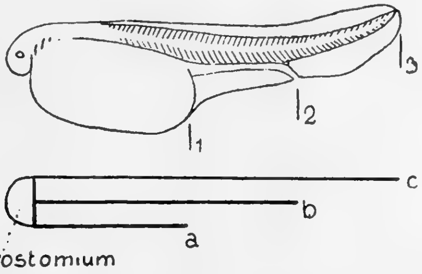

Fig. 5. Larva of the sturgeon after KUPFFER from HERTWIG's Handbuch. 1. limit of the gastrulation, 2. limit of the somatogenesis, 3. limit of the urogenosis. Beneath: Diagram of the interference of the gastrulation (*a*) with the action of the perianal (*b*) and the periporal (*c*) growing zones.

at another the second case might be realized (comp. DE LANGE and ISHIKAWA on *Megalobatrachus!*).

I have spoken above of the caudad movement of the neurenteric pore = blastopore stopping in front of the anus. In reality however there is no question of stopping. Although the anus seems to afford an insurmountable obstacle for the further backward growth of the stomodaeum = medullary tube, the activity of the periporal growing zone has not yet come to an end when the perianal growing zone has stopped working. There being no room however within the soma for further extension, a protuberance of the body wall in front of the anus results, into which the stomodaeum = medullary tube grows out: the tail-knob (fig. 4*c*, text). Thus we see the tail of vertebrates originating by the periporal growing zone continuing its activity after the perianal has stopped. In this way the position of the anus in vertebrates is not terminal, as in Annelids, but at the root of the tail, which overgrows it and which owes its origin simply to the presence of the anus. Phylogenetically we have to imagine that the longitudinal growth of the stomodaeum (medullary tube) surpasses that of the soma, so that the cardiac (neurenteric) pore overtakes the anus and passes it. Just as in Annelids the position of the anus in Vertebrates is terminal in regard to the soma proper, the tail is an outgrowth of the dorsal side of the latter in a backward direction. According to this conception the ventral side of the tail belongs to the dorsal side of the soma. In accordance with this the dorsal unpaired skinfold of the fish- and amphibia-larvae is continued over the tip and the underside of the tail as far as the anus. The mesoderm originating at the blastopore-border, and evidently being a product of the periporal growing zone, this too takes a considerable part in the tail-formation.

De Lange (1912) rightly emphasizes the difference between somatogenesis and urogenesis, though I cannot concur with him in his conceptions on gastrulation and mesoderm formation, as expressed by the words cephalo- and somatogenesis. From the foregoing results it appears that somatogenesis, just as the somatogenesis in Annelids, is produced by the perianal growing zone, which gives rise to the future somatic (not the neural, that is that of the medullary plate) ectoderm of the trunk, which, as long as the medullary plate is open, lies mainly ventrally and at the sides of the egg. Simultaneously, however, with the gastrulation the periporal growing zone is at work, which produces the backward movement of the blastopore and the backward extension of the originally crescentic rudiment of the medullary plate = the rudiment of the medullary tube. And both growing processes are combined with a third one, going on simultaneously: the gastrulation, manifesting itself at the surface in the contraction of the blastopore border.

The urogenesis however sets in after two of these three processes have finished, viz. the gastrulation and the activity of the perianal or somatic growing zone[1]), and accordingly is exclusively the result of the periporal growing zone, which causes an elongation of the medullary tube, disproportional to the length of the soma. The difference between somatogenesis and urogenesis herein finds an explanation. The activity of the periporal growing zone, manifesting itself in the backward movement of the blastopore resp. neurenteric pore, at first interferes with the gastrulation, which causes the backward directed, excentrical closure of the blastopore, then manifests itself in the backward movement of the slit-like blastopore, stated by us above, which stage lasts only a short time), and later in the urogenesis as longitudinal growth of the medullary tube.

There is then no question of stopping the backward movement of the blastopore viz. neurenteric pore in front of the anus (comp. fig. 4), and the difference between Anuran and Urodelan consequently does not lie in the fact that in the former the neurenteric pore stops a little before the anus is reached, in the latter only after

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[1]) While in Anurans both processes stop nearly at the same time, in fishes, as stated above, we fairly frequently find that somatogenesis continues after gastrulation has been completed, so that the anus eventually lies somewhere about halfway between the yolk-cell-mass and the tip of the tail.

this has occurred, but in that in Anurans the tube-formation, i. e. the closure of the medullary folds, occurs a little before the anus is reached, in Urodelans, Dipnoans and Cyclostomes only after this has occurred. And this, only graduated difference evidently again depends on the circumstance that in Urodelans the activity of the periporal growing zone is stronger than in Anurans, the activity of the perianal on the contrary weaker than in the latter. This manifests itself, as stated above, in the medullary plate in Urodelans being developed very strongly, the ventral side very little in comparison with the Anurans. The same holds for Dipnoans and Cyclostomes. Now, as we have seen, the perianal growing zone acts mainly ventrally and on both sides of the (future) anus, for the simple reason, that, as long as the medullary plate is open, the future trunk ectoderm also lies only ventrally and on both sides of the egg. But in front of the (future) anus too, there seems to be some feeble activity, directed against the ventral blastopore lip, which accordingly is developed more strongly where the perianal growing zone is most active (Anurans, fig. 1, plate), less so, where the perianal growing zone is less active (Urodelans etc., fig. 5).

Now the action of this dorsal part of the perianal growing zone is opposed by the periporal growing zone, which pushes the blastopore backwards. And it is no doubt due to the relative strength of the two growing zones that in Urodelans the blastopore is pushed back to the anus before the tube-formation [1]), in Anurans on the contrary it does not reach it till after the tube-formation. I hope that the brevity with which I am obliged to express myself will not militate too strongly against the clarity of this exposition. A more explicit review will doubtless be published later.

While I feel that the application of my theory has thus thrown light on a number of obscure problems, the facts and results recorded above afford yet further support to my theory of no inconsiderable value.

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[1]) The beginning of which is determined again by the end of gastrulation, just as in Protostomia the stomodaeal tube is formed directly after gastrulation. In Selachians, where the accomplishment of the gastrulation is so much retarded by the great yolk-richness, urogenesis actually sets in before the tube formation, the neurenteric canal thus originally being an open groove (sulcus neurentericus).

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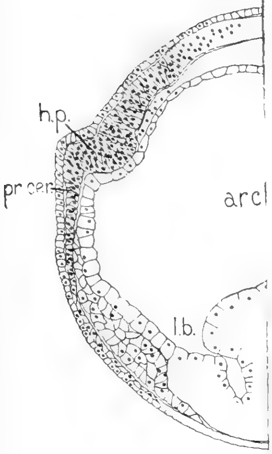

Fig.
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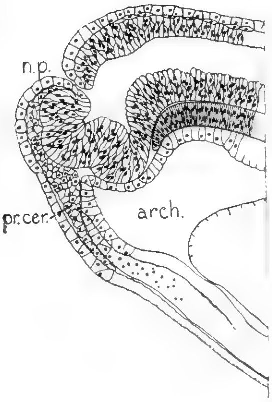

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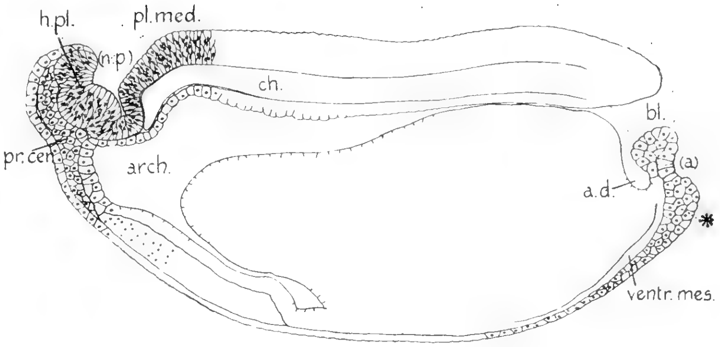

Fig. 3. *Rana esculenta.*
Median section through the egg of text fig. 1*c*.

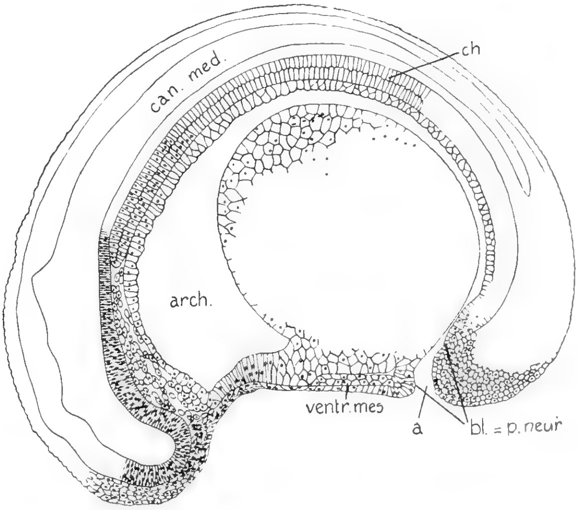

Fig. 7. *Amblystoma tigrinum.*
Median section through the egg of text fig. 2*d*.

ABBREVIATIONS.

a. anus, (*a*) anal pit, *a.d.* anal diverticulum of gut, *arch.* archenteron, *bl.* blastopore, *can. med.* medullary canal, *h.p.* cerebral plate, *l.b.* liver cove, *mes.* mesoderm, *n.p.* neuropore, (*n.p.*) place of the future neuropore, *p. neur.* neurenteric pore, *pl. med.* medullary fold, *pr. cer.* praecerebral thickening of the ectoderm, *ventr. mes.* ventral mesoderm.

H. C. DELSMAN. "On the relation of the anus to the blastopore and on the origin of the tail in vertebrates."

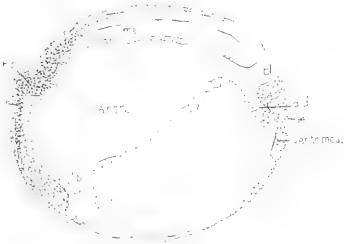

Fig. 1. *Rana esculenta.*
Median section through the egg of text fig. 1*a*.

Fig. 4. *Rana esculenta.*
Median section through an egg with closed medullary folds.

Fig. 2. *Rana esculenta.*
Median section through an egg as in text fig. 1*b*.

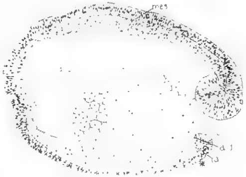

Fig. 5. *Amblystoma tigrinum.*
Median section through the egg of text fig. 2*a*.

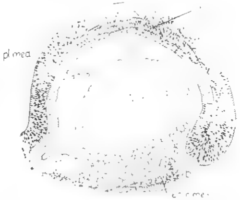

Fig. 6. *Amblystoma tigrinum.*
Median section through the egg of text fig. 2*b* and *c*.

Fig. 3. *Rana esculenta.*
Median section through the egg of text fig. 1*c*.

Fig. 7. *Amblystoma tigrinum.*
Median section through the egg of text fig. 2*d*.

ABBREVIATIONS.

a. anus, (*a*) anal pit, *a.d.* anal diverticulum of gut, *arch.* archenteron, *bl.* blastopore, *can. med.* medullary canal, *h.p.* cerebral plate, *l.b.* liver cove, *mes.* mesoderm, *n.p.* neuropore, (*n.p.*) place of the future neuropore, *p. neur.* neurenteric pore, *pl. med.* medullary fold, *pr. cer.* praecerebral thickening of the ectoderm, *ventr. mes.* ventral mesoderm.

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Microbiology. — *"The Enzyme Theory of Heredity."* By Prof. M. W. Beijerinck.

(Communicated in the meeting of March 31, 1917).

"Nothing is perfect at birth."

Combining the results of the enzymological researches of recent years with those obtained by the experiments on heredity, an insight is obtained into the nature of the thereby concerned substances which deserves attention.

The most acceptable theory of heredity is the conception that the living part of the protoplasm of the cell is built up from a great number of factors or bearers, different from one another, which determine the hereditary characters of the organism; at the cell division these bearers double or multiply, in consequence of which the characters, latent or unfolded, are transferred to the daughter-cells. They are called: *differirende Zellelemente* (Mendel), *gemmules* (Darwin), biophores, pangens, gens, character units, heredity units, Mendelian factors, or factors.¹)

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¹) G. J. Mendel, Versuche über Pflanzen-Hybriden. Verhandl. d. naturforschenden Vereines in Brünn, Bd. 4, Abh. Pag 42, 8 Februar u. 8 März 1865. — C. Darwin, Provisional hypothesis of Pangenesis. Domestication, 1st Ed. T. 2, Pag. 357, 1868. 2nd Ed. T. 2, 349, 1875. — Hugo de Vries, Intracellulare Pangenesis, Jena 1889, and the American edition, Intracellular Pangenesis, Chicago 1910. — V. Haecker,

How they appear in the cell, how they behave to nucleus, chromidia, chromosomes, and other cell-organs, and many questions more, form the subject of the heredity researches of to-day, which however start from the supposition that the said theory is in the main right. Nor does the observation that heredity units or factors may occur in latent condition and must then be activated by special kinds of food, by alcalies or acids, or other stimuli, touch the fact of their existence.

By the side of this view stands another, only apparently quite different, namely that the living part of the protoplasm is built up of a large number of various enzymes. A nearer consideration of these two views shows that "heredity units" and "enzymes" means the same. [1])

Hence the fundamental conception here to be proposed, that every hereditary character of an organism corresponds to one or more enzymes, which exert a reaction on specific substrates.

Long ago already I came to the conviction that the ontogenetic evolution of the higher plants and animals can be best explained by admitting that it is caused by a series of enzymes, for the greater part endoenzymes, which, becoming active in a fixed succession, determine the morphological and physiological properties gradually manifest in the development. These enzymes in the formation of plant-galls are likewise concerned, and in a study on the galls of the saw-fly *Nematus capreae* on the leaves of *Salix amygdalina,* I gave them the name of "growth enzymes".[2]) It is still my

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Allgemeine Vererbungslehre. Pag. 265, 1911. — M. W. BEIJERINCK, Mutation bei Mikroben. Folia microbiologica. Bd. 1, Pag 24, 1912. — W. JOHANNSEN, Elemente der exakten Erblichkeitslehre. 2nd Ed. Pag. 143, 1913. etc.

[1]) Younger physiologists (as E. ABDERHALDEN, Physiologische Chemie, 3te Aufl. Theil 2 Pag 997, 1915) wrongly use anew the old and equivocal word "ferment", instead of the practical and clear word "enzyme". The history of the introduction of the word enzyme is as follows. In "Verhandlungen des Naturhistor. und Medicin. Vereins zu Heidelberg", Sitzung am 4 Februar 1876, Bd. 1, N. F., the account of a lecture of KÜHNE begins thus: "Herr W. KÜHNE berichtet über das Verhalten verschiedener organisirter und sogenannter ungeformter Fermente. Um Missverständnissen vorzubeugen und lästige Umschreibungen zu vermeiden, schlägt Vortragender vor die ungeformten oder nichtorganisirten Fermente, deren Wirkung ohne Anwesenheit von Organismen und ausserhalb derselben erfolgen kann als Enzyme zu bezeichnen". This proposal is still acceptable. That KÜHNE only thought of exoenzymes was in accordance with the times. The term "endoenzyme" was introduced in 1900 by M. HAHN (Zeitschr. f. Biologie Bd. 40 Pag. 172, 1900). But the conception existed already long before. Enzyme comes from the Greek "en" in, and "zymè" leaven, and is related to "zeo" I boil.

[2]) Das Cecidium von *Nematus capreae* auf *Salix amygdalina*. Botan. Zeitung, 1888, Pag 1.

opinion that this view is in the main correct, but while I formerly thought that the growth enzymes partly derived from the gall-insect, I now recognize that they belong to the plant only and that the animal does not introduce enzymes into it.

Research material.

In the free living unicellular organisms morphological differentiation, joined with cell division, is quite or almost quite absent, which much simplifies the ontogenetic development. That in this case the properties must be represented just in the same way by specific factors, that is by heredity units or Mendelian factors, as in the cell protoplasm of the higher organisms, is beyond question. Although it would be erroneous to admit that the number of characters, and so of the heredity units or factors of the unicellular organisms must be small, we certainly have to deal here with a simpler case than in the multicellular. Hence it seemed probable that heredity experiments with the former would give some chance better to understand the nature of the heredity units in general.

But not all properties are equally well adapted to such a research. To show that some character of a cell corresponds to one or more units or Mendelian factors, that character must be able to change by mutability in such a way that the mutants prove to be hereditary constant races, distinctly different from the original form, for the conception of heredity units must also for the unicellulars start from the possibility of race formation.

The character to be studied must further be observable with ease and accuracy and it must be possible to cultivate the concerned organism in a simple way, so that in few days thousands of individuals can be examined and that no doubt is left as to their distinction from foreign infections. These requirements are very well answered by some pigment- and by the luminous bacteria as I repeatedly stated before. ¹) Especially the phosphorescence of the latter I have minutely examined, no character being better qualified to show the process of mutability and to enable us more quickly and precisely to judge of the vital energy of the culture material. Errors in the nutrition are in this way prevented, which so easily occur in microbiological experiments, in particular by too strong concentration and too alcaline reaction. Besides, the function of phosphorescence is not only found in certain luminous bacteria, but it is widely spread throughout the natural system and a remarkable

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¹) These Proceedings, 21 November 1900 and 9 February 1910.

similarity exists everywhere,¹) notwithstanding the enormous differences in the respective phosphorescent organs.

Another consideration which induced me to study with particular care the production of light by living microbes was the following.

I saw the great difficulty of explaining by the enzyme theory a function so obviously the attribute of the living protoplasm. Yet I had the conviction that if it were possible to account for this exceptional character by that theory, the same would be the case for any other character, physiological or morphological. Presently we shall see that the facts are in accordance with the expectation.

Not all luminous bacteria are equally well qualified for this investigation. *Photobacter splendidum,* common in the North Sea at the end of summer,²) and *Ph. phosphoreum* COHN, always present on sea-fish, whose properties are very different and in many respects complementary, are recommendable. *Ph. splendidum* produces trypsin, urease, diastase and invertase, and assimilates mannite with light production. *Ph. phosphoreum* has none of these enzymes and does not attack mannite.³)

The chief result of this study is that the function of phosphorescence may be ascribed as well to living protoplasm as to one or more enzymes.

I chose this function to elucidate the theory with regard to a physiological character; the production of the cell-wall shall be treated to test it from a morphological point of view, and also in the latter case it can be shown that the protoplasm as well as one or more enzymes may be regarded with the same right as the cause of its formation.

The subsequent considerations must be given in a short and somewhat aphoristic but I think not unclear form.

Enzymes considered as the bearers of phosphorescence. Irritability.

Already in 1898 RAFAËL DUBOIS endeavoured to demonstrate that phosphorescence should be considered as caused by an enzyme-action.⁴)

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¹) Perhaps with exception of the higher Fungi, where the luminosity seems to be in correlation with a state of collabescence.

²) Die Leuchtbakterien der Nordsee im August und September. Folia microbiologica, Bd. 4, Pag. 1, 1915.

³) Aliment photogène et aliment plastique des bactéries lumineuses. Archives Néerlandaises T. 24, P. 369, 1891 (Feeding of *Ph. phosphoreum* COHN.)

⁴) R. DUBOIS, Leçons de Physiologie générale, Pag 450 and 524. Paris 1898, Drawings of the phosphorescing organ of *Pholas* by ULRIC DAHLGREN: The production of light by animals. Franklin Institute, February 1916, Pag 38.

He experimented particularly with the luminous sipho-slime of *Pholas dactylus* and calls the enzyme, he thinks he has found "luciferase" and the unknown matter it acts upon "luciferine". The latter substance corresponds to what is called an "enzyme-substrate", but which might better be denominated "enzymoteel",¹) the word "enzyme-substrate" being evidently equivocal. To prepare a luciferase solution, free from luciferine, he leaves the luminous mucus till it becomes dark. He makes a solution of luciferine, free from luciferase, by slightly heating the mucus whereby the luciferase is destroyed. By mixing the two dark solutions light is evolved, from which he concludes that the luciferase acts as a catalysator similarly as other enzymes. The luminous slime consists of the cell-content of peculiar glands of the epiderm and flows from the cell through a fine canal; it seems not impossible that it contains protoplasm.

Various other sea animals as some Annelides, Cephalopodes and Coelenterates likewise secrete a luminous slime, which spreading in the sea-water illumines the surroundings of the animal.

E. NEWTON HARVEY has examined the phosphorescence of insects and comes to the same results as DUBOIS, but he calls the related substances "photogenine" and "photopheleine".²) It is also easy to show that the phosphorescent cells of our glow-worms, after mechanical destruction do not loose their luminosity. But these facts cannot be considered as proving incontestably the accuracy of the enzyme theory, it not being impossible that in all these cases not yet destroyed protoplasm is still active.

A better evidence for the view that the bearer of the phosphorescence consists of one or more endoenzymes is to be derived from the luminous bacteria. Here the production of light is inseparably bound to the bacterial body and secretion of a luminous slime never occurs.³) If thus there is question here of an enzyme as cause of the phosphorescence it can only be an endoenzyme, and that this supposition is in accordance with the facts may be shown by exposing the luminous bacteria to the influence of ultra-violet light. It is namely possible by means of the light of a quartzlamp, to bring them into the necrobiotic state, wherein they have lost their power of reproduction, but preserved their phosphorescence.⁴) If the time of the radiation is well chosen, the necrobiotic condition may last for

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¹) Of "telos"; aim.

²) Science N. S. T. 44, Pag. 208, 440, 652, 1916.

³) The slimy matter produced by some kinds of luminous bacteria is non-phosphorescent cell-wall substance.

⁴) For the particulars of this experiment see Folia microbiologica, Bd. 4, Pag. 10, 1915.

hours and it may be shown that the luminosity of *Ph. phosphoreum* during this period is greatly intensified by glucose. Hence the very same argument which leads us to consider the alcohol function of the necrobiotic yeast-cell as an enzyme action, caused by one or more enzymes, called zymase, holds likewise with regard to the connection between phosphorescence and its factor or factors the luciferase. The still unknown "luciferine" which, as said, can result in the case of *Ph. phosphoreum* from glucose, is the natural analogon of the "glucose-phosphoric-acid ester", i. e. the substrate or enzymoteel of the zymase.

The necrobiotic yeast-cells have lost their semi-permeability, as shown by the ease wherewith they are dyed by methylene-blue, their power of reproduction and certainly the motility of their protoplasm, whence they are considered as dead by several investigators. The same is probably the case with the necrobiotic luminous bacteria; but change of permeability could not be stated, since also in the condition of normal life they have a great affinity for pigments. I venture to think that the loss of the above properties when based, as is supposed, on the becoming inactive or on the destruction of the more sensitive heredity units or enzymes, can quite well go side by side with the continued activity of another part of the protoplasm, so that then it cannot be said that the cell is "dead" in the same sense as when all its functions are destroyed. The importance of this view is obvious if we bear in mind that the theory of the units of heredity consists in the very supposition that from their combination energies and activities may arise strange to the units separately. The demonstration of the properties to be ascribed to special factors and of those due to the co-operation of two or more factors is the chief subject of the heredity researches of to-day and the difficulties met with are well known. That the enzyme theory will here be useful is obvious.

About irritability I need not be long here, as for the lower immotile microbes this conception is only then based on observable facts if we think it coinciding with the power of metabolism and of reproduction.

In this connection I call to mind that the peculiarity of actions caused by stimuli, consists in their showing an optimum for certain intensities of these stimuli, which is also the chief character of enzyme action. So the influence of temperature and of different concentrations of poisons on the process of cell division and on that of amylolysis by diastase is analogous, and this is of course one of the best evidences for the correctness of the enzyme theory.

Phosphorescence considered as bound to protoplasm. Combination of the two views.

That the function of phosphorescence of the luminous bacteria is bound to the living protoplasm is supported by the following facts.

Anaesthetics, such as chloroform and aether, stop the light production almost completely, while after vaporisation of these substances it sets in anew, only slightly diminished. A short heating of temperatures near 40 ° to 45 ° C. of *Ph. splendidum* and of 30° to 35 ° C. of *Ph. phosphoreum,* with subsequent cooling, has the same effect. By the action of acids and alcalies the phosphorescence disappears and returns after neutralisation. A strong salt concentration darkens, after dilution the light is completely restored. Diminution of luminosity in these cases is caused by the dying of part of the germs. The phosphorescence of very active broth cultures, kept at rest for some time, undergoes a sudden and remarkable enhancement in its intensity by mechanical stimuli, such as shaking. The thus produced light reminds of the behaviour of higher luminous animals, possessing a nervous system, which by contact, or other mechanical stimuli, suddenly react with light production.

All these facts induced me already long ago [1]) to call the bearer of the phosphorescence "photoplasm" and its elementary units "photophores". Also for the Flagellate *Noctiluca miliaris* DE QUATREFAGES has demonstrated that the light issues from the protoplasmic threads that run from the nucleus to the cell-wall which, when seen under the microscope, presents a large number of minute light centres, corresponding to the ends of the threads, closely grouped near the flagellum, but farther on the surface at greater relative distances. [2]) The sudden radiance of *Noctiluca* by shaking the sea-water wherein it is suspended is well-known. When "fatigued" the cells become entirely luminous and DE QUATREFAGES called the so produced light "pathological light", but he does not say whether it originates from the cell-wall or the cavity.

A principal argument for the view that the photoplasm of the luminous bacteria possesses the properties of the protoplasm lies in the relation between food and luminosity. For if peptones are present in sufficient quantity the phosphorescence is considerably

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[1]) De Ingenieur, 15e Jaarg. Pag. 53, 27 Januari 1900.

[2]) Mémoire sur la phosphorescence de quelques invertebrés marins. Ann. d. sc. nat. Zoologie, 3me Sér. T. 14. Pag. 326, 1850. Vide also R. DUBOIS. Leçons de Physiologie générale, Pag. 498, Paris 1898.

increased by several carbon compounds either free from or containing nitrogen, as glucose, levulose, glycerin, malates, asparagin, and many others that do not act as stimuli, but as in the normal respiratory process are oxidised to carbonic acid and water. Peptones alone can also be broken off by the photoplasm, likewise under production of ammonium carbonate, carbonic acid, and water. Phosphorescence thus proves to be bound to the photoplasm in the same way as the respiratory process in general is bound to the protoplasm, so that it may be said that the photoplasm of the luminous bacteria forms part of their respiration protoplasm.

As now the chief criterion of enzyme action consists in the fact that enzymes act only on a specific substrate, in the case of phosphorescence this criterion at first sight seems to fail, and the process more reminds of a catabolism bound to the protoplasm as a whole and which is rather unanalysable.

But considering what should be understood by a catabolism we find in many cases that it is based on the co-operation of various factors of the nature of enzymes. The respiratory process itself supports this view, for recent enzymological investigations have shown that the respiration protoplasm is composed of different factors, in general called oxidases, with the specific distinction of peroxidases, oxigenases and oxidones.

These units possessing the character of enzymes, and only oxidising special substances, or but few nearly related ones, we must accept that in this case, too, a preformation of enzyme-substrates or enzymoteels takes place on which they exert their function. The composition of the photoplasm of several of such factors or oxidases is thereby rendered probable, and the ease wherewith by means of mutation experiments with the luminous microbes hereditary constant races arise of very unequal phosphorescence (but as it seems always of the same colour), is evidently connected with these facts.

That the factors of the photoplasm of the various species of luminous bacteria are not always the same follows from the before described experiments about the relation between nutrition and phosphorescence. ¹)

So, in the photoplasm of *Bacterium phosphoreum* an oxidase must exist associated with a substrate resulting from peptones only, and another oxidase whose substrate is an unknown matter, produced by peptone and sugars and perhaps by peptone and glycerin too. In the photoplasm of *Bacterium splendidum* another factor occurs

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¹) For *Ph. phosphoreum*, Aliment photogène, Archives Néerl. 1851. For *Ph splendidum,* Folia microb. 1915.

adapted to a still unknown substrate deriving from peptone and mannite. Really these still hypothetical substrates are but different "luciferines" in the sense of DUBOIS. It should be borne in mind here that DUBOIS knows nothing at all of his luciferine of the pholades, whereas regarding the photobacteria at least the substances are known from which they result.

By multiplying the nutrition experiments it will be possible to come to a complete "factor analysis" of the photoplasm. For other bacteria the difficulties will be greater, but for *B. prodigiosum*, where race formation easily occurs, a corresponding factor analysis of the "chromoplasm" will be possible, since, according to former demonstrations, it must quite like the photoplasm be regarded as a complex of heredity units possessing the character of oxidases.

