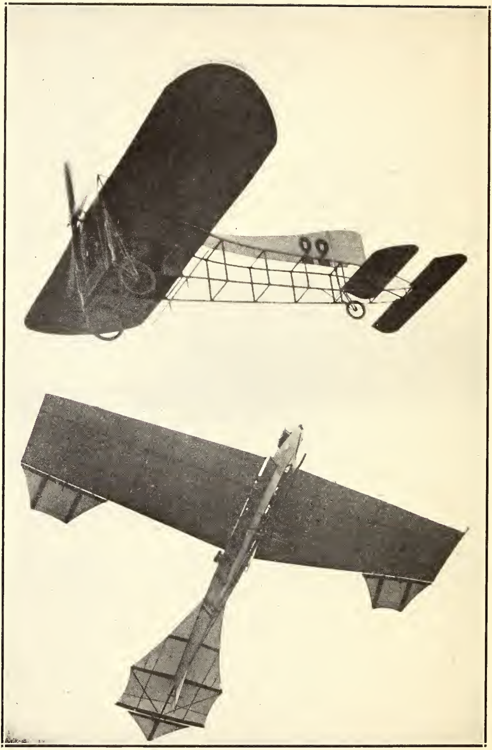

MONOPLANES AT RHEIMS.
Upper machine is that in which Bleriot crossed the English Channel; lower is that of Latham.

THE SCIENCE - HISTORY OF THE UNIVERSE

FRANCIS ROLT - WHEELER
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THE SCIENCE-HISTORY OF THE UNIVERSE

VOLUME VIII

PURE MATHEMATICS
By L. LELAND LOCKE

FOUNDATIONS OF MATHEMATICS
By PROFESSOR CASSIUS J. KEYSER

MATHEMATICAL APPLICATIONS
By DR. FRANZ BELLINGER

INTRODUCTION
By PROFESSOR CASSIUS J. KEYSER

CONTENTS

PART I—PURE MATHEMATICS

PART II—FOUNDATIONS OF MATHEMATICS

PART III—MATHEMATICAL APPLICATIONS

INTRODUCTION

The general reader, for whom this writing is primarily designed, though he be college-bred, and may thus have had a mathematical discipline extending possibly through an elementary course in the calculus, probably entertains very erroneous or very inadequate notions respecting the proper character of mathematics, and especially respecting alike its marvelous growth in modern times and the great range and variety of doctrines that the term has come to signify. With a view to correcting such errors, at least in some measure, if they exist, and in order to enhance the reader's interest and to enlighten his appreciation, it seems worth while to preface the exposition proper with some general indications—albeit they must needs be mainly of an exterior kind—of the nature and the extent of the science whose foundations are to be subsequently explained.

Let it be understood, then, that, while mathematics is the most ancient of the sciences, it is not surpassed by any of them in point of modernity, but is flourishing even to-day as never before, and at a rate unsurpassed by any rival. To compare it to a deep-rooted giant tree of manifold high and far-branching arms is not an adequate simile. Rather is the science like a mighty forest of such oaks. These, however, literally grow into and through each other, so that by the junction and intercrescence of limb with limb and root with root and trunk with trunk the manifold wood becomes a single living, organic, growing whole. The mathematical achievements of antiquity were great achievements. The works of Euclid and Archimedes, of Apollonius and Diophantus, will endure forever among the most glorious monuments of the human intellect. And just now, owing to Dr. Heath's superb English edition of Euclid's 'Elements'—a beautiful translation of the thirteen books from the definitive text of Heiberg, with rich bibliography and extensive commentary setting the whole matter in the composite light of ancient and modern geometric research—one sees even better than ever before how great, mathematically, was the age that produced the immortal Alexandrine classic. Yet the 'Elements' of Euclid is as small a part of Mathematics as the 'Iliad' is of Literature; as the 'Pandects' of Justinian is of human Jurisprudence; or as the sculpture of Phidias is of the world's total Art.

Not the age of Euclid, but our own, is the golden age of mathematics. Ours is the age in which no less than six international congresses of mathematics have been held in the course of ten years. To-day there exist more than a dozen mathematical societies, containing a growing membership of over two thousand men and women representing the centers of scientific light throughout the great culture nations of the world. In our time more than five hundred scientific journals are each devoted in part, while more than two score others are devoted exclusively, to the publication of mathematics. It is in our time that the 'Jahrbuch über die Fortschritte der Mathematik' ('Yearbook for the Progress of Mathematics'), tho it admits only condensed abstracts with titles and does not report upon all the journals, has, nevertheless, grown into nearly forty huge volumes in as many years. It requires no less than the seven ponderous tomes of the forthcoming 'Enkyclopädie der Mathematischen Wissenschaften' ('Encyclopedia of the Mathematical Sciences') to contain, not expositions, not demonstrations, but merely compact reports and bibliographic notices sketching developments that have taken place since the beginning of the nineteenth century. This great work is being supplemented and translated into the French language. Finally, to adduce yet another evidence of like kind, the three immense volumes of Moritz Cantor's 'Geschichte der Mathematik' ('History of Mathematics'), tho they do not aspire to the higher forms of elaborate exposition, and tho they are far from exhausting the period traversed by them, yet conduct the narrative down only to 1758. (A fourth volume in continuation of Cantor's work has recently appeared. It was composed mainly by other hands.) That date, however, but marks the time when mathematics, then schooled for over a hundred eventful years in the fast unfolding wonders of Analytic Geometry and the Calculus, and rejoicing in these the two most powerful instruments of human thought, had but fairly entered upon her modern career. And so fruitful have been the intervening years, so swift the march along the myriad tracks of modern Analysis and Geometry, so abounding and bold and fertile withal has been the creative genius of the time, that to record, even briefly, the discoveries and the creations since the closing date of Cantor's work would require an addition to his great volumes of a score of volumes more.

It is little wonder that so vital a spirit as that of Mathesis, increasing in intensity and more and more abounding as the ages have passed—it is small wonder that since pre-Aristotelian times it has challenged the mathematician and the philosopher alike to tell what it is—to define mathematics; and it is now not surprising that they should try in vain for many hundreds of years; for, naturally, conception of the science has had to grow with the growth of the science itself.

Cassius J. Keyser.

Editorial Note—Beginning on page 191 will be found a notable paper on "The Foundations of Mathematics," written especially for this series by Professor Keyser.

MATHEMATICS

CHAPTER I

NUMBER

The notion of number is extremely slow to develop, both in the individual and in the race, yet it has its origin at such a remote period in the evolution of man that only a possible reconstruction of its history may be given. Such an account may built up mainly from three sources, a study of the knowledge and use of number among peoples lowest in the scale of civilization at the present time, the genesis of the number concept in the mind of the child and a comparison of root words of the various languages, past and present.

Number is coeval with spoken language, and probably antedates by a long period any written language or symbolism. Primitive man recorded the results of hunting or fishing excursions, the number of warriors in the opposing camp, or the number of days' journey from home by the use of pebbles, shells, knots in cord, nicks in woods, scores on stone, and, most important for the present study, by the fingers and toes.

The mode of recording numbers by knots on cord gave rise to the term "quipu" reckoning, from the Peruvian language, quipu meaning knot. Edward Clodd, in 'The Story of the Alphabet,' has this reference: "The quipu has a long history, and is with us in the rosary upon which prayers are counted, in the knot tied in a handkerchief to help a weak memory, and in the sailor's log-line." Herodotus tells that when Darius bade the Ionians remain to guard the floating bridge which spanned the Ister, he "tied sixty knots in a thong, saying: 'Men of Ionia, do keep this thong and do as I shall say: so soon as ye shall have seen me go forward against the Scythians, from that time begin and untie a knot each day; and if within this time I am not here, and ye find that the days marked by the knots have passed by, then sail away to your own lands.'"

The quipu reached its more elaborate form among the ancient Peruvians. It consisted of a main cord, to which

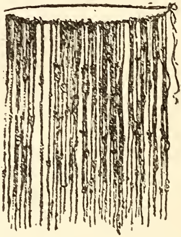


Fig. 1 —Quipu from Peru.

were fastened at given distances thinner cords of different colors, each cord being knotted in divers ways and each color having its own significance. Red strands stood for soldiers, yellow for gold, white for silver, green for corn, and so forth, while a single knot meant ten, two single knots meant twenty, double knots one hundred, two double knots two hundred. Each town had its officer whose special function was to tie and interpret the quipus. They were called Quipucamayocuna, or knot-officers (compare 'harpedonaptæ,' or rope-stretchers, in connection with the Geometry of the Egyptians).

The knot-reckoning is in use among the Puna herdsmen of the Peruvian plateaux. On the first strand of the quipu they register the bulls, on the second the cows, these again they divide into milch-cows and those that are dry; the next strands register the calves, the next the sheep, and so forth, while other strands record the produce; the different colors of the cords and the twisting of the knots giving the key to the several purposes. The Paloni Indians of California have a similar practice, concerning whom Dr. Hoffman reports that each year a certain number are chosen to visit the settlement at San Gabriel to sell native blankets. "Every Indian sending goods provided the salesman with two cords made of twisted hair or wool, on one of which was tied a knot for every real received, and on another a knot for each blanket sold. When the sum reached ten reals, or one dollar, a double knot was made. Upon the return of the salesman each person selected from the lot his own goods, by which he would at once perceive the amount due, and also the number of blankets for which the salesman was responsible." Hawaiian tax-gatherers kept accounts of the assessable property throughout the island on lines of cordage from four to five hundred fathoms long.

A method of keeping the accounts of the British exchequer before the use of writing paper was by means of tally sticks. These were of willow about 8 or 10 inches long. Notches were cut, a deep one for a pound, a small one for a shilling. The stick was then sawed half in two near one end and split down to this cut, each half bearing a record of the notches. The shorter piece was given to the depositor and the bank retained the longer.

A great mass of these sticks was still in the basement of the Parliament houses when it was decided to burn them in 1834. Samuel S. Dale describes the bonfire. He says, "A pile of little notched sticks bearing strange-looking inscriptions in abbreviated Latin and old English script, the evidence of thrift for a thousand years, tokens of all the motives that prompt men and women to save, love, hate, greed and sacrifice, hope and fear, frugality and fraud, the proceeds of honest toil and of crime, held for ages that the missing pieces carried away by successive generations might be redeemed, their presence a mute evidence of the blasted hopes of depositors for a thousand years. They were fed steadily to the flames from early morning until a few minutes before seven o'clock in the evening of Thursday, October 16, 1834, when suddenly a furnace flue, overheated by the unusual fire, started a blaze in a room above, and in a few hours the House of Lords and the House of Commons were in ashes, along with nearly all the old wooden tally sticks and all the basic standards of weight and measure for the British Empire." A few of the old tally sticks were saved.

When the savage in his first dim gropings for truth recognises that two objects are more than one, the first step is taken toward the formation of the number concept. That a long pause ensued before the next step was taken is evidenced by the number of cases, cited by various writers, of tribes whose only number words are for 'one' and 'many' or 'one,' 'two' and 'many.' This word for 'many' plays the same rôle in the language of the savage as 'infinity' in ordinary parlance, a number inexpressibly or inconceivably great. The growth of expressibility of number may be compared with the ever-widening ripples when a pebble is dropped into still water, the outer ripple representing the upper bound of conceivable number. All the region beyond would be, in the language of the savage, 'many.'

The Hindu number system is the first ever devised which has no outer bound. This fact has led to a more precise use of the word 'infinity' in modern mathematical terminology.

The possibilities of the Hindu system are well illustrated by the answers to the celebrated Archimedean "cattle problems." These answers, ten in number, were composed of 206,545 figures each. Such a number if printed in small pica type would be nearly a quarter of a mile in length.

The ability to form a definite conception of a number grows with intelligence, but in the presence of numbers of such magnitude it is opportune to ask what relation exists between the power to conceive the number and the ability to represent it. There seems to have been a curious crossing over of the two. The poverty of the aboriginal language should not be taken as evidence of inability to use larger numbers. It simply means that the verbal expression paused for a longer time after the number 'two' than did the number sense. Instances are given of peoples whose number names do not go beyond ten, but who reckon as far as one hundred. The number sense grows along with other mental development, but has not kept step with the verbal and symbolic expression of large numbers. It is questionable if the number 10,000 stands for a distinct conception if it is measured by units. One obtains an idea of such a number only by grouping it, say, into a hundred hundreds.

There are several distinct steps in the formation of a number system: The recognition of increase by adding, in succession, single objects to a group, counting, attaching a number name to the group counted, as 'three' sticks (such a number in which the object or unit is named is called a concrete number), the final separation of the number notion from the objects counted or abstraction (one asks how many sticks in the group and the answer is 'three,' an abstract number), the indicating of the number name by a symbol, the choosing of a method of grouping and finally the perfection of the system by arrangements and combinations of the number words and symbols. It is a long way from the 'mokenam,' 'one'; 'uruhu,' 'many,' of the Bococudos to the modern notion of number of the mathematician, "the class of all similar classes."

Number in its primitive sense answers the question, "How many?" It is a pure abstraction which results from counting. Cardinal number tells how many of the group, as 'seven' trees, while the ordinal number of any one of the objects indicates the position of the particular object in the series, as the 'sixth' tree. These two ideas are equally fundamental, each being derivable from the other. Counting is simply pairing off, or, in mathematical language, establishing a one-to-one correspondence between the individuals of a group of objects counted, as pebbles, the fingers, marks or scores, number names or the symbols for these number names.

In the first stages it would be comparatively easy to invent a word and a symbol for each number, but as the need for larger numbers grew some method of grouping became necessary. In 'Problemata,' attributed to Aristotle, the following discussion takes place: "Why do all men, barbarians as well as Greeks, numerate up to ten, and not to any other number, as two, three, four, or five, and then repeating one and five, two and five, as they do one and ten, two and ten, not counting beyond the tens, from which they again begin to repeat? For each of the numbers which precedes is one or two and then some other, but they enumerate, however, still making the number ten their limit. For they manifestly do it not by chance, but always. The truth is, what men do upon all occasions and always they do not from chance, but from some law of nature. Whether is it, because ten is a perfect number? For it contains all the species of number, the even, the odd, the square, the cube, the linear, the plane, the prime, the composite. Or is it because the number ten is a principle? For the numbers one, two, three, and four when added together produce the number ten. Or is it because the bodies which are in constant motion are nine? Or is it because of ten numbers in continued proportion, four cubic numbers are consummated (Euclid viii, 10), out of which numbers the Pythagoreans say that the universe is constituted? Or is it because all men from the first have ten fingers? As therefore men have counters of their own by nature, by this set, they numerate all other things."

Dr. Conant gives an illustration which typifies the beginnings of this grouping in 'The Number Concept.' "More than a century ago," he says, "travelers in Madagascar observed a curious but simple mode of ascertaining the number of soldiers in an army. Each soldier was made to go through a passage in the presence of the principal chiefs; and as he went through a pebble was dropped on the ground. This continued until a heap of ten was obtained, when one was set aside and a new heap begun. Upon the completion of ten heaps, a pebble was set aside to indicate one hundred, and so on until the entire army had been numbered."

That man carries in the fingers the natural counting machine is shown by the fact that the great majority of number systems have been based on five, ten or twenty. A typical case of such a number system is that of the Zuni scale:

1—töpinte	taken to start with.
2—kwilli	put down together with.
3—ha'ī	the equally dividing finger.
4—awite	all the fingers all but one done with.
5—öpte	the notched off.
6—topalĭk'ya	another brought to add to the done with.
7—kwillilĭk'ya	two brought to and held up with the rest.
8—hailĭk'ye	three brought to and held up with the rest.
9—tenalĭk'ya	all but all are held up with the rest.
10—ästem'thila	all the fingers.
11—āstem'thla topayä'thl'-tona	all the fingers and another over above held.

And so forth to 20.

20—kwillik'yënästem'thlan . two times all the fingers.
100—ässiästem'thlak'ya..... the fingers all the fingers.
1,000—ä s s i ä stem'thlanak'yë-nästem'thla .... the fingers all the fingers times all the fingers.

Arithmetic has been defined as the science of number and the art of computation. This twofold nature of the subject is due to the fact that the Greeks divided the subject into 'Arithmetic' proper, which is the science of numbers, a subject for the philosopher, and 'Logistic,' or computation, which was to be taught to the slave.

Notation and numeration are respectively the writing and reading of numbers. A theory of the building up of a number system is given by Dean Peacock in his article on arithmetic in the 'Encyclopedia Metropolitana': "The discovery of the mode of breaking up numbers into classes, the units in each class increasing in decuple proportion, would lead, very naturally, to the invention of a nomenclature for numbers thus resolved, which is simple and comprehensive. By giving names to the first natural numbers, or digits—*i.e.*, the first nine numbers, called digits, from counting on the fingers—and also to the units of each class in the ascending series by ten, we shall be enabled, by combining the names of the digits with those of the units possessing local or representative value, to express in words any number whatsoever. Thus the number, resolved by means of counters in the manner indicated by Fig. 2, would be expressed (supposing seven, six, five, and four denote the numbers of the counters, in A, B, C, D, and ten, hundred, and thousand, the value of each unit in B, C, and D) by seven, six tens, five hundreds, four thousands; or inverting the order, and making slight changes required by the existing form of the language, by four thousand, five hundred, and sixty-seven."

The successive columns A, B, C, D are called orders. The number of ones in any order required to make one of the next higher order, in this case ten, is called the radix, scale or base of the system. In the above formation when nine have been put in the column A, the tenth would be placed in column B and the nine removed from column A. Such a system is called a decimal or "ten times" system.

One of the earliest devices for reckoning consisted of a board strewn with sand on which parallel lines were drawn with the finger. These lines fulfil the same office as the compartments above marked A, B, C, D. Upon the lines the counters were laid. This reckoning board was called an abacus from an old Semitic word abaq, meaning sand. The development of the abacus from the sand-board to

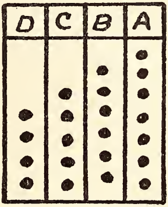


Fig. 2 —Old Method of Computation with Counters.

the swan pan of the Chinese and the counting frame of the kindergarten is to be considered in connection with reckoning.

It was the custom of the Romans to drive a nail in the temple of Minerva for each year. When, as with counters, the number of marks exceeded the power of the eye to grasp at a glance, grouping was used.

The simplest method of writing a number is by a mark or stroke for each unit, or one in the number, as ||||||| for seven. The stroke was universally used by primitive peoples as a symbol for one. The drawing of the tomb-board of Wabojeeg, a celebrated war chief who died on Lake Superior about 1793, shows this clearly. His totem, the reindeer, is reversed. The seven strokes number the war parties he led, the three upright strokes symbolize

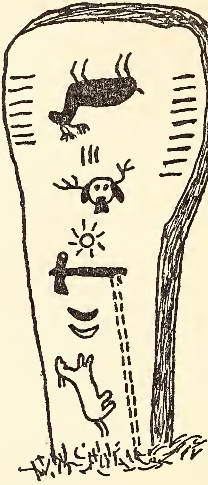


Fig. 3 —Tomb-board of Wabojeeg.

wounds received in battle. The horned head tells of a desperate fight with a moose.

The scoring of each fifth one counted may be regarded as the second step in the development of a satisfactory number symbolism. Such a method of recording succeeding events is not uncommon to-day. The thresher often so marks each sack of grain as it leaves the machine, and in loading and unloading vessels it is frequently the mode used by the tallyman. Thus twenty-two would be written

Of the numerous systems of notation which have been devised, three are distinctive from their mode of formation, from their logical completion, and from their extended use: The Greek, the Roman, and the Hindu, sometimes incorrectly called the Arabic. Consider a number formed by counters placed in the various compartments A, B, C, D (Fig. 2). The largest number of counters that may be put in any one compartment is nine; that is, there are nine numbers for each compartment. The Greeks adopted as their number symbols the letters of their alphabet in order, the first nine letters for nine numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, of column A; the next nine letters for the numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 of column B. As the alphabet consisted of but twenty-four letters, to fill out column C three obsolete letters were interpolated. In the accompanying scheme, taken from Gow's 'History of Greek Mathematics,' the starred letters are those not belonging to the alphabet.

The limit of the system with letters of the alphabet alone is 999. When it became necessary to write larger numbers, a stroke like an inverted prime was put before and usually somewhat below the letter, as seen in the number 1,000, to increase the value of the letter one thousandfold. For 10,000 a new letter was used, the M, the first letter of *μυρίοι* or myriad. The symbols were always written in descending order from left to right. The largest number now possible in the Greek notation is 99999999. The use of the alphabet as numerals seems to date from about 500 B.C. The Greek mode of writing fractions is quite simple, the denominator being written over the numerator, or the numerator is written with one ac-cent, followed by the denominator twice with two accents, as $\frac{\kappa\alpha}{\iota\varsigma}$ or ιϛ′κα″κα″. If the numerator is unity it is omitted. $^1/_{32}$ would be written $^{\lambda\beta'}$ or λβ″. Special signs were sometimes used for ½, ⅔, addition and subtraction.

Archimedes devised a plan by which the Greek number system might be prolonged indefinitely and which has been thought by some to contain the germ of the modern

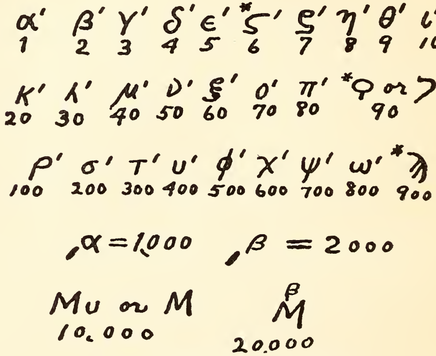


Fig. 4 —Greek Number System.

notion of logarithm. "In a pamphlet entitled *ψαμμίτης* (in Latin Arenarius, the sand-reckoner), addressed to Gelon, king of Syracuse," says Gow, "Archimedes begins by saying that some people think the sand cannot be counted, while others maintain that, if it can, still no arithmetical expression can be found for the number. 'Now I will endeavor,' he goes on, 'to show you, by geometrical proofs which you can follow, that the numbers which have been named by us and are included in my letter addressed to Zeuxippus, are sufficient to exceed not only the number of a sand-box as large as the whole earth, but of one which is as large as the universe. You understand, of course, that most astronomers mean by "the universe" the sphere of which the center is the center of the earth and the radius is a line drawn from the center of the earth to the center of the sun.' Assume the perimeter of the earth to be 3,000,000 stadia (a stadium was nearly 200 yards), and in all the following cases take extreme measurements. The diameter of the earth is larger than that of the moon and that of the sun is larger than that of the earth. The diameter of the sun is thirty times that of the moon and is larger than the side of a chiliagon (a polygon of 1,000 sides) inscribed in a great circle of the sphere of the universe. It follows from these measurements that the diameter of the universe is less than 10,000 times that of the earth and is less than 10,000,000,000 stadia.

"Now suppose that 10,000 grains of sand not $<$ 1 poppy-seed, and the breadth of a poppy-seed not $<$ $^1/_{40}$ of a finger-breadth. Further using the ordinary nomenclature, we have numbers up to a myriad myriads (100,000,000). Let these be called the first order and let a myriad myriads be a unit of the second order and let us reckon units, tens, etc., of the second order up to a myriad myriads; and let a myriad myriads of the second order be a unit of the third order and so on ad lib. If numbers be arranged in a geometrical series, of which 1 is the first term and 10 is the radix, the first eight terms of such a series will belong to the first order, the next eight to the second order and so on. Calling these orders octads and using these numbers, following the rule that spheres are to one another in the triplicate ratio of their diameters, Archimedes ultimately finds that the number of grains of sand which the sphere of the universe would hold is less than a thousand myriads or ten millions of the eighth octad. This number would be expressed in our notation as 1 with sixty-three ciphers annexed." There seems to have been no attempt to apply this method further, the ordinary system being sufficient for the needs of the time.

The main principle underlying the Roman system was to provide a symbol for each column or order, the symbol being repeated for each unit in the order. The following reconstruction of the Roman process is made for the purpose of comparison with the other two systems and is not offered as the probable historical course.

For each unit of column A a Roman I was used, it being the nearest to the primitive stroke or score | ; X was used for the second order, C for the order of hundreds, and M for thousands. These are called unit letters. So far the gap from 1 to 10 is too great, it being necessary to write I nine times for 9. A half-way symbol was then provided for each interval: V for 5, L for 50, and D for 500. These are called half-unit letters. It is altogether probable that the half-unit letter is a relic of the pause in finger reckoning when the first hand was completed. Many of the decimal systems still preserve this trace of a quinary base.

The half-unit symbol may have arisen in connection with the use of the reckoning board, placing counters on the spaces as well as upon the lines as the notes of the musical staff. Fig. 5 indicates the method of writing 7,868 on the sand-board. It is very probable that the use of the spaces was derived from the half-unit letter rather than in the reverse order.

So far the system is built upon an additive basis, the value of a symbol of equal or less value written at the right of a given symbol being added to the value of the given symbol; thus if 20 is to be written, another X is written at the right of the X for 10, as XX, while 16 would be written XVI. At this stage four would be written IIII, a form still to be seen on a clock face. A still further improvement, lessening the number of symbols, was the adoption of a subtractive principle. This means that a symbol of lesser value written at the left of a given symbol has its value taken from the value of the greater symbol. In this way 4 would be written IV. Two facts are here noticeable. The subtractive principle need be used but twice in each column; in the column A, for example, in writing 4 and 9, 3 might be written IIV with no advantage over III. A half-unit letter is never used in the subtractive sense; that is, L is used for 50 rather than LC.

The third and final step was the adoption of the multiplicative principle (also seen in the Greek notation). In the Roman scheme it appeared as a dash or vinculum

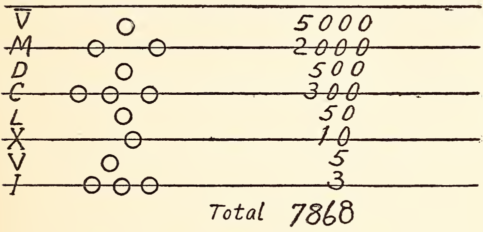


Fig. 5 —Roman Units and Half-units.

drawn over a letter to increase its value a thousandfold; as in Fig. 5, a V with a stroke across the top indicates 5,000. The Roman mind was not of a scientific cast and one would scarcely expect to find the number system worked out to logical perfection. In fact, there is a decided lack of uniformity in the manner of writing numbers used by various Roman authors.

The following set of rules compiled by Dr. French seems to be the logical working out of the system: "Affirmative Rules: (1) The value of a unit letter is repeated with every repetition of the letter; (2) the value of a letter written at the right of a letter of equal or greater value is added to that value; (3) the value of a unit letter written at the left of the next higher unit or half-unit letter is subtracted from the value of that letter; (4) a vinculum placed over a letter increases its value a thousandfold. Negative Rules: (1) A half-unit letter is never repeated; (2) a half-unit letter is never written before a letter of greater value; (3) a unit letter is never written before a letter of greater value except the next higher half-unit and unit letters—*i.e.,* 99 is never written IC; (4) the vinculum is never placed over I; (5) a letter is not used more than three times in any order."

Little may be said of the origin of the Roman Numerals. It is generally supposed that the system was inherited from the Etruscans. Various and interesting have been the theories advanced to explain the choice of the symbols. One is that the I is a sort of hieroglyphic form of the extended finger, V for the hand, and X for the double hand. Another theory is that decem is related to decussare, to cut across, and that the cutting across of a symbol multiplies its value by 10; thus I cut across is X. C is the initial letter of centum, one hundred.

Traces of the subtractive principle have been found on brick tablets from the Temple Library of Nippur, recently deciphered by Professor Hilprecht of the Babylonian Expedition of the University of Pennsylvania. These bricks probably date from about the twentieth century B.C.

Each of the wide symbols indicated a ten, the final straight wedge a one, the twenty and one being combined in a subtractive sense to give nineteen.

The fundamental principle of assigning a symbol to each column destined the Roman system of notation to ultimate disuse. By this principle an indefinitely large number would mean an indefinitely large number of columns, and hence an indefinitely large number of symbols. No difference how many symbols were in use, it would be easy to specify a number which could not be written.

Such a system must finally give way to another with no such limitations.

The Babylonian number system was based on 60, both for whole numbers and fractions. The possible explanation of this sexigesimal system is that the year was reckoned as 360 days, thus dividing the circle into 360 parts, and they were probably aware of the division of the circle into 6 parts by stepping off the radius 6 times on the circumference, and by so doing arriving at 60 parts of the circle in each part stepped off. 60 proved to be a particularly favorable base, being divisible by 2, 3, 4, 5, 6, 10, and 12. A large mass of information as to the mathematical accomplishments has recently been revealed by Professor Hilprecht, who has examined more than 50,000 cuneiform inscriptions from the Temple Library of Nippur.

The Babylonians had a strange custom of deriving their numbers from a large number which may be called a basal number. This basal number is 12,960,000 or 60^4. This number is, according to the theory of Professor Hilprecht, the famous "Number of Plato," Republic, Book VIII. "This number is constructed from 216, the minimal number of days of gestation in the human kind, and is called the lord of better and worse births." If the 216 be interpreted as days, together with 12,960,000, the latter number gives 36,000 years, the "great Platonic year," which was the length of the Babylonian cycle. Thus is implied that Plato's famous number and the idea of its influence upon the destiny of man originated in Babylonia.

The Aztec system of numeration had the score for its basis. There were special signs for the first five numerals; for twenty, for its square, four hundred, and for the cube, eight thousand. Certain combinations of signs symbolized the other numerals.

The Chinese had, from earliest times, constructed a system of numerals, similar in many respects to what the

Romans probably inherited from their Pelasgic ancestors. It is only to be observed that the Chinese mode of writing is the reverse of the Arabic, and that beginning at the top of the leaf it descends in parallel columns to the bottom, proceeding, however, from right to left, as practiced by

1	一	*Yĭh.*	10	十	*Shih.*
2	二	*Irr.*	100	百	*Pŭh.*
3	三	*San.*	1000	千	*Ts'hyen.*
4	四	*Sè.*	10,000	萬	*Wàn.*
5	五	*Ngóo.*	100,000	億	*Eè.*
6	六	*Lyeù.*	1,000,000	兆	*Chaò.*
7	七	*Ts'hĭh.*	10,000,000	京	*King.*
8	八	*Păh.*	100,000,000	垓	*Kyai.*
9	九	*Kyéu.*			

Fig. 6 —Chinese Number System.

most of the Oriental nations. Instead of the vertical lines used by the Romans, therefore, horizontal ones are found in the Chinese notation. Thus 'one' is represented by a horizontal stroke with a barbed termination, 'two' by a pair of such strokes. The mark for 'four' has four strokes with a flourish. Three horizontal strokes and two vertical ones form the mark for 'five,' and other symbols exhibit the successive strokes abbreviated as far as 'nine.' 'Ten' is figured by a horizontal stroke, crossed with a vertical score, to show that the first rank is completed, while a hundred has two vertical scores connected by three short horizontal ones.

The Hindu system was based on the principle of assigning a symbol to each of the nine numbers of the first column, 1 for one, 2 for two, 3 for three, 4 for four, 5 for five, 6 for six, 7 for seven, 8 for eight, and 9 for nine. The Hindu notation may be reconstructed as follows: It is required to write the number pictured in the accompanying cut. There are four in the A column, or four ones, three in the B column, or three tens, five in the C column, or five hundreds, one in the D column, or one thousand, and four in the E column, or four ten thousands. Using the symbols above, 4 is written in the A compartment, 3 in the B compartment, etc. So long as a box arrangement is used with the compartments named, the method would be considered complete. In fact, the above number could be written just as well without the cells, as 41534, and the order for which any symbol stands would be determined by its position with reference to the others. This is called the place-value property, and is the important feature of the system.

But one thing is lacking: the method fails when any column is empty. Suppose columns A and C above to be vacant; there would be then 4 E's, 1 D, 3 B's, and no A's nor C's. This could be written in cells, but could not be written without some scheme of labeling the columns. To avoid this difficulty a new symbol, O, was invented. It was called cipher from an Arabic word meaning empty. The above number may now be written 41,030.

In the Hindu notation each symbol has in addition to its intrinsic value an acquired value resulting from its position. Thus the 3, standing in the second place, has the value thirty; 3 being its intrinsic value and the ten being its acquired or place value. Thus both the multiplicative and additive principles are involved in place-value 325 is $3 \times 100 + 2 \times 10 + 5$. Writing two symbols, now called figures, side by side adds them after the left-hand figure has been multiplied by ten.

It is readily seen that there is no limit to the number of columns that may be used without increasing the number of symbols; that is, the Hindu notation begins at units' column and may be carried indefinitely to the left. The smallest number that may be written, so far, is Unity, or one. The two final steps in the perfecting of the system, the invention of the decimal point, which permits of the writing of numbers indefinitely small, striking off

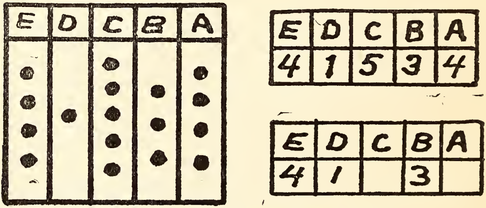


Fig. 7 —Hindu Notation Arrangement.

the right-hand barrier, and the discovery of the exponential notation and logarithms, which facilitate computations, will be considered later, together with the long struggle between the Roman and Hindu systems for supremacy.

The origin of the Hindu notation is shrouded in mystery. It is customary for Orientals to attribute any great discovery or invention to the direct revelation of the gods. Professor Hilprecht gives an illustration of this trait. "According to Berosus, a Babylonian priest who lived some time between 330 and 250 B.C., the origin of all human knowledge goes back to divine revelation in primeval times. 'In the first year there made its appearance

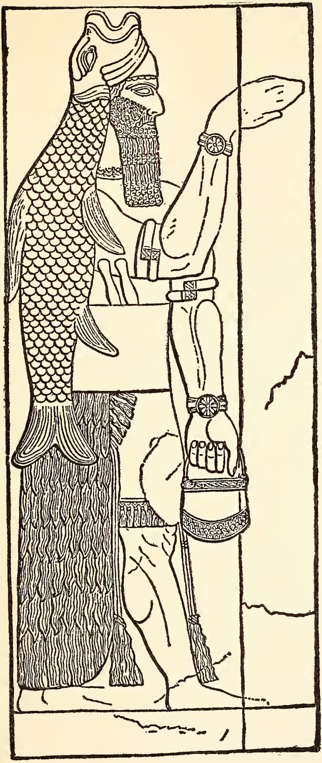


Fig. 8 —Oannes; Babylonian God of Mathematics and Learning.

from a part of the Erythraean Sea, which bordered upon Babylonia, a living being endowed with reason, who was called Oannes. According to this tradition, confirmed by Apollodorus, the whole body of this creature was like that of a fish, and it had under a fish's head another or human head, and feet similar to those of a man subjoined to the fish's tail, and it also had a human voice; and a representation of him is preserved even to this day. This being, it is said, in the day time used to converse with men, without, however, taking any food; he instructed men in the knowledge of writing, of sciences and every kind of art; he taught them how to settle towns, to construct temples, to introduce laws and to apply the principles of geometrical knowledge, he showed them how to sow and how to gather fruit; in short, he instructed men in everything pertaining to the culture of life. From that time [so universal were his instructions] nothing else has been added by way of improvement. But when the sun set, this being Oannes used to plunge again into the sea and abide all night in the deep; for he was amphibious.'"

Professor Florian Cajori thus sums up the leading conclusions due to Woepcke as to the historical development of the Hindu numeral system: "The Hindus possessed the nine numerals, without the zero or cipher, as early as the second century after Christ. It is known that about that time a lively commercial intercourse was carried on between India and Rome, by way of Alexandria. There arose an interchange of ideas as well as of merchandise. The Hindus caught glimpses of Greek thought, and the Alexandrians received ideas on philosophy and science from the east. The nine numerals, without the zero, thus found their way to Alexandria, where they may have attracted the attention of the Neo-Pythagoreans. From Alexandria they spread to Rome, thence to Spain and the western part of Africa.

"Between the second and eighth centuries the nine characters in India underwent changes in shape. A prom-

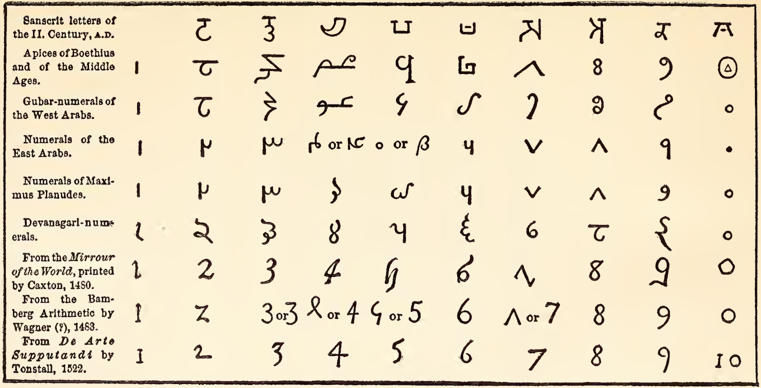


Fig. 9 —SHOWING THE DEVELOPMENT OF THE HINDU NUMERALS.
(Cajori.)

inent Arabic writer, Albîrûnî (died 1038), who was in India during many years, remarks that the shape of Hindu numerals and letters differed in different localities, and that when (in the eighth century) the Hindu notation was transmitted to the Arabs the latter selected from the various forms the most suitable. But before the East Arabs thus received the notation it had been perfected by the invention of the zero and the application of the principle of position.

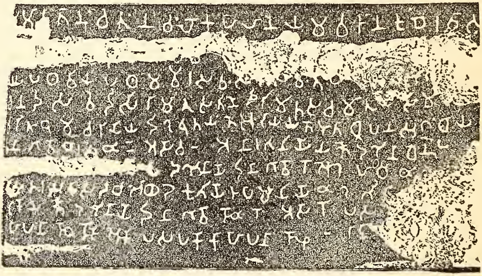


Fig. 10 —Nânâ Ghât Inscription, Containing One of the Earliest Forms of Hindu Numerals.

"Perceiving the great utility of the Columbus-egg, the zero, the West Arabs borrowed this epoch-making symbol from those in the East, but retained the old forms of the nine numerals which they had previously received from Rome. The reason for this retention may have been a disinclination to unnecessary change, coupled, perhaps, with a desire to be contrary to their political enemies in the East. The West Arabs remembered the Hindu origin of the old forms, the so-called Gubar or "dust" numerals. After the eighth century the numerals in India underwent further changes, and assumed the greatly modified forms of the modern Devanagari numerals." Professor Moritz Cantor recently expressed the opinion that the use of the zero was probably due to the Babylonians, 1700 B.C.

There are two methods of reading numbers in general use, in both of which the orders are grouped, beginning with the first order, or the order of units. In the French method each group consists of three orders, such a group being called a period. The names of the first three orders, beginning with the lowest, are units, tens and hundreds. These names are applied also to the three orders in each period followed by the name of the period. The names of the first 12 periods follow:

1. Units.	5. Trillions.	9. Septillions.
2. Thousands.	6. Quadrillions.	10. Octillions.
3. Millions.	7. Quintillions.	11. Nonillions.
4. Billions.	8. Sextillions.	12. Decillions.

In the English method each period consists of six orders, named units, tens, hundreds, thousands, ten thousands, and hundred thousands. The names of the periods follow:

1. Units.	3. Billions.	5. Quadrillions.
2. Millions.	4. Trillions.	6. Quintillions.

In both systems the number names are read in descending order from left to right, and in all cases compounds are formed in the same way, except in the interval from 10 to 20. Professor Brooks, in 'Philosophy of Arithmetic,' gives the following account of number naming: "A single thing is called 'one'; one and one more are 'two'; two and one are 'three'; and in the same manner we obtain 'four,' 'five,' 'six,' 'seven,' 'eight,' and 'nine,' and then adding one more and collecting in a group we have 'ten.' Now regarding the 'ten' as a single thing, and proceeding according to the principle stated, we have one and ten, two and ten, three and ten, and so on up to ten and ten, which we call two tens. When we arrive at ten tens we call this a new group, a 'hundred.' This is the actual method by which numbers were originally named;

but unfortunately, perhaps, for the learner and for science, some of these names have been so modified and abbreviated by the changes incident to use, that, with several of the smaller numbers at least, the principle has been so far disguised as not to be generally perceived. If, however, the ordinary language of arithmetic be carefully examined, it will be seen that the principle has been preserved, even if disguised so as not always to be immediately apparent. Instead of one and ten we have substituted 'eleven,' derived from an expression formerly supposed to mean one left after ten, but now believed to be a contraction of the Saxon 'endlefen,' or Gothic 'ainlif' (ain, one, and lif, ten); and instead of two and ten, we use the expression meaning, two left after ten, but now regarded as arising from the Saxon twelif, or Gothic tvalif (tva, two, and lif, ten). In the numbers following twelve, the stream of speech 'running day by day' has worn away a part of the primary form, and left the words that now exist. Thus, supposing the original expression to be three and ten, if we drop the conjunction we have three ten; changing the ten to teen and the three to thir, we have thirteen." In a similar manner twenty is a contraction of two tens. It is to be noticed that Professor Brooks has always used the form two and ten rather than ten and two. That such use leading to the forms from 10 to 20 is the exception rather than the rule is seen when it is recalled that from 20 on the larger number is always read first.

The word million seems to have been used first by Marco Polo (1254-1324). During the next 300 years it was used by writers in several senses, and not until the sixteenth century did it succeed in finally securing its place in the number system. Billion in the English system is equivalent to one thousand French billions, or a trillion.

An example will suffice to show the two methods of reading a number. Thus, 436,792,543,896,578, according to the French method, is read four hundred thirty-six trillion, seven hundred ninety-two billion, five hundred forty-three million, eight hundred ninety-six thousand, five hundred seventy-eight; while the English method would be four hundred thirty-six billion, seven hundred ninety-two thousand, five hundred forty-three million, eight hundred ninety-six thousand, five hundred seventy-eight.

The primitive form of the abacus was a board strewn with sand, upon which lines were drawn and pebbles were used as counters. On the Egyptian abacus the lines were at right angles to the operator, and Herodotus states that they "calculate with pebbles by moving the hand from right to left, while the Greeks move it from left to right," thus indicating that the units' column was taken with the Egyptians on the extreme left. The varying values of the counters when changed from one column to another is referred to in the comparison of Diogenes Laertius, "A person friendly with tyrants is like the stone in computation, which signifies now much, now little," which recalls Carlyle's ranking of men with the pieces on a chessboard. A single example of a Greek abacus is extant in the form of a marble table discovered on the island of Solamis in 1846, and now preserved in Athens. This table is 5 feet long and 2½ feet wide, and the lines, which are parallel to the operator, are in a good state of preservation.

Difficulty of calculation with Roman numerals rendered necessary the use of the abacus, inherited from the Greeks, and in turn, the ease with which the ordinary computations were performed with its aid prevented the perfecting or inventing of a usable system of notation. Horace (Sat. I, 6, 75) alludes to the practice of boys marching to school with the abacus and box of pebbles suspended from the left arm: "Quo puero magnis ex centurionibus orti, Lævo suspensi loculos tabulamque lacerto." In the time of greatest Roman luxury (Juvenal, Sat. II, 131) the counters were of ivory, silver, and gold.

A more serviceable form was developed under Roman usage, in which the table or board was replaced by a thin metal plate with grooves cut entirely through, in

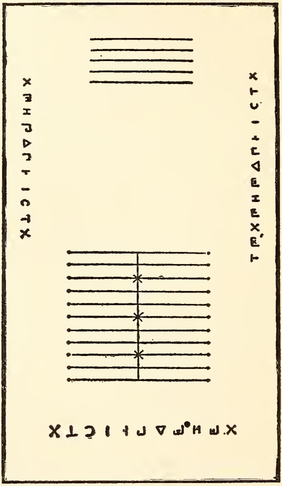


Fig. 11 —Salamis Abacus. The Only Known Early Greek Specimen.

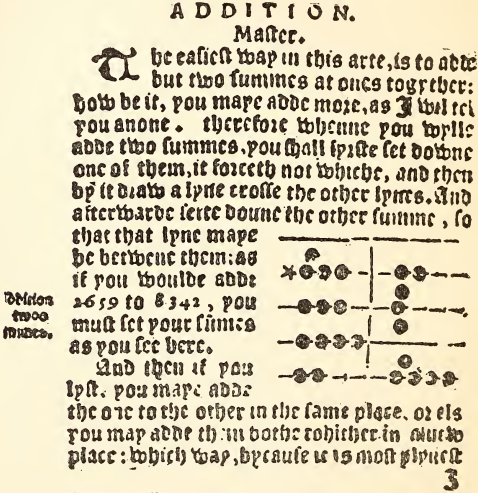
which were metal buttons which could be slid from one end of the groove to the other. If at one end, a button registered one in that groove; if at the other, it was valueless. In place of a long groove containing 9 buttons, a shorter groove registered 4, and a still shorter one, immediately above, had a value of 5. At the right of units' column were two short columns in which could be registered twelfths, the Roman fraction, still preserved in name, in ounce and inch. Several of these metal abaci are to be found in museums.

Another form of abacus still in general use in the Orient is that of a frame across which wires are strung, upon which are movable beads. This is the 'swanpan' of the Chinese and the 'tchotu' of the Russians. In 1812 the abacus was carried from Russia to France, in the form of the counting frame, as a device for teaching number in primary work, and is now found in all kindergartens, a slight evidence of belief in the "culture-epoch" theory that the training of the child mind should follow the steps in the mental development of the race.

At the decadence of Rome the Roman notation and abacus reckoning remained as an inheritance to central Europe. The Arabs being in possession of the Hindu numerals carried them to Spain, and they were used in the commercial towns bordering the eastern end of the Mediterranean Sea. Some of the more aspiring youths of England and France journeyed to Spain to acquire the learning of the Greeks and Hindus which had been preserved and cultivated assiduously by the Moors. Others, merchantmen of Italy, perceived the advantage gained in the use of these numerals in the Phoenician towns, and they in turn carried the knowledge home.

Of the former who visited Spain was Gerbert (d. 1003), afterward Pope Sylvester II. Gerbert's abacus was of leather, and contained 27 columns. In place of the old counters new ones of horn were used, upon each of which one of the first nine numerals was written. Thus the first step in the use of Hindu numerals was taken. Of the latter, merchantmen of Italy, was Leonardo of Pisa, who in 1202 wrote a treatise on mathematics called 'Liber Abaci.' It begins thus: "The nine figures of the Hindus are 9, 8, 7, 6, 5, 4, 3, 2, 1. With these nine figures and this sign, o, which in Arabic is called sifr, any number may be written."

The long struggle of 500 years for supremacy between the line-reckoning, or abacus, and the Hindu numerals, began. In one of the cuts is seen a page of line-reckoning


ADDITION.

Master.

The easiest way in this arte, is to adde but two summes at ones togyther: how be it, you maye adde more, as I wyl tell you anone. therefore whenne you wylle adde two summes, you shall fyrste set downe one of them, it forceth not whiche, and then by it drawe a lyne crosse the other lynes. And afterwarde sette downe the other summe, so that that lyne maye be betwene them: as if you woulde adde 2659 to 8342, you must set your summes as you see here.

Addition of two summes.

And then if you lyst, you maye adde the one to the other in the same place, or els you may adde them both togither in a newe place: which way, bycause it is most playnest

I

Fig. 12 —Reckoning on the Line (1558).

from an early English textbook, 'The Ground of Artes,' by Robert Recorde, 1558. This work, which ran through at least 28 editions, is in the form of a dialogue between master and pupil. The following extract concerns the difficulty the pupil has in multiplying by a fraction as to why the product should be less than the number multiplied. The master explains the definition of multiplication, but the scholar is not satisfied, and the master says:

> "*Master.*—If I multiply by more than one, the thing is increased; if I take it but once, it is not changed; and if I take it less than once, it cannot be as much as before. Then, seeing that a fraction is less than one, if I multiply by a fraction, it follows that I do take it less than once."
>
> "*Pupil.*—Sir, I do thank you much for this reason; and I trust that I do perceive the thing."

The use of counters had not disappeared in England and Germany before the middle of the seventeenth century.

Various methods of finger reckoning have been developed, and are commonly found in the older arithmetics. The accompanying cut is from Recorde's 'The Ground of Artes,' 1558, and gives a general idea of this practice.

According to Pliny the image of Janus or the Sun was cast with the fingers so bent as to indicate 365 days. Some have thought that Proverbs iii, 16, "Length of days in her right hand," alludes to such a form of expressing numbers.

An interesting illustration is given by Leslie: "The Chinese have contrived a very neat and simple kind of digital signs for denoting numbers, greatly superior to that of the Romans. Since each finger has three joints, let the thumbnail of the other hand touch these joints in succession, passing up one side of the finger, down the middle, and again up the other side, thus giving nine marks applicable to the decimal notation. On the little finger these signify units, on the next tens, on the next hundreds, etc. The merchants of China are accustomed, it is said, to conclude bargains with each other by help of these signs, and to conceal the pantomime from the knowledge of bystanders.

The Korean schoolboy carries to school a bag of counting-bones, each about 5 inches long, and somewhat thin-

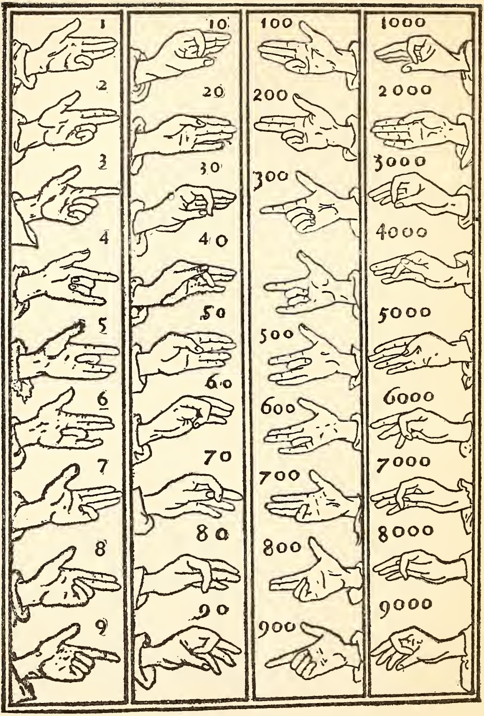


Fig. 13 —Finger Reckoning.

ner than the ordinary leadpencil. A box of square sticks, 4 inches in length and about ½ inch square, called sangi, is used in a very ingenious fashion by the Chinese for the solution of algebraic equations.

The form of reckoning board adopted in the Middle Ages has left some words and customs. Fitz-Nigel, writing about the middle of the twelfth century, describes the board as a table about ten feet long and five feet wide, with a ledge or border, and was surrounded by a bench, or 'bank,' for the officers. From this 'bank' comes the modern word bank as a place of money changing. The table was cov-

Fig. 14 —Chinese Digital Notation.

ered after the term of Easter each year with a new black cloth divided by a set of white lines about a foot apart, and across these another set which divided the table into squares. This table was called "scaccarium," which formerly meant chessboard, from which is the term exchequer, the Court of Revenue.

CHAPTER II

CALCULATION

Under the term 'logistic' the Greeks treated what is now ordinarily termed computation or calculation, the latter word coming from a Latin word meaning 'pebble,' inasmuch as the reckoning was done with counters or pebbles. Calculation is the process of subjecting numbers to certain operations now to be defined. There are six fundamental operations in arithmetic, all growing out of the first. Formerly these were differently classified, sometimes as high as nine being considered, the other three being special cases or complications of the fundamental six.

These six operations are divided into two groups, the direct operations, of which there are three, and the inverses, each of which has the effect of undoing one of the former three.

DIRECT	INVERSE
1. Addition.	4. Subtraction.
2. Multiplication.	5. Division.
3. Involution.	6. Evolution.

When one object is put with a group of like objects, forming thus a new group having one more object than the original group, the process is said to be that of addition, and is indicated by +. (This sign appears in a work by Grammateus in 1514, and in 1517 in a book by Gillis vander Hoecke. Thus, 1 apple added to 2 apples gives 3 apples, or with abstract numbers, $2 + 1 = 3$. The objects or numbers added are called 'addends' or 'summands,' and the resulting group or number is the 'sum.' The ending end or -and, so common in mathematical terminology, is Latin present passive participle; in this case

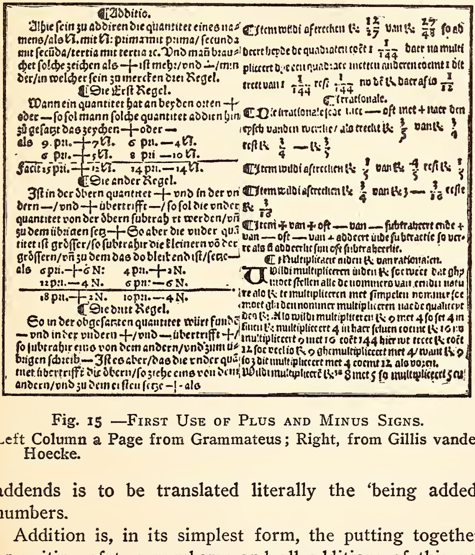

Additio.
Alhie sein zu addiren die quantitet eines namens/als N. mit N: prima mit prima/ secunda mit secunda/tertia mit tertia ꝛc. Und man braucht solche zeichen als + ist mehr/und — /minder/in welcher sein zu mercken drei Regel.
Die Erst Regel.
Wann ein quantitet hat an beyden orten + oder — so sol mann solche quantitet addiren hin zu gesatzt das zeychen + oder —
als 9 pri. + 7 N. 6 pri. — 4 N.
6 pri. + 5 N. 8 pri — 10 N.
Facit 15 pri. + 12 N. 14 pri. — 14 N.
Die ander Regel.
Ist in der obern quantitet + und in der undern — / und + ubertrifft — / so sol die under quantitet von der obern subtrahirt werden/und zu dem ubrigen setz + So aber die under quantitet ist grösser/so subtrahir die kleinern vö der grössern/vñ zu dem das do bleibt end ist/setz —
als 6 pri. + 6 N: 4 pri. + 2 N.
12 pri. — 4 N. 6 pri — 6 N.
18 pri. + 2 N. 10 pri. — 4 N.
Die dritt Regel.
So in der obgesatzten quantitet würt funden — und in der undern + / und — ubertrifft + / so subtrahir eins von dem andern/und zum ubrigen schreib — Ist es aber/das die under quantitet ubertrifft die obern/so ziehe eins von dem andern/und zu dem resten setze + / als

Fig. 15 —First Use of Plus and Minus Signs.
Left Column a Page from Grammateus; Right, from Gillis vander Hoecke.

addends is to be translated literally the 'being added' numbers.

Addition is, in its simplest form, the putting together or uniting of two numbers; and all additions of this nature may be broken up into a series of repetitions of the fundamental process of increasing a number by unity. Thus, if it be desired to add 3 apples to 5 apples, it may be done at a single step, or at three partial steps, which may be indicated thus:

$$
\begin{aligned}
5 \text{ apples} + 1 \text{ apple} &= 6 \text{ apples};\\
6 \text{ apples} + 1 \text{ apple} &= 7 \text{ apples};\\
7 \text{ apples} + 1 \text{ apple} &= 8 \text{ apples};\\
\text{or } 5 \text{ apples} + 3 \text{ apples} &= 8 \text{ apples};
\end{aligned}
$$

the three steps resulting the same as the single step given last, which justifies the statement above that addition rests upon the fundamental process of increasing a number by unity.

Like numbers are those in which the same unit is used; 7 apples and 3 apples are like numbers, as also 7 and 3; 4 trees and 9 stones are unlike numbers, as are 5 ones and 7 tens; that is, in a number 435, written in the Hindu notation, 4 in hundreds' order is not like 3 in tens' order, nor like 5 in units' order. It is fundamental that only like numbers may be added; before 3 tens is added to 5 ones, the 3 tens must be changed into 30 ones. This is a very simple matter, only being, as it were, a shift in thought, and it accounts in a great measure for the simplicity of the operations with Hindu numerals. In 435, the 4 may be thought of, in turn, as 4 hundred or as 40 tens or as 400 ones. The place-value feature permits of numbers being immediately broken up into parts, and these parts treated one at a time. Thus, in addition, like orders are written in the same column and the columns are added separately. This process is illustrated in the following example:

$$
\begin{aligned}
432 &= 4 \text{ hundred} + 3 \text{ tens} + 2 \text{ ones};\\
\underline{265} &= 2 \text{ hundred} + 6 \text{ tens} + 5 \text{ ones};\\
697 &= 6 \text{ hundred} + 9 \text{ tens} + 7 \text{ ones}.
\end{aligned}
$$

The sum of the ones, $5 + 2 = 7$, is first found, and written below the column of ones, and the other orders are added in succession.

A difficulty arises when the sum of a column is greater than 9, the largest number that may be written in a column. An example will make this clear:

$$
\begin{array}{rcl}
387 & = & 3 \text{ hundred} + 8 \text{ tens} + 7 \text{ ones;} \\
256 & = & 2 \text{ hundred} + 5 \text{ tens} + 6 \text{ ones;} \\
\hline
643 & = & 5 \text{ hundred} + 13 \text{ tens} + 13 \text{ ones;} \\
 & \text{or} & 5 \text{ hundred} + 14 \text{ tens} + 3 \text{ ones;} \\
 & \text{or} & 6 \text{ hundred} + 4 \text{ tens} + 3 \text{ ones.}
\end{array}
$$

The 13 ones is changed to 1 ten and 3 ones; the 3 is written in ones' column and the 1 ten is added in ("carried to") the tens' column. The 14 tens is treated in a similar way.

Addition obeys the commutative law; that is, the addition may be performed in any order. $5+3=3+5$. It is immaterial whether the 3 is added to the 5 or the 5 is added to the 3.

The associative law is also valid for addition. If 5 and 7 are to be added to 4, it does not matter whether the 5 be added and then the 7, or the 5 and 7 first united and then added to the 4. This is expressed by means of parentheses. The parentheses mean that the numbers within are first united: $4+5+7=4+(5+7)$. If two numbers are added, the sum is a number. This statement seems like mere verbiage, but will take on meaning when considered in the light of the other operations.

