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A STUDY OF THE TIME-DEPENDENT WIND-DRIVEN OCEAN CIRCULATION

by

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WHOI person

GRADUATE DIVISION OF APPLIED MATHEMATICS

BROWN UNIVERSITY

PROVIDENCE, R. I.

December, 1953

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Given in Loving Memory of

Raymond Braislin Montgomery
Scientist, R/V Atlantis maiden voyage
2 July - 26 August, 1931
****
Woods Hole Oceanographic Institution
Physical Oceanographer
1940-1949
Non-Resident Staff
1950-1960
Visiting Committee
1962-1963
Corporation Member
1970-1980
******
Faculty, New York University
1940-1944
Faculty, Brown University
1949-1954
Faculty, Johns Hopkins University
1954-1961
Professor of Oceanography,
Johns Hopkins University
1961-1975

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ERRATA

Page	Line	Replace	By
37	5	wind	wind-stress
37	7*	$\frac{\partial^2 U}{\partial y \partial \varepsilon}$	$\frac{\partial^2 U}{\partial y \partial \tau}$
38	4	(3.8)	(3.24)
38	6	(5) to (8)	(4) to (7)
38	9*	U	U_1
40	5*	(30)	(20)
44	12		Equation No. is (30)
45	3	$V_{ox\tau}$	$V_{obx\tau}$
48	4*		Equation No. is (43)
49	8*	$C_1(y)$	$C_1(y, \tau)$
51	3	sin ( )	$\sin(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2})$
63	13	$U_{20}\ V_{20} = 0$	$U_{20} = V_{20} = 0$
63	13	(49)	(7)
63	14	(3.22),(3.23),(3.26)	(4.22), (4.23), (4.26)
63	1*	$e^{-\frac{-x\varepsilon^{-1/3}}{2}}$	$e^{-\frac{x\varepsilon^{-1/3}}{2}}$
65	5	delete $e^{-\frac{x\varepsilon^{-1/3}}{2}}$ and y sin nsy	
66	6*	$\frac{(x-r-\varepsilon^{1/3}}{3}$	$(\frac{x-r-\varepsilon^{1/3}}{3})$
67	2	$\cos\frac{nys}{ns}$	$\frac{\cos nsy}{ns}$
67	3*	C_{33}	C_{32}
66	9	C_{23}	C_{32}

* Star after line indicates distance from bottom of page.

A Study of the Time-Dependent Wind-Driven Ocean Circulation[1]

by

G. Veronis[2] and G. W. Morgan[3]

Abstract. This investigation is concerned with the large-scale wind-driven motions of the ocean and their responses to a time variation in the wind. Starting from the equations of motion for an inhomogeneous fluid, a detailed formulation of the problem is presented, including the listing and discussion of the assumptions and simplifications necessary to reduce the general mathematical model to one which may be successfully attacked analytically.

Since the real ocean is baroclinic, the problem is formulated to include a non-uniform density distribution. Two special cases are considered.

(i) An ocean consisting of two superposed layers of constant density is assumed and the equations are integrated over each layer to simplify the analysis. Attempts at an analytical solution for this case were unsuccessful.

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[1] The results presented in this paper were obtained in the course of research conducted under Contract N7onr-35801.

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(ii) A more general density distribution is then assumed, but a more restrictive assumption is made concerning the vertical variation of velocity. In particular, it is assumed that there exists a (variable) depth below which the velocities are negligible. As a result of this assumption, a direct relation is found between the thermocline and the free surface. The equations are integrated from this depth up to the free surface. The linearized equations are then subjected to an analytical treatment consisting of a perturbation expansion in terms of a parameter which is proportional to the frequency of the wind variation. The resulting equations are solved by boundary layer technique.

Results are derived for the response of the mass transport to slowly varying winds, and the effect of the wind on the intensified stream near the western boundary is discussed in detail.

The two-layer steady problem is also solved and the steady position of the thermocline is determined.

1. Introduction. Much of the investigation, both theoretical and observational in the field of oceanography has centered around the dynamics of ocean currents - including the mass transport of the Gulf Stream and the Kuroshio Current, and the general oceanic circulation. Recently interest has developed regarding the response of the thermocline (the region of sharp vertical gradient of density) to a time-varying wind.

Since the time of Ekman's first paper [1]*, a large number of papers have appeared in some of the geophysical journals dealing with various aspects of ocean currents. However, analytical investigations of the problem of general oceanic circulation have met with success only in recent years. In the past decade various interesting and meaningful mathematical models have been suggested by numerous investigators. Sverdrup [2] and Reid [3] proposed a fairly simple model which seems to give very good qualitative results for a region with only one north-south boundary. Stommel [4] considered two linearized models with a simplified viscous term. His very important contribution to the overall problem is based on the difference between the results obtained with the two models. In one case, the Coriolis term was constant and the resulting streamline pattern is identical with the one in a model with no rotation. In the second case, the Coriolis term varied linearly with latitude and westward intensification resulted - a factor which

* Numbers in square brackets refer to the bibliography at the end of the paper.

was not present in the previous case. Since Stommel's paper all problems dealing with general circulation contain a varying Coriolis parameter. Munk [5] refined all the previous work and included the general viscous terms in the equations of motion. He solved the problem of a steady wind blowing over an enclosed ocean, taking account of many of the salient features which are present in the real ocean. Munk's work was extended by Munk and Carrier [6] to include oceans of various geometrical shapes, viz., triangular and semi-circular. It was further extended by Munk, Groves, and Carrier [7] to include the non-linear terms by means of a perturbation procedure.

Along with the American publications, a number of papers have appeared in Japan. Notable among the Japanese authors is Hidaka, who published a series of articles covering many of the interesting phenomena of oceanographic problems. Among his contributions are a series of three papers on drift currents in an enclosed ocean [12], [13], [14], and a contribution concerning the neglect of the non-linear terms in the solution of problems in dynamic oceanography [15].

Practically all of the work done so far in ocean current problems has been confined to motions which are independent of time. Each publication has treated some aspect of the general problem of oceanic circulation. This problem essentially consists of finding the dynamic pattern which results from a given distribution of winds acting on the ocean surface.

The complete problem contains a large number of features, such as large-scale oceanic circulation, surface waves, upwelling,

etc. To find all such motions one would have to take into account the effects of the wind, density and temperature distribution, the topography of the ocean bed and possibly even such features as salinity. Needless to say, a mathematical analysis including all these features is impossible. It is therefore necessary to decide what particular aspects of the problem one wishes to study. In this paper we shall confine our attention to large-scale wind-driven motions in the oceans and their responses to a prescribed time variation in the wind. In the Atlantic Ocean, such large-scale motions must include the Gulf Stream and its counter-currents, the Sargasso Sea, etc.

The time-dependent problem has also been considered by Ichiye [16]. We shall discuss his work later in the report.

It has been generally agreed upon by oceanographers that the type of phenomena we wish to consider can be adequately described by the dynamics of the problem alone, the temperature effects being included by way of an assumed semi-empirical density distribution. At the Woods Hole Oceanographic Institute, experiments with a model parabolic ocean basin verify the above conjecture. Hence, in the subsequent analysis, we shall neglect direct temperature dependency in the treatment of the problem and shall include only the effects of wind and gravitation.

A large part of our report is concerned with the formulation of the problem and the assumptions made to reduce the general problem to one which can be attacked mathematically. In the past a discussion of such assumptions has often been vague. It was felt therefore that an explicit and detailed analysis of

the simplifications involved in the formulation of the problem might be welcomed by workers in this field and that it might help to clear up any existing misconceptions concerning the validity of some of the assumptions.

2. Discussion of Results. At this point we shall discuss, without resorting to mathematical detail, the basic assumptions, the results, and the conclusions of the present investigation. In this manner we hope to convey a more integrated picture of the physics of the problem.

Mathematically, the motion which we want to study can be defined by the Navier-Stokes equations of motion with the viscous terms replaced by terms arising from a macroscopic viscosity, viz., an eddy viscosity. The complete non-linear equations are too difficult to solve, however, so that we are forced to make a number of simplifying assumptions which we shall list below.

1. The fluid is assumed to be incompressible, but it may be inhomogeneous.

2. The equations on a rotating sphere are approximated by equations in a rectangular Cartesian system. The effect of the sphericity of the earth is retained by allowing the Coriolis parameter to depend on the latitude. Since we shall consider a rectangular ocean in the Cartesian system, a few remarks must be made concerning the region of the sphere onto which the rectangle is mapped. The constant east-west distance of the rectangle is preserved in the mapping of the rectangle onto the

sphere. Such a mapping is not conformal since angles between lines are not preserved. The region under consideration must be well removed from the north pole.

3. The vertical acceleration terms and the viscous terms are neglected in the equation of vertical motion so that, in effect, hydrostatic pressure is assumed, i.e., $p = g\int_z^\eta \rho dz$, where η is the free surface height and $p = 0$ at $z = \eta$. The density ρ may, of course, be a function of the space coordinates. In Appendix 3 it is shown that for the problem which is independent of time, the hydrostatic pressure assumption is necessary only in the depths where there is no motion if one desires a solution for the components of the mass transport only. If it is necessary to find the shape of the free surface, however, or if the non-steady problem is considered, this assumption or some analogous one must be made.

4. As stated in the introduction, the thermodynamic effects are accounted for only empirically by stipulating a density distribution. We assume $\rho = \rho[z - T(x,y,t)]$ where the function ρ of the variable $(z - T)$ can be prescribed to fit observational data. This functional form for ρ makes the curves of constant density parallel.

5. The equations of motion are integrated over the vertical coordinate, z.

In order to perform this integration it is necessary that we specify the density distribution since ρ appears in some of the integrands. We consider two cases.

(i) The surface $z = T$ separates two layers of constant

density. The equations of motion in each layer are then integrated over the depths of the respective layers and the non-linear terms are neglected. We also neglect shear forces at the bottom of the lower layer and at the interface. No assumption is made concerning the vertical distribution of velocity*, but instead, we hope to solve for the integrated velocities (i.e., the transports) in each layer. This case is referred to as the two-layer problem. Unfortunately, it is much too difficult to handle analytically, and consequently we must consider a second problem.

(ii) The manner of performing the integration in this case will lead to a considerably simplified problem which allows us to stipulate a more general density distribution than that in (i). The density is specified as a continuous function of depth and the ocean is divided into three layers. A layer of constant density, ρ_o, lies above the surface $z = T(x,y,t)$. From $z = T$ down to $z = T - d$ (d is constant) the density increases linearly with depth from ρ_o to the value ρ_{-h}. Below $z = T - d$, the density has the constant value, ρ_{-h}.

We assume that there is a depth $z = - h(x,y,t)$ below which the velocities may be considered negligible (in some suitably defined sense). The pressure gradients will then also be negligible below $z = - h$. As a consequence of this assumption and the previous assumption of hydrostatic pressure, a relationship exists between the surface $z = T$ and the free surface

* Compare this with case (ii).

$z = \eta$, viz., $T = -\rho_o/\Delta\rho\ \eta - C$ (where $\Delta\rho = \rho_{-h} - \rho_o$ and $T = -C$ when $\eta = 0$). Thus, if the velocities are negligible in the depths of the ocean, the thermocline must respond immediately to a change in the shape of the free surface in order to maintain negligible pressure gradients at these depths.

The three assumptions, (a) hydrostatic pressure, (b) negligible velocities in the ocean depths, and (c) constant density below the thermocline, are crucial for the present case. It is, of course, possible that any one or a combination of these three assumptions may be incorrect. If this be the case, then the thermocline need not respond to the free surface immediately. The frequency of the wind variation which we shall consider later in our development will be small so that assumptions (a) and (b) seem plausible. Thus the only motion existing below the thermocline is caused by vertical shear and this motion decays exponentially with increasing depth according to Ekman [1].

The equations of motion are then integrated from the depth $z = -h$ to the free surface $z = \eta$. This problem will be called the one-layer problem because of the single integration. The depth, $z = -h$, does not appear explicitly in the integrated equations.

In both cases, the effect of the wind is represented by the shear stress at the ocean surface and appears in the evaluation of the vertical viscous terms at the upper limit of integration (free surface).

An additional difference between the two problems is

that the two-layer problem specifically restricts the fluid of the top layer to remain in the top layer and the fluid in the lower layer to remain in the lower layer. The one-layer problem has no such restriction and an interchange of fluid may result. However, because of the integration we have no information concerning this vertical motion.

6. The non-linear terms in the equations of horizontal motion are neglected. A plausibility argument for this assumption, based on the results of [7], is presented in Appendix 2. However, our results must now be considered tentative, since the case presented in the appendix for the neglect of the non-linear terms is a plausibility argument and not a justification. The primary motive for neglecting the non-linear terms is our inability to cope with them analytically.

7. The Coriolis parameter is linearized. In effect, this is comparable to linearizing the sine of an angle when the angle varies between 15° and 60°.

With the above assumptions and simplifications we are in a position to attempt a solution of the non-steady problem. The ocean is chosen to be rectangular with vertical walls as boundaries on the east and west. Because of the presence of viscosity, the boundary conditions on these walls are that the velocities vanish. The boundaries on the north and south are water boundaries.

The wind-stress is written as

$$\tau_x = -(W' + \Gamma' \sin \omega t) \cos ny$$

where W', Γ', ω, and n are constants and τ_x (Fig. 1) is the east-west component of the stress. The above form for the wind-stress may be considered as the general term of a Fourier series expansion so that the wind-stress may be generalized for the linear problem. However, for our numerical example, we have chosen ω to give a period of one year and n as $2\pi/s$ where s is the north-south length of the ocean ($0 \leq y \leq s$). The wind-stress component τ_y is assumed identically zero. Since the wind-stress is prescribed in such a manner that its y derivative vanishes at $y = 0, s$, it appears reasonable to demand that these boundaries be streamlines and that the normal derivatives of the velocities vanish there.

The one-layer problem is solved by the following procedure. The equations are non-dimensionalized. The non-dimensional velocities and free surface height are expanded in perturbation series with the non-dimensional time parameter as the perturbation parameter. Each resulting set of equations is then solved by application of the boundary layer technique.

The conditions for the validity of the expansion restrict the time variation to a maximum frequency of seasonal oscillation. In the numerical example, yearly frequency is assumed and the perturbation terms of second-order and higher are neglected. The error involved in neglecting the second-order term as compared to the zero-order term is about 5%, and it is about 20% as compared to the first-order term. The remaining physical parameters are given values which correspond roughly to those of the North Atlantic Ocean.

The following discussion will be based on the non-dimensional quantities defined in the body of the report. Whenever dimensional quantities are mentioned, we shall include the dimensions.

The graph of the north-south component, V, of the mass transport vs. the east-west coordinate x' near x' = 0, the western shore, is shown in Fig. 2 for the value y' = 0.25, i.e., where the Gulf Stream is most pronounced. The Gulf Stream region is the region of large positive V. The region of negative V adjacent to the Gulf Stream corresponds to the offshore counter-current.

The Gulf Stream responds to the wind in such a manner that the mass transport and the wind are in phase whenever the latter takes on its maximum or minimum value. At all other times the mass tramsport lags behind the wind with the greatest lag occurring when the wind reaches its steady position*. At this time the mass transport is about 9 days away from its steady value**. The length of this inverval, i.e., nine days, is independent of the frequency for slowly varying winds.

The wind (see Fig. 1) and the mass transport attain their maximum values at $\tau = \pi/2$. The mass transport now has a magnitude of $(1 + \Gamma'/W')$ times its steady value. Thus, within the accuracy of the present method of solution, the time at which maximum transport occurs and the magnitude of the maximum

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* We shall refer to the "steady position" whenever the time-dependent contribution of the wind is zero.

** i.e., the value due to its response to a steady wind $\tau_x = - W' \cos ny$.

transport are independent of the frequency. The magnitude of the out-of-phase effect (the second term in the perturbation series) which is largest when the wind has its steady value, is proportional to the frequency.

The time variation of the wind affects the Gulf Stream only by changing the mass transport through the Stream. It does not change the Stream's position.

As can be seen from Fig. 2, the relative importance of the out-of-phase effect is greatest in the counter-current.

Figure 3 is a graph of the north-south mass transport component near the eastern boundary of the rectangular ocean at the latitude $y' = 0.25$. The accompanying out-of-phase effect is shown at its maximum in the figure. V is negative on the eastern coast, i.e., the mass transport is toward the south.

Figures 4, 5, and 6 show the contour lines of the free surface in the southern half of the ocean for various times. With the values of the contour lines multiplied by -200 the three figures represent the contour lines of the thermocline. Qualitatively, the results agree fairly well with observation though some of the natural features are missing. It seems likely, however, that most missing features result from local effects which we have not taken into account.

Because of the lengthy computations involved, we have calculated numerical results for only one set of values of the parameters. It can be seen from the analytical results that if the average depth of the top layer be changed, the values for the deflection of the free surface and the out-of-phase

velocities will change. Specifically, if the depth is decreased, the free surface deflection is increased and all out-of-phase quantities are increased.

The above results appear to invalidate the solution of the problem as obtained by Ichiye [16]*. Ichiye neglected the contribution of the non-steady term in the integrated continuity equation. However, with the values of the parameters used in Section 4, the magnitude of this term in the interior of the ocean is as much as ten times that of the remaining non-steady terms which were retained in Ichiye's analysis.

We have computed the mass transport through the Gulf Stream for the one-layer steady problem. With the given wind distribution our result is 26.6 x 10^6 metric tons per second. This value is about three-fourths of Munk's value [5] and about one-third of the observed value. Munk used an empirical east-west wind distribution.

The two-layer steady problem is solved in Section 5 where it is shown that the mass transport streamline pattern is the same as in the one-layer problem. This is to be expected since, for the steady case, the same assumptions are made regarding negligible velocities below the thermocline. Thus, the height of the thermocline is shown to be proportional** to the free surface deflection. Since the free surface height is determined largely by the thickness of the top layer, the thermocline

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* In [16] the term corresponding to W' in the present paper was assumed to be identically zero. i.e., the wind had a zero mean value.

** The factor of proportionality is the reciprocal of the density difference.

variation depends on the choice of the two parameters, density difference and thickness of top layer.

By varying the two parameters we can get good qualitative agreement with observations of the shape of the thermocline. In Fig. 9, a cross-section of the computed thermocline is shown for four pairs of values of the parameters. Because of the rather vague definition of the actual thermocline, we cannot state specifically the extent of quantitative agreement between our computed results and the observed values. Consider, however, the curve in Fig. 9 with a depth of the top layer of 200 meters and a density difference of 0.0025. For that curve the results disagree by a factor of three when compared to some of the measurements of the thermocline off Chesapeake Bay [10].

The two-layer non-steady problem constitutes an attempt to drop the assumption made in the one-layer problem that the velocities vanish at some great depth. As a consequence the problem becomes much more complicated and it is necessary to introduce some other simplifying assumptions, viz., to neglect the shear forces at the bottom and at the thermocline. This may have far-reaching effects. These simplifications notwithstanding, we were unable to obtain a solution. A brief description of our attempts at such a solution follows.

First, the equations are non-dimensionalized as in the one-layer case. The integrated continuity equation for the top layer now contains the time derivative of the magnitude of the deviation of the thermocline from its equilibrium position. Since this term is very large, the perturbation method used in

the one-layer problem is restricted to a range of frequency values corresponding to less than one oscillation every hundred years. Since these results are not physically interesting no numerical results were computed.

A second method of attack is then attempted. The wind-stress term is first divided into its steady and non-steady parts and the two problems are treated separately without resorting to a perturbation in the time parameter. This method had been attempted for the one-layer problem with no success. In the present case, however, it was hoped that the new parameter involving the density difference could be used to advantage. Unfortunately, an analytic solution still appears to be quite hopeless.

The one interesting fact which seems to emerge from the attempts at the solution of our idealized, two-layer, non-steady problem concerns the magnitude of the lower layer transport. We must recall that, in the case treated, the solution is restricted to the frequency range for which the thermocline responds to the variation of the top surface in a quasi-steady manner; i.e., as a result of any change in the free surface, the thermocline assumes the same shape as it would for a steady problem with the given free surface, except for a small out-of-phase correction. In this case, the mass transport in the lower layer, excluding whatever transport may be caused by shear at the interface, is of the same order of magnitude as that portion of the transport in the upper layer which is out of phase with the wind. For a higher frequency this result does not necessarily hold true.

A final word should be said about the lack of quantitative agreement between our computed results and observation. The factor of three is not surprising when one considers the very idealized model which we have assumed. A number of more realistic assumptions may certainly affect our quantitative results by such a factor. The inclusion of the non-linear terms, a better representation of the wind effects on the water, a more natural topography, and a non-constant eddy viscosity may well alter the quantitative results and bring them into closer agreement with reality.

3. Formulation of the Problem. It is our aim to derive expressions for the velocity and the pressure satisfying the three equations of motion on a rotating sphere

$$\frac{\partial \underline{q}}{\partial t} + \underline{q} \cdot \nabla \underline{q} + 2\underline{\Omega} x \underline{q} + \underline{\Omega} x (\underline{\Omega} x \underline{r}) = -\frac{1}{\rho} \nabla p + \underline{F} + \frac{1}{\rho}(\nabla \cdot A_i \nabla)\underline{q}$$

the continuity equation

$$\nabla \cdot \underline{q} = 0$$

and the boundary condition that $\underline{q} = 0$ on a land-water boundary. Here,

$\underline{q} = (u,v,w)$* denotes the velocity vector relative to a coordinate system rotating with the sphere,

$\underline{\Omega}$ denotes the angular velocity vector representing the earth's rotation,

p denotes the pressure,

ρ denotes the density,

$\underline{F}$ denotes the external forces per unit mass (in our case, gravitation),

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* u,v,w are spherical components of velocity along the directions of the radius, the meridians, and the parallels of latitude respectively.

$(\nabla \cdot A_i\nabla)\underline{q}$ represents the eddy viscosity term (discussed below).

Let us consider the expression for the eddy viscosity term in a rectangular coordinate system, this being the system in which we shall later write our equations.

We define the operator $(\nabla \cdot A_i\nabla)$ as follows:

$$(\nabla \cdot A_i\nabla) \equiv \frac{\partial}{\partial x}(A_1 \frac{\partial}{\partial x}) + \frac{\partial}{\partial y}(A_2 \frac{\partial}{\partial y}) + \frac{\partial}{\partial z}(A_3 \frac{\partial}{\partial z}),$$

where A_1, A_2, A_3, may depend on the space coordinates. These three quantities (the coefficients of the lateral and vertical eddy viscosity) have been measured and are known to vary throughout the ocean. The definition of the viscous coefficients and our knowledge of their magnitudes, however, are rather vague. In view of this, and because of subsequent analytical simplifications, we assume that the lateral kinematic eddy viscosity coefficients are constant and equal, so that

$$\frac{1}{\rho}(\nabla \cdot A_i\nabla) \equiv A(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + \frac{1}{\rho}\frac{\partial}{\partial z}(A_3\frac{\partial}{\partial z})$$

where A is now a kinematic eddy viscosity and is constant. No simplification will be made concerning A_3.

Our continuity equation is valid for an incompressible fluid. In the steady problem the density may be more general and we have simply $\nabla \cdot (\rho\underline{q}) = 0$. In the non-steady problem, the assumption of incompressibility is imposed but the fluid may be homogeneous.

We shall want to make use of [7] regarding the effect of the non-linear terms. Because the results in [7] are discussed in terms of rectangular coordinates and because the use of

rectangular coordinates considerably simplifies the analysis, we shall first transform the equations of motion from spherical to rectangular coordinates in such a manner that the equilibrium free surface which establishes itself in the spherical system as a result of gravity and centripetal acceleration corresponds to the x-y plane of the rectangular system. The apparent gravitational force, i.e., the force which is the resultant of true gravity and centripetal acceleration, acts in a direction normal to this equilibrium surface.

In Appendix 1, it is shown that our original equations reduce to

$$\frac{\partial u'}{\partial t} + u' \frac{\partial u'}{\partial x} + v' \frac{\partial u'}{\partial y} - 2\Omega v' \sin(\frac{y}{R}) = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho}(\nabla \cdot A_i \nabla) u' \quad (1)$$

$$\frac{\partial v'}{\partial t} + u' \frac{\partial v'}{\partial x} + v' \frac{\partial v'}{\partial y} + 2\Omega u' \sin(\frac{y}{R}) = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{1}{\rho}(\nabla \cdot A_i \nabla) v' \quad (2)$$

$$- \frac{1}{\rho} \frac{\partial p}{\partial z} = g \quad (3)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \quad (4)$$

where

- x, u' denote the east-west coordinate and velocity respectively (x is positive eastward),
- y, v' denote the north-south coordinate and velocity respectively (y is positive northward),
- z, w' denote the vertical coordinate and velocity respectively (z is positive upward),
- R is the mean radius of the earth,
- g is the apparent gravitational acceleration on the earth's surface,
- $2\Omega \sin(\frac{y}{R})$ is the radial component of the angular velocity vector of the earth.

The rectangular coordinate system is oriented with the origin in the southwest corner of the ocean and with the equilibrium surface in the x-y plane.

A number of assumptions were made in the reduction of the four equations valid on a spherical earth to the four equations given above. These assumptions are listed here for the convenience of the reader who does not wish to go through the detail in Appendix 1.

(1) In the radial component of the equations of motion, the acceleration terms and the viscous terms are neglected in comparison to g, the gravitational acceleration. In essence, we assume hydrostatic pressure*.

(2) All terms involving radial velocity are neglected in the remaining two equations of motion on the supposition that the radial velocity is very small compared to the lateral velocities.