So we arrive also here at a result analogous to that already obtained for the alcohol function, which may be ascribed as well to "alcohol protoplasm" as to some enzymes, the zymase of BÜCHNER.

In consequence of the foregoing it is clear that conceptions such as "chromoplasm", "photoplasm", "alcoholprotoplasm" etc., are not in contradiction with the wider view that considers the protoplasm in general as composed of enzymes, as they themselves are built up of these.

There being nothing to object to the further generalisation of the view here forwarded, it is allowed to consider the heredity units as enzymes and these as heredity units, clearly two different names for the molecules or micells of the living part of the protoplasm.[1])

Cell-wallfactors are enzymes.

For the higher plants and animals factor analysis is based on crossing experiments between forms of which we wish to state by what and by how many heredity units they differ. For the bacteria and the other microbes, where for want of sexuality crossing is impossible, factor analysis is then possible when the factors of special properties can be recognised by race formation through mutation, which I already put forward before. The recognition of the heredity units as enzymes may likewise lead to factor analysis by applying the property of enzymes only to act on special substances.

We saw how this principle may be applied to a physiological

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[1]) This theory I first advanced, though with some doubt, in: Mutation bei Mikroben, Folia microbiologica, Bd. 1, Pag. 2, 1912, but now the difficulties are overcome.

function; that it can likewise lead to the factor analysis of a morphological character I will now endeavour to show with regard to the cell-wall.

The formation of the cell-wall is commonly considered as a function of the parietal protoplasm and must necessarily repose on the action of factors or heredity units. For some microbes this process is clearly caused by one or more enzymes and this is distinctly the case when the wall substance consists of levulan. This matter results from cane-sugar (and slower and less profusely from raffinose), but from no other substances. It forms the cell-wall of many species of sporulating bacteria, such as *B. megatherium* and also the common hay bacterium *B. mesentericus,* but only if fed with cane-sugar. The levulan arises in two ways: it either remains in contact and entirely united with the bacterial body as a slimy cell-wall, in which case on cane-sugar-agar plates strongly swelling colonies develop, or the levulan is deposited outside the bacterial body at some distance from the colony. If the latter takes place the remarkable reaction occurs which I have called the "emulsion reaction".¹) Its explanation was given by the discovery of a specific exoenzyme, viscosaccharase, which acts on cane-sugar and converts it into levulan slime, which is incapable of diffusion but attracts water, so that droplets are formed causing a strong swelling of the agar. This enzyme, acting synthetically and evidently polymerising the cane-sugar, might as well be called saccharo-levulanase and is obviously one factor of the factor-complex that governs the cell-wall formation. That it is not the only one follows from the fact that some levulan bacteria, for instance the hay bacterium itself, when fed with other sugars, produce another not slimy wall-substance, probably cellulose, which likewise derives from cane-sugar beside levulan, but only in slight quantity. If the production of cellulose is brought about by one or more factors is not yet known. As to the viscosaccharase, however, there is not the least doubt but that it consists of one single enzyme or factor.

Hence it may be concluded that it is quite well possible to become acquainted with the separate factors of a process at first sight so complicated as the formation of the cell-wall, and it may safely be predicted that further experiments will show whether the cellulose production also depends on one single or on more than one enzyme.

On the other hand, at the factor analysis by crossing experiments with higher plants and animals, without the guidance of the enzyme conception, we are continually in doubt whether a factor, thought to

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¹) These proceedings 9 February and 2 Mei 1910. Folia microbiologica Bd. 1 Pag. 382, 1912.

be elementary, will not, on continued examination, prove to be composed of other still unknown factors.

As to dextran I have stated elsewhere [1]) that it is a wall substance comparable to levulan, likewise only resulting from cane-sugar, but produced by some lactic acid ferments, belonging to the physiological genus *Lactococcus.* Dextran, however, never originates independently from the cell, as may occur with levulan, but exclusively at the surface of the outer layer of the protoplasm and in direct contact with it. But the knowledge of the relation between levulan and its producing enzyme, viscosaccharase, indicates clearly that dextran, whose properties are so analogous to those of levulan, must have a similar origin. It is therefore most probable that dextran also arises under the influence of one single factor or specific enzyme, which might be called saccharo-dextranase, but which, being an endoenzyme, cannot leave the cell.

The formation of the slime wall by *B. prodigiosum viscosum* [2]) must be brought about by at least two factors, differing from levulanase and dextranase since the slime produced by this bacterium, belongs to the celluloses or cellulan-slimes. That beside the slime factor, which might be called cellulanase and which produces cellulan from carbohydrates, still quite another factor operates here is proved by the following observations. By feeding this bacterium with glucose, cane-sugar, maltose or lactose, wall slime is readily yielded. In several other species, for instance *Aerobacter viscosus* and *Bacillus polymyxa* we find the same. But *B. prodigiosum* can besides produce slime from albuminous substances such as gelatin and peptone, which *B. polymyxa* and *A. viscosus* cannot. As now it is quite unacceptable that one and the same factor could be able to produce cellulose slime as well from proteids as from carbohydrates, *B. prodigiosum* must possess a specific factor able to split off from the albuminous matter an enzyme-substrate, converted into cellulose slime by the wall-forming factor. But this proteid-splitting factor does not exist in *B. polymyxa* and *A. viscosum. B. prodigiosum viscosum* is thus a mutant, distinct by at least two factors from *B. prodigiosum* itself, which produces no slime at all, neither from carbohydrates nor from proteids. It must thus be possible to detect another still unknown mutant lacking the factor to produce from proteids a substrate that

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[1]) Die durch Bakterien aus Rohrzucker erzeugten Wandstoffe. Folia microbiologica. Bd. 1. Page 392, 1912.

[2]) *B. prodigiosum viscosum* is no natural form but a mutant or race, easily obtained from *B. prodigiosum.* Folia microbiologica, Bd. 1, Pag. 35, 1912.

can be converted into slime, that is a mutant capable to produce slime from carbohydrates only and not from proteids.

A great number of other examples might be added demonstrating that the speculations about the heredity units or factors have relation to enzymes.

Limitation of the enzyme conception.

In my opinion the preceding may lead to a better enzyme conception than the existing. I will try to elucidate this by a few instances taken from the cecidia or galls and the substances called ferments in immunology.

Elsewhere I pointed out that the change of the plant at gall-formation is not hereditary. From the galls of *Nematus viminalis*, kept on moist sand, quite normal roots of the gall-bearer *Salix purpurea*, and from those of the gall-fly *Neuroterus lenticularis* on oak-leaves, quite normal oak roots may arise.[1].

From the axil-buds of the willow-rose, caused by *Cecidomya rosaria* on *Salix alba*, I have cultivated quite normal willow trees; likewise I grew normal plants of *Poa nemoralis* from the bud in the remarkable gall of *Cecidomya poae*, whose strange metamorphic roots readily develop into normal roots, when the whole gall is planted in earth.[2]) By strongly pruning the twigs of *Rosa canina* whereon Bedeguars developed, caused by the gall-fly *Rhodites rosae*, the wonderful appendices of this gall changed into long-petiolated, simple, green leaflets, whose anatomic structure and external appearance were quite identic with those of the leaf on which the gall originates.

These instances, to which I could easily add others, show that in the formation of galls two groups of substances are concerned: the protoplasm of the plant, consisting of the unchanged heredity units, and substances deriving from the egg of the gall-animal, or from the larva of *Cecidomyia*, which evidently have the character of enzymesubstrates. It is however clear that the heredity units concerned in the morphologically higher galls, multiply more intensely, in any case become more numerous under the influence of the gall-animal than under normal circumstances. Hence we come to the conclusion that either the enzyme-substrates may serve as food for the heredity units or enzymes to which they belong and may give rise to their multiplication, or that the gall-animal, beside the enzyme

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[1]) Only very few *Lenticularis*galls possess this disposition, which is probably connected with the spot where the gall grows on the leaf.

[2]) Botanische Zeitung 1886.

substrate, also supplies "enzymosites",[1]) that is to say a special "enzyme food". The latter supposition will probably be the right one, for the real enzymes are in their origin in no way dependent on their substrates, as we learn from almost every experiment with microbes.[2])

The enzymosites apparently correspond to ABDERHALDEN's "Bausteine" of the specific living proteids, that is of the protoplasm. That, in case these enzymosites differ, different heredity units or protoplasm micells will develop from the mixture of units from which the latter is built up, is to be expected. For if we remember in how remarkable a way in elective culture experiments with microbes, the thereby obtained floras depend on nutrition, we may safely conclude that the same will be the case in the subtle world of protoplasm molecules.

That from the gall-animal no enzymes pass into the plant, is in accordance with the fact that foreign exoenzymes commonly do not enter living cells. The diastase, which in the distilleries occurs in great quantity in the food of yeast, which consists for a great part of malt, does not penetrate into the yeast-cell. Experiments purposely carried out with other exoenzymes and various kinds of other microbes have invariably given the same result. The possibility of endoenzymes passing by diffusion from one living cell into another is of course wholly excluded.[3])

On the other hand, in the range of immunology, facts are known which prove that living cells sometimes take up enzymes from their surroundings.

In those cases namely when acquired immunity is hereditary the thereby concerned substances must needs belong to the heredity units, hence to the enzymes.

They give evidence that DARWIN's view, according to which the "gemmules" of his pangenesis hypothesis freely move within the

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[1]) Sitos, food.

[2]) Many diastatic bacteria for example produce diastase without the presence of amylum in their food. This must be ascertained by a special experiment, amylum being the only known reactive on diastase; the literature proves that this has sometimes been forgotten by the investigators.

[3]) It is not impossible that endoenzymes such as zymase are to some degree capable of ordinary diffusion (which is quite another thing than penetrating into living protoplasm). Gelatin can slightly penetrate into agar, likewise starch and even the carbon of Indian ink. Gold seems able to penetrate into lead. In the protoplasm of luminous bacteria no disposition for diffusion is to be observed. However the pathological light of *Noctiluca miliaris*, described by DE QUATREFAGES, seems to repose on the entering of the photoplasm or luciferase into the cell-sap in which the luciferine must then be dissolved,

organism, is true in certain cases, at least for the higher animals. Non-hereditary immunity might be caused by freely moving enzymes, unable to enter the cells.

Van Calcar's opinion that the anti-bodies of the serologists are ferments, that is enzymes, is thus undoubtedly right. He says: [1]) Whichever immunity reaction is examined, it is constantly found that the whole course of these reactions depends on the action of two substances, one of which having in all respects the character of a ferment, the other that of an enzyme-substrate to be decomposed by that ferment. The ferment-like substances are called "anti-bodies", the various substrates they act upon, "antigens".

In my opinion there is however no sufficient ground also to call the antigens and the complement "enzymes", as is done by several investigators.

If these substances are considered as enzymes only because of their action after injection into the blood of higher animals, it will be necessary, in order to be consistent, likewise to bring to the enzymes toxins and even some common coagulable proteids, which would make this word lose its real significance. Whereas in the descriptive sciences the necessity is felt to designate by special names even but slightly differing objects, it would be an error to attribute to the words enzyme and ferment a continually varying and wider meaning no more in accordance with the original conception. On the other hand it is clear that further knowledge about the enzymes or factors may necessitate the creation of new names to mark the vast differences between them, as now we are already compelled to use the words exo- and endoenzymes.

There is still another group of bodies worth being considered from the new point of view, namely the viri in general and in particular those of plant diseases, such as the mosaic disease of the tobacco. They clearly belong to the enzymes or factors, although commonly not hereditarily transported. But the further discussion of this point must be deferred to later.

The only place in literature, hitherto come to my knowledge, where an hypothesis is indicated somewhat corresponding to my view, is to be found in Bateson. He says [2]): "Ueber die physikalische Natur der Erbeinheiten können wir noch nichts aussagen; die Folgeerscheinungen ihrer Gegenwart sind aber in so vielen Fällen

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[1]) R. P. van Calcar, Voordrachten over algemeene biologie, Pag 182 and 188, Leiden 1915.

[2]) W. Bateson, Mendel's Vererbungstheorien, Pag. 269, 1914 (Translation of the English edition of 1909).

mit den durch Fermente hervorgerufenen Wirkungen vergleichbar, dass wir mit einiger Bestimmtheit annehmen, dass die Fähigkeiten einiger Erbeinheiten im wesentlichen in der Bildung bestimmter Substanzen bestehen, welche in der Art von Fermenten wirken".

Although the observations on which this statement is based are in accordance with the enzyme theory, it is clear that Bateson's view is quite different from mine.

Physics. — "*Contributions to the kinetic theory of solids.* I. *The thermal pressure of isotropic solids.* By Prof. L. S. Ornstein and Dr. F. Zernike. (Communicated by Prof. H. A. Lorentz).

(Communicated in the meeting of February 26, 1916).

P. Debije[1]) has in his Wolfskehl-lecture developed a theory of the equation of state of solid matter which has been elaborated by Dr. M. I. M. van Everdingen[2]). Debije assumes as a physical principle that the forces between the molecules in solid matter are not quasi-elastic, but depend also on higher powers of the deformations. He points out that only this principle enables us to understand the expansion of solid matter which gains energy under constant pressure. This assumption enables him to give a deduction of the Grüneisen-theorem about the connection between the coefficient of expansion and the specific heat.

Debije calculates the free energy of a solid body with the help of a canonical ensemble, using the method of normal vibrations, and introducing from the beginning the hypothesis of energy-quanta.

We shall indicate in this paper another way to find the equation of state with the aid of the physical principles of Debije. The quantum-theory will be applied to our final result if we wish to use it for low temperatures. Debije has taught us to replace in the calculations the space-lattice of molecules by a continuum, Born[3]) has shown this artifice to be right. Therefore, in considering the isotropic body, we shall use a continuum as a limiting case. For explanation we shall treat the case of a row of points and for this case we shall perform the transition to a continuous bar. Our method consists in determining the thermal pressure, i.e. the pressure that

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[1]) Vorträge über die kinetische Theorie der Materie, Leipzig 1914. "Zustandsgleichung und Quantenhypothese u. s. w.".

[2]) De toestandsvergelijkingen van het isotrope vaste lichaam. Diss. Utrecht 1914.

[3]) M. Born. Dynamik der Krystallgitter. Teubner. 1915.

is required to keep constant the volume of a solid body when gaining heat.

1. Let us consider a row of n equidistant points. Be the elongation in the direction x (the direction of the row being taken as an axis) for the ν^{th} point ξ_ν. Then the force exerted by the ν^{th} molecule on the $(\nu-1)^{\text{th}}$ will be represented by

$$S=f(\xi_\nu-\xi_{\nu-1})+\frac{g}{2}(\xi_\nu-\xi_{\nu-1})^2 \quad . \; . \; . \; . \; . \; . \quad (1)$$

The total potential energy, then, can be represented by

$$\varepsilon_q=\frac{f}{2}\,\Sigma_\nu(\xi_\nu-\xi_{\nu-1})^2+\frac{g}{6}\,\Sigma_\nu(\xi_\nu-\xi_{\nu-1})^3 \quad . \; . \; . \; . \; . \quad (2)$$

where the sum has to be extended over all molecules.

Now for a stationary state, $\overline{S}$, the time-average of S, will be equal for all points. Therefore, adding (1) for all points, we get n times the time-average of S. Thus

$$n\,\overline{S}=\frac{g}{2}\,\Sigma\,\overline{(\xi_\nu-\xi_{\nu-1})^2}=\frac{g}{2}\,\overline{\Sigma\,(\xi_\nu-\xi_{\nu-1})^2} \quad . \; . \; . \; . \; . \quad (3)$$

the mean of the first term in (1) being zero, as the mean length is invariable, and as the taking of the mean and of the sum may be interchanged.

For the mean value of ε_q we have

$$\overline{\varepsilon_q}=\frac{f}{2}\,\overline{\Sigma_\nu(\xi_\nu-\xi_{\nu-1})^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

the mean value of the second term being zero. We thus find

$$n\,\overline{S}=\frac{g}{f}\,\overline{\varepsilon_q}=\frac{g}{2f}\,\overline{\varepsilon}$$

for $\overline{\varepsilon_q}=\overline{\varepsilon_p}=\frac{1}{2}\,\overline{\varepsilon}$, where ε_p represents the kinetic and ε the total energy. Putting $g=n^2c_2$, $f=nc_1$, we find

$$\overline{S}=\frac{c_2}{2c_1}\,\overline{\varepsilon} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

For the dilatation taken from the absolute zero, we find

$$e=\frac{c_2}{2c_1^{\,2}}\,\overline{\varepsilon} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

being the relation of GRÜNEISEN.

2. We shall now consider the same problem, approximating this time the problem for a row of points by that of a continuum. Therefore we have to do with a bar in which the elastic qualities depart from HOOKE's law,

The force exercised in this bar by the part to the right of a section on the part to the left, will be represented by

$$S = c_1 \frac{\partial \xi}{\partial x} + \frac{c_2}{2} \left(\frac{\partial \xi}{\partial x}\right)^2 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

The total potential energy then amounts to

$$\varepsilon_q = \frac{c_1}{2} \int \left(\frac{\partial \xi}{\partial x}\right)^2 dx + \frac{e_2}{6} \int \left(\frac{\partial \xi}{\partial x}\right)^3 dx \quad . \; . \; . \; . \; . \; . \quad (8)$$

where the integration has to be extended over the length of the bar, which has been put equal to unity.

For this case the mean value $\overline{S}$ of the force is again equal for all points of the bar, and $\overline{\frac{\partial \xi}{\partial x}}$ being zero, we have

$$\overline{S} = \frac{c_2}{2} \overline{\left(\frac{\partial \xi}{\partial x}\right)^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

Integrating this result over the bar, we get

$$\overline{S} = \frac{c_2}{2} \int \overline{\left(\frac{\partial \xi}{\partial x}\right)}^2 dx = \frac{c_2}{2} \overline{\int \left(\frac{\partial \xi}{\partial x}\right)^2 dx}. \quad . \; . \; . \; . \quad (10)$$

as also in this case the integrating and the taking of the mean value may be interchanged.

Now, just as in the discrete problem, we determine ε_q, and $\overline{\left(\frac{\partial \xi}{\partial x}\right)}$ being zero we find

$$\overline{\varepsilon_q} = \frac{c_1}{2} \overline{\int \left(\frac{\partial \xi}{d x}\right)^2 dx} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

and from it

$$\overline{S} = \frac{c_2}{c_1} \overline{\varepsilon_q} = \frac{c_2}{2 c_1} \overline{\varepsilon} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

from which the relation of GRÜNEISEN again follows.

In calculating $\overline{\varepsilon}$ we can use the quantum-theory, but the formula (12) is evidently independent of it.

3. We will show that the same result is obtained by applying the method of normal vibrations. The differential equation for the motion of the bar is expressed by

$$\varrho \frac{\partial^2 \xi}{\partial t_2} = c_1 \frac{\partial^2 \xi}{\partial x^2} + c_2 \frac{\partial \xi}{\partial x} \frac{\partial^2 \xi}{\partial x^2}. \quad . \; . \; . \; . \; . \; . \; . \quad (14)$$

where ϱ represents the density. Properly speaking the equation (14), being non-linear, possesses no normal vibrations as a solution for a given bar with given conditions to be satisfied at the ends. But if c_2 is small enough, the normal vibrations of the equation without quadratic term will still have a physical meaning. These vibrations may be called quasi-normal vibrations, and the physical meaning of the constant c_2 is to effect a slow exchange of energy between the quasi-normal vibrations.

Now take the solution

$$\xi = \Sigma_k \left(P_k \sin kx + Q_k \cos kx\right) \cos kv\,(t - \varphi_k) \quad . \quad . \quad . \quad (15)$$

for a bar with the ends 0 and 2π; v being the velocity of propagation. The force in a point is represented by (7). Using (15) and calculating the time-average of S in the point 0, we find

$$\overline{S} = \frac{c^2}{4} \Sigma\, k^3\, P_k^2$$

The potential energy is expressed by

$$\varepsilon_q = \frac{2\pi\, c_1}{4} \Sigma\, k^2 \left(P_k^2 + Q_k^2\right) \cos^2 kv\,(t - \varphi_k)$$

its time-average by

$$\overline{\varepsilon_q} = \frac{2\pi\, c_1}{8} \Sigma\, k^2 \left(P_k^2 + Q_k^2\right)$$

Now the mean value of P_k is the same as that of Q_k; therefore we get

$$\overline{\varepsilon_q} = \frac{2\pi\, c_1}{4} \Sigma\, k^2\, P_k^{\ 2}$$

$$\overline{S} = \frac{c_2}{2\pi\, c_1} \overline{\varepsilon_q} = \frac{c_2}{2c_1} \frac{\overline{\varepsilon}}{2\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

As $\frac{\varepsilon}{2\pi}$ is equal to the energy per unit of length, the result agrees with (5) and (12).

4. We can determine the thermal pressure of an isotropic solid body in the same way as in 2. For this case, we have for the potential energy per unit of volume the expression [1])

$$\varepsilon_q = AI_1^{\,2} + BI_2 + CI_1^{\,3} + DI_1I_2 + EI_3 . \quad . \quad . \quad . \quad (17)$$

where the invariants I have the following forms

$$I_1 = e_1 + e_2 + e_3$$

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[1]) For the first time indicated by J. Finger, Wiener Sitzungsberichte **103**, 163 (1894), although in a less simple form (l. c. form (55)). Our notation is the one of v. Everdingen, l. c. p. 11, where no literature is mentioned. Cf. also P. Duhem, Recherches sur l'Elasticité, Paris 1906.

$$I_2 = e_2e_3 + e_3e_1 + e_1e_2 - \tfrac{1}{4}(e_4^2 + e_5^2 + e_6^2)$$
$$I_3 = e_1e_2e_3 + \tfrac{1}{4}(e_4e_5e_6 - e_1e_4^2 - e_2e_5^2 - e_3e_6^2)$$

$e_1 \dots e_6$ being the "components" of the strain (changes in length and angle).

The energy of a volume is found by multiplying the expression (17) with the element $d\tau$, and integrating.

From (17) the normal stress in the direction of x can be deduced, using the formula

$$S = \frac{\partial \varepsilon_q}{\partial e_1}\,{}^{1)}$$

We can only observe the time-average of this force and, taking the mean value, the linear parts issuing from the terms with A and B will fall out. We obtain therefore for the average value of the tension in the direction of x

$$\overline{S} = \overline{3CI_1^2 + DI_1(e_2 + e_3) + DI_2 + E(e_2e_3 - \tfrac{1}{4}e_4^2)}\,.$$

This force is again equal for all points, we can therefore integrate over the volume 1, and interchange the integration and the taking of the average. Taking into account the isotropy, it is easily seen that $\overline{e_2e_3 - \frac{1}{4}e_4^2}$ is equal to $\frac{1}{3}\overline{I_2}$, so that we get for $\overline{S}$

$$\overline{S} = \overline{\int \{(3C + \tfrac{2}{3}D)\, I_1^2 + (D + \tfrac{1}{3}E)\, I_3\}\, d\tau} \quad . \; . \; . \quad (18)$$

Now determining the mean value of the potential energy, the terms with C, D, and E will be found to fall out, and we get two parts, relating respectively to the longitudinal and transverse waves, as appears from the meaning of the invariants I_1 and I_2. These parts are

$$\overline{\varepsilon}_{q\,l} = A\overline{\int I_1^2\, d\tau} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (19a)$$

and

$$\overline{\varepsilon}_{q\,tr} = B\overline{\int I_2\, d\tau} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (19b)$$

In the stationary state the potential energy is distributed in a given way over these waves.

For the thermal pressure we now find

$$-\overline{S} = -\frac{3C+\frac{2}{3}D}{A}\,\overline{\varepsilon}_{q\,l} - \frac{D+\frac{1}{3}E}{B}\,\overline{\varepsilon}_{q\,tr} \quad . \; . \; . \; . \quad (20)$$

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1) This is the usual formula, which is, however, not correct if the second powers of the deformations are taken into account, as has been done in the above by introducing C, D, and E. In the third contribution we shall show that even in case of HOOKE's-law being true, a coefficient of expansion will be found, if the correct formula for S is used. As far as numerical values are known, they seem to indicate that the influence of the terms neglected here could sometimes be sensible.

In the notation of VOIGT we have $A = \frac{1}{2} c_{11}$, $B = -2c_{44}$.

If the temperature is high enough for the theorem of equipartition to be true, then

$$\overline{\varepsilon}_{q\,l} = \tfrac{1}{6}\,\varepsilon \text{ and } \overline{\varepsilon}_{q\,tr} = \tfrac{1}{3}\,\varepsilon$$

where ε is the total energy.

For the thermal pressure we then find

$$-\overline{S} = -\left\{\frac{9C+2D}{18A} + \frac{3D+E}{9B}\right\}\varepsilon \quad . \quad . \quad . \quad . \quad (20a)$$

We shall also use (20) for very low temperatures. According to BORN [1]) the proportion of the energy of the longitudinal and transversal waves can be put in the form

$$\varepsilon_{q\,l} : \varepsilon_{q\,tr} = \frac{1}{v_l^3} : \frac{2}{v_{tr}^3}$$

where v_l and v_{tr} are the velocity of propagation of these waves. Introducing the constants c_{11} and c_{44}, we thus have

$$\varepsilon_{q\,l} : \varepsilon_{q\,tr} = c_{44}^{3/2} : 2\,c_{11}^{3/2} .$$

Putting the total energy ε, we find

$$\varepsilon_{q\,l} = \frac{c_{44}^{3/2}}{2\,c_{44}^{3/2} + 4\,c_{11}^{3/2}}\,\varepsilon \qquad \varepsilon_{q\,tr} = \frac{2\,c_{11}^{3/2}}{2\,c_{44}^{3/2} + 4\,c_{11}^{3/2}}$$

and finally

$$\overline{S} = \frac{(3c + \frac{2}{3}D)\,c_{44}^{5/2} - (\frac{1}{2}D + \frac{1}{6}E)\,c_{11}^{5/2}}{c_{44}c_{11}\,(c_{44}^{3/2} + 2c_{11}^{3/2})}\,\varepsilon \quad . \quad . \quad . \quad . \quad (20b)$$

This special result agrees with the expression found by VAN EVERDINGEN [2]). The theorem of GRÜNEISEN can be immediately deduced from it.

The influence of temperature on the elastic constants can be examined in the same way, as we shall show in the third contribution.

Utrecht, Febr. 1916. *Institute for mathematical physics.*

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[1]) BORN l. c. p. 75.

[2]) VAN EVERDINGEN, l. c. p. 24 form (20) p. 53 form (37).

Physics. *"Contributions to the kinetic theory of solids. II. The unimpeded spreading of heat even in case of deviations from* HOOKE'S *law".* By Prof. L. S. ORNSTEIN and Dr. F. ZERNIKE. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of March 25, 1916.)

1. In a supplement to his lecture on the equation of state of the solid body, DEBIJE[1]) has endeavoured to "make a qualitative theoretical calculation of the coefficient of conduction of heat." There the author points out repeatedly that his estimations are only to be taken very approximately and should serve as a first orientation only. So, as we tried to obtain an accurate calculation of the conduction of heat, it did not seem desirable to us to deal with the problem in exactly the same way and to carry out only here and there some corrections and completions.

Now DEBIJE'S principle, which we therefore intended to work out otherwise, runs as follows. In an ideal solid body, i.e. a solid for which the elastic equations would be linear, various progressive waves may exist independently of each other, like the electromagnetic waves in a field of radiation. This implies that a heat-motion occurring on one side of the solid spreads unimpededly through the solid, so that the density of energy becomes equal in all parts of the solid. If the solid is in a stationary state, the temperature will thus be everywhere the same, even if continually a current of energy moves through the solid in a definite direction. Hence DEBIJE emphasizes this dictum: *the coefficient of heat conduction of the ideal solid body is infinitely great* (l.c. § 7, cf. the statement given there). Now in several regards it is preferable to formulate the rule in this way: the ideal solid body does not show any resistance of heat.

That a real solid body does show resistance of heat DEBIJE ascribes to the fact that the elastic equations are not perfectly linear. Therefore various normal vibrations strictly cannot be superposed and it is conceivable that waves running in different directions so to say oppose each other. DEBIJE has indeed succeeded in deducing indirectly a scattering and consequently a suppression of the running waves.