Subtraction is the inverse operation of addition. Addition is putting one number with another to form a third, and subtraction is taking one number from another to form a third. If addition has been stated in the form: given two numbers, to find their sum, subtraction would be stated: given the sum of two numbers and one of them, to find the other. The sum of two numbers is 8, and one of them is 5, what is the other? would be solved by taking 5 from 8, leaving 3. Subtraction is indicated by —. The number taken away is called the 'subtrahend,' and the number from which the subtrahend is taken is named 'minuend.' The resulting number is called 'remainder,' or 'difference,' depending upon which of the two phases of subtraction is considered. These two points of view may be brought out by concrete examples.

If A has \$10 and pays out \$7, how many dollars has he remaining? In this example the \$7, or subtrahend, was originally a part of the minuend \$10, and is taken away. The \$3 is then called 'remainder.' Again: If A has \$10 and B has \$7, how many dollars must B earn to have as many dollars as A? Here the \$10 of A and the \$7 of B are distinct numbers, and the resulting number is called the 'difference.'

In subtraction, the subtrahend is written before the minuend, with like orders in the same column. Each column is subtracted separately:

$$
\begin{array}{l}
476 = 4 \text{ hundred} + 7 \text{ tens} + 6 \text{ ones;} \\
263 = 2 \text{ hundred} + 6 \text{ tens} + 3 \text{ ones;} \\
\hline
213 = 2 \text{ hundred} + 1 \text{ ten} + 3 \text{ ones.}
\end{array}
$$

Two methods are in general use in the case that the number in an order of the subtrahend is too large to be taken from the number in the same order of the minuend. Both methods are inherited from the Hindus, having come down from the earliest printed textbooks, and seem to be of about equal difficulty.

The method of Decomposition, or Borrowing, consists of taking 1 unit from the next higher order, changing it to the order in question, adding to the number in that order, which makes the subtraction possible. 7 hundred + 2 tens + 4 units = 7 hundred + 1 ten + 14 units = 6 hundred + 11 tens + 14 units.

$$\begin{array}{r} 724 \\ -\ 269 \\ \hline 455 \end{array}$$

$$\begin{array}{rl} & 6 \text{ hundred} + 11 \text{ tens} + 14 \text{ units} \\ - & 2 \text{ hundred} + 6 \text{ tens} + 9 \text{ units} \\ \hline = & 4 \text{ hundred} + 5 \text{ tens} + 5 \text{ units.} \end{array}$$

The method of Equal Additions is based on the fact that the same number may be added to both minuend and subtrahend without changing the value of the difference; that is, $724 - 269 = (724 + 100 + 10) - (269 + 100 + 10)$. The 10 in the minuend is thought of as 10 ones, while in the subtrahend it is necessary to think of it as 1 ten. Similarly for the 100. The example used above is worked by means of equal additions, and will show the transformations involved:

$$\begin{array}{r} 724 \\ -\ 269 \\ \hline \end{array} \text{ is replaced by } \begin{array}{r} 724 + 100 + 10 \\ -\ 269 + 100 + 10 \\ \hline \end{array}$$

	hundreds		tens		ones		hundreds	tens	ones
724+10 tens+10 ones =	7	+	(2+10)	+	(4+10)	=	7	12	14
269+1 hundred+1 ten=	(2+1)	+	(6+1)	+	9	=	3	7	9
							4	5	5

In use with the first method it may be said

$$\begin{array}{r} 724 \\ -\ 269 \\ \hline \end{array}$$

9 from 14, 5;
6 from 11, 5;
2 from 6, 4.

With the second method,

9 from 14, 5;
7 from 12, 5;
3 from 7, 4.

Another mode of thinking of subtraction is called the

Austrian method, or the method of "making change." That the greater portion of subtractions in the business world is concerned with making change has led to a wide use of the method in the school-room. It consists in building to the subtrahend until the minuend is reached. That it is the natural method is evidenced by the fact that it is almost invariably used by those who have never had the benefit of, or have forgotten, school training:

$$\begin{array}{r} 987 \\ -\ 236 \\ \hline \end{array}$$

One says 6 and 1 are 7; writes 1;
3 and 5 are 8; writes 5;
2 and 7 are 9; writes 7.

Its introduction as a distinct method is due to Augustus de Morgan, England's foremost writer on arithmetic.

It is readily seen that subtraction does not obey the commutative law. One may subtract 5 from 8, but not 8 from 5. This leads to the query, If one number is subtracted from another, is the result always a number? The answer is 'yes,' if the minuend is larger than the subtrahend. Otherwise, that the result is not a number, such as those heretofore considered. These will be called natural numbers. If 5 is to be subtracted from 8 no difficulty arises; but if attempt be made to take 8 from 5, the fact arises that no such operation is possible. Such a condition brings the arithmetician face to face with one of the most important considerations in mathematics, one without which the complete structure, modern mathematics, would not be possible. It is the principle of continuity, or principle of no exception, due to Hankel. It may be stated in this form: There shall be no exception to the applicability of any operation. If the result is not found in such numbers as already belong to the system, call this result a number of a new kind and determine its properties.

Suppose a man has \$50 and spends \$40, he has left \$10. This operation is subtraction. Suppose he spends \$60 instead of \$40. This seems very much the same kind of an operation. It is agreed to call this subtraction also, and say that he has a debt of \$10, which is a new kind of number. The natural numbers may be represented by dots with any chosen interval between them:

1	2	3	4	5	6	7	8	9	10
.	.	.	.	.	.	.	.	.	.

If one goes 4 dots to the right from the third dot, he is at dot 7, or $3 + 4 = 7$. If one goes 5 dots to the left from dot 9 he is at dot 4. This going to the left is expressed by as — or subtraction, $9 - 5 = 4$. But if one starts at dot 5 and attempts to go 8 dots to the left, no dot is found to mark the stopping point. The fiat of the mathematician says, let there be a dot there. In this manner a series of dots is obtained extending to the opposite direction,

-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
.	.	.	.	.	.	.	.	.	.	.	.	.	.

These may be named or marked at pleasure. Call the first one, at the left of 1, 0, the second -1, the third -2, etc. The reason for the choice of these names is apparent. If a man has \$1 and spends \$1, he has no dollar remaining, and the symbol for an empty place is 0. If he now spends \$1 he is \$1 in debt. As this is the opposite of \$1 credit, it is appropriate to mark it -1, giving it a sign — to distinguish it from 1. If it is desired to mark the 1, a plus sign, +, is put before it, calling all numbers to the right of 0 positive numbers and those to the left negative numbers. Then $5 - 8 = -3$, while $8 - 5 = +3$. All the numbers, as now represented, are called whole numbers or 'integers.' If it is agreed always to mark the ones at the left of 0, one may mark the ones at the right, or not, at will, and no confusion will arise. 0 is now a number dividing the positives from the negatives. It is called zero.

The properties of a negative number which are most important are two: (1) A negative number may be represented by a dot as far to the left of 0 as the corresponding positive number is to the right. (2) A negative number destroys the effect of, or annuls, a positive number of the same value when added to it; thus, $+8 + (-5) = +3$, the -5 destroying $+5$ of the $+8$, leaving $+3$.

If in an addition example, all the addends are the same, as in $2 + 2 + 2 + 2 = 8$, the form is shortened into $4 \times 2 = 8$, the first number, or the 'multiplier,' indicating how many addends were taken. The second number, showing the addend, is called the 'multiplicand.' The St. Andrew's cross, indicating that the operation of multiplication is to be performed, was introduced by William Oughtred in 1631. Robert Recorde, about 1557, introduced = as the sign of equality, which he says is

> "A paire of parallels or Gemowe lines of one length, thus = becaufe noe 2 thyngs can be moare equalle."

Multiplication is, then, in essence, repeated addition.

The Commutative Law is seen to be valid in this operation: 7 rows of 3 dots is the same as 3 rows of 7 dots; or $3 \times 7 = 7 \times 3$.

Multiplication also obeys the associative law; that is, in a multiplication example where more than two numbers, or factors as they are called when used in multiplication, are involved, these factors may be grouped in any manner.

$$3 \times 7 \times 5 = 3 \times (7 \times 5) = (3 \times 5) \times 7.$$

The 3 may be multiplied by the 7, and this result, called a product, may then be multiplied by 5; or the 7 and 5 may first be multiplied and then the 3 used, etc.

A negative number multiplied by a positive gives a negative product. If in the line of dots one goes 5 dots to the left, 3 times, one arrives at dot -15, or $-5 \times 3 = -15$.

But if one attempts to multiply 3 by -5, no meaning is attached. One may perform a certain act 3 times, or 1 time, or 0 times (which means that the act is not performed), but to attempt to perform an act -5 times is meaningless. In keeping with the Principle of No Exception, such an operation must be given a meaning, and it is done by widening the definition of multiplication; but in doing so the old multiplication (repeated addition) must be kept as a special case.

It should be noted that this application of the Principle of Continuity is a purely arbitrary process. It may be said since the multiplication by a negative has no meaning, simply reject it and say it cannot be performed. Such was the usage for a long time, and had it continued so the whole system of mathematics would have been like an unsymmetrical tree, simply allowed to develop and branch in any manner. The filling out or completing the meaningless cases is like a process of grafting which rounds out and gives a symmetrical growth.

One method of procedure here would be as follows: $-5 \times 3 = -15$, and knowing that with positive numbers the commutative law holds, it is agreed to still let it be valid, from which, $-5 \times 3 = 3 \times -5$, but $-5 \times 3 = -15$; therefore, $3 \times -5 = -15$, and the conclusion is multiplication by a negative number changes the sign of the multiplicand and then multiplies it. Another and better method is to define the operation of multiplication in such a way that it will be applicable in all cases. Such a definition is the following: Multiplication is the performing that operation on the multiplicand which, if performed on unity (or one), produces the multiplier. To multiply 3 by -5.

The operation upon 1 which produces -5 is to change the sign of 1 and repeat it 5 times. Do the same with 3, -3, -3, -3, -3, -3, the sum of which is -15, as before. It will be seen that this definition of multiplication includes repeated addition as a special case. In the same manner it is seen that $-3 \times (-5) = +15$.

Considering the four cases,

$$+3 \times (+5) = +15, \qquad +3 \times (-5) = -15,$$
$$-3 \times (-5) = +15, \qquad -3 \times (+5) = -15,$$

it is clear that when the two signs are both + or both —, that is, alike, the product is +; when they are unlike, the product is —. The conclusion is, then, in multiplication, two like signs produce +, and two unlike signs produce —. The sign + is read plus, and — is read minus.

One of the commonest forms of the early methods for multiplication is the Tessellated Multiplication, very much akin to the usage of to-day.

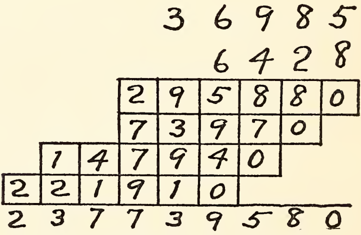


Fig. 16 —Tessellated Multiplication.

Another was the Quadrilateral Multiplication. In this form the partial products do not progress to the left, as in the Tessellated style, and are added obliquely, as shown by the arrows. These were not drawn in the work.

Lucas de Burgo called the third form Latticed Multiplication. The multiplicand is the outside top horizontal row, the multiplier the outside right vertical column. The product of any figure of the multiplicand by a figure of the multiplier is found in the square formed by the intersection of the column and row in which the figures multiplied are found; thus 9 $\times$ 2 is found in the third column from left, and second row from the bottom. These products are added in the oblique columns cut out by the

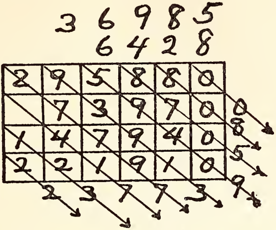


Fig. 17 —QUADRILATERAL MULTIPLICATION.

diagonal lines to the left. Less purely mental work is performed in this method than in either of the other two.

Napier, the inventor of logarithms, made use of this

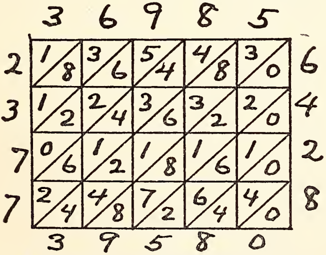


Fig. 18 —LATTICED MULTIPLICATION.

method in a device called Napier's rods, which were usually of bone, and enabled the operator to perform the multiplications mechanically.

From these methods was evolved the modern form. As in addition and subtraction, the numbers are broken up into orders:

$$\begin{array}{r} 437 \\ 56 \\ \hline 2622 \\ 2185 \\ \hline 24472 \end{array}$$

	Hundreds.	Tens.	Ones.
	4	3	7
		5	6
	24	18	42
20	15	35	

or,

42 ones = 4 tens + 2 ones;
18 tens + 4 tens = 22 tens = 2 hundred + 2 tens;
24 hundred + 2 hundred = 26 hundred = 2 thousand + 6 hundred.

In the second row of partial products,

35 tens = 3 hundred + 5 tens;
15 hundred + 3 hundred = 18 hundred = 1 thousand + 8 hundred;
20 thousand + 1 thousand = 21 thousand = 2 ten thousands + 1 thousand.

The two partial products then appear thus, and are added:

$$\begin{array}{r} 2622 \\ 2185 \\ \hline 24472 \end{array}$$

The product of any two whole numbers is a whole number. The product of 0 and any whole number is 0.

The inverse operation of multiplication is called division. In its simplest form it is repeated subtraction. If it is asked how many 2's in 8? the answer would be determined by subtracting 2 from 8 in succession as many times as possible, noting the number of times, 4, as the answer. Division has two phases. One may think of finding how many times one number is contained in another, which is 'Division,' proper, a species of measurement, or one may wish to divide a number into equal parts,

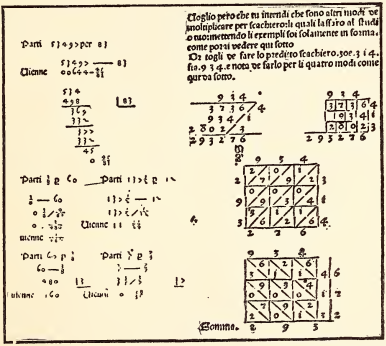


Fig. 19 —Right Column Is a Page from the First Printed Arithmetic (Treviso); Left Column Is a Page from Calandri, Showing Italian Long Division.

the number of such parts being given. This form is called 'Partition.' With abstract numbers no such distinction need be made, but with concrete numbers it is important.

The name of the number to be divided is 'Dividend,' of the dividing number 'Divisor,' and of the resulting number 'Quotient.' If any part of the Dividend is left undivided it is called 'Remainder.' There are various signs used to indicate division; $\frac{8}{2}$ or 8/2 may be regarded as indicating that 8 is to be divided by 2, as also 8 : 2. The sign in general use, ÷, was used by Dr. John Pell (1610-1685), altho this sign had been in use with other meaning by earlier German writers.

Three methods or algorithms for what is now termed long division deserve to be mentioned. One of the epoch-making works on arithmetic was written by Luca Paciuolo, a Franciscan monk. This book, published in Venice in 1494, gives the first of these methods, the galley or scratch

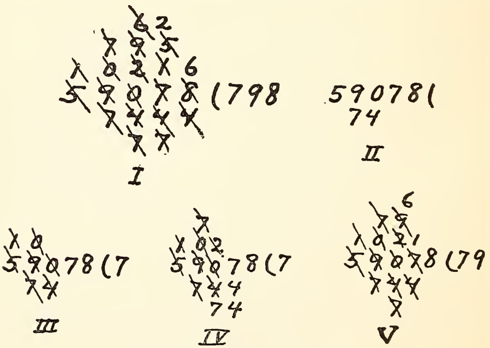


Fig. 20 —Galley Division.

I., Completed example; II., III., IV. and V., successive changes of same.

method, a dividing upward. It is a relic of the old method of reckoning on sand, where the figure is scratched out as soon as used. The above example of the method is from Purbach.

Thus to divide 59078 by 74. In the first step, 7 is divided into 59, and the quotient 7 is written, 7 7's are 49; 49 from 59 is 10, which is written above 59. The dividend is

10/

now /078. 7 4's are 28; 28 from 100 is 72, which is written still above the last dividend. The new dividend is

7/

/2/

now /78, and the division continues, each figure being scratched out as soon as used.

The first downward division, the present Italian method, appears in a printed arithmetic by Calandri (1491), altho it is found occasionally in manuscript form during the fifteenth century. See Fig. 19.

Consider the example following.

I

$$
\begin{array}{r}
74)\,59078\,(798\tfrac{26}{74} \\
518 \\
\hline
727 \\
666 \\
\hline
618 \\
592 \\
\hline
26 \text{ Rem.}
\end{array}
$$

I shows the completed form of solution, and II the successive steps, obtained by separating the number into orders.

II

Tens	Units		Ten thousands	Thousands	Hundreds	Tens	Units		Hundreds	Tens	Units
7	4	)	5	9	0	7	8	(	7	9	8
				49	28						
				51	8						
			5	1	8						
				7	2	7					
					63	36					
					66	6					
				6	6	6					
					6	1	8				
						56	32				
						59	2				
					5	9	2				
						2	6 Remainder				

The three lines show the partial product in the three stages of its reduction.

The third, or Austrian, method consists in omitting the partial products and performing the subtraction at once:

$$
\begin{array}{l}
74)59078(798\tfrac{24}{74} \\
\quad\;\; 727 \\
\quad\;\;\; 618 \\
\quad\;\;\;\; 26
\end{array}
$$

Comparing the three methods as to two points, (1) beginning on the left to subtract the partial product, (2) writing the partial product, the following scheme will show their relations:

	Galley.	Italian.	Austrian.
(1)	Yes	No	No
(2)	No	Yes	No

The Galley method is so called on account of the final form, which resembles a boat under full sail. The Aus-

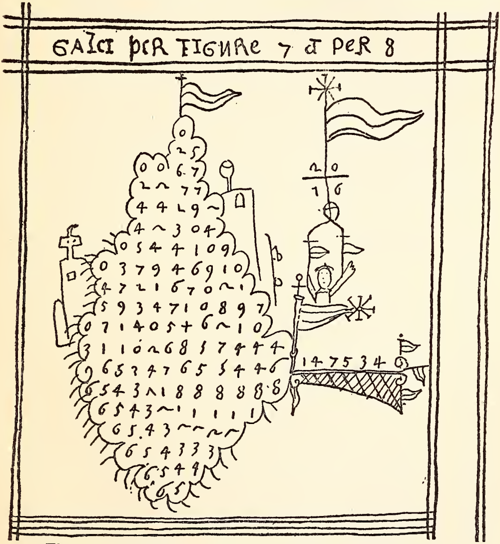
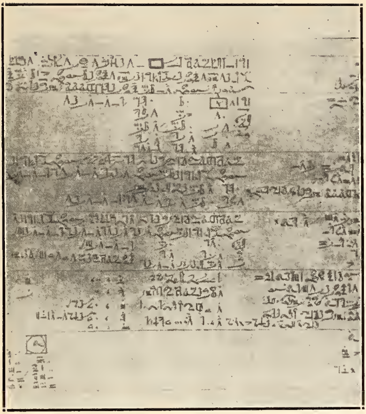


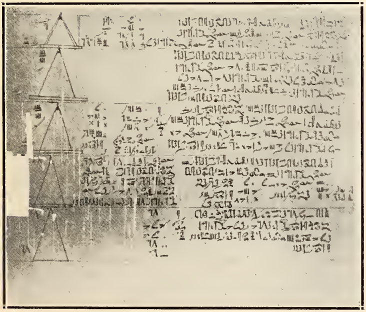
Fig. 21 —An Elaborate Form of Galley Division.

trian method, which probably will ultimately replace the Italian, is constructed from a combination of the best features of both the older methods (2) of the galley and (1) of the Italian.

As in the inverse process of subtraction it was found that the operation did not always result in a natural number, and it was necessary to create a new kind of number, the negative, thus widening the number system to form the class of whole numbers, or integers, it is to be expected that a like condition exists in the case of division. If 2 be divided by 1 the quotient is 2; but if one attempts to divide 1 by 2 no corresponding whole number is found. Considering the second phase of division, the separating of a number into 2 equal parts, it is agreed to let this quotient be a number such that it requires two of them to make 1, or unity. This new number is named one-half, and written by putting the number divided above a short horizontal line, and the divisor below the line, as $\frac{1}{2}$. The class of such numbers is called 'Fractions,' from the Latin, frangere, to break. The number below the line is called the 'denominator,' or namer, telling what the part is; the number above the line tells how many parts are taken, and is called the 'numerator,' or numberer. This function of the numerator will be apparent later.

The first widening of the number system, which arose in the case of the inverse operation, subtraction, created exactly as many new numbers as there were already in the system before the new numbers entered. Every combination of two numbers with a minus sign between them gives a positive or natural number, when the larger number appears before the sign; and a negative—that is, a new—number when the smaller number comes first. In division, the case is the exception rather than the rule where either order of the numbers results in a whole number, as $\frac{3}{2}$ and $\frac{2}{3}$, and if one order does so result the other does not, as $\frac{8}{2}$ and $\frac{2}{8}$. It is apparent, then, that the new numbers taken in under the name fraction are infinitely greater in number, when compared with the number already in.

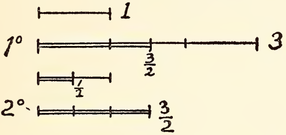




Oldest Geometrical Drawings Known—the Ahmes Papyrus

A fraction may be interpreted in any one of three ways: the fraction $\frac{3}{2}$ may be thought of as (1) 3 units divided into 2 equal parts, (2) 1 unit divided into 2 equal parts, and 3 of these parts taken, as, 3 times $\frac{1}{2}$; (3) an indicated division not yet performed. The distinction between (1) and (2) may be seen from a figure, where unity or 1 is represented by a line 1 centimeter in length.

If the numerator of a fraction is 1, it is called a unit fraction, as $\frac{1}{2}$, $\frac{1}{7}$, $\frac{1}{8}$. A 'proper' fraction has a numerator less than its denominator, as $\frac{1}{7}$, $\frac{2}{3}$, $\frac{3}{12}$. All other fractions are 'improper,' as $^8/_3$, $^5/_2$, $^4/_2$. Such a fraction can always be changed to either a whole number, as



$^4/_2 = 2$, or a whole number and a unit fraction, as $^3/_2 = 1^1/_2$.

The whole numbers were represented by dots arranged on a line at equal intervals, extending to the right and left indefinitely from a chosen dot, marked 0, or zero.

—6 —5 —4 —3 —2 —1 0 1 2 3 4

. . . .

½ 1½ 2½

The creation of the number ½ introduces a point midway between 0 and 1, and by combination with each of the whole numbers in the manner $^3/_2 = 1\frac{1}{2}$ also places a point midway in each interval. The fraction ¼ places a point half way from 0 to ½. By continuing this process it is seen that distance between the dots representing fractions is made smaller and smaller, as the various fractions take their places on the line. When all of the fractions have been represented, if one chooses a particular dot, say

$^3/_7$, one can always find another dot among those placed whose distance from the given dot, $^3/_7$, is less than any assigned length of line. The proof of this may be put in the form of conversation between A and B. If the dot 1 is 1 inch from the dot 0, A is to show that a dot may be found in the collection of dots which represent fractions which shall be nearer to the dot $^3/_7$ than any fractional part of an inch which B may name.

bedeüt diß figur der ſelben tayl ains.

$\frac{\mathrm{I}}{\mathrm{IIII}}$ Dieſſe figur iſt vñ bedeüt ain fiertel von ainez gantzen/alſo mag man auch ain fünfftail/ayn ſechſtail/ain ſybentail oder zwai ſechſtail 2c. vnd alle ander brüch beſchreiben/Als $\frac{\mathrm{I}}{\mathrm{V}}$ | $\frac{\mathrm{I}}{\mathrm{VI}}$ | $\frac{\mathrm{I}}{\mathrm{VII}}$ | $\frac{\mathrm{II}}{\mathrm{VI}}$ 2c.

$\frac{\mathrm{VI}}{\mathrm{VIII}}$ Diß ſein Secks achtail/das ſein ſechstail der acht ain gantz machen.

$\frac{\mathrm{IX}}{\mathrm{XI}}$ Diß Figur bezaigt ann newn ayilfftail das ſeyn IX tail/der XI ain gantz machen.

$\frac{\mathrm{XX}}{\mathrm{XXXI}}$ Diß Figur bezaichet/zwentzigk ainundreyſigk tail/das ſein zwentzigk tail der ainsundreiſſigk ain gantz machen.

$\frac{\mathrm{II^C}}{\mathrm{IIII^C.LX}}$ Diß ſein zwaihundert tail/der Fierhundert vnd ſechzigk ain gantz machen.

Fig. 22 —FRACTIONS WITH ROMAN NUMERALS.

B says, "Is there a dot nearer to $^3/_7$ than $^1/_{10}$ of an inch?"

A's reply is, "Choose the dot $^{31}/_{70}$, whose distance from $^3/_7$ is $^1/_{70}$."

B then says, "Find me a dot nearer than $^1/_{100}$ of an inch."

A's answer is, "The dot $^{301}/_{700}$ is only $^{1}/_{700}$ of an inch from $^{3}/_{7}$"; and so forth for any value B may name.

The dots are said to be 'dense,' and it might be thought that the whole line is filled up, that it has become a continuous line rather than a collection of discrete dots. But such is not the case; there are infinitely more dots on the line that do not represent fractions than there are dots that do represent them. The third of the inverse processes, evolution, will reveal the existence of the missing dots, and by its aid they, as a new type of number, will be included in the number system, which will then be represented by a continuous line.

Fractions are treated in the most ancient mathematical handbook known, written by an Egyptian scribe, Ahmes, or Moon-born, some time before 1700 B.C.. This papyrus, now preserved in the British Museum, is entitled 'Directions for obtaining the knowledge of all dark things," and covers practically the whole extent of Egyptian mathematics, no substantial advances being made until the time of Greek influence. Another papyrus, that found at Akhmim, written perhaps after 500 A.D., gives the same treatment of fractions as is found in the work of Ahmes. Thus Egyptian Mathematics was in its most flourishing condition when Abram left Ur of the Chaldees, and remained stationary for a thousand years. (See Frontispiece.)

The writer gives, in most cases, no general rule for obtaining results, simply a succession of like problems, the easy step of generalizing by induction seemingly being beyond his power. Whole numbers receive no treatment, the work beginning with fractions, which subject was evidently very difficult, as the author confines his attention solely to unit fractions and fractions with numerator 2. Fractions of the latter type are changed into the sum of two or more unit fractions. Thus Ahmes changed $^{2}/_{9}$ into $^{1}/_{6}$ and $^{1}/_{18}$, and gives a table of such changes of fractions between $\frac{2}{3}$ and $^2/_{99}$. By the aid of this table any fraction of odd denominator could be so broken up.

In this way Ahmes could solve such a problem as "Divide 5 by 21." The 5 is first broken into 2 and 2 and 1; from the table is found $^2/_{21} = {}^1/_{14}$ and $^1/_{42}$; $^5/_{21} = {}^1/_{21}$ and ($^1/_{14}$ and $^1/_{42}$) and ($^1/_{14}$ and $^1/_{42}$) $= {}^1/_{21}$ and ($^2/_{14}$ and $^2/_{42}$) $= {}^1/_{21}$ and $^1/_7$ and $^1/_{21} = {}^1/_7$ and $^2/_{21} = {}^1/_7$ and $^1/_{14}$ and $^1/_{42}$. The fractions were written side by side, with no sign for addition between them.

While the Egyptians met the difficulties of fractions by reducing them to fractions having a constant numerator, 1, the Babylonians avoided the same difficulties by treating only fractions with a fixed denominator, 60, and the Romans also used a single denominator, 12.

The usual rule for the division of fractions by inverting the divisor and then multiplying is not common in the textbooks of the sixteenth century. It is given as follows by Thierfeldern (1578):

"When the denominators are different invert the divisor (which you are to place at the right) and multiply the numbers above together and the numbers below together; then you have the correct result. As to divide $\frac{3}{4}$ by $\frac{5}{8}$, invert thus: $\frac{3}{4} \times {}^8/_5 = {}^{24}/_{20} = 1\frac{1}{5}$."

The close of the eighth century found the Hindu decimal notation practically perfect as a means of writing whole numbers. The final perfection of the method by applying it to fractions, in the form of decimals, did not occur until the time of Simon Stevin (1548-1620). In seven pages of his work, published in 1585, Stevin leaped what had been an impassable gap for 900 years. The reason for this pause is not difficult to determine.

In decimal fractions, or decimals, unity, or 1, is divided into ten equal parts, each part called a 'tenth'; a tenth is divided into ten equal parts and each part called a 'hundredth'; thus the orders on the right of units' column are symmetrically named, adding the suffix -th, with those on the right. As the number of orders on the left is unlimited, so the number of orders on the right is unbounded, and one is enabled to write numbers of unlimitedly small value; the smaller the value of the number (less than 1), the larger the number of orders required to express it. The units' column is marked by placing a period after it (sometimes the period is midway between the top and bottom lines of the type, as 3·8, but ordinarily it is written on the base line, as 3.8 for 3 and 8 tenths). In reading decimals the decimal point is always read 'and.'

In the first grouping of units there was no reason for putting ten in a group rather than any other number, the use of ten simply growing out of the use of the hands as a counting machine. In fact, it would have greatly simplified some applications of the number system if primitive mathematicians had been born with six fingers on each hand. A duodecimal, or 12 scale, would enable the writing of such common fractions, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{6}$, duodecimally, in the form .4, .8, .2; whereas decimally they have a continually recurring set of figures, $\frac{1}{3} = .3333$, etc; $\frac{1}{6} = .1666$, etc. Charles XII. of Sweden, a short time before his death, while lying in the trenches before a Norwegian fortress, seriously debated introducing the duodecimal system of numeration into his dominions.

On the other hand, there is a very decisive predetermining feature in the case of the division of the unit. Necessity arose for halving or dividing objects into two equal parts long before separation into ten parts was even thought of; while the difficulty of dividing into ten equal parts is apparent. The use of the period (or comma) to mark the unit order began with Pitiscus, 1612. With all the advantages of the decimal notation carried to the right of the units' column, it was not until the nineteenth century that decimals came into ordinary arithmetic.

CHAPTER III

POWERS OF NUMBERS

If in the product of several numbers, these numbers or factors happen to be a repetition of the same number, or, in other words, if the factors are equal, the product is called a power of the number which was repeatedly used to produce it. The process of finding a power of a number is involution. The term power was used by the early Greek writers in this sense. The powers are named following the ordinal names of the number of times the factor is used. If the factor 2 is used 5 times, as $2 \times 2 \times 2 \times 2 \times 2$, or 32, 32 is said to be the fifth power of 2. The second power is called the square of the number, as it was early known that the number of square units in a square is equal to the second power of the number of units of length in one side.

If a square is 5 inches on each side, its surface may be measured, using a small square 1 inch on each side. Such a unit is called a square unit or square inch, or a unit of square measure. This square inch may be laid along one edge 5 times, thus forming 1 row of 5 square inches; 5 such rows may be formed one above the other, completely using up or covering the original square. The area or surface of the square is then said to be $5 + 5$ square inches or 25 square inches. The number of square units in a square is then the second power of the number of units of length in one side. This fact, which was early known, led to the naming the second power of a number the square of the number. In a similar manner the volume of a cube is found by taking for the unit of cubical measure a cube 1 inch on each edge.

A cube is a solid figure in which all of the edges, meeting in a corner, are at right angles to each other, and in which all the edges are equal. In this cube each edge is 5 inches. Its volume is found by taking for a unit of cubical measure a cube 1 inch on each edge. This unit

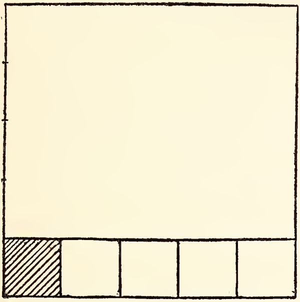


Fig. 23 —Square Measures.

or cubic inch is laid along one edge as many times as possible, or 5 times, thus forming a row of 5 cubic inches. On the bottom, 5 such rows may be formed, giving a layer of 5×5 cubic inches. It requires 5 such layers to fill up the given cube, or $5 \times 5 \times 5$ cubic inches. This use of the third power of the number of inches on the edge gives the name 'cube' of a number to the third power of the number. Since no solid figure exists with 4 edges at right angles, this process of naming the powers ceases with the third, or cube. In the figure, taken from a paper by Miss Benedict, are shown various symbols which have been devised for the indication of powers.

Writing the number of the power a little above and to the right of the number, as 7^3 for $7 \times 7 \times 7$, is due to the French mathematician and philosopher, Des Cartes. The 3, which indicates the number of times the 7 is used as a factor, is called an 'exponent,' while the 7 is termed the 'base.' The exponential notation permits the writing of very large (or very small) numbers much more com-

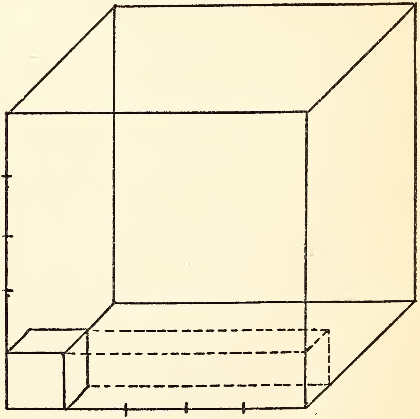


Fig. 24 —Cubic Measures.

pactly than can be done without its use. Modern researches in Astronomy and Physics have rendered necessary the use of extremely large numbers (as well as extremely small), the lower orders of which are either unknown or of small consequence. The number of vibrations per second of light waves in the visible spectrum vary from $3.94\ (10^{14})$ to $7.63\ (10)^{14}$. The wave lengths

Present Symbols		$\sqrt{\ }$	$\sqrt[3]{\ }$	$\sqrt[4]{\ }$	$\sqrt{\overline{\ \ }}$	1	x	x^2	x^3	x^4	x^5	+	−
Paciuolo	℞.p^a.	℞.2^a.	℞.3^a.	℞.4^a.	℞.v.	.n^o.	.co.	.ce	.cu.	.ce.ce.	.p.r.	.p̃.	.m̃.
Tartaglia		℞.	℞cu	℞℞	℞V		℞prima	ce	cu	ce.ce	rel	p̃	m̃
Rocha						N	Co	Ce	Cu	Cce	R	ma	me
Stifel		√	√ce	√zz	√.	ꝺ	ꝛe	z	ce	zz	ß	+	−
Recorde		√	√√	√√		ꝺ	ꝛe	z	ce	zz	ßz	+	−
Masterson		√z	√ce	√zz	√z.		ꝛe	z	ce	zz	ßz	+	−
Peletier							R	q	ce	qq	β	p	m
de la Roche		℞					ρ	ce	□	cce		p̃	m̃
Clavius		√z	√ce	√zz	√( )	N	ꝛe	z	ce	zz	β	+	−
Halcke						N	ꝛe	z	ce	zz	β	∔	−, ÷ }
Ghaligai		℞□	℞▭	℞□di□		n^o	c^o	□	▭	□di□	𐌁	piu	m^o
Cardan		℞	cu℞		℞V:	ˋ	res	q̃d	cub^9	q̃d q̃d	R^{o}p^o	p:	m:
Bombelli		R.q	Rc	RR.q	RL⌋		$\underset{\smile}{1}$	$\underset{\smile}{2}$	$\underset{\smile}{3}$	$\underset{\smile}{4}$	$\underset{\smile}{5}$	p	m
Vander Hoecke	h	℞^{2a}	℞^{3a}	℞^{4a}		n	Pri	2^a	3^a	4^a	5^a	+	−
Stevin		√	√③	√√	√bino	⓪	①	②	③	④	⑤	+	−
Vieta		√	√C.	√√	$\sqrt{\overline{\ \ }}$		A	Aquad	Acubus	Aquad-quad	Aquadcub.	+	= }
Harriot		√	√3)				a	aa	aaa	aaaa	aaaaa	+	−
Descartes		√	√C.		$\sqrt{\overline{\ \ }}$		x	xx	x^3	x^4	x^5	+	−
Newton		√	$\sqrt[3]{\ }$	$\sqrt[4]{\ }$	$\sqrt{\overline{\ \ }}$		x	xx	x^3	x^4	x^5	+	−

Fig. 25 —ALGERBAIC SYMBOLISM FROM PACIUOLO TO NEWTON. (Benedict.)

of the spectrum vary from .0000007621 meter to .0000003933825 meter. In the exponential notation these numbers would be written $7621 \times \frac{1}{10^{10}}$, and $3933.825 \times \frac{1}{10^{10}}$.

If two powers of the same number are to be multiplied, the exponents are added, as

$$7^3 \times 7^5 = 7^{3+5} = 7^8$$
$$(7 \times 7 \times 7) \times (7 \times 7 \times 7 \times 7 \times 7) = 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7$$

If two powers of the same number are to be divided, the exponent of the divisor is subtracted from the exponent of the dividend.

$$7^5 \div 7^3 = 7 \times 7 \times 7 \times 7 \times 7 \div 7 \times 7 \times 7 = 7 \times 7.$$

If a power of a number is itself to be raised to a power, as in finding the third power of 7^2, the result is obtained by multiplying the exponent 2 by the 3, exponent of the power to which 7^2 is to be raised:

$$(7^2)^3 = 7^2 \times 7^2 \times 7^2 = 7^6 = 7^{2 \times 3}.$$

A corresponding process takes place in extracting a root of a power:

$$\sqrt[3]{7^6} = \sqrt[3]{7^2 \times 7^2 \times 7^2} = 7^2 = 7^{6 \div 3}$$

As the exponent indicates the number of times the base is used as a factor, it must be a natural number, since using a number as a factor -3 times, or $\frac{1}{2}$ a time, is meaningless. The Principle of No Exception is here applied as before, and a meaning is given to exponents of the form, -3, $\frac{2}{3}$, 0, which will be at the same time consistent with the meaning of a whole number used as an exponent.

If 7^5 be divided by 7^5, the quotient is 1; but if the exponents be subtracted, as is done when division is performed, the quotient is $7^{5-5} = 7^0$, which should be equal to 1. In a similar manner it may be shown that any number with an exponent 0 is equal to 1. Is this reasonable in the light of the use of an exponent to tell how many times a factor appears? In 3×5, or 15, 7 is not used as a factor; or in other words, it is used zero times, which may be written $3 \times 5 \times 7^0 = 3 \times 5 \times 1 = 3 \times 5$.

Carrying the reasoning a step further,

$$7^5 \div 7^8 = (7\times7\times7\times7\times7) \div (7\times7\times7\times7\times7\times7\times7\times7) = \frac{1}{7^3}.$$

But subtracting exponents,

$$7^5 \div 7^8 = 7^{5-8} = 7^{-3}.$$

Therefore $7^{-3} = \frac{1}{7^3}$, which may be stated generally. The sign of an exponent may be changed by changing the position of the number from one side of the denominator line to the other.

The meaning to be attached to $7^{2/3}$ is determined in a similar manner. It will be assumed that ⅔, when used as an exponent, while as yet it has no meaning, will follow the law above for multiplication; that is, to multiply $7^{2/3}$ by itself, the exponents are added,

$$7^{\frac{2}{3}} \times 7^{\frac{2}{3}} = 7^{\frac{2}{3}+\frac{2}{3}} = 7^{\frac{4}{3}}$$

Repeat the process,

$$7^{\frac{2}{3}} \times 7^{\frac{2}{3}} \times 7^{\frac{2}{3}} = 7^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = 7^{\frac{6}{3}} = 7^2.$$

When a number is used as a factor 3 times it is said to be cubed. The inverse process, of finding the number when its cube is given, is called finding the cube root.

Since the cube of $7^{\frac{2}{3}} = 7^2$, 7 must be the cube root of 7^2; that is, the numerator of a fractional exponent tells the power that is to be taken, and the denominator tells the root to be taken. $32^{2/5}$ means that 32 is to be squared and its fifth root found, $32^2 = 1024$. The fifth root of 1024 is 4, since $4 \times 4 \times 4 \times 4 \times 4 = 1024$, whence $32^{2/5} = 4$; $16^{.5} = 16^{\frac{1}{2}} = 4$, since $4 \times 4 = 16$; 7^1 is, of course, 7.

The use of exponents in computations greatly facilitates the work. Exponents so used are called logarithms.

It will be agreed that 10 will be used as a base, and that every number is some power of 10, understanding by power, 10 with an exponent which is not necessarily a whole number. $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, etc. Since the exponent of 1 or 10^0 is 0, and of 10 or 10^1 is

1, any number between 1 and 10 must have for an exponent, or logarithm, a number between 0 and 1. In the same manner, any number lying between 10 and 100 will have a logarithm whose value is between 1 and 2. These facts may be put in a very brief form:

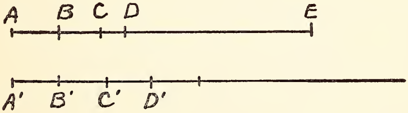
Logarithm of 1 is 0.
Logarithm of 10 is 1.
Logarithm of 100 is 2.
Logarithm of 1000 is 3.

Logarithm of 8 is a decimal lying between 0 and 1. The value of this decimal, found by an elaborate process of calculation, is .903090 + an unending decimal. Tables have been calculated of these exponents, to every number an exponent, or logarithm, and to every logarithm a number. If it be required to multiply one number by another, the logarithm of each number is found in the table; these two logarithms are added, giving, according to the method of adding exponents, the logarithm or exponent of the product. Opposite this logarithm is found the number or product desired. Thus, by the use of logarithmic tables the operation of multiplication is replaced by the much easier and shorter operation of addition, and division is replaced by subtraction. This final step in the perfecting of the methods of computation was the invention of John Napier, Baron of Merchiston (1550-1617). It seems to be an easy consequence of the exponential notation, but, curiously enough, was discovered by Napier before the invention of exponents by Des Cartes in 1637, altho the first steps toward this exponential notation are found in the works of Simon Stevin (1548-1620).

In October, 1608, Hans Lipperhey invented the telescope. In the summer of 1609 it was perfected by Galileo, and from this date began the conquest of the heavens. The next century, terminating with the death of Sir Isaac Newton, 1727, was the golden age of astronomy, in which the movements of the celestial bodies were subjected to mathematical law. It is a striking coincidence that the invention of the telescope, which so increased the need for tedious calculation, should occur almost simultaneously with the invention of logarithms, which to such a degree shortened these calculations. The greatest of French mathematicians and astronomers, La Place, paid this tribute to Napier: "The invention of logarithms, by shortening the labors, doubled the life of the astronomer."

"It is one of the greatest curiosities of science that Napier constructed logarithms before exponents were used," says Cajori, "and the fact that logarithms naturally flow from the exponential symbol was not observed until much later, by Euler."

Following is a description of Napier's method: "Let AE be a definite line (AE is taken to be 10^7, a proceeding very similar to the basing of the Babylonian number system, or 60^4), A′D′ a line extending from A′ indefinitely.



Imagine two points starting at the same moment, the one moving from A toward E, the other from A′ along A′D′. Let the velocity during the same moment be the same for both. Let that of the point on line A′D′ be uniform; but the velocity of the point on AE decreasing in such a way that when it arrives at any point C its velocity is proportional to the remaining distance CE. If the first point moves along a distance AC, while the second one moves over a distance A′C′, then Napier calls A′C′ the logarithm of CE." The adaptation to the number 10 was suggested at a meeting of Napier with Henry Briggs, who was professor of geometry in Gresham College, London. Briggs' own words indicated his admiration for the invention: "Napier, Lord of Markinston, hath set my head and hands at work with his new and admirable logarithms. I hope to see him this summer, if it please God, for I never saw a book which pleased me better and made me more wonder." Briggs was delayed in his journey to meet Napier, who said to a friend: "Ah, John, Mr. Briggs will not come." Just at that moment Briggs arrived, and it is said that almost one-quarter of an hour passed by, each beholding the other without speaking a word. Briggs at last spoke : "My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you first came to think of this most excellent help in astronomy, viz., the logarithms; but, my lord, being by you found out, I wonder nobody found it out before, when now known it is so easy."

Computations of logarithms to the base 10 soon followed, and are known to-day by the name Briggs logarithms.

In 1647 Gregory St. Vincent discovered that the use of a base denoted by $E = 2.718281828459046 + \ . \ . \ .$ had a peculiar relation to the equilateral hyperbola. Such logarithms are called hyperbolic, or natural, altho occasionally incorrectly termed Naperian, and are of immense service in Pure Mathematics. Since Napier did not use exponents, he cannot be said to have used a base in his system. If, however, his logarithms are expressed as exponents, the base or number which is raised to the power would be very nearly $^1/\varepsilon$, where $^1/\varepsilon$ is the base of the natural system.

The invention of logarithms was designed to simplify the labor of calculation. An attempt along another line has been to perform the calculations mechanically. Napier, with the "rods or bones," succeeded in a way with multiplication. The first successful attempt to perform

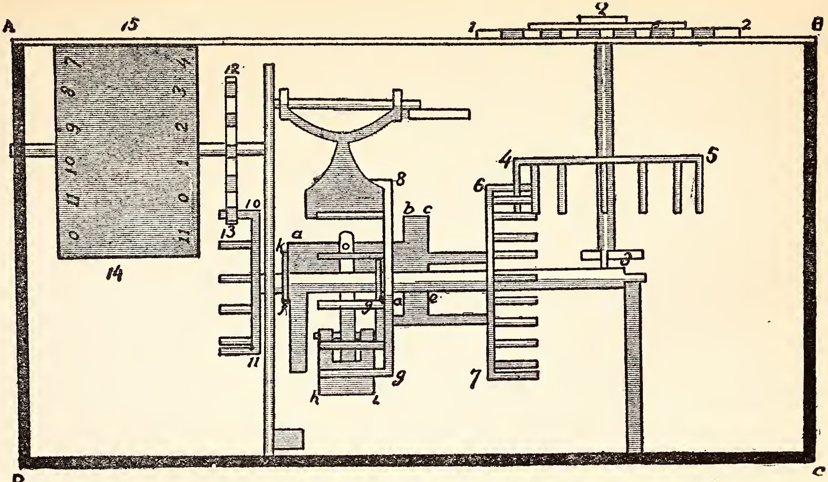


Fig. 26 —Pascal's Computing Machine.

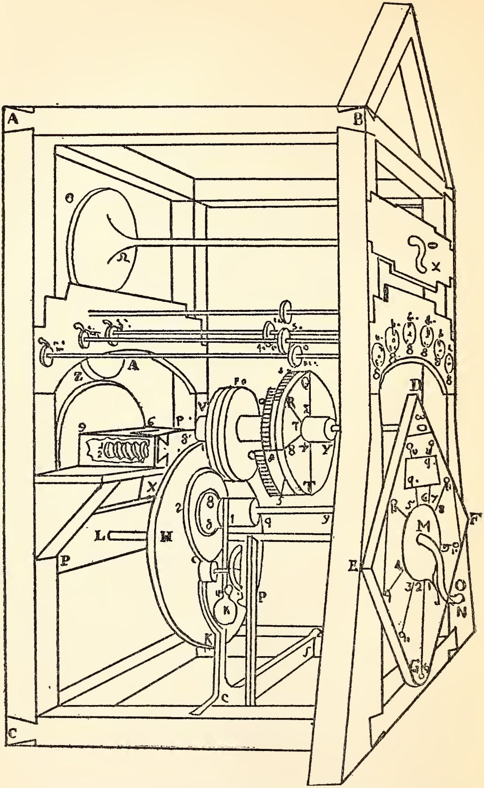


Fig. 27 —Leibnitz Calculating Machine.
From Poleni, the Earliest Treatise on Mechanical Calculation.

the first four operations by machinery alone was that of Blaisé Pascal (1623-1662), when a lad of eighteen. The close application to this work undermined a not over strong constitution, and he died at the early age of 39.

The Pascal machine, which is here illustrated, was constructed on the principle of a wheel upon the circumference of which were marked the first 9 numerals. One turn of this wheel caused the next wheel, similarly marked, to pass through a tenth of a revolution, and so forth. Pascal's machine was not built, however, strictly on a decimal scale, as it was designed for monetary work. A similar attempt was made by Leibnitz, the German mathematician.

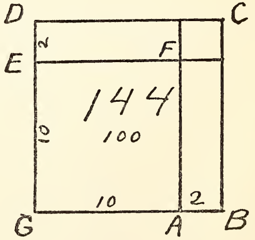
The most elaborate calculating engine ever attempted was designed by Charles Babbage (1791-1871), on which he expended a private fortune of over $100,000, and toward which the British Government contributed $80,000 and a fireproof building for its construction. While the machine was never completed, the work on it left an indelible stamp on British artizanship. The most successful machine was constructed by George and Edward Scheutz, who were inspired by the attempt of Babbage. This machine, which computes and prints logarithmic and other tables, finally came into the possession of the Dudley Observatory at Albany, N. Y. The last few years have seen a great advance in the art of constructing computing machines for purely commercial purposes.

The inverse process of involution is evolution, the problem of which is to determine one of a given number of equal factors when their product alone is given. The factors so found are called square root, cube root, fourth root, etc., depending upon the number of factors involved. The square root of 4 is 2, the cube root of 27 is 3. The simplest method of extracting a root is to divide the number by its lowest prime factor and continue the process. It may be illustrated in finding the cube root of 216. Since there are three factors 2, and three factors 3, there are three factors 2×3, or 6; or the cube root of 216 is 6.

$$\begin{array}{r|l} 2 & 216 \\ \hline 2 & 108 \\ \hline 2 & 54 \\ \hline 3 & 27 \\ \hline 3 & 9 \\ \hline 3 & 3 \\ \hline & 1 \end{array}$$

The symbol of evolution is $\sqrt{}$, an abbreviation, r, for root, followed by the vinculum; a figure is placed above the $\sqrt{}$ to indicate the root taken, except in the case of square root, when it is usually omitted.

The ordinary algorithm or scheme for finding square root is given in a paraphrase of the work of Theon, of Smyrna, who flourished about 139 A.D.: "We learn the process from Euclid, II, 4, where it is stated, 'If a straight



line be divided by any point, the square on the whole line is equal to the squares of both parts, together with twice the oblong which may be found from those segments.' So, with a number like 144, we take a lesser square, say 100, of which the root is 10. We multiply 10 by 2, because in the remaining gnomon, ABCDEF, there are two ob-

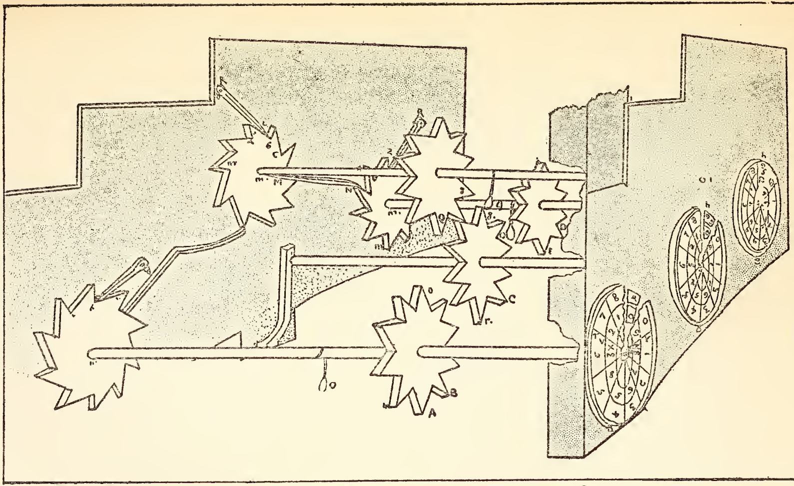


Fig. 28 —Disk Diagram of Leibnitz' Calculating Machine.

longs, and divide 44 by 20. The remainder, 4, is the square of AB, or 2."

Cube root is found in a similar manner, based on the cube instead of the square. Thus the cube on the sum of two lines, a and b, is equal to the cubes on a and b and 3 flat figures a on two edges and b on the third, together with 3 oblong figures b on each of 2 edges and a on the third; this is expressed by a formula: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

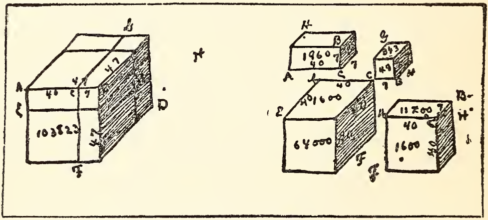
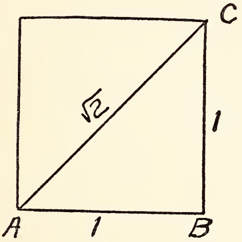


Fig. 29 —Illustration of Cube Root. (Trenchant's Arithmetic, 1566.)

That evolution does not always result in a number of our system, a fraction, which will now be called a 'rational' number, is seen if one attempts to find the square root of 2. This may be done with any degree of approximation by annexing ciphers on the right of units' column, resulting in an endless decimal, 1.4142 . . . . . . That this number cannot be expressed as a fraction is proved in Euclid's 'Elements of Geometry,' altho the proof is attributed to some commentator. Suppose $\sqrt{2} = \frac{m}{n}$, where m and n represent the numerator and denominator of a fraction, and have no common factor. Then multiplying this equation by itself, member by member, $2 = m^2/n^2$, which says that m^2 is divisible by n^2, which cannot be, since m and n have no common factor. In a square, side 1, the diagonal is represented by $\sqrt{2}$.

It is proved in Euclid I, 47, that the square of AC is equal to the sum of the squares on BC and AB. The square on AB is 1, on BC 1, and the sum of these is 2. The square on AC is 2, then AC is $\sqrt{2}$. If AC and AB have a common measure—that is, if a third line exists which is contained a whole number of times in AB and AC—$\frac{\sqrt{2}}{1}$ would be represented by the quotient of two whole numbers, as $\frac{m}{n}$, which is shown above to be im-



possible. If AB is taken as this third line, it is contained in itself once, and in AC more than once and not twice; or, the ratio of these two numbers, $\frac{m}{n}$, is less than 2 and more than 1. This may be put in the form, $1 < \frac{m}{n} < 2$. If $1/10$ of AB is taken, there results $1.4 < \frac{m}{n} < 1.5$. If $1/10$ of this is used, $1.41 < \frac{m}{n} < 1.42$. Continuing, $1.414 < \frac{m}{n} < 1.415$, $1.4142 < \frac{m}{n} < 1.4143$, and so on indefinitely. These two lines are said to be incommensurable; that is, they have no common measure. Euclid does not treat of incommensurables as such, as his mode of representing numbers by lines, which will be spoken of later, and the peculiar device used by him in dealing with ratios, avoided the difficulty. Theodorus (c. 400 B.C.) showed that the lines represented by $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{13}$, $\sqrt{14}$, $\sqrt{15}$, and $\sqrt{17}$, are incommensurable with the unit line.

Going back to the number system following division, it was found to be representable by a series of dots, between any two of which existed a third dot, yet the dots do not form a continuous line. If one chooses as the side of the above square the distance from dot 0 to dot 1, and then lays off AC from 0, the end C will give a dot which is not found in the system of rationals. The final widening of the number system, so far as arithmetic is concerned, takes place here when such expressions as $\sqrt{2}$, $\sqrt{3}$, π or the ratio of the circumference to the diameter of a circle $= 3.14159 \ldots$, ε, the base of the Naperian system of logarithms are called numbers, altho none of them is representable fully by any number of orders in the Hindu notation. Such numbers are called irrationals, and are divided into two classes: surds, which are expressible by a combination of root signs, and transcendentals, which are not, as π and ε. A transcendental is sometimes defined as a number which is not the root of any algebraic equation, with positive integral exponents and rational coefficients.

Irrationals were discovered by the Pythagoreans. The following story is told concerning irrationals: "It is said that the man who first made the theory of irrationals public died in a shipwreck because the unspeakable and invisible should always be kept secret, and that he who by chance first touched and uncovered this symbol of life was removed to the origin of things, where the eternal waves wash around him." Such is the reverence in which these men held the theory of irrational quantities.

Greek arithmetic, the science of numbers as distinguished from logistic, or calculation, has its beginnings with Pythagoras (circa 569-500 B.C.), who founded a brotherhood holding common philosophical beliefs, which were based on mathematics. The Pythagoreans did not commit their work to writing and held it secret from those outside their own circle, and the glory of any discovery was given to Pythagoras himself as the founder of the school.

The properties of numbers studied by the Pythagoreans may be classed under four heads which give rise to four types of numbers: 'Polygonal' numbers, or those numbers which if indicated by dots can be arranged in polygons or regular figures; 'factors' of numbers, numbers forming a 'proportion,' and numbers in 'series.'

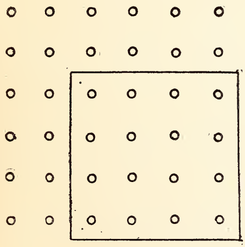


Fig. 30 —GNOMON.

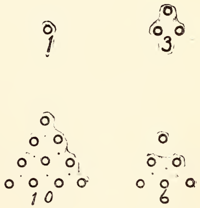


Fig. 31 —TRIANGULAR NUMBERS.

All numbers (whole) are divided into two classes, even and odd. The odd numbers, 1, 3, 5, 7, . . . are called 'gnomons'—that is, an odd number is always the difference between two square numbers, and can therefore be represented by the figure which remains when a square is cut from the corner of a larger square. Thus in the figure 36 is a square number, since it can be arranged in the form of a square with 6 dots on a side. The lower right-hand square 16 is taken from 36 and there remains the gnomon 20.

The product of two numbers is said to be plane, and if the number cannot be represented by a square it is called oblong. Triangular numbers are those which can be arranged in the form of a triangle:

In the triangular number 10, one side of the triangle is 4. The following passage from Lucian (given by Ball) has reference to this fact. A merchant asks Pythagoras what he can teach him. The following conversation ensues:

> *Pythagoras*—I will teach you how to count.
>
> *Merchant*—I know that already.
>
> *Pythagoras*—How do you count?
>
> *Merchant*—One, two, three, four——
>
> *Pythagoras*—Stop! What you take to be four is ten, a perfect triangle, and our symbol.