(3) The variation of the radial distance, r, over the depth of the ocean is neglected and we write $r \approx R$, the mean radius of the earth.
(Actually, the radial distance varies by about 1/1000 of its total length.)

(4) Terms which are divided by R are neglected in comparison with all other terms.

(5) The region considered must not lie close to the north pole since some terms which have been neglected

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* In Appendix 3, this assumption is discussed in more detail.

previously become infinite at the pole. In our problem the ocean is confined to a region lying south of latitude 70°.

(6) An appropriate interpretation of the results as applied to the spherical earth must be made, keeping in mind that the boundaries have been distorted. If we consider a rectangular ocean in the plane, the appropriate mapping onto the sphere would preserve the constant east-west length. Such a mapping is not conformal since angles are not preserved. (In the case of a Mercator projection, on the other hand, angles are preserved, but the east-west distance is distorted.)

Let us consider the simplified equation of vertical motion (3). In integrated form, this equation is

$$p = g \int_z^{\eta} \rho dz \qquad (3.a)$$

where η measures the deflection of the free surface from its equilibrium position and the scale of p is chosen in such a manner that $p = 0$ on $z = \eta$. Now, the density is a function of temperature and salinity. In our treatment of the problem, however, we wish to avoid the analytical difficulties introduced by including, explicitly, the energy equation and an equation of state. We propose instead to account for the thermodynamics of the problem empirically by prescribing a density distribution which roughly conforms to observation*. In particular, we

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* In Appendix 3 it is shown that a specification of the density distribution and the assumption of hydrostatic pressure are not necessary for the steady problem.

choose $\rho = \rho[z - T(x,y,t)]$, where the function ρ of the variable $(z - T)$ can be prescribed to fit observational data. We observe that this functional form for ρ makes the curves of constant density parallel to each other.

A complete analysis for the unknown quantities as functions of the four independent variables x,y,z,t is exceedingly difficult and we are forced to eliminate one variable by integrating our equations over the vertical coordinate, z, and then solving for suitably defined integrated quantities. In so doing, we lose information concerning the dependence of the unknowns on z. Since we are primarily concerned with general oceanic circulation and mass transport, however, and since the integration leads to a considerable reduction in diffculty, the advantages gained more than balance the loss of information involved.

Actually we cannot afford a complete loss of information concerning the vertical dependence of velocity. This will become apparent shortly.

The general density distribution must be specialized in order to permit integration of the equations over the vertical coordinate. Two cases will be considered.

First, let T be a surface which separates two layers of constant density so that

$$\rho[z - T(x,y,t)] = \rho_1 \quad \text{for} \quad z > T(x,y,t)$$

and

$$\rho[z - T(x,y,t)] = \rho_2 \quad \text{for} \quad z < T(x,y,t).$$

For this problem it is convenient to choose the coordinate system with the xy-planes parallel to the undisturbed

equilibrium surface and with the plane $z = 0$ at the bottom of the ocean, the bottom being assumed plane in this problem. A layer of constant density ρ_2 extends from the bottom of the ocean to the height $z = D_2 + \eta_2$ where the constant D_2 is the average height of the lower layer and η_2 is the height of the disturbed surface of this layer measured from the plane $z = D_2$. A layer of constant density ρ_1 extends from the height $z = D_2 + \eta_2$ to the free surface $z = D_1 + \eta_1$, where D_1 is the distance from $z = 0$ of the undisturbed equilibrium surface of the upper layer and η_1 is the height of the disturbed free surface of the upper layer measured from $z = D_1$.

Then equation (3.a) becomes

$$p_1 = g\rho_1(\eta_1 + D_1 - z) \qquad \text{for the upper layer} \tag{3.b}$$

$$p_2 = g\rho_1(\eta_1 + D_1 - \eta_2 - D_2) + g\rho_2(\eta_2 + D_2 - z) \qquad \text{for the lower layer} \tag{3.c}$$

If we denote all quantities in the upper and lower layers by subscripts 1 and 2, respectively, the equations (1), (2) and (4), with expressions (3.b) and (3.c) substituted for the pressure in the upper and lower layers, respectively, become

$$\frac{\partial u_1'}{\partial t} + u_1' \frac{\partial u_1'}{\partial x} + v_1' \frac{\partial u_1'}{\partial y} - 2\Omega v_1' \sin(\frac{y}{R}) = - g \frac{\partial \eta_1}{\partial x} + \frac{1}{\rho_1}(\nabla \cdot A_i \nabla) u_1' \tag{5}$$

$$\frac{\partial v_1'}{\partial t} + u_1' \frac{\partial v_1'}{\partial x} + v_1' \frac{\partial v_1'}{\partial y} + 2\Omega u_1' \sin(\frac{y}{R}) = - g \frac{\partial \eta_1}{\partial y} + \frac{1}{\rho_1}(\nabla \cdot A_i \nabla) v_1' \tag{6}$$

$$\frac{\partial u_1'}{\partial x} + \frac{\partial v_1'}{\partial y} + \frac{\partial w_1'}{\partial z} = 0 \tag{7}$$

$$\frac{\partial u_2'}{\partial t} + u_2' \frac{\partial u_2'}{\partial x} + v_2' \frac{\partial u_2'}{\partial y} - 2\Omega v_2' \sin\left(\frac{y}{R}\right) =$$

$$- g\left[b \frac{\partial \eta_2}{\partial x} + a \frac{\partial \eta_1}{\partial x}\right] + \frac{1}{\rho_i} (\nabla \cdot A_i \nabla) u_2' \qquad (8)$$

$$\frac{\partial v_2'}{\partial t} + u_2' \frac{\partial v_2'}{\partial x} + v_2' \frac{\partial v_2'}{\partial y} + 2\Omega u_2' \sin\left(\frac{y}{R}\right) =$$

$$- g\left[b \frac{\partial \eta_2}{\partial y} + a \frac{\partial \eta_1}{\partial y}\right] + \frac{1}{\rho_i} (\nabla \cdot A_i \nabla) v_2' \qquad (9)$$

$$\frac{\partial u_2'}{\partial x} + \frac{\partial v_2'}{\partial y} + \frac{\partial w_2'}{\partial z} = 0, \qquad (10)$$

where $a = \rho_1/\rho_2$, $b = (\rho_2 - \rho_1)/\rho_2 = \Delta\rho/\rho_2$.

The problem defined by equations (5) - (10) with appropriate boundary conditions is quite general in that no assumption has been made concerning the vertical distribution of velocity. As we shall see later, when the equations are integrated over z and linearized, the simplified problem is still too difficult to solve. For this reason we formulate a second problem which allows a more general density distribution but which is more restricted in other respects.

In this problem we retain, for the time being, the general form $\rho = \rho[z - T(x,y,t)]$. Then the pressure terms in equations (1) and (2) are*

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{g}{\rho} \int_z^\eta \frac{\partial \rho}{\partial x} dz + \frac{g}{\rho} \frac{\partial \eta}{\partial x} \rho_o \qquad (11.a)$$

$$\frac{1}{\rho} \frac{\partial p}{\partial y} = \frac{g}{\rho} \int_z^\eta \frac{\partial \rho}{\partial y} dz + \frac{g}{\rho} \frac{\partial \eta}{\partial y} \rho_o \qquad (11.b)$$

* For the present problem the plane z = 0 lies on the undisturbed equilibrium free surface.

where $\rho_o = \rho[\eta - T(x,y,t)]$, the density at the free surface.

If these terms be substituted into (1) and (2), we have

$$\frac{\partial u'}{\partial t} + u' \frac{\partial u'}{\partial x} + v' \frac{\partial u'}{\partial y} - 2\Omega v' \sin\left(\frac{y}{R}\right) = - \frac{g}{\rho} \int_z^\eta \frac{\partial \rho}{\partial x} dz - \frac{g}{\rho} \frac{\partial \eta}{\partial x} \rho_o + \frac{1}{\rho}(\nabla \cdot A_i \nabla) u' \qquad (12)$$

$$\frac{\partial v'}{\partial t} + u' \frac{\partial v'}{\partial x} + v' \frac{\partial v'}{\partial y} + 2\Omega u' \sin\left(\frac{y}{R}\right) = - \frac{g}{\rho} \int_z^\eta \frac{\partial \rho}{\partial y} dz - \frac{g}{\rho} \frac{\partial \eta}{\partial y} \rho_o + \frac{1}{\rho}(\nabla \cdot A_i \nabla) v'. \qquad (13)$$

As stated previously, the problem will be simplified by integrating the equations over the vertical coordinate, z.

Let us first consider the problem defined by the equations (4), (12), (13). We assume that there is a depth $z = - h(x,y,t)$ below which the velocities may be considered negligible* (in some suitably defined sense), and we integrate from $z = - h$ up to the free surface. The depth $z = - h(x,y,t)$ may, of course, vary from point to point in the ocean. Since the velocities are negligibly small below $z = - h$, the horizontal pressure gradients must also be negligibly small and we may therefore write

$$\left.\frac{1}{\rho} \frac{\partial p}{\partial x}\right|_{z=-h} = 0, \qquad \left.\frac{1}{\rho} \frac{\partial p}{\partial y}\right|_{z=-h} = 0. \qquad (14)$$

We must now specialize the general form of the density distribution because an integration involving ρ will actually

* This assumption is the fundamental difference between the two problems considered.

have to be carried out.

Define $\rho = \rho[z - T(x,y,t)]$ in such a way that

$$\left.\begin{aligned} \rho &= \rho_o, \text{ a constant} && \text{for } \eta \geq z > T \\ \rho &= [1 + c(T - z)]\rho_o && \text{for } T \geq z \geq T - d \text{ (c,d constants)} \\ \rho &= \rho_{-h} = (1 + cd)\rho_o && \text{for } T - d > z \,. \end{aligned}\right\} \quad (15)$$

With this definition the density is a continuous function of depth and the ocean is divided into three distinct layers. A layer of constant density, ρ_o, lies above a region in which the density increases linearly with depth from ρ_o to the value ρ_{-h}. Finally, at the bottom, there is a layer of constant density, ρ_{-h}. This prescribed distribution agrees well with the observed density distribution.

If ρ, as given by (15), be substituted into equation (14), we find that*

$$\frac{\partial T}{\partial x} = -\frac{\rho_o}{\Delta\rho}\frac{\partial \eta}{\partial x}, \quad \frac{\partial T}{\partial y} = -\frac{\rho_o}{\Delta\rho}\frac{\partial \eta}{\partial y} \quad (16)$$

where $\Delta\rho = \rho_{-h} - \rho_o$.

If we integrate equations (16), we obtain

$$T = -\frac{\rho_o}{\Delta\rho}\,\eta - C \quad (17)$$

where $z = -C$ is the constant depth of T when $\eta = 0$. Physically, $z = -C$ is an average depth of the top layer or the depth of T when the ocean surface is undisturbed (i.e., in the absence of winds). These two quantities are, of course, identical.

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* The algebraic manipulation is given in Appendix 4(a).

Let us next integrate equations (12) and (13) from $z = -h$ to $z = \eta$. The pressure terms become*

$$- \int_{-h}^{\eta} \frac{1}{\rho} \frac{\partial p}{\partial x} dz = - gD \frac{\partial \eta}{\partial x} - g \frac{\rho_{-h}}{\Delta\rho} \eta \frac{\partial \eta}{\partial x} \qquad (18.a)$$

$$- \int_{-h}^{\eta} \frac{1}{\rho} \frac{\partial p}{\partial y} dz = - gD \frac{\partial \eta}{\partial y} - g \frac{\rho_{-h}}{\Delta\rho} \eta \frac{\partial \eta}{\partial y} \qquad (18.b)$$

where $D = C + d/2$, and the complete equations are

$$\frac{\partial \bar{U}}{\partial t} + \bar{\rho} \int_{-h}^{\eta} u' \frac{\partial u'}{\partial x} dz + \bar{\rho} \int_{-h}^{\eta} v' \frac{\partial u'}{\partial y} dz - 2\Omega \bar{V} \sin \left(\frac{y}{R}\right)$$

$$= - gD \frac{\partial \eta \bar{\rho}}{\partial x} - g \frac{\rho_{-h}}{\Delta\rho} \eta \frac{\partial \eta \bar{\rho}}{\partial x} + A \Delta \bar{U} + \left(A_3 \frac{\partial u'}{\partial z}\right)\Big|_{-h}^{\eta **} \qquad (19)$$

$$\frac{\partial \bar{V}}{\partial t} + \bar{\rho} \int_{-h}^{\eta} u' \frac{\partial v'}{\partial x} dz + \bar{\rho} \int_{-h}^{\eta} v' \frac{\partial v'}{\partial y} dz + 2\Omega \bar{U} \sin \left(\frac{y}{R}\right)$$

$$= - gD \frac{\partial \eta \bar{\rho}}{\partial y} - g \frac{\rho_{-h}}{\Delta\rho} \eta \frac{\partial \eta \bar{\rho}}{\partial y} + A \Delta \bar{V} + \left(A_3 \frac{\partial v'}{\partial z}\right)\Big|_{-h}^{\eta} \qquad (20)$$

where

$$\bar{U} = \int_{-h}^{\eta} \bar{\rho} u' dz, \qquad \bar{V} = \int_{-h}^{\eta} \bar{\rho} v' dz,$$

$\bar{\rho}$ is a constant, average density,

and $$\lambda(x,y,z,t)\Big|_{-h}^{\eta} = \lambda(x,y,\eta,t) - \lambda(x,y,-h,t).$$

The non-linear terms, $u'(x,y,\eta,t)\partial\eta/\partial t$, etc., from

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* See Appendix 4(b) for the details.

** Since the viscous terms are, in any case, only approximations to the actual shear stresses, we have made the further approximation

$$\int_{-h}^{\eta} \frac{1}{\rho} \frac{\partial}{\partial z} \left(A_3 \frac{\partial u'}{\partial z}\right) dz \approx \frac{1}{\bar{\rho}} \int_{-h}^{\eta} \frac{\partial}{\partial z} \left(A_3 \frac{\partial u'}{\partial z}\right) dz = \frac{1}{\bar{\rho}} A_3 \frac{\partial u'}{\partial z}\Big|_{-h}^{\eta}.$$

the interchange of integrals and derivatives of the velocity terms have been neglected. We have defined $\overline{U}$ and $\overline{V}$ as mass transport components rather than as volume transport components (by simply including an average density in the definition) because we want to compare some of our quantitative results with observations and with the results of Munk, both of which are given in terms of mass transport.

The terms $A_3 \frac{\partial u'}{\partial z}\Big|_{-h}^{\eta}$ and $A_3 \frac{\partial v'}{\partial z}\Big|_{-h}^{\eta}$ must give the wind-stress terms since they represent the shear stress evaluated at the upper limits (the shear stress terms at $z = -h$ are negligible since $-h$ was chosen as the depth where the motion becomes negligible). Thus

$$A_3 \frac{\partial u'}{\partial z}\Big|_{-h}^{\eta} = \tau_x = x \text{ component of wind stress}$$

$$A_3 \frac{\partial v'}{\partial z}\Big|_{-h}^{\eta} = \tau_y = y \text{ component of wind stress.}$$

In the equation of continuity we shall want to make use of the kinematic free surface condition [9]

$$\frac{d}{dt}\left[z - \eta(x,y,t)\right] = 0 \qquad \text{at } z = \eta.$$

When expanded, this equation reads

$$w'|^{\eta} = \frac{\partial \eta}{\partial t} + u'|^{\eta} \frac{\partial \eta}{\partial x} + v'|^{\eta} \frac{\partial \eta}{\partial y}$$

where $w'|^{\eta}$ etc. denotes the value of $w'(x,y,z,t)$ at $z = \eta$.

Integration of the continuity equation (4) yields

$$\frac{\partial \overline{U}}{\partial x} + \frac{\partial \overline{V}}{\partial y} - \overline{\rho}\, u'|^{\eta} \frac{\partial \eta}{\partial x} - \overline{\rho}\, v'|^{\eta} \frac{\partial \eta}{\partial y} + \overline{\rho}\, w'|^{\eta} = 0$$

where $w'\big|_{-h}$ is negligible by definition of h(x,y,t). Substituting the free surface condition, we have

$$\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y} = - \frac{\partial \eta \bar{\rho}}{\partial t} . \qquad (21)$$

Equations (19) and (20) are now further simplified by neglecting the non-linear terms. The reader is referred to Appendix 2 for a detailed plausibility argument concerning this step*.

Two final simplifications will be made in equations (19) and (20). The Coriolis parameter $2\,\Omega \sin(\frac{y}{R})$ will be linearized by writing $2\,\Omega \sin(\frac{y}{R}) \approx \beta y$ where $\beta = 2\Omega/R$.

In addition, if the velocities are found in some manner, then the free surface shape can be obtained by integrating the equations (19) and (20) (neglecting the integrals of the non-linear terms) with respect to x and y respectively. This yields

$$(gD\bar{\rho}\,\eta + \frac{\bar{\rho}}{2}\,\frac{\rho_{-h}}{\Delta\rho}\,g\,\eta^2) = X$$

where X denotes a known function. The solution of this quadratic equation in η is

$$\eta = \frac{- D + D\sqrt{1 + \frac{2\rho_{-h}}{\Delta\rho}\,\frac{1}{g\bar{\rho}}\,\frac{X}{D^2}}}{\frac{\rho_{-h}}{\Delta\rho}}$$

* It must be emphasized that the argument presented in Appendix 2 is one of plausibility and not one of justification. In view of the desirability of obtaining an analytic solution, we neglect the non-linear terms in the hope that the results will agree qualitatively with observation and will so furnish a mathematical description of the ocean circulation.

But

$$\sqrt{1 + \frac{2\rho_{-h}}{\Delta\rho} \frac{1}{g\bar{\rho}} \frac{X}{D^2}} \approx 1 + \frac{\rho_{-h}}{\Delta\rho} \frac{1}{g\bar{\rho}} \frac{X}{D^2}$$

if

$$\frac{2\rho_{-h}}{\Delta\rho} \frac{1}{g\bar{\rho}} \frac{X}{D^2} < 1.$$

Hence

$$\eta \approx \frac{1}{g\bar{\rho}D} X$$

and the pressure term can be approximated by

$$- gD \frac{\partial \eta\bar{\rho}}{\partial x}$$

provided the above inequality holds. It will be shown in Section 5 that the values of the constants which are appropriate to our problem satisfy this condition.

Hence, the final equations take the form

$$\frac{\partial \bar{U}}{\partial t} - \beta y\bar{V} = - gD \frac{\partial \overline{\eta\rho}}{\partial x} + A\Delta\bar{U} + \tau_x \tag{22}$$

$$\frac{\partial \bar{V}}{\partial t} + \beta y\bar{U} = - gD \frac{\partial \overline{\eta\rho}}{\partial y} + A\Delta\bar{V} + \tau_y \tag{23}$$

$$\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y} = - \frac{\partial \eta\bar{\rho}}{\partial t} . \tag{24}$$

The boundary conditions are $\bar{U} = \bar{V} = 0$ on a land-water boundary. The wind-stress is prescribed to be

$$\tau_x = - (W' + \Gamma' \sin \omega t)\cos ny, \quad \tau_y = 0$$

where W', Γ' represent the magnitude of the mean wind-stress and the amplitude of the time variation of the wind-stress, respectively,

ω is the frequency of the wind variation,

n is the wave number associated with the wind distribution.

One can consider the above form for the wind as a typical term in a Fourier series for a more general wind distribution. The numerical results in this report are based on a value of ω corresponding to a period of one year and n is set equal to $2\pi/s$ where s is the north-south length of the ocean.

The problem defined by equations (22), (23), (24) together with the boundary conditions and the wind-stress term will be referred to as the one-layer problem or Problem 1; ("one layer" because the integration over z is carried out over the entire depth).

For the second problem in which the density stratification is specified as two constant density layers, we have equations (5) - (10). Each equation will be integrated over the vertical coordinate, z, with (5) - (7) integrated over the top layer, i.e., from $z = D_2 + \eta_2$ to $z = D_1 + \eta_1$, and (8) - (10) integrated over the lower layer, i.e., from $z = 0$ to $z = D_2 + \eta_2$.

As in problem 1, the non-linear terms, $u_1|^{\eta}\ \partial\eta/\partial t$ etc., resulting from the interchange of differentiation and integration, are neglected. The viscous terms are integrated in the same manner and the Coriolis parameter is again linearized. Then the integrated forms of (5) - (10) are

$$\frac{\partial \bar{U}_1}{\partial t} - \beta y \bar{V}_1 + g(D_1 - D_2 + \eta_1 - \eta_2)\frac{\partial \eta_1 \rho_1}{\partial x} = A\Delta \bar{U}_1 + \tau_{1x} - \tau_{2x} \quad (25)$$

$$\frac{\partial \bar{V}_1}{\partial t} + \beta y \bar{U}_1 + g(D_1 - D_2 + \eta_1 - \eta_2)\frac{\partial \eta_1 \rho_1}{\partial y} = A\Delta \bar{V}_1 + \tau_{1y} - \tau_{2y} \quad (26)$$

$$\frac{\partial \bar{U}_1}{\partial x} + \frac{\partial \bar{V}_1}{\partial y} = -\frac{\partial}{\partial t}(\rho_1\eta_1 - a\rho_2\eta_2) \quad (27)$$

$$\frac{\partial \bar{U}_2}{\partial t} - \beta y \bar{V}_2 + g(D_2 + \eta_2)\frac{\partial}{\partial x}[b\rho_2\eta_2 + \rho_1\eta_1] = A\Delta\bar{U}_2 + \tau_{2x} - \tau_{ox} \tag{28}$$

$$\frac{\partial \bar{V}_2}{\partial t} + \beta y \bar{U}_2 + g(D_2 + \eta_2)\frac{\partial}{\partial y}[b\rho_2\eta_2 + \rho_1\eta_1] = A\Delta\bar{V}_2 + \tau_{2y} - \tau_{oy} \tag{29}$$

$$\frac{\partial \bar{U}_2}{\partial x} + \frac{\partial \bar{V}_2}{\partial y} = - \frac{\partial}{\partial t}(\rho_2\eta_2) \tag{30}$$

where

$$\bar{U}_1 = \int_{D_2+\eta_2}^{D_1+\eta_1} \rho_1 u_1' dz, \quad \bar{V}_1 = \int_{D_2+\eta_2}^{D_1+\eta_1} \rho_1 v_1' dz,$$

$$\bar{U}_2 = \int_0^{D_2+\eta_2} \rho_2 u_2' dz, \quad \bar{V}_2 = \int_0^{D_2+\eta_2} \rho_2 v_2' dz,$$

τ_{1x}, τ_{1y} are the x and y components, respectively, of the wind-stress on the free surface

τ_{2x}, τ_{2y} are the x and y components, respectively, of the shear stress between the lower layer and the upper layer at the interface,

τ_{ox}, τ_{oy} are the x and y components, respectively, of the shear stress between water in the lower layer and the ocean bottom.

We specify τ_{1x} to take the same form as τ_x in Problem 1. The remaining shear stress terms are assumed to be negligible. The boundary conditions are $\bar{U}_1 = \bar{V}_1 = \bar{U}_2 = \bar{V}_2 = 0$ on a land-water boundary, i.e., vanishing mass transport in each layer. These conditions are much more restrictive than the boundary conditions of the one-layer problem since there can be no vertical interchange of transport across the interface at the boundaries.

Equations (25) - (30), together with the boundary conditions and the wind-stress, constitute Problem 2, or the two-

layer problem (the vertical integration being carried out in two steps).

It may seem to the reader at this point that, since we have integrated the equations of motion over the vertical coordinate z in both problems, there is nothing to be gained by considering Problem 2 in which the density distribution is more specialized than that of Problem 1. Because of the importance of this point, we shall discuss the significance of the two problems in more detail.

Needless to say, the problem of greatest interest includes the more general density distribution of Problem 1, the four independent coordinates x,y,z,t, and the full non-linear equations. The wind-stress components appear as the values of the vertical shear at the free surface $z = \eta(x,y,t)$. The solution of this problem would, of course, include complete information concerning the dependence of the motion on z. Being unable to attack this problem, we are forced to integrate the equations over z and to content ourselves with a solution for the transport components.

At first this integration over the vertical coordinate, z, appears to have only one shortcoming, viz., a loss of information concerning the vertical distribution of velocity. We cannot, however, completely afford such a loss of information in the formulation of the "transport" problem and some recourse to field evidence is necessary. Unfortunately, however, accurate observational data are extremely difficult to obtain. In particular, it is generally held that the motion in the deep

layers of the oceans is negligible, but no definite conclusions have been established to this effect. It is because of this uncertainty that we consider the two separate problems, 1 and 2. If the motion of deep water is really negligible, the pressure gradient in deep water is also negligible and the assumptions of Problem 1 are justified with the result that the thermocline responds instantaneously to a change in the free surface height provided the hydrostatic pressure assumption is also valid. Consequently, the only motion existing in the layer below the bottom of the thermocline is that due to the shear force exerted by the water at the depth $z = T - d$ onto the water below it. Vertical shear will extend the motion to lower depths but the velocities will decay exponentially in the vertical direction [1] until they become negligible.

If the motion of deep water is not negligible, then we must consider Problem 2 where no such assumption is made. In that case, the thermocline does not necessarily respond immediately to a change in the free surface and, consequently, a pressure gradient may result. Since the fluid in the bottom layer is homogeneous and since the wave length of the thermocline is large compared to the depth of the lower layer, a velocity with uniform vertical profile is set up, (hydrostatic pressure being again assumed). The shear stress, τ_{2x}, exerted by the water of the upper layer onto the surface of the lower layer also causes a velocity in the lower layer. This velocity is not uniform vertically. The problem including the effect of

τ_{2x} and, in addition, the stress of the ocean bottom on the lower layer, is so complex that an analytic solution is out of the question. We therefore assume that the effects of these shear stresses on the velocity in the lower layer are negligible when compared to the velocity resulting from the variation of the thermocline.