Our endeavours to state more directly the connection between resistance of heat and non-linear terms of the elastic equations of motion have failed. Therefore we will not report our considerations

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[1]) Mathematische Vorlesungen an der Universität Göttingen VI (WOLFSKEHL-Vorträge) pg. 19.

on that matter, except a small example in (2), but we will only mention how we have formulated the problem because that is important for another reason. First the problem was limited to one dimension for the sake of simplicity. Where DEBIJE already introduced the great simplification of considering longitudinal waves only, it is easy to understand that for a more strict calculation it is necessary to go as far as to consider provisionally only such waves as run in a special direction. Let us e.g. consider the longitudinal waves of a thin bar. The equation of motion for this case is *strictly linear* if the law of HOOKE is accepted. Generally this has been accepted, also in the case of the for the rest quite analogous discrete problem of a series of molecules with elastic forces between them. Therefore we may always represent the movement of the bar by a superposition of *normal-vibrations.* This is done in the case of the well-known calculations of the specific heat. It is necessary to suppose hereby a special statistical distribution of the energy over the different vibrations. At a high temperature e.g. one is inclined to take equipartition for it without troubling about how it occurs. But it is remarkable that in this case there can be no question of equipartition establishing itself as it occurs in a gas by the collisions. For during the movement the energy of each *normal-vibration* remains constant so that any method of division continues to exist permanently. We may put it statistically like this: time-ensemble and microcanonical ensemble are very different from each other and consequently are not practically equivalent.

This difficulty, which is essentially connected with the existence of *normal-vibrations* of the system, disappears as soon as non-linear terms of higher order are introduced into the equation of motion. When these terms are very small, and this is sufficient, then we can speak of the *quasi-normal vibrations* of the system. During a short time the quasi-normal-vibrations behave at first approximation like normal-vibrations. The non-linear terms however bring about *a slow exchange of energy between the quasi-normal-vibrations.* We intend to revert later on to the calculations regarding this subject which we have performed. As we mentioned already they did not produce a valuable result for the theory of heat conduction. It remains still to be examined which *other* phenomenon corresponds with the molecular action mentioned. Probably it will also be possible to prove later that by the terms mentioned the system will slowly approach the condition of equi-partition.

2. In contrast to the calculations mentioned above, we shall record

in this communication some positive results showing the non-existence of a heat-resistance, even if deviations from HOOKE's law appear. We shall confine ourselves to the linear problem, so that it still remains uncertain whether our conclusion holds really for three dimensions. This forces us to enter still into some principal errors in DEBIJE's calculation, which cause his results to be not only merely qualitative, but quite illusory in our opinion, so that they are no argument against the mentioned extension of our conclusion.

In consequence of his calculations on the linear problem, SCHRÖDINGER[1]) has doubted of the conduction of heat being infinitely great in the ideal case. Let us therefore in the first place mention a simple proof for this case.

Let x denote the place of a point on the bar in condition of rest, $x+\xi$ the same after deformation. According to HOOKE's law the stress would become proportional to $\frac{\partial \xi}{\partial x}$. As a next approximation we shall now put this proportional to $\frac{\partial \xi}{\partial x}+\frac{\alpha}{2}\left(\frac{\partial \xi}{\partial x}\right)^2$. The equation of motion now becomes:

$$\frac{\partial^2 \xi}{\partial t^2}=\frac{\partial^2 \xi}{\partial x^2}+\alpha\,\frac{\partial \xi}{\partial x}\cdot\frac{\partial^2 \xi}{\partial x^2}\quad \ldots\ldots\ldots \quad (1)$$

if for the moment we take unimportant coefficients $=1$.

If we represent for the sake of brevity the differential coefficients of ξ by $\dot{\xi}$ and ξ', the energy per unit of length is:

$$\varepsilon=\varepsilon_p+\varepsilon_q=\tfrac{1}{2}\,\dot{\xi}^2+\tfrac{1}{2}\,\xi'^2+\frac{\alpha}{6}\,\xi'^3.$$

In unit of time the tension of the bar performs in the point x a work

$$\dot{\xi}\left(\xi'+\frac{\alpha}{2}\,\xi'^2\right).$$

Now if a current of energy goes through the bar (conduction of heat) the time-average of this expression must differ from zero. Now we will examine whether a fall of temperature appears along the bar. For this purpose we calculate:

$$\frac{\partial \varepsilon}{\partial x}=\dot{\xi}\dot{\xi}'+\xi'\xi''+\frac{\alpha}{2}\,\xi'^2\xi''.$$

Applying the equation of motion we get:

$$\frac{\partial \varepsilon}{\partial x}=\dot{\xi}\dot{\xi}'+\xi'\ddot{\xi}''-\frac{\alpha}{2}\,\xi'^2\xi''=\frac{\partial}{\partial t}(\dot{\xi}\xi')-\frac{\alpha}{2}\,\xi'^2\xi''$$

We shall take the time average of the last form for a stationary

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[1]) Ann. de Physik. 42. 1914 p. 916.

state of motion that is to say: integrate that expression with respect to a long time T and divide the result by T. The contribution of the first term can be made as small as one wants by taking T larger and larger. So there remains:

$$\frac{\overline{\partial \varepsilon}}{\partial x} = \frac{\partial \overline{\varepsilon}}{\partial x} = -\frac{\alpha}{2}\overline{\xi'^2\xi''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The average energy is thus the same for all points of the bar if $\alpha = 0$ i. e. if HOOKE's law is satisfied, and this quite independent of the heat-current:

$$\overline{\dot{\xi}\left(\xi' + \frac{\alpha}{2}\xi'^3\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now, there appears to be a connection between the average (3) and the same in (2), causing for $\alpha \neq 0$ a difference in temperature proportional to the magnitude of the heat-current. However, that this connection does not exist appears in another way from the following considerations.

3. In direct connection with the treatment published by DEBIJE (l. c. § 9) we can examine the influence of deviations of density and elasticity on the vibrating movements of our bar. With density ϱ and elastic modulus E the equation of motion becomes:

$$\varrho \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial}{\partial x}\left(E \frac{\partial \xi}{\partial x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

as in this method the terms of higher order are not directly taken into account. Further we suppose that ϱ and E have everywhere on the bar the constant values ϱ_0 and E_0 except on a small part, from 0 to l, where they are $\varrho_0 + \varrho_1$ and $E_0 + E_1$. Then we can calculate in the usual way the secondary waves, appearing when primarily a wave $Ae^{ip(x-qt)}$ runs along the bar, for which

$$y = \sqrt{\frac{E_0}{\varrho_0}}$$

is the velocity of propagation. We shall only mention the result. From the disturbed little element 0 there goes in negative direction to l a wave

$$iAlp\,\frac{E_0\varrho_1 + E_1\varrho_0}{2E_0\varrho_0}\,e^{ip(x+qt)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and in positive direction a wave

$$iAlp\,\frac{\varrho_0 E_1 - E_0\varrho_1}{2E_0\varrho_0}\,e^{ip(x-qt)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This latter one has to be added to the primary wave. This gives a change of phase of that wave and it is easy to see that it corresponds with the change which one would deduce from the changed velocity of propagation between 0 and l. For our purpose only the "dispersed" wave (5) is of importance, so (5) is the only dispersed wave which is left. It will disappear if $\varrho_0 E_1 + E_0 \varrho_1 = 0$.

Now we must imagine the deviations in part 0 to l as caused by elastic deformation of the bar which is homogeneous in the condition of rest. One can imagine that for this purpose constant forces are used. Afterwards we will come back to the question whether the action of the accidental deviations of density can be described correctly in this way. In any case the problem is now sharply determined. It quite corresponds with the analogous treatment of DEBIJE.

Let a piece of the length of l_0 be stretched to l. The tension required therefore we will represent by the formula

$$S = E_0 \frac{l-l_0}{l_0} + \frac{a}{2} E_0 \left(\frac{l-l_0}{l_0}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in which again a denotes the deviation from HOOKE's law. In the new condition a small increase of length will bring about a change of tension, which can be calculated from a modulus E, if we take

$$E = l \frac{dS}{dl} = E_0 \frac{l}{l_0} + aE_0 \frac{l}{l_0} \cdot \frac{l-l_0}{l_0}$$

For the change E_1 of the modulus on the small portion between 0 and l we find approximately

$$\frac{E_1}{E_0} = \frac{l-l_0}{l_0}(1+a) = -\frac{\varrho_1}{\varrho_0}(1+a)$$

as the density ϱ changes inversely proportional to l. From this result follows that really the wave (5) disappears if the law of HOOKE is fulfilled. For the fraction in (5) one finds:

$$\frac{\varrho_0 E_1 + E_0 \varrho_1}{2E_0 \varrho_0} = -\frac{a}{2} \cdot \frac{\varrho_1}{\varrho_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We shall oppose to the given treatment the more direct method because it proves that the advantage of neglecting the terms with a in the equation of motion (4) is but apparent.

For the originally homogeneous bar the equation of motion, taking into account (7) and the applied-forces P is:

$$\varrho_0 \frac{\partial^2 \xi}{\partial t^2} = E_0 \frac{\partial^2 \xi}{\partial x^2} + aE_0 \frac{\partial \xi}{\partial x} \cdot \frac{\partial^2 \xi}{\partial x^2} + P \quad . \quad . \quad . \quad . \quad . \quad (9)$$

There we split the deviation ξ into three parts $\xi = \xi_0 + \xi_1 + \xi_2$ where ξ_0 is the statical deviation in consequence of the forces

P, ξ_1 the given primary wave, ξ_2 the secondary wave as a consequence of the deviations ξ_0 caused by P. Then ξ_0 can be found from:

$$E_0 \frac{\partial^2 \xi_0}{\partial x^2} + \alpha E_0 \frac{\partial \xi_0}{\partial x} \cdot \frac{\partial^2 \xi_0}{\partial x^2} = - P$$

Taking this into account, ξ_1 which should be independent of P and ξ_0 has to satisfy

$$\varrho_0 \frac{\partial^2 \xi_1}{\partial t^2} = E_0 \frac{\partial^2 \xi_1}{\partial x^2} + \alpha E_0 \frac{\partial \xi_1}{\partial x} \frac{\partial^2 \xi_1}{\partial x^2}.$$

Here the last term may be neglected, since it becomes proportional to the square of the amplitude of ξ_1 and this may be supposed to be very small. For ξ_2 we further find

$$\varrho_0 \frac{\partial^2 \xi_2}{\partial t^2} - E_0 \frac{\partial^2 \xi_2}{\partial x^2} + \alpha E_0 \left(\frac{\partial \xi_0}{\partial x} \cdot \frac{\partial^2 \xi_1}{\partial x^2} + \frac{\partial \xi_1}{\partial x} \cdot \frac{\partial^2 \xi_0}{\partial x^2} \right) \quad . \quad . \quad (10)$$

It appears at once from this equation that the amplitude of the secondary wave ξ_2 which has to be proportional to the disturbances ξ_0 and to the amplitude of ξ_1, is also proportional to α and that this wave therefore does not occur if HOOKE's law holds. In this case the former method still yielded the unimportant solution (6). This solution now does not appear at all, because here a different x-coördinate is used. The x used in the former treatment is evidently equal to what is represented here by $x + \xi_0$. For the rest the equation (10) as regards the form is equal to the equation from which, using the other method, the secondary wave is found, For the case which we considered the integration furnishes exactly the result likewise expressed in (5) and (8).

4. The problem of the scattering of elastic waves by accidental deviations of density is compared by DEBIJE to the scattering of light by those deviations. A great difference of course is that light has such a velocity that the molecular velocities, consequently also the velocity with which the deviations change, may be neglected. Therefore in the optical problem the deviations of density are rightly admitted to be at rest. If this is likewise done in the case of the analogous elastic problem, nothing but a qualitative conformity with reality will be expected.

In order to demonstrate this unnatural "keeping constant" of the deviations, we have spoken in point 3 about *statical* deviations caused by constant forces. Now let us imagine those forces to be removed suddenly.

Then the deviation between o and l gives origin to two disturbances of equilibrium moving with a velocity q, one to the left, the other to the right. The *dynamical* deviations occurring in reality move with the same velocity as the wave ξ_1, which they must disturb! Considered in this light we cannot expect at all that the replacing of the dynamical deviations by statical deviations will bring about anything useful.

Consequently the deviation ξ_0 we introduced before has also to be a progessive wave, and we should examine the interaction of two progressive waves ξ_0 and ξ_1, in either case they run along the bar in a different or in the same direction.

For this examination the equation of motion (9), where $P=0$, can be used. For the sake of brevity we shall take formula (1). Once more we put $\xi = \xi_0 + \xi_1 + \xi_2$. The waves ξ_0 and ξ_1 now satisfy each separately (1), so that for ξ_2 we have:

$$\frac{\partial^2 \xi_3}{\partial t^2} - \frac{\partial^2 \xi_2}{\partial x_2} = a\left(\frac{\partial \xi_0}{\partial x}\frac{\partial^2 \xi_1}{\partial x^2} + \frac{\partial \xi_1}{\partial x}\frac{\partial^2 \xi_0}{\partial x^2}\right) \quad . \quad . \quad . \quad . \quad (11)$$

Just as with the statical deviations we may neglect the terms with a in the equations ξ_0 and ξ_1. When we introduce the new variables

$$u = x + t \qquad\qquad v = x - t$$

then we can represent two waves moving in the same direction by

$$\xi_0 = f_0(v) \qquad\qquad \xi_1 = f_1(v).$$

After transformation (11) becomes:

$$\frac{\partial^2 \xi_2}{\partial u\, \partial v} = -\frac{a}{4}\left(f'_0 f''_1 + f'_1 f''_0\right).$$

Integrating this first over v and afterwards over u we find

$$\xi_3 = -\frac{a}{4} u f'_0(v) f'_1(v) + f_2(u) + f_3(v)$$

whereby the functions f_2 and f_3 can be used to satisfy the initial conditions. When these conditions are that ξ_2 and $\frac{\partial \xi_2}{\partial x}$ will be zero for $t=0$, then we find

$$\xi_2 = -\frac{a}{2} t f'_0(v) f'_1(v).$$

From this follows that the two waves ξ_0 and ξ_1 really react on each other and produce a secondary wave increasing with t — that is to say if f_0 and f_1 cover each other, not if they are two finite waves following each other. If one takes the special case of sine-waves, then ξ_2 becomes a "combination-wave", consisting of two terms, with the sum and the difference of the frequencies of ξ_0 and ξ_1. However, what is in the first place important for our purpose: ξ_2 always runs in the same direction as ξ^0 and ξ_1.

Where the waves run in opposite directions one has:

$$\xi_0 = f_0(u) \qquad \xi_1 = f_1(v)$$

and for ξ_2:

$$\frac{\partial_0 \xi_2}{du\, dv} = -\frac{\alpha}{4}\left(f''_0(u) f''_1(v) + f''_0(u) f'_1(v)\right)$$

which gives integrated

$$\xi_2 = -\frac{\alpha}{4}\left(f_0(u) f'_1(v) + f'_0(u) f_1(v)\right) + f_2(u) + f_3(v)$$

Now we imagine ξ_0 and ξ_1 given for a limited part of the bar and therefore restricted to it. Let for $t=0$ the wave ξ_0 only differ from zero between definite positive limits for u_1 and ξ_1, also between negative limits. Then the waves will meet. If this is the only wave motion on the bar for $t=0$, then in ξ_2, f_2 and f_3 must be zero. Therefore ξ_2 will remain zero until the waves f_0 and f_1 will cover each other. Only for such values of x_1, for which f_0 as well is f_1 — and consequently also their derivatives — differ from zero, there is a certain value for ξ_2, and when after some time the waves have entirely passed one another, ξ_2 will be again zero everywhere. *So waves in different directions run over each other without any exchange of energy.*

From what we have found it follows immediately that the linear body will show no fall of temperature even if a heat-current passes through it. If namely the heat-motion is dissolved into a great number of progressive waves, the presence of a heat-current will signify that the waves in one direction have on an average a little more energy than those in the opposite direction. The mean energy of either of the two wave motions however will be the same for all points of the bar. The co-operation of the waves running in the same direction does not in the least change this, for the arising combination-waves run in the same direction. Here we may also speak of the co-operation of a wave with itself. That also gives a combination-wave in the same direction. Such is the effect of the terms we repeatedly neglected. *Consequently on our bar there is no fall of energy at all, in other words no heat-resistance.*

5. For the present, as for the extension of what we discussed in points 3 and 4 to the three-dimensional problem and the further application to a real solid body, we are obliged to limit ourselves to a few remarks. Let us first consider the effect of statical deviations such as in 3. The first method given there has already been applied by Debije in exactly the same manner to a solid, however for longitudinal

waves only. A drawback of this method is that one cannot immediately see from the result how it depends on the fact whether the law of HOOKE prevails or not; further that terms such as (6) occur which would disappear in another choice of co-ordinates. The relation between the changement of ϱ and the compressibility DEBIJE expresses by a coefficient α. By a consideration analogous to the one given above (after equation (7)) one can make out what will be the value of that number α when the law of HOOKE holds. The scattered energy is found by DEBIJE to be proportional to $3\alpha^2 + 1$, and therefore does not vanish for any value of α. Whether this result is exact or not would be best settled by application to this problem of the second method of point 3. The equations of motion however become already very complicated. In this connection one should compare the results obtained by FINGER [1]). Anyhow, one obtains already non-linear terms if HOOKE's law holds and therefore probably also scattering.

The contradiction that in DEBIJE's final result the heat-conduction does not become ∞, if one takes $\alpha = 1$, is for that reason but an apparent one. Even in that case namely the solid is not "ideal". On the contrary one should say that such an *ideal solid* is a contradiction in terms, as the elastic equations of motion can become in no way strictly linear. Nevertheless one may yet say that a real solid at a very low temperature approaches to an ideal solid. For one can always take the different heat-motions so small that in the equations of motion the terms of higher order are negligible.

A remark, for the rest not in connection with the considerations we gave above, may yet follow. It is principally wrong to deduce the deviations of density in the elements of volume of a solid from the principle of BOLTZMANN, in the same way as this has been done by EINSTEIN for a gas. The correct way of course is with the aid of the normal vibrations. Now it is well-known how in the entirely comparable case of the radiation both ways give strongly different results. It has been tried at the time to explain the result of the first incorrect method with the aid of the light-quanta. The wrong point there, and this holds just as much for the solid, does not lie in the very principle of BOLTZMANN itself, but in the application. If namely the entropy of the whole is taken equal to the sum of the entropies of the parts, then this implies the independence of the probabilities of the conditions in those parts, which is in this

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[1]) J. FINGER, Wiener Sitzungsberichte, 103, 163 (1894). Also Conf. P. DUHEM, Recherches sur l'Elasticité. Paris 1916.

case certainly not so.[1] The correct deviations of density in the solid are much smaller and have another dependence of temperature than that which DEBIJE used.

In connection with what we found above in point 4, at last it remains to be said that especially the use of *statical* deviations of density instead of the *dynamic* ones is a great mistake. In some simple cases we have been able to demonstrate that also in three dimensions the latter do not produce any scattering. Our conclusion therefore is that the molecular theory of the heat-resistance still remains entirely open.

Physics. — "*Contributions to the kinetic theory of solids.* III. *The equation of state of the isotropic solid.*" By Prof. L. S. ORNSTEIN and Dr. F. ZERNIKE. (Communicated by Prof. H. A. LORENTZ.)

(Communicated in the meeting of June 24, 1916.)

In this contribution we shall use the method we developed in our first contribution for the determination of the expansion in order to deduce the equation of state, i.e. the connection between the strain and the stress in its dependence on the temperature. In contribution I we have treated only the simple case that the strains are zero, and have determined the stress resulting from heating (thermal pressure). A quite analogous deduction can be used in order to find the stresses of a solid, which has been deformed at the absolute zero (equation of state). The only difference with the former case lies in the fact, that by this strain the solid generally departs from exact isotropy. Hence a more ample calculation is necessary in the case of shearing.

Further we shall mention the terms which present themselves if we take into account the remark of the note on p. 1293.

Finally we shall show how the equation of state is also to be found with the aid of thermodynamic relations from the specific heat of solid bodies, which may be calculated from the formulae given by BORN. This method, for the present only mentioned in principle, is more analogous to that of DEBYE-EVERDINGEN than to our first deduction, which is purely dynamical.

1. Now we will calculate the force necessary to give the solid

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[1] See EPSTEIN, Physik. Zeitschr. XV.

body a homogeneous expansion at a given temperature, so that $e_1 = e_2 = e_3 = e$, $e_4 = e_5 = e_6 = 0$. This force consists of two parts. In the first place one may give the required expansion to the solid body at the absolute zero and thereupon supply at constant volume the energy necessary to give the solid body the required temperature. Therefore the thermal pressure as calculated in our first contribution has to be added to the elastic tension at the absolute zero.

Now in order to calculate the first part we use the expression for the energy

$$\varepsilon = AI_1^2 + BI_2 + CI_1^3 + DI_1I_2 + EI_3 \;{}^{1)} . \quad . \quad . \quad (1)$$

It is immediately clear that the invariants have the following values

$$I_1 = 3e \quad , \quad I_2 = 3e^2 \quad , \quad I_3 = e^3 .$$

so that the energy takes the form:

$$\varepsilon = (9A + 3B)\, e^2 + (27C + 9D + E)\, e^3$$

For the tension at the absolute zero we find consequently

$$S_0 = \frac{d\varepsilon}{d3e} = (6A + 2B)\, e + (27C + 9D + E)\, e^2 .$$

For the thermal pressure (S_{th}) we found in Contribution I the expression

$$S_{th} = \frac{9C + 2D}{6A}\, \overline{\varepsilon}_l + \frac{3D + E}{6B}\, \overline{\varepsilon}_{tr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now A, B, C, and D are constants which are relative to the non-deformed substance. In our case however we must replace these constants by their values in the strained condition when calculating the thermal pressure. But as the solid in that condition remains isotropic, the given formula still holds. We will neglect the variation of C and D with e as this variation is determined by terms of higher order in the energy, which were neglected by us. The average energy of the longitudinal and the transversal vibrations may change also. When we have to do with so high a temperature that the theorem of equipartition is true, that change is zero as the number of degrees of freedom does not change by the expansion. For lower temperatures where the quantum-theory must be taken into consideration, the change of ε_l and ε_{tr} has to be taken into account.

The variation of A and B with e can be found in the following manner. The strain can be represented by

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1) Conf. for the notations Contribution I,

$$e_1 = e + e'_1,\quad e_2 = e + e'_2,\quad e_3 = e + e'_3,\quad e_4 = e'_4,\quad e_5 = e'_5,\quad e_6 = e'_6.$$

We introduce this into the expression for the energy and determine the part of the second degree with respect to the quantities e'. As e'_1 etc. represent the deformations from the strained condition (e), the coefficients of the invariants I'^2_1 and I'_2 will give the new values of A and B. We find

$$A' = A + (9C + 2D)\, e$$
$$B' = B + (3D + E)\, e.$$

Now when we introduce these values for A' and B' into the expression for the thermal pressure and when we add to it the elastic tension at the absolute zero, we find for the total tension in the case of equipartition

$$\left.\begin{aligned} S = (6A+2B)\, e + (27C+9D+E)\, e^2 + \left(\frac{9C+2D}{18A} + \frac{3D+E}{9B}\right) \varepsilon \, - \\ - \left(\frac{(9C+2D)^2}{18A^2} + \frac{(3D+E)^2}{9B^2}\right) \varepsilon\, e. \end{aligned}\right\} \quad (3)$$

In this ε is the total energy which is proportional to T. Thus the equation of state for changes in which the solid body remains isotropic has been found.

To find the modulus of compression we determine $\dfrac{dS}{d3e}$; for this we find

$$m = \frac{6A+2B}{3} + \frac{54C+18D+2E}{3}\, e - \left(\frac{(9C+2D)^2}{54A^2} + \frac{(3D+E)^2}{27B^2}\right) \varepsilon \quad (4)$$

The factor e now still depends on the temperature; to find this factor we can apply (3), where the last term may be neglected as we will confine ourselves to a linear expression in ε. The e can then be found by considering the expansion at zero pressure. When we put in (3) $S = 0$ we find

$$e = -\frac{\left(\dfrac{9C+2D}{18A} + \dfrac{3D+E}{9B}\right)\varepsilon}{6A+2B}$$

$$\left.\begin{aligned} m = \frac{6A+2B}{3} - \left\{ \frac{27C+9D+E}{9A+3B} \left(\frac{9C+2D}{18A} + \frac{3D+E}{9B}\right) + \right. \\ \left. + \frac{(9C+2D)^2}{54A^2} + \frac{(3D+E)^2}{27B^2} \right\} \varepsilon \end{aligned}\right\} \quad (5)$$

Easily the form of the equation of state can be indicated in the case that the quantum-theory is introduced. We will confine ourselves to the case of the temperatures being so low that the upper limit of the integral in the expression (196), which is given by Born for the energy does not come into account. Then we have to take into consideration for the thermal pressure in formula (2) not only the variation of A and B with e, but also that of the longitudinal and the transversal energy.

By application of the formulae which Born has given in § 21, we find after a simple calculation as equation of state

$$\left.\begin{array}{l} S = (6A+2B)\,e + (27C+9D+E)\,e^2 - \\ \quad - e\left\{\dfrac{5(9C+2D)^2}{6A^2}\overline{\varepsilon_l} + \dfrac{5(3D+E)^2}{6B^2}\overline{\varepsilon_{tr}}\right\} \end{array}\right\} \quad \ldots \quad (6)$$

into which may be introduced the values for the energy of the longitudinal and the transversal vibration. From this the modulus of compression can afterwards be calculated. We find thus

$$\left.\begin{array}{l} m = \dfrac{6A+2B}{3} - \dfrac{5(9C+2D)^2}{18A^2}\overline{\varepsilon_l} - \dfrac{5(3D+E)^2}{18B^2}\overline{\varepsilon_{tr}} - \\ \qquad - \dfrac{(27C+9D+E)}{9A+3B}\left\{\dfrac{9C+2D}{3A}\overline{\varepsilon_l} + \dfrac{3D+E}{3B}\overline{\varepsilon_{tr}}\right\} \end{array}\right\} \quad (7)$$

2. Now we can try to deduce the shearing stresses and their dependence on temperature in a way analogous to the one that has been used for the thermal pressure. For this purpose we can determine the space-time average of the force $\dfrac{\partial \varepsilon}{\partial e_4}$.

We have

$$X_4 = \frac{\partial \varepsilon}{\partial e_4} = -\tfrac{1}{2} B\, e_4 + \tfrac{1}{2} E\,(e_5 e_6 - e_1 e_4) - \tfrac{1}{2} D\,(e_1 e_4 + e_2 e_4 + e_3 e_4).$$

Now we have to determine the mean force when the solid body has a given strain $e_4 = e$ in the initial condition. Thus, if we call again $e'_1 \ldots e'_6$ the deformations from the strained condition, we get

$$X_4 = -\tfrac{1}{2} B\,(e+e_4') + \tfrac{1}{2} E\,(e_5' e_6' - e_1'\,(e+e_4')) - \tfrac{1}{2} D\,(e_1' + e_2' + e_3')\,(e+e_4').$$

Now we have to find the space-time-average of this force. It should be taken into consideration that $\overline{e_4'} = \overline{e_1'} = \overline{e_2'} = \overline{e_3'} = 0$, whence

$$\overline{X_4} = -\tfrac{1}{2} Be + \tfrac{1}{2} E\,(\overline{e_5' e_6'} \quad \overline{e_1' e_4'}) - \tfrac{1}{2} D\,(\overline{e_1' e_4'} + \overline{e_2' e_4'} + \overline{e_3' e_4'}) \qquad (8)$$

The mean values $\overline{e'_5\, e'_6}$ etc. which vanish in the case the body remains isotropic will now, by the existence of the initial shearing $e_4 = e$, have values deviating from zero in consequence of the fact that the body presently behaves like a rhombic crystal. Thus far the calculation corresponds with that of the thermal pressure. The force calculated is indeed the force required to keep a certain deformation given at the absolute zero unchanged when the temperature is increased. But, whereas mean forces occurring at the thermical pressure could be deducted at once from the energy, this time this is not the case, and thus here it will be necessary to calculate the mean values $e'_1\, e'_4$ separately.

This calculation of which here only the general course will be indicated, is performed a little more easily for another case, i.e. for the case that the given deformation at the absolute zero is

$$e_1 = e_1,\ e_2 = e_2,\ e_3 = e_3,\ e_4 = e_5 = e_6 = 0.$$

Then only tensions X_x, Y_y and Z_z occur and the mean values

$$\overline{e_1'^2} \quad \overline{e_1'e_2'} \quad \overline{e_2'^2} \quad \overline{e_1'e_3'} \quad \overline{e_3'^2} \quad \overline{e_2'e_3'}$$

will have to be determined.

These mean values can be found by considering the progressive waves in the rhombic crystal into which the body has changed by the deformation.