It may be said that the whole treatment of numbers by the Greeks through the time of Euclid was geometrical. The ease with which numbers could be represented by lines led to a habitual linear symbolism such as is used by Euclid (circa 300 B.C.), where the second, seventh, eighth and ninth and tenth books either deal with magnitudes, which include lines as well as numbers, or numbers themselves which are represented by lines.

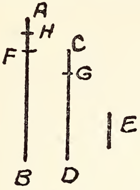
The first proposition of the seventh book of Euclid is taken from T. L. Heath's 'Euclid,' Vol. II, p. 296, the most valuable commentary that has appeared in English:

> Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another (that is, will contain no common factor).
>
> For, the less of two unequal numbers AB, CD being continually subtracted from the greater, let the number which is left never measure the one before it until

an unit is left. I say that AB, CD are prime to one another—that is, that an unit alone measures AB, CD.

For, if AB, CD are not prime to one another, some number will measure them.

Let a number measure them, and let it be E; let CD, measuring BF, leave FA less than itself; let AF, measuring DG, leave GC less than itself, and let GC, measuring FH, leave an unit HA.



Since, then, E measures CD, and CD measures BF, therefore E measures BF. But it also measures the whole BA; and therefore it will also measure the remainder AF.

But AF measures DG; and therefore E also measures DG. But it also measures the whole DC; therefore it will also measure the remainder CG.

But CG measures FH, therefore E also measures FH.

But it also measures the whole FA; therefore it will also measure the remainder, the unit AH, tho it is a number, which is impossible.

Therefore no number will measure the numbers AB, CD; therefore AB, CD are prime to one another.

This theorem leads to the usual method of determining the largest number which is a common factor of two given numbers. The smaller is divided into the larger, the remainder from this division into the former divisor. The final remainder which is contained without a remainder is the largest common divisor; if this last divisor is unity the numbers are said to be prime to each other.

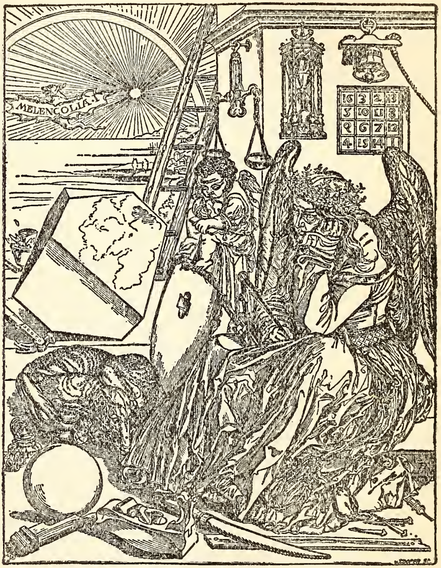


Fig. 32 —Albert Durer's Engraving Melancholy, Showing Magic Squares.

With the Greeks is found much mysticism, imbibed from the Egyptians. The Pythagoreans sought the origin of all things in number.

One is the essence of all things; four is the symbol of perfection corresponding to the human soul; five is the cause of color; six of cold; seven of mind, health and light; eight of love and friendship. A perfect number is equal to the sum of its factors: $28 = 1 + 2 + 4 + 7 + 14$. Other numbers are excessive or defective. Amicable numbers are those each of which is equal to the sum of the factors of the other, as $222 = 1 + 2 + 4 + 71 + 142$ and $284 = 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 45 + 55 + 110$.

To Eratosthenes is due a method of picking out prime numbers, numbers which have no factors except the number itself and unity. The even numbers, except 2, contain no primes. All the others, as far as one wished to go, were written upon a papyrus. Every third number contains 3 as a factor and was cut out of the papyrus, so with every fifth, seventh, and so forth. The remaining numbers on the papyrus are prime. The papyrus with the holes where the numbers were cut out was called Eratosthenes' sieve. The last important Greek writer on arithmetic was Diophantus of Alexandria, who flourished about 150 B.C. His work will be mentioned in connection with Algebra.

One of the famous theorems in the theory of numbers, due to Fermat, concerns the number of primes contained in the form $Fn = 2^{2n} + 1$ where n is any number. Fermat believed that every value of n gives a prime and showed this, for $n = 0, 1, 2, 3, 4$.

Euler in 1732 found that for $n = 5$ the number has a factor, 641. Factors have been found for each of the following values of n: 6, 7, 9, 11, 12, 18, 23, 36, 38.

Fermat asserted without proof that $n_n + y^n = z_n$ is unsolvable except in certain self-evident cases. Mathematicians have not as yet been able to prove or disprove this statement.

Dedekind's view of the irrational as a "schnitt," or cut, may be given in his own words, "If all points of the straight line fall into two classes, such that every point

10	26	6
24	1	17
8	15	19

8	15	19
12	25	5
22	2	18

10	24	8
23	7	12
9	11	22

23	3	16
7	14	21
12	25	5

24	1	17
7	14	21
11	27	4

26	1	15
3	14	25
13	27	2

9	13	20
11	27	4
22	2	18

10	26	6
23	3	16
9	13	20

6	17	19
16	21	5
20	4	18

Fig. 33 —The Nine Sections of a Magic Cube. (Andrews.)

47	10	23	64	49	2	59	6
22	63	48	9	60	5	50	3
11	46	61	24	1	52	7	58
62	21	12	45	8	57	4	51
19	36	25	40	13	44	53	30
26	39	20	33	56	29	14	43
35	18	37	28	41	16	31	54
38	27	34	17	32	55	42	15

Fig. 34 —Closed Knights Tour, Magic Square. (Wenzelides.)

A	B	C	D
E	F	G	H
I	K	L	M
N	O	P	Q

68	-29	41	-37
17	31	79	32
59	28	-23	61
-11	-77	8	49

Fig. 35 —Euler's Magic Square.

of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions." If the point represents a rational number, well and good; if not, the 'exists' posits such a point and it is said to represent an irrational number.

The formation of magic squares which reveal the wondrous symmetry of numbers has had a fascination for mathematicians of all lands. The earliest record of a magic square is found in Chinese literature of about 1125 A.D. ('Chinese Philosophy,' by Dr. Paul Carus, quoted by W. S. Andrews). The wood-cut by Albert Dürer contains the first magic square found in the Christian Occident. Successive numbers, beginning with 1, are to be so placed in square array that the sum of each column, the sum of each row and the sum of each diagonal shall be the same. A curious form of the magic square was worked out by a Moravian, Wenzelides, in which the numbers, in addition to having the arrangement of a magic square, follow the knight on a chessboard, one square forward and one square diagonally. (Fig. 34.)

Magic cubes have also been constructed, in which the numbers are arranged in cubical array.

An unsolved problem found among Euler's papers is to place a number in each of the sixteen squares A, B, C . . . such that the sum of the squares of the numbers shall fulfil the conditions of a magic square, and in addition the products of the numbers taken horizontally two at a time, and also vertically two at a time, shall be the same.

Euler stated that he had found a general means of solution which is not given. The particular case here given was found in the papers he left. (Fig. 35.)

CHAPTER III

ALGEBRA

There is no hard-and-fast dividing line between Algebra and Arithmetic. Algebra was called by Sir Isaac Newton Universal Arithmetic, a generalization of those processes which have to do with number. It is a generalization in the application of the processes rather than in the processes themselves. The most important generalization is in the notion of number itself. In arithmetic it was represented by a continuous line, indefinite in extent both to the right and left. A combined result of the three inverse processes, subtraction, division and evolution, widens this number system to cover the entire plane.

Algebra has been defined as the Science of the Equation, but the equation is also a valuable asset of arithmetic. When the savage first recognises that 2 is made up of 1 and 1, setting these ideas over against each other and balancing them, the equation has become a factor in his thought, altho it has had no symbolic or verbal expression. The algebraic use of the equation differs essentially from the general use to which it is put in arithmetic. In the latter it was arrived at after a process of thought and sums up that thought; that is, it becomes a formula in which are found only known terms. It is seen after an elaborate course of reason and experiment that the square described on the sum of two lines a and b is equivalent to two squares, one on a and another on b, and two rectangles or oblongs formed by a for one side and b for the other.

This is put in the shape of a formula, $(a + b)^2 = a^2 + 2ab + b^2$, where nothing is found in it except the known lines a and b. Thus in arithmetic the equation is the vehicle by which truth already discovered is expressed.

On the other hand, in Algebra the equation is the tool by which the discovery is made. The unknown number, or the number to be found, is represented by some symbol or word, and from the statement of the problem a balance is set up which the operator manipulates until such unknown is determined. The equation is the most useful and powerful tool in the hands of the algebraist, and this particular distinction just made may be said to be the important one. The main purpose of Algebra is to evolve a mechanism by which the equation may be so manipulated that it will reduce to a simple equation between the unknown number on the one hand and a known number on the other.

If the average school boy were asked for his notion of algebra his probable reply would be that it has something to do with x and y. In paging over a recent text-book on the subject the remark was made that the whole language seemed to be made up of x's and y's. While the development of a comprehensive symbolism is one of the important features of the Algebra of to-day it was not always so. The modern symbolism in Algebra did not reach its present perfection until the eighteenth century, and in the past ten years a new symbolism has sprung up in which words, which are ambiguous at best, are entirely replaced by symbols in the whole course of the reasoning.

However, Algebra to-day is characterized by a more general symbolism for number. The use of a single letter for the unknown number and of other letters for the known numbers involved greatly facilitates the operations with these numbers and enables the stating of a general law in a single step. In the formula cited above a is a line; it may be regarded as a number which is found by measuring the line by a unit. Two elements come in which make this a more general number than could be expressed in the Hindu notation. If the unit is changed the number a is changed. In this way a may be said to stand for any positive number whatsoever. Again a and b may be any two lines at will and the statement is still true. The Principle of Continuity or of No Exception, invoked in the widening of the number system, gave new numbers which in general obey the laws of the old. Thus the above statement, which originated with a and b as lines, is equally true if a and b are negative numbers. Summing up this point, it may be said that in addition to representing numbers by the Hindu method, Algebra represents numbers by means of letters, and while such numbers are regarded as known, yet it may be that no particular value is thought of in the discussion, and they may be given any value at will.

Again, a number which is in a constant state of flux or change may be the subject of thought, as the price of wheat on the exchange or the velocity of a railroad train. It would be exceedingly difficult to represent such a number with no more mechanism than arithmetic affords, but Algebra allows of its representation by a letter. The last letters of the alphabet are usually allotted to these variable numbers and the first letters to constants or numbers which do not vary. Another and in some ways parallel distinction is made in using the last letters for unknowns and the first letters for knowns. These are simply two phases of the same convention.

This use of a letter for a general number is found in the works of Aristotle, where he says in one place: "If A is the moving force, B that which is moved, G the distance and D the time," etc.

Still a more general representation of number may be arrived at through the idea of functionality. A number is said to be a function of one or more other numbers if it depends for its value upon the value of the other number or numbers. Thus the volume of a rectangular solid depends on the length of base, the width of base and the altitude, In some cases it is known exactly what the relation termed functionality is, but in the great majority of cases such functionality or dependence cannot be put in any more definite form.

If a, b are respectively the length and width of the base and c the altitude and V the volume of the rectangular solid, functionality is expressed by $V = F(a, b, c)$. This functionality may be more definitely expressed as $V = a \times b \times c$. One says the state of the weather, S, depends upon temperature, T, humidity H, direction and velocity of wind, D and V. But no more definite form can be written than $S = F(T, H, D, V)$.

If x, y are two variable quantities, with dependence of y upon x, this is put in the form $y = f(x)$.

The number system of arithmetic was developed from the simple process of counting and gave rise to an idea of number (the field of real numbers) which was associated with a line, a space notion. Real number may be thought as arising from sequence in time. 51 is thought of not as a collection of 51 units, but as an element in a series after 50 and before 52. In counting one arrives at 51 after 50 and before 52. In this way Algebra may be conceived of as the science of time series as opposed to geometry, the science of space, thus treating of the a priori elements of Kant, time and space. The two, Algebra and Geometry, have been closely interwoven in their historical development, especially in the beginnings of each. It has been seen how the Greeks built their theory of number upon its line representation, and it is a commonplace that if a relation can be pictured to the eye by means of a figure in space the reasoning is greatly assisted.

Such a view is sometimes misleading. If intuition alone had been trusted to determine whether or no all points had been used up by fractions, the answer would have been 'yes' and the irrationals would have been omitted.

Environment and racial conditions have been the determining factors in the growth of Algebra and Geometry.

Egypt was an agricultural country, land was of value and Geometry as the science of measurement began there. The Arabians were a nomadic people; land was only valuable at the time it was being grazed by the flocks and herds; the peculiarly clear atmosphere, resplendent with myriads of stars, nightly turned the Arab's attention to the celestial bodies as he tended the flock, and he was led to cultivate those branches of analysis and astronomy which he received as the product of the subtle, imaginative mind of the Hindu. Thus Geometry and Algebra, each arising from the needs and characteristics of a race peculiarly adapted to its cultivation, were developed side by side, each borrowing something from the other, but preserving its own distinctive qualities until the time of Descartes (1637), when by his invention of the analytic geometry the two streams converge again, each becomes in full the interpretative agency of the other.

Less than fifty years ago it began to be more and more realized that while Geometry always interpreted Algebra correctly if it itself were correctly interpreted, yet the notions of Geometry were only conventionally and approximately represented by a figure and that intuition guided by the eye was not always to be trusted. So a new movement sprang up to completely arithmetize Geometry. Its first and great apostle was Karl Weierstrass, the "father of precision," born at Ostendfelde October 13, 1815.

The investigations of the foundations of mathematics of the past ten or fifteen years, carried on by a host of mathematicians in Italy, France, Germany, England and the United States, has carried this work farther—to base all geometry upon number. Thus the continuity of the whole field of mathematics has been established and a complete symmetrical system has been built up or created, beginning with the simple notion of putting one with one, growing like a great oak from the acorn until to-day it is impossible for one mind in a lifetime to embrace it in all its ramifications.

A simple equation is one in which there is one unknown quantity and it is involved only in its first power or degree, as $x + 7 = 15$. It is easily seen that the only value of x for which this equation can be true is 8 or $x + 7 = 15$ if $x = 8$. A simple equation then may be looked upon as a single condition which is satisfied if a certain value is given to the unknown.

The Egyptian treatise on mathematics by Ahmes gives, after his treatment of unit fractions, eleven problems, each resulting in a simple equation. The equation given is quoted by Cajori.

Hā'	**neb-f**	**ma-f**	**ro sefeχ-f**	**hi-f**	**χeper-f**	**em sa sefeχ;**
Heap	**its $\frac{2}{3}$,**	**its $\frac{1}{2}$,**	**its $\frac{1}{7}$,**	**its whole,**	**it gives**	**37**
i.e.	$x(\frac{2}{3}$	$+ \frac{1}{2}$	$+ \frac{1}{7}$	$+ 1)$	$=$	**37**

Another problem reads "Heap, its $^2/_3$, its $^1/_2$, its $^1/_7$, its whole, it gives 33." Which put in modern form, omitting the sign of addition which was not used by Ahmes, $1\ ^2/_3\ ^1/_2\ ^1/_7\ x = 33$.

The method of solution is to determine by what $1\ ^2/_3\ ^1/_2\ ^1/_7$ must be multiplied to give 33 and the answer is $14\ ^1/_4\ ^1/_{97}\ ^1/_{56}\ ^1/_{679}\ ^1/_{776}\ ^1/_{194}\ ^1/_{388}$. Such was the laborious and awkward solution of a simple equation.

The mathematics of the Hindus from Brahmagupta, born 598 A.D., to Bhaskara, born 1114, was made known to the English-speaking world by H. T. Colebrooke (1817). These treatises are clothed in mystic and obscure language and are very difficult of translation. The story of the origin of the work by Bhaskara is given by Brooks: "The work is named for the author's daughter, Lilavati, who it appeared was destined to pass her life unmarried and without children. The father, however, having ascertained a lucky hour for contracting her in marriage, left an hour-cup on a vessel of water, intending that when the cup should subside the marriage should take place. It happened that the girl, from a curiosity natural to children, looked into the cup to see the water coming in at the hole, when, by chance, a pearl separated from her bridal dress, fell into the cup, and rolling down to the hole, stopped the influx of water. When the operation of the cup had been thus delayed, the father was in consternation, and examining, he found that the small pearl had stopped the flow of water, and the long expected hour had passed. Thus disappointed, the father said to the unfortunate daughter, 'I will write a book of your name, which shall remain to the latest times, for a good name is a second life and the groundwork of eternal existence.'"

The following problem from the Lilavati serves to show the poetic form in which they are garbed:

"Out of a heap of pure lotus flowers, a third part, a fifth, a sixth, were offered respectively to the gods Siva, Vishnu, and the Sun; a quarter was presented to Bhavani; the remaining six were given to the venerable preceptor. Tell me, quickly, the whole number of flowers."

"Out of a swarm of bees, one-fifth of them settled on the blossom of the cadamba and one-third on the flower of the silind'hri; three times the difference of these numbers flew to the bloom of a cutaja. One bee, which remained, hovered and flew about in the air, allured at the same moment by the pleasing fragrance of a jasmine and pandanus. Tell me, charming woman, the number of the bees."

The following examples are taken from The Ganita-Sara-Sangraha, previously quoted, translated by M. Rangacharya, of Madras. The source of this material is an article by Professor David Eugene Smith in 'Bibliotheca Mathematica' (December, 1908):

"One-fourth of a herd of camels was seen in the forest;

twice the square root (of that herd) had gone on to the mountain slopes; and three times five camels (were) however (found) to remain on the bank of a river. What is the numerical measure of that herd of camels?"

A quadratic equation is one in which appears as the highest power of the unknown the second power. Thus the equation $x^2 - 7x + 12 = 0$ contains the second power of x and is therefore a quadratic, yielding as the two values of x, 3 and 4. The question naturally arises, How can x be at the same time 3 and 4? The quadratic is the expression of a double condition; it is satisfied not by 3 and 4 at the same time, but by 3 or by 4.

As is seen by substituting 3 or x, giving $3^2 - 7 \times 3 + 12 = 0$, or, $9 - 21 + 12 = 0$, again 4, $4^2 - 7 \times 4 + 12 = 0$, or, $16 - 28 + 12 = 0$. The equation $x^2 - 7x + 12 = 0$ is true if x is 3, or if x is 4.

Various devices have been used to solve the quadratic, which may be written in the general form $ax^2 + bx + c = 0$, where a, b, c may have any values whatever except that a may not be 0 (if $a = 0$, the second degree term would vanish and the equation would no longer be quadratic). The simplest mode is by "completing the square." If the equation to be solved is $x^2 + 6x = 16$, it is seen by comparing with the expression for the square of $(a + b)$, $a^2 + 2ab + b^2$ that the left member of the equation in order to be a perfect square should have the term 9 added to it. Adding this to the other side also the balance is preserved.

$$x^2 + 6x + 9 = 16 + 9 = 25.$$

Now since both sides are perfect squares, the square roots may be found. The square root of $x^2 + 6x + 9 = x + 3$. And the square root of 25 may be $+5$, or -5, since $(+5) \times (+5) = 25$, and $(-5) \times (-5) = 25$. This two-fold condition is then expressed by writing $\sqrt{25} = \pm 5$. Where as above it is understood that, either $+5$, or -5 is to be taken. Equating the square roots of the two members $x + 3 = \pm 5$, and breaking this up into two conditions,

$$x + 3 = + 5 \text{ or } x + 3 = - 5$$
$$x = 2 \qquad x = - 8$$

Bhaskara, who solved such equations, says "the second value in this case is not to be taken, for it is inadequate, people do not approve of negative roots."

Such equations as the above were readily solved by the Hindus. Hankel says of them: "If one understands by Algebra the application of arithmetical operations to complex magnitudes of all sorts, whether rational or irrational numbers or space-magnitudes, then the learned Brahmins of Hindostan are the real inventors of Algebra."

About 150 years after Mohammed's flight from Mecca, the study of Hindu science was taken up at Bagdad in the court of Caliph Almansus. In 773 A.D. there appeared at his court a Hindu astronomer, with astronomical tables which were translated into Arabic. The first Arabic treatise now known is that of Muhammed ibn Mûsâ Alchwarizmi. The work, which was translated probably by Athelard of Bath, and which is the first work in which the word Algebra (or in the Arabic aldschebr walmukâbala) occurs, begins: "Spoken has Algorithmi. Let us give deserved praise to God, our leader and defender." The word Algorithmi is the Latin form of the author's name, from which comes the word algorithm, signifying a rule for computation. The two words used as a name for Algebra mean "restoration and opposition" and have reference to the transposing of the terms of an equation and discarding equal terms from both members.

An equation of the form $y = 2x + 5$ expresses a condition between two unknowns or variables. Such an equation is said to be indeterminate, since any number of pairs of values of x and y will satisfy it.

If $x = 1$, $y = 7$; if $x = 0$, $y = 5$; if $x = - 1$, y is 3;

if $x = -2$, $y = 1$; if $x = -3$, $y = -1$, and so on indefinitely. This relation between x and y may be shown graphically by a method which is the foundation of the Analytic Geometry, invented by Des Cartes (1637), from which date, it may safely be said, modern mathematics takes its rise. The principle upon which it is based is that a point in a plane may be located if its distances are known from two intersecting lines, called axes. These axes are chosen for convenience at right angles, altho this is immaterial except for simplicity.

The study of indeterminate equations is called Diophantine analysis, from Diophantos of Alexandria, the last great Greek mathematician, of whose work six books remain, which treat of such problems as:

To find a right-angled triangle such that the difference of its sides is a square, and also the greater alone is a square, and, thirdly, its area + the less side is a square. A solution to this problem is to take 1, 2 for the lengths of the sides. The Fermat equation $x^n + y^n = z^n$ is an indeterminate equation.

The most famous problem of this type is the "cattle" problem, attributed to Archimedes, the most celebrated problem of antiquity. It is in the form of an epigram, and has been translated by T. L. Heath as follows:

"Compute, O stranger! the number of cattle of Helios, which once grazed on the plains of Sicily, divided according to their color, to wit: (1) White Bulls = $\frac{1}{2} + \frac{1}{3}$ of the Black Bulls + Yellow Bulls; (2) Black Bulls = $\frac{1}{4}$ and $\frac{1}{5}$ of the Dappled Bulls + the Yellow; (3) Dappled Bulls = $\frac{1}{6} + \frac{1}{7}$ of the White + Yellow; (4) the White Cows = $\frac{1}{3}$ and $\frac{1}{4}$ of the Black Herd (Bulls and Cows = Herd); (5) the Black Cows = $\frac{1}{4}$ and $\frac{1}{5}$ of the Dappled Herd; (6) the Dappled Cows = $\frac{1}{5}$ and $\frac{1}{6}$ of the Yellow Herd; (7) the Yellow Cows = $\frac{1}{6} + \frac{1}{7}$ of the White Herd.

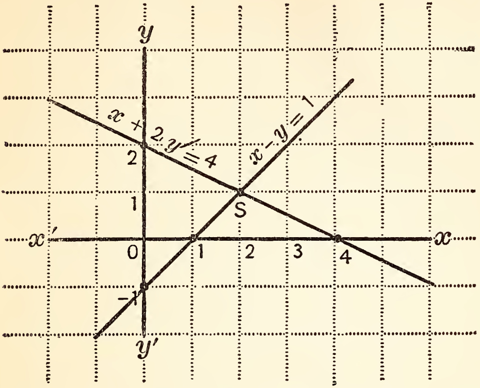
"He who can answer the above is no novice in numbers, nevertheless he is not yet skilled in wise calculations; but come, consider all the following numerical relations between the Oxen of the Sun: (8) If the White Bulls were combined in one total with the Black Bulls, they would be in a figure equal in depth and breadth, and the far-stretching plains of Thrinacia would be covered by the figure (square) formed by them; (9) should the Yellow and Dappled Bulls be collected in one place, they would stand, if they ranged themselves one after another, in the form of an equilateral triangle. If thou discover the solution of this at the same time; if thou grasp it with thy brain; and give correctly all the numbers; O stranger! go and exult as a conqueror; be assured that thou art by all means proved to have abundance of knowledge in this science."

The Hillsboro (Ill.) Mathematical Club worked on this problem from 1889 to 1893. The answer given for the number of white bulls will reveal the magnitude of the numbers involved: 1,596,510,804,671,144,531,435,526,194,-370, . . . 385,150,341,800.

Where the . . . indicate the omission of 68,834 periods of three figures each. Each of the ten answers is composed of 206,545 figures.

Another of these famous puzzles is attributed to Euclid: "A mule and a donkey were walking along laden with corn. The mule said to the donkey, 'If you gave me one measure I should carry twice as much as you; if I gave you one, we should both carry equal burdens.' Tell me their burdens, O most learned master of geometry."

If two equations, $x + 2y = 4$, and $x - y = 1$, are given, both x and y are determined. Such a system is called a linear system, and a single pair of values of x and y may be found which satisfies both conditions. The statement of the two equations may be thought of as requiring that the position be found in which the generating point of either line will simultaneously lie on its own line and also on the other. The graphical solution indicates that the point S ($x = 2$, $y = 1$), or more briefly put, S (2, 1), is the desired point.



In the study of such systems, Leibnitz (1646-1716) discovered a symmetrical arrangement of the known numbers, or the coefficients as they are called, which has been of immense service. This symmetrical array is called a determinant.

The system of three equations

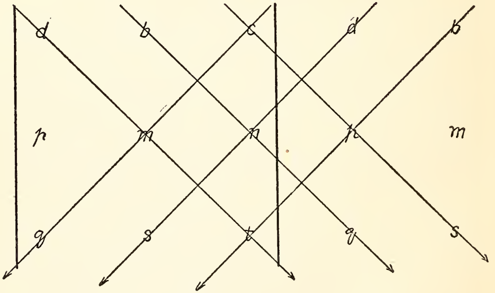
$$\begin{aligned} ax + by + cz &= d \\ lx + my + nz &= p \\ rx + sy + tz &= q \end{aligned}$$

may be solved for x by writing a fraction whose numerator is made up of the numbers on the right for a first column and the coefficients of y and of z for the other two, and the denominator is the three columns of coefficients of x and y and z.

The following is the arrangement:

$$x = \frac{\begin{matrix} d & b & c \\ p & m & n \\ q & s & t \end{matrix}}{\begin{matrix} a & b & c \\ l & m & n \\ r & s & t \end{matrix}}$$

The evaluation of this may be shown in the method used for finding the numerator.

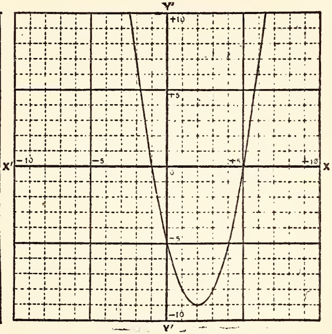


The numbers connected with each arrow to the right are multiplied and given the + sign, those connected with arrows pointing to the left are multiplied and given the — sign. The sum of the six terms is the numerator, or, $dmt + bnq + cps - smq - bpt - dsn$: — similarly for the denominator.

An equation of the form $x^2 - 4x - 5 = 0$, called a quadratic, or equation of the second degree, has been solved by "completing the square." Another method is by means of a graph $x^2 - 4x - 5$ is placed equal to y and the graph drawn by taking particular values for x, and from these determining the values of y which goes with each. A table of these values (taken from Boyd's Algebra) shows the process. It is required to find the values of x which makes y = o or which satisfy $x^2 - 4x - 5 = y$, when y = o. In the figure y = o when the curve crosses the x axis X′ X, or the values are — 1 + 5.

$x^2 - 4x - 5 = 0$, then, when $x = -1$, or $+5$.

For $y = x^2 - 4x - 5$	
x	y
0	− 5
+1	− 8
+2	− 9
+3	− 8
+4	− 5
+5	0
+6	+ 7
etc.	etc.
−1	0
−2	+ 7
−3	+16
etc.	etc.



Another figure taken from the same text shows the method of solving the simultaneous quadratic system.

$$x^2 + y^2 - 2xy - 4x - 8y - 20 = 0$$

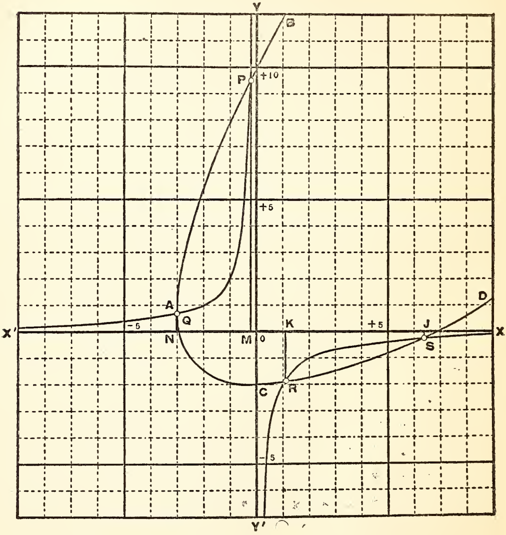
$$xy = -2.$$

P, Q, R, S are the points of intersection of the two curves, and the value of x and y for each can be read directly from the figure.

Solving the equation, $x^2 - 6x = -13$, by completing the square, adding 9 to both members, x is found to be equal to $3 \pm \sqrt{-4}$, and the question arises, What is the measuring of $\sqrt{-4}$? It is known that $(+2)^2 = +4$, and that $(-2)^2 = +4$. No number in the system so far considered will, when squared, give a negative number, and means must be devised by which such a number may be interpreted.

$\sqrt{-4}$ may be factored into $\sqrt{4}$, $\sqrt{-1}$ or $2\sqrt{-1}$.

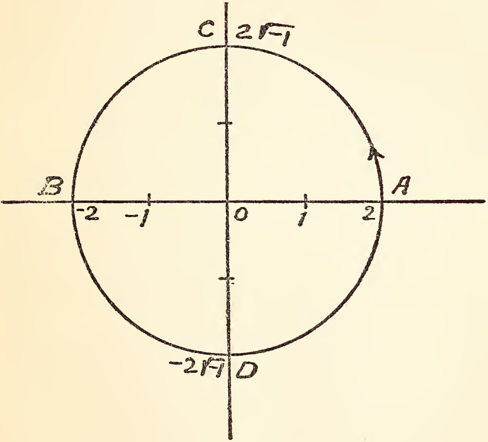
If 2 be multiplied by -1, the result is -2, or the point



A is changed over to the position B. $\sqrt{-1}$ multiplied by itself must produce -1, from the notion of square root, or $\sqrt{-1} \times \sqrt{-1} = -1$.

Then $2 \times \sqrt{-1} \times \sqrt{-1} = -2$. If 2 be multiplied twice in succession by $\sqrt{-1}$, the result is moving

A to B. Then it is reasonable to suppose that one multiplication or $2 \times \sqrt{-1}$ should move it halfway. All that is now necessary is to choose the path. If A should be moved along the line A B, half the motion would carry it to 0, or $\sqrt{-1} \times 2 = 0$. But $0 \times 2 = 0$, and that would require that $\sqrt{-1} = 0$. But this is not desirable. The next simplest path is a semicircle.



If two multiplications carry A to B, a single multiplication should carry it to C. This is found to be a satisfactory definition, for by 3 multiplications A is carried around to D.

$2 \times \sqrt{-1} \times \sqrt{-1} \times \sqrt{-1} = 2 \times (-1) \times \sqrt{-1} = -2\sqrt{-1}$, that is, D is marked with the — sign of C, which should be so, and a fourth multiplication gives 2, that is, 4 multiplications carries A through a complete revolution. The $\sqrt{-1}$ is indicated by i, which has the function of a sign, merely indicating that the number before which it is placed belongs on the vertical line CD, while a number without such a sign is on the horizontal line AB, that is, a real number. A number represented on AB is called a pure imaginary, the name 'imaginary' or 'fictitious' number being given to expressions of this kind which constantly arose in the solution of equations and to which no meaning had been attached. Bhaskara says: "The square of a positive as well as of a negative number is positive, and the square root of a positive number is double, positive and negative. There can be no square root of a negative number, for this is no square."

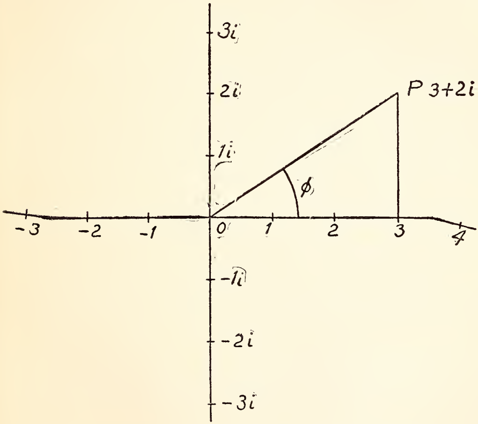
The Italian algebraists called them "impossible numbers." It was not until 1797 that Caspar Wessel devised a method of representation of imaginaries, but it did not attract particular attention. Again in 1806 Jean Robert Argand independently arrived at the representation given above. It is a curious fact that the entire known biography of Argand could be written in half a dozen lines, yet his work is the basis of one of the most extensive fields in all mathematics.

The number system now consists of real numbers represented on a horizontal line and pure imaginaries on a vertical line. The combination of these two classes forms the class complex numbers, which covers the entire plane. In the figure $3 + 2$ i is found by stepping off 3 units to the right of O and 2 units up, giving point P.

On the axis of real numbers, O4, the point marked 3 represents the number 3, but it was found to be sometimes more convenient to think of 3 as represented by the segment of line beginning with O and ending with 3. With the number $2 + 3$ i it will be thought of as represented by the point P or by the line segment OP at will.

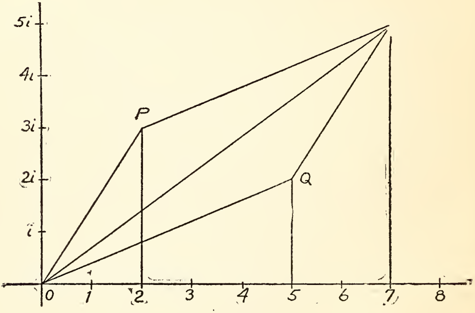
The angle MOP is called the amplitude of P, and is denoted by ϕ. The length of OP, which is $\sqrt{2^2 + 3^2} = \sqrt{13}$, is termed a modulus and indicated by mod P.

Complex numbers obey the laws laid down for real numbers. They may be subjected to the six operations of addition, subtraction, multiplication, division, involution and evolution. The mode of addition is the same as that employed in adding real numbers.



If the real numbers are thought of as line segments, and 2 is to be added to 3, it is done by placing the initial point O of O 2 on the terminal point 3 of O 3. The point then occupied by point 2 of O 2 in its new position is 5 and O 5 is the segment sum of O 3 and O 2. If the two complexes 2 + 3 i and 5 + 2 i are to be added they are represented as in the figure, the first by OP and the second by OQ. Starting at P, lay off OL, 5 units to the right and 2 units up. OR, which is the diagonal of a parallelogram on OP and OQ, is the sum of $2 + 3i$ and $5 + 2i$.

The number system now covers the entire plane; to every point in the plane there is a number and vice versa. The plane is two-dimensional, that is, by the Cartesian coordinates x y a point is determined by two values, x and y, or in the Argand diagram by the two real numbers a and b in the complex $a + bi$. Space is three-dimensional



in points. To locate a point in a room completely it is necessary to specify its distances respectively from, say, the floor and each of two intersecting walls, or by 3 numbers. To take in all points in space, a third line or axis would be drawn perpendicular to the plane of the paper in the Argand diagram at point O. Now if a third sign of direction j were used, and the number system extended to take in space, what would result? The apparent discrepancy between the number system, which is two-dimensional, and space, which is three-dimensional, has been a source of a great deal of study and involves some of the most important theorems of algebraic analysis.

A general equation of the form

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$$

is said to be of the nth degree, where the exponents are all whole numbers. It has been proved that if such an equation is satisfied by a single value of x it is satisfied by n values, that is, it has n roots. These roots may all be real or part real and part complex. If there are complex roots they enter in pairs which are conjugate, that is, if $a + bi$ is a root so also is $a - bi$. The condition that "if it is satisfied by a single root" is very important.

Why should it not be? It was found that the quadratic could be easily solved, and very many special equations of higher degree. The cubic or equation of the third degree taxed the powers of the algebraists, and it was not until 1545 that a general solution was found. It seems almost axiomatic that the general equation must have a root, but such things are not taken for granted. The first proof that the general equation with whole numbers for exponents and coefficients, real or complex, was given in Argand's memoirs. Since that time a number of proofs have been offered, the principal contributor being Cauchy. This is called the fundamental theorem of Algebra.

Now since the hypothesis is proved, the conclusion that there are n roots is easily proved, such proof being familiar to any schoolboy. The next concern is, what is the nature of the roots? Weierstrass proved that the roots all are of the form $a + bi$, that is, complex numbers of the two-dimensional system.

This at once settles the question raised above, whether or not it is possible to extend the number system to the three mensions of space. If the extension is made, such numbers would not be the roots of algebraic equations; in other words, such numbers would not be subject to the ordinary laws of Algebra.

Two diverging lines of thought begin here: if such extension of the number system be made what formal laws of Algebra shall be rejected? Having determined the nature of the roots of equations, to devise laws by which an equation may be solved. The second of the two will be considered first. It has been seen that the quadratic is solvable. Equations of higher degree have been solved in special cases. The general solution of the cubic next received attention.

The following account of the solution of the cubic is from Ball's History of Mathematics:

"Nicolo Fontana, generally known as Nicholas Tartaglia —that is, Nicholas the stammerer—was born at Brescia in 1500 and died in Venice on December 14, 1557. After the capture of the town by the French in 1512, most of the inhabitants took refuge in the cathedral and were there massacred by the soldiers. His father, who was a postal messenger at Brescia, was among the killed. The boy himself had his skull split through in three places, while both his jaws and his palate were cut open; he was left for dead, but his mother got into the cathedral, and finding him still alive managed to carry him off. Deprived of all resources, she recollected that dogs when wounded always licked the injured place, and to that remedy he attributed his ultimate recovery, but the injury to his palate produced an impediment in his speech from which he received his nickname. His mother managed to get sufficient money to pay for his attendance at school for fifteen days, and he took advantage of it to steal a copy-book from which he subsequently taught himself to read and write; but so poor were they that he tells us he could not afford to buy paper, and was obliged to make use of tombstones as slates on which to work his exercises.

"He commenced his public life by lecturing at Verona, but he was appointed at some time before 1535 to a chair of mathematics at Venice, where he was living when he became famous through his acceptance of a challenge from a certain Antonio del Fiori. Fiori had learned from his master, one Scipione Ferreo (who died at Bologna in 1526), an empirical solution of a cubic equation of the form $x^3 + qx = r$. This solution was previously unknown in Europe, and it is probable that Ferreo had found the result in an Arab work.

"Tartaglia, in answer to a request from Colla in 1530, stated that he would effect the solution of a numerical equation of the form $x^3 + px^2 = r$. Fiori, believing that Tartaglia was an impostor, challenged him to a contest. According to this challenge, each of them was to deposit a certain stake with a notary, and whoever could solve the most problems out of a collection of thirty propounded by the other was to get the stakes, thirty days being allowed for the solution of the questions proposed. Tartaglia was aware that his adversary was acquainted with the solution of a cubic equation of some particular form, and suspecting that the questions proposed to him would all depend on the solution of such cubic equations, set himself the problem to find a general solution, and certainly discovered how to obtain a solution of some if not all cubic equations. When the contest took place all the questions proposed to Tartaglia were, as he suspected, reducible to the solution of a cubic equation, and he succeeded within two hours in bringing them to particular cases of the equation $x^3 + qx = r$, of which he knew the solution. His opponent failed to solve any of the problems which were proposed to him, which as a matter of fact were all reducible to numerical equations of the form $x^3 + px^2 = r$ (notice that in this form the x^2 term is present, while in the other the x term appears). Tartaglia was therefore the conqueror, and he subsequently composed some verses commemorative of his victory."

Tartaglia, as was the custom in those days, did not reveal his method of solution. He hoped to publish a treatise on Algebra of which the crowning feature would be the making known to the world this newly discovered solution of the cubic; but in this he was to be disappointed through the treachery of Girolamo Cardan, the most famous astrologer of the time. This Cardan was a most strange admixture of genius and madness, a gambler if not a murderer, an ardent student of science, solving problems which had long baffled investigation. The elder of his two sons was executed for poisoning his wife, while it is said that Cardan cut off the ears of the younger in a fit of rage. In 1570 Cardan was imprisoned for heresy on account of having published the horoscope of Christ.

REGVLA.

Deducito tertiam partem numeri rerum ad cubum, cui addes quadratum dimidij numeri æquationis, & totius accipe radicem, ſcilicet quadratam, quam ſeminabis, uniq; dimidium numeri quod iam in ſe duxeras, adijcies, ab altera dimidium idem minues, habebisq; Binomium cum ſua Apotome, inde detracta ℞ cubica Apotomæ ex ℞ cubica ſui Binomij, reſiduũ quod ex hoc relinquitur, eſt rei æſtimatio.

Exemplum. cubus & 6 poſitiones, æquantur 20, ducito 2, tertiam partem 6, ad cubum, fit 8, duc 10 dimidium numeri in ſe, fit 100, iunge 100 & 8, fit 108, accipe radicem quæ eſt ℞ 108, & eam geminabis, alteri addes 10, dimidium numeri, ab altero minues tantundem, habebis Binomiũ ℞ 108 p: 10, & Apotomen ℞ 108 m: 10, horum accipe ℞[ta] cub[as] & minue illam quę eſt Apotomæ, ab ea quæ eſt Binomij, habebis rei æſtimationem, ℞ v: cub: ℞ 108 p: 10 m: ℞ v: cubica ℞ 108 m: 10.

cub⁹ p: 6 reb⁹ æq̃lis 20
2 20
8 —— 10
108
℞ 108 p: 10
℞ 108 m: 10
℞ v: cu. ℞ 108 p: 10
m: ℞ v: cu. ℞ 108 m: 10

Aliud, cubus p: 3 rebus æquetur 10, duc 1, tertiam partem 3, ad cubum, fit 1, duc 5, dimidium 10, ad quadratum, fit 25, iunge 25 & 1,

H 2 fiunt

Fig. 36 —FIRST PUBLISHED SOLUTION OF THE CUBIC EQUATION; FROM ARS MAGNA. (1545.)

He afterward settled at Rome, where he received a pension in order to secure his services as astrologer to the court. Having foretold that he should die on a particular day, he felt called upon to commit suicide to preserve his reputation.

In 1545 Cardan completed and published the Ars Magna, the most advanced treatise on Algebra which had appeared up to that time, and in which was given Tartaglia's solution of the cubic. This method has since been known as Cardan's method. Cardan also published the work of his pupil, Ferrari, on the biquadratic or equation of the fourth degree. This solution is sometimes known by Bombelli's name, to whom is due the credit of representing the three roots to the simplest form in the so-called irreducible case.

From this time on mathematicians devoted a great amount of time in attempting the solution of equations of higher degree. In his 'Reflections on the Resolution of Algebraic Equations,' Lagrange (1736-1813) gave a scientific classification of the methods already applied to the cubic and biquadratic, but was unable to apply them to the quintic or equation of the fifth degree. In this discussion the foundation was laid for the study of substitutions, but other matters pressing for attention made necessary the laying aside of this work. He determined to take up the subject at some future time, but never did so.

It was reserved for the brilliant young Norwegian, Neils Henrik Abel (1802-1829), to give a rigid demonstration of the impossibility of solving the quintic or higher equations by means of radicals.

The extension of the number system to three dimensions was attempted by Argand and resulted in failure. A corollary of Weierstrass' theorem that the root of an algebraic equation must be of the form $a + bi$ is that no further extension can be made and have the numbers still conform to the laws of algebra. In the formation of the complex number there are two units, 1, or the unit along the axis of reals, and i, the unit along the axis of pure imaginaries. If the system is to be extended to space a third unit is to be chosen—call it j—which will be measured on a perpendicular to the two axes already used. A number of this form would be $a + bi + cj$. When the negative number was introduced, it was assumed that in multiplication it should obey the commutative law that $1 \times i = i \times 1$. This was a pure assumption, made in order to give a meaning to multiplication by a negative. It was the subject of years of meditation with William Rowan Hamilton, as to what would be necessary in order to extend the system so as to include the new unit j. At last, on the 16th of October, 1843, while walking with his wife along the Royal Canal in Dublin, the discovery flashed upon him that the commutative law might be rejected, and he engraved with his knife on a stone in Brougham Bridge the fundamental formula of the new algebra which is called Quaternions. This bridge is since known as Quaternion Bridge.

In 1844 appeared a classic work on analysis, the 'Ausdehnungslehre' of Hermann Grassmann, in which the number system is carried to n dimensions. This work attracted so little notice on account of its "philosophische allgemeinheit" it is said that after eight years but one man had read it. In 1862 a new edition was published which received no more appreciation than the first, and at the age of fifty-three its author, with a heavy heart, gave up mathematics for the study of Sanskrit.

The generalization of algebra is carried out by assuming any number of units i, j, k, l, etc., forming numbers with them as $a + bi + cj + dk + el + \ldots$ and choosing to reject one or the other of the laws of ordinary algebra (for at least one must be rejected) and then building up a consistent algebra upon the remaining laws. In 1870 Benjamin Pierce, one of the foremost mathematicians that America has produced, published his 'Linear Associative Algebra,' givng the elements of 162 algebras, in which the numbers are linear functions of the units and obey the associative law.

CHAPTER IV

GEOMETRY

Geometry is the science of space and is concerned with relations which exist between its various elements, linear, superficial and solid. The earliest measurements were linear, and for the unit was taken some portion of the human body—for example, finger-breadth, palm, span, foot, ell, cubit and fathom—but the body does not possess any convenient unit for the measurement of either surface or solid. The oldest geometrical work known uses the square unit for areas and the cubical unit for solids. How and when the choice of such units was made is difficult to say. The study of primitive races made possible the reconstruction of the steps in the formation of the number concept, but such study is silent in regard to the beginnings of geometry.

The word geometry, from the Greek, meaning "to measure the earth," has its origin, as is the case with most sciences, in the needs of the human being at some particular time, as is indicated by Herodotus (II, 109) where he says that Sesostris (c. 1400 B.C.) divided the land of Egypt into rectangular plots for the purpose of more convenient taxation; that the annual floods, caused by the rising of the Nile, often swept away portions of a plot, and that surveyors were in such cases appointed to assess the necessary reduction in the tax. Hence, in my opinion, arose geometry and so came into Greece.

Ahmes gives a number of problems concerning the calculation of the contents of barns, but as the shapes are unknown it is impossible to interpret them. As with his work in arithmetic, no rules are given, but a number of problems solved in a similar manner. The method used in finding the contents of a barn is to multiply together two of the dimensions and this by one and one-half the third. He also finds the area of a square, of an oblong, of an isosceles triangle and of an isosceles trapezoid, the latter two being incorrectly found. In the isosceles triangle, a triangle with two equal sides, Ahmes takes half the product of the base and one of the equal sides and follows the analogous proceeding with the isosceles trapezoid. While the error is slight in the examples given, it is sufficient to show that the results were only empirical and that Ahmes was unable to extract the square roots which are necessary in an exact solution. The area of a circle is found by deducting from the diameter its one-ninth and squaring the remainder, which gives the value of the ratio of the circumference to the diameter of a circle, usually indicated by π, to be 3.1604, a value much more nearly correct than those used by many later writers.

Another glimpse of Egyptian geometry is given by Democritus (c. 460-370 B.C.): "In the construction of plane figures with proof no one has yet surpassed me, not even the Harpedonaptæ of Egypt. To Professor Cantor is due the credit of making clear the exact meaning of this word, which is a compound of two words, meaning "rope stretchers" or "rope fasteners." Cantor says: There is no doubt that the Egyptians were very careful about the exact orientation of their temples and other public buildings. But inscriptions seem to show that only the north and south lines were drawn by actual observation of the stars. The east and west lines were drawn at right angles to the others. Now it appears, from the practice of Heron of Alexandria and of the ancient Indian and probably also the Chinese geometers, that a common method of securing a right angle between two very long lines was to stretch round three pegs a rope measured in three portions, which were to one another in the ratio $3:4:5$. The triangle thus formed is right-angled. Further, the operation of rope-stretching is mentioned in Egypt, without explanation, at an extremely early time (Amenemhat I). If this be the correct explanation of it, then the Egyptians were acquainted 2,000 years B.C. with a particular case of the proposition now known as the Pythagorean theorem. Egyptian geometry, as well as the other sciences, was in the hands of the priestly caste, whose conservatism is illustrated by the fact that Egyptian doctors used only the recipes of the ancient sacred books, for fear of being accused of manslaughter if the patient died. That no progress was made beyond that of Ahmes is borne out by the Edfu inscriptions of 107-88 B.C., two hundred years after Euclid, in which the formula given by Ahmes for the isosceles trapezoid is still given but applied to any four-sided figure, a proceeding of which Ahmes himself would not have been guilty.

That the early Greek geometers derived their first knowledge from the Egyptians is derived from many sources. Eudemus (c. 330), pupil of Aristotle, wrote a history of geometry in which occurs this passage: "Geometry is said by many to have been invented among the Egyptians, its origin being due to the measurement of plots of land. This was necessary there because of the rising of the Nile, which obliterated the boundaries appertaining to separate owners. Nor is it marvelous that the discovery of this and other sciences should have arisen from such an occasion, since everything which moves in development will advance from the imperfect to the perfect. From mere sense-perception to calculation, and from this to reasoning, is a natural transition." The last step is the one taken by the Greeks—the Egyptian geometry was concrete, a thing of sense, and to Thales is due the honor of creating the beginnings of abstract geometry, a product of reason, the object of which is to establish precise relations between the parts of a figure, so that some of them could be found from others in a purely rigorous manner.

Thales of Miletus (640-546 B.C.) was a merchantman when his native city was in its most flourishing condition, and resided for a long period in Egypt, from whence he returned to his native city in his old age, bringing with him the knowledge of geometry and astronomy. Tradition informs us that he was one of the first gifted with the acumen to form a 'trust.' Learning from the stars that the crop of olives would be abundant during a certain year, Thales secured control of all of the oil-presses, and in the following fall made a large profit through his foresightedness. (Aristotle.) He announced beforehand an eclipse of the sun which happened May 28, 585 B.C., during a battle between the Medes and Lydians, and to this fact is attributed his inclusion in the ranks of the Seven Wise Men, for as Plutarch says, he "apparently was the only one of these whose wisdom stepped in speculation beyond the limits of practical utility; the rest acquired the name of wisdom in politics." In a conversation concerning Amasis, King of Egypt, between Niloxenus and Thales, given by Plutarch, the former says: "Altho he (Amasis) admired you (Thales) for other things, yet he particularly liked the manner by which you measured the height of the pyramid without any trouble or instrument; for by merely placing a staff at the extremity of the shadow which the pyramid casts, you formed two triangles by the contact of the sunbeams, and showed that the height of the pyramid was to the length of the staff in the same ratio as their respective shadows." From Proclus it is learned that Thales devised a method of determining the distance of ships at sea by a theorem which is now known as Euclid I, 26.

Pythagoras, concerning whose life there is a great deal of obscurity, was probably induced by Thales to visit

Egypt when a young man, where he lived many years, afterward visiting Crete and Tyre and perhaps Babylon. Returning to Samos, his home, he found it under the tyranny of Polycrates, and migrated to Italy, where he lived and taught for more than twenty years. His brotherhood falling under suspicion owing to its secrecy, Pythagoras fled to Metapontum; where it is supposed he was murdered in a popular outbreak about 500 B.C.

To Pythagoras, who raised geometry to the rank of a science, are many of the most important theorems. He is said to have introduced weights and measures among the Greeks, to have discovered the numerical relations of the musical scale, to have proved the theorem of squares on the sides of a right triangle, to have discovered that the plane around a point is filled by six equilateral triangles, four squares or three hexagons, to have found the construction of a figure upon a line which is similar to a given figure and equivalent to a second given figure. The word mathematics is due to the Pythagorean school, and to them is attributed the division of a line into extreme and mean ratio, called the Golden Section, so that the whole line is to the greater segment as this segment is to the lesser, from which construction is derived that of the inscription in a circle of the regular five and ten sided polygons.

Proclus says that Pythagoras discovered the "construction of the cosmic figures," "the five bodies in the sphere," concerning one of which Iamblichus says that Hippasus was drowned for the impiety of claiming its discovery, whereas the whole was HIS discovery, for "it is thus they speak of Pythagoras, and they do not call him by his name."

The five regular solids were alternately compared by the Pythagoreans with the five worlds and with the five senses of man. Kepler, led astray by the speculations of the philosophers, conjectured that they were in some way connected with the orbits of the five worlds. He accordingly arranged the five solids in order, each inscribed in a sphere, which in turn was inscribed in the next figure and with the sun at the center. The surfaces of the spheres carried the orbits of the planets. He found the

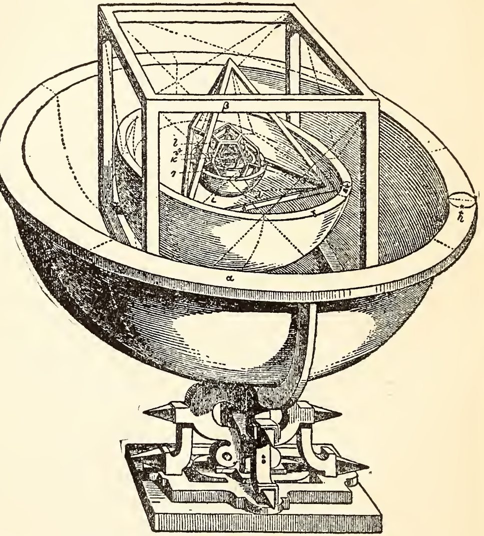


Fig. 37 —Kepler's Analogy of the Five Solids and the Five Worlds.

ratio of the distances to be remarkably near the ratio of the actual distances from the sun. He made known his remarkable pseudo-discovery in the 'Mysterium cosmographicum' (1596), which had at least one beneficial effect in that it brought him to the notice of Galileo and Tycho Brahe and opened the way for the future true discoveries which have placed his name in the galaxy of the immortals.

Plutarch, in relating the discovery of the construction of a figure similar to one and equivalent to another, says that Pythagoras offered a sacrifice in thanksgiving, thinking it finer and more elegant than the other concerning the squares on the sides of a right triangle. Pythagoras

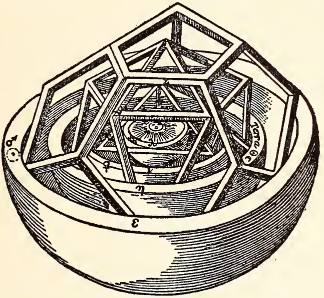


Fig. 38 —Inner Portion of Kepler's Cosmographicum. (See Fig. 37.)

thought that the distances of the heavenly bodies from the earth formed a musical progression, from which comes the expression "the harmony of the spheres."

The Pythagorean theorem that the square described on the hypothenuse of a right triangle is equivalent to the sum of the two squares described on the sides is the most famous theorem of geometry. It is said that over a thousand distinct proofs have been offered for it. The proof given by Pythagoras has never been found. He probably was led to the investigation of the figure from the observation of the special case which is common in flooring with square tiles, as in the figure. The Egyptians were familiar with the right angle property of the particular triangle with sides 3, 4, 5. Within the last few years it has not only been shown that the Hindus were familiar with

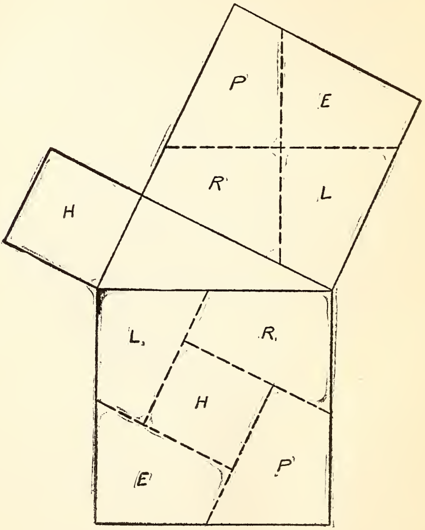
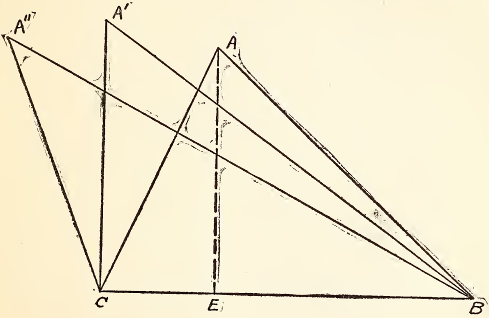


Fig. 39 —Perigal's Dissection.

the Pythagorean theorem in all its generality and the theory of the irrational long before the time of Pythagoras, but Bürk goes so far as to assert that the much-traveled Pythagoras obtained his knowledge from India. The proof given in the school text of to-day and is the classic one given by Euclid, which, notwithstanding the strictures of Schopenhauer as "a mouse-trap proof" and "a proof walking on stilts, nay, a mean, underhand proof," is one of the most beautiful ever offered.

One of the most celebrated forms of proof is known as Perigal's dissection, in which the squares are so cut that H + P + R + L + E in the figure may be arranged to form the large square. Another form of dissection is



given in the second figure in the shape of a puzzle, in which the parts A, B, C, D, E are to be cut out and arranged so as to exactly cover the large square.