If the two problems were now solved and the results compared with available observational data, it might be possible to determine whether or not sensible deep-water motion exists. As we shall see in Sec. 5, however, Problem 2 cannot be solved by the methods used in the present paper, and numerical methods of solution may have to be employed.

4. Solution to Problem 1. The solution to Problem 1 will be carried out by means of a boundary layer technique. For the convenience of the reader who is not familiar with this technique and who wishes to follow the details of the present section, a discussion of boundary layer analysis is presented in Appendix 5.

The solution of differential equations by boundary layer analysis can be carried out most conveniently if the equations are first put into non-dimensional form. Let the rectangular ocean have dimensions

$$0 \leq x \leq r_1, \quad 0 \leq y \leq s \quad \text{(Fig. 1).}$$

Choose as a reference length the north-south dimension, s, and define dimensionless coordinates x', y' by

$$y = sy', \quad x = sx'.$$

Then the east-west and north-south dimensions of the ocean in non-dimensional coordinates will be

$$0 \leq x' \leq \frac{r_1}{s} \equiv r, \quad 0 \leq y' \leq 1.$$

We shall assume that the ocean is bounded by land on $x' = 0, r$ and by water on $y' = 0, 1$.

Now differentiate equation (3.23) with respect to x and equation (3.22) with respect to y and subtract. Substituting for the prescribed wind-stress, τ_x, we then have

$$\frac{\partial}{\partial t}\left(\frac{\partial \bar{V}}{\partial x} - \frac{\partial \bar{U}}{\partial y}\right) + \beta y\left(\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y}\right) + \beta \bar{V} = A\,\Delta\left(\frac{\partial \bar{V}}{\partial x} - \frac{\partial \bar{U}}{\partial y}\right)$$

$$- [nW' + n\Gamma' \sin \omega t] \sin ny. \qquad (1)$$

Introducing

$$x' = \frac{x}{s}, \quad y' = \frac{y}{s}, \quad \tau = \omega t$$

and defining

$$nW' = W, \quad n\Gamma' = \Gamma, \quad \alpha = \frac{\Gamma'}{W'} = \frac{\Gamma}{W}.$$

equation (1) becomes

$$\frac{\omega}{s}\frac{\partial}{\partial \tau}\left(\frac{\partial \bar{V}}{\partial x'} - \frac{\partial \bar{U}}{\partial y'}\right) + \beta y'\left(\frac{\partial \bar{U}}{\partial x'} + \frac{\partial \bar{V}}{\partial y'}\right) + \beta \bar{V}$$

$$= \frac{A}{s^3}\left[\frac{\partial^3 \bar{V}}{\partial x'^3} + \frac{\partial^3 \bar{V}}{\partial x' \partial y'^2} - \frac{\partial^3 \bar{U}}{\partial x'^2 \partial y} - \frac{\partial^3 \bar{U}}{\partial y'^3}\right]$$

$$- W\,[1 + \alpha \sin \tau] \sin nsy' \qquad (2)$$

or

$$\frac{\omega}{sW}\frac{\partial}{\partial\tau}\left[\frac{\partial\overline{V}}{\partial x'} - \frac{\partial\overline{U}}{\partial y'}\right] + \frac{\beta y'}{W}\left(\frac{\partial\overline{U}}{\partial x'} + \frac{\partial\overline{V}}{\partial y'}\right) + \frac{\beta}{W}\overline{V}$$

$$= \frac{A}{Ws^3}\left[\frac{\partial^3\overline{V}}{\partial x'^3} + \frac{\partial^3\overline{V}}{\partial x'\,\partial y'^2} - \frac{\partial^3\overline{U}}{\partial x'^2\partial y'} - \frac{\partial^3\overline{U}}{\partial y'^3}\right]$$

$$- [1 + \alpha \sin \tau] \sin nsy'. \qquad (3)$$

Now, since the term $(1 + \alpha \sin \tau)\sin nsy'$ is of order unity*, and since this term represents the wind which generates the velocities, it is appropriate to choose a dimensionless velocity which will also be of order unity. Hence we select a non-dimensional term containing the velocity which is presumably of order one. The term suggested by an inspection of (3) is $\beta\overline{V}/W$ and we therefore put

$$V = \frac{\beta\overline{V}}{W} \quad \text{and} \quad U = \frac{\beta\overline{U}}{W}.$$

We shall drop the primes from the x' and y' coordinates and work in the non-dimensional system henceforth. With the definitions, $\varepsilon = A/\beta s^3$ and $\delta = \omega/\beta s$ equation (3) becomes

$$\delta[V_x - U_y]_\tau + y[U_x + V_y] + V = \varepsilon[V_{xxx} + V_{xyy} - U_{xxy} - U_{yyy}]$$

$$- (1 + \alpha \sin \tau)\sin nsy \qquad (4)$$

where $V_x \equiv \partial V/\partial x$, $(V_x - U_y)_\tau \equiv \partial^2 V/\partial x\partial\tau - \partial^2 U/\partial y\partial\tau$, etc.

If we non-dimensionalize the momentum equations (3.22) and (3.23) and the continuity equation (3.24) by means of the above definitions, we must introduce a new parameter Θ and a variable H defined by

$$\Theta = \frac{ngD}{\beta^2 s^3}, \qquad H = \frac{\overline{\rho}\eta\beta^2 s^2}{W}.$$

* As will be seen later, we shall choose α to be 0.2.

The equations become

$$ns\delta \frac{\partial U}{\partial \tau} - nsy\, V + \Theta \frac{\partial H}{\partial x} = ns\,\varepsilon \Delta U - (1 + \alpha \sin \tau)\cos nsy \qquad (5)$$

$$ns\delta \frac{\partial V}{\partial \tau} + nsy\, U + \Theta \frac{\partial H}{\partial y} = ns\,\varepsilon \Delta V \qquad (6)$$

and (3.8) becomes

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = -\,\delta \frac{\partial H}{\partial \tau}\,. \qquad (7)$$

Attempts to solve equations (5) to (8) in closed form were unsuccessful. We therefore resorted to seeking solutions by a perturbation expansion in the parameter δ.

Let

$$U = U_o + \delta U_1 + \delta^2 U_2 + \dots$$

$$V = V_o + \delta V_1 + \delta^2 V_2 + \dots$$

$$H = H_o + \delta H_1 + \delta^2 H_2 + \dots\,.$$

Our formal procedure is to regard the coefficients U_o, U_1, etc., as coefficients in a power series in δ.

Let us substitute the expansions into equations (4), (5), (6) and (7). We have

$$\delta[V_{ox} + \delta V_{1x} + \dots - U_{oy} - \delta U_{1y} - \dots]_\tau$$

$$+ y\,[U_{ox} + \delta U_{1x} + \dots + V_{oy} + \delta V_{1y} + \dots]$$

$$+ V_o + \delta V_1 + \dots = \varepsilon[V_{oxxx} + \delta V_{1xxx} + \dots$$

$$+ V_{oxyy} + \delta V_{1xyy} + \dots - U_{oxxy} - \delta U_{1xxy} - \dots$$

$$- U_{oyyy} - \delta U_{1yyy} - \dots] - (1 + \alpha \sin \tau)\sin nsy \qquad (8)$$

$$ns\delta\left[\frac{\partial U_o}{\partial \tau} + \delta \frac{\partial U_1}{\partial \tau} + \ldots\right] - nsy[V_o + \delta V_1 + \ldots]$$
$$+ \Theta\left[\frac{\partial H_o}{\partial x} + \delta \frac{\partial H_1}{\partial x} + \ldots\right] = ns\varepsilon\Delta[U_o + \delta U_1 + \ldots]$$
$$- (1 + \alpha \sin\tau)\cos nsy \qquad (9)$$

$$ns\delta\left[\frac{\partial V_o}{\partial \tau} + \delta \frac{\partial V_1}{\partial \tau} + \ldots\right] + nsy[U_o + \delta U_1 + \ldots]$$
$$+ \Theta\left[\frac{\partial H_o}{\partial y} + \delta \frac{\partial H_1}{\partial y} + \ldots\right] = ns\varepsilon\Delta[V_o + \delta V_1 + \ldots] \qquad (10)$$

$$\frac{\partial U_o}{\partial x} + \delta \frac{\partial U_1}{\partial x} + \ldots + \frac{\partial V_o}{\partial y} + \delta \frac{\partial V_1}{\partial y} + \ldots = -\delta\left[\frac{\partial H_o}{\partial \tau} + \delta \frac{\partial H_1}{\partial \tau} + \ldots\right]. \qquad (11)$$

If we regroup each of these equations so as to combine the coefficients of each power of δ, we have, upon retaining terms in δ^o and δ only:

$$\left\{y[U_{ox} + V_{oy}] + V_o - \varepsilon[V_{oxxx} + V_{oxyy} - U_{oxxy} - U_{oyyy}]\right.$$
$$\left. + (1 + \alpha \sin\tau)\sin nsy\right\} + \left\{V_{ox\tau} - U_{oy\tau} + y[U_{1x} + V_{1y}]\right.$$
$$\left. + V_1 - \varepsilon[V_{1xxx} + V_{1xyy} - U_{1xxy} - U_{1yyy}]\right\}\delta + \ldots = 0 \qquad (12)$$

$$\left\{-nsyV_o + \Theta\frac{\partial H_o}{\partial x} - ns\varepsilon\Delta U_o + (1 + \alpha\sin\tau)\cos nsy\right\}$$
$$+ \left\{ns\frac{\partial U_o}{\partial \tau} - nsyV_1 + \Theta\frac{\partial H_1}{\partial x} - ns\varepsilon\Delta U_1\right\}\delta + \ldots = 0 \qquad (13)$$

$$\left\{nsyU_o + \Theta\frac{\partial H_o}{\partial y} - ns\varepsilon\Delta V_o\right\} + \left\{ns\frac{\partial V_o}{\partial \tau} + nsU_1 + \Theta\frac{\partial H_1}{\partial y} - ns\varepsilon\Delta V_1\right\}\delta + \ldots = 0 \qquad (14)$$

$$\left\{\frac{\partial U_o}{\partial x} + \frac{\partial V_o}{\partial y}\right\} + \left\{\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y} + \frac{\partial H_0}{\partial \tau}\right\}\delta + \ldots = 0. \qquad (15)$$

Setting each of the coefficients of δ equal to zero we have as the zero order equations for (12) and (15)

$$\varepsilon[V_{oxxx} + V_{oxyy} - U_{oxxy} - U_{oyyy}] - V_o = (1 + \alpha \sin \tau)\sin nsy \quad (16)$$

$$U_{ox} + V_{oy} = 0 \quad (17)$$

and the boundary conditions

$$U_o = V_o = 0 \quad \text{on} \quad x = 0,\ x = r. \quad (18)$$

With the particular wind distribution prescribed we will also be able to satisfy the additional boundary conditions

$$V_o = \frac{\partial U_o}{\partial y} = 0 \quad \text{on} \quad y = 0,1. \quad (18.a)$$

We shall proceed to solve equations (16), (17) together with the boundary conditions (18), (18.a) for the velocities U_o and V_o.

Define a stream function

$$V_o = \frac{\partial \psi}{\partial x}, \quad U_o = -\frac{\partial \psi}{\partial y} \quad (19)$$

so that (17) is satisfied identically. Then (16) can be written

$$\varepsilon \Delta\Delta\psi - \psi_x = (1 + \alpha \sin \tau)\sin nsy \quad (20)$$

where $\Delta\Delta(\ )$ is the biharmonic operator $\frac{\partial^4(\ )}{\partial x^4} + 2\frac{\partial^4(\ )}{\partial x^2 \partial y^2} + \frac{\partial^4(\ )}{\partial y^4}$.

Equation (~~30~~ 20) is similar to the one solved by Munk [5] and Munk and Carrier [6]. In the present case, however, the non-dimensional time, τ, appears as a parameter, so that our problem corresponds to a quasi-steady problem.

Equation (20) together with the boundary conditions

$$\psi = \psi_x = 0 \quad \text{on } x = 0, r$$

$$\psi = \psi_{yy} = 0 \quad \text{on } y = 0, 1 \tag{20.a}$$

can be solved for ψ by applying the boundary-layer technique*
to the boundaries $x = 0, r$. The solution is

$$\psi = (1 + \alpha \sin \tau) \sin nsy \Big\{ -x + r - \varepsilon^{1/3} + \varepsilon^{1/3} e^{(x-r)\varepsilon^{-1/3}} + \Big[(\varepsilon^{1/3} - r) \cos\Big(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\Big) + \Big(\sqrt{3}\,\varepsilon^{1/3} - \frac{r}{\sqrt{3}}\Big) \sin\Big(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\Big) \Big] e^{-\frac{x\varepsilon^{-1/3}}{2}} \Big\}. \tag{21}$$

From (19) U_o and V_o are found to be

$$U_o = -ns(1 + \alpha \sin \tau) \cos nsy \Big\{ -x + r - \varepsilon^{1/3} + \varepsilon^{1/3} e^{(x-r)\varepsilon^{-1/3}} + \Big[(\varepsilon^{1/3} - r) \cos\Big(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\Big) + \Big(\sqrt{3}\,\varepsilon^{1/3} - \frac{r}{\sqrt{3}}\Big) \sin\Big(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\Big) \Big] e^{-\frac{x\varepsilon^{-1/3}}{2}} \Big\} \tag{22}$$

$$V_o = (1 + \alpha \sin \tau) \sin nsy \Big\{ -1 + e^{(x-r)\varepsilon^{-1/3}} + \Big[\cos\Big(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\Big) + \Big(\frac{2r\varepsilon^{-1/3}}{\sqrt{3}} - \sqrt{3}\Big) \sin\Big(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\Big) \Big] e^{-\frac{x\varepsilon^{-1/3}}{2}} \Big\}. \tag{23}$$

The zero-order equations derived from (13) and (14) are

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* The problem defined by equations (20), (20.a) is solved in detail in Appendix 5 by means of the boundary layer technique. The method used in the remainder of this paper is described in detail in that section. Munk and Carrier [6] used this method for solving the steady problem in a triangular ocean.

$$\Theta H_{ox} = nsyV_o + ns\varepsilon\Delta U_o - (1 + \alpha \sin\tau)\cos\ nsy \qquad (24)$$

$$\Theta H_{oy} = - nsyU_o + ns\varepsilon\Delta V_o. \qquad (25)$$

Solving for H_o, we have, (neglecting terms of order ε),

$$\Theta H_o = (1 + \alpha \sin\tau)(\cos nsy + nsy \sin nsy)(- x + r - \varepsilon^{1/3})$$
$$+ (1 + \alpha \sin\tau) nsy \sin nsy \Big\{ \varepsilon^{1/3} e^{(x-r)\varepsilon^{-1/3}}$$
$$+ [(\varepsilon^{1/3} - r)\cos(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}) +$$
$$+ (\sqrt{3}\,\varepsilon^{1/3} - \frac{r}{\sqrt{3}})\sin(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2})] e^{-\frac{x\varepsilon^{-1/3}}{2}} \Big\}. \qquad (26)$$

First-Order Solution

From equations (12) and (15) the terms of first order in δ are found to be

$$\varepsilon[V_{1xxx} + V_{1xyy} - U_{1xxy} - U_{1yyy}] - V_1 = [V_{ox} - U_{oy} - yH_o]\tau \qquad (27)$$

$$U_{1x} + V_{1y} = - H_{o\tau}. \qquad (28)$$

The boundary conditions are again $U_1 = V_1 = 0$ on $x = 0, r$.

In (27) and (28) the right sides of the equations provide the driving term as did $(1 + \alpha \sin\tau)\sin nsy$ in the zero-order equation. We shall proceed with the solution by means of the boundary layer technique.

For the interior solution we assume that the functions are smooth and hence that the derivatives are of the same order

of magnitude as the functions themselves. The terms multiplied by ε may therefore be neglected.

Let us rewrite equations (22), (23), and (26) as the sum of two parts - one part, with subscript i, having the same order of magnitude throughout the domain (the "interior solution"); the second part, with subscript b, sensibly large near the boundary and negligibly small in the interior, (the "boundary layer contribution")

$$U_{oi} = -\, ns(1 + \alpha \sin\tau)\cos nsy(-x + r - \varepsilon^{1/3})$$

$$U_{ob} = -\, ns(1 + \alpha \sin\tau)\cos nsy\left\{\varepsilon^{1/3} e^{(x-r)\varepsilon^{-1/3}} + \left[(\varepsilon^{1/3} - r)\cos\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right) + \left(\sqrt{3}\,\varepsilon^{1/3} - \frac{r}{\sqrt{3}}\right)\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\right] e^{-\frac{x\varepsilon^{-1/3}}{2}}\right\}$$

$$V_{oi} = -\,(1 + \alpha \sin\tau)\sin nsy$$

$$V_{ob} = (1 + \alpha \sin\tau)\sin nsy\left\{e^{(x-r)\varepsilon^{-1/3}} + \left[\cos\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right) + \left(\frac{2r\varepsilon^{-1/3}}{\sqrt{3}} - \sqrt{3}\right)\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\right] e^{-\frac{x\varepsilon^{-1/3}}{2}}\right\}$$

$$\Theta H_{oi} = (1 + \alpha \sin\tau)(\cos nsy + nsy \sin nsy)(-x + r - \varepsilon^{1/3})$$

$$\Theta H_{ob} = (1 + \alpha \sin\tau)nsy \sin nsy\left\{\varepsilon^{1/3} e^{(x-r)\varepsilon^{-1/3}} + \left[(\varepsilon^{1/3} - r)\cos\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right) + \left(\sqrt{3}\,\varepsilon^{1/3} - \frac{r}{\sqrt{3}}\right)\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\right] e^{-\frac{x\varepsilon^{-1/3}}{2}}\right\}$$

We expect the boundary layer thickness to have the same order of magnitude in the higher order solutions as in the zero-order solution, viz., $\varepsilon^{1/3}$. Thus, in order to find the first-order interior solution, we neglect all the terms with subscript b since they are negligible in the interior. Thus immediately, V_{1i}, the interior portion of V_1, is known and is (from (27))

$$V_{1i} = -[V_{oix} - U_{oiy} - yH_{oi}]_\tau$$
$$= \frac{\alpha \cos\tau}{\theta}(-x + r - \varepsilon^{1/3})[\cos nsy + (nsy - \theta n^2 s^2)\sin nsy] \qquad (29)$$

From (28) and (29) the interior portion of U_1, U_{1i}, can be computed directly, giving

$$U_{1i} = -\frac{\alpha\cos\tau}{\theta}[-\frac{x^2}{2} + x(r - \varepsilon^{1/3})][2nsy \sin nsy + (n^2s^2y^2 + \theta n^3 s^3 + 2)\cos nsy] + C_1(y,\tau) \qquad (30)$$

where $C_1(y,\tau)$ is arbitrary and must be evaluated by applying the boundary conditions to the complete solution, i.e., interior solution plus boundary layer contribution.

Before proceeding with the boundary layer analysis we can simplify equation (27) to some extent. Near $x = r$, $V_{ox} = O(\varepsilon^{-1/3})$, $U_{oy} = O(\varepsilon^{1/3})$, and $H_{oy} = \theta^{-1}O(\varepsilon^{1/3})$. Near $x = 0$, $V_{ox} = O(\varepsilon^{-2/3})$, $U_{oy} \doteq O(1)$, and $H_{oy} = \theta^{-1}O(1)$. Thus in each case we are justified in using only the contribution of the V_{ox} term provided $\varepsilon^{-2/3} >> 1$ and $\varepsilon^{-2/3} >> \theta^{-1}$. As will be shown later, when the appropriate dimensional constants are substituted, the error involved in neglecting the other terms is

extremely small. Thus for all practical purposes, equation (27), near the boundaries can be written

$$\varepsilon[V_{1bxxx} + V_{1bxyy} - U_{1bxxy} - U_{1byyy}] - V_{1b} = V_{0x\tau}$$
$$= \alpha \cos\tau \sin nsy \Big\{ \varepsilon^{-1/3} e^{(x-r)\varepsilon^{-1/3}} +$$
$$+ [(r\varepsilon^{-2/3} - 2\varepsilon^{-1/3})\cos(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}) +$$
$$+ \frac{r\varepsilon^{-2/3}}{\sqrt{3}} \sin(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2})] e^{-\frac{x\varepsilon^{-1/3}}{2}} \Big\}. \qquad (31)$$

Near x = 0, the inhomogeneous contribution which contains the term $e^{(x-r)\varepsilon^{-1/3}}$ can be neglected since its effect is felt only near the eastern boundary, i.e., near x = r. Similarly, near x = r, the terms multiplied by $e^{-\frac{x\varepsilon^{-1/3}}{2}}$ can be neglected. Thus for the region near x = 0,

$$\varepsilon[V_{1bxxx} + V_{1bxyy} - U_{1bxxy} - U_{1byyy}] - V_{1b}$$
$$= \alpha \cos\tau \sin nsy[(r\varepsilon^{-2/3} - 2\varepsilon^{-1/3})\cos(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}) +$$
$$+ \frac{r\varepsilon^{-2/3}}{\sqrt{3}} \sin(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2})] e^{-\frac{x\varepsilon^{-1/3}}{2}}. \qquad (32)$$

Now suppose the x coordinate is stretched by substituting $x = \varepsilon^k \xi$ ($k > 0$). Then (32) becomes

$$\varepsilon^{1-3k} V_{1b\xi\xi\xi} + \varepsilon^{1-k} V_{1b\xi yy} - \varepsilon^{1-2k} U_{1b\xi\xi y} - \varepsilon U_{1byyy} - V_{1b}$$
$$= \alpha \cos\tau \sin nsy[(r\varepsilon^{-2/3} - 2\varepsilon^{-1/3})\cos(\frac{\xi\sqrt{3}\,\varepsilon^{k-1/3}}{2})$$
$$+ \frac{r\varepsilon^{-2/3}}{\sqrt{3}} \sin(\frac{\xi\sqrt{3}\,\varepsilon^{k-1/3}}{2})] e^{-\frac{\xi\varepsilon^{k-1/3}}{2}}.$$

The term of highest order derivative in ξ is matched with the remaining largest term in the equation. Hence, we formally match $\varepsilon^{1-3k}V_{1b\xi\xi\xi}$ with V_{1b}. Then $k = 1/3$* and the equation becomes

$$V_{1b\xi\xi\xi} - V_{1b} = \alpha \cos\tau \sin nsy\Big[(r\varepsilon^{-2/3} - 2\varepsilon^{-1/3})\cos\Big(\frac{\xi\sqrt{3}}{2}\Big) + \frac{r\varepsilon^{-2/3}}{\sqrt{3}} \sin\Big(\frac{\xi\sqrt{3}}{2}\Big)\Big] e^{-\xi/2} + O(\varepsilon^{1/3}). \qquad (33)$$

The term V_{1b} can now be expanded in an asymptotic series in ε and only the first terms will be kept. Since the inhomogeneous term of (33) contains only exponential and trigonometric functions, let us try a solution of the form

$$V_{1b} = \alpha \cos\tau \sin nsy\, \varepsilon^{-2/3} \Big\{ V_1^o \cos\Big(\frac{\xi\sqrt{3}}{2}\Big) + \bar{V}_1^o \sin\Big(\frac{\xi\sqrt{3}}{2}\Big) \Big\} e^{-\xi/2} \qquad (34)$$

where V_1^o and $\bar{V}_1^o$ are the first terms of asymptotic expansions and are to be determined.

If V_{1b} as given by (34) be substituted into (33) and if coefficients of $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\Big(\frac{\xi\sqrt{3}}{2}\Big)$ be equated, two simultaneous differential equations with constant coefficients result.

$$-\frac{3}{2} V_{1\xi}^o - \frac{3}{2} V_{1\xi\xi}^o + V_{1\xi\xi\xi}^o - \frac{3\sqrt{3}}{2} \bar{V}_{1\xi}^o + \frac{3\sqrt{3}}{2} V_{1\xi\xi}^o = r - 2\varepsilon^{1/3} \qquad (35)$$

$$\frac{3\sqrt{3}}{2} V_{1\xi}^o - \frac{3\sqrt{3}}{2} V_{1\xi\xi}^o - \frac{3}{2} \bar{V}_{1\xi}^o - \frac{3}{2} \bar{V}_{1\xi\xi}^o + \bar{V}_{1\xi\xi\xi}^o = -\frac{r}{\sqrt{3}}. \qquad (36)$$

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* The fact that $k = 1/3$ indicates that the thickness of the boundary layer is of the same order of magnitude in the zero and first order solution, as was anticipated.