The total change can be characterized as follows. First the energy in the isotropic body is divided equally in all directions over the waves of the same frequency; for the crystal this is not the case. In the second place in the isotropic substance there are longitudinal and transversal waves; with the crystal the direction of the displacement is no longer so simple. Now e_1, e_2 and e_3 are small quantities, the change is therefore for both cases small; e.g. from the longitudinal wave arises a wave the elongation of which has a small inclination with respect to the wave-normal. As the effects are so small we are able to determine the influence on the averages separately. Indeed we may in calculating the influence of the new division of energy overlook the "declivity" of the waves, hence we may substitute the direction in the isotropic case for that of the vector of displacement. When we examine however the influence of the "declivity", we can take into account the division of the energy for an isotropic body, i.e. the homogeneous distribution over the directions.

As has been said above, we do not intend to reproduce here the calculation, but are going to show only how the elastic constants of the rhombic crystal are expressed in the magnitudes $e_1\, e_2\, e_3$.

We introduce the notation of VOIGT so that $c_{11}\, c_{12}\, c_{13}\, c_{22} c_{23}\, c_{33}\, c_{44}$ $c_{55}\, c_{66}$ are the constants of the rhombic crystal, i.e. the coefficients of $e_1^2, e_1 e_2$ etc. in the energy. When the strain at the absolute zero is represented by e_1, e_2, e_3, o, o, o, and the arbitrary strain which is superposed on it by $e'_1\, e'_2\, e'_3\, e'_4\, e'_5\, e'_6$, then the terms with C, D and E give quadratic parts in e'_1 etc. viz.

$$\left.\begin{array}{l} 3\,C\,(e_1 + e_2 + e_3)\,I_1'^2 + D\,(e_1 + e_2 + e_3)\,I_2' + \\ + D I_1'\,(e_1'\,(e_2 + e_3) + e_2'\,(e_1 + e_3) + e_3'\,(e_1 + e_2)) \\ E\,[e_1'e_2'e_3 + e_2'e_3'e_1 + e_1'e_3'e_2 - e_4'^2 e_1 - e_5'^2 e_2 - e_6'^2 e_3] \end{array}\right\} \qquad (9)$$

These parts should be added to $AI_1'^2 + BI_2'$, therefore the coefficient of $e_1'^2$ increases by

$$3C\,(e_1 + e_2 + e_3) + D\,(e_2 + e_3)$$

that of $e_1'\,e_2'$ by

$$(6C + 2D)\,(e_1 + e_2 + e_3) + (D + E)\,e_3$$

that of $e_4'^2$ by

$$-\tfrac{1}{4}\,D\,(e_1 + e_2 + e_3) - E\,e_1$$

from which the other follow by cyclic change. In order to introduce the notation of VOIGT also for the isotropic body one should remember that $c_{12} = c_{11} - c_{44}$, whereas $A = \frac{1}{2}\,c_{11}$, $B = -2c_{44}$. From this therefore the constants for the rhombic crystal are at once to be written down. From the elastic differential equation afterwards the determinant equation for the velocity of transmission in its dependence upon the direction is deduced, and also the frequency of the normal vibrations can be determined. The further, more detailed calculation will finally for the mean values in question, except the terms which appear also in the isotropic case, yield values which linearly depend on e_1, e_2 and e_3.

3. In a note to our contribution (I) already we have pointed to the fact that the ordinary formula which was used there for the tension was not exact. In the following manner the accurate form for finite deformations is found.

The elastic energy ε can also in that case be represented by the already often used formula (I), provided only that the correct signification is assigned to the quantities $e_1 \ldots e_6$. Let us now represent the differential quotients $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ etc. by a_{11}, a_{12} etc., then we must take [1])

$$\left.\begin{array}{l} e_1 = a_{11} + \frac{1}{2}\,(a_{11}^2 + a_{21}^2 + a_{31}^2) \\ e_4 = a_{23} + a_{32} + a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33}. \end{array}\right\} \quad . \quad . \quad . \quad (11)$$

From the energy ε it will be possible to find the tension X_x by means of a virtual elongation δ in the x-direction. This now has to be composed with the known deformation, which is determined by the magnitudes a_{11} etc. Hence a_{11}, a_{12} and a_{13} change and further also all magnitudes e. For the new values we find

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1) Vide e.g. LOVE.

$$e_1' = e_1 + (1 + a_{11})^2 \delta$$
$$e_2' = e_2 + a_{12}^2 \delta$$
$$e_4' = e_4 + 2a_{12}a_{13} \delta$$
$$e_5' = e_5 + a_{13}(1 + 2a_{11}) \delta$$

From these values the variation of the energy may be found. Apparently now also the terms AI_1^2 and BI_2 give parts which are quadratic in the quantities e (or a) and which therefore on the average do not drop off. After using the symmetry of the expression it is possible to put down the result as follows

$$\tfrac{2}{3} A \overline{I_1^2} + \tfrac{2}{3} B \overline{I_2}\,.$$

However, another correction of the same nature will still be required. Above it was taken as a matter of course that the averages of first powers of $e_1 \ldots e_6$ will be zero. Properly taken this is the case with the quantities $a_{11} = \frac{\partial u}{\partial x}$. On the other hand one finds from the relations (11)

$$\bar{e}_1 = \tfrac{1}{2}(\overline{e_1^2} + \tfrac{1}{2}\overline{e_4^2})\,.$$

This value should be taken into account if we take the mean value of the principal term of the tension X_x

$$2AI_1 + B(e_2 + e_3)$$

therefore a correction is found to an amount of

$$(A + \tfrac{1}{3}B)(\overline{I_1^2} - 2\overline{I_2})\,.$$

Consequently on the whole, to the thermical pressure we found before,

$$\frac{5A+B}{3}\overline{I_1^2}\,.$$

has to be added.

It will be permitted to neglect this term when the coëfficients C and D are large with respect to A and B.

Of course it would be likewise possible to indicate the corresponding terms at the farther calculations of the equation of state which we have mentioned in this contribution. To indicate the principle it seemed to us to be sufficient to treat only the thermal pressure in this way. Further we must point out that if once these terms are neglected it will have no sense to make any difference between the density before and after the strain, when we calculate the energy for a unit of volume, or to take into account other differences of the same kind. Van Everdingen has not always considered this (l.c. pg. 22—23); in consequence of this there occur terms in his results that are of the same order as the other neglected terms.

4. Besides the dynamic way we developed above, there is another method in which the results of Born's general theory about the specific heat as well as known thermodynamic relations are used. This method shows a certain conformity with the one used by Debye-v. Everdingen, but enables us to put the problem more strictly, whereas it has in common with the method given above the advantage that it can introduce or not the theory of the quanta of energy and may be easily extended to the temperatures where the approximations of v. Everdingen prevail no more. Moreover it can easily be extended to a theory of the equation of state of a crystal of any class of symmetry. Instead of using a characteristic temperature as v. Everdingen does — who also introduces the incorrect approximation that there is only *one* characteristic temperature — the specific heat itself is used. Hence the approximation which v. Everdingen introduces on p. 35 of his dissertation, the consequences of which it is impossible fully to survey, viz. the application of formulae which prevail for the isotropic body to a aeolotropic body, can be avoided.

Now the principle of our method is as follows. When the deformations $e_1\ e_2\ e_3$ are given to the body X_x may be represented (as is demonstrated in (2)) e.g. by

$$X_x = X_x^t + a\,e_1 + b\,(e_2 + e_3)$$

in which X_x^t is the thermical pressure for the isotropic body for $e_1 = e_2 = e_3 = 0$, and a and b are functions of temperature. On account of the isotropy in X_x the coefficient of e_2 has to be equal to that of e_3; Y_y and Z_z hence follow by cyclic interchange.

X_y etc., on account of the isotropy, are at the given deformation zero.

Now the specific heat can be calculated by application of the formulae of Born to the rhombiedric crystal with the e's given above. As we start from an isotropic body the development of the specific heat in terms of $e_1\ e_2\ e_3$ will only depend on the invariants. So we may put

$$C_v = C_{v0} + \alpha\,I_1 + \beta\,I_1^2 + \gamma\,I_2 + \ldots$$

in which C_v means the specific heat at constant $e_1\ e_2\ e_3$, C_{v0} that for $e_1 = e_2 = e_3 = 0$.

Now we can apply the thermodynamic relation

$$\frac{\partial C_v}{\partial e_1} = T\frac{d^2 X_x}{dT^2}$$

This gives

$$\alpha + 2\beta\,(e_1 + e_2 + e_3) + \gamma\,(e_2 + e_3) = T\left\{\frac{d^2 X_x{}^t}{dT^2} + \frac{d^2 a}{dT^2}\,e_1 + \frac{d^2 b}{dT^2}\,(e_2 + e_3)\right.$$

from which follows:

$$a = T \frac{d^2 X_x{}^t}{dT^2}$$

$$2\beta = T \frac{d^2 a}{dT^2}$$

$$2\beta + \gamma = T \frac{d^2 b}{dT^2} \cdot$$

The functions of temperature $\alpha \beta \gamma$ can be calculated from the formulae of BORN so that X^t_x a and b can be determined. In a more detailed communication we will communicate this calculation itself also.

Delft, June 1916. *Phys. Lab. of the T. H. S.*

Physics. — *"The influence of accidental deviations of density on the equation of state."* By Prof. L. S. ORNSTEIN and Dr. F. ZERNIKE. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of June 24, 1916).

In their article in the Encyclopädie der mathematischen Wissenschaften the following statement by Prof. KAMERLINGH ONNES and Dr, KEESOM is found:

"Da bei der Annäherung an den kritischen Punkt Liquid—Gas die von den BOLTZMANN-GIBBS'schen Prinzipien beherrschten Dichteunterschiede (Schwarmbildung Nr 48*f*), der bis ∞ ansteigenden Zusammendrückbarkeit der Substanz wegen, besonders hervortreten, ist zu erwarten, dasz bei der Entwicklung der Zustandsgleichung für die Umgebung des kritischen Punktes nach jenen Prinzipien Glieder auftreten werden, die mit der grossen Zusammendrückbarkeit in der Nähe des kritischen Punktes zusammenhängen. Diese Glieder werden wahrscheinlich durch die Art der Abweichung der Zusammendrückbarkeit in dem kritischen Gebiet (∞ im kritischen Punkt und von diesem aus, soweit sie das realisirbare homogene Gebiet betrifft, allseitig schnell abfallend) für dasselbe ein besondere Bedeutung erlangen, während sie für benachbarte Gebiete nicht mehr in Betracht kommen. Während eine allmählige Verschiebung oder Verzerrung, die sich durch das ganze Diagramm durchzieht, wie z. B. eine kontinuirliche Aenderung von a_w, b_w oder R_w, sich experimentell nicht besonders zeigen würde, werden die betreffende Glieder in der Zustandsgleichung in der Nähe des kritischen Punktes demgemäss zum Schluss führen können, dass die Eigenschaften in diesem Gebiet in beobachtbarer Weise abweichen von den Eigenschaften, die man durch Interpolation zwischen Zuständen, die um den kritischen herumliegen, aber weiter von ihm entfernt bleiben, erwarten sollte."

Kamerlingh Onnes and Keesom assert to have found deviations pointing to suchlike causes and they have tried to account for the special phenomena at the critical point by adding a function of disturbance to the equation of state which takes specially large values in the critical point.

In this contribution we will try to point out that there can be no question about a function of disturbance caused by the accidental deviations of density.

In deducing the equation of state we will use the method of the virial. The virial theorem takes the form

$$2\overline{L} + \Sigma\,\overline{(x_h X_h + y_k Y_h + z_h Z_k)} + \Sigma\Sigma\,\overline{r_{hk} F(r_{hk})} = 0$$

if L is the kinetic energy, the coordinates of the molecules are x_h, y_h, z_h, the distance between the centres of a pair of molecules h and k is r_{hk}, the external forces are $X_h\ Y_h\ Z_h$ and the force between the pair of molecules $h\,.\,k$ is $F(z_{hk})$. If the pressure is the only external force, the second average may be represented by $-3pV$, whereas the average kinetic energy depends on the absolute temperature in the known way. [1])

We will now take into consideration the second sum. Suppose the volume V containing n molecules to be divided into a large number of elements dv_k; let the mean density in these elements be ϱ. Now consider a system in which the density in the different elements can be represented by $\varrho + v_k$. Then the contribution by the elements dv_k and dv_h to the sum mentioned above amounts to

$$(\varrho + v_k)(\varrho + v_h)\, r_{hk}\, F(r_{hk})\, dv_k\, dv_h.$$

By integrating this expression with respect to dv_h and dv_k over the volume V we get, passing to the mean values, twice the required sum. Taking the averages the terms ϱv_h and ϱv_k will not contribute, the average of $v_h = v_k$ being zero.

The contribution

$$\iint \varrho^2\, F(r_{hk})\, r_{hk}\, dv_k\, dv_h$$

is the one taken into account by the usual theory, which neglects accidental deviations of density, and which therefore takes for ϱ the mean value, or what comes to the same, the most common value. Hence the correction for deviations of density appears to be

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[1]) Compare e.g. L. Boltzmann, Gastheorie II, p. 141.

$$\iint \overline{v_h v_k}\, r_{hk}\, F(r_{hk})\, dv_k\, dv_h.$$

Now this integral can be transformed by means of the function g, which we have introduced into our considerations on the opalescence in the critical point. [1])

According to the definition of the function g, the mean value of v_k if v_h is kept constant may be represented by

$$\overline{v_k} = v_h \;g(r_{hk})\, dv_h.$$

Further we have shown that:

$$\overline{v_k v_h} = g\,(r_{hk})\, \overline{v_h^2}\, dv_0 = \varrho g\,(r_{hk})$$

(l.c. p. 796) (8).

Introducing this we get

$$\iint \varrho\, r_{hk}\, F\,(r_{hk})\, g\,(r_{hk})\, dv_k\, dv_h.$$

For this we may write

$$n \int r_{hk}\, F\,(r_{hk})\, g\,(r_h k)\, dv_k$$

as $\int \varrho\, dv_h$ represents the whole number of molecules.

If we proceed to the usual first approximation regarding the terms of the first integral, then we find as equation of state

$$p + \frac{a}{v^2} = \frac{rT}{v}\left(1 + \frac{b}{v}\right) + \frac{1}{6v}\int m g\;(r_{hk})\,.\,r_{hk}\,.\,F\,(r_{hk})\, dv_k$$

where $v = \frac{V}{nm}$, and m is the mass of the molecule.

This formula shows that if we take into account the deviations of density, we find a new term in the equation of state. This term however is not equivalent to the function of disturbance of KAMERLINGH ONNES and KEESOM. The special character of the function g in the neighbourhood of the critical point causing the clustering tendency there to become very strong, consists in the fact that this function gets perceptible values for points far outside the sphere of action, so that for the critical state $\int_0^\infty g\, dv$ does not converge any more. Phenomena connected with this last integral become therefore specially strong at the critical point. For small values of r, inside the sphere of action, g hardly changes when we approach the critical point; and only these values have influence on the found term of correction.

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[1]) L. S. ORNSTEIN and F. ZERNIKE. The accidental deviations of density and the opalescence in the critical point. These proc. XVII, p. 793.

In this integral the function $F(r_{hk})$ is zero without the sphere of action, therefore this integral has only to be extended to this region.[1] Thus the abnormal character, stated by KAMERLINGH ONNES and KEESOM, cannot be explained by a function of disturbance as a consequence of the accidental deviations.

Physics. — "*Contributions to the research of liquid crystals.*" By Dr. W. J. H. MOLL and Prof. L. S. ORNSTEIN. (Communicated by Prof. W. H. JULIUS).

(Communicated in the meeting of October 28, 1916).

1. *The extinction of para-azoxyanisol in the magnetic field.*

VOIGT has devoted a circumstantial report to the liquid crystals in the Physikalische Zeitschrift XVII 1916. In this report he points specially to the fact that a great uncertainty still exists on the influence exercised by a magnetic field on the extinction. For this reason we further examined the extinction.

We will provisionally explain our method in this communication, and mention some of the results attained, and add a few remarks on a possible explanation of these results.

§ 1. *Method of observation.*

The extinction was measured with the aid of the galvanometer and the thermopile described by one of us formerly.[2] These sensitive instruments have the great advantage of indicating quickly within two seconds, and so they enable us to follow the changes in the liquid crystals proceeding slowly, but nevertheless so quickly that they must necessarily escape the observer's attention.

These very changes however are of great interest in order to understand the phenomena under consideration.

The substance is heated in a small electric oven, consisting of a strip of copper AA coiled at the two ends with resistance wire. By regulating the current sent through these coils each desired

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[1] According to the explicit calculation of g, executed by one of us, the solid character is denoted by $r^{-1}\, e^{-kr}$, where $k = 0$ for the critical point. Comp. ZERNIKE, The clustering tendency of the molecules in the critical state and the extinction of light caused thereby. These Proc. XVIII p. 1520.

[2] W. J. H. MOLL. Proc. Kon. Acad. v. Wetensch. May and Nov. 1913. Mrs. KIPP and SONS were so kind as to put the apparatus at our disposal.

temperature, up to above the second point of transition may be attained. In the middle of the strip between the coils a circular pole appears, the edge of which carries a round piece of glass, a second piece of glass lying on top of it with a ring of paper between them. The substance is put in the space between the two glasses and the ring.

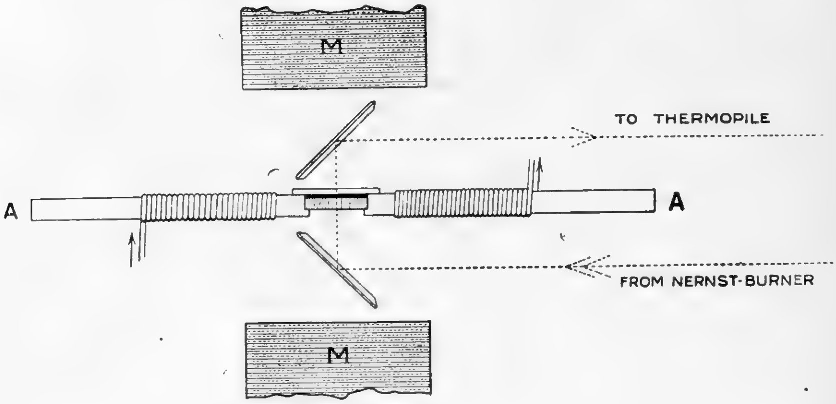

Fig. 1.

By the aid of two small mirrors a beam of light is sent through the substance from aside. During our measurements the small oven was placed horizontally between the poles M_1 and M_2 of a Dubois magnet. This magnet is fixed on a stand that allows a rotation round an axis lying in the direction of AA. Therefore, when the poles are placed as shown in fig. 1, the field of forces will be in a perpendicular position, parallel to the rays of light which penetrate the substance.

When the magnet is rotated 90° the lines of force are horizontal and will cross the rays of light perpendicularly.

A Nernst-burner connected to a battery of accumulators produces the required light. On the thermopile an image of the Nernst-burner is formed by a hollow mirror. The rays of light on this way from the hollow mirror to the thermopile penetrate the substance.

The slit in the thermopile has the same magnitude as the image of the Nernst-burner.

In the liquid-crystalline condition the opalescence of the matter confuses the image proportionally to the opacity. So the intensity of the thermocurrent gives us a measure for the extinction.

The galvanometer records were registered.

§ 2. *The measurements.*

It had appeared to us that the changes in the extinction caused by the magnetic field often persisted. So it became advisable to reduce the matter to its original ("virginal") state before the beginning of every series of measurings, which was effected by heating it up to a little above the second point of transition. The current was kept constant during the measurements.

Every series of measurements progressed as follows.

A, The radiation of the NERNST-burner is intercepted and consequently the galvanometer records the zero-condition.

B. The radiation is admitted, the substance being in the virginal condition.

C. The magnetic field is excited.
D. The magnetic field is removed.
E. The radiation is intercepted.
F. The radiation is admitted.
G. The magnetic field is excited.
H. The magnetic field is removed.
I. The radiation is intercepted.

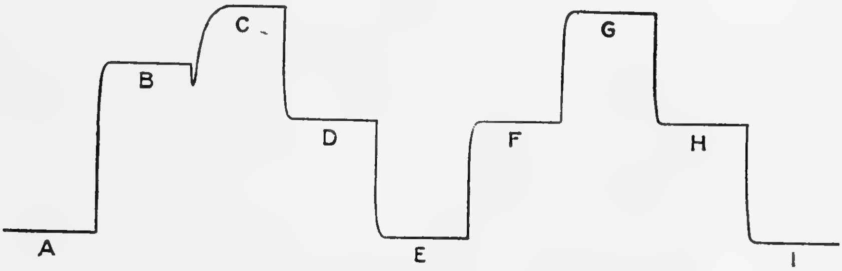

Fig. 2. Perpendicular magnetic field of 1100 Gauss.

The figures 2 and 3 show the phenomenon for a field of 1100 Gauss.

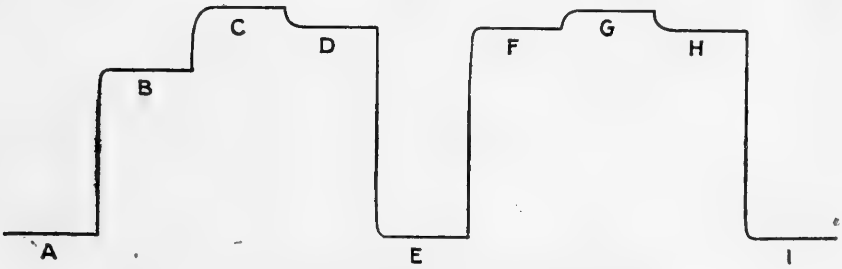

Fig. 3. Horizontal magnetic field of 1100 Gauss.

In order to understand their meaning it may be stated that to magnify the ordinates implies the diminishing of the extinction.

The difference between the two phenomena is very characteristic. Whereas the perpendicular field excited for the first time a temporary increase of the extinction, nothing of the sort is observed when the horizontal field is excited. Whereas after the breaking up of the perpendicular field the extinction is much larger than in the virginal state, the extinction is much smaller than before after the removal of the horizontal field. This condition of diminished extinction seems to be stable, it holds during some hours; the condition of increased extinction, as is left after removal of the perpendicular field, is of a transitory nature, gradually the virginal state being restored.

We expected from theoretical considerations which will be treated more fully hereafter that the effect of the temporary increase of the extinction when a vertical field is excited would appear more clearly in the case of weaker fields (compare the temporary drop of the curve between B and C in fig. 2). At the same time we expected that in the case of a sufficiently weak perpendicular field it might occur that the extinction of the field is on, would remain larger than before.

The experiments confirmed our expectation completely as is shown by the following figures

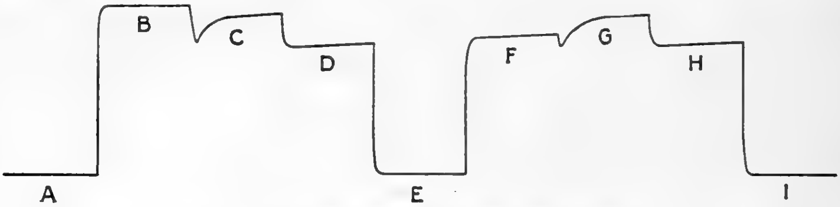

Fig. 4. Perpendicular field of 300 Gauss.

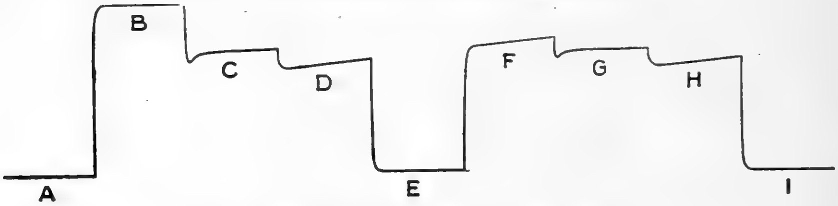

Fig. 5. Perpendicular field of 100 Gauss.

We have also examined whether commutation of the magnetic field has any influence on the extinction, but we could not state such an influence. ¹)

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¹) Considering the very different character which the influence of a magnetic field on the same preparation may have, it is easily understood that various observers, by visual examination of the extinction have come to quite contradictory results. Moreover, small impurities of the preparation and especially the state of the surface of the glass are of decisive influence. A further research on the influence of the surrounding surfaces is being prepared; we only want to remark that we performed these measurings with plates of glass which were not chemically cleared.

§ 3. *Attempt at a theory of the phenomena.*

Finely we want to make a few remarks containing a provisional elementary explanation of the observed phenomena, reserving to give afterwards a more mathematical theory.

We suppose that the molecules of para-azoxyanisol, as may be expected from the chemical constitution are of an oblong form, and that therefore a magnetic field will try to place their longitudinal axes parallel to the field.

Further we suppose the particles to undergo a directing couple from the glass-wall, so that the wall tries to direct them parallel to itself. The forces proceeding from the wall extend like all molecular influences only to a very small distance from the wall.

Let us admit further, that the particles influence each other, [1]) in such a way that particles try to turn their axes reciprocally parallel.

The result of the influences is that with given temperature and pressure two phases in equilibrium are possible, one of them being very sensible to a cause of outward direction, and the other not being so.

In the first (the liquid crystalline) phase, there will appear in consequence of the orientating influence of the molecules on each other regions, wherein the axes of the molecules are grouped around a direction of preference. In several parts of this phase the directions of preference will be divided accidentically, and as a consequence of irregular refraction such an unarranged condition will be opalescent.

In the other phase the effects of the molecules cannot cause suchlike directed regions and therefore there is no extinction. We hope to come back to the thermodynamics of these phases as well theoretically as experimentally.

Besides the forces mentioned above the influence of the molecular motion, which is always opposed to the directing effect has to be taken into consideration. In the following we will not go into the details of the optical problem of extinction, but we will use in our consideration the plausible supposition that the extinction will be the smaller according as the arrangement is more regular.

Let us begin our explication at the virginal state. By the influ-

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[1]) These latter influences cause a clustering tendency. For, if a molecule is at a certain point with a given direction of axis, this fact will influence the probability of the direction of the axis of a neighbouring molecule. Consequently the problem with which we are occupied at present shows an analogy with the problem of the clustering tendency in the neigbourhood of the critical point, treated by Zernike and one of us.

ence of the wall, added to the reciprocal effect of the molecules (correlation) the axes will show a preference for lying in horizontal planes. In the horizontal plane itself, however, each direction is equivalent.

Now, when we excite a perpendicular magnetic field, it tries to turn the molecules in a direction perpendicular to the plane of preference in virginal state; and the first result will be a disturbance of the order existing in the virginal state and therefore an increase of the extinction. Soon however a magnetic field of sufficient intensity will bring about a higher order i. e. a smaller extinction as is shown at C in fig. 2. A weak magnetic field however will only diminish the originally existing order i. e. increase of the extinction (see fig. 5). But a strong magnetic field has also to overcome the resistance of the reciprocal influence between the originally horizontally directed particles; therefore the slow rising towards C (fig. 2) is quite intelligible; whereas after F (fig. 2) a quicker rising can be explained, as the state of the matter there is such that no preference for directions parallel to the wall of glass is shown even at a very small distance from it.

Let us now proceed to consider the case represented in fig. 3. In the virginal state (B) the molecules were lying by preference in the horizontal planes, the exciting of the horizontal magnetic field (C) not only increases this preference, but moreover calls into existence in that horizontal plane a direction of preference. From this it follows that there can be no question of a temporary rising of the extinction at the exciting of a horizontal field.

The different conduct at the removal of the field can be explained too.

After the removal of the vertical field (*D* and *H* in fig. 2) the heat motion has free play as no direction of preference exists at some distance of the walls. The directing influence of the wall restores but slowly the original order by means of the mutual influences of the molecules. In the mean time (compare *FG* in fig. 4) if the field which has disturbed the order, is weak, the wall will sooner be able to recover the original state. Totally different is the case after the removal of the horizontal field (*D* and *H* in fig. 3). Here the influences proceeding from the wall, united with the mutual action of the molecules, practically maintain the higher order originated by the field.

The fact that the commutation of the magnetic field has no influence on the extinction shows that the particles themselves possess no polar peculiarities. A research which Mr. Rochell is working out in the Physical Laboratory, had proved already that the substance we had under consideration is paramagnetic.

For the moment these principal points may suffice. Perhaps we will discuss these questions when publishing further experimental results.

We gladly use the occasion to thank Prof. VAN ROMBURGH for his kindness in putting his preparation at our disposal.

Summary.