This theorem is the limiting case between two theorems which may be stated together: The square on the side opposite an acute (obtuse) angle is equal to the sum of the squares on the other two sides diminished (increased) by twice the rectangle of one of those sides and the projection of the other upon it. The figure of the Pythagorean theorem was called by the Persians the Princess, and other two figures were the Sisters of the Princess. The figure of one of these cases is here given which corresponds to the figure given by Euclid for the Pythagorean theorem. In the accompanying figure if the triangle in question is ABC, AB is the side opposite the acute angle BCA. CE is the projection of CA upon CB. If CA is allowed to revolve about point C to the position CA″,

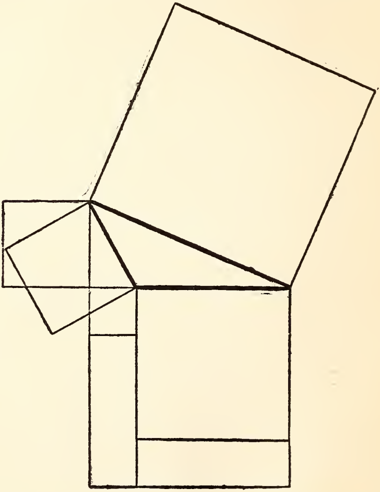
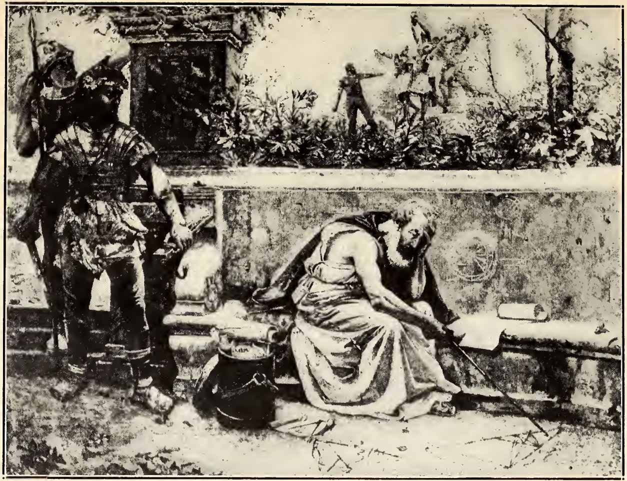


Fig. 40 —One of the 'Sisters of the Princess.'

A″B will have become a side opposite an obtuse (greater than a right) angle. But in the turning it passes through the condition of perpendicularity CA′, and the right triangle CA′B is the boundary between the two cases. When this condition occurs the projection CE is zero and the Pythagorean theorem results. The three cases are stated



The Death of Archimedes.

in a single law in trigonometry called the Law of Cosines, which in turn but one case of a general law in spherical, plane and pseudospherical geometry.

The third century B.C. produced the three greatest mathematicians of antiquity, Euclid, Archimedes and Apollonius, of which the earliest was Euclid. Very little is known of his life. Proclus gives this account of him: "Not much younger than these (Hermotimus and Philippus) is Euclid, who put together the Elements, collecting many of Eudoxus' theorems, perfecting many of Theætetus', and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors. This man lived in the time of the first Ptolemy. For Archimedes, who came immediately after the first Ptolemy, makes mention of Euclid: and further, they say that Ptolemy once asked him if there was in geometry any shorter way than that of the elements, and he answered that there was no royal road to geometry. He is younger than the pupils of Plato, but older than Eratosthenes and Archimedes; for the latter were contemporary with one another, as Eratosthenes somewhere says."

That Euclid founded a school at Alexandria is known from this passage from Pappus: "Apollonius spent a very long time with the pupils of Euclid at Alexandria, and it was thus that he acquired such a scientific habit of thought." Stobæus relates that "some one who had begun to read geometry with Euclid, when he had learned the first theorem, asked Euclid, 'But what shall I get by learning these things?' Euclid called his slave and said, 'Give him threepence, since he must make gain out of what he learns.'" The importance of Euclid's elements was recognised by the Greek philosophers, who posted on the doors of their schools: "Let no one enter here who is unacquainted with Euclid."

The purpose of the elements is to begin with a few common notions which are statements assumed to be evident to any reasoning being, and together with five assumptions from these build step by step a complete chain of theorems. That he succeeded is evidenced by the following passage from Brill: "Whatever has been said in praise of mathematics, of the strength perspicuity and rigor of its presentation, all is especially true of this work of the great Alexandrian. Definitions, axioms and conclusions are joined together link by link as into a chain, firm and inflexible, of binding force, but also cold and hard, repellent to a productive mind and affording no room for independent activity. A ripened understanding is needed to appreciate the classic beauties of this great monument of Greek ingenuity. It is not the arena for the youth eager for enterprise; to captivate him a field of action is better suited where he may hope to discover something new, unexpected."

The work of Euclid was so perfect that it has remained for 2,000 years the model from which text-books in elementary geometry have been written. It is safe to say that it is the greatest work that a single human mind has ever produced. The Elements was divided into thirteen books, best known to-day through three translators: Simson, Heiberg and T. L. Heath; the latter work appeared in 1908, and is of immense value in the realization of the great geometer's work.

Euclid defines a point as that which has no part, a line as breadthless length, and a straight line as a line which lies evenly with the points on itself. Five postulates and five common notions form the foundation upon which the superstructure is built. The following are granted:

1. That a straight line may be drawn from any point to any point.

2. That a finite straight line may be produced continuously in a finite straight line.

3. That a circle may be drawn with any center and any radius.

4. That all right angles are equal to one another.

5. That if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

It will be noticed that the plane geometry is built on three elements, the point, the straight line and the circle. This may be put otherwise: the three are only the circle and its two limiting forms, the point being the circle when its radius has become zero, and the straight line the form when the radius of the circle has increased to infinity. These three elements limit Euclidean geometry to two instruments, the undivided straight-edge and the compass. Euclid assumes that the circle may be drawn, but the straight line has been drawn. It is a significant fact that it was not until 1864 that an instrument was invented by Peaucillier by which a straight line could be drawn by mechanical means.

Postulate 2 implies that space is continuous, not discrete, and also assumes its infinitude.

The five common notions are:

1. Things which are equal to the same thing are equal to each other.

2. If equals be added to equals, the sums are equal.

3. If equals be subtracted from equals, the remainders are equal.

4. Things which coincide with one another are equal to one another.

5. The whole is greater than any part.

Common notion 4 implies the free mobility of rigid bodies in space. Bertrand Russell says what is called motion in geometry is merely the transference of attention from one figure to another, and actual superposition nominally employed by Euclid is not required. Common notion 5 separates the finite from the infinite. The modern definition of an infinite element is that which is equal to a part of itself.

According to Proclus, every problem and every theorem which is complete with all its parts perfect purports to contain in itself all of the following elements: enunciation, setting out, definition or specification, construction or machinery, proof and conclusion. The enunciation states what is given and what is sought. The setting out marks off what is given beforehand and adapts it to the investigation. The definition makes clear the particular thing sought. The construction adds what is needed for the purpose of finding out what is sought. The proof draws the required inference by reasoning scientifically from acknowledged facts. The conclusion reverts again to the enunciation, confirming what has been demonstrated.

The fifth proposition of Book I, asserting the equality of the base angles of an isosceles triangle, has been called the 'pons asinorum,' or bridge of asses, the inference being that if the youth had ability to master this theorem, his future career in geometry was assured. An important set of theorems in Book I is concerned with the conditions of equality of triangles, which may be stated as follows:

I. Two triangles are equal if the three sides of one are respectively equal to the three sides of the other.

II. Two triangles are equal if two sides and the included angle of one are respectively equal to the corresponding parts of the other.

II′. Two triangles are equal when a side and the two adjacent angles are equal respectively to the corresponding parts of the other.

III. Two triangles are equal when two sides and an angle opposite one of them are equal respectively to the corresponding parts of the other (containing, however, an ambiguous case).

III′. Two triangles are equal when two angles and a side opposite one of them are equal respectively to the corresponding parts of the other.

It will be noticed that these are arranged in pairs, with the exception of I, which would be paired with the theorem stating the equality of the triangles provided the corresponding angles are equal, which is not necessarily true in plane geometry.

The primed number of each pair may be gotten from the unprimed by changing side to angle and vice versa.

A side is determined by the two end points and an angle by the two including lines, the point and line being the two limiting cases, one on either side of the circle. Such a property of certain theorems is called reciprocity or duality, and enables one to think of such a theorem as a theorem in points or a theorem in lines as well. This statement well illustrates duality: two points (lines) determine a line (point). In the triangle theorems the breaking down of reciprocity in I is due to the fact that the three angles of a triangle are not independent, as in the case in sphere geometry. If two are given the third may be found by subtracting the sum of the two from two right angles. Three elements (a majority of the five which may be independent) are required for the determination of a triangue.

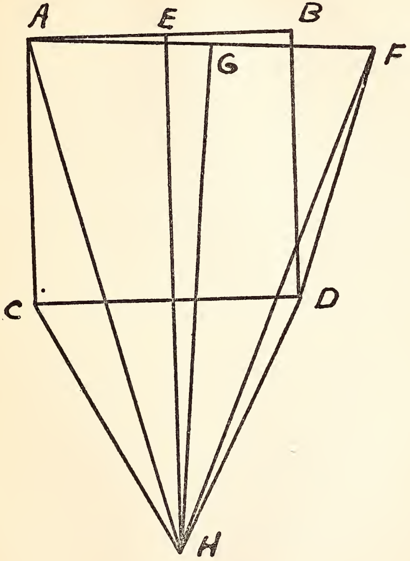
The most important of the remaining theorems of Book I are those treating of parallels (which will be considered later) and the Pythagorean theorem. Book III treats of circles. Book IV of the inscription of regular polygons in the circle, one of the famous problems of the ancients, and which leads to the usual method of determining the approximate value of the ratio of the circumference to the diameter of a circle. The remaining books through Book IX are mostly concerned with the geometry of lines—that is, arithmetic treated geometrically. The last three books are concerned with the geometry of space and culminate in the regular solid figures which may be inscribed in a sphere. While Euclid has been the guiding star of geometrical text-books for twenty centuries, yet the tides of darkness have been so dense at many times that only the faintest gleams of light were discernible. About 1570 Sir Henry Savile, warden of Merton College, strove to arouse an interest by a course of lectures on Greek geometry, which were published in 1621. Concluding, he says: "By the grace of God, gentlemen hearers, I have performed my promise; I have redeemed my pledge. I have explained, according to my ability, the definitions, postulates, axioms and the first eight propositions of the Elements of Euclid. Here, sinking under the weight of years, I lay down my art and my instruments." (Cajori.)

Savile says: "In the beautiful structure of geometry there are two blemishes, two defects; I know no more." These were the assumption of the fifth postulate and the theory of proportion. The non-Euclidean geometry has vindicated Euclid's position in the first, and it has taken 500 years from the time of Savile to appreciate the theory of proportion.

The purpose of Euclid was to build up, with a minimum of assumptions, a logical structure in which reason is the sole factor. In such a system the figure that is drawn is simply a guide to the thought and might be entirely dispensed with. Unless it is used with care, it may by subtly involving intuition ensnare one into error. The following example of the result of such misleading is well known: ABCD is a square. AB is bisected perpendicularly at E. DF is drawn equal to BD. AF is bisected perpendicularly at G. The two perpendiculars meet at H. CH, DH, AH, and FH are drawn in the triangles ACH and FDH. CH = DH, AC = FD, AH = FH. Therefore by the theorem of equality of two triangles having sides respectively equal, the triangles ACH and FDH are equal and the corresponding angles ACH and FDH are equals. But angle ACH = angle BDH, from which angle FDH = angle BDH; a magnitude equaling a part of itself which contradicts the fifth common notion, that a whole is greater than any part of it.

This elimination of observation from the geometry taught the schoolboy has led to attacks in recent years on the advisability of the use of Euclid as a school text. J. J.

Sylvester, one of England's two greatest mathematicians, in answer to Huxley's statement that "mathematics is that study which knows nothing of observation, nothing of experiment, nothing of induction, nothing of causation,"



gave voice to the following: "I should rejoice to see Euclid honorably shelved or buried 'deeper than e'er plummet sounded' out of the schoolboy's reach." The Perry movement, inaugurated in England by John Perry in 1901, has in a measure resulted in departing from Euclid so as to make geometry more of a subject of experiment and observation.

The second great mathematician of this period was Archimedes, born at Syracuse in 287 B.C., studied at Alexandria, returned to Sicily and died in his native city in 212 B.C. Aside from his mathematical contributions, his mechanical ability was marvelous.

Archimedes was killed during the sack of Syracuse by the Romans under Marcellus. A soldier found him in the garden tracing a geometrical figure in the sand as was customary in those days. Archimedes told him to get off the figure and not spoil it. The soldier, insulted, thrust him through with his dagger.

The figure of a sphere, inscribed in a cylinder, was cut on his tomb in commemoration of his favorite theorem that the volume of the sphere is two-thirds that of the cylinder and its surface is four times that of the base of the cylinder. Cicero rediscovered the tomb in 75 B.C. and gives a beautiful account of his search in Tusc. Disp., V. 23.

"Shall I not, then, prefer the life of Plato and Archytas, manifestly wise and learned men, to his (Dionysius'), than which nothing can possibly be more horrid, or miserable, or detestable?

"I will present you with an humble and obscure mathematician of the same city, called Archimedes, who lived many years after; whose tomb, overgrown with shrubs and briers, I in my quæstorship discovered when the Syracusans knew nothing of it, and even denied that there was any such thing remaining; for I remembered some verses which I had been informed were engraved on his monument, there was placed a sphere with a cylinder. When I had carefully examined all the monuments (for there are a great many tombs at the gate Achradmæ) I observed a small column standing out a little above the briers, with the figure of a sphere and cylinder upon it; whereupon I immediately said to the Syracusans—for there were some of their principal men with me there—that I imagined that was what I was inquiring for. Several men, being sent with scythes, cleared the way, and made an opening for us. When we could get at it, and were come near to the front of the pedestal, I found the inscription, tho the latter part of all the verses were effaced almost half way.

"Thus one of the noblest cities of Greece, and one which at one time likewise had been very celebrated for learning, had known nothing of the monument of its greatest genius, if it had not been discovered to them by a native of Arpinum."

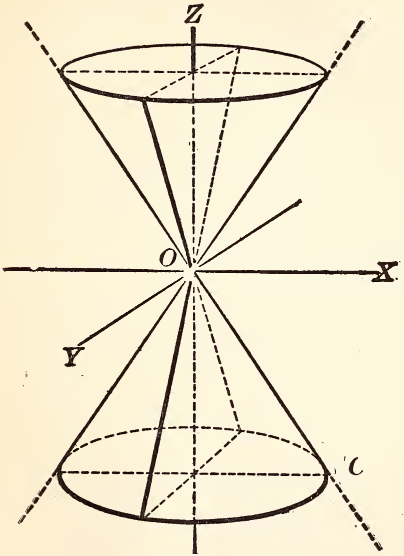
The work on the Quadrature (or finding the area of a segment) of the Parabola is one of the most important works of Archimedes. The proof of the principal theorem of this work depends upon the "method of exhaustions" invented by Eudoxus, and which is the forerunner of the modern powerful implement of analysis, the calculus. The lemma is thus stated by Archimedes: "The excess by which the greater of two unequal areas exceeds the less can, if it be continually added to itself, be made to exceed any finite quantity." The theorem itself asserts that the area of a segment of the parabola is equal to four-thirds of a certain triangle inscribed in it.

Another important work, 'The Sphere and the Cylinder,' containing sixty propositions, was sent to his friends in Alexandria, in which he purposely misstated some of his results, "to deceive those vain geometricians who say they have found everything but never give their proofs, and sometimes claim they have discovered what is impossible."

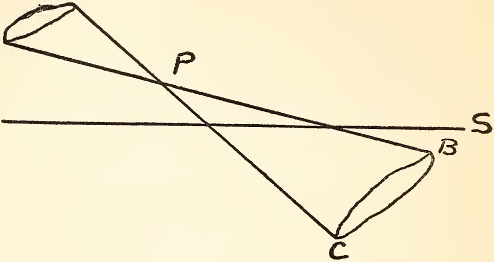
The work of Archimedes is of particular interest at the present time owing to the discovery of a lost work by Professor Heiberg in Constantinople during the summer of 1906. The purpose of this work, which is addressed to Eratosthenes, is well summed up in the following statement and makes clear the method by which Archimedes arrived at his discoveries: "I have thought it well to analyze and lay down for you in this same book a peculiar method by means of which it will be possible for you to derive instruction as to how certain mathematical questions may be investigated by means of mechanics. And I am convinced that this is equally profitable in demonstrating a proposition itself; for much that was made evident to me through the medium of mechanics was later proved by means of geometry, because the treatment by the former method had not yet been established by way of a demonstration. For of course it is easier to establish a proof, if one has in this way previously obtained a conception of the questions, than for him to seek it without such a preliminary notion. . . . Indeed, I assume that some one among the investigators of to-day or in the future will discover by the method here set forth still other propositions which have not yet occurred to us." Says Professor Smith: "Perhaps in all the history of mathematics no such prophetic truth was ever put into words. It would almost seem as if Archimedes must have seen as in a vision the methods of Galileo, Cavalieri, Pascal, Newton, and many of the other great makers of the mathematics of the Renaissance and the present time."

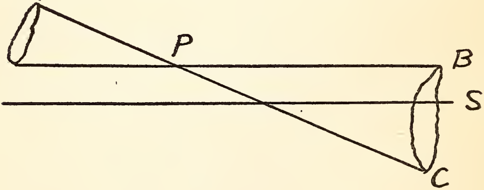
Very little is known of the life of the third member of this great trinity, Apollonius of Perga, "the great geometer." It is supposed that he was born about 260 B.C. and died about 200 B.C. He studied at Alexandria for many years and probably lectured there. His great work on the conic sections contains practically all of the theorems of the text-books of to-day. The work was divided into seven books, perhaps originally into eight, and while very tedious, is characterized by strict Euclidean rigor.

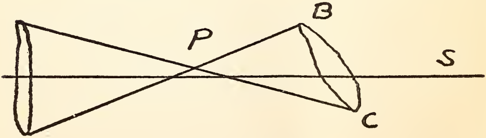
A cone is the figure generated by a line passing through a fixed point and constantly touching the circumference of a circle. If O is the point and C the circle, the line OC turns while still passing through O, so that point C traverses the circle. The complete cone consists of the symmetrical figure above O as well as the figure below and both are extended into space indefinitely. A conic section is a curve which is formed by passing a plane through the cone. One of the best methods of quickly constructing these sections is to immerse a wooden or tin cone in a



vessel of water. The line formed around the cone by the surface of the water will be the section. There are three general cases which arise, besides several special ones, as will be seen by the inspection of the figures, which are vertical cross-sections—that is, the eye is supposed to be on a level with the surface of the water and sees this surface as a line S.







In Fig. I, where the plane S cuts the two opposite generators PC and PB, an ellipse is formed. If the plane S happens to be at right angles to the axis of the cone as in Ia, a circle is the result.

In Fig. II the upper half or nappe of the cone has been lowered—that is, the cone has been revolved about P until the axis PB has become parallel with the plane S. The curve formed is an open curve and is called a parabola.

If the cone be still further turned until both nappes cut the water as in Fig. III, the hyperbola is the resulting

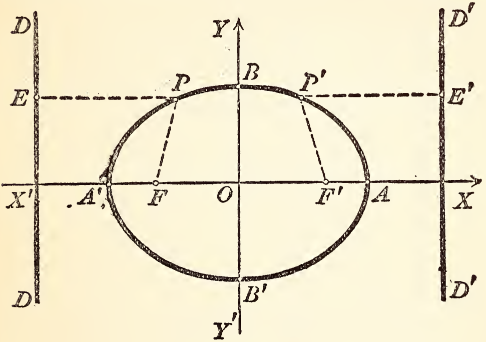


Fig. 41 —Ellipse.

curve. This curve consists of two branches, both of which are open.

If the plane S passes through the point P during this investigation, the degenerate conics are formed. I gives a degenerate circle or ellipse, which is a point where the radii have become zero; II gives a line which may be regarded as made up of two coincident lines; in III these lines become distinct and intersect at P.

It is thus seen that the parabola is the limiting case through which the varying ellipse passes as it merges into the hyperbola.

These three curves may be defined by a single law of motion of a point in a plane, and for purposes of study this is more convenient.

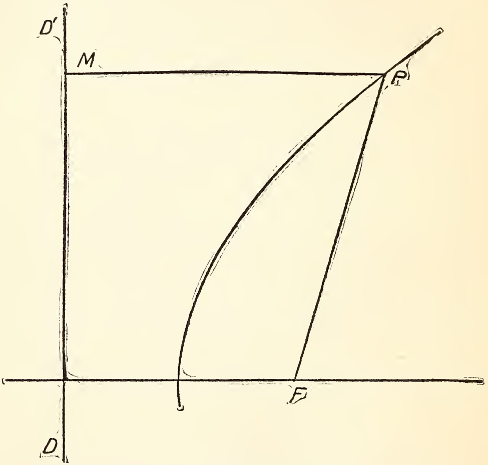


Fig. 42 —Parabola.

A point so moves that its distances from a fixed point, F, called the focus, and from a fixed line, DD′, called the directrix, are in a given ratio, e, the eccentricity of the curve. Now the form of the curve and the class to which it belongs, ellipse, parabola or hyperbola, depends upon the value given to e. In the figure F is the fixed point, P is the moving point on the curve and DD′ is the directrix or fixed line.

In Fig. I e is less than 1 and the curve is an ellipse. It is seen that it is symmetric to the line YY′ and therefore must have another directrix, DD′, on the right and also a second focus, F′.

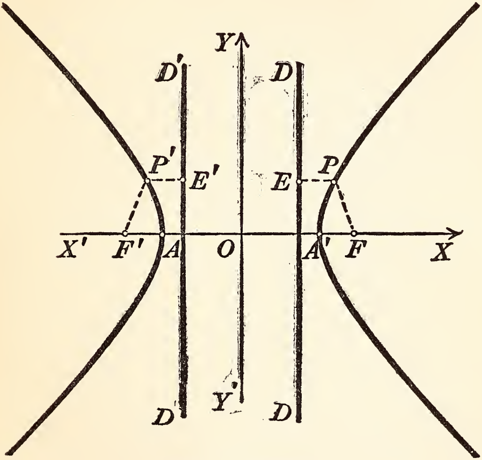


Fig. 43 —Hyperbola.

In Fig. II e = 1 and the curve is the parabola. This curve constantly recedes from the line, yet ever curves to it. It may be thought of as the left half of an ellipse of which the right focus has been pulled out to the right an infinite distance; it is an open curve—that is, the two arms of the curve never join again.

In Fig. III is seen the third case, where e is greater than 1; the hyperbola with two branches. In the generation of this curve the point starting at A′ recedes indefinitely downward to the right. It next appears coming back on the upper half of the left branch, passing along that branch to an infinite distance and finally coming back

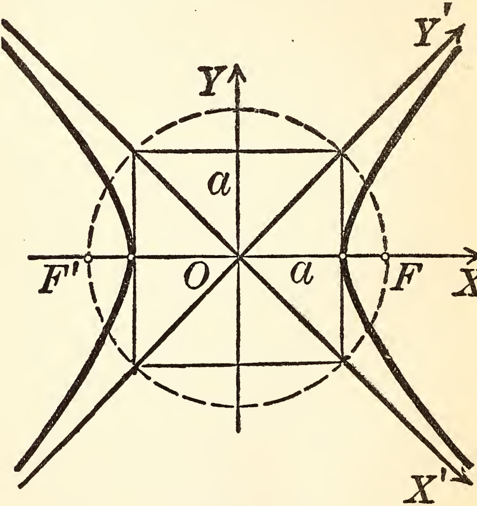


Fig. 44 —Asymptotes of Hyperbola.

along the upper right of the right branch. It is convenient sometimes to think of the two ends of the curve being joined by a single infinite point and thus preserve continuity in the motion of the moving point. The two branches of the hyperbola constantly approach without ever reaching the two intersecting lines OX′ and OY′ in the figure; that is, the curves are said to be asymptotic to these lines, which are called the asymptotes of the curve.

In the full-page figure is seen the relation which exists between the foci and directrices of the plane figure and the cone itself. The plane AB cuts the ellipse from the cone. If a sphere be dropped in the cone so that it will be in the cone and just touch the plane the point of touching or tangency will be a focus. Two such spheres are possible, the small one above the plane and the large one below; the foci are F and F′. These spheres touch the cone in circles. If planes be passed through these circles, as AC and BC, they will cut the original cutting plane AB in the lines AM and BN, which are the directrices.

The futility of the argument that it is vain to cultivate truth for truth's sake is well seen in the case of the Conics of Apollonius. This monumental work lay dormant and did not reach fruition until seventeen centuries after, when Kepler found the paths of the planets to be ellipses and Newton subjected to law the wanderer of the celéstial seas, the comet, whose path is an ellipse if it is a regular visitor of the solar system. If the path of the comiet is not an ellipse, it is a parabola, and it comes but once under the influence of the sun and then forever loses itself in the vastness of space.

Antiquity has left us three famous problems: The quadrature of the circle, the duplication of the cube, called the Delian problem, and the trisection of the angle, or more generally the problem of the inscription of the regular polygons in a circle.

The quadrature of the circle, popularly known as squaring the circle, is the problem of finding the side of a square which has the same area as a given circle. The philosopher Anaxagoras occupied himself with this problem in his prison. Hippocrates of Chios made one of the most

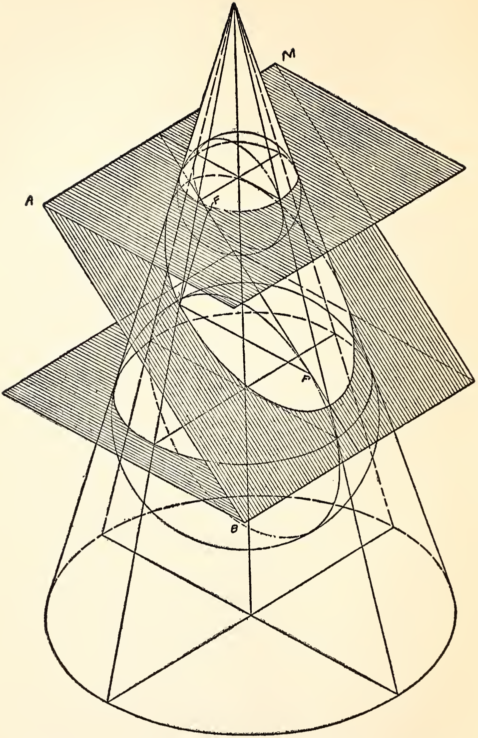


Fig. 45 —Relation Between Plane and Solid Figure.

famous attempts at its solution, which resulted in finding a lune or surface in the shape of a crescent bounded by two arcs, which was equal in area to a square. Archimedes showed that the problem is equivalent to finding the area of a right-angled triangle whose sides are respectively the perimeter of the circle and its radius, and further showed that the ratio of these two sides is more than $3^1/_7$ and less than $3^{10}/_{71}$. This ratio is indicated by the Greek letter π, introduced by W. Jones in 1706 and crystallized in use by Euler.

Archimedes' method of determining its value was by inscribing and cricumscribing polygons of 96 sides and by comparing the ratio of the perimeter of the circumscribed polygon to the radius determined a value greater than π, and by using the inscribed polygon he arrived at a value less than π. The present text-book method is to determine a formula or algorithm by which the perimeter of a polygon of 2n sides may be found from the perimeter of the polygon of n sides. By carrying this process on indefinitely the ratio may be found to any degree of approximation.

The ancient Egyptians took the value $^{256}/_{81}$, equal to 3, 1605; 3 was the value used by the early Babylonians and also by the Jews (I Kings vii, 23; II Chronicles iv, 2).

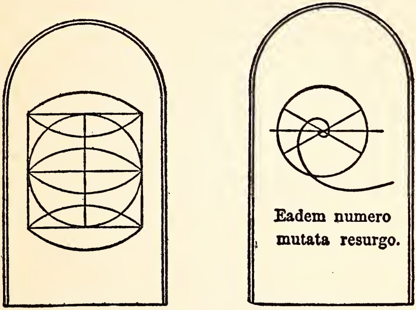
A quaint picture is found in the beginning of Halley's edition of Apollonius and again reproduced in Heath's volume. The legend below describes Aristiphus, the Socratic philosopher, shipwrecked on the island of Rhodes, where he found the sand of the seashore covered with geometrical drawings. His exclamation was, "Good cheer. I see evidences of the Man himself."

Ludolph van Ceulen devoted a considerable portion of his life to the computation of π. Dying in 1610, he requested that the result to 35 places, which he had obtained, be cut on his tombstone. Archimedes chose to have his favorite theorem graven on his tomb, as also James Bernoulli, who, while investigating the properties of the

Fig. 46. —Geometrical "Footprints in the Sand."

equiangular spiral, discovered the remarkable way in which curves deduced from it reproduced the original curve, and he requested that this figure should be carved on his tomb with the inscription "Eadem numero mutata resurgo."

Perhaps the limit of perseverance in this direction was reached by William Shanks, who in 1872 carried the result to 707 places. Some idea of the accuracy of this value



Bernoulli. Archimedes.

Fig. 47.—Design on Tombs.

may be inferred from Professor Newcomb's remark that if the circumference of the earth were a perfect circle, ten places of decimals would make its circumference known to a fraction of an inch.

In 1770 Lambert discussed the statement that π is irrational, that it cannot be expressed by a terminating decimal or the ratio of two whole numbers. In 1794 Legendre proved the irrationality of both π and π^2. Hermite in 1873 proved e, the base of the natural logarithms, to be transcendental—that is, it is inexpressible as a root of any algebraic equation with integral coefficients—and in 1882

Lindemann gave a similar proof for the transcendentalism of π. Euler derived the relation between e and π, expressed by the following formula, which is one of the most remarkable in mathematics:

$$\varepsilon^{\iota\pi} = -1.$$

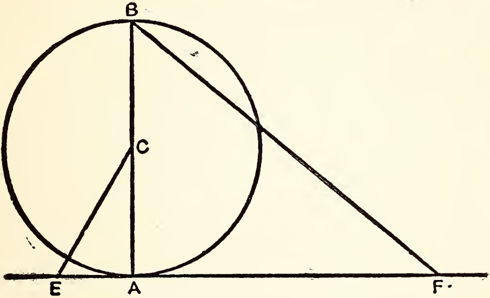
A method of approximating π is by the theory of probability. On a plane a number of straight lines are drawn parallel to each other and a units apart. If a stick of length l, less than a, is dropped at random on the plane of these lines, the probability that it will fall across one of the line is $\frac{2l}{\pi\alpha}$, from which, by a large number of trials in which the number of times is recorded that the stick crosses a line, an approximate value of π is obtained. In 1864 Captain Fox made 1,120 trials and obtained $\pi = 3.1419$. (Ball.)

In 1685 Kochausky gave a simple construction by which the length of a semicircle may be constructed with an accuracy correct to 4 decimal places. At the end point A of diameter BA draw tangent AF. Take the angle ACE, equal to 30°, and EF, equal to 3 times the radius. Draw BF and which is the required line? (Halsted.)

The value of π to 52 places of decimals is $\pi = 3.141,$-592, 653, 589, 793, 238, 462, 643, 383, 279, 502, 884, 197,-169, 399, 375, 105, 8.

"Circle squaring" has not entirely died out, but the mathematical knowledge of the cyclometer of to-day does not extend much beyond elementary arithmetic. For the lack of the requisite knowledge to appreciate the problem has been substituted a dogged perseverance which should achieve results if applied in a calling more befitting their abilities. Professor de Morgan, whose experience with the several cyclometers certainly puts him in a position to know their frailties, especially those of James Smith, of Liverpool, says: "The feeling which tempts persons to this problem is that which, in romance, made it impossible for a knight to pass a castle which belonged to a giant or an enchanter. This rinderpest of geometry cannot be cured when once it is seated in the system. All that can be done is to apply what the learned call prophylactics to those who are yet sound. When once the virus gets into the brain, the victim goes round the flame like a moth—first one way and then another, beginning again where it ended, and ending where he began."

Smith's value for π is $3\frac{1}{8}$, which he attributes to a French well-sinker, of which De Morgan says: "It does



the well-sinker great honor, being so near the truth, and he having no means of instruction." Further speaking of Smith, he says: "He is, beyond a doubt, the ablest head at unreasoning, and the greatest hand at writing it, of all who have tried in our day to attach their names to an error. Common cyclometers sink into puny orthodoxy by his side. The behavior of this singular character induces me to pay him the compliment Achilles paid Hector—to drag him around the walls again and again." Again: "As to Mr. James Smith, we can only say this: he is not mad. Madmen reason rightly upon wrong premises; Mr. Smith reasons wrongly on no premises at all. His procedures are not caricature of reasoning; they are caricature of blundering. The old way of proving $2 = 1$ is solemn earnest compared with his demonstration."

The origin of the Delian problem, which occupies a large space in the history of Greek geometry, is given in a letter from Eratosthenes to King Ptolemy Energetes:

"Eratosthenes to King Ptolemy greeting.

"There is a story that one of the old tragedians represented Minos as wishing to erect a tomb for Glaucus and as saying, when he heard that it was a hundred feet every way:

> " 'Too small thy plan to bound a royal tomb.
> Let it be double; yet of its fair form
> Fail not, but haste to double every side.' "

But he was clearly in error, for when the sides are doubled the area becomes four times as great and the solid content eight times as great. Geometers also continued to investigate the question in what manner one might double a given cube while it remained in the same form. And a problem of this kind was called doubling the cube, for they started from a cube and sought to double it. While then for a long time every one was at a loss, Hippocrates of Chios was the first to observe that if between two straight lines of which the greater is double of the less it were discovered how to find two mean proportionals in continued proportion, the cube would be doubled; and thus he turned the difficulty in the original problem into another difficulty no less than the former. Afterward, they say, some Delians attempting, in accordance with an oracle, to double one of the altars (to rid them of a pestilence) fell into the same difficulty. And they sent and begged the geometers who were with Plato in the Academy to find for them the required solution, and while they set themselves energetically to work and sought to find two means between two given straight lines, Archytas of Tarentum is said to have discovered them by means of half-cylinders and Eudoxus by means of so-called curved lines. It is, however, characteristic of them all that they indeed gave demonstrations, but were unable to make the actual construction or to reach the point of practical application, except to a small extent Menæchmus, and that with difficulty." Perhaps the most beautiful solution aside from that of Archytas is by means of the cissoid or "ivy-like" curve invented by Diocles. This curve is formed by drawing the horizontal diameter of a circle and drawing pairs of equal half chords perpendicular to this diameter. Through the upper extremity of one of these chords and the opposite end of the horizontal diameter is drawn a chord. The point of intersection of this chord with the other one of the pair of half chords is a point of the cissoid.

In discussing the possibility of a geometrical solution of a problem it has not always been clear just what is meant by possibility. Euclid limited his tools to the straight-edge and compass, so that every geometrical problem must ultimately reduce to a finite number of constructions which are of one or more of the three classes: finding the intersection of two straight lines, or a straight line and a circle, or of two circles. At first glance one would say that the impossibility of a construction by such methods could never be completely established, that perhaps some time some one would hit upon the happy combination necessary for the solution, and so far as geometry itself is concerned, it has as yet thrown no light on the subject. It is here that Algebra furnishes the clue. Since geometry admits of the construction of the square root of the product of two lines, it may be said that the necessary and sufficient condition that an analytic expression can be constructed with the straight-edge and compasses is that it can be derived from the known quantities by a finite number of rational operations and square roots. (Klein: 'Famous Problems in Elementary Geometry.')

It is at once seen that the Delian problem reduces to finding x where $x^3 = 2$ and therefore is unsolvable as a Euclidean problem.

The trisection of an arbitrary angle, while one of the famous unsolved problems, was not so enshrined in romance as was the Delian problem. The bisecting or dividing of an angle into two equal parts was very easy of solution, but not so the trisection. In very special cases, as that of the right angle, no difficulty is experienced. The earliest solutions were by means of the hyperbola and the conchoid of Nicomedes. Since that time many and various have been the solutions offered, all depending either on higher plane curves than the circle or upon mechanical instruments other than the ruler and compasses. Speaking of the latter, Plato says: "The good of geometry is set aside and destroyed, for we again reduce it to the world of sense, instead of elevating and imbuing it with the eternal and incorporeal images of thought, even as it is employed by God, for which reason He always is God."

It is easily shown that trisection cannot be reduced to the necessary conditions and therefore it must be classed as an unsolvable Euclidean problem.

Closely allied with this problem is the other of inscribing regular polygons in a circle. It has long been known that polygons may be inscribed if the number of sides is given by $n = 2^h$, 3, 5 or the product of any two or three of these numbers. Gauss showed that the operation is possible for every prime number of the form $p = 2^{2^n} + 1$, but impossible for all other primes.

Giving n the values 0, 1, 2, 3, 4, the primes 3, 5, 17, 257, 65537 result. With $n = 5, 6, 7$, primes do not result. Thus is seen that the regular polygon of 7, 9, 11, etc., sides are not constructable. The polygon of 17 sides has been constructed by many writers. One construction is given by Klein in 'Famous Problems.' To the investigation of the polygon of 65,537 sides Professor Hermes devoted ten years of his life.

The modification of the instruments used in constructions has been considered successfully by Mascheroni, who used compasses alone. All forms which involve rationals may be dealt with with the straight-edge, while Poncelet conceived the idea of using the straight-edge and a fixed circle.

That angle trisectors still exist is attested by the publication some years ago, with great éclat, that a Western school girl had succeeded where the mathematicians of twenty centuries had failed. Verily "fools venture in where angels fear to tread."

Euclid's definition of parallel lines is straight lines which, being in the same plane and produced indefinitely in both directions, do not meet one another in either direction. Euclid's fifth postulate differs from all the others, and as Staekel remarks, "It requires a certain courage to declare such a requirement alongside the other exceeding simple assumptions and postulates," and there is no better proof of the subtlety and power of the old Greek geometer than his assumption as undemonstrable that which required twenty-two centuries to prove as such. Euclid postpones the use of this postulate until nearly half of the first book is complete and then assumes it as the inverse of one already proved, the seventeenth, and uses it only to prove the inverse of another already proved, the twenty-seventh.

Proclus demanded a proof as the inverse was demonstrable and his time on it has been the bone of contention until the difficulty was cleared up in the nineteenth century by the most brilliant generalization in the whole field of mathematics. Playfair's form of this postulate (also stated by Proclus) is: Through a given point not on a straight line, one line and but one can be drawn which is parallel to the given line. Comparing this statement with the one that one and but one perpendicular can be drawn from a point to a line, they appear of equal difficulty. On this slippery ground many good and bad mathematicians have lost their footing. Lagrange at one time wrote a paper on parallels in which he hoped he had overcome the difficulty and began to read it before the Academy, but suddenly stopped and said: "Il faut que j'y songe encore" (I must think it over again). He put the paper in his pocket and never afterward referred to it.

Legendre showed that this assumption is equivalent to the statement that the sum of the angles of a triangle is equal to two right angles, and also proved that if ever a triangle is found in which the sum of the angles can be shown to be exactly two right angles, then this is true for any other triangle. Gerolamo Saccheri in 1733 in a work, 'Euclid Vindicated from All Faults,' obtained the first glimpse of the modern theory of parallels, and had it not been for his confidence in the existence of a parallel, would no doubt have had the credit which belongs now to others. He presents the curious spectacle of laboring to erect a structure for the purpose of afterward pulling it down on top of himself, constructing systems in which he sought for contradictions in order to prove the hypotheses false.

Wolfgang Bolyai, a Hungarian, was in his college days a friend of Gauss, the greatest mathematician Germany has ever produced. He was professor in the Reformed College of Maros Vasarhely. The son, Johann Bolyai de Bolyai, is best described by his father when he relates that the boy in mathematics "sprang before him like a demon." As soon as he enunciated a problem the child solved it and asked him to go farther. At the age of thirteen he lectured in his father's absence. Writing to his father November 3, 1823, at the age of 21, he says: "I have not got my object yet, but I have produced such stupendous things that I was overwhelmed myself, and it would be an eternal shame if they were lost. Now I can only say that I have made a new world out of nothing." And his discovery was nothing more nor less than to reject the postulate which had been intuitively accepted since the time of Euclid, and without this axiom builds up a non-self-contradictory geometry. It was published as an appendix of twenty-eight pages in a work of his father's. In 1829

Nicholaus Ivanovitch Lobatchewsky, a brilliant young Russian, issued his 'New Elements of Geometry with a Complete Theory of Parallels,' in which the same axiom is rejected. And so almost simultaneously the new field was created by two young men, one a Magyar and the other a Russian, in almost precisely the same manner.

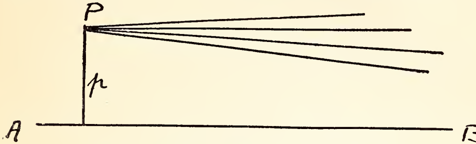


Fig. 48 —Lobatchewsky's Parallels.

If P is a point not on the line AB, the lines on the right of p are divided into two classes, those which cut AB and those which do not. The line which separates the two classes is said to be parallel to AB. On the left of p there is also a parallel. Euclid's axiom would say that one of these is the prolongation of the other, but such cannot be proved. Lobatchewsky began his geometry with the assumption that they are not one and the same line; in other words, through the point P there are two parallels to AB, one on either side of p.

In the figure PC and MB make equal angles with PM. Several cases arise.

I. PC meets AB in the two points, one on the right of PM and the other on the left (these points may be distinct or coincident).

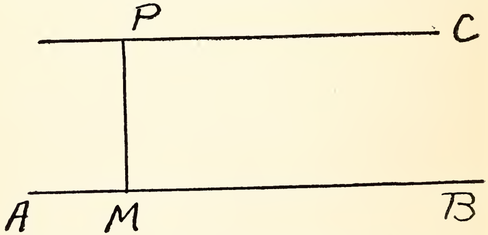
II. PC meets AB in one point, and

(a) there exists but the one line PC, which has this property, or

(b) there exists on the left of PM a second line having a similar property.

III. PC does not meet AB on either side, however far produced.

In order to give objective reality to these hypotheses, the geometry of a surface of a sphere will be considered. It will be necessary to inquire into the meaning of straight line, since obviously no line may be drawn on a spherical surface which has the property of straightness, in the common acceptance of the term. It is not always clear just what property is meant when the term straight is used. A very common conception of straightness is that property by which if a portion of the line terminated by two points A and B is placed on any part of the line so that A and B lie in the line, then the line is said to be straight



if, when this segment is rotated, keeping A and B in the line, all points between A and B lie evenly in the line. But this is an unnecessarily complicated statement. Another conception which is equally fundamental and much more fruitful is that the straight line is the minimum line between the two points. Such a line will be called a geodesic. It is the line which the navigator would naturally choose, other conditions being equal, when sailing between two points on the surface of the earth, or if a cord is stretched between two points on the surface of a sphere, without friction, it will mark a geodesic. It is easily shown that the only geodesic that may be drawn on the surface of a sphere is cut out by a plane passing through the center of the sphere, or the geodesic is a great circle.

It will be convenient to speak of the spherical surface as a sphere and the great circle as a straight line or geodesic.

A geodesic on a sphere is determined by two points (just as the geodesic or straight line in the plane), except in the special case of the two points being the extremities of a diameter. The sum of the angles of a triangle formed by three geodesics is greater than two right angles. The excess is denoted by E. The area of such a triangle is proportional to E. In the plane triangle the sum of the angles is exactly equal to two right angles and its area is entirely independent of the magnitude of the angles or their relations one with another. Two triangles are equal on the sphere if the three angles of one are equal respectively to the three angles of the other. This was the case of duality which broke down in the plane. The surface of the sphere is a two-dimensional manifold of points; in other words, it has extension in two ways, but has no thickness. If such a surface could be stripped from the sphere it could be folded and rolled up by bending one side inward. If such deformation be performed without tearing or stretching, it is evident that any theorem concerning lines on the surface would be still valid, and any figure could at will be moved freely about in the surface without in any way altering the relations of the various parts. Likewise the geometry of a portion of a plane is unaltered if it be rollsd up in the form of a cylinder, or cone. Such a property is said to belong to surfaces of constant curvature. If R is the radius of a circle, $\frac{1}{R}$ is called the curvature, since as R increases $\frac{1}{R}$ decreases and vice versa. If the radius becomes larger the curvature becomes smaller and the surface flattens out. Through any point of a surface let all the geodesics be drawn, and in the plane of any geodesic let that circle be drawn which most nearly conforms with the geodesic at the point. The geodesics form a pencil and the curvature of each geodesic is the curvature of its particular circle. Now if all the circles have the same radius, and this radius is the same for circles at any other point of the surface, it is said to have constant curvature. This may be put analytically. A certain expression is taken involving quantities that are known and which is fully determined when the line-element of the surface is given. This expression is an invariant of the surface—that is, it is independent of the coördinates used to define a point. This expression is indicated by K and called the Gaussian measure of curvature. When K is the same for all points of a surface, the surface is said to have constant curvature.

Suppose the radius of the sphere R to increase indefinitely; $\frac{1}{R}$, or the curvature, is positive and becomes indefinitely small. The surface flattens out and approaches, as a limit, the plane with curvature O; that is, the plane is the limiting case of a spherical surface as curvature or K approaches zero. Now allow K to pass through zero and become negative; since $K = \frac{1}{R}$ is negative the radius must be negative or turned in direction. Formerly it was directed inward, and for the moment it will be convenient to think of it as projecting outward from the surface. As K passes through O it is very small and R is very great but negative, or the surface first flattened into a plane and very slowly curves the other way, givng a saddle-shaped surface. A surface of this nature which has constant curvature is generated by the revolution of the tractrix about the axis to which it is asymptotic. The tractrix is a curve such that the tangent PT is always a constant. This curve is the projection on a plane of one of the curves of a skew arch. If this curve be revolved about the axis OX, it will give a saddle-shaped surface called the pseudosphere. On this pseudosphere a triangle has the appearance of Fig. 50 and the sum of the angles is less than two right angles. This

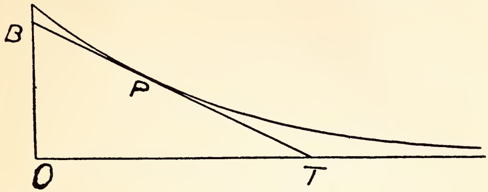


Fig. 49 —CURVE OF EQUAL TANGENTS.

deficiency is denoted by D and is proportional to the area of the triangle.

Going back to the hypotheses, it is seen that the spherical surface meets the conditions of I. Through a given point

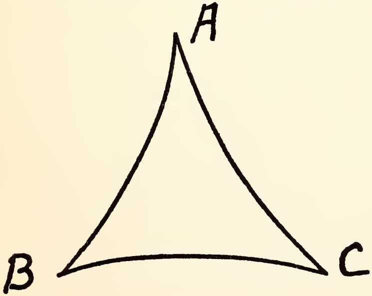


Fig. 50 —PSEUDO-SPHERICAL TRIANGLE.

P outside a line, no line can be drawn which does not intersect the given line in two distinct points.

The geometry of such a surface (of positive curvature) is called Riemannian or Gaussian.

The plane satisfies hypothesis III if it be assumed that no other such line may be drawn. The geometry of the plane is termed Euclidean, and IIb is true on the pseudosphere. Through a point outside a given line two parallels to the line may be drawn. The appearance of the parallels is indicated by the figure. The geometry of a surface of constant negative curvature is called Lobatschewskian.

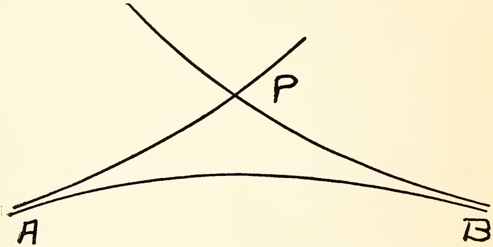


Fig. 51 —PARALLELS ON A PSEUDO-SPHERICAL SURFACE.

The curvature of a point (regarded as a sphere of zero radius) is infinite. Starting with a point, let the radius increase and curvature decrease. As the curvature runs continuously through the values from $+\infty$ down to zero the surface has a Riemannian geometry of no parallels. When curvature passes through zero, for an instant the surface is a plane with the property of one parallel, the curvature becoming negative. The Lobatschewskian geometry applies and there are two parallels. Continuing the curvature becomes larger and larger negatively, with radius becoming smaller until finally the surface closes up again into a point and the complete course has been run. Paralleling the case with the conic section, the parabola was seen to be the boundary between the ellipse and the hyperbola. So the Riemannian geometry is said to be elliptic, the plane parabolic and the pseudosphere hyperbolic; these terms come, however, from a different property of the spaces.

It is a curious fact that in the simple Riemannian plane the straight line cuts through the plane without cutting it in two. This cut cannot well be pictured, but an idea of its meaning may be got by thinking of the surface of a ring with a cut extending around the outside of it.

In Lobatchevskian space the unit of measure is a continuously decreasing length, while in Riemannian space it is continuously increasing.

Riemann, in his celebrated paper on 'The Hypotheses which Lie at the Basis of Geometry,' first advanced the theory that space might be unbounded without being infinite, in these words: In the extension of space-construction to the infinitely great, one must distinguish between unboundedness and infinite extent. That space is an unbounded three-fold manifoldness is an assumption which is developed by every conception of the outer world. The unboundedness of space possesses a greater empirical certainty than any external appearance. But its infinite extent by no means follows. On the other hand, if we assume independence of bodies from position, and therefore ascribe to space constant curvature, it must necessarily be finite, provided this curvature has ever so small a positive value. If we prolong all the geodesics from one point in a surface of constant curvature, this surface would take the form of a sphere.

The question as to whether the space of experience is Euclidean, Lobatchevskian or Riemannian is one which can never be determined. Are there two parallels, one or none? could only be settled in one of two ways, by reason or by measurement. A better form for the question is as to whether the sum of the angles of a triangle is less than, equal to, or greater than two right angles. As to reason, the geometry of one hypothesis is just as consistent as that of another. As to measurement, it is conceivable that an error in the measurement of the three angles of a triangle which may be drawn on this page would not show an error which would easily be detected if the triangle were drawn with sides 10 miles in length.

The largest triangles ever possible to measure have as a side the diameter of the earth's orbit, the opposite vertex being a celestial body. That no deviation from two right angles in the sum for this triangle is found is no evidence that if it were a million times as great the deviation would not be appreciable. The most that can be said is that if space is curved, the curvature is slight.

The study of non-Euclidean spaces enables one better to appreciate the insight of the old Greek geometer who 2,000 years ago realized that the proof of his fifth postulate was beyond his powers.

All measurement in mathematics is concerned either with that of lines or of angles. Euclid developed a complete theory of measurement of lines, but aside from the right angle and several of its exact divisors—as $\frac{1}{3}$ of a right angle, etc.—the only relations which he determined were those of greater and less; thus, if the sides of a triangle are 3, 4, 5, it is known by geometry that the angle opposite the side 5 is a right angle, and further that the angle opposite the side 4 is greater than that opposite 3, but exactly how much Euclid gives us no means of determining.

CHAPTER V

TRIGONOMETRY

Trigonometry is the science of the triangle with reference to the particular problem of finding the value of the unknown parts when three independent parts are given, as finding the angles when the three sides are given, etc. In a right triangle, ABC, lettered as in Fig. 49, six ratios are involved, which remain the same so long as the angles are not changed, the size of the triangle changing at will but preserving its shape. These six ratios are functions of the angles; that is, they depend for their value upon the values of the angles. They are named in the table below, with the abbreviations usually assigned to them given last.

$$\frac{a}{c} = \text{sine } A = \sin A \qquad \frac{a}{c} = \cos B$$

$$\frac{b}{c} = \text{cosine } A = \cos A \qquad \frac{b}{c} = \sin B$$

$$\frac{a}{b} = \text{tangent } A = \tan A = \text{tg } A \qquad \frac{a}{b} = \cot B$$

$$\frac{b}{a} = \text{cotangent } A = \cot A = \text{ctn } A \qquad \frac{b}{a} = \tan B$$

$\frac{c}{b}$ = secant A = sec A

$\frac{c}{a}$ = cosecant A = cs A

$\frac{c}{b}$ = csc B

$\frac{c}{a}$ = sec B

In the first column the functions are arranged in pairs, the second of the pair having its name from the first, with the prefix co. The origin of this prefix is from the rela-

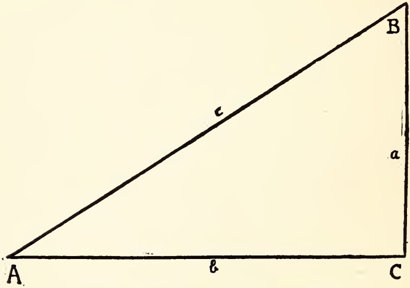
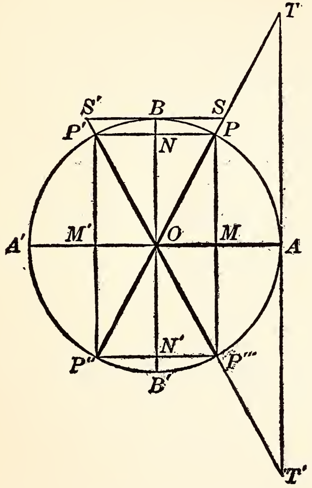


Fig. 52 —Ratios of a Triangle.

tion which exists between A and B, the sum of which is 1 right angle. It therefore takes B to fill a right angle together with A, or B is said to be the complement of A or co-A. Looking at the second column, one sees that the ratio $\frac{b}{c}$ is the sin B or sin of co-A or cosine A.

These six ratios were originally used in connection with a right triangle alone. When it became desirable to consider angles greater than 1 right angle, such angles not being found in a right triangle, the definitions for sine, cosine, etc., were so framed as to apply to any angle, positive or negative. This was done by means of a line representation. A circle of radius unity is chosen, and divided into 4 quadrants by means of a horizontal and a vertical line through the center. It is agreed that the angle shall begin at OA, and shall be considered positive



if it extends in a counter clockwise direction; directions of other lines are given by the arrows on the two axes. Take a point P on the terminal side of the angle and on the circumference of the circle; since the angle may be of any magnitude, the point P may be in any one of the 4 arcs AB, BA′,A′B′, or BA. The construction here given applies to any position of P. It will be supposed that P is in the arc AB, and the relations between the new and old definitions of the functions will be apparent. Draw OP, which will be directed outward from O. Drop a perpendicular from P to OA, calling the foot of the perpendicular M; then PM/OP = sin AOP, where the vertex of the angle is O, and OM/OP = cos AOP; but the circle was a unit circle, and OP = 1; whence MP = sin AOP and OM = cos AOP. From A erect a perpendicular cutting OP produced in T. Then AT/OA = tan AOP = AT, sec AOP = OT. From B draw a parallel to OA cutting OP produced at S. BS = cot AOP and OS = csc AOP.

If OP, beginning at OA, swings through a complete revolution about O, all angles from O to 4 right angles will be passed through.

There are two units employed in measuring angles: the degree, with its subdivisions minute and second, and the radian. The degree is $^1/_{360}$ of a complete circumference, due to the Babylonian year, which was made up of 360 days. The degree, symbolized by °, is divided into 60 equal parts, each called a minute (indicated by a single prime, ′), another Babylonian division; the minute is again divided by 60, giving the second, ″. The unit of radian measure is the angle which cuts off an arc equal to the radius of the circle. It is nearly 57.3 degrees. Since $2\pi r$ = circumference, 4 right angles = 2π radians, or $2\pi^r$, $90° = \frac{\pi^r}{2}$, $180° = \pi^r$, $270° = \frac{3\pi^r}{2}$. The number of radians is given by arc/radius.

In the figure of the line functions, if P returns, by making a complete revolution, or $2\pi^r$, to A, and continues turning in the same direction, an angle is formed which is greater than $2\pi^r$, but the functions of this angle are exactly those of the angle formed during the first revolution. This property of again passing through the same values with every complete turning is called periodicity. The periodicity of the six trigonometrical functions is well exhibited by a diagram in which distance along the horizontal line represents the magnitude of the angle measured in radians, and the perpendicular to this line at any point is the value of the function for the angle indicated by the point.

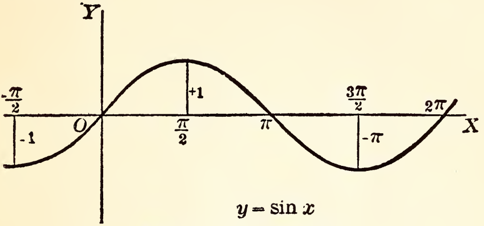


Fig. 53 —CURVE OF LINES.

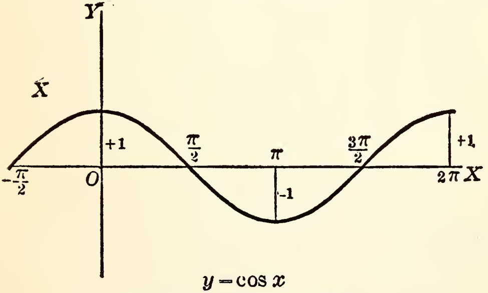


Fig. 54 —CURVE OF COSINES.

The simplest relation is that between the sine and cosine of an angle, which comes directly from the Pythagorean theorem, $\sin^2 A + \cos^2 A = 1$. One of the most important properties of these functions is that they have an addition law; that is, if two angles are added, the sine of the sum is not the sum of the sines of the two angles, but it may be expressed through functions of the angles. This is the most fruitful property. The addition theorem for sine and cosine follow where A and B are any two angles:

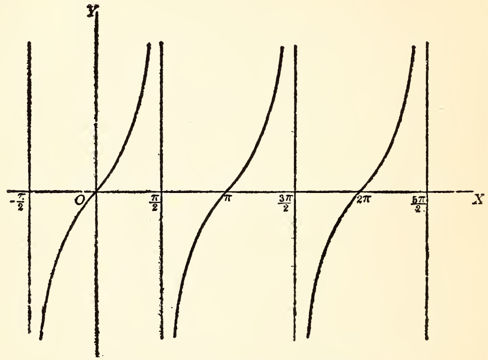


Fig. 55 —CURVE OF SECANTS.