Particular solutions of (35), (36) are

$$V_1^o = \frac{\varepsilon^{1/3} - r}{3}\xi, \qquad \bar{V}_1^o = \frac{3\varepsilon^{1/3} - r}{3\sqrt{3}}\xi.$$

The homogeneous solutions may be derived by letting

$$V_1^o = Ae^{\lambda\xi}, \quad \bar{V}_1^o = Be^{\lambda\xi}.$$

Then (35), (36) become

$$A(-\frac{3}{2}\lambda - \frac{3}{2}\lambda^2 + \lambda^3) - B[\frac{3\sqrt{3}}{2}\lambda - \frac{3\sqrt{3}}{2}\lambda^2] = 0 \qquad (37)$$

$$A(\frac{3\sqrt{3}}{2}\lambda - \frac{3\sqrt{3}}{2}\lambda^2) + B[-\frac{3}{2}\lambda - \frac{3}{2}\lambda^2 + \lambda^3] = 0. \qquad (38)$$

Hence, since the determinant of these two simultaneous equations must vanish, we have

$$(\lambda^3 - \frac{3}{2}\lambda^2 - \frac{3}{2}\lambda)^2 + \frac{27}{4}(\lambda - \lambda^2)^2 = 0. \qquad (39)$$

The roots are

$$\lambda = 0, 0, \ \frac{3 + \sqrt{3}\,i}{2}, \ \frac{3 - \sqrt{3}\,i}{2}, \ \sqrt{3}i, \ -\sqrt{3}i. \qquad (40)$$

Then,

$$V_1^o = \frac{\varepsilon^{1/3} - r}{3}\xi + A_5 e^{\frac{3+\sqrt{3}i}{2}\xi} + A_2 e^{\sqrt{3}i\xi} + A_3 e^{\frac{3-\sqrt{3}i}{2}\xi} + A_4 e^{-\sqrt{3}i\xi} + A_1$$

$$\bar{V}_1^o = \frac{3\varepsilon^{1/3} - r}{3\sqrt{3}}\xi + B_5 e^{\frac{3+\sqrt{3}i}{2}\xi} + B_2 e^{\sqrt{3}i\xi} + B_3 e^{\frac{3-\sqrt{3}i}{2}\xi} + B_4 e^{-\sqrt{3}i\xi} + B_1.$$

Hence, from (34)

$$V_{1b} = \alpha \cos\tau \sin nsy\, \varepsilon^{-2/3} \Big\{ \cos(\frac{\sqrt{3}\xi}{2})[\frac{\varepsilon^{1/3} - r}{3}\xi + A_2 e^{\sqrt{3}i\xi} + A_4 e^{-\sqrt{3}i\xi} + A_1]$$

$$+ \sin(\frac{\sqrt{3}\xi}{2})[\frac{3\varepsilon^{1/3} - r}{3\sqrt{3}}\xi + B_2 e^{\sqrt{3}i\xi} + B_4 e^{-\sqrt{3}i\xi} + B_1]\Big\} e^{-3/2} \qquad (41)$$

where we have set $A_5 = A_3 = B_5 = B_3 = 0$ since the contributions of the terms with those coefficients do not tend to zero as $\xi \rightarrow \infty$.

When (37) and (38) are used to get a relationship between the A_i and the B_i, then the final form for V_{1b} near $x = 0$ is found to be

$$V_{1b} = \alpha \cos\tau \sin nsy\, \varepsilon^{-2/3} \left\{ \left(\frac{\varepsilon^{1/3} - r}{3}\xi + C_2\right)\cos\left(\frac{\sqrt{3}\,\xi}{2}\right) + \left(\frac{3\varepsilon^{1/3} - r}{3\sqrt{3}}\,\xi + C_3\right)\sin\left(\frac{\sqrt{3}\,\xi}{2}\right)\right\} e^{-\xi/2} \quad (42)$$

where C_2 and C_3 are arbitrary functions of y and τ and must be found by applying the boundary conditions to the complete solution.

In a similar manner, if we make the following two substitutions for the right (eastern) boundary

$$(x - r) = e^{h\eta}$$

$$V_{1b} = \alpha \cos\tau \sin nsy\, \varepsilon^{-1/3} e^{\eta\varepsilon^{h-1/3}} [V_1^o + \ldots],$$

we find that $h = 1/3$* and

$$V_{1b} = \alpha \cos\tau \sin nsy\, \varepsilon^{-1/3}\left[\frac{\eta}{3} + A_1(y,\tau)\right]e^{\eta}. \quad (43)$$

We have used the fact that $V_{1b} \rightarrow 0$ as $\eta \rightarrow -\infty$. (As stated in the appendix, $\eta \rightarrow -\infty$ when the boundary on the right is under consideration, since the boundary layer solutions must become

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* The same remark applies to the value of h as previously made for the value of k.

negligibly small as the distance from the boundary increases, i.e., as η or x decreases.

If the three contributions (29), (42), (43) to the complete solution for V_1 be added, the final form for V_1 is

$$V_1 = \frac{\alpha \cos \tau}{\theta}(-x + r - \varepsilon^{1/3})[(y^2 ns + \theta n^2 s^2)\sin nsy + y \cos nsy]$$
$$+ \alpha \cos \tau \sin nsy\ \varepsilon^{-2/3}\left\{\left(\frac{1 - r\varepsilon^{-1/3}}{3} x + C_2(y)\right)\cos\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right) + \left(\frac{3 - r\varepsilon^{-1/3}}{3\sqrt{3}} x + C_3(y)\right)\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\right\} e^{-\frac{x\varepsilon^{-1/3}}{2}}$$
$$+ \alpha \cos \tau \sin nsy\ \varepsilon^{-1/3}\left\{\frac{x - r}{3}\varepsilon^{-1/3} + A_1(y)\right\} e^{(x-r)\varepsilon^{-1/3}} \quad . \tag{44}$$

By means of the continuity equation we then find

$$U_1 = -\frac{\alpha \cos \tau}{\theta}[2nsy \sin nsy + (y^2 n^2 s^2 + 2 + \theta n^3 s^3)\cos nsy]$$
$$\left[-\frac{x^2}{2} + x(r - \varepsilon^{1/3})\right] + C_1(y,\tau) - \frac{\alpha \cos \tau}{\theta} nsy \sin nsy\, \varepsilon^{2/3} e^{(x-r)\varepsilon^{-1/3}}$$
$$- \alpha \cos \tau\ ns \cos nsy\left[A_1 - \frac{1}{3} + \frac{(x-r)\varepsilon^{-1/3}}{3}\right] e^{(x-r)\varepsilon^{-1/3}}$$
$$- \alpha \cos \tau \sin nsy \frac{\partial A_1}{\partial y} e^{(x-r)\varepsilon^{-1/3}} - \frac{\alpha \cos \tau}{\theta} nsy \sin nsy \cdot$$
$$\cdot\left[(r\varepsilon^{1/3} - 2\varepsilon^{2/3})\cos\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right) - \frac{r\varepsilon^{1/3}}{\sqrt{3}}\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\right] e^{-\frac{x\varepsilon^{-1/3}}{2}}$$
$$- \alpha \cos \tau\ ns \cos nsy\, \varepsilon^{-2/3}\left[\left(-\frac{2x\varepsilon^{1/3}}{3} - \frac{\varepsilon^{2/3}}{3} + \frac{rx}{3}\right)\cos\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right) + \right.$$
$$\left. + \left(\frac{\varepsilon^{2/3}}{\sqrt{3}} - \frac{rx}{3\sqrt{3}} - \frac{2r\varepsilon^{1/3}}{3\sqrt{3}}\right)\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\right] e^{-\frac{x\varepsilon^{-1/3}}{2}}$$
$$+ \alpha \cos \tau \frac{\varepsilon^{-1/3}}{2}\frac{\partial}{\partial y}\left\{(C_2 + \sqrt{3}\, C_3)\sin nsy \cos\left(\frac{x\sqrt{3}\,\varepsilon^{1/3}}{2}\right) + \right.$$
$$\left. + (C_3 - \sqrt{3}\, C_2)\sin nsy \sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\right\} e^{-\frac{x\varepsilon^{-1/3}}{2}} \quad . \tag{45}$$

The arbitrary functions of y can be evaluated by means of the boundary conditions $U_1 = V_1 = 0$ on $x = 0, r$. We have

$$\sin nsy\, C_2 = \frac{\varepsilon - r\varepsilon^{2/3}}{\theta}[(nsy^2 + \theta n^2s^2)\sin nsy + y\cos nsy] \tag{46}$$

$$\sin nsy\, A_1 = \frac{\varepsilon^{2/3}}{\theta}[(nsy^2 + \theta n^2s^2)\sin nsy + y\cos nsy] \tag{47}$$

$$C_1 = \frac{\alpha\cos\tau}{\theta}\Big\{[2nsy\sin nsy + (y^2n^2s^2 + 2 + \theta n^3s^3)\cdot$$

$$\cdot\cos nsy][\frac{r^2}{2} - r\varepsilon^{1/3} + \varepsilon^{2/3}] - \varepsilon^{2/3}(\frac{\theta ns}{3} + 1)\cos nsy\Big\} \tag{48}$$

$$\sin nsy\, C_3 = \frac{1}{\sqrt{3}\,\theta}\Big\{[2\varepsilon^{1/3}(y^2ns + \frac{2}{ns} + \theta n^2s^2)(r\varepsilon^{1/3} - r^2 - \varepsilon^{2/3}) +$$

$$+ \frac{2\varepsilon}{3} - \frac{2\theta\varepsilon^{1/3}}{3}]\sin nsy + (5y\cos nsy - \frac{4}{ns}\sin nsy)r\varepsilon^{2/3} -$$

$$- (9y\cos nsy - \frac{10}{ns}\sin nsy)\varepsilon^{2/3} + (r\varepsilon^{2/3} - \varepsilon)(y^2ns + \theta n^2s^2)\sin nsy\Big\}. \tag{49}$$

The first-order contribution to H can be found from equations (12), (13). The first order equations are

$$ns\frac{\partial U_o}{\partial\tau} - nsy\, V_1 + \theta\frac{\partial H_1}{\partial x} = ns\varepsilon\Delta U_1$$

$$ns\frac{\partial V_o}{\partial\tau} + nsy\, U_1 + \theta\frac{\partial H_1}{\partial y} = ns\varepsilon\Delta V_1$$

from which H_1 is found to be

$$H_1 = -\frac{ns\alpha\cos\tau}{\theta^2}\Big\{[(\theta ns + y^2)\cos nsy + (y^3ns + y\theta n^2s^2)\sin nsy]\cdot$$

$$\cdot[\tfrac{1}{2}(x^2 + r^2) + (\varepsilon^{1/3} - r)(x + \varepsilon^{1/3})] + \frac{\theta}{ns}\cos nsy +$$

$$\varepsilon^{2/3}(\frac{\theta ns}{3} + 1)(\frac{y\sin nsy}{ns} + \frac{\cos nsy}{n^2s^2})\Big\} +$$

$$+ \frac{\alpha \cos \tau}{\theta} \text{ nsy} \sin \text{nsy}\ \varepsilon^{-2/3} \Big\{ \Big[\frac{\varepsilon^{2/3}}{\sqrt{3}} - \frac{2r\varepsilon^{1/3}}{3\sqrt{3}} - \frac{rx}{3\sqrt{3}} \Big] \sin\Big(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\Big)$$

$$+ \Big[\Big(\frac{r}{3} - \frac{2\varepsilon^{1/3}}{3}\Big)x + \frac{\varepsilon^{2/3}}{3} \Big] \cos\Big(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\Big) - \frac{\varepsilon^{1/3}}{2} \Big[(C_2 + \sqrt{3}C_3) \cos\Big(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\Big)$$

$$+ (C_3 - \sqrt{3}\,C_2) \sin\Big(\frac{\sqrt{3}}{2} x \varepsilon^{-1/3}\Big) \Big] \Big\} e^{-\frac{x\varepsilon^{-1/3}}{2}} + \frac{\alpha \cos \tau}{\theta} \text{ nsy} \sin \text{nsy} \cdot$$

$$\cdot \Big\{ \frac{1}{3} (x - r - \varepsilon^{1/3})\ {}^{-1/3} + A_1 \Big\} e^{(x-r)\varepsilon^{-1/3}} . \qquad (50)$$

The terms U_1 and V_1 do not satisfy the boundary conditions $V_1 = \frac{\partial U_1}{\partial y} = 0$ on $y = 0,1$. We must recall that these boundary conditions were chosen rather arbitrarily as being plausible ones for the type of wind distribution specified, and the y dependence of the zero-order solution was accordingly chosen as sin nsy. We cannot expect such a y dependence to satisfy all the conditions for each set of equations. The fact that U_1 and V_1 do not satisfy the boundary conditions does not seem to be very serious since we do not really know what conditions are appropriate.

If we next consider the equations resulting from equating the coefficients of δ^2 to zero, we obtain from (8) and (11),

$$\varepsilon[V_{2xxx} + V_{2xyy} - U_{2xxy} - U_{2yyy}] - V_2 = (V_{1x} - U_{1y} - yH_1)_\tau$$

$$U_{2x} + V_{2y} = - H_{1\tau} .$$

In the boundary layer, near $x = 0$, V_{1x} is of order ε^{-1}. Thus we can expect V_2 to be of order ε^{-1} in that region. By a similar argument, we can expect V_3 to be of order $\varepsilon^{-4/3}$, V_4 to be of order $\varepsilon^{-5/3}$, etc. If we therefore write out the series

$$V = V_0 + \delta V_1 + \delta^2 V_2 + \delta^3 V_3 + \delta^4 V_4 + \ldots$$

we have in terms of orders of magnitude near $x = 0$,

$$V = O(\varepsilon^{-1/3}) + \delta\varepsilon^{-1/3}\, O(\varepsilon^{-1/3}) +$$

$$+ \delta^2 \varepsilon^{-2/3}\, O(\varepsilon^{-1/3}) + \delta^3 \varepsilon^{-1}\, O(\varepsilon^{-1/3}) + \ldots$$

or factoring out the $O(\varepsilon^{-1/3})$, we have

$$V = O(\varepsilon^{-1/3})[1 + \delta\varepsilon^{-1/3} + (\delta\varepsilon^{-1/3})^2 + \ldots].$$

The perturbation scheme may be expected to be valid provided $\delta\varepsilon^{-1/3} < 1$. We can expect a fairly good approximation from only the first two terms provided the more stringent condition $\delta\varepsilon^{-1/3} << 1$ is imposed. If $\delta\varepsilon^{-1/3} = 1/5$, the error involved in neglecting the third term is no larger than 5% of the first term.

For yearly variation of the wind, $\delta\varepsilon^{-1/3} \approx 1/6$. Hence we shall keep only the first two terms of the series. It should be noted that α determines the magnitude of the effect of the perturbation but it has no bearing on the validity of the expansion.

Numerical Example

In order to discuss the above solution, we shall prescribe numerical values for the constants of the problem. Let

$r_1 = 6.5 \times 10^8$ cm　　　$\beta = 2 \times 10^{-13}$ cm^{-1} sec^{-1}

$s = 5 \times 10^8$ cm　　　$D = 5 \times 10^4$ cm(C = 200m., d = 600m.)

$A = 5 \times 10^7 cm^2 sec^{-1}$ $\omega = 2 \times 10^{-7} sec^{-1}$*

$\eta = 2\pi/s$ $W' = 0.65$ gm cm^{-1} sec^{-2} .

The magnitudes of r_1, s, A, D correspond roughly to the Atlantic Ocean parameters. The value of β is chosen so as to give the best approximation to the Coriolis parameter in the latitude of Cape Hatteras. The equality $n = 2\pi/s$ corresponds roughly to the east-west components of the trades and the westerlies. The value of ω corresponds to yearly frequency of the wind variation and $W' = 0.65$ gm $cm^{-1} sec^{-2}$ is the value used by Munk [5] for the wind stress.

Then the dimensionless constants have the values

$$\delta = \frac{\omega}{\beta s} = 2 \times 10^{-3} \qquad ns = 2\pi$$

$$\varepsilon = \frac{A}{\beta s^3} = 2 \times 10^{-6} \qquad \theta = \frac{ngD}{\beta^2 s^3} = 0.123$$

$$r = 1.3$$

Also Γ' has been chosen so that

$$\alpha = 0.2.$$

The results for this numerical example are shown in Figs. 2 - 6.

In Fig. 2 the non-dimensional, north-south component, V, of the mass transport is plotted against x' near x' = 0 for the value y' = 0.25. The region of large V corresponds to the Gulf

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* Corresponding to an annual period for the wind fluctuation.

Stream and the section adjacent to the Gulf Stream, with negative V, corresponds to the off-shore counter-current.

For the Gulf Stream, the extreme values of V are in phase with the extreme values of the wind. However, for the points between the maximum and minimum values of wind strength, the transport lags behind the wind.

During one cycle of wind variation the following result is found. The transport and wind both have maximum values at $\tau = \pi/2$. Immediately after $\tau = \pi/2$, the wind begins to decrease. The transport also decreases but it lags behind the wind. At $\tau = \pi$ the wind has reached its mean amplitude and the lag of the transport is greatest, viz., an interval of 9 days* elapses between the time the wind reaches its mean amplitude and the time at which the transport reaches its mean amplitude. After $\tau = \pi$, the transport begins to gain on the wind until at $\tau = 3\pi/2$, the two are again in phase. The wind and the transport now begin to increase and the transport again lags behind the wind. The maximum lag is reached at $\tau = 2\pi$ at which point the transport begins to catch up to the wind. They are in phase again at $\tau = 5\pi/2$. This cycle is repeated indefinitely.

The discussion presented here is based on the assumption that the first two terms of the series represent, in a sufficiently accurate manner, the complete solution. One result of this assumption is that transport reaches its maximum value at $\tau = \pi/2$.

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* It is shown later that the value 9 days is independent of the specific value of the frequency for slowly varying winds.

The perturbation contribution vanishes at that instant since its coefficient is cos τ. Thus, no matter what the value of δ (essentially, the frequency), as long as it lies within the limits necessary for the validity of the above method of solution, the maximum value of the transport will occur at $\tau = m\pi/2$, $m = 1, 5, 9 \ldots$, and its value is given by $1 + \alpha$ times the steady transport value.

The interval of 9 days between the time at which the wind reaches its mean amplitude and the time at which the transport reaches its mean amplitude is also independent of the frequency. To show this let $V_0 = (1 + \alpha \sin \tau)Q$ and $V_1 = \alpha L \cos \tau$. Then $V = (1 + \alpha \sin \tau)Q + \delta\alpha L \cos \tau$. Since the mean value of the transport is $V = Q$, we can find the time at which this occurs by setting

$$(1 + \alpha \sin \tau)Q + \delta\alpha L \cos \tau = Q$$

or

$$\tan \tau = - \frac{L\delta}{Q} .$$

Since τ is small, we can write $\tan \tau \approx \tau$ and therefore

$$\tau \approx - \frac{L\delta}{Q} .$$

Substituting $\tau = \omega t$ and $\delta = \omega/\beta s$, we have finally

$$t = - \frac{L}{Q} \frac{1}{\beta s}$$

which is independent of frequency and α.

It is apparent from Fig. 2 that the out-of-phase effect is of relatively greatest importance in the counter-current rather than in the main stream. The graph shows the various

effects only up to the eastern edge of the counter-current at $x' = 0.1$. For $x' > 0.1$ only the mean position of the transport is plotted since the deviations from this mean position are very small.

Near the eastern boundary of the ocean (Fig. 3) and in the counter-current region (Fig. 2), the absolute magnitude of the extreme values of the transport (which is now negative) are also in phase with the extreme values of the wind and the transport lags behind the wind at all other times.

Figures 4, 5, and 6 show surface contours* for the southern half of the rectangular ocean for $\tau = 0, \pi/2, \pi, 3\pi/2$. The contribution of δH_1 is very small throughout the ocean** and has therefore been neglected. Thus the graphs for $\tau = 0$ and $\tau = \pi$ coincide. This result is based on the assumption that D is 500 meters in thickness. If D were increased the above remarks would be even more appropriate. If D were decreased, the contribution of the perturbation term would be larger and we would therefore have to account for it. The value of the first-

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* If we define the thermocline as the surface at $z = T - d/2$, then the contour lines of Figs. 4, 5, and 6, multiplied by -200 represent the deviation of the thermocline from its equilibrium position at $z = - C - d/2 = - D$.

** If for any of the variables the magnitude of the coefficient of δ in the perturbation solution is of the same order as that of the zero-order term, the coefficient $\delta = 0.002$ renders such a correction negligible. Throughout the present example, the only sizable contribution of the out-of-phase term is found in the north-south transport V in the boundary layer where the function V increases by order $\varepsilon^{-1/3}$. However, H_0 and H_1 have the same order of magnitude throughout the ocean so that the first-order correction H_1 can be neglected throughout.

order velocities would also be altered when θ is changed. We shall consider several values of θ when we discuss the deflection of the thermocline in the steady two-layer ocean.

The mean mass transport of the Gulf Stream (corresponding to the steady problem) is 26.6×10^6 metric tons per second as compared to Munk's value [5] of 36×10^6 and the observed value of $72\text{-}80 \times 10^6$ metric tons per second. Munk [5] used the east-west component of an empirical wind system and the discrepancy is therefore due to the difference between the two wind systems. At the time of maximum (minimum) wind the transport is 20% higher (lower) in accord with the remarks made previously in this section. In the counter-current the steady mass tranport is 4.61×10^6 metric tons per second.

The difference between the computed and the observed values is not surprising when one considers the many idealizing assumptions made. Such features as the straight coast lines, the simplified theory of turbulence used, the neglect of the non-linear terms, and a more realistic stress-effect of the wind on the water could well change the quantitative results by a factor of two or three.

The problem as stated and solved by the above method gives no sensible east-west variation in the position of the Gulf Stream, but a careful investigation of the eastern boundary of the Gulf Stream shows a very small narrowing of the stream. How well such a result agrees with field evidence is uncertain since our solution yields no inshore counter-current.

It would be interesting to ascertain how well our predicted results agree with observation; specifically, if the mass transport of the Gulf Stream responds as indicated to variations in the wind and if the lag of the transport is independent of the frequency.

5. Methods of Solution for Problem 2. The equations (3.25) - (3.30) are non-dimensionalized below in order that boundary layer theory may be employed. Using the arguments of Section 4 for the method of non-dimensionalizing, we have

$$x = sx', \qquad \lambda = \frac{gW}{s^6 \rho_2 \beta^4},$$

$$y = sy',$$

$$\tau = \omega t, \qquad \theta = \frac{ng(D_1 - D_2)}{\beta^2 s^3},$$

$$V_1 = \frac{\bar{V}_1 \beta}{W}, \qquad \ell = \frac{gD_2}{\beta^2 s^4},$$

$$V_2 = \frac{\bar{V}_2 \beta}{W}, \qquad \varepsilon = \frac{A}{\beta s^3},$$

$$U_1 = \frac{\bar{U}_1 \beta}{W}, \qquad \delta = \frac{\omega}{\beta s},$$

$$U_2 = \frac{\bar{U}_2 \beta}{W}, \qquad \alpha = \frac{\Gamma}{W},$$

$$H_1 = \frac{\beta^2 s^2 \rho_1 \eta_1}{W}, \qquad a = \frac{\rho_1}{\rho_2},$$

$$H_2 = \frac{\beta^2 s^2 \rho_2 \eta_2}{W}, \qquad b = \frac{\rho_2 - \rho_1}{\rho_1}.$$

Then equations (3.25) - (3.30) become

$$ns\delta \frac{\partial U_1}{\partial \tau} - nsyV_1 = -\Theta \frac{\partial H_1}{\partial x} - \frac{ns\lambda}{a}[H_1 - aH_2]\frac{\partial H_1}{\partial x} +$$

$$+ ns\varepsilon\Delta U_1 - (1 + \alpha \sin \tau)\cos nsy \qquad (1)$$

$$ns\delta \frac{\partial V_1}{\partial \tau} + nsyU_1 = -\Theta \frac{\partial H_1}{\partial y} - \frac{ns\lambda}{a}[H_1 - aH_2]\frac{\partial H_1}{\partial y} + ns\varepsilon\Delta V_1 \qquad (2)$$

$$\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y} = -\delta\frac{\partial}{\partial \tau}[H_1 - aH_2] \qquad (3)$$

$$\delta \frac{\partial U_2}{\partial \tau} - yV_2 = -\ell\frac{\partial}{\partial x}(H_1 + bH_2) - \lambda H_2 \frac{\partial}{\partial x}(H_1 + bH_2) + \varepsilon\Delta U_2 \qquad (4)$$

$$\delta \frac{\partial V_2}{\partial \tau} + yU_2 = -\ell\frac{\partial}{\partial y}(H_1 + bH_2) - \lambda H_2 \frac{\partial}{\partial y}(H_1 + bH_2) + \varepsilon\Delta V_2 \qquad (5)$$

$$\frac{\partial U_2}{\partial x} + \frac{\partial V_2}{\partial y} = -\delta\frac{\partial H_2}{\partial \tau}. \qquad (6)$$

Steady Wind

Let us first treat the case of a steady wind, i.e., $\alpha = 0$ and $\partial/\partial t = \partial/\partial\tau = 0$, and let us assume that, in the case of steady motion, there are no velocities, and hence no horizontal pressure gradient, in the bottom layer. Equations (4) - (6) are then satisfied immediately by

$$U_2 = V_2 = 0, \quad H_2 = -\frac{1}{b}H_1 \qquad (7)$$

and equations (1) - (6) become

$$-nsyV_1 = -\frac{\partial}{\partial x}[\Theta H_1 + \frac{ns\lambda}{2b}H_1^2] + ns\varepsilon\Delta U_1 - \cos nsy \qquad (8)$$

$$nsyU_1 = -\frac{\partial}{\partial y}[\Theta H_1 + \frac{ns\lambda}{2b}H_1^2] + ns\varepsilon\Delta V_1 \qquad (9)$$

$$\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y} = 0. \qquad (10)$$

Differentiating (8) with respect to x, (9) with respect to y, and substracting, we have

$$\varepsilon[V_{1xxx} + V_{1xyy} - U_{1xxy} - U_{1yyy}] - V_1 = \sin nsy \qquad (11)$$

which is equation (4.16) with $\alpha = 0$.

Thus the transport distribution for the steady case is precisely the same as it is in Problem 1. The difference in behavior enters into the non-steady case when the motion of the interface affects the motion of the water in the top layer.