1. A new method is described to measure with the aid of thermopile and galvanometer the extinction of liquid-crystalline substances.
2. The very different influence on the extinction of a vertical (longitudinal) and a horizontal (transversal) magnetic field is traced.
3. An explanation of the observed phenomena is drawn in outline whereby the principal supposition is that the wall of glass directs the particles in planes parallel, directs then according the lines of force.

Physical Laboratory, Institute for Theoretical Physics.
Utrecht, October 1916.

Physics. — "*The clustering tendency of the molecules at the critical point*". By Prof. L. S. ORNSTEIN. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of May 27, 1916).

In a former communication by Dr. F. ZERNIKE and the author [1]) the arrangement of the molecules in space using a new method of probability is described, more accurately than this was possible in the considerations of VON SMOLUCHOWSKI and EINSTEIN; by this method it was also possible to calculate the opalescence at the critical point itself, which was impossible with the formulas of KEESOM-EINSTEIN.

We introduced a function f, defined in the following way. Suppose that space is divided into a great number of elements of volume $dV_0 dV_1 \dots dV_2$ etc. The numbers ν_0, ν_1, ν_2 etc. may represent the deviations of the average number of molecules in these elements. Then, if the *deviations* in all the surrounding elements of the working sphere are *given* ($\nu_1 \nu_2$ etc.), the average deviation in the element dV_0 may be represented by:

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[1]) Accidental deviations of density and the opalescence at the critical point of a single substance. These Proc, XVII, p. 793, 1914,

$$\overline{v}_0 = f_{01}\, v_1 dV_1 + f_{02} v_2 dV_2 + \dots \dots \quad (1)$$

The coëfficients f_{01}, etc. indicate the correlation of the elements. In a homogeneous phase they will only depend on the distance between the elements, whereas in a capillary transitory layer the function will be different.

As we published our paper we met with difficulties in the kinetic deduction of formula (1), owing to the fact that we tried to work with mathematically infinitely small elements of volume. As Dr. ZERNIKE [1]) has now solved another difficulty adhering to our considerations it is worth demonstrating that a deduction from statistical mechanics of (1) is possible. In this way it will be possible to indicate the physical meaning of function f and to prove that $\int f dV$ taken with respect to the working sphere is at the critical point equal to unity, which formerly was demonstrated by an artifice.

Starting from (12) of the cited paper the proof of formula (1) is very simple.

The number of the systems (3), for which in the elements dV_τ and dV_σ the deviations of the mean number of particles amount to v_τ and v_σ was found to be:

$$\zeta = C \omega^n \varrho^{-n}\, e^{\frac{1}{2\Theta} n\varrho \Sigma\, \varrho_{\sigma\tau} + \frac{1}{2} \sum_\sigma \left(-\frac{1}{v} + \frac{1}{v}\frac{d}{d\varrho} \varrho^2 \frac{d \log \omega}{d\varrho} + \frac{\varrho_{00}}{\Theta dV} \right) v_\sigma^2 + \Sigma \Sigma\, \varrho_{\sigma\tau} \frac{v_\sigma v_\tau}{dV}} \quad . \quad . \quad . \quad (2)$$

In this formula ϱ represents the number of molecules per unit of volume $\left(\frac{n}{V}\right)$, $\varphi_{\sigma\tau}$ is the mutual potential energy for a couple of molecules the one of which is lying in the element ϱ, the other in the element τ, ω is the function defined in my Thesis.

Now considering that the quantity $\frac{a}{\Theta dV}$ or $\Sigma \varrho \frac{\varphi_{\sigma\tau}}{\Theta}$ (as appears from the equation of state) is of the order of the unity, every term of the form $\frac{\varphi_{\sigma\tau}}{\Theta dv}$ will be small with respect to unity. Hence we may develop in the exponential function the part containing in v_τ and v_σ and write (summarising unimportant terms in the constant D):

$$\zeta = D\, e^{\frac{1}{2} \Sigma_0 \left(-\frac{1}{v} + \frac{1}{v}\frac{d}{d\varrho} \varrho^2 \frac{d \log}{d\varrho} \right) v_\sigma^2} \left(1 + \sum_\sigma \frac{\varphi_{00} v_\sigma^2}{2\Theta dV} + \sum_\sigma \sum_\tau \frac{\varphi_{\sigma\tau}}{\Theta dV} v_\sigma v_\tau \right) \quad (3)$$

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[1]) The clustering tendency of the molecules in the critical state and the extinction of light caused thereby. These Proc. XVIII, p. 1520. 1916.

The number of systems for which v_σ has every possible value and the other v's of the sphere of action have given values is found to be:

$$\bar{\zeta} = \frac{D\sqrt{2\pi}}{\sqrt{\frac{1}{v} - \frac{1}{v}\frac{d}{d\varrho}\varrho^2\frac{d\log\omega}{d\varrho}}}\, e^{\frac{1}{2}\sum\limits_{\tau}^{\tau\neq\sigma}\left(-\frac{1}{v}+\frac{1}{v}\frac{d}{d\varrho}\varrho^2\frac{d\log\omega}{d\varrho}\right)v_\tau^2} \quad . \quad (4)$$

The mean value of v_σ with given value of v_τ $(\tau \neq \sigma)$ in the sphere of action now becomes

$$\Sigma_\sigma v_\sigma \bar{\zeta}_\tau = D \sum_{\tau}^{\tau\neq\sigma} \frac{\varphi_{\sigma\tau}}{\Theta\, dV}$$

$$v_\tau \frac{\sqrt{2\pi}}{\sqrt{\left(\frac{1}{v} - \frac{1}{v}\frac{d}{d\varrho}\varrho^2\frac{d^3\log\omega}{d\varrho}\right)}}\, e^{\frac{1}{2}\,\Sigma_\tau\left(-\frac{1}{v}+\frac{1}{v}\frac{d}{d\varrho}\varrho^2\frac{d\log\omega}{d\varrho}\right)v_\tau^2} \quad . \quad (5)$$

From this follows

$$\bar{v}_\sigma = \frac{\Sigma v_\sigma \zeta}{\Sigma \zeta} = \Sigma_\tau v_\tau \frac{\varphi_{\sigma\tau}}{\Theta dV} \frac{v}{\left(1 - \frac{d}{d\varrho}\varrho^2\frac{d\log\omega}{d\varrho}\right)} = \Sigma \frac{\varrho\varphi_{\sigma\tau}}{\Theta\left(1 - \frac{d}{d\varrho}\varrho^2\frac{d\varrho}{d\log\omega}\right)} v_\tau \quad (6)$$

Thus we have now proved (1) whereas for $f_{\sigma\tau}$ we find

$$f_{\sigma\tau} = \frac{\varrho\varphi_{\sigma\tau}}{\Theta\left(1 - \frac{d}{d\varrho}\varrho^2\frac{d\log\omega}{d\varrho}\right)} \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

In my Thesis I have demonstrated that the pressure in an element may be represented by

$$\frac{p}{\Theta} = \varrho - \varrho^2\frac{d\log\omega}{d\varrho} - \frac{d\varrho^2}{2\Theta}\,.$$

Thus we have

$$f_{\sigma\tau} = \frac{\varrho\varphi_{\sigma\tau}}{\frac{dp}{d\varrho} + d\varrho} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

According to the meaning of $\varphi_{\sigma\tau}$, $\int \varphi_{\sigma\tau}\, dV$ taken with respect to the sphere of action equals a (in a homogeneous medium); we get therefore for F

$$F = \int f_{\sigma\tau}\, dV = \frac{\varrho a}{\frac{dp}{d\varrho} + \varrho a} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

For the critical point $\frac{dp}{d\varrho} = 0$ and therefore $F = 1$. It is worth noticing that in the deduction of (8) the fact that we have a homogeneous system has not been used. The given relation holds also in a capillary layer. However $f_{\sigma\tau}$ will depend on a parameter in the direction of the layer $\left(\text{for } \frac{dp}{d\varrho} \text{ depends on it}\right)$. The consideration may easily be extended to the case of a mixture and the capillary layers in a mixture. It will be possible then to develop MANDELSTAMM's [1]) considerations on the diffuse reflection at the layer of contact between two liquid phases in the critical point of mixture more exactly than he himself has done.

Utrecht, Mei 1916.

Physics. — *"The dilatation of solid bodies by heat."* By Prof. H. A. LORENTZ.

(Communicated in the meeting of October 30, 1915.)

When in the theory of specific heat the idea had been worked out that the heat motion of solid bodies consists in vibrations of the particles under the influence of the same forces that give rise to the phenomena of elasticity, DEBYE [2]) successfully attacked the problem of thermal dilatation. In his theory, which has been further developed by M. J. M. VAN EVERDINGEN [3]), it is shown that this phenomenon may be accounted for in a satisfactory way by adding in the expression for the potential energy of the body terms which are of the third order with respect to the displacements of the particles.

In the present paper considerations similar to those of DEBYE and VAN EVERDINGEN are presented in a form that is perhaps somewhat simpler.

§ 1. We shall suppose the body to be isotropic or crystallized in the regular system. Let S be its surface and v its volume at the temperature T and under a uniform pressure p. We can imagine that the particles lying on the surface are kept fixed in the positions about which they vibrate and that, when this has been done, the

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[1]) Ann. der Phys. 42.

[2]) P. DEBYE, Zustandsgleichung und Quantenhypothese, Wolfskehl-Vorträge, Göttingen, 1913, p. 17; Leipzig, Teubner.

[3]) M. I. M. VAN EVERDINGEN, De toestandsvergelijking van het isotrope, vaste lichaam. Proefschrift, Utrecht, 1914.

inner particles are likewise deprived of their heat motion. Then they will take definite positions of equilibrium P, P', P'' etc. This configuration of the system may be considered as the result of a dilatation equal in all directions, starting from the configuration that would exist at the absolute zero and in the absence of external pressure.

Now, always keeping fixed the outer particles, we may investigate the vibrations of the inner ones about their just mentioned positions of equilibrium $P, P', P''\ldots$ This is a perfectly definite problem. It can be simplified in the well known way by the introduction of a certain number of normal coordinates $q_1, q_2 \ldots q_s$, which we shall choose in such a manner that they are 0 in the position of equilibrium, so that they determine the deviation from that position. The corresponding velocities are $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_s$ and if the values of the coordinates are not too great we have for the potential energy U and the kinetic energy T expressions of the form

$$U = \tfrac{1}{2}\,(a_1 q_1^2 + a_2 q_2^2 + \ldots + a_s q_s^2)\,, \quad . \quad . \quad . \quad . \quad (1)$$

$$T = \tfrac{1}{2}\,(c_1 \dot{q}_1^2 + c_2 \dot{q}_2^2 + \ldots + c_s \dot{q}_s^2)\,, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

with positive, constant coefficients. Further there are s fundamental modes of vibration. In the first of these all the normal coördinates except q_1 are 0, in the second all except q_2 and so on. The frequencies of these fundamental vibrations are determined by

$$n_1 = \sqrt{\frac{a_1}{c_1}}\,, \quad n_2 = \sqrt{\frac{a_2}{c_2}}\,, \ldots \quad n_s = \sqrt{\frac{a_s}{c_s}}\,.$$

As to the deviation of a particle from the position of equilibrium, its components for the first particle may be represented by

$$\left.\begin{aligned}
\xi &= \alpha_1 q_1 + \alpha_2 q_2 + \ldots + \alpha_s q_s\,,\\
\eta &= \beta_1 q_1 + \beta_2 q_2 + \ldots + \beta_s q_s\,,\\
\zeta &= \gamma_1 q_1 + \gamma_2 q_2 + \ldots + \gamma_s q_s\,,\\
&\text{for the second by}\\
\xi' &= \alpha'_1 q_1 + \alpha'_2 q_2 + \ldots + \alpha'_s q_s\,,\\
\eta' &= \beta'_1 q_1 + \beta'_2 q_2 + \ldots + \beta'_s q_s\,,\\
\zeta' &= \gamma'_1 q_1 + \gamma'_2 q_2 + \ldots + \gamma'_s q_s\,,
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

and so on.

Here the coefficients α, β, γ have definite constant values. As the number s of the normal coordinates is equal to that of the degrees of freedom of the system, viz. to three times the number of the particles that are assumed to be movable, all possible displacements $\xi, \eta, \zeta, \xi', \eta', \zeta', \ldots$ may be represented by suitably chosen values of $q_1, q_2, \ldots q_s$.

§ 2. We shall now ascribe a greater mobility to the system by imagining that the outer particles too can be displaced, with the restriction, however, that for any instant their coordinates can be found by multiplying by the same factor $1 + q$ the coordinates which they had in the case considered in the preceding paragraph. Then, the quantity q, which we shall suppose to be very small compared with 1, will determine the position of the outer particles and by suitably extending the meaning of $q_1, \ldots . q_s$, these parameters may be made, together with q, to determine the position of the entire system.

Indeed, let $\overline{P}$, $\overline{P}'$, $\overline{P}''$, ... be the points which are found if the coordinates of P, P', P'', ... (§ 1) are altered in ratio of 1 to $1 + q$ simultaneously with the coordinates of the outer points, let $\xi, \eta, \zeta, \xi', \eta', \zeta'$ ... be the components of the deviations of the particles from *these* positions $\overline{P}$, $\overline{P}'$, $\overline{P}''$,... and let $q_1, q_2 \ldots q_s$ be quantities connected with $\xi, \eta, \zeta, \xi', \eta', \zeta'$ in the way shown by equations (3), if we continue to assign to α, β, γ the values we had to give them in the preceding paragraph. Then it is clear that the configuration of the whole system is really determined by $q, q_1 \ldots q_s$. The quantity q being now considered as variable, so that, though the places of the outer particles *in* the surface S be prescribed, this surface, keeping the same form, may dilate or contract as a whole, a constant value of q, i.e. a constant volume, can in general be maintained only by the application of an external force Q corresponding to that coordinate. It is precisely this force which we want to know, especially for the case $q = 0$, i. e. for the configuration of the body with which we began in § 1.

The value of Q is connected with that of the external pressure, for Q is defined by the condition that, for an infinitesimal variation δq of the coordinate q, the other coordinates remaining unchanged, the work of the external forces is $Q \delta q$. If now this change takes place starting from the value $q = 0$, all dimensions of the surface increase in ratio of 1 to $1 + \delta q$. The volume increases by $3v . \delta q$ and the work of the external pressure is $- 3pv . \delta q$. Hence

$$Q = - 3pv \quad \ldots \ldots \ldots \ldots \quad (4)$$

§ 3. The force Q may be determined by means of the equations of Lagrange, as soon as the potential energy U and the kinetic energy T are known as functions of all the coordinates q and the corresponding velocities.

Then we have

$$Q = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) \quad \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q},$$

where we must remark that the first term may be omitted. Indeed, whatever be the value of

$$\frac{\partial T}{\partial \dot{q}},$$

it certainly will be determined by the state of the body and its variations will therefore be limited to small changes on both sides of a certain mean value when the state is stationary. Then, however, the mean value of the differential coefficient $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right)$ taken over a time of sufficient length, will be equal to zero. We hardly need remark that it is such a mean value of the pressure p and therefore of the force Q, which we want to find.

So we may write

$$Q = -\frac{\partial T}{\partial q} + \frac{\partial U}{\partial q},$$

and, as we are seeking the value for the case $q = 0$, $\dot{q} = 0$, we may directly introduce this latter value into T and confine ourselves to terms with the first power of q when we represent U by a series.

By putting $\dot{q} = 0$, the points $\overline{P}, \overline{P}', \ldots$ of which we spoke in § 2 become immovable, so that we shall find the velocities of the particles by differentiating with respect to the time their deviations $\xi, \eta, \zeta, \xi', \eta', \zeta' \ldots$ from the positions $\overline{P}, \overline{P}', \ldots$ As now the coefficients α, β, γ in equations (3) are constants; the coordinate q does not appear in the expressions for the velocities and neither in T. This leads to a further simplification, viz.

$$Q = \frac{\partial U}{\partial q} \quad \ldots\ldots\ldots\ldots \quad (5)$$

§ 4. If, in the series for the potential energy, we confine ourselves, as we did in § 1, to the terms that are of the second order with respect to $q_1, q_2 \ldots q_s$, we may put

$$U = \tfrac{1}{2}(a_1 q_1^2 + a_2 q_2^2 + \ldots + a_s q_s^2) + (A_0 + A_1 + A_2)\, q \quad . \quad (6)$$

It is evident that the first term, which is to represent the potential energy for $q = 0$, must be the expression (1). Further A_0 is a constant, A_1 a homogeneous linear function of the coordinates $q_1, q_2 \ldots q_s$ and A_2 a homogeneous quadratic function of these same variables.

We have therefore, by (5)

$$Q = A_0 + A_1 + A_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In this equation we must take for $q_1, q_2, \ldots q_s$ the values as they are in the heat motion such as it really is. As now in the case of oscillations the mean value of each coordinate q over a long interval of time is 0, the term A_1 may be omitted.

As to A_0, this term represents the value of the force Q that would be required for maintaining the assumed value v of the volume, in case all the coordinates $q_1, \ldots q_s$ were 0, so that there would be no heat motion. For this force we may find an expression if we introduce the volume v_0 that the body would have at the absolute zero if it were free from external forces. In order to maintain at this same temperature the volume v, which we shall suppose to be greater than v_0, a negative pressure would have to be exerted on the body. It may be represented by

$$p = -\frac{v - v_0}{\varkappa v}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $\varkappa$ is a certain mean coefficient of cubical compressibility.

Substituting this in (4) we find

$$Q = \frac{3\,(v - v_0)}{\varkappa},$$

and this is the value of A_0. Thus, if there is a heat motion, we have according to (7)

$$Q = \frac{3\,(v - v_0)}{\varkappa} + A_2.$$

If finally we want to know what volume the body will occupy in the case of a heat motion, and in the absence of an external pressure, we have only to put $Q = 0$. We then find

$$v - v_0 = -\tfrac{1}{3}\,\varkappa A_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

for the connection between the heat motion and the volume, which it was our object to deduce.

§ 5. As to the meaning of A_2 we must remember that the part of the potential energy which contains terms of the second order with respect to $q_1, q_2 \ldots q_s$, will be

$$\tfrac{1}{2}\,(a_1 q_1^2 + a_2 q_2^2 + \ldots + a_s q_s^2) + q A_2$$

when the volume has increased to the extent determined by q.

After this expansion to the volume $(1 + 3q)v$ the coordinates $q_1, q_2, \ldots q_s$ need no longer be normal coordinates as they were for the volume v; so that A_2 may also contain products $q\ q_j$. As however the fundamental vibrations which constitute the heat motion, must be regarded as incoherent in phase, products of this kind will vanish from the mean value of A_2. So we obtain the right result, if we put

$$A_2 = \tfrac{1}{2}(a'_1 q_1^2 + a'_2 q_2^2 + \dots + a'_s q^2_s).$$

Therefore, according to (9), each harmonic mode of vibration contributes its part to the dilatation $v - v_0$, independently of the other modes.

The first of these parts is

$$-\tfrac{1}{6}\varkappa a'_1 q_1^2,$$

for which we may write

$$-\tfrac{1}{3}\varkappa \frac{\partial}{\partial q}(\tfrac{1}{2} a_1 q_1^2 + \tfrac{1}{2} q a'_1 q_1^2)$$

or on account of the connection between q and the volume

$$-\varkappa v \, . \, \frac{\partial}{\partial v}(\tfrac{1}{2} a_1 q_1^2 + \tfrac{1}{2} q a'_1 q_1^2) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now $\frac{1}{2} a_1 q_1^2$ is the value of the potential energy u_1 that belongs to the first coordinate during the heat motion in the state considered and $\frac{1}{2} a_1 q_1^2 + \frac{1}{2} q a'_1 q_1^2$ the value which this potential energy would have, if after the increase in volume determined by q the particles had the same deviations determined by q_1, from the positions $\overline{P}, \overline{P'}, \overline{P''}, \dots$ specified in § 2.

Thus we may write for (10)

$$-\varkappa v \frac{\partial u_1}{\partial v}.$$

To calculate the differential coefficient we must attend only to the first coordinate q_1, putting 0 for all the others.

Further, in performing the differentiation we must imagine that in the original volume v the particles have the deviations from their positions of equilibrium which, in the real heat motion, correspond to the first mode of vibration and that, after an infinitesimal increase of the volume they have the same deviations from the new positions of equilibrium $\overline{P}, \overline{P'}, \overline{P''}, \dots$

Proceeding in the same way with respect to the other coordinates, we obtain

$$v - v_0 = -\varkappa v \left(\frac{\partial u_1}{\partial v} + \frac{\partial u_2}{\partial v} + \dots + \frac{\partial u_s}{\partial v}\right) \quad . \quad . \quad . \quad . \quad (11)$$

§ 6. The calculation of the thermal dilatation by means of this formula will necessarily be a rather rough one. In the first place it is very questionable whether for somewhat high temperatures we may confine ourselves to terms of the second order with respect to the inner coordinates, and even if this were allowed, the difficulty would remain that we do not know enough about the forces acting between the particles to calculate the differential coefficients $\frac{\partial u}{\partial v}$. For the modes of vibration in which the wave-length is many times greater than the distances between neighbouring particles, these forces, so far as they have to be considered here, are determined by the ordinary elastic constants. If, however, the wave-length becomes of the same order of magnitude as those distances, this of course will no longer be so and unfortunately these very short waves are most prominent in the heat motion.

In his theory of specific heat however DEBYE, not withheld by this consideration, has applied the ordinary formulae of the theory of elasticity to all the modes of motion with which he was concerned, down to the shortest waves. Encouraged by his success we may avail ourselves of the same simplification in the theory of dilatation as has been done already by him and VAN EVERDINGEN. This enables us to continue the calculation of the right hand side of (11).

§ 7. We shall introduce the two constants of elasticity λ and μ, which are also used by DEBYE and which have been chosen in such a way that the potential energy per unit of volume is represented by the expression

$$\mu\,(x_x^2 + y_y^2 + z_z^2) + \tfrac{1}{2}\,\lambda\,(x_x + y_y + z_z)^2 + \tfrac{1}{2}\,\mu\,(x_y^2 + y_z^2 + z_x^2) \quad (12)$$

where

$$x_x = \frac{\partial \xi}{\partial x}, \ldots \qquad x_y = \frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}, \ldots$$

We remark that, if χ is any one of these six components of strain, or a homogeneous linear function of some of them, we may write

$$\frac{\partial \log \chi}{\partial \log v} = -\tfrac{1}{3} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

This is evident, if we keep in mind that, in the infinitesimal expansion determined by q, the quantities ξ, η, ζ are kept constant, so that their differential coefficients with respect to the coordinates are changed in ratio of 1 to $(1+q)^{-1}$.

The modes of vibration of which the heat motion consists, may be divided into two groups, that of the longitudinal and that of the transverse vibrations.

Now, if ω is an element of volume, the potential energy v_l contained in it and proper to a mode of motion of the first group, is proportional to an expression of the form

$$(\lambda + 2\mu)\, \chi^2 \omega,$$

while the potential energy v_t belonging to a mode of the second group is proportional to

$$\mu\, \chi^2\, \omega.$$

As ω changes proportionally to v, we have in virtue of (13)

$$\frac{\partial \log v_l}{\partial \log v} = \frac{d \log (\lambda + 2\mu)}{d \log v} + \tfrac{1}{3},$$

$$\frac{\partial \log v_t}{\partial \log v} = \frac{d \log \mu}{d \log v} + \tfrac{1}{3},$$

and this leads to similar relations for the potential energy u_l, u_t contained in the whole body. We may write them in the form

$$\frac{\partial u_l}{\partial \log v} = \left\{ \frac{d \log (\lambda + 2\mu)}{d \log v} + \tfrac{1}{3} \right\} u_l \,, \quad \ldots\ldots \quad (14)$$

$$\frac{\partial u_t}{\partial \log v} = \left\{ \frac{d \log \mu}{d \log v} + \tfrac{1}{3} \right\} u_t \quad \ldots\ldots\ldots \quad (15)$$

These formulae, of which the first may be used for all the terms in (11) that correspond to longitudinal motions and the second for all those that refer to transverse motions, also hold for the mean values which we have to take on the right hand side of (11). The mean values both of u_l and of u_t however are each half the total energy, and to this latter we must assign, both for the longitudinal and the transverse vibrations, the value ε_ν which depends on the frequency ν in the way specified in PLANCK's formula.

§ 8. Let us now first consider the terms on the right hand side of (11) that belong to modes of motion with frequencies between ν and $\nu + d\nu$. Let N be the total number of these modes, gN the number of those in which the vibrations are longitudinal and hN the number of those which consist in transverse vibrations, so that $g + h = 1$. To obtain

$$\Sigma \frac{\partial u}{\partial \log v} \quad \ldots\ldots\ldots\ldots \quad (16)$$

for this group of terms we must multiply (14) and (15) by gN and hN respectively and then take the sum, replacing at the same time u_l and u_t by their common value $\tfrac{1}{2}\, \varepsilon_\nu$. We shall also substitute for g and h the values that follow from DEBYE's calculations. He has found that the number of the longitudinal and that of the transverse modes of motion for which the frequency lies below an arbitrarily chosen limit are to each other in ratio of $(\lambda + 2\mu)^{-3/2}$ to $2\mu^{-3/2}$. As this is independent of ν it is also the ratio between the fractions g and h. Performing the calculations indicated we find for the sum (16)

$$\Sigma \frac{\partial u}{\partial \log v} = \left[-\frac{1}{3} \frac{d \log \{(\lambda+2\mu)^{-3/2}+2\mu^{-3/2}\}}{d \log v} + \frac{1}{6} \right] N\varepsilon_\nu .$$

To derive from this the sum

$$\frac{\partial u_1}{\partial \log v} + \frac{\partial u_2}{\partial \log v} + \ldots + \frac{\partial u_s}{\partial \log v}$$

which occurs in (11) we have still to extend the summation to all the modes of motion of different frequencies. As now $\Sigma N\varepsilon_\nu$ is the total energy E of the heat motion, (11) becomes

$$v - v_0 = \varkappa \left[\frac{1}{3} \frac{d \log \{(\lambda+2\mu)^{-3/2}+2\mu^{-3/2}\}}{d \log v} - \frac{1}{6} \right] E \quad . \quad . \quad (17)$$

In this formula we must give as well to $\varkappa$ as to the elastic constants λ and μ the values they would have if there were no heat motion, the volume being v, and strictly speaking it ought to be taken into account that these quantities and therefore the coefficient by which E is multiplied are more or less dependent on that volume; by this the equation becomes rather complicated. The simplest results will be obtained for very low temperatures. For these E is proportional to T^4. Hence, if we assume that the coefficient of E may be represented by a series

$$C_1 + C_2 (v - v_0) + \ldots$$

we may conclude that quite near the absolute zero $v - v_0$ is proportional to T^4 and the coefficient of dilatation $\frac{1}{v_0}\frac{dv}{dT}$ to T^3.

§ 9. The equation obtained for the dilatation can be still further simplified if one makes the assumption, rather arbitrary of course, that by an isotropic dilatation the coefficients λ and μ are made to change proportionally to each other. The coefficient of compressibility (for an infinitesimal change of volume) which has the value

$$\frac{3}{3\lambda + 2\mu}$$

and with which, in a rough approximation, we may identify the coefficient $\varkappa$ occurring in our formula, will then change in the inverse ratio to μ. We may also say that the quantity of which the logarithm appears in the numerator of the first fraction in (17) changes proportionally to $\varkappa^{3/2}$. Hence, denoting the pressure by p and using the relation

$$d \log v = - \varkappa \, dp$$

we have

$$v - v_0 = \left[- \tfrac{1}{2} \frac{d \log \varkappa}{dp} - \tfrac{1}{6} \varkappa \right] E ,$$

and if the coefficient is treated as independent of the temperature,

$$\frac{dv}{dT} = \left[- \tfrac{1}{2} \frac{d \log \varkappa}{dp} - \tfrac{1}{6} \varkappa \right] \frac{dE}{dT} .$$

If now ϱ is the density of the body, c the specific heat (the difference between c_p and c_v being considered as immaterial) expressed in calories and A the mechanical equivalent of heat, we have $\frac{dE}{dT} = A c \varrho v$, and for the coefficient of cubical expansion

$$\alpha = \frac{1}{v} \frac{dv}{dT} = \left[- \tfrac{1}{2} \frac{d \log \varkappa}{dp} - \tfrac{1}{6} \varkappa \right] A c \varrho , \quad . \quad . \quad . \quad . \quad (18)$$

a value that can well be positive, as the compressibility decreases with increasing pressure.

§ 10. An example may teach us, whether this result agrees with observation, at least as to the order of magnitude.