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$

In the practical application to the solving of triangles three laws are used, which may be expressed by formula:

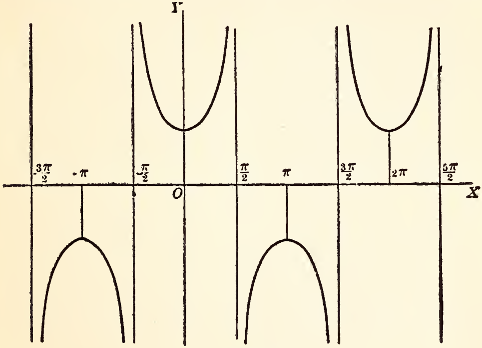
$$\text{Law of sines: } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\text{Law of cosines: } c^2 = a^2 + b^2 - 2\,ab \cos A$$

which is the law spoken of as summing up in one statement the Pythagorean theorem, with the acute and obtuse cases.

$$\text{Law of tangents: } \frac{a + b}{a - b} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}$$

Tables have been constructed by which the function of any angle, and conversely the angle of any function, may be obtained as accurately as the needs of science demand.



$y = \sec x$

Fig. 56 —a., Napier's Rules; b., Polar Triangles.

Spherical trigonometry is the science applied to a triangle on the surface of a sphere. The sides are now also expressed in angular measure. In the solution of the right triangle a mnemonic device, found by Napier, the inventor of logarithms, eliminates the necessity of committing to memory the relations of the functions. In the figure, C is a right angle, and before the parts A, c, B are written co-, which means that in the lines which follow that the complement of each part is to be taken rather than the part.

Napier's Rules of circular parts: S*i*n of m*i*ddle part is equal to the product of the c*o*sines of the *o*pp*o*site parts, or equal to the product of the t*a*ngents of the *a*dj*a*cent parts.

It is seen that omitting the right angle C, which is indicated by putting the C within the triangle, that there are five remaining parts; now choosing a part, and calling it a middle part, as a, there are two parts, b, co-B, adja-

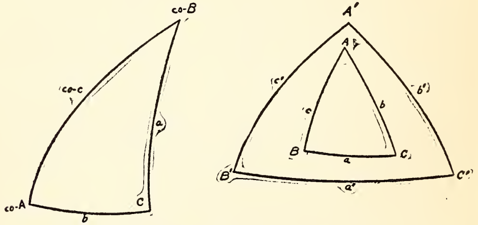


Fig. 57 —Section of a Model of a Cubic Surface. (Blythe.)

cent to a, and two parts, co-A, co-c, which are opposite to a. Apply the Rules above.

$$\begin{aligned} \sin a &= \cos \text{co-A} \cdot \cos \text{co-c} \\ &= \sin A \sin c \\ \sin a &= \tan b \cdot \tan \text{co-B} \\ &= \tan b \cot B \end{aligned}$$

In this way the ten necessary relations in the right triangle may be written at will.

There is a very interesting relation in spherical geometry concerning what are called polar triangles. If the angular points A, B, C of a triangle are used as centers, and the arc of 1 right angle is used as a radius, striking 3 arcs which form a triangle, this triangle, indicated by

$A'B'C'$, is called the polar triangle of ABC. The relation is reciprocal: ABC is polar of $A'B'C'$. The property which is to be noted is, that a side of a triangle (or angle) is the supplement of the opposite angle (or side) of the polar triangle.

$$A + a' = 180^\circ$$
$$a + A' = 180^\circ$$

The law of cosines in spherical trigonometry is the most general case of the universal law which is expressed in its simplest form by the Pythagorean Theorem:

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

If the radius of the sphere is allowed to become great without limit—that is, the spherical surface flattens out and approaches a plane, in the limit—this formula becomes the Law of Cosines in plane trigonometry:

$$c^2 = a^2 + b^2 - 2\,ab \cos C$$

If, now, the angle C becomes a right angle, the formula reduces to

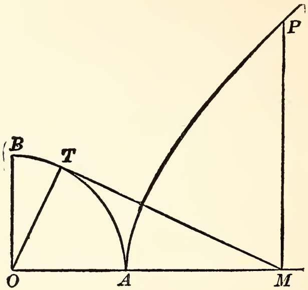
$$c^2 = a^2 + b^2$$

or the Pythagorean Theorem.

In the figure used in the definition of the trigometric functions by lines each function belonged to the angle AOP. Since the arc AP has the same measure as the angle, and the sector AOP—*i.e.*, the portion of the circle bounded by the two radii and the arc—is measured by the arc AP, it is convenient to say that the six ratios are functions of the sector as well as of the angle.

The circle was seen to be a particular case, with a fixed form or shape of the ellipse, which varied as the cone was turned. The hyperbola varies in shape also with the turning; there is a position of the cone which gives a form of the hyperbola analogous to the circle. This form is called the equilateral hyperbola. Its most familiar use is in representing the relation between the pressure and volume of a gas, which is expressed by $pv = a$ constant.

A set of functions belonging to the equilateral hyperbola has been devised which is distinguished from the set pertaining to the circle by calling the first set circular functions and the second hyperbolic functions. In the figure, the sector of the hyperbola bounded by OA, OP and the arc OP will be denoted by u. From the foot of the per-



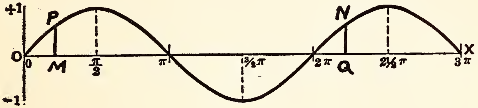
pendicular MP, MT is drawn tangent to the circle. The sector of the circle AOT will be called v. The hyperbolic functions of the sector AOP will be denoted by sinh u, cosh u, etc. v is said to be the gudermanian of u, or v = gd u. Some of the relations existing between the functions of u and v are:

cosh u = sec v;
sinh u = tan v;
tanh u = sin v, etc.

The discussion just given is of but a special case of these functions. The name hyperbolic was not originally given on account of the properties here stated.

One would expect that the term Elliptic function would be used for some similar relation in connection with the ellipse, but such is not the case. The desirable use of the word would be to denote the more general case of the circular functions. The term arose in connection with some expressions which appeared in the early attempts to rectify or measure an arc of the ellipse. They may, however, be regarded as an extension or branch of trigonometry, since they have two properties, analogous to two properties of trigonometric functions, namely: they admit of an addition theorem and periodicity.

The trigonometrical functions are simply periodic. In the sine curve let the angle be taken 30°. The value of the sine for 30° is indicated by the perpendicular line MP. If a point Q be taken 2π units from M, the sine line QN will be the same as MP; 4π will give the same sine. These points of periodicity are points of a line. The elliptic



functions are doubly periodic. It requires the entire plane to indicate the values of the independent variable.

Rudiments of trigonometry are found in the Ahmes papyrus, where the dimensions of square pyramids are to be found. In these computations appears a word, 'sept,' which has a value of about .75. This is the cosine of 41° 24′ 34″, which is very nearly the slope of the edges of the existing pyramids. In Ptolemy's 13 books of the Great Collection, or the Almagest, spherical trigonometry is developed and applied to astronomy. The names "minute" and "second" are from the Almagest. Half chords were first brought into favor by Al Battain, an Arab prince (c. 850-929), in whose work first appears the Law of Cosines for the spherical triangle. The greater part of the plan used in the trigonometry of to-day is the work of Regiomontanus or Johannes Müller (1436-1476).

CHAPTER VI

ANALYTIC GEOMETRY

The final union of algebra and geometry by means of the analytic geometry is usually attributed to Des Cartes. Algebra has been used at various times in connection with geometry by Apollonius and Vieta in particular, but in their works the idea of motion is wanting. Des Cartes, by introducing variables and constants, was enabled to represent curves by algebraic equations. A point in a plane is determined by its distances from two intersecting lines, which, for convenience, may be taken as perpendicular to each other. By allowing these two distances to vary, the point moves and generates a curve. By expressing the relation between these two variable distances in the form of an equation, the curve becomes subject to investigation following the laws of algebra. This is the great contribution by Des Cartes, and by it "the entire conic sections of Apollonius is wrapped up and contained in a single equation of the second degree." (Cajori.)

The plotting of an equation of the first degree which results in a straight line was spoken of in connection with algebra, as was also an equation of the second degree. The general equation of the second degree is written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

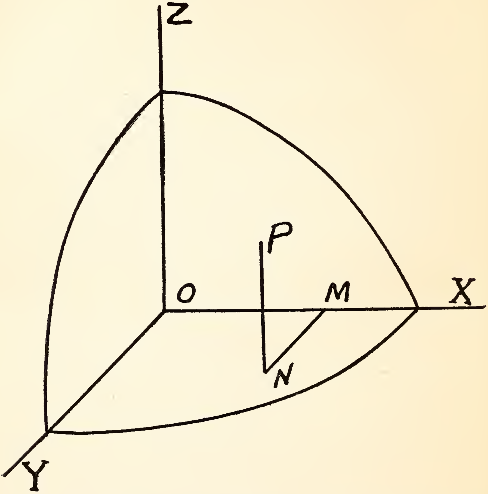
Two processes are applied to change the form of an equation, which evidently depends upon the axes chosen. One of these is to translate (or move parallel to themselves) the axes, and the other is to rotate them about the point of intersection, which is called the origin. If the general constants, A, B, C, D, E, F, are such that the equation can be reduced by one or both of these operations to the form $b^2x^2 + a^2y^2 = a^2b^2$, the curve is an ellipse; if to the form $x^2 + y^2 = r^2$, the circle, $b^2x^2 - a^2y^2 = a^2b^2$, is the equation of the hyperbola, and $y^2 = 2\ px$ is the parabola. If the left member of the equation can be factored, it is a degenerate conic. The equation of the third degree gives a curve which is called the cubic. Newton gave a classification of the cubic curves, the general form of which is a closed loop and an open branch. The curves of higher degree comprise some of the historic curves.

In addition to the algebraic curves there is a great class of curves called transcendentals. To this class belong the curves of the trigonometric functions given in p. 157. The most famous of the transcendentals is the cycloid, the path of a point on the rim of a carriage wheel as the wheel rolls on the ground. If the wheel rolls on the circumference of a circle, instead of on a line, the curve generated is called an epicycloid, and is one of the curves used in laying out gear wheels.

Some idea of the number of curves that have been investigated may be gathered from the fact that an Italian writer listed these curves, with a short description of each, filling a large book of about 700 pages.

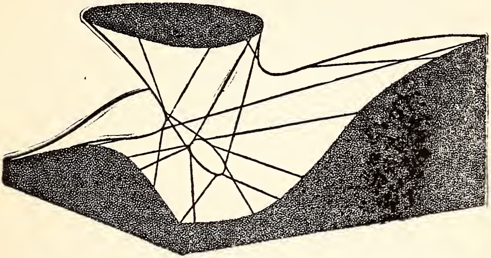
The method of Des Cartes is easily carried to three variables. An equation of this form might be $z = f\ (xy)$. The plane determined by the two perpendicular lines OY and OX is the old XY plane; perpendicular to it the new Z-axis, OZ. Since x and y are independent of each other, any value, as OM, may be laid off for x on the X-axis; perpendicular to this axis a value of y, say MN, is plotted. Putting these values in the equation, z is determined, which is laid off at right angles to the plane XOY, or NP; that is, P is one point of the surface represented by the equation. If a corresponding point is found for every point in the XY plane, the entire surface will be plotted.

An equation of the second degree in three variables, x, y, and z, represents one of what are called quadric surfaces. Such surfaces are of two classes; on a surface of



the first class, such as the ellipsoid, no straight lines may be drawn and the geodesics are all curved lines. The ellipsoid is generated by a variable ellipse moving parallel to itself. In the second class of surfaces, called the ruled surfaces, the geodesics are straight lines. The hyperboloid of one sheet may be generated by a line moving parallel to itself while constantly touching two circles in parallel planes, the planes being oblique to the moving line. Such a surface has two sets of line generators, one set inclined to the right, and the other to the left.

The cubic surface, or surface of the third degree, contains 27 straight lines, a fact discovered by Dr. Cayley in 1849. In the drawing of the section of one of these surfaces some of these lines are seen. The blackened portion indicates where the solid model is cut, only a part of the surface being shown.

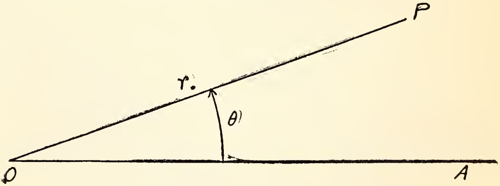


The principal advances in analytic geometry have been along three lines:

1. Changes in the system of coördinates.
2. Changes in the element used.
3. The introduction of the imaginary element.

In 1857 President Hill, of Harvard, gave a list of 22 systems of coördinates then in use, and since that time many more have been added. One of the most useful systems is known by the term polar coördinates, in which a point P is located by the distance $r = OP$ from the origin and the angle θ between OP and the initial line through O.

This system greatly simplifies some of the equations of the Cartesian system; for example, r = a constant is the equation of a circle in polar coördinates. The general equation of the straight line in Cartesian coördinates is $Ax + By + C = 0$. This equation is seen to lack homogeneity, or likeness, two of the terms containing variables and the third term being a constant. This unlikeness is removed if, in place of choosing as determining coördinates the distances from two intersecting lines, three lines are taken which intersect in pairs; that is, do not pass through the same point. Instead of using the three distances the three ratios of these distances are taken as the trilinear coördinates of a point.



In Euclid's choice of elements, the primary element is the point, with the circle and line as secondary, each of these being an aggregate of points. A point in motion generates a line or curve; the curve in motion, not along itself, generates a surface, which if moved outside of itself gives a solid. And the whole geometry is a point geometry, made up problems in which a certain point is to be found, the intersection of two lines, a line and a circle, or of two circles.

Looking at these elements from another viewpoint, they are but the circle which Euclid could draw and its two limiting cases, as the radius becomes indefinitely small, and becomes indefinitely great. The latter Euclid could not draw, whence he assumes straight-edge as one of his instruments. The symmetry of the three suggests that the line might just as well be taken as the point. A line is made up of an infinite number of points arranged in a certain way, and a point is made up of an infinite number of lines arranged in a definite manner.

A theorem which is thought of as a relation between points, it is evident, may be by simply interchanging the words 'point' and 'line' becomes the expression of a relation between lines. This principle of Duality was first worked out in its entirety by Jean Victor Poncelet, a brilliant young French lieutenant of engineers, who was made prisoner in the French retreat from Moscow in 1812. Fnding himself in prison, without books or any means of enjoyment, he occupied himself with investigations in geometry, and wrote his classic work on 'The Projective Properties of Figures,' in which the principle of Duality is completely worked out.

The analytical or algebraic investigations of geometry very often result in giving values which involve the imaginary element i. Every equation of the second degree represents a conic, and if two such equations are solved simultaneously for the points of intersection, four such points result. If the equations are those of circles, it is seen that two circles at most intersect in two real points. The other solutions result in imaginary solutions. The coördinates of these two points are conjugate imaginaries; one is of the form $a + ib$ and the other of the form $a - ib$. These two points are indicated by I and J and are called the two circular points at infinity, for it is found that every two circles, besides intersecting in two real or two imaginary points in the finite region of the plane, also intersect in I and J. Again, it requires five points to determine or pick out a conic section, and it is known that three points determine a circle. What about the two missing points? They are I and J, which lie on every circle in the plane. In this conception, a circle is the aggregate of all of the points in its circumference and the two points I and J.

If a circle has its radius indefinitely diminished it approaches as a limit a point, a degenerate conic which was its center. The equation of a circle with the center at the origin of coördinates is $x^2 + y^2 = r^2$. If r be made zero the equation is $x^2 + y^2 = 0$, which may be factored, giving $x = iy$ and $x = -iy$. These are the equations of two imaginary lines called isotropic lines, which have some interesting properties.

Through every point of the plane pass two isotropic lines.

These isotropic lines make the same angle with every real line through the point.

The distance between any two points on an isotropic line is zero, from which property they are called minimal lines.

The isotropic lines join the real point through which they pass with I and J respectively.

Perpendicularity between two real lines through the real point is a relation between the two lines and the two isotropic lines through the point.

The algebraic treatment of geometry permits the investigation of imaginary elements with exactly the same rigor as that of the real elements, and the only distinction between real and imaginary elements is not one of existence but of adaptability to the picturing processes of the mind. The term imaginary originally implied non-existence, but the development of algebraic processes has entirely swept away that meaning. The whole question of existence with the geometer is not one of material existence; points, lines and planes are but creations of thought without materiality. That which exists is that which is consistent in thought, coherent and non-contradictory. A real element is one which may be represented, as a line by a mark or string, a surface by a sheet of paper, and the imaginary is one of which no such picture or image may be formed.

The disposition to seek decision upon matters which do not come within the domain of present knowledge, that intuitive desire of mankind to rely upon the doctrine of chance, seems to be a universal trait with humanity. That such an instinct should arise and be cultivated in every branch of the human race is but a corollary of the fact that the future is hidden. Probability is more or less a factor in the life of every individual. It may be said that in no contingency which arises is there more than probable evidence upon which to proceed. Voltaire puts the case more strongly. "All life," says he, "rests on probability." As a moral guide it is said that the following theory was taught by 159 authors of the Church before 1667: 'If each of two opposite opinions in matters of moral conduct be supported by a solid probability, in which one is admittedly stronger than the other, we may follow our natural liberty of choice by acting upon the less probable.'

This gaming instinct has left as a heritage a number of games of great antiquity, varying from those in which skill and mental acuteness is the predominant factor down to those in which no element enters except that of pure chance. The best type of the first class is the game of chess, while perhaps midway comes cards and finally dice. Games akin to chess and checkers are represented in Egyptian drawings as early as 2000 B.C.

Professor Forbes puts the origin of chess "between three and four thousand years before the sixth century of our era." Altho this antiquity is to be doubted, it must be considered as extremely old. The game of chaturanga is said to have been invented by the wife of Ravana, King of Ceylon, when his capital, Lanka, was besieged by Rama. That the game was in some way connected with war seems evident. The Chinese name for chess is literally "the play of the science of war." The word chaturanga means the four divisions of the army, elephants, horses, chariots and foot soldiers.

The intricacies of the game are seen when it is known that there are as many as 197,299 ways of playing the first four moves, and nearly 72,000 different positions at the end of these moves.

The move of the knight is one move forward and one diagonally, and from this has been framed a famous problem: So to move the knight that it occupies but once each of the 64 squares of the board. This problem gives rise to some very odd geometrical designs on the board, if a

34	49	22	11	36	39	24	1
21	10	35	50	23	12	37	40
48	33	62	57	38	25	2	13
9	20	51	54	63	60	41	26
32	47	58	61	56	53	14	3
19	8	55	52	59	64	27	42
46	31	6	17	44	29	4	15
7	18	45	30	5	16	43	28

Fig. 55 —Knight's Move in Magic Square

straight line is drawn between each two successive positions. The solution here given is that of De Moivre. The number of possible solutions has been shown to be over 31,054,144.

The origin of cards is as uncertain as that of chess. They appeared in Europe about 1200. If one seeks to go back from this, one trail leads through Spain to Africa and Egypt, another over the Caucasus to Persia and India, and perhaps another is picked up in China. In the Chinese dictionary (1678) it is said that cards were invented in the reign of Sèun-ho, 1120 A.D., for the amusement of his various concubines. Tradition says that cards have existed in India from time immemorial and that they were invented by the Brahmans.

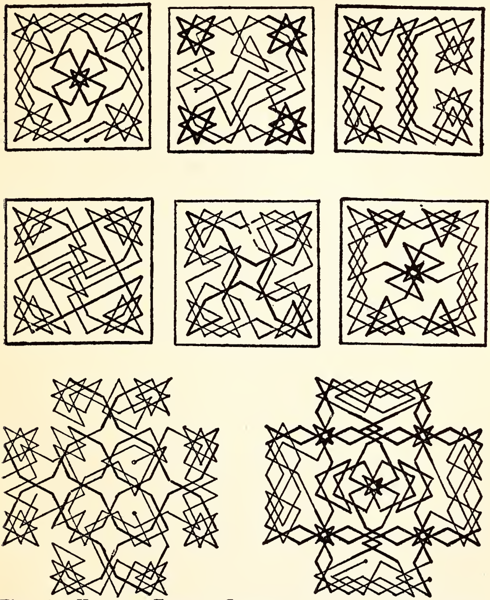


Fig. 56 —Knight's Tour on Single and Double Chess Boards. (Falkener.)

One form of cards, the Tarot card, was brought into Europe from the East by gipsies, who used them for divination purposes. They undoubtedly have been connected with witchery from the very beginning.

A number of famous problems have been devised with cards. The first to be spoken of is Gergonne's, or the three-pile problem. In this trick 27 cards are dealt face upward in three piles, dealing from the top of the pack, one card at a time to each pile. A spectator is requested to note a card and remember in which pile it is. Taking this pile between the other two the operation is repeated, and the third time is noted the middle card of each pack. Ask now for the pile and it is the card noted in this pile. Now if the three piles are taken up face down in the same order and dealt from the top it is the fourteenth card. Gergonne generalized the problem to a pack containing m^m cards.

The mouse-trap is another noted game with cards. A set of cards marked with consecutive numbers from 1 to n are dealt in any order face upward in the form of a circle. The player begins with any card and counts round the circle. If the kth card has the number k on it, a hit is scored and the player takes up the card and begins afresh. The player wins if he takes up all the cards. If he counts up to n without taking up a card, the cards win. In Tartaglia's work occurs a similar problem: A ship, carrying as passengers fifteen Turks and fifteen Christians, encounters a storm; and the pilot declares that in order to save the ship and crew one-half of the passengers must be thrown into the sea. To choose the victims, the passengers are arranged in a circle, and it is agreed to throw overboard every ninth man, reckoning from a certain point. In what manner must they be arranged that the lot will fall exclusively upon the Turks?

The number of combinations possible in various card games is enormous. With the whist deal this number is 53,644,737,765,488,792,839,237,440,000.

Dice and dolasses go back in history at least 3,000 years. Apollo taught their use to Hermes. These Greek gods probably got their knowledge from Egypt, where dice, and it is even said loaded ones, have been found in the tombs. Gaming with dice was common with the Romans, who had two forms, one like those of the present and the other oblong and numbered on but four sides. On these the deuce and the five were omitted. The convulsion of nature which overwhelmed Pompeii found a party of gentlemen at the gaming table, and they have been uncovered two thousand years after, with the dice firmly clenched in their fists. Seneca brings the gambling Emperor Claudius finally to Hades, where he is compelled to play constantly with a bottomless dice-box.

The two theories of choice and chance are very closely

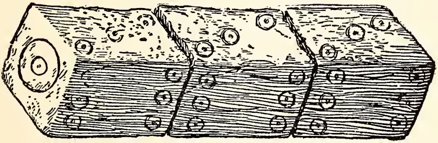


Fig. 57 —ROMAN DICE.

bound up together. Choice is made up of two branches, those problems which deal with arrangements and those with combinations alone. A problem of the first type is to find the number of ways in which 10 men may be seated at a round table. The first man has manifestly no choice; he may be seated anywhere; after he is seated the second man has 9 choices, the third 8 and so on until the tenth man, who has but 1 choice. It is a principle that if a thing may be done in a ways and another in b ways, the two together may be done in $a \times b$ ways. Therefore the 10 men may be seated in $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ ways, which is denoted by $9!$ or 9 factorial. The general expression for n things taken r at a time is $n! / (n - r)!$

If there is no distinction between the objects—that is, the order is immaterial—a choice is called a combination; as to find in how many ways a committee of 4 men may be chosen from 25 men. The mode of solution is to find in how many ways 25 men may be arranged if chosen 4 at a time, and divide by the number of arrangements possible with the 4 men.

If an event happens a times and fails b times, the probability of the event happening is $\frac{a}{a+b}$ and the probability of it failing is $\frac{b}{a+b}$. $\frac{a}{b}$ are the odds in favor and $\frac{b}{a}$ are the odds against the event happening. This may be illustrated in finding the probability of throwing at least 4 with 2 dice. The number of favorable cases is the number of cases in which 4, 5, 6, 7, 8, 9, 10, 11, 12 may be thrown. The number of unfavorable cases is the number of ways in which 2 and 3 can be thrown. 2 can be thrown in one way by throwing 1 and 1. 3 can be thrown in two ways, 2 and 1 and 1 and 2. The number of unfavorable cases is 3. The total number of cases is 6×6 or 36. The number of favorable cases is then $36 - 3$ or 33, and the probability of throwing at least 4 is $\frac{33}{36}$ or $\frac{11}{12}$.

If 52 cards be dealt to 4 players, the probability that a particular player will hold 4 aces is $\frac{11}{4165}$.

An application of the theory of probability may be given in determining the expectancy of a player in the ordinary "crap" game: A and B play with two dice, A throwing, and B being the "banker." If A throws 7 or 11 he wins; if he throws 3, or 2 aces, or 2 sixes, B wins. But if he throws 4, 5, 6, 8, 9 or 10 he continues throwing to duplicate his first throw, in which event he wins; if in throwing a 7 comes up, B wins. To determine the chances of the two players.

The chance of throwing 7 or 11 is $\frac{2}{9}$; of 2, 3 or 12 is $\frac{1}{9}$; of 4, 5, 6, 8, 9 or 10 is $\frac{2}{3}$. If A throws 4 his chance of winning the second throw is $\frac{1}{12} \cdot \frac{2}{3}$; of the third throw is

$$\tfrac{1}{12} \text{ of } \tfrac{2}{3} \text{ of } [1 - (\tfrac{1}{12} + \tfrac{1}{6})] \text{ or } \tfrac{1}{12} \text{ of } \tfrac{2}{3} \text{ of } \tfrac{3}{4}.$$

A's chance of winning on 4 is

$$\tfrac{2}{9} + \tfrac{1}{12} \text{ of } \tfrac{2}{3}\,[1 + \tfrac{3}{4} + (\tfrac{3}{4})^2 + (\tfrac{3}{4})^3 + \ldots] = \tfrac{4}{9}.$$

A's chance of winning on 5 is

$$\tfrac{2}{9} + \tfrac{1}{9} \text{ of } \tfrac{2}{3}\,[1 + \tfrac{13}{18} + (\tfrac{13}{18})^2 + (\tfrac{13}{18})^3 + \ldots] = \tfrac{22}{45}.$$

A's chance of winning on 6 is

$$\tfrac{2}{9} + \tfrac{5}{36} \text{ of } \tfrac{2}{3}\,[1 + \tfrac{25}{36} + (\tfrac{25}{36})^2 + (\tfrac{25}{36})^3 + \ldots] = \tfrac{52}{99}.$$

A's chance of winning on 8, 9 or 10 is the same as for 6, 5, or 4.

A's chance is then $\frac{1}{3}(\frac{4}{9} + \frac{22}{45} + \frac{52}{99}) = \frac{722}{1485}$.

B's chance is $1 - \dfrac{722}{1485} = \dfrac{763}{1485}$.

The odds in favor of B are $\dfrac{763}{722}$. (Zerr's solution.)

One very important application of probability is to determine the probable error in a number of observations. In 1805 Legendre gave his Law of Least Squares, which may be simply stated as follows: The most probable value of a measured quantity is that in which the sum of the squares of the differences between this quantity and the observed values, provided they are equally good, is a minimum.

Probability finds its greatest function, however, in determining the probable death-rate upon which are based insurance premiums. When it is recalled that at the present time the greatest amount of money that is involved in any single business is that in insurance, the words of Augustus de Morgan, penned in 1838, seem more than prophetic:

"The theory of insurance, with its kindred science of annuities, deserves the attention of the academic bodies. Stripped of its technical terms and its commercial associations, it may be presented from a point of view which will give it strong moral claims to notice. Tho based on self-interest, yet it is the most enlightened and benevolent form which the projects of self-interest ever took. It is, in fact, in a limited sense and a practicable method, the agreement of a community to consider the goods of its individual members as common. It is an agreement that those whose fortune it shall be to have more than the average success shall resign the overplus in favor of those who have less. And tho, as yet, it has only been applied to the reparation of the evils arising from storm, fire, premature death, disease, and old age, yet there is no placing a limit to the extensions which its application might receive, if the public were fully aware of its principles and of the safety with which they may be put in practice."

The science of probability had its origin in a problem proposed in 1654 to Blaisé Pascal by Chevalier de Méré, a professional gambler. It is now known as the problem of points. Two players want each a given number of points in order to win: if they separate, how should the stakes be divided? Pascal's solution is as follows: Two players play a game of 3 points and each player has staked 32 pistoles.

Suppose that the first player has gained 2 points and the second player 1 point; they have now to play for a point on this condition, that if the first player wins he takes all the money at stake, namely, 64 pistoles, and if the second player gains each player has 2 points, so that if they leave off playing each ought to take 32 pistoles. Thus, if the first player gains 64 pistoles belong to him, and if he loses 32 pistoles belong to him. If, then, the players do not wish to play this game, the first player would say to the second: "I am certain of 32 pistoles if I lose this game, and as for the 32 pistoles, perhaps I shall have them and perhaps you will have them: the chances are equal. Let us then divide these pistoles equally and give me also the 32 pistoles of which I am certain." Then the first player would have 48 pistoles and the second 16 pistoles.

Next, suppose that the first player has gained two points and the second player none, and that they are about to play for a point; the condition then is that if the first player wins this point he secures the game and the 64 pistoles, and if the second player gains this point they will be in the position just examined, in which the first player is entitled to 48 pistoles and the second to 16 pistoles. Thus, if they do not wish to play the first player would say to the second, "If I gain the point I gain 64 pistoles; if I lose I am entitled to 48 pistoles. Give me the 48 pistoles of which I am certain, and divide the other 16 equally, since our chances of gaining the point are equal." Thus the first player gets 56 pistoles and the second 8 pistoles.

Finally, suppose that the first player has gained one point and the second player none. If they proceed to play for a point the condition is that if the first player gains it the players will be in the position first examined, in which the first player is entitled to 56 pistoles; if the first player loses the point each player is then entitled to 32 pistoles. Thus if they do not wish to play, the first player would say to the second, "Give me the 32 pistoles of which I am certain and divide the remainder of the 56 pistoles equally—that is, divide 24 pistoles equally." Thus the first player will have the sum of 32 and 12 pistoles—that is, 44 pistoles—and consequently the second player will have 20 pistoles.

Thus the science which underlies the greatest business of the twentieth century had its origin at the gaming table. Pascal corresponded with his friend Fermat regarding the problem, and the subject continued to be developed to such an extent that Professor Todhunter's 'History of Probability,' from which the above problem is taken, covers 624 pages.

The theorem at the base of Probability is thus stated by James Bernoulli: "If a sufficiently large number of trials is made, the ratio of the favorable to the unfavorable events will not differ from the ratio of their respective probabilities beyond a certain limit in excess or defect, and the probability of keeping within these limits, however small, can be made as near certainty as we please by taking a sufficiently large number of trials." The inverse problem of reasoning from known events to probable causes is much more complicated. De Morgan thus states the principle of the inverse probability: "When an event has happened and may have happened in two or three different ways, that way which is most likely to bring about the event is most likely to have been the cause."

Another principle, due to Bayes, is thus stated: Knowing the probability of a compound event and that of one of its components, we find the probability of the other by dividing the first by the second.

Michell more than a century ago gave a classic attempt to apply the inverse theorem when he strove to find the probability that there is some cause for the fact that the stars are not uniformly distributed over the heavens.

The following witty dictum is from Poinsot:

"After having calculated the probability of an error, it is necessary to calculate the probability of an error in the calculations."

One thus gets in an endless regression by in turn calculating the probability of the correctness of the next preceding calculation.

Poincare closed his lectures on the calculus of probabilities with this skeptical statement: The calculus of probabilities offers a contradiction in the terms itself which serve to designate it, and if I would not fear to recall here a word too often repeated, I would say that it teaches us chiefly one thing—*i.e.,* to know that we know nothing.

An idea floating about in the minds of mathematicians for centuries, most nearly approached in the method of exhaustions used by Archimedes and in the method of indivisibles of Cavaleri, pupil of Galileo, was, by aid of the introduction of the notion of variable into geometry, finally evolved almost simultaneously and independently by the two greatest mathematicians of the period, Sir Isaac Newton and Gottfried Wilhelm Leibnitz, and has become the mighty engine of analysis, the first and only mathematical subject to be dignified by the article "The," The Calculus.

This subject is based upon two fundamental and comparatively easily understood operations: the direct operation, Differentiation, and its inverse, Integration. A few preliminary ideas are necessary.

A variable quantity is said to have a limit when it approaches a constant quantity in such a way that the difference between the variable and the constant quantity can be made to become and remain less than any previously assigned value. The constant quantity is called the limit of the variable. The condition is very often added that the variable never actually reaches its limit, but this is not necessary and very much narrows the application of the notion. Starting with the number 1, add to it its one-half, and continue the process indefinitely, each time adding one-half of the next preceding addition, thus: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

It is evident that this sum never reaches 2, but may, by proceeding far enough, be made to differ from 2 by as small a number as we please.

Inscribe in a circle a regular polygon; take the mid-point of each arc and join it with straight lines to the two adjacent vertices of the polygon. A new polygon is formed with double the number of sides of the original. Continuing indefinitely, a polygon may be formed which in area and perimeter differs from the circle by as little as we please; but the circle is never actually reached.

A quantity which aproaches zero as a limit is called an infinitesimal. An infinitesimal is not necessarily an exceedingly small quantity; the smallness is not the important matter, but the fact that it can be made small.

Zeno's paradox of Achilles and the tortoise rested upon the consideration of infinitesimals. Achilles was a certain distance behind the tortoise and attempting to overtake it. Zeno argues that he can never do so, for, says he, while Achilles travels half the distance between them the tortoise has traveled a certain distance; while Achilles is traveling half the remaining distance the tortoise has moved forward, etc. If these half distances were traveled in finite intervals of time Zeno's argument would be correct. But the intervals of time are approaching zero as well as the distances.

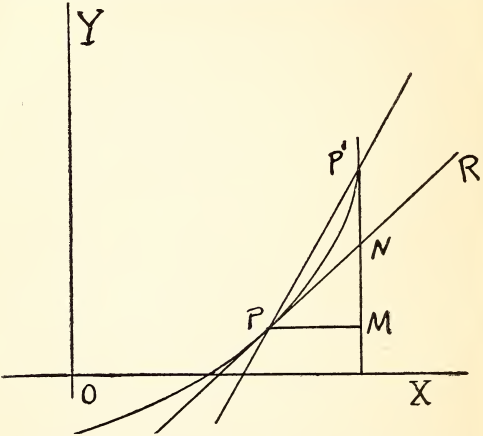
The differential calculus is based on finding the limit of the ratio of two infinitesimals. Suppose a train travels without stop from A to B, a distance of 100 miles, in 100 minutes, and it is required to find its speed. One says a mile a minute, but the train started from rest at A and comes to rest at B, whence there are points at which the speed is less than that given and at other points greater, so that the speed assigned is not the speed at every point, but what might be called an average speed. Suppose it is required to find the speed at a particular point, C; one would proceed in this manner: Measure a distance of say 1,000 feet along the track of which C is the middle point; time the train over this distance. The ratio of the distance to the time is the speed or rate, but it cannot be said that this is the rate at C; it is an average rate over the 1,000 feet. Take a shorter distance, say 500 feet; the ratio of this shorter distance to the shorter time is more nearly the rate at C than the former. Continue this process, and the ratio of the distance to the time as each becomes indefinitely small comes nearer and nearer to the exact rate at C. If the motion of the train was subjected to a law by which the limit of this ratio could be found, that limit would be the rate at C. Differentiation is this process of finding the limit of the ratio of two infinitesimals that are mutually dependent.

A geometric example will be given.

It is required to find the direction in which the point moves which generates the curve in the figure as it passes through the particular position, P. This direction will be along a tangent line, PR, since if the point were to continue in the direction in which it is moving at P, it would move in a straight line tangent to the curve at P. Take a second point, P′, on the curve and pass a line through P and P′. Now if P′ moves along the curve toward P this line swings around toward the limiting position PR. The direction of PP′ is fixed by the angle MPP′, of which the tangent is MP′/PM. As P′ approaches P, both MP′ and PM approach zero, but they have a limiting ratio which is equal to MN/PM, or the tangent of the angle MPN.

The mode of applying this operation algebraically is quite simple. The coördinates of P are given, say $(x_1\ y_1)$. A second point, P′, is chosen with coördinates $(x_1 + h,\ y_1 + k)$ and subjected to the condition that P′ lies on the curve. This is done by finding the relation of h and k by substituting $x_1 + h$ and $y_1 + k$ in the equation of the curve in place of x_1 and y_1. The limit of the ratio of h = PM and k = MP′ is then found as h and k approach zero.

In the parabola $y^2 = 8x$, find the direction or slope of the tangent at point P whose coördinates are 2 and 4. Take P′ $(2 + h, 4 + k)$, a point on the curve. Since it lies on the curve these coördinates must satisfy the equa-



tion of the curve. Putting $2 + h$ for x and $4 + k$ for y one has

$$(4 + k)^2 = 8\ (2 + h)$$
$$\text{or } 16 + 8\ k + k^2 = 16 + 8\ h$$
$$\text{or } 8\ k + k^2 = 8\ h$$

Solving for $\frac{k}{h}$ one has $\frac{k}{h} = \frac{8}{8 + k}$. As k and h approach zero together their ratio becomes 1, or the tangent of the angle which the line of direction makes with the X-axis is 1, from which the angle may be found, by consulting a table of tangents, to be 45°, or the line which is tangent to the parabola at the point (2, 4) makes an angle with the X-axis of 45°.

The sign of the operation of differentiation is $\frac{d}{dx}$.

The inverse operation, or integration, may be looked at from two viewpoints. If one chooses to consider it as simply the inverse operation, in order to perform it it would only be necessary to take cognizance of the steps in the direct process and reverse them. This would seem to be a very simple matter, but in practice frequently becomes extremely difficult or impossible. The second phase of integration is that of a summation of infinitesimals. $y = f(x)$ is the equation of a curve; if y is differentiated with respect to x, the result is a new function of x, say X. Then $\frac{d}{dx} = y$ or $\frac{dy}{dx} = X$, from which $dy = X\,dx$. This X being a function of x if plotted gives a curve as in the figure.

The y of any point of the curve is found by putting the corresponding value of x in the equation $y = X(x)$, as x gives y, x_2 gives y_2, etc.

In $dy = X\,dx$, take $dx_1 =$ the interval $(x_1\, x_2)$ and let $(x_1\, x_2) = (x_2\, x_3) = (x_3\, x_4)$, etc.

Then for $x = x_1$, $dy_1 = y_1 \times dx_1 = y_2 \times (x_1\, x_2) =$ area of rectangle x_1 R.

For $x = x_2$, $dy_2 = y_2 \times dx = y_2 \times (x_2\, x_3) =$ area of rectangle x_2 S.

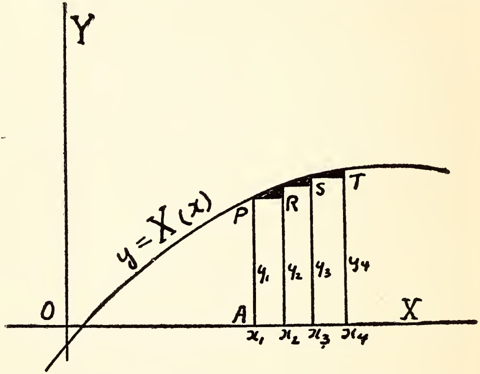
For $x = x_3$, $dy_3 = y_3 \times dx = y_3 \times (x_3\, x_4) =$ area of rectangle x_3 T.

Now if dx and dy each be made to approach zero and the sum of the dy's be taken to find y, this sum will be equal to the sum of the areas of these rectangles, as each rectangle has its base diminished toward zero. When this occurs the small shaded triangles approach zero and the sum of the rectangles approaches the area bounded by the curve, the X-axis, y_1, and y_4.

This is written $y = \Sigma\ X\,dx =$ area APQB.

Where Σ means the sum of all terms of the form x, dx as dx approaches zero.

If X be placed equal to Y and the curve plotted as



above, and also $y = f\ (x)$, the relations of the two curves is that the ordinate of any point of the second curve indicates the area under the first curve from a chosen point on the curve to the point for which the ordinate is taken.

When integration is regarded as above as a summation the sign Σ is sometimes used, altho it is customary to write the usual sign of integration S.

With the invention of the Analytic Geometry and the Calculus, modern mathematics begins. Speaking of its development from the date 1758, which closes the period covered by the third volume of Moritz Cantor's 'Geschichte der Mathematik,' Professor Keyser says: "That date, however, but marks the time when mathematics, then schooled for over a hundred eventful years in the unfolding wonders of Analytic Geometry and the Calculus and rejoicing in the possession of these the two most powerful among instruments of human thought, had but fairly entered upon her modern career. And so fruitful have been the intervening years, so swift the march along the myriad tracks of modern analysis and geometry, so abounding and bold and fertile withal has been the creative genius of the time, that to record even briefly the discoveries and the creations since the closing date of Cantor's work would require an addition to his great volumes of a score of volumes more."

And throughout all this wonderful growth nothing is lost or wasted, the achievements of the old Greek geometers are as admirable now as in their own days, and they remain the eternal heritage of man.

THE FOUNDATIONS OF MATHEMATICS

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THE FOUNDATIONS OF MATHEMATICS

A TRADITIONAL conception, still current everywhere except in critical circles, has held mathematics to be the science of quantity or magnitude, where magnitude including multitude (with its correlate of number) as a special kind, signified whatever was "capable of increase and decrease and measurement." Measurability was the essential thing. That conception of the science was a very natural one, for magnitude did appear to be a singularly fundamental notion, not only inviting, but demanding, consideration at every stage and turn of life. The necessity of finding out how many and how much was the mother of counting and measurement, and Mathematics, first from necessity and then from pure curiosity and joy, so occupied itself with these things that they very plausibly came to seem its whole enjoyment.

Nevertheless, numerous great events of a hundred years have been absolutely decisive against that view. For one thing, the notion of continuum—the "Grand Continuum," as Sylvester called it—that great central supporting pillar of modern Analysis, has been constructed by Weierstrass, Dedekind, Georg Cantor, and others, without any reference whatever to quantity, so that number and magnitude are seen to be more than independent—they are radically disparate. When the attempt is made to correlate the two, the ordinary concept of measurement as the repeated application of a constant finite unit, undergoes such refinement and generalization through the notion of Limit, or its equivalent, that counting no longer avails, and measurement retains scarcely a vestige of its original meaning. And when we add the further consideration that non-Euclidean geometry—primate among the emancipators of the human intellect—employs a scale in which the unit of angle and distance, tho it is a constant unit, nevertheless appears, from the Euclidean point of view, to suffer lawful change from step to step of its application, it is seen that to retain the old words, and call mathematics the science of quantity or magnitude, and measurement, cannot be accepted as telling either what the science has actually become or what its spirit is bent upon.

Moreover, the most striking measurements, as of the volume of a planet, the weight of a sunbeam, the growth of cells, the valency of atoms, rates of chemical change, the penetrative power of radium emanations, are none of them done by direct repeated application of a unit or by any direct method whatever. They are all of them accomplished by one form or another of indirection. It was perception of this fact that led the famous philosopher and respectable mathematician, Auguste Comte, to define mathematics as "the science of indirect measurement." But the thought is not yet sufficiently deep or comprehensive. For there is an immense range of admittedly mathematical activity that is not in the least concerned with measurement, whether direct or indirect. Consider, for example, that splendid creation of the nineteenth century, known as Projective Geometry. Here is a boundless domain of countless fields, where reals and imaginaries, finites and infinites, enter on equal terms, where the spirit delights in the artistic balance and symmetric interplay of a kind of conceptual and logical counterpoint—an enchanted realm where thought is double and flows throughout in curiously winding but parallel streams. In this domain there is no concern with number or quantity or magnitude, and metric considerations are entirely absent or completely subordinate. The fact, to take a simplest example, that two points determine a line uniquely, or that the intersection of a plane and a sphere is a circle, or that any configuration whatever —the reference is here to ordinary space—presents two reciprocal aspects according as it is viewed as an ensemble of points or as a manifold of planes, is not a metric fact at all: it is not a fact about size or quantity or magnitude of any kind. In this region of thought it was position, rather than size, that seemed to some the central matter, and so it was proposed to call mathematics the science of measurement and position.

The conception, thus mightily expanded, yet excludes many a mathematical realm of vast extent. Consider, for example, that limitless class of things known as operations—limitless alike in number and in kind. Now it so happens that there are many systems of operations such that any two operations of a given system, if they be thought as following one another, together thus produce the same effect as some single operation of the system. Such systems are infinitely numerous, and present themselves on every hand. For a simple illustration, think of the totality of possible straight motions in space. The operation of going from point A to point B, followed by the operation of going from B to point C, is equivalent to the single operation of going straight from A to C. Thus it is seen that the system of such operations is a closed system: that is, combination of any two of the operations yields a third one, not without, but within the system. The great notion of Group, thus simply exemplified, tho it had barely emerged into consciousness a hundred years ago, has meanwhile become a concept of fundamental importance and prodigious fertility. It not only affords the basis of an imposing mathematical doctrine—the Theory of Groups—but therewith serves also as a bond of union, a kind of connective tissue, uniting together a large number of widely dissimilar doctrines as organs of a single body. But—and this is the point to be noted here—the abstract operations of a group of operations, tho they are very real things, are neither magnitudes nor positions.

This way of trying to come at an adequate conception of what mathematics is, namely, by attempting to characterize in sucession its distinct domains, or its varieties of subject matter, or its modes of activity, in the hope of finding a common definitive mark, is not likely to prove successful. For it demands an exhaustive enumeration, not only of the fields now occupied by the science, but also of those destined to be conquered by it in the future, and such an achievement would require a prevision that none may claim.

Fortunately, there are other paths of approach that seem more promising. Every one has observed that mathematics, whatever it may be, possesses a certain mark, namely, a degree of certainty not found elsewhere. So it is, proverbially, the exact science par excellence. Exact, no doubt, but in what sense? An excellent answer is found in a definition given about one generation ago by a distinguished American mathematician, Professor Benjamin Peirce: "Mathematics is the science which draws necessary conclusions." This formulation is of like significance with the following, yet finer, mot, by that scholar of Leibnizian attainment and brilliance, Professor William Benjamin Smith: "Mathematics is the universal art apodictic." These statements, tho neither of them is adequate or final, are both of them telling approximations, wondrously penetrating insights, at once foreshadowing and neatly summarizing for popular use, the epochmaking thesis established mainly by the creators of modern logistic, namely, that mathematics is included in, and in a profound sense may be said to be identical with, Symbolic Logic. Observe that the emphasis falls on the quality of being "necessary"; that is, correct logically, or valid formally.

But why are mathematical conclusions correct? Is it that the mathematician has a reasoning faculty different in kind from that of other men? By no means. What, then, is the secret? Reflect that conclusion implies premises, that premises involve terms, that terms stand for ideas, concepts or notions, and these latter are the ultimate material with which the spiritual architect, called the Reason, designs and builds. Here, then, one may expect to find some light. The apodictic quality of mathematical thought is not due to any special kind of faculty peculiar to the mathematician, nor to any peculiar mode of ratiocination, but is rather due to the character of the concepts with which he deals. What is that distinctive characteristic? The answer is: precision and completeness of determination. But how comes the mathematician by such precision and completeness? There is no mystery or trick involved: some concepts admit of such precision and completeness, others do not—at least not yet; the mathematician is one who deals with those that do. The matter, however, is not so simple as it may now seem, and the attentive consideration of the reader is invited to what is yet to follow.

The Two Movements of Logico-mathematical Thought. —The foregoing thesis, which will be more narrowly examined in the latter part of this article, is the joint result of two modern movements of thought, which have had separate origins, have followed separate paths, and, having been carried on by two distinct and even alien groups of investigators, have recently converged, to the astonishment of both groups, upon the thesis in question. One of these movements originated at the very center of mathematics itself, and may be appropriately designated as the critico-mathematical movement. The other, which may be called the logistical movement, took its rise in other interests and in what seemed to logicians and mathematicians alike to be a very different and even a scientifically alien field—the interests and the field of what has come to be known as Logistic or Symbolic Logic.

The Critico-mathematical Movement.—For more than a century after the inventions (*i.e.,* the discoveries) of Analytical Geometry by Descartes and Fermat, and the Infinitesimal Calculus by Leibniz and Newton, mathematicians devoted themselves almost riotously to application of these powerful instruments to problems of physics, mechanics and geometry, without much concerning themselves about the nicer questions of fundamental principle, logical cogency and precision of concept and argumentation. In the latter part of the eighteenth century the efforts of "the incomparable Euler," of Lacroix, and others, to systematize results, served to reveal in a startling way the necessity of improving foundations. Constructive work was not, indeed, arrested by that disclosure. On the contrary, new doctrines continued to rise and old ones to expand and flourish. But a new spirit had begun to manifest itself. The science became increasingly critical as its towering edifices more and more challenged attention to their foundations. Manifest already in the work of Gauss and Lagrange, the new tendency, under the powerful impulse and leadership of Cauchy, rapidly developed into a momentous movement. The Calculus, while its instrumental efficacy was meanwhile marvelously improved, was itself advanced from the level of a tool to the rank and dignity of a science. The doctrines of the real and of the complex variable were grounded with infinite patience and care, so that, owing chiefly to the critical constructive genius of Weierstrass and his school, that stateliest of all the pure creations of the human intellect—the Modern Theory of Functions, with its manifold branches—came to rest on a basis not less certain and not less enduring than the very integers with which we count and tell the number of coins in the coffer or cattle in the field. The movement still sweeps on, not only extending to all the cardinal divisions of Analysis, but, through the agency of such as Lobachevski and Bolyai, Grassmann and Riemann, Cayley and Klein, Hilbert and Lie, Peano, Pieri and Pasch, recasting the foundations of Geometry also.

In the light of all this criticism of mathematics by mathematicians themselves, the science assumed the appearance of a great ensemble of theories, competent no doubt, interpenetrating each other in a wondrous way, yet all of them distinct, each built up by logical processes on its own appropriate basis of pure hypotheses, or assumptions, or postulates. As all the theories were thus seen to rest equally on hypothetical foundations, all were seen to be equally legitimate; and doctrines like those of Quaternions, non-Euclidean Geometry and Hyperspace, for a time suspected because based on postulates not all of them traditional, speedily overcame their heretical reputations and were admitted to the circle of the lawful and orthodox.

The Logistical Movement.—It is one thing, however, to deal with the principal divisions of mathematics severally, underpinning each with a foundation of its own; as, for example, the theory of the cardinal numbers (the positive integers) was assumed as the basis for the upbuilding of function theory. That, broadly speaking, was the plan and the effect of the critical movement above sketched. But it is a very different and a profounder thing to underlay all the divisions at once—both those that are and those that are yet to be—with a simple foundation, with a foundation that shall be such, not merely for the divisions but for something else, distinct from each and from the sum of all, namely, for the organic whole, the science itself, which they constitute. It is nothing less than that achievement—the founding, not of mathematical branches, but of mathematics—which, unconsciously at first, consciously at last, has been the aim and destined goal of the logistical movement—research in symbolic logic.

The advantage of employing symbols in the investigation and exposition of the formal laws of thought is not a recent discovery. As every one knows, symbols were thus employed to a small extent by the Stagirite himself. The advantage, however, was not pursued; because for two thousand years the eyes of logicians were blinded by the blazing genius of the "master of those that know." With the single exception of the reign of Euclid, the annals of science afford no match for the tyranny that has been exercised by the logic of Aristotle. Even the important logical researches of Leibniz and Lambert, and their daring use of symbolical methods, were powerless to break the spell. It was not till 1854, when George Boole, having invented an algebra to trace and illuminate the subtle ways of reason, published his symbolical 'Investigation of the Laws of Thought,' that the yet advancing revolution in logic really began. Altho it was neglected for a time by logicians and mathematicians, it was this work of Boole, who was both logician and mathematician, that inspired and inaugurated the scientific movement now known and honored throughout the world under the name of Symbolic Logic. Under the leadership of C. S. Peirce in America, of Bertrand Russell in England, of Schröder in Germany, of Couturat in France, and of Peano and his disciples and peers in Italy—supreme histologist of the human intellect—the deeps of logical reality have been explored in the present generation as never before in the history of the world. Not only have the foundations of the Aristotelian logic—the Calculus of Classes—been recast, but side by side with that everlasting monument of Greek genius there rise to-day two other structures, fit companions of the ancient edifice, namely, the Logic of Relations and the Logic of Propositions.

And now the base of this triune organon—the Calculus of Classes, the Calculus of Propositions, the Calculus of Relations—is surprising in its seeming meagerness, for it consists of a score or so of primitive propositions—the principles of logic—and less than a dozen primitive notions called logical constants. Yet more surprising, however, is the fact—justly described as "one of the greatest discoveries of our age"—that this foundation of logic is the foundation of mathematics also. So one may say: Symbolic logic is mathematics, mathematics is symbolic logic; the twain are one.

The Thesis.—The thesis, accordingly, which it is the purpose of the following paragraphs to explain with some detail, is this: All mathematical notions are definable directly or indirectly in terms of a few undefined or primitive notions (called logical constants), and in mathematical argumentation there enter as fundamental not more than about twenty undemonstrated or primitive propositions (called principles of logic).

What these primitives are will be seen presently. It is to be at once mentioned and to be constantly borne in mind that, if nothing be assumed, nothing can be deduced. Accordingly, in mathematics, as in any other science, the ideas that occur fall into two classes, the undefined and the defined; and the propositions fall into two classes, the undemonstrated and the demonstrated. In any case, the primitives—the undefined and the undemonstrated—are, to some extent, a matter of arbitrary choice and convenience. Simplicity is desirable, but not essential. What is necessary is that the set of notions chosen for primitives must be such that all other ideas of the science in question must be definable in terms of them; and whatever system of propositions be chosen for primitives must be such that all the other propositions of the science are demonstrable in terms of them. The set of primitive propositions must be compatible among themselves, and it is desirable, though not necessary, that the system should be non-tautological or irreducible; that is, that none of them be logically deducible from the others. The primitives contemplated in the foregoing thesis constitute the foundation of modern logic. It is to be shown that no new primitives are required in mathematics. This done, it follows that mathematics, instead of being a science that merely uses logic, is really a prolongation of it —a proper part, and, indeed, the principal part, of the superstructure of logic. If, then, an edifice includes both the basal masonry and that which is built upon it—and such appears to be the better use of the term—the propriety of identifying mathematics and logic is sufficiently evident.

It remains a moot question which of the three abovementioned branches of modern logic, if any one of them is entitled to the distinction, is logically prior to the others. As, however, discourse would seem to be quite impossible without propositions, in the following sketch we adopt the obvious recommendation of common sense, and begin with the calculus or logic of propositions. The set of primitive notions and propositions here presented is that which at present seems most likely to be finally adopted with least modification. Tho it is the result of the thought of numerous investigators, it may be called the Peano-Russell system, as suggesting the two men who have done most to produce it.

The Logic or Calculus of Propositions.—In this logic, besides the notion of truth, which remains undefined and constantly employed, the primitive ideas are two: (1) material implication and (2) formal implication. The notion of implication is not defined. We know, however, that it is a relation that, when it is found, is found to subsist between propositions. The idea of proposition is, however, defined. It is, namely, any thing that is true or false, or any thing that implies any thing. It is important to distinguish between a genuine proposition, as Xerxes was a soldier, from what has merely the form of a proposition, as y was a soldier, this last being, as it stands, neither true nor false. Such forms as the last, containing a variable (y or some other), are known as propositional functions, the notion being one of the primitives of the logic of classes. The distinction between material and formal implication is to be acquired very much as a child learns to distinguish cats from dogs. And the very young logician often confounds them. For one thing, material implication subsists only between genuine propositions, while formal implication is the kind that holds between propositional functions. Thus, 'Xerxes was a soldier implies Xerxes was a man' is an example of material implication, but 'A was a soldier implies A was a man' is an example of formal implication. If, in the last, one replaces the variable A by some constant, as Columbus or Cesar, the function is replaced by a proposition, and formal by material implication. In actual discourse, as it runs in the world, the distinction in question is often disguised. If p and q are propositions, then the proposition, p implies q, asserts a material implication, and means that either q is true or p is false. The proposition—if 2 is 4, 5 is 10—states a material implication, but the implication in the statement—if x is twice x, any multiple of x is twice that multiple—is formal. To borrow Mr. Russell's illustration: "The fifth proposition of Euclid follows from the fourth: if the fourth is true, so is the fifth; while if the fifth is false, so is the fourth. This is a case of material implication, for both propositions are absolute constants. . . . But each of them states a formal implication. The fourth states that if x and y be triangles fulfilling certain conditions, then x and y are triangles fulfilling certain other conditions: and the fifth states that if x is an isosceles triangle, x has the angles at the base equal." [Cf. Russell's 'Principles of Mathematics,' Vol. I, p. 14.]

The primitive propositions of propositional logic are ten in number, and are as follows:

(1) 'p implies q' implies 'p implies q';

(2) 'p implies q' implies 'p implies p';

(3) 'p implies q' implies 'q implies q';

(4) If p implies q and if p is true, then p may be dropped and q asserted;

(5) 'p implies p and q implies q' implies 'pq implies p'—the expression "p and q" being denoted by the symbol pq;

(6) 'p implies q and q implies r' implies 'p implies r';

(7) 'q implies q and r implies r and p implies (q implies r)' implies 'pq implies r';

(8) 'p implies p and q implies q and pq implies r' implies 'p implies (q implies r)';

(9) 'p implies q and p implies r' implies 'p implies qr'; and

(10) 'p implies p and q implies q' implies '('p implies q' implies p) implies p'.