If we set $\alpha = 0$, then equations (4.22) and (4.23) are the solutions for the present U_1, V_1. Similarly with $\alpha = 0$, from equations (8) and (9) above

$$\frac{ns\lambda}{2b} H_1^2 + \theta H_1 = \theta H_o$$

where H_o is given by (4.26). Then H_1 may be written

$$H_1 = -\frac{\theta + \theta\sqrt{1 + \frac{2ns\lambda}{\theta b} H_o}}{\frac{ns\lambda}{b}} . \qquad (12)$$

However, if $2ns\lambda/\theta b\ H_o < 1$, then H_1 may be written approximately

$$H_1 \approx -\frac{\theta + [\theta + \frac{ns\lambda}{b} H_o]}{\frac{ns\lambda}{b}} = H_o . \qquad (12.a)$$

H_2 can then be evaluated by

$$H_2 = -\frac{1}{b} H_1 . \qquad (7)$$

If the dimensional constants* which were used in Problem

* The depth (D_1-D_2) is given the same value as D in Problem 1.

1 are used here, and if we put b = .005, then (12.a) is correct to $O(10^{-2})$. The streamlines and the thermocline, H_2 are shown in Figs. 7 and 8.

In Fig. 8 it can be seen from the contour lines of the thermocline that there is not much deviation of the thermocline from its equilibrium position. In particular, if the initial depth be 500 meters, the thermocline does not fall more than 35 meters below its average depth in the southern half of the ocean.

In checking our results with observation, we find that quantitatively this result is in poor agreement with field evidence. The definition of the thermocline in the real ocean is vague, however, and hence the two parameters Θ (corresponding to the average thickness of the top layer) and b (the density difference) are not clearly determined. In fact, they may vary over a wide range giving rise to a very considerable variation in the deflection of the thermocline.

In Fig. 9, the vertical cross section of the ocean at $y' = 0.25$ is shown for four combinations of Θ and b. If we consider the curve with $\Theta = 0.0492$ ($D_1 - D_2 = 200$ m.) and b = 0.0025, our result is in good qualitative agreement with measurements of the thermocline off Chesapeake Bay [10]. Quantitatively, the values are out by a factor of (approximately) three.

Our solution shows a tendency for the thermocline to approach the surface in the northern part of the ocean (Fig. 8). As a matter of fact, if Θ and b be chosen small enough, the

interface lies above the free surface! Such a result is absurd, of course, but the tendency of the thermocline to approach the surface in the northern part of the ocean is clearly indicated. This fact agrees with observation since the thermocline actually reaches the surface in the north.

Non-Steady Wind

In the treatment of the non-steady, two-layer problem, we shall neglect the terms with coefficient λ in equations (1), (2), (4), (5). For the steady problem, if Θ and b are chosen appropriately, it has been shown (equation (12.a) that the error involved herein is small.

Two methods of attack have been applied to the linearized equations of (1) - (6). Our first procedure is that used in Problem 1, viz, a perturbation in δ followed by a boundary layer analysis.

The difficulty in the first method of solution arises from the fact that the quantities with coefficient δ are no longer small, i.e., the magnitude of the terms is no longer governed by δ. In particular, in the continuity equation (3), the term on the right hand side has magnitude $\delta/b\ H_1$ (based on the steady solution). In the interior of the ocean where U_1 and V_1 are O(1) and $H_1 = O(\Theta^{-1})$, in order for the perturbation in δ to be valid, we must have $\delta << 1/\Theta b$. With the dimensional constants of Problem 1, this means $\delta << 10^{-4}$. Such a value corresponds to a wind period of one hundred years or more.

If the above results were the only objection to the

analysis, the problem as defined thus far might still have some qualitative value. Unfortunately, for such a small value of δ, the terms in the equations of motion which involve a time-derivative become very small, and we are wholly unjustified in neglecting the non-linear terms while still retaining these time dependent terms.

In spite of these objections, the analysis for Problem 2 by the first method was carried through but the results were not computed numerically. The analytical results are listed in the next few pages.

$$U_1 = U_{10} + \delta U_{11}, \quad V_1 = V_{10} + \delta V_{11}, \quad H_1 = H_{10} + \delta H_{11}$$

$$U_2 = U_{20} + \delta U_{21}, \quad V_2 = V_{20} + \delta V_{21}, \quad H_2 = H_{20} + \delta H_{21}$$

where $U_{20} = V_{20} = 0$ by equation (7), U_{10}, V_{10}, H_{10} are given by equations (3.22), (3.23) and (3.26) and the remaining values are given below.

$$V_{11} = \frac{\alpha \cos \tau}{\theta}(-x + r - \varepsilon^{1/3})\left[\left(\theta n^2 s^2 + \frac{nsy^2}{b}\right)\sin nsy + \frac{y}{b}\cos nsy\right]$$

$$+ \alpha \cos \tau \sin nsy\left[\left(\varepsilon^{-2/3} - \frac{nsy^2}{b\theta}\right)\frac{x - r}{3} + A\right] e^{(x-r)\varepsilon^{-1/3}}$$

$$+ \alpha \cos \tau \sin nsy\, \varepsilon^{-2/3}\left\{\left(\frac{1}{3} - \frac{r\varepsilon^{-1/3}}{3}\right) x \cos\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\right.$$

$$\left. + \left(\frac{1 - r\varepsilon^{-1/3}}{3}\right)\frac{x}{\sqrt{3}}\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-\frac{1}{3}}}{2}\right)\right\} e^{-\frac{x\varepsilon^{-1/3}}{2}} - \alpha \cos \tau \frac{nsy^2}{\theta b}\sin nsy \cdot$$

$$\cdot \left\{\frac{x}{3}\cos\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right) + \left(\frac{2r\varepsilon^{-1/3}}{3} - 1\right)\frac{x}{\sqrt{3}}\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\right\} e^{-\frac{x\varepsilon^{-1/3}}{2}}$$

$$+ \alpha \cos \tau \sin nsy\left\{C_2 \cos\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right) + C_3 \sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\right\} e^{-\frac{x\varepsilon^{-1/3}}{2}}$$

$$U_{11} = -\frac{\alpha\cos\tau}{\theta}\left(-\frac{x^2}{2} + xr - x\varepsilon^{1/3}\right)\left\{\left(\theta n^3s^3 + \frac{n^2s^2y^2+2}{b}\right)\cos nsy\right.$$

$$\left.+\frac{2yns}{b}\sin nsy\right\} + \alpha\cos\tau\, C(y) - \frac{\alpha\cos\tau}{\theta b}nsy\sin nsy\left\{\varepsilon^{2/3}e^{(x-r)\varepsilon^{-\frac{1}{3}}}\right.$$

$$\left.+\left[\frac{-r\varepsilon^{1/3}}{\sqrt{3}}\sin\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)+(r\varepsilon^{1/3}-2\varepsilon^{2/3})\cos\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)\right]e^{-\frac{x\varepsilon^{-1/3}}{2}}\right\}$$

$$-\alpha\cos\tau\frac{\partial}{\partial y}\left\{\sin nsy\left[\left(\varepsilon^{-1/3}-\frac{nsy^2\varepsilon^{1/3}}{b\theta}\right)\left(\frac{x-r-\varepsilon^{1/3}}{3}\right)\right.\right.$$

$$\left.\left.+\varepsilon^{1/3}A\right]\right\}e^{(x-r)\varepsilon^{-1/3}} - \alpha\cos\tau\frac{\partial}{\partial y}\left\{\sin nsy\left[\left(\frac{-xr\varepsilon^{-1/3}}{3}\right.\right.\right.$$

$$+1-\frac{2r\varepsilon^{-1/3}}{3}\Big)\frac{\sin}{\sqrt{3}}\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)+\left(\frac{xr\varepsilon^{-2/3}}{3}-\frac{2x\varepsilon^{-1/3}}{3}-\frac{1}{3}\right)\cos\cdot$$

$$\left.\left.\cdot\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)\right]\right\}e^{-\frac{x\varepsilon^{-1/3}}{2}} + \alpha\cos\tau\frac{\partial}{\partial y}\left\{\frac{nsy^2}{\theta b}\sin nsy\left[\left(\frac{2\varepsilon^{2/3}}{3}\right.\right.\right.$$

$$-\frac{r\varepsilon^{1/3}}{3}+\frac{x\varepsilon^{1/3}}{3}-\frac{xr}{3}\Big)\cos\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)$$

$$\left.\left.+\left(\frac{r\varepsilon^{1/3}}{3}+x\varepsilon^{1/3}-\frac{xr}{3}\right)\frac{\sin}{\sqrt{3}}\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)\right]\right\}e^{-\frac{x\varepsilon^{-1/3}}{2}}$$

$$-\alpha\cos\tau\frac{\varepsilon^{1/3}}{2}\frac{\partial}{\partial y}\left\{\sin nsy\left[(\sqrt{3}C_2-C_3)\sin\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)\right.\right.$$

$$\left.\left.-(\sqrt{3}C_3+C_2)\cos\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)\right]\right\}e^{-\frac{x\varepsilon^{-1/3}}{2}}\,.$$

$$H_{11}\frac{1}{\alpha\cos\tau}\frac{\theta}{ns} = \frac{1}{\theta}\left[-\frac{1}{2}(x^2+r^2)+(r-\varepsilon^{1/3})(x+\varepsilon^{1/3})\right]\left\{\left(\frac{nsy^3}{b}\right.\right.$$

$$\left.+\theta n^2s^2y\Big)\sin nsy + \left(\theta ns + \frac{y^2}{b}\right)\cos nsy\right\} + \frac{\cos nsy}{3ns} +$$

$$+\frac{y\sin nsy}{3} - \frac{\varepsilon^{2/3}}{3b\theta}\left[4y^2\cos nsy - \frac{11y\sin nsy}{ns} - \frac{11\cos nsy}{n^2s^2}\right.$$

$$\left.+nsy^3\sin nsy\right] + y\sin nsy\left\{\left(\varepsilon^{-1/3}-\frac{nsy^2\varepsilon^{1/3}}{\theta b}\right)\left(\frac{x-r-\varepsilon^{1/3}}{3}\right)\right.$$

$$+ \varepsilon^{1/3}A \Big\} e^{(x-r)\varepsilon^{-1/3}} + y \sin nsy \Big\{ [(\frac{r}{3} - \varepsilon^{1/3}) \frac{nsy^2}{b\Theta}$$

$$- \frac{r\varepsilon^{-2/3}}{3}] \frac{x}{\sqrt{3}} \sin (\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}) + [\frac{r\varepsilon^{-2/3}}{3} - \frac{2\varepsilon^{-1/3}}{3}$$

$$+ \frac{nsy^2}{b\Theta}(\frac{r}{3} - \frac{\varepsilon^{1/3}}{3})] x \cos (\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}) + [1 - \frac{2r\varepsilon^{-1/3}}{3}$$

$$- \frac{nsy^2}{b\Theta} \frac{r\varepsilon^{1/3}}{3}] \frac{\sin}{\sqrt{3}} (\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}) + [\frac{nsy^2}{b\Theta}(\frac{r\varepsilon^{1/3}}{3} - \frac{2\varepsilon^{2/3}}{3}) - \frac{1}{3}] \cos \cdot$$

$$\cdot (\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}) - \frac{\varepsilon^{1/3}}{2} [(C_2 + \sqrt{3}\, C_3) \cos \cdot$$

$$\cdot (\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}) + (C_3 - \sqrt{3}\, C_2)\sin(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2})] \Big\} e^{-\frac{x\varepsilon^{-1/3}}{2}} .$$

$$V_{21} = \frac{\alpha \cos \tau}{\Theta b} (y \cos nsy + nsy^2 \sin nsy)(x - r + \varepsilon^{1/3})$$

$$+ \alpha \cos \tau \frac{nsy^2}{\Theta b} \sin nsy \Big\{ \frac{x - r}{3} + A_2(y) \Big\} e^{(x-r)\varepsilon^{-1/3}}$$

$$+ \alpha \cos \tau \frac{nsy^2}{\Theta b} \sin sy \Big\{ \frac{x}{3} \cos(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2})$$

$$+ (\frac{2r\varepsilon^{-1/3}}{3} - 1) \frac{x}{\sqrt{3}} \sin (\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}) \Big\} e^{-\frac{x\varepsilon^{-1/3}}{2}}$$

$$+ \alpha \cos \tau \frac{\sin nsy}{\Theta b} \Big\{ C_{22}\cos(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}) + C_{32}\sin(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}) \Big\} e^{-\frac{x\varepsilon^{-\frac{1}{3}}}{2}}$$

$$U_{21} = \frac{\alpha \cos \tau}{\Theta b}(-\frac{x^2}{2} - rx - x\varepsilon^{1/3})[(2+y^2n^2s^2)\cos nsy + 2yns \sin nsy]$$

$$+ \alpha \cos \tau \frac{C_{21}(y)}{\Theta b} - \alpha \cos \tau \frac{\partial}{\partial y} \Big\{ \frac{nsy^2}{\Theta b} \sin nsy \, [(\frac{2\varepsilon^{2/3}}{3}$$

$$- \frac{r\varepsilon^{1/3}}{3})\cos(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}) + \frac{r\varepsilon^{1/3}}{3\sqrt{3}} \sin(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}$$

$$+ (\varepsilon^{1/3} - \frac{r}{3}) \frac{x}{\sqrt{3}} \sin(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}) + (\frac{\varepsilon^{1/3}}{3} - \frac{r}{3}) x \cos \cdot$$

$$\cdot \left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)]\Big\} e^{-\frac{x\varepsilon^{-1/3}}{2}} - \alpha \cos\tau \frac{\partial}{\partial y}\Big\{\frac{\varepsilon^{1/3}}{2}\frac{\sin nsy}{\Theta b} \cdot$$

$$\cdot [(\sqrt{3}\, C_{22} - C_{32})\sin\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right) - (C_{22} + \sqrt{3}\, C_{32})\cos \cdot$$

$$\cdot \left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)]\Big\} e^{-\frac{x\varepsilon^{-1/3}}{2}} - \alpha \cos\tau \frac{\partial}{\partial y}\Big\{\frac{nsy^2}{\Theta b} \sin nsy\left(\frac{x - r - \varepsilon^{1/3}}{3}\right.$$

$$\left. + A_2\right)\varepsilon^{1/3}\Big\} e^{(x-r)\varepsilon^{-1/3}} .$$

$$bH_{21} = -\frac{ns}{\Theta}\alpha\cos\tau\Big\{ y \sin nsy[\varepsilon^{-1/3}\left(\frac{x - r - \varepsilon^{1/3}}{3}\right)$$

$$+ \varepsilon^{1/3}A_2]e^{(x-r)\varepsilon^{-1/3}} + y \sin nsy\ [(\frac{1}{\sqrt{3}} - \frac{2r\varepsilon^{-1/3}}{3\sqrt{3}}$$

$$- \frac{r\varepsilon^{-2/3}}{3\sqrt{3}}x)\sin\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right) + \left(\frac{xr\varepsilon^{2/3}}{3} - \frac{2x\varepsilon^{-1/3}}{3} - \frac{1}{3}\right)$$

$$\cos\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)]\, e^{-\frac{x\varepsilon^{-1/3}}{2}} - \frac{\varepsilon^{1/3}}{2}\ y \sin nsy\ [(C_{22} + \sqrt{3}\, C_{32})$$

$$\cos\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right) + (C_{32} - \sqrt{3}C_{22})\sin\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)]e^{-\frac{x\varepsilon^{-1/3}}{2}}\Big\}$$

$$+ \frac{\alpha\cos\tau\, nsy^3 \sin nsy}{b\Theta}\ \varepsilon^{1/3}\left(\frac{x - r - \varepsilon^{1/3}}{3}\right)[\frac{1}{\ell}$$

$$+ \frac{ns}{\Theta}]\, e^{(x-r)\varepsilon^{-1/3}} + \left(\frac{1}{\ell} + \frac{ns}{\Theta}\right)\frac{\alpha\cos\tau\, nsy^3\sin nsy}{b\Theta}$$

$$\Big\{[(\varepsilon^{1/3} - \frac{r}{3})\frac{x}{\sqrt{3}} + \frac{r\varepsilon^{1/3}}{3\sqrt{3}}]\sin\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)$$

$$+ [\frac{2\varepsilon^{2/3}}{3} - \frac{r\varepsilon^{1/3}}{3} + \left(\frac{\varepsilon^{1/3} - r}{3}\right)x]\cos\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)\Big\}\, e^{-\frac{x\varepsilon^{-1/3}}{2}}$$

$$- \frac{\varepsilon^{1/3}}{2}\frac{y \sin nsy}{b\Theta}\Big\{(C_{22} + \sqrt{3}\, C_{32})\cos\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)$$

$$+ (C_{32} - \sqrt{3}\, C_{22})\sin\left(\frac{x\sqrt{3}\varepsilon^{-1/3}}{2}\right)\Big\}\, e^{-\frac{x\varepsilon^{-1/3}}{2}}$$

$$+ \frac{ns\,\alpha\,\cos\tau}{\theta} \left\{ \left[\frac{1}{2}(x^2 + r^2) + (\varepsilon^{1/3} - r)(x + \varepsilon^{1/3}) \right] \cdot \right.$$

$$\left. \cdot \left[yn^2s^2 \sin nsy + ns \cos nsy \right] - \frac{1}{3}\left(\frac{\cos nys}{ns} + y \sin nsy\right) \right\}$$

$$+ \frac{\alpha \cos\tau}{b\theta} \left(\frac{1}{\ell} + \frac{ns}{\theta}\right) \left[\frac{1}{2}(x^2 + r^2) + (\varepsilon^{1/3} - r)(x + \varepsilon^{1/3}) \right] \cdot$$

$$\cdot \left[nsy^3 \sin nsy + y^2 \cos nsy \right] + \frac{\alpha \cos\tau}{3b\theta} \left(\frac{1}{\ell} + \frac{ns}{\theta}\right) \cdot$$

$$\cdot \left[4y^2 \cos nsy + nsy^3 \sin nsy - \frac{11 \sin nsy}{ns} - \frac{11 \cos nsy}{n^2s^2} \right] .$$

The functions A, A_2, C, C_1, C_2, C_3, C_{22}, C_{32} are determined by applying the boundary conditions $U_1 = V_1 = U_2 = V_2 = 0$ on $x = 0, r$.

$$A \sin nsy = \frac{\varepsilon^{1/3}}{\theta}\left\{(\theta n^2 s^2 + \frac{y^2 ns}{b})\sin nsy + \frac{y \cos nsy}{b}\right\}$$

$$C_2 \sin nsy = \frac{\varepsilon^{1/3-r}}{\theta}\left\{(\theta n^2 s^2 + \frac{y^2 ns}{b})\sin nsy + \frac{y \cos nsy}{b}\right\}$$

$$C(y) = \frac{1}{\theta}(\frac{r^2}{2} - r\varepsilon^{1/3} + \varepsilon^{2/3})\left\{(\theta n^3 s^3 + \frac{y^2 n^2 s^2 + 2}{b})\cos nsy\right.$$

$$+ \frac{2yns}{b}\sin nsy - \frac{yns\,\varepsilon^{2/3}}{3b\theta}\sin nsy + (\frac{n^2 s^2 y^2 \varepsilon^{2/3}}{3b\theta} - \frac{\varepsilon^{2/3}}{\theta b}$$

$$\left. - \frac{ns}{3})\cos nsy\right\}$$

$$\frac{\varepsilon^{1/3}}{2}\sqrt{3}\sin nsy\, C_3 = -\frac{1}{\theta}(\frac{r^2}{2} - r\varepsilon^{1/3} + \varepsilon^{2/3})\left\{\theta n^2 s^2 + \frac{1}{b}(nsy^2\right.$$

$$\left. + \frac{2}{ns})\right\}\sin nsy + \frac{\varepsilon^{2/3}}{\theta}\left\{(-\frac{3}{2}\frac{nsy^2}{b} - \frac{\theta n^2 s^2}{2})\sin nsy + \frac{y}{2b}\cos nsy\right\}$$

$$+ \frac{r\varepsilon^{1/3}}{\theta}\left\{(\frac{1}{bns} + \frac{5nsy^2}{b} + \frac{\theta n^2 s^2}{2})\sin nsy - \frac{y}{2b}\cos nsy\right\}$$

$$nsy^2 \sin nsy\, A_2 = -\varepsilon^{1/3}(y \cos nsy + nsy^2 \sin nsy)$$

$$\sin nsy\, C_{22} = -(r - \varepsilon^{1/3})(y \cos nsy + nsy^2 \sin nsy)$$

$$C_1 = -(\frac{r^2}{2} - r\varepsilon^{1/3} + \varepsilon^{2/3})[(2 + y^2 n^2 s^2)\cos nsy + 2yns \sin nsy]$$

$$+ \varepsilon^{2/3}[\frac{nsy}{3}\sin nsy + \cos nsy - \frac{n^2 s^2 y^2}{3}\cos nsy]$$

$$\frac{\sqrt{3}\,\varepsilon^{1/3}}{2}\sin nsy\, C_{32} = (\frac{r^2}{2} - r\varepsilon^{1/3})[nsy^2 \sin nsy + \frac{2 \sin nsy}{ns}]$$

$$+ \varepsilon^{2/3}(\frac{y \cos nsy}{2} + \frac{3}{2} nsy^2 \sin nsy)$$

$$+ \frac{r\varepsilon^{1/3}}{6}(nsy^2 \sin nsy + 3y \cos nsy).$$

The second method of attack on the non-steady two-layer problem consists of separating the expression for the wind-stress into its steady and periodic parts, i.e., $(1 + \alpha \sin \tau) \cos nsy = \cos nsy + \alpha \sin \alpha \cos nsy$, and treating each problem separately. This method of solution was also attempted in the one-layer problem. The resulting equations could not be solved, however, without recourse to numerical methods. In the present case, we hope to make use of the smallness of the parameter b in seeking a solution.

In equation (3) the right hand side may be approximated by $\partial/\partial\tau\,(- H_1 + aH_2) \approx \partial/\partial\tau\,(- H_1 + H_2) \approx \partial H_2/\partial\tau$.

The steady problem with cos nsy as the wind-stress term has been solved previously. For the time-dependent problem, we write

$$\overline{H}_1 = \Theta H_1, \quad \overline{H}_2 = \Theta b H_2, \quad \overline{\ell} = \frac{\ell}{\Theta}, \quad \overline{\gamma} = \frac{\delta}{\Theta b} .$$

Then, with the time-dependent part of the wind-stress only, (1) - (6) become

$$ns\delta \frac{\partial U_1}{\partial \tau} - nsyV_1 = - \frac{\partial \overline{H}_1}{\partial x} + ns\varepsilon\Delta U_1 - \alpha \sin \tau \cos nsy$$

$$ns\delta \frac{\partial V_1}{\partial \tau} + nsyU_1 = - \frac{\partial \overline{H}_1}{\partial y} + ns\varepsilon\Delta V_1$$

$$\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y} = \overline{\gamma} \frac{\partial \overline{H}_2}{\partial \tau}$$

$$\delta \frac{\partial U_2}{\partial \tau} - yV_2 = - \overline{\ell} \left(\frac{\partial \overline{H}_1}{\partial x} + \frac{\partial \overline{H}_2}{\partial x}\right) + \varepsilon\Delta U_2$$

$$\delta \frac{\partial V_2}{\partial \tau} + yU_2 = -\bar{\ell}\left(\frac{\partial \bar{H}_1}{\partial y} + \frac{\partial \bar{H}_2}{\partial y}\right) + \varepsilon \Delta V_2$$

$$\frac{\partial U_2}{\partial x} + \frac{\partial V_2}{\partial y} = -\bar{\gamma}\,\frac{\partial \bar{H}_2}{\partial \tau}\ .$$

Next let us write the wind-stress as the imaginary part of $\alpha e^{i\tau}$ cos nsy. Then if we take only the imaginary terms in the remaining parts of the equation, the results will be the same as those above.

Define

$$U_{1,2} = \alpha e^{i\tau} u_{1,2}(x,y), \quad V_{1,2} = \alpha e^{i\tau} \bar{v}_{1,2}(x,y), \quad \bar{H}_{1,2} = \alpha e^{i\tau} h_{1,2}(x,y).$$

The equations become

$$ins\delta u_1 - nsy\bar{v}_1 = -\frac{\partial h_1}{\partial x} + ns\varepsilon \Delta u_1 - \cos nsy$$

$$ins\delta \bar{v}_1 + nsyu_1 = -\frac{\partial h_1}{\partial y} + ns\varepsilon \Delta v_1$$

$$u_{1x} + \bar{v}_{1y} = i\bar{\gamma}\, h_2$$

$$i\delta u_2 - y\bar{v}_2 = -\bar{\ell}\left(\frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial x}\right) + \varepsilon \Delta u_2$$

$$i\delta \bar{v}_2 + yu_2 = -\bar{\ell}\left(\frac{\partial h_1}{\partial y} + \frac{\partial h_2}{\partial y}\right) + \varepsilon \Delta \bar{v}_2$$

$$u_{2x} + \bar{v}_{2y} = -i\bar{\gamma}\, h_2\ .$$

The above equations must be solved for the six unknowns. The difficulty arises in trying to match the boundary layer contirbution with the interior solution. To conserve space, we shall not give the entire analysis here, but shall confine

ourselves to the determination of the boundary layer contribution and to an indication of the ensuing difficulties.

Carry out the following three steps:

(a) Let $x = \varepsilon^{1/3}\xi$, i.e., stretch x coordinate near $x = 0$.