According to the measurements of LUSSANA [1]) the compressibility of lead decreases by about $\frac{1}{30}$ of its value when the pressure is raised to 1000 atmospheres. Therefore we have, taking the atmosphere as unit of pressure

$$\frac{d \log \varkappa}{dp} = - 3{,}3.10^{-5}$$

and if p is expressed in dynes per cm²

$$\frac{d \log \varkappa}{dp} = - 3{,}3.10^{-11} .$$

For the compressibility itself LUSSANA's value is $\varkappa = 3{,}9.10^{-12}$, so that the coefficient of $A c \varrho$ in (18) becomes equal to $1{,}6.10^{-11}$. With $A = 4{,}18.10^7$; $c = 0{,}03$ and $\varrho = 11$ we find $\alpha = 0{,}00022$, while in reality the coefficient of expansion is 0,00008.

For tin LUSSANA's observations lead to the numbers

$$\frac{d \log \varkappa}{dp} = - 3{,}7.10^{-11} , \qquad \varkappa = 2{,}7.10^{-12} .$$

Here $c = 0{,}05$ and $\varrho = 7{,}3$. This gives $\alpha = 0{,}00027$. The coefficient of expansion is 0,00006.

It is seen that the agreement is scarcely satisfactory.

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[1]) Taken from W. SCHUT, Piëzochemie der gecondenseerde systemen, p. 72. Proefschrift, Utrecht, 1912.

§ 11. For a few metals the value of $\frac{d\mu}{d \log v}$ can be derived from measurements made by POYNTING.[1]) This physicist has investigated the changes in length and diameter caused by the torsion of a wire. We shall shortly discuss this phenomenon, not only with a view to the numerical value that follows from it, but also because the theory shows a certain analogy with that of the dilatation by heat.

Let us consider a cylindrical wire, the axis of which we take for the z-axis, and let us suppose that, starting from the unstrained state, it is subjected to the following three deformations: 1. a homogeneous stretch in the direction of the length, 2. a displacement of the particles in radial direction, so that the distance r of a particle from the axis changes by sr, s being a function of r, and 3. a torsion, by which each cross-section normal to the axis is turned over an angle ϑz about its point of intersection with that line; then ϑ is the angle of torsion per unit of length.

Supposing the temperature to be kept constant we shall seek the free energy of the body in the final state reached by these three steps. Assuming it to be 0 in the original state we can calculate its changes by means of (12) or of similar expressions.

As the second of the three changes produces a stretch $\frac{d(sr)}{dr}$ in radial and a stretch s in tangential direction, we obtain the free energy that exists per unit of volume after the first two steps if we replace x_x, y_y, z_z in the first two terms of (12) by $s, s + r\frac{ds}{dr}$ and q.

The result is

$$\mu \left\{ 2s^2 + 2rs\frac{ds}{dr} + r^2\left(\frac{ds}{dr}\right)^2 + q^2 \right\} + \tfrac{1}{2}\lambda \left(2s + r\frac{ds}{dr} + q\right)^2 . \quad (19)$$

A point that originally was at a distance r from the axis, has now shifted to the distance $r' = (1+s)r$, while an element of the length dl has become $dl' = (1+q)\,dl$.

By the first two changes an annular element between two cylindric surfaces described about the axis with the radii r and $r + dr$, and further limited by two cross sections at a distance dl from each other, will have taken a volume for which with the approximation required for our calculation we may write

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[1]) POYNTING, On the changes in the dimensions of a steel wire when twisted, and on the pressure of distortional waves in steel, Proc. Royal Soc. (A) **86** (1912), p. 534.

$$2\pi\left(1+2s+r\frac{ds}{dr}+q\right)r\,dr\,dl \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Now to obtain the free energy in the state S that is reached by the first two deformations we should have to multiply (19) by this expression (20), and then to integrate it with respect to r and l. For *this* calculation however we may replace (20) by $2\pi r\,dr\,dl$, because in the expression for the free energy we shall omit terms that are of an order higher than the second with respect to q and s.

§ 12. To calculate now the change of the free energy accompanying the third deformation specified in § 11, we shall consider the state S as the original one and introduce elastic constants referring to it. On account of the preceding deformations determined by q and s, these constants are a little different from the values λ and μ introduced into (12). To find an expression for them we regard the quantities q and s as infinitely small and neglect their second and higher powers. The change we are investigating being proportional to ϑ^2, we obtain in this way terms with $q\vartheta^2$ and $s\vartheta^2$.

A point which in the state S has the coordinates x, y, z and lies at a distance r' from the axis, is displaced by the torsion ϑ over the distances

$$\xi = -\vartheta yz, \qquad \eta = +\vartheta xz, \qquad \zeta = 0,$$

to which correspond the components of strain

$$x_x = 0, \quad y_y = 0, \quad z_z = 0, \quad x_y = 0, \quad x_z = -\vartheta y, \quad y_z = +\vartheta x.$$

Let us now consider an element of volume which in the state S lies at a distance r' from the axis and for which $x=0$, $y=r'$. The preceding changes have given to this element the stretches $\mathbf{x} = s$, $\mathbf{y} = s + r\frac{ds}{dr}$ and $\mathbf{z} = q$ in the direction of the axes, without other changes of form. By the torsion it is now further subjected to a shear $x_z = -\vartheta r'$.

It is evident that the change in free energy per unit of volume caused by this shear will be obtained by multiplying $\frac{1}{2}x_z^2$ by a coefficient μ', which is the coefficient of rigidity μ as it has been modified by the dilatations $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. In calculating this modification we may treat $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ as infinitely small. It can be shown that

$$\mu' = \mu\,(1+2\mathbf{z}) + a\,(\mathbf{x}+\mathbf{z}) + b\,\mathbf{y}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where a and b are two constants depending on the nature of the material[1]). In this way we find for the change of the free energy

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[1]) In my original paper I had used a wrong formula, in which the term $2\mu\mathbf{z}$ was wanting, an error that has been pointed out by Mr. Tresling in his paper: On the use of third degree terms in the energy of a deformed elastic body. (These

per unit of volume caused by the torsion

$$\tfrac{1}{2}\left\{\mu(1+2q+2s) + a(q+s) + b\left(s+r\frac{ds}{dr}\right)\right\} \vartheta^2 r^2,$$

Proceedings, **19** (1916), p. 281). I shall avail myself of the occasion of this translation for introducing the corrections necessitated by his remark.

In deducing the new equation (21) we need not occupy ourselves with the term $b\,\mathbf{y}$; we have only to show that

$$\mu' = \mu(1+2\mathbf{z}) + a(\mathbf{x}+\mathbf{z}),$$

if $\mathbf{y}=0$. By this latter assumption the problem is reduced to one in two dimensions, which may be treated as follows.

Let x, z be the coordinates of a point in the original state and $x+\xi$, $z+\zeta$ its coordinates in the strained state, the displacements ξ, ζ being functions of x, z. We shall consider the free energy per unit of volume at the point $x+\xi$, $z+\zeta$; as compared with the free energy which we had originally in unit of volume.

The difference ψ must be a function of the quantities

$$\frac{\partial \xi}{\partial x} = a_1\,,\quad \frac{\partial \zeta}{\partial z} = a_2\,,\quad \frac{\partial \xi}{\partial z} = b_1\,,\quad \frac{\partial \zeta}{\partial x} = b_2$$

and can be developed in ascending powers of these, the series beginning with quantities of the second order and terms of the third order being necessary for our purpose.

As we may assume that the free energy is the same in the body considered and in a second body that is the image of the first with respect to a plane perpendicular to one of the axes of coordinates, the expansion can contain no terms that are of an odd order with respect to b_1 and b_2. Moreover the value of ψ must remain the same when the axes are rotated in their plane. These considerations lead to the formula

$$\psi = f(a_1+a_2)^2 + g(b_1-b_2)^2 + h(a_1 a_2 - b_1 b_2) + k(a_1+a_2)^3 + \\ + l(a_1+a_2)(b_1-b_2)^2 + m(a_1+a_2)(a_1 a_2 - b_1 b_2)$$

with six constants f, g, h, k, l, m, which can be easily verified. Indeed it can be shown that the values of a_1+a_2, b_1-b_2 and $a_1 a_2 - b_1 b_2$ are not altered by a rotation of the axes.

Let us next suppose the body, strained already in the way determined by a_1, a_2, b_1, b_2, to be rotated about OY through an infinitely small angle ω. This rotation, which must leave the value of ψ unchanged, leads to the variations

$$\delta\xi = -\omega(z+\zeta)\,,\quad \delta\zeta = \omega(x+\xi),$$

$$\delta(a_1+a_2) = \omega(b_1-b_2),\quad \delta(b_1-b_2) = -2\omega - \omega(a_1+a_2),$$

$$\delta(a_1 a_2 - b_1 b_2) = -\omega(b_1-b_2).$$

Substituting these values in $\delta\psi$ and putting equal to 0 the coefficients of the terms that are of the first and the second order with respect to a_1, a_2, b_1, b_2, one is led to the relations

$$h = -4g,\quad m = 2f - 2g - 4l,$$

so that

where r' has been replaced by $(1+s)\,r$ and where, as to q and s we have neglected terms of orders higher than the first.

Multiplying this by (20), integrating over the cylinder and adding the result to the free energy in the state S, of which the value has been found already, we obtain for the free energy in the final state

$$\psi = 2\pi l \int_0^R \left[\mu \left\{ 2s^2 + 2rs\frac{ds}{dr} + r^2\left(\frac{ds}{dr}\right)^2 + q^2 \right\} + \tfrac{1}{2}\lambda \left(2s + r\frac{ds}{dr} + q \right)^2 \right] r\,dr$$

$$+ \frac{\pi\Theta^2}{l} \int_0^R \left[\mu\left(1 + 4s + r\frac{ds}{dr} + q\right) + a(q+s) + b\left(s + r\frac{ds}{dr}\right) \right] r^3\,dr \quad . \quad (22)$$

where the original length and radius are denoted by l and R, and the total angle of torsion by Θ, so that $\vartheta = \dfrac{\Theta}{(1+q)l}$.

§ 13. Now Θ, q and the value $\mathbf{s}$, which s assumes for $r = R$, may be regarded as the parameters upon which external forces can act directly. If these parameters are kept fixed, we can determine the values of s within the wire by means of the condition that, for an arbitrary infinitesimal variation δs given to them, $\delta\psi$ must be 0.

For constant values of Θ and q we have by (22)

$$\delta\psi = \int_0^R G\,\delta s\,dr + \int_0^R F\,\frac{d\,\delta s}{dr}\,dr, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where

$$\psi = f(a_1 + a_2)^2 + g(b_1 - b_2)^2 - 4g(a_1 a_2 - b_1 b_2) + k(a_1 + a_2)^3 +$$
$$+ l(a_1 + a_2)(b_1 - b_2)^2 + (2f - 2g - 4l)(a_1 + a_2)(a_1 a_2 - b_1 b_2).$$

In the case considered in the text the final values of ξ and ζ (after the application of the torsion) are

$$\xi = \mathbf{x}\,x - \vartheta(1 + \mathbf{z})\,r'z = \mathbf{x}\,x + x_z(1 + \mathbf{z})\,z, \quad \zeta = \mathbf{z}\,z,$$

if x, z are the coordinates in the original state (before the application of the dilatations $\mathbf{x}$, $\mathbf{z}$). Hence

$$a_1 = \mathbf{x},\ a_2 = \mathbf{z},\ b_1 = (1 + \mathbf{z})\,x_z,\ b_2 = 0.$$

If these values are substituted in the expression for ψ, the coefficient of x_z^2 will give us the value of $\frac{1}{2}\,\mu'$. Hence

$$\mu' = 2g(1 + 2\mathbf{z}) + 2l(\mathbf{x} + \mathbf{z}),$$

or, if we replace $2l$ by a and observe that, for $\mathbf{x} = 0$, $\mathbf{z} = 0$, μ' must be equal to μ,

$$\mu' = \mu(1 + 2\mathbf{z}) + a(\mathbf{x} + \mathbf{z}).$$

$$G = 2\pi l \left[4(\lambda+\mu) rs + 2(\lambda+\mu) r^2 \frac{ds}{dr} + 2\lambda rq \right] + \frac{\pi\Theta^2}{l} (4\mu + a + b) r^3 . \quad (24)$$

$$F = 2\pi l \left[2(\lambda+\mu) r^2 s + (\lambda+2\mu) r^3 \frac{ds}{dr} + \lambda r^2 q \right] + \frac{\pi\Theta^2}{l} (\mu + b) r^4 \quad . \quad (25)$$

By partial integration of the second term (23) becomes

$$\delta\psi = \left| F\delta s \right|_0^R + \int_0^R \left(G - \frac{dF}{dr} \right) \delta s \, dr .$$

As for $r = 0$ the dilatations s and $r \frac{ds}{dr}$ must have finite values, the function F vanishes for $r = 0$, so that we obtain

$$\delta\psi = F_{r=R} \, \delta\mathbf{s} + \int_0^R \left(G - \frac{dF}{dr} \right) \delta s \, dr \, . \quad . \quad . \quad . \qquad (26)$$

If now we put $\delta\mathbf{s} = 0$, only the last term remains, so that we are led to the condition

$$G = \frac{dF}{dr} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (27)$$

or after some transformation

$$\frac{d}{dr} \left(r^3 \frac{ds}{dr} \right) = \frac{\Theta^2}{2l^2} \frac{a - 3b}{\lambda + 2\mu} r^3 .$$

But for $r = 0$ we may put $r^3 \frac{ds}{dr} = 0$. We find therefore

$$s = \frac{\Theta^2}{16 l^2} \cdot \frac{a - 3b}{\lambda + 2\mu} (r^2 - R^2) + \mathbf{s} \, . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

If this value is substituted in (22) we obtain ψ as a function of the external parameters Θ, $\mathbf{s}$ and q. By differentiation with respect to these variables we may calculate the external forces corresponding to them. We need only the two last ones, S and Q, of which S is immediately determined by (26). For according to this formula we have

$$\delta\psi = F_{r=R} \, \delta\mathbf{s} ,$$

so that

$$S = \frac{\partial\psi}{\partial\mathbf{s}} = F_{r=R},$$

which can be calculated by means of (25) and (28). As to

$$Q = \frac{\partial\psi}{\partial q}$$

this quantity is found if, after differentiating in (22) under the signs of integration we substitute the value (28) and then perform the integrations.

The result is

$$S = 2\pi l\,[\lambda R^2 q + 2\,(\lambda + \mu)\,R^2\mathbf{s}] + \frac{\pi\Theta^2}{4l}\,(4\mu + a + b)\,R^4$$

$$Q = \pi l\,[(\lambda + 2\mu)\,R^2 q + 2\lambda R^2\mathbf{s}] + \frac{\pi\Theta^2}{4l}\,(\mu + a)\,R^4\,.$$

If no stretching forces act on the ends of the wire, nor any forces on the surface, we have $Q = 0$, $S = 0$, so that

$$2\lambda q + 4\,(\lambda + \mu)\,\mathbf{s} = -\frac{\Theta^2 R^2}{4l^2}\,(4\mu + a + b)\,, \quad . \quad . \quad . \quad (29)$$

$$(\lambda + 2\mu)\,q + 2\lambda\mathbf{s} = -\frac{\Theta^2 R^2}{4l^2}\,(\mu + a)\,. \quad . \quad . \quad . \quad . \quad . \quad (30)$$

When the coefficients of elasticity λ, μ and the coefficients a and b are given, we can derive from these equations the changes in length and diameter (q and **s**) caused by a torsion.

§ 14. We shall use formulae (29) and (30) to calculate the coefficients a and b from Poynting's measurements.

Poynting has worked with two steel wires and one copper wire, for which he has determined in the first place Young's modulus $\dfrac{\lambda + \mu}{\mu\,(3\lambda + 2\mu)}$ and Poisson's ratio $\dfrac{\lambda}{2(\lambda + \mu)}$. From these quantities we can calculate λ and μ. Further he has measured q and **s**, so that a and b can be found. The results are given in the following table, in which everything has been expressed in C.G.S. units. The length of the wire was in all cases

$$l = 160{,}5 \text{ cm}$$

and the numbers given for q and **s** refer to the value $\Theta = 2\pi$; so they indicate by what part of the original value the length and the diameter change, if one end of the wire is once twisted round.

	R	Young's modulus	Poisson's ratio	λ	μ	q	$\mathbf{s}$	a	b
Steel 1	0,0493	$2{,}12 . 10^{12}$	0,270	$9{,}77 . 10^{11}$	$8{,}35 . 10^{11}$	$1{,}71 . 10^{-6}$	$-3{,}19 . 10^{-7}$	$-5{,}03 . 10^{12}$	$0{,}58 . 10^{12}$
Steel 2	0,0605	$2{,}12 . 10^{12}$	0,287	$11{,}09 . 10^{11}$	$8{,}24 . 10^{11}$	$2{,}90 . 10^{-6}$	$-5{,}24 . 10^{-7}$	$-5{,}70 . 10^{12}$	$0{,}70 . 10^{12}$
Copper	0,06095	$1{,}31 . 10^{12}$	0,331	$9{,}64 . 10^{11}$	$4{,}92 . 10^{11}$	$4{,}25 . 10^{-6}$	$1{,}75 . 10^{-6}$	$-3{,}94 . 10^{12}$	$3{,}37 . 10^{12}$

§ 15. We can further calculate $\frac{d \log \mu}{d \log v}$ by means of the values found for a and b. Let us suppose the metal to be stretched equally in all directions, so that there is an infinitely small cubical dilatation $d \log v$. Then we have according to (21) in which expression we must put $\mathbf{x} = \mathbf{y} = \mathbf{z} = \frac{1}{3}\, d \log v$,

$$d\mu = \tfrac{1}{3}(2\mu + 2a + b)\, d \log v,$$

$$\frac{d \log \mu}{d \log v} = \frac{2\mu + 2a + b}{3\mu}.$$

To calculate from this the coefficient of dilatation, we shall suppose that, when the volume is increased, λ and μ change proportionally to each other.

The differential coefficient in (17) then becomes

$$-\tfrac{3}{2}\frac{d \log \mu}{d \log v},$$

and the formula itself

$$v - v_0 = \varkappa \left[-\tfrac{1}{2}\frac{d \log \mu}{d \log v} - \tfrac{1}{6} \right] E.$$

Treating the coefficient of E as a constant (comp. § 9) we find from this for the coefficient of cubical expansion

$$\alpha = \varkappa \left[-\tfrac{1}{2}\frac{d \log \mu}{d \log v} - \tfrac{1}{6} \right] A c \varrho.$$

If the coefficient of compressibility $\varkappa$ is derived from λ and μ, this equation gives the following results:

	$\frac{d \log \mu}{d \log v}$	$\varkappa$	c	ϱ	α	
					calc.	obs.
Steel 1	— 3,1	$6{,}5.10^{-13}$	0,11	7,8	$3{,}2.10^{-5}$	$3{,}3.10^{-5}$
" 2	— 3,7	$6{,}0.10^{-13}$	0,11	7,8	$3{,}6.10^{-5}$	$3{,}3.10^{-5}$
Copper	— 2,4	$7{,}7.10^{-13}$	0,093	8,9	$2{,}8.10^{-5}$	$5{,}1.10^{-5}$

The only inaccuracy in the above calculation of the terms in (11) corresponding to transverse vibrations is the application of the ordinary formulae of the theory of elasticity to very short waves. For the determination of the terms referring to the longitudinal vibrations, however, we had to make the assumption that λ changes proportionally to μ. As however the transverse vibrations have a greater part in the heat motion than the longitudinal ones we may perhaps hope that the error introduced by *this* assumption will not be considerable[1]).

We mentioned already the analogy between the problem treated in §§ 11—13 and that of the thermal expansion. In the one case the torsion plays the same part as the heat motion in the other and the quantities that have been indicated by q in the two problems are comparable with each other; the similarity of the mathematical treatment in the two cases is likewise evident. Poynting remarks that a dilatation of the wire will also take place when it executes torsional vibrations or when vibrations of this kind are propagated in it. With similar phenomena we are generally concerned, when an elastic body is traversed by waves, and when we consider the very short waves especially, this leads us directly to an insight into the nature of thermal dilatation.

Finally it deserves our attention that, though the phenomena discussed in this paper are chiefly determined by the change of the elastic constants caused by a previous deformation, yet there are as well in equation (17) as in (29) and (30) terms that are independent of this change.

Physics. – "*On* Einstein's *Theory of gravitation.*" I. By Prof. H. A. Lorentz.

(Communicated in the meeting of February 26, 1916).

§ 1. In pursuance of his important researches on gravitation Einstein has recently attained the aim which he had constantly kept in view; he has succeeded in establishing equations whose form is not changed by an arbitrarily chosen change of the system of coordinates[2]). Shortly afterwards, working out an idea that had been expressed already in one of Einstein's papers, Hilbert[3]) has shown the use that may be made of a variation law that may be regarded as Hamilton's principle in a suitably generalized form. By these results the "general theory of relativity" may be said to have taken a definitive form, though much remains still to be done in further

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[1]) This paper had already gone to press, when an article of Försterling came under my notice (Ann. d. Phys. **47** (1915) p 1127) in which considerations similar to those here developed are put forward.

[2]) A. Einstein, Zur allgemeinen Relativitätstheorie, Berliner Sitzungsberichte 1915, pp. 778 799; Die Feldgleichungen der Gravitation, ibid. 1915, p. 844.

[3]) D. Hilbert, Die Grundlagen der Physik I, Göttinger Nachrichten, Math.-phys. Klasse, Nov. 1915.

developing it and in applying it to special problems. It will also be desirable to present the fundamental ideas in a form as simple as possible.

In this communication it will be shown that a four-dimensional geometric representation may be of much use for this latter purpose; by means of it we shall be able to indicate for a system containing a number of material points and an electromagnetic field (or eventually only one of these) the quantity H, which occurs in the variation theorem, and which we may call the *principal function*. This quantity consists of three parts, of which the first relates to the material points, the second to the electromagnetic field and the third to the gravitation field itself.

As to the material points, it will be assumed that the only connexion between them is that which results from their mutual gravitational attraction.

§ 2. We shall be concerned with a four-dimensional extension R_4, in which "space" and "time" are combined, so that each point P in it indicates a definite place A and at the same time a definite moment of time t. If we say that P refers to a material point we mean that at the time t this point is found at the place A. In the course of time the material point is represented every moment by a new point P; all these points lie on the "world-line", which represents the state of motion (or eventually the state of rest) of the material point[1]. In the same sense we may speak of the world-line of a propagated light-vibration. An intersection of two world-lines means that the two objects to which they belong meet at a certain moment, that a "coincidence" takes place[2]. Now EINSTEIN has made the striking remark[3] that the only thing we can learn from our observations and with which our theories are essentially concerned, is the existence of these coincidences. Let us suppose e.g. that we have observed an occultation of a star by the moon or rather the reappearance of a star at the moon's border. Then the world-line of a certain light-vibration starting from a point on the world-line of the star has in its further course intersected the world-line of a

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[1] It will be known that in the theory of relativity MINKOWSKI was the first who used this geometric representation in an extension of four dimensions. The name "world-line" has been borrowed from him.

[2] For the sake of simplicity we shall imagine the two motions not to be disturbed by this coincidence, so that e.g. two material points penetrate each other or pass each other at an extremely small distance without any mutual influence.

[3] In a correspondence I had with him.

point of the border of the moon and finally that of the observer's eye. A similar remark may be made when the moment of reappearance is read on a clock. Let us suppose that the light-vibration itself lights the dial-plate, reaching it when the hand is at the point a; then we may say that three world-lines, viz. that of the light-vibration, that of the hand and that of the point a intersect.

§ 3. We may imagine that, in order to investigate a gravitation field as e.g. that of the sun, a great number of material points, moving in all directions and with different velocities, are thrown into it, that light-beams are also made to traverse the field and that all coincidences are noted [1]). It would be possible to represent the results of these observations by world-lines in a four-dimensional figure — let us say in a "field-figure" — the lines being drawn in such a way that each observed coincidence is represented by an intersection of two lines and that the points of intersection of one line with a number of the others succeed each other in the right order.

Now, as we have to attend only to the intersections, we have a great degree of liberty in the construction of the "field-figure". If, independently of each other, two persons were to describe the same observations, their figures would probably look quite different and if these figures were deformed in an arbitrary way, without break of continuity, they would not cease to serve the purpose.

After having constructed a field-figure F we may introduce "coordinates", by which we mean that to each point P we ascribe four numbers x_1, x_2, x_3, x_4, in such a way that along any line in the field-figure these numbers change continuously and that never two different points get the same four numbers. Having done this we may for each point P seek a point P' in a four-dimensional extension R'_4, in which the numbers $x_1, \ldots x_4$ ascribed to P are the Cartesian coordinates of the point P'. In this way we obtain in R'_4 a figure F', which just as well as F can serve as field-figure and which of course may be quite different according to the choice of the numbers $x_1 \ldots x_4$ that have been ascribed to the points of F.

If now it is true that the coincidences only are of importance it must be possible to express the fundamental laws of the phenomena by geometric considerations referring to the field-figure, in such a way that this mode of expression is the same for all possible field-figures; from our point of view all these figures can be considered as being the *same*. In such a geometric treatment the introduction of

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[1]) In other terms, that the data procured by astronomical observations can be extended arbitrarily and unboundedly.

coordinates will be of secondary importance; with a single exception (§ 13) it only serves for short calculations which we have to intercalate (for the proof of certain geometric propositions) and for establishing the final equations, which have to be used for the solution of special problems. In the discussion of the general principles coordinates play no part; and it is thus seen that the formulation of these principles can take place in the same way whatever be our choice of coordinates. So we are sure beforehand of the general covariancy of the equations that was postulated by EINSTEIN.

§ 4. EINSTEIN ascribes to a line-element PQ in the field-figure a length ds defined by the equation

$$ds^2 = \Sigma\,(ab)\, g_{ab}\, dx_a\, dx_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$(g_{ab} = g_{ba})$$

Here $dx_1 \ldots dx_4$ are the changes of the coordinates when we pass from P to Q, while the coefficients g_{ab} depend in one way or another on the coordinates. The gravitation field is known when these 10 quantities are given as functions of $x_1 \ldots x_4$. Here it must be remarked that in all real cases the coordinates can be chosen in such a way that for one point arbitrarily chosen (1) becomes

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2.$$

This requires that the determinant g of the coefficients of (1) be always negative. The minor of this determinant corresponding to the coefficient g_{ab} will be denoted by G_{ab}.

Around each point P of the field-figure as a centre we may now construct an infinitesimal surface [1]), which, when P is chosen as origin of coordinates, is determined by the equation

$$\Sigma\,(ab)\, g_{ab}\, x_a x_b = \varepsilon^2\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where ε is an infinitely small positive constant which we shall fix once for all. This surface, which we shall call the *indicatrix*, is a hyperboloid with one real axis and three imaginary ones. We shall also introduce the surface determined by the equation

$$\Sigma\,(ab)\, g_{ab}\, x_a x_b = -\varepsilon^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which differs from (2) only by the sign of ε^2. We shall call this the *conjugate indicatrix*. It is to be understood that the indicatrices and conjugate indicatrices take part in the changes to which the field-figure may be subjected. As these surfaces are infinitely small,

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[1]) A "surface" determined by one equation between the coördinates is a three-dimensional extension. It will cause no confusion if sometimes we apply the name of "plane" to certain two-dimensional extensions, if we speak e.g. of the "plane" determined by two line-elements.

they always remain hyperboloids of the said kind. The gravitation field will now be determined by these indicatrices, which we can imagine to have been constructed in the field-figure without the introduction of coordinates. When we have occasion to use these latter, we shall so choose them that the "axes" x_1, x_2, x_3 intersect the conjugate indicatrix constructed around their starting point, while the indicatrix itself is intersected by the axis x_4. This involves that the coefficients g_{11}, g_{22}, g_{33} are negative and that g_{44} is positive.

§ 5. The indicatrices will give us the units in which we shall express the length of lines in the field-figure and the magnitude of two-, three or four-dimensional extensions. When we use these units we shall say that the quantities in question are expressed in *natural measure*.

In the case of a line-element PQ the unit might simply be the radius-vector in the direction PQ of the indicatrix or the conjugate indicatrix described about P. It is however desirable to distinguish the two cases that PQ intersects the indicatrix itself or the conjugate indicatrix. In the latter case we shall ascribe an imaginary length to the line-element[1]). Besides, by taking as unit not the radius-vector itself but a length proportional to it, the numerical value of a line-element may be made to be independent of the choice of the quantity ε.