Experience has shown that it is in various ways advantageous without compensating disadvantages to reduce all matter, whenever it is possible, to symbolic form. In case of the foregoing primitives such reduction is readily accomplished by employing the symbol $\supset$ (an inverted c) to denote the word "implies," and by using periods or dots in place of the word "and," as well as to indicate the relative ranks of the various copulas $\supset$ of a same proposition. A very little practice suffices to enable one both to translate into symbolic forms and to interpret them. Thus, in symbolic form the primitives in question stand as follows:

(1) $p \supset q . \supset . p \supset q$;

(2) $p \supset q . \supset . p \supset p$;

(3) $p \supset q . \supset . q \supset q$;

(4) If $p \supset q$, and if p be true, p may be dropped and q asserted;

(5) $p \supset p . q \supset q . \supset . pq \supset p$;

(6) $p \supset q . q \supset r . \supset . p \supset r$;

(7) $q \supset q . r \supset r . p \supset . q \supset r : \supset . pq \supset r$;

(8) $p \supset p . q \supset r . pq \supset r . \supset : p \supset . q \supset r$;

(9) $p \supset q . p \supset r . \supset . p \supset qr$;

(10) $p \supset p . q \supset q . \supset :. p \supset q . \supset p : \supset p$.

Of these, (1) means that p⊃q is a proposition; (2) means that what implies something is a proposition; (3) means that what is implied is a proposition; (4) is peculiarly interesting as illustrating the limits of formalism —it does not admit of symbolic statement, a fact not to surprise or mystify since it is a priori obvious that discourse is essentially prior to symbolism and is necessary to tell the meaning of it. The meaning of (4) may be made clear by a familiar example. If Socrates is a man, and if all men are mortal, then Socrates is mortal. Let the two premises be granted true, how justify the assertion of the conclusion as a true proposition—to be henceforth so taken? The answer is (4)—an exceedingly subtle principle introduced into logic by Peano. Couturat calls it the principle of deduction. (5) means that the joint assertion of two propositions, p and q, implies the assertion of the former; (6) is evidently the familiar principle of the syllogism; (7) states that, if a certain proposition implies that a second one implies a third, then the third is implied in the joint assertion of the other two. Thus, the proposition—Socrates lived in Athens—implies that, if Athens was then a city of Greece, the population of Greece once contained a philosopher. Now all this, says (7), implies that the proposition—the population and so on—is implied in the joint assertion of the two propositions: Socrates lived in Athens; Athens was then a city of Greece. (8) is the converse of seven; (9) means that, if a proposition implies each of two propositions, it implies both of them; that is, that the assertion of the first carries implicitly the joint assertion of the other two. The reader can easily illustrate. Finally, (10) tells us that if p is implied by the proposition, p implies q, then p is implied by the proposition that 'p implies q' implies p.

The reader will have observed that the foregoing principles differ in respect to simplicity and obviousness. He must be reminded that they were not selected because they were simple or obvious, but because they were found to be expedient. It is their serviceability that recommends them. They shine in their agency and use.

That the primitive propositions are true propositions the reader may convince himself by means of a test now to be explained. The proposition, p implies q, means that q is true or p is false and nothing more. This consideration readily serves to justify the remarkable statement: in respect to material implication, every false proposition implies all propositions, and every true proposition is implied by every proposition. Let us now apply this proposition as a criterion to test the truth of one of the primitives, say (8). Suppose, first, that p, q, r are all true. Then $q \supset r$ is true, hence $p \supset . q \supset r$ is true, and hence (8) is true. Next suppose p is false, and q and r true. Then $p \supset . q \supset r$ is true, and hence (8) is true. Again, suppose p true, q false, and r true. Then $p \supset . q \supset r$ is true, and hence (8) is true. Once more, suppose p and q true, and r false. Then $p \supset p$ is true, $q \subset r$ is false, and hence the joint assertion preceding the third dot is false; hence (8) is true. A like result follows under all the other possible suppositions respecting the elements p, q and r. And in like maner for the remaining nine primitives.

The conception of a science in a state of perfection requires that all other notions entering the structure of the propositional calculus be defined in terms of implication (and truth), and that all the other propositions of that calculus be demonstrated as theorems by means of the above-given primitive propositions. Among such superstructural notions and theorems are the following cardinal ones:

The logical product of two propositions, p and q, is their joint assertion, and is symbolized by $p \frown q$ or simply by pq. In terms of implication and truth, the definition is: if p implies p and if q implies q—*i.e.*, if p and q are propositions—pq signifies that r is true if p implies that q implies r.

The logical sum of p and q is denoted by writing $p \smile q$. It is a proposition s implied by p and by q and implying every proposition that is implied both by p and by q. The sum of p and q is the same as the disjunction or alternation, p or q.

The negative, $-p$, of a proposition p is defined to be such a proposition that, if r be any proposition whatever, $-p$ implies that p implies r.

Two propositions are said to be equivalent when and only when each of them implies the other; that is, if $p \supset q$ and $q \supset p$, then $p = q$, and conversely.

The fact that the product of a proposition by the same proposition is equivalent to the proposition—$pp = p$—is called the law of tautology for propositional multiplication. And for addition it is the fact that $p \smile p = p$.

Cardinal among the theorems of the propositional calculus are the following:

The product, $p \frown -p$, of a proposition and its negative is false—the law of contradiction.

The sum, $p \smile -p$, of a proposition and its negative is true—the law of excluded middle or third.

The negative of the negative of a proposition is equivalent to the proposition; that is, $-(-p) = p$. Such is the law of double negation.

Logical multiplication of propositions is commutative, associative and distributive; that is, $p \frown q = q \frown p$, $p \frown (q \frown r) = (p \frown q) \frown r$, and $p \frown (q \frown r) = (p \frown q) \frown (p \frown r)$.

The same three laws hold for logical summation of propositions.

The Calculus or Logic of Classes.—This logic is characterized by three primitive or undefined ideas or notions and by two primitive or undemonstrated propositions. The primitive ideas are: (1) Propositional function; (2) the relation of an individual to a class containing it; (3) and the notion expressed by the phrase such that, or its equivalent in the same or another language. The notion (1) is denoted by such symbols as $\phi(x)$, $\Psi(x)$, f(y), etc. It is familiar to everybody. Of it Mr. Russell ('Principles of Mathematics,' Vol. I, p. 19) says: "We may explain (but not define) this notion as follows: $\phi(x)$ is a propositional function if, for every value of x, $\phi(x)$ is a proposition, determinate when x is given." Thus $x + 2 = 0$ is a propositional function, for it yields a proposition, true or false, on replacing the variable x by any constant, as 1, 5, —2, Socrates, Wednesday, or love. Again, $\tan 45^\circ = 1$, $\tan 60^\circ = 4$, are propositions, while $\tan x = 1$, $\tan x = x$, are propositional functions. Once more, x is a triangle, is a propositional function, but 'John Jones is a triangle' is a proposition. Primitive (2) is denoted by the letter ; thus, to say that the individual k belongs to the class a, we write $k \varepsilon a$. The important distinction between the relation denoted by and that of part to whole was first pointed out by Peano. To say that a class a is a part of or is included in a class b, we write: $a \supset b$, the symbol $\supset$ being that which in the logic of propositions denotes "implies." Thus the syllogism, $a \supset b \,.\, x \varepsilon a \,.\, \supset .\, x \varepsilon b$, means: the class a belongs (as a part) to the class b, the individual x belongs to the class a, therefore the individual x belongs to the class b. But if a, b, and x are all classes, the syllogism is: $a \supset b \,.\, x \supset a \,.\, \supset .\, x \supset b$. The third primitive, such that, is denoted by the symbol $\ni$ (inverse of ε). Thus, to say the ensemble of x-values that render the function $\phi(x)$ a true proposition, or verify or satisfy it, we write: $x \ni [\phi(x)]$, which may be read "the x's such that $\phi(x)$ is true."

The two primitive propositions of this calculus are as follows:

(1) $\phi(x)$ is true when and only when x belongs to the ensemble of terms satisfying $\phi(x)$.

(2) If $\phi(x)$ and $\Psi(x)$ are equivalent propositions for all values of x, then the class of x's such that $\phi(x)$ is true is identical with the class of x's such that $\Psi(x)$ is true.

These primitives may be stated symbolically as follows:
(1) $K \varepsilon [x \ni \phi(x)] \supset \phi(k)$;
(2) $\phi(x) = \Psi(x) \;.\supset:\; x \ni \phi(x) \;.= x \ni \Psi(x)$.

The chief among defined ideas and proved propositions of class-logic are the following:

A class of terms is composed of the constants that satisfy a propositional function.

A propositional function that is false for every value of the variable in it defines a null-class.

An individual x is identical with an individual y if and only if y belongs to every class that contains x; otherwise x and y are diverse.

The class a is said to be included in the class b, and then we write $a \supset b$, when and only when every proposition of the form $x \varepsilon a$ implies, for the same x, that $x \varepsilon b$.

The classes a and b are said to be identical if each includes the other.

A class a is said to exist when and only when the logical sum of all propositions of the form $x \varepsilon a$ is true.

The logical product of two classes a and b is the class of terms x such that the logical product of the two propositions, $x \varepsilon a$, $x \varepsilon b$, is true.

The logical sum of two classes a and b is the class of terms x such that the logical sum of the two propositions, $x \varepsilon a$, $x \varepsilon b$, is true.

The logical product of a class c of classes is the class of terms x such that $u \varepsilon c$ implies $x \varepsilon u$.

The logical sum of a class c of classes is the class of terms x such that, if $u \varepsilon c$ implies $u \varepsilon k$ for all u's, then $x \varepsilon k$ for all k's.

When, as often happens, it is necessary to distinguish formally between a singular class (one having but one term) and its term, it is customary to place the Greek letter ι before the symbol for the class. Thus, if a be a singular class, ιa is its term. Also the inverse ɿ of the Greek letter, if placed before the symbol for a term, gives a symbol for the singular class having that term for sole term. Thus, if x be a term, ιx denotes the corresponding singular class.

The laws of tautology for class multiplication and summation are the facts that the logical product of a class by the class is the same class, and the sum of a class and the class is again just the class.

If we write $x - \varepsilon a$ for the negative of $x \varepsilon a$, then the negative, $-a$, of the class a, is defined by $-a . = . x \varepsilon (x - \varepsilon a)$; that is the class, $-a$, is the class of terms x such that x is an a is false.

The negative of the negative of a class is this class: $-(-a) = a$, the law of double negation for classes.

The laws of commutation, association and distribution are valid for the logical multiplication and addition of classes; thus, if a, b, c be classes, then $ab = ba$, $a + b = b + a$, $a(bc) = (ab)c$, $a + (b + c) = (a + b) + c$, $a(b + c) = ab + ac$, $a + (b + c) = (a + b) + (a + c)$.

The foregoing class-logic definitions in terms of ideas in the propositional logic serve to exhibit the close connection and interpenetration of the two logics. Their parallelism, too, is striking. Thus to the propositional syllogism, 'p implies q and q implies r' implies 'p implies r,' corresponds the class syllogism; if a is included in b, and b in c, then a is included in c. The parallelism is not, however, thoro-going, and may not be incautiously employed. For example, if p, q and r denote propositions and if a, b, c denote classes, then we have $pq \supset r = pcr \smile \supset q \supset r$, but not $ab \supset c = a \supset c \smile b \supset c$.

The Calculus or Logic of Relations.—We come now to the latest (in point of development), the subtlest, the profoundest, and, for mathematics, the most significant division of modern logistic. Founded by Charles S. Peirce upon the extensional view of relations—the view that a relation consists of the class of couples between which it subsists—elaborated and expounded on the same view by Schröder ('Algebra der Logik'), the Calculus of Relations was then refounded by Bertrand Russell, in

1900, upon the intensional view of relations, and by him dressed in the garb of a slightly modified Peano symbolism. It is this last theory, mainly due to Mr. Russell, of which the following account is a sketch:

This logic is characterized by two primitive ideas and eleven primitive propositions.

The primitive ideas are: (1) the notion of relation—symbolized by R and written rel; (2) the notion of identity—denoted by the symbol |'.

The primitive propositions are as follows:

(1) R being a relation, xRy means for all x's and y's that x has the relation R to y.

(2) Given any R, there is a relation R′—called the converse of R—such that xR′y is equivalent to yRx.

(3) If x and y be any two definite terms, there is a relation that x has to y and that does not subsist between any other couple of terms.

(4) If K be a class of relations, the logical sum of the relations of K is a relation, where by logical sum is meant the class of relations R such that, if an R relates an x to a y, there is in K a relation R′ relating that x to that y and that, if an R′ of K relates an x to a y, that x is related to that y by an R.

(5) If K be a class of relations, the logical product of the relations of the class is a relation, where by this product is meant the class of relations R such that if an R relate an x to a y, then each relation R′ of K relates that x to that y and that, if an x be related to a y by each R′ of K, that x is related to that y by one of the R's.

(6) If any term x is related to a term y by a relation R_1, and if y is related to z by R_2, x is related to z by a relation R_1R_2, called the relative product of R_1 and R_2.

(7) The negative, — R, of a relation R is a relation, where — R means that the proposition, x — Ry, is equivalent to the proposition x is not related to y by R.

(8) The symbol (as employed in the class-logic) is, or expresses, a relation.

(9) Identity (the primitive notion) is a relation.
(10) Any term x is identical with that term x.
(11) Identity implies identity.

If we denote the assertion that a thing exists by writing before its symbol the symbol $\exists$ (inverse of the letter E), denote the logical sum and product of a class K of relations respectively by the symbols, $\smile$'K and $\frown$'K, and denote by $\breve{\rho}$ the class of terms that may stand before an R—*i.e.*, its domain—and by $\breve{\rho}$ the codomain or class of terms that may come after R, then the foregoing primitive propositions may be written in symbollic form as follows:

(1) $R \epsilon \text{rel} \, . \supset : xRy \, . = .$ x has the relation R to y;
(2) $R \epsilon \text{rel} \, . \supset . \, \exists \, \text{rel} \frown R' \ni (xR'y \, . = . \, yRx)$;
(3) $\exists \, \text{rel} \frown R \ni (\breve{\rho} = \iota x \, . \, \breve{\rho} = \breve{\iota} y)$;
(4) $\smile$'K ϵ rel;
(5) $\frown$'K ϵ rel;
(6) $R_1R_2 \; \epsilon$ rel;
(7) $-R \; \epsilon$ rel;
(8) $\epsilon \; \epsilon$ rel;
(9) $1' \; \epsilon$ rel;
(10) $x \; \epsilon$ rel;
(11) $1' \supset 1'$.

It will be observed that a relation has sense; that is, xRy means to assert that R relates the antecedent x to the consequent y, and not y to x.

The class of the antecedents is the domain of the relation; that of the consequents is the co-domain; and the logical sum of the domain and the co-domain is the field of the relation.

Relations admit of important classifications. Thus a relation R is uniform if each of its antecedents has the relation to one, and but one, of the consequents. A relation R is biuniform if R is uniform and its converse $\breve{R}$ is also uniform. R is symmetric if xRy implies yRx; it is non-symmetric if xRy and yRx are both true for some but not all pairs of values of x and y; and asymmetric if, when xRy is true, yRx is false. R is transitive if the logical product of xRy and yRz implies xRz; non-transitive if the three statements are true for some but not all triplets of values of x, y, z; and intransitive if xRz is false when xRy and yRz are both true. Thus the relation of equality is both symmetric and transitive; the relations, greater than and less than, are transitive but asymmetric; the relation, implies, is non-symmetric but transitive; and the relation ε is asymmetric and non-transitive.

A relation R is reflexive if, like equivalence, for example, it holds between an x and that x.

The relation R is included in the relation R′ if xRy implies xR′y for all values of x and y; and R and R′ are equivalent if each includes the other.

Among the theorems that enter the logic of relations the two following ones, which are converses of one another, are specially noteworthy:

(1) The relative product of a relation and the converse relation is a symmetric and transitive relation;

(2) Every relation that is symmetric and transitive is equivalent to the relative product of a uniform relation and the converse relation.

The last states the principle of the so-called definition "by abstraction."

The Thesis Justified.—A sketch of modern logic having been premised, the above-stated thesis regarding the connection of mathematics with symbolic logic remains now to be justified by taking up serially the ideas upon which the chief divisions of mathematics have been built up, and presenting them in terms of the primitives (above given) of logic. Conceiving mathematics as falling into Analysis and Geometry, we may begin with the former, tho in this connection some ideas, as that of order, belong as well to geometry as to analysis. The reader should note that all definitions are given directly or indirectly in terms of the above-given logical ideas.

The Cardinal Theory of Cardinals.—The cardinal numbers may be defined either with or without use of the notion of order, giving rise to two theories of the cardinals, namely, the cardinal and the ordinal. It will be instructive to present the cardinal theory first.

Two classes, a and b, are said to have the same cardinal number when there is a biuniform relation whose domain includes a and such that the class of consequents of the terms of a is identical with b. It follows that two null-classes have the same number. This is called zero, and denoted by the symbol 0. Plainly, too, two singular classes have the same number. It is called one, and denoted by the symbol 1. It is to be noted that we have defined sameness of number of two classes, but have not yet defined number of a (given) class. Two classes having the same number are said to be equivalent. Now equivalence is a reflexive, transitive and symmetric relation, so that, a class a being given, there is a class of classes each equivalent to a and to any other class in the class of classes. The number of the class a is defined to be the class of classes each equivalent to a. Two classes without a common term are said to be disjoint. If a and b are two disjoint classes, and if α and β are their cardinal numbers, then the arithmetic sum of α and β is the cardinal number γ of the logical sum of a and b. The commutativity ($\alpha + \beta = \beta + \alpha$) of arithmetic addition is evident in the fact that the notion of order does not enter the definition of such addition. Arithmetic multiplication (of cardinals) is definable as follows: Let k be a class of disjoint classes of which none is a null class. The class of classes formed by taking (to compose a class) one, and but one, term of each of the classes k, is named the multiplicative class of the classes k. The cardinal number of this multiplicative class is named product of the cardinal numbers of the classes k. The notion of order being absent, the validity of the commutative law ($\alpha\beta = \beta\alpha$) is obvious. And the laws of distribution and association are readily shown to be valid.

It is noteworthy that in the foregoing there enters no distinction of finite and infinite class or number, and that the theory is applicable, therefore, alike to finite and to infinite cardinals. A class is said to be infinite or finite according as it contains or does not contain a part, or sub-class, such that a biuniform relation (a one-to-one correspondence) subsists or does not subsist between the terms of the class (the whole) and the sub-class (the part). And the number of a class is said to be infinite or finite according as the class is infinite or finite.

The Ordinal Theory of Cardinals.—This begins by adjoining to the foregoing definitions of zero (0) and one (1), the two definitions: (1) The successor of a cardinal n is the cardinal n + 1, the arithmetic sum (already defined in logical terms) of n and 1; (2) N is the class of cardinals that belong to every class c that contains both zero and the successor of every cardinal that it contains. This last definition states the principle of mathematical induction. It readily admits of proof that N is an infinite class, but that all the cardinals in N are finite, so that, unlike the cardinal theory, the ordinal theory of cardinals applies only to finite cardinals. It is not difficult to establish the propositions that zero is in N; that, if a is in N, the successor of a is in N; that, if a is in N, the successor of a is not zero; that, if the successor of a is identical with the successor of b, a and b are themselves identical; and, without using other than logical primitives, to erect the entire arithmetic of the finite integers.

The Notion of Order.—The definition of this exceedingly important notion is a notable achievement of recent investigation. Whatever order is, it was noticed that it might be linear, any two terms of the ordered class being the one before, the other after, with or without a term between, the class so ordered being called an open series; or it might be circular, of which a term cannot be said to be before or after another, but of which we are enabled to say merely that a pair of terms, a, b, is separated by a pair, c, d, if the four terms are arranged thus: a c b d a . . . or a d b c a . . ., a class thus ordered being described as a closed series. The sense of the disposition ab is disregarded, so that ab and ba are the same; accordingly, a triplet of terms is essential to linear order; thus abc (or cba) differs from acb (or bca), and enables us to say that one of the terms is between the other two. Similarly, disregarding sense, three terms cannot be in circular order, for abca is then the same as acba. Hence four terms are the element in case of circular order.

What order is has been ascertained by inductive study of the various relations that generate order. These, which reduce apparently to six distinct varieties, cannot be here presented. It is found, however, that any order, no matter by what relation it is generated, is generable by a transitive asymmetric relation. That is to say, if we have any ordered class of terms, the order, whatever it may be, is regardable as being set up by some asymmetrical transitive relation R, such that, x and y being any two terms of the class xRy, or else yRx is true but one of them is false; that, R being transitive, the logical product of xRy and yRz implies xRz; that the converse of R is also transitive and asymmetric; and that, given any term x of the class the remaining terms fall into two classes y and z such that xRy and zRx; and thus, of any three terms, x, y, z of the class, one of them, as y, is between the other two—*i.e.*, xRy and yRz, or zRy and yRx. A simple example is that of the class N of finite cardinals ordered by the relation greater than. Another example is that of the class of points of a line of unit length extending from 0 to 1, the points 0 and 1 being both included; the points being taken in their so-called natural order of increasing distance from 0; the order may be regarded as established by the asymmetric transitive relation, farther from 0.

Ordinal Numbers.—We are now prepared to define ordinal numbers, or types of order, which must not be confounded with the terms of the familiar series, first, second, third, and so on. Two series, u and v, are said to be like when there is between them a biuniform relation such that for every pair of terms a_1, b_1 of u and their correspondents a_2, b_2 of v, if a_1 precedes b_1, a_2 precedes b_2; or, the likeness may be affirmed of the two relations by which the series u and v are generated. It is noteworthy that likeness is to series or their generating relations analogous to equivalence in case of classes. Like equivalence, the relation of likeness is reflexive, symmetric and transitive.

The ordinal number, or order-type, of a series u is the class of series each like to u. If a series be a finite class, its ordinal number is uniquely determined by its cardinal number, the two numbers obey the same laws of operation and are (owing to the failure of man to distinguish between them) denoted by the same symbol. Thus the cardinal three and the ordinal three are both denoted by 3; yet they are radically different things; for the cardinal three contains, for example, the class composed of the individuals a, b, c, but not the series a, b, c as such; while the ordinal three contains that series and the distinct series b, a, c, among others. In the field of infinites, the difference between the concept of ordinal number and that of cardinal not only may, but must, be observed. For the laws of operation are then no longer the same for the two kinds of numbers. For cardinals, whether finite or not, the commutative law of addition holds without exception; not so, however, for ordinals. For example, denote by α the infinite ordinal number of the endless series $a_1, a_2, a_3, \ldots$, and by 3 the ordinal number of the series b_1, b_2, b_3; then the ordinal number of the series $b_1, b_2, b_3, a_1, a_2, a_3, \ldots$ is naturally $3 + \alpha$; that of the series $a_1, a_2, a_3, \ldots b_1, b_2, b^3$, is $\alpha + 3$; but the last two series are not similar, so that $3 + \alpha$ is not the same number as a $+ 3$; hence not all ordinals obey the commutative law of addition. And so for other laws of operations. The calculus of infinite cardinals and the distinct calculus of infinite ordinals are among the most beautiful and inspiring creations of mathematics. Philosophers and theologians have yet to learn to appreciate the significance of these doctrines, both of which are due mainly to the subtle creative genius of Georg Cantor, tho others have made important contributions to their development and refinement.

Rational Numbers.—Rational numbers, or fractions, are defined to be certain relations between the integers or cardinal numbers. This may be made clear as follows: Let the small letters, a, b, c, d, e, . . . denote integers. Suppose that $ab = c$, $db = e$, . . . It is obvious that to b there corresponds a relation, conveniently denoted by B, which consists in the fact that $ab = c$, $db = e$, ... Similarly, to any other integer, as m, there corresponds a relation M such that pMq means that $pm = q$. Now suppose that $ab = cd$; then we may write $ab = p$, $cd = p$, whence aBp and dCp. From the last follows pCd, while from this and aBp follows aBCd. The compound relation BC, uniquely determined by the integers b and c, is named fraction, and denoted by the familiar symbol $^b/_c$. All such relations together constitute the class of fractions or so-called rational numbers. Rational numbers having the cardinal 1 for denominator are usually denoted by the symbol for the numerator, and are thus made to appear as cardinals. Cardinals, however, they are not, as is evident by comparing definitions: a cardinal is a class; a rational is a relation. Upon this relational basis the entire theory of rationals is easily built up.

Positive and Negative.—It is to be noted and kept in mind that cardinal numbers and rational numbers are neither positive nor negative. Each of them is signless. Numbers having sign (+ or —) are defined as follows: If two integers are consecutive, there is a relation between them, the same for every pair of consecutives, by virtue of which one of them precedes and the other follows. Denote this relation by R. Then, a and b being integers, the proposition aRb means that $a + 1 = b$. The relation R is asymmetric but intransitive. If aRb and bRc, then aRRc or aR^2c, and so on. The powers of R are also asymmetric relations. The converse of R^p is $\breve{R}^p$, that is $(\breve{R})^p$; s that $aR^ps = s\breve{R}^pa$, the left-hand member signifying simply that $a + p = s$, and the right-hand member that $s - p = a$. The relations R^p and $\breve{R}^p$ are defined to be respectively the positive and negative integers, commonly denoted by $+p$ and $-p$. Next let a, b, c, . . . denote rational numbers or fractions. Suppose that the sum of a and b is c, then corresponding to b there is a relation B such that aBc means that $a + b = c$, that mBn means $m + b = n$, and so on. This relation B is defined to be a positive fraction, and is denoted by $+b$. The converse relation B is named negative fraction, is denoted by $-b$, and is such that mBn means $n - b = m$. The reader should not fail to discriminate the integer a and the positive integer $+a$; the former is a class, the latter a relation. Similarly, the fraction a and the positive fraction $+a$ are distinct: both are relations, but the relations are by no means the same.

Real Numbers.—Consider the ensemble E_1 of all the rational numbers less than the rational number 1, and the ensemble E_2 of all rationals whose squares are less than the rational 2. Each of the ensembles possesses the properties: it does not contain all the rational numbers; it contains every rational number that is less than any rational whatever (any variable rational) contained by it; that is, if it contains the rational x, it contains every rational less than x; it contains no number greater than all the other numbers in it. Any class of rational numbers that has the three properties stated is named segment (of rationals). Given any segment, s, the class composed of all other rationals may be conveniently denominated cosegment of s (complement of s). A segment of rational numbers is called a real number, which is thus a class. The real number E_1 is named one, and denoted by 1. The real number E_2 is named square root of 2, and denoted by the usual symbol. Segments fall into two classes, according as their cosegments contain or do not contain a minimum number, one that is smaller than every other number in the cosegment. The segments, or reals, of the latter kind are called irrationals. Those of the former kind are commonly called rational numbers, tho they are obviously very different from the rationals, merely corresponding to them. Thus the symbol 2, for example, denotes the cardinal two, the positive integer two, the rational two, and the real number two, all different ideas manageable by the same laws of operation. The theory of real numbers, as thus defined, turns out to be identical with that arising from the usual definition of reals, and has the advantage of not having to assume a limit where there is none, as, for example. in case of the foregoing segment E_2. The notion of limit, not yet employed, will be defined in the following section.

The Concept of Continuum.—This most important concept is definable in terms of order and without use of metric or magnitudinal considerations. The process is due to that primate among subtile thinkers, Georg Cantor. Denote by η the order-type represented by the ensemble of rational numbers taken in order of magnitude. Any series of this type has the following three properties: (1) It is denumerable; (2) it has neither a first nor a last term; (3) it is compact; that is, between any two of its terms there is another term of it. By calling it denumerable it is meant that a biuniform relation subsists between its terms and the terms of the series 1, 2, 3, 4, . . . That it is denumerable may be shown easily. Arrange the rationals in a series by beginning with $^1/_1$, following this with those having 3 for sum of numerator and denominator, these with the fractions having 4 for sum of terms, and so on, omitting any fraction that is equal to a predecessor in the series. The series is: $^1/_1$, $^1/_2$, $^2/_1$, $^1/_3$, $^3/_1$, $^1/_4$, $^2/_3$, $^3/_2$, $^4/_1$, . . . , the fractions having same number for sum of terms being arranged according to increasing magnitude. It is now plain that we can correlate the first term of the series with 1, the second with 2, the third with 3, and so on, so that each term gets paired with an integer, and conversely; hence the series of rationals or any other series of type η is denumerable.

A series of the type of the series 1, 2, 3, . . . is named progression. A progression all of whose terms are terms of a series η is called a fundamental progression of η; an ascending progression of its terms follow in the same order or sense as those of η, but descending if in the contrary sense. A class of terms belonging to a series is said to have a limit x when and only when x immediately follows (or precedes) the class but does not immediately folow (or precede) any one term of the class. A fundamental progression of a series η has a limit x if x be in η and immediately follows (or precedes) all the terms of the progression. Again, a series is said to be perfect when and only when all of its fundamental progressions have limits and all of its terms are limits of fundamental progressions.

These definitions premised, we are now prepared to define continuum. A series is said to be continuous if it is perfect and contains a series of type η. It admits of proof that an ensemble that belongs to a perfect series, is denumerable and has a term between every pair of terms of the containing series, is of type η. Hence we may say that a series S is continuous if it is perfect and if it contains a denumerable class having a term between every two terms of S. A standard example of a continuum is the class of the real numbers equal to or greater than zero and equal to or less than 1. This continuum is commonly represented by the class of points of a line segment of unit length, it being assumed that the series of such points and the mentioned series of real numbers are like.

Multiple Series and Geometry.—The remainder of this article, which aims at merely sketching modern thought on the foundations of mathematics, will be devoted to Geometry. For many centuries, indeed down to the early part of the last century, the term geometry meant Euclidean geometry, and the propositions constituting it—the axioms and postulates, together with the theorems deduced therefrom—were regarded, not merely as a set of assumptions and deductions from them, thus constituting a coherent body of doctrine suspended in the intellectual air, but as true statements about actual space. And so geometry has often been said to be the science of space, where "space" was used to denote actual or sensuous space, and not, as in recent years, merely the ensemble of elements, whether existent or not, about which geometry discourses. One of the Euclidean premises, however, namely, the so-called parallel axiom, seemed to critical minds to be not sufficiently "self-evident," and yet baffled all attempts (of which there is a vast literature, and still increasing by occasional contributions of the ill informed) to deduce it as a theorem from the other Euclidean axioms. At length appeared the geometries of Lobachevski and Bolyai, in which the axiom in question was denied. The fact that these geometries contradict the Euclidean at many points (for example, regarding the sum of the angles of a triangle) and are at the same time both free from interior contradiction and from contradictability by experimental measurement or other experience, lead first to the suspicion and then, through the discovered possibility of manifold geometries each consistent with itself but inconsistent with the others, to the conviction that the attempt to describe space results in an experimental science like physics or biology, that the so-called geometry thus arising is but a branch of what is commonly denominated applied mathematics (tho there is, strictly speaking, no such thing as applied mathematics), and that geometry, regarded as a branch of mathematics, is to be regarded and justified, not as a description of actual space, but, like every other branch of mathematics, as a hypothetico-deductive system. A given geometry consists of certain assumptions A and certain theorems T deducible from A. The truth of the geometry resides in the implication of the theorems T by the assumptions A, and not in their practical usability in the business of the work-a-day world—not in any applications to the concrete facts of the universe.

In recent years numerous memoirs on the foundations of geometry have been produced by European and American mathematicians. A striking result of such many-sided investigation is that the subject matter of what is called geometry is multiple series; that is, series of two or more dimensions. These terms may be explained as follows: A series s_1 generated by an asymmetrical transitive relation R is said to be simple, no matter what the nature of the terms of s_1. Suppose, now, that each term of s_1 is itself a simple series or an asymmetric transitive relation (for the relation, and not the terms, is the essence of a series). The class of all the terms in all the fields of the terms of s_1 is said to be a series of two dimensions. Call it s_2. For an image, the reader may think of s_1, as the series of the lines of a plane that are parallel to a given line. Each line (term of s_1) is a simple series (asymmetric relation) of points. The plane s_2 is the field of all the points of all the lines of s_1. Next suppose the terms of s_2 to be each of them an asymmetric transitive relation. Thus arises a three-dimensional series s_3, the field of the fields of the fields of the terms of s_1. The process here indicated, or its reverse, will, if continued, lead to the concept of a series of n dimensions. It is noteworthy that the ordinary complex numbers of the type $x + iy$, where x and y are real numbers and i is the square root of -1, constitute a double series, and that the result of assigning to y the value zero is, contrary to customary speech, not a real number.

Projective Geometry.—The study of such multiple series, or of the relations generating them, has yielded three grand types of geometry: Projective, Descriptive and Metric. These agree in the fact that they are concerned with multiple series of what are called points. But the terms of the series might as well be called roints, "slithy toves," "wabes," or any other names, for, as will be seen, "point" is to be merely the name of a class-concept, no matter what, whose individuals satisfy certain relations prescribed by the hypotheses or assumptions or postulates or so-called axioms (all the terms are in use) that are chosen for undemonstrated propositions of whatever geometry is being built up. In what respects the three grand divisions differ fundamentally will appear in the sequel. For each of the varieties in question there have been found various systems of basal hypotheses, so that an (undemonstrated) proposition of one system may be a theorem, a proposition demonstrated on the basis of another system serving for a basis of the same geometry.

The following system of basal assumptions for projective geometry is due to Pieri: 'I principii della Geometria di posizione composti in sistema logico deduttivo' ['Memorie della R. Accad. delle Scienze di Torino,' second series, vol. XLVIII, 1898]. An analysis of the system is found in Russell's 'Principles of Mathematics,' and also in Couturat's 'Les Principes des Mathématiques.' The basis upon which Pieri erects the beautiful edifice of projective geometry consists of the following assumed (undemonstrated) propositions:

I. Points form a class.

II. There is at least one point.

III. If a is a point, there exists a point other than a.

IV. If a and b are two different points, the straight line ab is a class.

V. Each term of this class is a point.

VI. If a and b are two different points, the straight line ab is contained in the straight line ba.

VII. If a and b are different points, a belongs to the straight line ab.

VIII. If a and b are distinct points, the straight line ab contains at least one point distinct from a and from b.

IX. If a and b are distinct points, and if c, a point of the straight line ab, is distinct from a, then b is a point of the straight line ac.

X. Under the hypothesis of IX, the straight line ac is contained in the straight line ab.

XI. If a and b are distinct points, there exists at least one point not belonging to the straight line ab.

XII. If a, b and c are three non-collinear points, and if a′ is a point of bc other than b and c, and b′ a point of ac other than a and c, then the straight lines aa′ and bb′ have a point in common.

XIII. If a, b and c are non-collinear points, there exists at least one point that does not belong to the plane abc.

XIV. If a, b, c are collinear points, their fourth harmonic does not coincide with c.

XV. If a, b, c are three distinct points of a straight line, then if d, a point of the line, be distinct from a and from c, and does not belong to the segment abc, it belongs to the segment bca.

XVI. If a, b, c are three distinct collinear points, then if the point d belongs to both of the segments bca and cab, it cannot belong to the segment abc.

XVII. If a, b, c are distinct collinear points, and if d belongs to the segment abc, and e to the segment adc, the point e belongs to the segment abc.

XVIII. If the segment abc is divided into parts X and Y such that each of them contains at least one point and that every point x of X precedes every point y of Y in the order abc, there exists at least one point z of the segment abc such that every point of abc that precedes it belongs to X and every point of abc that succeeds it belongs to Y.

Some of these propositions plainly presuppose certain definitions. These are now to be given, along with some commentaries designed to indicate the spirit and course of the author's thought. Certain diagrams, which the reader may readily construct, tho they are not essential, will serve to make clear. Such propositions as II and III show that no more points are to be assumed than are indispensable. The existence of others is to be proved. Thus, in the matter of fundamental assumptions, William of Occam's famous dictum is regulative: 'Entia non sunt multiplicanda praeter necessitatem.' The meaning of IV and V is that two points a and b determine a class of points, named straight line, and denoted by ab, where by "determine" is meant that, given any pair of points, there is a certain definite relation R that holds between the pair and a corresponding unique class of points. The offices of a and b being indistinguishable, it follows from VII that b, too, belongs to ab. From X it readily follows that a straight line is completely determined by any two of its points. Number XI, with preceding postulates, implies the existence of at least several straight lines. Number XII, which is not valid in either the Euclidean or the Lobachevskian (called by Klein hyperbolic) geometry, leads to the conception of the (projective) plane. The class of points on the straight lines containing a, and each of them a point of bc, is named plane, and denoted by abc. It is then proved that the planes abc, acb, bac, bca, cab, and cba, are one and the same; also that a plane is determined by any three of its non-collinear points, whence it follows that a plane containing two points of a straight line contains the entire line. The term fourth harmonic of XIV is defined as follows: The fourth harmonic of three collinear points a, b, c, or (as it is often called) the harmonic conjugate of c with respect to a and b, is a point x of ab such that there exist two distinct points u and v collinear with c, but not on ab, and such that x is collinear with the intersections of au with bv and av with bu. The point x is constructed by means of a figure (indicated in the foregoing definition) known as the von Staudt Quadrilateral. It is noteworthy that the definition implies neither the existence nor the unicity of x. The former is readily demonstrable by means of the first twelve postulates, but the latter requires XIII; for the unicity depends upon the theorem of homologous triangles (found in every book of projective geometry), and it is a most notable fact that this plane theorem does not admit of proof except by the help of points outside the plane—a most suggestive fact. What is true in a given domain of experience may, nevertheless, not be provable within that domain.

The straight line has been introduced as a whole, as an orderless class. Pieri endows it with order, thus giving it the character of a series of points, as follows: Given a, b, c, three collinear points. Let y be any other point of the line, and z the harmonic conjugate of y with respect to a and c. Let x be the harmonic conjugate of b with respect to y and z. By taking a new y, and hence a new z, a new x is obtained. The class of x's thus obtainable is named segment abc. It is shown that b belongs to the segment, that its extremities a and c do not, and that the segment abc is the same as the segment cba. The segment has the property: if a, b, c, d be four points of a straight line, and if a′, b′, c′, d′ be four points so situated on another straight line that the lines aa′, bb′, cc′, dd′ have a point in common, then d′ belongs to the segment a′b′c′ when and only when d belongs to the segment abc. If d does not belong to the segment abc, and is distinct from a and c, then, the four points being collinear, the points a and c are said to separate the points b and d. It is proved that the relation of separation is symmetric; that is, that the points a and c are also separated by b and d; furthermore, that the statement is valid if in it we exchange the points of either couple. The ordering of the points of a line is then completed by means of the postulates XV, XVI and XVII. Continuity is introduced by number XVIII. The effect of the postulate XIX, namely, if a, b, c, d are four non-complanar points, and e a point in none of the planes, abc, abd, acd, bcd, then there exists a point common to the line ae and the plane bcd, is to restrict the geometry to a space of three dimensions. This restriction is essential to the duality of ordinary projective geometry in virtue of which the notions point and plane may be interchanged. If we wish to pass to projective geometry of hyperspace, postulate XIX must be omitted and other suitable postulates added. One such, for example, would be: if a, b, c and d be four points not belonging to a same plane, there exists at least one point not in the hyperplane abcd, where by hyperplane is meant the class of points on the lines determined by the points of a plane and a point not in the plane.

If, now, a definition of projective geometry (of three dimensions) be required, the answer is: it is the theory consisting of the foregoing nineteen postulates (or an equivalent set), together with the propositions logically deducible from them. And, similarly, projective space (of three dimensions) is any class of things (for convenience called points) that are related as prescribed by the foregoing or an equivalent set of postulates.

The one undefined notion in projective geometry, as above founded, is that of straight line. In order that the doctrine shall be quite expressible in terms of logical constants, it is necessary and sufficient that the straight line be defined in such terms explicitly. Such a definition is: A projective straight line ab is a relation R between the points a and b, R being symmetric, aliorelative (not subsisting between a point and that point) and transitive, in so far as transitivity is not restricted by aliorelativity.

Descriptive Geometry.—The doctrine of which some account is to be rendered here is not the descriptive geometry commonly so called, created by Gaspard Monge, and in elementary form presented to technological students as the semi-practical art of graphically representing space configurations by means of their projections on a plane. This last is about identical with projective geometry, or the geometry of position, as popularly understood. The descriptive geometry to be dealt with here is a new theory, having been created by Pasch ('Vorlesungen über neuere Geometrie,' 1882) and formulated in the symbols of modern logic by Peano ('I principii di Geometria logicamenta esposti,' 1889, and 'Sui fondamenti della Geometria in Rivista di Matematica,' Vol. IV, 1894). How it differs from projective geometry in procedure and fundamentals will appear in the light of the following postulates (as given by Peano) and commentaries upon them. For fuller analyses of the postulates, the reader may consult the above-cited works of Russell and Couturat. The Peano postulates (undemonstrated propositions) of descriptive geometry are as follows. The meaning of some of them will be clear only by aid of definitions to follow.

I. There is at least one point.

II. Given a point a, there is a point x distinct from a.

III. Between two coincident or identical points there is no point.

IV. Between two distinct points there is a point.

V. The segment ab is contained in the segment ba.

VI. The point a is not between a and b.

VII. If a and b are two distinct points, there are points that belong to a'b.

VIII. If c is a point of the segment ab, and if d is a point of the segment ac, d is also a point of the segment ab.

IX. If c and d belong to a segment ab, they coincide, or d is between a and c or is between c and b.

X. If c and d belong to the ray a'b, they coincide, or d is between b and c or c is between b and d.

XI. If b is between a and c, and c between b and d, c is between a and d.

XII. If r is a straight line, there exists at least one point outside of r.

XIII. If a, b, c are non-collinear points, and if d is between b and c, and e between a and d, then there is a point common to ac and the prolongation of be.

XIV. If a, b, c are three non-collinear points, and if d is between b and c, and f between a and c, the segments ad and bf have a common point.

XV. Given any plane, there exists at least one point outside the plane.

XVII. If p is a plane, a a point outside the plane, and b a point on the prolongation of one of the segments joining a to points of p, then, if x is any point, it belongs to p, or else p and the segment ax or else the segment bx have a common point.

XVIII. Let k be a class of points in the segment ab; there exists a point x of the segment ab, or coinciding with b, such that no point of k is between x and b, and that, y being any point taken between a and x, there exist points of k between y and b.

Such are the basal assumptions of descriptive geometry. A few explanatory words will make their meaning clear and will serve to show the concept of descriptive space and the corresponding geometry in the process of gradually coming into being.

By segment ab is meant the class of points between the points a and b. In this geometry the notion of segment is central like that of straight line in projective geometry. By III the segment aa or xx is a null-segment, one void of points, an empty class. By IV a segment ab is null if its extremities a and b are identical (coincident). V shows that segments ab and ba are one and the same: to be between a and b is the same as to be between b and a; a segment is without direction, or sense. By VI the extremities of a segment are not points of it. By the symbol a′b (in number VII), called the prolongation of ab beyond b, is meant the class of points x such that b is between a and x. VII postulates the existence of such prolongation. The existence of ab′ is a consequence, as is also the fact that a′b = ba′ and that ab′ = b′a. Such prolongations, which are not segments, are called rays. Number VIII enables us to prove that segment ab contains the segments ac, bc and cd; that the ray a′c contains the ray a′b; that the logical product of the propositions, b is between a and c, c is between a and b, is false; and that, consequently, the segment ab and the rays a′b and ab′ have no common point. By help of IX it is demonstrable that the segment ab is the logical sum of the segments ac and cb and the point c; that, if c is between a and b and d between c and b, then c is between a and d; that, if c is between a and b, d between a and c, and e between c and b, then c is between d and e; that, under the same hypothesis, the segments ac and cb have no common point; and that, if c and d belong to the segment ab, the segment cd is contained in the segment ab. Such are properties of segments. Those of rays are found by means of X and XI to be that, under the hypothesis of X, the ray a′b is the logical sum of the segment bc, the point c and the ray a′c; under the same hypothesis, the segment cd is contained in the ray a′b; and by XI, if b belongs to the segment ac or to the ray ac′, the rays a′c and b′c coincide.

The straight line ab (a term occurring in XII) is defined to be the logical sum of the points a and b, the segment ab and the rays a′b and b′a. The first eleven postulates suffice to show that the straight lines ab and ba are identical; that, if c is different from a and belongs to the straight line ab, the straight lines ab and ac are identical; and that, if c and d are distinct points of the straight line ab, the straight lines ab and cd are one and the same; or, what is equivalent, that a straight line is determined by any two distinct points of it. Postulates XII and XIII provide for the concept of plane, as will presently be seen. If h and k be two classes of points, the symbol hk will denote the class of all the points on the segments joining the points of h to those of k; h′k the class of points on the prolongations of the segments each beyond its k point, whence the meaning of hk′ is also clear, and that, too, of such symbols as a(bc), a′(bc), etc. From XIII follows that a(bc) = b(ac). This figure or class of points is named triangle and denoted by triangle abc. The plane abc is defined to be the class composed of the (non-collinear) points a, b and c, the segments ab, bc, ca, the prolongations ab′, ba′, bc′, cb′, ca′, ac′, the triangle abc, and the figures a′(bc), b′(ca), c′(ab), c(a′b′), a(b′c′), b(c′a′). Postulate XIV is essential to prove that a plane is uniquely determined by any three non-collinear points of it. And numbers XV and XVII are respectively necessary that space shall have three dimensions and that it shall be continuous.

Obvious among the notable differences of projective geometry and descriptive geometry are the following. In the former the straight line is a closed series of points (like the circumference of a circle); in the latter the straight line is an open series of points. Two projective straight lines of a (projective) plane, or a projective line and plane, always have a point in common; but a descriptive plane contains many pairs of non-intersecting straight lines and a descriptive line and a descriptive plane may or may not have a common point. One point of a descriptive line divides it into two parts, and a pair of points divide it into three parts one of which is a segment determined by the two points. It requires three points to determine a segment of a projective straight line, two points separate the line into two portions, and one does not divide it into parts. Two projective planes have a line in common but two descriptive planes may or may not have a common line, tho they have a common line or no common point.

It is an interesting and instructive fact that upon the foregoing descriptive postulates it is possible by suitable choice of elements to build up a projective space and geometry. This may be done as follows, and the process further reveals the differences and relationships of the two varieties of space. Let a and b be any two given lines of a descriptive plane π., and let P be any given point of descriptive space. The two planes determined by P and a, and P and b, have a common line L. The class of lines L thus determined by allowing P to take all positions in descriptive space is named sheaf of lines. These will have a common point (called the vertex of the sheaf) or not according as a and b have a common point or not. Again, if S_1 and S_2 be two sheaves and P a point (not on the common line of the sheaves if they have one) . P, S_1 and S_2 determine a plane π, namely, that containing those lines of S_1 and S_2 that contain P. The class of planes π thus obtainable by varying P, is named pencil of planes. The planes of the pencil will have a common line (called the axis of the pencil) or not according as S_1 and S_2 have a common line or not. Finally, let S_1, S_2, and S_3 be any three sheaves whose lines are not all in the planes of a same pencil, and let S_4 be a sheaf such that there is a sheaf S whose lines are common to the pencils S_1S_3 and S_2S_4. The class of sheaves S_4 that fulfil the condition will be named hyperpencil of sheaves. If now we denote the new entities, sheaves, pencils and hyperpencils, respectively by the names, points, lines and planes, it can be shown that these points, lines and planes constitute a projective space, altho as seen the new elements are defined in terms of descriptive space.

Metric Geometry.—In recent years various investigators, American and European, have proposed various logically equivalent systems of postulates for this the most ancient form of geometry. Of such systems, that found in Hilbert's 'Grundlagen der Geometrie' (also in English and French) is the most famous. We prefer, however, to present here that of Pieri as being more interesting and not less profound. In this system there are two undefined terms, namely, point and movement. It will be seen that point is merely a name for the element of any system of elements (if such there be) that satisfy the postulates. And movement does not mean ordinary motion, but only a transformation, or change of attention from one thing to another. Even so the process is disregarded, only the initial and the final stages and not any passage are regarded. The postulates are as follows. Subsequent explanations will make them clear.

I. Point and movement are genuine concepts or classes.

II. There exists at least one point.

III. If p is a point, there exists a point different from p.

IV. Every movement is a biuniform correlation between two figures.

V. Whatever be the movement μ which makes the point y, for example, correspond to the point x, there is a movement u that makes x correspond to y.

VI. Two movements, μ and ν, effected successively the one on the result of the other, are equivalent to a single movement.

VII. For each pair of distinct points there is an effective movement that leaves them fixed.

VIII. If a, b and c are three distinct points, and if there exists an effective movement that leaves them fixed, every other movement that leaves a and b fixed leaves c fixed.

IX. If a, b and c are three collinear points, and if d is a point of (the line) bc other than b, the plane abd is contained in the plane abc.

X. If a and b are distinct points there exists a movement that leaves a fixed and transforms b into another point of the straight line ab.

XI. If a and b are distinct points, and if two movements that leave a fixed transform b into another point of the straight line ab, this point is the same in both movements.

XII. If a and b are distinct points, there is a movement that transforms a into b and that leaves one point of the straight line ab fixed.

XIII. If a, b and c are three non-collinear points, there is a movement that leaves a and b fixed and transforms c into another point of the plane abc.

XIV. If a, b and c are three non-collinear points, and if d and e are points of the plane abc common to the spheres c_a and c_b, and different from c, then d and e coincide.

XV. If a, b and c are distinct non-collinear points, there exists at least one point outside the plane abc.

XVI. If a, b, c and d are four non-complanar points, there exists a movement that leaves a and b fixed and transforms d into a point of the plane abc.

XVII. If a, b, c and d are four distinct collinear points, the point d cannot be upon only one of the segments ab, ac, bc.

XVIII. If a, b and c are three collinear points, and if c is between a and b, no point can be at once between a and c and between b and c.

XIX. If a, b and c are three non-collinear points, every straight line of the plane abc that has a point in the segment ab has a point in the segment ac or in the segment bc, or it contains one of the points a, b, c.

XX. If k is a class of points in the segment ab, there exists in the segment, or coincides with b, a point x, such that no point of k is between x and b, and that for every point y between a and x there is a point k between y and x or coincident with x.

Two figures (classes of points) coincide when and only when they are composed of the same points. IV means that a movement is a one-to-one relation between two figures. The movements μ and u (V) are each the other's converse; they are mutually converse biuniform relations. By VI the relative product of the movements μ and is a movement. The relative product μu leaves every point fixed, or, as we say, transforms all points each into itself. In contradistinction from such movements, others are described as effective. VII provides for rotation of a figure about two of its points. A straight line ab is defined to be the class of all points that remain fixed in case of every movement leaving a and b fixed. It is a matter of proof that a straight line is determined by any two distinct points of it. VIII is not valid in space of four or more dimensions, and hence no special postulate restricting our geometry to three dimensions is necessary. It is readily proved that any movement whatever transforms any and every triplet of collinear points into such a triplet; in other words, a movement is a collineation. By plane abc is meant the figure composed of the points of the lines joining a to points of bc, or b to points of ac, or c to points of ab, it being assumed that a, b and c are non-collinear points. It is a theorem that every movement converts a plane into a plane. Postulate IX is necessary to prove that a plane is determined by any three non-collinear points of it.

By the sphere $\mathbf{b}_a$ is meant the class of points such that for each of them there is a movement transforming it into b while leaving a fixed. The point a is the center of the sphere. It is demonstrable that every movement transforms spheres into spheres; that any movement that leaves the center of a sphere fixed transforms the sphere into itself; and that, if two spheres have but one common point, that point is collinear with the centers of the spheres. X, XI, and XII provide for transforming a line into itself; and XIII and XIV make the like provision for the plane. A circle is the logical product of a sphere and a plane containing its center. The center of the circle is that of the sphere. The notion of perpendicularity is introduced by the definition: the pair (a, c) of points is said to be perpendicular to the pair (a, b) when and only when there is a movement that leaves a and b fixed and transforms c into another point of the straight line ac. The notion is readily extensible to straight lines. XV provides for a plurality of planes, and XVI for the transformation of one plane upon another. The notion of equidistance is introduced by the definition: a point a is equidistant from two points b and c when and only when it is the center of a sphere containing b and c. It is demonstrable that, in a plane containing the distinct points a and b, the class of points equidistant from a and b is the straight line perpendicular to the straight line ab and containing the mid-point of the segment ab; that a straight line perpendicular to two straight lines ab and ac is perpendicular to every straight line that contains a and is contained in the plane abc; and other theorems respecting perpendicularity are readily proved.

A point is interior to a sphere if it is the mid-point of two distinct points of the sphere. If not, it is exterior, or else is a point of the sphere. A point of a plane containing a circle is interior or exterior to the circle according as it is interior or exterior to the sphere having the same center as the circle and containing the circle. A sphere having for center the mid-point of two points a and b, and containing them, is called the polar sphere of the points a and b. The notion between is introduced by the definition: a point x is between points a and b if it is contained in the straight line ab and is interior to the polar sphere of a and b. The class of points between two points a and b is named segment ab. The segment ab is less than the segment cd when and only when there exists a movement that transforms a into c and b into a point between c and d. Two segments (or other figures) are congruent if there exists a movement transforming one of them into the other. It is demonstrable that if two segments are not congruent, one of them is less than the other. The notion angle is defined and to it are extended the ideas of less than and congruence. If a, b and c are non-collinear points, the triangle abc is the figure composed of the points of the segments, each joining a and a point of the segment bc. The three theorems regarding congruence are proved; and so on and on. By XX, which provides for continuity, is deduced the Archimedian "axiom" as a theorem. Thence follows the idea of measurability of segments.

General Remarks.—No geometry involves ideas not found in logic or definable in terms of logical constants, and no geometry contains other undemonstrated propositions than the primitive propositions of logic. The name point is merely that of a class of things (if there be such things) that satisfy a certain set of postulates, but geometry does not assert the actual existence of any such class and does not assert the truth of the postulates. What it does assert is that, if such a class exists, then such and such a body of theorems are valid regarding the class. Geometry is thus a body of implications. It says merely "if so and so, then so and so." This important fact is somewhat disguised by the categorical form in which postulates are often stated.

Bibliography.—Instead of giving a list of the works constituting the vast and rapidly growing modern literature dealing with the foundations of mathematics in general, with the foundations of special branches, and with modern logic, it will be sufficient to refer the reader to Russell's 'Principles of Mathematics,' Vol. I (Cambridge, University Press) and to Couturat's 'Les Principes des Mathématiques' (Paris, Félix Alcan) and 'Traité de Logistique' (Alcan), wherein nearly all the important works are cited in connections showing the bearings of them. Most of the works are too technical for the general reader, who will naturally begin with the mentioned treatises of Russell and Couturat, extending his reading gradually according to increasing ability and interest.

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MATHEMATICAL APPLICATIONS

Dr. FRANZ BELLINGER

MATHEMATICAL APPLICATIONS

CHAPTER I

EARLY NON-MECHANICAL APPLICATIONS

Modern life, as a whole, lies under a debt to Mathematics far beyond calculation. Science has shown many underlying principles which govern matter, life and mind in their several environments and in their relation each to each, but it has required the mathematical faculty and the mathematical knowledge to transpose those principles into productive value. Mathematics may be termed the spirit, practical application the flesh, of a single and indivisible entity. Hence the term "applied mathematics" is to be used with caution, since it is inherent in the nature of mathematics that it shall not be divorced from any of its subsidiary uses, but remain as a vigorously vital and governing law.

Mechanical principles, for example, are mainly mathematical deductions from principles enunciated by 'Pure Science,' even as that same Pure Science finds itself dependent upon mathematical expression for the enunciation of those principles. The very words that are spoken or written bear a definite relation each with the other, and no more mathematical concept than 'Relation' could well be thought of. From the abstruse and remote questions of the affirmation of a stellar parallax in Astronomy to the 'multiplication' of yeast cells in making a loaf of bread, from the lofty flights into the regions of the mathematically infinite to the counting of change over a counter, Mathematics is applied and practical. It does not always appear mechanical, because it has not always been transliterated into such forms, and these non-mechanical applications existed in antiquity as they do now. Applied Mathematics, in that sense, is as old as Applied Thought, and Applied Thought is coeval with Man.

"To think aright," says Prof. Cassius Keyser in an illuminative recent lecture on 'Mathematics,' "is no characteristic striving of a class of men; it is a common aspiration; and Mechanics, Mathematical Physics, Mathematical Astronomy, and the other chief 'Anwendungsgebiete' (spheres of application) of mathematics, as Geodesy, Geophysics, and Engineering in its various branches, are all of them but so many witnesses to the truth of Riemann's saying that 'Natural science is the attempt to comprehend nature by means of exact concepts.' A gas molecule regarded as a minute sphere or other geometric form, however complicate; stars and planets conceived as ellipsoids or as points, and their orbits as loci; time and space, mass and motion and impenetrability; velocity, acceleration and energy; the concepts of norm and average—what are these but mathematical notions? And the wondrous garment woven of them in the loom of logic—what is that but mathematics?

"Indeed, every branch of so-called applied mathematics is a mixed doctrine, being thoroly analyzable into two disparate parts: one of these consists of determinate concepts formally combined in accordance with the canons of logic—*i.e.,* it is mathematics and not natural science viewed as matter of observation and experiment—the other is such matter, and is natural science in that conception of it, and not mathematics. No fiber of either component is a filament of the other.

"It is a fundamental error to regard the term Mathematicization of thought as the importation of a tool into a foreign workshop. It does not signify the transition of mathematics conceived as a thing accomplished over into some outlying domain like physics, for example. Its significance is different radically, far deeper and far wider. It means the growth of mathematics itself, its extension and development from within; it signifies the continuous revelation, the endlessly progressive coming into view, of the static universe of logic; or, to put it dynamically, it means the evolution of intellect, the upward striving and aspiration of thought everywhere, to the level of cogency, precision and exactitude.

"It is the aggregate of things thinkable logically that constitutes the mathematician's universe, and it is inconceivably richer in mathetic content than can be any outer world of sense, such as the physical universe according to which we chance to have our physical being."

The term 'practical,' in its common acceptation, often denotes shorter methods of obtaining results than are indicated by science. It implies a substitution of natural sagacity and mother wit for the results of hard study and laborious effort. It implies the use of knowledge before it is acquired—the substitution of the results of mere experiment for the deductions of science, and the placing of empiricism above philosophy. But if to "practical" be given its true and right signification, then it becomes a word of real import and definite value. In its right sense it denotes the best means of making the true ideal the actual; that is, of applying the principles of science in all the practical business of life and of bodying forth in material form the conception of taste and genius.