(b) Substitute $v_{1,2} = \varepsilon^{1/3}\,\bar{v}_{1,2}$, $\gamma = \varepsilon^{1/3}\bar{\gamma}$.

(c) Keep the leading terms of the equations.

The equations then reduce to

$$nsyv_1 = h_{1\xi}$$

$$ins\delta\varepsilon^{-1/3}v_1 + nsyu_1 = -\,h_{1y} + nsv_{1\xi\xi}$$

$$u_{1\xi} + v_{1y} = i\gamma h_2$$

$$yv_2 = \bar{\ell}\,(h_{1\xi} + h_{2\xi})$$

$$i\delta\varepsilon^{-1/3}v_2 + yu_2 = -\,\bar{\ell}\,(h_{1y} + h_{2y}) + v_{2\xi\xi}$$

$$u_{2\xi} + v_{2y} = -\,i\gamma h_2\,.$$

Eliminating all the unknowns except h_2, we find

$$h_{2\xi\xi\xi\xi} - h_{2\xi\xi} - h_{2\xi} + L\,h_2 = 0$$

where

$$\delta_1 = i\delta\varepsilon^{-1/3}, \quad L = i\gamma y^2(ns + \frac{1}{\bar{\ell}})\,.$$

Solutions are

$$h_2 = \sum_{i=1}^{4} C_i\,e^{D_i\xi}$$

where the D_i are the roots of

$$D^4 - \delta_1 D^2 - D + L = 0.$$

They are

$$D_{1,2} = \frac{-(\frac{4}{3}\delta_1+2A+2B) \pm \sqrt{(\frac{4}{3}\delta_1+2A+2B)^2 - 2(A+B-\frac{\delta}{3}+2)}}{2}$$

$$D_{3,4} = \frac{\frac{4}{3}\delta_1+2A+2B \pm \sqrt{(\frac{4}{3}\delta_1+2A+2B)^2 - 2(A+B-\frac{\delta}{3}-2)}}{2}$$

where

$$A = \left\{\frac{1}{2} + \frac{4\delta_1 L}{3} - \frac{\delta_1^3}{27} + \sqrt{-[\frac{1}{3}(4L+\frac{\delta_1^2}{3})]^3 + [\frac{1}{2} + \frac{4\delta_1 L}{3} - \frac{\delta_1^3}{27}]^2}\right\}^{1/3}$$

$$B = \left\{\frac{1}{2} + \frac{4\delta_1 L}{3} - \frac{\delta_1^3}{27} - \sqrt{-[\frac{1}{3}(4L+\frac{\delta_1^2}{3})]^3 + [\frac{1}{2} + \frac{4\delta_1 L}{3} - \frac{\delta_1^3}{27}]^2}\right\}^{1/3}.$$

The above solution for h_2 must now be substituted into the previous six equations and the boundary layer contributions for $u_{1,2}$, $v_{1,2}$, $h_{1,2}$ can be derived by keeping the parts which $\longrightarrow 0$ as $\xi \longrightarrow \infty$. If the interior and boundary layer solutions are added, the C_i can be evaluated by means of the boundary conditions $U_1 = V_1 = U_2 = V_2 = 0$ on $x = 0, r$.

Practically, this is an almost impossible task, and numerical methods must be employed for the whole procedure. In view of this fact, nothing is gained by the analysis and the entire solution might as well be carried out numerically from the very beginning.

Since we have been unable to arrive at a useful solution for the non-steady ocean circulation without assuming negligible velocities in the bottom layer, we have no assurance that our analysis is valid. Reliable observational data which might guide us in this matter are not available. We may perhaps gain a little more confidence in the results of this investigation by the following considerations.

For the formulation of Problem 1 it was assumed that the velocities, and hence the horizontal pressure gradient, vanish in the bottom layer. This, together with the hydrostatic pressure law, immediately led to the conclusion that the thermocline responds instantaneously to any motion of the free surface. Naturally, this can hold, if at all, only for sufficiently slowly varying circulation.

Some investigators are of the opinion that the very opposite situation actually exists, i.e., the thermocline remains essentially fixed and does not respond to wind variations of, say, seasonal or annual periods. This is perhaps a more reasonable assumption because it is based on the idea that the frequency of wind variation is much greater than the important frequencies of free oscillations of the bottom layer.

Let us assume, therefore, that the shape of the thermocline remains roughly fixed in such a manner as to result in a vanishing time-average horizontal pressure gradient in the bottom layer. That is to say, the thermocline adjusts itself to the mean wind distribution so as to give zero pressure

gradient for the case of a steady wind having this mean distribution. If we now have a time-dependent wind, we will have non-vanishing pressure gradients in the bottom layer as a result of changes in the free surface shape. The resultant velocities in the bottom layer will tend to be uniform vertically (except as influenced by friction) provided the bottom layer has fairly uniform density so that the pressure gradient is independent of depth.

Suppose we have a two-layer ocean and integrate over the top layer only. If we make use of the assumption of a stationary thermocline, and if the effect of friction at the thermocline on the transport in the top layer is negligible, then the resulting transport equations are essentially the same as those attained in Problem 1. Hence, the distribution of mass transport obtained in Problem 1 may be expected to be valid now, provided it is interpreted as the distributions of transport above the thermocline. Since this is the transport usually measured, we may still hope that the results are useful.

6. <u>Conclusions</u>. If the velocities in the depths of the ocean are negligible, then the horizontal pressure gradients are also negligible and the thermocline responds immediately to a change in the free surface height provided the hydrostatic pressure equation is valid. For such a case, the following results appear to be valid (within the framework of subsequent approximations made in this report):

(i) For a varying wind with a period of three months or

more, the mass transport through the Gulf Stream responds to the wind but lags behind it at all times except at the instants of extreme wind variation when the two are in phase.

(ii) The maximum lag appears when the wind is in its mean position and an interval of about nine days elapses between the time at which the wind reaches its mean value and the time at which the transport reaches its mean value. The actual length of the interval, i.e., nine days, is independent of the frequency of the wind variation.

(iii) The value of the maximum mass transport through the Gulf Stream does not depend on the frequency but only on the maximum strength of the wind.

(iv) The Gulf Stream does not undergo any noticeable east-west shift nor is its width altered because of the wind variation.

For the steady two-layer problem, the streamline pattern coincides with that of the one-layer case. The computed steady position of the thermocline can be made to agree qualitatively with the position of the observed thermocline provided the two parameters (a) the thickness of the top layer and (b) the density difference, are chosen appropriately.

At the outset of our investigation we had hoped to solve the linearized, non-steady, two-layer problem with no a priori assumption concerning the vertical distribution of velocity. However, we were unsuccessful in doing so except for the case of a wind with a period of oscillation of 100 years or more. For such a low frequency, the retention of the time derivative

terms in favor of the non-linear terms seems wholly unjustified. The only conclusion (which may not be justified because of the previous statement) resulting from this last investigation is that the transports in the lower layer are of the same order of magnitude as the out-of-phase transports of the upper layer.

In view of the statements made at the end of Section 5, the results listed for the one-layer problem are approximately valid for the non-steady two-layer problem provided:

(a) The thermocline adjusts itself to the mean wind distribution and remains fixed.

(b) The mass transports of Problem 1 are interpreted as the transports in the upper layer.

The assumption of hydrostatic pressure is not necessary for the solution of the mass transports in the steady problem.

Wherever the results of this analysis permit a comparison with observation, good qualitative agreement is achieved, but the quantitative results are off by a factor of about three. In view of the many idealizing assumptions made, however, no more than qualitative agreement could be hoped for.

A number of features have been left out of the present model. Changing topography, non-linear terms, variable eddy viscosity and many other features could combine to change the results noticeably. However, the analysis of the problem including most of the features which were omitted in our model would probably require a numerical treatment.

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Appendix 1. Transformation of the Differential Equations from Spherical to Rectangular Coordinates.

Consider a rotating spherical coordinate system; let r be the radial distance from the center of the sphere, θ the colatitude, φ the meridianal angle. The equations of motion are*

$$\frac{Du}{Dt} - \Omega^2 r \sin^2\theta - 2w\Omega \sin\theta = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g' + \frac{1}{\rho}(\nabla \cdot A_i \nabla)\, u$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial v}{\partial \varphi} + \frac{uv}{r} - \frac{w^2 \cot\theta}{r} + \Omega^2 r \sin\theta\cos\theta$$

$$- 2w\Omega\cos\theta = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{\rho}(\nabla \cdot A_i \nabla)\, v$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial w}{\partial \varphi} + \frac{wu}{r} - \frac{vw\cot\theta}{r} + 2v\Omega\cos\theta$$

$$+ 2u\Omega\sin\theta = -\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial \varphi} + \frac{1}{\rho}(\nabla \cdot A_i \nabla) w$$

where $\frac{Du}{Dt}$ is the material derivative of the radial velocity in terms of spherical coordinates

g' denotes the gravitational force

$\frac{1}{\rho}\nabla \cdot A_i \nabla \equiv A\nabla^2 + \frac{1}{\rho}\frac{\partial}{\partial r}(A_3 \frac{\partial}{\partial r})$ and ∇^2 denotes the Laplacian operator for the two dimensions θ and φ.

We shall neglect the radial acceleration and shear terms arising as a result of the velocities relative to the rotating

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* We shall not consider the non-linear terms or the viscous terms in the radial equation of motion; hence, this equation is written in operator form only.

sphere. We then have

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = - g \qquad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{w}{r \sin\theta}\frac{\partial v}{\partial \varphi} + \frac{uv}{r} - \frac{w^2 \cot\theta}{r} + \Omega^2 r \sin\theta\cos\theta$$

$$-2w\Omega\cos\theta = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + A\nabla^2 v + \frac{1}{\rho}\frac{\partial}{\partial r}\left(A_3\frac{\partial v}{\partial r}\right) \qquad (2)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial w}{\partial \varphi} + \frac{wu}{r} - \frac{vw\cot\theta}{r} + 2v\Omega\cos\theta$$

$$-2u\Omega\sin\theta = -\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial \varphi} + A\nabla^2 w + \frac{1}{\rho}\frac{\partial}{\partial r}\left(A_3\frac{\partial w}{\partial r}\right) \qquad (3)$$

where $g = g' - \frac{\partial}{\partial r}(\frac{1}{2}\Omega^2 r^2 \sin^2\theta)$ is the apparent gravitational force. The viscous terms for equations (2) and (3) are

$$A\nabla^2 v = \frac{A}{r^2}\left\{\cot\theta\frac{\partial v}{\partial \theta} + \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial \varphi^2} - \frac{v}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial w}{\partial \varphi}\right\}$$

$$A\nabla^2 w = \frac{A}{r^2}\left\{\cot\theta\frac{\partial w}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial \varphi^2} - \frac{w}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial v}{\partial \varphi}\right\}$$

Since the region of interest to us consists of a very thin layer on the surface of the globe, we shall approximate r by R, the mean radius of the earth, whenever r appears in undifferentiated form. At the same time let us define a new east-west coordinate by $x = \varphi R \sin\theta$, a north-south coordinate by $y = R(\frac{\pi}{2} - \theta)$ and a vertical coordinate by $z = r$. Then equations (1)-(3) become

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = g \qquad (4)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial z} - v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial x} + \frac{uv}{R} - \frac{w^2 \cot\theta}{R} + \Omega^2 r \sin\theta\cos\theta$$

$$- 2w\Omega\cos\theta = \frac{1}{\rho}\frac{\partial p}{\partial y} + A\left\{-\frac{\cot\theta}{R}\frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} - \frac{v}{R^2\sin^2\theta}\right.$$

$$\left.- \frac{2\cot\theta}{R}\frac{\partial w}{\partial x}\right\} + \frac{1}{\rho}\frac{\partial}{\partial z}\left(A_3\frac{\partial v}{\partial z}\right) \qquad (5)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial z} - v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial x} + \frac{wu}{R} - \frac{uw\cot\theta}{R} + 2v\Omega\cos\theta - 2u\Omega\sin\theta$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + A\left\{-\frac{\cot\theta}{R}\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2} - \frac{w}{R^2\sin^2\theta} - \frac{2\cot\theta}{R}\frac{\partial v}{\partial x}\right\}$$

$$+ \frac{1}{\rho}\frac{\partial}{\partial z}\left(A_3\frac{\partial w}{\partial z}\right) \qquad (6)$$

Since R is very large, we shall neglect terms divided by R. We can do this provided the region is sufficiently far removed from the poles ($\theta = 0, \pi$) where cot θ becomes infinite. The velocity component u is assumed to be much smaller than the components v and w so that we can neglect u throughout the equations of motion.

Ordinarily, one uses the velocity components u,v,w to correspond to the directions x,y,z respectively. In equations (4)-(6), u,-v,w correspond to z,y,x respectively. The negative sign was carried over from the definition of v which was defined positive southward. If we revert to the more familiar notations and write $u' = w$, $v' = -v$, $w' = u$, we have for equations (4)-(6)(with the terms with coefficient $\frac{1}{R}$ and all terms containing w' neglected)

$$\frac{\partial u'}{\partial t} + u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} - 2\Omega v' \sin\left(\frac{y}{R}\right)$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + A\left\{\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2}\right\} + \frac{1}{\rho}\frac{\partial}{\partial z}\left(A_3\frac{\partial u'}{\partial z}\right) \qquad (7)$$

$$\frac{\partial v'}{\partial t} + u'\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y} + 2\Omega u' \sin\left(\frac{y}{R}\right) - \Omega^2 r \sin\Theta\cos\Theta$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial y} + A\left\{\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2}\right\} + \frac{1}{\rho}\frac{\partial}{\partial z}\left(A_3\frac{\partial v'}{\partial z}\right) \qquad (8)$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g \qquad (9)$$

If the above procedure be carried out for the continuity equation, the latter becomes

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \qquad (10)$$

In making the transformation from spherical to rectangular coordinates, we must consider the distortion of the spherical surface as a result of the mapping process. Specifically, a rectangle in the rectangular system maps into a region on the sphere in such a manner that the east-west distance remains constant and the right angles between the lines x = const. and y = const. map into obtuse angles between the lines on the sphere corresponding to x = const. and y = const. Thus, the mapping is not conformal.

With the above transformation we have mapped a spherical surface onto the plane. Our real aim, however, is to map the equilibrium surface which establishes itself as a

result of the interaction of centripetal acceleration and gravity, onto the plane. We shall, therefore, neglect the small difference between the true equilibrium surface and the sphere.

The apparent gravity, g, in (1) acts perpendicular to the spherical surface. We shall now consider g to act perpendicular to the equilibrium surface. We must then drop the term $\Omega^2 r \sin \Theta \cos \Theta$ from the Θ equation since, in reality, this force combines with g acting normal to the spherical surface, to give rise to a resultant normal to the equilibrium surface. Finally, $g = g' - \frac{\partial}{\partial r}(\frac{1}{2}\Omega^2 r^2 \sin^2\Theta)$ is assumed constant. The final result of the approximate transformation is to map the equilibrium free surface of the ocean onto the x-y plane, with the apparent force of gravity acting normal to this plane.

Appendix 2. Neglect of the Non-Linear Terms.

Consider the integrated equations of motion of section 3.

$$\int_{-h}^{\eta} \frac{\partial u'}{\partial t} dz + \int_{-h}^{\eta} u' \frac{\partial u'}{\partial x} dz + \int_{-h}^{\eta} v' \frac{\partial u'}{\partial y} dz - \beta y \int_{-h}^{\eta} v' dz$$

$$= - g D \frac{\partial \eta}{\partial x} + A \int_{-h}^{\eta} \Delta u' \, dz + \tau_x \qquad (1)$$

$$\int_{-h}^{\eta} \frac{\partial u'}{\partial t} dz + \int_{-h}^{\eta} u' \frac{\partial v'}{\partial x} dz + \int_{-h}^{\eta} v' \frac{\partial v'}{\partial y} dz + \beta y \int_{-h}^{\eta} u' dz$$

$$= - g D \frac{\partial \eta}{\partial y} + A \int_{-h}^{\eta} \Delta v' \, dz + \tau_y \qquad (2)$$

where we have linearized the pressure term in accordance with remarks to be made later in sections 3 and 5. τ_x, τ_y are now the wind-stress components of section 3 divided by ρ. Assume

$$u' = \bar{u}(x,y,t)\, e^{kz} \ , \ v' = \bar{v}(x,y,t)\, e^{kz}$$

i.e. the velocities decay exponentially with depth.

Then,

$$\frac{\partial \bar{u}}{\partial t} e^{kz} \Big|_{-h}^{\eta} + \frac{1}{2} \bar{u} \frac{\partial \bar{u}}{\partial x} e^{2kz} \Big|_{-h}^{\eta} + \frac{1}{2} \bar{v} \frac{\partial \bar{u}}{\partial y} e^{2kz} \Big|_{-h}^{\eta}$$

$$- 2\Omega \sin(\tfrac{y}{R})\, \bar{v}\, e^{kz} \Big|_{-h}^{\eta} = - g D \frac{\partial \eta}{\partial x} k + A \Delta \bar{u}\, e^{kz} \Big|_{-h}^{\eta} + \tau_x k \qquad (3)$$

$$\frac{\partial \bar{v}}{\partial t} e^{kz} \Big|_{-h}^{\eta} + \frac{1}{2} \bar{u} \frac{\partial \bar{v}}{\partial x} e^{2kz} \Big|_{-h}^{\eta} + \frac{1}{2} \bar{v} \frac{\partial \bar{v}}{\partial y} e^{2kz} \Big|_{-h}^{\eta}$$

$$+ 2\Omega \sin(\tfrac{Y}{R}) \bar{u} e^{kz} \Big|_{-h}^{\eta} = - g D \frac{\partial \eta}{\partial y} k + A\Delta \bar{v} e^{kz} \Big|_{-h}^{\eta} + \tau_y k \qquad (4)$$

Approximate the exponentials at their limits by $e^{k\eta} \approx 1$, $e^{-kh} \approx 0$. Then (3) and (4) become

$$\frac{\partial \bar{u}}{\partial t} + \frac{1}{2} \bar{u} \frac{\partial \bar{u}}{\partial x} + \frac{1}{2} \bar{v} \frac{\partial \bar{u}}{\partial y} - 2\Omega \sin(\tfrac{Y}{R}) \bar{v} = - g D \frac{\partial \eta}{\partial x} k + A\Delta \bar{u} + \tau_x k \qquad (5)$$

$$\frac{\partial \bar{v}}{\partial t} + \frac{1}{2} \bar{u} \frac{\partial \bar{v}}{\partial x} + \frac{1}{2} \bar{v} \frac{\partial \bar{v}}{\partial y} + 2\Omega \sin(\tfrac{Y}{R}) \bar{u} = - g D \frac{\partial \eta}{\partial y} k + A\Delta \bar{v} + \tau_y k \qquad (6)$$

Linearize the Coriolis parameter by $2\Omega \sin(\frac{Y}{R}) \simeq \beta y$ where $\beta = \frac{2\Omega}{R}$. Taking the derivative of (6) with respect to x and the derivative of (5) with respect to y and subtracting, we have

$$\frac{\partial}{\partial t} \left(\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y}\right) + \frac{1}{2} \left[\frac{\partial \bar{u}}{\partial x} \frac{\partial \bar{v}}{\partial x} + \bar{u} \frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial \bar{v}}{\partial x} \frac{\partial \bar{v}}{\partial y} + \bar{v} \frac{\partial^2 \bar{v}}{\partial x \partial y} - \frac{\partial \bar{u}}{\partial x} \frac{\partial \bar{u}}{\partial y} - \bar{u} \frac{\partial^2 \bar{u}}{\partial x \partial y} \right.$$

$$\left. - \frac{\partial \bar{v}}{\partial y} \frac{\partial \bar{u}}{\partial y} - \bar{v} \frac{\partial^2 \bar{u}}{\partial y^2} \right] + \beta y \left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) + \beta \bar{v} = A\Delta \left(\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y}\right)$$

$$+ k\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right) \qquad (7)$$

Choose $\tau_y \equiv 0$, $\tau_x = - (W' + \Gamma' \sin \omega t) \cos ny$.

We shall non-dimensionalize the velocities so that they are of order unity in the interior of the ocean. It is convenient* to choose

*The choice of the non-dimensional quantities is motivated in section 4.

$$V = \frac{\beta \bar{v}}{kW}, \quad U = \frac{\beta \bar{u}}{kW}, \quad x = sx', \quad y = sy', \quad \tau = \omega t, \quad \delta = \frac{W}{\beta s}, \quad \alpha = \frac{\Gamma'}{W'},$$

$$\Upsilon = \frac{kW}{2\beta^2 s^2}, \quad \varepsilon = \frac{A}{\beta s^3}, \quad H = \frac{\beta^2 s^2 n}{W}, \quad W = nW', \quad \Gamma = n\Gamma'$$

In this notation and with the prescribed form for τ_x and τ_y, equation (7) becomes

$$\delta \frac{\partial}{\partial \tau}\left[\frac{\partial V}{\partial x'} - \frac{\partial U}{\partial y'}\right] + \Upsilon\left[\frac{\partial U}{\partial x'}\frac{\partial V}{\partial x'} + U\frac{\partial^2 V}{\partial x'^2} + \frac{\partial V}{\partial x'}\frac{\partial V}{\partial y'} + V\frac{\partial^2 V}{\partial x'\partial y'}\right.$$

$$\left. - \frac{\partial U}{\partial x'}\frac{\partial U}{\partial y'} - U\frac{\partial^2 U}{\partial x'\partial y'} - \frac{\partial V}{\partial y'}\frac{\partial U}{\partial y'} - V\frac{\partial^2 U}{\partial y'^2}\right]$$

$$+ y\left[\frac{\partial U}{\partial x'} + \frac{\partial V}{\partial y'}\right] + V = \varepsilon\Delta'\left[\frac{\partial V}{\partial x'} - \frac{\partial U}{\partial y'}\right] - (1 + \alpha \sin\tau)\sin nsy' \tag{8}$$

The integrated, non-dimensionalized continuity equation becomes

$$\frac{\partial U}{\partial x'} + \frac{\partial V}{\partial y'} = -\delta\frac{\partial H}{\partial \tau} \tag{9}$$

If we expand the velocities and the height, H, in a series in δ, then the solution can be looked upon as the sum of a quasi-steady part plus a number of out-of-phase contributions. If δ is small enough we may be justified in keeping only the first two terms of such a series as a fairly accurate representation of the complete series.

Hence, let

$$U = U_o + \delta U_1 + \delta^2 U_2 + \ldots, \quad V = V_o + \delta V_1 + \delta^2 V_2 + \ldots$$

$$H = H_o + \delta H_1 + \delta^2 H_2 + \ldots$$

Then for the equations of zero-order in δ, we have*

$$\gamma \left[\frac{\partial U_o}{\partial x'}\frac{\partial V_o}{\partial x'} + \ldots - V_o \frac{\partial^2 U_o}{\partial y'^2}\right] + V_o = \varepsilon\Delta \left[\frac{\partial V_o}{\partial x'} - \frac{\partial U_o}{\partial y'}\right]$$

$$- (1 + \alpha \sin \tau)\sin nsy' \qquad (10)$$

$$\frac{\partial U_o}{\partial x'} + \frac{\partial V_o}{\partial y'} = 0 \qquad (11)$$

The first order equations in δ are:

$$\frac{\partial}{\partial \tau}\left(\frac{\partial V_o}{\partial x'} - \frac{\partial U_o}{\partial y'}\right) + \gamma \left[\frac{\partial U_o}{\partial x'}\frac{\partial V_1}{\partial x'} + \frac{\partial U_1}{\partial x'}\frac{\partial V_o}{\partial x'} + U_o \frac{\partial^2 V_1}{\partial x'^2} + U_1 \frac{\partial^2 V_o}{\partial x'^2}\right.$$

$$\left. + \ldots\right] - y \frac{\partial H_o}{\partial \tau} + V_1 = \varepsilon\, \Delta' \left(\frac{\partial V_1}{\partial x'} - \frac{\partial U_1}{\partial y'}\right) \qquad (12)$$

$$\frac{\partial U_1}{\partial x'} + \frac{\partial V_1}{\partial y'} = - \frac{\partial H_o}{\partial \tau}\,. \qquad (13)$$

Munk, Groves, and Carrier [7] have shown that the effect of the non-linear terms in [10] is quantitative and that these non-linear terms can be neglected as compared to the Coriolis term, V_o. The relationship of the non-linear terms to the Coriolis term in equation (12) is essentially the same as that in equation (10). This fact can be shown by considerations based on orders of magnitude. We choose a typical non-linear term in each equation, $\gamma \frac{\partial U_o}{\partial x'}\frac{\partial V_o}{\partial x'}$ in (10) and $\gamma \frac{\partial U_o}{\partial x'}\frac{\partial V_1}{\partial x'}$ in (12), and compare it to the Coriolis terms in that equation, V_o in (10) and V_1 in (12).

In the solution it is shown that U_o, V_o, U_1, V_1 and all their derivatives are of order unity in the interior of the

*Equation (10) with $\alpha = 0$ is the same as that of Munk, Groves, Carrier [7].

ocean. Near the boundary $x' = 0$, it is shown that $U_o = O(1)$, $V_o = O(\varepsilon^{-1/3})$, $U_1 = O(\varepsilon^{-1/3})$, $V_1 = O(\varepsilon^{-2/3})$ and $\frac{\partial}{\partial x'}$ has the effect of multiplying the magnitude of a term by $O(\varepsilon^{-1/3})$.

Based on these results the terms to be compared are given in the table below.