These considerations lead us to define the length that will be ascribed to line-elements by the assumption that each radius-vector of the indicatrix has in natural measure the length ε, while each radius-vector of the conjugate indicatrix has the length $i\varepsilon$. [2])

It will now be clear that the length of an arbitrary line in the field-figure can be found by integration, each of its elements being measured by means of the indicatrix or the conjugate indicatrix belonging to the position of the element. In virtue of our definitions a deformation of the field-figure will not change the length of lines expressed in natural measure and a geodetic line will remain a geodetic line.

§ 6. We are now in a position to indicate the first part H_1 of the principal function (§ 1). Let σ be a closed surface in the field-figure and let us confine ourselves to the principal func-

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[1]) This corresponds to the negative value which (1) gives for ds^2.

[2]) For a radius-vector on the asymptotic cone we may take either of these values; this makes no difference, as the numerical value of a line-element in the direction of such a radius-vector becomes 0 in both cases.

tion so far as it belongs to the space Ω enclosed by that surface. Then the quantity H_1 is the sum, taken with the negative sign, of the lengths of all world-lines of material points so far as they lie within Ω, each length multiplied by a constant m, characteristic of the point in question and to be called its *mass.*[1])

It must be remarked that the elements of the world-lines of material points intersect the corresponding indicatrices themselves. The lengths of these lines are therefore real positive quantities.

A deformation of the field-figure leaves H_1 unchanged.

§ 7. We shall now pass on to the part of the principal function belonging to the gravitation field. The mathematical expression for this part was communicated to me by EINSTEIN in our correspondence. It is also to be found in HILBERT's paper in which it is remarked that the quantity in question may be regarded as the measure of the *curvature* of the four-dimensional extension to which (1) relates. Here we have to speak only of the interpretation of this quantity. To find this the following geometrical considerations may be used.

Let PQ and PR be two line-elements starting from a point P of the field-figure, QR the line-element joining the extremities Q and R. If then the lengths of these elements in natural measure are

$$PQ = ds', \quad PR = ds'', \quad QR = ds,$$

we define the angle (s', s'') between PQ and PR by the well known trigonometric formula

$$ds^2 = ds'^2 + ds''^2 - 2ds'ds'' \cos(s', s''),$$

$$\cos(s', s'') = \frac{ds'^2 + ds''^2 - ds^2}{2ds'ds''} \quad \ldots \ldots \ldots (4)$$

from which one can derive

$$\cos(s', s'') = \Sigma\,(ab)\, g_{ab} \frac{dx'_a}{ds'} \frac{dx''_b}{ds''}. \quad \ldots \ldots (5)$$

By means of this formula we are able to determine the angle between any two intersecting lines. Of course the two other angles of the triangle PQR can be calculated in the same way.

Now two cases must be distinguished.

a. The plane of the triangle PQR cuts the conjugate indicatrix, but not the indicatrix itself. Then the three sides have positive imaginary values. Moreover each of them proves to be smaller than

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[1]) This agrees with the value of the LAGRANGIAN function, which is to be found e.g. in my paper on "HAMILTON's principle in EINSTEIN's theory of gravitation." These Proceedings **19** (1916), p. 751.

the sum of the others, from which one finds that the angles have real values and that their sum is π.

b. The plane PQR cuts both the indicatrix and the conjugate indicatrix. In this case different positions of the triangle are still possible. We can however confine ourselves to triangles the three sides of which are real. These are really possible, for in the plane of a hyperbola we can draw triangles the sides of which are parallel to radius-vectors drawn from the centre to points of the curve (and not of the conjugate hyperbola).

By a closer consideration of the triangles now in question it is found however that by the choice of our "natural" units one side is necessarily longer than the sum of the other two. Formula (4) then shows that the cosines of the angles are real quantities, greater than 1 in absolute value, two of them being positive, and the third negative. We must therefore ascribe to the angles imaginary or complex values. If for $p > +1$ we put

$$arc\ cos\ p = i\ log\ (p + \sqrt{p^2 - 1})$$

and

$$arc\ cos\ (-p) = \pi - arc\ cos\ p\ ,$$

we find for the three angles expressions of the form

$$i\,\alpha,\ i\,\beta \text{ and } \pi - i\,(\alpha + \beta)\ ,$$

so that the sum is again π.

From the cosine calculated by (4) or (5) the sine can be derived by means of the formula

$$sin\ \varphi = \sqrt{1 - cos^2\ \varphi}\ ,$$

where for the case $cos^2\ \varphi > 1$ we can confine ourselves to the value

$$sin\ \varphi = i \sqrt{cos^2\ \varphi - 1}$$

with the positive sign.

It deserves special notice that two conjugate radius-vectors of the indicatrix and the conjugate indicatrix are perpendicular to each other and that a deformation of the field-figure does not change the angle between two intersecting lines determined according to our definitions.

§ 8. Before proceeding further we must now indicate the natural units (§ 5) for two-, three-, or four-dimensional extensions in the field-figure. Like the unit of length, these are defined for each point separately, so that the numerical value of a finite extension is found by dividing it into infinitely small parts.

A two-dimensional extension cuts the conjugate indicatrix in an ellipse, or the indicatrix itself and the conjugate indicatrix in two

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conjugate hyperbolae. In both cases we derive our unit from the area of a parallelogram described on conjugate radius-vectors.

A three-dimensional extension cuts the conjugate indicatrix in an ellipsoid, or the indicatrix and its conjugate in two conjugate hyperboloids. Now our unit will be derived from the volume of a parallelepiped described on three conjugate radius-vectors.

In a similar way the magnitude of four-dimensional extensions will be determined by comparison with a parallelepiped the edges of which are four conjugate radius-vectors of the indicatrix and the conjugate indicatrix.

It must here be kept in mind that, according to well known theorems, the area of the parallelogram and the volume of the parallelepipeds in question are independent of the special choice of the conjugate radius-vectors.

We shall further specify the units in such a way (comp. § 5) that the numerical magnitude of a parallelogram or a parallelepiped described on conjugate radius-vectors is found by multiplying the numbers by which the edges are expressed in natural measure.

From what has been said it follows that the area of the parallelogram described on two line-elements is given by the product of the lengths of these elements and the sine of the enclosed angle. Similarly the area of an infinitely small triangle is determined by half the product of two sides and the sine of the angle between them.

We need hardly add that the numerical value of any two-, three- or four-dimensional domain expressed in natural measure is not changed by a deformation of the field-figure.

§ 9. Let, at any point P of the field-figure, 1, 2, 3, 4 be four arbitrarily chosen conjugate radius-vectors of the indicatrix. Two of these determine an infinitely small part V of a two-dimensional extension. We may prolong this part to finite distances from P by drawing from this point geodetic lines whose initial directions lie in the plane V. In this way we obtain six two-dimensional extensions (1,2), (2,3), (3,1), (1,4), (2,4) and (3,4). Let us now consider in one of these e. g. (a, b) an infinitesimal triangle near the point P, the sides of which are geodetic lines (viz. geodetic lines *in* (a, b)). If in calculating the angles of this triangle we go to quantities of the second order with respect to the sides and to the distances from P, the sum s of the angles proves to have no longer the value π (comp. § 7). The "excess" $e = s - \pi$ is proportional to the area Δ of the triangle, independently of the length of the sides, of their ratios and of the position of the triangle in the extension (a, b). For the three extensions (1,2) (2,3), (3,1), which do not intersect the indicatrix itself but the conjugate indicatrix, this proposition follows from a well-known theorem of GAUSS in the theory of curvature of surfaces; for the other three (1,4), (2,4), (3,4), which cut the indicatrix itself, the proof can be given by direct calculation. The considerations necessary for this, and some other calculations with which we shall be concerned further on will be communicated in a later paper.

In considering the three last-mentioned extensions I have confined myself to triangles with real sides (§ 7, b).

The quotient

$$\frac{e}{\Delta} = K_{ab}$$

is now for each extension a definite number, which we may consider as a measure of the *curvature* of the two-dimensional extension (a, b); the sum K of the six numbers K_{ab} may be called the *curvature of the field-figure* at the point P in question. This quantity is the same that has been introduced by HILBERT; this results from the calculation of its value, which at the same time shows K to be independent of the special choice of the directions 1, 2, 3, 4 introduced in the beginning of this §.

The numbers K_{ab} are all real and have a meaning that can be indicated without the introduction of coordinates; moreover their sum K is not changed by a deformation of the field-figure.

If now $d\Omega$ is an element of the four-dimensional extension of the field-figure, expressed in natural measure, the part of the principal function belonging to the gravitation field is

$$H_3 = \frac{i}{\varkappa}\int K d\Omega \,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where the integration is extended to the domain considered (§ 6) while $\varkappa$ is the gravitation constant. H_3 too is not changed by a deformation of the field-figure.

The factor i has been introduced in order to obtain a real value for H_3, the element $d\Omega$ being represented in natural measure by a negative imaginary number (§ 8).

§ 10. What we have to say of the electromagnetic field must be preceded by some considerations belonging to what may be called the "vector theory" of the field-figure.

A line-element PQ, taken in a definite direction (indicated by the order of the letters), may be called a *vector*. Such vectors can be compounded or decomposed by means of parallelograms or parallelepipeds. Especially, when coordinates $x_1, \ldots x_4$ have been chosen, a vector may be resolved into four components which have the directions of the coordinates, viz. such directions that a shift along the first e.g. changes only x_1, while x_2, x_3, x_4 remain constant. The four components in question are determined by the differentials $dx_1, \ldots dx_4$ corresponding to PQ. We shall say that by these they are expressed in "x-measure". Their values in natural measure are found by multiplying $dx_1, \ldots dx_4$ by certain factors. If we keep in mind that the radius-vectors of the conjugate indicatrix and the indicatrix in the directions of the axes are expressed in "x measure" by

$$\frac{\varepsilon}{\sqrt{-g_{11}}}, \quad \frac{\varepsilon}{\sqrt{-g_{22}}}, \quad \frac{\varepsilon}{\sqrt{-g_{33}}}, \quad \frac{\varepsilon}{\sqrt{g_{44}}},$$

and in natural units by

$$i\varepsilon, \quad i\varepsilon, \quad i\varepsilon, \quad \varepsilon$$

we find for the reducing factors

$$l_1 = i\sqrt{-g_{11}}, \; l_2 = i\sqrt{-g_{22}}, \; l_3 = i\sqrt{-g_{33}}, \; l_4 = \sqrt{g_{44}} \quad . \quad (7)$$

In the language of vector-analysis the vector obtained by the composition of two or more vectors is also called the *sum* of these vectors.

We shall also speak of *finite* vectors, i.e. of directed quantities which can be represented on an infinitely reduced scale by line-elements in the field-figure. If ω is the constant "reduction factor" chosen for this purpose, a vector A will be represented by a line-element $\omega\mathsf{A}$, the direction of which is also ascribed to A. It will now be evident that two finite vectors, as well as two infinitely small ones, determine an infinitesimal two-dimensional extension and that finite vectors can be compounded and resolved by means of parallelograms and parallelepipeds. Also that we may speak of the "magnitude" of such figures, that e.g. the rule given in § 8 applies to the parallelogram described on two vectors.

The components of a vector in the directions of the coordinates expressed in x-measure will be called X_1, X_2, X_3, X_4. This means that $\omega X_1, \ldots \omega X_4$ are equal to the differentials $dx_1, \ldots dx_4$ corresponding to the infinitely small vector $\omega\mathsf{A}$.

If we want to know the components of A in natural units we must multiply $X_1, \ldots X_4$ by the factors (7).

§ 11. Two vectors A and B starting from a point P of the field-figure and lying in a plane V, determine what we shall call a *rotation* R in that plane. We ascribe to it the direction indicated by the order AB and a value given by the parallelogram described on

A and B and expressed in natural measure[1]). This involves that the same rotation may be represented in many different ways by two vectors in the plane V.

For the rotation R we shall also use the symbol $[A . B]$.

By the *vector product* $[A . B . C]$ of three vectors A, B, C at a point of the field-figure and not lying in one plane we shall understand a vector D the direction of which is conjugate with each of the three vectors (and therefore with the three-dimensional extension A, B, C), the direction of D corresponding to those of A, B and C in a way presently to be indicated, while the magnitude of D, expressed in natural measure, is equal to that of the parallelepiped described on A, B and C and expressed in the same measure. This definition involves that the value 0 is ascribed to the vector product of three vectors lying in one and the same plane.

A further statement about the direction of D is necessary because *two* opposite directions are conjugate with A, B, C. For one set of three directions A_0, B_0, C_0 we shall choose arbitrarily which of its two conjugate directions will be said to correspond to it. If this is the direction D_0, then the direction D corresponding to A, B, C will be determined by the rule that D_0 passes into D by a gradual passage of the first three vectors from A_0, B_0, C_0 into A, B, C, this latter passage being effected in such a way that during the change the vectors never come to lie in one plane.

The vector product $[A . B . C]$ takes the opposite direction when one of the vectors is reversed as well as when two of them are interchanged. We must therefore always attend to the order of the symbols in $[A . B . C]$.

The vector product possesses the distributive property with respect to each of the three vectors, so that e.g. if A_1 and A_2 are vectors,

$$[(A_1 + A_2) . B . C] = [A_1 . B . C] + [A_2 . B . C].$$

From this we can infer that $[A . B . C]$ depends only on C and the rotation R determined by A and B. For this reason we write for the vector product also $[R . C]$; in calculating it we are free to replace the rotation R by any two vectors by means of which it can be represented.

If R, R_1 and R_2 are rotations in the same plane, such that the value and direction of R are found by adding R_1 and R_2 algebraically, we have, in virtue of the distributive property

$$[R_1 . C] + [R_2 . C] = [R . C]$$

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[1]) If, according to circumstances, different signs are given to R, the angle whose sine occurs in the formula for the area of a parallelogram must be understood to be positive in one case and negative in the other

§ 12. In what precedes we were concerned with the volumes of parallelepipeds expressed in natural units. When we have introduced coordinates $x_1, \ldots x_4$ we may also express these volumes in the "x-units" corresponding to the coordinates chosen.

Let us consider e.g. the three-dimensional extension $x_4 = \text{const.}$, which cuts the conjugate indicatrix in the ellipsoid

$$g_{11}x_1^2 + g_{22}x_2^2 + g_{33}x_3^2 + 2g_{12}x_1x_2 + 2g_{23}x_2x_3 + 2g_{31}x_3x_1 = -\varepsilon^2.$$

If we agree that in x-measure spaces in this extension will be represented by positive numbers and that a parallelepiped with the positive edges dx_1, dx_2, dx_3 will have the volume $dx_1\, dx_2\, dx_3$, we find for that of the parallelepiped on three conjugate radius-vectors

$$\frac{\varepsilon^3}{\sqrt{-G_{44}}},$$

where it has been taken into consideration that G_{44} is negative.

The volume of the same parallelepiped being expressed in natural measure by $-i\varepsilon^3$ (§ 8), we have to multiply by

$$l_{123} = -i\sqrt{-G_{44}} \quad \ldots\ldots\ldots \quad (8)$$

if we want to pass from the expression in x-measure to that in natural measure.

For the extension (x_2, x_3, x_4), i.e. $x_1 = 0$ the corresponding factor is

$$l_{234} = -\sqrt{G_{11}} \quad \ldots\ldots\ldots \quad (9)$$

§ 13. In the theory of electromagnetic phenomena we are concerned in the first place with the electric charge and the convection current. So far as these quantities belong to a definite element $d\Omega$ of the field-figure they may be combined into

$$\mathsf{q}\, d\Omega$$

where q is a vector which we may call the *current vector*. When it is resolved into four components having the directions of the axes, the first three components determine the convection current, while the fourth component gives the density of the electric charge.

As to the electric and the magnetic force, these two taken together can be represented at each point of the field-figure by two rotations

$$\mathsf{R}_e \text{ and } \mathsf{R}_h$$

in definite, mutually conjugate two-dimensional extensions. These quantities are closely connected with the current vector, for after having introduced coordinates $x_1, \ldots x_4$ we have for each closed surface σ the vector equation

$$\int \{ [\mathsf{R}_e \cdot \mathsf{N}] + [\mathsf{R}_h \cdot \mathsf{N}] \}_x \, d\sigma = i \int \{ \mathsf{q} \}_x \, d\Omega , \quad . \quad . \quad . \quad (10)$$

where the second integral has to be taken over the domain Ω enclosed by σ. On the left hand side $d\sigma$ represents a three-dimensional surface-element expressed in natural units and N a vector of the magnitude 1 in natural measure conjugate with or perpendicular to that element (§ 7) and directed towards the outside of the domain Ω. The index x shows that the vector $[\mathsf{R}_e.\mathsf{N}]+[\mathsf{R}_h.\mathsf{N}]$ must be expressed in x-measure. At each point of the surface we must resolve the vector along the four directions of the coordinates, express each component in x-measure (§ 10) and finally, after multiplication by $d\sigma$, we must add algebraically all x_1-components; similarly all x_2-components and so on.

It must be expressly remarked that if an equation like (10) in which we are concerned with the composition of vectors at *different* points of the field-figure, shall have a definite meaning we must know which components are to be considered as having the same direction, so that they can be added. This has been determined by the introduction of coordinates.

On the right hand side of the equation the index x means that the vector q must be expressed in x-measure and the factor i had to be introduced because $d\Omega$ is imaginary.

One can prove that equation (10) is equivalent to the differential equations which in Einstein's theory serve for the same purpose and further that when the equation holds for one choice of coordinates it will also be true for any other choice.

§ 14. The proof for these assertions must be deferred to the second part of this communication. For the present we shall only add that the part of the principal function referring to the electromagnetic field is given by

$$H_2 = i \int \tfrac{1}{2} (\mathsf{R}_e^2 + \mathsf{R}_h^2) \, d\Omega ,$$

where R_e and R_h are, expressed in natural units, the two rotations that are characteristic of the field. Like the two other parts of the principal function, H_2 is not changed by a deformation of the field-figure. In this statement it is to be understood that the parallelograms by which R_e and R_h are represented take part in the deformation.

Some remarks on the way in which, starting from the principal function, we may obtain the fundamental equations of the theory must also be deferred. I shall conclude now by remarking that, as an immediate consequence of HAMILTON's principle, the world-line of a material point which is acted on only by a given gravitation field, will be a geodetic line, and that the equations which determine the gravitation field caused by material and electromagnetic systems will be found by the consideration of infinitely small variations of the indicatrices, by which the numerical values of all quantities that are measured by means of these surfaces will be changed.

Physics. — "*On* EINSTEIN'*s Theory of gravitation.*" II. By Prof. H. A. LORENTZ.

(Communicated in the meeting of March 25, 1916).

§ 15. In the first part of this communication the connexion between the electric and the magnetic force on one hand and the charge and the convection current on the other was expressed by the equation

$$\int \{[\mathsf{R}_e \,.\, \mathsf{N}] + [\mathsf{R}_h \,.\, \mathsf{N}]\}_x d\sigma = i \int \{\mathsf{q}\}_x \, d\boldsymbol{\Omega}, \quad . \quad . \quad . \quad . \quad (10)$$

which has been discussed in § 13. It will now be shown that this formula is equivalent to the differential equations by which the connexion in question is expressed in the theory of EINSTEIN. For this purpose some further geometrical considerations must first be developed. They refer to the special case that the quantities g_{ab} have the same values at every point of the field-figure.

If this condition is fulfilled, considerations which generally may be applied to infinitesimal extensions only are valid for finite extensions too.

§ 16. The factor required, in the measurement of four-dimensional domains, for the passage from x-units to natural units has now the same value at every point of the field-figure. Similarly, when any one-, two- or three-dimensional extension in the field-figure that is determined by linear equations ("linear extensions") is considered, the factor by means of which the said passage may be effected for parts of that extension, will be the same for all those parts. Moreover the factor in question will be the same for two "parallel" extensions of this kind, i.e. for two extensions the determining equations of which can be written in such a way that the coefficients of $x_1, \ldots x_4$ are the same in them.

It is obvious that linear one-dimensional extensions can be called "straight lines", also it will be clear what is to be understood by a "prism" (or "cylinder"). This latter is bounded by two mutually parallel linear three-dimensional extensions σ_1 and σ_2 and by a lateral surface which may be extended indefinitely to both sides and in which mutually parallel straight lines ("generating lines") can be drawn. We need not dwell upon the elementary properties of the prism.

§ 17. A vector may now be represented by a straight line of finite length; the quantities $X_1, \ldots X_4$, which have been introduced in § 10, are the changes of the coordinates caused by a displacement along that line. The magnitude of the vector, expressed in natural units, will be denoted by S. It is given by a formula similar to (1), viz. by

$$S^2 = \Sigma\,(ab)\, g_{ab}\, X_a\, X_b\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

A vector may be regarded as being the *same* everywhere in the field-figure, if $X_1, \ldots X_4$ have constant values. In the same way a rotation R (§ 11) may be said to be the same everywhere, if it can be represented by two vectors of this kind.

If from a point P two vectors PQ and PR issue, denoted by $X_1', \ldots X_4'$, S' and $X_1'', \ldots X_4''$, S'' resp., the angle between them (comp. (5)) is defined by

$$S'\,S''\cos(S', S'') = \Sigma\,(ab)\, g_{ab}\, X'_a\, X''_b\,. \quad . \quad . \quad . \quad . \quad (12)$$

We remark here that X_a', X_b'' are real, positive or negative quantities and that S' and S'' are expressed in the way indicated in § 5 ("absolute" values). It is to be understood that S does not change when the signs of $X_1, \ldots X_4$ are reversed at the same time.

If S''' is the value of the vector RQ and if the angle between this vector and RP is denoted by (S'', S'''), it follows further from (11) and (12) that

$$S'' = S'\cos(S', S'') + S'''\cos(S'', S''').$$

In the special case of a right angle R we have

$$S'' = S'\cos(S', S''),$$

an equation expressing the connexion between a vector PQ and its "projection" on a line PR. The angle (S', S'') is the angle between the vector and its projection, both reckoned from the same point P.

§ 18. Let us now return to the prism P mentioned in § 16. From a point A_2 of the boundary of the "upper face" σ_2 we can draw a line perpendicular to σ_2 and σ_1. Let B_1 be the point, where it cuts this last plane, the "base", and A_1 the point where this plane is encountered by the generating line through A_2. If then $\angle A_1 A_2 B_1 = \vartheta$, we have

$$\overline{A_2 B_1} = \overline{A_2 A_1} \cos \vartheta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The strokes over the letters indicate the absolute values of the distances $A_2 B_1$ and $A_2 A_1$.

It can be shown (§ 8) that, all quantities being expressed in natural units, the "volume" of the prism P is found by taking the product of the numerical values of the base σ_1 and the "height" $A_2 B_1$.

Let now linear three-dimensional extensions perpendicular to $A_1 A_2$ be made to pass through A_1 and A_2. From these extensions the lateral boundary of the prism cuts the parts σ_1' and σ_2' and these parts, together with the lateral surface, enclose a new prism P', the volume of which is equal to that of P. As now the volume of P' is given by the product of $\overline{A_2 A_1}$ and σ_1', we have with regard to (13)

$$\sigma'_1 = \sigma_1 \cos \vartheta .$$

If now we remember that, if a vector perpendicular to σ_1 is projected on the generating line, the ratio between the projection and the vector itself (viz. between their absolute values) is given by $\cos \vartheta$ and that a connexion similar to that which was found above between a normal section σ'_1 of the prism and σ_1 also exists between σ'_1 and any other oblique section, we easily find the following theorem:

Let σ and $\overline{\sigma}$ be two arbitrarily chosen linear three-dimensional sections of the prism, N and $\overline{\mathsf{N}}$ two vectors, perpendicular to σ and $\overline{\sigma}$ resp. and of the same length, S and $\overline{S}$ the absolute values of the projections of N and $\overline{\mathsf{N}}$ on a generating line. Then we have

$$S\sigma = \overline{S}\overline{\sigma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

§ 19. After these preliminaries we can show that the left hand side of (10) is equal to 0, if the numbers g_{ab} are constants and if moreover both the rotation R_e and the rotation R_h are everywhere the same. For the two parts of the integral the proof may be given in the same way, so that it suffices to consider the expression

$$\int [\mathsf{R}_e \cdot \mathsf{N}]_x \, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Let $X_1, \ldots X_4$ be the components of the vector N, expressed in x-units. From the distributive property of the vector product it then follows that each of the four components of

$$[\mathsf{R}_c \,.\, \mathsf{N}]_x$$

is a homogeneous linear function of $X_1, \ldots X_4$. Under the special assumptions specified at the beginning of this § these are everywhere the same functions. Let us thus consider a definite component of (15) e.g. that which corresponds to the direction of the coordinate x_a. We can represent it by an expression of the form

$$\int (\alpha_1 X_1 + \ldots + \alpha_4 X_4)\, d\sigma,$$

where $\alpha_1, \ldots . \alpha_4$ are constants. It will therefore be sufficient to prove that the four integrals

$$\int X_1\, d\sigma \ldots \int X_4\, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

vanish.

In order to calculate $\int X_1\, d\sigma$ we consider an infinitely small prism, the edges of which have the direction x_1. This prism cuts from the boundary surface σ two elements $d\sigma$ and $\overline{d\sigma}$. Proceeding along a generating line in the direction of the positive x_1 we shall enter the extension Ω bounded by σ through one of these elements and leave it through the other. Now the vectors perpendicular to σ, which occur in (15) and which we shall denote by N and $\overline{\mathsf{N}}$ for the two elements, have the same value.[1] If, therefore, S and $\overline{S}$ are the absolute values of the projections of N and $\overline{\mathsf{N}}$ on a line in the direction x_1, we have according to (14)

$$S\, d\sigma = \overline{S}\, \overline{d\sigma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Let first the four directions of coordinates be perpendicular to one another. Then the components of the vector obtained by projecting N on the above mentioned line are $X_1, 0, 0, 0$ and similarly those of the projection of $\overline{\mathsf{N}}$: $\overline{X}_1, 0, 0, 0$. But as, proceeding in the direction of x_1, we enter Ω through one element and leave it through the other, while N and $\overline{\mathsf{N}}$ are both directed outward, X_1 and $\overline{X}_1$ must have opposite signs. So we have

$$S : \overline{S} = X_1 : -\overline{X}_1$$

and because of (17) we may now conclude that the elements $X_1\, d\sigma$

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[1] From § 10 it follows that if the length of a vector A that is represented by a line (§ 17) coincides with a radius-vector of the conjugate indicatrix, it is always represented by an imaginary number. We may however obtain a vector which in natural units is represented by a real number e.g. by 1 (§ 13) if we multiply the vector A by an imaginary factor, which means that its components and also those of a vector product in which it occurs are multiplied by that factor.

and $\overline{X_1}\ \overline{d\sigma}$ in the first of the integrals (16) annul each other. It will be clear now that the whole integral vanishes and that similar considerations may be applied to the other three.

So we have proved that under the special assumptions made the left hand side of (10) will vanish in the special case that the directions of the coordinates are perpendicular to each other. This conclusion likewise holds for an other set of coordinates if only the assumption made at the beginning of this § is fulfilled. This is obvious, as we can pass from mutually perpendicular coordinates $x_1, \ldots x_4$ to arbitrarily chosen other ones $x'_1, \ldots x'_4$ which fulfil this latter condition by linear transformation formulae with constant coefficients. The x- and the x'-components of the vector

$$[\mathsf{R}_e\ .\ \mathsf{N}] + [\mathsf{R}_h\ .\ \mathsf{N}]$$

are then connected by homogeneous linear formulae with coefficients which have the same value at all points of the surface σ. Hence if, as has been shown above, the four x-components of the vector

$$\int \{ [\mathsf{R}_e\ .\ \mathsf{N}] + [\mathsf{R}_h\ .\ \mathsf{N}] \} d\sigma$$

vanish, the four x'-components are now seen to do so likewise. [1])

§ 20. The above considerations were intended to prepare a corollary which will be of use in the treatment of the integral on the left hand side of (10), if we now leave the special assumptions made above and suppose the quantities g_{ab} to be functions of the coordinates while also the rotations R_e and R_h may change from point to point.

This corollary may be formulated as follows: If all dimensions of the limiting surface σ are infinitely small of the first order, the integral

$$\int \{ [\mathsf{R}_e\ .\ \mathsf{N}] + [\mathsf{R}_h\ .\ \mathsf{N}] \}_x d\sigma$$

will be of the *fourth* order.

In order to make this clear let us suppose that in the calculation of the integral we confine ourselves to quantities of the third order. The surface σ being already of that order we may then omit all infinitesimal values in the quantities by which $d\sigma$ is multiplied;

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[1]) In the above considerations difficulties might arise if the vector N lay on the asymptotic cone of the indicatrix, our definition of a vector of the value 1 would then fail (comp. note 2, p. 1345). With a view to this we can choose the for mof the extension Ω (§ 13) in such a way that this case does not occur, a restriction leading to a boundary with sharp edges.

we may therefore neglect the infinitesimal changes of the quantities g_{ab} over the extension considered, and also those of R_e and R_h. By this we just come to the case considered in § 19. Thus it is evident, that as regards quantities of the third order the first part of (10) is 0. From this it follows that in reality it is at least of the fourth order.