Beyond the obvious application of simple and known principles, the whole problem of the practical lies in the measurement, modification and best uses of the forces of nature. The uses and applications of these must be fashioned according to certain forms indicated by scientific formulæ. These formulæ are constructed from the laws which regulate the cohesion of the particles of the substance employed—the nature of the force to be applied

—the amount of that force and the ultimate end to be attained. All these fixed laws of force—all their combinations—and all the forms of the material employed in using them for practical purposes can only be reached through the processes and language of mathematics.

The language of Geometry and Number furnished the architect with all the signs and instruments of thought necessary to a perfect ideal of his work before he took the first step in its execution. It also enabled him, by drawings and figures, so to direct the hand of labor as to form the actual after its pattern—the ideal. The various parts may be constructed by different mechanics, at different places, but the law of science is so certain that every part will have its right dimensions, and when all are put together they form a perfect whole.

The influence of mathematical investigations on physical theories is not restricted to any single stage, but makes itself apparent throughout the whole course of their evolution. Numbers form the connecting link between theory and verification, and they always imply mathematical formulæ, however simple these may be.

There seems to be historical evidence that a practical acquaintance with certain rules of number and form was acquired by ancient peoples, especially by the Egyptians, before there was any knowledge of mathematics as a pure science. In Babylonia geometrical figures were used in augury. Herodotus, Plato and Strabo ascribe the origin of geometry to the changes which annually took place from the inundation of the Nile, and to the consequent necessity of settling disputes as to the extent of property, and of determining the tax due to the government. There was a well-developed system of mensuration in the time of the traditional biblical Joseph; and besides the extraordinary mechanical ability of the Egyptians in handling stone, they were able to construct accurately leveled canals, to ascertain the various elevations of the country, and, tradition says, to deflect the course of the Nile.

At the time of King Menes, who is supposed to have performed this extraordinary feat, dykes had been built and sluices invented, with all the mechanism pertaining to them. The water supply into plains of various levels was regulated, and a report was made of the exact quantity of land irrigated, the depth of the water, and the time it remained upon the surface. All this required much mathematical skill, and it was not likely to be carelessly carried on, since the amount of taxes and the price of provisions for the ensuing year were ascertained at the time of the inundation. Nilometers—instruments for measuring the gradual rise or fall of the river—were in use in various parts of Egypt as early as the twelfth dynasty.

"The employment of squared granite block and the beauty of the masonry of the interior of the Pyramids," says Geo. Rawlinson, "which has not been surpassed, if even equaled, at any subsequent age, also prove the degree of skill the Egyptians had reached at a time long anterior to the rudest attempts at masonry in Italy or Greece. We may well conclude that the principles of construction were known to them, as well as the engineering skill required for changing the course of the Nile, even before the reign of Menes."

The immense weight of the blocks of stone used in building shows that the Egyptians were well acquainted with mechanical powers and a method of applying force with wonderful success. The largest obelisk in Egypt is calculated to weigh about 297 tons, is more than 70 feet in height, and was carried 138 miles from the quarry.

The Egyptians could not only move immense weights; they could erect obelisks, lift large stones to a considerable height and adjust them with the utmost precision; and this sometimes in spaces that would not admit the introduction of the inclined plane.

Pliny mentions that one obelisk, built by Rameses, was 99 feet in height. He adds: "And, fearing lest the engineer should not take sufficient care to proportion the power of the machinery to the weight he had to raise, he ordered his son to be bound to the apex, more effectually to guarantee the safety of the monument."

Of the science of arithmetic the Egyptians early were in need, both in their domestic economy and in the application of geometrical theorems; but its greatest utility was in the cultivation of astronomical studies. Indeed, mathematics was the handmaid of astronomy among the Assyrians, Babylonians and Egyptians. An ancient writer says: "The orders and motions of the stars are observed at least as industriously by the Egyptians as by any people whatever; and they keep a record of the motions of each for an incredible number of years, the study of this science having been, from remotest times, an object of national ambition with them."

There is record in Egypt of the solid contents of barns before the calculation of areas. In the papyrus of Ahmes, reaching back to about 2500 B.C., there are problems relating to the pyramids which disclose some knowledge not only of geometrical figures but the principles of proportion, and possibly trigonometry. Cantor is of the opinion that the Egyptians were familiar with the properties of the right triangle in case of sides with the ratio 3:4:5 as early as 2000 B.C. This opinion is based on the orientation of the temples and early records of the "rope-stretching" method of laying out the land.

The Arabs developed the notion of "specific gravity," and gave experimental methods for its determination. Al Biruni used for this purpose a vessel with a spout slanting downward. It was filled with water up to the spout, then the solid was immersed, and the weight of the overflow determined. This, together with the weight of the solid in air, yielded the specific gravity. Al Khazini, in his Book of the Balance of Wisdom, written 1137 B.C.,

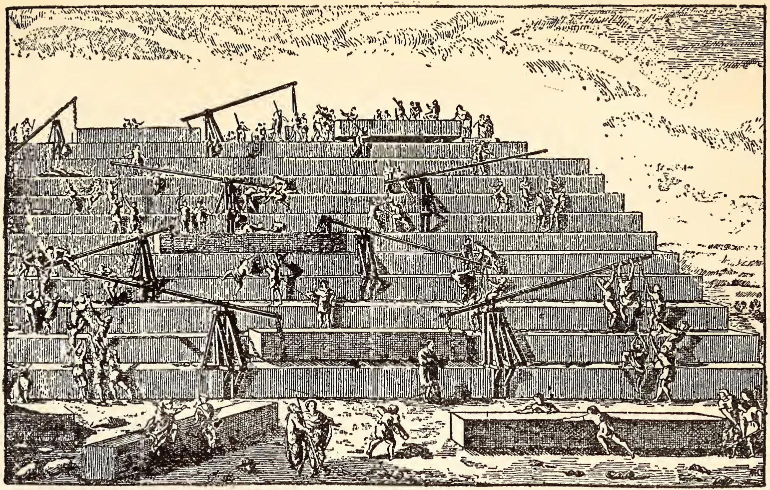


Fig. 1 —Use of Lever in Building Pyramids.

describes a curious beam balance, with five pans, for weighing in air and in water. One pan was movable along the graduated beam. He points out that air, too, must exert a buoyant force, causing bodies to weigh less.

Thales, in his pyramid and ship measurements, was probably the first to apply theoretical geometry to practical uses. He was able to predict an eclipse of the sun in 585 B.C., and several practical applications of geometry are attributed to him. But the illustrious name among the Greeks, in respect to both mathematical and mechanical science, is that of Archimedes. The most important services of Archimedes were rendered in the science of pure mathematics, but his popular fame rests chiefly on his application of mathematical theory tc mechanics.

Heron of Alexandria, called Heron the Elder, was a mathematician and also a practical surveyor who lived in the second century B.C. His teacher, Ctesibus, was celebrated for his mechanical inventions, such as the water-clock, the hydraulic organ and catapult. Heron himself was the inventor of the æolipile, which contains the germ of the steam-engine, and a curious mechanism known as "Heron's Fountain."

It is, however, in architecture that the Greeks and Romans made the most marked advance upon the achievements of the Egyptians, mechanically as well as artistically. The three principles of the beam, the arch and the truss were known to the Greeks and Romans; indeed, it is the opinion of H. W. Desmond that they possessed all the technical knowledge of the medieval builders. It is evident that they adopted from the Egyptians whatever they needed. The construction of the arch dates from an early period. Mathematical skill is a great factor in the development of architecture; the very term implies tools and force at command and instruments for supplementing the labor of the hands. The draftsman, in designing a structure, should be conversant not only with the nature of his material, but also with the forces to which it is to be subjected, their magnitude, direction, points of application and their effects. The ancient Romans not only constructed arches, but the largest domes of brick now in existence. These structures rest on all sides of the space to be covered, but there is also the simple or wagon-head vault, which rests on only two sides of the covered rectangle, leaving the other two free from all pressure. Further than this, the Romans invented that highly ingenious contrivance, the cross-vault, which exerts its whole pressure solely on the angles of the apartment, leaving all the sides free.

The origin of this construction is simply the crossing of two vaulted passages lying at right angles to each other and each corridor required to be left perfectly free. The crossway is covered by a ceiling that rests solely on the four angles or corners; the elliptic lines that form the internal ridges, called groins, can support not only themselves but the whole of the upper ceiling. The beauty and advantages of this kind of vaulting led the Romans to use it not only over crossways, but over corridors and long apartments with a boldness of construction that has never been equaled.

With the decline of Roman power this art of vaulting was lost, and for centuries the basilicas of Italy and the churches of all Roman Christendom remained with nothing but timber roofs. The Byzantine Greeks, however, retained or else reinvented another mode of vaulting possessing many of the advantages of groining, but not all of them. This system depended on two simple geometrical principles: First, that every section of a sphere by a plane is circular, and, second, that every intersection of two spheres is a plane curve and therefore circular.

The Greek vaulting then consists wholly of spherical surfaces. A hemispherical dome may be supposed whose base circumscribes the plane of any apartment or compartment, square, rectangular, triangular or polygonal.

Imagine the sides of this plane continued upward, as vertical planes, till they meet the hemispheric surface. This meeting line must in every case be a semi-circle and may therefore be made an open arch, and the portions of the dome thus cut off from every side of its base may be omitted altogether, provided their office as buttresses to the

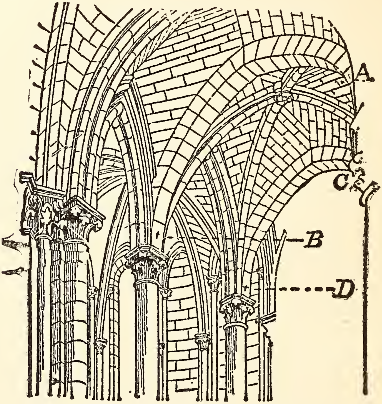


Fig. 2 —Groin Device to Support Weight of Building.

remaining portion above be replaced by the pressure of some other vault, which may be of any kind, if it be applied against the semi-circular arch. Hence no walls are required on the sides of the supposed compartment, all the weight of the pendentive dome, as it is called, being thrown on the angles of its plane. Thus this dome serves for covering an open crossway and is so applied at the mosque of St. Sophia, at Constantinople. The covered crossway, a 115-foot square, might well be esteemed, in the barbarous age of its erection, a wonder of the world.

The same idea repeated without end—the same sprouting of domes out of domes, continues to characterize the Byzantine style, both in Greek churches and Turkish mosques, down to the present day. Hope describes them as a congeries of globes of various sizes growing one out of another. This system of vaulting has been adopted by two great modern architects, Sir Christopher Wren at St. Paul's, in London, and by Soufflot at Ste. Geneviève, Paris; by the former with great success and in both made to harmonize well with the Roman style.

There is no more striking and beautiful example of the application of mathematical principles to practical affairs than in the history of architecture. The close reasoning of the mathematician has been behind and above the work of the draftsman and artisan; his imagination has reached out boldly to the projection of new designs, restrained always by the immutable laws of science; his achievement it is to unite strength and durability with beauty and geometric truth with grandeur.

"In architecture," says Fergusson, "there is still to be taken into consideration not only that subtler and complexer force, the personal genius of the architect, but also the native genius of the people in which he is a sharer, that spirituality or temper of mind which is obvious enough in its stronger manifestations." Thus the nations that showed a talent for mathematics were building nations, since here was a science which could be definitely and immediately applied to practical use.

It is necessary to discharge from the mind many unconsciously implied conditions before an exact picture of the 'pre-mechanical age' can be gained. All the raw material of mechanical science was at hand, as much before as after the magical words of Newton or of Helmholtz, but mathematical genius had not yet touched the spring which dissipated the inertia of established habit.

But before the civilized world could be transformed from a world utilizing, as one might say, only the more obvious natural forces to a world filled with devices for multiplying hands and feet, for increasing the value of eye and ear, a news-gathering world where oceans are neighborly high roads and warfare a contest of scientific equipment—before this transformation could happen the mathematician had need to direct his analytic and speculative powers to the natural phenomena of the universe.

Concerning this stage of the development of mathematics, Cassius J. Keyser writes: "A traditional conception, still current everywhere except in critical circles, has held mathematics to be the science of quantity or magnitude, where magnitude, including multitude (with its correlate of number) as a special kind, signified whatever was 'capable of increase and decrease and measurement.' Measurability was the essential thing. That definition of the science was a very natural one, for magnitude did appear to be a singularly fundamental notion, not only inviting but demanding consideration at every stage and turn of life. The necessity of finding out how many and how much was the mother of counting and measurement; and mathematics, first from necessity and then from pure curiosity and joy, so occupied itself with these things that they came to seem its whole employment.

"Indeed, for direct beholding, for immediate discerning of the things of mathematics there is none other light but one—namely, psychic illumination—but mediately and indirectly they are often revealed or at all events hinted by their sensuous counterparts, by indications within the radiance of day, and it is a great mistake to suppose that the mathetic spirit elects as its agents those who, having eyes, yet see not the things that disclose themselves in solar light. To facilitate eyeless observation of his sense-transcending world the mathematician invokes the aid of physical diagrams and physical symbols in endless variety and combination; the logos is thus drawn into a kind of diagrammatic and symbolical incarnation, gets itself externalized, made flesh, so to speak; and it is by attentive physical observation of this embodiment, by scrutinizing the physical frame and make-up of his diagrams, equations and formulæ, by experimental substitutions in and transformations of them, by noting what emerges as essential and what as accidental, the things that vanish and those that do not, the things that vary and the things that abide unchanged as the transformations proceed and trains of algebraic evolution unfold themselves to view—it is thus, by the laboratory method, by trial and by watching that often the mathematician gains his best insight into the constitution of the invisible world thus depicted by visible symbols.

"Indeed, the time is at hand when at least the academic mind should discharge its traditional fallacies regarding the nature of mathematics and thus in a measure promote the emancipation of criticism from inherited delusions respecting the kind of activity in which the life of the science consists. Mathematics is no more the art of reckoning and computation than architecture is the art of making bricks or hewing wood, no more than painting is the art of mixing colors on a palette, no more than the science of geology is the art of breaking rocks or the science of anatomy the art of butchering.

"Pernicious, because deeply embedded and persistent, is the fallacy that the mathematician's mind is but a syllogistic mill and that his life resolves itself into a weary repetion of A is B, B is C, therefore A is C, and Q.E.D. That fallacy is the 'Carthago delenda' of regnant methodology. Reasoning, indeed, in the sense of compounding propositions into formal arguments, is of great importance at every stage and turn, as in the deduction of consequences, in the testing of hypotheses, in the detection of error, in purging out the dross from crude material, in chastening the deliverances of intuition, and especially in the final stages of a growing doctrine in welding together and concatenating the various parts into a compact and coherent whole. But, indispensable in all such ways as syllogistic undoubtedly is, it is of minor importance and minor difficulty compared with the supreme matters of Invention and Construction.

"When the late Sophus Lie, great comparative anatomist of geometric theories, creator of the doctrines of Contact Transformations and Infinite Continuous Groups and revolutionizer of the Theory of Differential Equations, was asked to name the characteristic endowment of the mathematician, his answer was the following quaternion: 'Phantasie,' 'Energie,' 'Selbstvertrauen,' 'Selbstkritik.' Not a word, you observe, about ratiocination. Phantasie, not merely the fine frenzied fancy that gives to airy nothings a local habitation and a name, but the creative imagination that conceives ordered realms and lawful worlds in which our own universe is as but a point of light in a shining sky; Energie, not merely endurance and doggedness, not persistence merely, but mental vis viva, the kinetic, plunging, penetrating power of intellect; Selbstvertrauen and Selbstkritik, self-confidence aware of its ground, deepened by achievement and reinforced until in men like Richard Dedekind, Bernhard Bolzano and especially Georg Cantor it attains to a spiritual boldness that dares leap from the island shore of the Finite over into the all-surrounding boundless ocean of Infinitude itself, and thence brings back the gladdening news that the shoreless vast of Transfinite Being differs in its logical structure from that of our island home only in owning the reign of more generic law."

CHAPTER II

CHRONOLOGY AND HOROLOGY

ALTHO the ancients gave so much of their attention to understanding and recording the facts of astronomy, yet there was very little systematic attention given to the computation of time or to the chronological aspect of history. Chronology is comparatively a modern science, yet a highly important one. Accurate chronology is essential to all reasoning from historical facts; the mutual dependence and relations of events cannot be traced without it; with great propriety it has been called one of the eyes of history, while geography with equal propriety has been said to be the other.

Present acquaintance with the truths of astronomy would have been as deep had Eastern philosophers never turned their eyes to the realms of space or watched the harmonious movements of the worlds in the firmament above. "The moment," says Sir John Herschel, "astronomy became a branch of mechanics, a science essentially experimental—that is to say, one in which any principle laid down can be subjected to immediate and decisive trial and where the experience does not require to be waited for—its progress acquired a tenfold acceleration; nay, to such a degree that were the results of all the observations from the earliest ages annihilated, leaving only those made in Greenwich Observatory during the single lifetime of Maskelyne, the whole of this most perfect of sciences might, from those data and as to the objects included in them, be at once reconstructed and appear precisely as it stood at their conclusion. The operation, indeed, of Arabian knowledge of astronomy in the early ages was perhaps principally to lend a plausibility to astrology; the observers of stars, like Columbus predicting the eclipse, had the power of astonishing when they prepared to delude."

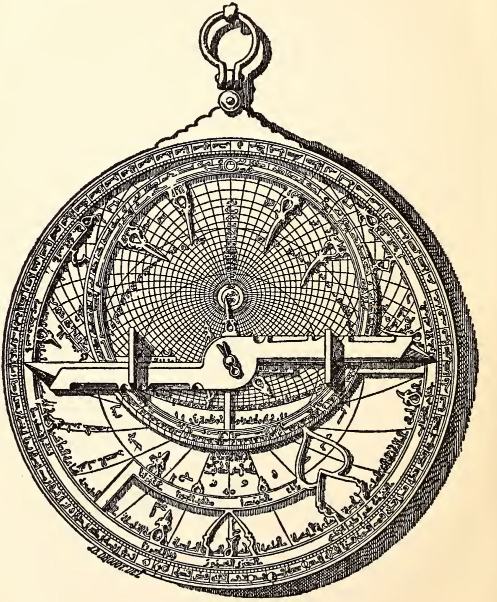


Fig. 3 —Arabic Astrolabe.

The most obvious measures and divisions of time are those suggested to all men by the revolutions of the heavenly bodies. These are three—days, months and years; the day from the revolution of the earth on her axis or the apparent revolution of the sun around the earth; the month from the periodical changes in the moon; the year from the annual motion of the earth in her orbit round the sun. These three divisions are not commensurate, and this has caused the chief embarrassment in the science of chronology; it has, in point of fact, been difficult so to adjust them with each other in a system of measurement as to have the computed time and the actual time perfectly in agreement or coincidence.

The day was undoubtedly the earliest division and originally was distinguished, it is likely, from the night and extended from sunrise to sunset. It was afterward considered as including also the night and was marked as the time from sunrise to sunrise. But the beginning of the day has been reckoned differently by different nations for civil purposes; at sunrise by the Babylonians, Persians, Syrians and inhabitants of India; at sunset by the Jews, Athenians, ancient Gauls and Chinese; at midnight by the Egyptians, Romans and moderns generally. Astronomers in their calculations consider the day as beginning at noon, after the manner of the Arabians.

There have also been various modes of subdividing the day. The division of time into hours is very ancient, the oldest hour being the twelfth part of a day. Herodotus observes that the Greeks learned from the Egyptians, among other things, the method of dividing the day into twelve parts, and the "astronomers of Cathaya" still retained that method at the time of Herodotus. The division of the day into twenty-four hours was not known to the Romans before the Punic War.

The Greeks, in the time of Homer, seem not to have used the division into hours; his poems present the more obvious parts of the day, morning, noon and evening. But before the time of Herodotus they were accustomed to the division of the day and of the night also into twelve parts. They were acquainted also with the division of the day and night into four parts each, according to the Jewish and Roman custom.

The Romans subdivided the day and night each into four parts, which were called vigils or watches. They also considered the day and night as each divided into twelve hours; three hours, of course, were included in a vigil. The day vigils were designated simply by the numerals first, second, third, fourth, but as the second vigil commenced with the third hour, the third vigil with the sixth hour and the fourth with the ninth hour, the terms first, third, sixth and ninth are also used to signify the four vigils of the day. The night vigils were designated by the names vesper, evening, midnight and cockcrow. The first hour of the day began with sunrise and the twelfth ended at sunset; the first hour of the night began at sunset and the twelfth ended at sunrise. Of course, therefore, the hours of the day in summer were longer than those of the night and in the winter they were shorter.

The division of time into months, without much doubt, had its origin in the various phases or changes of the moon. It included the time of the moon's revolution round the earth, or between two new moons, or two successive conjunctions of the sun and moon. The mean period is 29 days, 12 hours, 44 minutes. It was considered to be 29½ days, and the ancients commonly reckoned the month as consisting alternately of 29 and 30 days.

The Greeks thus reckoned their months. Twelve lunations so computed formed the year, but it fell short of the true solar year by about 11¼ days, making in four years about 45 days. To reconcile this and bring the computation by months and years to coincide more exactly another month was intercalated every two years; in the first two years a month of 22 days and in the next two a month of 23 days. Thus, after a period of four years, the lunar and solar years would begin together. But the effect of this system was to change the place of the months relatively to the seasons, and another system was adopted. This was based on the supposition that the solar year was 365¼ days, while the lunar was 354, which would in a period of eight years give a difference of 90 days. The adjustment was made by intercalating, in the course of the period, three months of 30 days each. Its invention was attributed to Cleostratus of Tenedos; it was universally adopted and was followed in civil matters even after the more perfect cycle of Meton was known.

With the Romans the case was somewhat different. Under Romulus they are said to have had only ten months, but Numa introduced the division into twelve, according to that of the Greeks. But, as has been seen, this formed only a lunar year, a little more than eleven days short of the solar year; therefore an "extraordinary month" was to be inserted every other year. The intercalating of this and the whole charge of dividing the year was intrusted to the Pontifices, and they managed, by inserting more or fewer days, to make the current year longer or shorter, as they for any reason might choose. This finally caused the months to be transposed from their stated seasons, so that the winter months were carried back into autumn and the autumnal into summer. Julius Cæsar put an end to this disorder by abolishing the intercalation of months and by adopting a system which was available by the more accurate division of the year.

A consideration of the division of the year takes the historian back into the twilight of history. It is well known that the Babylonians had a system of notation called the sexagesimal, which reveals a high degree of mathematical insight. It was used chiefly in the construction of a system of weights and measures and reveals some knowledge of geometrical progressions, but the indications are that it was in the possession of few and was used but little. The base of this system was the number 60. The Babylonians reckoned the year at 360 days.

The Grecian year, however, which was established by Solon and continued to the time of Meton and even after, consisted of 365¼ days. This division was probably not formed until considerable advance had been made in astronomical science, and it was long after its first adoption before it attained to anything like an accurate form.

The Roman year seems to have consisted of 365 days until the time of Julius Cæsar, who attempted to remedy the confusion resulting from the method employed by the Romans to adjust their computations by lunar months to the solar year. Cæsar instituted a year of 365 days and 6 hours. To remove the error of 80 days, which computed time had gained of actual time, he ordered one year of 445 days, which was called the Year of Confusion. To secure a proper allowance for the six hours which had been disregarded, but which would amount in four years to a day, he directed that one additional day should be intercalated in the reckoning of every fourth year. Thus each fourth year should have 366 days, the others 365. This is called the Julian year and begins to show some of the familiar landmarks of modern chronology. But even in the plan of the great Julius there was still a fault, owing to an error in computed time. The extra day was intercalated too soon—that is, computed time, instead of gaining six hours a year as was supposed, gained only 5 hours, 48 minutes and 57 seconds, so that a whole day was not gained in four years. The intercalated day was inserted too soon by 44 minutes and 12 seconds, and of course computed time by this plan lost 44 minutes and 12 seconds every four years or 11 minutes and 3 seconds every year. In 131 years this makes a loss of computed time of one day or computed time would be one day behind actual time. In 1582 A.D. this loss had amounted to ten days, and Pope Gregory XIII. attempted to remedy the evil by a new expedient. This was to drop the intercalary day every hundredth year except the four hundredth. The Gregorian year was immediately adopted in Spain, Portugal and Italy and during the same year in France, in Catholic Germany in 1583, in Protestant Germany and Denmark in 1700, in Sweden in 1753. In England it was adopted in 1752 by act of Parliament directing the 3d of September to be styled the 14th, as computed time had lost eleven days. This was called the change from Old to New Style. The Julian calendar, or Old Style, is still retained in Russia and Greece, whose dates consequently are now 12 days in arrear of those of other countries of the western hemisphere. It is also retained in the Greek and Armenian churches.

Different nations have begun the year at different seasons or months. The Romans at one time considered it as beginning in March, but afterward in January. The Greeks placed its commencement at the summer solstice. The Christian clergy used to begin it at the 25th of March, and this style was practiced in England and in the American colonies until 1752 A.D., on the change from Old Style to New, when the 1st of January was adopted.

In adjusting the different methods of computing time, or the division of time into days, months and years, great advantage is derived from the invention of cycles. These are periods of time so denominated from the Greek word meaning a circle, because in their compass a certain revolution is completed. Under the term cycle may be included the Grecian Olympiad, a period of four years; the Octæteris, or period of eight years; the Roman Lustrum, a period of five years, and also the Julian year, or period of four years. The period of 400 years, comprehended in the system of Gregory, may justly be termed the Cycle of Gregory. Besides these, there are the Lunar Cycle, the Solar Cycle, the Cycle of Indiction and the Julian Period.

The Lunar Cycle is a period of 19 years. Its object is to accommodate the computation of time by the moon to the computation by the sun, or to adjust the solar and lunar years. The nearest division of the year by months is into twelve, but twelve lunations fall short of the solar year by about eleven days. Of course, every change in the moon in any year will occur eleven days earlier than it did in the preceding year, but at the expiration of nineteen years they occur again nearly at the same time. This cycle was invented by Meton, an Athenian astronomer who lived about 430 B.C. The improvement was at the time received with universal approbation, but not being perfectly accurate it was afterward corrected by Eudoxus and subsequently by Calippus. The Cycle of Meton was employed by the Greeks to settle the time of their festivals, and the use of it was discontinued when these festivals ceased to be celebrated. The Council of Nice, however, wishing to establish some method for adjusting the new and full moons to the course of the sun, with a view to determining the time for Easter, adopted again the Meton Cycle, and from its great utility they caused the numbers of it to be written on the calendar in golden letters, which has obtained for it the name of the Golden Number. This name is still applied to the current year of the Lunar Cycle and is always given in the almanacs.

The Solar Cycle is a period of 28 years. Its use is to adjust the days of the week to the days of the month and the year. As the year consists of 52 weeks and 1 day, it is plain that it must begin and end on the same day, and if 52 weeks and 1 day were the exact year, or if there were no leap year, the year would, after seven years, begin again on the same day. But the leap year, consisting of 52 weeks and 2 days, interrupts the regular succession every fourth year, and the return to the same day of the week is not effected until four times seven, or twenty-eight years. This cycle is employed particularly to furnish a rule for finding Sunday or to ascertain the Dominical Letter. Chronologers employ the first seven letters of the alphabet to designate the seven days of the week, and the Dominical Letter for any year is the letter which represents Sunday for that year. Tables are given for the purpose of finding it in chronological and astronomical books.

The Cycle of Indiction is a period of 15 years. The origin and primary use of this has been the subject of various conjectures and discussions. It seems to have been established by Constantine the Great, in the fourth century, as a period at the end of which a certain tribute should be paid by the different provinces of the empire. Public acts of the emperors were afterward dated by the years of this cycle.

The Paschal Cycle is a period of 532 years, after which Easter falls on the same day of the year.

The cycle which has been perhaps most celebrated is that termed the Julian Period and was invented by Joseph Scaliger. Its object was to furnish a common language for chronologers by forming a series of years, some term of which should be fixed, and to which the various modes of reckoning might be easily applied. To accomplish this he combined three cycles of the moon, sun and indiction, multiplying 19, 28 and 15 into one another, which produces 7,980, after which all three cycles will return in the same order, every year taking again the same number of each cycle as before. This invention would be of great importance if there was no universally acknowledged epoch, or fixed year, from which to compute, but its use is almost entirely superseded by the general adoption of the Christian era as a fixed standard.

It is essential to correct and exact chronology that there should be some fixed epoch to which all events may be referred and be measured by their distance from it. It is of comparatively little consequence what the epoch is, provided it is fixed and acknowledged, as it is perfectly easy to compute in a retrograde manner the time before it, as well as in a direct manner the time after it. The Greeks for a long time had no fixed epoch, but afterward they reckoned by Olympiads, periods of four years. These began 776 B.C. The Romans often reckoned by lustrums, often by the year of the consul or emperor. The building of the city was their grand epoch, which began 753 B.C. The present era began to be used about 360 A.D., according to some writers, but others state that it was invented by Dionysius, a monk, about 527 A.D.

The Mohammedan Era, or Hegira, was founded on the flight of Mohammed from Mecca to Medina, 622 A.D. One of the interesting vagaries of chronological history is found in the Era of the French Republic, which the revolutionists attempted to establish. This was introduced in 1793, with a formal rejection of the Sabbath and of the hebdominal week and a novel arrangement and pedantic nomenclature of the months. The 22d of September was fixed as the beginning of the year. The year consisted of twelve months of thirty days each, which were divided, not by weeks, but into three decades or periods of ten days. As this would comprise but 360 days, five were added at the close of the last month of the year, called complementary days, and at the close of every fourth year a sixth day was added, called the Day of the Republic. The cycle of the four years was termed the Franciade. This calendar was used about twelve years. The Gregorian calendar was restored on January 1, 1806.

The mechanical instruments that have been made for the measurement of time present in themselves an interesting pictorial commentary upon the more abstract science of chronology. Horology, the art of measuring the hours or any definite small portions of time, began when man first marked the shadow of any upright object and noted its movements in relation to the apparent movement of the sun. The next step came when he noted that a staff placed in the ground and pointed toward the north will always at a particular hour of the day throw a shadow in the same direction. This fact, undoubtedly observed by the Babylonians in the most ancient times, suggested the idea of the sun-dial. This instrument consists of two parts, the "gnomon," or upright staff or "style," usually a piece of metal, always placed parallel to the earth's axis and therefore pointing to the north star, and the dial, another plate of metal or stone, usually horizontal, on which are marked the directions of the shadow for the several hours, their halves and quarters and sometimes smaller divisions.

Fig. 4 —Simple Sun-dial Face, Measured for Latitude of New York.

Sun-dials were generally known in ancient times. It is suggested that the circular rows of stones built by the Druids were used to mark the sun's path and to indicate the times and seasons. Obelisks are also supposed by some writers to have been used for measuring sun shadows. The Greeks were perfectly acquainted with the method of making sun-dials with inclined styles. Small portable sun-dials were much prized before the introduction of watches, and were provided with compasses by which they could be turned round so that the style pointed to the north. Sun-dials have been found in the ruins of ancient cities

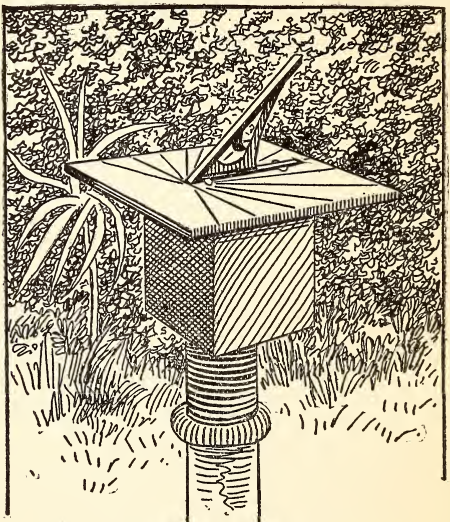


Fig. 5 —Dial Set Up with South Exposure.

of Greece, in Rome, in the excavations of Pompeii and Herculaneum, and many medieval specimens are well known.

The objections to a sun-dial are that the shadow of the style is not sufficiently well defined to give very accurate results and that refraction, which always makes the sun appear a little too high, throws the shadow a trifle toward noon at all times. That is, the time is a little too fast in the morning and too slow in the afternoon. More than that, a correction is always necessary in order to find civil, or clock, time.

The simplest form of sun-dial is the best, and as a regulator of clocks the dial is good within one or two minutes. The "noon mark" is simply a north-and-south line marked on a horizontal plane and the style is any object fixed to

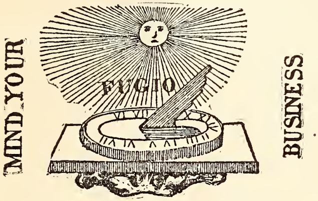


Fig. 6 —Early Currency of the United States, Showing Sun-dial.

the dial and slanted so as to point to the north pole. On four days of the year the sun is right with mean time and the shadow mark may be set on those days, or on other days the noon mark may be set by consulting the table in the almanac which shows the variation of the sun from civil time in even minutes. Thus on October 10, 1909, the noon mark could be made by the shadow of the style at 11.47 by the clock and it would be right for all time to come.

A device less dependent upon the climatic conditions was the water-clock, or clepsydra. It is said that this instrument was in use among the Chaldeans and ancient Hindus. Sextus Empiricus says that the Chaldees used such a vessel for finding their astrological data, but remarks that the unequal flowing of the water and the alterations of atmospheric temperature rendered their calculations inaccurate.

In this instrument the water, which falls drop by drop from the orifice of one vessel into another, floats a light body that marks the height of the water as it rises against a graduated scale and thus denotes the time that has elapsed. As a measure of hours of the day in countries such as Egypt, where the hours were always equal and thus where the longer days contained more hours, the water-clock was very suitable, but in Greece and Rome, where the day, whatever its length, was always divided into twelve hours, the simple water-clock was as unsuitable as a modern clock would be, for it always divided the hours equally and took no account of the fact that by such a system the hours in summer were longer than in winter.

In order, therefore, to make the water-clock available in Greece and Italy it became necessary to make the hours unequal and to arrange them to correspond with unequal hours in the Greek day. This plan was accomplished by placing a float upon the water in the vessel that measured the hours, and on the float stood a figure made of thin copper, with a wand in its hand. This wand pointed to an unequally divided scale. A separate scale was provided for every day in the year, and these scales were mounted on a drum which revolved so as to turn round once in the year. Thus as the figure rose each day by means of a cogwheel it moved the drum round one division or one-three-hundred-and-sixty-fifth part of a revolution. By this means the scale corresponding to any particular day of winter or summer was brought opposite the wand of the figure, and thus the scale of hours was kept true. In fact, the water-clock, which kept true time, was made by artificial means to keep untrue time, in order to correspond with the unequal hours of the Greek days. One of the more complicated forms of the water-clock was probably invented by Ctesibus of Alexandria. In the Athenian courts a speaker was allowed a certain number of amphoræ of water for his speech, the quantity dependent on the importance of his suit. Both the simple and more elaborate forms of clepsydræ were introduced into Rome in the second century B.C.

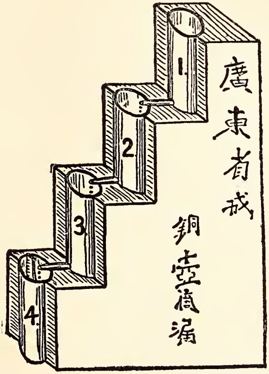


Fig. 7 —CHINESE WATER CLOCK.

second century B.C. A Chinese water-clock, reputed to be over 3,000 years old, consisted of four copper jars, on ascending steps, with small openings and filled every morning. The purpose of the series was to obviate the irregularity in dropping which would be caused by the greater weight in the first jar at the beginning of the day.

The running of fine sand from one vessel into another was found to afford a still more certain measure of time, so the hour-glass came into being. This instrument consists of two bulbs of glass united by a narrow neck; one of the bulbs is nearly filled with dry sand, fine enough to run freely through the orifice in the neck, and the quantity of sand is just as much as can run through the orifice in an hour, if the instrument is to be really an hour-glass; in a minute, if a minute-glass. It is said that King Alfred observed the lapse of time by noting the gradual shortening of a lighted candle.

The pendulum is the mechanical basis of modern clocks and was first scientifically investigated by Galileo in the latter half of the sixteenth century. The story runs that while he was praying one day in the cathedral at Pisa his attention was arrested by the motion of the great lamp which, after being lighted, had been left swinging. Galileo proceeded to time its oscillations by the only watch in his possession—namely, his own pulse. He found the times, as near as he could tell, to remain the same, even after the motion had greatly diminished. Thus was discovered the isochronism of the pendulum. Later experiments carried out by Galileo showed that the time of oscillation was independent of the mass and material of the pendulum and varied as the square root of its length.

Galileo's invention did not become generally known at that time, and fifteen years later, in 1656, Christian Huygens independently invented a pendulum clock which met with general and rapid appreciation. The honor of this invention belongs, therefore, to both Galileo and Huygens.

Wheel-work had been known long before the time of Galileo and had been skilfully applied by Archimedes. When therefore some sort of wheel mechanism was needed to keep the pendulum oscillating, the mechanical means were at hand. Galileo saw that if the pendulum could be kept swinging, a timepiece could be constructed which would be mathematically perfect. There must be some reservoir of force such that when a pendulum comes back and touches it the touch shall allow some pent-up power to escape and to drive the pendulum forward. An arrangement of this kind was contrived by Galileo. He provided a wheel with a number of pins around it. The pendulum had an arm attached to it and there was a ratchet with a projecting arm which engaged with the pins. This arrangement is called an escapement.

The type of escapement invented by Galileo was, for practical purposes, full of imperfections, and it was left for later inventors to modify his ideas and to improve on

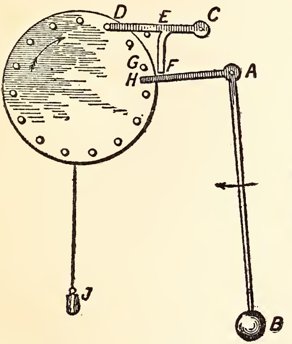


Fig. 8 —Escapement Principle.

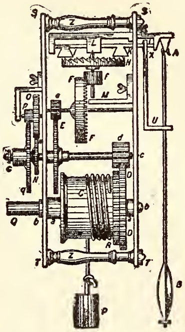


Fig. 9 —Mechanism of 'Grandfather's Clock.'

them until an accurate timepiece was achieved. The balance-wheel was invented, which does the work of the pendulum, and various escapements, such as the crown or verge escapement, the anchor-and-crutch escapement, the dead-beat escapement and the gravity escapement, have all taken their place in the development of the timepiece. The prime requisite of a good escapement is that the impulse communicated to the pendulum be invariable, notwithstanding any irregularity or foulness in the train of wheels. The compensating balance-wheel is a balance-wheel whose rim is formed of two metals of different expansive powers, so arranged that the change of size of the wheel, as the temperature rises or falls, is compensated for by the change in position of the parts of the rim.

The anchor escapement was employed in that popular and excellent timepiece used throughout the eighteenth and in the early part of the nineteenth century and now known as the Grandfather Clock. In this clock the pendulum is hung from a strip of thin steel spring, which allows it to oscillate and supports it without friction. This manner of supporting pendulums is now very much in use.

The watch differs from the original clock in that it has a vibrating wheel instead of a vibrating pendulum. As in a clock gravity is always pulling the pendulum down to the bottom of its arc, but does not fix it there, because the momentum acquired during its fall from one side carries it up to an equal height on the other, so in a watch a spring, generally spiral, surrounding the axis of the balance-wheel, is always pulling this toward a middle position of rest, but does not fix it there, because the momentum acquired during its approach to the middle position from either side carries it just as far past on the other side, and the spring has to begin its work again. The balance-wheel at each vibration allows one tooth of the adjoining wheel to pass, as the pendulum does in a clock, and the record of the beats is preserved by the wheel which follows. A main spring is used to keep up the motion of the watch, instead of the weight used in a clock, and as a spring acts equally well whatever be its position, the watch keeps time altho carried in the pocket or in a moving ship. In winding up a watch one turn of the axle on which the watch is fixed is rendered equivalent by the train of wheels to about 400 turns or beats of the balance-wheel, and thus the exertion during a few seconds of the hand which winds up gives motion for twenty-four or thirty hours.

The laws of the mechanism of the clock can easily be understood. The experiments with the pendulum and with springs revealed certain principles which were early reduced to six and can be stated thus:

(1) A harmonic motion is one in which the accelerating force increases with the distance of the body from some fixed point.

(2) Bodies moving harmonically make their swings about this point in equal times.

(3) A spring of any sort or shape always has a restitutional force proportional to the displacement,

(4) And therefore masses attached to springs vibrate in equal times, however large the vibration may be.

(5) The bob of a pendulum, oscillating backward and forward, acts like a weight under the influence of a spring and is therefore isochronous.

(6) The time of vibration of a pendulum is uninfluenced by changes in the weight of the bob, but is influenced by changes in the length of the pendulum rod. The time of vibration of a mass attached to a spring is influenced by changes in the mass.

Early attempts were made to use a pendulum clock at sea, suspending it so as to avoid disturbance to its motion by the rocking of the ship. These proved vain. It therefore became desirable that a watch with a balance-wheel should be contrived to go with a degree of accuracy in some respects comparable with the accuracy of a pendulum clock. To encourage inventors an Act of Parliament was passed in the thirteenth year of Queen Anne's reign promising a large reward to any one who would invent a method of finding the longitude at sea true to half a degree—that is, true to thirty geographical miles. If the finding of the longitude were to be accomplished by the invention of an accurate watch, then this involved the use of a watch that should not, in several months' going, have an error of more than two minutes, or the time the earth takes to turn through half a degree of longitude.

This was the problem which John Harrison, a carpenter of Yorkshire, made it his life business to solve. His efforts lasted over forty years, but at the end he succeeded in winning the prize. His instruments have been much improved by subsequent inventors and have resulted in the construction of the modern ship's chronometer, a large watch about six inches in diameter, mounted on axles, in a mahogany box. The marine chronometer differs from the ordinary watch in the principle of its escapement, which is so constructed that the balance is free from the wheels during the greater part of its vibration, and also in being fitted with a compensation adjustment similar to that in the balance-wheels of the finer clocks and watches. The balance-spring of the chronometer is helicoidal, that of the watch spiral.

One of the inventions of modern times is the pneumatic clock, which is one of a series of clocks governed by pulsations of air sent at regular intervals to them through tubes by a central clock or regulator. The movement of the central clock compresses the air in the tube and causes a bellows to expand on each dial, thus moving the hands.

Another recent invention is a clock without wheels or pendulum. It consists solely of two inclined plates with zigzag tracks and the clock framework supporting them. A perforated disk connected with the shaft which journals in the frame and two ball weights suspended in each tower and connected by means of a cord to the shaft successfully furnish the motive power. These weights are raised daily.

So the ingenuity of man goes on measuring this earthly element of time. Laplace said that "Time is to us the impression left on the memory by a series of events," and that motion, and motion only, can be used in measuring it. Thus it is motion, whether of the shadow on the grass, the dropping of water or the continuous oscillations of a swinging body, which is the necessary and unvarying element in all the measurements of time.

CHAPTER III

SURVEYING AND NAVIGATION

One of the earliest necessities of civilization was a system of ascertaining by measurement the shape and size of any portion of the earth's surface and representing the results on a reduced scale on maps. This is the surveyor's art and is supposed to have originated in Egypt, where property boundaries were annually obliterated by the inundations of the Nile. In Rome surveying was considered one of the liberal arts, and the measurement of lands was entrusted to public officers, who enjoyed certain privileges.

Julius Cæsar conceived the idea of a complete survey of the whole empire. For this purpose three geometers were employed: Theodotus, entrusted with the survey of the northern provinces; Zenodoxus, with the survey of the eastern, and Polycletus, of the southern. It is stated that a partial survey was finished 19 B.C. and the whole completed in 6 A.D. The materials collected were lodged in the public archives, receiving from time to time marks and notes to designate the various changes in the provinces. It was consulted by Pliny. The numerous changes at length required the construction of another chart with corrected measurements, which was effected about 230 A.D. under Alexander Severus. Of this chart the celebrated document Tabula Peutingerianæ is supposed by some modern critics to be an imperfect copy.

The mathematicians of the Alexandrian school made a distinct contribution to the art of surveying. Most authorities believe Heron of Alexandria to be the author of "Dioptra," tho some writers have attributed it to another mathematician of a later date by the name of Heron. "Dioptra," says Venturi, "were instruments resembling the modern theodolites. The instrument consisted of a rod, four yards long, with little plates at the end for aiming. This rested upon a circular disk. The rod could be moved horizontally and also vertically. By turning the rod around until stopped by two suitably located pins on the circular disk, the surveyor could work off a line perpendicular to a given direction. The level and plumb line were also used." Heron explains, with the aid of these instruments and of geometry, a large number of surveying problems, such as to find the distance between two points, only one of which is accessible, or between two points which are visible but both inaccessible; from a given point to run a perpendicular to a line which cannot be approached; to find the difference of level between two points, and to measure the area of a field without entering it. The "Dioptra" discloses considerable mathematical ability, but it gives rules and directions without proof.

The higher development of the art of surveying, like so many other mechanical arts depending on mathematics, is of comparatively recent date. The enormous areas of new land opened for habitation in the New World, the construction of railroads, bridges and water works have employed the keenest practical minds in solving large surveying and engineering problems, of which the Government does a large part.

Surveys may be divided into three classes: First, those made for general purposes, or information surveys, which may be exploratory, geodetic, geographic, topographic or geologic; second, those made for jurisdictional purposes, or cadastral surveys, which define political boundaries and those of private property and determine the enclosed areas; third, there are surveys made for construction purposes, or engineering surveys, on which are based estimates of the cost of public and private works, such as canals, railways, water supplies and the like and their construction and improvement.

The topographic survey, one of those in the first class, is made for military, industrial and scientific purposes. The topographic map, made directly from nature by measurements and sketches on the ground, is the mother map from which all others are derived. It shows with accuracy all the drainage, relief and cultural features which it is practicable to represent on the scale chosen. These features are numerous and important, if the government maps of the advanced modern nations are taken as a model. On the topographical maps issued by the United States Geological Survey are exhibited hydrography, or water features, such as ponds, streams, lakes and swamps; hypsography, or relief of surface, as hills, valleys and plains, and the features constructed by man, as cities, roads and villages, with the names and boundaries.

The uses of topographic maps are many. For the purposes of a national government or a State they are invaluable, as they furnish data from which may be determined the value of projects for highway improvement, for railways, for city water supply and sewerage and for the subdivision into counties, townships and the like. They serve the military department in locating encampment grounds, in planning practice or actual operations in the field and, during war, in indicating the precise situations of ravines, ditches, buildings, hills and streams. The Post-office Department utilizes them in considering all problems connected with the changing of mail routes, star routes, and especially in connection with contracts and assignments of rural free-delivery routes. In the future wooded areas are to be indicated on the United States Government maps, so that foresters will find them useful, as well as those people who are investigating mineral resources, water power and land reclamation.

The operations involved in surveying are the measurement of distances, level, horizontal, vertical and inclined, and of angles, horizontal, vertical and inclined, and the necessary drawing and computing to represent properly on paper the information obtained by the field work. If the tract to be surveyed is so large that the curvature of the earth's surface must be taken into account, it is a geodetic survey.

The practical basis of surveying is the mathematical theory of the triangle and the solution of the various problems of the triangle by means of geometrical formulæ and

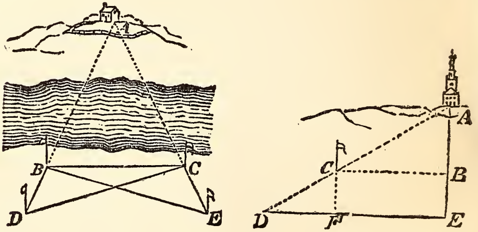
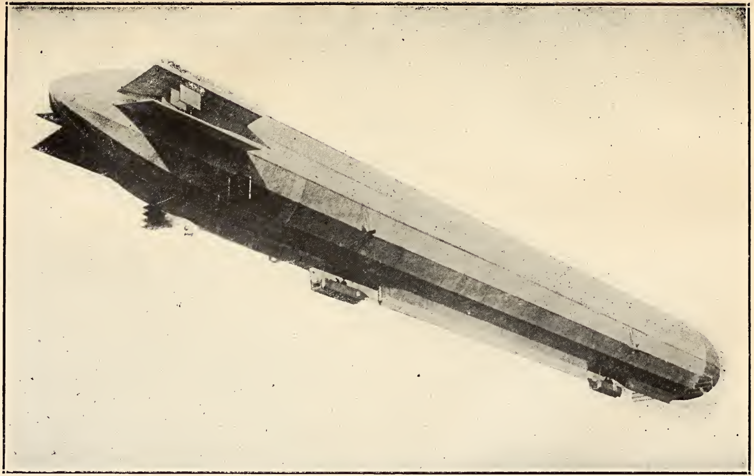


Fig. 10 —Examples of Triangulation.

logarithms. If two angles and one side of a triangle are known, the third angle and the length of the other two sides can be computed by easy geometrical rules. The use of logarithms, which are artificial numbers so devised that they shorten the processes of multiplication and division, reduces the work of computing the long tables of angles and measurements which often falls to the work of the surveyor.

Now an actual measurement of a portion of the earth's surface can be made by any one by means of a rope, a tape or a chain, thus insuring actual knowledge of the length of one side, called the base line, of the future triangle. By means of a telescope and a level, together with other in-



Zeppelin Dirigible Balloon; Pioneer of Three Long-distance Directed Flights.

genious devices, placed at the end of the base line, two objects in a given area are sighted, as, for instance, a church steeple in one direction and a signal placed at the other end of the base line. The three points are the apexes of the triangle, formed by connecting lines. The angles can be measured by the instruments at the surveyor's hand, the length of the base line is known; therefore the length of the other two sides can be computed.

This principle of triangulation has many variations, and in actual practice there are many complicating elements. The topography of an area of any size hangs, not on one, but on a system of triangles. In the preliminary work an arbitrary line, or meridian, is established, from which to compute the measurements. But if the actual position is required—that is, the location on the earth's surface according to latitude and longitude—observations of the sun or of the fixed stars must be made and the measurements recorded. The elevation of the pole measures the distance of the observer from the equator, and this distance is the latitude of a place, north or south, the pole lying midway between the highest and lowest positions of the pole star. In practice other means, not quite so accurate, but useful, may be used for determining the latitude. One of the common methods, exact enough for ordinary geographical reconnaissances, is to measure the angular altitude of the sun when on the meridian, and from this altitude, with the aid of the declination taken from the Nautical Almanac, and with correction for refraction, the latitude is obtained. This method on land requires the use of an artificial horizon in place of the natural.

But to fix the position of any place on the globe it is necessary to know at what point on the circle of latitude it lies, or its longitude. This is a more difficult matter and one that requires for its determination, astronomically, the introduction of the element of time. Strictly speaking, longitude is the angle at the pole contained between two meridians, one of which, called the prime meridian, passes through some conventional point from which the angle is measured. The longitude of the conventional point is zero, and longitudes are reckoned east and west from it to 180 degrees in arc and to 12 hours in time, 15 degrees being equal to one hour. In Great Britain universally and in the United States generally geographers reckon from the meridian of the transit circle at the Royal Observatory of Greenwich in England; the meridian of Washington is also used occasionally in the United States. On shore the most accurate method is to compare the time of the two places by means of the electric telegraph; while at sea, the local time being determined by observation of some celestial object, it is compared with Greenwich time, as shown by a chronometer carefully set and regulated before sailing.

The instruments used in surveying are numerous, but the more important are the measuring chain, the vernier, the level, the barometer and compass, the transit, the sextant and theodolite.

The instruments commonly used in the measurement of angles are the compass, which determines directions and, indirectly, angles, and the transit, which determines angles and, indirectly, directions. The sextant is an angle-measuring instrument, the use of which is confined to certain particular operations, such as the location of soundings taken offshore and angular measurements at sea.

The compass consists of a line of sight attached to a graduated circular box, in the center of which is hung, on a pivot, a magnetic needle. At any place on the earth's surface the needle, if allowed to swing freely, will assume a position in what is called the magnetic meridian of the place. If the direction of any line is required, the compass may be placed at one end of the line and the line of sight may be made to coincide with the line. The needle lying in the magnetic meridian and the zero of the graduations of the circular needle-box being in the line of sight, the angle that the line on the ground makes with the magnetic meridian is read on the graduated circle.

At a very few places on the earth's surface the needle points to the true north. When it does not point thus, the angle that the magnetic meridional plane makes at any point with the true meridional plane is called the magnetic declination. This declination is subject at every place to changes, regular and irregular, so that the magnetic bearings of lines run with the compass are required to be reduced to the true bearings.

The sextant is an important instrument in surveying and navigation, used for measuring the angular distance of two stars or other objects, or the altitude of a star above the horizon, the two images being brought into coincidence by reflection from the transmitting horizon-glass. In the hands of a competent observer, the work of the sextant is extremely accurate. The first inventor of the sextant (quadrant) was Newton. A description of this instrument was found among his papers after his death, not, however, until after its reinvention by Thomas Godfrey, of Philadelphia, in 1730. This is the instrument used by seamen for observations for finding latitude and longitude.

The transit is used for measuring horizontal angles, and resembles a theodolite, but is not intended for very precise measurements. The theodolite has appeared in a variety of forms. Its purpose is to measure horizontal, and sometimes vertical, angles. It consists essentially of a telescope which has a motion about a horizontal axis which rests in two pillars which are perpendicular to the axis of rotation of the telescope. These pillars are fixed at right angles to a plate, which turns upon a vertical axis and to which is attached a vernier. Around this is a second plate, graduated, and concentric with the first.

It may also be provided with a vertical circle, and if this is not very much smaller than the horizontal circle the instrument is called an altazimuth. If it is provided with a delicate striding level and is in every way convenient for astronomical work, it is called a universal instrument. A small altazimuth with a concentric mag-

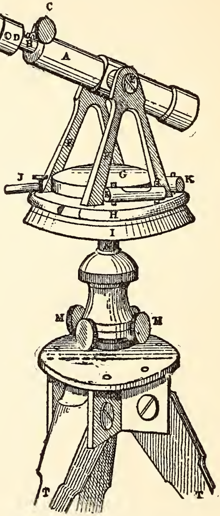


Fig. 11 —Essential Parts of Theodolite.

A., telescope; B., eye tube; C., Ratchet and pinion for moving eye tube; D., Screw for adjustment of cross wires; E., axis of rotation; F., pillars supporting axis; G., compass; H., upper plate carrying vernier; I., lower (graduated) plate; J., clamp and tangent screws for upper plate; K., levels; M., ball and socket joint with four leveling screws; N., spindle axis of rotation of azimuth plate; T., tripod.

netic compas is called a surveyor's transit. A theodolite in which the whole instrument, except the feet and their connections, turns relatively to the latter, and can be clamped in different positions, is called a repeating circle.

A hydrographic survey is one that has to do with any body of water, and may be undertaken for any one of a number of purposes. One of the most important uses of hydrographic surveying is to supply maps of the bed of the sea, or harbor, or bay, or river for the information of seamen. In this case it is necessary to locate the channels, dangerous rocks and shoals. In many cases the work of the hydrographic surveyor goes much further than this, and determines the cross-sections of streams, their velocities, their discharge, the direction of their currents, and the character of their beds.

The topography of the bed of a body of water is determined by sounding—that is, measuring the depth of the water. If many points are observed a contour map of the bottom may be drawn, the water surface being the plane of reference. For depths less than 15 or 20 feet a pole is used. Soundings made in moderately deep water are made with a weight, known as a lead, attached to a suitable line. There is a deep-sea sounding machine, by the aid of which soundings may be made to great depths, with a close approach to accuracy. This result has been attained by a combination of improvements in which great ingenuity has been displayed and in which the inventive genius of Sir William Thomson has been particularly conspicuous. The principal features of the most perfect sounding-machine are: (1) The sinker, which is a cannon-ball through which passes a cylinder provided with a valve to collect and retain a specimen of the bottom, the cylinder being, by an ingenious mechanical arrangement, detached from the shot, which remains at the bottom; (2) the line, made of steel wire, weighing about 14½ pounds to the nautical mile; (3) machinery for regulating the lowering of the sinker and for reeling in the wire with the cylinder attached, in such a manner that the irregular strain due to the motion of the ship

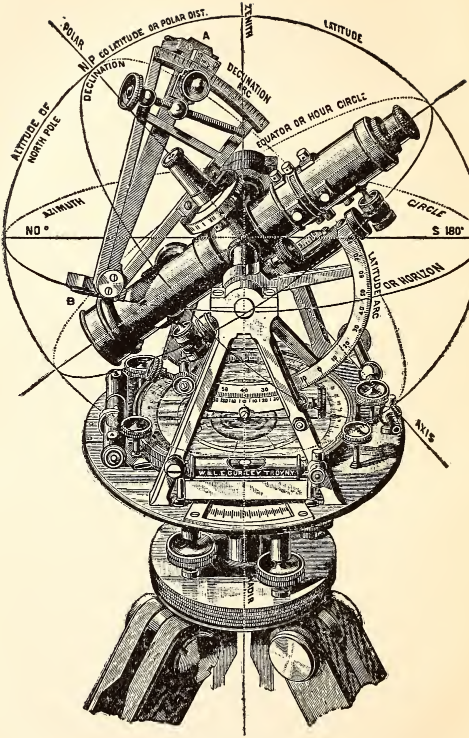


Fig. 12 —The Solar Transit.

may be guarded against and the danger of breakage thus reduced to a minimum. In the deepest accurate sounding yet made the bottom was reached at the depth of 4,655 fathoms.