Interior	Near $x' = 0$
$V_o = O(1)$	$V_o = O(\varepsilon^{-1/3})$
$\gamma \frac{\partial U_o}{\partial x'} \frac{\partial V_o}{\partial x'} = \gamma O(1)$	$\gamma \frac{\partial U_o}{\partial x'} \frac{\partial V_o}{\partial x'} = \gamma O(\varepsilon^{-1})$
$V_1 = O(1)$	$V_1 = O(\varepsilon^{-2/3})$
$\gamma \frac{\partial U_o}{\partial x'} \frac{\partial V_1}{\partial x'} = \gamma O(1)$	$\gamma \frac{\partial U_o}{\partial x'} \frac{\partial V_1}{\partial x'} = \gamma O(\varepsilon^{-4/3})$

Thus, in the interior in each case we have $O(1)$ vs. $\gamma O(1)$. Near the boundary $x' = 0$, in each case we must compare $O(1)$ vs. $\gamma O(\varepsilon^{-2/3})$. Hence, the relationship of the non-linear terms to the Coriolis terms is essentially the same in the two sets of equations. It would seem therefore that, if the non-linear terms can be neglected in the steady equation (10), they can also be neglected in the first-order, non-steady equation (12).

Appendix 3. Hydrostatic Pressure Assumption.

The results in the main body of the report are based on the assumption that the vertical equation of motion can be approximated by the hydrostatic pressure equation. Although this approximation is probably sufficiently accurate for the problem under consideration, it may warrant a few further remarks.

Consider the steady, linearized problem. The equations of motion with a linearized Coriolis term are

$$-\beta y v' = -\frac{1}{\rho}\frac{\partial p}{\partial x} + A\Delta u' + \frac{1}{\rho}\frac{\partial}{\partial z}\left(A_3 \frac{\partial u'}{\partial z}\right) \quad (1)$$

$$\beta y u' = -\frac{1}{\rho}\frac{\partial p}{\partial y} + A\Delta v' + \frac{1}{\rho}\frac{\partial}{\partial z}\left(A_3 \frac{\partial v'}{\partial z}\right) \quad (2)$$

and the continuity equation is

$$\frac{\partial(\rho u')}{\partial x} + \frac{\partial(\rho v')}{\partial y} + \frac{\partial(\rho w')}{\partial z} = 0 \quad (3)$$

Equations (1) and (2) can be multiplied by the density to yield

$$-\beta y v'\rho = -\frac{\partial p}{\partial x} + A\Delta(\rho u') + \frac{\partial}{\partial z}\left(A_3 \frac{\partial u'}{\partial z}\right)$$

$$\beta y u'\rho = -\frac{\partial p}{\partial y} + A\Delta(\rho v') + \frac{\partial}{\partial z}\left(A_3 \frac{\partial v'}{\partial z}\right)$$

where we have written $A\Delta(\rho u')$ for $A\rho\Delta u'$ and $A\Delta(\rho v')$ for $A\rho\Delta v'$. This approximation is certainly permissible since these terms represent, in the first instance, only very rough approximations to the true state of affairs in turbulent motion.

If we integrate (1)-(3) from a depth $z = -h(x,y,t)$ where the motion is assumed negligible to the free surface $z = \eta(x,y,t)$, then

$$-\beta_y \bar{V} = -\int_{-h}^{\eta} \frac{\partial p}{\partial x} dz + A\Delta\bar{U} + A_3 \frac{\partial u'}{\partial z}\Big|_{-h}^{\eta} \qquad (4)$$

$$+\beta_y \bar{U} = -\int_{-h}^{\eta} \frac{\partial p}{\partial y} dz + A\Delta\bar{V} + A_3 \frac{\partial v'}{\partial z}\Big|_{-h}^{\eta} \qquad (5)$$

$$\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y} = 0 \qquad (6)$$

where the non-linear terms resulting from the interchange of derivatives and integrals in the viscous terms have been neglected.

The terms $A_3 \frac{\partial u'}{\partial z}\Big|_{-h}^{\eta} = \tau_x$ and $A_3 \frac{\partial v'}{\partial z}\Big|_{-h}^{\eta} = \tau_y$

provide the wind-stress components at the free surface (see sec. 3 of report). The depth $z = -h$ has been chosen as that depth where the velocities are negligible so that the contributions of the above terms at the lower limit are negligible. When the $\frac{\partial w'}{\partial z}$ term in the continuity equation is integrated, it provides a contribution involving a time-derivative, viz., $\frac{\partial(\eta\rho)}{\partial t}$, so that it vanishes in the present problem.

The pressure terms are

$$\int_{-h}^{\eta} \frac{\partial p}{\partial x} dz = \frac{\partial}{\partial x}\int_{-h}^{\eta} p\, dz - \frac{\partial \eta}{\partial x} p_{\eta} - \frac{\partial h}{\partial x} p_{-h}$$

$$\int_{-h}^{\eta} \frac{\partial p}{\partial y} dz = \frac{\partial}{\partial y}\int_{-h}^{\eta} p\, dz - \frac{\partial \eta}{\partial y} p_{\eta} - \frac{\partial h}{\partial y} p_{-h}$$

where p_η is p evaluated at $z = \eta$, and p_{-h} is p evaluated at $z = -h$.

If the free surface be considered a surface of zero pressure, then $p_\eta = 0$.

Defining

$$P = \int_{-h}^{\eta} p \, dz$$

we have for equation (4) and (5)

$$-\beta_y \bar{V} = - \frac{\partial P}{\partial x} + \frac{\partial h}{\partial x} p_{-h} + A\Delta\bar{U} + \tau_x \quad (7)$$

$$\beta_y \bar{U} = - \frac{\partial P}{\partial y} + \frac{\partial h}{\partial y} p_{-h} + A\Delta\bar{V} + \tau_y \quad (8)$$

A stream function ψ can be defined by $\bar{U} = - \frac{\partial\psi}{\partial y}$, $\bar{V} = + \frac{\partial\psi}{\partial x}$ so that (6) is satisfied identically. Taking the derivative of (7) with respect to y and (8) with respect to x and subtracting, we obtain

$$A \, \Delta\Delta \, \psi - \beta\psi_x = \frac{\partial h}{\partial x} \frac{\partial p_{-h}}{\partial y} - \frac{\partial h}{\partial y} \frac{\partial p_{-h}}{\partial x} + \frac{\partial \tau_x}{\partial y} - \frac{\partial \tau_y}{\partial x} \quad (9)$$

Since $z = -h$ is the depth where the velocities are negligible, the third equation of motion below this depth reduces to the hydrostatic pressure equation, $- \frac{\partial p}{\partial z} = g\rho$, if ρ is constant along $z = -h(x,y,t)$. Then $\frac{\partial p_{-h}}{\partial y} = g\rho\frac{\partial h}{\partial y}$, $\frac{\partial p_{-h}}{\partial x} = g\rho\frac{\partial h}{\partial x}$. With these results substituted into (9), we have

$$A\Delta\Delta\psi - \beta\psi_x = \frac{\partial \tau_x}{\partial y} - \frac{\partial \tau_y}{\partial x} \quad (10)$$

If boundary conditions are imposed and if τ_y and τ_x are specified, the problem defined by (10) can be solved (see Appendix 5). Thus for the analysis of the steady state problem, the only necessary assumption concerning the pressure and the density is that the density be constant along the surface below which the velocities are negligible.

If the height $z = -h$ is approximated by a constant, then the derivatives of the pressure terms in (9) vanish and no assumption need be made concerning the density along the surface $z = -h$.

Appendix 4(a). Derivation of Relationship Between T and η.

With the density distribution given by

$$\rho = \rho_o \qquad \eta \geq z > T$$

$$\rho = \rho_o [1 + c(T-z)] \qquad T \geq z \geq T - d$$

$$\rho = \rho_{-h} = \rho_o [1 + cd] \qquad T - d > z$$

we can find a relationship between T and η by considering the conditions

$$\frac{1}{\rho}\frac{\partial p}{\partial x}\bigg|_{z=-h} = 0 , \quad \frac{1}{\rho}\frac{\partial p}{\partial y}\bigg|_{z=-h} = 0.$$

The hydrostatic pressure equation is

$$p = g\int_z^\eta \rho d\zeta$$

$$\therefore \qquad \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{g}{\rho}\int_z^\eta \frac{\partial \rho}{\partial x} d\zeta + \frac{g}{\rho}\frac{\partial \eta}{\partial x}\rho_o$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x}\bigg|_{z=-h} = \frac{g}{\rho_{-h}}\int_{-h}^\eta \frac{\partial \rho}{\partial x} d\zeta + \frac{g}{\rho_{-h}}\frac{\partial \eta}{\partial x}\rho_o = 0$$

But

$$\frac{\partial \rho}{\partial x} = 0 \qquad \eta \geq z > T$$

$$= \rho_o c\frac{\partial T}{\partial x} \qquad T \geq z \geq T-d$$

$$= 0 \qquad T-d > z$$

Hence,

$$\frac{1}{\rho}\frac{\partial p}{\partial x}\bigg|_{z=-h} = \frac{g}{\rho_{-h}}\int_{T-d}^{T}\rho_o c\,\frac{\partial T}{\partial x}\,d\zeta + \frac{g\rho_o}{\rho_{-h}}\frac{\partial \eta}{\partial x} = 0$$

or

$$cd\,\frac{\partial T}{\partial x} + \frac{\partial \eta}{\partial x} = 0$$

$$\frac{\partial T}{\partial x} = -\frac{1}{cd}\frac{\partial \eta}{\partial x} = -\frac{\rho_o}{\Delta\rho}\frac{\partial \eta}{\partial x} \qquad (1)$$

where $\Delta\rho = \rho_{-h} - \rho_o$.

Similarly,

$$\frac{\partial T}{\partial y} = -\frac{\rho_o}{\Delta\rho}\frac{\partial \eta}{\partial y} \qquad (2)$$

Integrating (1) and (2), we have finally

$$T = -\frac{\rho_o}{\Delta\rho}\eta - C$$

where $T = -C$ when $\eta = 0$.

Appendix 4(b). Derivation of Integrated Pressure Terms.

In order to compute the terms $\int_{-h}^{\eta}\frac{1}{\rho}\frac{\partial p}{\partial x}\,dz$, $\int_{-h}^{\eta}\frac{1}{\rho}\frac{\partial p}{\partial y}\,dz$, we must divide the region of integration into three separate parts, viz.,

$$\int_{-h}^{\eta} = \int_{-h}^{T-d} + \int_{T-d}^{T} + \int_{T}^{\eta}$$

$$\therefore \int_{-h}^{\eta}\frac{1}{\rho}\frac{\partial p}{\partial x}\,dz = g\int_{-h}^{T-d}\frac{1}{\rho}\left[\int_{z}^{\eta}\frac{\partial \rho}{\partial x}\,d\zeta\right]dz + g\int_{T-d}^{T}\frac{1}{\rho}\left[\int_{z}^{\eta}\frac{\partial \rho}{\partial x}\,d\zeta\right]dz$$

$$+\, g\int_{T}^{\eta}\frac{1}{\rho}\left[\int_{z}^{\eta}\frac{\partial \rho}{\partial x}\,d\zeta\right]dz + g\rho_o\frac{\partial \eta}{\partial x}\int_{-h}^{\eta}\frac{1}{\rho}\,dz \qquad (1)$$

Using the values of $\frac{\partial \rho}{\partial x}$ for the three layers listed in Appendix 4(a), we have

$$\int_z^{\eta} \frac{\partial \rho}{\partial x}\, d\zeta = \rho_o\, cd\, \frac{\partial T}{\partial x} \qquad z < T-d$$

$$= \rho_o c\, \frac{\partial T}{\partial x}\,(T-z) \qquad T-d \leq z \leq T$$

$$= 0 \qquad T < z$$

Then

$$g\int_{-h}^{T-d} \frac{1}{\rho}\Big[\int_z^{\eta} \frac{\partial \rho}{\partial x}\, d\zeta\Big]dz = g\,\frac{cd}{1+cd}\,\frac{\partial T}{\partial x}\,[T-d+h]$$

$$g\int_{T-d}^{T} \frac{1}{\rho}\Big[\int_z^{\eta} \frac{\partial \rho}{\partial x}\, d\zeta\Big]dz = gd\,\frac{\partial T}{\partial x} + g\,\frac{\partial T}{\partial x}\,\frac{1}{c}\,\log\,\frac{1}{cd+1}$$

$$g\int_{T}^{\eta} \frac{1}{\rho}\Big[\int_z^{\eta} \frac{\partial \rho}{\partial x}\, d\zeta\Big]dz = 0$$

$$g\rho_o\,\frac{\partial \eta}{\partial x}\int_{-h}^{\eta}\frac{1}{\rho}\,dz = g\,\frac{\partial \eta}{\partial x}\,\frac{1}{1+cd}\,[T-d+h] - g\,\frac{\partial \eta}{\partial x}\,\frac{1}{c}\,\log\Big(\frac{1}{cd+1}\Big)$$

$$+\, g\,\frac{\partial \eta}{\partial x}\,(\eta-T).$$

Let us put these values into (1) and at the same time use

$$\frac{\partial T}{\partial x} = -\,\frac{\rho-o}{\Delta\rho}\,\frac{\partial \eta}{\partial x} = -\,\frac{1}{cd}\,\frac{\partial \eta}{\partial x} \text{ and } T = -\,\frac{\rho-o}{\Delta\rho}\,\eta - C$$

$$\therefore \int_{-h}^{\eta} \frac{1}{\rho}\,\frac{\partial p}{\partial x}\,dz = -\,g\,\frac{\partial \eta}{\partial x}\,\frac{1}{1+cd}\,[T-d+h] + g\,\frac{\partial \eta}{\partial x}\,\frac{1}{1+cd}\,[T-d+h]$$

$$-\,\frac{g}{c}\,\frac{\partial \eta}{\partial x} - g\,\frac{\rho o}{\Delta\rho}\,\frac{1}{c}\,\frac{\partial \eta}{\partial x}\,\log\Big(\frac{1}{cd+1}\Big) - g\,\frac{1}{c}\,\frac{\partial \eta}{\partial x}\,\log\Big(\frac{1}{cd+1}\Big)$$

$$+ g\eta \frac{\partial \eta}{\partial x} + g(\frac{\rho_o}{\Delta\rho}\ \eta + D)\ \frac{\partial \eta}{\partial x}$$

$$= - \frac{g}{c} \frac{\partial \eta}{\partial x} + g \frac{\rho_{-h}}{\Delta\rho} \eta \frac{\partial \eta}{\partial x} + g\ C \frac{\partial \eta}{\partial x} - \frac{\rho_{-h}}{\Delta\rho} \frac{g}{c} \frac{\partial \eta}{\partial x} \log \frac{\rho_o}{\rho_{-h}} \qquad (2)$$

But,

$$\log \frac{\rho_o}{\rho_{-h}} = - \log \frac{\rho_{-h}}{\rho_o} = - \log (\frac{\Delta\rho + \rho_o}{\rho_o}) = - \log (\frac{\Delta\rho}{\rho_o} + 1)$$

Since the term $\frac{\Delta\rho}{\rho_o}$ is small we can write

$$\log(1 + \frac{\Delta\rho}{\rho_o}) \approx \frac{\Delta\rho}{\rho_o} - \frac{1}{2} (\frac{\Delta\rho}{\rho_o})^2$$

Hence (2) becomes

$$\int_{-h}^{\eta} \frac{1}{\rho} \frac{\partial p}{\partial x} dz = g\ (C + \frac{\rho_{-h}}{\Delta\rho}\ \eta + \frac{d}{2})\ \frac{\partial \eta}{\partial x}$$

Similarly,

$$\int_{-h}^{\eta} \frac{1}{\rho} \frac{\partial p}{\partial y} dz = g\ (C + \frac{\rho_{-h}}{\Delta\rho}\ \eta + \frac{d}{2})\ \frac{\partial \eta}{\partial y}$$

$$= g\ (D + \frac{\rho_{-h}}{\Delta\rho}\ \eta\ )\ \frac{\partial \eta}{\partial y}$$

where $D \equiv C + \frac{d}{2}$.

Appendix 5. An Example of Boundary Layer Technique.*

In this section we shall discuss the application of the boundary layer technique to the solution of the problem defined by the equation

$$\varepsilon \Delta\Delta\psi - \psi_x = (1+\alpha \sin\tau) \sin nsy \qquad (1)$$

and the boundary conditions

$$\psi = \psi_x = 0 \text{ on } x = 0, r \qquad (2)$$

$$\psi = \psi_{yy} = 0 \text{ on } y = 0, 1.$$

The nature of the boundary layer problem is characterized by three features: (1) the problem is non-dimensionalized so that the size of the domain has lengths of order unity; (2) the coefficient of the most highly differentiated term is small compared to unity; (3) the remaining terms have coefficients of order unity. The problem to be considered here has already been put into a suitable non-dimensional form.

If ψ were everywhere a smooth** function of its arguments and of order unity, then it should be possible to determine a good approximation to the solution by neglecting the term with coefficient $\varepsilon (\varepsilon << 1)$ and by considering the remaining equation

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* For an interesting account of boundary layer technique, including the treatment of non-linear problems, the reader is referred to [8] .

** By "smooth" we mean that ψ has no large derivatives, i.e., ψ, ψ_x, ψ_{xxxx}, etc. are all of the same order of magnitude.

$$\psi_x = -(1+\alpha \sin \tau) \sin nsy \quad (3)$$

Thus, a possible solution is

$$\psi_i = (1+\alpha \sin \tau) \sin nsy \; [-x + C_{11}(y,\tau)]. \quad (4)$$

We are now faced with a dilemna, however. ψ as given in (4) provides one arbitrary function of y and τ to satisfy the four conditions on the boundaries x = 0, x = r. If our assumption that ψ is everywhere a smooth function is correct, then we are at a loss to find a complete answer to the problem. For if ψ and its derivatives have the same order of magnitude everywhere, the only possible solution is of the form $\psi_i + O(\varepsilon)$ and it is not possible to satisfy all boundary conditions.

It is obvious, therefore, that ψ cannot be smooth everywhere. In particular, in order for the full solution to be different from $\psi_i + O(\varepsilon)$, at least one of the terms, ψ_{xxxx}, ψ_{xxyy}, or ψ_{yyyy} must be of order ε^{-1} in some part of the domain under consideration so that the approximation of neglecting terms of order ε will not reduce the order of the differential equation. If ψ is smooth away from the boundaries and if derivatives with respect to x are large, so that ψ_{xxxx} is of order ε^{-1}, near x = 0,r, then the problem is one of the boundary layer type. We shall proceed formally on the assumption that this is true, realizing that if it is not the case, we shall be led to a contradiction.

The solution may now be written as the sum of two parts- ψ_i given by (4)(the "interior solution"), ψ_b being sensibly large only near the boundary and negligibly small in the interior, (the "boundary layer contribution"). We must now try to determine the boundary layer contribution.

The nature of the total solution itself is the guiding factor in the investigation. We have supposed that near the boundaries $x = 0, r$, ψ_b has large derivatives with respect to x while ψ_i is everywhere smooth and of order unity. Thus, if we write our solution in two parts, i.e., $\psi_i + \psi_b$, the differential equation can be written in the form

$$\varepsilon\Delta\Delta\psi_i + \underline{\varepsilon\Delta\Delta\,\psi_b} - \underline{\underline{\psi_{ix}}} - \underline{\psi_{bx}} = \underline{\underline{(1+\alpha \sin \tau)\sin nsy}}.$$

Now the term $\varepsilon\Delta\Delta\psi_i$ is of order ε, the terms underlined twice are of order unity and the order of magnitude of the terms underlined once is as yet undetermined. Since the terms in ψ_b are to have derivatives with respect to x which are (assumed) large, we have $\psi_{bx} \gg 1$. Hence at least one of the terms of $\varepsilon\Delta\Delta\psi_b$ must be as large as ψ_{bx} in order to balance this term. The equation will then be satisfied approximately if we write

$$-\psi_{ix} = (1+\alpha \sin \tau) \sin nsy$$

and

$$\varepsilon\Delta\Delta\psi_b - \psi_{bx} = 0 \qquad (5)$$

We must now integrate these equations and then add the two solutions ψ_i and ψ_b to form the complete solution ψ.

The solution to the first of the two equations is given by (4). Since the complete solution will only be approximate, in that terms of order ε have already been neglected, ψ_b need only be determined approximately.

It is suggested by the above considerations that we find a formal method for writing our equation so that the magnitudes of the terms are expressed by the coefficients and that the derivatives, etc., be of order unity. We can do this by stretching the x coordinate near the boundary i.e., by defining a new x coordinate so that a particular distance in x becomes a much larger distance in the new coordinate.

Formally, we operate as follows. Let x be replaced by the coordinate ξ such that

$$x = \varepsilon^{n}\xi$$

where n is to be determined. Then the equation (5) becomes

$$\varepsilon^{-4n+1}\,\psi_{b} \qquad + 2\varepsilon^{-2n+1}\,\psi_{b\xi\,yy} + \varepsilon\psi_{byyyy} - \varepsilon^{-n}\,\psi_{b\xi} = 0.$$

In choosing n we note that it must be positive if the x coordinate is to be stretched. Thus of the terms which originally had coefficient ε, $\varepsilon^{-4n+1}\,\psi_{b\xi\xi\xi\xi}$ is the largest since it has the largest coefficient (n.b. $\psi_{b\xi}$, $\psi_{b\xi\xi\xi\xi}$, $\psi_{b\xi\xi yy}$, ψ_{byyyy} are the same order of magnitude). This term is matched with $\varepsilon^{-n}\,\psi_{b\xi}$, the remaining large term in the differential equation, and by equating the coefficients of the above two terms, we have n = 1/3.

Thus we get

$$\varepsilon^{-1/3}\,\psi_{b\xi\xi\xi\xi} + 2\varepsilon^{1/3}\,\psi_{b\xi\xi yy} + \varepsilon\,\psi_{byyyy} - \varepsilon^{-1/3}\,\psi_{b\xi} = 0$$

or

$$\psi_{b\xi\xi\xi\xi} - \psi_{b\xi} = O(\varepsilon^{2/3}).$$

Now if ψ_b be expanded into an asymptotic series and if we keep only the first term in the series (for all practical purposes, this amounts to neglecting the $O(\varepsilon^{2/3})$ terms), we have

$$\psi_{b\xi\xi\xi\xi} - \psi_{b\xi} = 0 \qquad (6)$$

The solution to (6) is

$$\psi_b = C_{12}(y,\tau) + C_{22}(y,\tau)e^{\xi} + C_{32}(y,\tau)e^{\xi e^{\frac{2\pi i}{3}}} + C_{42}(y,\tau)e^{\xi e^{\frac{4\pi i}{3}}}$$

We have specified that this solution is to become negligibly small as the distance from the boundary increases. Thus letting $\xi \to \infty$, we note that it is necessary that $C_{12} = C_{22} = 0$ since neither C_{12} nor ε^{ξ} tends to zero. Hence, for the region near $x = 0$, we have

$$\psi_b = C_{32}(y,\tau)e^{\xi e^{\frac{2\pi i}{3}}} + C_{42}(y,\tau)e^{\xi e^{\frac{4\pi i}{3}}}$$

or, changing our coordinates back to x by means of $x = \varepsilon^{1/3}\xi$,

$$\psi_b = C_{32}(y,\tau)e^{x\varepsilon^{-1/3}\, e^{\frac{2\pi i}{3}}} + C_{42}(y,\tau)e^{x\varepsilon^{-1/3}\, e^{\frac{4\pi i}{3}}}$$

For the boundary near $x = r$, we now define ξ by

$$(x-r) = \varepsilon^n \xi$$

and specify that the solution vanish as $\xi \to -\infty$, i.e., as the distance into the interior part of the ocean increases. By a similar analysis, we find that near $x = r$,

$$\psi_b = C_{13}(y,\tau) + C_{23}(y,\tau)e^{\xi} + C_{33}(y,\tau)e^{\xi e^{\frac{2\pi i}{3}}}$$

$$+ C_{43}(y,\tau)e^{\xi e^{\frac{4\pi i}{3}}}$$

In order for ψ_b to tend to zero as $\xi \to -\infty$, it is necessary that $C_{13} = C_{33} = C_{43} = 0$. Hence

$$\psi_b = C_{23}(y,\tau)e^{\xi} = C_{23}(y,\tau)e^{(x-r)\varepsilon^{-1/3}}$$

The total boundary layer solution can be written

$$\psi_b = C_2(y,\tau)e^{(x-r)\varepsilon^{-1/3}} + C_3(y,\tau)e^{x\varepsilon^{-1/3}\, e^{\frac{2\pi i}{3}}}$$

$$+ C_4(y,\tau)e^{x\varepsilon^{-1/3}\, e^{\frac{4\pi i}{3}}} \qquad (7)$$

The solution throughout the domain consists of (4) and (7) or

$$\psi = \psi_i + \psi_b = (1 + \alpha \sin\tau)\sin nsy \Big\{ [- x + C_1(y,\tau)]$$

$$+ C_2(y,\tau)e^{(x-r)\varepsilon^{-1/3}} + C_3(y,\tau)e^{x\varepsilon^{-1/3} e^{\frac{2\pi i}{3}}}$$

$$+ C_4(y,\tau)e^{x\varepsilon^{-1/3} e^{\frac{4\pi i}{3}}} \Big\} \qquad (8)$$

An application of the boundary conditions, $\psi = \psi_x = 0$ on $x = 0, r$, yields

$$\psi = (1+\alpha \sin \tau) \sin nsy \Big\{ \underbrace{-x+r-\varepsilon^{1/3}}_{1} + \underbrace{\varepsilon^{1/3} e^{(x-r)\varepsilon^{-1/3}}}_{2}$$

$$+ \underbrace{\left[(\varepsilon^{1/3}-r)\cos\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right) + \left(\sqrt{3}\,\varepsilon^{1/3} - \frac{r}{\sqrt{3}}\right)\sin\left(\frac{x\sqrt{3}\,\varepsilon^{-1/3}}{2}\right)\right] e^{-\frac{x\varepsilon^{-1/3}}{2}}}_{3} \Big\}$$

The term 1 is valid throughout the ocean. Near $x = 0$, 3 becomes as important as 1 and gets negligibly small as x increases. Near $x = r$, 2 and 1 together form the solution but 2 tends to zero as x decreases.