§ 21. Let us now return to the general case that the extension Ω to which equation (10) refers, has finite dimensions. If by a surface $\overline{\sigma}$ this extension is divided into two extensions Ω_1 and Ω_2, the quantities on the two sides in (10) each consist of two parts referring to these extensions. For the right hand side this is immediately clear and as to the quantity on the left hand side, it follows from the consideration that the contributions of $\overline{\sigma}$ to the integrals over the boundaries of Ω_1 and Ω_2 are equal with opposite signs. In the two cases namely we must take for N equal but opposite vectors.

Also, if the extension Ω is divided into an arbitrary number of parts, each term in (10) will be the sum of a number of integrals, each relating to one of these parts.

By surfaces with the equations $x_1 = \text{const.}, \ldots x_4 = \text{const.}$ we can divide the extension Ω into elements which we shall denote by $(dx_1, \ldots dx_4)$. As a rule there will be left near the surface σ certain infinitely small extensions of a different form. From the preceding § it is evident that, in the calculation of the integrals, these latter extensions may be neglected and that only the extensions $(dx_1, \ldots dx_4)$ have to be considered. From this we can conclude that equation (10) is valid for any finite extension, as soon at it holds for each of the elements $(dx_1, \ldots dx_4)$.

§ 22. We shall now show what equation (10) becomes for one element $(dx_1, \ldots dx_4)$. Besides the infinitesimal quantities $x_1, \ldots x_4$, occurring in the equation

$$F = \Sigma (ab)\, g_{ab}\, x_a\, x_b = \varepsilon^2$$

of the indicatrix we introduce four other quantities $\xi_1, \ldots \xi_4$, which we define by

$$\xi_a = \frac{1}{2} \frac{\partial F}{\partial x_a}, \quad \ldots \ldots \ldots \quad (18)$$

or

$$\left.\begin{array}{l} \xi_1 = g_{11}\, x_1 + g_{12}\, x_2 + \ldots + g_{14}\, x_4 \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ \xi_4 = g_{41}\, x_1 + g_{42}\, x_2 + \ldots + g_{44}\, x_4 \end{array}\right\} \quad \ldots \quad (19)$$

with the equalities $g_{ba} = g_{ab}$.

To each of these quantities corresponds a definite direction, viz. that in which we have to proceed in order to make the considered quantity change in positive sense while the other three remain constant. If we denote these directions by 1*, 2*, 3*, 4* and in the same way the directions of the coordinates x_1, x_2, x_3, x_4 by 1, 2, 3, 4, it is evident that 1* is conjugate with 2, 3 and 4, 2* with 3, 1 and 4, and so on; inversely 1 with 2*, 3*, 4*; 2 with 3*, 1*, 4*, and so on. From what has been said above about the algebraic signs of g_{11}, g_{22}, g_{33}, g_{44} it follows further that, if directions opposite to 1, 1* etc. are denoted by -1, -1^* etc., the directions -1 and 1* will point to the same side of an extension $x_1 =$ const. The same may be said of the directions -2 and 2* or -3 and 3* with respect to extensions $x_2 =$ const. or $x_3 =$ const., while with respect to an extension $x_4 =$ const. the directions 4 and 4* point to the same side.

Finally, we shall fix (§ 11) as far as is necessary, which direction corresponds to three others. For that purpose we shall imagine the directions of coordinates 1, . . . 4 to pass into mutually conjugate directions, which will also be called 1, . . . 4, by gradual changes, in such a way that never three of them come to lie in one plane. We shall agree that after this change -4 corresponds to 1, 2, 3.

Let a, b, c, d be the numbers 1, 2, 3, 4 in an order obtained from the natural one by an *even* number of permutations. Then the rule of § 11 teaches us that the direction $-d$ corresponds to a, b, c. It is clear that this would be the case with d, if a, b, c, d were obtained from 1, 2, 3, 4 by an odd number of permutations. If further it is kept in mind that, always in the new case, the directions 1*, 2*, 3*, 4* coincide with -1, -2, -3, 4, we come to the conclusion that the directions 1, 2, 3 and 4 correspond to the sets 2*, 3*, 4*; 3*, 1*, 4*; 1*, 2*, 4* and 1*, 2*, 3* respectively. The rule of gradual change (§ 11) involves that this holds also for the original case, in which 1, 2, 3, 4 were not yet mutually conjugate.

This is all that has to be said about the relations between the different directions. It must only be kept in mind, that whenever two of the first three directions are interchanged, the fourth must be reversed.

§ 23. In the neighbourhood of a point P of the field-figure we may introduce as coordinates instead of $x_1, \ldots x_4$ the quantities $\xi_1, \ldots \xi_4$ defined by (19). Line-elements or finite vectors can be resolved in the directions of these coordinates, i. e. in the directions

1*, 2*, 3*, 4*. Their components and the magnitudes of different extensions can now be expressed in ξ-units in the same way as formerly in x-units. So the volume of a three-dimensional parallelepiped with the positive edges $d\xi_1$, $d\xi_2$, $d\xi_3$ is represented by the product $d\xi_1 \; d\xi_2 \; d\xi_3$.

Solving $x_1, \ldots x_4$ from (19) we obtain expressions of the form

$$\left.\begin{array}{l} x_1 = \gamma_{11}\,\xi_1 + \gamma_{21}\,\xi_2 + \ldots + \gamma_{41}\,\xi_4 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_4 = \gamma_{14}\,\xi_1 + \gamma_{24}\,\xi_2 + \ldots + \gamma_{44}\,\xi_4 \\ \qquad \gamma_{ba} = \gamma_{ab} \end{array}\right\} \quad . \; . \; . \; . \; . \quad (20)$$

If we use the coordinates ξ the coefficients γ_{ab} play the same part as the coefficients g_{ab} when the coordinates x are used. According to (18) and (20) we have namely

$$F = \Sigma\,(a)\,\xi_a\,x_a = \Sigma\,(ab)\,\gamma_{ba}\,\xi_a\,\xi_b,$$

so that the equation of the indicatrix may be written

$$\Sigma\,(ab)\,\gamma_{ba}\,\xi_a\,\xi_b = \varepsilon^2.$$

§ 24. Let the rotations R_e and R_h of which we spoke in § 13 be defined by the vectors A^{I}, A^{II} and $\mathsf{A}^{\mathrm{III}}$, A^{IV} respectively, the resultants of the vectors $\mathsf{A}_{1*}{}^{\mathrm{I}}, \ldots \mathsf{A}_{4*}{}^{\mathrm{I}}$, etc. in the directions 1*, ... 4*. Then, according to the properties of the vector product that were discussed in § 11,

$$\begin{aligned}[\mathsf{R}_c \,.\, \mathsf{N}] &= [(\mathsf{A}_{1*}{}^{\mathrm{I}} + \ldots + \mathsf{A}_{4*}{}^{\mathrm{I}}) \,.\, (\mathsf{A}_{1*}{}^{\mathrm{II}} + \ldots + \mathsf{A}_{4*}{}^{\mathrm{II}}) \,.\, \mathsf{N}] \\ &= \Sigma\,(\overline{ab})\,\{\,[\mathsf{A}_{a*}{}^{\mathrm{I}} \,.\, \mathsf{A}_{b*}{}^{\mathrm{II}} \,.\, \mathsf{N}] - [\mathsf{A}_{a*}{}^{\mathrm{II}} \,,\, \mathsf{A}_{b*}{}^{\mathrm{I}} \,.\, \mathsf{N}]\,\},\end{aligned}$$

where the stroke over ab indicates that each combination of two different numbers a, b contributes one term to the sum. For the vector product $[\mathsf{R}_h \,.\, \mathsf{N}]$ we have a similar equation. Now two or more rotations in one and the same plane, e.g. in the plane a^*b^*, may be replaced by one rotation, which can be represented by means of two vectors with arbitrarily chosen directions in that plane, e.g. the directions a^* and b^*. We may therefore introduce two vectors B_{a*} and B_{b*} directed along a^* and b^* resp., so that

$$\begin{aligned}[\mathsf{B}_{a*} \,.\, \mathsf{B}_{b*}] = [\mathsf{A}_{a*}{}^{\mathrm{I}} \,.\, \mathsf{A}_{b*}{}^{\mathrm{II}}] - [\mathsf{A}_{a*}{}^{\mathrm{II}} \,.\, \mathsf{A}_{b*}{}^{\mathrm{I}}] + \\ + [\mathsf{A}_{a*}{}^{\mathrm{III}} \,.\, \mathsf{A}_{b*}{}^{\mathrm{IV}}] - [\mathsf{A}_{a*}{}^{\mathrm{IV}} \,.\, \mathsf{A}_{b*}{}^{\mathrm{III}}] \quad . \quad (21)\end{aligned}$$

Then we must substitute in (10)

$$[\mathsf{R}_e \,.\, \mathsf{N}] + [\mathsf{R}_h \,.\, \mathsf{N}] = \Sigma\,(\overline{ab})\,[\mathsf{B}_{a*} \,.\, \mathsf{B}_{b*} \,.\, \mathsf{N}] \quad . \; . \; . \quad (22)$$

Here it must be remarked that the magnitude and the sense of one of the vectors B may be chosen arbitrarily; when this has been done, the other vector is perfectly determined.

In the following calculations the vector N has one of the directions $1^*, \ldots 4^*$. As this is also the case with the vectors B_{a^*} and B_{b^*}, the vector product occurring in (22) can easily be expressed in ξ-units. After that we may pass to natural units and finally, as is necessary for the substitution in (10), to x-units.

In order to pass from ξ-units to natural units we have to multiply a vector in the direction a^* by a certain coefficient λ_a, and a part of the extension a^*, b^*, c^* by a coefficient λ_{abc}. These coefficients correspond to l_a (§ 10) and l_{abc} (§ 12). The factors λ_{abc} e.g. can be expressed by means of the minors Γ_{ab} of the determinant γ of the quantities γ_{ab}. If this is worked out and if the equations

$$\gamma_{ab} = \frac{G_{ab}}{g} \quad , \quad g_{ab} = \frac{\Gamma_{ab}}{\gamma} \quad . \quad g\gamma = 1$$

are taken into consideration, we obtain the following corollary, which we shall soon use:

Let a, b, c, d and also $a', b' c', d'$ be the numbers 1, 2, 3, 4 in any order, a' being not the same as a, then we have, if none of the two numbers a and a' is 4,

$$\frac{l_{bcd}\, \lambda_{b'c'd'}}{l_{a'}\, \lambda_a} = -1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and if one of the two is 4

$$\frac{l_{bcd}\, \lambda_{b'c'd'}}{l_{a'}\, \lambda_a} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

§ 25. We shall now suppose (comp. § 24) that in ξ-units the vector B_{a^*} has the value $+1$, and we shall write χ_{ab} for the value that must then be given to B_{b^*}. If the ξ-components of the vectors A^{I} etc. are denoted by $\Xi_1{}^{\mathrm{I}}, \ldots \Xi_4{}^{\mathrm{I}}$ etc., we find from (21)

$$\chi_{ab} = (\Xi_a{}^{\mathrm{I}}\, \Xi_b{}^{\mathrm{II}} - \Xi_a{}^{\mathrm{II}}\, \Xi_b{}^{\mathrm{I}}) + (\Xi_a{}^{\mathrm{III}}\, \Xi_b{}^{\mathrm{IV}} - \Xi_a{}^{\mathrm{IV}}\, \Xi_b{}^{\mathrm{III}}) \quad . \quad (25)$$

This formula involves that

$$\chi_{ba} = -\chi_{ab}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

It may be remarked that χ_{ba} is the value that must be given to the vector B_{a^*} if B_{b^*} is taken to be 1.

The quantities χ_{ab} may be said to represent the rotations $[B_{a^*} \,.\, B_{b^*}]$. At the end of our calculations we shall introduce instead of χ_{ab} the quantities ψ_{ab} defined by

$$\psi_{ab} = \chi_{a'b'}\, (a \mp b) \quad , \quad \psi_{aa} = 0 \quad . \quad . \quad . \quad . \quad . \quad (27)$$

In the first of these equations a, b, a', b' are supposed to be the numbers 1, 2, 3, 4, in an order obtained from 1, 2, 3, 4 by an even number of permutations.

§ 26. We have now to calculate the left hand side of equation (10) for the case that σ is the surface of an element $(dx_1, \ldots dx_4)$. For this purpose we shall each time take together two opposite sides, calculating for each pair the contributions due to the different terms on the right hand side of (22), or as we may say to the different rotations χ_{ab}. It is convenient now to denote by a, b, c the numbers $1, 2, 3$ either in this order or in any other derived from it by a cyclic permutation, while the x-components of the vector we are calculating and which stands on the left hand side of (10) will be represented by $X_1, \ldots X_4$.

a. Let us first consider that one of the sides (dx_a, dx_b, dx_c) which faces towards the side of the positive x_4. The vector N drawn outward has the direction 4^* and in ξ-units the magnitude $\frac{1}{\lambda_4}$. As the direction c corresponds to $a^*, b^*, 4^*$, the rotation χ_{ab} gives with N a vector product represented by a vector in the direction c. The magnitude of this vector is in ξ-units

$$\frac{1}{\lambda_4} \chi_{ab}$$

and in natural units

$$\frac{\lambda_{ab4}}{\lambda_4} \chi_{ab}.$$

This must be multiplied by $l_{abc}\, dx_a\, dx_b\, dx_c$, the magnitude of the side under consideration in natural units, and finally by $\frac{1}{l_c}$ to express the vector product in x-units. Because of (24) we may write for the result

$$\chi_{ab}\, dx_a\, dx_b\, dx_c = \psi_{c4}\, dx_a\, dx_b\, dx_c .$$

The opposite side gives a similar result with the opposite sign (N having for that side the direction -4^*), so that together the sides contribute the term

$$\frac{\partial \psi_{c4}}{\partial x_4} dW$$

to the component X_c. For shortness' sake we have put here

$$dx_1\, dx_2\, dx_3\, dx_4 = dW.$$

Finally we may take $c = 1, 2, 3$.

b. Secondly we consider a side (dx_a, dx_b, dx_4) facing towards the positive x_c. The vector N has now the direction $-c^*$. We consider the vector products of this vector with the rotations χ_{b4}, χ_{4a} and χ_{ba}, which vector products have the directions a, b and 4. A calculation exactly similar to the one we performed just now gives the contributions to X_a, X_b, X_4. For these we thus find the products of $dx_a\, dx_b\, dx_4$ by

$$\frac{l_{ab4}\,\lambda_{bc4}}{l_a\,\lambda_c}\,\chi_{b4} = \chi_{4b} = \psi_{ac},$$

$$\frac{l_{ab4}\,\lambda_{ac4}}{l_b\,\lambda_c}\,\chi_{4a} = \chi_{a4} = \psi_{bc},$$

$$\frac{l_{ab4}\,\lambda_{abc}}{l_4\lambda_c}\,\chi_{ba} = \chi_{ba} = \psi_{4c}.$$

Taking also into consideration the opposite side (dx_a, dx_b, dx_4) we find for X_a, X_b, X_4 the contributions

$$\frac{\partial \psi_{ac}}{\partial x_c}\,dW,\quad \frac{\partial \psi_{bc}}{\partial x_c}\,dW\quad \frac{\partial \psi_{4c}}{\partial x_c}\,dW.$$

This may be applied to each of the three pairs of sides not yet mentioned under a; we have only to take for c successively 1, 2, 3.

Summing up what has been said in this § we may say: the components of the vector on the left hand side of (10) are

$$X_a = \Sigma\,(b)\,\frac{\partial \psi_{ab}}{\partial x_b}\,dW.$$

§ 27. For the components of the vector occurring on the right hand side of (10) we may write

$$i\,\mathfrak{q}_a\,d\Omega,$$

if $\mathfrak{q}_a$ is the component of the vector $\mathfrak{q}$ in the direction x_a expressed in x-units, while $d\Omega$ represents the magnitude of the element $(dx_1, \ldots dx_4)$ in natural units. This magnitude is

$$-\,i\,\sqrt{-g}\,dW,$$

so that by putting

$$\sqrt{-g}\,\mathfrak{q}_a = w_a \quad \ldots\ldots\ldots\ldots \quad (28)$$

we find for equation (10)

$$\Sigma\,(b)\,\frac{\partial \psi_{ab}}{\partial x_b} = w_a \quad \ldots\ldots\ldots\ldots \quad (29)$$

The four relations contained in this equation have the same form as those expressed by formula (25) in my paper of last year[1]). We shall now show that the two sets of equations correspond in all respects. For this purpose it will be shown that the transformation formulae formerly deduced for w_a and ψ_{ac} follow from the way in which these quantities have been now defined. The notations from the former paper will again be used and we shall suppose the transformation determinant p to be positive.

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[1]) Zittingsverslag Akad. Amsterdam, **23** (1915), p. 1073; translated in Proceedings Amsterdam, **19** (1916), p. 751. Further on this last paper will be cited by l. c.

§ 28. Between the differentials of the original coordinates x_a and the new coordinates x'_a which we are going to introduce we have the relations

$$dx'_a = \Sigma\,(b)\,\pi_{ba}\,dx_b \quad . \; . \; . \; . \; . \; . \; . \; . \quad (30)$$

and formulae of the same form (comp. § 10) may be written down for the components of a vector expressed in x-measure. As the quantities q_a constitute a vector and as

$$\sqrt{-g'} = p\sqrt{-g},$$

we have according to (28) [1])

$$\frac{1}{\sqrt{-g'}}\,w'_a = \frac{1}{\sqrt{-g}}\,\Sigma\,(b)\,\pi_{ba}\,w_b,$$

or

$$w'_a = p\,\Sigma\,(b)\,\pi_{ba}\,w_b\,.$$

Further we have for the infinitely small quantities ξ_a [2]) defined by (19)

$$\xi'_a = \Sigma\,(b)\,p_{ba}\,\xi_b\,.$$

and in agreement with this for the components of a vector expressed in ξ-units

$$\Xi'_a = \Sigma\,(b)\,p_{ba}\,\Xi_b,$$

so that we find from (25) [3])

$$\chi'_{ab} = \Sigma\,(cd)\,p_{ca}\,p_{db}\,\chi_{cd}\,.$$

Interchanging here c and d, we obtain

$$\chi'_{ab} = \Sigma\,(cd)\,p_{da}\,p_{cb}\,\chi_{dc} = -\,\Sigma\,(cd)\,p_{da}\,p_{cb}\,\chi_{cd}$$

and

$$\chi'_{ab} = \tfrac{1}{2}\,\Sigma\,(cd)\,(p_{ca}\,p_{db} - p_{da}\,p_{cb})\,\chi_{cd} \quad . \; . \; . \; . \quad (31)$$

The quantity between brackets on the right hand side is a second order minor of the determinant p and as is well known this minor

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[1]) Comp. § 7, l. c.

[2]) For the infinitesimal quantities x_a occurring in (19) we have namely (comp. (30))

$$x'_a = \Sigma\,(b)\,\pi_{ba}\,x_b$$

and taking into consideration (19) and (20), i e.

$$\xi_a = \Sigma\,(b)\,g_{ab}\,x_b\;,\;\; x_a = \Sigma\,(b)\,\gamma_{ba}\,\xi_b$$

and formula (7) l. c., we may write (comp. note 2, p. 758, l. c.)

$$\xi'_a = \Sigma\,(b)\,g'_{ab}\,x'_b = \Sigma\,(bcde)\,p_{ca}\,p_{db}\,\pi_{eb}\,g_{cd}\,x_e =$$
$$= \Sigma\,(cd)\,p_{ca}\,g_{cd}\,x_d = \Sigma\,(cdf)\,p_{ca}\,g_{cd}\,\gamma_{fd}\,\xi_f = \Sigma\,(c)\,p_{ca}\,\xi_c\,.$$

[3]) Put $\Xi_a{}^{\mathrm{I}}\,\Xi_b{}^{\mathrm{II}} = \vartheta_{ab}$. Then we have

$$\vartheta'_{ab} = \Xi_a{}^{\mathrm{I}'}\,\Xi_b{}^{\mathrm{II}'} = \Sigma\,(cd)\,p_{ca}\,p_{db}\,\Xi_c{}^{\mathrm{I}}\,\Xi_d{}^{\mathrm{II}} = \Sigma\,(cd)\,p_{ca}\,p_{db}\,\vartheta_{cd}$$

and similar formulae for the other three parts of (25).

is related to a similar minor of the determinant of the coefficients π_{ab}. If $a'b'$ corresponds to ab in the way mentioned in § 25, and $c'd'$ in the same way to cd, we have

$$p_{ca}\, p_{db} - p_{da}\, p_{cb} = p\, (\pi_{ca'}\, \pi_{d'b'} - \pi_{d'a'}\, \pi_{c'b'}),$$

so that (31) becomes

$$\chi'_{ab} = \tfrac{1}{2}\, p\, \Sigma\, (cd)\, (\pi_{c'a'}\, \pi_{d'b'} - \pi_{d'a'}\, \pi_{c'b'})\, \chi_{cd}.$$

According to (27) this becomes

$$\psi'_{a'b'} = \tfrac{1}{2}\, p\, \Sigma\, (cd)\, (\pi_{c'a'}\, \pi_{d'b'} - \pi_{d'a'}\, \pi_{c'b'})\, \psi_{c'd'},$$

for which we may write

$$\psi'_{ab} = \tfrac{1}{2}\, p\, \Sigma\, (cd)\, (\pi_{ca}\, \pi_{db} - \pi_{da}\, \pi_{cb})\, \psi_{cd}.$$

Interchanging c and d in the second of the two parts into which the sum on the right hand side can be decomposed, and taking into consideration that

$$\psi_{dc} = -\psi_{cd},$$

as is evident from (26) and (27), we find [1])

$$\psi'_{ab} = p\, \Sigma\, (cd)\, \pi_{ca}\, \pi_{db}\, \psi_{cd}.$$

§ 29. Finally it can be proved that if equation (10) holds for one system of coordinates $x_1, \ldots. x_4$, it will also be true for every other system $x'_1, \ldots. x'_4$, so that

$$\int \{[\mathsf{R}_e \,.\, \mathsf{N}] + [\mathsf{R}_h \,.\, \mathsf{N}]\}_{x'}\, d\sigma = i \int \{\mathsf{q}\}_{x'}\, d\Omega. \quad . \quad . \quad . \quad (32)$$

To show this we shall first assume that the extension Ω, which is understood to be the same in the two cases, is the element $(dx_1, \ldots. dx_4)$.

For the four equations taken together in (10) we may then write

$$\int u_1\, d\sigma = v_1\, d\Omega, \ldots \int u_4\, d\sigma = v_4\, d\Omega \quad . \quad . \quad . \quad . \quad (33)$$

and in the same way for the four equations (32)

$$\int u'_1\, d\sigma = v'_1\, d\Omega, \ldots \int u'_4\, d\sigma = v'_4\, d\Omega \quad . \quad . \quad . \quad . \quad (34)$$

We have now to deduce these last equations from (33). In doing so we must keep in mind that $u_1, \ldots. u_4$ are the x-components and $u'_1 \ldots. u'_4$ the x'-components of one definite vector and that the same may be said of $v_1, \ldots. v_4$ and $v'_1 \ldots. v'_4$.

Hence, at a definite point (comp. (30))

$$v'_a = \Sigma\, (b)\, \pi_{ba}\, v_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

We shall particularly denote by π_{ba} the values of these quantities belonging to the angle P from which the edges $dx_1, \ldots. dx_4$ issue

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[1]) Comp. (28) l. c.

in positive directions. To the right hand sides of the equations (34) we may apply transformation (35) with *these* values of π_{ba}, $d\Omega$ being infinitely small of the fourth order and it being allowed to confine ourselves to quantities of this order.

On the left hand sides of (34), however, we must take into consideration, the surface being of the third order, that the values of π_{ba} change from point to point. Let $\mathbf{x}_1, \ldots \mathbf{x}_4$ be the changes which $x_1, \ldots x_4$ undergo when we pass from P to any other point of the surface. Then we must write for the value of the coefficient at this last point

$$\pi_{ba} + \Sigma\,(c)\,\frac{d\pi_{ba}}{dx_c}\,\mathbf{x}_c.$$

We thus have

$$\int u'_a\,d\sigma = \Sigma\,(b)\,\pi_{ba}\int u_b\,d\sigma + \Sigma\,(b)\int u_b\,\Sigma\,(c)\,\frac{\partial\pi_{ba}}{\partial x_c}\,\mathbf{x}_c\,d\sigma.$$

It will be shown presently that the last term vanishes. This being proved, it is clear that the relations (34) follow from (33); indeed, multiplying equations (33) by $\pi_{1a}, \ldots \pi_{4a}$ respectively and adding them we find

$$\int u'_a\,d\sigma = v'_a\,d\Omega.$$

§ 30. The proof for

$$\Sigma\,(b)\int u_b\,\Sigma\,(c)\,\frac{\partial\pi_{ba}}{\partial x_c}\,\mathbf{x}_c\,d\sigma = 0 \quad \ldots\ldots \quad (36)$$

rests on the relations

$$\frac{\partial\pi_{ba}}{\partial x_e} = \frac{\partial\pi_{ea}}{\partial x_b}, \quad \ldots\ldots\ldots \quad (37)$$

which follow from

$$\pi_{ba} = \frac{\partial x'_a}{\partial x_b}\,, \quad \pi_{ea} = \frac{\partial x'_a}{\partial x_e}.$$

The integral which occurs in (36) differs from

$$\int u_b\,d\sigma \quad \ldots\ldots\ldots \quad (38)$$

by the infinitely small factor under the sign of integration

$$\Sigma\,(c)\,\frac{\partial\pi_{ba}}{\partial x_c}\,\mathbf{x}_c.$$

Now we have calculated in § 26 integrals like (38) by taking together each time two opposite sides, one of which Σ_1 passes through P while the second Σ_2 is obtained from the first by a shift in the direction of one of the coordinates e. g. of x_e over the distance dx_e. We had then to keep in mind that for the two sides the values of u_b, which have opposite signs, are a little different; and it was precisely this difference that was of importance. In the calculation of the integral

$$\int u_b \, \Sigma (c) \frac{\partial \pi_{ba}}{\partial x_c} \mathfrak{x}_c \, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

however it may be neglected. Hence, when we express the components u_b in terms of the quantities ψ_{ab}, we may give to these latter the values which they have at the point P.

Let us consider two sides situated at the ends of the edges dx_e, and whose magnitude we may therefore express in x-units by $dx_j \, dx_k \, dx_l$ if j, k, l are the numbers which are left of 1, 2, 3, 4 when the number e is omitted. For the part contributed to (38) by the side Σ_2 we found in § 26

$$\psi_{be} \, dx_j \, dx_k \, dx_l \; .$$

We now find for the part of (39) due to the two sides

$$\psi_{be} \, \Sigma (c) \frac{\partial \pi_{ba}}{\partial x_c} \left[\int\limits_2 \mathfrak{x}_c \, d\sigma - \int\limits_1 \mathfrak{x}_c \, d\sigma \right]$$

where the first integral relates to Σ_2 and the second to Σ_1. It is clear that but one value of c, viz. e has to be considered. As everywhere in $\Sigma_1 : \mathfrak{x}_e = 0$ and everywhere in $\Sigma_2 : \mathfrak{x}_e = dx_e$ it is further evident that the above expression becomes

$$\psi_{eb} \frac{\partial \pi_{ba}}{\partial x_e} \, dW.$$

This is one part contributed to the expression (36). A second part, the origin of which will be immediately understood, is found by interchanging b and e. With a view to (37) and because of

$$\psi_{eb} = - \psi_{be}$$

we have for each term of (36) another by which it is cancelled. This is what had to be proved.

§ 31. Now that we have shown that equation (32) holds for each element $(dx_1, \ldots dx_4)$ we may conclude by the considerations of § 21 that this is equally true for any arbitrarily chosen magnitude and shape of the extension Ω. In particular the equation may be applied to an element $(dx'_1 \ldots dx'_4)$ and by considerations exactly similar to those presented in § 26 we see that in the new coordinates as well as in the original ones we have equations of the form (29).

Whatever be our choice of the coordinates the part of the principal function indicated in § 14 can therefore be derived for a given current vector q.

In a sequel to this paper some conclusions that may be drawn from HAMILTON's principle will be considered.

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