The determination of the coast line is accomplished by a general scheme of triangulation, just as the topographical map of land areas is determined by it; but the necessity of taking observations from a ship makes the practice somewhat different. A map of a section of coast is the double product of the measurements of angles and base lines and the soundings taken to determine the depth

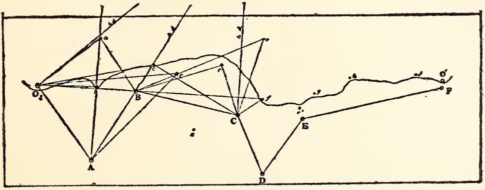


Fig. 13 —Survey of Coast-line.

of the water. The survey is made by two parties, one on shore and one in a boat sailing along the coast.

If the reckoning of a ship could be accurately kept as she runs along a coast, a very good chart could be made simply by taking exact bearings of various points on the shore line and noting the time. The track of the ship would be a base line, and the intersections of the bearings would fix the positions of the shore line. The latitude and longitude would be determined accurately at intervals of forty or sixty miles, and the intervening points could be plotted by plane surveying methods. The bearing of any terrestrial object can be determined from a ship by astronomical methods, but owing to currents, leeway, and difficulties in steering, the accuracy of the track base cannot be depended upon. Therefore the astronomical observations are made on shore with the transit and zenith telescope.

The ship and shore parties proceed along the coast by carefully determined stages, each party taking angular measurements from three points and soundings. Both parties take angular measurements from some fixed object farther inshore, and by comparing observations, determining the exact position of the ship at certain intervals, and establishing a system of triangles not only with the shore party, but with new fixed objects at each stage, the data for coast line are obtained. The work can be plotted on a polyconic chart to include the coast, the scale depending on its extent.

The art of the land surveyor is closely allied to that of the seaman, who is obliged to find his course, in any extended voyage, by angular observations of the heavenly bodies and the mathematical solution of the problems thus offered. The mariner has more than an academic interest in determining his position—it is a matter of life and death to him, and navigation depends mainly upon the acquisition of that knowledge.

Navigation is the art or science of directing the course of vessels as they sail from one part of the world to another. The management of the sails, or as it may be of machinery, the holding of the assigned course by proper steering, and the working of the ship generally pertain rather to seamanship. The two fundamental problems of navigation are the determination of the ship's position at a given moment and the decision of the most advantageous course to be steered in order to reach a given point. The methods of solving the first are, in general, four: (1) By reference to one or more known and visible landmarks; (2) by ascertaining through soundings the depth and character of the bottom; (3) by calculating the direction and distance sailed from a previously determined position; and (4) by ascertaining the latitude and longitude by observations of the heavenly bodies. The places of the sun, moon, planets and fixed stars are deduced from observation and calculation, and are published in nautical almanacs, the use of which, together with logarithmic and other tables computed for the purpose, is necessary in reducing observations taken to determine latitude, longitude, and the error of the compass.

The calculation of a ship's place at sea, independently of observations of the heavenly bodies, is called dead-reckoning. The ship's position is calculated simply from the distance she has run by the log and the courses steered by the compass, this being rectified by due allowances for drift and leeway. In very early times dead-reckoning was an important branch of knowledge, in which the instruments for measuring time, such as the sand-glass, played a considerable part. The sand-glass is still found on many sailing ships using the old-fashioned 'log.' The earliest mode of measuring the speed of a vessel at sea was by throwing overboard a heavy piece of wood, so shaped that it resisted being dragged through the water, and with a line tied to it. The block of wood was the log, and the string had knots in it, so arranged that when one knot ran through a sailor's fingers in half a minute, measured by the sand-glass, the vessel was going at the speed of one nautical mile an hour, ten knots on the line ten miles, and so forth. The nautical mile is of such a length that 60 of them constitute one degree on a great circle of the earth; therefore the knots are 50 feet and 7 inches apart.

Patent logs are generally used now at sea, those most commonly found on vessels being either the harpoon or the taffrail log. The harpoon log is shaped like a torpedo, and has at one end a metal loop to which the log-line is fastened, and at the other fans which cause the machine to spin round as it is drawn through the water. The spinning of the instrument sets a clockwork machinery in motion, which records the speed of the vessel upon dials, the rotation of the instrument being, of course, dependent upon the rate at which it is dragged through the water. In the taffrail log the recording machinery is secured to the taffrail, and the fan is towed astern at the end of a long line.

If the sea were a smooth plane surface, without currents or tides, it would be a simple matter to fix accurately the position of a vessel, and to take her from one place to another on the earth's surface by dead-reckoning only; but as it is in constant motion, influenced by irregular currents and tides and the drift of the waves, it becomes necessary to have some more accurate method to insure safe navigation, and this is to be found in the system of observation of the heavenly bodies, or, in other words, in the science of nautical astronomy.

Thus the angular measurement of the sun and the fixed stars, by means of the sextant, becomes a necessity, and also the solution of the triangle problems by means of logarithms and trigonometrical formulæ. Since the sailor always has the horizon and zenith with him, he can find his latitude at any time by taking the meridian altitude of the sun and correcting that by the declination found in his nautical almanac. His longitude will be found by the aid of the sun and a chronometer. The apparent time at sea he will find by observing the sun's hour-angle; apparent time must be turned into mean time by applying the equation of time; and mean time at ship must be compared with mean time at Greenwich, as ascertained by the chronometer. The difference between these two is the ship's longitude.

Nautical almanacs are published by the governments of Great Britain, the United States, and most other maritime powers. These are almanacs for the use of navigators and astronomers, in which are given the ephemerides of all the bodies of the solar system, places of the fixed stars, predictions of astronomical phenomena, and the angular distances of the moon from the sun, planets, and fixed stars.

The laws of the tides and of storms must also be studied by the seaman, especially the "law of storms" in a navigational sense. This expression generally means the law of circular storms or cyclones, and should be understood by all who are responsible for the safe conduct of foreign-going ships. Owing to the nature of the cyclone, very fair general rules can be made which assist the mariner in steering a course away from the storm center. A good many generalizations have been made in regard to winds in a wide sense. Airey found that the wind never blows steadily for any period of time except from eight points of the compass. When in any other quarter it is merely shifting round to one of these points. It never blows at all directly from the south. The two most prevalent winds are south-southwest and west-southwest. The first serious study of the circulation of winds on the earth's surface was instituted at the beginning of the second quarter of this century by W. H. Dove, William C. Redfield, and James P. Espy, followed by researches of W. Reid, Piddington, and Elias Loomis. But the deepest insight into the wonderful correlations that exist among the varied motions of the atmosphere was obtained by William Ferrel (1817-1891). He was born in Fulton County, Pa., and brought up on a farm. In 1885 appeared his Recent Advances in Meteorology. In the opinion of a leading European meteorologist, Julius Hann, of Vienna, Ferrel has "contributed more to the advance of the physics of the atmosphere than any other living physicist or meteorologist."

Ferrel teaches that the air flows in great spirals toward the poles, both in the upper strata of the atmosphere and on the earth's surface beyond the 30th degree of latitude; while the return current blows at nearly right angles to the above spirals, in the middle strata as well as on the earth's surface, in a zone comprized between the parallels 30° N. and 30° S. The idea of three superposed currents blowing spirals was first advanced by James Thomson, but was published in very meager abstract.

Another theory of the general circulation of the atmosphere was propounded by Werner Siemens, of Berlin, in which an attempt is made to apply thermodynamics to aerial currents. Important new points of view have been introduced recently by Helmholtz, who concludes that when two air currents blow one above the other in different directions, a system of air waves must arise in the same way as waves are formed on the sea. He and A. Oberbeck showed that when the waves on the sea attain lengths of from 16 to 33 feet, the air waves must attain lengths of from 10 to 20 miles and proportional depths. Superposed strata would thus mix more thoroly and their energy would be partly dissipated. From hydrodynamical equations of rotation Helmholtz established the reason why the observed velocity from equatorial regions is much less in a latitude of, say, 20° or 30° than it would be were the movements unchecked.

Another science bearing directly on navigation is the construction of vessels, both in its architectural aspect and in its relation to magnetism. The earth being a magnet, it induces magnetism in all things on its surface. When an iron ship is being built, the hammering which she undergoes causes magnetism of a more or less permanent character to be induced in her. This is known as sub-permanent magnetism, because tho a ship rarely loses it altogether, it alters very much after the vessel is launched, through change of position, through being knocked about in a heavy sea, and from other causes.

In the case of a ship built head south in northern latitudes her blue polarity will be in her bow, and the north point of her compass needle will be attracted to it. This will cause westerly deviations as the ship's head passes through the western half of the compass and easterly when through the eastern. If her head is north when

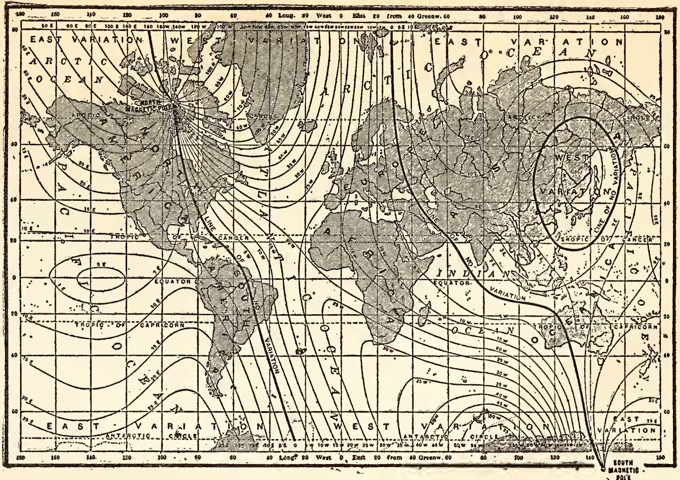


Fig. 14 —Chart of Magnetic Variation.

building her stern will have blue polarity, and she will have easterly deviation with her head in the western semicircle of the compass and westerly deviation with her head in the eastern semicircle. With her head east when building she will have more blue polarity in her starboard side than in her port, and with her head west when building there will be easterly deviation on southerly courses and westerly deviation on northerly.

A ship, like everything else, has its center of gravity, tho this center is not a fixed point. It varies with every change in the position and quantity of the weights in her. A ship has also her center of buoyancy. This is the geometrical center of her immersed portion, and its position can be arrived at with great certainty. Thus, a vessel floating upright and at rest will fulfil certain conditions. First, she will displace a weight of water equal to her own weight; secondly, her center of gravity will lie in one and the same vertical line with the center of gravity of the volume of water displaced, and in that line is the center of buoyancy.

If weights are moved in a vessel laterally the position of her center of gravity is changed laterally, too; but when she is heeled by wind or sea no change occurs in it. The buoyancy, acting upward through the center of buoyancy, shifting as it does from side to side as a ship is heeled over or rolls through the action of wind or sea, is the upward righting force mainly to be relied upon to keep a vessel from capsizing.

The knowledge of mathematical laws and principles is necessary to good seamanship, but perhaps in no art is the practical and actual handling of apparatus more useful than in that of the mariner. Theory can but lead the learner to the edges of the subject; science and practice must go hand in hand before any substantial acquirements can be gained.

CHAPTER IV

MECHANICAL PRINCIPLES

It is the privilege of the modern to make the most of an environment of mechanism, a development consequent upon the growing complexity of society. This, while it adds greatly to the luxury of the whole, reduces the sphere of the individual, making it no longer possible to be well versed in many lines; the day of the Jack-at-all-trades is past and the day of the expert has come.

Numbers form the connecting link between theory and the application of theory to practical arts. In every mechanical principle mathematical formulæ are implied, tho they may be extremely simple. It is for the mathematician to find out how far experimental confirmation of a theory can be pushed and where a new hypothesis is necessary. Facts apparently unconnected are found to have their origin in a common source, and often only the mathematician can trace their connection. More than this, the mathematician is able to draw corollaries and secondary truths from a given principle which the experimentalist alone does not discover.

"Mechanical science," said William J. M. Rankine, "enables its possessor to plan a structure or machine for a given purpose without the necessity of copying some existent example; to compute the theoretical limit of the strength and stability of a structure or the efficiency of a machine of a particular kind; to ascertain how far an actual structure or a machine fails to attain that limit, and to discover the cause and remedy of such shortcoming; to determine to what extent, in laying down principles for practical use, it is advantageous for the sake of simplicity to deviate from the exactness required by pure science; and to judge how far an existing practical rule is founded on reason, how far on custom, and how far on error."

A signal illustration of the truth of these words is offered in the famous instance of falling bodies. Aristotle proved to his own satisfaction, it seemed, and told the world at large, that heavy bodies fall to earth faster than lighter ones; and it was left for Galileo, more than a thousand years later, to disprove a statement whose truth or falsity, it would seem, might have been established by any one. It required mathematical science to confute experimental error.

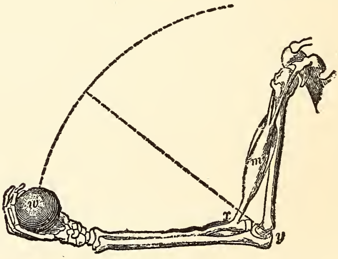


Fig. 15 —ARM AS A LEVER.

Weight is raised by shortening of muscle. m., muscle; w., weight; x., point of application; y., fulcrum.

Not only has mechanical nomenclature been largely taken from animals, but many of the principal mechanical devices have preëxisted in them. Examples of levers of all three orders are to be found in the bodies of animals. The human foot contains instances of the first and second and the fore arm of the third order of lever. The knee cap is practically a part of a pulley. There are several hinges and some ball-and-socket joints, with perfect lubricating arrangements. Lungs are bellows, and the vocal organs comprize every requisite of a perfect musical instrument. The heart is a combination of four force pumps acting harmoniously together. The wrist, ankle and spinal vertebræ form universal joints. The eyes may be regarded as double-lens cameras, with power to adjust local length, and able, by their stereoscopic action, to gage size, solidity and distance. The nerves form a complete telegraph system, with separate up-and-down lines and a central exchange. The circulation of the blood is a double-line system of canals, in which the canal liquid and canal boats move together, making the complete circuit twice a minute, distributing supplies wherever needed, and taking up return loads wherever ready without stopping. It is also a heat-distributing apparatus, carrying heat from wherever it is generated, or in excess, to wherever it is deficient, and establishing a general average.

Archimedes was almost the only philosopher among the ancients, so far as is known, who formed clear and correct notions concerning the simple machines. He acquired firm possession of the idea of pressure, which lies at the root of mechanical science, and of equilibrium. The proof of the properties of the lever given in Archimedes' "Equiponderance of Planes" holds its place in text-books to this day. His estimate of the efficiency of the lever is expressed in the saying attributed to him, "Give me a fulcrum on which to stand, and I will move the earth." The "Equiponderance" treats of solids, while the book on "Floating Bodies" treats of hydrostatics, or the equilibrium of fluids.

It was long a common practice for mechanicians to recognise six simple machines, or six devices representing the first principles of mechanics. These are the pulley, the lever, the wedge, the screw, the inclined plane and the wheel and axle. In the latter part of the eighteenth century, however, La Grange simplified the mechanical principles, including them all under two, the principle of the lever and the principle of the inclined plane. Every machine that exists, from the egg-beater to the escalator, is constructed by the application of these principles or a combination of them.

The lever consists of a bar or rigid piece of any shape, acted upon at different points by two forces, which severally tend to rotate it in opposite directions about a fixed axis. It was beautifully demonstrated by Archimedes that the power at one end and the weight, or resistance, at the other are in equilibrium under certain conditions, the simplest being the case in which the load is ten times as great as the power, but the power is ten times as far from the fulcrum as the load is from the fulcrum; or, stated otherwise, the two forces are in equilibrium when they are inversely as the length of their respective arms. There are three different kinds of levers, according to the relative positions of the three points, the fulcrum, the point of application of power, and the point of application of the load. The handle of a common pump is a lever of the first class, in which the fulcrum is between the other two points. The piston and the water are the weight, the hand of the worker is the power, while the pivot on which the handle turns is the fulcrum. The ordinary steelyard is another example of a lever of this class.

The second class is formed by levers in which the weight is between the fulcrum and the power, as is illustrated by the wheelbarrow. The axle of the wheel is the fulcrum in this case, the load in the barrow is the weight, and the handles of the barrow are the levers. The boat with its oars is another example of this class of levers.

In the third class of levers the point of application of the power lies between the fulcrum and the load, and is illustrated by the lifting of a ladder when one end is resting on the ground. These distinctions are of slight importance, however, since they become confused as the machines to which they are applied become more complicated. The Archimedean laws, however, which apply to levers are extremely simple, and illustrate the beauty with which physical or mechanical phenomena, of apparently diverse types, may often be reduced to law. First, the two extreme forces must always act in the same direction; secondly, the middle one must act in the opposite direction and be equal to the sum of the other two; and thirdly, the magnitude of the extreme forces is inversely proportional to their distance from the middle one.

Probably of all devices of man none is more frequently in evidence than the rope tackle used in hoisting, and known as the pulley. This is a contrivance for balancing a great force against a small one, or for lifting a big load with a small power. Its sole use is to produce equilibrium. It does not save work, unless indirectly in some unmechanical way. The pulley is a lever with equal arms; but when it turns, the attachments of the forces are moved.

The wheel and axle, also one of the simple machines, works indirectly on the principle of the lever. In its primary form it consists of a cylindrical axle on which a wheel, concentric with the axle, is firmly fastened. A rope is usually attached to the wheel, and the axle is turned by means of a lever; the rope acts as in the pulley; that is, upon the principle of the lever, which explains all the possible phenomena exhibited by the pulley and the wheel and axle, just as the principle of the inclined plane explains all the phenomena of the wedge and the screw.

The inclined plane in mechanics is a plane inclined to the horizon, or forming with a horizontal plane any angle whatever except a right angle. It is one of the two fundamental machines, the other being the lever. The power necessary to sustain any weight on an inclined plane is to the weight as the height of the plane to its

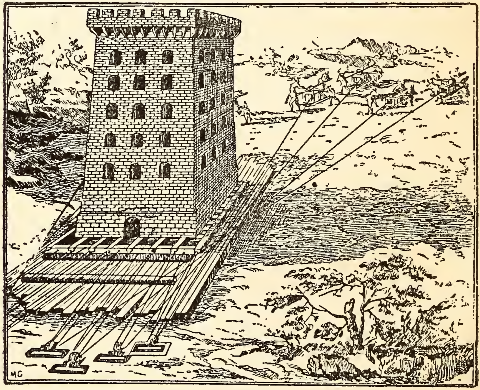


Fig. 16 —Tower Moved by Windlass and Pulleys. (From a sixteenth-century print.)

length. This was first proved by Stevin in the sixteenth century. If the inclined plane, with its horizontal plane as a base, and the line connecting the two planes be considered as a right-angled triangle, the weights proportional to the hypotenuse and height of the triangle balance.

The screw and the wedge, both called simple machines, are special applications of this principle. The wedge consists of a very acute-angled triangular prism of some hard material, which is driven in between objects to be separated, or into anything to be split. It is, of course, one of the commonest of implements, as is also the screw; but in the apparently simple action of these two devices lie the germs of some of the most effective instruments for increasing man's "natural" power. It is necessary to understand the exact function of each part of this apparently innocuous machine, the screw, in order to follow its development in the more complicated inventions.

The screw is a cylinder of wood or metal having a spiral ridge, the thread, running round it, usually turning in a hollow cylinder in which a spiral channel is cut corresponding to the ridge. The convex and concave spirals, with their supports, are often called the screw and nut, and also the external or male screw and the internal or female screw, respectively. The screw is virtually a spiral inclined plane; only the inclined plane is commonly used to overcome gravity, while the screw is more often used to overcome some other resistance. Screws are right and left, according to the direction of the spiral.

Screws have a variety of uses, the most important of which are two. First, they are used for balancing forces, as the jack screw against gravity, the propeller screw against the resistance of water, and the screw-press against elasticity. Secondly, they are used for magnifying a motion and rendering it easily manageable and measurable, as in the screw-feet of instruments, micrometer screws, and the like. Hunter's screw is a double screw consisting of a principal male screw that turns in a nut, but in the cylinder of which, concentric with its axis, is formed a female screw of different pitch that turns on a secondary but fixed male screw. The device furnishes an instrument of slow but enormous lifting power without the necessity of finely cut and consequently frail threads. Everything else being equal, the lifting power of this screw increases exactly as the difference between the pitches of the principal male screw and the female screw diminishes, in accordance with the principle of virtual velocities.

Archimedes himself made several experimental applications of his screw, among which were a railway and a machine for lifting water. In the railway a continuous shaft rotates on pillars between two lines of rails, and propels the car by means of a screw which engages in a pedestal attached to the car. The instrument for lifting water, technically called the "Archimedean screw," is

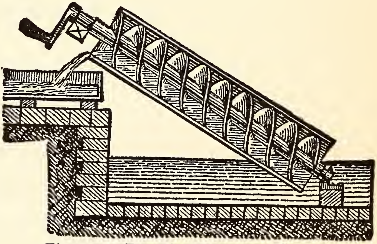


Fig. 17 —The Archimedean Screw.

made by forming a spiral tube within or by winding a flexible tube spirally without a cylinder. When the cylinder is placed in an inclined position and the lower end is immersed in water, its revolution will cause the water to move upward through the spiral chambers.

The mechanical powers, as the six simple machines have long been called, are often in evidence in modern inventions almost in their original simplicity. The screw propeller, for instance, consists of a continuous spiral vane on a hollow core running lengthwise of a vessel. This is but an extension and amplification of the screw and was also devised by Archimedes. The modern screw propeller is attached to the exterior end of a shaft protruding through the hull of a vessel at the stern. It consists of a number of spiral metal blades either cast together in one piece or bolted to a hub. In some special cases, as in ferry-boats, there are two screws, one at each end of the vessel. In some war-vessels transverse shafts with small propellers have been used to assist in turning quickly. An arrangement of screws now common is the twin-screw system, in which two screws are arranged at the stern, each on one of two parallel shafts, which are driven by power independently one of the other. By stopping or slowing up one shaft while the other maintains its velocity, very rapid turning can be effected by twin screws, which have, moreover, the advantage that, one being disabled, the vessel can still make headway with the other. Some vessels designed to attain high speed have been constructed with three screws. A very great variety of forms have been proposed for screw-propeller blades; but the principle of the original true screw is still in use. Variations in pitch and modifications of the form of the blades have been adopted with success by individual constructors.

The actual area of the screw propeller is measured on a plane perpendicular to the direction in which the ship moves. The outline of the screw projected on that plane is the actual area, but the effective area is, in good examples, from 0.2 to 0.4 greater than this; and it is the effective area and the mean velocity with which the water is thrown astern that determine the mass thrown backward. The mass thrown backward and the velocity with which it is so projected determine the propelling power. A kind of feathering propeller has also been used, but has not been generally approved.

The mechanism of nature has offered suggestions for many inventions, one of which provides an illustration of many others. The pedrail, for instance, which is a rail moving on feet, is constructed on the principle of the horse. A horse has practically two wheels, its front legs one, its back legs the other. The shoulder and hip joints form the axes and the legs the spokes. So the pedrail has wheels the spokes of which, to any number, are connected at their outer ends by flat plates. As each angle of the plates is passed, the wheel falls plumb on to the next plate. The greater the number of spokes, the less will be each successive jar, or step; and consequently the perfect wheel is theoretically one in which the sides have been so much multiplied as to be infintely short.

With the exception of Archimedes and a few mathematicians of the Alexandrian school, the ancients and the generations of the Middle Ages slept, so far as mechanical science was concerned, in an untroubled peace. Not until the seventeenth century were some of the Aristotelian myths of science banished, when Galileo aroused the mechanical and scientific genius of the age.

Among the curious vagaries of imagination which have deluded the human mind, none is more interesting than the idea of perpetual motion, which has been followed for centuries with fatuous hope. Perpetual motion, in a mechanical sense, is a motion that is preserved and continuously renewed of itself without the aid of any external cause. It is, however, one of the chimeras of the brain which has its aspects of plausibility for the tyro.

Many historic machines purporting to display the power of perpetual motion have brought their inventors to poverty, if not to despair. One authoritative writer says: "In order to produce a perpetual motion, we have only to remove all the obstacles which oppose that motion; and it is obvious that if we could do this, any motion whatever would be a perpetual motion. But how are we to get rid of these obstacles? Can the friction between two touching bodies be entirely annihilated? Or has any substance yet been found that is void of friction? Can we totally remove all the resistance of the air, which is a force continually varying? And does the air at all times retain its impeding force? These things cannot be re-

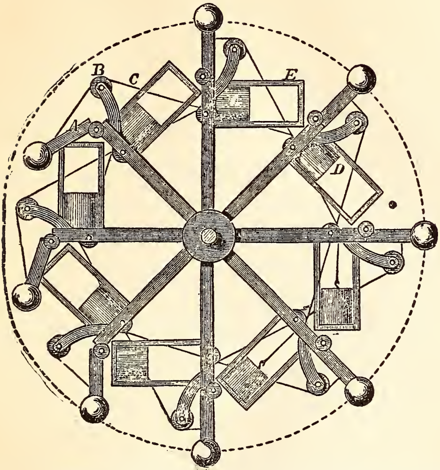


Fig. 18 —Ferguson's Machine to Show the Fallacy of Perpetual Motion Schemes.

The axle is placed horizontally and the spokes turn in a vertical position. The spokes are jointed, as shown, and to each of them is fixed a frame in whi^h a weight, D, moves. When any spoke is in a horizontal position, the weight, D, in it falls down, and pulls the weighted arm, A, of the then vertical spoke straight out, by means of a cord, C, going over the pulley, B, to the weight, D. But when the spokes come about to the left hand, their weights fall back and cease pulling, so that the spokes then bend at their joints and the balls at their ends come nearer the center on the left side as the balls or weights at the right-hand side are farther from the center than they are on the left. It might be supposed that this machine would turn round perpetually, but it is a mere balance.

moved so long as the present laws of nature continue to exist.

"Every attempt to produce a self-moving machine has been in open defiance to the coördinated relations of force and motion; and any man who comprehends this law of velocity will no sooner attempt to solve the problem of perpetual motion than to climb upon his own shoulders as a higher point of observation.

"But in the search for an impossibility so many valuable and practical certainties have been demonstrated that perhaps no time has been absolutely thrown away. 'As alchemy fostered and developed chemistry, so the search after perpetual motion has taught scientists how to apply force through complicated machinery, and has given rise to many new devices."

In treating of perpetual motion—"that grand secret for the discovery of which those dictators of philosophy, Democritus, Pythagoras, Plato, did travel unto the Gymnosophists and Indian priests"—its history would be a fascinating but tragic tale. Every contrivance hitherto planned or experimented upon has been proved fallible. Paracelsus built a "little world," Cornelius Dreble invented a planetarium for King James, and Peregrinus suggested the "magnetical globe of Terella," which he thought might be kept in motion by pieces of steel and loadstones; and Bishop Wilkins himself made an application of Archimedes' screw, but all were alike "found inadequate to the grand end for which they were designed."

CHAPTER V

MACHINES

In a general mechanical sense a machine is any instrument which converts motion, or rather force, into motion, as, for instance, a machine designed to convert rapid motion into slow motion, as a windlass, or, vice versa, as the connection of a large wheel to a small increases the velocity of the latter. The ordinary tools consisting of a single device, such as the hammer, or a simple combination of moving parts, such as shears or tongs, are machines in the strict technical definition of the term. Many writers have used the word in a sense other than the strictly technical one, as Huxley when he says: "The human body, like all living bodies, is a machine, all the operations of which will, sooner or later, be explained on physical principles."

Among the most ancient machines were those that employed wind or flowing water as a motor power for turning wheels. In medieval times even bellows were adapted to this purpose. The windmill is a familiar device for raising water from a well or spring for grinding and other purposes. There are two types of these wind-motors, the vertical being the most common. The vertical motor consists essentially of a horizontal wind-shaft, with a combination of sails or vanes fixed at the end of the shaft and suitable gearing for conveying the motion of the wind-shaft to the pump or to the other machinery.

The typical Dutch windmill was provided with four vanes or sail-frames, called whips, covered with canvas and provided with arrangements for reefing the sails in a high wind. To present the vanes to the wind, the whole

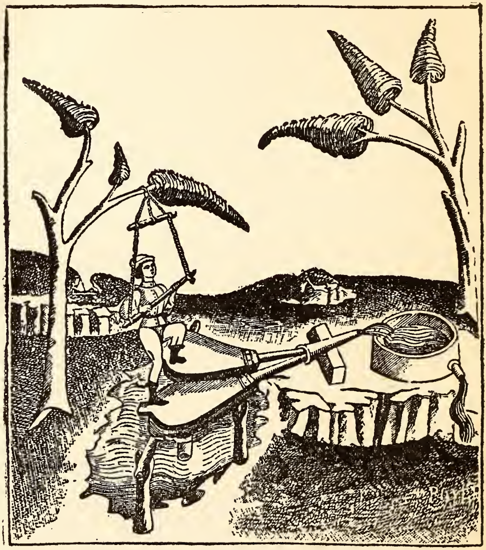


Fig. 19 —Bellows for Raising Water. (From a fifteenth-century print.)

structure or tower was at first turned round by means of a long lever. Later the top of the tower, or cap, was made movable. Windmills are now made with many wooden vanes forming a disk exposed to the winds and fitted with automatic feathering and steering machinery, governors for regulating the speed and apparatus for closing the vanes in storms. These improved windmills are chiefly an American invention and are used for pumping water.

Water power is perhaps, after wind power, the most natural and the most truly economic source of energy. The term "water power" is not exact, since the real agent in water machines is gravity, the fluid itself being only the medium through which the action of gravity is transmitted to the prime motor. In order that water may be available for doing work, it must be in such a position that it can fall from a higher to a lower level or must be under pressure produced by some external force, such as that of a weight or spring acting on the surface of the fluid through a piston or plunger. Under the former condition its utmost capacity for doing work is the product of the height through which it can fall into the weight of the water falling.

For practical purposes there are three ways by which water power can be applied to the performance of work: through the velocity of the fluid itself, by weight or by pressure. Each of these three methods requires a different type of motor for its application. An illustration of the first is the turbine, which is moved by the force of projected water; the second, the water-wheel, which is moved by the weight of the falling water; the third, the hydraulic pressure engine, which operates by the application of the hydraulic law of equal pressure.

The old-fashioned mill for grinding flour or corn, which was once the center of nearly every New England village, was run by water flowing over the upper wings of a clumsy wooden wheel. These overshot wheels are now nearly obsolete, but have been constructed in the past on rather gigantic plans. The water falls from a sluice or pen trough near the top and moves the wheel by falling into floats or shallow buckets. It is regulated by a gate and falls into the third or fourth bucket from the summit, thus utilizing as much as possible of the gravitational force. The undershot wheel turns by having the force of the stream of water act at its lowest point instead of its summit.

But the numerous disadvantages of the water-wheels described have caused them to be almost entirely super-

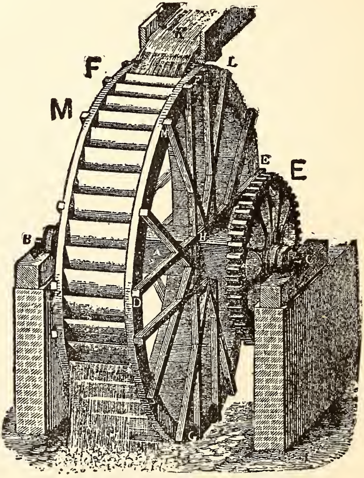


Fig. 20 —Overshot Wheel.

seded by the turbine. This is a water-wheel driven by the impact or reaction of a stream of water flowing against a series of radial buckets or by impact and reaction combined. Turbines are usually horizontally rotating wheels on vertical shafts. They are of various constructions and may be classed as reaction turbines, whose buckets move in a direction opposite to that of the flow; impulse turbines, whose buckets move with the flow, and the combined reaction and impulse wheels, which include the best modern types of turbines. In these a very high percentage of the potential energy of water is converted into work while passing through the wheels.

Impulse wheels, constructed as large as 18½ feet in diameter, have been employed to work air compressors in mines. A wheel of this size weighs 10,000 pounds and runs at 110 revolutions per minute; it has energy equal to 300 horse-power. The wheel is made of iron plates riveted together, which are held concentric with the shaft by radial spokes. There is a variable nozzle operated by an automatic hydraulic regulator, through which the water is applied to the wheel. It will run at uniform speed with varying loads. Turbines are now made from 6 to 80 feet in diameter and are so cheap, durable and highly effective that they are fast superseding other types of wheels.

Two other important applications of water power are found in the hydraulic press and the hydraulic ram. The hydraulic press is operated by the pressure of a liquid, under the action either of gravity or of some mechanical device such as a force pump. It depends on the law of hydrostatics that any pressure upon a body of water is distributed equally in all directions throughout the whole mass, whatever its shape. In the more common forms of hydraulic press the pressure of a piston upon a body of water in a cylinder of small area is distributed through pipes or openings to a piston or a larger area. The statical force is thus multiplied in the direct ratio of the areas of the pistons. Therefore if the diameter of a small piston is one inch and of a larger piston in the cylinder is one foot, the area of the larger piston will be 144 times the area of the smaller, and if a load of one ton is applied to the smaller, the larger will exert an upward statical force of 144 tons.

This interesting machine is used as the basis of a great number of inventions, such as the hydraulic block, jack, crane, hoist, lift and others, and for the pressing of paper and other materials.

The hydraulic ram is a self-contained and automatic pump, operated partly by the pressure of a column of water in a pipe and partly by the living force acquired by the intermittent motion of the column. This machine can be used to raise water to a height many times greater than the available head, and it is also adapted to draw water from a source independent of that which supplies the power for operating it.

Hydraulic machines are very wonderful to people who observe their action for the first time. With a common hydraulic press a laborer, without any other help, can raise a load of a hundred tons, which is the weight of a long railway train. At large ship docks any boy can, by the manipulation of a few handles, lift heavy weights rapidly from a ship and place them on the dock.

No single invention in the history of the world has had so deep and wide an influence as the steam engine. This truism is one which deserves consideration, even in days when there is all too much exploitation of the mechanical inventiveness of the age. If the possibility of travel which the locomotive has brought within the reach of nearly every one be considered, apart from any other uses of the steam engine, its extraordinary influence on the life of the century is startlingly apparent. Until within fifty years travel and acquaintance with foreign peoples, historic monuments and all the artistic accumulations of other nations and other generations were the privilege of very few; in these days traveling is the universal epidemic. More than that, with better acquaintance nation has reacted upon nation, so that political and military problems have taken on a wholly different aspect.

The germ of the steam engine existed in Heron's eolipile, invented in the second century B.C. This illustrated perfectly the expansive force of steam generated in a closed vessel and escaping by a narrow aperture. It consisted of a hollow ball containing water and two arms bent in opposite directions, from the narrow apertures of which steam issued with such force that the air, reacting on it, caused a

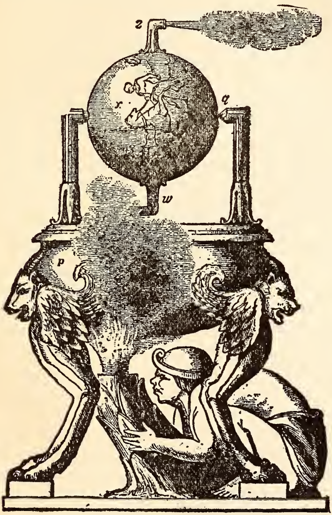


Fig. 21 —Heron's Eolipile.

circular or rotary motion of the ball. Several attempts have been made to apply the principle of the eolipile to rotating machinery.

In 1705 there was invented the first important device for the practical application of steam power. For about

1,500 years after Heron's eolipile no progress had been made. During the seventeenth century steam fountains were designed, but they were merely modifications of Heron's engine, and were probably applied only for ornamental purposes. Some effort was also made by Morland, Papin and Savery to construct practical machines for the raising of water or driving of mill-works. The first successful attempt to combine the principles and forms of mechanism then known into an economical and convenient machine was made by Thomas Newcomen, a blacksmith of Dartmouth, England. It is probable that he knew of Savery's engine, as Savery lived only fifteen miles away.

Assisted by John Calley, Newcomen constructed an engine—an "atmospheric steam-engine," for which a patent was secured in 1705. In 1711 such a machine was set up at Wolverhampton for the raising of water. Steam passing from the boiler into the cylinder held the piston up against the external atmospheric pressure until the passage between the cylinder and boiler was closed by a cock. Then the steam in the cylinder was condensed by a jet of water. A partial vacuum was formed and the air above pressed the piston down. This piston was suspended from one end of an overhead beam, the other end of the beam carrying the pump-rod. Desaguliers tells the story that a boy, Humphrey Potter, who was charged with the duty of opening and closing the stopcock between the boiler and cylinder for every stroke, contrived by catches and strings an automatic motion of the cock. The fly-wheel was introduced in 1736 by Jonathan Hulls. The next great improvements were introduced in Scotland by James Watt in the latter half of the eighteenth century. Watt was educated as a maker of mathematical instruments, and in 1760 he opened a shop in Glasgow. Becoming interested in the steam-engine and its history, he began to experiment in a scientific manner. He took up chemistry and was assisted in his studies by Dr. Black, the discoverer of "latent heat." Observing the great loss of heat in the Newcomen engine, due to the cooling of the cylinder by the jet of water at every stroke, he began to think of means to keep the cylinder "always as hot as the steam that entered it." He has told us how, finally, the happy thought securing this end occurred to him: "I had gone to take a walk on a fine Sabbath afternoon. I had entered the Green by the gate at the foot of Charlotte Street, and had passed the old washing-house. I was thinking upon the engine at the time, and had gone as far as the herd's house when the idea came into my mind that as steam was an elastic body it would rush into a vacuum, and if a communication were made between the cylinder and an exhausted vessel, it would rush into it, and might be there condensed without cooling the cylinder." Through this invention the piston was now moved by the expansion of steam, not by air pressure, as in Newcomen's engine. Watt introduced a separate condenser, a steam-jacket and other improvements. He deservedly commands a preëminent place among those who took part in the development of the steam-engine. The expiration of Watt's vital patent occurred in 1800, and he himself then retired from the active supervision of his engineering business, having virtually finished his life's work on the last year of the century.

One of the first and most obvious uses of the steam-engine was to apply its power to locomotion, both on sea and land. Before steam lent its power to the propulsion of ships, navigation was, like the windmill, subject to the intermittent character of the winds or limited to the man-power of rowers. The method of moving vessels by paddle-wheels was adopted by the Romans, probably borrowed from the Egyptians; but the wheels were turned by handles within the vessels, operated by men. There are several obscure references in annals of the seventeenth century to what is supposed to be the propulsion of paddle-wheels by steam. Among the rest there is a description of a steam propeller, invented by one Genevois, a pastor at Berne, which was formed like the foot of a duck. This was made to expand and present a large surface to the water when moved against it and to close up into a small compass when moved in the opposite direction. In 1774 there is a tradition of a boat which, when tried upon the Seine near Paris, moved against the stream, tho slowly, "the engine being of insufficient power." The construction of this engine is attributed to the Count D'Auxiron, a French nobleman.

Many attempts to apply the force of the steam-engine to the propulsion of paddle-wheels were made in the latter part of the eighteenth century, and it is said that William Symington, an English inventor, accomplished a certain form of steam navigation. But it was left for Robert Fulton, an American artist as well as inventor, to bring the trials to a successful issue. In 1809 Fulton's steam vessel, the Clermont, made her first voyage from New York to Albany, a distance of about 140 miles, at the rate of five miles an hour. To those who viewed this spectacle this first steamer "had a most terrific appearance." She used dry pine wood for fuel and sent forth a column of ignited vapor for a distance of many feet above the flue, and whenever the fire was stirred showers of sparks flew off into the air. One of the chroniclers states: "Notwithstanding the wind and tide were adverse to its approach, they saw with astonishment that the vessel was rapidly coming toward them, and when it came so near that the noise of the machinery and paddles was heard the crews, in some instances, shrank beneath their decks from the terrific sight and left their vessels to go ashore, while others prostrated themselves and besought Providence to protect them from the approach of the horrible monster, which was marching on the tide and lighting its path by the fire which it vomited."

The Clermont was of 160 tons burden, the paddle-wheels were 15 feet in diameter and dipped 2 feet in the water. She was impelled by a machine of four-foot stroke and a two-foot cylinder. Within a few weeks after the appearance of the Clermont, Stevens, of Hoboken, launched a steam vessel. She could not ply on the waters of the Hudson, in consequence of the exclusive patent of Fulton and Livingston, so she was taken to the Delaware. This was the first steamer that ever sailed the ocean. From that time steamboats have multiplied till every water in the civilized portion of the earth was marked with these agents of rapid intercourse.

For the purpose of comparison the Cunard steamer Lusitania, launched in 1907, may be placed beside that of the Clermont. The Lusitania is 790 feet long and 88 feet broad. She has a displacement of 45,000 tons and is propelled by four screws rotated by turbine engines of 68,000 horse-power. Placed in perspective, her length would outreach the angular height of the Great Pyramid.

The history of locomotion on land presents a parallel tale of simple beginnings and extraordinarily rapid progress. The "Stourbridge Lion" was the first locomotive brought to America and was tried on the road at Honesdale, Pa., on the 8th of August, 1829. Its boiler was 16½ feet long, the two cylinders were three-foot stroke and its weight was 7 tons. It was operated around a curve and up the road for about two miles and then was returned to the place of starting. The experiment demonstrated that the track was not substantial enough for so heavy an engine, and it was housed beside the track, where it remained for fifteen years. It was then removed to Carbondale, where the boiler was used for stationary purposes and the remainder was sold for old iron.

This ignominious end to the first attempt to utilize the steam-engine for locomotion on land in America did not discourage other people from making other trials. Peter Cooper, having an interest in the Baltimore and Ohio road, in 1829 built an engine known as the "Tom Thumb," to demonstrate that a locomotive could be built that would run round short curves. This engine had an upright boiler 20 inches in diameter by 5 feet high, fitted with gun barrels for flues. The engine drove a large gear which fitted into a smaller gear on the axle. The fire was urged by a fan driven by a belt. The driving-wheels were 2½ feet in diameter. In August, 1830, the first railroad car in America propelled by a locomotive was tested on the Baltimore and Ohio road. The wheels were "coned," which was the first use of this principle as applied to car-wheels. Cooper's engine was coupled to a car in front of it containing a load of 4½ tons, including 24 passengers. The trip of 13 miles was made in 1 hour and 15 minutes and the return trip in 57 minutes. This was the first locomotive built in America.

In the locomotive engines used at the present time it is not unusual to see engines for passenger service which have a total weight of about 185,000 pounds, cylinders 22 inches in diameter and a piston stroke of 30 inches. The locomotive will now at least double the speed of the race horse and will carry not only itself, but three or four times its own weight in addition, and will go a hundred miles without stopping, if only the road ahead be clear.

The fastest mechanism of any size, which has ever cut its way through the water for any considerable distance is the torpedo boat Ariete, made by a London firm in 1887. This little craft has a displacement of 110 tons and machinery capable of exerting 1,290 effective horse-power. The speed accomplished at the trial tests was 30 miles per hour, this being the average of six one-mile tests.

CHAPTER VI

AVIATION

For more than two centuries man has been trying to invent a means whereby he might navigate the air, but it is only since the beginning of the twentieth century that any degree of success has been attained.

The apparatus used in aviation divides, roughly, into two classes, dirigible balloons and the so-called "gasless," or heavier-than-air machines, represented by the biplane, the ornithopter, or beating-wing machine, and the helicopter, or direct lift machine.

The dirigible balloon has already, relatively speaking, arrived at some degree of perfection, insomuch as the serious difficulties connected with this type of aërial locomotive have been largely overcome. The gas-bag, with the volume of gas employed, has been brought to its smallest practicable size, and the weight of subsidiary material and machinery has, it is believed, been brought to its lowest limit of safety. With the inventions of Count Zeppelin Germany has been in the lead, so far as actual progress in the making of dirigible balloons is concerned, but France is a close second. As long ago as 1907 the Zeppelin dirigible, 413 feet in length, attained a speed of 34 miles an hour and covered more than 200 miles in one ascent which lasted eight hours. "La Patrie," a dirigible owned by the French Government, traveled without rest from Paris to Verdun, 142.8 miles, at a mean speed of more than 20 miles an hour.

Great Britain, Italy, Spain and the United States have also produced dirigibles, but no essential advance in the principle has been made. The American Baldwin dirigible has a gas-bag of 84 feet in length, with a capacity of 18,000 cubic feet. The frame is 66 feet long; the 12-foot propeller, placed on the forward end of the frame, has a speed of 450 revolutions a minute. The ship is kept on an even keel and is lowered or raised by a number of box-like planes near the forward end, operated by the aviator. It is driven by a 20-horse-power Curtiss engine. The frame is almost as long as the gas-bag and is attached to it by means of a fine strong netting, while the operators are carried in two cars. The Baldwin is distinctly an American machine, but bears a general resemblance to the enormous German dirigibles.

Germany, represented by Count Zeppelin, has made significant contributions to aëronautics. August, 1909, was commemorated by a recording-breaking flight of the dirigible Zeppelin III. from Friedrichshafen to Berlin. It was a triumph of Count Zeppelin's scientific skill and his patient courage and perseverance. At the end of the remarkable journey the roofs, streets and parks of the German capital swarmed with people, singing and cheering, as the airship sailed round the palace and cathedral and landed in the Tempelhof parade ground, where the Emperor, Empress and many leading officials were waiting to receive the aged Count.

The dirigible, as at present designed, consists of a huge skeleton framework of aluminium alloy, over which is stretched continental rubberized fabric. The ship is sixteen-sided, with long, latticework girders springing out from the solid central prow, giving the ship the required shape. It is something more than 440 feet long and has seventeen separate gas envelopes. It can be used over water, owing to its floating cars; it can mount duplicate engines of considerable horse-power, and it has a far wider range of action and utility than any other aërial vessel. Already it holds every record in distance, altitude and duration in the air.

The helicopter is a machine with an upright shaft and revolving blades, which can rise nearly vertically or at a steep angle and has other points of advantage over the aëroplane, tho it has not yet been perfected for practical use. It is said that the helicopter was first suggested four hundred years ago by the artist, Leonardo da Vinci, "as a practical, comparatively simple and inexpensive flying device." One of the most successful helicopters has two superposed propellers in horizontal parallel planes, mounted on concentric hollow shafts, revolving in opposite directions and driven by an eight-cylinder 40-horsepower air-cooled Curtiss motor. The propellers are 17 feet in diameter and the platform is 16 feet square. The machine possesses in a marked degree the desiderata of initial stability and flexibility of movement. It has attained a speed of thirty miles an hour.

The aëroplane, it is evident, has not nearly attained its possible limit of perfection. The great originator of the flying machine was Lilienthal, who, after exhaustive study and experimentation with specially designed apparatus modeled after the wings of birds, was the first man to glide with large wing-like surfaces through the air. Lilienthal was compelled to use his machine merely as an aërial coaster, as there was no light motor then in existence.

Several distinguished aërial engineers have emulated Lilienthal's zeal, among whom are Herring, the Wright brothers and Glen H. Curtiss in America, Henri Bleriot in France, Henry Farman and Latham in England. Herring improved on Lilienthal's machine, changing his design and providing the glider with a wonderful mechanism which performed most of Lilienthal's acrobatic feats automatically. To one of these machines he later applied stored power in the shape of compressed air. Applying this to two large wooden screw propellers, he was able to fly horizontally, instead of coasting downward for the short time his power would last. Since then Curtiss has invented a light motor of great ingenuity, which has successfully been applied to the aëroplane and the helicopter.

The actual methods by which practical progress is made in the equipment and operation of these machines is more or less shrouded in mystery so far as the public is concerned, but results are evident. The Wright brothers began work on the Lilienthal basis, as did Herring. They also worked out their own methods of controlling the glider by mechanical means. The chief feature of the Wright aëroplane lies in the application of the petrol motor to the propelling blades. It is the lightness of this motor that has made progress possible in this direction. The propellers force the machine through the air and the two planes, from which the machine gets its name—biplane—support it. The two planes are rigid at their tips, which can be twisted in order to prevent too much tilting when turning. It is guided by a horizontal rudder in front and another ordinary rudder at the rear. The length of the planes had become difficult to handle, therefore it was cut in two and one plane placed above the other. The whole mechanism is handled by a single operator, who is seated in the center of the lower plane.

The Aërial Experiment Association operating at Hammondsport, New York, has contributed interesting chapters to the history of aviation. The June Bug, a very efficient type of aëroplane, was constructed by this body. In winning the trophy on July 4, 1908, the machine rose rapidly to a height of 20 feet and sped on, traversing a distance of one mile in 1 minute and 42 seconds, corresponding to an average speed of $35^1/_{10}$ miles per hour. The first trans-oceanic flight was that of Bleriot, the French experimenter, who performed in August, 1909, the feat of crossing the English Channel in a monoplane.

During the last week of August, 1909, the first international aviation race-meet held anywhere in the world took place near the city of Rheims, France. It was there that the best achievements of the heavier-than-air machines were exhibited and practically every contribution to the science of aviation by motor placed before the public. The exhibitions of aërial skill were such as to make the week a memorable one in the history of aviation. New records were made and broken every day and the safety of the flying machines was as remarkable as their efficiency. Flights were made during rain and when the wind was blowing twenty-five miles an hour.

Altogether there were thirty-eight aëroplanes entered in the various contests and races, for which $40,000 in cash prizes was offered. The machines which made flights were divided about equally between the monoplane and the biplane types, altho the latter type was rather more in favor. Of the machines of this kind five were Wright biplanes, five were biplanes of the Voisin cellular type with a tail and three of the Farman type with a tail, but without vertical partitions between the main planes. The Curtiss biplane, which is modeled closely after the pattern used by the Wright brothers, represented America.

These machines were entered in contests for speed in long-distance flights, for "sprints," for passenger-carrying power and for duration of flight. Flights of half an hour, an hour, an hour and a half became common early in the meeting, and on Tuesday M. Paulhan, driving a Voisin biplane, broke the record made by Wilbur Wright at Le Mans, France, in 1908, by flying for 2 hours and 43 minutes. In that time he covered 83 miles and only descended when his fuel was exhausted. The next day his record, in point of distance, was promptly superseded by M. Latham, the French aërialist, who made the first, tho unsuccessful, attempt to fly across the English Channel. In an Antoinette monoplane M. Latham circled the course fifteen times, covering a distance of 96 miles in 2 hours and 18 minutes. This is about the same time that Mr. Wright remained in the air on his record flight in 1908, but during that time he covered only 77 miles. On the 28th (Friday) Mr. Farman, an Englishman, flying in a biplane of his own design, once more set the mark at a higher point. He flew about 118 miles, remaining in the air more than three hours, breaking the records made both by M. Latham and M. Paulhan. His performance won for him the Champagne Grand Prize. Bleriot made the best time for a single round of the course during the first part of the week, covering the distance of $6^1/_5$ miles in almost exactly 8 minutes and 4 seconds.

In the middle of the week the International Aviation Trophy was contested. France was represented by two monoplanes, a Bleriot and an Antoinette and a Wright biplane, while America was represented by one tiny biplane with an eight-cylinder motor, designed and operated by Glen H. Curtiss. The real race was between Bleriot and Curtiss, the champions of the biplane and monoplane types of flying machines, respectively. The morning of the contest, August 28, was mild, calm and hazy at Rheims. Curtiss, after a preliminary round of the course, circled round once in front of the grand stand and crossed the line at full speed. The aëroplane pitched perceptibly and the turns were at first rather wide. Nevertheless he made the two rounds in record time, the second being $4^1/_5$ seconds faster than the first. The total time of the rounds was 15 minutes $50^3/_5$ seconds, corresponding to an average speed of 47.04 miles an hour.

Bleriot was unable to better this record, tho his monopalne flew splendidly, without any rolling or pitching. His time was $5^3/_5$ seconds more than that of Curtiss. The third place in the competition was secured by Latham, who flew at a height of about 150 feet and covered the course in 17 minutes 32 seconds. Lefebvre, the third French representative, with a Wright biplane fitted with a 40-horse-power motor, was fourth, making the course in 20 minutes 47 seconds.

The passenger-carrying competition was won by Henry

Wright Biplane Machine on Ground, with Glenn Curtiss Flying Overhead. (Taken at Rheims.)

Farman, who, after making a round with one passenger in 9 minutes $53^4/_5$ seconds, carried two people around the course at a speed of 34.96 miles an hour. The total live weight lifted by his machine was in the neighborhood of 450 pounds. Farman's biplane was the only machine that succeeded in carrying three people. Bleriot's "No. 12" monoplane, however, was the first aëroplane to accomplish this feat, which it did at Douai in June, 1909. At that time a total weight of 1,234 pounds was carried at about 30 miles an hour with a 30-horse-power motor.

The chief event of the meet at Rheims, however, was the contest for the James Gordon Bennett cup for the fastest flight of 30 kilometers. Early in the week it was evident that Bleriot and Curtiss were the two serious candidates for this prize, and the excitement over the two contestants was intense. Bleriot started on his journey, crossed the line and made the first turn at a rapid rate, flying at a low elevation. He disappeared from sight, however, at the far end of the long course, and presently it was found that his machine had suddenly dived to the ground, caught fire and was rapidly being consumed. This unfortunate accident eliminated serious rivalry to the American machine, which had already proved its remarkable powers. Curtiss made the three rounds of the course in his 60-horse-power biplane in 23 minutes 29 seconds, or at a speed corresponding to 47.6 miles an hour. The second lap of the course was made at a speed of 47.73 miles an hour. Latham; with the Antoinette monoplane, was second in this contest and the Wright biplane third. Thus the Prix de la Vitesse also fell to Curtiss, bringing to America the lion's share of the honors of the meeting.

The Curtiss biplane carries an eight-cylinder water-cooled motor, weighing 200 pounds. All valves are mechanically operated and the ignition is by magnet. The weight of the aëroplane loaded is 700 pounds; the total surface is 225 square feet. The thrust developed by the propellers is 280 pounds and its greatest speed is 47.73

miles an hour. The machine is, in comparison to the other types of biplane, compact and small, weighing less than half as much as those of his competitors.

The contest seems to have settled many of the moot questions concerning stability, landing and manipulation of the machines. The most important factor appears to be the reliability of the motor. The spectacle during the week's contests was an unprecedented one, for at times six machines were in the air at once.

The last few years have seen the revolutionary triumph of the flying machine over gravity; the coming years will see its evolutionary subjugation of the treacherous element into which it has launched itself.

"Flight is a new mental and physical experience," says Thos. S. Baldwin, the inventor of the U. S. military dirigible balloon, in a recent article. "It transposes one to a world of action and emotion in direct contrast to much of what one feels and lives on the hard surface of the globe. It tends to exhilarate and exalt the mind; it changes the registry and the workings of a number of the human senses; and it breathes into the body an overflowing measure of health, endurance and power. The elimination of the force of gravity affects the habits of gravity. The mind's freedom is denoted by an enormous increase of energy and power of action. The gravity of every square inch of the plane on which one stands or sits, and of every ounce of one's body, have been neutralized by a buoyancy of a gas lighter than air or by mechanical force and pressure upon the air.

The aëronaut brings a measure of this power from the heavens down to the earth with him as he alights from his ship. After a long voyage one touches the ground with the feeling that he can step over tall buildings, leap broad rivers and fly from place to place. His tread upon the ground is like walking upon bags of wool. This fact explains why so small a percentage of persons who fall in flight are killed. This apparent lightness and buoyancy remains in the very bones for many hours after one has made a protracted aërial voyage, and lures one back to the height of the air. It is a sensation of pleasure that the great majority of humanity have yet to know.

"First we shall fly a step in a crude machine—we have begun to do that—then in time we shall sail the air in great ships, and in some remote day man will pass through the air in his own body solely. No one who has keenly felt the joy and triumph of flight in his own person can fail to believe in this last prediction."

But it would be doing Mathematics a grievous injustice to level its applicative value to mechanical inventiveness, for if there is one thing that is more sure than another it is that the development of machinery, marvelous tho it has been, is but one—and a small—part of the heritage that Modern Mathematics has given. The scope of logistics is immeasurable, and there are not wanting evidences that abstruse subjects supposed to be inherently psychologic may come under the magic spell of number.

Whether imagination itself shall ever be reduced to a fourth dimension in space, man cannot yet know; but regarding that spiritual essence of man, the mathematician has always his fixed idea. Cassius J. Keyser couples the science with what was once known as "the queen of all sciences," and makes mathematics the key to a vaster realm than it has hitherto conquered.

"I do not believe," he says, "that the present declined state of Theology is destined to be permanent. The present is but an interregnum in her reign and her fallen days will have an end. She has been deposed mainly because she has not seen fit to avail herself promptly and fully of the dispensations of advancing knowledge. The aims, however, of the ancient mistress are as high as ever, and when she shall have made good her present lack of modern education and learned to extend a generous and eager hospitality to modern light, she will reascend and will occupy with dignity as of yore an exalted place in the ascending scale of human interests and the esteem of enlightened men. And mathematics, by the character of her inmost being, is especially qualified, I believe, to assist in the restoration.

"It was but little more than a generation ago that the mathematician, philosopher and theologian, Bernhard Bolzano, dispelled the clouds that throughout all the foregone centuries had developed the notion of Infinitude in darkness, completely sheared the great term of its vagueness without shearing it of its strength, and thus rendered it forever available for the purposes of logical discourse. Whereas, too, in former times the Infinite betrayed its presence not indeed to the faculties of Logic but only to the spiritual Imagination and Sensibility, mathematics has shown, even during the life of the elder men here present —and the achievement marks an epoch in the history of man—that the structure of Transfinite Being is open to exploration by the organon of Thought.

"Again, it is in the mathematical doctrine of Invariance, the realm wherein are sought and found configurations and types of being that, amid the swirl and stress of countless hosts of transformations, remain immutable, and the spirit dwells in contemplation of the serene and eternal reign of the subtile law of Form, it is there that Theology may find, if she will, the clearest conceptions, the noblest symbols, the most inspiring intimations, the most illuminating illustrations and the surest guarantees of the object of her teaching and her quest, an Eternal Being, unchanging in the midst of the universal flux."