Perhaps a few remarks should be made as to the specific choice of sin nsy for the total y dependence of the solution. The particular choice of sin nsy satisfies the boundary conditions $\psi = \psi_{yy} = 0$ on $y = 0$, $y = 1$, and is supported by the specified wind distribution. Thus we were not forced to resort to a boundary layer analysis to satisfy the four boundary conditions. Of course, such a simple choice is not always possible, and one might have to resort to methods for refining the interior solution in other problems in order to satisfy the necessary boundary conditions.

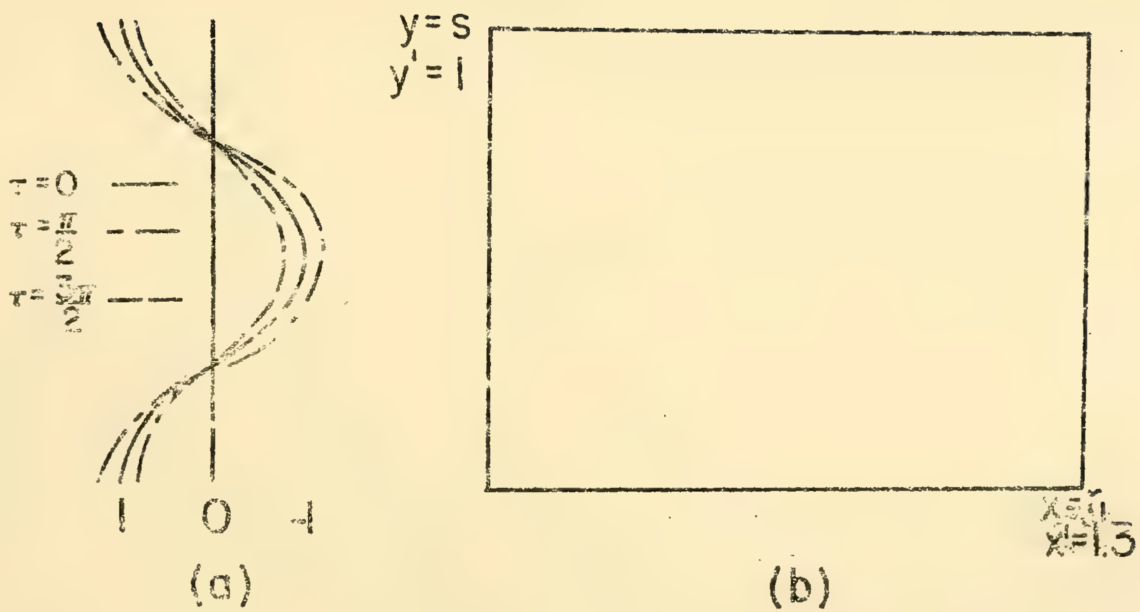

Fig. 1(a) The wind stress distribution vs. y coordinate.

(b) The rectangular ocean with dimensions in both the dimensional and the non-dimensional coordinate system.

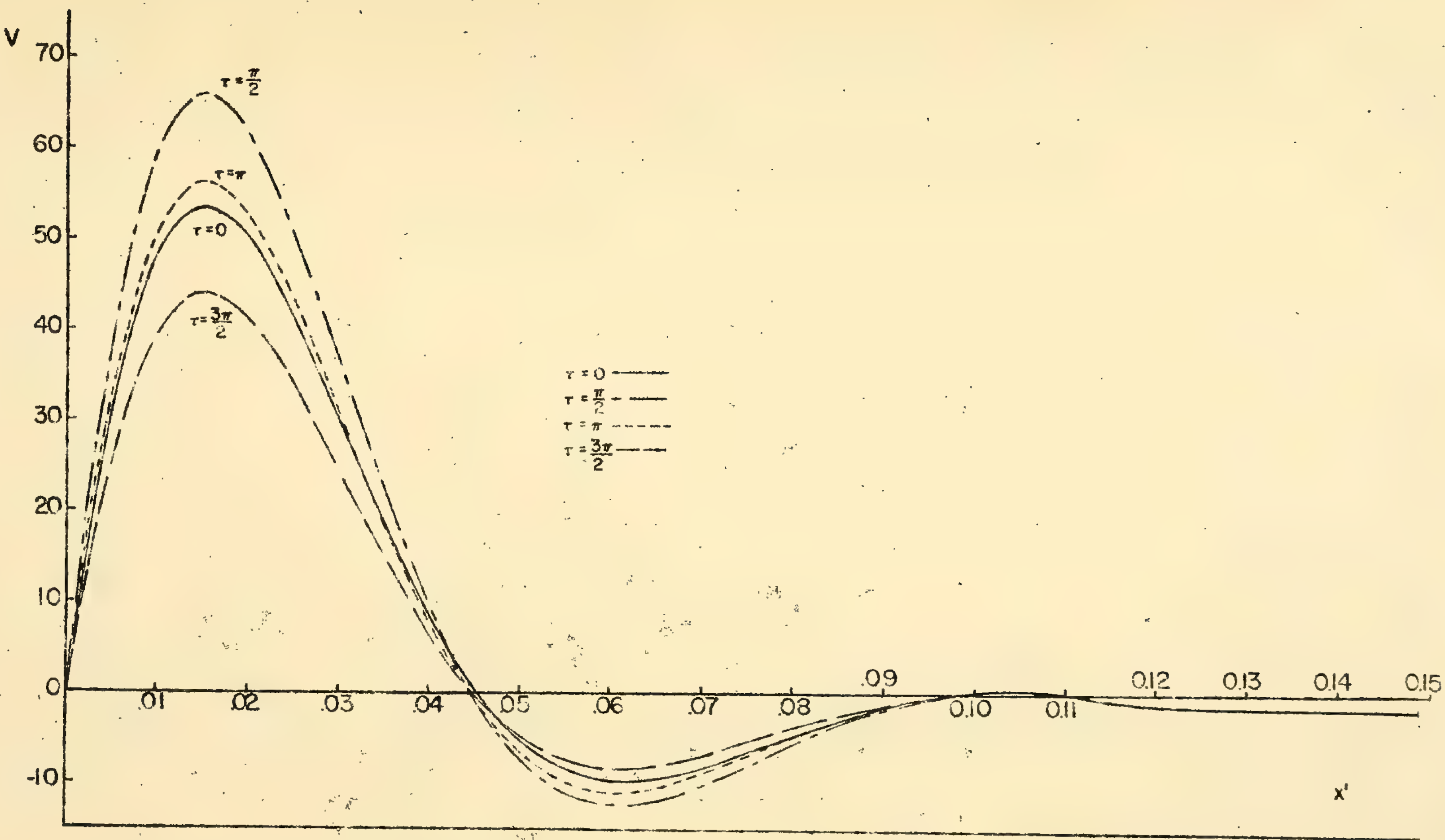

Fig. 2 The non-dimensional mass transport function V near x' = 0 (western boundary) showing gulf stream and counter current.

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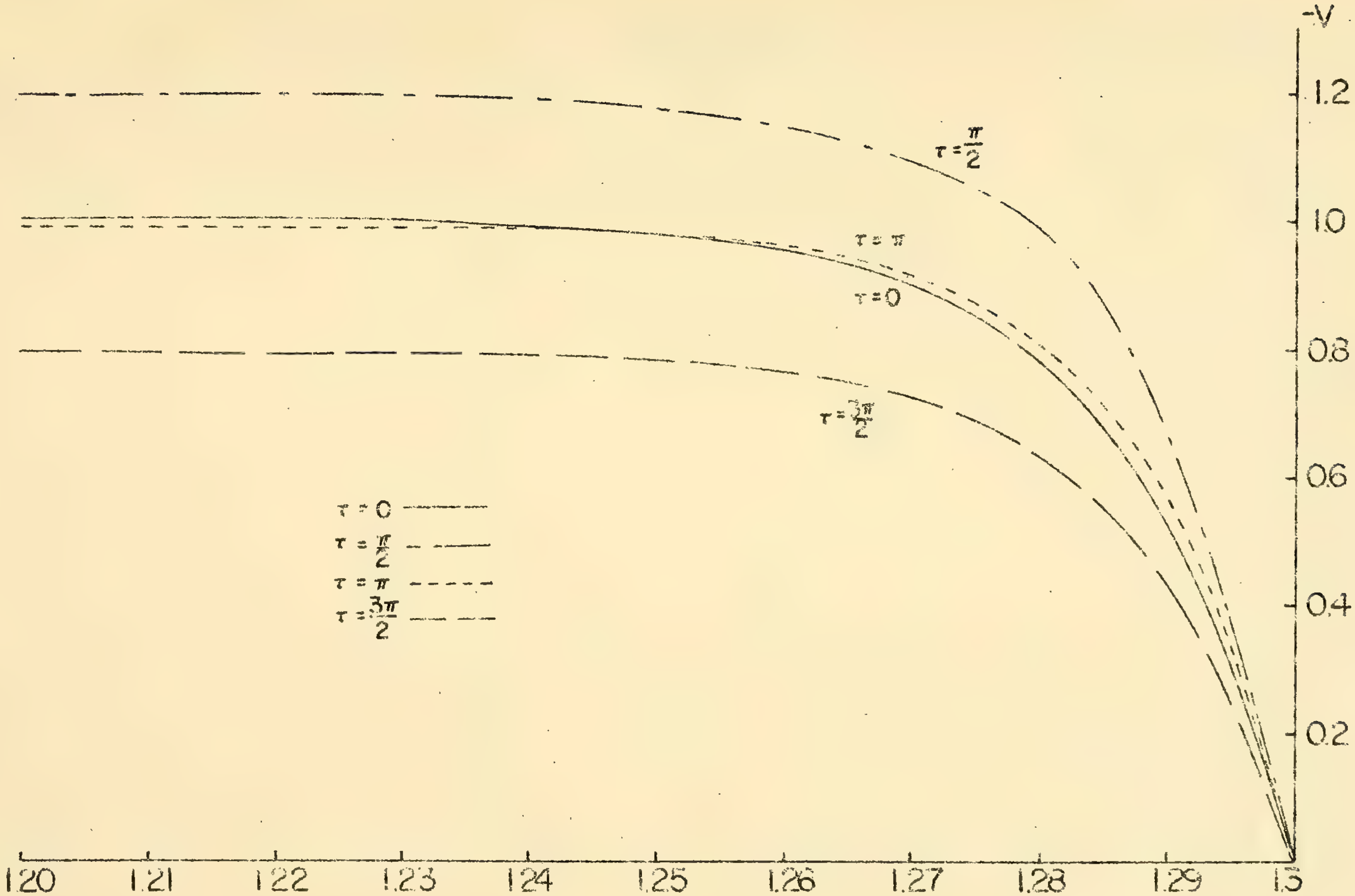
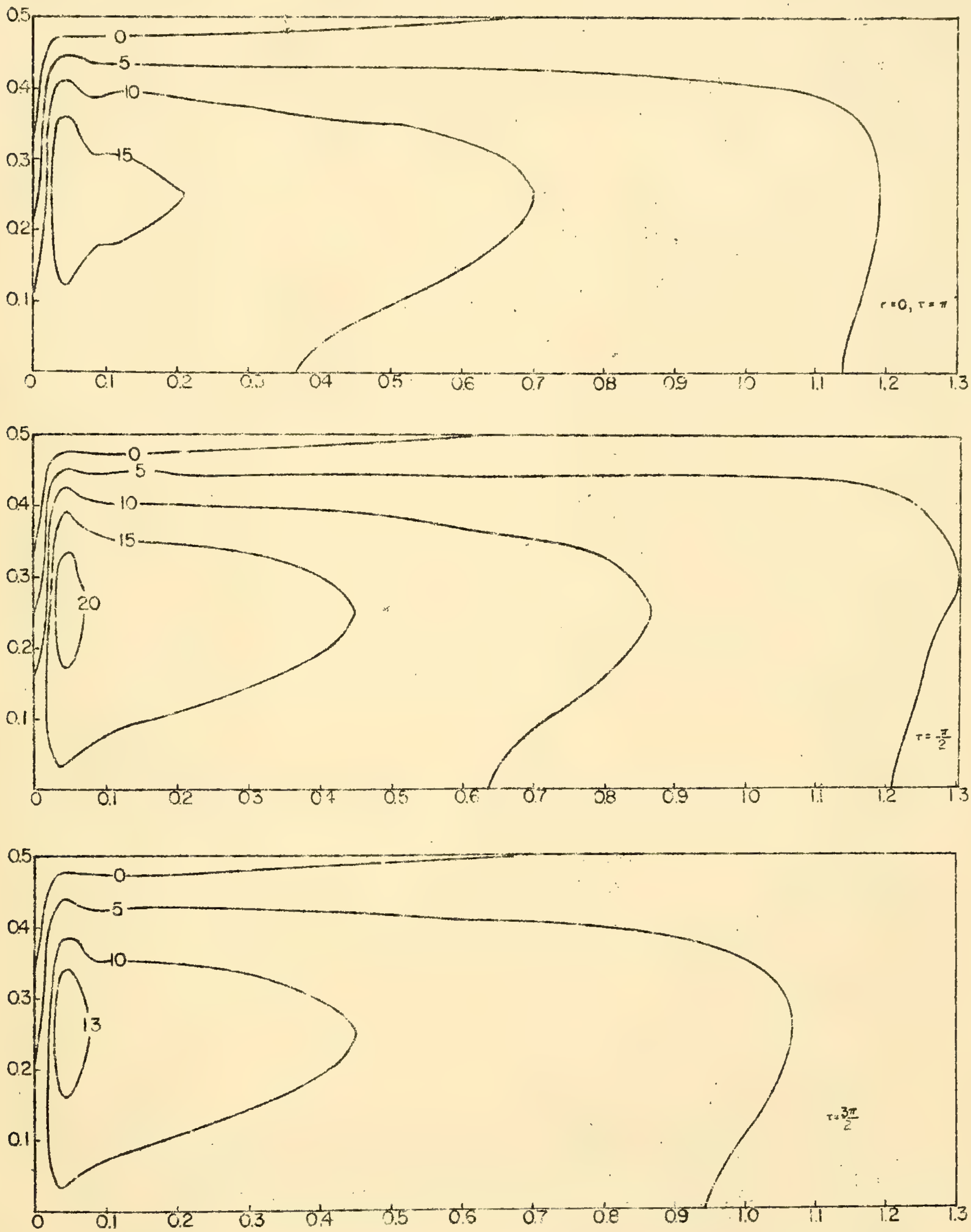


Fig. 3. Non-dimensional mass transport function V near x' = r (eastern boundary).



Figs. 4, 5, 6 Height in cms. of free surface for southern half of rectangular ocean at different "times" $\tau = 0$, $\tau = \frac{\pi}{2}$, $\tau = \pi$, $\tau = \frac{3\pi}{2}$. The correction of the perturbation terms is negligible.

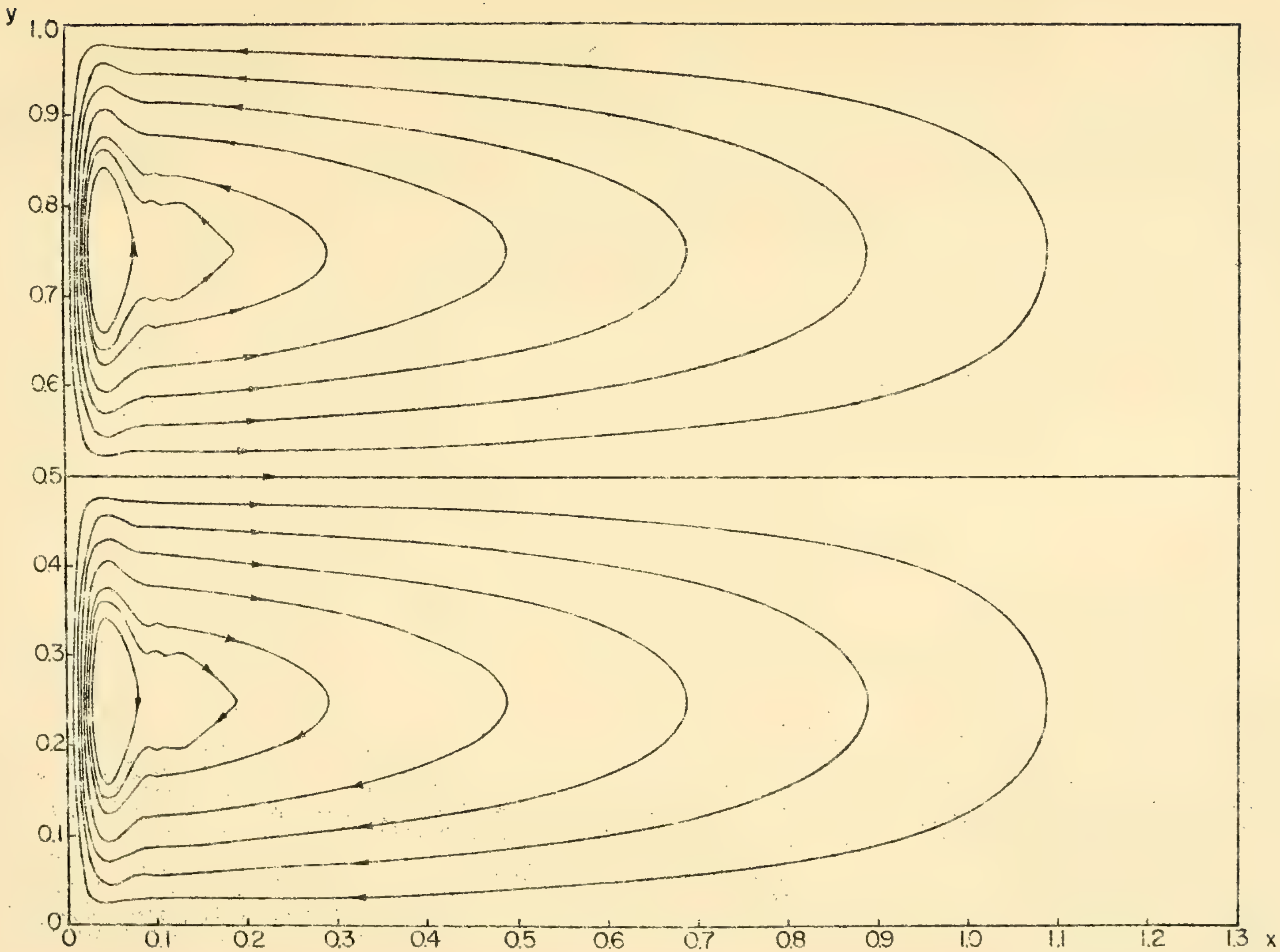


Fig. 7 Streamlines for the steady problem for both one-layer and two-layer oceans.

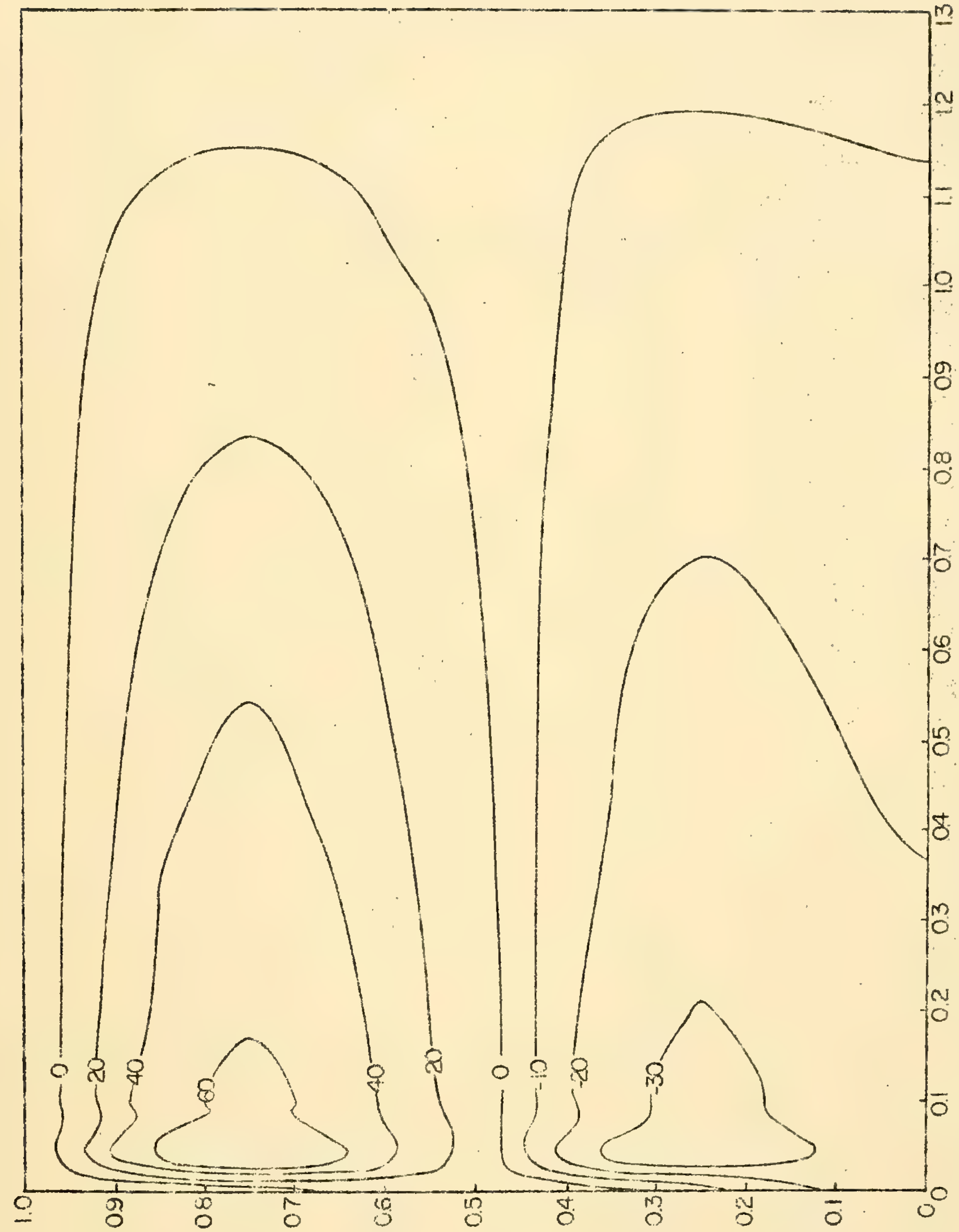


Fig 8. Contour lines of the thermocline for the steady problem. Numbers denote meters above equilibrium position.

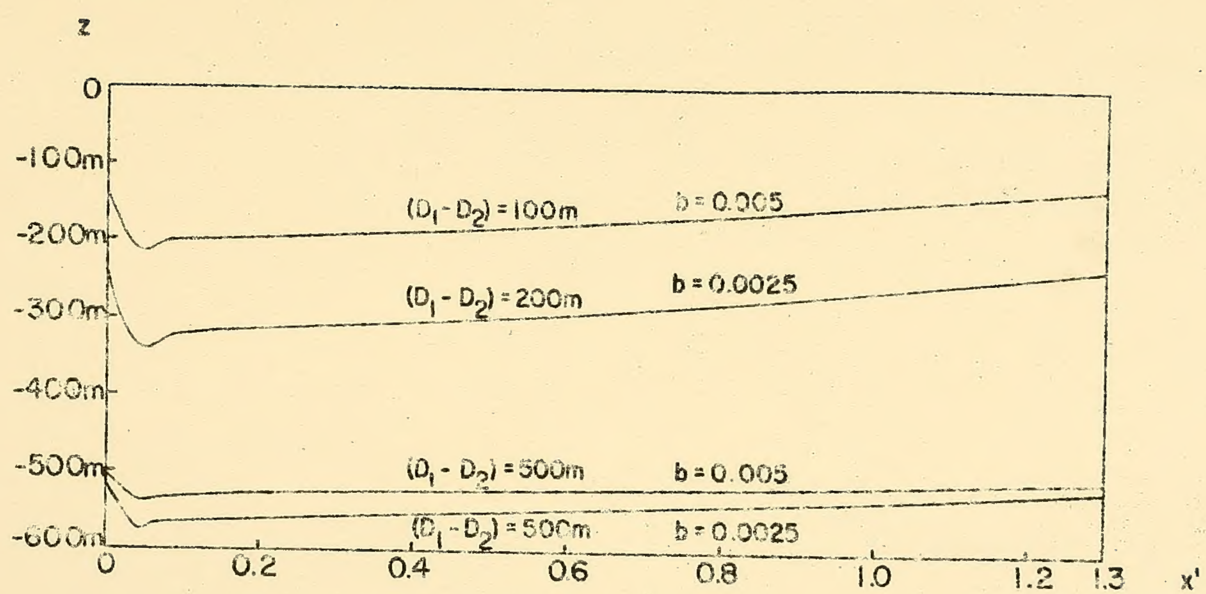


FIG. 9 The deviation of the thermocline from a constant depth for four combinations of θ = const x $(D_1 - D_2)$ and b at a vertical cross section extending across the entire ocean at $y' = 0.25$.